Имеем такие наблюдения:

$$I_k = A_k \sum_{l=0}^{L-1} \cos(\omega_0 t_{k,l} + \omega_{\partial\, k} l T_d + \varphi_k) \cdot \cos(\omega_0 t_{k,l} + \breve{\omega}_{\partial\, k} l T_d + \breve{\varphi}_k) + n_{I\,k}$$
$$Q_k = A_k \sum_{l=0}^{L-1} \cos(\omega_0 t_{k,l} + \omega_{\partial\, k} l T_d + \varphi_k) \cdot \sin(\omega_0 t_{k,l} + \breve{\omega}_{\partial\, k} l T_d + \breve{\varphi}_k) + n_{Q\,k}$$

Введем обозначения:

$$\bar{I}_k = A_k \sum_{l=0}^{L-1} \cos(\Phi_{k,l}) \cdot \cos(\breve{\Phi}_{k,l})$$
$$\bar{Q}_k = A_k \sum_{l=0}^{L-1} \cos(\Phi_{k,l}) \cdot \sin(\breve{\Phi}_{k,l})$$
$$\Phi_{k,l} = \omega_0 t_{k,l} + \omega_{\partial\, k} l T_d + \varphi_k$$
$$\breve{\Phi}_{k,l} = \omega_0 t_{k,l} + \breve{\omega}_{\partial\, k} l T_d + \breve{\varphi}_k$$

Тогда наблюдения перепишем так

$$I_k = \bar{I}_k + n_{I\,k}$$
$$Q_k = \bar{Q}_k + n_{Q\,k}$$

Запишем функцию правдоподобия $p(I_k, Q_k \mid \boldsymbol{\lambda}_k)$, где $\boldsymbol{\lambda}_k = \begin{vmatrix} \varphi_k \\ \omega_{\partial\, k} \end{vmatrix}$

$$p(I_k, Q_k \mid \boldsymbol{\lambda}_k) = \frac{1}{\sqrt{2\pi\sigma_{IQ}^2}} \exp\left(-\frac{(I_k - \bar{I}_k(\boldsymbol{\lambda}_k))^2}{2\sigma_{IQ}^2} \right) \cdot \frac{1}{\sqrt{2\pi\sigma_{IQ}^2}} \exp\left(-\frac{(Q_k - \bar{Q}_k(\boldsymbol{\lambda}_k))^2}{2\sigma_{IQ}^2} \right) =$$
$$= \frac{1}{2\pi\sigma_{IQ}^2} \exp\left(-\frac{1}{2\sigma_{IQ}^2} \left[(I_k - \bar{I}_k(\boldsymbol{\lambda}_k))^2 + (Q_k - \bar{Q}_k(\boldsymbol{\lambda}_k))^2 \right] \right)$$
$$\left\{ \frac{1}{2\pi\sigma_{IQ}^2} = B \right\}$$

Возьмем ln:

$$\ln p(I_k, Q_k \mid \boldsymbol{\lambda}_k) = -\frac{\ln B}{2\sigma_{IQ}^2} \cdot \left(I_k^2 - 2 I_k \bar{I}_k(\boldsymbol{\lambda}_k) + \bar{I}_k(\boldsymbol{\lambda}_k)^2 + Q_k^2 - 2 Q_k \bar{Q}_k(\boldsymbol{\lambda}_k) + \bar{Q}_k(\boldsymbol{\lambda}_k)^2 \right)$$

Теперь нужно взять производную по φ_k. Учитываем, что $\bar{I}_k^2(\boldsymbol{\lambda}_k)+\bar{Q}_k^2(\boldsymbol{\lambda}_k)$ - мощность «отсчета» в корреляторе, от фазы не зависит.

Получилось так:

$$u_{\partial\ \varphi_k}(t_{k,l})=\frac{\partial}{\partial\varphi_k}\ln p(I_k,Q_k\,|\,\boldsymbol{\lambda}_k)\,|_{\varphi_k=\tilde{\varphi}_k}=-\frac{\ln B}{2\sigma_{IQ}^2}\cdot\left(-2I_k\frac{\partial\bar{I}_k\left(\boldsymbol{\lambda}_k\right)}{\partial\varphi_k}-2Q_k\frac{\partial\bar{Q}_k\left(\boldsymbol{\lambda}_k\right)}{\partial\varphi_k}\right)=$$

$$=\frac{\ln B}{\sigma_{IQ}^2}\left(I_k\frac{\partial\bar{I}_k\left(\boldsymbol{\lambda}_k\right)}{\partial\varphi_k}+Q_k\frac{\partial\bar{Q}_k\left(\boldsymbol{\lambda}_k\right)}{\partial\varphi_k}\right)=$$

