Magnetic propeties of a two dimentional electron gas strongly coupled to lights

Kosala Herath

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1 Schrödinger problem for Landau levels in dressed 2DEG

Our analysis start with considering 2 dimentional free electronic gas which has been distrubuted in confined (x, y) plane in configuration space.

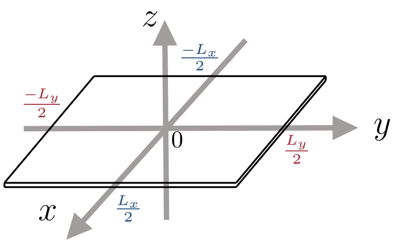

Figure 1: Confined 2DEG in configuration space with the size of $A = L_x L_y$.

We are going to examine the properties of 2DEG with stationary magnetic field

$$\mathbf{B} = (0, 0, B)^T \tag{1.1}$$

which directed on z axis and a linearly y-polarized strong electomagnetic wave (dressing field) with electric field given by

$$\mathbf{E} = (0, E\sin(\omega t), 0)^T \tag{1.2}$$

which also propagate in z direction. Here B and E represent the amplitude of the stationary magnetic field and electric field of dressing field.

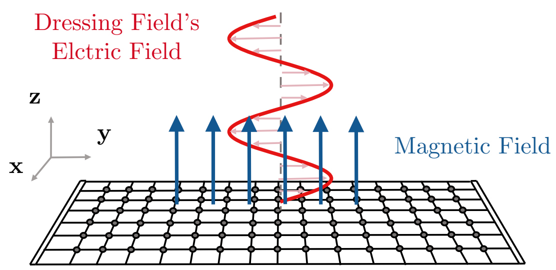

Figure 2: Stationary magnetic filed (blue color) and Strong EM wave (red color) applied to the 2DEG.

Using Landau gauge for the stationary magnetic field we can represent it using vector potential as

$$\mathbf{A}_s = (-By, 0, 0)^T \tag{1.3}$$

and choosing Coulomb gauge the dressing field can be present as the following vector potential

$$\mathbf{A}_d(t) = (0, [E/\omega]\cos(\omega t), 0)^T. \tag{1.4}$$

Now the Hamiltonian of an electron in 2DEG can be reads as

$$\hat{H}_e(t) = \frac{1}{2m_e}\Big[\hat{\mathbf{p}} - e\big(\mathbf{A}_s + \mathbf{A}_d(t)\big)\Big]^2 \tag{1.5}$$

where m_e is the effective mass of the electron and e is the magnitude (without considering the sign of the charge) of the electron charge. This can be simplified to

$$\hat{H}_e(t) = \frac{1}{2m_e}\Big[(\hat{p}_x + eBy)\mathbf{e}_x + (\hat{p}_y - \frac{eE}{\omega}\cos(\omega t))\mathbf{e}_y\Big]^2 \tag{1.6}$$

where $\mathbf{e}_x$ and $\mathbf{e}_y$ are unit vectors along x and y directions respectively. Moreover,

$$\hat{H}_e(t) = \frac{1}{2m_e}\Big[(\hat{p}_x + eBy)^2 + (\hat{p}_y - \frac{eE}{\omega}\cos(\omega t))^2\Big] \tag{1.7}$$

Since $[\hat{H}_e(t), \hat{p}_x] = 0$ both operators share same (simultaneous) eigen functions which are free electron wave functions ($\frac{1}{\sqrt{L_x}}\exp\big(\frac{ip_x x}{\hbar}\big)$). Therefore we can modify the Hamiltonian as follows

$$\hat{H}_e(t) = \frac{1}{2m_e}\Big[(p_x + eBy)^2 + (\hat{p}_y - \frac{eE}{\omega}\cos(\omega t))^2\Big]. \tag{1.8}$$

Using momentum operator definition

$$\hat{p}_y = -i\hbar\frac{\partial}{\partial y} \tag{1.9}$$

we can modify Eq. (1.8) as

$$\begin{aligned}\hat{H}_e(t) &= \frac{1}{2m_e}\Big[(p_x + eBy)^2 + \Big(-i\hbar\frac{\partial}{\partial y} - \frac{eE}{\omega}\cos(\omega t)\Big)^2\Big] \\ &= \frac{1}{2m_e}\Big[(p_x + eBy)^2 + \Big(i\hbar\frac{\partial}{\partial y} + \frac{eE}{\omega}\cos(\omega t)\Big)^2\Big].\end{aligned} \tag{1.10}$$

Define the *center of the cyclotron orbit* along y axis as

$$y_0 \equiv \frac{-p_x}{eB} \tag{1.11}$$

and the *cyclotron frequency* as

$$\omega_0 \equiv \frac{eB}{m_e}. \tag{1.12}$$

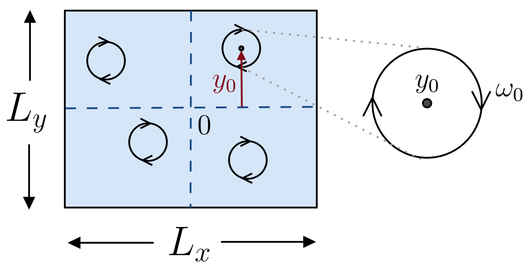

Figure 3: Paramters of the cyclotron orbits in the classical interpretation.

Then the Hamiltonian will leads to

$$\hat{H}_e(t) = \frac{m_e\omega_0^2}{2}(y - y_0)^2 + \frac{1}{2m_e}\Big(i\hbar\frac{\partial}{\partial y} + \frac{eE}{\omega}\cos(\omega t)\Big)^2 \tag{1.13}$$

$$\begin{aligned}\hat{H}_e(t) = \frac{m_e\omega_0^2}{2}(y - y_0)^2 + \frac{1}{2m_e}\Big(-\hbar^2\frac{\partial^2}{\partial y^2} + i\hbar\frac{\partial}{\partial y}\Big[\frac{eE}{\omega}\cos(\omega t)\Big] \\ + \frac{i\hbar eE}{\omega}\cos(\omega t)\frac{\partial}{\partial y} + \frac{e^2E^2}{\omega^2}\cos^2(\omega t)\Big)\end{aligned} \tag{1.14}$$

$$\hat{H}_e(t) = \frac{m_e\omega_0^2}{2}(y-y_0)^2 + \frac{1}{2m_e}\Big(-\hbar^2\frac{\partial^2}{\partial y^2} + \frac{2i\hbar eE}{\omega}\cos(\omega t)\frac{\partial}{\partial y} + \frac{e^2E^2}{\omega^2}\cos^2(\omega t)\Big). \tag{1.15}$$

Let

$$\tilde{y} = (y-y_0) \longrightarrow dy = d\tilde{y} \tag{1.16}$$

and then this becomes

$$\hat{H}_e(t) = \frac{m_e\omega_0^2}{2}\tilde{y}^2 + \frac{1}{2m_e}\Big(-\hbar^2\frac{\partial^2}{\partial \tilde{y}^2} + \frac{2i\hbar eE}{\omega}\cos(\omega t)\frac{\partial}{\partial \tilde{y}} + \frac{e^2E^2}{\omega^2}\cos^2(\omega t)\Big). \tag{1.17}$$

Now assume that the solution for the time-dependent schrödinger equation

$$i\hbar\frac{\mathrm{d}\psi}{\mathrm{d}t} = \hat{H}_e(t)\psi \tag{1.18}$$

can be represent by the following form

$$\psi(\mathbf{r},t) = \frac{1}{\sqrt{L_x}}\exp\left(\frac{ip_x x}{\hbar} + \frac{ieE(y-y_0)}{\hbar\omega}\cos(\omega t)\right)\phi(y-y_0,t). \tag{1.19}$$

Using the same subtution from Eq. (1.16) this becomes

$$\psi(x,\tilde{y},t) = \frac{1}{\sqrt{L_x}}\exp\left(\frac{ip_x x}{\hbar} + \frac{ieE\tilde{y}}{\hbar\omega}\cos(\omega t)\right)\phi(\tilde{y},t). \tag{1.20}$$

