

Show that X is Hausdorff \Leftrightarrow

$\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$

Let $a, b \in X$, $a \neq b$ then $(a, b) \in \Delta^c$

we have \mathcal{U}, \mathcal{V} s.t. $a \in \mathcal{U} \in \mathcal{T}$, $b \in \mathcal{V} \in \mathcal{T}$
s.t. $\mathcal{U} \cap \mathcal{V} = \emptyset$ then $x \times x \notin \mathcal{U} \times \mathcal{V}$ for any $x \in X$
thus Δ^c is open so Δ is closed

" \Leftarrow "

Δ^c is open thus for $(a, b) \in \Delta^c$

$\exists \mathcal{U} \times \mathcal{V}$ s.t. $(a, b) \in \mathcal{U} \times \mathcal{V} \subset \Delta^c$

This means there are no points $(x, x) \in \mathcal{U} \times \mathcal{V}$
thus $\mathcal{U} \cap \mathcal{V} = \emptyset$ and X is Hausdorff.