Let X, Y be complete metric spaces: $f : X \times Y \to \mathbb{C}$ be a function (continuous in each variable) show that there exists a point of continuity of f. Conclude that the set of points of continuity is dense $X \times Y$.

Hint: for $\varepsilon > 0$ consider $A_n \subset X \times Y$ consisting of (x_0, y_0) s.t $|f(x_0, y_0) - f(x, y_0)| < \varepsilon$ and $|f(x_0, y_0) - f(x_0, y)| < \varepsilon$ wheneveever $d(x, x_0) < \frac{1}{n}$ and $d(y, y_0) < \frac{1}{n}$ and observe that $\bigcup_{n \in \mathbb{N}} A_n = X \times Y$

Want to show that $\exists (x_0, y_0) \in X \times Y$ s.t for $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t

$|f(x, y) - f(x_0, y_0)| < \varepsilon$ when $d((x, y), (x_0, y_0)) < \delta$

$X \times Y$ is complete w.r.t the metric

$d((x_1, y_1), (x_2, y_2)) = d_x(x_1, x_2) + d_y(y_1, y_2)$

Let A_n be as described since f is continuous in ~~some~~ both variables each point $(x, y) \in X \times Y$ belongs to some A_N, $N \in \mathbb{N}$.

thus $X \times Y = \bigcup_{n \in \mathbb{N}} A_n$ also $A_{n-1} \subset A_n$

since $X \times Y$ is complete there must be some A_N that is not nowhere dense

then for $(x_0, y_0) \in A_N$ $\exists$ $r > 0$ s.t

$B_N = B((x_0, y_0), r) \subset A_N$ we can assume $r < \frac{1}{N}$ as we can always make r smaller. let $(x, y) \in B_N$

then $d((x, y), (x_0, y_0)) = d(x, x_0) + d(y, y_0) < \frac{1}{N}$

therefore (x, y_0) and $(x_0, y) \in B_N$

and $d((x,y_0),(x,y)) = d(y_0,y) < \frac{1}{N}$

then a $d((x_0,y_0),(x,y_0)) = d(x_0,x) < \frac{1}{N}$

$$|f(x_0,y_0) - f(x,y)| \leq |f(x_0,y_0) - f(x,y_0)| + |f(x,y_0) - f(x,y)|$$

$< 2\varepsilon$

Hence we have at least one point of continuity

Show that it is dense.

Assumme not dense. then we have a point (x_0,y_0), $r>0$ s.t $B((x_0,y_0),r)$ contains no points of continuity.

Then $\overline{B((x_0,y_0),r)}$ contains no points of continuity furthermore. $\overline{B((x_0,y_0),r)}$ is complete, but then by the argument before it does contain a point of continuity $\blacksquare$