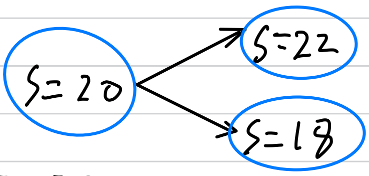
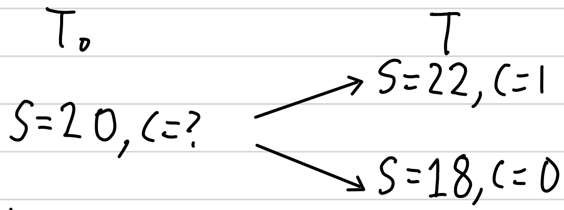
<Binary Trees>

옵션가격 계산법?
- 이항모형, 상항모형, 블랙숄츠 모형(SDE, PDE)
- 몬테카를로 시뮬레이션, 유한 차분법

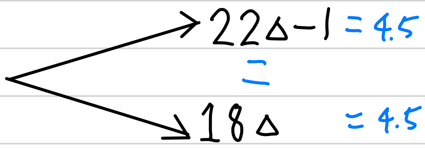


이항모형은 주식의 가격이 오르거나 내리는 것만을 생각한다.
그리고 각각의 노드 별 옵션의 가치를 구하는 것이다.
이항 모형 -> 아메리칸 옵션을 계산하는데 유용하다.

위험 중립을 가정한다.



$K=21$



같게 하여 리스크를 줄인다.

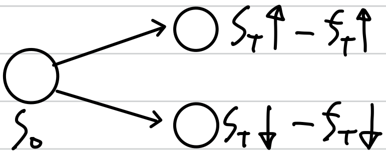
$22\Delta - 1 = 18\Delta$

$4\Delta = 1$

$\therefore \Delta = 0.25$

$r = 0.12$라고 가정.
기초자산 $S_0 * 0.25$ 매수
콜옵션 1개 매도

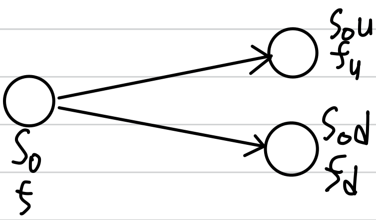
3개월 뒤 포트폴리오의 가치:
$22 * 0.25 - 1 = 4.5$
현재시점으로 할인된 포트폴리오의 가치:
$4.5e^{-0.12 \cdot \frac{3}{12}} = 4.3670$



$$(S_0\cdot\Delta - f_0) = (S_T\uparrow\cdot\Delta - f_T\uparrow)e^{-rT}$$

$$f_0 = S_0\cdot\Delta - (S_T\uparrow\cdot\Delta - f_T\uparrow)e^{-rT}$$

일반화) Δ만큼 매수하고 옵션을 1계약 매도하는 경우



$$\Delta = \frac{f_u - f_d}{S_0u - S_0d} \quad (\because S_0u\cdot\Delta - f_u = S_0d\cdot\Delta - f_d),\ (u, d\text{는 Parameter})$$

$$f = S_0\Delta - (S_0u\Delta - f_u)e^{-rT}$$

$$f = [p\cdot f_u + (1-p)\cdot f_d]e^{-rT}$$

$$p = \frac{e^{rT} - d}{u - d}$$

proof)

$$[(S_0\cdot u)\Delta - f_u]e^{-rT} = S_0\Delta - f$$

$$f = S_0\Delta - (S_0\cdot u\cdot\Delta - f_u)e^{-rT}$$

$$= S_0\Delta - S_0\cdot u\cdot\Delta\cdot e^{-rT} + f_u\cdot e^{-rT}$$

$$= S_0\Delta(1 - ue^{-rT}) + f_u\cdot e^{-rT}$$

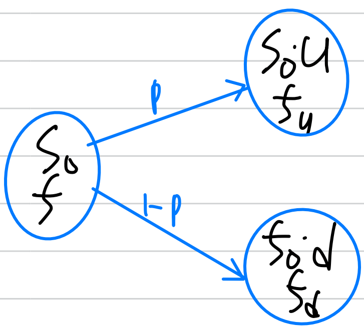
$$= S_0\cdot\frac{f_u - f_d}{S_0\cdot u - S_0\cdot d}(1 - ue^{-rT}) + f_u e^{-rT}$$

$$= \frac{f_u - f_d}{u - d}(1 - ue^{-rT}) + f_u e^{-rT}$$

$$= \frac{1 - d\cdot e^{-rT}}{u - d}f_u + \frac{ue^{-rT} - 1}{u - d}f_d$$

$$= e^{-rT}\left[\frac{e^{rT} - d}{u - d}f_u + \frac{u - e^{rT}}{u - d}f_d\right] \quad (\text{where } p = \frac{e^{rT} - d}{u - d})$$

$$= e^{-rT}(p\cdot f_u + (1-p)\cdot f_d)$$



p는 위험중립을 가정한 상태에서 기초자산이 상승할 확률로 해석된다.