Defining

$$\varphi(x,\tilde{y},t) \equiv \frac{1}{\sqrt{L_x}}\exp\left(\frac{ip_x x}{\hbar} + \frac{ieE\tilde{y}}{\hbar\omega}\cos(\omega t)\right) \tag{1.21}$$

we can simply the the Eq. (1.20) as

$$\psi(x,\tilde{y},t) = \varphi(x,\tilde{y},t)\phi(\tilde{y},t). \tag{1.22}$$

Let's subtitue Eq. (1.20) and Eq. (1.17) into Eq. (1.18) and we can observe that

$$\begin{aligned} \text{L.H.S} = i\hbar\frac{\mathrm{d}\psi}{\mathrm{d}t} &= i\hbar\Big(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\phi + \frac{\mathrm{d}\phi}{\mathrm{d}t}\varphi\Big) = i\hbar\Big(\Big[\frac{-ieE\tilde{y}}{\hbar}\sin(\omega t)\Big]\varphi\phi + \varphi\frac{\mathrm{d}\phi}{\mathrm{d}t}\Big) \\ &= \big[eE\tilde{y}\sin(\omega t)\big]\varphi\phi + i\hbar\varphi\frac{\mathrm{d}\phi}{\mathrm{d}t} \end{aligned} \tag{1.23}$$

and

$$\begin{aligned} \text{R.H.S} &= \hat{H}_e(t)\psi \\ &= \left[\frac{m_e\omega_0^2}{2}\tilde{y}^2 + \frac{1}{2m_e}\Big(-\hbar^2\frac{\partial^2}{\partial \tilde{y}^2} + \frac{2i\hbar eE}{\omega}\cos(\omega t)\frac{\partial}{\partial \tilde{y}} + \frac{e^2E^2}{\omega^2}\cos^2(\omega t)\Big)\right]\varphi\phi \end{aligned} \tag{1.24}$$

where we will can calculate this part by part as follows:

$$\begin{aligned} \frac{-\hbar^2}{2m_e}\frac{\partial^2}{\partial \tilde{y}^2}(\varphi\phi) &= \frac{-\hbar^2}{2m_e}\frac{\partial}{\partial \tilde{y}}\left[\Big(\frac{ieE}{\hbar\omega}\cos(\omega)t\Big)\varphi\phi + \varphi\frac{\partial\phi}{\partial \tilde{y}}\right] \\ &= \frac{-\hbar^2}{2m_e}\left[\Big(\frac{ieE}{\hbar\omega}\cos(\omega)t\Big)^2\varphi\phi + \Big(\frac{ieE}{\hbar\omega}\cos(\omega)t\Big)\varphi\frac{\partial\phi}{\partial \tilde{y}} + \Big(\frac{ieE}{\hbar\omega}\cos(\omega)t\Big)\varphi\frac{\partial\phi}{\partial \tilde{y}} + \varphi\frac{\partial^2\phi}{\partial \tilde{y}^2}\right] \\ &= \Big(\frac{e^2E^2}{2m_e\omega^2}\cos^2(\omega)t\Big)\varphi\phi - \Big(\frac{ieE\hbar}{m_e\omega}\cos(\omega)t\Big)\varphi\frac{\partial\phi}{\partial \tilde{y}} - \frac{\hbar^2}{2m_e}\varphi\frac{\partial^2\phi}{\partial \tilde{y}^2} \end{aligned} \tag{1.25}$$

and

$$\begin{aligned} \frac{2i\hbar eE}{2m_e\omega}\cos(\omega t)\frac{\partial}{\partial \tilde{y}}(\varphi\phi) &= \frac{i\hbar eE}{m_e\omega}\cos(\omega t)\left[\Big(\frac{ieE}{\hbar\omega}\cos(\omega)t\Big)\varphi\phi + \varphi\frac{\partial\phi}{\partial \tilde{y}}\right] \\ &= \Big(\frac{-e^2E^2}{m_e\omega^2}\cos(\omega)t\Big)\varphi\phi + \frac{i\hbar eE}{m_e\omega}\cos(\omega t)\varphi\frac{\partial\phi}{\partial \tilde{y}}. \end{aligned} \tag{1.26}$$

Therefore we can derive that

$$\text{R.H.S} = \left[\frac{m_e\omega_0^2}{2}\tilde{y}^2\varphi\phi - \frac{\hbar^2}{2m_e}\varphi\frac{\partial^2\phi}{\partial\tilde{y}^2}\right]. \tag{1.27}$$

To satisfy the condition L.H.S=R.H.S we need to find a function $\phi(\tilde{y},t)$ such that

$$\big[eE\tilde{y}\sin(\omega t)\big]\varphi\phi + i\hbar\varphi\frac{\mathrm{d}\phi}{\mathrm{d}t} = \left[\frac{m_e\omega_0^2}{2}\tilde{y}^2\varphi\phi - \frac{\hbar^2}{2m_e}\varphi\frac{\partial^2\phi}{\partial\tilde{y}^2}\right] \tag{1.28}$$

by removing φ this can be simplyfied as

$$\left[\frac{m_e\omega_0^2}{2}\tilde{y}^2 - eE\tilde{y}\sin(\omega t) - \frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial\tilde{y}^2} - i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\right]\phi(\tilde{y},t) = 0. \tag{1.29}$$

If we turn off the external dressing field, this equation leads to simple harmonic oscillator Hamiltonian as follows

$$\left[\frac{m_e\omega_0^2}{2}\tilde{y}^2 - \frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial\tilde{y}^2} - i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\right]\phi(\tilde{y},t) = 0 \tag{1.30}$$

$$i\hbar\frac{\mathrm{d}\phi(\tilde{y},t)}{\mathrm{d}t} = \left[\frac{\hat{p}_{\tilde{y}}^2}{2m_e} + \frac{1}{2}m_e\omega_0^2\tilde{y}^2\right]\phi(\tilde{y},t). \tag{1.31}$$

Therefore we can identify the $S(t) \equiv eE\sin(\omega t)$ part as a external force act on the harmonic oscillator and we can solve this as a forced harmonic oscillator in $\tilde{y}$ axis.

$$i\hbar\frac{\mathrm{d}\phi(\tilde{y},t)}{\mathrm{d}t} = \left[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial\tilde{y}^2} + \frac{1}{2}m_e\omega_0^2\tilde{y}^2 - \tilde{y}S(t)]\right]\phi(\tilde{y},t). \tag{1.32}$$

This system can be extacly solvable and we can solve this equation using the methods explained by Husimi [*Ref:1] as follows.
First we can introduce the time dependent shifted corrdinte as

$$\tilde{y} \to y' = \tilde{y} - \zeta(t) \quad \Rightarrow \quad \tilde{y} = y' + \zeta(t) \tag{1.33}$$

and this implies that

$$\frac{\mathrm{d}\phi(y',t)}{\mathrm{d}t} = \frac{\partial\phi(y',t)}{\partial t} + \frac{\partial\phi(y',t)}{\partial y'}\frac{\partial y'}{\partial t} = \frac{\partial\phi(y',t)}{\partial t} - \dot{\zeta}(t)\frac{\partial\phi(y',t)}{\partial y'} \tag{1.34}$$

where $\dot{\zeta}(t) = \frac{\partial\zeta(t)}{\partial t}$. Therefore, Eq. (1.32) will be modified to

$$i\hbar\frac{\partial\phi(y',t)}{\partial t} = \left[i\hbar\dot{\zeta}\frac{\partial}{\partial y'} - \frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y'^2} + \frac{1}{2}m_e\omega_0^2(y'+\zeta)^2 - (y'+\zeta)S(t)\right]\phi(y',t). \tag{1.35}$$

Let's tranform the wave function using following unitary trasnform

$$\phi(y',t) = \exp\left(\frac{im_e\dot{\zeta}y'}{\hbar}\right)\varphi(y',t) \tag{1.36}$$

and subtitte this into the Eq. (1.35) and we will get the following

$$\text{L.H.S} = \left[i\hbar\frac{\partial}{\partial t} - i\hbar\Big(\frac{im_e\ddot{\zeta}y'}{\hbar}\Big)\right]\exp\left(\frac{-im_e\dot{\zeta}y'}{\hbar}\right)\varphi(y',t) \tag{1.37}$$

and

$$\begin{aligned}\text{R.H.S} = \Bigg[&i\hbar\dot{\zeta}\Big(\frac{im_e\dot{\zeta}}{\hbar}\Big) + i\hbar\dot{\zeta}\frac{\partial}{\partial y'}\\ &- \frac{\hbar^2}{2m_e}\Big[\Big(\frac{im_e\dot{\zeta}}{\hbar}\Big)^2 + \Big(\frac{2im_e\dot{\zeta}}{\hbar}\Big)\frac{\partial}{\partial y'} + \frac{\partial^2}{\partial y'^2}\Big]\\ &+ \frac{1}{2}m_e\omega_0^2y'^2 + \frac{1}{2}m_e\omega_0^2\zeta^2 + m_e\omega_0^2y'\zeta\\ &- y'S(t) - \zeta S(t)\Bigg]\exp\left(\frac{-im_e\dot{\zeta}y'}{\hbar}\right)\varphi(y',t).\end{aligned} \tag{1.38}$$

Combining these two and removing exponential terms we can derive that

$$i\hbar\frac{\partial\varphi(y',t)}{\partial t} = \Big[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y'^2} + \frac{1}{2}m_e\omega_0^2 y'^2 + \Big[m_e\ddot{\zeta} + m_e\omega_0^2\zeta - S(t)\Big]y' + \Big[-\frac{1}{2}m_e\dot{\zeta}^2 + \frac{1}{2}m_e\omega_0^2\zeta^2 - \zeta S(t)\Big]\Big]\varphi(y',t). \tag{1.39}$$

Then we can restrict our $\zeta(t)$ function such that

$$m_e\ddot{\zeta} + m_e\omega_0^2\zeta = S(t) \tag{1.40}$$

and that leads to

$$i\hbar\frac{\partial\varphi(y',t)}{\partial t} = \Big[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y'^2} + \frac{1}{2}m_e\omega_0^2 y'^2 - L(\zeta,\dot{\zeta},t)\Big]\varphi(y',t) \tag{1.41}$$

where

$$L(\zeta,\dot{\zeta},t) \equiv \frac{1}{2}m_e\dot{\zeta}^2 - \frac{1}{2}m_e\omega_0^2\zeta^2 + \zeta S(t) \tag{1.42}$$

is the largrangian of a classical driven oscillator.

