Topics: Duality

1. Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & x^2+1 \\ \text{subject to} & (x-2)(x-4) \leq 0 \end{array}$$

with variable $x \in \mathbb{R}$

   (a) Give the feasible set, the optimal value, and the optimal solution.

   (b) Plot the objective versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ. Verify the lower bound property $p^* \geq \inf_x L(x, \lambda)$ for $\lambda \geq 0$. Derive and sketch the Lagrange dual function g.

   (c) State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution λ^*. Does strong duality hold?

2. Find the dual function of the LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \leq h \\ & Ax = b \end{array}$$

Also give the dual problem,

3. Derive a dual problem for

$$\text{minimize} \quad \sum_{i=1}^{N} \|A_i x + b_i\}_2 + (1/2)\|x - x_0\|_2^2$$

where $A_i \in \mathbb{R}^{m_i \times n}$, $b_i \in \mathbb{R}^{m_i}$ and $x_0 \in \mathbb{R}^n$. First introduce new variables $y_i = A_i x + b_i$.

4. We consider the convex piecewise-linear minimization problem

$$\text{minimize} \quad \max_{i=1,\ldots,m} (a_i^T x + b_i) \tag{1}$$

with variable $x \in \mathbb{R}^n$

   (a) Derive a dual problem based on the equivalent problem

$$\begin{array}{ll} \text{minimize} & \max_{i=1,\ldots,m} y_i \\ \text{subject to} & a_i^T x + b_i = y_i, \; i = 1, \ldots, m \end{array}$$

   (b) Alternatively, formulate the problem (1) as an LP, and find the dual of that LP. Is that dual problem same as the result in (a)?

(c) Suppose that, instead of hard-pointwise maximum in (1), we use a soft (smooth) max function given by

$$\text{minimize} \quad \log\left(\sum_{i=1}^{m} \exp(a_i^T x + b_i)\right) \tag{2}$$

We would like to derive a dual for problem (2) Show that the dual problem is given by

$$\begin{array}{ll} \text{maximize} & b^T\nu - \sum_{i=1}^{m} \nu_i \log \nu_i \\ \text{subject to} & \mathbf{1}^T\nu = 1,\ \nu \geq 0,\ A^T\nu = 0 \end{array}$$

You may consider problem

$$\begin{array}{ll} \text{minimize} & \log\left(\sum_{i=1}^{m} \exp(y_i)\right) \\ \text{subject to} & y = Ax + b \end{array}$$

(d) Denote the optimal value of (1) and (2) by p^* and p^*_{soft} respectively. Show that

$$0 \leq p^*_{\text{soft}} - p^* \leq \log m$$

You may want to use the fact that the optimal value of entropy maximization problem

$$\begin{array}{ll} \text{maximize} & -\sum_{i=1}^{m} z_i \log z_i \\ \text{subject to} & \mathbf{1}^T z = 1 \end{array}$$

is given by $\log m$ (try to solve the entropy maximization using KKT condition!)

5. Consider the QCQP

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1 \\ & (x_1 - 1)^2 + (x_2 + 1)^2 \leq 1 \end{array}$$

with variable $x \in \mathbb{R}^2$.

(a) Sketch the feasible set and level sets of the objective. Find the optimal point x^* and optimal value p^*.

(b) Give the KKT conditions. Do there exist Lagrange multipliers λ_1^* and λ_2^* that prove that x^* is optimal?

(c) Derive and solve the Lagrange dual problem. Does strong duality hold?

6. Consider the equality constrained least-squares problem

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & Gx = h \end{array}$$

where $A \in \mathbb{R}^{m \times n}$, with rank $A = n$,and $G \in \mathbb{R}^{p \times n}$ with rank $G = p$

(a) Find Lagrange dual function

(b) Find KKT condition.

(c) Derive expressions for the primal solution x^* and the dual solution ν^*.