TQG NOTES

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Contents

1. Recalls and conventions

Be very carefull when reading the document : there is a lot of subtils notations like K_γ^2 and K^2.

If you're lost, see the "What we are doing ?" section (page 5).

Sometime in the document we will use $\partial_t, \partial_x, \partial_y$ instead of $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ to avoid reading confusions. For the whole document we state here the writing convention in **Tab. 1.**,

	This document	Version 1	Version 2
Vector/Matrix	$\underline{u}$ / $\underline{\underline{A}}$	$\vec{u}$ / $\vec{\vec{A}}$	$\mathbf{u}$ / A
Divergence	$\mathbf{div}\ \underline{u}$	$\vec{\nabla}.\vec{u}$	$\nabla.\mathbf{u}$
Gradient	$\underline{\underline{\mathbf{grad}\ \underline{u}}}$	$\overline{\overline{\nabla \vec{u}}}$	$\nabla\mathbf{u}$
Curl	$\underline{\mathbf{curl}\ \underline{u}}$	$\vec{\nabla} \wedge \vec{u}$	$\nabla \times \mathbf{u}$
Laplacian	$\mathbf{div}\left(\underline{\underline{\mathbf{grad}\ \underline{u}}}\right)$	$\vec{\nabla} \cdot \left(\overline{\overline{\nabla \vec{u}}}\right)$	$\nabla \cdot (\nabla\mathbf{u})$

Tab. 1. Writing conventions for this document. Note that the identity matrix Id is written without the $\underline{\underline{...}}$ because it's a remarkable matrix.

For wave hypothesis we shall introduce a quantity λ that can be described with waves with $\omega = c.k$ and k, l the wave number of x, y directions,

$$\lambda = \widehat{\lambda}.\mathbf{exp}(i.(k.x + l.y - \omega.t))$$

Subsequently we can introduce the time derivative and also the space derivative of this function,

$$\begin{aligned}
\partial_t \lambda &= -i.\omega.\widehat{\lambda}.\mathbf{exp}(i.(k.x + l.y - \omega.t)) \\
\partial_x \lambda &= i.k.\widehat{\lambda}.\mathbf{exp}(i.(k.x + l.y - \omega.t)) \\
\partial_y \lambda &= i.l.\widehat{\lambda}.\mathbf{exp}(i.(k.x + l.y - \omega.t)) \\
\mathbf{div}(\underline{\mathbf{grad}\ \lambda}) &= (-k^2 - l^2).\mathbf{exp}(i.(k.x + l.y - \omega.t)) \\
&= -(k^2 + l^2).\mathbf{exp}(i.(k.x + l.y - \omega.t))
\end{aligned}$$

Generally we will state $\aleph = i.(k.x + l.y - \omega.t)$ that gives $\lambda = \widehat{\lambda}.\exp(\aleph)$. And when it's possible, normalise by $\exp(\aleph)$ to avoid reading confusions.

We note also that,

$$\lambda = \mathbf{Re}\left\{\widehat{\lambda}.\exp(\aleph)\right\}$$
$$||\widehat{\lambda}|| = ||\lambda||$$

We use the the notation **Id** to make appear the identity matrix of $n \times n$ size with $n \in \mathbb{N}$.

$$\mathbf{Id}_{\mathrm{n\times n}} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & 0 \\ 0 & & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{n\times n}$$

2. *Thermal Rotating Shallow Water (TRSW) model*

- History of the TRSW and TQG equations (primitive thermal model) : [Ripa, 1991] [Ripa, 1993] [Ripa, 1995]
- Other usefull articles : [Gouzien et al., 2017] ; the [Lahaye et al., 2020] solves and propose visualisation of some TQG solution.

The Thermal Shallow Water model can be founded in the Zeitlin's book [Zeitlin, 2018] (Chapter 14 page 409) and also Roullet's notes [Roullet, 2021] is defined by,

$$\boxed{\frac{\partial \underline{u}}{\partial t} + \underline{u}.\underline{\mathbf{grad}}\ \underline{u} = -\frac{g}{2.h}.\underline{\mathbf{grad}}(\Theta.h)} \tag{1a}$$
$$\boxed{\frac{\partial h}{\partial t} + \underline{u}.\mathbf{div}(h.\underline{u}) = 0} \tag{1b}$$
$$\boxed{\frac{\partial \Theta}{\partial t} + \underline{u}.\underline{\mathbf{grad}}\ \Theta = 0} \tag{1c}$$

Where the Temperature Θ modifies the "classical" version of the shallow water model. We note that in our case the conservation of the temperature over time $\frac{D\Theta}{Dt} = 0$ is true. We assume a 1 layer model where there is no variation of the velocity folowing z : $\frac{\partial u}{\partial z} = 0$ and so we get rid of the thermal wind balance.

We can cite also [Wang and Xu, 2024] that provides a good statement of the TRSW and TQG models. For a complete description we can see also [Warneford and Dellar, 2013].

3. *Thermal Quasi-Geostrophic (TQG) model*

The fundamental article that drives a QG analysis is [Flierl et al., 1987].

3.1. What we are doing ?

→ 1. From **page 6** to **page 10** we derive a linear model with a linear $\overline{\psi} = \overline{U}.y$.

2. Note that originally the system is 2D : $J(\psi, \lambda)$ is x, y dependent (λ is a parameter like q or Θ).

3. **The solving with a linear $\overline{\psi} = \overline{U}.y$ gives us a scalar solution of c.** There is no t, x, y dependency anymore because we divide by $i.k(x + y - c.t)$.

That's a linear TQG, scalar model, with a linear $\overline{\psi} = \overline{U}.y$

→ 1. From **page 19** to **page 24** we discretise the initial TQG problem into a 1D problem.

2. But we conserved the y dependency due to the non-linearity

That's a linear TQG 1D in y model, with a non-linear $\overline{\psi} = f(y)$

3.2. The system for $\overline{\psi} = -\overline{U}.y$ and $\overline{\Theta} = M^2.y$

We recall that the streamfunction ψ is defined as $\frac{\partial\psi}{\partial x} = v$ and $-\frac{\partial\psi}{\partial y} = u$. We consider the QG 1 layer Potential vorticity and it's conservation,

$$\boxed{\beta.y + \mathbf{div}(\underline{\mathbf{grad}\ \psi}) - \frac{\psi}{R_d^2} = q} \tag{2a}$$

$$\boxed{\frac{\partial q}{\partial t} + J(\psi, q) = 0} \tag{2b}$$

If we force the system with a parameter such like the buoyancy b or the Temperature Θ we can write a new PV conservation. We shall now introduce also the conservation of this quantity in the QG equations,

$$\boxed{\beta.y + \mathbf{div}(\underline{\mathbf{grad}\ \psi}) - \frac{\psi}{R_d^2} = q} \tag{3a}$$

$$\boxed{\frac{\partial q}{\partial t} + J(\psi, q - \Theta) = 0} \tag{3b}$$

$$\boxed{\frac{\partial \Theta}{\partial t} + J(\psi, \Theta) = 0} \tag{3c}$$

Here we don't introduce the temperature's dissipation for this model. And we note also that the Jacobian of a function $g(x, y)$ is equivalent to $J(\psi, g) = \underline{u}.\underline{\mathbf{grad}\ g} = \frac{\partial\psi}{\partial x}.\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}.\frac{\partial\psi}{\partial y}$.

3.2.1. Linearisation

We start by the linearisation of the previous set of equation. We introduce the quantities $q = \overline{q} + q'$, $\psi = \overline{\psi} + \psi'$ and $\Theta = \overline{\Theta} + \Theta'$. We recall that all time derivatives of means $\overline{q}, \overline{\psi}, \overline{\Theta}$ are equals to 0. Let's do it for the equation with Θ (it will be exactly the same developpement for the $q - \Theta$ equation). We shall now introduce $\overline{\psi} = -\overline{U}.y$, we assume $\overline{q} = \left(\frac{\overline{U}}{R_d^2} + \beta\right).y$ and a $\overline{\Theta} = M^2.y$ with $M^2 = \frac{\mathrm{d}\bar{b}}{\mathrm{d}y}$ or $\frac{\mathrm{d}\Theta}{dz}$ a constant. From (3c), we immediatly see that,

$$\frac{\partial(\overline{\Theta} + \Theta')}{\partial t} + J(\overline{\psi} + \psi', \overline{\Theta} + \Theta') = 0$$

$$\underbrace{\cancel{\frac{\partial\overline{\Theta}}{\partial t}}}_{0} + \frac{\partial\Theta'}{\partial t} + J(\overline{\psi}, \overline{q}) + J(\overline{\psi}, q') + J(\psi', \overline{q}) + \underbrace{\cancel{J(\psi', q')}}_{\text{Small}} = 0$$

> **Note also that**
>
> Taking into account that $\overline{\psi} = -\overline{U}.y$ and $\Theta = M^2.y$ we see that the Jacobian of the 2 means is 0 because,
>
> $$\begin{aligned} J(\overline{\psi}, \overline{\Theta}) &= \partial_x\overline{\psi}.\partial_y\overline{\Theta} - \partial_x\overline{\Theta}.\partial_y\overline{\psi} \\ &= \underbrace{\cancel{\partial_x(-\overline{U}.y)}}_{0}.\partial_y(M^2.y) - \underbrace{\cancel{\partial_x(M^2.y)}}_{0}.\partial_y(-U.y) \\ &= 0 \end{aligned}$$

So if we detail the equation above we get,

$$\begin{aligned} \frac{\partial\Theta'}{\partial t} + \underbrace{\cancel{J(\overline{\psi}, \overline{\Theta})}}_{0} + J(\overline{\psi}, \Theta') + J(\psi', \overline{\Theta}) + \underbrace{\cancel{J(\psi', \Theta')}}_{\text{Small}} = 0 \\ \frac{\partial\Theta'}{\partial t} + J(\overline{\psi}, \Theta') + J(\psi', \overline{\Theta}) = 0 \end{aligned}$$

The same result can be founded with the equation (3b) : $\frac{\partial q}{\partial t} + J(\psi, q - \Theta) = 0$ (we just have to replace Θ by $q - \Theta$ and as long as $\overline{\Theta}, \overline{q} \not\propto (t, x)$ but $\propto y$ there is no risk about doing that) and we get the linearised equations,

$$\boxed{\frac{\partial q'}{\partial t} + J(\overline{\psi}, q' - \Theta') + J(\psi', \overline{q} - \overline{\Theta}) = 0} \tag{4a}$$

$$\boxed{\frac{\partial\Theta'}{\partial t} + J(\overline{\psi}, \Theta') + J(\psi', \overline{\Theta}) = 0} \tag{4b}$$

3.2.2. Wave hypothesis

We will start the with 2 equation (4a) and (4b) to find the dispersion relation c of the system.

