a) $\{\tau_\alpha\}$ is a family of topologies on X show that $\cap \tau_\alpha$ is a topology on X

1) $\emptyset \in \tau_\alpha$

2) let $U_1, \dots, U_n$ be open sets in $\cap \tau_\alpha$. Then $U_i \in \tau_\alpha$ for any α

then $\bigcap_{i=1}^{n} U_i \in \tau_\alpha$ for any α

then $\bigcap_{i=1}^{n} U_i \in \bigcap_\alpha \tau_\alpha$

3) let $\{U_i\}_{i\in I}$ be a collection of sets from $\cap \tau_\alpha$

then $U_i \in \tau_\alpha \to \bigcup_{i\in I} U_i \in \tau_\alpha$

$\to \bigcup_{i\in I} U_i \in \bigcap_\alpha \tau_\alpha$

b) let $\{\tau_\alpha\}$ be a family of topologies on X. show that there is a unique smallest topology on X containg all τ_α and a unique largest contained in all.

~ largest $\bigcap_\alpha \tau_\alpha$ if $U \in \tau_\alpha$ $\forall \alpha$ then

$U \in \bigcap_\alpha \tau_\alpha$ by (a) it is a topology

Let $\mathcal{A}$ be the set of all topologies containing $\{\mathcal{T}_\alpha\}$ then let $\mathcal{T} = \bigcap_{A \in \mathcal{A}} A$. by (a) this is a topology and the smallest (since it intersects the smallest)

Need to show that this is nonempty

Let $\mathcal{B}_\alpha$ be a basis for $\mathcal{T}_\alpha$ consider $S = \bigcup_\alpha \mathcal{B}_\alpha$ as a subbasis (it's union is X) this generates a topology $\mathcal{T}$

let $\mathcal{U} \in \mathcal{T}_\alpha$ then $\exists\, B \in \mathcal{B}_\alpha \subset \mathcal{B}$ s.t. $B \subset \mathcal{U}$ thus $\mathcal{U} \in \mathcal{T}$ and the set is nonempty $\blacksquare$

c) $X = \{a, b, c\}$, $\mathcal{T}_1 = \{\phi, X, \{a\}, \{a, b\}\}$

$\mathcal{T}_2 = \{\phi, X, \{a\}, \{b, c\}\}$

$\mathcal{T}_4 = \{\phi, X, \{a\}\}$ ~ largest in $\mathcal{T}_1$ and $\mathcal{T}_2$

$\mathcal{T}_5 = \{\phi, X, \{a\}, \{a, b\}, \{b, c\}, \{b\}\}$