Suppose $g(T)$ were nameable in the system. Then by complementation $\overline{g(T)}$ would also be nameable. Then by the diagonal lemma, there would be a Gödel sentence X for $\overline{g(T)}$, and we would have $X \in T$ if and only if $g(X) \in \overline{g(T)}$, but $g(X) \in \overline{g(T)}$ if and only if $X \notin T$, and so we would have the absurdity that X is in T if and only if X is not in T. Therefore $g(T)$ is not one of the nameable sets.

It should be of interest to note that this result of Tarski provides an alternative proof of Gödel's theorem: Suppose (S) is rich, complemented, and diagonalizable. By Tarski's result, the set $g(T)$ is not nameable in the system, but by richness, the set $g(P)$ is nameable in the system. Therefore P, T must be different sets. Since $P \subseteq T$, then T must contain a sentence not in P, which alternatively proves Theorem A.

3. AN ABSTRACT RECURSION THEOREM

Consider now a denumerable set of any objects whatsoever arranged in an infinite sequence $E_1, E_2, \ldots, E_n, \ldots$. Let Σ be a collection of functions from the positive integers to the positive integers. Σ is said to be closed under composition if for any functions f, g in Σ, there is a function h in Σ such that for all (positive integers) x, $h(x) = f(g(x))$. We shall also consider a function $F(x,y)$ from the set of ordered pairs of positive integers to the positive integers.

Theorem 2

Suppose the following three conditions hold:

C_1: Σ is closed under composition.

C_2: The function $F(x,x)$ is in Σ.

C_3: For any $f \in \Sigma$, there is a positive integer a such that for all x, $E_{F(a,x)} = E_{f(x)}$.

Conclusion: For any $f \in \Sigma$ there is at least one positive integer i such that $E_i = E_{f(i)}$.

Proof

Take any function f in Σ. By C_2, the function $F(x,x)$ is in Σ, and so by C_1, the function $f[F(x,x)]$ is in Σ. Then by C_3, there is a number a such that for all x, $E_{F(a,x)} = E_{f(F(x,x))}$. Taking a for x, it follows that $E_{F(a,a)} = E_{f(F(a,a))}$. We take $F(a,a)$ for i, and so $E_i = E_{f(i)}$.

Discussion

In applications to Recursion Theory, Σ is the class of recursive functions of one argument. This class is closed under composition, so C_1 holds. For one form of the recursion theorem, we take E_i to be the i^{th} partial recursive function of one argument (in a standard enumeration). By a result known as the iteration theorem there is a recursive function $F(x,y)$ satisfying C_3, and condition C_2 is automatic (because for a recursive function $G(x,y)$, the function $G(x,x)$ is recursive). Then by Theorem 2, for any recursive function $f(x)$ there is an i such that the partial recursive function E_i is the same as the partial recursive function $E_{f(i)}$; this is one form of the Recursion Theorem.