<Convex Optimization>

Date No.

Homework 1

2018320161_송대선

1. We are going to use induction on k.

if) $n=2$, for $\theta_1 x_1 + \theta_2 \theta_2$, satisfying $\theta_1+\theta_2=1$, $\theta_i \geq 0$, $\theta \in \mathbb{R}$, $x_i \in C$

then $\theta_2 = 1-\theta_1$, $\theta_2 \geq 0$

$1-\theta_1 \geq 0$

$1 \geq \theta_1 \quad \therefore 0 \leq \theta_1 \leq 1$

$\theta_1 = 1-\theta_2$, $\theta_1 \geq 0$

$1-\theta_2 \geq 0$

$1 \geq \theta_2 \quad 0 \leq \theta_2 \leq 1$

$\theta_1 x_1 + \theta_2 x_2 = \theta_1 x_1 + (1-\theta_1) x_2 \in C$ by the definition of convex set.

Assume that if $n=k$, $\alpha_1 x_1 + \cdots + \alpha_k x_k \in C$ with $x_1, \ldots, x_k \in C$, and let $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ satisfy $\alpha_i \geq 0$, $\alpha_1 + \cdots + \alpha_k = 1$.

We are going to show if $n=k+1$, $\beta_1 x_1 + \cdots + \beta_k x_k + \beta_{k+1} x_{k+1} \in C$

with $x_1, \ldots, x_{k+1} \in C$, and let $\beta_1, \ldots, \beta_{k+1} \in \mathbb{R}$ satisfy $\beta_i \geq 0$, $\beta_1 + \cdots + \beta_{k+1} = 1$.

let $\alpha_i = \frac{\beta_i}{1-\beta_{k+1}}$ (if $\beta_{k+1}=1$, $\sum_{i=1}^{k+1} \beta_i x_i = x_{k+1} \in C$, so we don't lose generality although we assume $\beta_{k+1} \neq 1$)

$\beta_i = \alpha_i (1-\beta_{k+1})$

(if $\beta_{k+1} = 0$, $\sum_{i=1}^{k+1} \beta_i x_i = \sum_{i=1}^{k} \beta_i x_i \in C$, ($\because \sum_{i=1}^{k+1} \beta_i = \sum_{i=1}^{k} \beta_i = 1$) So we don't lose generality although we assume $\beta_{k+1} \neq 0$)

$\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + \beta_{k+1} x_{k+1}$

$= (1-\beta_{k+1})(\alpha_1 x_1 + \cdots + \alpha_k x_k) + \beta_{k+1} x_{k+1}$

$x_{k+1} \in C$, $\alpha_1 x_1 + \cdots + \alpha_k x_k \in C$ by Assumtion

and $(1-\beta_{k+1}) + \beta_{k+1} = 1$

$\therefore (1-\beta_{k+1})(\alpha_1 x_1 + \cdots + \alpha_k x_k) + \beta_{k+1} x_{k+1} \in C$ by definition of Convex Sets

$\sum_{i=1}^{k+1} \beta_i x_i \in C$

by induction on k, $\sum_{i=1}^{k} \theta_i x_i \in C$

2. Consider all arbitrary Solutions ~~of~~ of linear equation given by $C = \{x \mid Ax = b\}$, ~~$x_1, \cdots x$~~ that are $x_1, x_2, \ldots, x_n$

Then $Ax_1 = b,\ Ax_2 = b,\ \ldots,\ Ax_n = b$

$A\theta_1 x_1 = \theta_1 b,\ A\theta_2 x_2 = \theta_2 b,\ \ldots,\ A\theta_n x_n = \theta_n b \ (\sum_{i=1}^{n} \theta_i = 1,\ \theta_i \in \mathbb{R})$

Sum of them is $A\theta_1 x_1 + A\theta_2 x_2 + \cdots + A\theta_n x_n = \theta_1 b + \theta_2 b + \cdots + \theta_n b$

$A(\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n) = b(\theta_1 + \theta_2 + \cdots + \theta_n)$