$$=-A_k\frac{\ln B}{\sigma_{IQ}^2}\left(I_k\sum_{l=0}^{L-1}\sin(\tilde{\Phi}_{k,l})\cdot\cos(\breve{\Phi}_{k,l})+Q_k\sum_{l=0}^{L-1}\sin(\tilde{\Phi}_{k,l})\cdot\sin(\breve{\Phi}_{k,l})\right)=$$

$$=-A_k\frac{\ln B}{\sigma_{IQ}^2}\left(I_k\sum_{l=0}^{L-1}\left[\frac{\sin(\tilde{\Phi}_{k,l}-\breve{\Phi}_{k,l})+\sin(\tilde{\Phi}_{k,l}+\breve{\Phi}_{k,l})}{2}\right]+\right.$$

$$\left.+Q_k\sum_{l=0}^{L-1}\left[\frac{\cos(\tilde{\Phi}_{k,l}-\breve{\Phi}_{k,l})-\cos(\tilde{\Phi}_{k,l}+\breve{\Phi}_{k,l})}{2}\right]\right)$$

Домножим на $\frac{T_d}{T_d}$, тогда, с учетом малости T_d, сумма переходит в интеграл.

$$u_{\partial\ \varphi_k}(t)=-A_k\frac{\ln B}{\sigma_{IQ}^2\cdot T_d}\left(I_k\int_0^T\left[\frac{\sin(\tilde{\Phi}_k-\breve{\Phi}_k)+\sin(\tilde{\Phi}_k+\breve{\Phi}_k)}{2}\right]dt+\right.$$

$$\left.+Q_k\int_0^T\left[\frac{\cos(\tilde{\Phi}_k-\breve{\Phi}_k)-\cos(\tilde{\Phi}_k+\breve{\Phi}_k)}{2}\right]dt\right)$$

Пренебрегаем слагаемыми с двойной частотой:

$$u_{\partial\ \varphi_k}(t)=-A_k\frac{\ln B}{\sigma_{IQ}^2\cdot T_d}\left(I_k\int_0^T\left[\frac{\sin(\tilde{\Phi}_k-\breve{\Phi}_k)}{2}\right]dt+Q_k\int_0^T\left[\frac{\cos(\tilde{\Phi}_k-\breve{\Phi}_k)}{2}\right]dt\right)=$$

$$=-A_k\frac{\ln B}{\sigma_{IQ}^2\cdot T_d}\left(I_k\int_0^T\left[\frac{\sin(\tilde{\delta}\omega_{\partial\ k}t+\tilde{\delta}\varphi_k)}{2}\right]dt+Q_k\int_0^T\left[\frac{\cos(\tilde{\delta}\omega_{\partial\ k}t+\tilde{\delta}\varphi_k)}{2}\right]dt\right)$$

Где обозначено $\tilde{\delta}\omega_{\partial\ k}=\tilde{\omega}_{\partial\ k}-\breve{\omega}_{\partial\ k}$, $\tilde{\delta}\varphi_k=\tilde{\varphi}_k-\breve{\varphi}_k$

$$u_{\partial\,\varphi_k}(t) = -A_k \frac{\ln B}{\sigma_{IQ}^2 \cdot T_d}\left(\frac{I_k}{2}\cdot\left(-\left.\frac{\cos(\tilde{\delta}\omega_{\partial\,k}t+\tilde{\delta}\varphi_k)}{\tilde{\delta}\omega_{\partial\,k}}\right|_0^T\right)+\frac{Q_k}{2}\left(\left.\frac{\sin(\tilde{\delta}\omega_{\partial\,k}t+\tilde{\delta}\varphi_k)}{\tilde{\delta}\omega_{\partial\,k}}\right|_0^T\right)\right)=$$

$$=-A_k \frac{\ln B}{\sigma_{IQ}^2 \cdot T_d}\left(\frac{I_k}{2}\cdot\left(\frac{\cos(\tilde{\delta}\varphi_k)-\cos(\tilde{\delta}\omega_{\partial\,k}T+\tilde{\delta}\varphi_k)}{\tilde{\delta}\omega_{\partial\,k}}\right)+\frac{Q_k}{2}\left(\frac{\sin(\tilde{\delta}\omega_{\partial\,k}T+\tilde{\delta}\varphi_k)-\sin(\tilde{\delta}\varphi_k)}{\tilde{\delta}\omega_{\partial\,k}}\right)\right)=$$

$$=-A_k \frac{\ln B}{\sigma_{IQ}^2 \cdot T_d}\left(\frac{I_k}{2}\cdot\left(\frac{-2\sin\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2}+\tilde{\delta}\varphi_k\right)\sin\left(-\frac{\tilde{\delta}\omega_{\partial\,k}T}{2}\right)}{\tilde{\delta}\omega_{\partial\,k}}\right)+\right.$$

$$\left.+\frac{Q_k}{2}\left(\frac{2\cos\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2}+\tilde{\delta}\varphi_k\right)\sin\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2}\right)}{\tilde{\delta}\omega_{\partial\,k}}\right)\right)$$

