2.5.3 Logical Consequence

Definition 2.48 Let U be a set of formulas and A a formula. A is a *logical consequence* of U, denoted $U \models A$, iff every model of U is a model of A. ■

The formula A need not be true in every possible interpretation, only in those interpretations which satisfy U, that is, those interpretations which satisfy every formula in U. If U is empty, logical consequence is the same as validity.

Example 2.49 Let $A = (p \vee r) \wedge (\neg q \vee \neg r)$. Then A is a logical consequence of $\{p, \neg q\}$, denoted $\{p, \neg q\} \models A$, since A is true in all interpretations $\mathscr{I}$ such that $\mathscr{I}(p) = T$ and $\mathscr{I}(q) = F$. However, A is not valid, since it is not true in the interpretation $\mathscr{I}'$ where $\mathscr{I}'(p) = F$, $\mathscr{I}'(q) = T$, $\mathscr{I}'(r) = T$. ■

The caveat concerning $\leftrightarrow$ and $\equiv$ also applies to $\rightarrow$ and $\models$. Implication, $\rightarrow$, is an operator in the object language, while $\models$ is a symbol for a concept in the metalanguage. However, as with equivalence, the two concepts are related:

Theorem 2.50 *$U \models A$ if and only if $\models \bigwedge_i A_i \rightarrow A$.*

Definition 2.51 $\bigwedge_{i=1}^{i=n} A_i$ is an abbreviation for $A_1 \wedge \cdots \wedge A_n$. The notation $\bigwedge_i$ is used if the bounds are obvious from the context or if the set of formulas is infinite. A similar notation $\bigvee$ is used for disjunction. ■

Example 2.52 From Example 2.49, $\{p, \neg q\} \models (p \vee r) \wedge (\neg q \vee \neg r)$, so by Theorem 2.50, $\models (p \wedge \neg q) \rightarrow (p \vee r) \wedge (\neg q \vee \neg r)$. ■

The proof of Theorem 2.50, as well as the proofs of the following two theorems are left as exercises.

Theorem 2.53 *If $U \models A$ then $U \cup \{B\} \models A$ for any formula B.*

Theorem 2.54 *If $U \models A$ and B is valid then $U - \{B\} \models A$.*

2.5.4 Theories *

Logical consequence is the central concept in the foundations of mathematics. Valid logical formulas such as $p \vee q \leftrightarrow q \vee p$ are of little mathematical interest. It is much more interesting to assume that a set of formulas is true and then to investigate the consequences of these assumptions. For example, Euclid assumed five formulas about geometry and deduced an extensive set of logical consequences. The formal definition of a mathematical theory is as follows.

Definition 2.55 Let $\mathscr{T}$ be a set of formulas. $\mathscr{T}$ is *closed under logical consequence* iff for all formulas A, if $\mathscr{T} \models A$ then $A \in \mathscr{T}$. A set of formulas that is closed under logical consequence is a *theory*. The elements of $\mathscr{T}$ are *theorems*. ■