Show that the euclidian metric on $\mathbb{R}^n$ is a metric, as follows: If, $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$ define

$$x+y = (x_1+y_1, \dots, x_n+y_n)$$
$$cx = (cx_1, \dots, cx_n)$$
$$x \cdot y = x_1y_1 + \dots + x_ny_n$$

a) $x \cdot (y+z) = x_1(y_1+z_1) + \dots + x_n(y_n+z_n)$

$= x_1y_1 + \dots + x_ny_n + x_1z_1 + \dots + x_nz_n$

$= x \cdot y + x \cdot z$

b) Show that $|x \cdot y| \leq \|x\| \|y\|$

clearly if x or $y = 0$ then it is true

$(ax+by) \cdot (ax+by) = a^2\|x\|^2 + 2ab(x \cdot y) + b^2\|y\|^2 \iff$

$-ab(x \cdot y) \leq \frac{1}{2}(a^2\|x\|^2 + b^2\|y\|^2)$

similarly:

$(ax-by) \cdot (ax-by) = a^2\|x\|^2 - 2ab(x \cdot y) + b^2\|y\|^2$

$ab(x \cdot y) \leq \frac{1}{2}(a^2\|x\|^2 + b^2\|y\|^2)$

pick $a = \frac{1}{\|x\|}$, $b = \frac{1}{\|y\|}$ then from the equations above:

$\frac{1}{\|x\|\|y\|} |(x \cdot y)| \leq \frac{1}{2}\left(\frac{\|x\|^2}{\|x\|^2} + \frac{\|y\|^2}{\|y\|^2}\right) = 1 \iff$

$|(x \cdot y)| \leq \|x\| \|y\|$

c) $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2x \cdot y$

$\leq \|x\|^2 + \|y\|^2 + \|x\|\|y\| = (\|x\| + \|y\|)^2$

$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$

$$d(x,z) = \|x-z\| = \|x-y+y-z\|$$

$$\leq \|x-y\| + \|y-z\|$$

$$= d(x,y) + d(y,z)$$