a) what are the components and path components of $\mathbb{R}^\omega$ (in the product topology)

$\mathbb{R}^\omega$ is connected in the product topology so the components are $\mathbb{R}^\omega$ (and $\emptyset$)

- Path connectedness: consider $x, y \in \mathbb{R}^\omega$

let $f: [0,1] \to \mathbb{R}^\omega$ by $f(t) = (x_n + t(y_n - x_n))_n$

continuous in product topology so path connected

b) Consider $\mathbb{R}^\omega$ in the uniform topology. Show that x and y live in the same component $\iff$ $x-y$ is bounded.

[Hint suffices to consider the case when $y=0$]

we showed in ex 23.8 that $\mathbb{R}^\omega$ has a separation consisting of bounded and unbounded sequences.

"$\Rightarrow$" ~~ce~~ true for $y=0$ by ex 23.8. if $y \neq 0$ then consider the separation $x-y$ is bounded $x-y$ is unbounded.

"$\Leftarrow$" for bounded x, consider

$f:[0,1] \to \mathbb{R}^\omega$ by $f(t) = tx$ $\to$ this is continuous (not cont when x is unbounded)

so this is path connected. for some $y \in \mathbb{R}^\omega$ the case is similar for $x-y$ bounded. Thus they are in the same components.