Finite-size effects in quantum thermodynamics

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TEAM-NET

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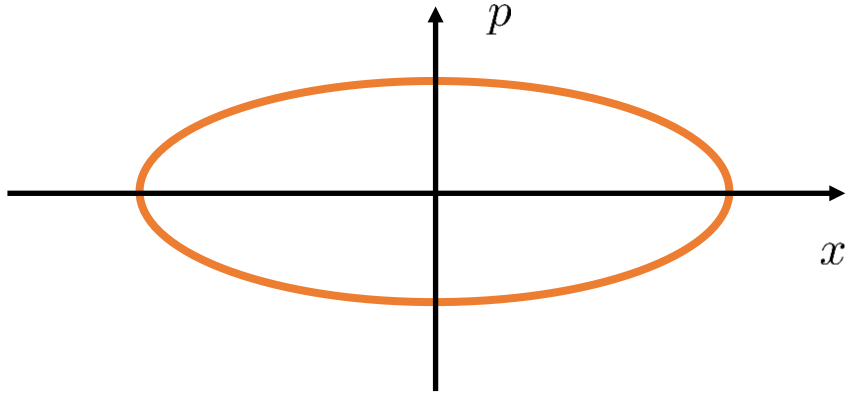
Motivation & background

Motivation

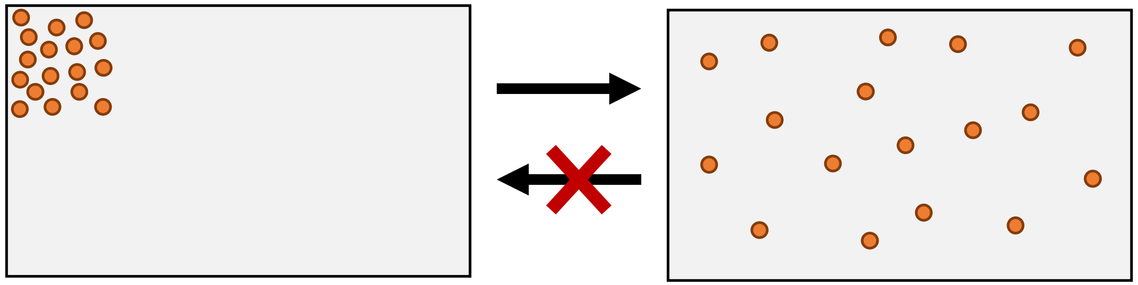
What can we say about the dynamics without solving equations of motion?

Closed systems



Energy conservation

Open systems



Entropy growth

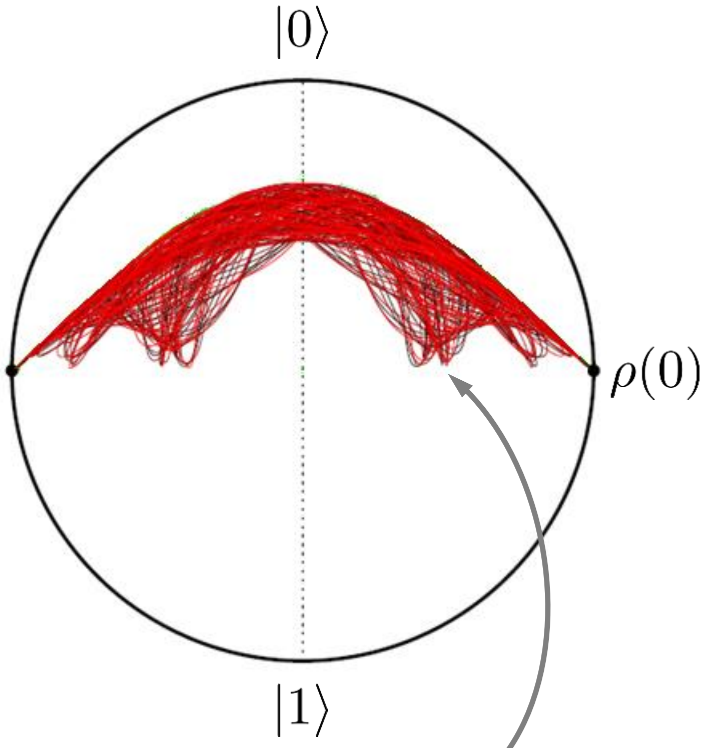
Quantum thermodynamics:

Using minimal assumptions of the quantum theory, find constraints on the evolution of a quantum system interacting with thermal baths

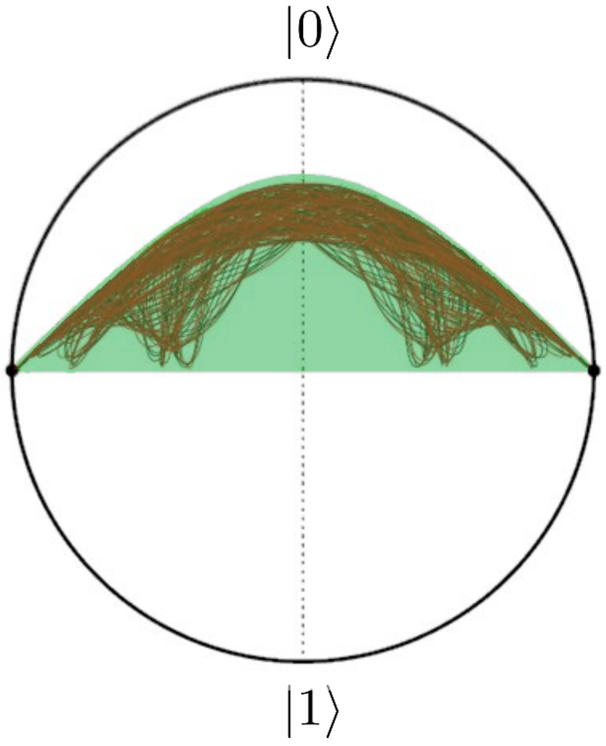
Motivation

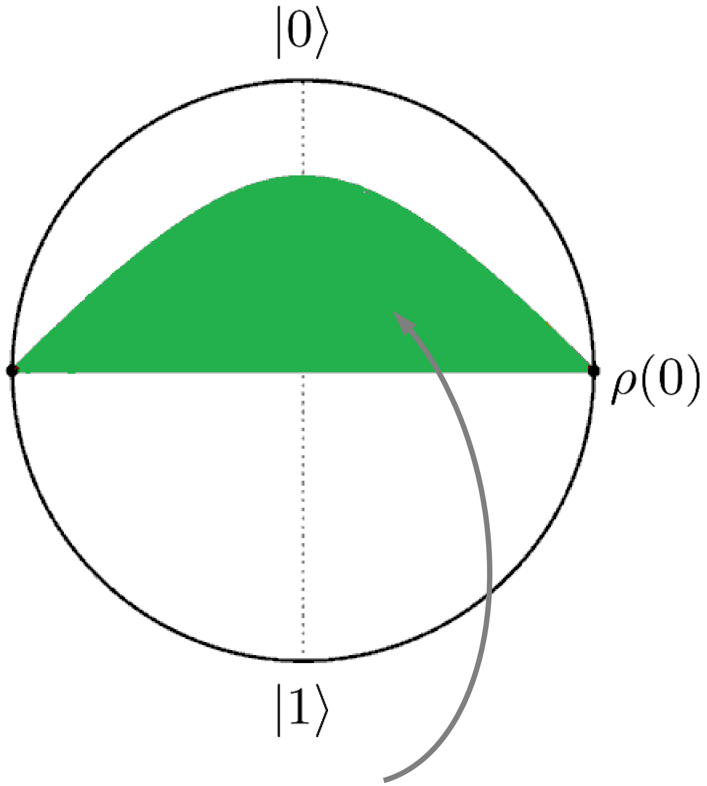
Open dynamics approach:

