On time evolution of coherences and populations

Kamil Korzekwa

Controlled Quantum Dynamics Centre Doctoral Training, Imperial College, London, UK

Quantum Science Research Group, University of Sydney, Australia

Team

Terry Rudolph

Antony Milne Matteo Lostaglio

David Jennings

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What?

Why?
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Motivation

Why?
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Why?

Historical example

Quantum measurement of a general observable

$$X^A = \sum_{\nu} x_\nu^A |x_\nu^A\rangle\langle x_\nu^A|$$

Von Neumann measurement process

$$|\psi^A\rangle \otimes |\zeta^B\rangle \xrightarrow{U} \sum_{\nu} \langle x_\nu^A|\psi^A\rangle \; |x_\nu^A\rangle \otimes |\varphi_\nu^B\rangle$$

Measured system

Measurement instrument

Macroscopically distinguishable states of the measuring instrument

$$\langle \varphi_\nu^B | \varphi_{\nu'}^B \rangle = \delta_{\nu,\nu'}$$

$\frac{\uparrow+\downarrow}{\sqrt{2}} \otimes$

$\rightarrow \frac{\uparrow}{\sqrt{2}} \otimes$

$+ \frac{\downarrow}{\sqrt{2}} \otimes$

Historical example

Introduce a conserved quantity: $Z = Z^A \otimes \mathbb{I}^B + \mathbb{I}^A \otimes Z^B, \quad [U, Z] = 0$

$$Z^A|n^A\rangle = z_n^A|n^A\rangle \qquad Z^B|n^B\rangle = z_n^B|n^B\rangle$$

What if the measured observable X^A does not commute with Z^A?

$$[X^A, Z^A] \neq 0$$

E.P. Wigner,
Z. Phys. **133**, 101 (1952)
English translation: arXiv:1012.4372

Consider a typical operator X^A not commuting with Z^A with the following eigenstates

$$|+^A\rangle = \frac{|0^A\rangle + |1^A\rangle}{\sqrt{2}} \qquad |-^A\rangle = \frac{|0^A\rangle - |1^A\rangle}{\sqrt{2}}$$

Von Neumann measurement process reads

$$|+^A\rangle \otimes |\zeta^B\rangle \longrightarrow |+^A\rangle \otimes |\varphi_+^B\rangle$$

$$|-^A\rangle \otimes |\zeta^B\rangle \longrightarrow |-^A\rangle \otimes |\varphi_-^B\rangle$$

With: $\langle\varphi_+^B|\varphi_-^B\rangle = 0$

Historical example

Sum and subtract both sides

$$|0^A\rangle \otimes |\zeta^B\rangle \;\longrightarrow\; |0^A\rangle \otimes |\varphi_0^B\rangle + |1^A\rangle \otimes |\varphi_1^B\rangle$$

$$|1^A\rangle \otimes |\zeta^B\rangle \;\longrightarrow\; |0^A\rangle \otimes |\varphi_1^B\rangle + |1^A\rangle \otimes |\varphi_0^B\rangle$$

Impossible! The amount of conserved quantity on the RHS is equal, but differs on LHS.

Possible in an approximate way, but only if $|\zeta^B\rangle$ is a superposition over many eigenstates of the conserved quantity

$$|\zeta^B\rangle = \sum_{n=1}^{N} c_n |n^B\rangle, \quad c_n \neq 0, \qquad N \to \infty$$

Wigner, **A**raki, **Y**anase – WAY Theorem

E.P. Wigner,
Z. Phys. **133**, 101 (1952)
English translation: arXiv:1012.4372

H. Araki, M.M. Yanase,
Phys. Rev. **120**, 622 (1960)

M.M. Yanase,
Phys. Rev. **123**, 666 (1961)

Modern QI approach
(using resource theory)

I. Marvian, R.W. Spekkens,
arXiv:1212.3378 (2012)

M. Ahmadi, D. Jennings, T. Rudolph,
New J. Phys. **15**, 013057 (2013)

General constrained dynamics

1. Constrained dynamics

General CPTP operation

$$\mathcal{E}(\rho^A) = Tr_B\left[U(\rho^A \otimes \sigma^B)U^\dagger\right]$$

With constraints:

Conservation of energy

$$[U, H^A + H^B] = 0$$

No superposition in the ancillary system

$$[\sigma^B, H^B] = 0$$

2. Symmetric dynamics

Time-translation covariant operation

$$\mathcal{E}\left(U_t(\rho^A)U_t^\dagger\right) = U_t\mathcal{E}(\rho^A)U_t^\dagger$$

where: $U_t = e^{-iH^A t}$

1. ⇔ 2.

⇒ Straightforward

⇐ M. Keyl, R.F. Werner, *J. Math. Phys.* **40**, 3283 (1999)

Applications in current research

1. Resource theory of thermodynamics

Allowed thermal operations: $\mathcal{E}_T(\rho^A) = Tr_B\left(U(\rho^A \otimes \gamma^B)U^\dagger\right)$

