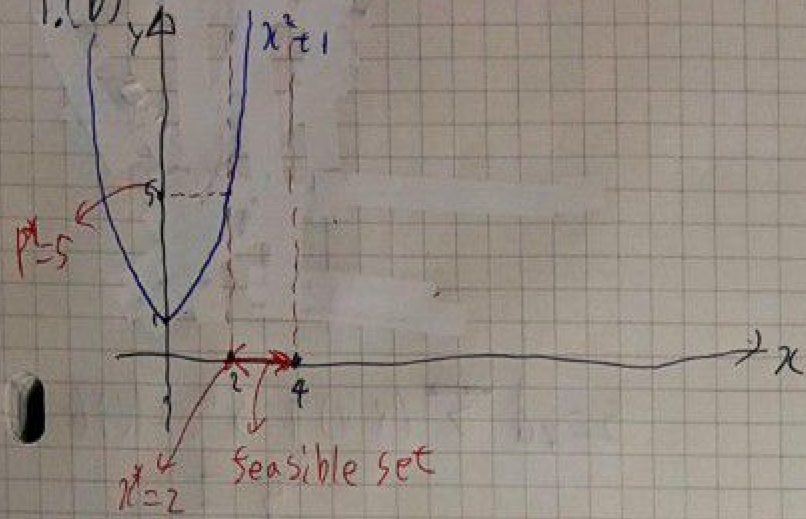
1.(a)

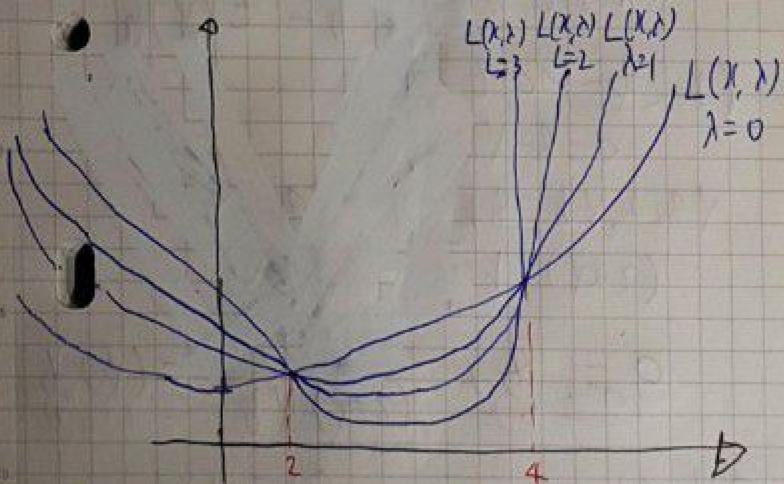
feasible set $= \{x \mid 2 \le x \le 4\}$

$x^* = 2, \quad p^* = 2^2 + 1 = 5$

1.(b)



$$L(x, \lambda) = x^2 + 1 + \lambda (x-2)(x-4)$$



$$L'(x, \lambda) = 0$$

$$2x + \lambda(2x - 6) = 0$$

$$x + \lambda(x - 3) = 0$$

$$(1+\lambda)x - 3\lambda = 0$$

$$x = \frac{3\lambda}{1+\lambda}$$

$$g(\lambda) = \begin{cases} -\frac{9\lambda^2}{1+\lambda} + 8\lambda + 1 & \text{if } \lambda > -1 \\ -\infty & \text{if } \lambda \leq -1 \end{cases}$$

$$g'(\lambda) = -9\frac{\lambda(\lambda+2)}{(\lambda+1)^2} + 8 = 0$$

$$8(\lambda+1)^2 = 9\lambda(\lambda+2)$$

$$8\lambda^2 + 16\lambda + 8 = 9\lambda^2 + 18\lambda$$

$$\lambda^2 + 2\lambda - 8 = 0$$

$$\lambda^* = 2 \quad (\lambda > -1)$$

$$g(\lambda^*) = -\frac{9 \cdot 2^2}{1+2} + 8 \cdot 2 + 1$$

$$= 5$$

$$g(\lambda) = \inf_x L(x, \lambda) \leq p^* = 5$$

1, (c)

dual Problem:

$$\text{maximize} \quad -9\frac{\lambda^2}{1+\lambda}+8\lambda+1$$
$$\text{subject to} \quad \lambda \geq 0$$

Let $f(\lambda) = -9\frac{\lambda^2}{1+\lambda}+8\lambda+1$

$f'(\lambda) = -\frac{\lambda^2+2\lambda-8}{(\lambda+1)^2}$

$f''(\lambda) = -\frac{18}{(\lambda+1)^3}$

< 0

$\therefore f(\lambda)$ is concave

따라서 제시된 dual Problem은 concave maximum problem이다.

