Show that $D \subset C[a,b]$ of functions differentiable at least one point is meager (in the supremum norm), that is, it is a countable union of nowhere dense sets. Conclude that the continuous functions not differentiable at any point are dense in $C[a,b]$

Hint: observe that $D \subset \bigcup_n D_n$ where

$$D_n = \{ f \mid |f(x) - f(x_0)| \le n|x - x_0| \text{ for some } x_0 \text{ and all } x \}$$

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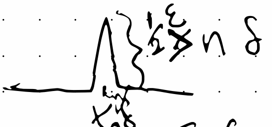
if $\bigcup D_n$ is meager then D is certainly meager. consider $f_1 \in D_n$, then for some $x_0 \in [a,b]$

$|f_1(x) - f_1(x_0)| \le n|x - x_0| \quad \forall \ x \in [a,b]$

let $\varepsilon > 0$ want to show that $\exists \ g \in C[a,b]$ s.t. $|f - g| < \varepsilon$ but $g \notin D_n$



$\to$ pick $\delta < \frac{\varepsilon}{2n}$

$$g(x) = \begin{cases} f(x), & |x - x_0| > \delta \\ f(x) + |x - x_0| \frac{\varepsilon}{\delta 2}, & |x - x_0| \le \delta \end{cases}$$

then $|g(x_0 + \delta) - g(x_0)| = \delta \frac{\varepsilon}{2\delta} = \frac{\varepsilon}{2} > n\delta = n|x + \delta - x_0|$

thus D_n is nowhere dense and D is meager.

$C[a,b]$ is complete and thus not meager

thus D^c is dense in $C[a,b]$