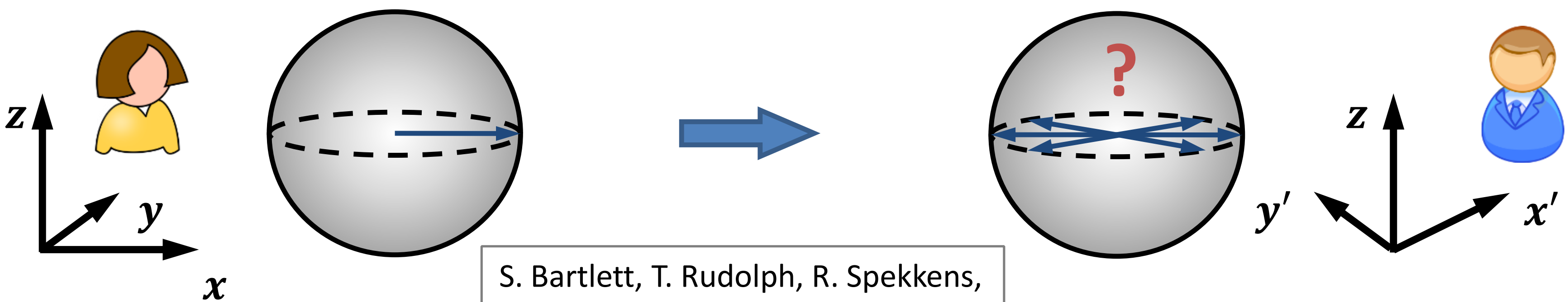
"Encoding" 1st Law: $[U, H^A + H^B] = 0$

"Encoding" 2nd Law: $\gamma^B \sim e^{-\beta H^B}$

F. Brandao, M. Horodecki, N.Ng, J. Oppenheim, S. Wehner, PNAS **112**, 3275 (2015)

M. Lostaglio, D. Jennings, T. Rudolph, Nat. Commun. **6**, 6383 (2015)

2. Quantum communication without a shared reference frame



S. Bartlett, T. Rudolph, R. Spekkens, *Rev. Mod. Phys.* **79** 555 (2007)

3. Quantum metrology, quantum optics (with rotating wave approximation), open quantum systems (within secular approximation)…

How?
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Mathematical framework

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Modes: Decomposing a density matrix

System described by nondegenerate Hamiltonian:

$$H = \sum_x \hbar\omega_x \, |x\rangle\langle x| \qquad \rho(0) = \sum_{x,y} \rho_{xy} \, |x\rangle\langle y|$$

Free evolution of the system described by:

$$\rho(t) = e^{-iHt}\rho(0)e^{iHt} = \sum_{x,y} \rho_{xy} \, e^{-iHt}|x\rangle\langle y|e^{iHt} = \sum_{x,y} \rho_{xy}|x\rangle\langle y|e^{-i\hbar\omega_{xy}t}$$

With: $\omega_{xy} = \omega_x - \omega_y$

Modes of coherence $\rho^{(\omega)}$:

$$\rho^{(\omega)} := \sum_{\substack{x,y \\ \omega_{xy}=\omega}} \rho_{xy}|x\rangle\langle y|;$$

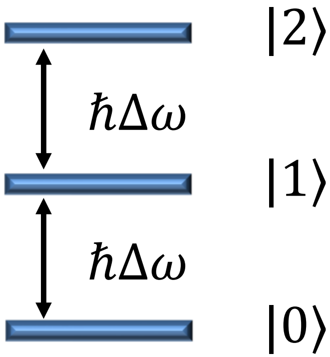
$$\boxed{f(t) = \sum_n f^{(\omega_n)} \, e^{-i\omega_n t}}$$

Similar to Fourier series decomposition of a real/complex function

$$\rho(0) = \sum_\omega \rho^{(\omega)} \qquad \rho(t) = \sum_\omega \rho^{(\omega)} e^{-i\hbar\omega t}$$

I. Marvian, R. Spekkens,
Phys. Rev. A **90**, 062110 (2014)

Modes: Qutrit example

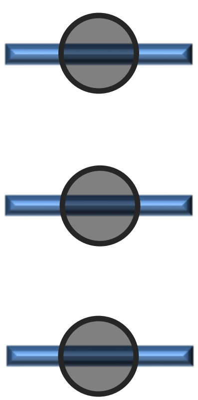


$$H = \sum_{x=0}^{2} x\, \hbar\Delta\omega\, |x\rangle\langle x|$$

$$\rho = \begin{pmatrix} p_0 & c_{01} & c_{02} \\ c_{10} & p_1 & c_{12} \\ c_{20} & c_{21} & p_2 \end{pmatrix}$$

Mode 0:

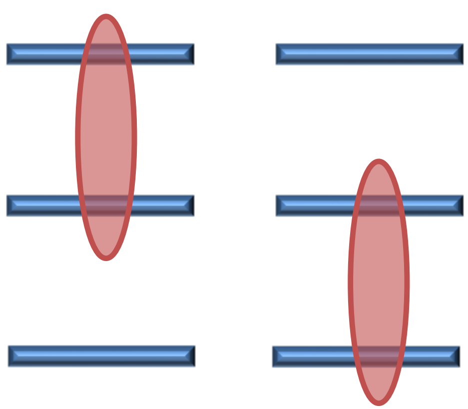
$$\rho^{(0)}(t) = \rho^{(0)}$$



Mode $\Delta\omega$ and $-\Delta\omega$:

$$\rho^{(\Delta\omega)}(t) = \rho^{(\Delta\omega)} e^{-i\Delta\omega t}$$

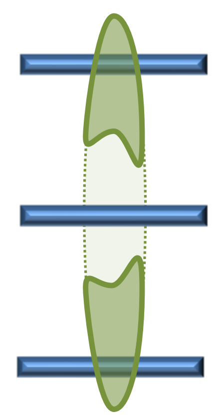
$$\rho^{(-\Delta\omega)}(t) = \rho^{(-\Delta\omega)} e^{i\Delta\omega t}$$