$f'(\lambda^*)=0$

$-\frac{(\lambda^*)^2+2\lambda^*-8}{(\lambda^*+1)^2}=0$

$(\lambda^*)^2+2\lambda^*-8=0$

$(\lambda^*-2)(\lambda^*+4)=0$

$\therefore \lambda^*=2. \ (\because \lambda^* \geq 0)$

$d^* = -9\cdot\frac{2^2}{(1+2)}+8\cdot 2+1$

$= -12+16+1$

$\therefore = 4$

$p^*=4$ 이므로, $p^*=d^*$, 따라서 이는 strong duality 이다.

2.

$$L(x, \lambda, \nu) = c^T x + \lambda^T (Gx - h) + \nu^T (Ax - b)$$

$$= (c^T + \lambda^T G + \nu^T A)x - \lambda^T h - \nu^T b$$

$$g(\lambda, \nu) = \inf_x (c^T + \lambda^T G + \nu^T A)x - \lambda^T h - \nu^T b$$

$$= \begin{cases} -\lambda^T h - \nu^T b & \text{if } c^T + \lambda^T G + \nu^T A = 0 \\ -\infty & \text{otherwise} \end{cases}$$

dual problem:

$$\begin{array}{ll} \text{maximize} & -\lambda^T h - \nu^T b \\ \text{subject to} & c^T + \lambda^T G + \nu^T A = 0, \\ & \lambda \succeq 0 \end{array}$$

4. (c)

$$\text{minimize } \log\left(\sum_{i=1}^{m} \exp(y_i)\right)$$
$$\text{subject to } y = Ax + b$$

$$L(x, v) = \log\left(\sum_{i=1}^{m} \exp(y_i)\right) + v^T(Ax + b - y)$$
$$= \log\left(\sum_{i=1}^{m} \exp(y_i)\right) + v^T Ax + v^T b - v^T y$$

$v^T A = 0$ 이어야 한다. $\quad v^T Ax = \begin{cases} 0 & \text{if } v^T A = 0 \\ -\infty & \text{otherwise} \end{cases}$

$$L(x, v) = \log\left(\sum_{i=1}^{m} \exp(y_i)\right) + v^T b - v^T y$$

$$L'(x, v) = \frac{e^{y_i}}{\sum_{i=1}^{m} \exp(y_i)} - v_i = 0$$

$$\frac{e^{y_i}}{\sum_{i=1}^{m} \exp(y_i)} = v_i \longrightarrow \frac{\sum_{i=1}^{m} e^{y_i}}{\sum_{i=1}^{m} e^{y_i}} = \sum_{i=1}^{m} v_i$$

$$y_i - \log\left(\sum_{i=1}^{m} \exp(y_i)\right) = \log(v_i) \qquad \therefore \sum_{i=1}^{m} v_i = 1.$$

$$y_i = \log(v_i) + \log\left(\sum_{i=1}^{m} \exp(y_i)\right) \qquad v \cdot 1 = 1$$

$$v^T y = \sum_{i=1}^{m} v_i y_i$$
$$= \sum_{i=1}^{m} v_i \cdot \log(v_i) + v_i \log\left(\sum_{i=1}^{m} \exp(y_i)\right)$$

$$L(x, v) = \log\left(\sum_{i=1}^{m} \exp(y_i)\right) + v^T b - \sum_{i=1}^{m} v_i \log(v_i) - \sum_{i=1}^{m} v_i \log\left(\sum_{i=1}^{m} \exp(y_i)\right)$$

$$= v^T b - \sum_{i=1}^{m} v_i \log(v_i)$$

dual problem : $\text{maximize } b^T v - \sum_{i=1}^{m} v_i \log v_i$

$\qquad\qquad\qquad\text{subject to } 1^T v = 1,\ v \geq 0,\ A^T v = 0$

dual problem

$$\text{maximize} \quad \sum_{i=1}^{m} v_i b_i$$

$$\text{subject to} \quad A^T v = 0, \quad \sum_{i=1}^{m} v_i = 1, \quad v \geq 0$$

