Show that if X has a countable dense subset, every collection of disjoint open sets in X is countable.

Let $Y = \{y_n\}_{n \in \mathbb{N}}$ be a dense countable subset of X.

Let $\{U_\alpha\}_{\alpha \in J}$ be a collection of disjoint open sets as $\overline{Y} = X$ we know that any U_α contains some element $y_n \in Y$, as $U_\alpha \cap U_\beta = \emptyset$ $y_n \notin U_\beta$ then for any $\alpha \in J$ pick $y_n \in Y$. $\beta \neq \alpha$

then we have an injection into a countable set thus $\{U_\alpha\}_{\alpha \in J}$ is countable.