Mode $2\Delta\omega$ and $-2\Delta\omega$:

$$\rho^{(2\Delta\omega)}(t) = \rho^{(2\Delta\omega)} e^{-i2\Delta\omega t}$$

$$\rho^{(-2\Delta\omega)}(t) = \rho^{(-2\Delta\omega)} e^{i2\Delta\omega t}$$



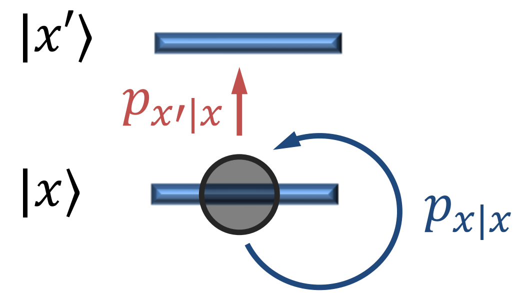
Mode structure of time-translation covariant maps

Each mode transforms **independently** and its *intensity* cannot increase:

Given: $\sigma = \mathcal{E}(\rho)$ We have: $\sigma^{(\omega)} = \mathcal{E}\left(\rho^{(\omega)}\right)$ and $\left\|\sigma^{(\omega)}\right\| \leq \left\|\rho^{(\omega)}\right\|$

Action of $\mathcal{E}$ on mode zero (populations)

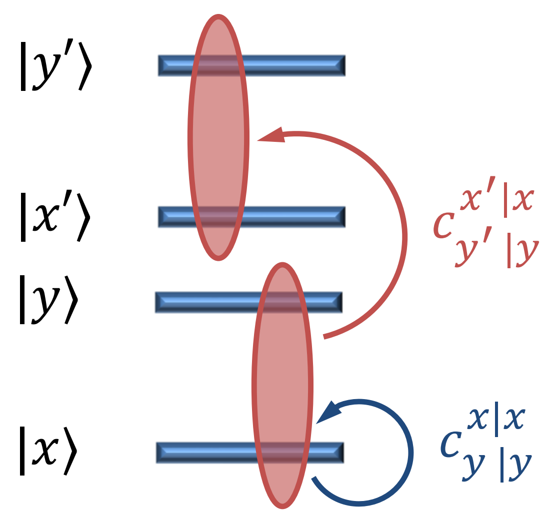
$$p_{x'|x} := \langle x'|\mathcal{E}(|x\rangle\langle x|)|x'\rangle$$



Action of $\mathcal{E}$ on non-zero modes (coherences)

$$c^{x'|x}_{y'|y} := \langle x'|\mathcal{E}(|x\rangle\langle y|)|y'\rangle$$

With: $c^{x'|x}_{y'|y} = 0$ unless $\omega_{xy} = \omega_{x'y'}$

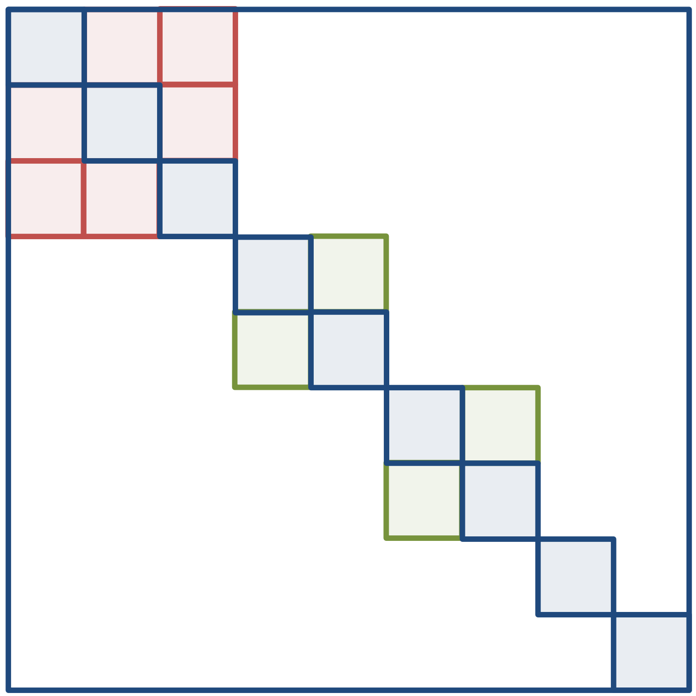


Block-diagonal Choi-Jamiołkowski state

Choi-Jamiołkowski isomorphism: $J[\mathcal{E}] = [\mathcal{E} \otimes I]|\Omega\rangle\langle\Omega|$ with: $|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{x=0}^{d-1} |xx\rangle$

Enforcing $\mathcal{E}$ to be CPTP $\Leftrightarrow$ Enforcing $J[\mathcal{E}] \geq 0$

$J[\mathcal{E}]$



Evolution of **populations** described by the diagonal terms

Depletion of **coherences** described by the first-block off-diagonal terms

Transfer of **coherences** described by particular off-diagonal terms

The positivity of $J[\mathcal{E}]$ connects **population** transfer with **coherence** depletion and transfer.

What?

