

Magnetic propeties of a two dimentional electron gas strongly coupled to lights

Kosala Herath

April 21, 2021

1 Schrödinger problem for Landau levels in dressed 2DEG

Our analysis is consider on 2 dimentional electronic gas which has distrubuted in (x, y) plane in configuration space. We are going to examine the properties of 2DEG with stationary magnetic field

$$\mathbf{B} = (0, 0, B)^T \quad (1.1)$$

which directed on z axis and a linearly y -polarized strong electromagnetic wave (dressing field) with electric field given by

$$\mathbf{E} = (0, E \sin(\omega t), 0)^T \quad (1.2)$$

which also propagate in z direction. Here B and E represent the amplitude of the stationary magnetic field and electric field of dressing field.

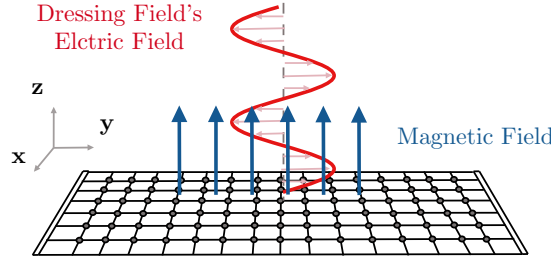


Figure 1: Stationary magnetic filed (blue color) and Strong EM wave (red color) applied to the 2DEG.

Using Landau gauge for the stationary magnetic field we can represent it using vector potential as

$$\mathbf{A}_s = (-By, 0, 0)^T \quad (1.3)$$

and choosing Coulomb gauge the dressing field can be present as the following vector potential

$$\mathbf{A}_d(t) = (0, [E/\omega] \cos(\omega t), 0)^T. \quad (1.4)$$

Now the Hamiltonian of an electron in 2DEG can be reads as

$$\hat{H}_e(t) = \frac{1}{2m_e} \left[\hat{\mathbf{p}} - e(\mathbf{A}_s + \mathbf{A}_d(t)) \right]^2 \quad (1.5)$$

where m_e is the effective mass of the electron and e is the magnitude (without considering the sign of the charge) of the electron charge. This can be simplified to

$$\hat{H}_e(t) = \frac{1}{2m_e} \left[(\hat{p}_x + eBy)\mathbf{e}_x + \left(\hat{p}_y - \frac{eE}{\omega} \cos(\omega t) \right) \mathbf{e}_y \right]^2 \quad (1.6)$$

where \mathbf{e}_x and \mathbf{e}_y are unit vectors along x and y directions respectively. Moreover,

$$\hat{H}_e(t) = \frac{1}{2m_e} \left[(\hat{p}_x + eBy)^2 + \left(\hat{p}_y - \frac{eE}{\omega} \cos(\omega t) \right)^2 \right] \quad (1.7)$$

Since $[\hat{H}_e(t), \hat{p}_x] = 0$ both operators share same eigenvalue and eigen functions which are free electron wave functions. Therefore we can modify the Hamiltonian as follows

$$\hat{H}_e(t) = \frac{1}{2m_e} \left[(p_x + eBy)^2 + \left(\hat{p}_y - \frac{eE}{\omega} \cos(\omega t) \right)^2 \right]. \quad (1.8)$$

Using momentum operator definition

$$\hat{p}_y = -i\hbar \frac{\partial}{\partial y} \quad (1.9)$$

we can modify Eq. (1.8) as

$$\begin{aligned} \hat{H}_e(t) &= \frac{1}{2m_e} \left[(p_x + eBy)^2 + \left(-i\hbar \frac{\partial}{\partial y} - \frac{eE}{\omega} \cos(\omega t) \right)^2 \right] \\ &= \frac{1}{2m_e} \left[(p_x + eBy)^2 + \left(i\hbar \frac{\partial}{\partial y} + \frac{eE}{\omega} \cos(\omega t) \right)^2 \right]. \end{aligned} \quad (1.10)$$

Define the *center of the cyclotron orbit* along y axis as

$$y_0 \equiv \frac{-p_x}{eB} \quad (1.11)$$

and the *cyclotron frequency* as

$$\omega_0 \equiv \frac{eB}{m_e}. \quad (1.12)$$

Then the Hamiltonian will leads to

$$\hat{H}_e(t) = \frac{m_e \omega_0^2}{2} (y - y_0)^2 + \frac{1}{2m_e} \left(i\hbar \frac{\partial}{\partial y} + \frac{eE}{\omega} \cos(\omega t) \right)^2 \quad (1.13)$$

$$\begin{aligned} \hat{H}_e(t) &= \frac{m_e \omega_0^2}{2} (y - y_0)^2 + \frac{1}{2m_e} \left(-\hbar^2 \frac{\partial^2}{\partial y^2} + i\hbar \frac{\partial}{\partial y} \left[\frac{eE}{\omega} \cos(\omega t) \right] \right. \\ &\quad \left. + \frac{i\hbar eE}{\omega} \cos(\omega t) \frac{\partial}{\partial y} + \frac{e^2 E^2}{\omega^2} \cos^2(\omega t) \right) \end{aligned} \quad (1.14)$$

$$\hat{H}_e(t) = \frac{m_e \omega_0^2}{2} (y - y_0)^2 + \frac{1}{2m_e} \left(-\hbar^2 \frac{\partial^2}{\partial y^2} + \frac{2i\hbar eE}{\omega} \cos(\omega t) \frac{\partial}{\partial y} + \frac{e^2 E^2}{\omega^2} \cos^2(\omega t) \right). \quad (1.15)$$

Let

$$(y - y_0) \rightarrow y \quad (1.16)$$

and then this becomes

$$\hat{H}_e(t) = \frac{m_e \omega_0^2}{2} y^2 + \frac{1}{2m_e} \left(-\hbar^2 \frac{\partial^2}{\partial y^2} + \frac{2i\hbar eE}{\omega} \cos(\omega t) \frac{\partial}{\partial y} + \frac{e^2 E^2}{\omega^2} \cos^2(\omega t) \right). \quad (1.17)$$

Now assume that the solution for the time-dependent schrödinger equation

$$i\hbar \frac{d\psi}{dt} = \hat{H}_e(t) \psi \quad (1.18)$$

can be represent by the following form

$$\psi(\mathbf{r}, t) = \frac{1}{\sqrt{L_x}} \exp \left(\frac{ip_x x}{\hbar} + \frac{ieE(y - y_0)}{\hbar\omega} \cos(\omega t) \right) \phi(y - y_0, t). \quad (1.19)$$

Using the same subttution from Eq. (1.16) this becomes

$$\psi(x, y, t) = \frac{1}{\sqrt{L_x}} \exp \left(\frac{ip_x x}{\hbar} + \frac{ieEy}{\hbar\omega} \cos(\omega t) \right) \phi(y, t). \quad (1.20)$$

