Quantum resource theories

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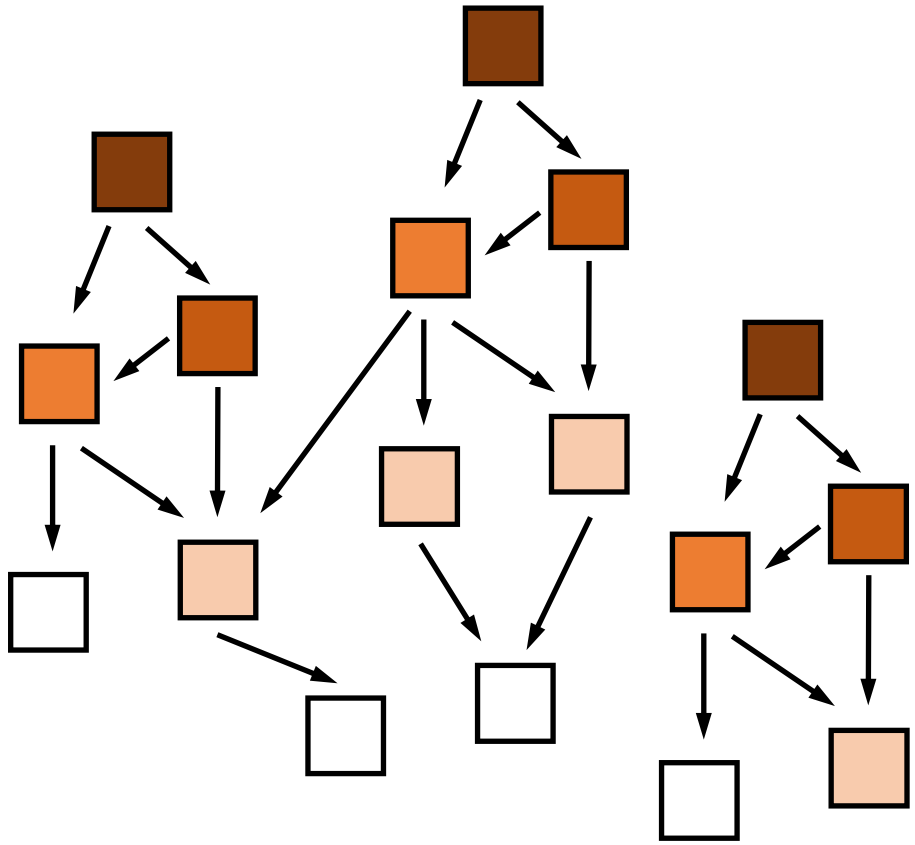


Motivation

Motivation (entanglement)

Quantum entanglement has clear operational meanings:

- Teleportation
- Super-dense coding
- E91 cryptographic protocol
- Bell inequalities violation
- ...

But:

*There is no Hermitian operator such that the value of an entanglement measure could be given by its expectation value for any state of a composite quantum system**

So how can we quantify the amount of entanglement in a given state ρ ?

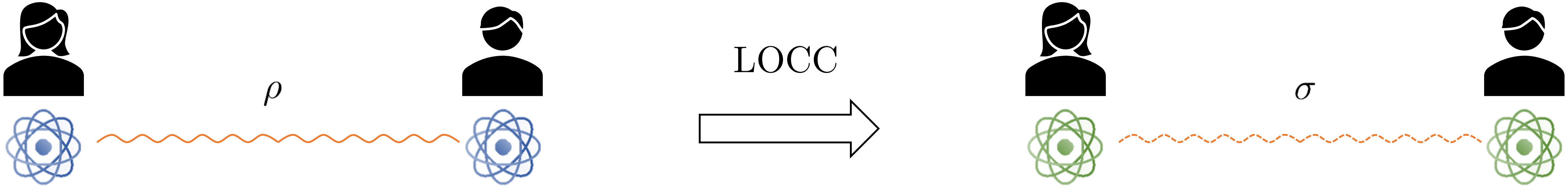
App. Phys. B* **89, 493 (2007)

Motivation (entanglement)

1. Identify the restricted setting for which entanglement has operational meaning as a resource

   Local operations & classical communication (LOCC)

2. For pairs of states, verify whether you can transform them into each other using LOCC



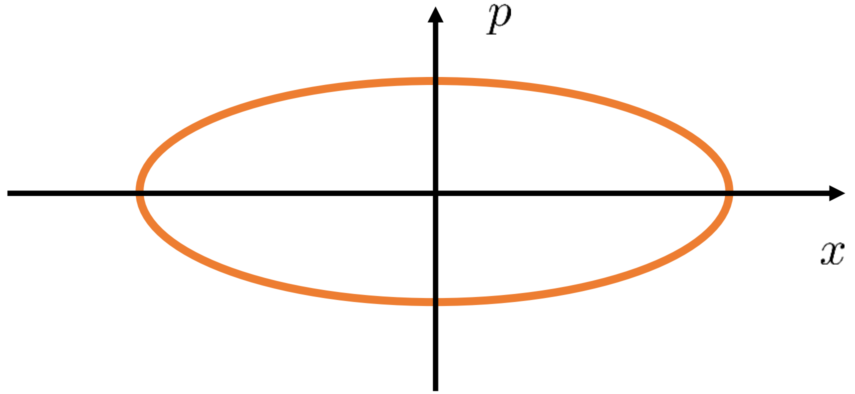
3. Entanglement measure: any real-valued function preserving the LOCC order structure

$$\rho \xrightarrow{\text{LOCC}} \sigma \qquad \Rightarrow \qquad E(\rho) \geq E(\sigma)$$

Motivation (thermodynamics)

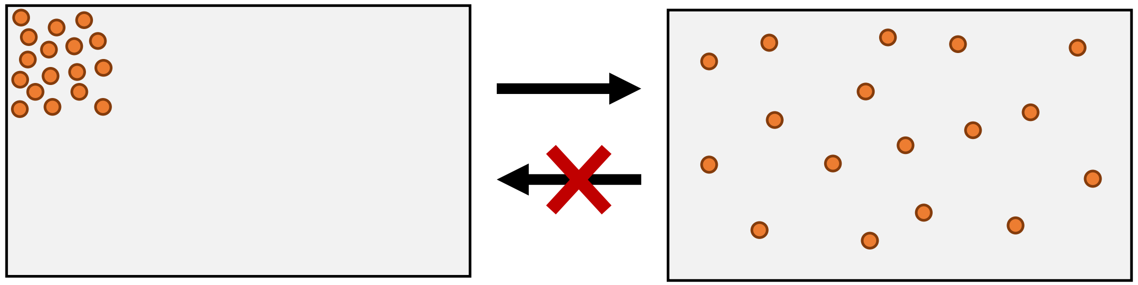
What can we say about the dynamics without solving equations of motion?

Closed systems



Energy conservation

Open systems



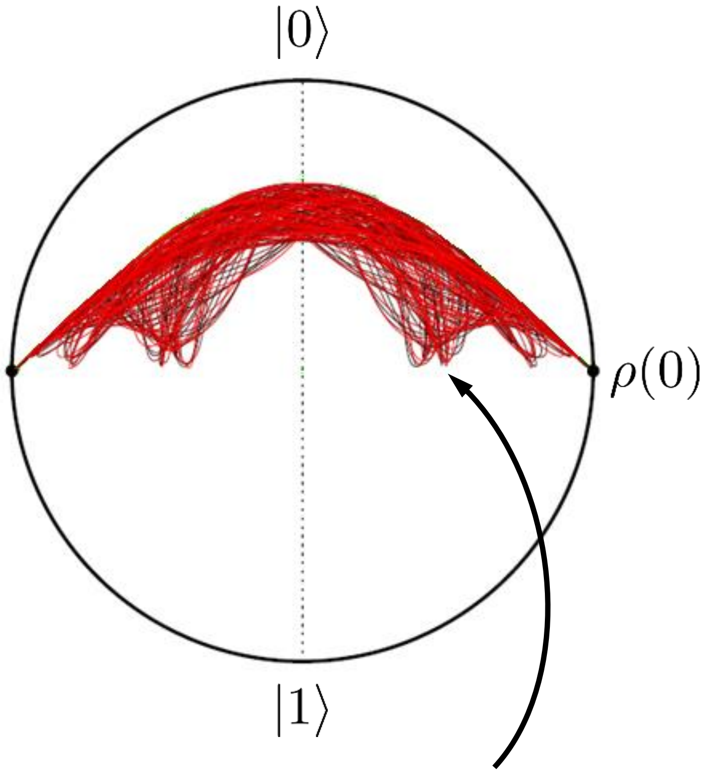
Entropy growth

Resource theoretic approach to thermodynamics:
Using minimal assumptions of the quantum theory, find constraints on the evolution of a quantum system interacting with thermal baths