4. (b)

LP로 바꾸면

$$\text{minimize} \quad t$$

$$\text{subject to} \quad Ax + b \leq T \quad (T = (t, t, \dots, t))$$

$$L(x, \lambda) = t + \lambda (Ax + b - t \cdot 1)$$

$$= (\lambda^T A)x + t - t \cdot 1 + \lambda^T b$$

$\lambda^T A \neq 0$ 이면 $-\infty$ 가능

$\lambda^T A = 0$ 이어야 한다.

$$L(x, \lambda) = \lambda^T b$$

$\sum_{i=1}^{m} \lambda_i = 1$, $\lambda \geq 0$ 이어야 한다.

dual Problem:

$$\text{maximize} \quad \lambda^T b$$

$$\text{subject to} \quad \lambda^T A = 0, \ \sum_{i=1}^{m} \lambda_i = 1, \ \lambda \geq 0$$

이것은 4. (a)과 일치한다.

dual Problem:

$$\text{maximize} \ \sum_{i=1}^{N} v_i^T (A_i x_0 + b_i) - \frac{1}{2} \left\| \sum_{i=1}^{N} A_i^T v_i \right\|_2^2$$

$$\text{subject to} \ \|v_i\| \le 1, \ i = 1, 2, \cdots, N$$

4. (a)

$$L(x,v) = \max_{i=1,\dots,m} y_i + \sum_{i=1}^{m} v_i (-y_i + a_i^T x + b_i)$$

$$\inf_{x,y} L(x,v) = g(v)$$

x에 대하여 보면 $\sum_{i=1}^{m} v_i a_i = 0$ 이어야만 $g(v)$가 ∞가 되지 않는다

$$\inf_{y} L(x,v) = \inf_{y} \max_{i=1,\dots,m} y_i - \sum_{i=1}^{m} v_i y_i + \sum_{i=1}^{m} v_i b_i$$

$\inf_{y} \max_{i=1,\dots,m} y_i - \sum_{i=1}^{m} v_i y_i$ 만 보자.

$$= \inf_{y} \max_{i=1,\dots,m} y_i - v^T y$$

if $v \ge 0$, $\sum_{i=1}^{m} v_i = 1$ then

$$\max_{i=1,\dots,m} y_i = \sum_{i=1}^{m} v_i \max_{i=1,\dots,m} y_i$$

$$\ge \sum_{i=1}^{m} v_i y_i$$

$$= v^T y$$

if $v < 0$, then

$$\max_{i=1,\dots,m} y_i - v^T y \to -\infty$$

if $\sum_{i=1}^{m} v_i \ne 1$, then

$$\max_{i=1,\dots,m} y_i - v^T y \to -\infty$$

$$\therefore g(\lambda) = \begin{cases} \sum_{i=1}^{m} v_i b_i & \text{if } v \ge 0, \sum_{i=1}^{m} v_i = 1 \\ -\infty & \text{otherwise} \end{cases}$$

3.

$$\text{minimize } \sum_{i=1}^{N} \|A_i x + b_i\|_2 + \frac{1}{2}\|x - x_0\|_2^2$$

$$\text{subject to } y_i = A_i x + b_i$$

$$L(x, v) = \sum_{i=1}^{N} \| y_i \|_2 + \frac{1}{2}\|x - x_0\|_2^2 - \sum_{i=1}^{N} v_i^T (y_i - A_i x - b_i)$$

(v는 부호상관 없다)

$$= \sum_{i=1}^{N} \|y_i\|_2 - \sum_{i=1}^{N} v_i^T y_i + \frac{1}{2}\|x - x_0\|_2^2 + \sum_{i=1}^{N} v_i^T A_i x + \sum_{i=1}^{N} v_i^T b_i$$

$$= \sum_{i=1}^{N} (\|y_i\|_2 - v_i^T y_i) + \frac{1}{2}\|x - x_0\|_2^2 + \sum_{i=1}^{N} v_i^T A_i x + \sum_{i=1}^{N} v_i^T b_i)$$

y_i에 대해 minimize 시킨다

$L_2(y, v) = \|y_i\|_2 - v_i^T y_i$ 이면 이 문제는 $\|y\|_2^2$ minimize 시키는 문제인 것이다

$$\inf_y L_2(y, v) = \begin{cases} 0 & \text{if } \|v_i\|_2 \le 1 \\ -\infty & \text{otherwise} \end{cases}$$ 이다.