Now introduce new unitary transformation for the wavefunction as follows

$$\varphi(y',t) = \exp\Big(\frac{i}{\hbar}\int_0^t dt' L(\zeta,\dot{\zeta},t')\Big)\chi(y',t) \tag{1.43}$$

and subtite this into the Eq. (1.41) and gets

$$i\hbar\Big[\exp\Big(\frac{i}{\hbar}\int_0^t dt' L(\zeta,\dot{\zeta},t')\Big)\frac{\partial}{\partial t} + i\hbar L(\zeta,\dot{\zeta},t)\exp\Big(\frac{i}{\hbar}\int_0^t dt' L(\zeta,\dot{\zeta},t')\Big)\Big]\chi(y',t) = \Big[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y'^2} + \frac{1}{2}m_e\omega_0^2 y'^2 - L(\zeta,\dot{\zeta},t)\Big]\exp\Big(\frac{i}{\hbar}\int_0^t dt' L(\zeta,\dot{\zeta},t')\Big)\chi(y',t) \tag{1.44}$$

removing exponential terms finally we can derive that

$$i\hbar\frac{\partial}{\partial t}\chi(y',t) = \Big[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y'^2} + \frac{1}{2}m_e\omega_0^2 y'^2\Big]\chi(y',t). \tag{1.45}$$

This is the well known Schrodinger equation of a stationary quantum harmonic oscillator. In terms of the eigenvalues

$$E_n = \hbar\omega_0(n + \frac{1}{2}) \tag{1.46}$$

of well-known harmonic eigenfucntions (using Gauss-Hermite functions ϑ)

$$\chi_n(x) \equiv \sqrt{\kappa}\vartheta(\kappa x) \quad \text{where} \quad \vartheta(x) = \frac{1}{\sqrt{2^n n!}}e^{-x^2/2}\mathcal{H}_n(x) \quad \text{with} \quad \kappa = \sqrt{\frac{m_e\omega_0}{\hbar}} \tag{1.47}$$

being propositional to the Hermite functions $\mathcal{H}_n$, the solutions of Eq. (1.32) can be represent as

$$\phi_n(\tilde{y},t) = \chi_n(\tilde{y} - \zeta(t))\exp\Big(\frac{i}{\hbar}\Big[-E_n t + m_e\dot{\zeta}(t)(\tilde{y} - \zeta(t)) + \int_0^t dt' L(\zeta,\dot{\zeta},t')\Big]\Big) \tag{1.48}$$

The set $\{\chi_n(x)\}$ forms a complete set and thus any general solution $\phi_(\tilde{y},t)$ can be expaned in terms of the solutions in Eq. (1.48).

Next we consider special case where we assumed

$$S(t) = eE\sin(\omega t) \tag{1.49}$$

and one can derive the Eq. (1.40) for $\zeta(t)$

$$m_e\ddot{\zeta} + m_e\omega_0^2\zeta = eE\sin(\omega t) \tag{1.50}$$

and using Green function method the solution can be write as

$$\zeta(t) = \frac{eE}{m_e(\omega_0^2 - \omega^2)}\sin(\omega t). \tag{1.51}$$

form this solutions we are able to derive the final solutions $\alpha = (n, m)$ where $n \in \mathbb{Z}_0^+ andm \in \mathbb{Z}$ are two quantum numbers that describe the state of the electron, can be present as

$$\begin{aligned}\psi_\alpha(x, \tilde{y}, t) =& \frac{1}{\sqrt{L_x}}\chi_n\big(\tilde{y} - \zeta(t)\big)\\ &\times \exp\left(\frac{i}{\hbar}\left[-E_n t + p_x x + \frac{eE\tilde{y}}{\omega}\cos(\omega t) + m_e\dot{\zeta(t)}\big[\tilde{y} - \zeta(t)\big] + \int_0^t dt' L(\zeta, \dot{\zeta}, t')\right]\right)\end{aligned} \tag{1.52}$$

and the exponential phase shifts represent the effect done by the stationary magnetic field and strong dressing field. In here p_x is qunatized with the quantum number m due to the spacial confinemnet in x direction.

$$p_x = m\frac{2\pi\hbar}{L_x} \quad , \quad m = 0, \pm1, \pm2, ... \tag{1.53}$$

Therefore we can assume that the magnetitranport properties of 2DEG will be renormalized by the magnetic field as well as the dressing field. ■

2 Floquet theory

Since we describe the lifetime of an electron in certain Landau level using conventianal perturbation theeory, now we can apply the Floquet theory to identify the difference of these methods.
First we need to identify the *quasienergies* and periodic *Floquet modes* for derived wavefunctions (1.52) for a 2DEG system with both stationary magnetic field and strong dressing filed.
Let's consider the following paramter which is lineraly increasing in time

$$\Delta_E t \equiv \frac{t}{T}\int_0^T dt'\ L(\zeta,\dot{\zeta},t') \tag{2.1}$$

where we can calculate this using Eq. (1.42) and (1.51) as follows

$$\begin{aligned}\Delta_E t = \frac{t}{T}\int_0^T dt'\ \frac{1}{2}m_e\frac{(eE\omega)^2}{m_e^2(\omega_0^2-\omega^2)^2}\cos^2(\omega t') - \frac{1}{2}m_e\omega_0^2\frac{(eE)^2}{m_e^2(\omega_0^2-\omega^2)^2}\sin^2(\omega t')\\ + \frac{eE}{m_e(\omega_0^2-\omega^2)}\sin(\omega t')eE\sin(\omega t')\end{aligned} \tag{2.2}$$

$$\begin{aligned}\Delta_E t = \frac{t\omega}{2\pi}\times\frac{(eE)^2}{2m_e(\omega_0^2-\omega^2)^2}\Bigg[\omega^2\int_0^T dt'\ \cos^2(\omega t') - \omega_0^2\int_0^T dt'\ \sin^2(\omega t')\\ + 2(\omega_0^2-\omega^2)\int_0^T dt'\ \sin^2(\omega t')\Bigg]\end{aligned} \tag{2.3}$$

$$\Delta_E t = \frac{t\omega}{2\pi}\times\frac{(eE)^2}{2m_e(\omega_0^2-\omega^2)^2}\left[\omega^2\frac{\pi}{\omega} - \omega_0^2\frac{\pi}{\omega} + 2(\omega_0^2-\omega^2)\frac{\pi}{\omega}\right] \tag{2.4}$$

$$\Delta_E t = \frac{t\omega}{2}\times\frac{(eE)^2}{2m_e(\omega_0^2-\omega^2)^2}(\omega_0^2-\omega^2) = \frac{(eE)^2}{4m_e(\omega_0^2-\omega^2)}t \tag{2.5}$$

Since this is the continuous increasing part of the Laggrangian integral in Eq. (1.52) we can make this as 2ω periodic function as follows

$$\Lambda \equiv \int_0^t dt'\ L(\zeta,\dot{\zeta},t') - \frac{t}{T}\int_0^T dt'\ L(\zeta,\dot{\zeta},t') \tag{2.6}$$

which can be proved as follows. First consider the first term of the Λ

$$\begin{aligned}\int_0^t dt'\ L(\zeta,\dot{\zeta},t') = \frac{(eE)^2}{2m_e(\omega_0^2-\omega^2)^2}\Bigg[\omega^2\int_0^t dt'\ \cos^2(\omega t') - \omega_0^2\int_0^t dt'\ \sin^2(\omega t')\\ + 2(\omega_0^2-\omega^2)\int_0^t dt'\ \sin^2(\omega t')\Bigg]\end{aligned} \tag{2.7}$$

$$\begin{aligned}\int_0^t dt'\ L(\zeta,\dot{\zeta},t') = \frac{(eE)^2}{2m_e(\omega_0^2-\omega^2)^2}\Bigg[\omega^2\left[\frac{t}{2}+\frac{\sin(2\omega t)}{4\omega}\right] - \omega_0^2\left[\frac{t}{2}-\frac{\sin(2\omega t)}{4\omega}\right]\\ + 2(\omega_0^2-\omega^2)\left[\frac{t}{2}-\frac{\sin(2\omega t)}{4\omega}\right]\Bigg]\end{aligned} \tag{2.8}$$

$$\begin{aligned}\int_0^t dt'\ L(\zeta,\dot{\zeta},t') = \frac{(eE)^2}{2m_e(\omega_0^2-\omega^2)^2}\Bigg[\frac{t}{2}\left[\omega^2-\omega_0^2+2\omega_0^2-2\omega^2\right]\\ + \frac{\sin(2\omega t)}{4\omega}\left[\omega^2+\omega_0^2-2\omega_0^2+2\omega^2\right]\Bigg]\end{aligned} \tag{2.9}$$

$$\int_0^t dt'\ L(\zeta,\dot{\zeta},t') = \frac{(eE)^2}{4m_e(\omega_0^2-\omega^2)}t + \frac{(eE)^2(3\omega^2-\omega_0^2)}{8m_e\omega(\omega_0^2-\omega^2)^2}\sin(2\omega t) \tag{2.10}$$

then using Eq.(2.5) we can write this as

$$\int_0^t dt'\; L(\zeta,\dot{\zeta},t') = \Delta_E t + \frac{(eE)^2\big(3\omega^2-\omega_0^2\big)}{8m_e\omega(\omega_0^2-\omega^2)^2}\sin(2\omega t). \tag{2.11}$$

