let $\bar{\rho}$ be the uniform metric on $\mathbb{R}^\omega$.

Given $x = (x_1, x_2, \ldots) \in \mathbb{R}^\omega$ and given $0 < \varepsilon < 1$ let $U(x, \varepsilon) = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon) \times \cdots$

a) show that $U(x, \varepsilon)$ is not equal to the ε-ball $B_{\bar{\rho}}(x, \varepsilon)$

first pick N s.t $\frac{1}{N} < \varepsilon$ the define $\tilde{x}$ s.t $\tilde{x}_n = x_n + \varepsilon - \frac{1}{N+n}$, $\tilde{x} \in U(x, \varepsilon)$

but $d(x, \tilde{x}) = \varepsilon$ so $\tilde{x} \notin B_{\bar{\rho}}(x, \varepsilon)$

b) show that $U(x, \varepsilon)$ is not even open in the box topology

take any $\tilde{\varepsilon}$-ball of $\tilde{x}$ then $B(\tilde{x}, \tilde{\varepsilon}) \not\subset B(x, \varepsilon)$

in particular there are no neighborhoods of $\tilde{x}$

c) show that $B_{\bar{\rho}}(x, \varepsilon) = \bigcup_{\delta < \varepsilon} U(x, \delta)$

let $y \in \bigcup_{\delta < \varepsilon} U(x, \delta)$ then $\bar{\rho}(x, y) \leq \delta < \varepsilon \to$

$y \in B_{\bar{\rho}}(x, \varepsilon)$

let $y \in B_{\bar{\rho}}(x, \varepsilon)$ then $\sup|x_i - y_i| < \varepsilon$

so $y \in U(x, \sup|x_i - y_i|) \subset \bigcup_{\delta < \varepsilon} U(x, \delta)$ $\blacksquare$