$L_3(x, v) = \frac{1}{2}\|x - x_0\|_2^2 + \sum_{i=1}^{N} (A_i^T v_i)^T x$ 의 경우,

$$(L_3(x, v))' = 0$$

$$x - x_0 + \sum_{i=1}^{N} A_i^T v_i = 0$$

$$x = x_0 - \sum_{i=1}^{N} A_i^T v_i$$

$$\inf_x L_3(x, v) = \frac{1}{2}\|\sum_{i=1}^{N} A_i^T v_i\|_2^2 - \|\sum_{i=1}^{N} A_i^T v_i\|_2^2 + \sum_{i=1}^{N} (A_i^T v_i)^T x_0$$

$$\inf_x L(x, v) = \sum_{i=1}^{N} (A_i^T v_i)^T x_0 - \frac{1}{2}\|\sum_{i=1}^{N} A_i^T v_i\|_2^2 + \sum_{i=1}^{N} v_i^T b_i$$

$$= \sum_{i=1}^{N} v_i^T (A_i x_0 + b_i) - \frac{1}{2}\|\sum_{i=1}^{N} A_i^T v_i\|_2^2$$

4.(d) ν^*_{soft}를 (2)의 dual optimal solution 이라고하자

$$p^*_{soft} = b^T \nu^*_{soft} - \sum_{i=1}^{m} \nu^*_{soft,i} \log \nu^*_{soft,i}$$

$$b^T \nu^*_{soft} = p^*_{soft} + \sum_{i=1}^{m} \nu^*_{soft,i} \log \nu^*_{soft,i}$$

(1)을 생각해보면, p^*가 $\lambda^T b$ 미만일 수 없다는 bound가 있다.

$$p^* \geq p^*_{soft} + \underline{\sum_{i=1}^{m} \nu^*_{soft,i} \log \nu^*_{soft,i}} \quad , \quad p^*_{soft} \geq p^*$$

$$p^*_{soft} - p^* \geq 0$$

$\inf_{1^T k=1} \sum_{i=1}^{m} k_i \log k_i$를 생각해보자.

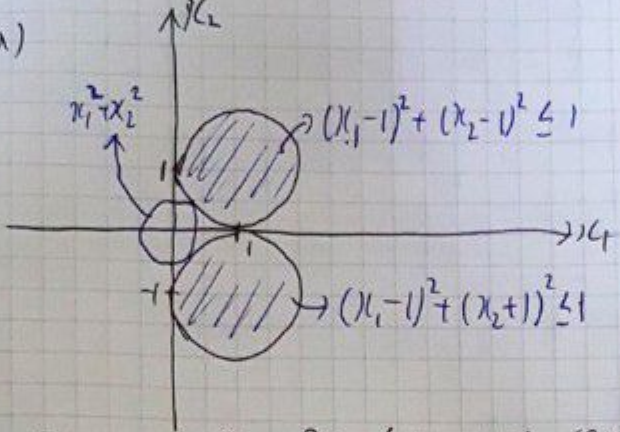
$$\inf_{1^T k=1} \sum_{i=1}^{m} k_i \log k_i = -\log m$$

$$p^* \geq p^*_{soft} - \log m$$

$$p^*_{soft} - p^* \leq \log m$$

$$\therefore \ 0 \leq p^*_{soft} - p^* \leq \log m$$

5.(a)



feasible set은 점 (1,0)이 유일하다.

따라서, $x^* = (1,0)$, $p^* = 1^2+0^2$
$= 1$

5.(b) $L(x_1, x_2, \lambda_1, \lambda_2) = x_1^2+x_2^2+\lambda_1((x_1-1)^2+(x_2-1)^2-1)$
$+\lambda_2((x_1-1)^2+(x_2+1)^2-1)$

KKT conditions,

$$(x_1-1)^2+(x_2-1)^2-1 \le 0$$
$$(x_1-1)^2+(x_2+1)^2-1 \le 0$$
$$\lambda_1 \ge 0, \ \lambda_2 \ge 0$$
$$\lambda_1((x_1-1)^2+(x_2-1)^2-1)=0$$
$$\lambda_2((x_1-1)^2+(x_2+1)^2-1)=0$$
$$(L(x_1, x_2, \lambda_1, \lambda_2))' = 0$$
$$\rightarrow 2x_1 + \lambda_1(2x_1-2)+\lambda_2(2x_1-2)=0$$
$$\rightarrow 2x_2 + \lambda_1(2x_2-2)+\lambda_2(2x_2+2)=0$$