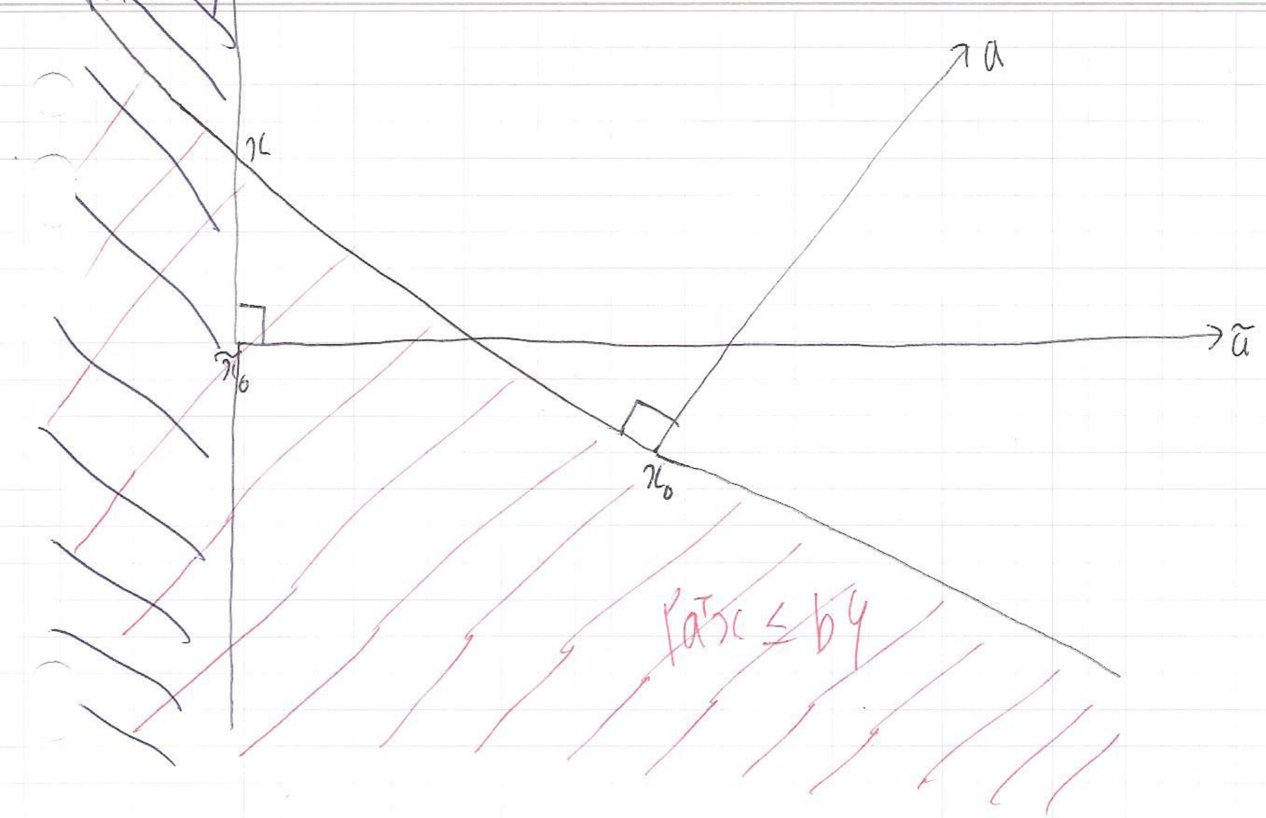
$A(\sum_{i=1}^{n} \theta_i x_i) = b \ (\because \sum_{i=1}^{n} \theta_i = 1)$

$\sum_{i=1}^{n} \theta_i x_i$ is a solution of C,

$\sum_{i=1}^{n} \theta_i x_i \in C$ for all $x_1, \ldots, x_n \in C$

$\therefore C$ is a affine set.

3. $\{\tilde{a}^T x \le \tilde{b}\}$



Let $b = a^T x_0$, $\tilde{b} = \tilde{a}^T \tilde{x}_0$, and a and $\tilde{a}$ are not parallel.

Then $\{x \mid a^T x \le b\} \nsubseteq \{\tilde{a}^T x \le \tilde{b}\}$ is a and $\tilde{a}$ are not parallel

