Topics: Convex functions

1. **Running average of convex functions.** Suppose f is convex with $\mathcal{D}(f) = \{x|x > 0\}$. Consider running average F such that
$$F(x) = \frac{1}{x}\int_0^x f(t)dt$$
for $x > 0$. Show that F is convex.
Hint: For some s, is $f(sx)$ convex? Also then is $\int_0^1 f(sx)ds$ convex?

2. For $0 < \alpha \leq 1$ let
$$u_\alpha(x) = \frac{x^\alpha - 1}{\alpha}$$
with $\mathcal{D}(u_\alpha) = \{x|x \geq 0\}$. We also define $u_0(x) = \log x$ for $x > 0$
   (a) Show that
$$u_0(x) = \lim_{\alpha \to 0} u_\alpha(x)$$
   (b) Show that for u_α are concave, monotone increasing and $u_\alpha(1) = 0$

3. For each of the following function determine whether it is convex, concave, quasiconvex or quasiconcave?
   (a) $f(x) = e^x - 1$ on $\mathbb{R}$
   (b) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$
   (c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$
   (d) $f(x_1, x_2) = x_1/x_2$ on $\mathbb{R}^2_{++}$
   (e) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbb{R} \times \mathbb{R}_{++}$
   (f) $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, where $0 \leq \alpha \leq 1$ on $\mathbb{R}^2_{++}$

   (Hint: You would need to work out Hessian for most of the problems. For example for part (f), can you express Hessian as a rank-1 matrix)?

4. Show that the following functions are convex.
   (a) $f(x) = -\log(-\log(\sum_{i=1}^m e^{a_i^T x + b_i}))$ with $\mathcal{D}(f) = \left\{x | \sum_{i=1}^m e^{a_i^T x + b_i} < 1\right\}$.
   (b) $f(x, u, v) = -\sqrt{uv - x^T x}$ with $\mathcal{D}(f) = \left\{(x, u, v) | uv > x^T x, u, v, > 0\right\}$. Use the fact that $x^T x/u$ is convex in (x, u) and that $-\sqrt{x_1 x_2}$ is convex.
   (c) $f(x, u, v) = -\log(uv - x^T x)$ with $\mathcal{D}(f) = \left\{(x, u, v) | uv > x^T x, u, v, > 0\right\}$

5. In general, the product or ratio of two convex functions is not convex. However, there are special cases, and prove the following (assume f and g are functions on $\mathbb{R}$ and are twice differentiable)

   (a) Suppose f, g are convex, both nondecreasing (or both nonincreasing) and positive functions. Show that fg is convex

   (b) Suppose f, g are concave, positive and one nondecreasing and the other nonincreasing. Show that fg is concave.

   (c) Suppose f is convex, nondecreasing, and positive, and g is concave, nonincreasing, and positive. Show that f/g is convex

6. Let X be a real-value random variable which takes value in $\{a_1, \ldots, a_n\}$ with $a_1 < a_2 < \cdots < a_n$ with $\mathbb{P}(X = a_i) = p_i$. Consider the following as functions of $p = (p_1, \ldots, p_n)$ with $p \in \left\{p \in \mathbb{R}^n_+ | \sum_i p_i = 1\right\}$. Determine whether the function is convex or concave in p.

   (a) $f_1(p) = \mathbb{E}[X] := \sum_i a_i p_i$

   (b) $f_2(p) = \mathbb{P}(X > \alpha)$

   (c) $f_3(p) = \mathbb{P}(\alpha \leq X \leq \beta)$

   (d) $f_4(p) = \sum_{i=1}^n p_i \log p_i$ which is called the negative entropy of the distribution.

   (e) $f_5(p) = \text{Var}(X) := \mathbb{E}[(X - \mathbb{E}[X])^2]$.

7. Show that

$$f(x) = \sum_{i=1}^{r} \alpha_i x_{[i]}$$

   is convex in x where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r \geq 0$. *Hint:* Use that $f(x) = \sum_{i=1}^r x_{[i]}$ is convex in x.

8. Show that the following functions are convex

   (a) $f(x) = \max_{i=1,\ldots,k} \|A^{(i)}x - b^{(i)}\|$ where $A^{(i)} \in \mathbb{R}^{m \times n}$, $b^{(i)} \in \mathbb{R}^m$ where $\|\cdot\|$ is a norm in $\mathbb{R}^m$

   (b) $f(x) = \sum_{i=1}^r |x|_{[i]}$ where $|x|$ denotes the vector with $|x|_i = |x_i|$, that is $|x|$ is the component-wise absolute value of x and $|x|_{[i]}$ is the $i$the largest component of $|x|$.

9. Suppose $p(x) \geq 0$ is convex, and $q(x) > 0$ is concave function. Show that

$$f(x) = \frac{p(x)}{q(x)}$$

   is quasiconvex. That is, for any $t \geq 0$, the set $\{x | f(x) \leq t\}$ is a convex set.

10. Show that the following functions are log-concave.

    (a) *Logistic function.* $f(x) = e^x/(1 + e^x)$ for $x \in \mathbb{R}$.

    (b) *Harmonic mean*

    $$f(x) = \frac{1}{\sum_{i=1}^n x_i^{-1}}$$

    for $x_i > 0$.

    (c) *Product over sum*

    $$f(x) = \frac{\prod_{i=1}^n x_i}{\sum_{i=1}^n x_i}$$

    for $x_i > 0$.