Домножим на $\frac{L/2}{L/2}$, где L – число отсчетов на интервале T, $T = L\cdot T_d$.

Продолжаем:

$$u_{\partial\,\varphi_k}(t) = -A_k \frac{\ln B}{\sigma_{IQ}^2}\frac{L}{2}\left(I_k\cdot\left(\frac{\sin\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2}+\tilde{\delta}\varphi_k\right)\sin\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2}\right)}{\frac{\tilde{\delta}\omega_{\partial\,k}T}{2}}\right)+\right.$$

$$\left.+Q_k\left(\frac{\cos\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2}+\tilde{\delta}\varphi_k\right)\sin\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2}\right)}{\frac{\tilde{\delta}\omega_{\partial\,k}T}{2}}\right)\right)=$$

$$=-A_k \frac{\ln B}{\sigma_{IQ}^2}\frac{L}{2}\left(I_k\cdot\sin\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2}+\tilde{\delta}\varphi_k\right)\mathrm{sinc}\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2}\right)+Q_k\cdot\cos\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2}+\tilde{\delta}\varphi_k\right)\mathrm{sinc}\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2}\right)\right)$$

$$=-A_k \frac{\ln B}{\sigma_{IQ}^2}\frac{L}{2}\mathrm{sinc}\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2}\right)\left[I_k\cdot\sin\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2}+\tilde{\delta}\varphi_k\right)+Q_k\cdot\cos\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2}+\tilde{\delta}\varphi_k\right)\right]$$

Итого:

$$u_{\partial\,\varphi_k}(t) = -A_k \frac{\ln B}{\sigma_{IQ}^2}\frac{L}{2}\operatorname{sinc}\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2}\right)\left[I_k \cdot \sin\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2} + \tilde{\delta}\varphi_k\right) + Q_k \cdot \cos\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2} + \tilde{\delta}\varphi_k\right)\right]$$

Рассчитаем дискриминационную характеристику:

$U(\delta\varphi_k) = M\{u_{\partial\,\varphi_k}(t)\}$, где будем обозначать $C = A_k \dfrac{\ln B}{\sigma_{IQ}^2}\dfrac{L}{2}\operatorname{sinc}\left(\dfrac{\tilde{\delta}\omega_{\partial\,k}T}{2}\right)$

$\breve{\delta}\varphi_k = \varphi_k - \breve{\varphi}_k,\ \breve{\delta}\omega_{\partial k} = \omega_{\partial k} - \breve{\omega}_{\partial\,k},\ \delta\varphi_k = \varphi_k - \tilde{\varphi}_k,\ \delta\omega_{\partial k} = \omega_{\partial k} - \tilde{\omega}_{\partial\,k}.$

$$U(\delta\varphi_k) = -C\left[M\{I_k\}\cdot\sin\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2} + \tilde{\delta}\varphi_k\right) + M\{Q_k\}\cdot\cos\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2} + \tilde{\delta}\varphi_k\right)\right] =$$

$$= -C\left[\bar{I}_k \cdot \sin\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2} + \tilde{\delta}\varphi_k\right) + \bar{Q}_k \cdot \cos\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2} + \tilde{\delta}\varphi_k\right)\right] =$$

$$= -C\left[A_k \sum_{l=0}^{L-1} \cos(\Phi_{k,l})\cdot\cos(\breve{\Phi}_{k,l})\cdot\sin\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2} + \tilde{\delta}\varphi_k\right) + \right.$$

$$\left. + A_k \sum_{l=0}^{L-1} \cos(\Phi_{k,l})\cdot\sin(\breve{\Phi}_{k,l})\cdot\cos\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2} + \tilde{\delta}\varphi_k\right)\right]$$

Домножим на $\dfrac{T_d}{T_d}$.