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Results

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Optimal coherence processing

Given: $\sigma = \mathcal{E}(\rho)$ and the evolution of populations described by $p_{x'|x}$, the evolution of coherences is bounded by:

$$|\sigma_{x'y'}| \leq \sum_{x,y}^{(\omega_{x'y'})} |\rho_{xy}| \sqrt{p_{x'|x} p_{y'|y}}$$

Sum only over elements (x, y) satisfying $\omega_{xy} = \omega_{x'y'}$

$\sqrt{p_{x|x} p_{y|y}}$ - bound on preserving coherence ρ_{xy}

$\sqrt{p_{x'|x} p_{y'|y}}$ - bound on transferring coherence from ρ_{xy} to $\rho_{x'y'}$

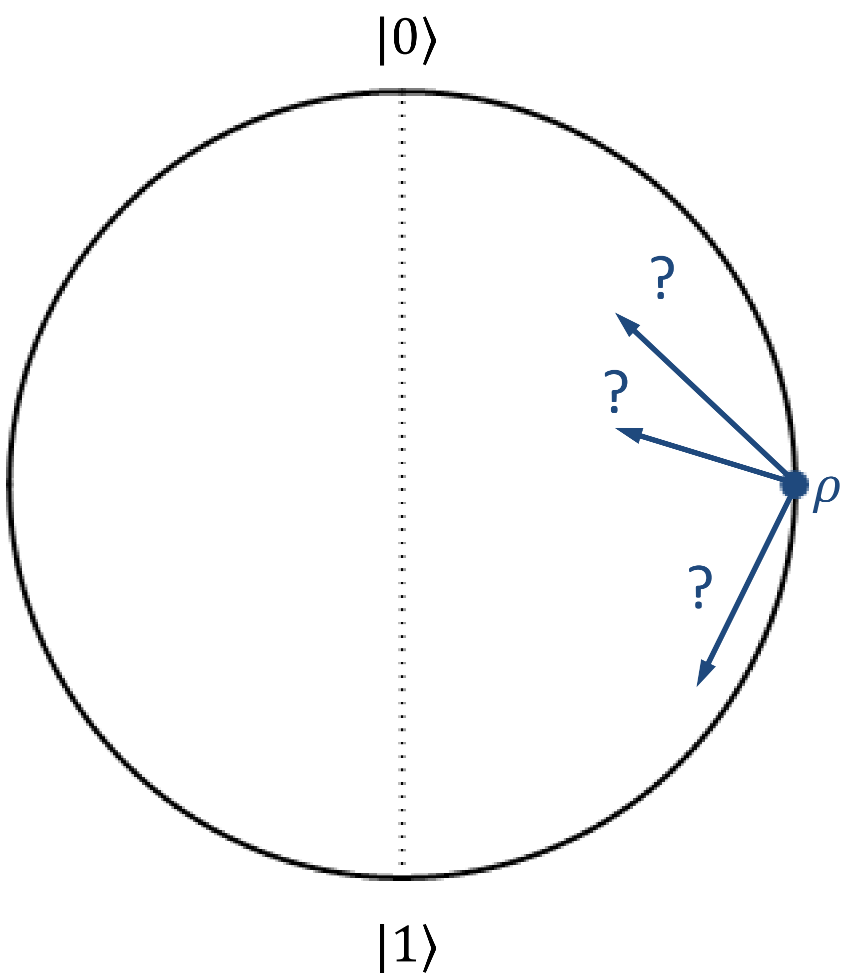
The bound can be saturated when

- The initial state is pure
- The initial state is mixed, but with positive coherence terms
- The system has only 1- and 2-dimensional modes

M. Lostaglio, K. Korzekwa, D. Jennings, T. Rudolph, Phys. Rev. X **5**, 021001 (2015)

Optimal coherence preservation: Qubit

State interconversion problem: Given initial state ρ what is the set of final states $\mathcal{E}(\rho)$?



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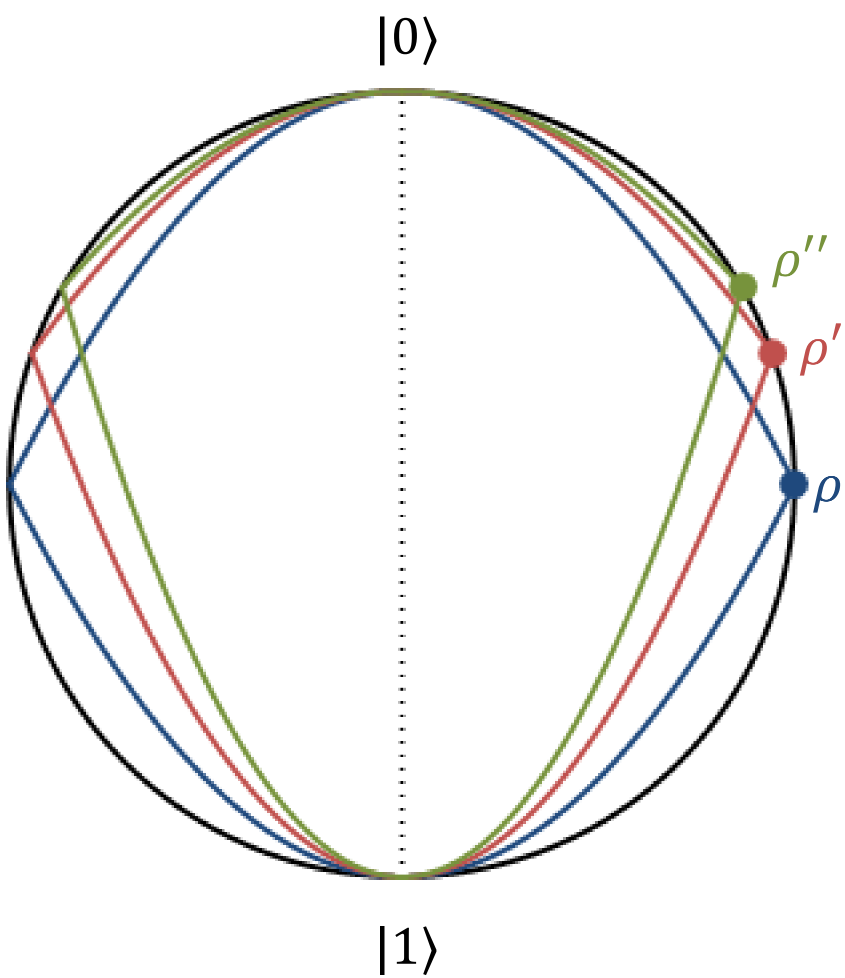


