Consider the product, uniform, and box topologies on $\mathbb{R}^\omega$

a) in which topologies are the following functions from $\mathbb{R}$ to $\mathbb{R}^\omega$ continuous.

$f(t) = (t, 2t, 3t, \ldots$

first consider any basis element of the uniform metric if $x \neq y$ then $\bar{d}(x,y) = \underbrace{\sup\{\min\{|x_\alpha - y_\alpha|, 1\}\}}_{=1}$

thus the single points are the basis elements

but then $(t, 2t, \ldots) \in \tau^\omega \; \forall t \in \mathbb{R}$ but

$f^{-1}(t, 2t, \ldots) = t \notin \tau$

thus f is not continuous on the box either

on the product topology we have the basis element

$\mathcal{U} = \prod_{n=1}^{\infty} \mathcal{U}_n$ where $\mathcal{U}_n$ is open in $\mathbb{R}$ and $\mathcal{U}_n \neq \mathbb{R}$ for all but finitely many n

lis these as $n_1, n_2, \ldots, n_N$ then

$f^{-1}(\mathcal{U}) = f^{-1}\left(\bigcap_{k=1}^{N} \frac{\mathcal{U}_{n_k}}{n_k}\right) = \bigcap_{k=1}^{N} f^{-1}\left(\frac{\mathcal{U}_{n_k}}{n_k}\right)$ open

so continuous

$g(t) = (t, t, t \ldots$

for the uniform topology we get all basis elements of $\mathbb{R}$ from the euclidean topology where $\varepsilon \not\leq 1$ but this is a basis for $\mathbb{R}$ so continuous

$\to$ product topology is also continuous.

not box: take $\mathcal{U} = \prod B_n$, $B_n = B(0, \frac{1}{n})$

$f^{-1}(\mathcal{U}) = \{0\}$

$h(t) = (t, \frac{t}{2}, \frac{t}{3}, \ldots)$

continuous in uniform as before

$\to$ also product

product $B_n = B(0, \frac{1}{n^2})$

b) in which do following sequences converge

$w_1 = (1, 1, \ldots), w_2 = (0, 1, 1, \ldots), w_3 = (0, 0, 1, 1, \ldots)$

$\to$ clearly to $(0, 0, \ldots)$ in product

not uniform

not box

$x_1 = (1, 1, \ldots), x_2 = (0, \frac{1}{2}, \ldots), x_3 = (0, 0, \frac{1}{3}, \ldots)$

converges in unifor, product, not box $\mathcal{U} = \prod B(0, \frac{1}{n^2})$

y and z also converges in all 3