Now we can express

$$\Lambda = \Delta_E t + \frac{(eE)^2\big(3\omega^2-\omega_0^2\big)}{8m_e\omega(\omega_0^2-\omega^2)^2}\sin(2\omega t) - \Delta_E t = \frac{(eE)^2\big(3\omega^2-\omega_0^2\big)}{8m_e\omega(\omega_0^2-\omega^2)^2}\sin(2\omega t) \tag{2.12}$$

which is a periodic function in time with 2ω frequency.
Now using this parmaters we can factorize the wavefunction (1.52) as linearly time dependend part and periodic time dependend part as follows

$$\begin{aligned}\psi_\alpha(x,y,t) =& \exp\left(\frac{i}{\hbar}[-E_n t+\Delta_E t]\right)\frac{1}{\sqrt{L_x}}\chi_n\big(y-\zeta(t)\big)\\ &\times\exp\left(\frac{i}{\hbar}\left[p_x x+\frac{eEy}{\omega}\cos(\omega t)+m_e\dot{\zeta}(t)\big[y-\zeta(t)\big]+\int_0^t dt' L(\zeta,\dot{\zeta},t')-\Delta_E t\right]\right)\end{aligned} \tag{2.13}$$

where we can identify(let $\alpha \to (n,m)$) the *quasienergies* as

$$\varepsilon_\alpha \equiv \varepsilon_n = \hbar\omega_0\left(n+\frac{1}{2}\right)-\Delta_E \quad \text{where} \quad n=0,1,2,\ldots \quad \text{for any given } m \tag{2.14}$$

which is only depend on one quantum number (n) and *Floquet modes* as

$$\phi_\alpha(x,\tilde{y},t) \equiv \frac{1}{\sqrt{L_x}}\chi_n\big(\tilde{y}-\zeta(t)\big)\exp\left(\frac{i}{\hbar}\left[p_x x+\frac{eE\tilde{y}}{\omega}\cos(\omega t)+m_e\dot{\zeta}(t)\big[\tilde{y}-\zeta(t)\big]+\Lambda\right]\right) \tag{2.15}$$

with

$$\zeta(t) = \frac{eE}{m_e(\omega_0^2-\omega^2)}\sin(\omega t) \quad \text{and} \quad \dot{\zeta}(t) = \frac{eE\omega}{m_e(\omega_0^2-\omega^2)}\cos(\omega t) \tag{2.16}$$

where *Floquet modes* are time-periodic functions that also create a complete orthonormal set. ■

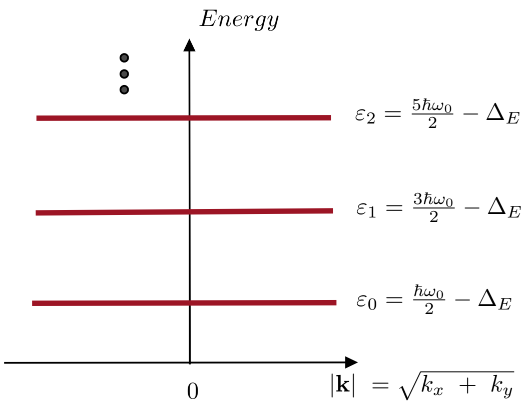

Figure 4: Quasienergies for each Landau levels against magnitude of momentum.

Therefore using Floquet theory, the solutions (Floquet states) for the periodic Hamiltonian (1.5) can be written in position space as

$$\psi_\alpha(x,\tilde{y},t) = \exp\left(-\frac{i}{\hbar}\varepsilon_\alpha t\right)\phi_\alpha(x,\tilde{y},t) \tag{2.17}$$

where

$$\varepsilon_\alpha \equiv \left(\frac{eB\hbar}{m_e}\right)\left(n+\frac{1}{2}\right) - \frac{(eE)^2}{4m_e(\omega_0^2-\omega^2)} \quad \text{where} \quad n=0,1,2,\ldots \tag{2.18}$$

and

$$\begin{aligned}\phi_\alpha(x,\tilde{y},t) \equiv& \frac{1}{\sqrt{L_x}}\chi_n\left(\tilde{y}-\frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}\right)\\ &\times \exp\left(\frac{i}{\hbar}\left[p_x x+\frac{eE\tilde{y}}{\omega}\cos(\omega t)+\frac{eE\omega\tilde{y}}{(\omega_0^2-\omega^2)}\cos(\omega t)\right]\right)\\ &\times \exp\left(\frac{i}{\hbar}\left[-\frac{(eE)^2\omega}{2m_e(\omega^2-\omega_0^2)^2}\sin(2\omega t)+\frac{(eE)^2\left(3\omega_0^2-\omega^2\right)}{8m_e\omega(\omega_0^2-\omega^2)^2}\sin(2\omega t)\right]\right)\end{aligned} \tag{2.19}$$

Now we can write this by more simplying and considering spacial dependencies and using previous subtituting done in Eq. (1.16) and now χ function depend on both quantum numbers because y_0 gives the p_x dependence and we can present as

$$\begin{aligned}\phi_\alpha(x,y,t) \equiv \frac{1}{\sqrt{L_x}}\chi_n\left(y-y_0-\frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}\right)\exp\left(\frac{ip_x}{\hbar}x\right)\exp\left(\frac{i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right](y-y_0)\right)\\ \times\exp\left(\frac{-i}{\hbar}\left[\frac{(eE)^2\left(\omega_0^2+\omega^2\right)}{8\omega m_e(\omega_0^2-\omega^2)^2}\right]\sin(2\omega t)\right)\end{aligned} \tag{2.20}$$

Now we can transform this solution in spacial variable into the momentum space using Fourier trasform over the considering confined space $A = L_xL_y$.

$$\begin{aligned}\phi_\alpha(k_x,k_y,t) = &\int_{-L_y/2}^{L_y/2} dy\ \exp(-ik_y y)\left[\frac{1}{\sqrt{L_x}}\chi_n\left(y-y_0-\frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}\right)\exp\left(\frac{i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y\right)\right]\\ &\times\int_{-L_x/2}^{L_x/2} dx\ \exp(-ik_x x)\left[\exp\left(\frac{ip_x}{\hbar}x\right)\right]\\ &\times\exp\left(\frac{-i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y_0\right)\times\exp\left(\frac{-i}{\hbar}\left[\frac{(eE)^2\left(\omega_0^2+\omega^2\right)}{8\omega m_e(\omega_0^2-\omega^2)^2}\right]\sin(2\omega t)\right)\end{aligned} \tag{2.21}$$

Then this can be re-write as follows

$$\phi_\alpha(k_x,k_y,t) = \exp\left(\frac{-i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y_0\right)\exp\left(\frac{-i}{\hbar}\left[\frac{(eE)^2\left(\omega_0^2+\omega^2\right)}{8\omega m_e(\omega_0^2-\omega^2)^2}\right]\sin(2\omega t)\right)\Theta_\alpha(k_y,t)\delta_{k_x,\frac{p_x}{\hbar}} \tag{2.22}$$

where we used

$$\int_{L_x} dx\ \exp\left(-ik_x x+\frac{ip_x}{\hbar}x\right) = L_x\delta_{k_x,\frac{p_x}{\hbar}} \tag{2.23}$$

and

$$\Theta_\alpha(k_y,t) \equiv \int_{-L_y/2}^{L_y/2} dy\ \exp(-ik_y y)\left[\sqrt{L_x}\chi_n\left(y-y_0-\frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}\right)\exp\left(\frac{i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y\right)\right] \tag{2.24}$$

and this can be simplied as

$$\Theta_\alpha(k_y,t) = \sqrt{L_x}\int_{-L_y/2}^{L_y/2} dy\ \chi_n\left(y-y_0-\frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}\right)\exp\left(-ik_y y+\frac{i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y\right). \tag{2.25}$$

Then by defining

$$\mu(t) \equiv \frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}+y_0 \tag{2.26}$$

and

$$\gamma(t) \equiv \frac{eE\omega_0^2\cos(\omega t)}{\hbar\omega(\omega_0^2-\omega^2)} \tag{2.27}$$

we can re-write this by neglecting time dependencies as

$$\Theta_\alpha(k_y,t) = \sqrt{L_x}\int_{-\infty}^{\infty} dy\ \chi_n(y-\mu)\exp(-i(k_y-\gamma)y). \tag{2.28}$$

We can subtitute following variables

$${k_y}' = k_y - \gamma \quad \text{and} \quad y' = y - \mu \tag{2.29}$$

and for $L_y \to \infty$ this leads to

$$\Theta_\alpha({k_y}',t) = \sqrt{L_x}e^{-i{k_y}'\mu}\int_{-\infty}^{\infty} dy'\ \chi_n(y')\exp\left(-i{k_y}'y'\right) = \sqrt{L_x}e^{-i{k_y}'\mu}\sqrt{\kappa}\int_{-\infty}^{\infty} dy'\ \vartheta_n(\kappa y')\exp\left(-i{k_y}'y'\right) \tag{2.30}$$