Defining

$$\varphi(x, y, t) \equiv \frac{1}{\sqrt{L_x}} \exp \left(\frac{ip_x x}{\hbar} + \frac{ieEy}{\hbar\omega} \cos(\omega t) \right) \quad (1.21)$$

we can simplify the the Eq. (1.20) as

$$\psi(x, y, t) = \varphi(x, y, t)\phi(y, t). \quad (1.22)$$

Let's substitute Eq. (1.20) and Eq. (1.17) into Eq. (1.18) and we can observe that

$$\begin{aligned} \text{L.H.S} &= i\hbar \frac{d\psi}{dt} = i\hbar \left(\frac{d\varphi}{dt} \phi + \varphi \frac{d\phi}{dt} \right) = i\hbar \left(\left[\frac{-ieEy}{\hbar} \sin(\omega t) \right] \varphi \phi + \varphi \frac{d\phi}{dt} \right) \\ &= [eEy \sin(\omega t)] \varphi \phi + i\hbar \varphi \frac{d\phi}{dt} \end{aligned} \quad (1.23)$$

and

$$\begin{aligned} \text{R.H.S} &= \hat{H}_e(t)\psi \\ &= \left[\frac{m_e \omega_0^2}{2} y^2 + \frac{1}{2m_e} \left(-\hbar^2 \frac{\partial^2}{\partial y^2} + \frac{2i\hbar eE}{\omega} \cos(\omega t) \frac{\partial}{\partial y} + \frac{e^2 E^2}{\omega^2} \cos^2(\omega t) \right) \right] \varphi \phi \end{aligned} \quad (1.24)$$

where we will calculate this part by part as follows:

$$\begin{aligned} \frac{-\hbar^2}{2m_e} \frac{\partial^2}{\partial y^2} (\varphi \phi) &= \frac{-\hbar^2}{2m_e} \frac{\partial}{\partial y} \left[\left(\frac{ieE}{\hbar \omega} \cos(\omega t) \right) \varphi \phi + \varphi \frac{\partial \phi}{\partial y} \right] \\ &= \frac{-\hbar^2}{2m_e} \left[\left(\frac{ieE}{\hbar \omega} \cos(\omega t) \right)^2 \varphi \phi + \left(\frac{ieE}{\hbar \omega} \cos(\omega t) \right) \varphi \frac{\partial \phi}{\partial y} + \left(\frac{ieE}{\hbar \omega} \cos(\omega t) \right) \varphi \frac{\partial \phi}{\partial y} + \varphi \frac{\partial^2 \phi}{\partial y^2} \right] \\ &= \left(\frac{e^2 E^2}{2m_e \omega^2} \cos^2(\omega t) \right) \varphi \phi - \left(\frac{ieE\hbar}{m_e \omega} \cos(\omega t) \right) \varphi \frac{\partial \phi}{\partial y} - \frac{\hbar^2}{2m_e} \varphi \frac{\partial^2 \phi}{\partial y^2} \end{aligned} \quad (1.25)$$

and

$$\begin{aligned} \frac{2i\hbar eE}{2m_e \omega} \cos(\omega t) \frac{\partial}{\partial y} (\varphi \phi) &= \frac{i\hbar eE}{m_e \omega} \cos(\omega t) \left[\left(\frac{ieE}{\hbar \omega} \cos(\omega t) \right) \varphi \phi + \varphi \frac{\partial \phi}{\partial y} \right] \\ &= \left(\frac{-e^2 E^2}{m_e \omega^2} \cos(\omega t) \right) \varphi \phi + \frac{i\hbar eE}{m_e \omega} \cos(\omega t) \varphi \frac{\partial \phi}{\partial y}. \end{aligned} \quad (1.26)$$

Therefore we can derive that

$$\text{R.H.S} = \left[\frac{m_e \omega_0^2}{2} y^2 - \frac{\hbar^2}{2m_e} \varphi \frac{\partial^2 \phi}{\partial y^2} \right] \varphi \phi. \quad (1.27)$$

To satisfy the condition L.H.S=R.H.S we need to find a function $\phi(y, t)$ such that

$$[eEy \sin(\omega t)] \varphi \phi + i\hbar \varphi \frac{d\phi}{dt} = \left[\frac{m_e \omega_0^2}{2} y^2 - \frac{\hbar^2}{2m_e} \varphi \frac{\partial^2 \phi}{\partial y^2} \right] \varphi \phi \quad (1.28)$$

which can be simplified as

$$\left[\frac{m_e \omega_0^2}{2} y^2 - eEy \sin(\omega t) - \frac{\hbar^2}{2m_e} \frac{\partial^2}{\partial y^2} - i\hbar \frac{d}{dt} \right] \phi(y, t) = 0. \quad (1.29)$$

If we turn off the external dressing field, this equation leads to simple harmonic oscillator Hamiltonian as follows

$$\left[\frac{m_e \omega_0^2}{2} y^2 - \frac{\hbar^2}{2m_e} \frac{\partial^2}{\partial y^2} - i\hbar \frac{d}{dt} \right] \phi(y, t) = 0 \quad (1.30)$$

$$i\hbar \frac{d\phi(y, t)}{dt} = \left[\frac{\hat{p}_y^2}{2m_e} + \frac{1}{2} m_e \omega_0^2 y^2 \right] \phi(y, t). \quad (1.31)$$

Therefore we can identify the $S(t) \equiv eE \sin(\omega t)$ part as a external force act on the harmonic oscillator and we can solve this as a forced harmonic oscillator in y axis.

$$i\hbar \frac{d\phi(y, t)}{dt} = \left[-\frac{\hbar^2}{2m_e} \frac{\partial^2}{\partial y^2} + \frac{1}{2} m_e \omega_0^2 y^2 - yS(t) \right] \phi(y, t). \quad (1.32)$$

This system can be exactly solvable and we can solve this equation using the methods explained by Husimi [*1] as follows.

First we can introduce the time dependent shifted coordinate as

$$y \rightarrow y' = y - \zeta(t) \quad \Rightarrow \quad y = y' + \zeta(t) \quad (1.33)$$

and this implies that

$$\frac{d\phi(y', t)}{dt} = \frac{\partial\phi(y', t)}{\partial t} + \frac{\partial\phi(y', t)}{\partial y'} \frac{\partial y'}{\partial t} = \frac{\partial\phi(y', t)}{\partial t} - \dot{\zeta}(t) \frac{\partial\phi(y', t)}{\partial y'} \quad (1.34)$$

where $\dot{\zeta}(t) = \frac{\partial\zeta(t)}{\partial t}$. Therefore, Eq. (1.32) will be modified to

$$i\hbar \frac{\partial\phi(y', t)}{\partial t} = \left[i\hbar \dot{\zeta} \frac{\partial}{\partial y'} - \frac{\hbar^2}{2m_e} \frac{\partial^2}{\partial y'^2} + \frac{1}{2} m_e \omega_0^2 (y' + \zeta)^2 - (y' + \zeta) S(t) \right] \phi(y', t). \quad (1.35)$$

