Let $X_n$ be a metric space with metric $d_n$ for $n \in \mathbb{Z}$

a) Show that $\rho(x,y) = \max[d_1(x_1,y_1), \ldots, d_n(x_n,y_n)]$ is a metric for the product space $X_1 \times \cdots \times X_n = X$

- let $x,y \in X$ then $\rho(x,y) \geq d_k(x_k,y_k) \geq 0$ $\forall\, 1 \leq k \leq n$
  and $\rho(x,y) = 0 \iff \underset{1 \leq k \leq n}{d_k(x_k,y_k) = 0} \iff x = y$

- clearly $\rho(x,y) = \rho(y,x)$
- $\rho(x,z) = d_\alpha(x_\alpha, z_\alpha) \leq d_\alpha(x_\alpha,y_\alpha) + d_\alpha(y_\alpha, z_\alpha)$
  (for some $\alpha \in \{1, \ldots, n\}$)

$\leq \sup\limits_{1 \leq k \leq n}\{d_k(x_k,y_k)\} + \sup\limits_{0 \leq k \leq n}\{d_k(y_k,z_k)\}$

$= \rho(x,y) + \rho(y,z)$

b) Let $\bar{d}_i = \min\{d_i, 1\}$. Show that $D(x,y) = \sup\{\bar{d}_i(x_i,y_i)/i\}$ is a metric for the product space $\prod X_i$.

- $D(x,y) \geq 0$ since $\bar{d}_i(x_i,y_i) \geq 0$
  $D(x,y) = 0 \iff x_i = y_i\ \forall i \iff x = y$
- $D(x,y) = D(y,x)$ is clearly true.

- $D(x,y) = \frac{\overline{d}_\alpha(x_\alpha, y_\alpha)}{\alpha}$ for some $\alpha$

$$\overline{d}_i(x_i, y_i) = \min(d_i(x_i, y_i), 1) \leq \min(d_i(x_i, z_i) + d_i(z_i, y_i), 1)$$

$$\leq \min(d_i(x_i, z_i), 1) + \min(d_i(z_i, y_i), 1)$$

$$= \overline{d}_i(x_i, z_i) + \overline{d}_i(z_i, y_i)$$

so $\frac{\overline{d}_\alpha(x_\alpha, y_\alpha)}{\alpha} \leq \frac{\overline{d}_\alpha(x_\alpha, z_\alpha)}{\alpha} + \frac{\overline{d}(z_\alpha, y_\alpha)}{\alpha}$

$\leq D(x,z) + D(z,x)$