On estimation of the CES production function—Revisited[☆]

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ABSTRACT

Estimation of the non-linear Constant Elasticity of Scale (CES) function is generally considered problematic due to convergence problems and unstable and/or meaningless results. These problems often arise from a non-smooth objective function with large flat areas, the discontinuity of the CES function where the elasticity of substitution is one, and possibly significant rounding errors where the elasticity of substitution is close to one. We suggest three (combinable) solutions that alleviate these problems and improve the reliability and stability of the results.

1. Introduction

The Constant Elasticity of Substitution (CES) function[1]

$$y = \gamma \left(\delta x_1^{-\rho} + (1 - \delta) x_2^{-\rho} \right)^{-\frac{\nu}{\rho}} \qquad (1)$$

was developed by the Stanford group (Arrow et al., 1961) to alleviate restrictive assumptions implied by the Cobb–Douglas function. Furthermore, nested CES functions have been developed to model more than two inputs in a flexible way. Their general specification is[2]

$$y = \gamma \left(\sum_{i=1}^{n} \delta_i z_i^{\rho/\rho_i} \right)^{-\nu/\rho}$$

$$\text{with } z_i = \sum_{j=1}^{n_i} \delta_{j,i} x_{j,i}^{-\rho_i} \ \forall i = 1, \dots, n. \qquad (2)$$

Recently CES functions have gained popularity mainly in macroeconomic programming models, particularly in the areas of economic growth (e.g. Papageorgiou and Saam, 2008), international trade (e.g. Lloyd and MacLaren, 2002), and energy economics (e.g. McFarland et al., 2004), but the parameters are usually guesstimated and calibrated rather than based on empirical evidence. This arises from the low popularity of CES functions in empirical work, which mostly results from serious problems inherent in the econometric estimation. As CES functions are non-linear and cannot be linearised, standard linear estimation methods cannot be applied. Conversely, the estimation of the non-linear CES function frequently performs poorly. The main reason for this is the surface of the objective function, which often has a tendency to large flat areas and local minima. Further problems are discontinuities of CES functions and considerable rounding errors at specific parameter values.

Although this paper concentrates on non-linear least squares estimations, the presented solutions can be applied regardless of

[1] Variable y is the output quantity, x_1 and x_2 are the input quantities, and γ, δ, ρ, and ν are parameters, where $\gamma \in (0, \infty)$ determines the productivity, $\delta \in [0, 1]$ determines the optimal distribution of the inputs, $\rho \in [-1, 0) \cup (0, \infty)$ determines the (constant) elasticity of substitution, which is $\sigma = 1/(1 + \rho)$, and $\nu \in (0, \infty)$ is equal to the elasticity of scale.

[2] The inputs are subdivided into n groups, where n_i denotes the number of inputs in the ith group and $x_{j,i}$ denotes the jth input in the ith group. This specification requires the normalisations $\sum_{i=1}^{n} \delta_i = 1$ and $\sum_{j=1}^{n_i} \delta_{j,i} = 1 \forall i = 1, \dots, n$.

whether the model is estimated in log or levels (see Sun et al., 2011); the applied estimation method (e.g. the Poisson pseudo-maximum likelihood estimation, see Sun et al., 2011), and which extensions of the CES functions are used (e.g. Hicks neutral or factor augmented technical progress, see Klump et al., 2007).

Section 2 exemplifies these problems and presents solutions that result in more reliable estimation results. Section 3 concludes.

2. Challenges when estimating the CES by non-linear least squares

The two standard approaches to estimating the parameters of CES functions are the linear Taylor-series approximation developed by Kmenta (1967) and the non-linear least squares estimation. However, the applicability of the so-called Kmenta approximation is limited, because it cannot be used to linearise CES functions with more than two inputs (Hoff, 2004), whilst it only returns reliable results if ρ is close to its point of approximation (i.e. zero, see Thursby and Lovell, 1978). Conversely, non-linear optimisation algorithms very often face convergence problems or return varying, unreliable, or economically meaningless parameter estimates.

2.1. Removable discontinuities

One problem of non-linear optimisation algorithms may be caused by the discontinuities of the CES function where at least one of the substitution parameters $(\rho, \rho_1, \ldots, \rho_n)$ is zero. However, as these discontinuities are "removable", we can make CES function continuous by calculating the output quantities at these discontinuities by the limits for the substitution parameters approaching zero. For the two-input CES function, this limit is:

$$y = \gamma x_1^{\nu\delta} x_2^{\nu(1-\delta)}. \tag{3}$$

The limits of the three-input and four-input nested CES functions for all possible combinations of substitution parameters approaching zero are given in Henningsen and Henningsen (2011a).