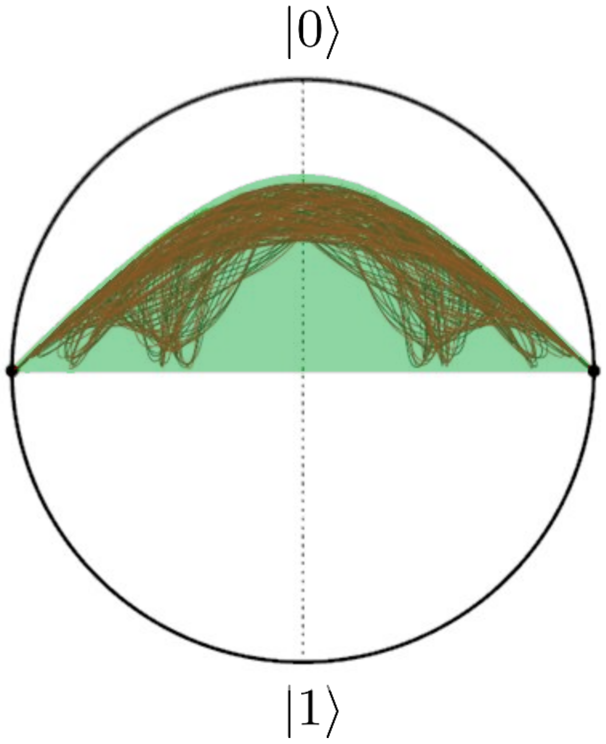
Motivation (thermodynamics)

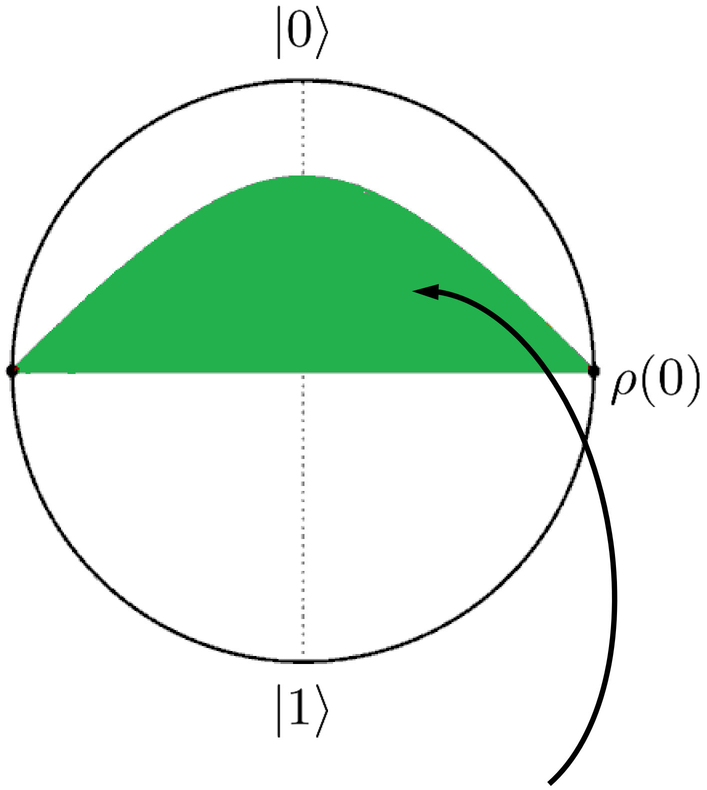
Open dynamics approach:

Resource-theoretic approach:



Exact time evolution for a given model





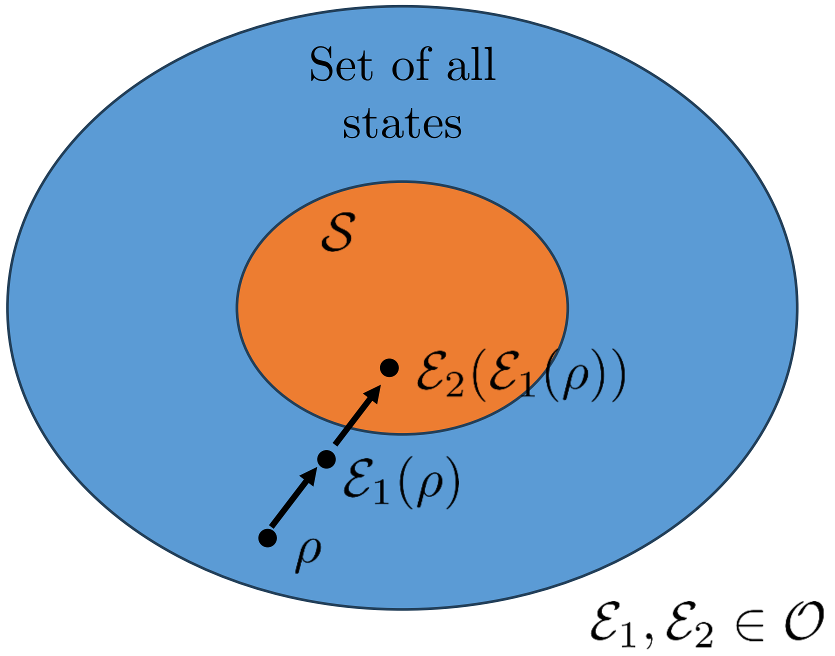
Allowed final states compatible with the laws of thermodynamics

Resource theories

Definition and examples

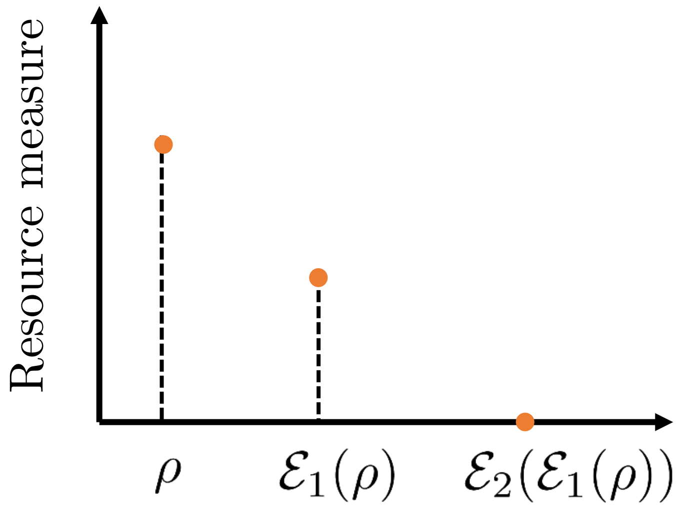
1. Define the set of free operations $\mathcal{O}$ (e.g., LOCC).
2. Define the set of free states $\mathcal{S}$ (e.g., separable states).
3. These imply resource measures (e.g., entropy of entanglement).

The set of free states needs to be closed under free operations.



Resource measures axioms:

- Faithfulness
- Monotonicity
- Convexity
- Subadditivity
- Asymptotic continuity



Definition and examples

Resource	Physical constraint	Free operations	Free states	Reference
Entanglement	Local quantum control	Local operations & classical communication	Separable	Rev. Mod. Phys. **81**, 865 (2009)
Asymmetry	Conservation laws / lack of a shared reference frame	G-covariant operations	Symmetric	I. Marvian's PhD thesis (2012)
Thermodynamics	No control over the degrees of freedom of environment	Thermal operations	Thermal	Rep. Prog. Phys. **82**, 114001 (2019)
Coherence	No experimental control to prepare certain superpositions	Incoherent operations	Incoherent	Rev. Mod. Phys. **89**, 865 (2017)
Magic	Restriction to stabilizer circuits	Stabilizer operations	Stabilizer	New J. Phys. **16**, 013009 (2014)
Non-Gaussianity	Typical quantum optics setup	Gaussian operations	Gaussian	Phys. Rev. A **82**, 052341 (2010)
...	...	...	...	...

