Let x_0 and x_1 be points of the path connected space X show that $\pi_1(X, x_0)$ is abelian $\Leftrightarrow$ for every pair α and β of paths from x_0 to x_1 we have $\hat{\alpha} = \hat{\beta}$.

We have that $\alpha * \bar{\beta}$, $\beta * \bar{\alpha}$ are both in $\pi_1(X_0, X_0)$

"$\Rightarrow$" $\hat{\alpha}(f) * \hat{\beta}(\bar{f}) = \bar{\alpha} * f * (\alpha * \bar{\beta}) * \bar{f} * \beta$

$= \bar{\alpha} * (\alpha * \bar{\beta}) * f * \bar{f} * \beta$

$= (\bar{\alpha} * \alpha) * (\bar{\beta} * \beta) = e$

"$\Leftarrow$"

let $f \in \pi_1(X, x_0)$, $\gamma = f * \alpha$ is a path from x_0 to x_1, for $g \in \pi_1(X, x_0)$:

$\hat{\gamma}(g) = \hat{\alpha}(g) \Rightarrow \overline{f * \alpha} * g * f * \alpha = \bar{\alpha} * g * \alpha$

$\Rightarrow \bar{\alpha} * \bar{f} * g * f * \alpha = \bar{\alpha} * g * \alpha \Rightarrow$

$\bar{f} * g * f = g \Rightarrow g * f = f * g$ ∎