We know that $\{\chi_\alpha\}$ are well-known harmonic eigenfunctions(with Gauss-Hermite functions) as given in the Eq. (1.47). However, the equation in (2.30) represents the Fourier transform of the these Gauss-Hermite functions. Due to the symmetric condition [*Ref:E.Celeghini] the Fourier transform of these functions can be represent as

$$\mathcal{FT}[\vartheta_n(\kappa x), x, k] = \frac{i^n}{|\kappa|}\vartheta_n(k/\kappa) \tag{2.31}$$

Therefore

$$\Theta_\alpha({k_y}',t) = \sqrt{L_x}e^{-i{k_y}'\mu} \times \frac{i^n}{\sqrt{\kappa}}\vartheta_n\left(\frac{{k_y}'}{\kappa}\right) = \sqrt{L_x}e^{-i{k_y}'\mu}\tilde{\chi}_n\left({k_y}'\right) \tag{2.32}$$

where

$$\tilde{\chi}_n(k) = \frac{i^n}{\sqrt{2^n n!\sqrt{\pi}}}\left(\frac{1}{\kappa}\right)^{1/2} e^{-\frac{k^2}{\kappa^2}}\mathcal{H}_\alpha\left(\frac{k}{\kappa}\right). \tag{2.33}$$

Using Eq. (2.32) and Eq. (2.22) we can derive that

$$\begin{aligned}\phi_\alpha(k_y,t) = \exp\left(\frac{-i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y_0\right)\exp\left(\frac{-i}{\hbar}\left[\frac{(eE)^2\left(\omega_0^2+\omega^2\right)}{8\omega m_e(\omega_0^2-\omega^2)^2}\right]\sin(2\omega t)\right) \\ \times\sqrt{L_x}e^{-i(k_y-\gamma)\mu}\tilde{\chi}_n(k_y-\gamma)\end{aligned} \tag{2.34}$$

where we included the k_x dependence into α quantum number using m value and this can be re-write subtituting μ and γ values as follows

$$\begin{aligned}\phi_\alpha(k_y,t) = \sqrt{L_x}\exp\left(\frac{-i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y_0\right)\exp\left(\frac{-i}{\hbar}\left[\frac{(eE)^2\left(\omega_0^2+\omega^2\right)}{8\omega m_e(\omega_0^2-\omega^2)^2}\right]\sin(2\omega t)\right) \\ \times\exp\left(-ik_y\frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}\right)\exp\left(\frac{i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]\frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}\right)\exp(-ik_y y_0) \\ \times\exp\left(i\frac{1}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y_0\right)\tilde{\chi}_n(k_y-\gamma)\end{aligned} \tag{2.35}$$

and

$$\begin{aligned}\phi_\alpha(k_y,t) = \sqrt{L_x}\exp\left(\frac{i}{\hbar}\left[\frac{(eE)^2\left(3\omega_0^2-\omega^2\right)}{8\omega m_e(\omega_0^2-\omega^2)^2}\right]\sin(2\omega t)\right) \\ \times\exp\left(-ik_y\left[\frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}+y_0\right]\right)\tilde{\chi}_n(k_y-\gamma).\end{aligned} \tag{2.36}$$

For notation convinient we can introduce few constant as follows

$$b \equiv \frac{(eE)^2\left(3\omega_0^2-\omega^2\right)}{8\hbar\omega m_e(\omega_0^2-\omega^2)^2} \tag{2.37}$$

and

$$d \equiv \frac{eE}{m_e(\omega_0^2-\omega^2)} \tag{2.38}$$

with

$$g \equiv \frac{eE\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)}. \tag{2.39}$$

Therefore we can write Eq. (2.36) as

$$\phi_\alpha(k_y, t) = \sqrt{L_x} e^{ib\sin(2\omega t)} e^{-ik_y[d\sin(\omega t)+y_0]} \tilde{\chi}_n(k_y - g\cos(\omega t)). \tag{2.40}$$

3 Floquet Fermi Goldern Rule

In this section we are going to derive the Floquet Fermi goldern rule for above derived quantum Floquet states using $t - t'$ formalism.

The Floquet states (2.17) fullfills the $t - t'$ Schrödinger equation [*Ref:myReport] as follows

$$i\hbar\frac{\partial}{\partial t}\left|\psi_\alpha(t,t')\right\rangle = H_F(t')\left|\psi_\alpha(t,t')\right\rangle \tag{3.1}$$

where Floquet Hamiltonian given by

$$H_F(t') \equiv H_e(t) - i\hbar\frac{\mathrm{d}}{\mathrm{d}t} \tag{3.2}$$

and

$$\left|\psi_\alpha(t,t')\right\rangle = \exp\left(-\frac{i}{\hbar}\varepsilon_\alpha t\right)\left|\phi_\alpha(t')\right\rangle \tag{3.3}$$

Now for the Eq. (3.1) corresponding time evolution operator satisfy the Schrödinger equation

$$U_0(t,t_0;t') = \exp\left(-\frac{i}{\hbar}H_F(t')[t-t_0]\right) \tag{3.4}$$

Consider a time-independent total perturbation $V(\mathbf{r})$ switched on at the reference time $t = t_0$, then Schrödinger equation becomes

$$i\hbar\frac{\partial}{\partial t}\left|\Psi_\alpha(t,t')\right\rangle = [H_F(t') + V(\mathbf{r})]\left|\Psi_\alpha(t,t')\right\rangle \tag{3.5}$$

and when $t \leq t_0$ both solutions of the Schrödinger equation coincide

$$\left|\psi_\alpha(t,t')\right\rangle = \left|\Psi_\alpha(t,t')\right\rangle \quad \text{when} \quad t \leq t_0 \tag{3.6}$$

Now, we can introduce the interaction picture representation of the $t - t'$ Floquet state as

$$\left|\Psi_\alpha(t,t')\right\rangle_I = U_0^\dagger(t,t_0;t')\left|\Psi_\alpha(t,t')\right\rangle \tag{3.7}$$

and the perturbation in the interaction picture will be

$$V_I(\mathbf{r}) = U_0^\dagger(t,t_0;t')V(\mathbf{r})U_0(t,t_0;t') = V(\mathbf{r}). \tag{3.8}$$

This leads to the Schrödinger eqution in the interction picture

$$i\hbar\frac{\partial}{\partial t}\left|\Psi_\alpha(t,t')\right\rangle_I = V_I(\mathbf{r})\left|\Psi_\alpha(t,t')\right\rangle_I \tag{3.9}$$

with the recursive solution

$$\left|\Psi_\alpha(t,t')\right\rangle_I = \left|\Psi_\alpha(t_0,t')\right\rangle_I + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\; V_I(\mathbf{r})\left|\Psi_\alpha(t_1,t')\right\rangle_I \tag{3.10}$$

Iterating the solution only upto first order (Born approximation) this leads to

$$\left|\Psi_\alpha(t,t')\right\rangle_I \approx \left|\psi_\alpha(t_0,t')\right\rangle + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\; V_I(\mathbf{r})\left|\psi_\alpha(t_0,t')\right\rangle \tag{3.11}$$

and multiply it by $\left\langle\psi_\beta(t_0,t')\right|$ and we will get

$$\left\langle\psi_\beta(t_0,t')\middle|\Psi_\alpha(t,t')\right\rangle_I = \left\langle\psi_\beta(t_0,t')\middle|\psi_\alpha(t_0,t')\right\rangle + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\; \left\langle\psi_\beta(t_0,t')\right| V_I(\mathbf{r})\left|\psi_\alpha(t_0,t')\right\rangle. \tag{3.12}$$

Then introdusing unitory operator U_0 we can re-write this as

$$\begin{aligned}\langle\psi_\beta(t_0,t')|U_0^\dagger(t,t_0;t')|\Psi_\alpha(t,t')\rangle = & \ \langle\psi_\beta(t_0,t')|U_0^\dagger(t,t_0;t')U_0(t,t_0;t')|\psi_\alpha(t_0,t')\rangle \\ & + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1 \ \langle\psi_\beta(t_0,t')|\, U_0^\dagger(t_1,t_0;t')V(\mathbf{r})U_0(t_1,t_0;t')\,|\psi_\alpha(t_0,t')\rangle\end{aligned} \tag{3.13}$$

and this can be simplied as

$$\langle\psi_\beta(t,t')|\Psi_\alpha(t,t')\rangle = \langle\psi_\beta(t,t')|\psi_\alpha(t,t')\rangle + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1 \ \langle\psi_\beta(t_1,t')|\, V(\mathbf{r})\,|\psi_\alpha(t_1,t')\rangle\,. \tag{3.14}$$

Since our $t-t'$ Floquet states are orthonormal [*Ref:myReport- t-t' formalism] we can derive that

$$\langle\psi_\beta(t,t')|\Psi_\alpha(t,t')\rangle = \delta_{\alpha\beta}\exp(i\omega[t'-t]) + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1 \ \langle\psi_\beta(t_1,t')|\, V(\mathbf{r})\,|\psi_\alpha(t_1,t')\rangle\,. \tag{3.15}$$