Resource-theoretic approach:



Exact time evolution
for a given model



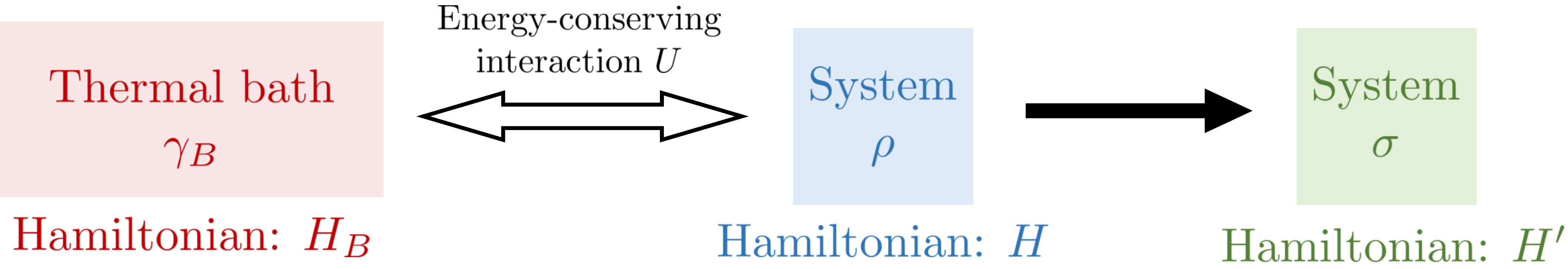


Allowed final states
compatible with the laws
of thermodynamics

Thermodynamic setting

Thermodynamic transformations modelled by **thermal operations***:

$$\mathcal{E}^{T}(\cdot) = \mathrm{Tr}_{B'}\left(U\left(\cdot \otimes \gamma_B\right) U^{\dagger}\right) \quad \text{with} \quad [U, H + H_B] = 0$$



Gibbs state γ of the system at temperature T: $\gamma = e^{-\frac{H}{T}}/\mathcal{Z}, \quad \mathcal{Z} = \mathrm{Tr}\left(e^{-\frac{H}{T}}\right)$

Note: all results with units such that $k_B = 1$.

*M. Horodecki, J. Oppenheim Nature Commun. **4**, 2059 (2013)

State interconversion and related problems

State interconversion: Initial state ρ, target state σ, background temperature T

Single-shot interconversion: Does there exist $\mathcal{E}^T$ such that $\mathcal{E}^T(\rho) = \sigma$?

Many-copies interconversion: (large but finite n) Does there exist $\mathcal{E}^T$ such that $\mathcal{E}^T\left(\rho^{\otimes n}\right) \approx_\epsilon \sigma^{\otimes R_n n}$?

Optimal rate R_n for error ϵ?

Incoherent interconversion: $[\rho, H] = [\sigma, H'] = 0$

(states represented by: $\boldsymbol{p} = \mathrm{eig}(\rho)$, $\boldsymbol{q} = \mathrm{eig}(\sigma)$)

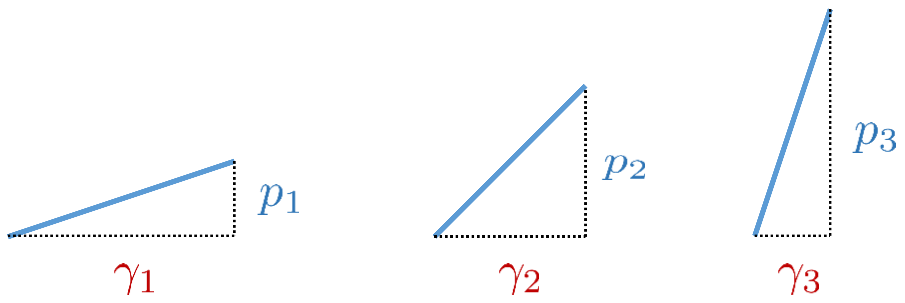
$[\gamma, H] = 0$

(thermal state represented by: $\boldsymbol{\gamma} = \mathrm{eig}(\gamma)$)

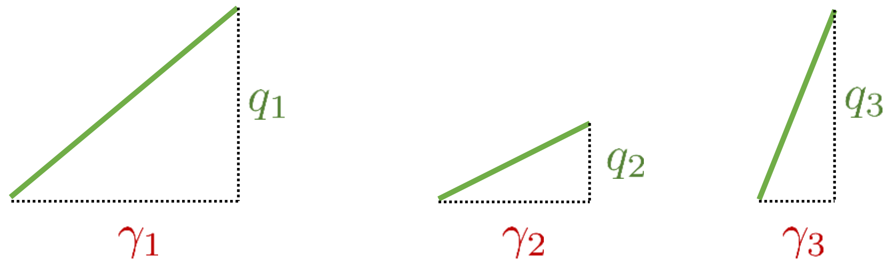
State interconversion and related problems

Incoherent interconversion completely described by **thermomajorisation***:

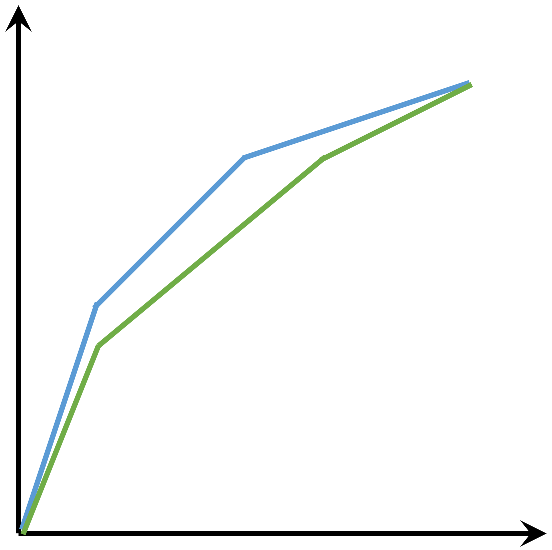
Lorenz curve segments for the initial state $\boldsymbol{p}$:



Lorenz curve segments for the target state $\boldsymbol{q}$:



Form convex Lorenz curves



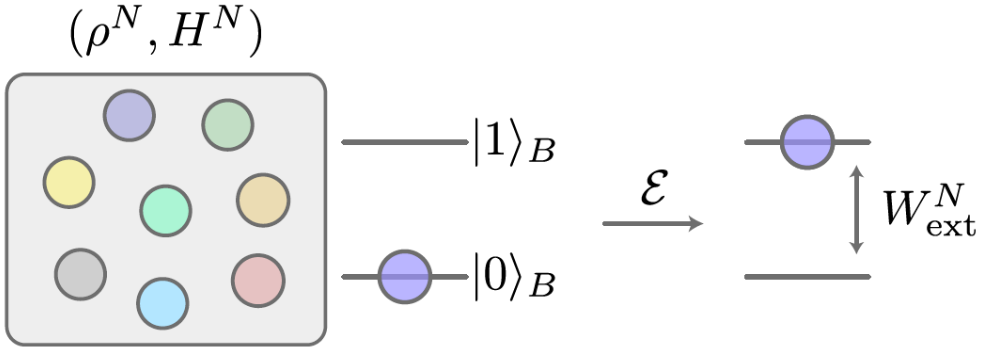
Interconversion possible iff the initial curve is always above the target curve

*M. Horodecki, J. Oppenheim Nature Commun. **4**, 2059 (2013)