$f = (S_0\cdot d\cdot\Delta - f_d)e^{-rT}$로 가정했을 때를 생각해봐라

$f = (S_0 \cdot d \cdot \Delta - f_d) e^{-rT}$ 로 가정했을 때를 생각해봐라

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$$[(S_0 \cdot d)\Delta - f_d] e^{rT} = S_0 \Delta - f$$

$$f = S_0 \Delta - (S_0 \cdot d \cdot \Delta - f_d) e^{-rT}$$

$$= S_0 \Delta - S_0 \cdot d \cdot \Delta \cdot e^{-rT} + f_d \cdot e^{-rT}$$

$$= S_0 \Delta (1 - d e^{-rT}) + f_d \cdot e^{-rT}$$

$$= S_0 \cdot \frac{f_u - f_d}{S_0 \cdot u - S_0 \cdot d} (1 - d e^{-rT}) + f_d e^{-rT}$$

$$= \frac{f_u - f_d}{u - d} (1 - d e^{-rT}) + f_d \cdot e^{-rT}$$

$$= \frac{1 - d e^{-rT}}{u - d} f_u - \frac{1 - d e^{-rT}}{u - d} f_d + e^{-rT} \cdot f_d$$

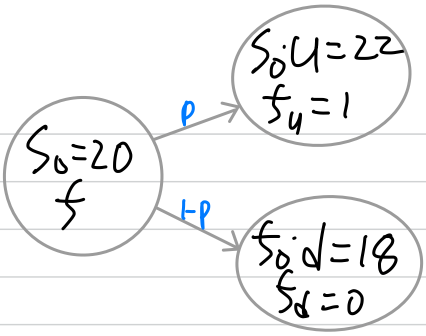
$$= \frac{1 - d e^{-rT}}{u - d} f_u + \frac{d e^{-rT} - 1}{u - d} f_d + \frac{u e^{-rT} - d e^{-rT}}{u - d} f_d$$

$$= \frac{1 - d e^{-rT}}{u - d} f_u + \frac{u e^{-rT} - 1}{u - d} f_d$$

$$= e^{-rT} \left(\frac{e^{rT} - d}{u - d} f_u + \frac{u - e^{rT}}{u - d} f_d \right)$$

$$= e^{-rT} (p \cdot f_u + (1-p) f_d) \quad (\text{where } p = \frac{e^{rT} - d}{u - d})$$

p는 위험중립을 가정한 상태에서 기초자산이 상승할 확률로 해석된다.



$20e^{-0.12\times0.25} = 22\cdot p + 18(1-p)$, $p = 0.6523$

$f = e^{-rT}[p\cdot f_u + (1-p)f_d]$

$= e^{-0.12\cdot0.25}(0.6523\cdot 1 + 0.3477\cdot 0)$

$= 0.633$

예제) $S_0 = 100$, $k = 100$, $u = 1.2$, $d = 0.8$, $r = 0.1$, $T = 1$, call option?

$f_u = \max(S_0\cdot u - k, 0)$ $\quad f_d = \max(S_0\cdot d - k, 0)$

$= \max(20, 0)$ $\quad = \max(-20, 0)$

$= 20$ $\quad = 0$

$S_0\cdot u\cdot\Delta - f_u = S_0\cdot d\cdot\Delta - f_d$

$120\Delta - 20 = 80\Delta$

$40\Delta = 20$

$\Delta = 0.5$

$S_0\cdot\Delta - f = (S_0\cdot u\cdot\Delta - f_u)e^{-rT}$

$100\cdot 0.5 - f = (120\cdot 0.5 - 20)e^{-0.1\cdot 1}$

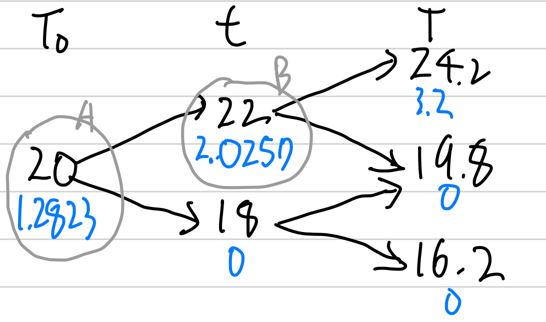
$50 - f = 40e^{-0.1\cdot 1}$

$f = 50 - 40e^{-0.1\cdot 1}$

$\simeq 13.8065$

call option의 가격은 약 13.8065이다.

