B_t is one-dimensional brownian $c > 0$

prove that

$$\hat{B}_t = \frac{1}{c} B_{c^2 t} \quad \text{is brownian}$$

- $\hat{B}_0 = \frac{1}{c} B_0 = 0$
- $\hat{B}_t - \hat{B}_s = \frac{1}{c}\left(B_{c^2 t} - B_{c^2 s}\right) \sim \frac{1}{c} N(0, c^2(t-s))$
  $= N(0, t-s)$

- let $n \in \mathbb{N}$, $i, j \in \mathbb{N}$ s.t $i, j \leq n$, $i \neq j$

$0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$

then $0 \leq c^2 t_1 \leq c^2 t_2 \leq \ldots \leq c^2 t_n$

consider $\hat{B}_{t_i} - \hat{B}_{t_{i-1}} = \frac{1}{c}\left(B_{c^2 t_i} - B_{c^2 t_{i-1}}\right)$ and

$$\hat{B}_{t_j} - \hat{B}_{t_{j-1}} = \frac{1}{c}\left(B_{c^2 t_j} - B_{c^2 t_{j-1}}\right)$$

the increments on the right are independent of each other and thus the increment on the left are too. so $\hat{B}_t$ has independent increments.