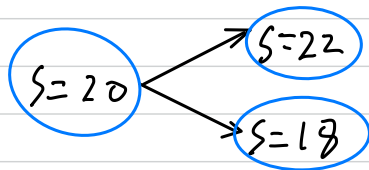


# <Binary Trees>

옵션가격 계산법?

- 이항모형, 상향모형, 블랙숄츠 모형(SDE, PDE)
- 몬테카를로 시뮬레이션, 유한 차분법

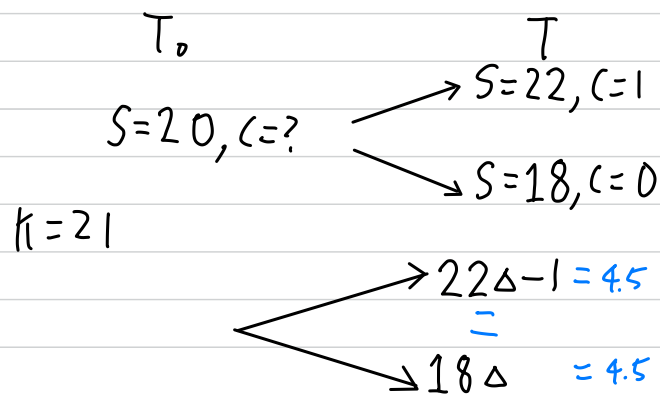


이항모형은 주식의 가격이 오르거나 내리는 것만을 생각한다.

그리고 각각의 노드 별 옵션의 가치를 구하는 것이다.

이항 모형 -> 아메리칸 옵션을 계산하는데 유용하다.

위험 중립을 가정한다.



같이 하여 리스크를 줄인다.

$$22\Delta - 1 = 18\Delta$$

$$4\Delta = 1$$

$$\therefore \Delta = 0.25$$

$r = 0.12$ 라고 가정.

기초자산  $S_0$  \* 0.25 매수

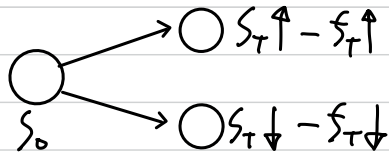
콜옵션 1개 매도

3개월 뒤 포트폴리오의 가치:

$$22 * 0.25 - 1 = 4.5$$

현재시점으로 할인된 포트폴리오의 가치:

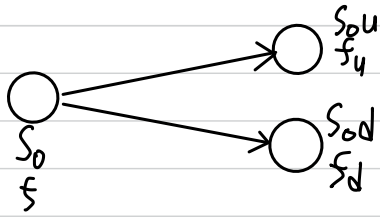
$$4.5e^{-0.12 \cdot \frac{3}{12}} = 4.3670$$



$$(S_0 \cdot \Delta - f_0) = (S_T^up \cdot \Delta - f_T^up) e^{-rT}$$

$$f_0 = S_0 \cdot \Delta - (S_T^up \cdot \Delta - f_T^up) e^{-rT}$$

일반화)  $\Delta$ 만큼 매수하고 옵션을 1계약 매도하는 경우



$$\Delta = \frac{f_u - f_d}{S_0u - S_0d} \quad (\because S_0u \cdot \Delta - f_u = S_0d \cdot \Delta - f_d), (u, d \in \text{Parameter})$$

$$f = S_0 \Delta - (f_u u \Delta - f_u) e^{-rT}$$

$$\begin{cases} f = [p \cdot f_u + (1-p) \cdot f_d] e^{-rT} \\ p = \frac{e^{rT} - d}{u - d} \end{cases}$$

proof)

$$\begin{aligned} [(S_0 \cdot u) \Delta - f_u] e^{-rT} &= S_0 \Delta - f \\ f &= S_0 \Delta - (S_0 \cdot u \Delta - f_u) e^{-rT} \\ &= S_0 \Delta - S_0 \cdot u \cdot \Delta \cdot e^{-rT} + f_u \cdot e^{-rT} \\ &= S_0 \Delta (1 - u e^{-rT}) + f_u \cdot e^{-rT} \\ &= S_0 \cdot \frac{f_u - f_d}{S_0 u - S_0 d} (1 - u e^{-rT}) + f_u e^{-rT} \end{aligned}$$

