Consider $X = (\mathbb{R} \setminus \{0\}) \cup \{p, q\}$. Give X the topology with basis containing all ~~elements of~~ open sets of $\mathbb{R}$ not containing 0 along with all sets

$(-a, 0) \cup \{q\} \cup (0, a)$ and $(-a, 0) \cup \{p\} \cup (0, a)$ $a > 0$

a) check this is basis for a topology.

let $x \in X$ if $x \in \mathbb{R} \setminus \{0\}$, clearly there is an element containing x. similar for $x = q$ or $x = p$

let x be in the intersection of 2 basis elements
if $x \in \{p, q\}$ then just pick lower value of a.
if B_1, B_2 are of the standard open sets there is $B_3 \subset B_1 \cap B_2$ containing x.
suppose $x \in (b, c) \cap ((-a, 0) \cup \{p\} \cup (0, a))$ and $x \neq p$ ~~the~~
then again there is $B_3 \subset \cap$ containing x so it is a basis

b) we show for $X \setminus \{p\}$ by symmetry this holds for $X \setminus \{q\}$

define $f: \mathbb{R} \to X \setminus \{p\}$ by

$$f(x) = \begin{cases} q, & x = 0 \\ x & \text{otherwise} \end{cases}$$

clearly bijective and continuous

Show that X satisfies the T_1 axiom but is not Hausdorff

clearly $\{p\}, \{q\}$ are closed it follows that all other points are closed and thus finite point sets. Not hausdorf since no disjoint open sets containing $\{p\}$ and $\{q\}$

d) show that satisfies all conditions for a 1 manifold except hausdorf

as $X \setminus \{q\}$ and $X \setminus \{p\}$ are homeomorphic to $\mathbb{R}$ there are neighborhoods of p, q homeomorphic to open sets of $\mathbb{R}$. this is clearly true for all other Points as well

Let B be a countable basis for $\mathbb{R} \setminus \{0\}$

$B_p = \bigcup_{n=1}^{\infty} (-\frac{1}{n}, 0) \cup \{p\} \cup (0, \frac{1}{n})$ similar B_q then $B \cup B_p \cup B_q$ is a countable basis for X