So a and $\tilde{a}$ are parallel, $\therefore \tilde{a} = \theta a \ (\theta \in \mathbb{R}, \theta \ne 0)$

if $\theta < 0$, $\{\tilde{a}^T x \le \tilde{b}\}$

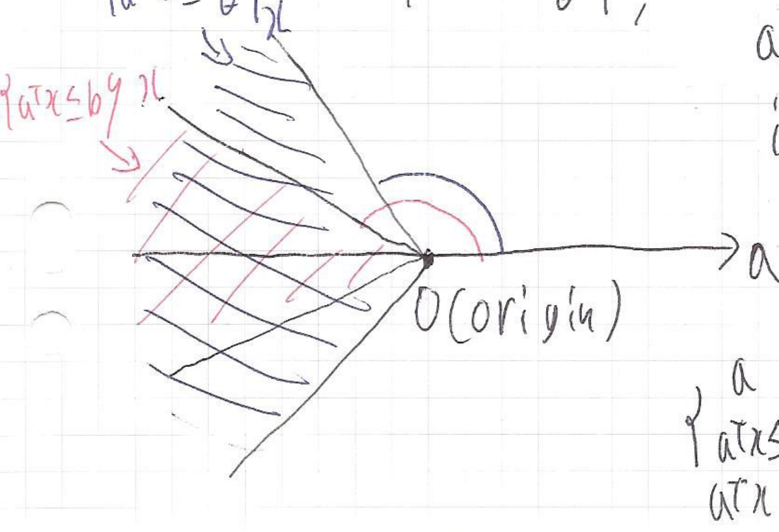
$= \{\theta a^T x \le \tilde{b}\}$

$= \{a^T x \ge \frac{\tilde{b}}{\theta}\}$, If $\theta < 0$, $\{\tilde{a} x \le \tilde{b}\}$ is not a halfspace.

So $\theta > 0$.

$\{\tilde{a}^T x \le \tilde{b}\} = \{\theta a^T x \le \tilde{b}\}$

$= \{a^T x \le \frac{\tilde{b}}{\theta}\}$, if $\{a^T x \le b\} \subseteq \{a^T x \le \frac{b}{\theta}\}$, Possible angle between x and a in $\{a^T x \le b\}$ is larger than possible angle between x and a in $\{a^T x \le \frac{\tilde{b}}{\theta}\}$



$a^T x$ is a inner product of a and x, so maximum $a^T x$ in $\{a^T x \le b\}$ shouldn't be larger then $a^T x$ in $\{a^T x \le \frac{\tilde{b}}{\theta}\}$, So, $b \le \frac{\tilde{b}}{\theta}$, $\theta b \le \tilde{b}$

When $\tilde{a} = \theta a$, $\tilde{b} \geq \theta b$ $(\theta \in \mathbb{R}, \theta > 0)$, $\{x \mid a^T x \leq b\} \subseteq \{x \mid \tilde{a}^T x \leq \tilde{b}\}$

When $\tilde{a} = \theta a$, $\tilde{b} = \theta b$ $(\theta \in \mathbb{R}, \theta > 0)$ $\{x \mid a^T x \leq b\} = \{x \mid \tilde{a}^T x \leq \tilde{b}\}$

4. (a) $\|x-a\|_2 \le \|x-b\|_2$

$$\sqrt{\sum_{i=1}^{n}(x_i-a_i)^2} \le \sqrt{\sum_{i=1}^{n}(x_i-b_i)^2}$$ (x_i is i th element of x)

$$\sum_{i=1}^{n}(x_i-a_i)^2 \le \sum_{i=1}^{n}(x_i-b_i)^2$$ (a_i is i th element of a)

(b_i is i th element of b)

$$\sum_{i=1}^{n}(x_i)^2 - 2\sum_{i=1}^{n}a_i x_i + \sum_{i=1}^{n}(a_i)^2 \le \sum_{i=1}^{n}(x_i)^2 - 2\sum_{i=1}^{n}b_i x_i + \sum_{i=1}^{n}(b_i)^2$$

$$2\left(\sum_{i=1}^{n}b_i x_i - \sum_{i=1}^{n}a_i x_i\right) \le \sum_{i=1}^{n}(b_i)^2 - \sum_{i=1}^{n}(a_i)^2$$