Let's transform the wave function using following unitary transform

$$\phi(y', t) = \exp\left(\frac{im_e \dot{\zeta} y'}{\hbar}\right) \varphi(y', t) \quad (1.36)$$

and substitute this into the Eq. (1.35) and we will get the following

$$\text{R.H.S} = \left[i\hbar \frac{\partial}{\partial t} - i\hbar \left(\frac{im_e \ddot{\zeta} y'}{\hbar} \right) \right] \exp\left(\frac{-im_e \dot{\zeta} y'}{\hbar}\right) \varphi(y', t) \quad (1.37)$$

and

$$\begin{aligned} \text{L.H.S} = & \left[i\hbar \dot{\zeta} \left(\frac{im_e \dot{\zeta}}{\hbar} \right) + i\hbar \dot{\zeta} \frac{\partial}{\partial y'} \right. \\ & - \frac{\hbar^2}{2m_e} \left[\left(\frac{im_e \dot{\zeta}}{\hbar} \right)^2 + \left(\frac{2im_e \dot{\zeta}}{\hbar} \right) \frac{\partial}{\partial y'} + \frac{\partial^2}{\partial y'^2} \right] \\ & + \frac{1}{2} m_e \omega_0^2 y'^2 + \frac{1}{2} m_e \omega_0^2 \zeta^2 + m_e \omega_0^2 y' \zeta \\ & \left. - y' S(t) - \zeta S(t) \right] \exp\left(\frac{-im_e \dot{\zeta} y'}{\hbar}\right) \varphi(y', t). \end{aligned} \quad (1.38)$$

Combining these two we get derive that

$$\begin{aligned} i\hbar \frac{\partial\varphi(y', t)}{\partial t} = & \left[-\frac{\hbar^2}{2m_e} \frac{\partial^2}{\partial y'^2} + \frac{1}{2} m_e \omega_0^2 y'^2 + [m_e \ddot{\zeta} + m_e \omega_0^2 \zeta - S(t)] y' \right. \\ & \left. + \left[-\frac{1}{2} m_e \dot{\zeta}^2 + \frac{1}{2} m_e \omega_0^2 \zeta^2 - \zeta S(t) \right] \right] \varphi(y', t). \end{aligned} \quad (1.39)$$

Then we can restrict our $\zeta(t)$ function such that

$$m_e \ddot{\zeta} + m_e \omega_0^2 \zeta = S(t) \quad (1.40)$$

and that leads to

$$i\hbar \frac{\partial\varphi(y', t)}{\partial t} = \left[-\frac{\hbar^2}{2m_e} \frac{\partial^2}{\partial y'^2} + \frac{1}{2} m_e \omega_0^2 y'^2 - L(\zeta, \dot{\zeta}, t) \right] \varphi(y', t) \quad (1.41)$$

where

$$L(\zeta, \dot{\zeta}, t) \equiv \frac{1}{2} m_e \dot{\zeta}^2 - \frac{1}{2} m_e \omega_0^2 \zeta^2 + \zeta S(t) \quad (1.42)$$

is the Lagrangian of a driven oscillator.

Now introduce new unitary transformation for the wavefunction as follows

$$\varphi(y', t) = \exp\left(\frac{i}{\hbar} \int_0^t dt' L(\zeta, \dot{\zeta}, t')\right) \chi(y', t) \quad (1.43)$$

and substitute this into the Eq. (1.41) and gets

$$\begin{aligned} i\hbar \left[\exp\left(\frac{i}{\hbar} \int_0^t dt' L(\zeta, \dot{\zeta}, t')\right) \frac{\partial}{\partial t} + i\hbar L(\zeta, \dot{\zeta}, t) \exp\left(\frac{i}{\hbar} \int_0^t dt' L(\zeta, \dot{\zeta}, t')\right) \right] \chi(y', t) \\ = \left[-\frac{\hbar^2}{2m_e} \frac{\partial^2}{\partial y'^2} + \frac{1}{2} m_e \omega_0^2 y'^2 - L(\zeta, \dot{\zeta}, t) \right] \exp\left(\frac{i}{\hbar} \int_0^t dt' L(\zeta, \dot{\zeta}, t')\right) \chi(y', t) \end{aligned} \quad (1.44)$$

and finally we can derive that

$$i\hbar \frac{\partial}{\partial t} \chi(y', t) = \left[-\frac{\hbar^2}{2m_e} \frac{\partial^2}{\partial y'^2} + \frac{1}{2} m_e \omega_0^2 y'^2 \right] \chi(y', t). \quad (1.45)$$

This is the well known Schrodinger equation of a stationary harmonic oscillator. In terms of the eigenvalues

$$E_n = \hbar \omega_0 \left(n + \frac{1}{2}\right) \quad (1.46)$$

of well-known harmonic eigenfunctions

$$\chi_n(y') = \frac{1}{\sqrt{2^n n!}} \cdot \left(\frac{m_e \omega_0}{\pi \hbar}\right)^{1/4} \cdot e^{-\frac{m_e \omega_0 y'^2}{2\hbar}} \cdot \mathcal{H}_n\left(\sqrt{\frac{m_e \omega_0}{\hbar}} y'\right) \quad (1.47)$$

being proportional to the Hermite functions \mathcal{H}_n , the solutions of Eq. (1.32) can be represent as

$$\phi_n(y, t) = \chi_n(y - \zeta(t)) \exp\left(\frac{i}{\hbar} \left[-E_n t + m_e \dot{\zeta}(t)(y - \zeta(t)) + \int_0^t dt' L(\zeta, \dot{\zeta}, t') \right]\right) \quad (1.48)$$

The set $\chi(y)$ forms a complete set and thus any general solution $\phi(y, t)$ can be expanded in terms of the solutions in Eq. (1.48).

Next we consider special case where we assumed

$$S(t) = eE \sin(\omega t) \quad (1.49)$$

and one can derive the Eq. (1.40) for $\zeta(t)$

$$m_e \ddot{\zeta} + m_e \omega_0^2 \zeta = eE \sin(\omega t) \quad (1.50)$$

and using Green function method the solution can be write as

$$\zeta(t) = \frac{eE}{m_e(\omega_0^2 - \omega^2)} \sin(\omega t). \quad (1.51)$$

from this solutions we are able to derive the final solutions ($n = 0, 1, \dots$) would be

$$\begin{aligned} \psi_n(x, y, t) = \frac{1}{\sqrt{L_x}} \chi_n(y - \zeta(t)) \\ \times \exp\left(\frac{i}{\hbar} \left[-E_n t + p_x x + \frac{eEy}{\omega} \cos(\omega t) + m_e \dot{\zeta}(t)[y - \zeta(t)] + \int_0^t dt' L(\zeta, \dot{\zeta}, t') \right]\right) \end{aligned} \quad (1.52)$$

and the exponential phase shifts represent the effect done by the stationary magnetic field and strong dressing field. Therefore we can assume that the magnetitranport properties of 2DEG will be renormalized by the magnetic field as well as the dressing field. ■

2 Scattering theory

Since in a real metal there would be many scatters that can behave as obstacles for electron that have free wave functions. Therefore we need to calculate them to analyse the real behaviour of the electrons.

