ing five properties:

1. $(S, +, 0)$ is a *monoid*, that is, it is *closed* under $+$ [i.e., $a + b \in S$ for all a and b in S], $+$ is *associative* [i.e., $a + (b + c) = (a + b) + c$ for all a, b, c in S], and 0 is an *identity* [i.e., $a + 0 = 0 + a = a$ for all a in S]. Likewise, $(S, \cdot, 1)$ is a monoid. We also assume 0 is an *annihilator*, i.e., $a \cdot 0 = 0 \cdot a = 0$.
2. $+$ is *commutative*, i.e., $a + b = b + a$, and *idempotent*, i.e., $a + a = a$.
3. $\cdot$ *distributes* over $+$, that is, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$.
4. If $a_1, a_2, \ldots, a_i, \ldots$ is a countable sequence of elements in S, then $a_1 + a_2 + \cdots + a_i + \cdots$ exists and is unique. Moreover, associativity, commutativity, and idempotence apply to infinite as well as finite sums.
5. $\cdot$ must distribute over countably infinite sums as well as finite ones (this does not follow from property 3). Thus (4) and (5) imply

$$\left(\sum_i a_i\right) \cdot \left(\sum_j b_j\right) = \sum_{i,j} a_i \cdot b_j = \sum_i \left(\sum_j (a_i \cdot b_j)\right).$$

Example 5.9. The following three systems are closed semirings.

1. Let $S_1 = (\{0, 1\}, +, \cdot, 0, 1)$ with addition and multiplication tables as follows:

$+$	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

Then properties 1–3 are easy to verify. For properties 4 and 5, note that a countable sum is 0 if and only if all terms are 0.

2. Let $S_2 = (R, \text{MIN}, +, +\infty, 0)$, where R is the set of nonnegative reals including $+\infty$. It is easy to verify that $+\infty$ is the identity under MIN and 0 the identity under $+$.
3. Let Σ be a finite alphabet (i.e., a set of symbols), and let $S_3 = (F_\Sigma, \cup, \cdot, \emptyset, \{\epsilon\})$, where F_Σ is the family of sets of finite-length strings of symbols from Σ, including ϵ, the empty string (i.e., the string of length 0). Here the first operator is set union and $\cdot$ denotes set concatenation.† The $\cup$ identity is $\emptyset$ and the $\cdot$ identity is $\{\epsilon\}$. The reader may verify properties 1–3. For properties 4 and 5, we must observe that countable unions behave as they should if we define $x \in (A_1 \cup A_2 \cup \cdots)$ if and only if $x \in A_i$ for some i. □

† The *concatenation* of sets A and B, denoted $A \cdot B$, is the set $\{x | x = yz, y \in A \text{ and } z \in B\}$.

A unary operation, denoted $*$ and called *closure*, is central to our analysis of closed semirings. If $(S, +, \cdot, 0, 1)$ is a closed semiring, and $a \in S$, then we define a^* to be $\sum_{i=0}^{\infty} a^i$, where $a^0 = 1$ and $a^i = a \cdot a^{i-1}$. That is, a^* is the infinite sum $1 + a + a \cdot a + a \cdot a \cdot a + \cdots$. Note that property 4 of the definition of a closed semiring assures that $a^* \in S$. Properties 4 and 5 imply $a^* = 1 + a \cdot a^*$. Note that $0^* = 1^* = 1$.

Example 5.10. Let us refer to the semirings S_1, S_2, and S_3 of Example 5.9. For S_1, $a^* = 1$ for $a = 0$ or 1. For S_2, $a^* = 0$ for all a in R. For S_3, $A^* = \{\epsilon\} \cup \{x_1 x_2 \cdots x_k | k \geq 1 \text{ and } x_i \in A \text{ for } 1 \leq i \leq k\}$ for all $A \in F_\Sigma$. For example, $\{a, b\}^* = \{\epsilon, a, b, aa, ab, ba, bb, aaa, \ldots\}$, that is, all strings of a's and b's including the empty string. In fact, $F_\Sigma = \mathscr{P}(\Sigma^*)$, where $\mathscr{P}(X)$ denotes the power set of set X. □

Now, let us suppose we have a directed graph $G = (V, E)$ in which each edge is labeled by an element of some closed semiring $(S, +, \cdot, 0, 1)$.† We define the *label of a path* to be the product $(\cdot)$ of the labels of the edges in the path, taken in order. As a special case, the label of the path of zero length is 1 (the $\cdot$ identity of the semiring). For each pair of vertices (v, w), we define $c(v, w)$ to be the sum of the labels of all the paths between v and w. We shall refer to $c(v, w)$ as the *cost* of going from v to w. By convention, the sum over an empty set of paths is 0 (the $+$ identity of the semiring). Note that if G has cycles, there may be an infinity of paths between v and w, but the axioms of a closed semiring assure us that $c(v, w)$ will be well defined.

Example 5.11. Consider the directed graph in Fig. 5.17, in which each edge has been labeled by an element from the semiring S_1 of Example 5.9. The label of the path v, w, x is $1 \cdot 1 = 1$. The simple cycle from w to w has label $1 \cdot 0 = 0$. In fact, every path of length greater than zero from w to w has label 0. However, the path of zero length from w to w has cost 1. Consequently, $c(w, w) = 1$. □

We now give an algorithm to compute $c(v, w)$ for all pairs of vertices v and w. The basic unit-time steps of the algorithm are the operations $+$, $\cdot$, and

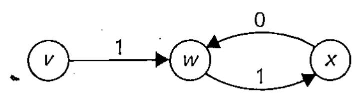

Fig. 5.17 A labeled directed graph.

† The reader should not miss the analogy between such a situation and a nondeterministic finite automaton (see Hopcroft and Ullman [1969] or Aho and Ullman [1972]), as we shall discuss in Section 9.1. There, the vertices are states and the edge labels are symbols from some finite alphabet.