Noise and disturbance of quantum measurement: measures and uncertainty relations

Kamil Korzekwa, David Jennings and Terry Rudolph

Department of Physics, Imperial College London, London SW7 2AZ, United Kingdom

Imperial College London

Controlled Quantum Dynamics

EPSRC

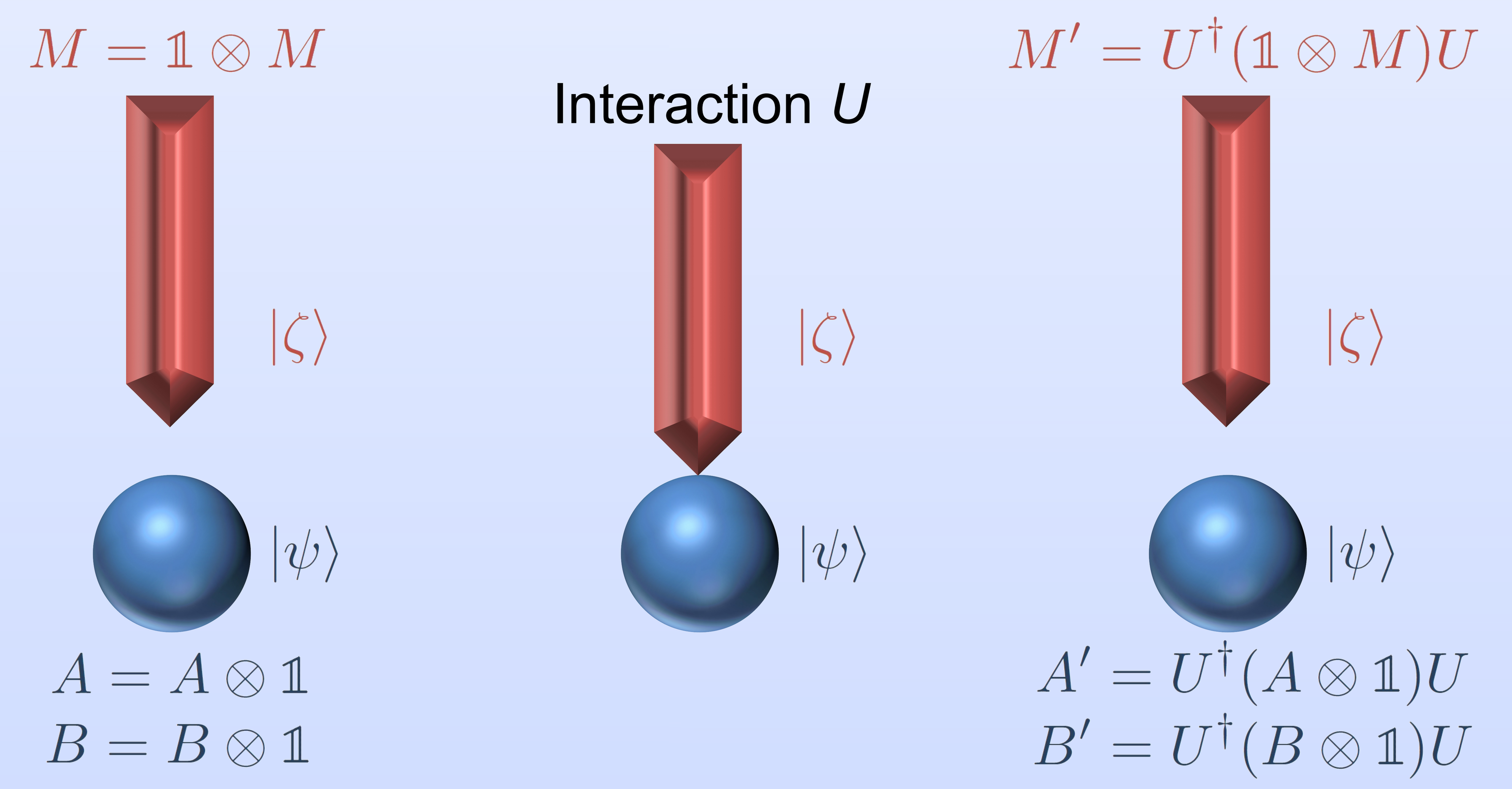
Motivation

- In spite of its great importance in quantum physics, the famous Heisenberg uncertainty principle has only recently been studied with a mathematically rigorous approach. This lead to the reformulation of the uncertainty principle for noise and disturbance in measurement and soon after its experimental verification.
- Linking the notion of noise and disturbance to the asymmetry resources of the system and the measuring device (with respect to the symmetry generated by measured observables) can shed a new light on the noise-disturbance relation and allow one to use quantum information tools to further investigate this subject.

