Let $B_t$ be brownian motion $t_0 \geq 0$

Prove that $\tilde{B}_t := B_{t_0+t} - B_{t_0}$ is brownian motion.

- $\tilde{B}_0 = B_{t_0} - B_{t_0} = 0$
- $\tilde{B}_t - \tilde{B}_s = B_{t_0+t} - B_{t_0+s} \sim N(0, t-s)$
- let $n \in \mathbb{N}$, $0 \leq t_1 \leq t_2 \leq \dots \leq t_n$

let $i, j \in \mathbb{N}$ s.t. $i, j \leq n$, $i \neq j$

consider the increments

$\tilde{B}_{t_i} - \tilde{B}_{t_{i-1}} = B_{t_0+t_i} - B_{t_0+t_{i-1}}$ and

$\tilde{B}_{t_j} - \tilde{B}_{t_{j-1}} = B_{t_0+t_j} - B_{t_0+t_{j-1}}$

Since $B_t$ is brownian these are independent (they do not overlap).

Since the ~~intere~~ increments were arbitrary $\tilde{B}_t$ has independent increments. thus $\tilde{B}_t$ is brownian.