Step 1 : $q - \Theta$

$$\frac{\partial q'}{\partial t} + \cancelto{0}{\partial_x\overline{\psi}.\partial_y.(q'-\Theta')} - \partial_x(q'-\Theta').\partial_y.\overline{\psi} + \partial_x\psi'.\partial_y.(\overline{q}-\overline{\Theta}) - \cancelto{0}{\partial_x(\overline{q}-\overline{\Theta}).\partial_y\psi'} = 0$$

$$\frac{\partial q'}{\partial t} - \partial_x(q'-\Theta').\partial_y.\overline{\psi} + \partial_x\psi'.\partial_y.(\overline{q}-\overline{\Theta}) = 0$$

We replace some values with $q' = \mathbf{div}(\underline{\mathbf{grad}\ \psi'}) - \frac{\psi'}{R_d^2} + \beta.y$, that will allows us to make the wave hypothesis only on ψ and Θ,

$$\frac{\partial}{\partial t}.\left[\mathbf{div}(\underline{\mathbf{grad}\ \psi'}) - \frac{\psi'}{R_d^2} + \beta.y\right] - \partial_x\left(\left[\mathbf{div}(\underline{\mathbf{grad}\ \psi'}) - \frac{\psi'}{R_d^2} + \beta.y\right] - \Theta'\right).\partial_y.(-\overline{U}.y)$$
$$+\partial_x\psi'.\partial_y.\left(\left(\frac{\overline{U}}{R_d^2} + \beta\right).y - M^2.y\right) = 0$$

$$\frac{\partial}{\partial t}.\left[\mathbf{div}(\underline{\mathbf{grad}\ \psi'}) - \frac{\psi'}{R_d^2} + \beta.y\right] + \partial_x\left(\left[\mathbf{div}(\underline{\mathbf{grad}\ \psi'}) - \frac{\psi'}{R_d^2} + \beta.y\right] - \Theta'\right).\overline{U}$$
$$+\partial_x\psi'.\left(\frac{\overline{U}}{R_d^2} + \beta - M^2\right) = 0$$

We introduce the folowing wave-hypothesis : $\psi' = \widehat{\psi}'.\mathbf{exp}(\aleph)$ with $\aleph = i.(k.x + l.y - \omega.t)$, the same wave hypothesis can be done for Θ' (note that $c = \frac{\omega}{k}$), (we have divided by $\mathbf{exp}(\aleph)$ because it should have been too heavy to write) we note that,

$$-i.\omega.\left[-\left(k^2 + l^2 + \frac{1}{R_d^2}\right)\right].\widehat{\psi}' + i.k.\left(\left[-(k^2 + l^2 + \frac{1}{R_d^2}).\widehat{\psi}'\right] - i.k.\widehat{\Theta}'\right).\overline{U}$$
$$+i.k.\widehat{\psi}'.\left(\frac{\overline{U}}{R_d^2} + \beta - M^2\right) = 0 \quad || \times \frac{1}{i.k}$$

$$-c.\left[-\left(k^2 + l^2 + \frac{1}{R_d^2}\right)\right].\widehat{\psi}' - \left(k^2 + l^2 + \frac{1}{R_d^2}\right).\widehat{\psi}'.\overline{U} - \widehat{\Theta}'.\overline{U}$$
$$+\widehat{\psi}'.\left(\frac{\overline{U}}{R_d^2} + \beta - M^2\right) = 0$$

We introduce $K_\gamma^2 = k^2 + l^2 + \frac{1}{R_d^2}$ and $\alpha = \beta - M^2 + \frac{\overline{U}}{R_d^2}$,

$$-c.\left[-K_\gamma^2.\widehat{\psi}'\right] - K_\gamma^2.\widehat{\psi}'.\overline{U} - \widehat{\Theta}'.\overline{U} + \widehat{\psi}'.\alpha = 0$$
$$[-(U-c).K_\gamma^2 + \alpha].\widehat{\psi}' - \overline{U}.\widehat{\Theta}' = 0$$

Step 2 : Θ

$$\frac{\partial \Theta'}{\partial t} + \partial_x\overline{\psi}.\partial_y.\Theta' - \partial_x\Theta'.\partial_y.\overline{\psi} + \partial_x\psi'.\partial_y\overline{\Theta} - \partial_x\overline{\Theta}.\partial_y\psi' = 0$$

$$\frac{\partial \Theta'}{\partial t} + \cancelto{0}{\partial_x(-\overline{U}.y).\partial_y.\Theta'} - \partial_x\Theta'.\partial_y.(-\overline{U}.y) + \partial_x\psi'.\partial_y(M^2.y) - \cancelto{0}{\partial_x(M^2.y).\partial_y\psi'} = 0$$

$$\frac{\partial \Theta'}{\partial t} - \partial_x\Theta'.\partial_y.(-\overline{U}.y) + \partial_x\psi'.\partial_y(M^2.y) = 0$$

$$\frac{\partial \Theta'}{\partial t} + \partial_x\Theta'.\overline{U} + \partial_x\psi'.M^2 = 0$$

We introduce the folowing wave-hypothesis : $\psi' = \widehat{\psi}'.\mathbf{exp}(\aleph)$ with $\aleph = i.(k.x + l.y - \omega.t)$, the same wave hypothesis can be done for Θ' (note that $c = \frac{\omega}{k}$),

$$\begin{aligned}
\frac{\partial}{\partial t}.\Theta'.\mathbf{exp}(\aleph) + \partial_x.\Theta'.\mathbf{exp}(\aleph).\overline{U} + \partial_x\psi'.\mathbf{exp}(\aleph).M^2 &= 0 \\
-i.\omega.\widehat{\Theta}'.\mathbf{exp}(\aleph) + i.k.\widehat{\Theta}'.\mathbf{exp}(\aleph).\overline{U} + i.k.\widehat{\psi}'.\mathbf{exp}(\aleph).M^2 &= 0 \qquad \Bigg\| \times \frac{1}{i.k.\mathbf{exp}(\aleph)} \\
-\frac{\omega}{k}.\widehat{\Theta}' + \widehat{\Theta}'.\overline{U} + \widehat{\psi}'.M^2 &= 0 \\
(\overline{U} - c).\widehat{\Theta}' + M^2.\widehat{\psi}' &= 0
\end{aligned}$$

So the 2 equations are,

$$\boxed{\left[-(\overline{U} - c).K_\gamma^2 + \alpha\right].\widehat{\psi}' - \overline{U}.\widehat{\Theta}' = 0} \tag{5a}$$

$$\boxed{(\overline{U} - c).\widehat{\Theta}' + \widehat{\psi}'.M^2 = 0} \tag{5b}$$

3.2.3. Solving for $\Theta' = 0$

We suppose $\Theta = 0, \Theta' = 0, \overline{\Theta} = 0$, so the previous set of equations (5a) and (5b) becomes,

$$\begin{aligned}
\left[-(\overline{U} - c).K_\gamma^2 + \alpha\right].\widehat{\psi}' - \cancelto{0}{\overline{U}.\widehat{\Theta}'} &= 0 \\
\cancelto{0}{(\overline{U} - c).\widehat{\Theta}'} + \widehat{\psi}'.M^2 &= 0
\end{aligned}$$

ψ' can't be 0, so we deduce that $M^2 = 0$, we recall that $\alpha = \beta - \cancelto{0}{M^2} + \frac{\overline{U}}{R_d^2}$ and so,

$$\begin{aligned}
\left[-(\overline{U} - c).K_\gamma^2 + \alpha\right].\widehat{\psi}' - \cancelto{0}{\overline{U}.\widehat{\Theta}'} &= 0 \\
\left[-(\overline{U} - c).K_\gamma^2 + \beta + \frac{\overline{U}}{R_d^2}\right].\widehat{\psi}' &= 0 \\
-(\overline{U} - c).K_\gamma^2 &= -\left(\beta + \frac{\overline{U}}{R_d^2}\right)
\end{aligned}$$

Which is the dispersion relation for a neutral Rossby wave with $K_\gamma^2 = k^2 + l^2 + \frac{1}{R_d^2}$.

$$\boxed{c = \overline{U} - \frac{\left(\beta + \frac{\overline{U}}{R_d^2}\right)}{K_\gamma^2}} \tag{6}$$

3.2.4. Solving for $\Theta' \neq 0$

Now we restart from our system presented in (5a) and (5b) and if $\Theta, \Theta', \overline{\Theta} \neq 0$ we can solve a 2D matrix system (we recall that we set $\overline{\Theta} = M^2.y$). From equations,

$$\begin{aligned}\left[-(\overline{U}-c).K_\gamma^2+\alpha\right].\widehat{\psi}'-\overline{U}.\widehat{\Theta}'=0\\(\overline{U}-c).\widehat{\Theta}'+\widehat{\psi}'.M^2=0\end{aligned}$$

We get,

$$\underbrace{\begin{pmatrix}-(\overline{U}-c).K_\gamma^2+\alpha & -\overline{U}\\ M^2 & \overline{U}-c\end{pmatrix}}_{\underline{\underline{A}}}\times\underbrace{\begin{pmatrix}\widehat{\psi}'\\ \widehat{\Theta}'\end{pmatrix}}_{\underline{v}}=\begin{pmatrix}0\\0\end{pmatrix}$$

This system $\underline{\underline{A}} \times \underline{v} = \underline{0}$ can be solved by using the null-determinant method,

$$\begin{aligned}\textbf{det}\left(\underline{\underline{A}}\right)&=0\\ \left[-(\overline{U}-c).K_\gamma^2+\alpha\right].\left[\overline{U}-c\right]+M^2.\overline{U}&=0\\ -\left[\overline{U}^2-2.\overline{U}.c+c^2\right].K_\gamma^2+\alpha.\left[\overline{U}-c\right]+M^2.\overline{U}&=0\\ -K_\gamma^2(\overline{U}-c)^2+\alpha.\left[\overline{U}-c\right]+M^2.\overline{U}&=0\end{aligned}$$

The previous equation is a polynomial 2nd order equation : to make it more visible let us introduce $R = \overline{U} - c$, we see

$$\boxed{-K_\gamma^2.R^2+\alpha.R+M^2.\overline{U}=0} \tag{7}$$

The coefficients of the equation in $\underline{X} = \overline{U} - c$, are $k_1 = -K_\gamma^2, k_2 = \alpha, k_3 = M^2.\overline{U}$ so the discriminant is $\Delta = k_2^2 - 4.k_1.k_3$,

$$\Delta=\alpha^2+4.K_\gamma^2.M^2.\overline{U}>0$$

And the solutions are, with $\alpha = \beta - M^2 + \frac{\overline{U}}{R_d^2}$ and $K_\gamma^2 = k^2 + l^2 + \frac{1}{R_d^2}$

$$\begin{aligned}\overline{U}-c&=-\frac{b}{2.a}\pm\frac{\sqrt{\Delta}}{2.a}\\&=-\frac{\alpha}{-2.K_\gamma^2}\pm\frac{\sqrt{\alpha^2+4.K_\gamma^2.M^2.\overline{U}}}{-2.K_\gamma^2}\end{aligned}$$

Now we just expand the coefficients,

$$\boxed{\overline{U}-c=\frac{\beta-M^2+\frac{\overline{U}}{R_d^2}}{2.\left(k^2+l^2+\frac{1}{R_d^2}\right)}\pm\frac{\sqrt{\left(\beta-M^2+\frac{\overline{U}}{R_d^2}\right)^2+4.\left(k^2+l^2+\frac{1}{R_d^2}\right).M^2.\overline{U}}}{-2.\left(k^2+l^2+\frac{1}{R_d^2}\right)}} \tag{8}$$

3.2.5. Verification of the solution

We are now able to compare this result with the article [Beron-Vera, 2021] that derives the same kind of problem. To verify that we can set $U_\sigma = -\frac{M^2}{R_d^2}$ so $M^2 = -U_\sigma.R_d^2$ to translate our solution into their solution and note that $|\mathbf{k}|^2 = k^2 + l^2$.

$$
\begin{aligned}
\overline{U} - c &= \frac{\beta - M^2 + \frac{\overline{U}}{R_d^2}}{2.\left(k^2 + l^2 + \frac{1}{R_d^2}\right)} \pm \frac{\sqrt{\left(\beta - M^2 + \frac{\overline{U}}{R_d^2}\right)^2 + 4.\left(k^2 + l^2 + \frac{1}{R_d^2}\right).M^2.\overline{U}}}{-2.\left(k^2 + l^2 + \frac{1}{R_d^2}\right)} \\
&= \frac{\beta + U_\sigma.R_d^2 + \frac{\overline{U}}{R_d^2}}{2.\left(|\mathbf{k}|^2 + \frac{1}{R_d^2}\right)} \pm \frac{\sqrt{\left(\beta + U_\sigma.R_d^2 + \frac{\overline{U}}{R_d^2}\right)^2 - 4.\left(|\mathbf{k}|^2 + \frac{1}{R_d^2}\right).U_\sigma.\overline{U}.R_d^2}}{-2.\left(|\mathbf{k}|^2 + \frac{1}{R_d^2}\right)} \\
&= \frac{\beta.R_d^2 + U_\sigma + \overline{U}}{2.|\mathbf{k}|^2.R_d^2 + 2} \pm \frac{\sqrt{\left(\beta.R_d^2 + U_\sigma + \overline{U}\right)^2 - 4.\left(|\mathbf{k}|^2.R_d^2 + 1\right).U_\sigma.\overline{U}.R_d^2}}{-2.|\mathbf{k}|^2.R_d^2 + 2}
\end{aligned}
$$

After that we juste have to multiply by $\times -1$ to convert $c - \overline{U}$ in the article into $\overline{U} - c$ for us. **Both solutions are equivalent**.