Now, set $t_0 = 0$ and for a case $\alpha \neq \beta$ where we can represent $\alpha = (n_\alpha, m_\alpha)$ and $\beta = (n_\beta, m_\beta)$ and this will simplied to

$$\langle\psi_\beta(t,t')|\Psi_\alpha(t,t')\rangle = -\frac{i}{\hbar}\int_{0}^{t} dt_1 \ \langle\psi_\beta(t_1,t')|\, V(\mathbf{r})\,|\psi_\alpha(t_1,t')\rangle\,. \tag{3.16}$$

In addition, since our Floquet states create a basis for composite space we can represent any solution using our Floquet states

$$|\Psi_\alpha(t,t')\rangle = \sum_\beta a_{\alpha\beta}(t,t')\,|\psi_\beta(t,t')\rangle\,. \tag{3.17}$$

Therefore we can derive a equation for this *scattering amplitude* as

$$a_{\alpha\beta}(t,t') = \langle\psi_\beta(t,t')|\Psi_\alpha(t,t')\rangle = -\frac{i}{\hbar}\int_{0}^{t} dt_1 \ \langle\psi_\beta(t_1,t')|\, V(\mathbf{r})\,|\psi_\alpha(t_1,t')\rangle\,. \tag{3.18}$$

Now lets assume a scattering event from a $t-t'$ Floquet state $|\psi_\beta(t,t')\rangle$ into another $t-t'$ Floquet state $|\Psi_\alpha(t,t')\rangle$ with constant quansienergy ε given as follows

$$|\Psi_\alpha(t,t')\rangle = \exp\left(-\frac{i}{\hbar}\varepsilon t\right)|\Phi_\alpha(t')\rangle \tag{3.19}$$

Now consider a scattering event

$$\psi_\beta(\mathbf{k}',t,t') = \exp\left(-\frac{i}{\hbar}\varepsilon_\beta t\right)\phi_\beta(\mathbf{k}',t') \longrightarrow \Psi_\alpha(\mathbf{k},t,t') = \exp\left(-\frac{i}{\hbar}\varepsilon t\right)\Phi_\alpha(\mathbf{k},t') \tag{3.20}$$

Here we need to undestand a state of this considering system only be represented by two independent quantum numbers which are n energy eigen states and m quantum number which represents the qunatized momentum in x direction values. Lets calculate the scattering amplitutde of the above mentioned scattering scenario using the equation derived in (3.18).

$$\begin{aligned}a_{\alpha\beta}(\mathbf{k}',\mathbf{k},t,t') &= -\frac{i}{\hbar}\int_{0}^{t} dt_1 \ \langle\psi_{\beta,\mathbf{k}'}(t_1,t')|\, V(\mathbf{r})\,|\psi_{\alpha,\mathbf{k}}(t_1,t')\rangle \\ &= -\frac{i}{\hbar}\int_{0}^{t} dt_1 \ e^{\frac{i}{\hbar}(\varepsilon_\beta-\varepsilon)t_1}\,\langle\phi_{\beta,\mathbf{k}'}(t')|\, V(\mathbf{r})\,|\phi_{\alpha,\mathbf{k}}(t')\rangle\end{aligned} \tag{3.21}$$

Next assuimg this scenario for long time $t \to \infty$ we can turn this integral into a delta distrubution as follows

$$\begin{aligned}a_{\alpha\beta}(\mathbf{k}',\mathbf{k},t,t') &= -\frac{i}{\hbar}\lim_{t\to\infty}\left[\int_{-t/2}^{t/2} dt_1 \ e^{\frac{i}{\hbar}(\varepsilon_\beta-\varepsilon)t_1}\,\langle\phi_{\beta,\mathbf{k}'}(t')|\, V(\mathbf{r})\,|\phi_{\alpha,\mathbf{k}}(t')\rangle\right] \\ &= -2\pi i\delta(\varepsilon_\beta-\varepsilon)\,\langle\phi_{\beta,\mathbf{k}'}(t')|\, V(\mathbf{r})\,|\phi_{\alpha,\mathbf{k}}(t')\rangle\end{aligned} \tag{3.22}$$

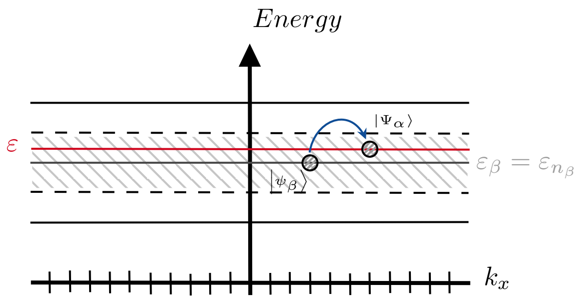

Figure 5: Scattering from $|\psi_\beta(t,t')\rangle$ to constant energy state $|\Psi_\alpha(t,t')\rangle$ due to scattering potential created by impurities.

Now lets consider about the inner product of the above derivation. Using completeness properties we can write that as follows

$$\begin{aligned}Q &\equiv \langle\phi_{\beta,\mathbf{k}'}(t')|\, V(\mathbf{r})\, |\phi_{\alpha,\mathbf{k}}(t')\rangle \\ &= \sum_{\mathbf{k}}\sum_{\mathbf{k}'} \langle\phi_{\beta,\mathbf{k}'}(t')|\mathbf{k}'\rangle\, \langle\mathbf{k}'|V(\mathbf{r})|\mathbf{k}\rangle\, \langle\mathbf{k}|\phi_{\alpha,\mathbf{k}}(t')\rangle\end{aligned} \tag{3.23}$$

and seperating x and y directional momentums we can modify this as follows (Assuming $L_y \to \infty$) and then using $\frac{1}{L_y}\sum_{k_y} = \frac{1}{2\pi}\int k_y$

$$\begin{aligned}Q &\equiv \langle\phi_{\beta,\mathbf{k}'}(t')|\, V(\mathbf{r})\, |\phi_{\alpha,\mathbf{k}}(t')\rangle \\ &= \frac{L_y^2}{4\pi^2}\sum_{k_x}\sum_{k'_x}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_y dk'_y\ \phi_\beta(\mathbf{k}',t')\,\langle\mathbf{k}'|V(\mathbf{r})|\mathbf{k}\rangle\,\phi_\alpha(\mathbf{k},t').\end{aligned} \tag{3.24}$$

For a random white scattering potential we can represent the inner product of scattering potential with momentum as a constant value as

$$V_{\mathbf{k}',\mathbf{k}} \equiv \langle\mathbf{k}'|V(\mathbf{r})|\mathbf{k}\rangle\,. \tag{3.25}$$

In this study, the perturbation potential is assumed to be formed by an ensemble of randomly distributed impurities, since randomimpurities iin a disorded metal is a better approximation for experimental results.

Consider N_{imp} identical impurities positioned at the randomly distributed but fixed potions $\mathbf{r}_i$. The elastic scattering potential $V(\mathbf{r})$ is then given by the sum over uncorrelated single impurity potentials $\upsilon(\mathbf{r})$

$$V(\mathbf{r}) \equiv \sum_{i=1}^{N_{imp}} \upsilon(\mathbf{r}-\mathbf{r}_i). \tag{3.26}$$

Now assume that the perturbation $V(\mathbf{r})$ is a Gaussian random potential where one can choose the zero of enerrgy such that the potential is zero on average. This model characterized by [*Ref: e.Akkermans G. Montambaux]

$$\langle\upsilon(\mathbf{r})\rangle_{imp} = 0 \tag{3.27}$$

$$\langle\upsilon(\mathbf{r})\upsilon(\mathbf{r}')\rangle_{imp} = \Upsilon(\mathbf{r}-\mathbf{r}') \tag{3.28}$$

where $\langle\cdot\rangle_{imp}$ denoted the average over realizations of the impurity disorder. In addition, this model assume that $\upsilon(\mathbf{r}-\mathbf{r}')$ only dependes on the position difference $|\mathbf{r}-\mathbf{r}'|$ and it decays with a characteristic leangth r_c. Since the study considers the case where the waveleagth of radiation or scattering electrons is much faster than r_c, it is good approximation to make two-point correlation function to be

$$\langle\upsilon(\mathbf{r})\upsilon(\mathbf{r}')\rangle_{imp} = \Upsilon_{imp}^2\delta(\mathbf{r}-\mathbf{r}') \tag{3.29}$$

and a random potential $V(\mathbf{r})$ with this property is called white noise [*Ref: e.Akkermans G. Montambaux]. Then we can choose approximately total scattering potential as

$$V(\mathbf{r}) = \sum_{i=1}^{N_{imp}} \Upsilon_{imp}\delta(\mathbf{r}-\mathbf{r}_i). \tag{3.30}$$

Now we can calculate the Eq. (3.25) using this assumption as follows

$$\begin{aligned} V_{\mathbf{k}',\mathbf{k}} &= \langle \mathbf{k}'|V(\mathbf{r})|\mathbf{k}\rangle \\ &= \left\langle \mathbf{k}' \middle| \sum_{i=1}^{N_{imp}} \Upsilon_{imp}\delta(\mathbf{r}-\mathbf{r}_i) \middle| \mathbf{k} \right\rangle \\ &= \left\langle \mathbf{k}' \middle| \sum_{i=1}^{N_{imp}} \Upsilon_{imp}\delta(x-x_i)\delta(y-y_i) \middle| \mathbf{k} \right\rangle \\ &= \left\langle k'_x \middle| \sum_{x_i=1}^{N_{imp}} \Upsilon_{imp}\delta(x-x_i) \middle| k_x \right\rangle \sum_{y_i=1}^{N_{imp}} \int_{-\infty}^{\infty} dy\ e^{ik'_y y}\delta(y-y_i)e^{-ik_y y} \\ &= \left\langle k'_x \middle| \sum_{x_i=1}^{N_{imp}} \Upsilon_{imp}\delta(x-x_i) \middle| k_x \right\rangle \sum_{y_i=1}^{N_{imp}} e^{i\left(k'_y-k_y\right)y_i} \end{aligned} \tag{3.31}$$

