X is the space of sequenses in $\mathbb{R}^\omega$ s.t

$\sum x_i^2$ converges

a) show that if $x, y \in X$ then $\sum |x_i y_i|$ converges

let $S_n^2 = \left(\sum_{i=1}^{n} |x_i y_i|\right)^2 \leq \sum_{i=1}^{n} |x_i|^2 \sum_{i=1}^{n} |y_i|^2 \leq \|x\|^2 \|y\|^2$

since S_n is bounded and monotone it converges.

b) Let $c \in \mathbb{R}$. show that if $x, y \in X$ then so are $x+y$ and cx.

clearly $\|cx\|^2 = \sum_{i=1}^{\infty} c^2 x_i^2 = c^2 \sum x_i^2 < \infty$

$cx \in X$

$\left(\sum_{i=1}^{n} |x_i + y_i|^2\right)^{1/2} \leq \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} + \left(\sum_{i=1}^{n} |y_i|^2\right)^{1/2} \leq \|x\| + \|y\| \quad \forall n \in \mathbb{N}$

thus

$\lim \left(\sum_{i=1}^{n} |x_i + y_i|^2\right)^{1/2} \leq \|x\| + \|y\|$

c) show that $d(x, y) = \left(\sum (x_i - y_i)^2\right)^{1/2}$ is a well defined metric on X

- $d(x, y) \geq 0$. Clearly $d(x,y) = 0 \Rightarrow x_i = y_i \quad \forall i$

  thus $d(x,y) = 0 \Leftrightarrow x = y$

- $d(x,y) = d(y,x)$.

- $$d(x,z) = \lim_{n\to\infty}\left(\sum_{i=1}^{n}(x_i - z_i)^2\right)^{1/2} \le \lim_{n\to\infty}\left(\left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2} + \left(\sum_{i=1}^{n}(y_i - z_i)^2\right)^{1/2}\right)$$

  $x, z, y \in X$. true for all $i \in \mathbb{N}$

  $$= d(x,y) + d(y,z)$$