$$2(b^T x - a^T x) \le b^T b - a^T a$$

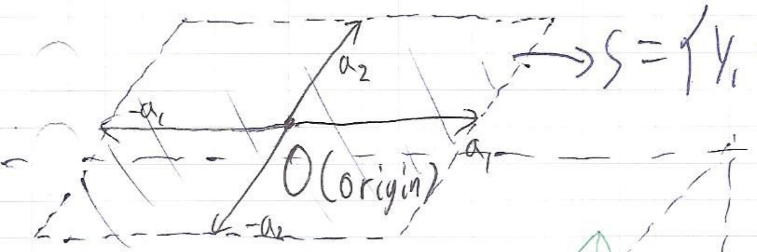
$$2(b^T - a^T)x \le b^T b - a^T a$$

$$(b-a)^T x \le \frac{b^T b - a^T a}{2}$$

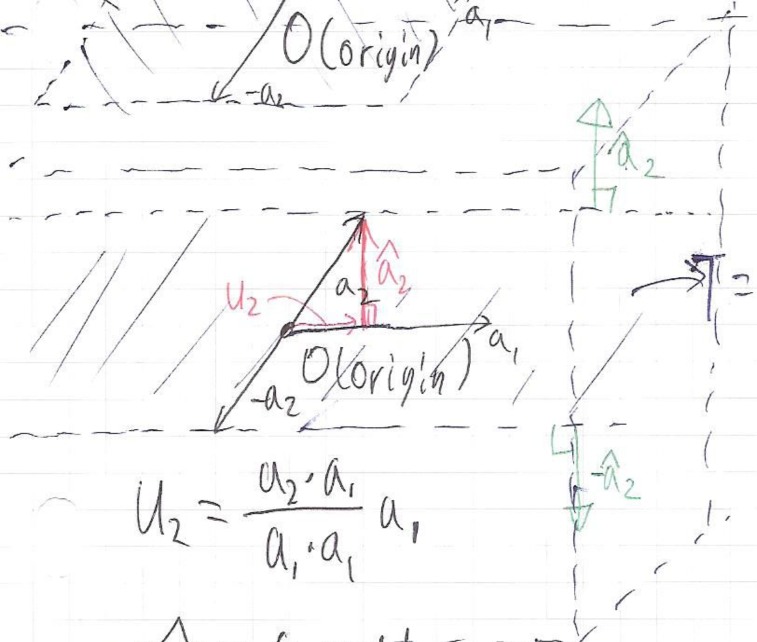
(b) Using (a), $V = \{x \in \mathbb{R}^n \mid (x_i - x_0)^T x \le (x_i^T x_i - x_0^T x_0)/2, i=1,\dots,K\}$

$$V = \left\{ x \;\middle|\; \begin{bmatrix} x_1 - x_0 \\ x_2 - x_0 \\ \vdots \\ x_K - x_0 \end{bmatrix} x \le \begin{bmatrix} (x_1^T x_1 - x_0^T x_0)/2 \\ (x_2^T x_2 - x_0^T x_0)/2 \\ \vdots \\ (x_K^T x_K - x_0^T x_0)/2 \end{bmatrix} \right\}$$

5. (a).



$S = \{ y_1 a_1 + y_2 a_2 \mid -1 \le y_1 \le 1, -1 \le y_2 \le 1 \}$

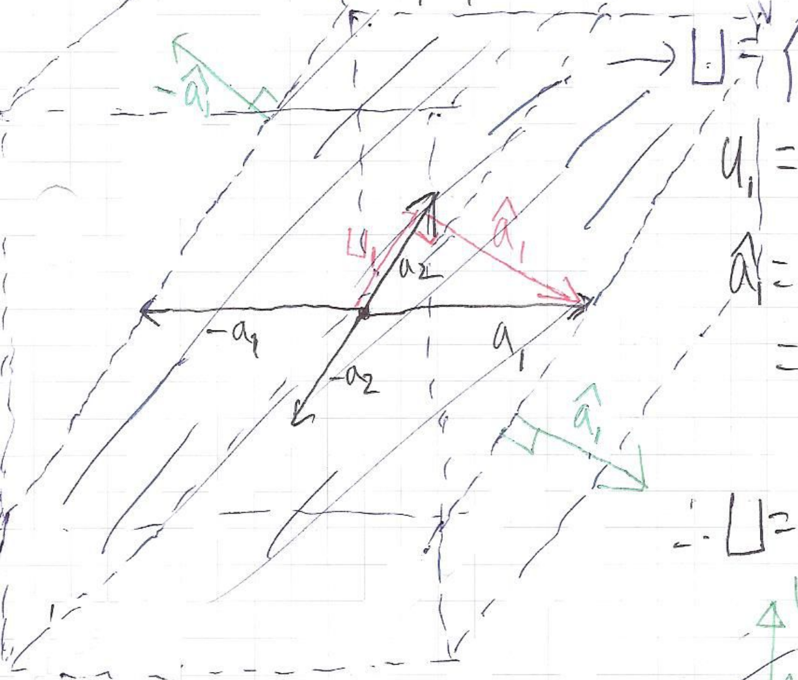


$T = \{ y_1 a_1 + y_2 a_2 + z \mid -1 \le a_2 \le 1 \}$
$(z \perp a_2,\ z \perp a_1)$

$$u_2 = \frac{a_2 \cdot a_1}{a_1 \cdot a_1} a_1$$

$$\hat{a}_2 = a_2 - u_2$$
$$= a_2 - \frac{a_2 \cdot a_1}{a_1 \cdot a_1} a_1$$

$T = \{ x \mid -|a_2 \cdot \hat{a}_2| \le \hat{a}_2 \cdot x \le |a_2 \cdot \hat{a}_2| \}$