Then the wave function of the electron in a real metal $\Psi(\mathbf{r}, t)$ should satisfy the following time-dependent Schrodinger equation

$$i\hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = [H_e(t) + U(\mathbf{r})]\Psi(\mathbf{r}, t) \quad (2.1)$$

where $U(\mathbf{r})$ is the total scattering potential. We have represented the all scatters using this potential. Since the solutions (1.52) create a complete orthonormal basis we can represent this wave function using those as follows

$$\Psi(\mathbf{r}, t) = \sum_j a_j(t) |\psi_j(t)\rangle \quad (2.2)$$

where the difference indices j corresponding to the different sets of all quantum numbers p_x and n

$$j \rightarrow (m, n) \quad \text{where} \quad m, n = 0, 1, 2, \dots \quad (2.3)$$

with m is defined for quantized momentum in x direction

$$p_x = m \frac{2\pi\hbar}{L_x} \quad (2.4)$$

Now we can use the conventional perturbation theory to calculate scattering process of electron at a state $|\psi_j\rangle$ to a state $|\psi_{j'}\rangle$. For that assume an electron be in the j state at the time $t = 0$ and corresponding $a'_j(0) = \delta_{j,j'}$.

First substitute a general electron state $\Psi(\mathbf{r}, t)$ at time t as the incoming electron to the Schrodinger equation given in Eq. (2.1)

$$i\hbar \frac{\partial}{\partial t} \sum_j a_j(t) |\psi_j(t)\rangle = [H_e(t) + U(\mathbf{r})] \sum_j a_j(t) |\psi_j(t)\rangle \quad (2.5)$$

$$i\hbar \sum_j \dot{a}_j(t) |\psi_j(t)\rangle + a_j(t) \frac{\partial}{\partial t} |\psi_j(t)\rangle = [H_e(t) + U(\mathbf{r})] \sum_j a_j(t) |\psi_j(t)\rangle \quad (2.6)$$

since all the $|\psi(t)\rangle$ satisfy the Schrodinger equation (1.18)

$$i\hbar \sum_j \dot{a}_j(t) |\psi_j(t)\rangle = \sum_j U(\mathbf{r}) a_j(t) |\psi_j(t)\rangle. \quad (2.7)$$

Then take inner product with state with the state $|\psi_{j'}(t)\rangle$

$$i\hbar \sum_j \dot{a}_j(t) \langle \psi_{j'}(t) | \psi_j(t) \rangle = \sum_j a_j(t) \langle \psi_{j'}(t) | U(\mathbf{r}) | \psi_j(t) \rangle \quad (2.8)$$

But using the *Born approximation* we can assume that this incoming wave have the initial state of the electron at $t = 0$ and therefore this equation will modified to

$$i\hbar \sum_j \dot{a}_j(t) \langle \psi_{j'}(t) | \psi_j(t) \rangle = \langle \psi_{j'}(t) | U(\mathbf{r}) | \psi_j(t) \rangle \quad (2.9)$$

due to orthonormality this becomes

$$i\hbar \dot{a}_{j'}(t) = \langle \psi_{j'}(t) | U(\mathbf{r}) | \psi_j(t) \rangle \quad (2.10)$$

and finally this leads to first order perturbation theory for Scattering as follows

$$a_{j'}(t) = -\frac{i}{\hbar} \langle \psi_{j'}(t) | U(\mathbf{r}) | \psi_j(t) \rangle \quad (2.11)$$

where

$$a_{j'}(t) = -\frac{i}{\hbar} \int_0^t dt' \int_S d\mathbf{r} \psi_{j'}^*(\mathbf{r}, t') U(\mathbf{r}) \psi_j(\mathbf{r}, t') \quad (2.12)$$

where the integration should be performed over the 2DEG area $S = L_x L_y$. Then we can calculate this using the equation we derived in (1.52) as follows

$$\begin{aligned} a_{j'}(t) = & -\frac{i}{\hbar} \int_0^t dt' \int_S d\mathbf{r} \left[\frac{1}{\sqrt{L_x}} \chi_{n'}^*(y - y'_0 - \zeta(t)) \right. \\ & \times \exp \left(\frac{i}{\hbar} \left[E_{n'} t' - m' \frac{2\pi \hbar x}{L_x} - \frac{eE(y - y'_0)}{\omega} \cos(\omega t') - m_e \dot{\zeta}(t) [y - y'_0 - \zeta(t')] - \int_0^{t'} dt'' L(\zeta, \dot{\zeta}, t'') \right] \right) \\ & \times U(\mathbf{r}) \\ & \times \frac{1}{\sqrt{L_x}} \chi_n(y - y_0 - \zeta(t')) \\ & \times \exp \left(\frac{i}{\hbar} \left[-E_n t' + m \frac{2\pi \hbar x}{L_x} - \frac{eE(y - y_0)}{\omega} \cos(\omega t') - m_e \dot{\zeta}(t') [y - y_0 - \zeta(t')] - \int_0^{t'} dt'' L(\zeta, \dot{\zeta}, t'') \right] \right) \Big] \end{aligned} \quad (2.13)$$

then this will be simplified to

$$\begin{aligned} a_{j'}(t) = & -\frac{i}{\hbar} \int_0^t dt' \int_S d\mathbf{r} \left[\frac{1}{\sqrt{L_x}} \chi_{n'}^*(y - y'_0 - \zeta(t')) U(\mathbf{r}) \frac{1}{\sqrt{L_x}} \chi_n(y - y_0 - \zeta(t')) \right. \\ & \times \exp \left(\frac{i}{\hbar} \left[E_{n'} t' - m' \frac{2\pi \hbar x}{L_x} - \frac{eE(y - y'_0)}{\omega} \cos(\omega t') - m_e \dot{\zeta}(t') [y - y'_0 - \zeta(t')] - \int_0^{t'} dt'' L(\zeta, \dot{\zeta}, t'') \right] \right) \\ & \times \exp \left(\frac{i}{\hbar} \left[-E_n t' + m \frac{2\pi \hbar x}{L_x} + \frac{eE(y - y_0)}{\omega} \cos(\omega t') + m_e \dot{\zeta}(t') [y - y_0 - \zeta(t')] + \int_0^{t'} dt'' L(\zeta, \dot{\zeta}, t'') \right] \right) \Big] \end{aligned} \quad (2.14)$$

$$\begin{aligned} a_{j'}(t) = & -\frac{i}{\hbar} \int_0^t dt' \int_S d\mathbf{r} \left[\frac{1}{\sqrt{L_x}} \chi_{n'}^*(y - y'_0 - \zeta(t')) U(\mathbf{r}) \frac{1}{\sqrt{L_x}} \chi_n(y - y_0 - \zeta(t')) \exp \left(\frac{2\pi i(m - m') \hbar x}{L_x} \right) \right. \\ & \times \exp \left(\frac{i}{\hbar} \left[E_{n'} t' + \frac{eE y'_0}{\omega} \cos(\omega t') + m_e \dot{\zeta}(t') y'_0 \right] \right) \exp \left(\frac{i}{\hbar} \left[-E_n t' - \frac{eE y_0}{\omega} \cos(\omega t') - m_e \dot{\zeta}(t) y_0 \right] \right) \Big]. \end{aligned} \quad (2.15)$$

