a) X positive r.v and $f: \mathbb{R}^+ \to \mathbb{R}$ a differentiable function with continuous derivative s.t $f(X)$ is integrable then

$$E[f(X)] = f(0) + \int_0^\infty f'(t) P(X \geq t)\, dt$$

$$E[f(X)] = \int_{\mathbb{R}} f(X(\omega))\, P(d\omega) = \int_{[0,\infty]} f(x)\, \mu_X(dx)$$

$$f(x) = f(0) + \int_0^x f'(t)\, d\cancel{t}$$

$$\int_{[0,\infty]} f(x)\, \mu_X(dx) = \int_{[0,\infty]} \mu_X(dx) \left(f(0) + \int_0^{\cancel{x}} f'(t)\, d\mu(t) \right)$$

$$= \int_{[0,\infty]} f(0)\, d\mu(x) + \int_{[0,\infty]} \mu_X(dx) \int_0^x f'(t)\, d\mu(t)$$

$$= f(0) + \int_0^\infty f'(t)\, d\mu(t) \int_t^\infty d\mu(x)$$

$$= f(0) + \int_0^\infty f'(t)\, d\mu(t)\, P(X \geq t)$$

b) X is positive with integer values

$$E[X] = \int_{\Omega} X(x)\, dP = \sum_{k=1}^{\infty} k\, \mu_x(k)$$

$$\mu_x(k) = P(X=k) = P(X \geq k) - P(X \geq k+1)$$

$$\sum_{k=1}^{\infty} k\left(P(X \geq k) - P(X \geq k+1)\right)$$

$$= \sum_{k=1}^{\infty} k\, P(X \geq k) - \sum_{k=1}^{\infty} k\, P(X \geq k+1)$$

$$= \sum_{k=1}^{\infty} k\, P(X \geq k) - \sum_{k=2}^{\infty} (k-1) P(X \geq k)$$

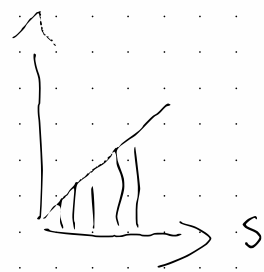
$$= P(X \geq k) + \sum_{k=2}^{\infty} k\, P(X \geq k) = \sum_{k=1}^{\infty} P(X \geq k)$$

$$E[f(X)] = \int_{\Omega} f(X(\omega))\, dP = \int_{[0,\infty]} f(x)\, d\mu(x)$$

$$f(x) = f(0) + \int_0^x f'(s)\, \cancel{ds}\, \mu(ds)$$

$$E[f(X)] = \int_{[0,\infty]} \left(f(0) + \int_0^x f'(s)\, ds \right) d\mu(x)$$

$$= \int_0^\infty f(0)\, d\mu(x) + \int_0^\infty \int_0^x f'(s)\, ds\, d\mu(x)$$



$$= f(0) + \int_0^\infty \int_s^\infty f'(s)\, d\mu(x)\, ds$$

$$= f(0) + \int_0^\infty f'(s) \int_s^\infty d\mu(x)\, ds$$

$$= f(0) + \int_0^\infty f'(s)\, P(X \geq s)\, ds$$