$f: A \to B$, $A_i \subset A$, $B_i \subset B$, $i = 0, 1$

Show that f^{-1} preserves inclusions, unions intersections and differences of sets:

a) $B_0 \subset B_1 \Rightarrow f^{-1}(B_0) \subset f^{-1}(B_1)$

let $b \in B_0$ then for any $a \in A$ s.t

$f(a) = b$, $f(a) \in B_1$ $\blacksquare$

b) $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$

"$\subset$" $a \in A$ s.t $f(a) \in B_0 \cup B_1$

then $f(a) \in B_0$ or $f(a) \in B_1$ thus

$a \in f^{-1}(B_0)$ or $a \in f^{-1}(B_1)$

"$\supset$" $a \in f^{-1}(B_0) \cup f^{-1}(B_1)$ if $f(a) \in B_0$

then $f(a) \in B_0 \cup B_1$. if $f(a) \in B_1$ then

$f(a) \in B_0 \cup B_1 \to a \in f^{-1}(B_0 \cup B_1)$

c) $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$

"$\subset$" a s.t $f(a) \in B_0 \cap B_1$ then $f(a) \in B_0$ and

$f(a) \in B_1$ thus $a \in f^{-1}(B_0)$ and $a \in f^{-1}(B_1)$

$\to a \in f^{-1}(B_0 \cap B_1)$

"$\supset$" $a \in f^{-1}(B_1) \cap f^{-1}(B_2)$, $f(a) \in B_1 \cap B_2$

$\to a \in f^{-1}(B_1 \cap B_2)$

d) $f^{-1}(B_0 - B_1) = f^{-1}(B_0) - f^{-1}(B_1)$

"$\subset$" $a \in f^{-1}(B_0 - B_1) \to f(a) \in B_0 - B_1$

$\to f(a) \in B_0$ but $f(a) \notin B_1$ thus

$a \in f^{-1}(B_0)$ but $a \notin f^{-1}(B_1) \to a \in f^{-1}(B_0) - f^{-1}(B_1)$

"$\supset$" $a \in f^{-1}(B_0) - f^{-1}(B_1)$

$f(a) \in B_0$ but $f(a) \notin B_1$ so $f(a) \in B_0 - B_1$

$a \in f^{-1}(B_0 - B_1)$

Show that f preserves inclusions and unions only:

e) $A_0 \subset A_1 \Rightarrow f(A_0) \subset f(A_1)$

$a \in A_0$ then $f(a) \subset f(A_1)$

f) $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$ "$\subset$"

let $b \in f(A_0 \cup A_1)$ then $\exists$ $a \in A_0 \cup A_1$ s.t

$f(a) = b$. if $a \in A_0$ then $b \in f(A_0)$ if $a \in A_1$

then $b \in f(A_1) \to b \in f(A_1) \cup f(A_0)$

"$\supset$": $b \in f(A_0) \cup f(A_1)$, if $b \in f(A_0)$ then $a \in A_0$

$A_0 \subset A_0 \cup A_1$ thus $b \in f(A_0 \cup A_1)$ same

g) $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$

$b \in f(A_0 \cap A_1)$, $\exists\ a \in A_0 \cap A_1$ s.t $f(a) = b$

$a \in A_0$ thus $b \in f(A_0)$, same for A_1 so

$b \in f(A_0) \cap f(A_1)$

assume f is injective and $b \in f(A_0) \cap f(A_1)$

$\exists\ a_0 \in A_0$ s.t $f(a_0) = b$ and $a_1 \in A_1$ s.t $f(a_1) = b$

since f is injective $a_0 = a_1 = a \in A_0 \cap A_1$

thus $b \in f(A_0 \cap A_1)$

h) $f(A_0 - A_1) \supset f(A_0) - f(A_1)$

$b \in f(A_0) - f(A_1)$. $\exists\ a_0 \in A_0$ s.t $f(a_0) = b$ but

no $a_1 \in A_1$ s.t $f(a_1) = b$ thus $a_0 \in A_0 - A_1$

and $b \in f(A_0 - A_1)$.

Assume f is injective; $b \in f(A_0 - A_1)$

then $\exists$ a unique $a \in A$ s.t $f(a) = b$

since $b \in f(A_0 - A_1)$, $a \in A_0 - A_1$ so $a \notin A_1$ but $a \in A_0$

thus $f(a) \in f(A_0)$ but by uniqueness $f(a) \notin f(A_1)$

so $b \in f(A_0) - f(A_1)$