The time dependence of the $\chi_n(y)$ can neglect since it is integrate over all the values of the y and we can write this as

$$\begin{aligned} a_{j'}(t) = & -\frac{i}{\hbar} \int_S d\mathbf{r} \frac{1}{\sqrt{L_x}} \chi_{n'}^*(y - y'_0 - \zeta(t')) U(\mathbf{r}) \frac{1}{\sqrt{L_x}} \chi_n(y - y_0 - \zeta(t')) \exp \left(\frac{2\pi i(m - m') \hbar x}{L_x} \right) \\ & \times \int_0^t dt' \left[\exp \left(\frac{i}{\hbar} \left[(E_{n'} - E_n) t' + \frac{eE(y'_0 - y_0) \omega_0^2}{\omega(\omega_0^2 - \omega^2)} \cos(\omega t') \right] \right) \right]. \end{aligned} \quad (2.16)$$

Using Jacobi-Anger expansion

$$e^{iz \cos(\theta)} = \sum_{l=-\infty}^{\infty} i^l J_l(z) e^{in\theta} \quad (2.17)$$

above equation can be modified as

$$a_{j'}(t) = -\frac{i}{\hbar} U_{j'j} \int_0^t dt' \left[\sum_{l=-\infty}^{\infty} i^l J_l \left[\frac{eE(y'_0 - y_0) \omega_0^2}{\hbar \omega(\omega_0^2 - \omega^2)} \right] \exp \left(\frac{i}{\hbar} (E_{n'} - E_n + l \hbar \omega) t' \right) \right] \quad (2.18)$$

where

$$U_{j'j} \equiv \langle \Phi_{j'}(\mathbf{r}) | U(\mathbf{r}) | \Phi_j(\mathbf{r}) \rangle \quad (2.19)$$

with bare electron eigen states (without dressing field)

$$\Phi_j(\mathbf{r}) = \frac{1}{\sqrt{L_x}} \exp \left(\frac{2\pi i m \hbar x}{L_x} \right) \chi_n(y). \quad (2.20)$$

Considering time evaluation from negative values we can write the same expression as follows

$$a_{j'}(t) = -\frac{i}{\hbar} U_{j'j} \int_{-t/2}^{t/2} dt' \left[\sum_{l=-\infty}^{\infty} i^l J_l \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \exp\left(\frac{i}{\hbar}(E_{n'} - E_n + l\hbar\omega)t'\right) \right]. \quad (2.21)$$

To calculate scattering probability we can use this scattering amplitude's square value

$$\begin{aligned} |a_{j'}(t)|^2 &= \frac{|U_{j'j}|^2}{\hbar^2} \int_{-t/2}^{t/2} dt' \left[\sum_{l=-\infty}^{\infty} -i^l J_l \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \exp\left(\frac{-i}{\hbar}(E_{n'} - E_n + l\hbar\omega)t'\right) \right] \\ &\quad \times \int_{-t/2}^{t/2} dt'' \left[\sum_{k=-\infty}^{\infty} i^k J_k \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \exp\left(\frac{i}{\hbar}(E_{n'} - E_n + k\hbar\omega)t''\right) \right] \end{aligned} \quad (2.22)$$

Considering long time $t \rightarrow \infty$ we can make the integral into a delta function as follows

$$\begin{aligned} |a_{j'}(t)|^2 &= 4\pi^2 |U_{j'j}|^2 \left[\sum_{l=-\infty}^{\infty} -i^l J_l \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \delta(-E_{n'} + E_n - l\hbar\omega) \right] \\ &\quad \times \left[\sum_{k=-\infty}^{\infty} i^k J_k \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \delta(E_{n'} - E_n + k\hbar\omega) \right] \end{aligned} \quad (2.23)$$

and this implies $l = k$ and this leads to

$$|a_{j'}(t)|^2 = 4\pi^2 |U_{j'j}|^2 \left[\sum_{l=-\infty}^{\infty} J_l^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \delta^2(E_{n'} - E_n + l\hbar\omega) \right]. \quad (2.24)$$

Then using the famous the square δ function transformation method

$$\delta^2(\epsilon) = \delta(\epsilon) \delta^2(0) \lim_{t \rightarrow \infty} \int_{-t/2}^{t/2} e^{i0 \times t' / \hbar} dt' = \frac{\delta(\epsilon)t}{2\pi\hbar} \quad (2.25)$$

we can calculate the probability of electron scattering between states j and j' per unit time as

$$\mathcal{W}_{j'j} \equiv \frac{d|a_{j'}(t)|^2}{dt} = |U_{j'j}|^2 \sum_{l=-\infty}^{\infty} J_l^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \times \frac{2\pi}{\hbar} \delta(E_{n'} - E_n + l\hbar\omega) \quad (2.26)$$

To avoid the energy exchange between a high-frequency field and electrons, the field should be purely dressing. We can achieve that by using the field with off-resonant and high frequency. Therefore, the only effect of the dressing field on 2DEG is the renormalization of the probability of elastic electron scattering within the same Landau level ($E_{n'} = E_n$), which described by the term with $l = 0$ the Eq. (2.26) leads to

$$\mathcal{W}_{j'j} = J_0^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \mathcal{W}_{j'j}^{(0)} \quad (2.27)$$

where

$$\mathcal{W}_{j'j}^{(0)} = \frac{2\pi}{\hbar} |U_{j'j}|^2 \delta(E_{n'} - E_n) \quad (2.28)$$

is the probability of scattering of a *bare electron*. It is important to notice that the Bessel function factor depend on both the dressing field and stationary magnetic field. This factor is responsible for all the effects discussed in this article.

One can define the lifetime of the dressed electron at the Landau level τ is renormalized by the Bessel function as below

$$\frac{1}{\tau} \equiv \sum_{j'} \mathcal{W}_{j'j} = \sum_{j'} J_0^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \mathcal{W}_{j'j}^{(0)} \quad (2.29)$$

where we have consider all possibilities that electron can jump to the state j' . Then rewrite the delat function as follows

$$\delta(\epsilon) = \frac{1}{\pi} \lim_{\Gamma \rightarrow 0} \frac{\Gamma}{\Gamma^2 + \epsilon^2} \quad (2.30)$$

where in this study we can assume that the paramater $\Gamma \equiv \hbar/\tau$ as scattering induced broading of the Landau level. But for the elestic scatteing within the same Landau level, we can write the δ function as

$$\delta(E_{n'} - E_n) \approx \frac{1}{\pi\Gamma}. \quad (2.31)$$

Therefore Eq. (2.29) will change to

$$\frac{1}{\tau} = \sum_{j'} J_0^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \times \frac{2\pi}{\hbar} |U_{j'j}|^2 \times \frac{1}{\pi\Gamma} \quad (2.32)$$

$$\frac{1}{\tau} = \sum_{j'} J_0^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \times \frac{2}{\hbar} |U_{j'j}|^2 \times \frac{\tau}{\hbar} \quad (2.33)$$

and finally this can be modified to

$$\frac{1}{\tau} = \left[\frac{2}{\hbar^2} \sum_{j'} J_0^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] |U_{j'j}|^2 \right]^{1/2} \quad (2.34)$$

where the summation is performed over electron states j' within the same Landau level.

