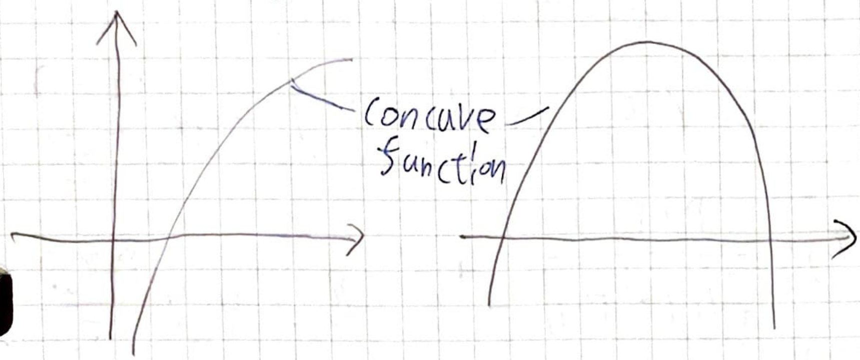
(1), $\nabla f(x) = 2Px$

$\nabla^2 f(x) = 2P < 0 \quad (\because P \ngeq 0)$

$\therefore f(x)$ is concave

$\therefore \nabla f(x)$ is decreasing function $(\because \nabla^2 f(x) < 0)$

Concave 함수를 minimize한다면, 그 결과값이 $-\infty$이 될 수 밖에 없다. $(\, f(\lambda x + (1-\lambda)y) > \lambda f(x) + (1-\lambda) f(y) \,)$



$P<0$ 이기에, $x^T P x < 0$ 이면, $(tx)^T P (tx) < 0$

$t^2(x^T P x) < 0$을 만족하기도 한다

$f(x) = t^2(x^T P x) + t \cdot q^T x$ 에서

$t \to \infty$, $f(x) \to \infty$ 이다. $(\because x^T P x < 0$,
t^2의 증가 폭이
t의 증가폭보다 크다)

2. $f(x) = \dfrac{\|Ax-b\|_2^2}{c^T x + d}$ dom $f = \{x \mid c^T x + d > 0\}$

$\therefore f(x)$ is convex ($\because \|Ax-b\|_2^2$ is 2-norm, $c^T x + d > 0$)

$\nabla f(x) = 0$을 만족하는 x^*를 찾자.

$$\frac{2(Ax-b)^T A}{c^T x+d} - \frac{\|Ax-b\|_2^2}{(c^T x+d)^2} c = 0$$

$$\frac{2}{c^T x+d}(A^T A x - A^T b) - \frac{\|Ax-b\|_2^2}{(c^T x+d)^2} c = 0$$

$$\frac{2}{c^T x+d} A^T A (x - (A^T A)^{-1} A^T b) - \frac{\|Ax-b\|_2^2}{(c^T x+d)^2} c = 0$$

$$\frac{2}{c^T x+d} A^T A (x - (A^T A)^{-1} A^T b) - \frac{\|Ax-b\|_2^2}{(c^T x+d)^2} (A^T A)(A^T A)^{-1} c = 0$$

$$\frac{2}{c^T x+d} A^T A \left(x - (A^T A^{-1}) A^T b - \frac{\|Ax-b\|_2^2}{c^T x+d} (A^T A)^{-1} c\right) = 0$$

$$\frac{2}{c^T x+d} A^T A \left(x - x_1 - \frac{\|Ax-b\|_2^2}{c^T x+d} x_2\right) = 0$$

$$\frac{2}{c^T x+d} A^T A (x - x_1 - t\, x_2) = 0 \quad (t \in \mathbb{R})$$

$$x - x_1 - t x_2 = 0$$

$$\therefore x^* = x_1 + t x_2$$

3. exact line search를 이용한다.

