a) A r.v. X has exponential law with parameter λ if it has density

$$f(x) = \lambda e^{-\lambda x} \mathbb{1}_{[0,\infty)}$$

what is the p.f of X ? Compute mean and variance

$$\text{P.f}: \quad P(X \le t) = \int_0^t \lambda e^{-\lambda x}\, dx = \lambda \cdot \frac{e^{-\lambda x}}{-\lambda}\Big|_0^t = -e^{-\lambda x}\Big|_0^t$$

$$= 1 - e^{-\lambda t}$$

$$\text{mean}: \quad E[X] = \int_0^\infty x \lambda e^{-\lambda x}\, dx = \int e^{-\lambda x}\, dx = -\frac{1}{\lambda} e^{-\lambda x}\Big|_0^\infty$$

$$= \frac{1}{\lambda}$$

$$Var(X) = E[X^2] - E[X]^2$$

$$E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x} dx = \left[x^2 \left(-\frac{e^{-\lambda x}}{\lambda}\right)\right]_0^\infty + \int_0^\infty 2 \cancel{x}\, e^{-\lambda x}\, dx$$

$$= \frac{2}{\lambda} \int_0^\infty \lambda x e^{-\lambda} dx = \frac{2}{\lambda^2}$$

$$Var(X) = \frac{1}{\lambda}$$

b) Let U be a r.v uniformly distributed on $[0,1]$ i.e having density $\mathbb{1}_{[0,1]}(x)$

compute the mean and variance of U

$$E[U] = \int_{(-\infty,\infty)} U \, dP = \int_0^1 x \, dx = \frac{1}{2}$$

$$E[U^2] = \int_0^1 x^2 \, dx = \frac{1}{3}$$

$$-\ln\left(e^{-\lambda a}\right) = \lambda a$$

$$Var(U) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

c) U as in (b), compute law of $Z = \alpha U$, $\alpha > 0$

c_2) compute law of $W = -\frac{1}{\lambda} \log U$, $\lambda > 0$

$$\mu_Z(A) = P(Z^{-1}(A))$$

$$\mu_Z((-\infty, 0]) = 0, \quad \mu_Z((\alpha, \infty]) = 0$$

$$\mu_Z((a,b)) = P(Z^{-1}(a,b)) = P\left(\left(\frac{a}{\alpha}, \frac{b}{\alpha}\right)\right)$$

$$= \frac{1}{\alpha}(a,b) = \frac{1}{\alpha}(b-a)$$

$$P(W \leq t) = P\left(-\frac{1}{\lambda} \ln(U) \leq t\right) = P\left(U \geq e^{-\lambda t}\right)$$

$$= \int_{e^{-\lambda t}}^1 1 \, dx = 1 - e^{-\lambda t}, \quad P(W \leq 0) = 0 \rightarrow P(0 \leq a \leq W \leq b) = e^{-\lambda a} - e^{-\lambda b}$$

$$\mu_W(-\infty, 0) = 0$$

$$\mu_W(a,b)$$