Now lets specify more on the scattering potential where we can model them as randomly distributed delta fucntions as follows

$$U(\mathbf{r}) \equiv \sum_{i=1}^{N_s} U_0 \delta(\mathbf{r} - \mathbf{r}_i) \quad (2.35)$$

where N_s is the total number of scatters in the considering matel. Now we can calculate $|U_{j'j}|^2$ as follows

$$\begin{aligned} |U_{j'j}|^2 &= \sum_{i=1}^{N_s} \frac{1}{L_x^2} \int \int dx_1 dy_1 \exp\left(\frac{-p'_x x_1}{\hbar}\right) \chi_n^*(y_1 - y'_0) U_0 \delta(x_1 - x_i) \delta(y_1 - y_i) \exp\left(\frac{p_x x_1}{\hbar}\right) \chi_n(y_1 - y_0) \\ &\quad \times \int \int dx_2 dy_2 \exp\left(\frac{p'_x x_2}{\hbar}\right) \chi_n(y_2 - y'_0) U_0 \delta(x_2 - x_i) \delta(y_2 - y_i) \exp\left(\frac{-p_x x_2}{\hbar}\right) \chi_n^*(y_2 - y_0) \end{aligned} \quad (2.36)$$

and considering only non-zero values for x_1 and x_2 integrals we can re-write this as

$$\begin{aligned} |U_{j'j}|^2 &= \sum_{i=1}^{N_s} \frac{U_0^2}{L_x^2} \int dy_1 \exp\left(\frac{-p'_x x_i}{\hbar}\right) \chi_n^*(y_1 - y'_0) \delta(y_1 - y_i) \exp\left(\frac{p_x x_i}{\hbar}\right) \chi_n(y_1 - y_0) \\ &\quad \times \int dy_2 \exp\left(\frac{p'_x x_i}{\hbar}\right) \chi_n(y_2 - y'_0) \delta(y_2 - y_i) \exp\left(\frac{-p_x x_i}{\hbar}\right) \chi_n^*(y_2 - y_0) \end{aligned} \quad (2.37)$$

and this will be simplified to

$$\begin{aligned} |U_{j'j}|^2 &= \sum_{i=1}^{N_s} \frac{U_0^2}{L_x^2} \int dy_1 \chi_n^*(y_1 - y'_0) \delta(y_1 - y_i) \chi_n(y_1 - y_0) \\ &\quad \times \int dy_2 \chi_n(y_2 - y'_0) \delta(y_2 - y_i) \chi_n^*(y_2 - y_0). \end{aligned} \quad (2.38)$$

Again considering only non-zero values for y_1 and y_2 integrals we can re-write this as

$$|U_{j'j}|^2 = \sum_{i=1}^{N_s} \frac{U_0^2}{L_x^2} \chi_n^*(y_i - y'_0) \chi_n(y_i - y_0) \chi_n(y_i - y'_0) \chi_n^*(y_i - y_0). \quad (2.39)$$

$$|U_{j'j}|^2 = \frac{U_0^2}{L_x^2} \sum_{i=1}^{N_s} \chi_n^2(y_i - y'_0) \chi_n^2(y_i - y_0). \quad (2.40)$$

Now substituting this derivation into the Eq. (2.34) we will get

$$\frac{1}{\tau} = \left[\frac{2U_0^2}{\hbar^2 L_x^2} \sum_{y'_0} J_0^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \sum_{i=1}^{N_s} \chi_n^2(y_i - y'_0) \chi_n^2(y_i - y_0) \right]^{1/2} \quad (2.41)$$

where j' reduced to p'_x (since $n' = n$) and we can represent it by y'_0 . Then this will modified to

$$\frac{1}{\tau} = \left[\frac{2U_0^2}{\hbar^2 L_x^2} \sum_{y'_0} \sum_{i=1}^{N_s} J_0^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \chi_n^2(y_i - y'_0) \chi_n^2(y_i - y_0) \right]^{1/2}. \quad (2.42)$$

Now considering large size of sample and a macroscopically large N_s scatters we can promote the summation to integrations as follows

$$\frac{1}{\tau} = \left[\frac{2U_0^2}{\hbar^2 L_x^2} \frac{eBL_x}{2\pi\hbar} \int dy'_0 \frac{N_s}{L_x} \int dy_i J_0^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \chi_n^2(y_i - y'_0) \chi_n^2(y_i - y_0) \right]^{1/2}. \quad (2.43)$$

Assuming $L_x = L_y$ we can define the area of the 2D material as

$$S \equiv L_x L_y = L_x L_x \quad (2.44)$$

and then we can re-write the above as

$$\frac{1}{\tau} = \left[\frac{eBN_s U_0^2}{\pi\hbar^3 S} \int dy'_0 \int dy_i J_0^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \chi_n^2(y_i - y'_0) \chi_n^2(y_i - y_0) \right]^{1/2}. \quad (2.45)$$

Define the *density of scatters* per unit area of 2DEG

$$n_s \equiv \frac{N_s}{S} \quad (2.46)$$

and the *magnetic length* as

$$l_0 \equiv \sqrt{\frac{\hbar}{eB}}. \quad (2.47)$$

Now our Eq. (2.45) leads to

$$\frac{1}{\tau} = \sqrt{\frac{n_s U_0^2}{\pi l_0^2 \hbar^2}} \left[\int dy'_0 \int dy_i J_0^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \chi_n^2(y_i - y'_0) \chi_n^2(y_i - y_0) \right]^{1/2} \quad (2.48)$$

and now define new dummy variables as follows (since y_0 is a paramter)

$$(y'_0 - y_0) \rightarrow y \quad \text{and} \quad (y_i - y'_0) \rightarrow y' \quad (2.49)$$

and finally we will get the equation for the dressed electron lifetime at the n th Landau level as

$$\frac{1}{\tau} = \sqrt{\frac{n_s U_0^2}{\pi l_0^2 \hbar^2}} \left[\int \int dy dy' J_0^2 \left[\frac{eE y \omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \chi_n^2(y') \chi_n^2(y + y') \right]^{1/2} \quad (2.50)$$

■

3 Floquet theory

Since we describe the lifetime of an electron in certain Landau level using conventional perturbation theory, now we can apply the Floquet theory to identify the difference of these methods.

First we need to identify the *quasienergies* and periodic *Floquet modes* for derived wavefunctions (1.52) for a 2DEG system with both stationary magnetic field and strong dressing filed.

Let's consider the following paramter which is lineraly increasing in time

$$\Delta_E t \equiv \frac{t}{T} \int_0^T dt' L(\zeta, \dot{\zeta}, t') \quad (3.1)$$

where we can calculate this using Eq. (1.42) and (1.51) as follows

$$\begin{aligned} \Delta_E t = \frac{t}{T} \int_0^T dt' \frac{1}{2} m_e \frac{(eE\omega)^2}{m_e^2(\omega_0^2 - \omega^2)^2} \cos^2(\omega t') - \frac{1}{2} m_e \omega_0^2 \frac{(eE)^2}{m_e^2(\omega_0^2 - \omega^2)^2} \sin^2(\omega t') \\ + \frac{eE}{m_e(\omega_0^2 - \omega^2)} \sin(\omega t') eE \sin(\omega t') \end{aligned} \quad (3.2)$$

$$\begin{aligned} \Delta_E t = \frac{t\omega}{2\pi} \times \frac{(eE)^2}{2m_e(\omega_0^2 - \omega^2)^2} \left[\omega^2 \int_0^T dt' \cos^2(\omega t') - \omega_0^2 \int_0^T dt' \sin^2(\omega t') \right. \\ \left. + 2(\omega_0^2 - \omega^2) \int_0^T dt' \sin^2(\omega t') \right] \end{aligned} \quad (3.3)$$

$$\Delta_E t = \frac{t\omega}{2\pi} \times \frac{(eE)^2}{2m_e(\omega_0^2 - \omega^2)^2} \left[\omega^2 \frac{\pi}{\omega} - \omega_0^2 \frac{\pi}{\omega} + 2(\omega_0^2 - \omega^2) \frac{\pi}{\omega} \right] \quad (3.4)$$

$$\Delta_E t = \frac{t\omega}{2} \times \frac{(eE)^2}{2m_e(\omega_0^2 - \omega^2)^2} (\omega_0^2 - \omega^2) = \frac{(eE)^2}{4m_e(\omega_0^2 - \omega^2)} t \quad (3.5)$$

