Topics: Convex optimization problems

1. Suppose we would like to solve the following optimization problem:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a^T x \leq b \end{array}$$

Basically, we would like to minimize linear function of x, subject to that x is in a halfspace.

(a) Show that, if vectors c and a are not parallel, the optimal value of the problem is $-\infty$. (Intuitively if c and a are not parallel, we can always find a hyperplane $c^T x = d$ with d negative with arbitrarily large magnitude, such that the hyperplane intersects with halfspace $\{x | a^T x \leq b\}$.) *Hint:* Because a and c are not parallel, c can be decomposed as sum of two vectors where one is parallel to a, and the other is orthogonal to a. Assume c can be decomposed as $c = \lambda a + \hat{a}$, where λ is some constant and $\hat{a}$ is orthogonal to a. Now consider $x = ba/\|a\|_2^2 + d\hat{a}$, and see what happens if we make d arbitrarily negative.

(b) Suppose c and a are parallel, and so we can write $c = \lambda a$. Denote the optimal value of the problem by p^*. Show that

$$p^* = \begin{cases} -\infty & \lambda > 0 \\ \lambda b & \text{otherwise} \end{cases}$$

2. Find the optimal solution x^* for the following optimization problems:

(a)

$$\begin{array}{c} \text{minimize } c^T x \\ \text{subject to } l \leq x \leq u \end{array}$$

where vectors l and u satisfy $l \leq u$.

(b)

$$\begin{array}{c} \text{minimize: } c^T x \\ \text{subject to } \mathbf{1}^T x = 1,\ x \geq 0 \end{array}$$

where $\mathbf{1} = (1, 1, \ldots, 1)$ is the vector of all ones. Assume $c_1 < c_2 < \cdots < c_n$.

(c)

$$\begin{array}{c} \text{minimize: } c^T x \\ \text{subject to } \mathbf{1}^T x = \alpha,\ \mathbf{0} \leq x \leq \mathbf{1} \end{array}$$

where α is an integer and $1 \leq \alpha \leq n$. Assume $c_1 < c_2 < \cdots < c_n$.

(d)

$$\begin{array}{c} \text{minimize: } c^T x \\ \text{subject to } d^T x = \alpha,\ \mathbf{0} \leq x \leq \mathbf{1} \end{array}$$

with $d > \mathbf{0}$ and $0 \leq \alpha \leq \mathbf{1}^T d$.

3. Consider the LP

$$\begin{aligned} &\text{minimize } c^T x \\ &\text{subject to } Ax \le b \end{aligned}$$

with A square and nonsingular. Show that the optimal value is given by

$$p^* = \begin{cases} c^T A^{-1} b & A^{-T} c \le 0 \\ -\infty & \text{otherwise} \end{cases}$$

Hint: Transform the problem by making substitution $y = Ax$, and try to solve the transformed problem directly.

4. Formulate the following problems as LP.

   (a)

$$\text{minimize } \|Ax - b\|_\infty$$

   (b)

$$\text{minimize } \|Ax - b\|_1$$

5. Find the optimal solution of the following optimization problem

$$\begin{aligned} &\text{minimize } \frac{x^T x}{2} \\ &\text{subject to } a^T x = b \end{aligned}$$

Approach: First write down the first-order optimality condition with equality constraints, introducing dual variable ν. Then try to find out ν using the constraint.

6. Find the optimal solution of the following quadratic problems.

   (a)

$$\begin{aligned} &\text{minimize: } c^T x \\ &\text{subject to } x^T A x \le 1 \end{aligned}$$

   where $A \in \mathbb{S}^n_{++}$ and $c \neq 0$.

   (b)

$$\begin{aligned} &\text{minimize: } x^T B x \\ &\text{subject to } x^T A x = 1 \end{aligned}$$

   where $A \in \mathbb{S}^n_{++}$ and $B \in \mathbb{S}^n$. Note that the problem does not have the form of convex optimization (still you can solve it, think from eigenvalue perspective).

7. Formulate the following problem as QCQP:

$$\text{minimize: } \|Ax - b\|_4 = \left(\sum_{i=1}^m (a_i^T x - b_i)^4 \right)^{1/4}$$

for $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.