1)

if $\mathrm{Cov}(\bar{X}, X_i - \bar{X}) = 0$

then $Cov(\bar{X}, S^2) = 0 \rightarrow \bar{X}$ and S^2 are independent

$$Cov(\bar{X}, X_i - \bar{X})$$
$$= Cov(\bar{X}, X_i) - Cov(\bar{X}, \bar{X})$$
$$= \frac{1}{n}\sum_{j=1}^{n} Cov(X_j, X_i) - Var(\bar{X})$$
$$= \frac{1}{n} Cov(X_i, X_i) - \frac{\sigma^2}{n} = 0$$

2)

$$\sum_{i=1}^{n}(X_i - \overline{X})^2 = \sum_{i=1}^{n}(X_i - \mu)^2 - n(\overline{X} - \mu)^2$$

$$\sum_{i=1}^{n}(X_i - \mu)^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2 + n(\overline{X} - \mu)^2$$

$$\frac{\sum_{i=1}^{n}(X_i - \mu)^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{\sigma^2} + \frac{n(\overline{X} - \mu)^2}{\sigma^2}$$

$$\Rightarrow \sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2 = \frac{(n-1)S^2}{\sigma^2} + \left(\frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}\right)^2$$

$$\sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi_n^2 \quad \text{and} \quad \left(\frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}\right)^2 \sim \chi_1^2.$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2. \quad \begin{pmatrix} \mathrm{X} = \mathrm{Y} + \mathrm{Z} \\ \mathrm{M}_X(t) = M_Y(t) M_Z(t) \\ M_Y(t) = \frac{M_X(t)}{M_Z(t)} \end{pmatrix}$$

Corollary 6.5.2

$$\frac{\overline{X} - \mu}{\frac{S}{\sqrt{n}}} \sim t_{n-1}$$

proof

$$\frac{\overline{X} - \mu}{\frac{S}{\sqrt{n}}} = \frac{\frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}}{\frac{S}{\sigma}} = \frac{\frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}}{\sqrt{\frac{\frac{(n-1)S^2}{\sigma^2}}{n-1}}} \sim \frac{Z}{\sqrt{\frac{\chi_{n-1}^2}{n-1}}} \sim t_{n-1}$$