À re-vérifier pour être bien sûr

3.2.6. Growth rate (I) when β is large

We know, from the equation (7), that the discriminant $\Delta = b^2 - 4.a.c$ can be written,

$$
\begin{aligned}
\Delta &= \alpha^2 + 4.K_\gamma^2.M^2.\overline{U} \\
&= \beta - T_0 + \frac{\overline{U}}{R_d^2} + 4.K_\gamma^2.M^2.\overline{U}
\end{aligned}
$$

That can be re-written with $\alpha = \beta - T_0 + \frac{\overline{U}}{R_d^2}$. We shall now assume that $\beta >>$ to other terms so we can simplify it. And assuming $M^2 = T_0$ we get

$$
\Delta = \beta^2 + 4.K_\gamma^2.T_0.\overline{U}
$$

A critical value is reached at $\Delta = 0$ so we get the following value of K^2,

$$
\boxed{{K_\gamma}_{\textbf{critical}}^2 = -\frac{\beta^2}{4.T_0.\overline{U}}} \tag{9}
$$

This last equation will be the mean state of K_γ^2 when assuming $K_\gamma = K_{\gamma_{\textbf{critical}}} + K_\gamma{}'$
When $\Delta < 0$ we get $K_\gamma^2 > -\frac{\beta^2}{4.T_0.\overline{U}}$ and the roots are basically

$$
\begin{aligned}
c &= -\frac{b}{2.a} \pm \frac{\sqrt{-\Delta}}{2.a} \\
0 < \textbf{Im}\{c\} &= \frac{\sqrt{-\Delta}}{2.a}
\end{aligned}
$$

We want the Imaginary part of the roots of c then we multiply by k to find the imaginary part of $\sigma = \sigma_i$

$$\sigma_i = \frac{\sqrt{-\Delta}}{2.K_\gamma^2}.k$$

We know that $K_\gamma^2 = k^2 + l^2 + \frac{1}{R_d^2}$ and we choose to set $\frac{1}{R_d^2} = 0$ so we get K^2 instead of K_γ^2,

$$\boxed{\sigma_i = \frac{\sqrt{-\Delta}}{2.K^2}.k} \tag{10}$$

Here we set $K = K_{\textbf{critical}} + K'$ so in the "normal" discriminant,

$$\begin{aligned}
\Delta &= \beta^2 + 4.K_\gamma^2.T_0.\overline{U} \\
&= \beta^2 + 4.\left(K_{\textbf{critical}} + K'\right)^2.T_0.\overline{U} \\
&= \beta^2 + 4.T_0.\overline{U}.(K_{\textbf{critical}}^2 + 2.K_{\textbf{critical}}.K' + \overset{\text{small}}{\cancel{K'^2}})
\end{aligned}$$

(By definition ?)

$$\Delta = 2.K_c.K'$$

We re-write the growth rate σ_i, and be also considerate that $K = k = k_{\textbf{critical}} + k'$

$$\sigma_i = \frac{\sqrt{-2.K_{\textbf{critical}}.k'}}{2.K_{\textbf{critical}}}$$

3.2.7. Limits study of the growth rate

We know now that our growth rate σ_i can be written as,

$$\sigma_i = \frac{\sqrt{-2.K_{\textbf{critical}}.k'}}{2.K_{\textbf{critical}}}.k \quad \text{or} \quad \sigma_i = \frac{\sqrt{\frac{\beta^2}{2.T_0.U_0}.k'}}{-\frac{\beta^2}{2.T_0.U_0}}.k_{\textbf{critical}} = -\frac{2.T_0.U_0}{\beta^2}.\sqrt{\frac{\beta^2.k'}{2.T_0.U_0}}.k_{\textbf{critical}}$$

Or,
[RE REDIGER]
For all cases, by default we assume (if not precised as a limit),

$$\begin{aligned}
\beta &= 0 \\
U_0 &= 1 \\
F_1^* = \frac{1}{R_d^2} &= 0 \\
\frac{\Theta_0}{U_0} &= 1 \\
K_{\textbf{critical}}^2 &= -\frac{\beta^2}{4.T_0.U_0}
\end{aligned}$$

For $\beta \gg$ paramaters

$$\lim_{\beta \to 1} \sigma_i = \lim_{\beta \to 1} \quad -\frac{2.T_0.U_0}{\beta^2}.\sqrt{\frac{\beta^2.k'}{2.T_0.U_0}}.k_{\textbf{critical}}$$
$$= -2.T_0.U_0.\sqrt{\frac{k'}{2.T_0.U_0}}.k_{\textbf{critical}}$$

3.2.8. Growth rate (II)

To find growth rates of c (that is a scalar) given by (8), weed to look for,

$$\sigma_i = k.\mathbf{Im}\{c\} \tag{11}$$

Where k is the wavenumber, and c eigenvalues. The growth rate gives us an indication of the energy of the little perturbations. When growth rates increase, the energy is distributed into small perturbations. We show in **Fig. 1.** an example of 2 groths rates : QG and TQG. computed with the numerical model presented from **page 19**.

To simplify the analysis, we will explore the stability of the system with growth rates numerically.

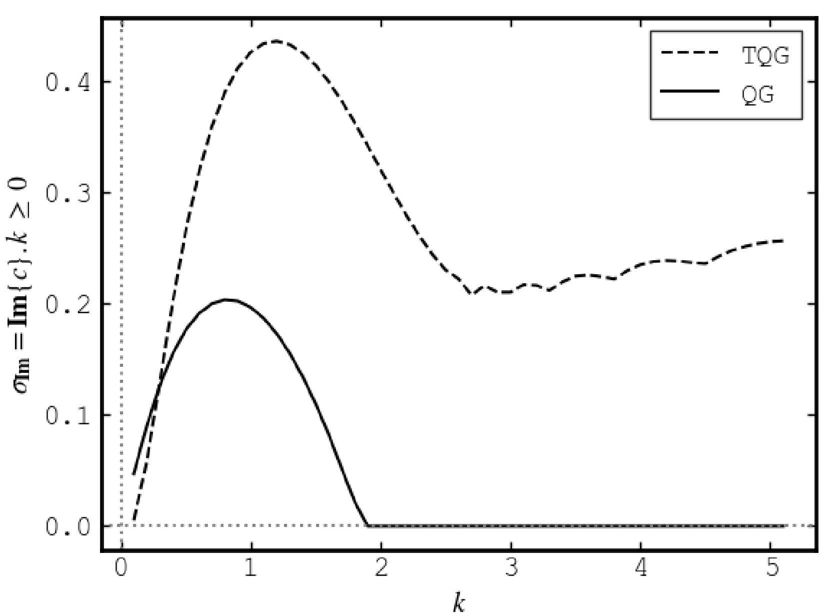

Fig. 1. A random example of growth rates between QG and TQG models. We see the difference of amplitude and shape. We see that the QG in very smooth along wavenumber k and it becomes null when $k \to 2$. The TQG one is not that smooth after $k = 2.75$ and becomes bouncy : that is typically observed when we use thermal forcing.

We know,

$$\sigma_i = -\frac{2.T_0.U_0}{\beta^2}.\sqrt{\frac{\beta^2.k'}{2.T_0.U_0}}.k_{\textbf{critical}}$$

We can compute the derivative of this function with respect to β. We know $u = \frac{2.T_0.U_0}{\beta^2}$ and $v = \sqrt{\frac{\beta^2.k'}{2.T_0.U_0}}.k_{\textbf{critical}}$ so we deduce : $u' = -\frac{2.T_0.U_0}{\beta^3}$ and $v' = \frac{2.k'.\beta}{2.T_0.U_0}.\frac{k_{\textbf{critical}}}{2.\sqrt{\frac{\beta^2.k'}{2.T_0.U_0}}}$

$$
\begin{aligned}
\frac{\partial \sigma_i}{\partial \beta} &= -\frac{\partial}{\partial \beta}\frac{2.T_0.U_0}{\beta^2}.\sqrt{\frac{\beta^2.k'}{2.T_0.U_0}}.k_{\textbf{critical}} \\
&= -\frac{2.T_0.U_0}{\beta^3}.\sqrt{\frac{\beta^2.k'}{2.T_0.U_0}}.k_{\textbf{critical}} + \frac{2.T_0.U_0}{\beta^2}.\frac{2.k'.\beta}{2.T_0.U_0}.\frac{k_{\textbf{critical}}}{2.\sqrt{\frac{\beta^2.k'}{2.T_0.U_0}}} \\
&= -\frac{2.T_0.U_0}{\beta^3}.\sqrt{\frac{\beta^2.k'}{2.T_0.U_0}}.k_{\textbf{critical}} + \frac{2.k'.k_{\textbf{critical}}}{2.\beta.\sqrt{\frac{\beta^2.k'}{2.T_0.U_0}}}
\end{aligned}
$$

Vérifié numériquement : cohérent

3.3. The system for $\overline{\psi} = f(y)$ and $\overline{\Theta} = M^2.y$

Instead of considering $\overline{\psi} = -\overline{U}.y$ we will assume that it's a function depending on y : $\overline{\psi} = f(y)$. Some details of the previous equations will change ...

3.3.1. Linearisation

The linearisation presented in **page 6** at section 3.2.1 is valid for this case. We have the same set of equation that are (4a) and (4b) that are,

$$
\begin{aligned}
\frac{\partial q'}{\partial t} + J(\overline{\psi}, q' - \Theta') + J(\psi', \overline{q} - \overline{\Theta}) = 0 \\
\frac{\partial \Theta'}{\partial t} + J(\overline{\psi}, \Theta') + J(\psi', \overline{\Theta}) = 0
\end{aligned}
$$

3.3.2. Wave hypothesis

As usual we consider that $\overline{\Theta} = M^2.y$, $\overline{q} = \left(\frac{\overline{U}}{R_d^2} + \beta\right).y$ and now $\overline{\psi} = f(y)$,

Step 1 : $q - \Theta$

$$
\begin{aligned}
\frac{\partial q'}{\partial t} + \cancelto{0}{\partial_x \overline{\psi}.\partial_y.(q' - \Theta')} - \partial_x(q' - \Theta').\partial_y.\overline{\psi} + \partial_x \psi'.\partial_y.(\overline{q} - \overline{\Theta}) - \cancelto{0}{\partial_x(\overline{q} - \overline{\Theta}).\partial_y \psi'} = 0 \\
\frac{\partial q'}{\partial t} - \partial_x(q' - \Theta').\partial_y.\overline{\psi} + \partial_x \psi'.\partial_y.(\overline{q} - \overline{\Theta}) = 0
\end{aligned}
$$

And replacing the known terms into the equation we get (we don't forget that $q' = \text{div}(\underline{\textbf{grad}\ \psi'}) - \frac{\psi'}{R_d^2} + \beta.y$),

$$\frac{\partial}{\partial t}.\left(\mathbf{div}(\underline{\mathbf{grad}\ \psi'}) - \frac{\psi'}{R_d^2} + \beta.y\right) - \partial_x\left(\mathbf{div}(\underline{\mathbf{grad}\ \psi'}) - \frac{\psi'}{R_d^2} + \beta.y - \Theta'\right).\partial_y.f(y)$$
$$+\partial_x\psi'.\partial_y.\left(\left(\frac{\overline{U}}{R_d^2} + \beta\right).y - M^2.y\right) = 0$$
$$\frac{\partial}{\partial t}.\left(\mathbf{div}(\underline{\mathbf{grad}\ \psi'}) - \frac{\psi'}{R_d^2}\right) - \partial_x\left(\mathbf{div}(\underline{\mathbf{grad}\ \psi'}) - \frac{\psi'}{R_d^2} - \Theta'\right).\frac{\mathrm{d}f(y)}{\mathrm{d}y}$$
$$+\partial_x\psi'.\left(\frac{\overline{U}}{R_d^2} + \beta - M^2\right) = 0$$

