a) in $\mathbb{R}^n$ define

$$d'(x,y) = |x_1 - y_1| + \dots + |x_n - y_n|$$

show that d' is a metric that induces the usual topology of $\mathbb{R}^n$. sketch the basis elements for d' when $n=2$

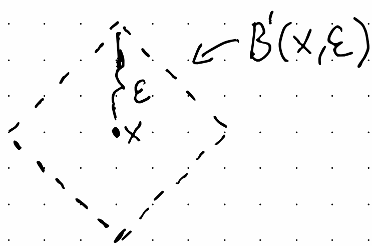
Consider the usual topology and a basis element $B(x,\varepsilon)$ if $y \in B(x,\varepsilon)$ want to show that there is a ball in the d' metric contained in this consider $B'(x, \frac{\varepsilon}{\sqrt{n}})$ w.r.t the d' metric. let $y \in B'(x, \frac{\varepsilon}{\sqrt{n}})$

$$d(x,y) = \left(\sum_{k=1}^{n} |x_k - y_k|^2 \right)^{1/2} \leq \left(\sum_{k=1}^{n} \frac{\varepsilon^2}{n} \right)^{1/2} = \varepsilon$$

So $y \in B(x,\varepsilon)$

by lemma 13.2 the open balls in the d' metric are then a basis for the standard topology

sketch:



Show d' is a metric

- clearly $d'(x,y) \geq 0$ since it is a sum of nonnegative numbers. if $d'(x,y)=0$ then $x_i = y_i \ \forall 1 \leq i \leq n$ $\Leftrightarrow x = y$
- $d'(x,y) = d'(y,x)$ since $|x_i - y_i| = |y_i - x_i|$
- $d'(x,z) = \sum_{i=1}^{n} |x_i - z_i| = \sum_{i=1}^{n} |x_i - y_i + y_i - z_i| \leq \sum_{i=1}^{n} |x_i - y_i| + |y_i - z_i|$

$= d'(x,y) + d'(y,z)$

b) more generally, given $p \geq 1$, define

$d'(x,y) = \left[\sum_{i=1}^{n} |x_i - y_i|^p\right]^{1/p}$. Assume this is a metric. Show that it induces the usual topology on R^n

consider a ball $B(x,\varepsilon)$ with the standard metric. Want again to find $B'(x,\delta)$ s.t $B'(x,\delta) \subset B(x,\varepsilon)$

if $y \in B'(x,\delta)$ then $d'(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p} < \delta$

so $|x_i - y_i| < \delta \ \forall 1 \leq i \leq n$. pick again $\delta = \frac{\varepsilon}{\sqrt{n}}$

then $d(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2} \leq \left(\sum_{i=1}^{n} \frac{\varepsilon^2}{n}\right)^{1/2} = \varepsilon$

so $y \in B(x,\varepsilon)$ thus we are done ■