Making the CES function continuous also makes the sum of squared residuals – the objective function of the non-linear least-squares estimation – continuous so that optimisation routines are not misled at these discontinuities. Furthermore, by removing the discontinuities of the (analytical) gradients of the RSS regarding the coefficients in a similar way, gradient-based solvers are also undisturbed by the discontinuities.

2.2. Rounding errors

Rounding errors are unavoidable on digital computers, but they are usually negligible. However, given the construction of the CES function, rounding errors become larger when at least one of the substitution parameters $(\rho, \rho_1, \ldots, \rho_n)$ approaches zero. In this case, the calculation of the predicted output $\hat{y}$ becomes increasingly imprecise, because first very small (in absolute terms) exponents (e.g. $-\rho$ or $-\rho_i$), then very large (in absolute terms) exponents (e.g. $-\nu/\rho$ or ρ/ρ_i) are applied. The imprecise calculation of the predicted output $\hat{y}$ distorts the calculation of the sum of squared residuals $\left(\text{RSS} = \sum_{t=1}^{T} \left(y_t - \hat{y}_t\right)^2\right)$, which is the objective function of the non-linear least squares estimation (see left panel of Fig. 1). In absolute terms, these rounding errors are rather small (see scale of vertical axis), but given the extremely flat surface of the objective function, they can cause a significant deformation. The rounding errors in the calculations of the (analytical) gradients of the objective function with respect to the coefficients – particularly with respect to the substitution parameters (e.g. $\partial \text{RSS}/\partial \rho$) – are even more severe (see right panel of Fig. 1). Hence, if the technology is close to a

Cobb–Douglas function, i.e. with an elasticity of substitution close to one, estimating the CES function with non-linear optimisation algorithms may be problematic, particularly when using gradient-based optimisation algorithms.

To avoid a potentially imprecise calculation of the output of the two-input CES function, which would lead to an imprecise calculation of the sum of squared residuals (RSS), we use a first-order Taylor series approximation at the point $\rho = 0$ (i.e. the Kmenta approximation), if ρ is close to zero (e.g. if $|\rho| < 10^{-5}$):

$$y = \gamma x_1^{\nu\delta} x_2^{\nu(1-\delta)} \exp\left(-0.5\rho\nu\delta\left(1-\delta\right)\left(\ln x_1 - \ln x_2\right)^2\right). \tag{4}$$

Analogously, we use first-order Taylor series approximations to calculate the gradients of the RSS with respect to the coefficients (see Henningsen and Henningsen, 2011a) if ρ is close to zero.

In case of nested CES functions, the derivations of the Taylor series approximations (particularly of the gradients) are extremely laborious and cumbersome. Therefore, we suggest use of linear interpolations to avoid rounding errors. We can calculate the output quantities for two different values of each substitution parameter that is close to zero, i.e. zero (using the formula for the limit of this parameter approaching zero) and a small positive or negative number that has the same sign as this substitution parameter, but that is large enough to avoid rounding errors (e.g. $\pm 10^{-5}$). Depending on the number of substitution parameters $(\rho, \rho_1, \ldots, \rho_n)$ that are close to zero, a one or multi-dimensional linear interpolation must be applied.

2.3. Surface of the objective function

The most dominant problem in the estimation of the CES function is the surface of the objective function, which often encounters non-convexity, local minima, rills, valleys, and extremely flat surfaces, e.g. around the minimum. Fig. 2 demonstrates this problem for a three-input nested CES function (capital, labour, energy)[3] by plotting the sum of squared residuals (RSS) depending on the substitution parameters ρ and ρ_1 in a three dimensional plot. Using global optimisation algorithms can alleviate problems caused by non-convexity, local minima, rills, and valleys, but they are usually rather slow and do not alleviate the problem of flat surfaces. Therefore, we suggest using a grid search procedure, where a grid of values for the substitution parameters $(\rho, \rho_1, \ldots, \rho_n)$ is pre-selected and the remaining parameters are estimated by non-linear least-squares, whilst holding the substitution parameters constant at each combination of the pre-defined values. While two-input CES functions only require a one-dimensional grid search for different values of ρ, the grid search for nested CES functions can be multi-dimensional, because these functions have more than one substitution parameter. An example of the outcome of a two-dimensional grid search for the two substitution parameters of a three-input nested CES function (ρ and ρ_1) is given in Fig. 2.