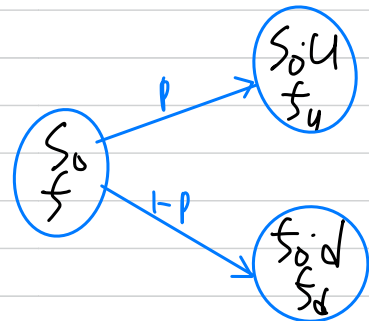
$$\begin{aligned} &= \frac{f_u - f_d}{u - d} (1 - u e^{-rT}) + f_u e^{-rT} \\ &= \frac{1 - d \cdot e^{-rT}}{u - d} f_u + \frac{u e^{-rT} - 1}{u - d} f_d \end{aligned}$$

$$= e^{-rT} \left[ \frac{e^{rT} - d}{u - d} f_u + \frac{u - e^{rT}}{u - d} f_d \right] \quad (\text{where } p = \frac{e^{rT} - d}{u - d})$$

$$= e^{-rT} (p \cdot f_u + (1-p) \cdot f_d)$$

$p$ 는 위험중립을 가정한 상태에서 기초자산이 상승할 확률로 해석된다.

$f = (S_0 \cdot d \cdot \Delta - f_d) e^{-rT}$ 로 가정했을 때를 생각해봐라



$f = (S_0 \cdot d \cdot \Delta - f_d) e^{-rT}$  로 가정했을 때를 생각해봐라

proof)

$$[(S_0 \cdot d) \Delta - f_d] e^{-rT} = S_0 \Delta - f$$

$$f = S_0 \Delta - (S_0 \cdot d \cdot \Delta - f_d) e^{-rT}$$

$$= S_0 \Delta - S_0 \cdot d \cdot \Delta \cdot e^{-rT} + f_d \cdot e^{-rT}$$

$$= S_0 \Delta (1 - d e^{-rT}) + f_d \cdot e^{-rT}$$

$$= S_0 \frac{f_u - f_d}{S_0 u - S_0 d} (1 - d e^{-rT}) + f_d e^{-rT}$$

$$= \frac{f_u - f_d}{u - d} (1 - d e^{-rT}) + f_d e^{-rT}$$

$$= \frac{1 - d e^{-rT}}{u - d} f_u - \frac{1 - d e^{-rT}}{u - d} f_d + e^{-rT} f_d$$

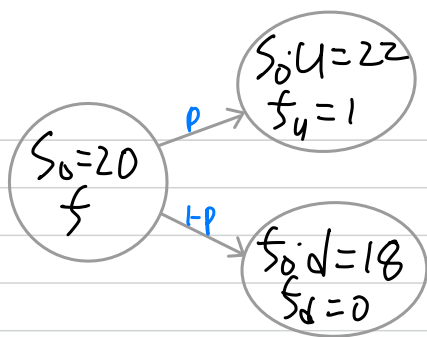
$$= \frac{1 - d e^{-rT}}{u - d} f_u + \frac{d e^{-rT} - 1}{u - d} f_d + \frac{u e^{-rT} - d e^{-rT}}{u - d} f_d$$

$$= \frac{1 - d e^{-rT}}{u - d} f_u + \frac{u e^{-rT} - 1}{u - d} f_d$$

$$= e^{-rT} \left( \frac{e^{rT} - d}{u - d} f_u + \frac{u - e^{rT}}{u - d} f_d \right)$$

$$= e^{-rT} (p f_u + (1-p) f_d) \quad (\text{where } p = \frac{e^{rT} - d}{u - d})$$