Resource-theoretic problems

Single-shot transformations

Find a set of necessary and sufficient conditions for $\rho \xrightarrow{\mathcal{O}} \sigma$

This full characterization for all states is usually very hard to obtain!

- Finding only necessary conditions yields constraints
- Finding only sufficient conditions yields protocols

Notable exceptions:

Bipartite pure entanglement	Incoherent thermodynamics	U(1)-covariant pure states
Majorization	**Thermomajorization**	**Cyclic majorization**
Phys. Rev. Lett. **83**, 436 (1999)	*Nature Commun. 4, 2059 (2013)*	*New J. Phys.* **10**, 033023 (2008)

Resource-theoretic problems

Majorization intermission

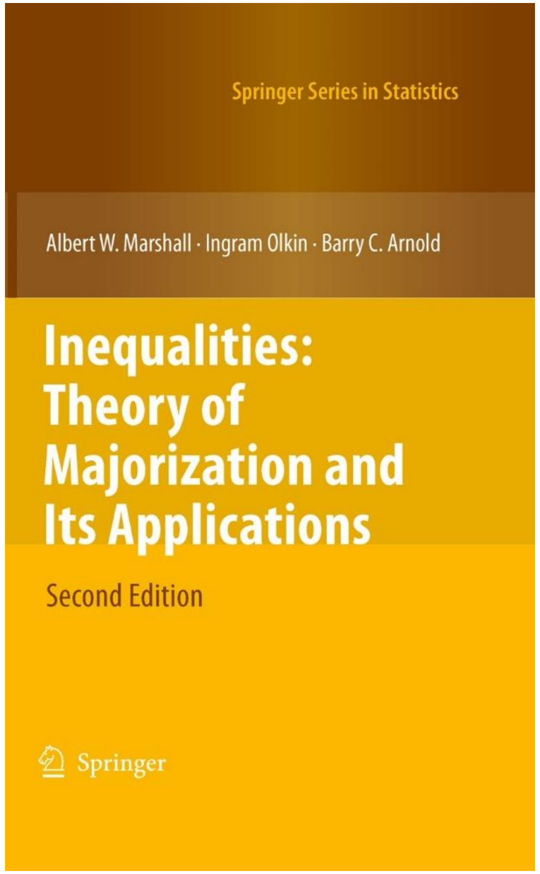
...

Consider two entangled states of two spin-1 systems:

$$|\psi_{AB}\rangle = \sqrt{\frac{3}{8}}|\uparrow\uparrow\rangle + \frac{1}{2}|00\rangle + \sqrt{\frac{3}{8}}|\downarrow\downarrow\rangle \qquad \boldsymbol{p}(\psi)^{\downarrow} = [3/8, 3/8, 1/2]$$

$$|\phi_{AB}\rangle = \frac{1}{2}|\uparrow\uparrow\rangle + \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|\downarrow\downarrow\rangle \qquad \boldsymbol{p}(\phi)^{\downarrow} = [1/2, 1/4, 1/4]$$

Open problem: Transformation laws in the resource theory of thermodynamics at infinite temperature are exactly the reverse transformation laws in the resource theory of entanglement. Coincidence?



Resource-theoretic problems

Asymptotic transformations

Maybe $\rho \xrightarrow{\mathcal{O}} \sigma$ is not possible, but $\rho^{\otimes 2} \xrightarrow{\mathcal{O}} \sigma$ is?

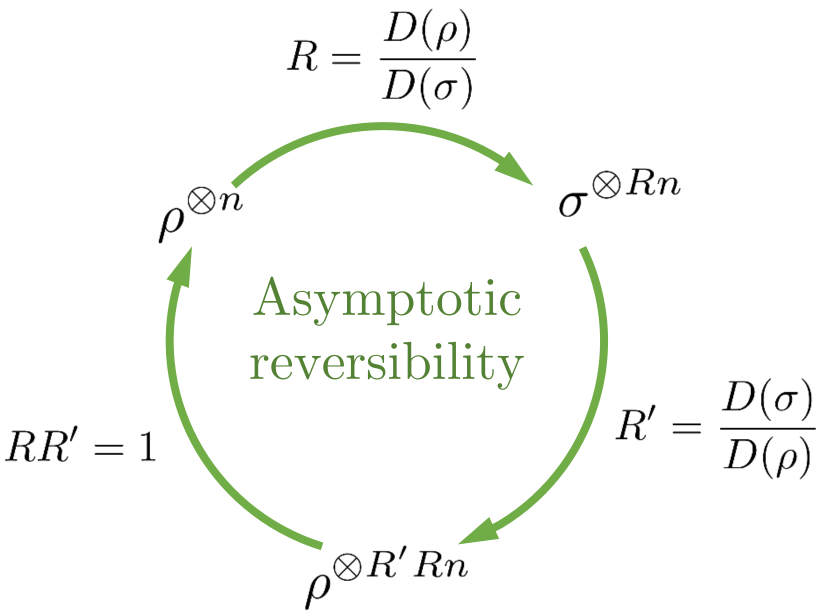
If so, maybe $\rho^{\otimes 3} \xrightarrow{\mathcal{O}} \sigma^{\otimes 2}$ is possible?

In general, we ask for the maximal rate R for which

$\rho^{\otimes N} \xrightarrow{\mathcal{O}} \sigma^{\otimes RN}$ is possible when N goes to infinity.

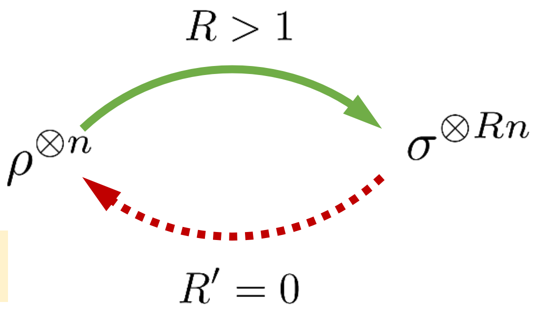
For a range of resource theories: $R = \dfrac{D(\rho)}{D(\sigma)}$

where: $D(\rho) := \min_{\tau \in \mathcal{S}} D(\rho \| \tau)$



Bound entanglement
completely breaks
asymptotic reversibility

Phys. Rev. Lett. **80**, 5239 (1998)



Resource-theoretic problems

Catalytic transformations

Maybe $\rho \xrightarrow{\mathcal{O}} \sigma$ is not possible, but $\rho \otimes \tau \xrightarrow{\mathcal{O}} \sigma \otimes \tau$ is?

Surprisingly it is sometimes the case!

Entanglement catalysis:

$$|\psi_1\rangle = \sqrt{0.4}\,|00\rangle + \sqrt{0.4}\,|11\rangle + \sqrt{0.1}\,|22\rangle + \sqrt{0.1}\,|33\rangle\,,$$

$$|\psi_2\rangle = \sqrt{0.5}\,|00\rangle + \sqrt{0.25}\,|11\rangle + \sqrt{0.25}\,|22\rangle\,.$$

$$|\phi\rangle = \sqrt{0.6}\,|44\rangle + \sqrt{0.4}\,|55\rangle\,.$$

Phys. Rev. Lett. **83**, 3566 (1999)

Check for more applications:

Accepted Paper

Catalysis in quantum information theory

Rev. Mod. Phys.

Patryk Lipka-Bartosik, Henrik Wilming, and Nelly H. Y. Ng

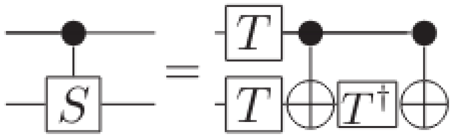
Accepted 27 February 2024

E.g., catalytically improved teleportation fidelity

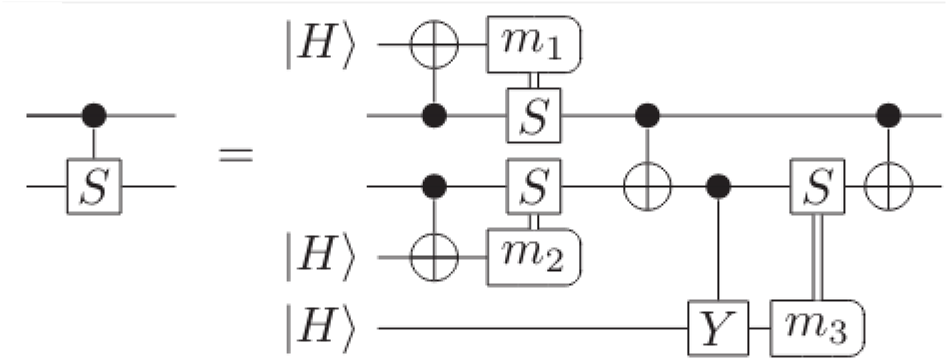
Resource-theoretic problems

Simulation of non-free operations

Decompose a non-free operation into free ones and some representative resource ones:



Replace non-free operations with non-free states (here: use gate teleportation)



How do we know that we found a decomposition that uses the smallest number of representative resources needed?

