Let X be a metric space with metric d

a) show that $d: X \times X \to \mathbb{R}$ is continuous

let $x \times y \in X \times X$, $r = d(x,y) \in \mathbb{R}$ consider a neighborhood V of r. We can assume V is of the form (a,b), $a < r < b$, since V contains such an interval. Let $m = \min\{|a-r|, |b-r|\}$ then let $B_1 = B(x, \frac{m}{2})$, $B_2 = B(y, \frac{m}{2})$ then for $\tilde{x} \in B_1$, $\tilde{y} \in B_2$

$$d(\tilde{x}, \tilde{y}) \leq d(x, \tilde{x}) + d(x,y) + d(y, \tilde{y}) < r + m$$

thus $d(B_1, B_2) \subset (r-m, r+m) \subset (a,b)$

by Theorem 18.1, d is then continuous

b) Let X' denote a space having the same underlying set as X. Show that if $d: X' \times X' \to \mathbb{R}$ is continuous $\mathcal{T}'$ is finer than $\mathcal{T}$.

let $\mathcal{U} \in \mathcal{T}$ then for $x \in \mathcal{U}$ $\exists$ $\varepsilon > 0$ s.t. $B(x, \varepsilon) \subset \mathcal{U}$. Since d is continuous, it is continuous in each variable. Then $d': X' \to \mathbb{R}$ by $d'(y) = d(x,y)$ is continuous thus $d'^{-1}((-1, \varepsilon)) = B(x, \varepsilon) \in \mathcal{T}'$ so $\mathcal{U} \in \mathcal{T}'$ ∎

$d(x', y') \in (r - \frac{m}{2}, r + \frac{m}{2}) =: I \quad , \quad d(x,y) = r$

$r \leq d(x, x') + d(x', y')$

$|d(x', x) - d(x, y)| \leq d(x', y) \leq r + \frac{m}{2}$

$|d(x', x) - r| \leq r + \frac{m}{2}$