Operator-induced measures – Ozawa's approach

1. Indirect measuring scheme

- Instead of measuring the system observable *A* directly the system undergoes joint evolution *U* with the probe and then the probe observable *M* is measured. The measuring interaction *U* and the probe observable *M* are chosen so that the outcome statistics of the probe measurement coincides with the outcome statistics of the original measurement of *A*.
- Description in the Heisenberg picture: states of the system and the probe (ψ and ζ) are constant throughout the measuring unitary evolution *U* and the operators (*A*, *B* and *M*) evolve.



2. Noise and disturbance operators and the uncertainty relation

- The noise operator *N* is defined as the difference between the observable *M′* that one actually measures and the observable *A* that one wants to measure. The measure of noise ε in a state ρ is defined as the root-mean-square value of the noise operator:

$$N(A) = M' - A\,, \qquad \epsilon(A,\rho) = \left(\mathrm{Tr}\left(\rho N^2(A)\right)\right)^{\frac{1}{2}}$$

- The disturbance operator *D* is defined as the difference between the observable *B′* after the measuring process of *A* and the observable *B* before the measurement of *A*. The measure of disturbance η in a state ρ is defined as the root-mean-square value of the disturbance operator:

$$D(B) = B' - B\,, \qquad \eta(B,\rho) = \left(\mathrm{Tr}\left(\rho D^2(B)\right)\right)^{\frac{1}{2}}$$

- The following inequality, which improves the original Heisenberg error-disturbance uncertainty relation by adding terms proportional to standard deviation of the measured operators, $V^{1/2}$, was recently proven to be satisfied[1] for all states ρ:

$$\epsilon(A)\eta(B) + \epsilon(A)V^{\frac{1}{2}}(B) + \eta(B)V^{\frac{1}{2}}(A) \geq \frac{1}{2}|\mathrm{Tr}\left(\rho[A,B]\right)|$$

- Special case of the above relation[2] allows to find the lower bound for the error of measuring the WAY scenario observable, i.e. an observable *A* not commuting with the additive conserved quantity $L=L_1+L_2$:

$$\epsilon^2(A) \geq \frac{|\mathrm{Tr}\left(\rho[A,L_1]\right)|^2}{4V(L_1)+4V(L_2)}$$

3. Tightening the bound for mixed states

- The derivation of the uncertainty relation for the noise and disturbance of the quantum measurement obtained by Ozawa is based on the Heisenberg-Robertson uncertainty relation which bounds the variances *V* (measures of uncertainty) of two operators:

$$V(A)V(B) \geq \frac{1}{4}|\mathrm{Tr}\left(\rho[A,B]\right)|^2$$

- The uncertainty of the mixed state comes from both the classical and quantum uncertainty. One may separate these components, e.g. in the following ways:

$$V(A) = I(A) + C(A)$$

The measure of the total uncertainty | The measure of the quantum uncertainty | The measure of the classical uncertainty

$$V^2(A) = U^2(A) + C^2(A)$$

Definitions of uncertainty measures

$$\begin{aligned} V(A) &= \mathrm{Tr}\left(\rho A_0^2\right) \\ I(A) &= \mathrm{Tr}\left(\rho A_0^2\right) - \mathrm{Tr}\left(\sqrt{\rho}A_0\sqrt{\rho}A_0\right) \\ C(A) &= \mathrm{Tr}\left(\sqrt{\rho}A_0\sqrt{\rho}A_0\right) \end{aligned}$$

where: $A_0 = A - \mathrm{Tr}\left(\rho A\right)$

- These particular separations allow one to find the uncertainty relations that bound the quantum components of the uncertainties of two operators, instead of bounding the total uncertainties

$$U(A)U(B) \geq \frac{1}{4}|\mathrm{Tr}\left(\rho[A,B]\right)|^2, \qquad V(A)I(B) \geq \frac{1}{8}|\mathrm{Tr}\left(\rho[A,B]\right)|^2$$

- The first of the above uncertainty relations can be used to tighten the bound of the error-disturbance uncertainty relation for mixed states:

$$\epsilon(A)\eta(B) + \epsilon(A)U^{\frac{1}{2}}(B) + \eta(B)U^{\frac{1}{2}}(A) \geq \frac{1}{2}|\mathrm{Tr}\left(\rho[A,B]\right)|$$

- The second of the above uncertainty relations can be used to modify the WAY scenario bound (tighten it if $V>2I$):

$$\epsilon^2(A) \geq \frac{|\mathrm{Tr}\left(\rho[A,L_1]\right)|^2}{8I(L_1)+8I(L_2)}$$