We can compare resource measures!

$$\mathcal{R}(|H^{\otimes 2}\rangle) < \mathcal{R}(|CS\rangle) < \mathcal{R}(|H^{\otimes 3}\rangle)$$

Also another optimality proof:

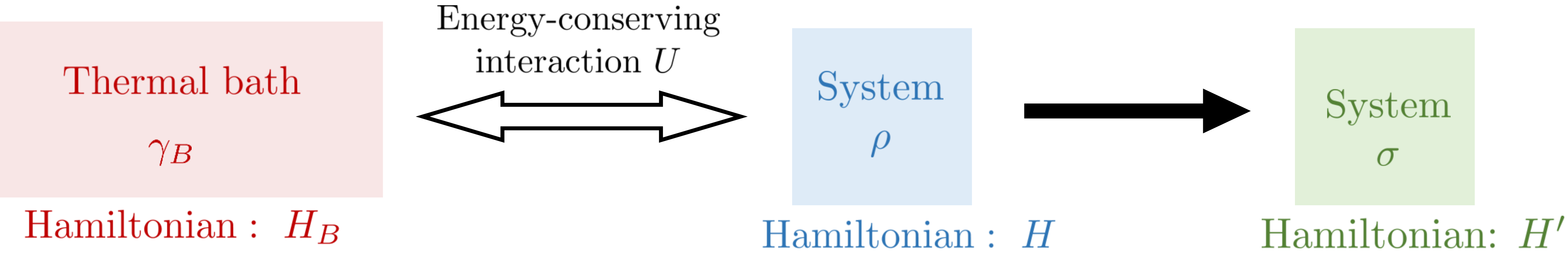
$$\mathcal{R}(|H^{\otimes 3}\rangle) < \mathcal{R}(|CCZ\rangle) < \mathcal{R}(|H^{\otimes 4}\rangle).$$

Phys. Rev. Lett. **118**, 090501 (2017)

Example: thermodynamics

Thermodynamic transformations modelled by **thermal operations***:

$$\mathcal{E}^T(\cdot) = \mathrm{Tr}_{B'}\left(U\left(\cdot \otimes \gamma_B\right)U^\dagger\right) \qquad \text{with} \quad [U, H + H_B] = 0$$



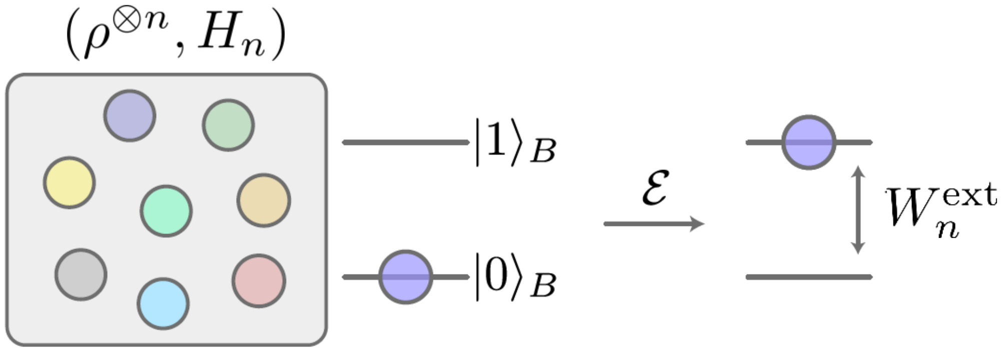
Gibbs state γ of the system at temperature T: $\gamma = e^{-\frac{H}{T}}/\mathcal{Z}, \quad \mathcal{Z} = \mathrm{Tr}\left(e^{-\frac{H}{T}}\right)$

