

for an m -dimensional $\mathbf{r}, \mathbf{u}, \mathbf{x}$ it's

laplace-transforming $\mathbb{R}^m \ni \theta \rightarrow E[e^{\langle \theta, \mathbf{x} \rangle}]$

(possibly $+\infty$)

Prove that if $\mathbf{X} \sim N(\mathbf{b}, \mathbf{\Gamma})$ then

$$E[e^{\langle \theta, \mathbf{x} \rangle}] = e^{\langle \theta, \mathbf{b} \rangle} e^{\frac{1}{2} \langle \mathbf{\Gamma} \theta, \theta \rangle}$$

$$\int_{-\infty}^{\infty} e^{\langle \theta, \mathbf{x} \rangle} \frac{1}{(2\pi)^{m/2}} e^{-\frac{1}{2} \|\mathbf{x}\|^2} d\mathbf{x} = \frac{1}{(2\pi)^{m/2}} \int_{-\infty}^{\infty} e^{\langle \theta, \mathbf{x} \rangle - \frac{1}{2} \|\mathbf{x}\|^2} d\mathbf{x}$$

$$\begin{aligned} \langle \theta - \mathbf{x}, \theta - \mathbf{x} \rangle &= \|\theta\|^2 + \|\mathbf{x}\|^2 - 2\langle \theta, \mathbf{x} \rangle \\ \frac{\|\theta - \mathbf{x}\|^2}{2} &= \frac{\|\theta\|^2}{2} + \frac{\|\mathbf{x}\|^2}{2} - \langle \theta, \mathbf{x} \rangle \end{aligned}$$

$$\int_{-\infty}^{\infty} e^{\frac{\|\theta\|^2}{2} - \frac{\|\theta - \mathbf{x}\|^2}{2}} d\mathbf{x} = \frac{1}{(2\pi)^{m/2}} \int_{-\infty}^{\infty} e^{\frac{\|\theta\|^2}{2} - \frac{\|\theta - \mathbf{x}\|^2}{2}} d\mathbf{x}$$

$$= e^{\frac{\|\theta\|^2}{2}} \frac{1}{(2\pi)^{m/2}} \int_{-\infty}^{\infty} e^{-\frac{\|\theta - \mathbf{x}\|^2}{2}} d\mathbf{x} = e^{\frac{\|\theta\|^2}{2}}$$

$$\mathbf{\Gamma} = \mathbf{A} \mathbf{A}^*, \quad \mathbf{x} = \mathbf{A} \mathbf{z} + \mathbf{b}$$

$$E[e^{\langle \theta, \mathbf{x} \rangle}] = e^{\langle \theta, \mathbf{b} \rangle} E[e^{\langle \theta, \mathbf{A} \mathbf{z} \rangle}] = e^{\langle \theta, \mathbf{b} \rangle} E[e^{\langle \mathbf{A}^* \theta, \mathbf{z} \rangle}]$$

$$\begin{aligned} &= e^{\langle \theta, \mathbf{b} \rangle} e^{\frac{1}{2} \|\mathbf{A}^* \theta\|^2} = e^{\langle \mathbf{A}^* \theta, \mathbf{A}^* \theta \rangle} = \langle \mathbf{A} \mathbf{A}^* \theta, \theta \rangle \\ &= e^{\langle \theta, \mathbf{b} \rangle} e^{\frac{1}{2} \langle \mathbf{\Gamma} \theta, \theta \rangle} = e^{\langle \theta, \mathbf{b} \rangle} e^{\frac{1}{2} \langle \mathbf{\Gamma} \theta, \theta \rangle} \end{aligned}$$

