# Coherifying quantum channels

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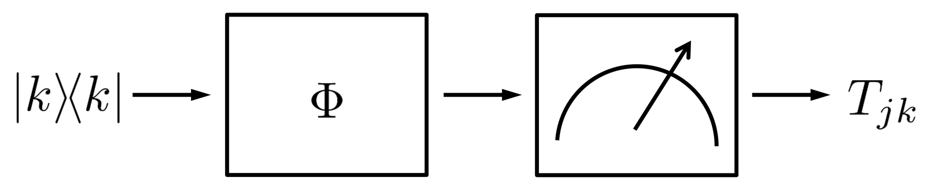
## Motivation

### Classical action of a quantum channel

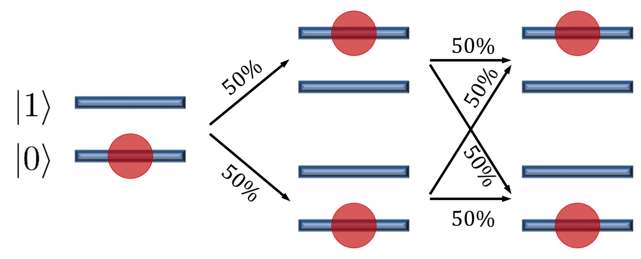



$\{|k\rangle\}$ - distinguished orthonormal basis

$T_{jk}=\langle j|\Phi(|k\rangle\langle k|)|j\rangle$ - classical action

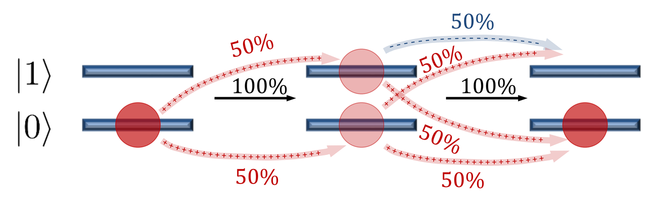
What does $T$ tell us about $\Phi$?

### Channels with fixed classical action




$\Phi_1(\cdot)=\frac{1}{2}$

$\Phi_1$ is irreversible




$\Phi_2(\cdot)=H(\cdot)H^\dagger$

$\Phi_2$ is reversible

$H=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$

### Questions

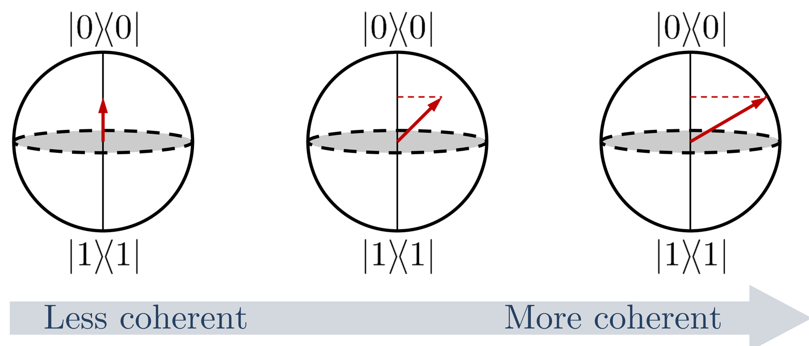
1. Is it always possible to explain random stochastic transitions as arising from the underlying deterministic quantum evolution?
2. If not, what is the minimal amount of randomness required by quantum theory to explain a given stochastic process?
3. And can there exist perfectly distinguishable quantum processes that nevertheless lead to the same classical evolution?

