$p_1, \ldots, p_n$ are seminorms on a vector space X. Assume f is a linear functional on X s.t $|f(x)| \leq \sum_{k=1}^{n} p_k(x)$ for all $x \in X$.

Show that $\exists$ linear functionals f_k s.t $f = \sum f_k$ and $|f_k(x)| \leq p_k(x)$ $\forall$ $x \in X$.

Consider X^n and A the supspace consisting of vecters $(x, x, \ldots, x)$ $x \in X$

want to check that $p = \sum p_k(x_k)$ is a seminorm:

$$p(\lambda v) = \sum p_k(\lambda v_k) = |\lambda| \sum p_k(v_k) \quad \checkmark$$

$$\begin{aligned} p(v+w) = \sum p_k(v_k + w_k) &\leq \sum(p_k(v_k) + p_k(w_k)) \\ &= \sum p_k(v_k) + \sum p_k(w_k) \\ &= p(v) + p(w) \end{aligned}$$

So p is a seminorm on X^n.

let $a_x = (x, \cdots, x) \in A$. Define $\tilde{f}: A \to \mathbb{F}$ by $\tilde{f}(a_x) = f(x)$. then $\tilde{f}$ is also a linear functional and $|\tilde{f}(a_x)| \leq p(a_x)$ $\forall$ $a_x \in A$.

By Hanach-Bahn $\exists$ a linear functional $F: \mathbb{R}^n \to \mathbb{F}$ s.t $F|_A = \tilde{f}$ and $|F(x)| \leq p(x)$ $\forall$ x

define $f_k(x)$ by $F((0,\dots,0,\underset{\substack{\uparrow\\ k}}{x},0,\dots,0))$

then $f(x) = F(a_x) = \sum f_k(x)$

and $|f_k(x)| = |F(0,\dots,0,x,0,\dots,0)| \leq P(0,\dots,0,x,0,\dots,0)$

$= P_k(x)$