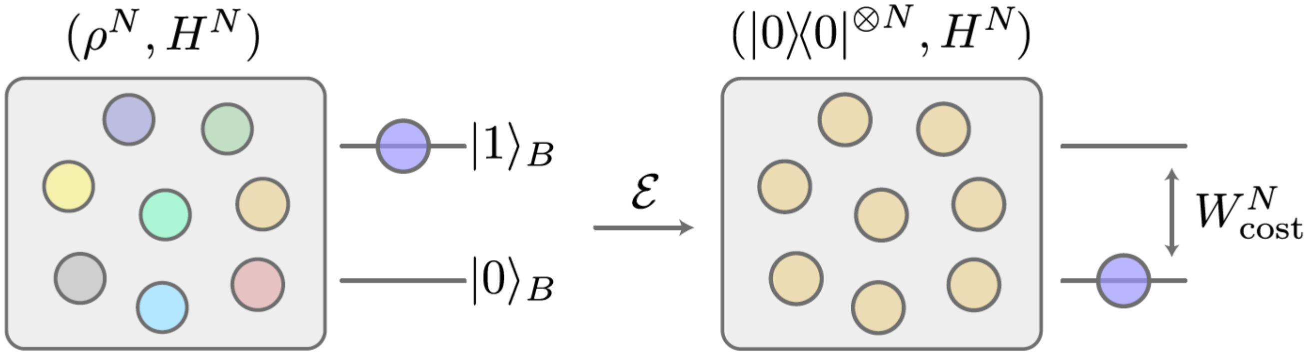
State interconversion and related problems

Thermodynamic protocols are various instances of state interconversion problem

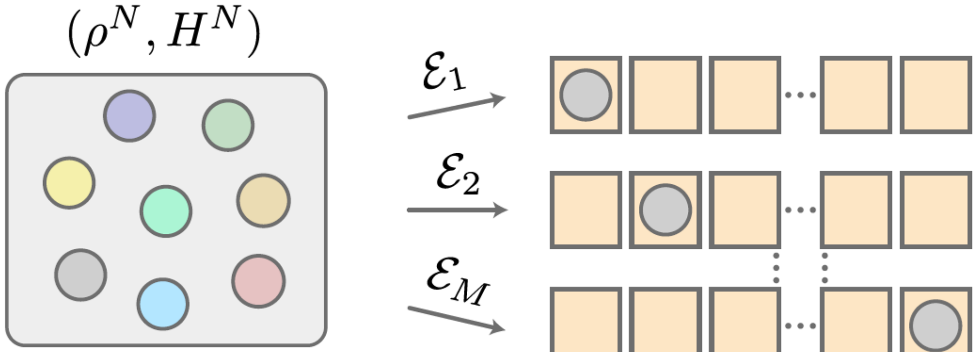
Work extraction



Information erasure



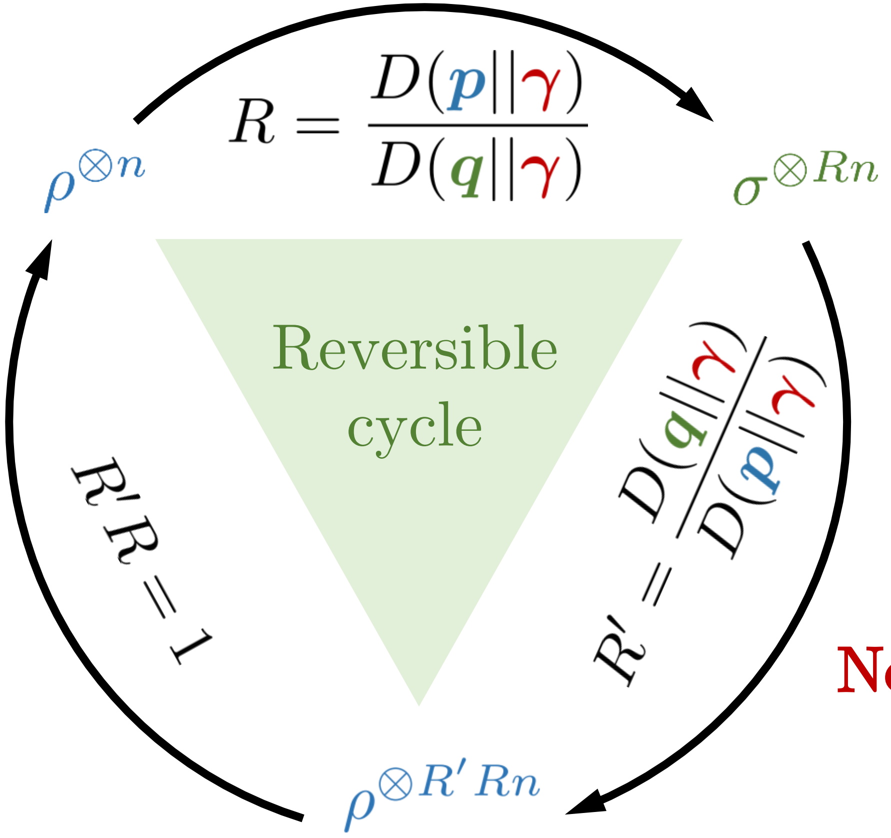
Thermodynamically-free communication



Results on incoherent thermodynamics

Asymptotic reversibility

Asymptotic rate for $n \to \infty$*: $$R_\infty(\boldsymbol{p} \to \boldsymbol{q}) = \frac{D(\boldsymbol{p}\|\boldsymbol{\gamma})}{D(\boldsymbol{q}\|\boldsymbol{\gamma})}$$



Relative entropy: $$D(\boldsymbol{p}\|\boldsymbol{\gamma}) := \sum_{i=1}^{d} p_i \log \frac{p_i}{\gamma_i}$$

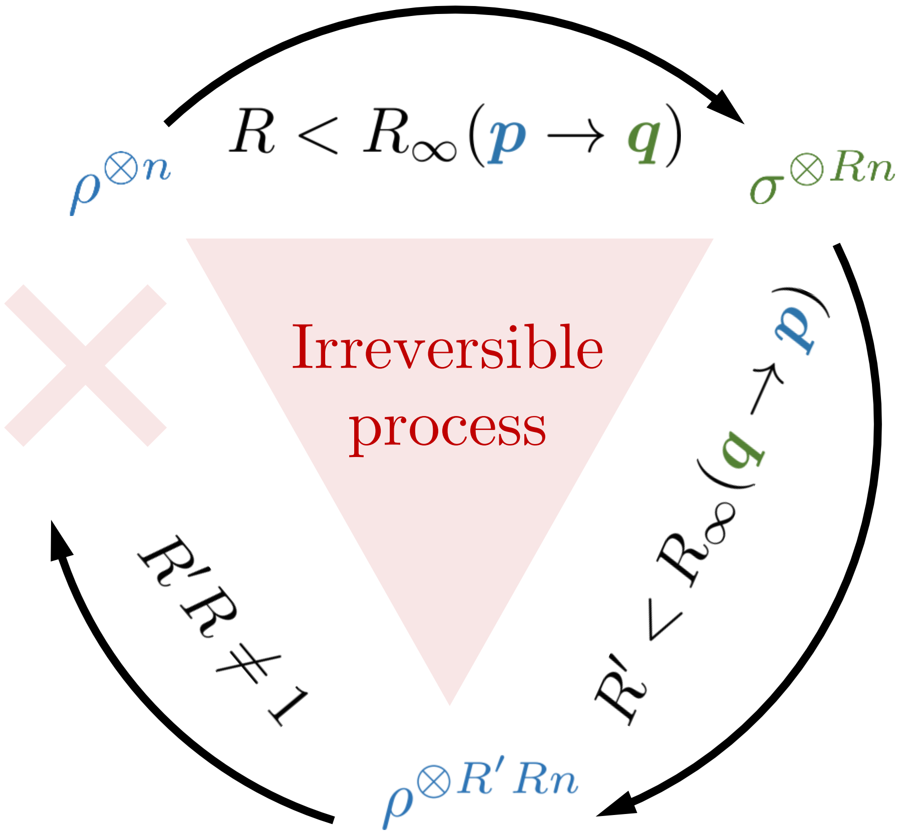
Physical interpretation: $$\frac{1}{T}\Big[\underbrace{\langle E\rangle_{\boldsymbol{p}} - TH(\boldsymbol{p})}_{\substack{\text{Free energy}\\ F = U - TS}} - \underbrace{(-T\log\mathcal{Z})}_{\text{Free energy of } \gamma}\Big]$$

No dissipation of free energy in the thermodynamic limit!

