for an m-dimensional r.v. X it's laplace-transform is $\mathbb{R}^m \ni \theta \to E[e^{\langle \theta, X\rangle}]$ (possibly $+\infty$)

Prove that if $X \sim N(b, \Gamma)$ then

$$E[e^{\langle \theta, X\rangle}] = e^{\langle \theta, b\rangle} e^{\frac{1}{2}\langle \Gamma\theta, \theta\rangle}$$

$$\int_{-\infty}^{\infty} e^{\langle \theta, x\rangle} \frac{1}{(2\pi)^{m/2}} e^{-\frac{|x|^2}{2}} dx = \frac{1}{(2\pi)^{m/2}} \int_{-\infty}^{\infty} e^{\langle \theta, x\rangle - \frac{|x|^2}{2}} dx$$

$$\langle \theta - x, \theta - x\rangle = |\theta|^2 + |x|^2 - 2\langle \theta, x\rangle$$

$$\cancel{|\theta - x|^2} \; \frac{|\theta|^2}{2} + \frac{|x|^2}{2} - \frac{|\theta - x|^2}{2} = \langle \theta, x\rangle$$

$$\to = \frac{1}{(2\pi)^{1/m}} \int_{-\infty}^{\infty} e^{\frac{|\theta|^2}{2} - \frac{|\theta - x|^2}{2}} dx$$

$$= e^{\frac{|\theta|^2}{2}} \frac{1}{(2\pi)^{m/2}} \int_{-\infty}^{\infty} e^{-\frac{|\theta - x|^2}{2}} dx = e^{\frac{|\theta|^2}{2}}$$

$$\Gamma = AA^*, \quad X = AZ + b$$

$$E[e^{\langle \theta, X\rangle}] = e^{\langle \theta, b\rangle} E[e^{\langle \theta, AZ\rangle}] = e^{\langle \theta, b\rangle} E[e^{\langle A^*\theta, Z\rangle}]$$

$$= e^{\langle \theta, b\rangle} e^{\frac{|A^*\theta|^2}{2}} \qquad |A^*\theta|^2 = \langle A^*\theta, A^*\theta\rangle = \langle AA^*\theta, \theta\rangle = \langle \Gamma\theta, \theta\rangle$$

$$= e^{\langle \theta, b\rangle} e^{\langle \Gamma\theta, \theta\rangle}$$