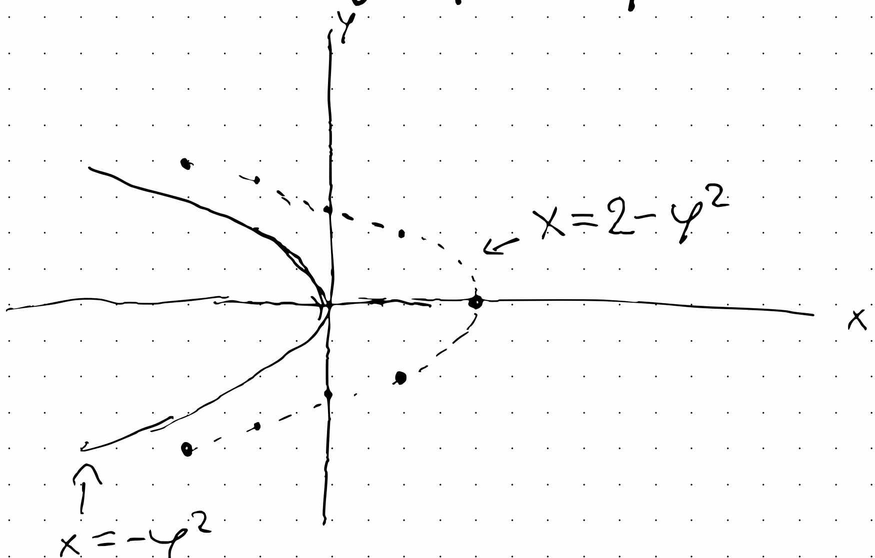
a) define an equivalence relation on the plane $X = \mathbb{R}^2$ as follows:

$x_0 \times y_0 \sim x_1 \times y_1$ if $x_0 + y_0^2 = x_1 + y_1^2$

Let X^* be the corresponding Quotient space. Is it homeomorphic to a familiar space and what is it? Hint: Set $g(x \times y) = x + y^2$



The shape of these horisontal parabolas is constant so our intuition tells us tha this is homeomorphic to $\mathbb{R}$.

Let $g: \mathbb{R}^2 \to \mathbb{R}$ by $g(x \times y) = x + y^2$. g is surjective and continuous by theorem 21.5.

let $A \subset \mathbb{R}$ be s.t $g^{-1}(A)$ is open

$g^{-1}(A) = \{x \times y : x + y^2 \in A\}$. consider $r \in A$, $\varepsilon > 0$

x_0, y_0 s.t $x_0 + y_0^2 = r$ since $g^{-1}(A)$ is open there is a basis element containing $x_0 \times y_0$

Consider $B_1 = (x_0, \delta)$, $B_2 = (y_0, \delta)$

then for $x \times y \in B_1 \times B_2$

$|x + y^2 - x_0 - y_0^2| \leq |x - x_0| + |y^2 - y_0^2|$

$< \delta + |(y - y_0)(y + y_0)| < \delta(1 + \delta) < \varepsilon$

let δ_1 s.t $B_1 \times B_2 \subset g^{-1}(A)$

then pick $\delta = \min\{\delta_1, \frac{\varepsilon}{2}, 1\}$ so $\delta(1+\delta) \leq 2\delta \leq \varepsilon$

thus g is a quotient map.

by corollary 22.3 g induces a homeomorphism

$f: X^* \to \mathbb{R}$

b) **do the same for $x_0^2 + y_0^2 = x_1^2 + y_1^2$**

These are circles of radius $r \in [0, \infty)$ so we guess that to be the homomorphic space

let $g: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $g(x \times y) = x^2 + y^2$. Clearly surjective and continuous. Assume $A \subset \mathbb{R}$ s.t $g^{-1}(A)$ is open

as before get B_1, B_2 then for $x \times y \in B_1 \times B_2$

we get $|x^2 - x_0^2 + y^2 - y_0^2| \leq |x^2 - x_0^2| + |y^2 - y_0^2| < 2\delta^2$

let $\delta \to 0$ and we see that A is open. Thus g is a quotient map and induces a homeomorphism $f: X^* \to [0, \infty)$ by corollary 22.3.