Assuming the total umber of scatterers N_{imp} is macroscopically large we can achieve following expression

$$\begin{aligned} V_{\mathbf{k}',\mathbf{k}} &= \left\langle k'_x \middle| \sum_{x_i=1}^{N_{imp}} \Upsilon_{imp}\delta(x-x_i) \middle| k_x \right\rangle \frac{N_{imp}}{L_y} \int_{-\infty}^{\infty} dy_i\ e^{i\left(k'_y-k_y\right)y_i} \\ &= \frac{N_{imp}}{L_y} V_{k'_x,k_x}\delta(k'_y-k_y) \end{aligned} \tag{3.32}$$

where

$$V_{k'_x,k_x} \equiv \left\langle k'_x \middle| \sum_{x_i=1}^{N_{imp}} \Upsilon_{imp}\delta(x-x_i) \middle| k_x \right\rangle \tag{3.33}$$

Therefore, using the Eq. (2.36), the Eq. (3.24) modified to (we can change varable $t' \to t$)

$$\begin{aligned} Q = \sum_{k_x}\sum_{k'_x} & \frac{N_{imp}L_y V_{k'_x,k_x}}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_y dk'_y\ \delta(k'_y-k_y) \\ &\times \sqrt{L_x}\exp(-ib\sin(2\omega t))\exp(ik'_y[d\sin(\omega t)+y'_0])\tilde{\chi}_{n_\beta}(k'_y-g\cos(\omega t)) \\ &\times \sqrt{L_x}\exp(ib\sin(2\omega t))\exp(-ik_y[d\sin(\omega t)+y_0])\tilde{\chi}_{n_\alpha}(k_y-g\cos(\omega t)) \end{aligned} \tag{3.34}$$

and we can simplify this as

$$\begin{aligned} Q = \sum_{k_x}\sum_{k'_x} & \frac{N_{imp}L_x L_y V_{k'_x,k_x}}{4\pi^2} \int_{-\infty}^{\infty} dk_y \\ &\times \exp(ik_y y'_0)\tilde{\chi}_{n_\beta}(k_y-g\cos(\omega t))\exp(-ik_y y_0)\tilde{\chi}_{n_\alpha}(k_y-g\cos(\omega t)) \end{aligned} \tag{3.35}$$

and this can re-write as

$$Q = \sum_{k_x}\sum_{k'_x} \frac{N_{imp}L_x L_y V_{k'_x,k_x}}{4\pi^2} I \tag{3.36}$$

where

$$I \equiv \int_{-\infty}^{\infty} dk_y\ \tilde{\chi}_{n_\beta}(k_y-g\cos(\omega t))\tilde{\chi}_{n_\alpha}(k_y-g\cos(\omega t))\exp(-ik_y[y_0-y'_0]). \tag{3.37}$$

To avoid the energy tramision from external high-frequency field and electrons in the system, the applied radiation should be purely dressing field. Therefore, the only effect of the dressing field on 2DEG is the renormalization of the probability of elastic electron scattering within same Landau level ($n_\alpha = n_\beta$). Therefore Eq. (3.37) can be modified to

$$I \equiv \int_{-\infty}^{\infty} dk_y \ \tilde{\chi}^2_{n_\beta}(k_y - g\cos(\omega t)) \exp(-ik_y[y_0 - {y'}_0]). \tag{3.38}$$

Lets consider about this integral and we can calculate it as using the following subtitution. Let

$$k_y - g\cos(\omega t) = \bar{k}_y \longrightarrow dk_y = d\bar{k}_y \tag{3.39}$$

and this leads to

$$I \equiv 2\pi \times \frac{1}{2\pi}\int_{-\infty}^{\infty} d\bar{k}_y \ \tilde{\chi}^2_{n_\alpha}(\bar{k}_y) \exp\big(-i\big(\bar{k}_y + g\cos(\omega t)\big)(y_0 - {y'}_0)\big). \tag{3.40}$$

Using Fourier transform of Gauss-Hermite functions and convolution theorem we can write this as

$$I \equiv 2\pi \exp(g[{y'}_0 - y_0]\cos(\omega t)) \int_{-\infty}^{\infty} dy \ \chi_{n_\beta}(y)\chi_{n_\beta}(y_0 - {y'}_0 - y). \tag{3.41}$$

Therefore the scatterng amplitude (3.22) will modified to

$$a_{\alpha\beta}({k'}_x, k_x, t) = -2\pi i\delta(\varepsilon_\beta - \varepsilon)\sum_{k_x}\sum_{{k'}_x} \frac{N_{imp}L_xL_yV_{{k'}_x,k_x}}{4\pi^2} I \tag{3.42}$$

Consideirng qunatized momentum given in x direction derived in Eq. (**??**), we can identify the non-zero values for scattering amplitude using following conditions

$${k'}_x = \frac{p_{x_\beta}}{\hbar} = m'\frac{2\pi}{L_x} \quad \text{and} \quad k_x = \frac{p_{x_\alpha}}{\hbar} = m\frac{2\pi}{L_x}. \tag{3.43}$$

Then we can simplfied scattering amplitude for given ${k'}_x$ and k_x as

$$\begin{aligned} a_{\alpha\beta}({k'}_x, k_x, t) = \delta(\varepsilon_\beta - \varepsilon)\left[\frac{-iN_{imp}L_xL_yV_{{k'}_x,k_x}}{2\pi}\right] &\exp(g[{y'}_0 - y_0]\cos(\omega t)) \\ &\times \int_{-\infty}^{\infty} dy \ \chi_{n_\beta}(y)\chi_{n_\beta}(y_0 - {y'}_0 - y) \end{aligned} \tag{3.44}$$

Since this scattering amplitude is time-periodic we can write this as a Fourier series expansion

$$a_{\alpha\beta}({k'}_x, k_x, t) = \sum_{l=-\infty}^{\infty} a^l_{\alpha\beta}({k'}_x, k_x)e^{-il\omega t}. \tag{3.45}$$

In addition, using Jacobi-Anger expansion

$$e^{iz\cos(\theta)} = \sum_{l=-\infty}^{\infty} i^l J_l(z)e^{-il\theta} \tag{3.46}$$

we can re-write the Eq.(3.44) as folllows

$$\begin{aligned} a_{\alpha\beta}({k'}_x, k_x, t) = \delta(\varepsilon_\beta - \varepsilon)\left[\frac{-iN_{imp}L_xL_yV_{{k'}_x,k_x}}{2\pi}\right] &\sum_{l=-\infty}^{\infty} i^l J_l(g[{y'}_0 - y_0])e^{-il\omega t} \\ &\times \int_{-\infty}^{\infty} dy \ \chi_{n_\beta}(y)\chi_{n_\beta}(y_0 - {y'}_0 - y) \end{aligned} \tag{3.47}$$

$$\begin{aligned} a_{\alpha\beta}({k'}_x, k_x, t) = \sum_{l=-\infty}^{\infty} \delta(\varepsilon_\beta - \varepsilon)\left[\frac{-i^{l+1}N_{imp}L_xL_yV_{{k'}_x,k_x}}{2\pi}\right] &J_l(g[{y'}_0 - y_0]) \\ \times \int_{-\infty}^{\infty} dy \ \chi_{n_\beta}(y)\chi_{n_\beta}(y_0 - {y'}_0 - y)&e^{-il\omega t} \end{aligned} \tag{3.48}$$

Then we can identified the Fourier series component as

$$a^l_{\alpha\beta}(k'_x, k_x) = \delta(\varepsilon_\beta - \varepsilon)\left[\frac{-i^{l+1}N_{imp}L_xL_yV_{k'_x,k_x}}{2\pi}\right]J_l(g[y'_0 - y_0])\int_{-\infty}^{\infty}dy\ \chi_{n_\beta}(y)\chi_{n_\beta}(y_0 - y'_0 - y) \tag{3.49}$$

Now one can introduce the definition of the *transition probability matrix* as

$$(A_{\alpha\beta}(k'_x, k_x))_{l,l'} \equiv a^l_{\alpha\beta}(k'_x, k_x)\left[a^{l'}_{\alpha\beta}(k'_x, k_x)\right]^* \tag{3.50}$$

and this becomes

$$\begin{aligned}(A_{\alpha\beta}(k'_x, k_x))_{l,l'} = &\left[\frac{N^2_{imp}A^2|V_{k'_x,k_x}|^2}{4\pi^2}\right]J_l(g[y'_0 - y_0])J_{l'}(g[y'_0 - y_0])\delta^2(\varepsilon_\beta - \varepsilon) \\ &\times\int_{-\infty}^{\infty}dy\ \chi_{n_\beta}(y)\chi_{n_\beta}(y_0 - y'_0 - y)\int_{-\infty}^{\infty}d\bar{y}\ \chi_{n_\beta}(\bar{y})\chi_{n_\beta}(y_0 - y'_0 - \bar{y}).\end{aligned} \tag{3.51}$$