*F. Brandão *et al.*,
Phys. Rev. Lett. 111, 250404 (2013)

Finite-size irreversibility

Rate for large but finite n: $\quad R_n = R_\infty - f(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\gamma}, n, \epsilon)$



Relevant quantity quantifying irreversibility:

Relative entropy variance:

$$V(\boldsymbol{p}\|\boldsymbol{\gamma}) := \sum_{i=1}^{d} p_i \left(\log \frac{p_i}{\gamma_i} - D(\boldsymbol{p}\|\boldsymbol{\gamma})\right)^2$$

Physical interpretation:

$$V(\boldsymbol{\gamma}'\|\boldsymbol{\gamma}) = \underbrace{\frac{\partial \langle E \rangle_{\boldsymbol{\gamma}'}}{\partial T'}}_{\text{Specific heat capacity}} \cdot \underbrace{\left(1 - \frac{T'}{T}\right)^2}_{\text{Carnot factor}}$$

Quantum **2**, 108 (2018)

Finite-size irreversibility

Optimal conversion rate R_n with constant error ϵ:

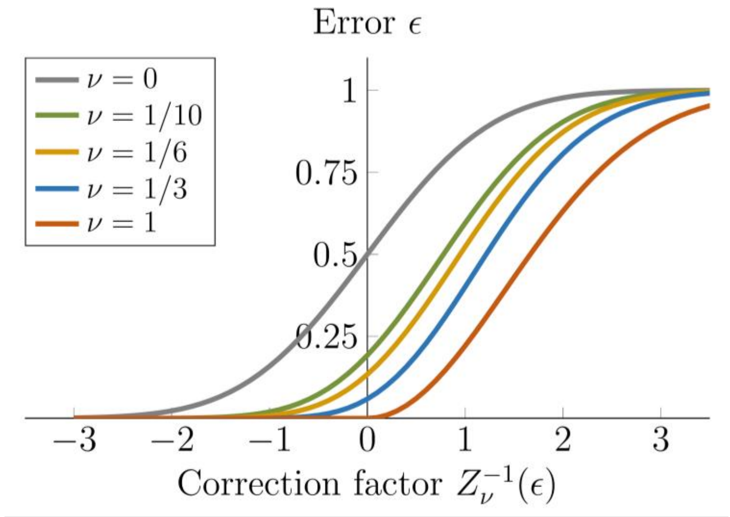
$$R_n(\epsilon) = R_\infty + \sqrt{\frac{V(\mathbf{p}\|\boldsymbol{\gamma})}{D(\mathbf{q}\|\boldsymbol{\gamma})^2}}\,\frac{Z_\nu^{-1}(\epsilon)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)$$

Reversibility parameter:

$$\nu = \frac{V(\mathbf{q}\|\boldsymbol{\gamma})/D(\mathbf{q}\|\boldsymbol{\gamma})}{V(\mathbf{p}\|\boldsymbol{\gamma})/D(\mathbf{p}\|\boldsymbol{\gamma})}$$

Rayleigh-normal distribution Z_ν*:

Quantum **2**, 108 (2018)



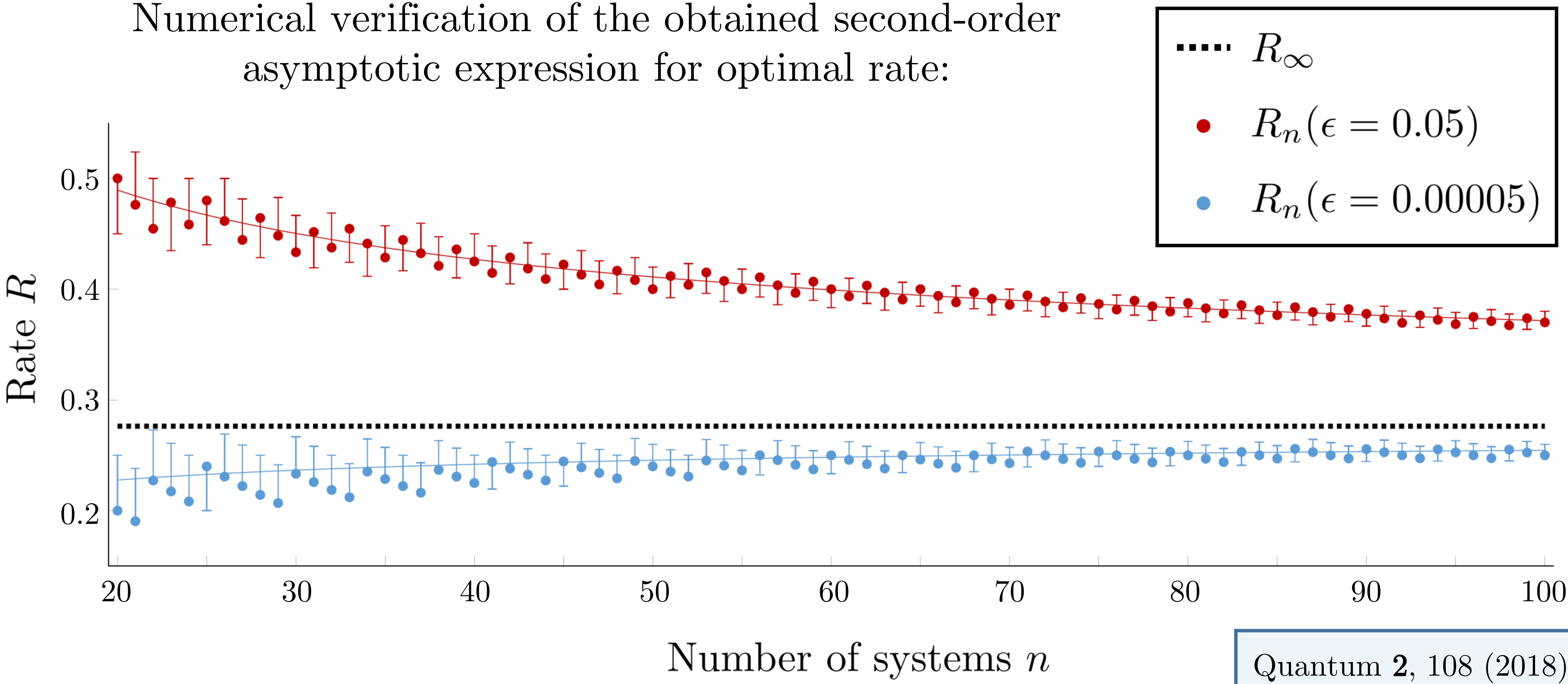
Z_0 - standard normal distribution Φ

Z_1 - Rayleigh distribution ($Z_1(x) = 0$ for $x \leq 0$)

*W. Kumagai *et al.*, IEEE Trans. Inf. Theory **63**, 1829–1857 (2017)

Finite-size irreversibility

Numerical verification of the obtained second-order asymptotic expression for optimal rate:



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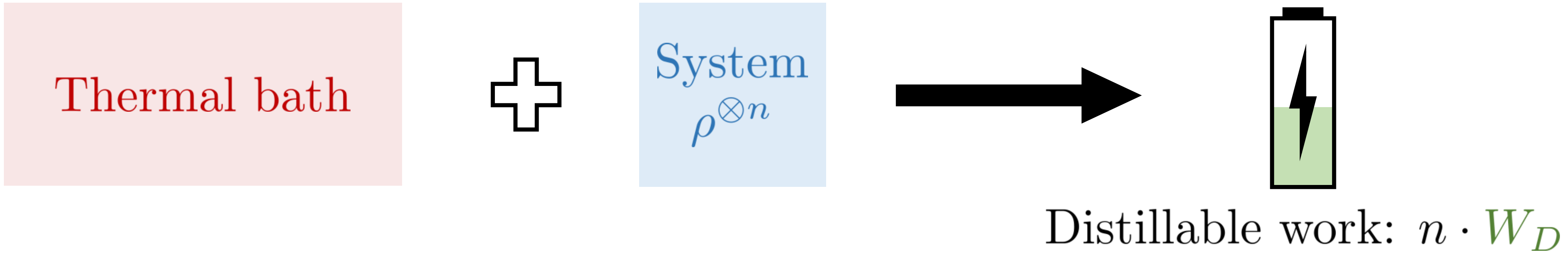
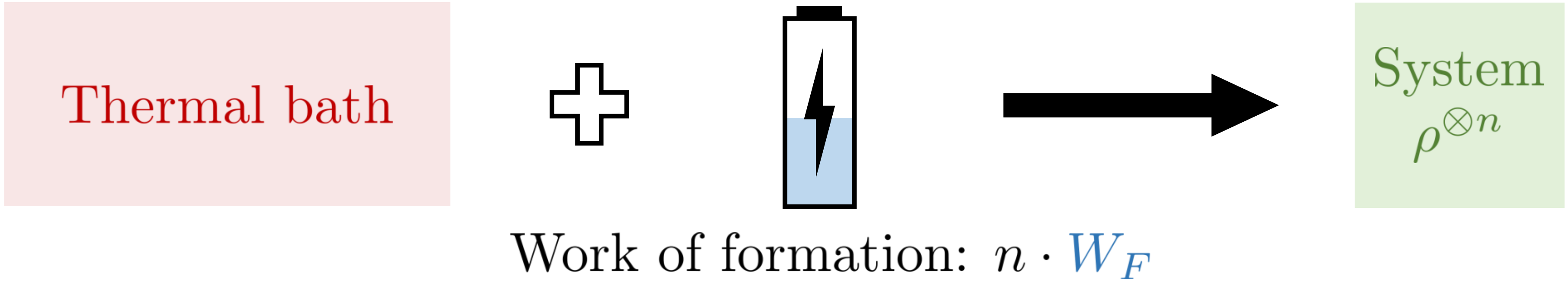
Finite-size irreversibility

Effects of finite-size irreversibility on work distillation and dilution processes:

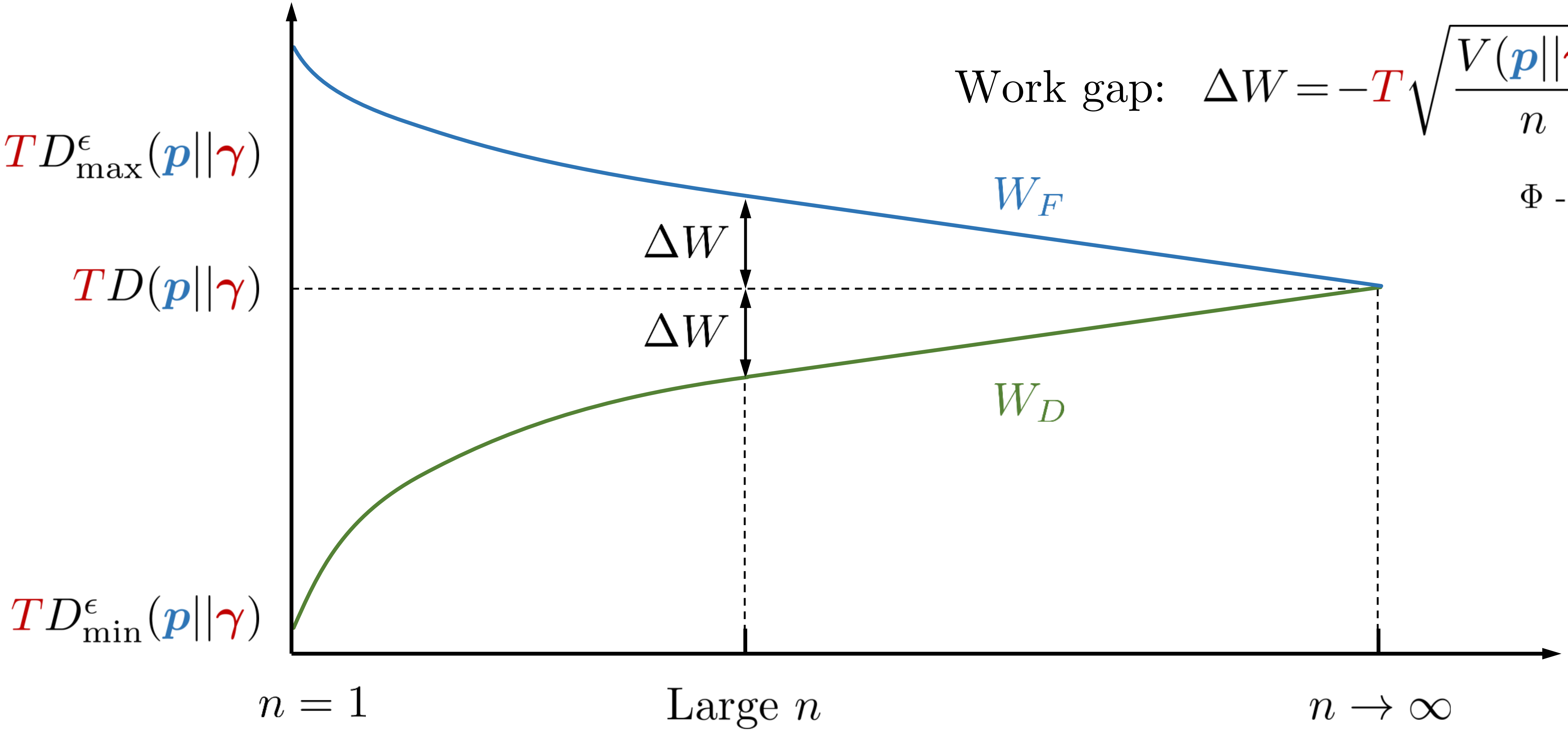
Work distillation process:



Work dilution process:



Finite-size irreversibility



Work gap: $\Delta W = -T\sqrt{\frac{V(\boldsymbol{p}||\boldsymbol{\gamma})}{n}}\Phi^{-1}(\epsilon)$

Φ - Normal cdf

Dissipation of free energy beyond the thermodynamic limit!