As usual, we use the wave-hypothesis : $\psi' = \widehat{\psi}'.\mathbf{exp}(i.(k.x + l.y - \omega.t)) = \widehat{\psi}'.\mathbf{exp}(\aleph)$ and the same for Θ' (as usual we remove the $\mathbf{exp}(\aleph)$ by dividing for lisibility ...),

$$-i.\omega.\left(-(k^2 + l^2 + \frac{1}{R_d^2}).\widehat{\psi}'\right) - i.k.\left(-\left(k^2 + l^2 + \frac{1}{R_d^2}\right).\widehat{\psi}' - \Theta'\right).\frac{\mathrm{d}f(y)}{\mathrm{d}y}$$
$$+i.k.\psi'.\left(\frac{\overline{U}}{R_d^2} + \beta - M^2\right) = 0 \quad || \times \frac{1}{i.k}$$
$$-c.\left(-(k^2 + l^2 + \frac{1}{R_d^2}).\widehat{\psi}'\right) - \left(-\left(k^2 + l^2 + \frac{1}{R_d^2}\right).\widehat{\psi}' - \Theta'\right).\frac{\mathrm{d}f(y)}{\mathrm{d}y}$$
$$+\widehat{\psi}'.\left(\frac{\overline{U}}{R_d^2} + \beta - M^2\right) = 0$$

We recall that $K_\gamma^2 = k^2 + l^2 + \frac{1}{R_d^2}$ and $\alpha = \beta - M^2 + \frac{\overline{U}}{R_d^2}$ and so,

$$-c.\left(-K_\gamma^2.\widehat{\psi}'\right) + \left(K_\gamma^2.\widehat{\psi}' + \Theta'\right).\frac{\mathrm{d}f(y)}{\mathrm{d}y} + \psi'.\alpha = 0$$
$$\left(\left(c + \frac{\mathrm{d}f(y)}{\mathrm{d}y}\right).K_\gamma^2 + \alpha\right).\widehat{\psi}' + \frac{\mathrm{d}f(y)}{\mathrm{d}y}.\widehat{\Theta}' = 0$$

Step 2 : Θ

$$\frac{\partial\Theta'}{\partial t} + \cancelto{0}{\partial_x\overline{\psi}.\partial_y.\Theta'} - \partial_x\Theta'.\partial_y.\overline{\psi} + \partial_x\psi'.\partial_y\overline{\Theta} - \cancelto{0}{\partial_x\overline{\Theta}.\partial_y\psi'} = 0$$
$$\frac{\partial\Theta'}{\partial t} - \partial_x\Theta'.\partial_y.\overline{\psi} + \partial_x\psi'.\partial_y\overline{\Theta} = 0$$

Now we replace with known values that are $\Theta = M^2.y$ and $\overline{\psi} = f(y)$,

$$\frac{\partial\Theta'}{\partial t} - \partial_x\Theta'.\partial_y.f(y) + \partial_x\psi'.\partial_y M^2.y = 0$$
$$\frac{\partial\Theta'}{\partial t} - \partial_x\Theta'.\frac{\mathrm{d}f(y)}{\mathrm{d}y} + \partial_x\psi'.M^2 = 0$$

And now we set the wave hypothesis $\psi' = \widehat{\psi}'.\mathbf{exp}(\aleph)$ for ψ' and Θ',

$$\frac{\partial\Theta'}{\partial t} - \partial_x\Theta'.\frac{\mathrm{d}f(y)}{\mathrm{d}y} + \partial_x\psi'.M^2 = 0$$

$$-i.\omega.\widehat{\Theta}'.\mathbf{exp}(\aleph) - i.k.\widehat{\Theta}'.\mathbf{exp}(\aleph).\frac{\mathrm{d}f(y)}{\mathrm{d}y} + i.k.\widehat{\psi}'.\mathbf{exp}(\aleph).M^2 = 0 \;\; || \times \frac{1}{i.k.\mathbf{exp}(\aleph)}$$

$$\left(\frac{\mathrm{d}f(y)}{\mathrm{d}y} - c\right).\widehat{\Theta}' + \widehat{\psi}'.M^2 = 0$$

We have a new set of equations (12a) and (12b) for the case where $\psi' = f(y)$,

$$\boxed{\left(\left(c + \frac{\mathrm{d}f(y)}{\mathrm{d}y}\right).K_\gamma^2 + \alpha\right).\widehat{\psi}' + \frac{\mathrm{d}f(y)}{\mathrm{d}y}.\widehat{\Theta}' = 0} \tag{12a}$$

$$\boxed{\left(\frac{\mathrm{d}f(y)}{\mathrm{d}y} - c\right).\widehat{\Theta}' + \widehat{\psi}'.M^2 = 0} \tag{12b}$$

3.3.3. Solving for $\Theta' = 0$

From (12a) and (12b) and considering $\Theta' = 0$: the set of equation, for $\Theta' = 0$ is,

$$\left(\left(c + \frac{\mathrm{d}f(y)}{\mathrm{d}y}\right).K_\gamma^2 + \alpha\right).\widehat{\psi}' = 0$$
$$\widehat{\psi}'.M^2 = 0$$

From the second equation we deduce that $M^2 = 0$ because $\widehat{\psi}'$ can't be 0 (unless if we are looking for 0-solutions). We separate the c value that gives a simple equation to solve,

$$\left(c + \frac{\mathrm{d}f(y)}{\mathrm{d}y}\right).K_\gamma^2 + \alpha = 0$$

And so,

$$\boxed{c = -\frac{\alpha}{K_\gamma^2} - \frac{\mathrm{d}f(y)}{\mathrm{d}y}} \tag{13}$$

3.3.4. Solving for $\Theta' \neq 0$

From (12a) and (12b) :

$$\left(\left(c + \frac{\mathrm{d}f(y)}{\mathrm{d}y}\right).K_\gamma^2 + \alpha\right).\widehat{\psi}' + \frac{\mathrm{d}f(y)}{\mathrm{d}y}.\widehat{\Theta}' = 0$$
$$\left(\frac{\mathrm{d}f(y)}{\mathrm{d}y} - c\right).\widehat{\Theta}' + \widehat{\psi}'.M^2 = 0$$

If $\Theta' \neq 0$ we see a matrix system $\mathrm{A} \times v = 0$ that can be written as,

$$\underbrace{\begin{pmatrix} \left(c + \frac{\mathrm{d}f(y)}{\mathrm{d}y}\right).K_\gamma^2 + \alpha & \frac{\mathrm{d}f(y)}{\mathrm{d}y} \\ M^2 & \frac{\mathrm{d}f(y)}{\mathrm{d}y} - c \end{pmatrix}}_{\underline{\underline{A}}} \times \underbrace{\begin{pmatrix} \widehat{\psi}' \\ \widehat{\Theta}' \end{pmatrix}}_{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

As usual we can try a null determinant method to solve this system, that gives,

$$\begin{aligned}
\mathbf{det}\ \underline{\underline{A}} &= 0 \\
\left[\left(c + \frac{\mathrm{d}f(y)}{\mathrm{d}y}\right).K_\gamma^2 + \alpha\right].\left(\frac{\mathrm{d}f(y)}{\mathrm{d}y} - c\right) - M^2.\frac{\mathrm{d}f(y)}{\mathrm{d}y} &= 0 \\
\left(c + \frac{\mathrm{d}f(y)}{\mathrm{d}y}\right).K_\gamma^2.\frac{\mathrm{d}f(y)}{\mathrm{d}y} + \alpha.\frac{\mathrm{d}f(y)}{\mathrm{d}y} - \left(c + \frac{\mathrm{d}f(y)}{\mathrm{d}y}\right).K_\gamma^2.c - c.\alpha - M^2.\frac{\mathrm{d}f(y)}{\mathrm{d}y} &= 0 \\
\cancel{c.K_\gamma^2.\frac{\mathrm{d}f(y)}{\mathrm{d}y}} + K_\gamma^2\left(\frac{\mathrm{d}f(y)}{\mathrm{d}y}\right)^2 + \alpha.\frac{\mathrm{d}f(y)}{\mathrm{d}y} - K_\gamma^2.c^2 - \cancel{\frac{\mathrm{d}f(y)}{\mathrm{d}y}.K_\gamma^2.c} - c.\alpha - M^2.\frac{\mathrm{d}f(y)}{\mathrm{d}y} &= 0 \\
-c^2.K_\gamma - c.\alpha + \underbrace{\frac{\mathrm{d}f(y)}{\mathrm{d}y}.(\alpha - M^2) + \left(\frac{\mathrm{d}f(y)}{\mathrm{d}y}\right)^2.K_\gamma^2}_{\eta} &= 0 \quad \Big\|\times(-1)
\end{aligned}$$

That's a 2nd order polynomial equation,

$$c^2 + c.\alpha - \eta = 0$$

With $k_1 = 1$; $k_2 = \alpha = \beta - M^2 + \frac{\overline{U}}{R_d^2}$; $k_3 = \eta$ and c will depends of the sign of the discriminant $\Delta = k_2^2 - 4.k_1.k_3$,

$$\begin{aligned}
\Delta &= \alpha^2 - 4 \times 1 \times \eta \\
&= \alpha^2 - 4.\eta
\end{aligned}$$

We need an imaginary part, so to have complex solutions, we must have $\Delta < 0$. If $\alpha^2 > 0$ that implies,

$$\alpha^2 - 4.\eta < 0$$

$$\boxed{\eta > \frac{1}{4}.\alpha^2 \ \text{ with } \ \alpha = \beta - M^2 + \frac{\overline{U}}{R_d^2}} \tag{14}$$

The conditions where we can solve for $\Theta' \neq 0$ is $\eta > \frac{1}{4}.\alpha^2$. And this equation is a stability criteria for $\eta = \frac{\mathrm{d}f(y)}{\mathrm{d}y}.(\alpha - M^2) + \left(\frac{\mathrm{d}f(y)}{\mathrm{d}y}\right)^2.K_\gamma^2$

3.4. Stability of the system

Re rediger la stabilité du modèle,

3.4.1. Set the system

We will derive a stability criteria for the folowing system derived from (4a) and (4b) that are,

$$\frac{\partial q'}{\partial t} + J(\psi', \overline{q} - \overline{\Theta}) + J(\overline{\psi}, q' - \Theta') = 0$$
$$\frac{\partial \Theta'}{\partial t} + J(\overline{\psi}, \Theta') + J(\psi', \overline{\Theta}) = 0$$

We recall the following parameters $\overline{\psi} = -\overline{U}.y$; $\overline{q} = \left(\frac{U}{R_d^2} + \beta\right).y$; $\overline{\Theta} = f(y)$. And we recall that for any pertubations $\lambda' = \widehat{\lambda}.\mathbf{exp}(\aleph)$, always with $\aleph = i.(k.x + l.y - \omega.t)$.