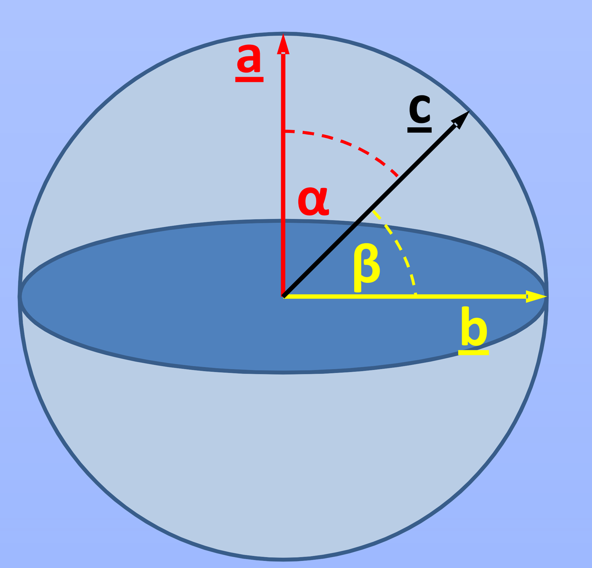
Criticism of operator-induced measures and proposal for a new approach

1. Repeatability condition embedded in the definitions of noise and disturbance

- One of the features of the operator-induced measures of noise and disturbance is that measurements of *M′* and *B′* (approximate measurement of *A* and disturbed measurement of *B*) that perfectly reproduce the outcome statistics of the original measurements (*A* and *B*) may be attributed with nonzero noise/disturbance.
- As an illustration of this fact consider the simplest measurement scenario: measuring qubit observables along two directions on a Bloch sphere, $A=a\sigma$ and $B=b\sigma$. If instead of measuring *A* one measures $M'=c\sigma$, then the error and disturbance of measurements of *A* and *B* are given by:

$$\epsilon(A) = 2\left|\sin\frac{\alpha}{2}\right|$$

$$\eta(B) = \sqrt{2}\left|\sin\beta\right|$$



Since error and disturbance are independent of the system state:

- Measuring a system in an eigenstate of *M′* leads to disturbance if $\sin(\beta)\neq 0$ (even though such a measurement does not modify the system state).
- Every measurement of $M'\neq A$ will be treated as noisy, even if it gives the same outcome statistics.
- The reason for this is that while one measurement has the same statistics as the other, it does not have to collapse the system to the same quantum state. As a result any subsequent exact (not noisy/disturbed) measurements will have different outcome statistics.
- The conclusion is that the operator-induced measures of noise and disturbance not only quantify how noisy/disturbed the measurement outcome statistics is, but also how the subsequent exact measurement outcome statistics differ from the obtained one (i.e. how much the subsequent measurements fail to satisfy the repeatability condition).

2. Measures of noise and disturbance based on asymmetry monotones

- We propose to use the measures of noise and disturbance that are solely based on the difference between outcome probability distributions of the original observables and the noisy/disturbed ones.
- We note that a projective measurement of an observable *A* of a system in a state ρ is equivalent to the *G*-twirling of the state ρ over the *U(1)* group generated by observable *A*:

$$\rho' = \sum_n \langle a_n|\,\rho\,|a_n\rangle\,|a_n\rangle\,\langle a_n| = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{i\theta A}\rho e^{-i\theta A} = \mathcal{G}_A(\rho)$$

- The states that form the G-equivalence[3] class with respect to the *U(1)* group generated by *A* have the same outcome statistics for the measurement of *A* (the G-equivalence class is defined a set of states that can be reversibly interconverted with the use of symmetric quantum operation, i.e. the quantum operation that commutes with every element of the group representation).
- As states with different outcome statistics for the measurement of *A* lie in different G-equivalence classes, we propose to use the distance *D* between the G-equivalence classes to quantify error/disturbance. If an approximate measurement of *A* is described by quantum operation $\mathcal{E}$, then the error of measuring *A* and the disturbance of the measurement of *B* can be quantified by:

$$\epsilon(A) = D_A(\mathcal{G}_A(\rho),\mathcal{E}(\rho)), \quad \eta(B) = D_B(\mathcal{G}_B(\rho),\mathcal{G}_B(\mathcal{E}(\rho)))$$

- Asymmetry monotones (quantities non-increasing under symmetric quantum operations) can be used to define the distance between *G*-equivalence classes.
- Skew information is an asymmetry monotone that may be of particular importance in the investigations on the error-disturbance relation, as the skew information for pure states coincides with the variance appearing in the current bound for the error-disturbance product.

References:

1. M. Ozawa, Physical Review A **67**, 042105 (2003).
2. M. Ozawa, Phys. Rev. Lett. **88**, 050402 (2002).
3. I. Marvian and R. W. Spekkens, New Journal of Physics **15**, 033001 (2013).