Let X be a metric space with the metric d.
let $A \subset X$ be nonempty.

a) show that $d(x, A) = 0 \iff x \in \overline{A}$

"$\Rightarrow$" $d(x, A) = \inf_{a \in A} d(x, a)$. Suppose $x \notin \overline{A}$ then $\exists\ \varepsilon > 0$ s.t $B(x, \varepsilon) \cap A = \phi$ thus $d(x, A) \geq \varepsilon$

"$\Leftarrow$" for any $\varepsilon > 0$ $B(x, \varepsilon) \cap A \neq \phi$ thus we can find a_ε s.t $d(x, a_\varepsilon) < \varepsilon$ so $\inf_{a \in A} d(x, a) = 0$

b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.

for any x, the function $d_x(a) = d(x, a)$ is continuous. As A is compact there is a_0 in A s.t $d_x(a_0) \leq d(a)$ $\forall\ a \in A$ thus

$d(x, a_0) = \inf_{a \in A} d(x, a) = d(x, A)$.

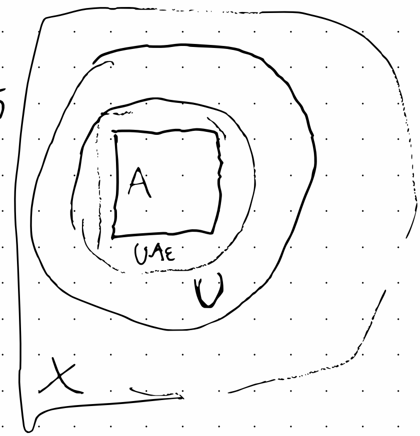
c) Define the ε-neighborhood of A in X to be the set $U(A, \varepsilon) = \{x \mid d(x, A) < \varepsilon\}$

Show that $U(A, \varepsilon)$ equals the union of open balls $B_d(a, \varepsilon)$ for $a \in A$

let $\cancel{x} \in U(A, \varepsilon)$ then $\inf_{a \in A} d(x, a) < \varepsilon$ then $\exists$ som $a \in A$ s.t $d(x, a) < \varepsilon$ (otherwise $d(x, A) \geq \varepsilon$). so $x \in B(a, \varepsilon)$.

let $x \in B_d(a, \varepsilon)$ then x is clearly in $U(A, \varepsilon)$

d) Assume that A is compact. Let U be an open set containing A. Show that some ε-neighborhood of A is contained in U.



$d_{U^c}(a) = d(a, U^c)$ is continuous and thus by the extreme value theorem there is $a_0 \in A$ s.t.

$$d(a_0, U^c) \le d(a, U^c) \quad \forall a \in A$$

$d(a_0, U^c) > 0$ since $a_0 \in U$, U open

take then $\bigcup_{a \in A} B(a, d(a_0, U^c))$ which is contained in U.

e) Show that (d) need not hold if A is closed but not compact.

$A = \{x \times 1/x : 0 < x \le 1\}$ has an open covering

$U = \{x \times (1/x - \varepsilon, 1/x + \varepsilon) / 0 < x < 2\}$ but no ε-neighborhood in $\mathbb{R}^2$