Step 1 : $q - \Theta$,

$$\frac{\partial q'}{\partial t} + \partial_x \psi'.\partial_y(\overline{q} - \overline{\Theta}) - \cancelto{0}{\partial_x(\overline{q} - \overline{\Theta})}.\partial_y \psi' + \cancelto{0}{\partial_x \overline{\psi}.\partial_y(q' - \Theta')} - \partial_x(q' - \Theta').\partial_y \overline{\psi} = 0$$
$$\frac{\partial q'}{\partial t} + \partial_x \psi'.\frac{\mathrm{d}}{\mathrm{d}y}(\overline{q} - \overline{\Theta}) + \partial_x(q' - \Theta').\overline{U} = 0$$
$$-i.\omega.\widehat{q}'.\mathbf{exp}(\aleph) + i.k.\widehat{\psi}'.\mathbf{exp}(\aleph).\frac{\mathrm{d}}{\mathrm{d}y}(\overline{q} - \overline{\Theta}) + i.k.\mathbf{exp}(\aleph).(\widehat{q}' - \widehat{\Theta}').\overline{U} = 0$$

As usual, we multiply by $\times\frac{1}{i.k.\mathbf{exp}(\aleph)}$ that gives (and we recall that $c = \frac{\omega}{k}$),

$$-c.\widehat{q}' + \widehat{\psi}'.\frac{\mathrm{d}}{\mathrm{d}y}(\overline{q} - \overline{\Theta}) + (\widehat{q}' - \widehat{\Theta}').\overline{U} = 0$$
$$(\overline{U} - c).\widehat{q}' - \overline{U}.\widehat{\Theta}' + \widehat{\psi}'.\frac{\mathrm{d}}{\mathrm{d}y}(\overline{q} - \overline{\Theta}) = 0$$

Step 2 : Θ

$$\frac{\partial \Theta'}{\partial t} + \partial_x \psi'.\partial_y \overline{\Theta} - \cancelto{0}{\partial_x \overline{\Theta}.\partial_y \psi'} + \cancelto{0}{\partial_x \overline{\psi}.\partial_y \Theta'} - \partial_x \Theta'.\partial_y \overline{\psi} = 0$$
$$\frac{\partial \Theta'}{\partial t} + \partial_x \psi'.\partial_y \overline{\Theta} - \partial_x \Theta'.\partial_y \overline{\psi} = 0$$
$$-i.\omega.\widehat{\Theta}'.\mathbf{exp}(\aleph) + i.k.\widehat{\psi}'.\mathbf{exp}(\aleph).\frac{\mathrm{d}\overline{\Theta}}{\mathrm{d}y} + i.k.\widehat{\Theta}'.\mathbf{exp}(\aleph).\overline{U} = 0$$

We multiply by $\times\frac{1}{i.k.\mathbf{exp}(\aleph)}$,

$$-c.\widehat{\Theta}' + \widehat{\psi}'.\frac{\mathrm{d}\overline{\Theta}}{\mathrm{d}y} + \widehat{\Theta}'.\overline{U} = 0$$
$$(\overline{U} - c).\widehat{\Theta}' + \widehat{\psi}'.\frac{\mathrm{d}\overline{\Theta}}{\mathrm{d}y} = 0$$

The 2 equations that we need to study are,

$$\boxed{(\overline{U}-c).\widehat{q}'-\overline{U}.\widehat{\Theta}'+\widehat{\psi}'.\frac{\mathrm{d}}{\mathrm{d}y}(\overline{q}-\overline{\Theta})=0} \tag{15a}$$

$$\boxed{(\overline{U}-c).\widehat{\Theta}'+\widehat{\psi}'.\frac{\mathrm{d}\overline{\Theta}}{\mathrm{d}y}=0} \tag{15b}$$

3.4.2. Stability criteria[RE-REDIGER]

We set

We shall now introduce the following relations always with $\aleph = i.(k.x+l.y-\omega.t)$

$$q'=\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2}-k_d^2.\phi\right).\mathbf{exp}(\aleph)\quad;\quad k_d^2=k^2+\frac{1}{R_d^2}$$
$$\frac{\mathrm{d}\overline{q}}{\mathrm{d}y}=-\frac{\mathrm{d}^2\overline{U}}{\mathrm{d}y^2}+\frac{\overline{U}}{R_d^2}+\beta\quad;\quad \phi(y)=(\overline{U}-c)^{\frac{1}{2}}.\chi(y)$$

We deduce the derivatives for ϕ, (as long as $u^{\frac{1}{2}}=\sqrt{u}$ we know the derivative $\frac{du}{2.\sqrt{u}}$) we set for χ and for other quantities $\chi_y=\frac{\mathrm{d}\chi}{\mathrm{d}y}$.

$$\frac{\mathrm{d}\phi}{\mathrm{d}y}=(\overline{U}-c)^{\frac{1}{2}}.\chi_y+\frac{\overline{U}_y}{2.(\overline{U}-c)^{\frac{1}{2}}}.\chi$$
$$\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2}=(\overline{U}-c)^{\frac{1}{2}}.\chi_{yy}+2.\left[\frac{1}{2}.\frac{\overline{U}_y.\chi_y}{(\overline{U}-c)^{\frac{1}{2}}}\right]+\frac{1}{2}.\frac{\overline{U}_{yy}.\chi}{(\overline{U}-c)^{\frac{1}{2}}}-\frac{1}{4}.\frac{(\overline{U}_y)^2.\chi}{(\overline{U}-c)^{\frac{3}{2}}}$$

That we first replace in the equation (15a),

$$(\overline{U}-c).\left[\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2}-k_d^2.\phi\right).\mathbf{exp}(\aleph)\right]-\overline{U}.\widehat{\Theta}'+\widehat{\psi}'.\frac{\mathrm{d}}{\mathrm{d}y}(\overline{q}-\overline{\Theta})=0$$

[DETAILLER + QUESTIONS]
We get the following equation,

$$(\overline{U}-c).\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-k_d^2\right).\phi+\frac{\overline{U}}{\overline{U}-c}.\frac{\mathrm{d}\overline{\Theta}}{\mathrm{d}y}.\phi+\frac{\mathrm{d}(\overline{q}-\overline{\Theta})}{\mathrm{d}y}=0$$

And we set a new value called $\alpha_*=(\overline{U}-c)^{\frac{1}{2}}$. Now we replace the values of ϕ, $\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2}$ into the main equation,

$$\alpha_*^2.\left[\chi_{yy}+\frac{\overline{U}_y.\overline{\chi}_y+\frac{1}{2}.\chi.\overline{U}_{yy}}{\alpha_*^2}-\frac{1}{4}.\frac{\chi.(\overline{U}_y)^2}{\alpha_*^3}-k_d^2.\alpha_*\right]+\frac{\overline{U}}{\alpha_*^2}.\frac{\mathrm{d}\overline{\Theta}}{\mathrm{d}y}.\alpha_*.\chi+\alpha_*.\chi.\frac{\mathrm{d}(\overline{q}-\overline{\Theta})}{\mathrm{d}y}=0$$
$$=0$$

$$\boxed{\text{If}\quad \overline{U}.\frac{\mathrm{d}\Theta}{\mathrm{d}y}\begin{cases} > \dfrac{1}{4}.(\overline{U}.y)^2 & \rightarrow \text{Stable} \\ < \dfrac{1}{4}.(\overline{U}.y)^2 & \rightarrow \text{Unstable}\end{cases}} \tag{16}$$

4. Numerical investigation of the TQG model

4.1. TQG Numerical solution

We are now able to propose a method to solve numerically the equations (15a) and (15b). As references the Roullet's courses provides elements of numerical methods with [Roullet, 2023a] and [Roullet, 2023b].

4.1.1. Before the numerical analysis

We set

Now we will set that (with $\aleph = i.(k.x + l.y - \omega.t)$),

$$q' = \left(\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2} - k_d^2.\phi\right).\mathbf{exp}(\aleph) \;\;;\;\; k_d^2 = k^2 + \frac{1}{R_d^2} \;\;;\;\; K^2 = k_d^2.\Delta_y^2$$

$$\frac{\mathrm{d}\overline{q}}{\mathrm{d}y} = -\frac{\mathrm{d}^2\overline{U}}{\mathrm{d}y^2} + \frac{\overline{U}}{R_d^2} + \beta \;\;;\;\; G_{11} = \frac{\mathrm{d}}{\mathrm{d}y}(\overline{q} - \overline{\Theta}) \;\;;\;\; G_{12} = \frac{\mathrm{d}\overline{\Theta}}{\mathrm{d}y}$$

We assume that $\psi' = \phi.\mathbf{exp}(\aleph)$ and $\Theta' = \widehat{\Theta}'.\mathbf{exp}(\aleph)$ so from (15a) we get

$$(\overline{U} - c).\widehat{q}' - \overline{U}.\widehat{\Theta}' + \widehat{\psi}'.\frac{\mathrm{d}}{\mathrm{d}y}(\overline{q} - \overline{\Theta}) = 0$$

$$(\overline{U} - c).\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - k_d^2\right).\phi.\mathbf{exp}(\aleph) - \overline{U}.\widehat{\Theta}'.\mathbf{exp}(\aleph) + \phi.\mathbf{exp}(\aleph).G_{11} = 0 \;\; || \times \frac{1}{\mathbf{exp}(\aleph)}$$

$$(\overline{U} - c).\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2} - \phi.k_d^2\right) - \overline{U}.\widehat{\Theta}' + \phi.G_{11} = 0$$

From (15b) we get,

$$(\overline{U} - c).\widehat{\Theta}' + \widehat{\psi}'.\frac{\mathrm{d}\overline{\Theta}}{\mathrm{d}y} = 0$$

$$(\overline{U} - c).\widehat{\Theta}'.\mathbf{exp}(\aleph) + \phi.\mathbf{exp}(\aleph).\frac{\mathrm{d}\overline{\Theta}}{\mathrm{d}y} = 0 \;\; || \times \frac{1}{\mathbf{exp}(\aleph)}$$

$$(\overline{U} - c).\widehat{\Theta}' + \phi.G_{12} = 0$$

If we drop the prime $'$ and hat $\widehat{}$ symbols we get,

$$\boxed{(\overline{U}-c).\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2}-\phi.k_d^2\right)-\overline{U}.\Theta+\phi.G_{11}=0} \tag{17a}$$

$$\boxed{(\overline{U}-c).\Theta+\phi.G_{12}=0} \tag{17b}$$

4.1.2. The analytical principle of an eigenvalue problem

We shall now introduce the following system that will drives the numerical solution of the TQG model,

$$\begin{aligned}
\underline{\underline{\mathrm{A}}}.X &= c.\underline{\underline{\mathrm{B}}}.X \\
\underbrace{\underline{\underline{\mathrm{B}}}^{-1}.\underline{\underline{\mathrm{A}}}}_{\mathrm{P}}.X &= \underline{c}.X \\
\underline{\underline{\mathrm{P}}}.X &= \underline{c}.X
\end{aligned}$$

And $\underline{\underline{\mathrm{P}}}=\underline{\underline{\mathrm{N}}}.\Lambda.\underline{\underline{\mathrm{N}}}^{-1}$, we note **Id** the identity matrix so,

$$\begin{aligned}
\mathrm{P}.\underline{X} &= c.\underline{X} \\
(\mathrm{P}-c).\mathbf{Id}.\underline{X} &= 0
\end{aligned}$$

We get the following system,

$$\mathbf{det}(\mathrm{P}-c.\mathbf{Id})=0$$

That gives eigenvalues in an eigenvector after a few lines of algebra.

4.1.3. Discretisation

Using finite differences we can transform the 2 equations presented in (17a) and (17b),

$$\begin{aligned}
(\overline{U}_n-c).\left[\frac{\phi_{j+1}-2.\phi_n+\phi_{j-1}}{\Delta y^2}-\phi_n.k_d^2\right]-\overline{U}_n.\Theta_n+\phi_n.G_{11n} &= 0 \\
(\overline{U}_n-c).\Theta_n+\phi.G_{12_n} &= 0
\end{aligned}$$

We set

Some usefull definitions,

$$K^2 = k_d^2.\Delta y^2 = \left(k^2 + \underbrace{\frac{1}{R_d^2}}_{F_1^*}\right).\Delta y^2 \quad \text{Be aware ! Not the same than} \quad K_\gamma^2$$

$$F_{11_n} = \Delta y^2.G_{11_n}$$
$$F_1^* = \frac{1}{R_d^2}$$
$$\overline{V}_n = \overline{U}_n.\Delta y^2$$

We can re-write the previous equation by multiplying it by Δy^2.
For (17a) we get,

$$(\overline{U}_n - c).\left[\frac{\phi_{j+1} - 2.\phi_n + \phi_{j-1}}{\Delta y^2} - \phi_n.k_d^2\right] - \overline{U}_n.\Theta_n + \phi_n.G_{11n} = 0 \quad \Big|\Big| \times \Delta y^2$$
$$(\overline{U}_n - c).\left[\phi_{j+1} - 2.\phi_n + \phi_{j-1} - \phi_n.k_d^2.\Delta y^2\right] - \overline{U}_n.\Theta_n.\Delta y^2 + \phi_n.G_{11n}.\Delta y^2 = 0$$
$$(\overline{U}_n - c).\left[\phi_{j+1} - 2.\phi_n + \phi_{j-1} - \phi_n.K^2\right] - \overline{V}_n.\Theta_n + \phi_n.F_{1_n} = 0$$