$p$ 는 위험중립을 가정한 상태에서 기초자산이 상승할 확률로 해석된다.



$$20e^{-0.12 \times 0.25} = 22 \cdot p + 18(1-p), \quad p = 0.6523$$

$$f = e^{-rT} [p \cdot f_u + (1-p) \cdot f_d]$$

$$= e^{-0.12 \times 0.25} (0.6523 \cdot 1 + 0.3477 \cdot 0)$$

$$= 0.633$$

예제)  $S_0 = 100$ ,  $K = 100$ ,  $u = 1.2$ ,  $d = 0.8$ ,  $r = 0.1$ ,  $T = 1$ , call option?

$$f_u = \max(S_0 \cdot u - K, 0) \quad f_d = \max(S_0 \cdot d - K, 0)$$

$$= \max(120, 0) \quad = \max(-20, 0)$$

$$= 120 \quad = 0$$

$$S_0 \cdot u \cdot \Delta - f_u = S_0 \cdot d \cdot \Delta - f_d$$

$$120\Delta - 120 = 80\Delta$$

$$40\Delta = 120$$

$$\Delta = 0.5$$

$$S_0 \cdot \Delta - f = (S_0 \cdot u \cdot \Delta - f_u) e^{-rT}$$

$$100 \cdot 0.5 - f = (120 \cdot 0.5 - 120) e^{-0.1 \cdot 1}$$

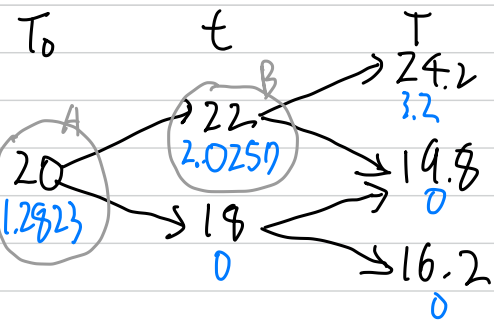
$$50 - f = 40 e^{-0.1 \cdot 1}$$

$$f = 50 - 40 e^{-0.1 \cdot 1}$$

$$\approx 13.8065$$

call option의 가격은 약 13.8065이다.

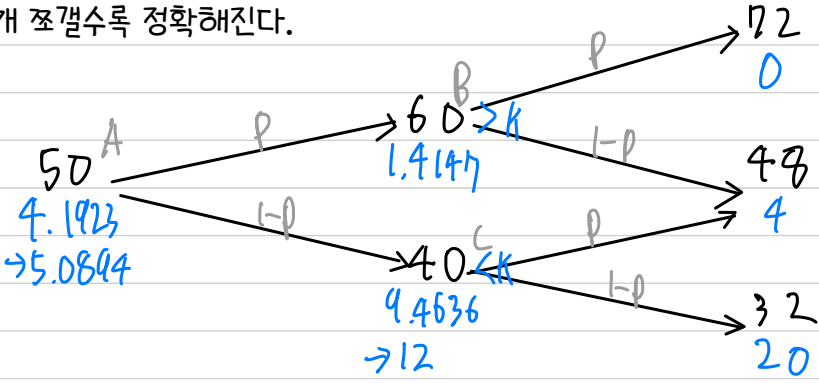
2 step)



$k=21, r=0.12, \text{time step} = \frac{3}{12}$

$B : e^{-0.12 \cdot 0.25} (0.6523 \cdot 3.2 + 0.3477 \cdot 0) = 2.0257$   
 $A : e^{-0.12 \cdot 0.25} (0.6523 \cdot 2.0257 + 0.3477 \cdot 0) = 1.2823$

잘개 쪼갤수록 정확해진다.