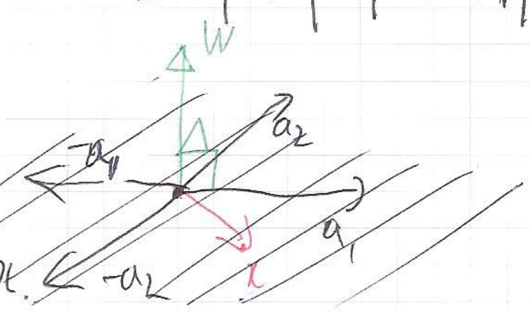


$U = \{ y_1 a_1 + y_2 a_2 + z \mid -1 \le a_1 \le 1 \}$
$(z \perp a_2,\ z \perp a_1)$

$$u_1 = \frac{a_1 \cdot a_2}{a_2 \cdot a_2} a_2$$

$$\hat{a}_1 = a_1 - u_1$$
$$= a_1 - \frac{a_1 \cdot a_2}{a_2 \cdot a_2} a_2.$$

$\therefore U = \{ x \mid -|a_1 \cdot \hat{a}_1| \le \hat{a}_1 \cdot x \le |a_1 \cdot \hat{a}_1| \}$



$V = \{ y_1 a_1 + y_2 a_2 \}$

$w \perp a_1,\ w \perp a_2,\ w \perp x.$

$V = \{ x \mid w \cdot x = 0 \}$

$S = T \cup \sqcup \cup V$

$T: -|a_2 \cdot \hat{a}_2| \leq \hat{a}_2 \cdot x \leq |a_2 \cdot \hat{a}_2|$

$$\hat{a}_2^T x \leq |a_2 \cdot \hat{a}_2|$$

$$-\hat{a}_2^T x \leq |a_2 \cdot \hat{a}_2|$$

$\sqcup: -|a_1 \cdot \hat{a}_1| \leq \hat{a}_1 \cdot x \leq |a_1 \cdot \hat{a}_1|$

$$\hat{a}_1^T x \leq |a_1 \cdot \hat{a}_1|$$

$$-\hat{a}_1^T x \leq |a_1 \cdot \hat{a}_1|$$

$V: w, x = 0$

$$\therefore S = \left\{ x \,\middle|\, \begin{bmatrix} \hat{a}_2^T \\ -\hat{a}_2^T \\ \hat{a}_1^T \\ -\hat{a}_1^T \end{bmatrix} x = \begin{bmatrix} |a_2 \cdot \hat{a}_2| \\ |a_2 \cdot \hat{a}_2| \\ |a_1 \cdot \hat{a}_1| \\ |a_1 \cdot \hat{a}_1| \end{bmatrix}, \; wx = 0 \right\}$$

5.(b).

$$Ix = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1\cdot x_1 + 0\cdot x_2 + \cdots + 0\cdot x_n \\ 0\cdot x_1 + 1\cdot x_2 + \cdots + 0\cdot x_n \\ \vdots \\ 0\cdot x_1 + 0\cdot x_2 + \cdots + 1\cdot x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x \geq 0$$

$$\therefore Ix \geq 0$$

$$1^T x = [1\,1\cdots 1]\,x = 1.$$

$$\sum_{i=1}^n x_i a_i = [a_1\ a_2 \cdots a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [a_1\ a_2 \cdots a_n]\,x = b_1$$

$$\sum_{i=1}^n x_i a_i^2 = [a_1^2\ a_2^2 \cdots a_n^2] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [a_1^2\ a_2^2 \cdots a_n^2]\,x = b_2.$$

$$\therefore \begin{bmatrix} [1\,1\cdots 1] \\ [a_1\, a_2 \cdots a_n] \\ [a_1^2\, a_2^2 \cdots a_n^2] \end{bmatrix} x = \begin{bmatrix} 1 \\ b_1 \\ b_2 \end{bmatrix}$$

