Prove theorem 19.4

Theorem 19.4: If each space X_α is Hausdorff space then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies.

let $(x_\alpha)_{\alpha\in J}$ and $(y_\alpha)_{\alpha \in J}$ be different points in $\prod X_\alpha$. Then for at least one α, say $\tilde{\alpha}$, $y_{\tilde{\alpha}} \neq x_{\tilde{\alpha}}$ then $\exists$ $\mathcal{U}, \mathcal{V}$ open in $X_{\tilde{\alpha}}$ s.t $x_{\tilde{\alpha}} \in \mathcal{U}$, $y_{\tilde{\alpha}} \in \mathcal{V}$ but $\mathcal{U} \cap \mathcal{V} = \emptyset$ let $\{\mathcal{U}_\alpha\}_\alpha$ be an indexed family of sets s.t $x_\alpha \in \mathcal{U}_\alpha$ and $\mathcal{U}_\alpha$ open in X_α and $\mathcal{U}_{\tilde{\alpha}} = \mathcal{U}$. Similar for $\{\mathcal{V}_\alpha\}_{\alpha\in J}$ w.r.t y_α. Then $\prod \mathcal{U}_\alpha$ and $\prod \mathcal{V}_\alpha$ are both open in $\prod X_\alpha$ but $(\prod \mathcal{U}_\alpha) \cap (\prod \mathcal{V}_\alpha) = \emptyset$ so $\prod X_\alpha$ is Hausdorff in the box topology.

For the product topology we let $\mathcal{U}_\alpha = X_\alpha$ when $\alpha \neq \tilde{\alpha}$ and $\mathcal{U}_{\tilde{\alpha}} = \mathcal{U}$, similar for $\mathcal{V}_\alpha$. These are still open in $\prod X_\alpha$ so it is hausdorff in the product topology as well.