Show that if X has a countable basis $\{B_n\}$ then every basis $\mathcal{C}$ for X contains a countable basis for X. Hint: for every pair of indices n,m for which it is possible choose $C_{n,m} \in \mathcal{C}$ s.t $B_n \subset C_{n,m} \subset B_m$

We follow the hint and pick 1 such $C_{n,m}$ when possible.

for any n let $\mathcal{C}_n = \{C_{n,m} : m \in \mathbb{Z}_+\}$ then let $\tilde{\mathcal{C}} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$. $\tilde{\mathcal{C}}$ is countable being a countable union of countable sets.

Let $\mathcal{U}$ be an open set, $x \in \mathcal{U}$. Then there is some m s.t $x \in B_m \subset \mathcal{U}$. then ther is $C \in \mathcal{C}$ s.t $x \in C \subset B_m$. then there is n s.t $x \in B_n \subset C \subset B_m$ so we can assume $C \in \tilde{\mathcal{C}}$. thus $\tilde{\mathcal{C}}$ is a countable basis.