2 step)

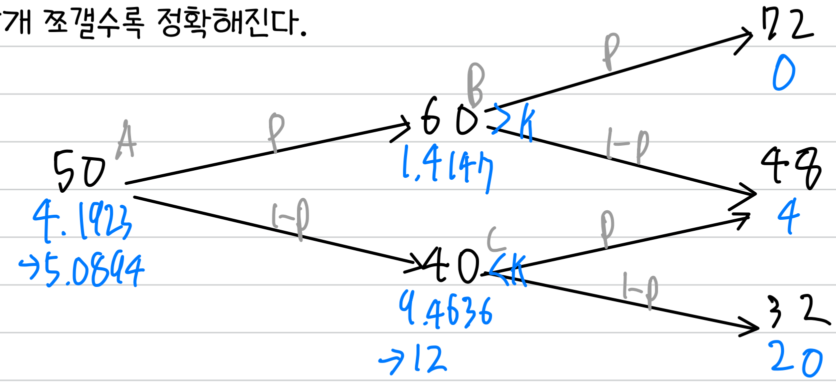


$K=21$, $r=0.12$, time step $= \frac{3}{12}$

B : $e^{-0.12 \cdot 0.25}(0.6523*3.2 + 0.3477*0) = 2.0257$

A : $e^{-0.12 \cdot 0.25}(0.6523*2.0257 + 0.3477*0) = 1.2823$

잘게 쪼갤수록 정확해진다.



$k = 52$, time step $= 1$,

$r = 0.05$, $u = 1.32$, $d = 0.8$, $p = 0.6282$

B : $e^{-0.05 \cdot 1}(0.6282*0 + (1-0.6282)*4)$

≈ 1.4147

A : $e^{-0.05 \cdot 1}(0.6282*1.4147 + (1-0.6282)*12)$

≈ 5.0894

Delta Δ: "수량"이다.

S가 변하더라도 델타는 그대로 있기를 원하는 것이다.

$u = e^{\sigma\sqrt{\Delta t}}$
$d = \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}}$

연기준 분산 : σ^2, 연기준 표준편차 : $\sigma \rightarrow \Delta t$ 기간 분산 : $\sigma^2 \Delta t$, Δt기간 표준편차 : $\sigma \cdot \sqrt{\Delta t}$

확률변수 X

$E[X], Var[X] = E[X^2] - (E[X])^2, E[X+Y] = E[X] + E[Y]$

$p \cdot \cancel{S_0} \cdot u + (1-p) \cdot \cancel{S_0} \cdot d = \cancel{S_0} \cdot e^{r \cdot \Delta t}$

X

$p \cdot u + (1-p) \cdot d = e^{r \cdot \Delta t}$
$p \cdot u + d - p \cdot d = e^{r \cdot \Delta t}$

$p = \frac{e^{r \cdot \Delta t} - d}{u-d}, 1-p = \frac{u - e^{r \cdot \Delta t}}{u-d}$

$E[X^2] - (E[X])^2$
$= pu^2 + (1-p)d^2 - (pu + (1-p)d)^2$
$= pu^2 + (1-p)d^2 - p^2u^2 - 2pu(1-p)d - (1-p)^2 d^2$
$= pu^2 + \cancel{d^2} - pd^2 - p^2u^2 - 2pud + 2p^2ud - \cancel{d^2} + 2pd^2 - p^2d^2$
$= p(u^2 - d^2 - 2ud + 2d^2) - p(u^2 - 2ud + d^2)$
$= p(u-d)^2 - p^2(u-d)^2$
$= p(1-p)(u-d)^2$
$= \frac{e^{r\Delta t} - d}{u-d} \cdot \frac{u - e^{r \cdot \Delta t}}{u-d} (u-d)^2$
$= (e^{r \cdot \Delta t} - d)(u - e^{r \cdot \Delta t})$
$= (u+d)e^{r \cdot \Delta t} - e^{2 \cdot r \cdot \Delta t} - ud$
$= \sigma^2 \Delta t$

$\therefore \sigma^2 \cdot \Delta t = (u+d)e^{r \cdot \Delta t} - e^{2 \cdot r \cdot \Delta t} - ud \quad (u = \frac{1}{d})$
$\Rightarrow u = e^{\sigma\sqrt{\Delta t}}, d = e^{-\sigma \cdot \sqrt{\Delta t}}$
$\rightarrow u = 1 + \sigma\sqrt{\Delta t} + \frac{\sigma^2}{2}\Delta t$
$d = 1 - \sigma\sqrt{\Delta t} + \frac{\sigma^2}{2}\Delta t$