## Setting the scene

### Coherence of quantum states

$\langle j|\rho|j\rangle$ - occupations $p_j$, $\langle j|\rho|k\rangle$ - coherences

Fixed $\boldsymbol{p}$:



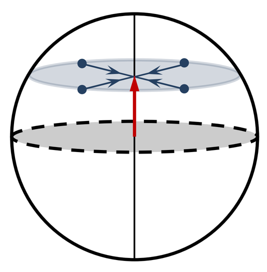

Coherence measures
(distance from incoherent states):

$$C_{\mathrm{e}}(\rho):=S(\rho\|\mathcal{D}(\rho))=S(\boldsymbol{p})-S(\boldsymbol{\lambda}(\rho))$$

$$C_2(\rho):=\|\rho-\mathcal{D}(\rho)\|_2=\boldsymbol{\lambda}(\rho)\cdot\boldsymbol{\lambda}(\rho)-\boldsymbol{p}\cdot\boldsymbol{p}$$

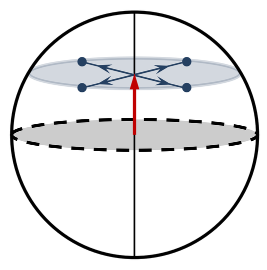
$\mathcal{D}$ - decohering channel, $\boldsymbol{\lambda}(\rho)$ - eigenvalues of $\rho$

### Decoherence and coherification



Decohering channel $\mathcal{D}$:

$\rho$ with $\langle j|\rho|j\rangle=p_j \xrightarrow{\mathcal{D}} \rho^{\mathcal{D}}=\mathrm{diag}(\boldsymbol{p})$



Coherification $\mathcal{C}$
(non-unique inverse of $\mathcal{D}$):

$\rho=\mathrm{diag}(\boldsymbol{p}) \xrightarrow{\mathcal{C}} \rho^{\mathcal{C}}$ with $\langle j|\rho|j\rangle=p_j$

One can always completely coherify $\boldsymbol{p}$:

$\mathrm{diag}(\boldsymbol{p}) \xrightarrow{\mathcal{C}} |\psi\rangle\langle\psi|$ with $|\psi\rangle=\sum_j\sqrt{p_j}e^{i\phi_j}|j\rangle$

$C_{\mathrm{e}}(|\psi\rangle\langle\psi|)=S(\boldsymbol{p})$, $C_2(|\psi\rangle\langle\psi|)=1-\boldsymbol{p}\cdot\boldsymbol{p}$

### Coherence of quantum channels

Choi-Jamiołkowski isomorphism: $J_\Phi=\frac{1}{d}(\Phi\otimes\mathcal{I})\,|\Omega\rangle\langle\Omega|$, $|\Omega\rangle=\sum_j|jj\rangle$

Classical action on diagonal: $\langle jk|J_\Phi|jk\rangle=\frac{1}{d}T_{jk}$

Optimising coherence of $\Phi \Longleftrightarrow$ optimising $\boldsymbol{\lambda}(J_\Phi)$

Relating randomness of $\Phi$ with $\boldsymbol{\lambda}(J_\Phi)$:

$|\psi\rangle \xrightarrow{\Phi} \frac{1}{\sqrt{q_j}}K_j|\psi\rangle$ with probability $q_j$,

$\Phi(\cdot)=\sum_j K_j(\cdot)K_j^\dagger$, $q_j=\mathrm{Tr}\left(K_j\,|\psi\rangle\langle\psi|\,K_j^\dagger\right)$

Path probability averaged over all pure states: $\langle q_j\rangle_\psi=\lambda_j(J_\Phi)$

## Coherifying quantum channels

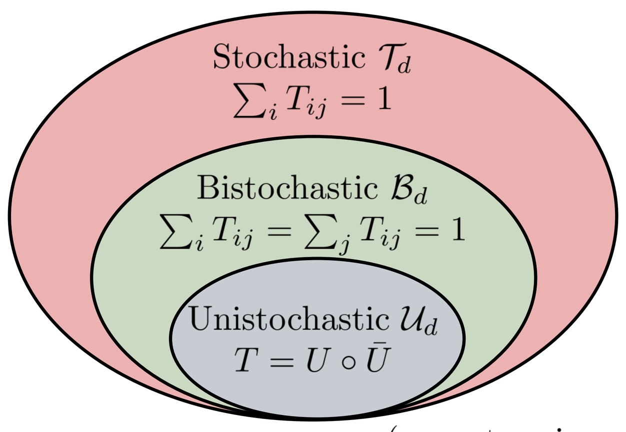
### Limitations on perfect coherification

Can one always completely coherify $T$?

