Let (X,d) be a metric space. If f satisfies the condition $d(f(x),f(y))<d(x,y)$ $\forall$ $x,y \in X$ then f is called a <u>Shrinking map</u>. If there is $\alpha<1$ s.t $d(f(x),f(y)) \le \alpha d(x,y)$, then f is called a <u>contraction</u>. A <u>fixed point</u> is s.t $f(x)=x$.

a) If f is a contraction and X is compact, show f has a unique fixed point.

let $f^1=f$, $f^{n+1}=f^1 \circ f^n$. f is clearly continuous thus $f(X)$ is compact and thus closed. By induction $f^n(X)$ is compact and closed. Clearly $f(X)\subset X$ and by induction it follows that $f^{n+1}(X)\subset f^n(X)$.

$C=\{f^n(X)\}_{n=0}^{\infty}$ is a collection of closed sets with the finite intersection property thus $\bigcap_{n=0}^{\infty} f^n(X)$ is nonempty. These are the points s.t $f(x)=x$. assuming this set has more than one element take 2: x,y then

$$d(f(x),f(y))=d(x,y)\le d(x,y) \quad \text{↯}$$

or $D=\text{diam}(X)$, $\text{diam}(f^n(X))\le \alpha^n D \to 0$

b) Show more generally that if f is a shrinking map and X is compact, then f has a unique fixed point.

Let $A = \bigcap_{n=0}^{\infty} f^n(X)$ A is closed and thus compact Nonempty as before. let $x \in A$, choose x_n s.t $x = f^{n+1}(x_n)$. Let $y_n = f^n(x_n)$ this has some subsequence converging to $a \in X$. Want to show that $a \in A_n$ $\forall$ n. We know again that $f^{n+1}(X) \subset f^n(X)$. thus $y_n \in f^N(X)$, $n \geq N$. As $f^N(X)$ is closed, a is in A_n for all n so $a \in A$. We have that for $\varepsilon > 0$

$$d(f(a), x) < d(a, y_{n_k}) < \varepsilon, \quad k \geq N$$

$f(A) = A$, diam $A = 0$ $\rightarrow$ $x = a$

c) Let $X = [0,1]$. Show that $f(x) = x - x^2/2$ maps X into X and is a schrinking map that is not a contraction.

$$d(f(x), f(y)) = x - \frac{x^2}{2} - y + \frac{y^2}{2} = x - y - \underbrace{\left(\frac{x^2}{2} - \frac{y^2}{2}\right)}_{>0 \text{ for } x > y}$$

$$f(x) = -\frac{1}{2}(x-1)^2 + \frac{1}{2}$$

$f'(x) = 1 - x$

for $b > 0$ by mvt there is $0 < c < b$ s.t $f(b) - f(0) = f(b) = f'(c)$ i.e $b - \frac{b^2}{2} = 1 - c$. If f was a contraction then for some $\alpha < 1$

We would have

$f(b) \leq \alpha b \quad \forall \ b \in [0,1]$

so $1-b \leq \alpha \ \forall \ b$ which is clearly not true

d) The result in (a) holds if X is a complete metric space such as $\mathbb{R}$. The result in (b) does not. Show that $f: \mathbb{R} \to \mathbb{R}$ given by

$f(x) = [x + (x^2+1)^{1/2}]/2$ is a shrinking map that is not a contraction and has no fixed points.

Let $g(x) = (x^2+1)^{1/2}$, $g'(x) = \frac{x}{\sqrt{x^2+1}}$, $|g'(x)| < 1$ increasing

for $x, y \in \mathbb{R}$, $x > y$, $x - y > \frac{x}{\sqrt{x^2+1}} - \frac{y}{\sqrt{y^2+1}}$ (by mvt)

So $d(f(x), f(y)) = \left| \frac{x-y}{2} + \frac{(x^2+1)^{1/2} - (x^2+1)^{1/2}}{2} \right|$

$\leq \frac{|x-y|}{2} + \frac{|(x^2+1)^{1/2} + (y^2+1)^{1/2}|}{2} < \frac{|x-y| + |x-y|}{2} = d(x,y)$

to see that it is not a contraction we note that $g'(x) \xrightarrow[x \to \pm\infty]{} \pm 1$. to see that there is no fix point note that

$f(x) > x \ \forall \ x$ ■