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Resource resonance

Optimal conversion rate R_n with vanishing error $\epsilon = e^{-n^\alpha}$ and $\alpha \in (0,1)$:

$$R_n(\epsilon) = R_\infty - \sqrt{\frac{V(\boldsymbol{p}\|\gamma)}{D(\boldsymbol{q}\|\gamma)^2}} \frac{\left|\sqrt{1/\nu} - 1\right|}{\sqrt{n^{1-\alpha}}} + o\left(\frac{1}{\sqrt{n^{1-\alpha}}}\right)$$

When $\nu = 1$ correction term disappears for every error ϵ

$$\Downarrow$$

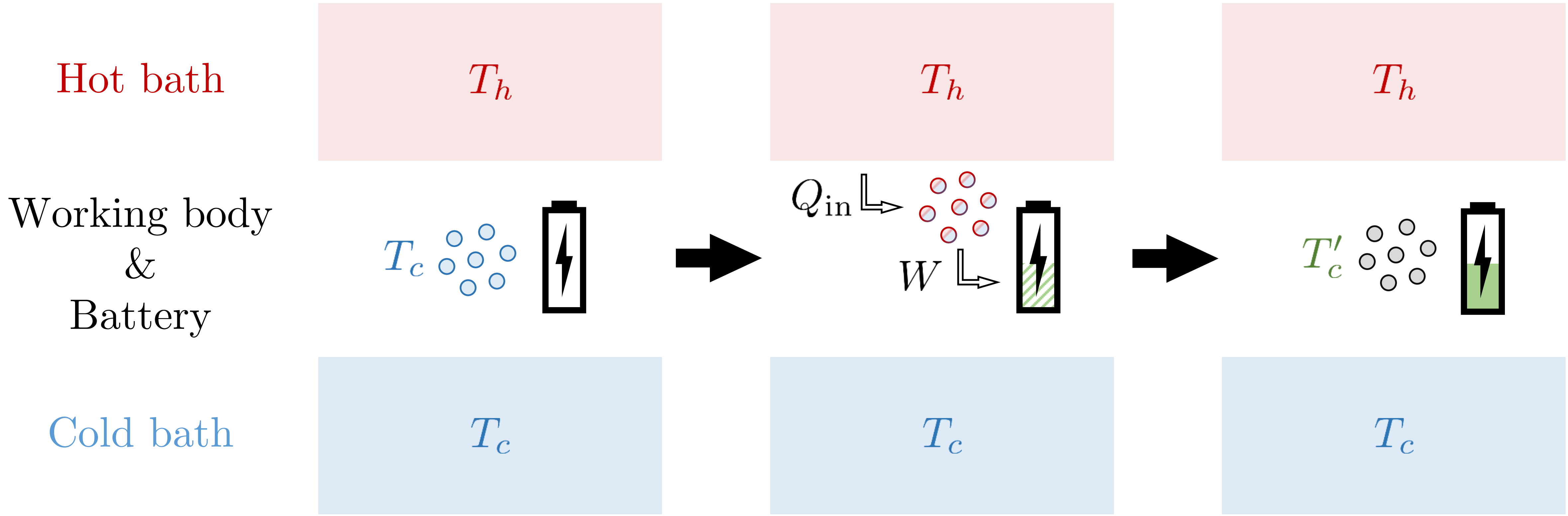
No free energy dissipation!
(at least up to second order asymptotics)

(recall that $\nu = 1$ means that the relative fluctuations of free energy are the same for the initial state ρ and target state σ)

Phys. Rev. A **99**, 032332 (2019)

