Let X be a ~~$f_n: X \to R$ be a~~ set and let $f_n : X \to \mathbb{R}$ be a sequence of functions. let $\bar{\rho}$ be the uniform metric on $\mathbb{R}^X$. Show that (f_n) converges uniformly to $f: X \to \mathbb{R} \iff$ the sequence (f_n) converges to f as elements of the metric space $(\mathbb{R}^X, \bar{\rho})$

" $\implies$ "

$f \in \mathbb{R}^X$ can be written as a tuple $(f^{(\alpha)})_{\alpha \in X}$

then if $f, g \in \mathbb{R}^X$, $\bar{\rho}(f, g) = \sup_{\alpha \in X} \{ \bar{d}(f^\alpha, g^\alpha) \}$

$$= \sup_{\alpha \in X} \{ \bar{d}(f(\alpha), g(\alpha)) \}$$

let f_n be a sequence converging to f then for any $\varepsilon > 0$ $\exists$ N s.t $d(f_n(x), f(x)) < \varepsilon$ $\forall$ $x \in X$ then clearly $\bar{\rho}(f_n, f) < \varepsilon$ when $n \geq N$

" $\impliedby$ "

for $\varepsilon > 0$ $\exists$ N s.t $\bar{\rho}(f_n, f) = \sup_{x \in X} \{ \bar{d}(f_n(x), f(x)) \} < \varepsilon$

So for any x: $d(f_n(x), f(x)) < \varepsilon < 1$ when $n \geq N$ ∎