B_t is 1-d brownian, $\sigma \in \mathbb{R}$ is constant
$0 \le s < t$ use (2.2.2) to prove that

$$E[\exp(\sigma(B_s - B_t))] = \exp(\tfrac{1}{2}\sigma^2(t-s))$$

1. using properties of ~~normal distri~~ brownian we see that

$$\sigma(B_s - B_t) \sim N(0, \sigma^2(t-s))$$

then using the laplace transform (proven in Baldi exercise 1.6): $E[e^{\langle\theta, X\rangle}] = e^{\langle\theta, b\rangle} e^{\frac{1}{2}\langle\Gamma\theta, \theta\rangle}$

$$E[e^{N(0,\sigma^2(t-s))}] = e^{\frac{1}{2}\sigma^2(t-s)}$$

2) using 2.2.2.

$$E[e^{\sigma(B_s - B_t)}] = \int e^{\sigma x} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{x^2}{2(t-s)}} dx$$

$$= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} \exp\left(\sigma x - \frac{x^2}{2(t-s)}\right) dx$$

$$\sigma x - \frac{x^2}{2(t-s)} = \left(\frac{x}{\sqrt{2(t-s)}} - \sigma\sqrt{\frac{(t-s)}{2}}\right)^2 + \sigma^2 \frac{(t-s)}{2}$$

$$= -\left(\frac{x - \sigma(t-s)}{\sqrt{2(t-s)}}\right)^2 + \sigma^2 \frac{(t-s)}{2}$$

$$\frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty}\exp\left(\sigma x-\frac{x^2}{2(t-s)}\right)dx$$

$$=\frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty}\exp\left(\frac{\sigma^2(t-s)}{2}\right)\exp\left(-\left(\frac{x-\sigma(t-s)}{\sqrt{2(t-s)}}\right)^2\right)dx$$

$$=\frac{1}{\sqrt{2\pi(t-s)}}\exp\left(\frac{\sigma^4(t-s)}{2}\right)\int_{-\infty}^{\infty}\exp\left(-\left(\frac{x}{\sqrt{2(t-s)}}\right)^2\right)dx$$

$$=\exp\left(\frac{\sigma^2(t-s)}{2}\right)\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{x^2}{2(t-s)}}dx$$

$$=\exp\left(\frac{\sigma^2(t-s)}{2}\right)$$