***Nature Commun.* **4**, 2059 (2013)

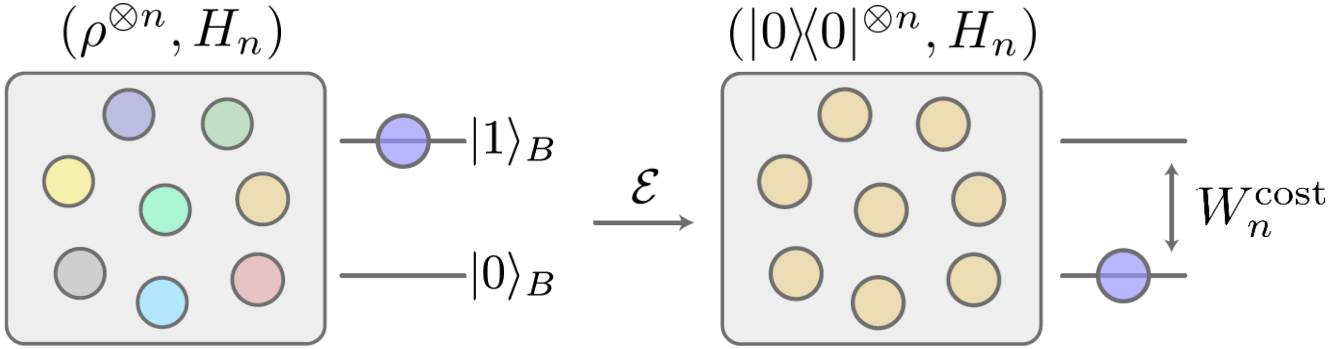
Example: thermodynamics

Thermodynamic protocols are various instances of state interconversion problem

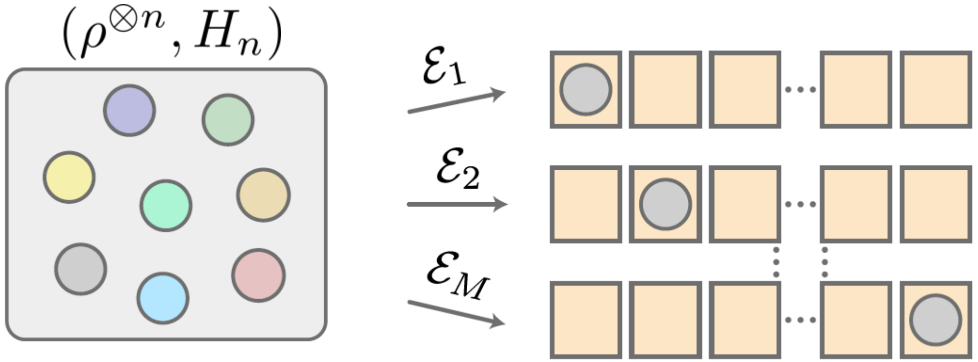
Work extraction



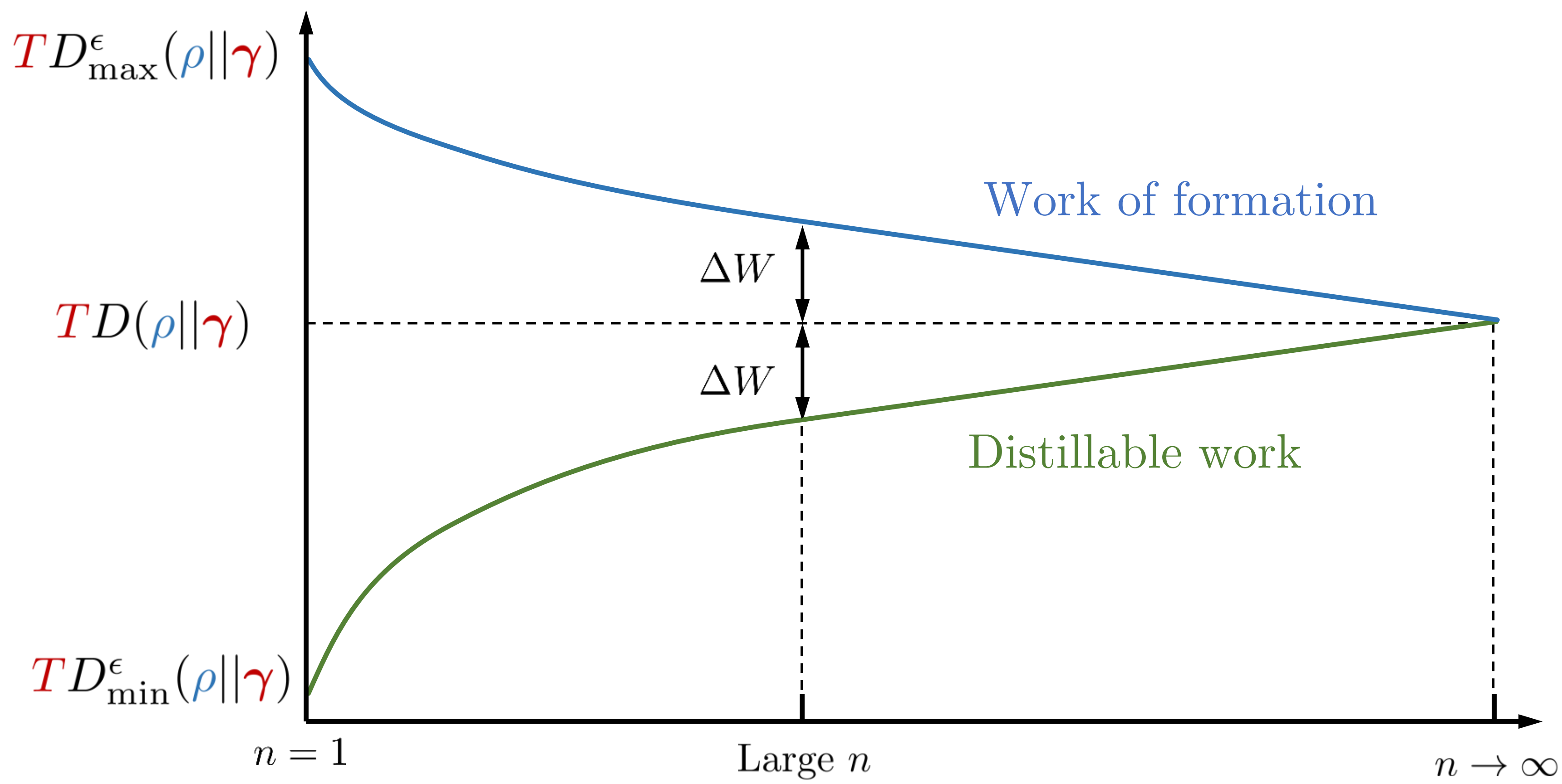
Information erasure



Thermodynamically-free communication



Example: thermodynamics



Physical interpretation:

$$D(\rho||\gamma) := \mathrm{Tr}\left(\rho(\log\rho - \log\gamma)\right)$$

$$\frac{1}{T}\Big[\underbrace{\langle E\rangle_\rho - TH(\rho)}_{\substack{\text{Free energy}\\ F=U-TS}} - \underbrace{(-T\log\mathcal{Z})}_{\text{Free energy of }\gamma}\Big]$$

Work gap:

$$\Delta W = -T\sqrt{\frac{V(\rho||\gamma)}{n}}\,\Phi^{-1}(\epsilon)$$

Φ - Normal cdf

Dissipation of free energy beyond the thermodynamic limit!

Quantum **2**, 108 (2018)

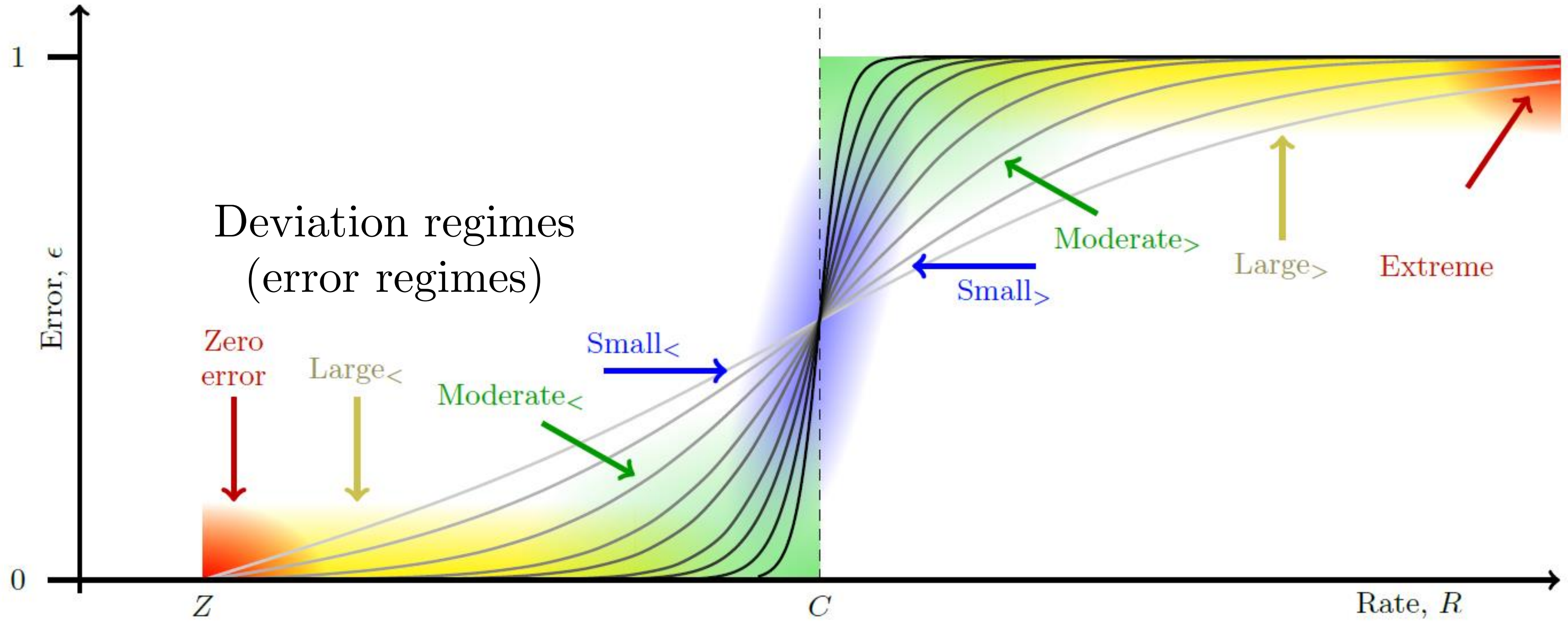
Selected research problems

Irreversibility beyond asymptotics

Single-shot interconversion: Does there exist $\mathcal{E}^T$ such that $\mathcal{E}^T(\rho) = \sigma$?

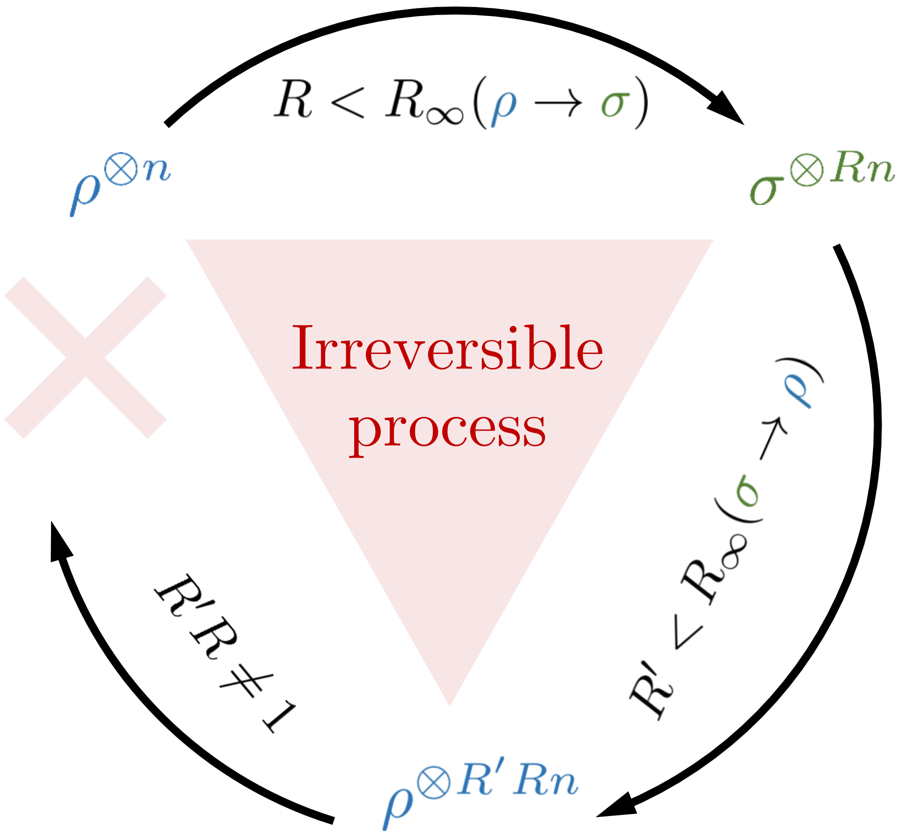
Many-copies interconversion: Does there exist $\mathcal{E}^T$ such that $\mathcal{E}^T\left(\rho^{\otimes n}\right) \approx_\epsilon \sigma^{\otimes R_n n}$?

