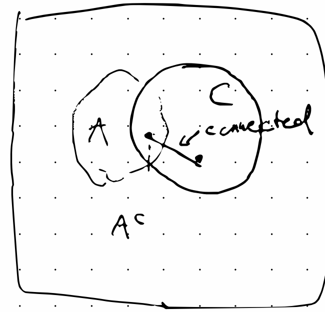
Let $A \subset X$ show that if C is a connected subspace of X that intersects both A and $X \setminus A$ then C intersects $\mathrm{Bd}\, A$

$$\mathrm{Bd}\, A = \overline{A} \cap \overline{A^c}$$



Argue contrapositively.

assume that C does not intersect $\mathrm{Bd}\, A$

$\overline{A} = \mathrm{int}\, A \cup \mathrm{Bd}\, A$ so $C \cap \mathrm{int}\, A \neq \emptyset$

similarly $C \cap \mathrm{int}\, A^c \neq \emptyset$

but $X = \mathrm{int}\, A \cup \mathrm{int}\, A^c \cup \mathrm{Bd}\, A$

$$\text{so } C = (C \cap \mathrm{int} A) \cup (C \cap \mathrm{int} A^c) \cup (C \cap \mathrm{Bd}\, A)$$

$$= (C \cap \mathrm{int} A) \cup (C \cap \mathrm{int}\, A^c)$$

but these are disjoint open sets so C is not connected