Topics: Convex sets

1. Let $C \subset \mathbb{R}^n$ be a convex set. This means that for any two points x_1 and x_2 in C, we have $\theta x_1 + (1-\theta)x_2 \in C$ for any $0 \leq \theta \leq 1$. Extend this to k points, that is, with $x_1, \ldots, x_k \in C$, and let $\theta_1, \ldots, \theta_k \in \mathbb{R}$ satisfy $\theta_i \geq 0,\ \theta_1 + \cdots + \theta_n = 1$. Show that $\theta_1 x_1 + \cdots + \theta_n x_n \in C$. Hint. Use induction on k.

2. A set C is affine if it contains the line that connects two points arbitrarily chosen out of C. An equivalent formal definition is as follows: Consider n points $x_1, \ldots, x_n \in C$, and some n real numbers $\theta_1, \ldots, \theta_n$ such that $\sum_{i=1}^n \theta_i = 1$ (no restriction on positiveness). Then C is affine if and only if $\sum_{i=1}^n \theta_i x_i \in C$ for all $x_1, \ldots, x_n \in C$. Now the problem: consider a set of solutions to linear equations given by

$$C = \{x | Ax = b\}$$

Show that C is an affine set.

3. *When does a halfspace contain another?* Given conditions under which

$$\left\{x | a^T x \leq b\right\} \subseteq \left\{x | \tilde{a}^T x \leq \tilde{b}\right\}$$

where $a, \tilde{a} \in \mathbb{R}^n$ are nonzero vectors and $b, \tilde{b} \in R$. Also find the conditions under which the two halfspaces are equal.

4. *Voronoi description of spaces*

   (a) Let a and b be distinct points in $\mathbb{R}^n$. Show that the set of all points that are closer in 2-norm (or Euclidean norm) to a than b, i.e., $\{x | \|x-a\|_2 \leq \|x-b\|_2\}$, is a halfspace. Describe it explicitly as an inequality of the form $c^T x \leq d$.

   (b) Let $x_0, \ldots, x_k \in \mathbb{R}^n$. Consider the set of points that are closer (in Euclidean norm distance) to x_0 than the other x_i, i.e.,

   $$V = \{x \in \mathbb{R}^n | \|x - x_0\|_2 \leq \|x - x_i\|_2, i = 1, \ldots, k\}.$$

   V is called the *Voronoi region* around x_0 with respect to $x_1, \ldots, x_k$. Show that V is a polyhedron and express V in the form $V = \{x | Ax \leq b\}$.

5. Show that following set S is polyhedra.If possible, express S in the form $S = \{x | Ax \succeq b, Fx = g\}$. Also note we define a vector $\mathbf{1}$ as a vector whose entries are all ones.

   (a) $S = \{y_1 a_1 + y_2 a_2 | -1 \leq y \leq 1, -1 \leq y_2 \leq 1\}$ where $a_1, a_2 \in \mathbb{R}^n$.

   (b) $S = \left\{x \in \mathbb{R}^n | x \succeq 0, \mathbf{1}^T x = 1, \sum_{i=1}^n x_i a_i = b_1, \sum_{i=1}^n x_i a_i^2 = b_2\right\}$ where $a_1, \ldots, a_n \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}$.

6. Which of the following sets are convex?

   (a) A *slab*, i.e., a set of the form $\left\{x \in \mathbb{R}^n | \alpha \leq a^T x \leq \beta\right\}$.

   (b) A *rectangle*, i.e., a set of the form $\{x \in \mathbb{R}^n | \alpha_i \leq x_i \leq \beta_i, i = 1, \ldots, n\}$. A rectangle is sometimes called a hyperrectangle when $n > 2$.

   (c) A wedge ,i.e., $\left\{x \in \mathbb{R}^n | a_1^T x \leq b_1, a_2^T x \leq b_2\right\}$.

   (d) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to B, i.e., the set $\{x | \|x - a\|_2 \leq \theta \|x - b\|_2\}$. You can assume $a \neq b$ and $0 \leq \theta \leq 1$.

7. *Some sets of probability distributions* Let x be a real-valued random variable with $\mathbb{P}(x = a_i) = p_i$, $i = 1, \ldots, n$, wgere $a_1 < a_2 < \cdots < a_n$ and $\mathbb{P}(E)$ denotes the probability that event E happens. In this problem, we will consider probability vector $p = (p_1, \ldots, p_n)$. First, we define a set P such that

$$P = \left\{x \in \mathbb{R}^n | \mathbf{1}^T x = 1, x \geq 0\right\}.$$

That is P is set of vectors such that, the elements are nonnegative and sum up to 1. Because p is a probability vector, p must be in P. In addition, suppose p satisfies the following condition. $\alpha \leq \mathbb{E}[f(x)] \leq \beta$ where $\mathbb{E}[f(x)]$ is the *expected value* of $f(x)$, i.e., $\mathbb{E}[f(x)] = \sum_{i=1}^n p_i f(a_i)$ for a given function $f : \mathbb{R} \to \mathbb{R}$. Show that, p satisfying all the above conditions, must be in a polyhedron.