Let A_0 be the closed interval $[0,1]$ in $\mathbb{R}$. Let A_1 be the set obtained from A_0 by deleting its "middle third" $(\frac{1}{3}, \frac{2}{3})$. Let A_2 be the set obtained from A_1 by deleting its "middle thirds" $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. In general define A_n By:

$$A_n = A_{n-1} \setminus \left(\bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right) \right). \qquad \begin{array}{l} 3k \le 3^n - 3 \\ k \le 3^{n-1} - 1 \end{array}$$

$C = \bigcap_{n \in \mathbb{Z}^+} A_n$ is called the cantor set

a) show that C is totally disconnected

want to show that any subspace containing more than one element has a separation.

(non rigorous) Let X be a subspace of C. Let (a,b) be an interval in $[0,1]$ s.t a and b are both in some "middle thirds" and $(a,b) \cap X$ is a nonempty, proper subset of X then

$$((a,b) \cap X)^c = (a,b)^c \cap X = ((-\infty, a] \cup [b, \infty)) \cap X$$
$$= ((-\infty, a) \cup (b, \infty)) \cap X$$
as $a, b \notin C \supset X$

So $((a,b) \cap X)^c$ and $(a,b) \cap X$ are a separation

Show that C is compact

C is an intersection of closed sets so closed
C is a closed subset of a compact space
so compact.

c) Show that each set A_n is a union of finitely many disjoint closed intervals of length $1/3^n$ and show that the endpoints of these intervals lie in C.

d) show that C has no isolated points.

let $x \in C$. then x is in all A_n. for $\varepsilon > 0$ we have N st $\frac{1}{3^n} < \varepsilon$, $n \geq N$ then x in some interval of length less than ε. Since the endpoints of this interval is in C we have that they are included in $B(x, \varepsilon)$. thus any basis element containing x contains at least 2 elements. Therefore x have no isolated points.

e) Conclude that C is uncountable.

By 27.7 C is uncountable.