Optimal rate R_n for error ϵ?



Irreversibility beyond asymptotics

Rate for large but finite n: $R_n = R_\infty - f(\rho, \sigma, \gamma, n, \epsilon)$



Relevant quantity quantifying irreversibility:

Relative entropy variance:

$$V(\rho\|\gamma) := \mathrm{Tr}\left(\rho(\log\rho - \log\gamma - D(\rho\|\gamma))^2\right)$$

Physical interpretation:

$$V(\gamma'\|\gamma) = \underbrace{\frac{\partial\langle E\rangle_{\gamma'}}{\partial T'}}_{\text{Specific heat capacity}} \cdot \underbrace{\left(1 - \frac{T'}{T}\right)^2}_{\text{Carnot factor}}$$

Quantum **2**, 108 (2018)

Irreversibility beyond asymptotics

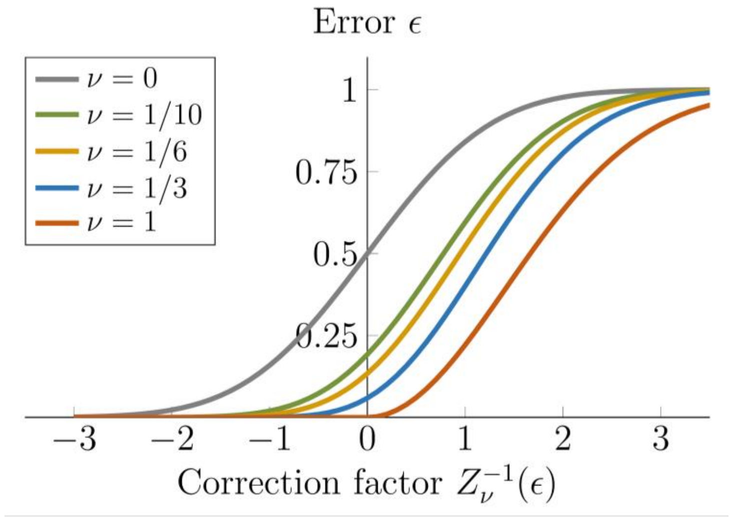
Optimal conversion rate R_n with constant error ϵ:

$$R_n(\epsilon) = R_\infty + \sqrt{\frac{V(\rho\|\gamma)}{D(\sigma\|\gamma)^2}}\,\frac{Z_\nu^{-1}(\epsilon)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)$$

Reversibility parameter:

$$\nu = \frac{V(\sigma\|\gamma)/D(\sigma\|\gamma)}{V(\rho\|\gamma)/D(\rho\|\gamma)}$$

Rayleigh-normal distribution Z_ν*:



Z_0 - standard normal distribution Φ

Z_1 - Rayleigh distribution ($Z_1(x) = 0$ for $x \leq 0$)

IEEE Trans. Inf. Theory **63**, 1829–1857 (2017)

Resource resonance

Optimal conversion rate R_n with vanishing error $\epsilon = e^{-n^\alpha}$ and $\alpha \in (0,1)$:

$$R_n(\epsilon) = R_\infty - \sqrt{\frac{V(\rho\|\gamma)}{D(\sigma\|\gamma)^2}}\,\frac{\left|\sqrt{1/\nu}-1\right|}{\sqrt{n^{1-\alpha}}} + o\left(\frac{1}{\sqrt{n^{1-\alpha}}}\right)$$

When $\nu = 1$ correction term disappears for every error ϵ

$$\Downarrow$$

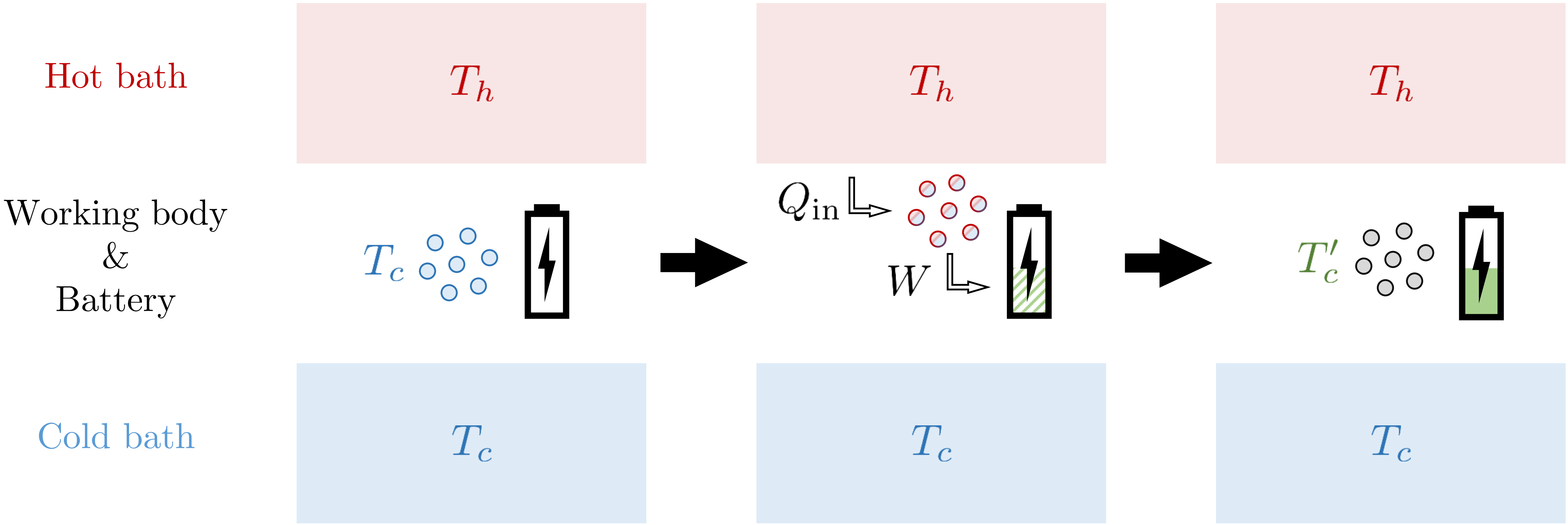
No free energy dissipation!
(at least up to second order asymptotics)

(recall that $\nu = 1$ means that the relative fluctuations
of free energy are the same for the
initial state ρ and target state σ)

Phys. Rev. A **99**, 032332 (2019)

