Let X be an inf dim normed space

i) construct by induction vectors $x_n \in X$ and linear functionals $f_n \in X^*$ s.t $\|x_n\| = \|f_n\| = f_n(x_n) = 1$ $\forall n$ and $f_n(x_m) = 0$ $\forall n < m$

let $x \in X$, $x_1 = \frac{x}{\|x\|}$ $M_1 = \text{Span}\{x_1\}$

$\tilde{x}_2 \in X \setminus M_1$, $x_2 = \frac{\tilde{x}_2}{\|\tilde{x}_2\|}$, $M_2 = \text{Span}\{x_1, x_2\}$

Continue picking $\tilde{x}_n \in X \setminus M_{n-1} \to x_n = \frac{\tilde{x}_n}{\|\tilde{x}_n\|}$

then for all n, $\|x_n\| = 1$.

for any n we know that $\exists$ f s.t

$f(x_n) = \|x_n\| = 1$ and $\|f\| = 1$ for any x

we can write $x = x_m + \tilde{x}$, $x_m \in M_n$, $\tilde{x} \in M_n^c$

then let $f_n(x) = f_n(x_m + \tilde{x}) = f(x_m)$ ∎

ii) Show that if X is complete then $\exists$ a linear injective map $T: \ell^\infty \to X$

let $f \in \ell^\infty$ $x = \sum_{n=1}^{\infty} f(n) x_n$

this is linear and injective

(but is $x \in X$?)

iii) Show that ℓ^∞ has a ~~contin~~ continuum of linearly independent vectors:

assume ℓ^∞ only has countable independent vectors list these as $\{f_k\}_{k\in\mathbb{N}}$ then let

$g\in\ell^\infty$ s.t $g(n)=f_n(n)$

Consider any $k\in\mathbb{N}$ then

$ag(k)+bf_k(k)=0 \iff a=-b$

since f_k is independent of f_n, $n\neq k$

there must be some $i\in\mathbb{N}$ s.t

$f_k(i)\neq f_n(i)$ thus $g(i)-f_k(i)\neq 0$ and

so g and f_k are independent $\forall k$ $\lightning$