For (17b) we just have to set the indices n on the equation without modify it. We get the final system discretised with (18a) and (18b),

$$\boxed{(\overline{U}_n - c).\left[\phi_{j+1} - 2.\phi_n + \phi_{j-1} - \phi_n.K^2\right] - \overline{V}_n.\Theta_n + \phi_n.F_{1_n} = 0} \tag{18a}$$
$$\boxed{(\overline{U}_n - c).\Theta_n + \phi_n.G_{12_n} = 0} \tag{18b}$$

That can be re-writed,

$$(\overline{U}_n - c).\left[\phi_{n+1} - 2.\phi_n + \phi_{n-1} - \phi_n.K^2\right] - \overline{V}_n.\Theta_n + \phi_n.F_{1_n} = 0$$
$$(\overline{U}_n - c).\Theta_n + \phi.G_{12_n} = 0$$

And,

$$\boxed{\overline{U}_n.\left[\phi_{j+1} - 2.\phi_n + \phi_{j-1} - \phi_n.K^2\right] - \overline{V}_n.\Theta_n + \phi_n.F_{1_n} = c.\left[\phi_{j+1} - 2.\phi_n + \phi_{j-1} - \phi_n.K^2\right]} \tag{19a}$$
$$\boxed{\overline{U}_n.\Theta_n + \phi.G_{12_n} = c.\Theta_n} \tag{19b}$$

Now we set $X = \begin{pmatrix}\phi_n \\ \Theta_n\end{pmatrix}$ and we find a a system defined by,

$$\boxed{\underline{\underline{A}}.X = \underline{c}.\underline{\underline{B}}.X} \tag{20}$$

4.1.4. Numerical proposition

And from (19a) and (19b) we detail the equation (20),

$$\underbrace{\begin{pmatrix} \overbrace{\mathrm{A}_{11}}^{\times\phi_n} & \overbrace{\mathrm{A}_{12}}^{\times\Theta_n} \\ \mathrm{A}_{21} & \mathrm{A}_{22} \end{pmatrix}}_{\underline{\underline{\mathrm{A}}}} \times \underbrace{\begin{pmatrix} \phi_n \\ \Theta_n \end{pmatrix}}_{\underline{\underline{X}}} = \underline{c}. \underbrace{\begin{pmatrix} \mathrm{B}_{11} & \mathrm{B}_{12} \\ \mathrm{B}_{21} & \mathrm{B}_{22} \end{pmatrix}}_{\underline{\underline{\mathrm{B}}}} \times \underbrace{\begin{pmatrix} \phi_n \\ \Theta_n \end{pmatrix}}_{\underline{\underline{X}}}$$

And the associated matrix, for a 3×3 example are,

$$\mathrm{A}_{11} \text{ or } \mathbf{div}(\mathbf{grad})_U = \begin{pmatrix} -\overline{U}_n.(2+K^2)+F_{1n} & \overline{U}_n & 0 \\ \overline{U}_n & -\overline{U}_n.(2+K^2)+F_{1n} & \overline{U}_n \\ 0 & \overline{U}_n & -\overline{U}_n.(2+K^2)+F_{1n} \end{pmatrix}$$

$$\mathrm{A}_{12} = \begin{pmatrix} -\overline{V}_n & 0 & 0 \\ 0 & -\overline{V}_n & 0 \\ 0 & 0 & \overline{V}_n \end{pmatrix}$$

$$\mathrm{A}_{21} = \begin{pmatrix} G_{12n} & 0 & 0 \\ 0 & G_{12n} & 0 \\ 0 & 0 & G_{12n} \end{pmatrix} \quad \rightarrow \text{that we substitute into } A = \begin{pmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} \\ \mathrm{A}_{21} & \mathrm{A}_{22} \end{pmatrix}$$

$$\mathrm{A}_{22} = \begin{pmatrix} \overline{U}_n & 0 & 0 \\ 0 & \overline{U}_n & 0 \\ 0 & 0 & \overline{U}_n \end{pmatrix}$$

With the same method we deduce the matrix B,

$$\mathrm{B} = \begin{pmatrix} \mathrm{B}_{11} & \mathrm{B}_{12} \\ \mathrm{B}_{21} & \mathrm{B}_{22} \end{pmatrix} = \begin{pmatrix} \overbrace{\begin{bmatrix} -(2+K^2) & 1 & 0 \\ 1 & -(2+K^2) & 1 \\ 0 & 1 & -(2+K^2) \end{bmatrix}}^{\mathbf{div}(\mathbf{grad})_c} & \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{\underline{\underline{0}}} \\ \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\underline{\underline{0}}} & \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{Id}} \end{pmatrix}$$

We recall that,

$$
\begin{aligned}
\overline{V}_n &= \overline{U}_n.\Delta y^2 \\
G_{12n} &= \frac{\mathrm{d}\overline{\Theta}_n}{\mathrm{d}y} = \frac{\mathrm{d}}{\mathrm{d}y} M_n^2.y \\
&= M_n^2 \\
G_{11n} &= \frac{\mathrm{d}(\overline{q}_n - \overline{\Theta}_n)}{\mathrm{d}y} = \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\overline{U}_n}{R_d^2+\beta} - M_n^2\right).y = \\
&= \frac{\overline{U}_n}{R_d^2+\beta} - M_n^2 \\
F_{1n} &= G_{11n}.\Delta y^2 \\
K^2 &= k_d^2.\Delta y^2 = k^2 + \frac{1}{R_d^2}
\end{aligned}
$$

Note that the system $\underline{\underline{A}}.X = \underline{c}.\underline{\underline{B}}.X$ can be quickly solved with python modules : for instance `c, X = scipy.linalg.eig(A,B)` that returns eigenvalues and eigenvectors.

4.1.5. Tools

We want to study the stability of the system, and for that we can plot some charts that will helps us to interprete what we are simulating. We will first draw a chart that represents,

$$
\sigma_{\mathbf{Im}} = \mathbf{Im}\{c\}.k \tag{21a}
$$
$$
\sigma_{\mathbf{Re}} = \mathbf{Re}\{c\}.k \tag{21b}
$$

Where c are the eigenvalues computed with the numerical solving of $A.\underline{X} = c.B.\underline{X}$, and k is the wavenumber. These parameters are called "growth rates".

And we shall study also the stability of the system by using the following criteria (that is nothing else than equation (16)),

$$
\text{If} \quad \overline{U}.\frac{\mathrm{d}\Theta}{\mathrm{d}y}
\begin{cases}
> \frac{1}{4}.(\overline{U}.y)^2 & \rightarrow \text{Stable} \\
< \frac{1}{4}.(\overline{U}.y)^2 & \rightarrow \text{Unstable}
\end{cases}
$$

$\frac{1}{4}.(\overline{U}.y)^2$ and $\overline{U}.\frac{\mathrm{d}\Theta}{\mathrm{d}y}$ are proportional to a value $V \propto \overline{U}.y^2$.

If we want to visualise the $\phi(t,y)$ and $\Theta(t,y)$ values, we can use the following definitions. First we extract the mode that we want on the $\underline{X}$ eigenvectors. Then we separate $\phi(y)$ and $\Theta(y)$ (if TQG)

$$
\phi_y = X_{\mathbf{mode}}^{0\rightarrow N} \quad \text{and} \quad \omega_n^{\phi} = c^{0\rightarrow N}.k_n \tag{22a}
$$
$$
\Theta_y = X_{\mathbf{mode}}^{N\rightarrow 2.N} \quad \text{and} \quad \omega_n^{\Theta} = c^{N\rightarrow 2.N}.k_n \tag{22b}
$$

Then,

$$\phi_n(t,y) = \mathbf{Re}\Big\{\phi_y.\mathbf{exp}(-i.\omega_n^{\phi}.t)\Big\} \tag{23a}$$

$$\Theta_n(t,y) = \mathbf{Re}\Big\{\Theta_y.\mathbf{exp}(-i.\omega_n^{\Theta}.t)\Big\} \tag{23b}$$

4.2. Non-TQG Numerical solution

- Here's some values to set,

$$\overline{U}(y) = U_0.\mathbf{exp}\left(-y^2\right)$$
$$G_{12} = \frac{d\overline{\Theta}}{dy} = -2.y.\Theta_0.\mathbf{exp}(-y^2)$$
$$G_{11} = \frac{d(\overline{q}-\overline{\Theta})}{dy} = \Big[2.\overline{U}.(1-2.y^2) + F_1^*.\overline{U} + \beta\Big] - \Big[-2.y.\Theta_0.\mathbf{exp}(-y^2)\Big]$$

And associated to these values,

$$F_{11} = G_{11}.\Delta y^2$$
$$F_1^* = \frac{1}{R_d^2}$$

Be aware : Do not make the confusion bewteen F_1^* and F_{11} !

We can simplify equations (19a) and (19b) to find the non-TQG equation that is,

$$\overline{U}_n.\Big[\phi_{j+1} - 2.\phi_n + \phi_{j-1} - \phi_n.K^2\Big] + \phi_n.F_{1_n} = c.\Big[\phi_{j+1} - 2.\phi_n + \phi_{j-1} - \phi_n.K^2\Big]$$

That's really tempting to use A_{11} and B_{11} but ...

- **WARNING :** The $F_{1_n} = G_{11}.\Delta y^2 = \frac{\mathrm{d}(\overline{q}-\overline{\Theta})}{\mathrm{d}y}.\Delta y^2$ term contains thermal terms that aren't in the QG problem : that makes the matrix A_{11} incorrect for this.
- **TO FIX IT :** We need to erase the $\frac{\mathrm{d}\overline{\Theta}}{\mathrm{d}y}$ that is equals to G_{12}. We add $G_{12}.\Delta y^2$ to F_{1_n} because the $\frac{\mathrm{d}\overline{\Theta}}{\mathrm{d}y}$ was initially substracted to $\frac{\mathrm{d}\overline{q}}{\mathrm{d}y}$ in the TQG model. By addition, G_{12} vanishes. Here we disconnect the thermal term for the QG-solving.

So we get,

$$\boxed{\begin{aligned}\overline{U}_n.\Big[\phi_{j+1} - 2.\phi_n + \phi_{j-1} - \phi_n.K^2\Big] + \phi_n.\left(F_{1_n} + G_{12}.\Delta y^2\right) \\ = c.\Big[\phi_{j+1} - 2.\phi_n + \phi_{j-1} - \phi_n.K^2\Big]\end{aligned}} \tag{24}$$

This is still an eigenvalue problem that can be solved using $\underline{X} = \phi_n$. We shall re-arrange A_{11} (let us call it A_{11}^*) when solving the QG model, but we can use the B_{11} matrix that is still the same (due to **WARNING** section).

$$\underline{\underline{A}}^*_{11}.\underline{X} = c.\underline{\underline{B}}_{11}.\underline{X}$$

And in the same way than before we can propose a matrix construction that gives,

$$\underbrace{\mathbf{div}(\mathbf{grad})_U}_{\mathrm{A}} \times \underbrace{\phi_n}_{\underline{X}} = c_n . \underbrace{\mathbf{div}(\mathbf{grad})_c}_{\mathrm{B}} \times \underbrace{\phi_n}_{X}$$

Here's a 3×3 example,

$$\begin{pmatrix} -\overline{U}_n.(2+K^2) + F_{1n} + G_{12}.\Delta y^2 & \overline{U}_n & 0 \\ \overline{U}_n & -\overline{U}_n.(2+K^2) + F_{1n} + G_{12}.\Delta y^2 & \overline{U}_n \\ 0 & \overline{U}_n & -\overline{U}_n.(2+K^2) + F_{1n} + G_{12}.\Delta y^2 \end{pmatrix}$$
$$\times \phi_n = c_n \times \begin{pmatrix} -(2+K^2) & 1 & 0 \\ 1 & -(2+K^2) & 1 \\ 0 & 1 & -(2+K^2) \end{pmatrix} \times \phi_n$$

That is even simpler to solve and construct than the previous eigenvalue problem. And the numerical problem can be still solved with python modules :

For instance `c, X = scipy.linalg.eig(`A^*_{11}`,`B_{11}`)`. that will returns different eigenvalues and eigenvectors compared to the TQG-problem.