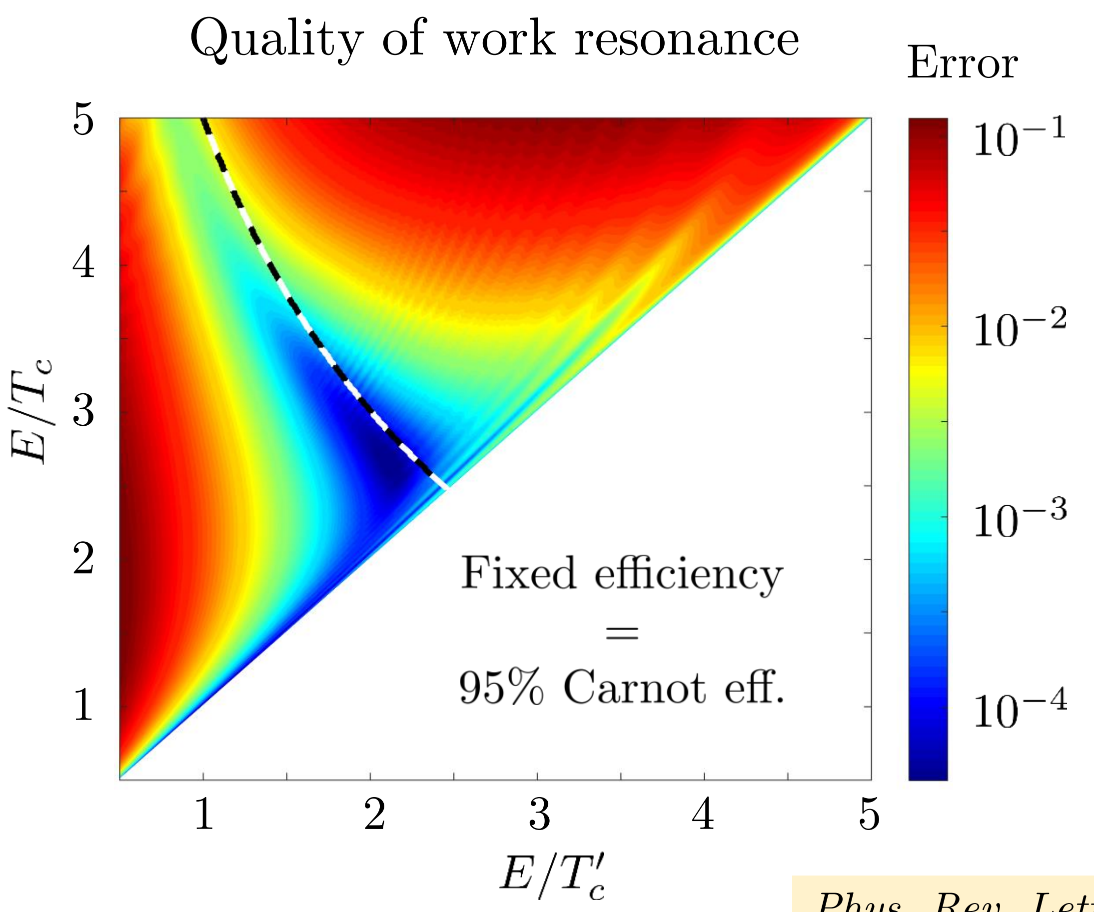
Resource resonance

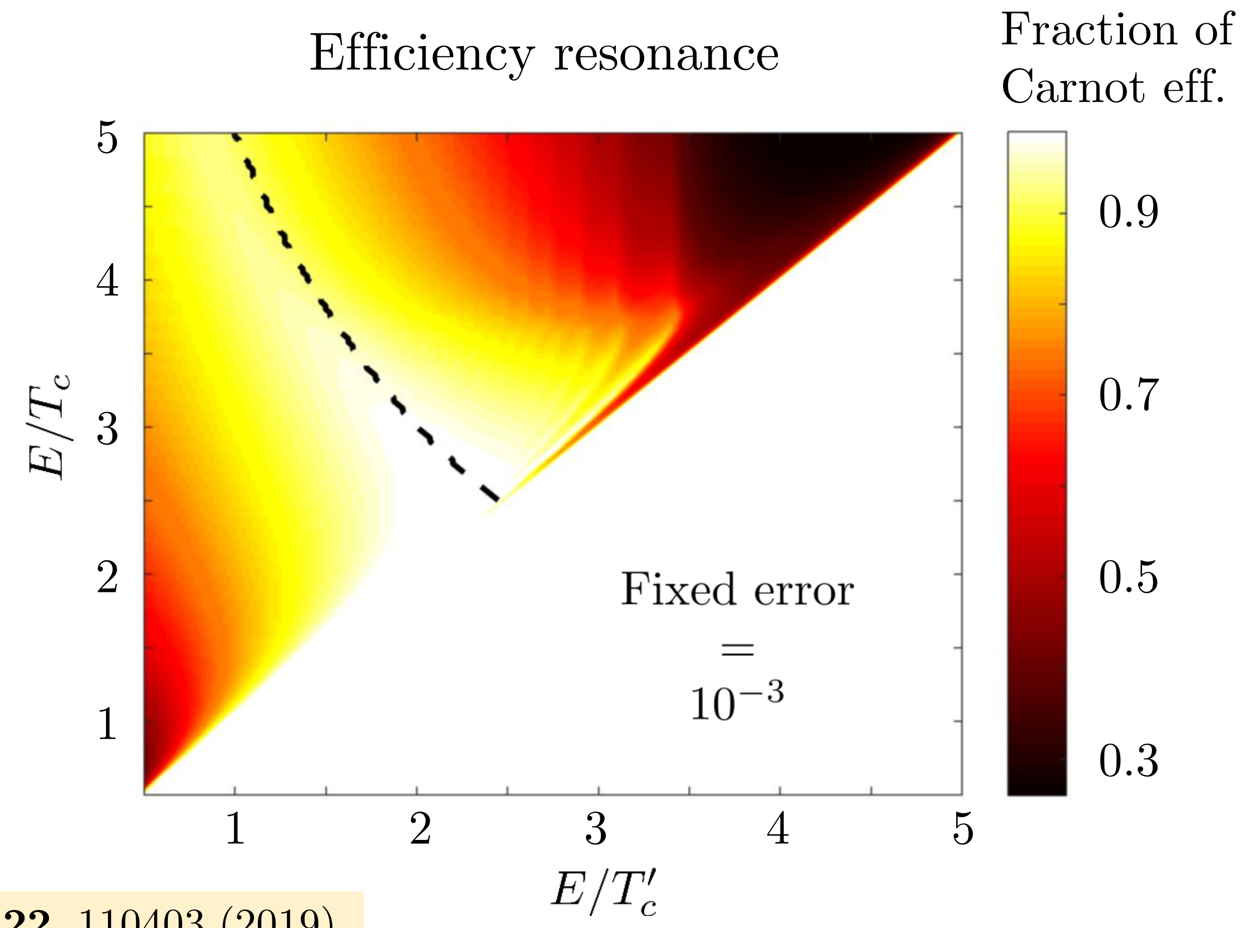
Resonance example: Heat engine with a finite-size working body:



