a) r.v. X has exponential law with parameter λ if it has density

$$f(x) = \lambda e^{-\lambda x} \mathbb{1}_{[0,\infty]}(x)$$

What is p.f of X, calculate variance and mean

$$F(t) = P(X \le t) = P\{\omega \in \Omega : X(\omega) \le t\}$$

$$= \int_{[0,t]} dP = \int_0^t \lambda e^{-\lambda x} \, d\cancel{t}u = -\lambda \frac{e^{-\lambda x}}{\lambda}\Big|_0^t = -e^{-\lambda x}\Big|_0^t$$

$$= 1 - e^{-\lambda t}$$

$$E[X] = \int_{[0,\infty]} X \, dP = \int_{[0,\infty]} X \lambda e^{-\lambda x} du$$

$$= \lambda \int_0^\infty \underset{f}{\hat{x}} \underset{g'}{\hat{e}}^{-\lambda x} du = \lambda\left(\left(-x \frac{e^{-\lambda x}}{\lambda}\right)\Big|_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-\lambda x} dx \right)$$

$$= \int_0^\infty e^{-\lambda x} dx = \frac{1}{\lambda} = \lambda \cdot \frac{1}{\lambda^2}$$

$$E[X^2] = \int_{[0,\infty]} X^2 dP = \int_0^\infty x^2 \lambda e^{-\lambda x} dx = \lambda \int \underset{f}{\hat{x}^2} \underset{g'}{\hat{e}^{-\lambda x}} dx$$

$$= \lambda\left(-x^2 \frac{e^{-\lambda x}}{\lambda}\Big|_0^\infty + \frac{2}{\lambda}\int_0^\infty x e^{-\lambda x} dx \right) = 2\int_0^\infty x e^{-\lambda x} dx$$

$$= 2/\lambda^2 \rightarrow \quad Var[X] = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

b) U is an r.v with density

$f(x) = \mathbb{1}_{[0,1]}$ $\qquad\qquad P(X=A) = \mu(A)$

$$E[U] = \int U \, dP = \int_{[0,1]} U \, d\mu = \int_0^1 x \, dx = \frac{1}{2}$$

$$E[U^2] = \int_0^1 x^2 \, dx = \frac{1}{3}, \quad \rightarrow \quad Var(U) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

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c_1) compute the law of $Z = \alpha U$, $\alpha > 0$

$$P(\cancel{\xi}\, Z \leq t) = P(\alpha U \leq t) = P(U \leq \tfrac{1}{\alpha} t)$$

$$= \int_0^{\frac{1}{\alpha} t} \mathbb{1}_{[0,1]} dP = \begin{cases} \frac{1}{\alpha} t, & 0 \leq t \leq \alpha \\ 1, & t > \alpha \end{cases} = \int_0^t \frac{1}{\alpha} \, dt$$

$$\mu_Z = \begin{cases} \alpha, & 0 \leq t \leq \alpha \\ 0, & t > \alpha \end{cases}$$

c_2) $P(W \leq t) = P(-\frac{1}{\lambda} \ln U \leq t) = P(\ln U \geq -\lambda t)$

$$= P(U \geq e^{-\lambda t}) = \int_{e^{-\lambda t}}^1 u \, du = 1 - e^{-\lambda t}$$ ~ exponential law