Since this is the continuous increasing part of the Laggrangian integral in Eq. (1.52) we can make this as 2ω periodic function as follows

$$\Lambda \equiv \int_0^t dt' L(\zeta, \dot{\zeta}, t') - \frac{t}{T} \int_0^T dt' L(\zeta, \dot{\zeta}, t') \quad (3.6)$$

which can be proved as follows. First consider the first term of the Λ

$$\begin{aligned} \int_0^t dt' L(\zeta, \dot{\zeta}, t') = \frac{(eE)^2}{2m_e(\omega_0^2 - \omega^2)^2} \left[\omega^2 \int_0^t dt' \cos^2(\omega t') - \omega_0^2 \int_0^t dt' \sin^2(\omega t') \right. \\ \left. + 2(\omega_0^2 - \omega^2) \int_0^t dt' \sin^2(\omega t') \right] \end{aligned} \quad (3.7)$$

$$\begin{aligned} \int_0^t dt' L(\zeta, \dot{\zeta}, t') = \frac{(eE)^2}{2m_e(\omega_0^2 - \omega^2)^2} \left[\omega^2 \left[\frac{t}{2} + \frac{\sin(2\omega t)}{4\omega} \right] - \omega_0^2 \left[\frac{t}{2} - \frac{\sin(2\omega t)}{4\omega} \right] \right. \\ \left. + 2(\omega_0^2 - \omega^2) \left[\frac{t}{2} - \frac{\sin(2\omega t)}{4\omega} \right] \right] \end{aligned} \quad (3.8)$$

$$\begin{aligned} \int_0^t dt' L(\zeta, \dot{\zeta}, t') = \frac{(eE)^2}{2m_e(\omega_0^2 - \omega^2)^2} \left[\frac{t}{2} [\omega^2 - \omega_0^2 + 2\omega_0^2 - 2\omega^2] \right. \\ \left. + \frac{\sin(2\omega t)}{4\omega} [\omega^2 + \omega_0^2 - 2\omega_0^2 + 2\omega^2] \right] \end{aligned} \quad (3.9)$$

$$\int_0^t dt' L(\zeta, \dot{\zeta}, t') = \frac{(eE)^2}{4m_e(\omega_0^2 - \omega^2)^2} t + \frac{(eE)^2 (3\omega^2 - \omega_0^2)}{8m_e\omega(\omega_0^2 - \omega^2)^2} \sin(2\omega t) \quad (3.10)$$

then using Eq.(3.5) we can write this as

$$\int_0^t dt' L(\zeta, \dot{\zeta}, t') = \Delta_E t + \frac{(eE)^2 (3\omega^2 - \omega_0^2)}{8m_e\omega(\omega_0^2 - \omega^2)^2} \sin(2\omega t). \quad (3.11)$$

Now we can express

$$\Lambda = \Delta_E t + \frac{(eE)^2(3\omega^2 - \omega_0^2)}{8m_e\omega(\omega_0^2 - \omega^2)^2} \sin(2\omega t) - \Delta_E t = \frac{(eE)^2(3\omega^2 - \omega_0^2)}{8m_e\omega(\omega_0^2 - \omega^2)^2} \sin(2\omega t) \quad (3.12)$$

which is a periodic function in time with 2ω frequency.

Now using this parameters we can factorize the wavefunction (1.52) as linearly time dependent part and periodic time dependent part as follows

$$\begin{aligned} \psi_n(x, y, t) = & \exp\left(\frac{i}{\hbar}[-E_n t + \Delta_E t]\right) \frac{1}{\sqrt{L_x}} \chi_n(y - \zeta(t)) \\ & \times \exp\left(\frac{i}{\hbar}\left[p_x x + \frac{eE y}{\omega} \cos(\omega t) + m_e \dot{\zeta}(t)[y - \zeta(t)] + \int_0^t dt' L(\zeta, \dot{\zeta}, t') - \Delta_E t\right]\right) \end{aligned} \quad (3.13)$$

where we can identify (let $n \rightarrow \alpha$) the *quasienergies* as

$$\varepsilon_\alpha \equiv \hbar\omega_0\left(\alpha + \frac{1}{2}\right) - \Delta_E \quad \text{where } \alpha = 0, 1, 2, \dots \quad (3.14)$$

and *Floquet modes* as

$$\Phi_\alpha(x, y, t) \equiv \frac{1}{\sqrt{L_x}} \chi_\alpha(y - \zeta(t)) \exp\left(\frac{i}{\hbar}\left[p_x x + \frac{eE y}{\omega} \cos(\omega t) + m_e \dot{\zeta}(t)[y - \zeta(t)] + \Lambda\right]\right) \quad (3.15)$$

with

$$\zeta(t) = \frac{eE}{m_e(\omega_0^2 - \omega^2)} \sin(\omega t) \quad \text{and} \quad \dot{\zeta}(t) = \frac{eE\omega}{m_e(\omega_0^2 - \omega^2)} \cos(\omega t) \quad (3.16)$$

where *Floquet modes* are time-periodic functions that also create a complete orthonormal set. ■

Therefore the solutions (Floquet states) for the periodic Hamiltonian (1.5) can be written in position space as

$$\psi_\alpha(x, y, t) = \exp\left(-\frac{i}{\hbar}\varepsilon_\alpha t\right) \Phi_\alpha(x, y, t) \quad (3.17)$$

where

$$\varepsilon_\alpha \equiv \left(\frac{eB\hbar}{m_e}\right)\left(\alpha + \frac{1}{2}\right) - \frac{(eE)^2}{4m_e(\omega_0^2 - \omega^2)} \quad \text{where } \alpha = 0, 1, 2, \dots \quad (3.18)$$

and

$$\begin{aligned} \Phi_\alpha(x, y, t) \equiv & \frac{1}{\sqrt{L_x}} \chi_\alpha\left(y - \frac{eE \sin(\omega t)}{m_e(\omega_0^2 - \omega^2)}\right) \\ & \times \exp\left(\frac{i}{\hbar}\left[p_x x + \frac{eE y}{\omega} \cos(\omega t) + \frac{eE\omega y}{(\omega_0^2 - \omega^2)} \cos(\omega t)\right]\right) \\ & \times \exp\left(\frac{i}{\hbar}\left[-\frac{(eE)^2\omega}{2m_e(\omega^2 - \omega_0^2)^2} \sin(2\omega t) + \frac{(eE)^2(3\omega_0^2 - \omega^2)}{8m_e\omega(\omega_0^2 - \omega^2)^2} \sin(2\omega t)\right]\right) \end{aligned} \quad (3.19)$$