Resource resonance

Working body: $n = 200$ qubits, energy gap E
Background (hot) bath: $T_h = 10E$

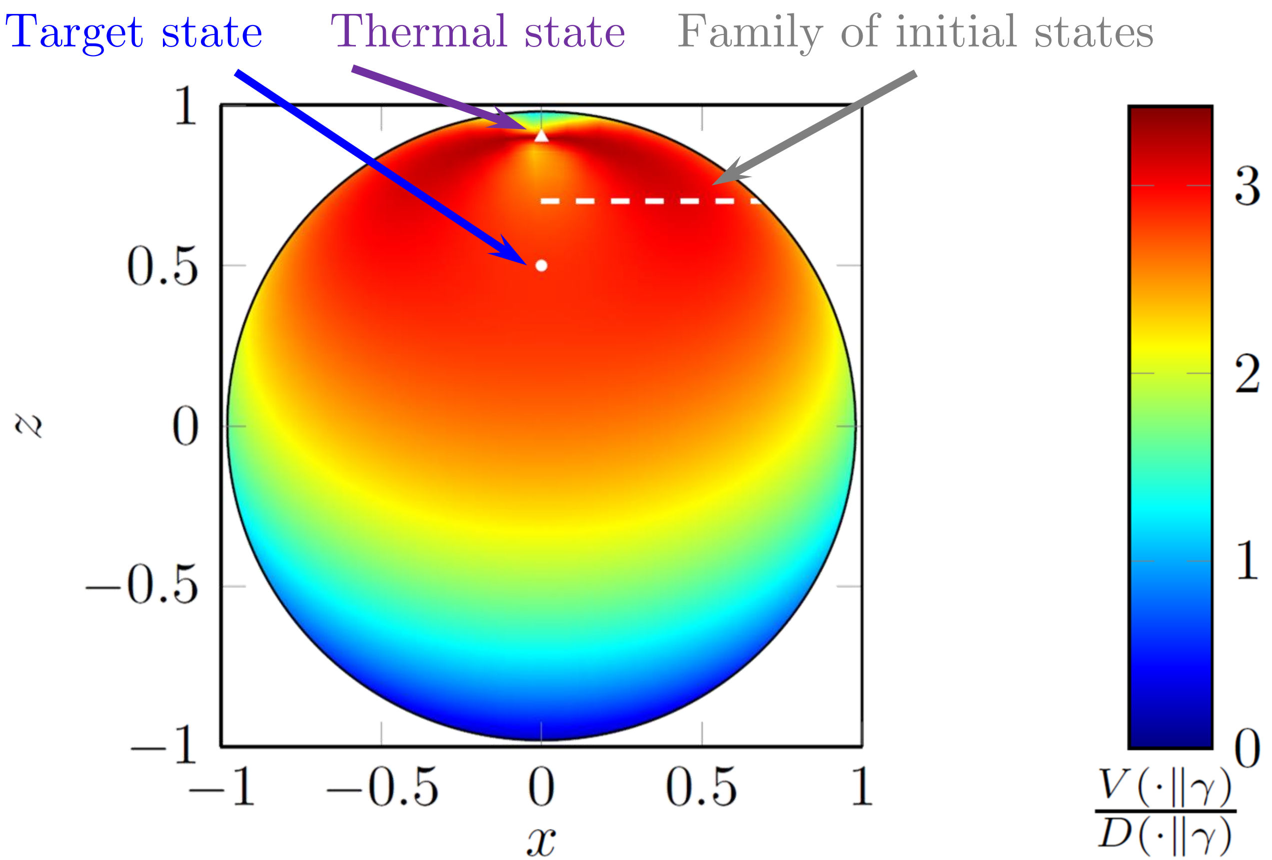




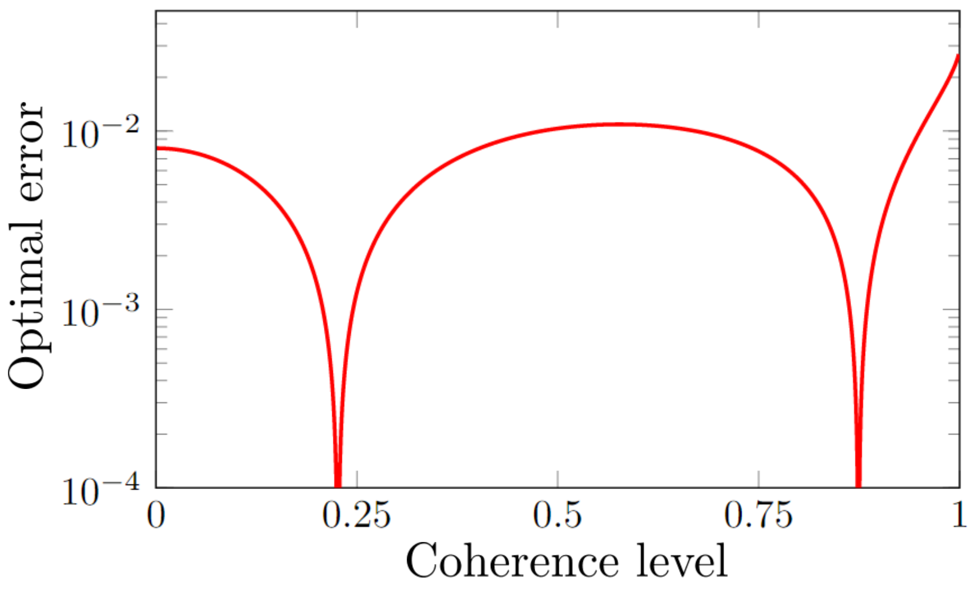
Phys. Rev. Lett. **122**, 110403 (2019)

Resource resonance

Predicting coherent resonance phenomenon:



Transformation with the asymptotic rate



Recall reversibility parameter:

$$\nu = \frac{V(\sigma\|\gamma)/D(\sigma\|\gamma)}{V(\rho\|\gamma)/D(\rho\|\gamma)}$$

PRX Quantum **5**, 020335 (2024)

Fluctuation-dissipation relations

Thermodynamic distillation process

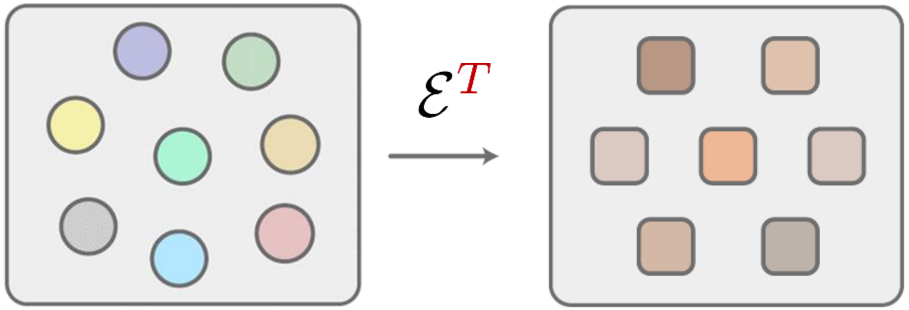
Non-zero free energy:

$$F := nD(\rho\|\gamma)/\beta$$

Non-zero free energy fluctuations:

$$\sigma^2(F) := nV(\rho\|\gamma)/\beta^2$$

$(\rho^{\otimes n}, H_n)$ $\xrightarrow{\mathcal{E}^T}$ $(|\tilde{E}_k\rangle\langle\tilde{E}_k|, \tilde{H})$

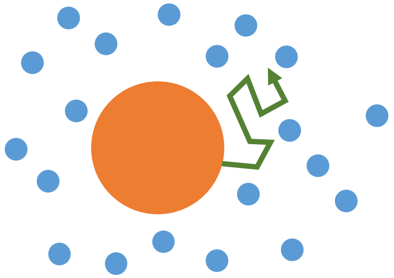


Non-zero free energy, but vanishing free energy fluctuations

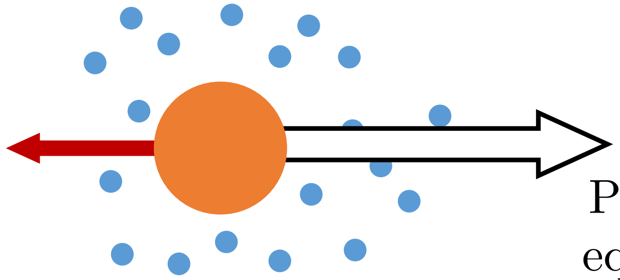
Free energy fluctuations $\overset{?}{\sim}$ Free energy dissipated in the process

Einstein-Smoluchowski relation for a Brownian particle:

In thermal equilibrium



Position fluctuations $\sim$ Dissipative friction



Pull out of equilibrium

Fluctuation-dissipation relations

Optimal error in thermodynamic distillation process:

$$\epsilon = \lim_{n\to\infty} \Phi\left(-\frac{\Delta F}{\sigma(F)}\right)$$

ΔF - Free energy difference between initial and **target** state

Minimal amount of free energy dissipated in the optimal distillation process:

$$F_{\text{diss}} \simeq a(\epsilon)\sigma(F)$$

F_{diss} - Free energy difference between initial and **final** state

$$a(\epsilon) = -\Phi^{-1}(\epsilon)(1-\epsilon) + \exp(-[\Phi^{-1}(\epsilon)]^2/2)/\sqrt{2\pi}$$

Three regimes:

$$\lim_{n\to\infty} \frac{\Delta F}{\sqrt{n}} = \begin{cases} \infty, & \Longrightarrow \quad \epsilon = 0, \quad F_{\text{diss}} = \Delta F \\ -\infty, & \Longrightarrow \quad \epsilon = 1, \quad F_{\text{diss}} = 0 \\ \alpha \in \mathbb{R} & \end{cases}$$

Also holds for initial pure states with coherence!

Phys. Rev. E ***105***, 054127 (2022)

Outlook

Outlook

- Resource theories provide a unified framework for studying accross fields
- With the raise of quantum hardware optimization of scarce quantum resources becomes crucial
- Allows both for practical (how many T gates needed) and fundamental investigations (existence of bound entanglement)
- Many other: resource thoeries of channels, Markovianity, mixed reosurce theories, etc.
- Important: not to get too abstract!
- Read more: Rev. Mod. Phys. **91**, 025001 (2019)

Thank you!