$e^{r\Delta t} = 1 + r\Delta t$
$e^{2r\Delta t} = 1 + 2r\Delta t$

$(a+b+c)(a-b+c)$
$= a^2 - b^2 + c^2 - \cancel{ab} + ac + \cancel{ab} + \cancel{bc} + ac - \cancel{bc}$

$\therefore \sigma^2 \cdot \Delta t = (1 + r\Delta t)(2 + \sigma^2\Delta t) - 1 - (1 + 2r\Delta t)$
$= \cancel{2} + \cancel{\sigma^2 \cdot r \cdot \Delta t^2} + \cancel{2r \cdot \Delta t} + \sigma^2\Delta t - \cancel{1} - \cancel{1} - \cancel{2r\Delta t}$
$\sigma^2 \cdot \Delta t = \sigma^2 \Delta t$

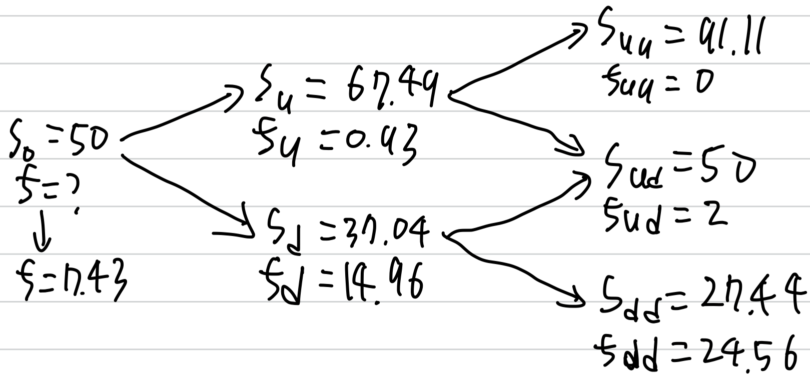
$S_0 = 50, K = 52, r = 0.02, T = 2, \Delta t = 1, N = 2, \sigma = 0.3$

$P^A = ?$

$u = e^{\sigma\sqrt{\Delta t}}$
$= e^{0.3\cdot\sqrt{1}}$
≈ 1.3499

$d \approx 0.7408$

$\Rightarrow p = \frac{e^{0.05\cdot 1} - 0.7408}{1.3499 - 0.7408} = 0.5097$



배당금이 있는 경우?

$a = e^{(r-q)\Delta t}$

기초자산의 배당률 (q)가 주어진 경우

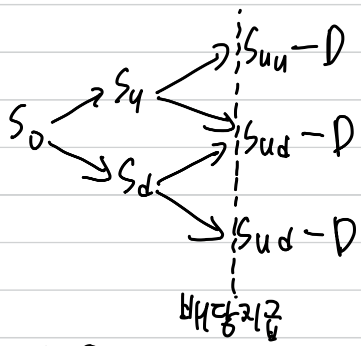
$[(S_0 * u * \Delta)e^{qT} - f_u]e^{-rT} = S_0 * \Delta - f_0$

$[(S_0 * d * \Delta)e^{qT} - f_d]e^{-rT} = S_0 * \Delta - f_0$

$f_0 = e^{-rT}[p\cdot f_u - (1-p) f_d]$

Where $p = \frac{e^{(r-q)T} - d}{u - d}$

현금 배당 D를 지급하는 경우



왜 배당을 지급라면 주가가 하락하는가?

1. 배당주 : 주가 100, 주식수 100, 배당 10 -> 시가 총액 : 100 * 100

주가가격 : 91

배당지급 : 시가총액이 고정되어 있는 상황에서 10000 = 110*x

2. 배당금 : 배당일 이후 주식을 사는 사람은 배당금을 받을 권리가 없으므로 그 주식의 가치가 하락함

N step)

$$f_0 = e^{-rT}\left[\sum_{i=0}^{N} {}_N C_i \, p^i (1-p)^{N-i} f_{u^i d^{N-i}}\right]$$

$$N=2: \; f_0 = e^{-rT}\left[{}_2C_0 \, p^2 (1-p)^0 f_{uu} + {}_2C_1 \, p^1 (1-p)^1 f_{ud} + {}_2C_2 \, p^0 (1-p)^2 f_{dd}\right]$$
$$= e^{-rT}\left[p^2 f_{uu} + 2p(1-p) f_{ud} + (1-p)^2 f_{dd}\right]$$