The final state σ:

1. Can have arbitrary populations $\{\sigma_{00}, \sigma_{11}\}$

2. Coherence σ_{01} is (tightly) bounded by:

$$|\sigma_{01}| \leq \sqrt{\alpha}\ |\rho_{01}|$$

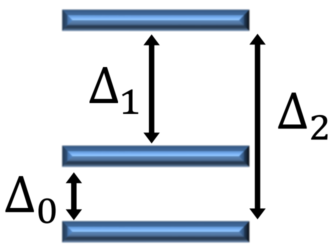
with

$$\alpha = \min\left(\frac{\sigma_{00}}{\rho_{00}}, \frac{\sigma_{11}}{\rho_{11}}\right)$$

No maximally coherent state!

Optimal coherence preservation: Non-degenerate Bohr spectrum

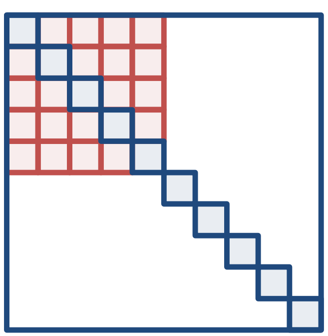
When all energy differences are distinct, e.g.,



$$\Delta_0 \neq \Delta_1 \neq \Delta_2$$

Choi-Jamiołkowski state is given by:

$J[\mathcal{E}]$



No coherence transfer terms

Only coherence depletion terms

The final state σ:

1. Can have arbitrary populations σ_{xx}

2. Coherences σ_{xy} are (tightly) bounded by:

$$|\sigma_{xy}| \leq |\rho_{xy}| \sqrt{p_{x|x} p_{y|y}}$$

with

$$p_{x|x} = 1 \quad \text{if} \quad \sigma_{xx} \geq \rho_{xx}$$

$$p_{x|x} = \frac{\sigma_{xx}}{\rho_{xx}} \quad \text{if} \quad \sigma_{xx} < \rho_{xx}$$

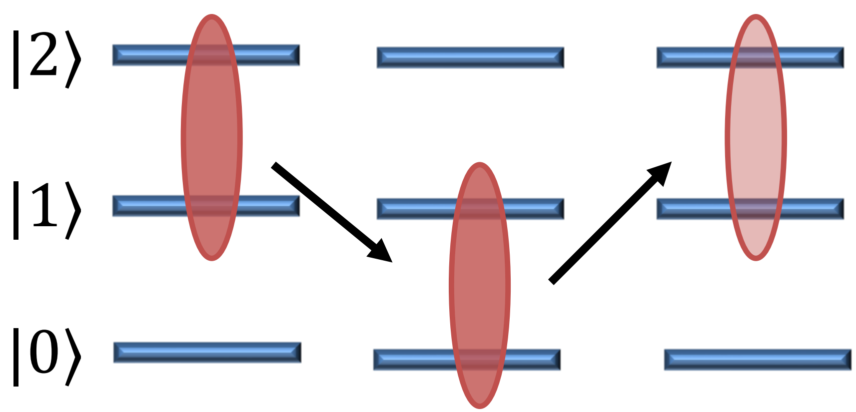
Irreversible coherence transfer

In thermodynamic considerations when system interacts with bath at thermal equilibrium, the evolution has a fixed point γ:

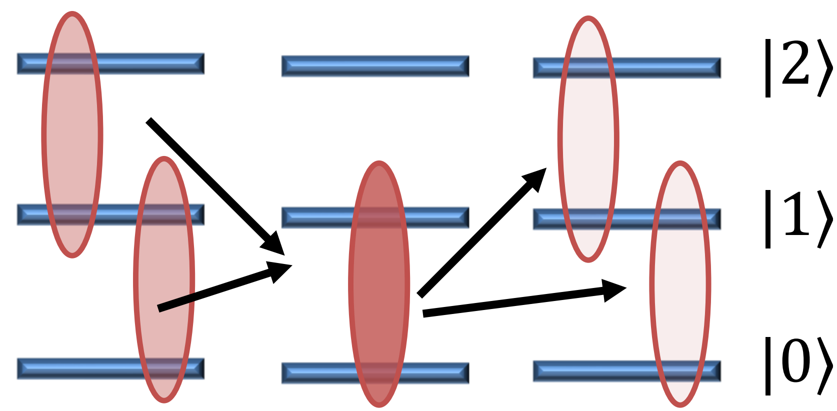
$$\mathcal{E}(\gamma) = \gamma, \qquad \gamma = \frac{e^{-\beta H}}{Tr(e^{-\beta H})}$$

This condition constrains the evolution of populations (described by $p_{x'|x}$) and, as a result, leads to irreversibility in coherence processing:

Coherence shifting



Coherence merging



M. Lostaglio, K. Korzekwa, D. Jennings, T. Rudolph, Phys. Rev. X **5**, 021001 (2015)

Markovian covariant dynamics

Markovian (memoryless) dynamics $\mathcal{E}_t$ generated by Lindbladian $\mathcal{L}$:

$$\mathcal{E}_t = e^{\mathcal{L}t}, \qquad \mathcal{L}(\cdot) = \Phi(\cdot) - \tfrac{1}{2}\{\Phi^\dagger(\mathbb{I}),\cdot\}_+ - i[\cdot, H],$$

Φ - CP map
$H = H^\dagger$

When $\mathcal{E}_t$ is covariant (e.g., within secular approximation) then $\mathcal{L}$ and Φ also are.