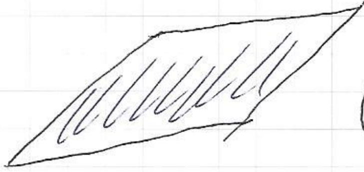
$$S = \left\{ x \;\middle|\; Ix \geq 0,\ \begin{bmatrix} [1\,1\cdots 1] \\ [a_1\ a_2 \cdots a_n] \\ [a_1^2\, a_2^2 \cdots a_n^2] \end{bmatrix} x = \begin{bmatrix} 1 \\ b_1 \\ b_2 \end{bmatrix} \right.$$

6. (a)

a slab is convex

($\because$ It's intersection of convex sets)

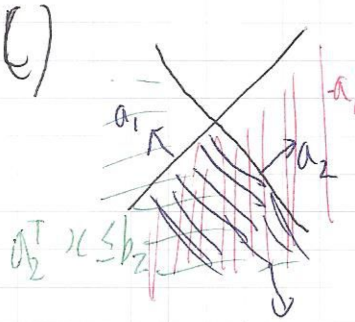
(b)



a rectangle is convex

($\because$ It's intersection of convex sets)

(c)



a wedge is convex

($\because$ It's intersection of convex sets)

$a_1^T x \le b_1$ and $a_2^T x \le b_2$.

(d) $\|x-a\|_2 \le \theta \|x-b\|_2$

$$\sqrt{\sum_{i=1}^{n}(x_i-a_i)^2} \le \theta \sqrt{\sum_{i=1}^{n}(x_i-b_i)^2}$$

$$\sum_{i=1}^{n}(x_i-a_i)^2 \le \theta^2 \sum_{i=1}^{n}(x_i-b_i)^2$$

$$x^Tx - 2a^Tx + a^Ta \le \theta^2 x^Tx - 2\theta^2 b^Tx + \theta^2 b^Tb$$

$$(1-\theta^2)x^Tx - 2(a-\theta^2 b)^T x \le -a^Ta + \theta^2 b^Tb, \text{ if } 1-\theta^2 > 0,\ \theta < 1.$$

$$x^Tx - 2\frac{(a-\theta^2 b)}{(1-\theta^2)}x \le \frac{-a^Ta+\theta^2 b^Tb}{1-\theta^2}$$

Let $\frac{a-\theta^2 b}{1-\theta^2} = x_0$, $x^Tx - 2x_0^T x \le \frac{-a^Ta+\theta^2 b^Tb}{1-\theta^2}$

$$x^Tx - 2x_0^Tx + x_0^Tx_0 \le \frac{-a^Ta+\theta^2 b^Tb}{1-\theta^2} + x_0^Tx_0$$

Let $R^2 = \frac{-a^Ta+\theta^2 b^Tb}{1-\theta^2} + x_0^Tx_0$, $x^Tx - 2x_0^Tx + x_0^Tx_0 \le R^2$.

$$(x-x_0)^T(x-x_0) \le R^2.$$

$\rightarrow$ This is a ball, ball is convex.

if $\theta = 1$, then

$$-2(a-b)^T x \leq -a^T a + b^T b$$
$$2(b-a)^T x \leq b^T b - a^T a.$$

This is halfspaces, so is convex

$\therefore \{x \mid \|x-a\|_2 \leq \theta \|x-b\|_2\}$ is convex

17. $\alpha \leq E[f(x)] \leq \beta$

$$\alpha \leq \sum_{i=1}^{n} P_i f(a_i) \leq \beta$$
$$\alpha \leq [f(a_1) \ f(a_2) \cdots f(a_n)] \, p \leq \beta$$

Then, $[f(a_1) \ f(a_2) \cdots f(a_n)] \, p \geq \alpha$

and

$$[f(a_1) \ f(a_2) \cdots f(a_n)] \, p \leq \beta$$

$$-[f(a_1) \ f(a_2) \cdots f(a_n)] \, p \leq -\alpha$$
$$[-f(a_1) \ -f(a_2) \cdots -f(a_n)] \, p \leq -\alpha$$

$$\therefore \begin{bmatrix} -f(a_1) & -f(a_2) & \cdots & -f(a_n) \\ f(a_1) & f(a_2) & \cdots & f(a_n) \end{bmatrix} p \leq \begin{bmatrix} -\alpha \\ \beta \end{bmatrix}$$