Let X be the subset of $\mathbb{R}^\omega$ consisting of all sequences x s.t $\sum x_i^2$ converges. Then the formula

$$d(x,y) = \left(\sum (x_i - y_i)^2\right)^{1/2}$$

defines a metric on X. On X we have the three topologies it inherits from the box, uniform and product topologies on $\mathbb{R}^\omega$.
We also have the topology given by the metric $d \sim \ell^2$-topology

a) show that box topology $\supset \ell^2$-topology $\supset$ uniform topology

let $B_u(x,\varepsilon)$ be a basis element in the uniform topology. the $B_d(x,\varepsilon) \subset B_u(x,\varepsilon)$ as $|x_i - y_i| < \varepsilon$ and $\lim |x_i - y_i| = 0$.

let $B_d(x,\varepsilon)$ be a basis element in the ℓ^2-topology the the basis element $\prod B(x_i, \frac{\varepsilon}{2}\frac{1}{2^{i/2}})$ is in $B_d(x,\varepsilon)$ as if $y \in \prod B(x_i, \frac{\varepsilon}{2}\frac{1}{2^{i/2}})$

$$d(x,y) = \left(\sum (x_i - y_i)^2\right)^{1/2} < \left(\sum \frac{1}{4}\frac{\varepsilon^2}{2^i}\right)^{1/2} = \frac{\varepsilon}{2}\left(\sum_{i=1}^{\infty} \frac{1}{2^i}\right)^{1/2}$$

$$= \varepsilon$$

choosing c s.t this is $< \varepsilon$

But $\prod B(x_i, \frac{\varepsilon}{2^{i+1}})$ is a basis element of the box-topology.

(b) the set $\mathbb{R}^\infty$ of all sequences that are eventually zero is contained in X. Show that the four topologies that $\mathbb{R}^\infty$ inherits as a subspace of X are all distinct.

find a basis element in box $\notin \ell^2$

a basis element is the intersection by ~~an~~ basis element in X want one that is not in ℓ^2