A. Holevo,
Rep. Math. Phys. **32**, 211 (1993)

Given population transfer rates:

$$l_{x'|x} := \langle x'|\mathcal{L}(|x\rangle\langle x|)|x'\rangle \quad \text{then} \quad \frac{d}{dt}\rho_{x'x'}(t) = \sum_x l_{x'|x}\rho_{xx}(t)$$

We find optimal Markovian evolution of coherences (using block-diagonality of J[Φ]):

$$\frac{d}{dt}\tilde{\rho}_{x'y'}(t) = -\Gamma_{x'y'}\tilde{\rho}_{x'y'}(t) + \sum_{x,y}^{(\omega_{x'y'})} \sqrt{l_{x'|x}l_{y'|y}}\,\tilde{\rho}_{xy}(t)$$

With: $\Gamma_{xy} = \frac{|l_{x|x}|+|l_{y|y}|}{2}$

Coherence damping rates   Coherence transfer rates

For all times we have: $\rho_{xy}(t) \le \tilde{\rho}_{xy}(t)$

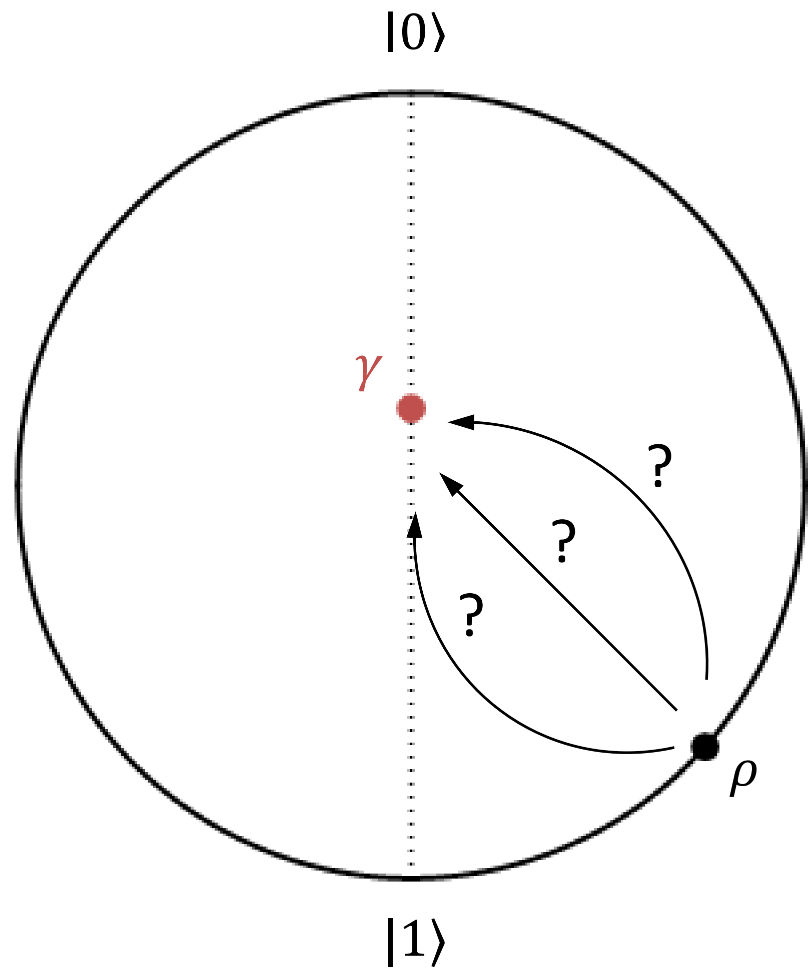
M. Lostaglio, K. Korzekwa, A. Milne,
arXiv expected in January 2017

Markovian vs non-Markovian: Qubit

Restriction to maps with a given fixed point γ leads to evolution towards γ

$$\rho = \begin{pmatrix} p & c \\ c & 1-p \end{pmatrix} \longrightarrow \gamma = \begin{pmatrix} g & 0 \\ 0 & 1-g \end{pmatrix}$$

