Let $x_1, x_2, \ldots$ be a sequence of points of the product space $\prod X_\alpha$. Show that this sequence converges to a point $x \Leftrightarrow$ the sequence $\pi_\alpha(x_1), \pi_\alpha(x_2), \ldots$ converges to $\pi_\alpha(x)$ $\forall \alpha$. Is this true if one uses the box topology instead of the product topology?

That ~~the~~ the sequence converges means that for any neighborhood $\mathcal{U}$ $\exists N \in \mathbb{N}$ s.t. $x_n \in \mathcal{U}$ $\forall n \geq N$

"$\Rightarrow$" Let $\mathcal{U}$ be a neighborhood of x. We can assume $\mathcal{U}$ is a basis element since any neighborhood contains a basiselement containing x and any basiselement containing x is a neighborhood of x. We can thus write $\mathcal{U}$ as $\prod \mathcal{U}_\alpha$ where $\mathcal{U}_\alpha$ is open in X_α $\forall \alpha$ and $\mathcal{U}_\alpha = X_\alpha$ for all but finitely many values of α. let $N \in \mathbb{N}$ s.t. $x_n \in \mathcal{U}$ $\forall n \geq N$ then $x_n(\alpha) \in \mathcal{U}_\alpha$ $\forall n \geq N$ $\forall \alpha$. Thus $\pi_\alpha(x_1), \pi_\alpha(x_2), \ldots$ converges to $\pi_\alpha(x)$ $\forall \alpha$

"$\Leftarrow$" let $\mathcal{U}$ be a neighborhood of x, assume again $\mathcal{U}$ is a basis element, $\mathcal{U} = \prod \mathcal{U}_\alpha$.
if $\mathcal{U}_\alpha = X_\alpha$ then $x_n(\alpha) \in \mathcal{U}_\alpha$ $\forall n$. For the last values of α (finitely many) $\exists N_\alpha$ s.t.
$x_n(\alpha) \in \mathcal{U}_\alpha$ when $n \geq N_\alpha$

We can then let $N = \max\{N_\alpha : \alpha \text{ s.t. } U_\alpha \neq X_\alpha\}$.
then $X_n(\alpha) \in U_\alpha$ when $n \geq N$ $\forall \alpha$. Thus

$X_n \in \prod U_\alpha = U$ when $n \geq N$ ∎

Notice that for the "$\Rightarrow$" part we did not use Property of the product topology so this should work for box topology as well.