$$U(\delta\varphi_k) = -\frac{CA_k}{T_d}\left[\sin\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2} + \tilde{\delta}\varphi_k\right)\int_0^T \cos(\Phi_k)\cdot\cos(\breve{\Phi}_k)dt + \right.$$

$$\left. + \cos\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2} + \tilde{\delta}\varphi_k\right)\int_0^T \cos(\Phi_k)\cdot\sin(\breve{\Phi}_k)dt\right] =$$

$$= -\frac{CA_k}{2T_d}\left[\sin\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2} + \tilde{\delta}\varphi_k\right)\int_0^T (\cos(\Phi_k - \breve{\Phi}_k) + \cos(\Phi_k + \breve{\Phi}_k))dt + \right.$$

$$\left. + \cos\left(\frac{\tilde{\delta}\omega_{\partial\,k}T}{2} + \tilde{\delta}\varphi_k\right)\int_0^T (\sin(\breve{\Phi}_k - \Phi_k) + \sin(\breve{\Phi}_k + \Phi_k))dt\right]$$

Пренебрегаем слагаемыми с двойной частотой:

$$U(\delta\varphi_k) = -\frac{CA_k}{2T_d}\left[\sin\left(\frac{\tilde{\delta}\omega_{\partial\, k}T}{2}+\tilde{\delta}\varphi_k\right)\int_0^T \cos(\breve{\delta}\omega_{\partial\, k}t+\breve{\delta}\varphi_k)dt+\right.$$

$$\left.+\cos\left(\frac{\tilde{\delta}\omega_{\partial\, k}T}{2}+\tilde{\delta}\varphi_k\right)\int_0^T \sin(-\breve{\delta}\omega_{\partial\, k}t-\breve{\delta}\varphi_k)dt\right]=$$

$$=-\frac{CA_k}{2T_d}\left[\sin\left(\frac{\tilde{\delta}\omega_{\partial\, k}T}{2}+\tilde{\delta}\varphi_k\right)\frac{\sin\left(\breve{\delta}\omega_{\partial\, k}T+\breve{\delta}\varphi_k\right)-\sin(\breve{\delta}\varphi_k)}{\breve{\delta}\omega_{\partial\, k}}-\right.$$

$$\left.-\cos\left(\frac{\tilde{\delta}\omega_{\partial\, k}T}{2}+\tilde{\delta}\varphi_k\right)\frac{\cos(\breve{\delta}\varphi_k)-\cos\left(\breve{\delta}\omega_{\partial\, k}T+\breve{\delta}\varphi_k\right)}{\breve{\delta}\omega_{\partial\, k}}\right]=$$

$$=-\frac{CA_k}{2T_d}\left[\sin\left(\frac{\tilde{\delta}\omega_{\partial\, k}T}{2}+\tilde{\delta}\varphi_k\right)2\frac{\sin\left(\frac{\breve{\delta}\omega_{\partial\, k}T}{2}\right)\cos(\frac{\breve{\delta}\omega_{\partial\, k}T}{2}+\breve{\delta}\varphi_k)}{\breve{\delta}\omega_{\partial\, k}}-\right.$$

$$\left.-\cos\left(\frac{\tilde{\delta}\omega_{\partial\, k}T}{2}+\tilde{\delta}\varphi_k\right)2\frac{\sin\left(\frac{\breve{\delta}\omega_{\partial\, k}T}{2}\right)\sin(\frac{\breve{\delta}\omega_{\partial\, k}T}{2}+\breve{\delta}\varphi_k)}{\breve{\delta}\omega_{\partial\, k}}\right]$$

Домножим на $\dfrac{L/2}{L/2}$.

$$U(\delta\varphi_k) = -\frac{CA_k L}{2}\left[\sin\left(\frac{\tilde{\delta}\omega_{\partial\, k}T}{2}+\tilde{\delta}\varphi_k\right)\frac{\sin\left(\dfrac{\breve{\delta}\omega_{\partial\, k}T}{2}\right)\cos\left(\dfrac{\breve{\delta}\omega_{\partial\, k}T}{2}+\breve{\delta}\varphi_k\right)}{\dfrac{\breve{\delta}\omega_{\partial\, k}T}{2}}-\right.$$

$$\left.-\cos\left(\frac{\tilde{\delta}\omega_{\partial\, k}T}{2}+\tilde{\delta}\varphi_k\right)\frac{\sin\left(\dfrac{\breve{\delta}\omega_{\partial\, k}T}{2}\right)\sin\left(\dfrac{\breve{\delta}\omega_{\partial\, k}T}{2}+\breve{\delta}\varphi_k\right)}{\dfrac{\breve{\delta}\omega_{\partial\, k}T}{2}}\right]=$$