$$\begin{array}{ccc} 0 & t & \infty \\ \hline 0 & \lambda & 1 \end{array}$$



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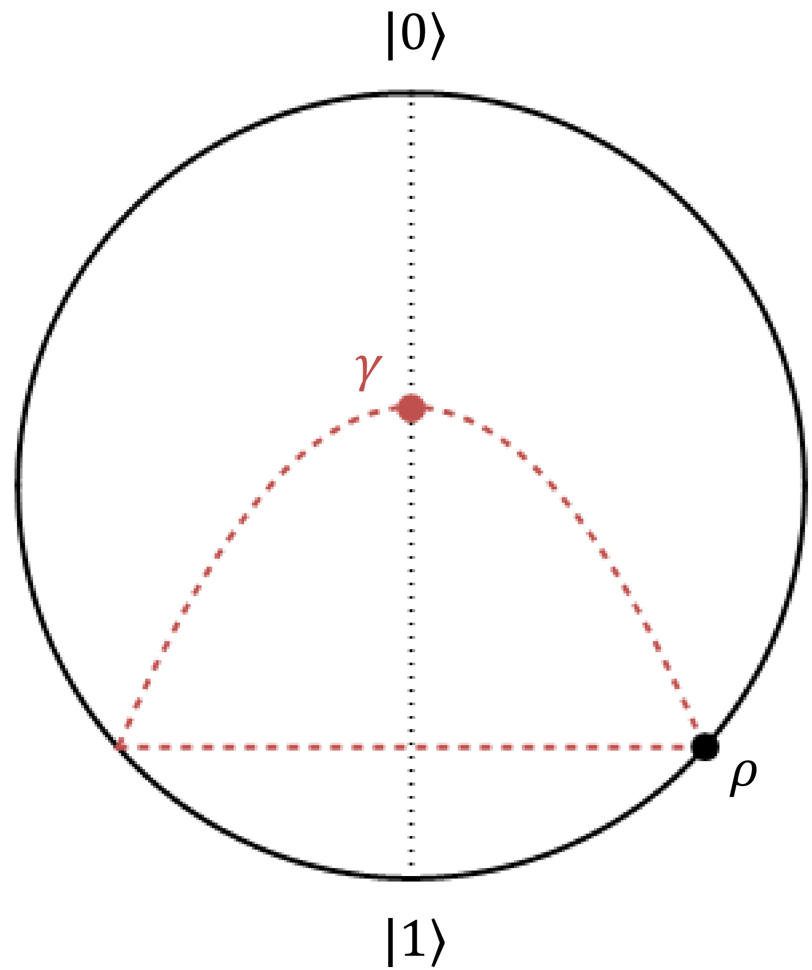

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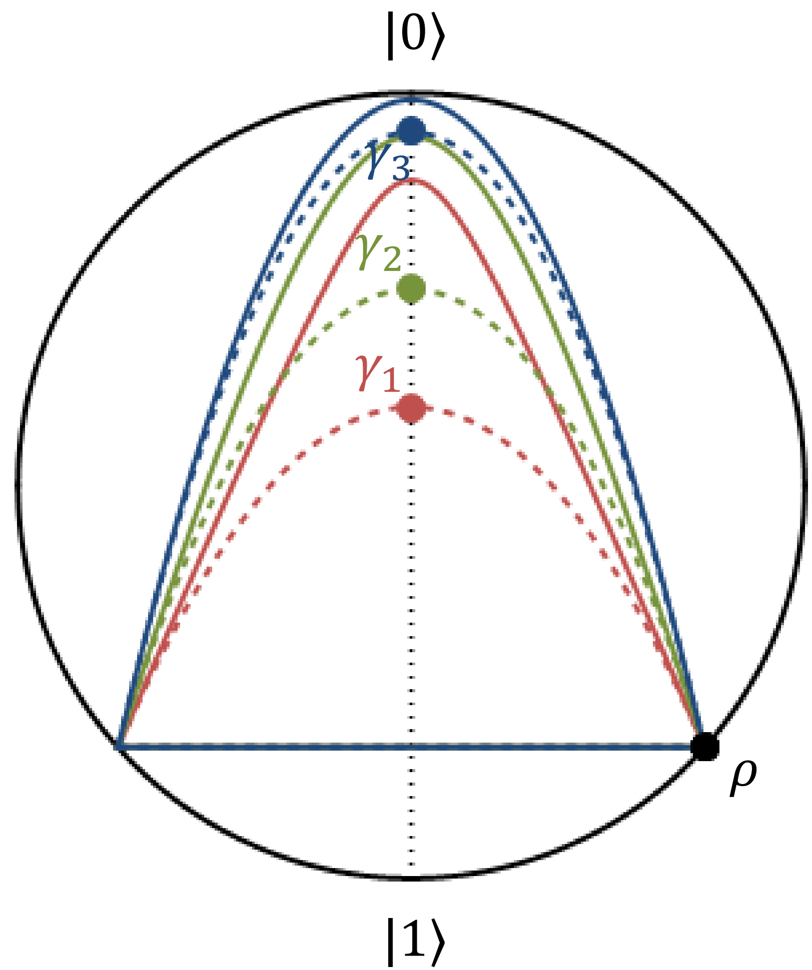
$$\rho = \begin{pmatrix} p & c \\ c & 1-p \end{pmatrix} \longrightarrow \gamma = \begin{pmatrix} g & 0 \\ 0 & 1-g \end{pmatrix}$$

0	t	∞
0	λ	1

Ratio of optimally preserved coherence under non-Markovian and Markovian dynamics

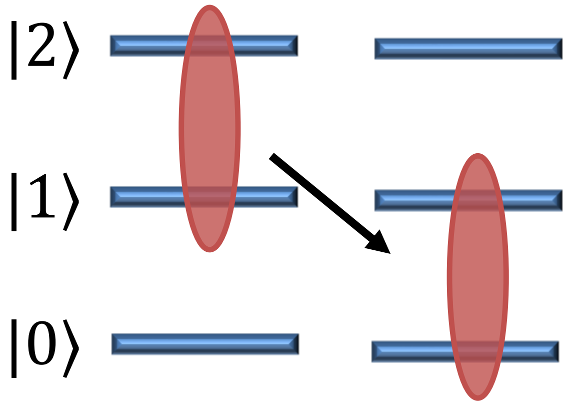
$$\frac{|c_{NM}|^2}{|c_M|^2} = 1 + \frac{\lambda^2}{1-\lambda}(1-g)g$$

If no restriction to fixed-point maps then Markovian and non-Markovian maps have the same power to process coherence.