$k = 52, \text{time step} = 1,$   
 $r = 0.05, u = 1.32, d = 0.8, p = 0.6282$

$B : e^{-0.05 \cdot 1} (0.6282 \cdot 0 + (1-0.6282) \cdot 4)$   
 $\approx 1.4147$   
 $A : e^{-0.05 \cdot 1} (0.6282 \cdot 1.4147 + (1-0.6282) \cdot 12)$   
 $\approx 5.0894$

Delta  $\Delta$ : "수량"이다.  
S가 변하더라도 델타는 그대로 있기를 원하는 것이다.

$$u = e^{\sigma \sqrt{\Delta t}}$$

$$d = \frac{1}{u} = e^{-\sigma \sqrt{\Delta t}}$$

연기준 분산:  $\sigma^2$ , 연기준 표준편차:  $\sigma \rightarrow \Delta t$  기간 분산:  $\sigma^2 \Delta t$ ,  $\Delta t$  기간 표준편차:  $\sigma \cdot \sqrt{\Delta t}$

확률변수  $X$

$$E[X], \text{Var}[X] = E[X^2] - (E[X])^2, E[X+Y] = E[X] + E[Y]$$

$$P \cdot S_0 \cdot u + (1-P) \cdot S_0 \cdot d = S_0 \cdot e^{r \cdot \Delta t}$$

$$P \cdot u + (1-P) \cdot d = e^{r \cdot \Delta t}$$

$$P \cdot u + d - P \cdot d = e^{r \cdot \Delta t}$$

$$P = \frac{e^{r \cdot \Delta t} - d}{u - d}, 1-P = \frac{u - e^{r \cdot \Delta t}}{u - d}$$

$$E[X^2] - (E[X])^2$$

$$= P u^2 + (1-P) d^2 - (P u + (1-P) d)^2$$

$$= P u^2 + (1-P) d^2 - P^2 u^2 - 2P u (1-P) d - (1-P)^2 d^2$$

$$= P u^2 + d^2 - P d^2 - P^2 u^2 - 2P u d + 2P^2 u d - d^2 + 2P d^2 - P^2 d^2$$

$$= P(u^2 - d^2 - 2u d + 2d^2) - P(u^2 - 2u d + d^2)$$

$$= P(u-d)^2 - P^2(u-d)^2$$

$$= P(1-P)(u-d)^2$$

$$= \frac{e^{r \cdot \Delta t} - d}{u - d} \cdot \frac{u - e^{r \cdot \Delta t}}{u - d} (u-d)^2$$

$$= (e^{r \cdot \Delta t} - d)(u - e^{r \cdot \Delta t})$$

$$= (u+d) e^{r \cdot \Delta t} - e^{2 \cdot r \cdot \Delta t} - u d$$

$$= \sigma^2 \Delta t$$

$$\therefore \sigma^2 \cdot \Delta t = (u+d) e^{r \cdot \Delta t} - e^{2 \cdot r \cdot \Delta t} - u d \quad (u = \frac{1}{d})$$

$$\Rightarrow u = e^{\sigma \sqrt{\Delta t}}, d = e^{-\sigma \sqrt{\Delta t}}$$

$$\rightarrow u = 1 + \sigma \sqrt{\Delta t} + \frac{\sigma^2}{2} \Delta t$$

$$d = 1 - \sigma \sqrt{\Delta t} + \frac{\sigma^2}{2} \Delta t$$

$$e^{r \Delta t} = 1 + r \Delta t$$

$$e^{2r \Delta t} = 1 + 2r \Delta t$$

$$\therefore \sigma^2 \cdot \Delta t = (1+r \Delta t)(1+\sigma^2 \Delta t) - 1 - (1+2r \Delta t)$$

$$= 1 + \sigma^2 \cdot r \cdot \Delta t^2 + 2r \cdot \sigma^2 \Delta t + \sigma^2 \Delta t - 1 - 1 - 2r \Delta t$$

$$\sigma^2 \cdot \Delta t = \sigma^2 \Delta t$$

$$(a+b+c)(a-b+c)$$

$$= a^2 - b^2 + c^2 - ab + ac + ab + bc + ac - bc$$

$S_0 = 50, K = 52, r = 0.02, T = 2, \Delta t = 1, N = 2, \sigma = 0.3$   
 $P^A = ?$