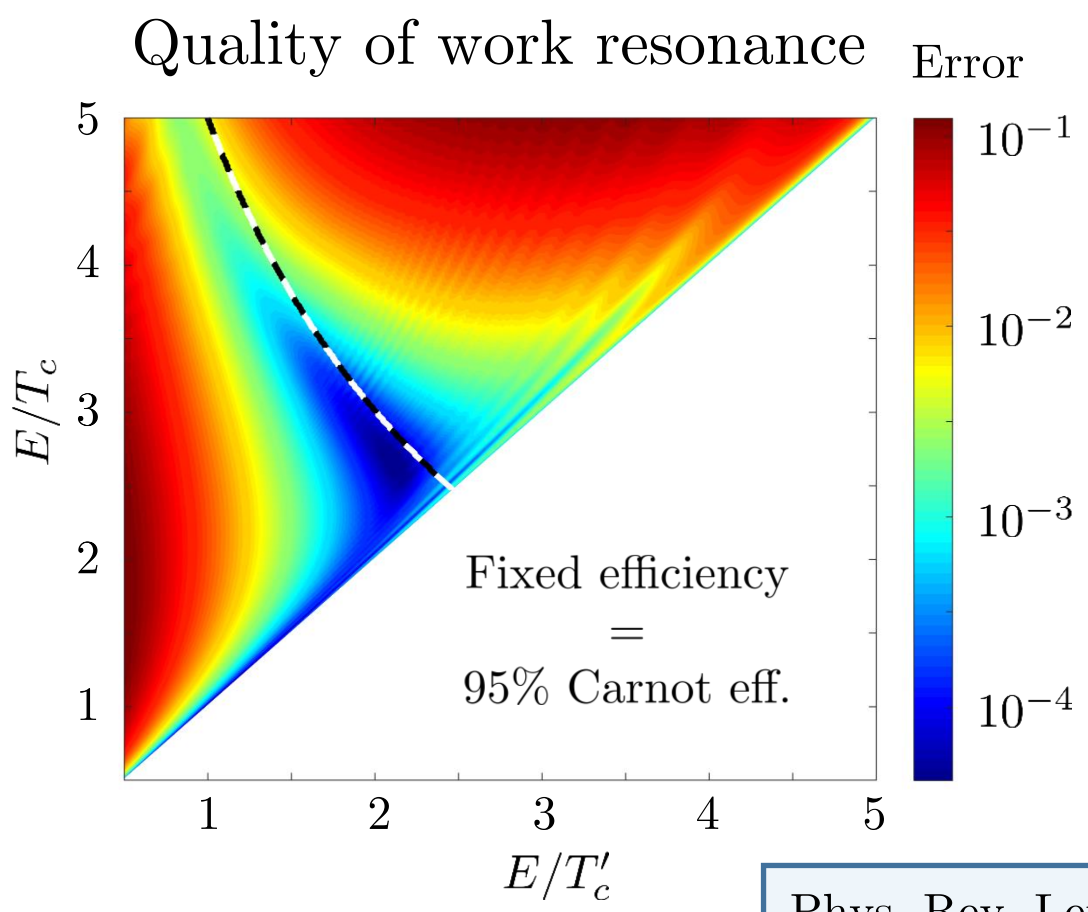
Resource resonance

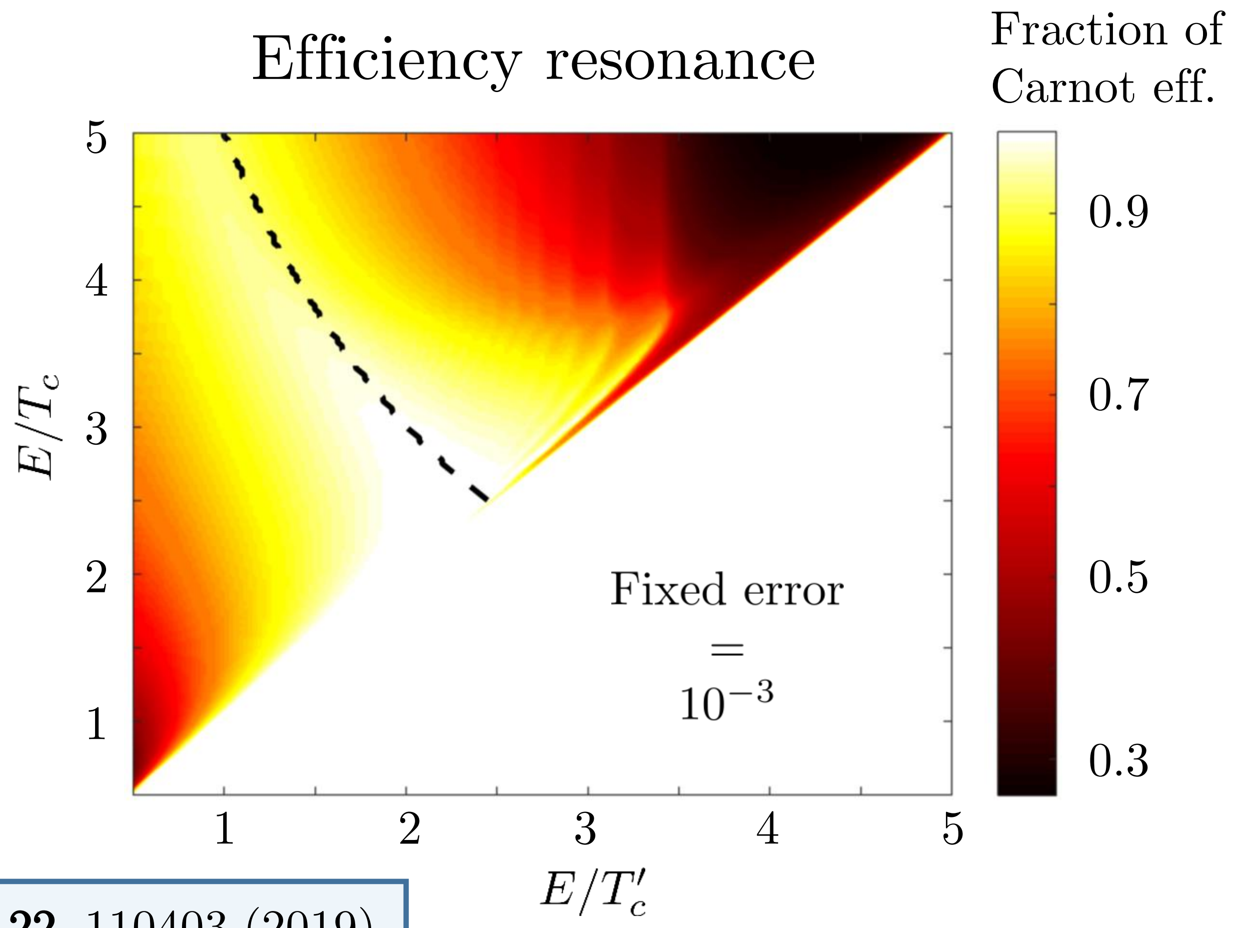
Resonance example: Heat engine with a finite-size working body:



Resource resonance

Working body: $n = 200$ qubits, energy gap E
Background (hot) bath: $T_h = 10E$





Phys. Rev. Lett. **122**, 110403 (2019)

Results on coherent thermodynamics

Fluctuation-dissipation relations

Thermodynamic distillation process

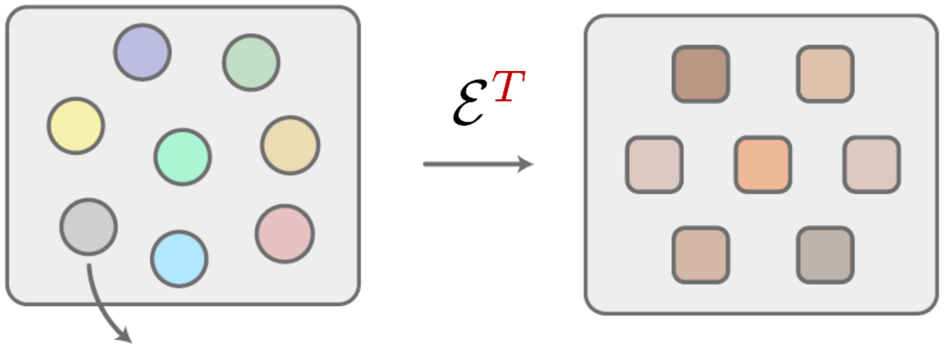
Non-zero free energy:

$$F^N := \frac{1}{\beta}\sum_{n=1}^{N} D(\rho_n^N \| \gamma_n^N)$$

Non-zero free energy fluctuations:

$$\sigma^2(F^N) := \frac{1}{\beta^2}\sum_{n=1}^{N} V(\rho_n^N \| \gamma_n^N)$$

(ρ^N, H^N) $\qquad$ $(\tilde{\rho}^N, \tilde{H}^N)$



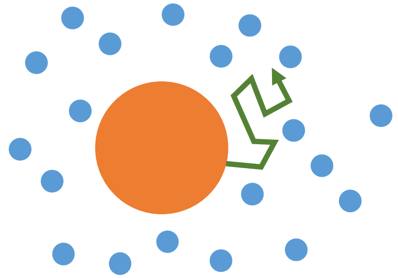
(ρ_n^N, H_n^N) $\qquad$ $\tilde{\rho}^N = |\tilde{E}_k^N\rangle\langle\tilde{E}_k^N|$

Non-zero free energy, but vanishing free energy fluctuations

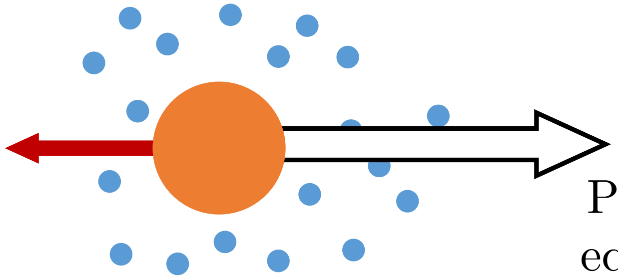
Free energy fluctuations $\overset{?}{\sim}$ Free energy dissipated in the process

Einstein-Smoluchowski relation for a Brownian particle:

In thermal equilibrium



Position fluctuations $\sim$ Dissipative friction



Pull out of equilibrium

Fluctuation-dissipation relations

Optimal error in thermodynamic distillation process:

$$\lim_{N\to\infty} \epsilon_N = \lim_{N\to\infty} \Phi\left(-\frac{\Delta F^N}{\sigma(F^N)}\right)$$

ΔF^N - Free energy difference between initial and **target** state

Minimal amount of free energy dissipated in the optimal distillation process:

$$F^N_{\text{diss}} \simeq a(\epsilon_N)\sigma(F^N)$$

F^N_{diss} - Free energy difference between initial and **final** state

$$a(\epsilon) = -\Phi^{-1}(\epsilon)(1-\epsilon) + \exp(-[\Phi^{-1}(\epsilon)]^2/2)/\sqrt{2\pi}$$

Three regimes:

$$\lim_{N\to\infty} \frac{\Delta F^N}{\sqrt{N}} = \begin{cases} \infty, & \Longrightarrow \quad \epsilon = 0, \quad F^N_{\text{diss}} = \Delta F^N \\ -\infty, & \Longrightarrow \quad \epsilon = 1, \quad F^N_{\text{diss}} = 0 \\ \alpha \in \mathbb{R} & \end{cases}$$

Also holds for initial pure states with coherence!

Phys. Rev. E **105**, 054127 (2022)

Converting coherent to incoherent states

Consider a coherent qubit state: $\rho = \begin{pmatrix} p & c \\ c^* & 1-p \end{pmatrix}$

Then, dephasing many copies means:

Incoherent state

$$\rho^{\otimes 3} = \begin{pmatrix} \boxed{k=0} & & & \\ & \boxed{k=1} & & \\ & & \boxed{k=2} & \\ & & & \boxed{k=3} \end{pmatrix} \xrightarrow[\text{energy-preserving unitaries}]{\text{Each block can be diagonalised with}} \begin{pmatrix} \lambda_0^1 & & & & & & & \\ & \lambda_1^1 & & & & & & \\ & & \lambda_1^2 & & & & & \\ & & & \lambda_1^3 & & & & \\ & & & & \lambda_2^1 & & & \\ & & & & & \lambda_2^2 & & \\ & & & & & & \lambda_2^3 & \\ & & & & & & & \lambda_3^1 \end{pmatrix} =: \boldsymbol{\lambda}$$

As $n \to \infty$ such dephasing pre-processing "kills" only $O(\log n)$ of free energy!