At the end of the grid search procedure, we select the values of the substitution parameters (and the corresponding estimates of the other parameters) that result in the smallest sum of squared residuals (RSS). These selected parameter estimates can be used either directly or as starting values for a non-linear least-squares estimation, e.g. using a gradient-based optimisation routine. In the latter, substitution parameters that are between the grid points can also be estimated.

Removing the discontinuities of the CES function (see Section 2.1) is particularly useful in combination with the grid search procedure, because it allows the inclusion of zeros as pre-selected values of the substitution parameters. Hence, the grid search can
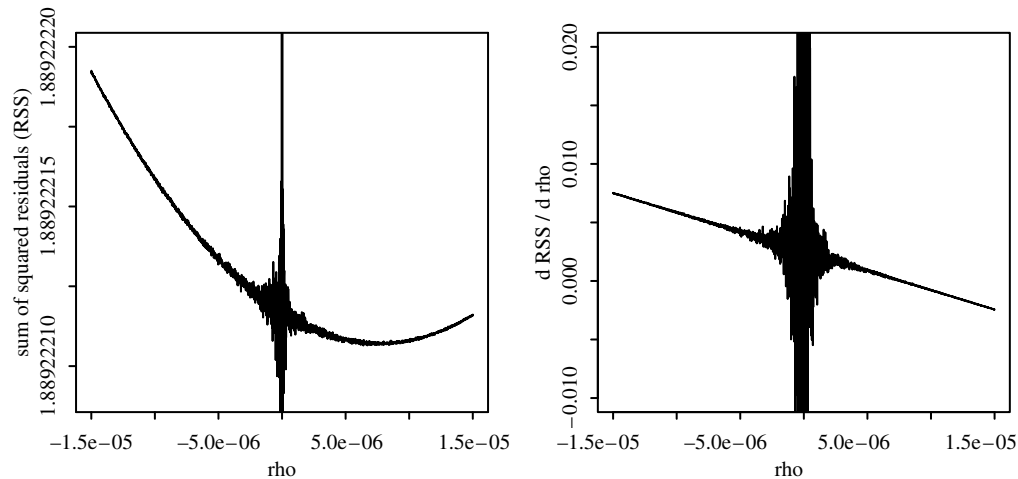
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[3] Data is available in Kemfert (1998).

Fig. 1. Rounding errors when ρ is close to zero.
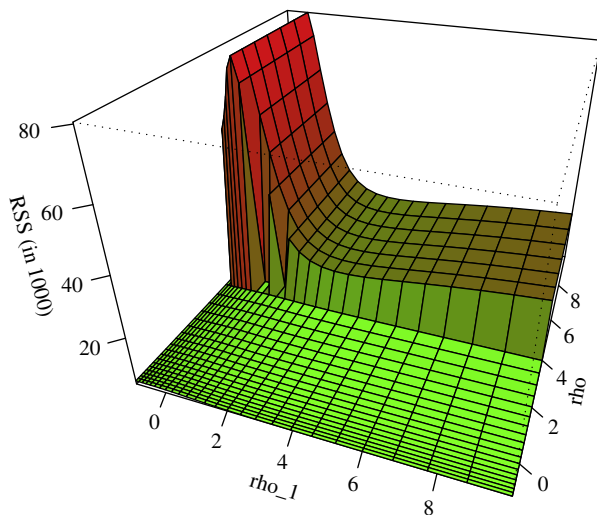

Fig. 2. Minimum RSS for different combinations of ρ_1 and ρ.

also include (nested) CES functions with the elasticities of substitution being unity. Furthermore, if the grid search procedure suggests a substitution parameter of zero and this is used as a starting value for subsequent estimation, the avoidance of rounding errors (see Section 2.2) allows the optimisation routine to smoothly move away from zero.

3. Conclusion

We have distinguished three factors that often cause severe problems when estimating the CES function by non-linear optimisation algorithms and we have presented three different methods for alleviating or solving these problems: (i) removing discontinuities by using limits, (ii) removing rounding errors by using local linear approximations, and (iii) overcoming problems with an ill-behaved objective function by using a grid search procedure.

All three solutions have been implemented for non-linear least squares estimations in the R package "micEconCES" (Henningsen and Henningsen, 2011b). Henningsen and Henningsen (2011a)

demonstrate, with two replication studies, that estimations that use these solutions produce stable and reliable results. We are optimistic that these tools will make the CES function more attractive to empirical economists. Furthermore, we hope that these tools will contribute to the reliability of economic programming models by increasing the availability of CES parameters that are based on empirical evidence rather than on guesstimation and calibration.

Acknowledgements

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