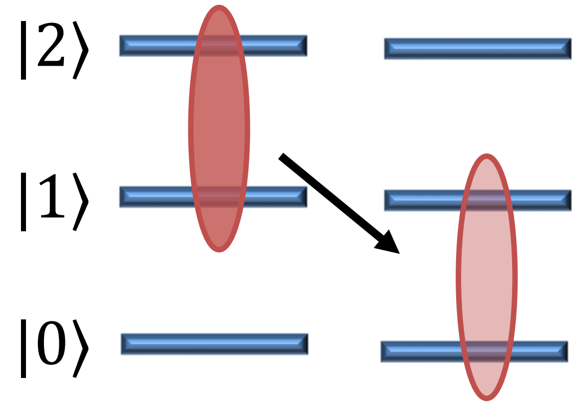


Markovian vs non-Markovian: Qutrit

Perfect transfer possible using non-Markovian maps



Perfect transfer impossible using Markovian maps



Dynamics described by a damped harmonic oscillator equation. Optimal parameter choice yields ≈ 0.5474 transfer.

Explanation: Due to continuity of the process shifting coherence actually also involves merging coherence.

Comment: Only happens when 2 components of a mode involve the same state.

Generalising relaxation-decoherence times relation

Probability vector describing populations: $\boldsymbol{p}(t) = \left(\rho_{00}(t), \rho_{11}(t), \dots, \rho_{d-1\,d-1}(t)\right)$

Covariant Markovian evolution generated by Lindbladian $\mathcal{L}$ yields:

$$\boldsymbol{p}(t) = \boldsymbol{\pi} + \sum_{i=1}^{d-1} c_i\, \boldsymbol{v}_i \exp\left(-\frac{t}{T_1^{(i)}}\right)$$

Stationary population

Dependent on $\boldsymbol{p}(0)$

Dependent on $\mathcal{L}$

For distinct energy differences coherences evolve according to:

$$\left|\rho_{ij}(t)\right| = \left|\rho_{ij}(0)\right| \exp\left(-\frac{t}{T_2^{(ij)}}\right)$$

Introducing: $\boldsymbol{T}_1 = \left(T_1^{(1)}, \dots, T_1^{(d-1)}\right)$ and $\boldsymbol{T}_2 = \left(T_2^{(01)}, \dots, T_2^{(d-1\ d)}\right)$

We have: $\boxed{\langle \boldsymbol{T}_2 \rangle_h \le \frac{d}{d-1} \langle \boldsymbol{T}_1 \rangle_h}$

For $d = 2$ we recover well-known relation for qubits

$$T_2^{(01)} \le 2\, T_1^{(1)}$$

Where $\langle \cdot \rangle_h$ denotes harmonic mean:

$$\langle \boldsymbol{x} \rangle_h = \frac{d}{\sum_{i=1}^{d} x_i^{-1}}$$

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Where $\langle \cdot \rangle_h$ denotes harmonic mean: $\langle \boldsymbol{x} \rangle_h = \frac{d}{\sum_{i=1}^{d} x_i^{-1}}$

For $d = 3$ we get

$$\left(\frac{1}{T_2^{(01)}} + \frac{1}{T_2^{(12)}} + \frac{1}{T_2^{(02)}}\right)^{-1} \leq \left(\frac{1}{T_1^{(1)}} + \frac{1}{T_1^{(2)}}\right)^{-1}$$

Generalising relaxation-decoherence times relation

Probability vector describing populations: $\boldsymbol{p}(t) = \left(\rho_{00}(t), \rho_{11}(t), \dots, \rho_{d-1\,d-1}(t)\right)$

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Stationary population

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Introducing: $\boldsymbol{T}_1 = \left(T_1^{(1)}, \dots, T_1^{(d-1)}\right)$ and $\boldsymbol{T}_2 = \left(T_2^{(01)}, \dots, T_2^{(d-1\ d)}\right)$

We have: $$\boxed{\langle \boldsymbol{T}_2 \rangle_h \leq \frac{d}{d-1} \langle \boldsymbol{T}_1 \rangle_h}$$

Where $\langle \cdot \rangle_h$ denotes harmonic mean: $$\langle \boldsymbol{x} \rangle_h = \frac{d}{\sum_{i=1}^{d} x_i^{-1}}$$

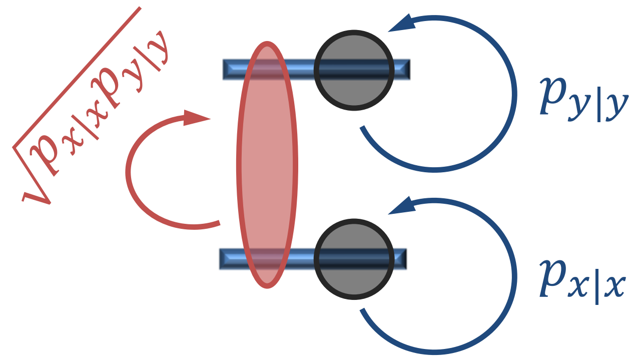
For $d \to \infty$ we get that optimally

$$\langle \boldsymbol{T}_2 \rangle_h \to \langle \boldsymbol{T}_1 \rangle_h$$

Conclusions

Coherences under time-translation covariant dynamics:

- Cannot increase.
- Can only be transferred between pairs of states with the same energy difference.
- Depend on population transfer between corresponding states.



Markovian covariant dynamics:

- Allows to preserve less coherence while performing the same transformation of populations (compared to non-Markovian dynamics).
- Does not allow for a perfect transfer of coherence within a given mode.
- Is characterised by decoherence times that are neatly bounded by relaxation times.

Thank you!