If $f: X_1 \to X_2$ is a homeomorphism of locally compact Hausdorff spaces, show that f extends to a homeomorphism of their 1-point compactifications.

Let Y_1, Y_2 be the 1-point compactifications of X_1, X_2. let φ_1, φ_2 be the points not in X_1, X_2 define $\tilde{f}$ by:

$$\tilde{f}: Y_1 \to Y_2 \quad \text{by} \quad \tilde{f}(x) = \begin{cases} \varphi_2, & x = \varphi_1 \\ f(x), & \text{otherwise} \end{cases}$$

Let $\mathcal{U}$ be open in Y_1. if $\varphi_1 \notin \mathcal{U}$ then $\tilde{f}(\mathcal{U})$ is open in X_2. thus it is the intersection of some openset $\mathcal{U}_2$ with X_2 but this set is just $\tilde{f}(\mathcal{U})$ so it is open in Y_2

if $\mathcal{U}$ contains φ_1 we know that $\mathcal{U}^c$ is closed and thus compact. Therefore $f(\mathcal{U}^c)$ is compact and thus closed so $\underbrace{f(\mathcal{U}^c)^c = f(\mathcal{U})}_{\text{Bijective}}$ is open