a) X is $N(\mu, \sigma^2)$, compute density of e^X

$$F(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} e^x \, dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{t} e^{-\frac{(x-\mu)^2}{2\sigma^2} + x} \, dx$$

for normal

$$P(e^X \le t) = P(X \le \log t) = F(\log t)$$

$$\frac{\partial}{\partial t} F(\log t) = \frac{1}{t} F'(\log t)$$

$$= \frac{1}{t} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(\log t - \mu)^2}{2\sigma^2}} = \cancel{\frac{1}{t} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(\log t)^2}{2\sigma^2}} e^{-\frac{\mu^2}{2\sigma^2}} e^{\frac{\mu \log t}{\sigma^2}}}$$

not so pretty

b) show that a lognormal law has finite moments of all orders and compute them

$E[(e^X)^\alpha]$, $X = \sigma Z + \mu$

$$E[e^{\langle \theta, X\rangle}] = e^{\langle \theta, b\rangle} e^{\frac{1}{2}\langle \Gamma\theta, \theta\rangle}$$

$$E[e^{\alpha X}] = e^{\alpha\mu} e^{\frac{1}{2}\sigma^2\alpha^2}$$, $\alpha = 1$ gives mean

$$= e^{\mu} e^{\frac{1}{2}\sigma^2}$$, $\mathrm{Var} = e^{2\mu} e^{\sigma^2} - e^{2\mu} e^{2\sigma^2}$