$$t = \arg\min_{t>0} f(x + t\Delta x) \qquad (x_1^{(0)}, x_2^{(0)}) = (\gamma, 1)$$

$$x^{(k)} = x^{(k-1)} - t\nabla f(x^{(k-1)})$$

$$\nabla f(x) = (x_1, \gamma x_2)$$

$$\nabla f(x^{(k-1)}) = (x_1^{(k-1)}, \gamma x_2^{(k-1)})$$

$$x^{(k)} = \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \end{bmatrix} - \begin{bmatrix} t x_1^{(k-1)} \\ t\gamma x_2^{(k-1)} \end{bmatrix}$$

$$= \begin{bmatrix} (1-t)x_1^{(k-1)} \\ (1-t\gamma)x_2^{(k-1)} \end{bmatrix}$$

$$x_1^{(k)} = (1-t)x_1^{(k-1)} = (1-t)^k x_1^{(0)} = (1-t)^k \gamma$$

$$x_2^{(k)} = (1-t\gamma)x_2^{(k-1)} = (1-t\gamma)^k x_2^{(0)} = (1-t\gamma)^k$$

$$t = \arg\min_{t>0} f(x + t\Delta x)$$

$$= \arg\min_{t>0} f(x - t\cdot\nabla f(x))$$

$$= \arg\min_{t>0} f((1-t)x_1, (1-t\gamma)x_2)$$

$$= \arg\min_{t>0} \frac{1}{2}\left((1-t)^2 x_1^2 + \gamma(1-t\gamma)^2 x_2^2\right)$$

$$= \arg\min_{t>0} \frac{1}{2}\left((x_1^2 + \gamma^3 x_2^2)t^2 - 2(x_1^2 + \gamma^2 x_2^2)t + x_1^2 + \gamma x_2^2\right)$$

미분하여 0이 되는 지점을 찾으면,

$$(x_1^2 + \gamma^3 x_2^2)t - (x_1^2 + \gamma^2 x_2^2) = 0$$

$$t = \frac{x_1^2 + \gamma^2 x_2^2}{x_1^2 + \gamma^3 x_2^2} = \frac{\gamma^2 + \gamma^2\cdot 1^2}{\gamma^2 + \gamma^3} = \frac{1+1}{1+\gamma} = \frac{2}{1+\gamma}$$

$x_1^{(k)} = (1-t)^k v \quad = \left(1-\frac{2}{1+v}\right)^k v \quad = \left(\frac{v-1}{v+1}\right)^k v$

$x_2^{(k)} = (1-tv)^k \quad = \left(1-\frac{2}{1+v}\cdot v\right)^k \quad = \left(\frac{1-v}{1+v}\right)^k$

$= \left(-\frac{v-1}{v+1}\right)^k$

4(b) $(a^T x + b)$ is affine set

그래서 $\frac{e^x}{1+e^x}$이 convex 인지 아닌지가 중요한 것이다.

$$\nabla_x \frac{e^x}{1+e^x} = \frac{e^x(1+e^x) - e^x \cdot e^x}{(1+e^x)^2}$$

$$= \frac{e^x}{(1+e^x)^2} > 0$$

$$\nabla_x^2 \frac{e^x}{1+e^x} = \frac{e^x(1+e^x)^2 - e^x \cdot 2(1+e^x)e^x}{(1+e^x)^4}$$

$$= \frac{-(e^x-1)e^x}{(1+e^x)^2}$$

$\frac{e^x}{1+e^x}$은 비록 convex function은 아니지만,

$\nabla_x \frac{e^x}{1+e^x} > 0$ 임으로 증가함수이다.

따라서 $\frac{e^x}{1+e^x}$에서 x의 역할을 하는 $a^T x + b$를 maximize 시키면,

$p = \frac{\exp(a^T x + b)}{1+\exp(a^T x + b)}$도 maximize 된다.

따라서

$$\begin{aligned} &\text{maximize} \quad a^T x + b \\ &\text{subject to} \quad Fx \preceq g \end{aligned}$$

($a^T x + b$ is concave, so maximize (concave function) is convex.
So It's convex optimization)

4.(b)

$$P(c^T x+d) = \frac{\exp(a^T x+b)}{1+\exp(a^T x+b)}(c^T x+d)$$ 이것을 maximize 시키는 것은,

$$\log(P(c^T x+d)) = a^T x+b - \log(1+\exp(a^T x+b)) + \log(c^T x+d)$$

을 maximize 하는 것과 동일한 문제이다.

각각의 요소들을 보면,

$a^T x+b$은 affine set 이기에, concave 하다.

$\log(1+\exp(a^T x+b))$은 log-sum-exp function의 일종이라 convex 한데,

$-\log(1+\exp(a^T x+b))$은 부호가 음수라서 concave 하다.

$\log(c^T x+d)$은 log 함수의 일종이라 concave 하다.

concave한 함수들의 합이기에, $\log(P(c^T x+d))$은 concave 하다.