$T \xrightarrow{\mathcal{C}} |\psi\rangle\langle\psi|$ with $|\psi\rangle=\frac{1}{\sqrt{d}}\sum_{j,k}\sqrt{T_{jk}}e^{i\phi_{jk}}|jk\rangle$

No! TP condition requires $\mathrm{Tr}_1\left(|\psi\rangle\langle\psi|\right)=\frac{\mathbb{1}}{d}$

Types of classical action:




($\circ$ - entrywise product)

$\Phi$ can be competely coherified $\Leftrightarrow$ $T$ is unistochastic

### Coherification upper-bound

Look for $\boldsymbol{\mu}^{\succ}(T)$ s. t. $\forall J_\Phi:\ \boldsymbol{\mu}^{\succ}(T)\succ\boldsymbol{\lambda}(J_\Phi)$

$\boldsymbol{\mu}^{\succ}$ yields upper bounds for $C_{\mathrm{e}}(J_\Phi)$ and $C_2(J_\Phi)$

Procedure to obtain upper-bounding $\boldsymbol{\mu}^{\succ}(T)$:

$$T=\begin{bmatrix}0.7 & 0.2 & 0.6\\ 0.1 & 0.6 & 0.4\\ 0.2 & 0.2 & 0\end{bmatrix}\xrightarrow{\text{sum over columns}}\begin{bmatrix}1.5\\ 1.1\\ 0.4\end{bmatrix}\xrightarrow{\text{distribute into parts}\leq 1}\begin{bmatrix}1 & 0.5 & 0\\ 1 & 0.1 & 0\\ 0.4 & 0 & 0\end{bmatrix}\xrightarrow{\text{sum over columns and normalize}}$$

$$\boldsymbol{\mu}^{\succ}(T)=\left[0.8,0.2,0\right]$$

### Coherification lower-bound

Look for $\boldsymbol{\mu}^{\prec}(T)$ s. t. $\exists J_\Phi:\ \boldsymbol{\mu}^{\prec}(T)\prec\boldsymbol{\lambda}(J_\Phi)$

$\boldsymbol{\mu}^{\prec}$ yields lower bounds for $C_{\mathrm{e}}(J_\Phi)$ and $C_2(J_\Phi)$

Procedure to obtain lower-bounding $\boldsymbol{\mu}^{\prec}(T)$:

$$T=\begin{bmatrix}0.7 & 0.2 & 0.6\\ 0.1 & 0.6 & 0.4\\ 0.2 & 0.2 & 0\end{bmatrix}\xrightarrow[\text{within rows}]{\text{order}}\begin{bmatrix}0.7 & 0.6 & 0.2\\ 0.6 & 0.4 & 0.1\\ 0.2 & 0.2 & 0\end{bmatrix}\xrightarrow{\text{sum over rows and normalize}}$$

