Let A be a set: let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of spaces and let $\{f_\alpha\}_{\alpha \in J}$ be an indexed family of ~~set~~ functions $f_\alpha : A \to X_\alpha$

a) Show there is a unique coarsest topology on A relative to which each function f_α is continuous

$\mathcal{T} = \bigcap \mathcal{T}_i$, $\mathcal{T}_i$ s.t f_α is continuous on X_α

by definition the coarsest

We can then define $\mathcal{T}_\alpha = \mathcal{T}(f_\alpha^{-1}(U) : U \in X_\alpha)$

then take take a topology generated by their union thus we have at least one where f_α is continuous $\to$ coarsest.

b) Let $S_\beta = \{f_\beta^{-1}(U_\beta) : U_\beta \text{ open in } X_\beta\}$ and $S = \bigcup S_\beta$ show that S is a subbasis for $\mathcal{T}$.

In exercise 13.6 it was showed that the topology generated by a subbasis is equal to the intersection of all topologies containing the subbasis. Thus we will show that f_α is continuous w.r.t a topology $\mathcal{T}' \Leftrightarrow \mathcal{T}'$ contains S

"$\Rightarrow$" consider $V \in S$ then $V = f_\alpha^{-1}(\mathcal{U})$ for some α, $\mathcal{U}$ open in X_α. Since f_α is continuous $V \in \mathcal{T}'$

"$\Leftarrow$" let $\mathcal{U} \in X_\alpha$ for some α then $f_\alpha^{-1}(\mathcal{U}) \in S \subset \mathcal{T}'$ so all f_α are continuous ∎

c) Show that the map $g: Y \to A$ is continuous relative to $\mathcal{T} \Leftrightarrow$ each map $f_\alpha \circ g$ is continuous.

"$\Rightarrow$" since f_α is continuous $f_\alpha \circ g$ is continuous

"$\Leftarrow$" since S is a subbasis of $\mathcal{T}$ any element of $\mathcal{T}$ can be written as a union of finite intersections from S that is

$$\mathcal{U} \in \mathcal{T} \to \mathcal{U} = \bigcup_{i \in I} \left(\bigcap_{k=1}^{N_i} V_k^{(i)} \right) \quad \text{wher } V_k^{(i)} \in S$$

$$\text{that is } \mathcal{U} = \bigcup_{i \in I} \left(\bigcap_{k=1}^{N_i} f_{\beta_k}^{-1}\left(\tilde{V}_{\beta_k}^{(i)}\right) \right) \quad \tilde{V}_{\beta_k}^{(i)} \in X_{\beta_k}$$

$$\text{so } g^{-1}(\mathcal{U}) = g^{-1} \circ \left(\bigcup_{i \in I} \left(\bigcap_{k=1}^{N_i} f_{\beta_k}^{-1}\left(V_{\beta_k}^{(i)}\right) \right) \right)$$

$$= \bigcup_{i \in I} \left(\bigcap_{k=1}^{N_i} \left(g^{-1} \circ f_{\beta_k}^{-1}\left(\tilde{V}_{\beta}^{(i)}\right) \right) \right) = \bigcup_{i \in I} \bigcap_{k=1}^{N_i} (f_{\beta_k} \circ g)^{-1}\left(\tilde{V}_{\beta_k}^{(i)}\right)$$

which is open since $(f_{\beta_k} \circ g)^{-1}$ is continuous

d) Let $f: A \to \prod X_\alpha$ be defined by $f(a) = (f_\alpha(a))_{\alpha \in J}$. Let Z denote the subspace $f(A)$ of the product space $\prod X_\alpha$. Show that the image under f of each element of $\mathcal{T}$ is an open set of Z.

by (c) if we can show that $f_\alpha \circ f^{-1}$ is continuous for all α we are done

that is if $\mathcal{U}_\beta$ is open in X_β then $f \circ f^{-1}_\cdot(\mathcal{U}_\beta)$ is open in Z. we note that this amounts to showing $f(S_\beta)$ open in Z where $S \in \mathcal{S}$. let x be any point in S_β clearly $f_\beta(X_\beta) \in \mathcal{U}_\beta$ and for $\alpha \neq \beta$ $f_\alpha(X_\alpha) \in f_\alpha(A)$ but $\prod_{\alpha \in J} (\mathcal{U}_\alpha \cap f_\alpha(A))$, $\mathcal{U}_\alpha = X_\alpha$, $\alpha \neq \beta$

is a basis element of Z. thus $f(S_\beta)$ is open in Z $\blacksquare$