$X: \Omega \to \mathbb{R}$ is a function which assumes countably many values in $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$

a) show that X is a r.v $\Leftrightarrow X^{-1}(a_n) \in \mathcal{F}$

"$\Rightarrow$" X is measurable so $X^{-1}(a_n) \in \mathcal{F}$

"$\Leftarrow$" $X^{-1}(a_n) \in \mathcal{F}$ thus for Any $A \in \mathcal{B}(\mathbb{R})$ either there are some or none $a_n \in A$ if none then $X^{-1}(A) = \emptyset \in \mathcal{F}$ or if several then $X^{-1}(A) = \bigcup_{n_i=1}^{\infty} X^{-1}(a_{n_i}) \in \mathcal{F}$ where $a_{n_i} \in \{a_n\}_{n\in\mathbb{N}}$

b) $E[|X|] = \int_\Omega |X(\omega)| dP$ define $A_n = \{\omega \in \Omega \,|\, X(\omega) = a_n\}$

$\bigcup A_n = \Omega$

then $X_n = \sum \mathbb{1}_{A_n} X$ is measurable and

$X_n \to X$ pointwise. $\liminf \int |X_n| dP \geq \int \liminf |X_n| dP$
$\underset{\text{fatou}}{=} \int |X| dP$

but $\int |X| dP \geq \liminf \int |X_n| dP$

so $\int |X| dP = \liminf \int |X_n| dP = \liminf \sum |a_n| P(X = a_n)$
$= \sum^{\infty} |a_n| P(X = a_n)$.

c) X_n is bounded by $|X|$ thus by dominated convergence $\int X dP = \int \lim X_n dP = \lim \int X_n dP =$

$= \sum a_n P(X = a_n)$

d) Let $f_n = f(X_n)$. $|f_n| \le \sup|f| = M$

$\int M \, dP = M$ thus by dominated convergence

$$\int f(X) \, dP = \int \lim f_n \, dP = \lim \int f_n \, dP$$

$$= \lim \sum f(a_n) P(X = a_n) = \sum^{\infty} f(a_n) P(X = a_n)$$