$$=-\frac{CA_k L}{2}\operatorname{sinc}\left(\frac{\breve{\delta}\omega_{\partial\, k}T}{2}\right)\left[\sin\left(\frac{\tilde{\delta}\omega_{\partial\, k}T}{2}+\tilde{\delta}\varphi_k\right)\cos\left(\frac{\breve{\delta}\omega_{\partial\, k}T}{2}+\breve{\delta}\varphi_k\right)-\right.$$

$$\left.-\cos\left(\frac{\tilde{\delta}\omega_{\partial\, k}T}{2}+\tilde{\delta}\varphi_k\right)\sin\left(\frac{\breve{\delta}\omega_{\partial\, k}T}{2}+\breve{\delta}\varphi_k\right)\right]=$$

$$=-\frac{CA_k L}{2}\operatorname{sinc}\left(\frac{\breve{\delta}\omega_{\partial\, k}T}{2}\right)\sin\left(\frac{\tilde{\delta}\omega_{\partial\, k}T}{2}+\tilde{\delta}\varphi_k-\frac{\breve{\delta}\omega_{\partial\, k}T}{2}-\breve{\delta}\varphi_k\right)=$$

$$=-\frac{CA_k L}{2}\operatorname{sinc}\left(\frac{\breve{\delta}\omega_{\partial\, k}T}{2}\right)\sin\left(\frac{-\delta\omega_{\partial\, k}T}{2}-\delta\varphi_k\right)$$

Итого:

$$U(\delta\varphi_k) = -\frac{CA_k L}{2}\operatorname{sinc}\left(\frac{\breve{\delta}\omega_{\partial\, k}T}{2}\right)\sin\left(\frac{-\delta\omega_{\partial\, k}T}{2}-\delta\varphi_k\right)=$$

$$=\frac{CA_k L}{2}\operatorname{sinc}\left(\frac{\breve{\delta}\omega_{\partial\, k}T}{2}\right)\sin\left(\frac{\delta\omega_{\partial\, k}T}{2}+\delta\varphi_k\right)$$

Найдем крутизну:

$$S_{\partial} = \frac{\partial}{\partial \delta\varphi_k}\left(-\frac{CA_k L}{2}\,\mathrm{sinc}\left(\frac{\breve{\delta\omega}_{\partial\, k} T}{2} \right) \sin\left(\frac{-\delta\omega_{\partial\, k} T}{2} - \delta\varphi_k \right) \right)\Bigg|_{\substack{\delta\varphi_k = 0 \\ \delta\omega_{\partial\, k} = 0 \\ \breve{\delta\omega}_{\partial\, k} = 0}} =$$

$$= \frac{CA_k L}{2}$$

Флуктуационная характеристика:

$$D_{\eta_\varphi} = M\left\{ \left(u_{\partial\ \varphi_k}(t_k) - U(\delta\varphi_k) \right)^2 \right\} = M\left\{ n^2_{\partial\ \varphi_k} \right\}$$

С учетом принятых ранее обозначений, запишем:

$$D_{\eta_\varphi} == M\left\{ n^2_{\partial\ \varphi_k} \right\} = M\left\{ \left(-C\left[n_I \sin\left(\frac{\tilde{\delta\omega}_{\partial\, k} T}{2} + \tilde{\delta\varphi}_k \right) + n_Q \cos\left(\frac{\tilde{\delta\omega}_{\partial\, k} T}{2} + \tilde{\delta\varphi}_k \right) \right] \right)^2 \right\} =$$

$$= C^2 M\left\{ n_I^2 \sin^2\left(\frac{\tilde{\delta\omega}_{\partial\, k} T}{2} + \tilde{\delta\varphi}_k \right) + n_Q^2 \cos^2\left(\frac{\tilde{\delta\omega}_{\partial\, k} T}{2} + \tilde{\delta\varphi}_k \right) + \right.$$

$$\left. + 2 n_I n_Q \sin\left(\frac{\tilde{\delta\omega}_{\partial\, k} T}{2} + \tilde{\delta\varphi}_k \right) \cos\left(\frac{\tilde{\delta\omega}_{\partial\, k} T}{2} + \tilde{\delta\varphi}_k \right) \right\} =$$

$$= C^2\left[\sigma_{IQ}^2 \sin^2\left(\frac{\tilde{\delta\omega}_{\partial\, k} T}{2} + \tilde{\delta\varphi}_k \right) + \sigma_{IQ}^2 \cos^2\left(\frac{\tilde{\delta\omega}_{\partial\, k} T}{2} + \tilde{\delta\varphi}_k \right) \right] = C^2 \sigma_{IQ}^2$$