따라서 convex optimization problem은

$$\begin{aligned} &\text{maximize} \quad a^T x+b - \log(1+\exp(a^T x+b)) + \log(c^T x+d) \\ &\text{subject to} \quad Fx \preceq g \end{aligned}$$

5. (a)

Assume that $t^* > 0$,

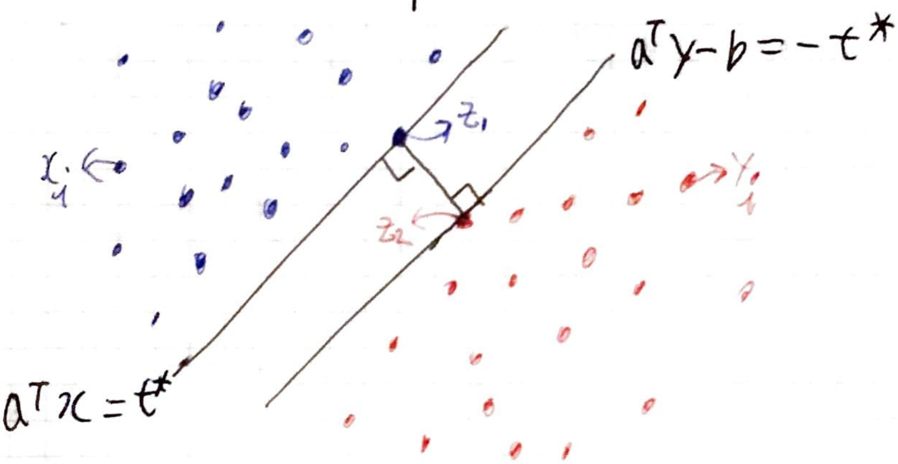
then $a^T x_i - b \geq t^*$

$a^T y_i - b \leq -t^*$

$$a^T x_i - b \geq t^* > -t^* \geq a^T y_i - b$$

$$a^T x_i - b > a^T y_i - b$$

$\therefore$ two sets of points are linearly separated.



$z_1 - z_2 = ca$ $(c > 0)$ 이라고 할 수 있다. (c is constant)

$a^T z_1 - b = t^*$

$a^T z_2 - b = -t^*$

$a^T(z_1 - z_2) = 2t^*$

$a^T ca = 2t^*$

$$t^* = \frac{c\|a\|_2^2}{2}$$

maximize t^*이 되려면, $\|a\|_2^2$이 maximized 되어야한다.

$\therefore \|a^*\|_2 = 1$ $(\because \|a\|_2 \leq 1)$

5.(b)

$$\begin{aligned} \text{maximize} \quad & t \\ \text{subject to} \quad & a^T x_i - b \ge t \\ & a^T y_i - b \le -t \\ & \|a\|_2 \le 1 \end{aligned}$$

$a = t\tilde{a}$
$b = t\tilde{b}$

$$\begin{aligned} \text{maximize} \quad & t \\ \text{subject to} \quad & t\tilde{a}^T x_i - t\cdot\tilde{b} \ge t \\ & t\tilde{a}^T y_i - t\cdot\tilde{b} \le -t \\ & \|t\tilde{a}\|_2 \le 1 \end{aligned}$$

$$= \begin{aligned} \text{maximize} \quad & t \\ \text{subject to} \quad & \tilde{a}^T x_i - \tilde{b} \ge 1 \quad (\because t > 0 \text{ by } 5.(a)) \\ & \tilde{a}^T y_i - \tilde{b} \le -1 \\ & t \le \frac{1}{\|\tilde{a}\|_2} \end{aligned}$$

5.(a)의 결과에 의하면, inequality $\|a\|_2 \le 1$ is tight 하기에 $\|t\tilde{a}\|_2 = 1$, $t = \frac{1}{\|\tilde{a}\|_2}$ 이다.

$$= \begin{aligned} \text{maximize} \quad & \frac{1}{\|\tilde{a}\|_2} \\ \text{subject to} \quad & \tilde{a}^T x_i - \tilde{b} \ge 1 \\ & \tilde{a}^T y_i - \tilde{b} \le -1 \end{aligned} \qquad = \begin{aligned} \text{minimize} \quad & \|\tilde{a}\|_2 \\ \text{subject to} \quad & \tilde{a}^T x_i - \tilde{b} \ge 1 \\ & \tilde{a}^T y_i - \tilde{b} \le -1 \end{aligned}$$