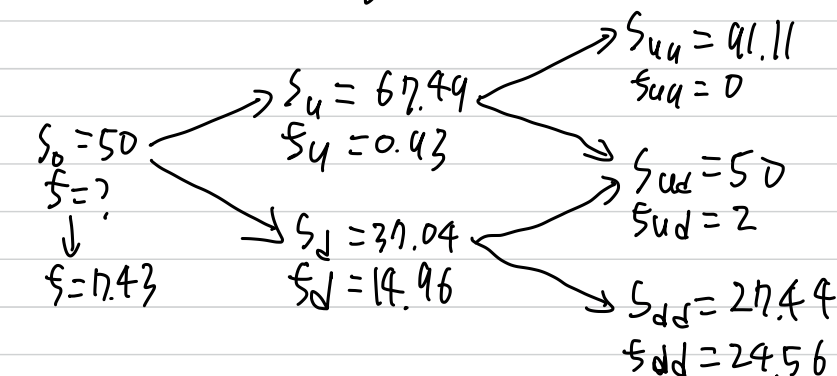
$$u = e^{\sigma \sqrt{\Delta t}}$$

$$= e^{0.3 \cdot \sqrt{1}}$$

$$\approx 1.3444$$

$$d \approx 0.7406$$

$$\Rightarrow p = \frac{e^{0.05 \cdot 1} - 0.7406}{1.3444 - 0.7406} = 0.5097$$



배당금이 있는 경우?

$$a = e^{(r-q)\Delta t}$$

기초자산의 배당률 ( $q$ )가 주어진 경우

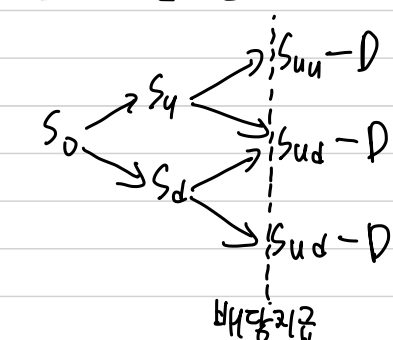
$$[(S_0 * u * \Delta) e^{qT} - f_u] e^{rT} = S_0 * \Delta - f_0$$

$$[(S_0 * d * \Delta) e^{qT} - f_d] e^{rT} = S_0 * \Delta - f_0$$

$$f_0 = e^{-rT} [p \cdot f_u - (1-p) f_d]$$

$$\text{Where } p = \frac{e^{(r-q)T} - d}{u - d}$$

현금 배당  $D$ 를 지급하는 경우



왜 배당을 지급하면 주가가 하락하는가?

1. 배당주 : 주가 100, 주식수 100, 배당 10  $\rightarrow$  시가 총액 :  $100 * 100$

배당지급 : 시가총액이 고정되어 있는 상황에서  $10000 = 110 * X$

주가가격: 91

2. 배당금 : 배당일 이후 주식을 사는 사람은 배당금을 받을 권리가 없으므로 그 주식의 가치가 하락함

N step)

$$f_0 = e^{-rT} \left[ \sum_{i=0}^N \binom{N}{i} p^i (1-p)^{N-i} f_{u^i d^{N-i}} \right]$$

$$\begin{aligned} N=2: f_0 &= e^{-rT} [ \binom{2}{0} p^2 (1-p)^0 f_{uu} + \binom{2}{1} p^1 (1-p)^1 f_{ud} + \binom{2}{2} p^0 (1-p)^2 f_{dd} ] \\ &= e^{-rT} [ p^2 f_{uu} + 2p(1-p) f_{ud} + (1-p)^2 f_{dd} ] \end{aligned}$$