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Considering orthonormality of Gusee-Hermite functions we can reduce these intragal into one variable and derive

$$\begin{aligned}(A_{\alpha\beta}(k'_x, k_x))_{l,l'} = &\left[\frac{32\pi^5N^2_{imp}|V_{k'_x,k_x}|^2}{L^2_xL^4_y}\right]J_l(g[y'_0 - y_0])J_{l'}(g[y'_0 - y_0])\delta^2(\varepsilon_\beta - \varepsilon) \\ &\times\int_{-\infty}^{\infty}dy\ \tilde{\chi}^2_{n_\beta}(y)\tilde{\chi}^2_{n_\beta}(y_0 - y'_0 - y).\end{aligned} \tag{3.52}$$

Then desribing the square of the delta distribution using following procedure

$$\delta^2(\varepsilon) = \delta(\varepsilon)\delta(0) = \frac{\delta(\varepsilon)}{2\pi\hbar}\int_{-t/2}^{t/2}e^{i0\times t'/\hbar}dt' = \frac{\delta(\varepsilon)t}{2\pi\hbar} \tag{3.53}$$

one can modify our derivation in Eq. (3.51) as

$$\begin{aligned}(A_{\alpha\beta}(k'_x, k_x))_{l,l'} = &\left[\frac{32\pi^5N^2_{imp}|V_{k'_x,k_x}|^2}{L^2_xL^4_y}\right]J_l(g[y'_0 - y_0])J_{l'}(g[y'_0 - y_0])\delta(\varepsilon_\beta - \varepsilon)\frac{t}{2\pi\hbar} \\ &\times\int_{-\infty}^{\infty}dy\ \tilde{\chi}^2_{n_\beta}(y)\tilde{\chi}^2_{n_\beta}(y_0 - y'_0 - y).\end{aligned} \tag{3.54}$$

Then performing thetime derivation of each matrix element yeild the *transition amplitude matrix* as folllows

$$\begin{aligned}\Gamma^{ll'}_{\alpha\beta}(k'_x, k_x) &\equiv \frac{d(A_{\alpha\beta}(k'_x, k_x))_{l,l'}}{dt} \\ &= \Lambda|V_{k'_x,k_x}|^2\delta(\varepsilon_\beta - \varepsilon)J_l(g[y'_0 - y_0])J_{l'}(g[y'_0 - y_0])\int_{-\infty}^{\infty}dy\ \tilde{\chi}^2_{n_\beta}(y)\tilde{\chi}^2_{n_\beta}(y_0 - y'_0 - y)\end{aligned} \tag{3.55}$$

where

$$\Lambda \equiv \frac{16\pi^4N^2_{imp}}{L^2_xL^4_y\hbar} \tag{3.56}$$

Now using defintion of y_0 given in Eq. (1.11) we can write that

$$y_0 - y'_0 = -\frac{p_{x_\alpha}}{eB} + \frac{p_{x_\beta}}{eB} = \frac{\hbar k'_x}{eB} - \frac{\hbar k_x}{eB} = \frac{\hbar}{eB}[k'_x - k_x] \tag{3.57}$$

and this leads Eq. (3.56) to

$$\begin{aligned}\Gamma^{ll'}_{\alpha\beta}(k'_x, k_x) = \Lambda|V_{k'_x,k_x}|^2\delta(\varepsilon_\beta - \varepsilon)&J_l\left(\frac{g\hbar}{eB}[k_x - k'_x]\right)J_{l'}\left(\frac{g\hbar}{eB}[k_x - k'_x]\right) \\ &\times\int_{-\infty}^{\infty}dy\ \tilde{\chi}^2_{n_\beta}(y)\tilde{\chi}^2_{n_\beta}\left(\frac{\hbar}{eB}[k'_x - k_x] - y\right)\end{aligned} \tag{3.58}$$

An impurity average of white noise potential allows to identify $\langle |V_{k'_x,k_x}|^2 \rangle = V_{imp}$ and the inverse scattering time matrix is the sum over all momentum over the transition probability matrix

$$\left(\frac{1}{\tau(\varepsilon, k_x)}\right)^{ll'}_{\alpha\beta} \equiv \frac{1}{L_x L_y} \sum_{k'_x} \langle \Gamma^{ll'}_{\alpha\beta}(k'_x, k_x) \rangle_{imp} \tag{3.59}$$

and this implies

$$\begin{aligned}\Gamma^{ll'}_{\alpha\beta}(k'_x, k_x) = \frac{\Lambda V_{imp}}{L_x L_y} \sum_{k'_x} \delta(\varepsilon_\beta - \varepsilon) J_l\left(\frac{g\hbar}{eB}[k_x - k'_x]\right) J_{l'}\left(\frac{g\hbar}{eB}[k_x - k'_x]\right) \\ \times \int_{-\infty}^{\infty} dy\, \tilde{\chi}^2_{n_\beta}(y) \tilde{\chi}^2_{n_\beta}\left(\frac{\hbar}{eB}[k'_x - k_x] - y\right)\end{aligned} \tag{3.60}$$

For the 2-dimentional case introduce the momentum continuum limit as folllows

$$\frac{1}{L_x L_y} \sum_{k'_x} \longrightarrow \frac{1}{4\pi^2} \int dk'_x \tag{3.61}$$

and this leads to

$$\begin{aligned}\left(\frac{1}{\tau(\varepsilon, k_x)}\right)^{ll'}_{\alpha\beta} = \frac{\Lambda V_{imp}}{4\pi^2} \delta(\varepsilon_\beta - \varepsilon) \int_{-\infty}^{\infty} dk'_x J_l\left(\frac{g\hbar}{eB}[k_x - k'_x]\right) J_{l'}\left(\frac{g\hbar}{eB}[k_x - k'_x]\right) \\ \times \int_{-\infty}^{\infty} dy\, \tilde{\chi}^2_{n_\beta}(y) \tilde{\chi}^2_{n_\beta}\left(\frac{\hbar}{eB}[k'_x - k_x] - y\right)\end{aligned} \tag{3.62}$$

Using following subtitution

$$y = \frac{\hbar \bar{k}}{eB} \longrightarrow dy = \frac{\hbar}{eB} d\bar{k} \tag{3.63}$$

we can modify above derivation as

$$\begin{aligned}\left(\frac{1}{\tau(\varepsilon, k_x)}\right)^{ll'}_{\alpha\beta} = \frac{\Lambda V_{imp}}{4\pi^2} \delta(\varepsilon_\beta - \varepsilon) \int_{-\infty}^{\infty} dk'_x J_l\left(\frac{g\hbar}{eB}[k_x - k'_x]\right) J_{l'}\left(\frac{g\hbar}{eB}[k_x - k'_x]\right) \\ \times \frac{\hbar}{eB} \int_{-\infty}^{\infty} d\bar{k}\, \tilde{\chi}^2_{n_\beta}\left(\frac{\hbar}{eB}\bar{k}\right) \tilde{\chi}^2_{n_\beta}\left(\frac{\hbar}{eB}\left[k'_x - k_x - \bar{k}\right]\right).\end{aligned} \tag{3.64}$$

Since squared Guess-Hermite functions are even function around zero we can re-write above derived expression as

$$\begin{aligned}\left(\frac{1}{\tau(\varepsilon, k_x)}\right)^{ll'}_{\alpha\beta} = \frac{\Lambda V_{imp}}{4\pi^2} \delta(\varepsilon_\beta - \varepsilon) \int_{-\infty}^{\infty} dk'_x J_l\left(\frac{g\hbar}{eB}[k_x - k'_x]\right) J_{l'}\left(\frac{g\hbar}{eB}[k_x - k'_x]\right) \\ \times \frac{\hbar}{eB} \int_{-\infty}^{\infty} d\bar{k}\, \tilde{\chi}^2_{n_\beta}\left(\frac{\hbar}{eB}\bar{k}\right) \tilde{\chi}^2_{n_\beta}\left(\frac{\hbar}{eB}\left[\bar{k} - (k_x - k'_x)\right]\right)\end{aligned} \tag{3.65}$$

and finally we can derive our expression for the *inverse scattering time matrix* for Nth Landau level (let $n_\alpha = n_\beta = N$)

$$\begin{aligned}\left(\frac{1}{\tau(\varepsilon, k_x)}\right)^{ll'}_{N} = \frac{4\pi^2 N^2_{imp} V_{imp}}{eB L_x^2 L_y^4} \delta(\varepsilon - \varepsilon_N) \int_{-\infty}^{\infty} dk'_x \int_{-\infty}^{\infty} d\bar{k}\; J_l\left(\frac{g\hbar}{eB}[k_x - k'_x]\right) J_{l'}\left(\frac{g\hbar}{eB}[k_x - k'_x]\right) \\ \times \tilde{\chi}^2_{N}\left(\frac{\hbar}{eB}\bar{k}\right) \tilde{\chi}^2_{N}\left(\frac{\hbar}{eB}\left[\bar{k} - (k_x - k'_x)\right]\right).\end{aligned} \tag{3.66}$$