$X_t \in \mathbb{R}^d$ is gaussian if $\forall t_i$

~~X_i~~ $(X_{t_1}, X_{t_2}, \dots, X_{t_n}) \in \mathbb{R}^{dn}$ is Gaussian distributed

equivalently if $\forall n, \gamma_i \in \mathbb{R}^d$

$$\sum_{k=1}^{n} \langle X_{t_i}, \gamma_i \rangle \text{ is Gaussian:}$$

$n=1$, $\langle B_t, \gamma \rangle = \sum_{i=1}^{d} B_t^{(i)} \gamma^{(i)}$

we know that $B_t - B_0$ is normally distributed $N(0, tI)$ distributed

this means that $B_t^{(i)}$ is $N(0,t)$ distributed

and independent of $B_t^{(k)}$, $i \neq k$

then $\gamma^{(i)} B_t^{(i)} = B^{(i)}_{t/(\gamma^{(i)})^2}$ is $N(0, t/(\gamma^{(i)})^2)$ distributed

and independent of the others

So $\sum_{i=1}^{d} B_t^{(i)} \gamma^{(i)} \sim N(0, t \sum \frac{1}{\gamma^{(i)2}})$

this is true so

$\langle \gamma_k, X_k \rangle$ is normally distributed

thus $(\langle \gamma_1, X_{t_1} \rangle, \ldots, \langle \gamma_n, X_n \rangle)$ is normally distributed.

thus $\sum \langle \gamma_k, X_{t_k} \rangle$ is normally distributed

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$$\sum_{k=1}^{n} \langle \gamma_k, X_{t_k} \rangle = \sum_{k=1}^{n} \sum_{i=1}^{d} \gamma_k^{(i)} X_{t_k}^{(i)} = \sum_{i=1}^{d} \sum_{k=1}^{n} \gamma_k^{(i)} X_{t_k}^{(i)}$$

$$\sum_{k=1}^{n} \gamma_k^{(i)} X_{t_k}^{(i)} = \cancel{\sigma} \sum_{k=1}^{n} \cancel{\sigma} X_{t_k / \gamma_k^{(i)2}}$$

we ~~&~~ can order these $t_k / \gamma_k^{(i)2}$ increasingly

~~by~~ as $0 \leq \tilde{t}_1 \leq \tilde{t}_2 \leq \ldots \leq \tilde{t}_n$

so $\sum_{k=1}^{n} X_{t_k / \gamma_k^{(i)2}} = \sum_{k=1}^{n} X_{\tilde{t}_k}$ the variables

$$\tilde{X}_{t_k} = \begin{cases} X_{\tilde{t}_k}, & k = 1 \\ X_{\tilde{t}_k} - X_{\tilde{t}_{k-1}} \end{cases}$$ are gaussian distributed and independent

so $(\tilde{X}_{t_1}, \tilde{X}_{t_2}, \ldots, \tilde{X}_{t_n})$ ~~er~~ is gaussian

$$X_{t_k} = \tilde{X}_{t_k} + \tilde{X}_{t_{k-1}} + \ldots + \tilde{X}_{t_1} =$$

so

$$\begin{pmatrix} X_{t_n} \\ \vdots \\ X_{t_1}\end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ & 1 & \cdots & & 1 \\ & & 1 & \cdots & 1 \\ 0 & & & \ddots & \vdots \\ & & & 1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{X}_{t_n} \\ \vdots \\ \tilde{X}_{t_1}\end{pmatrix}$$ is gaussian

thus $\sum_{k=1}^{n} \cancel{\gamma} X^{(i)}_{t_k} \cancel{\tilde{\gamma}^{(i)2}_{t_k}}$ is gaussian these

are independent for each i so $\sum_{k=1}^{n} \langle \gamma_k, X_{t_k} \rangle$ is gaussian

a smarter way

suppose $\sum_{k=1}^{n} \langle \gamma_k, X_{t_k} \rangle$ is gaussian for some

n, What about $\sum_{k=1}^{n+1} \langle \gamma_k, X_{t_k} \rangle$

$$\langle \gamma_n, X_{t_n} \rangle + \langle \gamma_{n+1}, X_{t_{n+1}} \rangle = \langle \gamma_n + \gamma_{n+1} - \gamma_{n-1}, X_{t_n} \rangle + \langle \gamma_{n+1}, X_{t_{n+1}} \rangle$$

$$= \langle \tilde{\gamma}_n, X_{t_n} \rangle + \langle \gamma_{n+1}, X_{t_{n+1}} - X_{t_n} \rangle$$

$\langle \gamma, B_t \rangle = \langle \gamma, B_t - B_0 \rangle$

$\langle \gamma, B_t - B_s \rangle$

B_t

$\sum \gamma_i B_t^{(i)}$

d-dim

(B_t)

$\gamma \begin{pmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_n \end{pmatrix} B_t$

$(B_{t_1}, B_{t_2}$

$\langle 1, \gamma B_t \rangle$

$B_t^{(i)}$