Chapter 6. Distribution of Sampling Statistics

6.2 The sample mean

- $\mathbb{E}[\overline{X}] = \frac{1}{n}\mathbb{E}[X_1 + \cdots + X_n] = \frac{1}{n} \cdot n \cdot \mathbb{E}X = \mu$
- $\mathrm{Var}\overline{X} = \frac{1}{n^2}\mathrm{Var}(X_1 + \cdots + X_n) = \frac{1}{n^2} \cdot n\sigma^2 = \frac{\sigma^2}{n}$

6.3 The central limit theorem

$$X_1 + X_2 + \cdots + X_n \approx N(n\mu, n\sigma^2)$$

$$\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sqrt{n}\sigma} \longrightarrow Z \text{ in distribution, as } n \to \infty.$$

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sqrt{n}\sigma} \leq x\right) = \Phi(x)$$

$$\approx P\{Z < x\}$$

proof of central limit theorem

μ=0, σ^2 =1 이라고 가정하면

$$\frac{X_1 + X_2 + \ldots + X_n - n\mu}{\sqrt{n}\,\sigma} = \frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}} \rightarrow z\sim N(0, 1)$$

정규화를 각각의 X_i에 시키면 일반적인 경우에도 사용가능하다.

z의 mgf는 $e^{\frac{t^2}{2}}$ 이다.

$\frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}}$ 의 mgf가 $e^{\frac{t^2}{2}}$이라는 것을 보이겠다.

$$M(t) \equiv E\left[e^{tX_i}\right], \quad L(t) = \log M(t)$$

$$E\left[e^{t\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i}\right] = \left[M\left(\frac{t}{\sqrt{n}}\right)\right]^n = M_n(t)$$

$$\begin{aligned}
L(0) &= 0 \\
L'(0) &= \frac{M'(0)}{M(0)} \\
&= \mu \\
&= 0 \\
L''(0) &= \frac{M(0)M''(0) - [M'(0)]^2}{[M(0)]^2} \\
&= E[X^2] \\
&= 1
\end{aligned}$$

$\log M_n(t) = n \cdot \log M(\frac{t}{\sqrt{n}}) = \frac{L(t/\sqrt{n})}{n^{-1}}$

$$\begin{aligned}
\lim_{n\to\infty} \frac{L(t/\sqrt{n})}{n^{-1}} &= \lim_{n\to\infty} \frac{-L'(t/\sqrt{n})n^{-3/2}t}{-2n^{-2}} \\
&= \lim_{n\to\infty} \left[\frac{L'(t/\sqrt{n})t}{2n^{-1/2}}\right] \\
&= \lim_{n\to\infty} \left[\frac{-L''(t/\sqrt{n})n^{-3/2}t^2}{-2n^{-3/2}}\right] \\
&= \lim_{n\to\infty} \left[L''\left(\frac{t}{\sqrt{n}}\right)\frac{t^2}{2}\right] \\
&= \frac{t^2}{2}
\end{aligned}$$