X, Y are independent r.v's assume they are bounded for simplicity.

Show that $E[XY] = E[X]E[Y]$

- Assume first that X, Y are nonnegative simple functions

$X = \sum a_n \mathbb{1}_{A_n}$, $Y = \sum b_n \mathbb{1}_{B_n}$

$$XY = \sum_{j=1}^{n} a_j \mathbb{1}_{A_j} \sum_{k=1}^{m} b_k \mathbb{1}_{B_k} = \sum_{j=1}^{n} \sum_{k=1}^{m} a_j b_k \mathbb{1}_{A_j \cap B_k}$$

$$E[XY] = \int \sum_{j=1}^{n} \sum_{k=1}^{m} a_j b_k \mathbb{1}_{A_j \cap B_k} \, dP$$

$$= \sum^{n} \sum^{m} a_j b_k P(A_j \cap B_k) \underset{\text{independence}}{=} \sum_{j=1}^{n} a_j P(A_j) \sum b_j P(B_j)$$

$$= E[X] E[Y]$$

- for nonnegative measurable X, Y exists increasing sequences of measurable functions $X_n Y_n \to XY$

$$E[XY] = \int \lim X_n Y_n \, dP = \lim \int X_n Y_n \, dP = \lim (E[X_n] E[Y_n])$$
$$= E[X] E[Y]$$

• for general XY let X^+, X^-, Y^+, Y^-

so $XY = X^+Y^+ + X^-Y^- - X^+Y^- - X^-Y^+$

define as before

then $E[XY] = \int X^+Y^+ \, dP + \int X^-Y^- \, dP - \int X^+Y^- \, dP - \int X^-Y^+ \, dP$

$$= E[X^+]E[Y^+] + E[X^-]E[Y^-] - E[X^+]E[Y^-] - E[X^-]E[Y^+] = E[X^+](E[Y^+] - E[Y^-])$$

$$+ E[X^-](E[Y^+] - E[Y^-])$$

$$= E[X^+](E[Y]) + E[X]E[Y]$$

$$= E[X]E[Y]$$