$x_1=1$, $x_2=0$ 을 적용하면,

$0 \le 0$, $0 \le 0$, $\lambda_1 \ge 0$, $\lambda_2 \ge 0$, $\lambda_1(0)=0$, $\lambda_2(0)=0$, $2=0$, $-2\lambda_1+2\lambda_2=0$

정리하자면, $\lambda_1 \ge 0$, $\lambda_2 \ge 0$, $2=0$, $-2\lambda_1+2\lambda_2=0$

$2=0$로 인하여 KKT fails.

5.(c)

$$g(\lambda_1, \lambda_2) = \inf_{x_1, x_2} L(x_1, x_2, \lambda_1, \lambda_2)$$

$$= \inf_{x_1, x_2} x_1^2 + x_2^2 + \lambda_1\left((x_1-1)^2 + (x_2-1)^2 - 1\right) + \lambda_2\left((x_1-1)^2 + (x_2+1)^2 - 1\right)$$

x_1로 나눌 때

$$2x_1 + \lambda_1(2x_1 - 2) + \lambda_2(2x_1 - 2) = 0$$

$$(1+\lambda_1+\lambda_2)x_1 = \lambda_1 + \lambda_2$$

$$x_1 = \frac{\lambda_1+\lambda_2}{1+\lambda_1+\lambda_2}$$

x_2로 나눌 때

$$2x_2 + \lambda_1(2x_1 - 2) + \lambda_2(2x_2 + 2) = 0$$

$$x_2 = \frac{\lambda_1-\lambda_2}{1+\lambda_1+\lambda_2}$$

대입하면,

$$g(\lambda_1, \lambda_2) = \begin{cases} -\dfrac{(\lambda_1+\lambda_2)^2 + (\lambda_1-\lambda_2)^2}{1+\lambda_1+\lambda_2} + \lambda_1 + \lambda_2 & \text{if } 1+\lambda_1+\lambda_2 \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

dual Problem:

$$\text{maximize } \frac{\lambda_1+\lambda_2-(\lambda_1-\lambda_2)^2}{1+\lambda_1+\lambda_2}$$

$$\text{Subject to } \lambda_1 \geq 0,\ \lambda_2 \geq 0$$

5.(b)에서 TTK condition fails 했다.

strong duality $\longrightarrow$ TTK

not TTK $\rightarrow$ not strong duality

따라서 Strong duality가 아니다.

6.(a)

$$L(x,v) = \|Ax-b\|_2^2 + v^T(Gx-h)$$

$$= x^TA^TAx - 2(A^Tb)^Tx + b^Tb + (G^Tv)^Tx - v^Th$$

$$= x^TA^TAx + (G^Tv - 2A^Tb)^Tx - v^Th + b^Tb$$

$$L'(x,v) = 2A^TAx + G^Tv - 2A^Tb = 0$$

$$A^TAx = -\frac{1}{2}(G^Tv - 2A^Tb)$$

$$x^* = -\frac{1}{2}(A^TA)^{-1}(G^Tv - 2A^Tb)$$

$$g(v) = -\frac{1}{4}(G^Tv - 2A^Tb)^T(A^TA)^{-1}(G^Tv - 2A^Tb) - v^Th + b^Tb$$

6.(b)

KKT condition

$$Gx - h = 0$$

$$2A^TAx + G^Tv - 2A^Tb = 0$$

6.(c)

$$\therefore x^* = -\frac{1}{2}(A^TA)^{-1}(G^Tv^* - 2A^Tb)$$

$$Gx^* = -\frac{1}{2}G(A^TA)^{-1}(G^Tv^* - 2A^Tb)$$

$$h = -\frac{1}{2}G(A^TA)^{-1}G^Tv^* + G(A^TA)^{-1}A^Tb$$

$$h - G(A^TA)^{-1}A^Tb) = -\frac{1}{2}G(A^TA)^{-1}G^Tv^*$$

$$\therefore v^* = -2\left(G(A^TA)^{-1}G^T\right)^{-1}\left(h - G(A^TA)^{-1}A^Tb\right)$$