Let (X,d) be a metric space. If $f:X\to X$ Satisfies the condition

$$d(f(x),f(y))=d(x,y)$$

for all $x,y\in X$, then f is called an isometry of X. Show that if f is an isometry and X is compact, then f is bijective and hence a homeomorphism.

if $x\neq y$ then $d(f(x),f(y))>0 \iff f(x)\neq f(y)$

so f is injective.

assume f is not surjective. f is continuous thus $f(X)$ is compact and thus closed.

$f(X)^c$ is open pick $a\in f(X)^c$, $\varepsilon>0$ s.t.

$B(a,\varepsilon)\cap f(X)=\emptyset$.

let $x_0=a$, $x_{n+1}=f(x_n)$ let $f_n=\underbrace{f\circ\cdots\circ f}_{n}$

for $n>m$ $d(x_n,x_m)=d(f_n(a),f_m(a))$

$=d(f_{n-m}(a),a)>\varepsilon$. thus there is no convergent subsequence so X is not compact $\lightning$

Thus f is bijective and a homeomorphism.