a) is the product of path-connected spaces necessarily path connected

Yes

b) If $A \subset X$ and A is path-connected is $\bar{A}$ necesarilly path connected.

no, Topologists sine curve (not closed)

c) if $f: X \to Y$ is continuous and X is path connected, is $f(X)$ necessarily path connected?

yes. take $y_1, y_2 \in f(X)$, $x_1, x_2 \in X$ s.t $f(x_1) = y_1$, $f(x_2) = y_2$. $g: [a,b] \to X$ be a path connecting x_1, x_2. Then $f \circ g: [a,b] \to f(X)$ is continuous and connects y_1, y_2.

d) If $\{A_\alpha\}$ is a collection of path connected subspaces of X and if $\bigcap A_\alpha \neq \emptyset$ is $\bigcup A_\alpha$ necessarily path connected

let $x_0 \in \bigcap A_\alpha$. consider $x, y \in \bigcup A_\alpha$. Then $x \in A_k$, $y \in A_j$, $f_1: [a,b] \to A_k$ is a path from $x \to x_0$, $f_2: [b,c] \to A_j$ is a path $x_0 \to y$ so $\bigcup A_\alpha$ is path connected