X is $N(0, \sigma^2)$ distributed

Compute $E[e^{tX^2}]$

$$E[e^{tX^2}] = \int_{-\infty}^{\infty} e^{tX^2} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2(1-2\sigma^2 t)}{2\sigma^2}} \, dx$$

$$u = x\sqrt{(1-2\sigma t)}$$

$$tx^2 - \frac{x^2}{2\sigma} \cdot x^2 = -\frac{x^2 - 2\sigma t x^2}{2\sigma} = -\frac{x^2(1-2\sigma t)}{2\sigma}$$

assuming $1-2\sigma t > 0$

$$= \int \frac{1}{\sqrt{2\pi}\sigma} \cdot \frac{1}{\sqrt{1-2\sigma t}} e^{-\frac{u^2}{2\sigma}} \, du$$

$$= \frac{1}{\sqrt{1-2\sigma t}} \int \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{u^2}{2\sigma^2}} \, du = \frac{1}{\sqrt{1-2\sigma t}}$$

so if $t < \frac{1}{2\sigma}$ we have $\searrow$ other wise it diverges