4.3. Flow stability

4.3.1. Linear : 2 cases [A RE REDIGER]

- **Configuration 1 :**

$$\overline{U}(y) = U_0.\mathbf{exp}(-y^2) \tag{25a}$$
$$\overline{\Theta}(y) = \Theta_0.\mathbf{exp}(-y^2) \tag{25b}$$

- **Configuration 2 :**

$$\overline{U}(y) = U_0.\mathbf{exp}(-y^2) \tag{26a}$$
$$\overline{\Theta}(y) = \Theta_0.\mathbf{exp}\left(-\frac{y^2}{{L_*}^2}\right) \tag{26b}$$

We first study the variation of 3 parameters (and their effect on the growth rates) with the **configuration 1** that are,

$$\beta \in [0\ ;\ 3]$$
$$F_1^* = \frac{1}{R_d^2} \in [0\ ;\ 12]$$
$$\frac{\Theta_0}{U_0} \in [0\ ;\ 2]$$

Then for the **configuration 2** we need to make L_* vary and re-computes growth rates for each variation of paramters,

$$\left.\begin{aligned} \beta &\in [0\ ;\ 3] \\ F_1^* = \frac{1}{R_d^2} &\in [0\ ;\ 12] \\ \frac{\Theta_0}{U_0} &\in [0\ ;\ 2] \end{aligned}\right\} \quad \textbf{For} \quad L_* = [0.5\ ;\ 1\ ;\ 1.5\ ;\ 2]$$

Observations :

- Effect of L_* :
  1. Seems to regulate $\Delta = \sigma_{\mathbf{TQG}} - \sigma_{\mathbf{QG}}$.
  2. Critical values.
- Effect of β and F_1^* :
  1. Linearise σ and creates bulges (sometimes).
  2. Seems to control the rapidity of the decreasing.
- Effect of $\frac{\Theta_0}{U_0}$:
  1. Control the amplitude of the growth of the growth rates.
  2. Subsequently it controls also $\Delta = \sigma_{\mathbf{TQG}} - \sigma_{\mathbf{QG}}$.

4.4. 2D TQG numerical model [IN PROGRESS]

Inspired by (17a) and (17b) we can re-write the 2D equations for the TQG model,

$$\left(\overline{U} - c\right).\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \frac{\mathrm{d}^2\phi}{\mathrm{d}y^2} - \phi.k_d^2\right) - \overline{U}.\Theta + \phi.G_{11} = 0 \tag{27a}$$

$$\left(\overline{U} - c\right).\Theta + \phi.G_{12} = 0 \tag{27b}$$

If we discretise these equations we get,

$$\left(\overline{U}_n^j - c\right).\left(\frac{\phi_{n+1}^j - 2.\phi_n^j + \phi_{n-1}^j}{\Delta x^2} + \frac{\phi_n^{j+1} - 2.\phi_n^j + \phi_n^{j-1}}{\Delta y^2} - \phi_n^j.k_d^2\right) - \overline{U}_n^j.\Theta + \phi_n^j.G_{11} = 0$$

$$\left(\overline{U}_n^j - c\right).\Theta + \phi.G_{12} = 0$$

Then re-arrange,

$$\begin{aligned} \overline{U}_n^j.\left(\frac{\phi_{n+1}^j - 2.\phi_n^j + \phi_{n-1}^j}{\Delta x^2} + \frac{\phi_n^{j+1} - 2.\phi_n^j + \phi_n^{j-1}}{\Delta y^2} - \phi_n^j.k_d^2\right) - \overline{U}_n^j.\Theta + \phi_n^j.G_{11} \\ = c.\left(\frac{\phi_{n+1}^j - 2.\phi_n^j + \phi_{n-1}^j}{\Delta x^2} + \frac{\phi_n^{j+1} - 2.\phi_n^j + \phi_n^{j-1}}{\Delta y^2} - \phi_n^j.k_d^2\right) \end{aligned}$$

$$\overline{U}_n^j.\Theta + \phi.G_{12} = c.\Theta$$

4.4.1. $\Delta x = \Delta y = \Delta h$

$$\overline{U}^j_n.\left(\frac{\phi^j_{n+1} - 2.\phi^j_n + \phi^j_{n-1}}{\Delta h^2} + \frac{\phi^{j+1}_n - 2.\phi^j_n + \phi^{j-1}_n}{\Delta h^2} - \phi^j_n.k_d^2\right) - \overline{U}^j_n.\Theta + \phi^j_n.G_{11}$$
$$= c.\left(\frac{\phi^j_{n+1} - 2.\phi^j_n + \phi^j_{n-1}}{\Delta h^2} + \frac{\phi^{j+1}_n - 2.\phi^j_n + \phi^{j-1}_n}{\Delta h^2} - \phi^j_n.k_d^2\right)$$
$$\overline{U}^j_n.\Theta + \phi.G_{12} = c.\Theta$$

We multiply by Δh^2 (only the 1st equation) and we get,

$$\overline{U}^j_n.\left(\phi^j_{n+1} - 2.\phi^j_n + \phi^j_{n-1} + \phi^{j+1}_n - 2.\phi^j_n + \phi^{j-1}_n - \phi^j_n.k_d^2.\Delta h^2\right) - \overline{U}^j_n.\Delta h^2.\Theta + \phi^j_n.G_{11}.\Delta h^2$$
$$= c.\left(\phi^j_{n+1} - 2.\phi^j_n + \phi^j_{n-1} + \phi^{j+1}_n - 2.\phi^j_n + \phi^{j-1}_n - \phi^j_n.k_d^2.\Delta h^2\right)$$
$$\overline{U}^j_n.\Theta + \phi.G_{12} = c.\Theta$$

We shall now introduce : $\overline{U}^j_n.\Delta h^2 = \overline{V}^j_n$, $G_{11}.\Delta h^2 = F_{11}$, $k_d^2.\Delta h^2 = K^2$ and so,

$$\overline{U}^j_n.\left(\phi^j_{n+1} - 4.\phi^j_n + \phi^j_{n-1} + \phi^{j+1}_n + \phi^{j-1}_n - \phi^j_n.K^2\right) - \overline{V}^j_n.\Theta + \phi^j_n.F_{11} \tag{28a}$$
$$= c.\left(\phi^j_{n+1} - 4.\phi^j_n + \phi^j_{n-1} + \phi^{j+1}_n + \phi^{j-1}_n - \phi^j_n.K^2\right) \tag{28b}$$
$$\overline{U}^j_n.\Theta + \phi.G_{12} = c.\Theta \tag{28c}$$

So that's an eigenvalue problem such that $\underline{\underline{A}}.X = c.\underline{\underline{B}}.X$ and $X = \begin{pmatrix}\phi \\ \Theta\end{pmatrix}$,

$$\begin{pmatrix}\mathrm{A}_{11} & \mathrm{A}_{12} \\ \mathrm{A}_{21} & \mathrm{A}_{22}\end{pmatrix} \times \begin{pmatrix}\phi \\ \Theta\end{pmatrix} = c.\begin{pmatrix}\mathrm{B}_{11} & \mathrm{B}_{12} \\ \mathrm{B}_{21} & \mathrm{B}_{22}\end{pmatrix} \times \begin{pmatrix}\phi \\ \Theta\end{pmatrix} \tag{29}$$

With the matrix A

$$
\mathrm{A} = \begin{pmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} \\ \mathrm{A}_{21} & \mathrm{A}_{22} \end{pmatrix}
$$

$$
\mathrm{A}_{11} = \begin{bmatrix} -\overline{U}^j_n.(4+K^2+F_{11}) & 1 & 0 & 1 \\ 1 & -\overline{U}^j_n.(4+K^2+F_{11}) & 1 & 0 \\ 0 & 1 & -\overline{U}^j_n.(4+K^2+F_{11}) & 1 \\ 1 & 0 & 1 & -\overline{U}^j_n.(4+K^2+F_{11}) \end{bmatrix}
$$

$$
\mathrm{A}_{12} = \begin{bmatrix} -\overline{V}^j_n & 0 & 0 & 0 \\ 0 & -\overline{V}^j_n & 0 & 0 \\ 0 & 0 & -\overline{V}^j_n & 0 \\ 0 & 0 & 0 & -\overline{V}^j_n \end{bmatrix}
$$

$$
\mathrm{A}_{21} = \begin{bmatrix} G_{12} & 0 & 0 & 0 \\ 0 & G_{12} & 0 & 0 \\ 0 & 0 & G_{12} & 0 \\ 0 & 0 & 0 & G_{12} \end{bmatrix}
$$

$$
\mathrm{A}_{22} = \begin{bmatrix} \overline{U}^j_n & 0 & 0 & 0 \\ 0 & \overline{U}^j_n & 0 & 0 \\ 0 & 0 & \overline{U}^j_n & 0 \\ 0 & 0 & 0 & \overline{U}^j_n \end{bmatrix}
$$

And with the matrix B

$$
\mathrm{B} = \begin{pmatrix} \mathrm{B}_{11} & \mathrm{B}_{12} \\ \mathrm{B}_{21} & \mathrm{B}_{22} \end{pmatrix}
= \left(\begin{array}{cc} \begin{bmatrix} -(4+K^2) & 1 & 0 & 1 \\ 1 & -(4+K^2) & 1 & 0 \\ 0 & 1 & -(4+K^2) & 1 \\ 1 & 0 & 1 & -(4+K^2) \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \right)
$$

4.4.2. $\Delta x \neq \Delta y$

5. *Code : TQG solve*

5.1. Validation of TQG_solve_v2_bis.py

Experience : "Taranis"

Here we propose to compare the code detailed below with the Fortran code developped by Xavier Carton (as a reference). The comparison of the 2 codes is presented in **Fig. 2.** and **Fig. 3.**

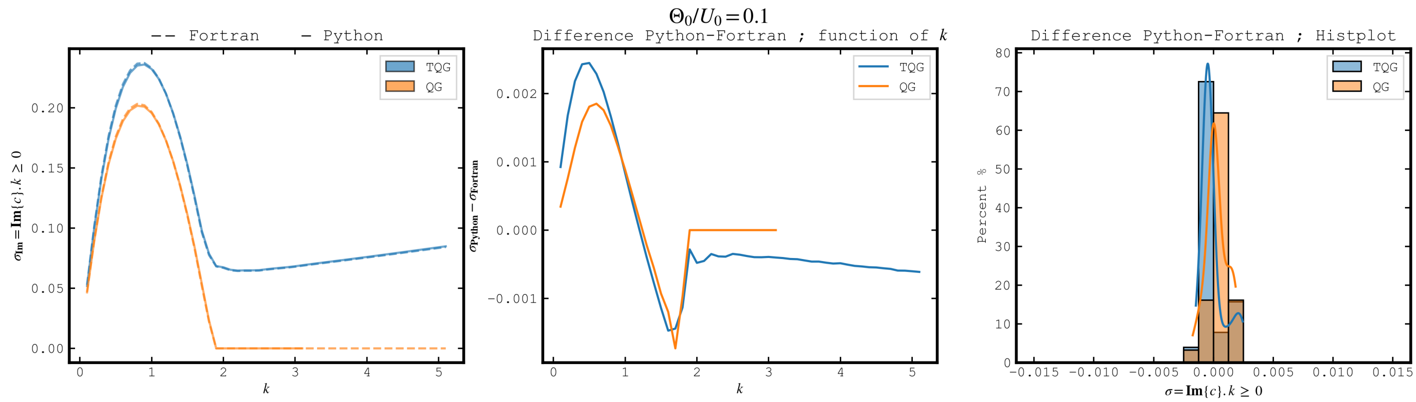

Fig. 2. Experience Taranis : Case where $\Theta_0/U_0 = 0.1$. Left : growth rates comparison between the fortran (straight line) and the python (dashed line) code for TQG and QG cases. Center : Difference between Python and Fortran codes for TQG and QG growth rates. Right : Histplot of the difference where we see a Gaussian tendency for the two cases.