(proof using hypothesis testing approach to the interconversion problem)

In preparation (2023)

Converting coherent to incoherent states

Optimal conversion rate R_n with constant error ϵ:

$$R_n(\epsilon) = R_\infty + \sqrt{\frac{V(\rho\|\gamma)}{D(\sigma\|\gamma')^2}}\,\frac{S_\nu^{-1}(\epsilon)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)$$

Previous incoherent result

$$\left[R_n(\epsilon) = R_\infty + \sqrt{\frac{V(\mathbf{p}\|\boldsymbol{\gamma})}{D(\mathbf{q}\|\boldsymbol{\gamma})^2}}\,\frac{Z_\nu^{-1}(\epsilon)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right]$$

Optimal performance of thermodynamic protocols employing interference effects:

Extractable work: $$w \simeq \frac{1}{\beta}\left(D(\rho\|\gamma) + \sqrt{\frac{V(\rho\|\gamma)}{n}}\Phi^{-1}(\epsilon)\right)$$

Work cost of information erasure: $$w_{\text{cost}} \simeq \frac{1}{\beta}\left(S(\rho) - \sqrt{\frac{V(\rho)}{n}}\Phi^{-1}(\epsilon)\right)$$

Number of bits that can be communicated without a thermodynamic cost:

$$\frac{\log M(\rho^{\otimes n},\epsilon)}{n} \simeq D(\rho\|\gamma) + \sqrt{\frac{V(\rho\|\gamma)}{n}}\Phi^{-1}(\epsilon),$$

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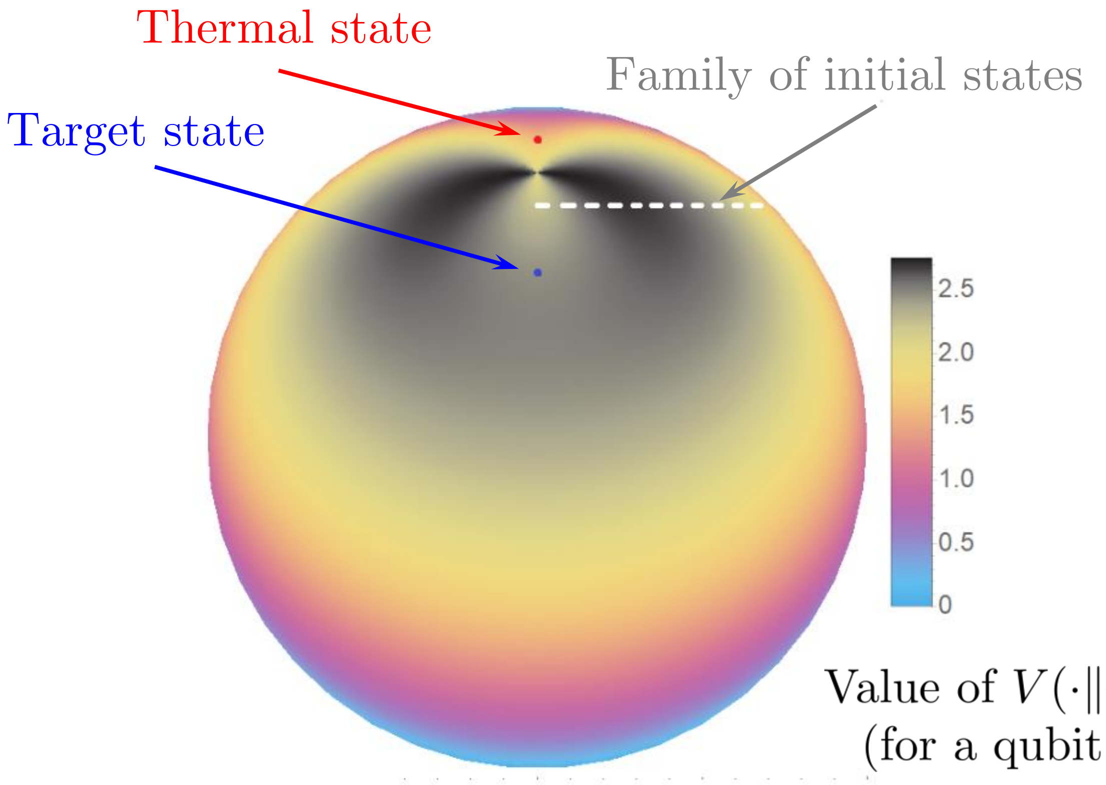
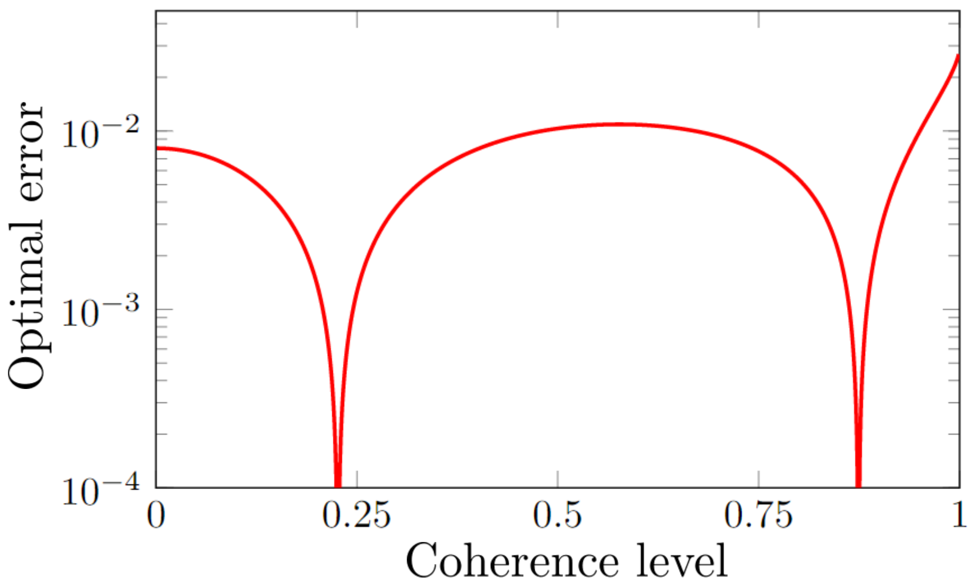
Converting coherent to incoherent states

Predicting coherent resonance phenomenon:



Value of $V(\cdot\|\gamma)/D(\cdot\|\gamma)$
(for a qubit system)

Transformation with the asymptotic rate



Recall reversibility parameter: $\nu = \dfrac{V(\sigma\|\gamma)/D(\sigma\|\gamma)}{V(\rho\|\gamma)/D(\rho\|\gamma)}$

In preparation (2023)

Outlook

- Extend finite-size analysis to other resource-theories (asymmetry, contextuality).
- Design experimental protocols employing the resonance phenomenon.
- Generalise the formalism to include target quantum states with coherence.
- Look for resonance phenomena in other quantum information processing tasks.
- Extend resource-theoretic fluctuation-dissipation theorem to continuous variable systems

Quantum **2**, 108 (2018)
Phys. Rev. A **99**, 032332 (2019)
Phys. Rev. Lett. **122**, 110403 (2019)
Phys. Rev. E **105**, 054127 (2022)
In preparation (2023)

Thank you!