$B_t$ is brownian motion on $\mathbb{R}$, $B_0 = 0$

$E = E^0$

a) use (2.2.3) to prove that

$$E\left[e^{iuB_t}\right] = e^{-\frac{1}{2}u^2 t}$$

$$B_t = B_t - B_0 \sim N(0,t)$$

using characteristic of normal distribution

$$E\left[e^{iuN(0,t)}\right] = e^{iu\cdot 0}\, e^{-\frac{1}{2}u^2 t} = e^{-\frac{1}{2}u^2 t}$$

b) Using power series exp of exponential determine $E\left[B_t^4\right]$

$$e^{iuB_t} = \sum_{k=0}^{\infty} \frac{(iuB_t)^k}{k!} \to \frac{\partial^n}{\partial u^n} e^{iuB_t} = \sum_{k=n}^{\infty} (iB_t)^k \frac{(iuB_t)^{k-n}}{(k-n)!}$$

$$e^{-\frac{1}{2}u^2 t} = \sum_{k=0}^{\infty} \frac{(-\frac{1}{2}u^2 t)^k}{k!}$$

for $B^4$: $E\left[\frac{u^4 B_t^4}{4!}\right] = \frac{(-\frac{1}{2}u^2 t)^2}{2!} = \frac{\frac{1}{4}u^4 t^2}{2}$

$$E\left[B_t^4\right] = 3t^2$$

in general substituting ' we get

$$E\left[\frac{(iuB_t)^{2k}}{(2k!)}\right] = \cancel{E}\,\frac{(-\frac{1}{2}u^2t)^k}{k!} =$$

$$= \frac{(-1)^k u^{2k}}{(2k)!} E[B_t^{2k}] = \frac{(-1)^k u^{2k} t^k}{2^k k!}$$

$$\rightarrow E[B_t^{2k}] = \frac{(2k)!\, t^k}{2^k k!}$$

c) $$E[f(X)] = \int f(X(\omega))\, P(d\omega)$$

$$= \int f(x) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

for $f(x) = x^{2k}$ start ~~k~~ $k=1$

$$\int_{-\infty}^{\infty} \underset{f'}{x^2} \underset{g}{e^{-x^2}} dx = \frac{x^3}{3} e^{-x^2}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{x^3}{3} e^{-x^2} \cdot (-2x)\, dx$$

$$= \frac{\overset{2k}{2}}{\underset{2k+1}{3}} \int_{-\infty}^{\infty} \cancel{x}\, x^4 e^{-x^2} dx$$

assume $$\int_{-\infty}^{\infty} x^{2k} e^{-x^2} dx = \frac{2}{2k+1} \int_{-\infty}^{\infty} x^{2(k+1)} e^{-x^2} dx$$

check if holds for further up

i.e $\int_{-\infty}^{\infty} x^{2(k+1)} e^{-x^2} dx = \frac{2}{2(k+1)+1} \int_{-\infty}^{\infty}$

$$\int x^{2k} e^{-x^2} dx = \frac{x^{2k+1}}{2k+1} e^{-x^2} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{x^{2k+1}}{2k+1} e^{-x^2}(-2x)\, dx$$

$$= \frac{2}{2k+1} \int_{-\infty}^{\infty} x^{2(k+1)} e^{-x^2} dx \qquad \int x^{2(k+1)} e^{-x^2} dx = \frac{2k+1}{2} \int x^{2k} e^{-x^2} dx$$

$-ax^2$, $u = \sqrt{a}\, x$, $\frac{du}{\sqrt{a}} = dx$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \left( \int\!\!\int e^{-x^2+y^2} dx\, dy \right)^{1/2} = \left( \int_0^{2\pi} \int_0^{\infty} r e^{-r^2} dr\, d\theta \right)^{1/2}$$

$u = r^2$, $\frac{du}{2r} = dr$

$R^2 = \infty$

$$= \left( \pi \int_0^{\infty} e^{-u} du \right)^{1/2} = \left( \pi \left( -e^{-u} \right) \Big|_0^{\infty} \right)^{1/2}$$

$$= \sqrt{\frac{\pi}{a}} \quad , \quad k = 1 \Rightarrow \frac{\cancel{2k+1}}{2} \sqrt{\pi}$$

$$k = 2 \Rightarrow \frac{1 \cdot 3}{2^2} \sqrt{\pi} =$$

$$= \left(\cancel{\frac{\pi}{a}}\right) \sqrt{\pi}\,(a)^{-1/2} \rightarrow \frac{1}{2}\sqrt{\pi}\,(a)^{-\frac{3}{2}} \rightarrow \frac{3}{2^2} \rightarrow$$

$$\frac{(2k)!}{k!} \quad \frac{4 \cdot 3 \cdot 2}{2} = 4 \cdot 3 \, , \quad 6 \cdot 5 \cdot 4$$

for $k+1$ ? $\frac{2(k+1)!}{(k+1)!} = \frac{(2k+2)(2k+1)}{(k+1)} \frac{(2k)!}{k!}$

$\frac{(2k+2)(2k+1)}{k+1} =$