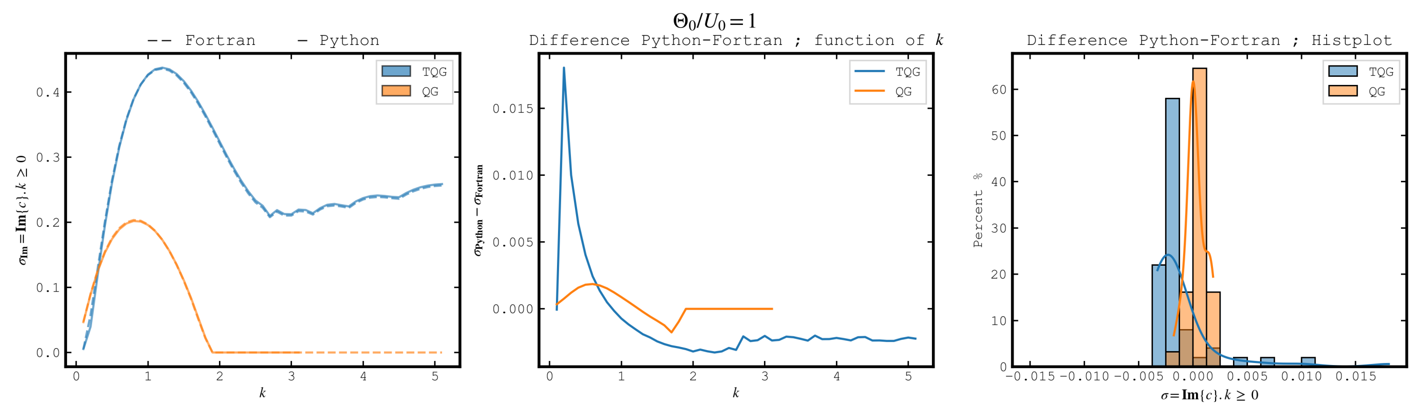

Fig. 3. Experience Taranis : Case where $\Theta_0/U_0 = 1$. Left : growth rates comparison between the fortran (straight line) and the python (dashed line) code for TQG and QG cases. Center : Difference between Python and Fortran codes for TQG and QG growth rates. Right : Histplot of the difference where we see a Gaussian tendency for the two cases.

<u>Conclusion</u> : The 2 codes are giving almost the same solution. The maximum difference between the 2 codes is around $2.5 \times 10^{-3}\ \sigma_{\mathbf{Im}}$.

```
import os
import numpy as np
import matplotlib as mpl
import matplotlib.pyplot as plt

import imageio.v2 as imageio
from PIL import Image
from io import BytesIO
import os

from tqdm import tqdm
```

```
from scipy.linalg import eig

mpl.rcParams['font.size'] = 14
mpl.rcParams['mathtext.fontset'] = 'stix'
mpl.rcParams['font.family'] = 'Courier New'
mpl.rcParams['legend.edgecolor'] = '0'

print('~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~')
print('~~~~~~~~~~~~~~~~~~TQG_SOLVE_2_BIS~~~~~~~~~~~~~~~~~~~~')
print('~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~')
print('-----------------------------------------------------')



# cf TQG notes : A.X = c.B.X is the system that is solved here
# and also the non thermal system
# @uthor : dimitri moreau 20/05/2025


save_png = False  # create a folder im_para and a folder per
     # experience with the "name_exp" variable
     # save also the used parameters and Real
     # and Imaginary sigma
font_size = 17
choice_plot_name = 'max_sigma_im'

name_exp = input('Name of the experience ?')

print('-----------------------------------------------------')

##################################
# VARIABLES, SPACE ...
##################################

Ny, Nk = 60, 51
dk = 0.1
ymin, kmin, Ly, Lk = 0.1, 0.1, np.pi, 0.1+dk*Nk
dy = (Ly - ymin)/Ny

y_l, k = np.linspace(ymin,Ly,Ny), np.arange(kmin,Nk*dk,dk)


beta = 0
F1star = 0 # 1/Rd**2

U0 = 1
Theta0_U0 = 1 # ratio
Theta0 = Theta0_U0 *U0


Un = U0*np.exp(-y_l**2)
#Un = 1/(1+np.exp(-y_l)) # sigmoide

Vn = Un*(dy**2)
G12 = -2*y_l*Theta0*np.exp(-y_l**2) # dThetabar/dy

```

```python

G11 = 2.0*Un*(1-2*y_l**2) + F1star*Un + beta - G12
F11 = G11*dy**2

print('/////////////////////////////////////////////////////')
print('PARAMS : OK')


# Save all parameters in a txt file

if save_png == True:

  print('/////////////////////////////////////////////////////')


  # Create the full directory path
  folder_path = os.path.join("im_para", name_exp)
  os.makedirs(folder_path, exist_ok=True)  # Create directories if they don't exist

  # Create full file path
  file_path = os.path.join(folder_path, 'variables_used_' + name_exp + '.txt')

  # Open a file in write mode
  with open('im_para/'+name_exp+'/variables_used_'+name_exp+'.txt', 'w') as file:
      file.write('Used variables for : '+name_exp+'\n')
      file.write('~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~\n')
      file.write(f"Ny = {Ny}\n")
      file.write(f"Nk = {Nk}\n")
      file.write(f"ymin = {ymin}\n")
      file.write(f"Ly = {Ly}\n")
      file.write(f"kmin = {kmin}\n")
      file.write(f"Lk = {Lk}\n")
      file.write(f"dy = {dy}\n")
      file.write(f"dk = {dk}\n")
      file.write(f"F1star = {F1star}\n")
      file.write(f"beta = {beta}\n")
      #file.write(f"Rd = {Rd}\n")
      file.write(f"U0 = {U0}\n")
      file.write(f"Theta0 = {Theta0}\n")
      file.write(f"ratio_Theta0_U0 = {Theta0_U0}\n")
      file.write('~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~\n')

  print('Variables stored into : variables_used_'+name_exp+'.txt')




print('/////////////////////////////////////////////////////')
print('COMPUTATION...')

sigma_matrix = np.zeros((len(k),2*Ny))
sigmaNT_matrix = np.zeros((len(k),Ny))

sigma_matrix_ree = np.zeros((len(k),2*Ny))
sigmaNT_matrix_ree = np.zeros((len(k),Ny))
```

```
sigma_tot = np.zeros(Ny*Ny) # for the eigenfrequencies later

# loop for each case of k
for ik in tqdm(range(len(k))):


  K2 = (k[ik]**2 + F1star)*dy**2


  ##################################
  # CONSTRUCTION OF THE B MATRIX
  ##################################


  B11 = np.zeros((Ny, Ny))


  for i in range(Ny):
    B11[i,i] = -(2 + K2)
    if i>0:
      B11[i,i-1] = 1.
    if i<Ny-1:
      B11[i,i+1] = 1.


  # Construct other blocks
  B12 = np.zeros((Ny, Ny))
  B21 = np.zeros((Ny, Ny))
  B22 = np.eye(Ny, Ny)


  B = np.block([[B11,B12],[B21,B22]])


  ##################################
  # CONSTRUCTION OF THE A MATRIX
  ##################################



  A11 = np.zeros((Ny,Ny))
  A11_star = np.zeros((Ny,Ny)) # same B11 without the thermal
  # term that is F11 for the non-TQG solving

  # Block A11

  for i in range(Ny):

    A11[i,i] = -Un[i] * (2 + K2) + F11[i]
    A11_star[i,i] = -Un[i] * (2 + K2) + F11[i] + G12[i]*dy**2
    if i>0:
      A11[i,i-1] = Un[i]
      A11_star[i,i-1] = Un[i]
    if i<Ny-1:
      A11[i,i+1] = Un[i]
```

```
    A11_star[i,i+1] = Un[i]



# Block A12
A12 = np.diag(-Vn)
# Block A21
A21 = np.diag(G12)
# Block A22
A22 = np.diag(Un)

# Final block matrix A

A = np.block([[A11,A12],[A21,A22]])




# velocity odd
A[0,1] = 2.0*A[0,1]
B[0,1] = 2.0*B[0,1]


# velocity not odd
#A[0,1]=0.0
#B[0,1]=0.0

A[2*Ny-1,2*Ny-1] = 0.0
B[2*Ny-1,2*Ny-1] = 0.0



A11[0,1] = 2.0*A11[0,1]
A11_star[0,1] = 2.0*A11_star[0,1]
B11[0,1] = 2.0*B11[0,1]

A11[Ny-1,Ny-1] = 0.0
A11_star[Ny-1,Ny-1] = 0.0
B11[Ny-1,Ny-1] = 0.0







#################################
# SOLUTION
#################################
# A.X = c.B.X


###### THERMAL SOLVING (TQG)

c, X = eig(A,B)

```

```
    sigma = np.imag(c) * k[ik]
    sigma_matrix[ik,:] = sigma

    sigma_ree = np.real(c) * k[ik]
    sigma_matrix_ree[ik,:] = sigma_ree

    sigma_tot = c * k[ik]



    ###### NON THERMAL SOLVING (QG)

    c_NT, X_NT = eig(A11_star,B11)

    sigma_NT = np.imag(c_NT) * k[ik]
    sigmaNT_matrix[ik,:] = sigma_NT

    sigma_NT_ree = np.real(c_NT) * k[ik]
    sigmaNT_matrix_ree[ik,:] = sigma_NT_ree






val_c = np.max(sigma_matrix, axis=1)
val_cNT = np.max(sigmaNT_matrix, axis=1)

val_c_ree = np.max(sigma_matrix_ree, axis=1)
val_cNT_ree = np.max(sigmaNT_matrix_ree, axis=1)


print('COMPUTATION : OK')


##################################
# PLOT
##################################


print('/////////////////////////////////////////////////////')



print('PLOT...')





if save_png==True:
    # Save GIF
    frames_pil = [Image.fromarray(frame) for frame in frames]
    frames_pil[0].save('output/phi_theta_evolution.gif',
```

```
      save_all=True,
      append_images=frames_pil[1:],
      duration=150,  # milliseconds per frame
      loop=0
  )

  print("GIF saved to output/phi_theta_evolution.gif")






fig, (ax) = plt.subplots(1,1)

ax.plot(k, val_c, 'k--', label='TQG')
ax.plot(k, val_cNT, 'k-', label='QG')
ax.set_xlabel(r'$k$')
ax.set_ylabel(r'$\sigma_\mathbf{Im} = \mathbf{Im}\{c\}.k ~\geq~ 0$')
ax.tick_params(top=True,right=True,direction='in',size=4,width=1)
ax.legend(fancybox=False)
ax.axhline(0, color='gray', linestyle=':')
ax.axvline(0, color='gray', linestyle=':')
for spine in ax.spines.values():
    spine.set_linewidth(2)








if save_png == True:
  plt.savefig('im_para/'+name_exp+'/fig1_'+name_exp+choice_plot_name+'.png',dpi
    =300)




  # save outputs
  np.savetxt('im_para/'+name_exp+'/sigma_TQG.txt',
       np.column_stack([k, val_c, val_c_ree]),
       fmt='%.18e',
       header='k  sigmaIm   sigmaRe')

  np.savetxt('im_para/'+name_exp+'/sigma_QG.txt',
       np.column_stack([k, val_cNT, val_cNT_ree]),
       fmt='%.18e',
       header='k  sigmaIm   sigmaRe')





plt.show()
```

```python

print('END')
print('~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~')
```

5.2. Sensibility tests

5.2.1. Routines

To realise sensibility tests, we have regrouped the kernel of TQG_solve_v2_bis.py into a python function :

```
fig, (ax) = compute_sigmas(Ny, Nk, dk, ymin, kmin, Ly, Lk, Lstar,
                    beta, F1star, U0, Theta0_U0, config)
```

that supports arrays (to make variables variates). We just have to call this function in launcher and compute all growth rates. Note that the function returns a figure that are the growth rates for each mode of k. The constant paramters are,

```python
Ny, Nk = 60, 51
dk = 0.1
ymin, kmin, Ly, Lk = 0.1, 0.1, np.pi, 0.1+dk*Nk
U0 = 1
```

5.2.2. Parameters

And we make these 3 parameters variates,

```python
# choice 1
beta = 0
F1star = 0
Theta0_U0 = np.round(np.linspace(0., 1., 15), 3)

# choice 2
beta = 0
F1star = np.round(np.linspace(0., 4., 15), 3)
Theta0_U0 = 1

# choice 3
beta = np.round(np.linspace(0., 1., 15), 3)
F1star = 0
Theta0_U0 = 1
```

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