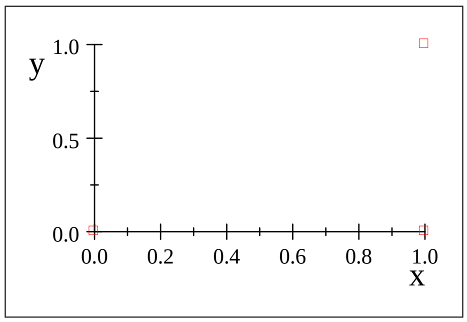
Teorema de Green

$\oint\limits_C Pdx + Qdy = \int\int_D \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dA$

donde $C$ corre en sentido horario

---

Evaluar $\oint\limits_C x^2 dx - xydy$ donde $C$ es el triángulo que forman los puntos $(0,0), (1,0)$ y $(1,1)$.




Primer tramo:

$\int_0^1 x^2 dx - x(0)(0) = \int_0^1 x^2 dx = \frac{1}{3}$

Segundo tramo:

$\int_0^1 (1)^2 (0) - 1ydy = \int_0^1 -ydy = -\frac{1}{2}$

Tercer tramo:

$\int_1^0 x^2 dx - xxdx = 0$

El resultado final es la suma de los 3:

$\frac{1}{3} - \frac{1}{2} + 0 = -\frac{1}{6}$

---

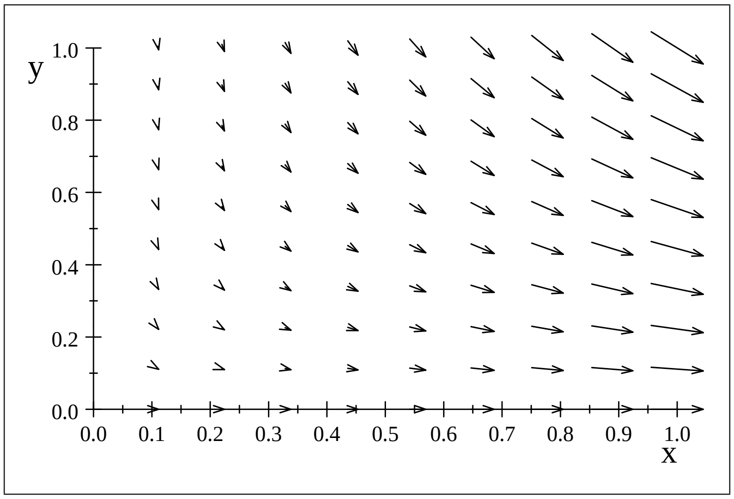
$\oint\limits_C Pdx + Qdy = \int\int_D \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dA$

$\oint\limits_C x^2 dx - xydy = \int\int_D (0 - (-y)) dA = \int\int_D ydA$

$\int_0^1 \int_0^x ydydx = \frac{1}{6}$

Como $C$ corre antihorario el verdadero resultado es $-\frac{1}{6}$

$F = \begin{bmatrix} x^2 \\ -xy \end{bmatrix}$




---

Evalúe $\oint\limits_C x^2ydx + 3xdy$ donde $C$ es el círculo en el origen de radio 2 en sentido horario. $\quad x^2 + y^2 = r^2$

$x^2 + y^2 = 4$

$y = \pm\sqrt{4 - x^2}$

$dy = \pm\frac{1}{2}\left(4 - x^2\right)^{-\frac{1}{2}}(-2x) = \pm - \frac{x}{\sqrt{4-x^2}}dx$

$\int_{-2}^{2}\left(x^2\sqrt{4-x^2} + 3x\left(-\frac{x}{\sqrt{4-x^2}}\right)\right)dx = -4\pi$

$\int_{2}^{-2}\left(-x^2\sqrt{4-x^2} + 3x\left(\frac{x}{\sqrt{4-x^2}}\right)\right)dx = -4\pi$

$-4\pi - 4\pi = -8\pi$

Ahora con Green en $(x, y)$

$\oint\limits_C x^2ydx + 3xdy = \int\int_D \left(x^2 - 3\right)dA$

$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(x^2 - 3\right)dydx = -8\pi$

Ahora haremos ambas (línea y área) en coordenadas polares:

$x = 2\cos\theta \qquad y = 2\sin\theta$

$dx = -2\sin\theta d\theta \qquad dy = 2\cos\theta d\theta$

$\oint\limits_C x^2ydx + 3xdy = \int_{2\pi}^{0}(2\cos\theta)^2(2\sin\theta)(-2\sin\theta d\theta) + 3(2\cos\theta)(2\cos\theta d\theta)$

$= \int_{2\pi}^{0}\left((2\cos\theta)^2(2\sin\theta)(-2\sin\theta) + 3(2\cos\theta)(2\cos\theta)\right)d\theta = -8\pi$

Ahora Green en polares:

$x = r\cos\theta \qquad y = r\sin\theta$

$dA = rdrd\theta$

$\int\int_D \left(x^2 - 3\right)dA = \int_0^{2\pi}\int_0^2\left((r\cos\theta)^2 - 3\right)rdrd\theta = -8\pi$

---

Operaciones sobre funciones vectoriales y escalares

- Gradiente $\nabla f = F$ Esc->Vect
  $\operatorname{grad} f$
- Divergencia $\nabla \cdot F = f$ Vect->Esc
  $\operatorname{div} F$
- Rotacional $\nabla \times F = F$ Vect->Vect
  $\operatorname{curl} F$ $\operatorname{rot} F$
- Laplaciano $\nabla^2 f = f$ Esc->Esc
  $\operatorname{lapl} f$

---

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

$$\nabla f = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} f = \begin{bmatrix} \frac{\partial}{\partial x} f \\ \frac{\partial}{\partial y} f \\ \frac{\partial}{\partial z} f \end{bmatrix}$$

$$\nabla \cdot F = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3$$

$$\nabla \times F = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial y} F_3 - \frac{\partial}{\partial z} F_2 \\ \frac{\partial}{\partial z} F_1 - \frac{\partial}{\partial x} F_3 \\ \frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1 \end{bmatrix}$$

$$\nabla^2 f = \nabla \cdot (\nabla f) = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} f = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial x} f \\ \frac{\partial}{\partial y} f \\ \frac{\partial}{\partial z} f \end{bmatrix} = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f + \frac{\partial^2}{\partial z^2} f$$

---

$\operatorname{grad}(\operatorname{div} F)$ Si, vector

$\operatorname{div}(\operatorname{div} F)$ No, la segunda divergencia no es posible

$\nabla(\nabla \cdot (\nabla \times F)) = \operatorname{grad}(\operatorname{div}(\operatorname{curl} F))$ Si, vector

$\nabla \cdot (\nabla(\nabla \times F))$ No, porque no se puede calcular un gradiente de un vector

---

$F = x^2 z\mathrm{i} + \sin y\mathrm{j} - 3xz\mathrm{k}$

$$F = \begin{bmatrix} x^2 z \\ \sin y \\ -3xz \end{bmatrix}$$

Calcular

$$\operatorname{curl} \begin{bmatrix} x^2 z \\ \sin y \\ -3xz \end{bmatrix} = \begin{bmatrix} 0 \\ x^2 + 3z \\ 0 \end{bmatrix}$$

$$\operatorname{div} \begin{bmatrix} x^2 z \\ \sin y \\ -3xz \end{bmatrix} = \cos y - 3x + 2xz$$

$$\nabla \left( \nabla \cdot \begin{bmatrix} x^2 z \\ \sin y \\ -3xz \end{bmatrix} \right) = \begin{bmatrix} 2z - 3 \\ -\sin y \\ 2x \end{bmatrix}$$