$$\boldsymbol{\mu}^{\prec}(T)=\left[0.5,0.4,0.1\right]$$

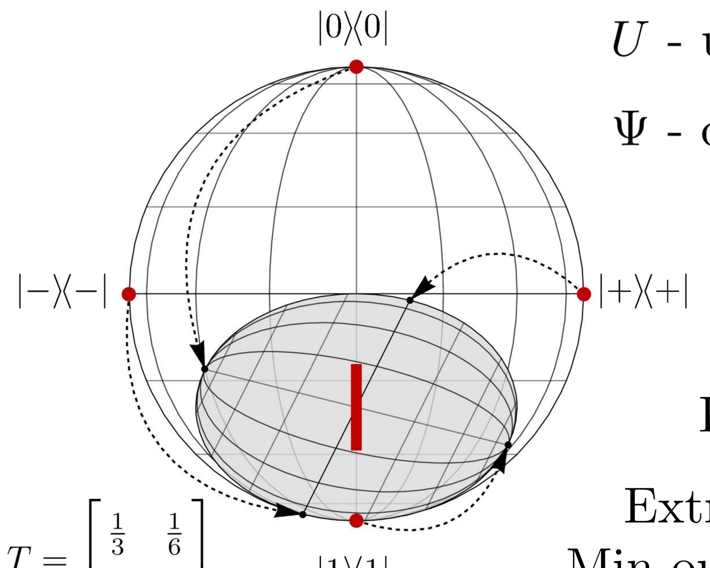
## Example: qubit channel

Classical action: $T=\begin{bmatrix}a & 1-b\\ 1-a & b\end{bmatrix}$

Optimal coherification: $\Phi^{\mathcal{C}}(\cdot)=\Psi(U(\cdot)U^\dagger)$

$U$ - unitary channel,

$\Psi$ - decaying channel




$T=\begin{bmatrix}\frac{1}{3} & \frac{1}{6}\\ \frac{2}{3} & \frac{5}{6}\end{bmatrix}$

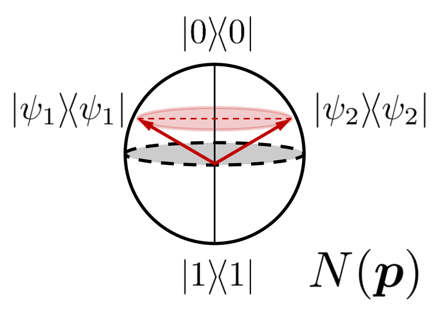
Properties:

Extremal channel
Min output entropy=0

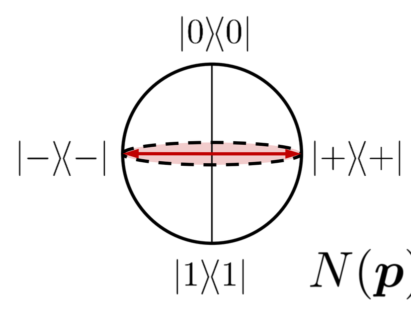
## Number of distinct coherifications

### Quantum states

What is the number $N(\boldsymbol{p})$ of perfectly distinguishable states with classical version $\boldsymbol{p}$?



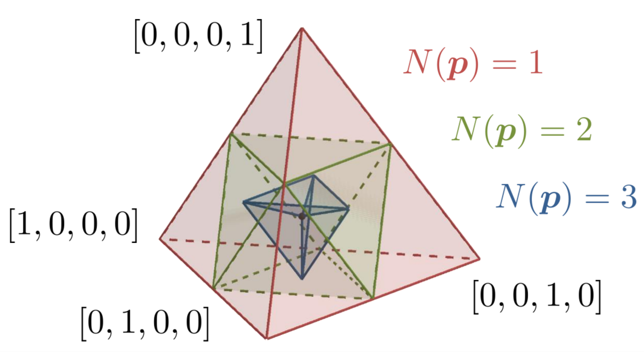

$N(\boldsymbol{p})=1$




$N(\boldsymbol{p})=2$

Simple bound: $N(\boldsymbol{p})\leq\dfrac{1}{\max_i p_i}$

But things get more complicated. E.g. for $d=4$:




### Quantum channels

What is the number $N(T)$ of perfectly distinguishable channels with classical version $T$?

$1\leq N(T)\leq d^2$, both limits achieveable

| Classical action | $N(T)$ | Requires entanglement? |
|---|---|---|
| Unistochastic | $d$ | No |
| Unistochastic | $d+1,\dots,d^2$ | Yes |
| Bistochastic | 2 | Yes |
| Circulant | $d$ | No |
| S.t. $T_{ij}\leq\frac{1}{2}$ | 2 | No |

More soon on arXiv! Look for: *Distinguishing classically indistinguishable states and channels*

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