Theorem. Let $x_n \to x$, $y_n \to y$ in $\mathbb{R}$

Then:

$$x_n + y_n \to x + y$$
$$x_n - y_n \to x - y$$
$$x_n y_n \to xy$$

and for $\cancel{y_n} \neq 0 \neq y$: $\frac{x_n}{y_n} \to \frac{x}{y}$

$x_n + y_n \to x + y$:

$\exists N_x$ s.t $d(x, x_n) < \frac{\varepsilon}{2}$ when $n \geq N$

N_y s.t $d(y, y_n) < \varepsilon/2$

$$d(x_n + y_n, x + y) = |x + y - (x_n + y_n)| \leq |x - x_n| + |y - y_n|$$

$< \varepsilon$ when $n \geq \max\{N_x, N_y\}$

for $x_n - y_n \to x - y$ substitute y_n by $-y_n$ in the argument above.

$x_n y_n \to xy$

$$d(x_n y_n, xy) = |xy - x_n y_n| = |xy - x_n y + x_n y - x_n y_n|$$

$$\leq |xy - x_n y| + |x_n y - x_n y_n| = |y||x - x_n| + |x_n||y - y_n| \xrightarrow[n \to \infty]{} 0$$

as $\lim |x_n| < \infty$

assume $Y_n \neq 0 \neq Y$

$$d\left(\frac{X_n}{Y_n}, \frac{X}{Y}\right) = \left|\frac{X}{Y} - \frac{X_n}{Y_n}\right| = \left|\frac{X}{Y} - \frac{X_n}{Y} + \frac{X_n}{Y} - \frac{X_n}{Y_n}\right|$$

$$\leq \left|\frac{1}{Y}\right| |X - X_n| + \left|\frac{X_n Y_n - X_n Y}{Y Y_n}\right|$$

$$= \left|\frac{1}{Y}\right| |X - X_n| + \left|\frac{X_n}{Y}\right| \left|1 - \frac{Y}{Y_n}\right| \xrightarrow[n \to \infty]{} 0$$

as $|X - X_n| \to 0$ $\lim \left|\frac{X_n}{Y}\right| < \infty$ and $\frac{Y}{Y_n} \to 1$, see

since $Y_n \neq 0$, $Y_n \to Y$ we know that

$$-\varepsilon < Y_n - Y < \varepsilon \iff Y - \varepsilon < Y_n < Y + \varepsilon \iff$$

$$\frac{Y - \varepsilon}{Y_n} < 1 < \frac{Y + \varepsilon}{Y_n} \quad \text{so} \quad \frac{Y}{Y_n} \to 1$$