Let $p: X \to Y$ be a quotient map. Show that if each set $p^{-1}(\{y\})$ is connected and if Y is connected then X is connected

Suppose U, V is a separation of X. Since $p^{-1}(\{y\})$ is connected we know that for any y, $p^{-1}(\{y\})$ is contained entirely in either U or V. Thus for any $u \in U$, $v \in V$, $p(u) \neq p(v)$ but $p(U \cup V) \overset{\text{surjective}}{=} Y$ so $U = p^{-1}(Y_1)$, $V = p^{-1}(Y_2)$ for sets Y_1, Y_2 s.t. $Y_1 \cap Y_2 = \emptyset$.

but then Y_1, Y_2 are open in Y as p is a quotient map. This contradicts our assumption that Y is connected