Now we can write this by more simplying and considering spacial dependencies and using previous substituting done in Eq. (1.16) we can derive that

$$\begin{aligned} \Phi_\alpha(x, y, t) \equiv & \frac{1}{\sqrt{L_x}} \chi_\alpha\left(y - y_0 - \frac{eE \sin(\omega t)}{m_e(\omega_0^2 - \omega^2)}\right) \exp\left(\frac{ip_x}{\hbar} x\right) \exp\left(\frac{i}{\hbar}\left[\frac{eE\omega_0^2 \cos(\omega t)}{\omega(\omega_0^2 - \omega^2)}\right](y - y_0)\right) \\ & \times \exp\left(\frac{-i}{\hbar}\left[\frac{(eE)^2(\omega_0^2 + \omega^2)}{8\omega m_e(\omega_0^2 - \omega^2)^2}\right] \sin(2\omega t)\right) \end{aligned} \quad (3.20)$$

Now we can transform this solution in spacial variable into the momentum space using Fourier transform over the considering space.

$$\begin{aligned}\Phi_\alpha(k_x, k_y, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy \exp(-ik_y y) \left[\frac{1}{\sqrt{L_x}} \chi_\alpha \left(y - y_0 - \frac{eE \sin(\omega t)}{m_e(\omega_0^2 - \omega^2)} \right) \exp \left(\frac{i}{\hbar} \left[\frac{eE\omega_0^2 \cos(\omega t)}{\omega(\omega_0^2 - \omega^2)} \right] y \right) \right] \\ &\times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \exp(-ik_x x) \left[\exp \left(\frac{ip_x}{\hbar} x \right) \right] \\ &\times \exp \left(\frac{-i}{\hbar} \left[\frac{eE\omega_0^2 \cos(\omega t)}{\omega(\omega_0^2 - \omega^2)} \right] y_0 \right) \times \exp \left(\frac{-i}{\hbar} \left[\frac{(eE)^2(\omega_0^2 + \omega^2)}{8\omega m_e(\omega_0^2 - \omega^2)^2} \right] \sin(2\omega t) \right)\end{aligned}\quad (3.21)$$

Then this can be re-write as follows

$$\Phi_\alpha(k_x, k_y, t) = \exp \left(\frac{-i}{\hbar} \left[\frac{eE\omega_0^2 \cos(\omega t)}{\omega(\omega_0^2 - \omega^2)} \right] y_0 \right) \exp \left(\frac{-i}{\hbar} \left[\frac{(eE)^2(\omega_0^2 + \omega^2)}{8\omega m_e(\omega_0^2 - \omega^2)^2} \right] \sin(2\omega t) \right) \delta \left(k_x - \frac{p_x}{\hbar} \right) \Theta(k_y, t) \quad (3.22)$$

where we used

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \exp \left(-ik_x x + \frac{ip_x}{\hbar} x \right) = \sqrt{2\pi} \delta \left(k_x - \frac{p_x}{\hbar} \right) \quad (3.23)$$

and

$$\Theta(k_y, t) \equiv \int_{-\infty}^{\infty} dy \exp(-ik_y y) \left[\frac{1}{\sqrt{L_x}} \chi_\alpha \left(y - y_0 - \frac{eE \sin(\omega t)}{m_e(\omega_0^2 - \omega^2)} \right) \exp \left(\frac{i}{\hbar} \left[\frac{eE\omega_0^2 \cos(\omega t)}{\omega(\omega_0^2 - \omega^2)} \right] y \right) \right] \quad (3.24)$$

and this can be simplified as

$$\Theta(k_y, t) = \frac{1}{\sqrt{L_x}} \int_{-\infty}^{\infty} dy \chi_\alpha \left(y - y_0 - \frac{eE \sin(\omega t)}{m_e(\omega_0^2 - \omega^2)} \right) \exp \left(-ik_y y + \frac{i}{\hbar} \left[\frac{eE\omega_0^2 \cos(\omega t)}{\omega(\omega_0^2 - \omega^2)} \right] y \right). \quad (3.25)$$

Then by defining

$$\mu(t) \equiv \frac{eE \sin(\omega t)}{m_e(\omega_0^2 - \omega^2)} + y_0 \quad (3.26)$$

and

$$\gamma(t) \equiv \frac{eE\omega_0^2 \cos(\omega t)}{\hbar\omega(\omega_0^2 - \omega^2)} \quad (3.27)$$

we can re-write this by neglecting time dependencies as

$$\Theta(k_y, t) = \frac{1}{\sqrt{L_x}} \int_{-\infty}^{\infty} dy \chi_\alpha(y - \mu) \exp(-i(k_y - \gamma)y). \quad (3.28)$$

We can substitute following variables

$$k_y' = k_y - \gamma \quad \text{and} \quad y' = y - \mu \quad (3.29)$$

and this leads to

$$\Theta(k_y', t) = \frac{e^{-ik_y' \mu}}{\sqrt{L_x}} \int_{-\infty}^{\infty} dy' \chi_\alpha(y') \exp(-ik_y' y'). \quad (3.30)$$

We know that $\{\chi_\alpha\}$ are well-known harmonic eigenfunctions as given in the Eq. (1.47). Due to the symmetric condition the Fourier transform of these functions are same with the position eigenfunctions by changing

$$y \rightarrow k_y \quad \text{and} \quad \frac{m\omega_0}{\hbar} \rightarrow \frac{\hbar}{m\omega_0} \quad (3.31)$$

Therefore

$$\Theta(k_y', t) = \frac{e^{-ik_y' \mu}}{\sqrt{L_x}} \tilde{\chi}_\alpha(k_y') \quad (3.32)$$

where

$$\tilde{\chi}_\alpha(k_y') = \frac{1}{\sqrt{2^n \alpha!}} \cdot \left(\frac{\hbar}{\pi m_e \omega_0} \right)^{1/4} \cdot e^{-\frac{\hbar}{2m_e \omega_0} (k_y')^2} \cdot \mathcal{H}_\alpha \left(\sqrt{\frac{\hbar}{m_e \omega_0}} k_y' \right) \quad (3.33)$$

Using Eq. (3.32) and Eq. (3.22) we can derive that

$$\begin{aligned} \Phi_\alpha(k_x, k_y, t) = & \exp \left(\frac{-i}{\hbar} \left[\frac{eE\omega_0^2 \cos(\omega t)}{\omega(\omega_0^2 - \omega^2)} \right] y_0 \right) \exp \left(\frac{-i}{\hbar} \left[\frac{(eE)^2 (\omega_0^2 + \omega^2)}{8\omega m_e (\omega_0^2 - \omega^2)^2} \right] \sin(2\omega t) \right) \delta \left(k_x - \frac{p_x}{\hbar} \right) \\ & \times \frac{e^{-i(k_y - \gamma)\mu}}{\sqrt{L_x}} \tilde{\chi}_\alpha(k_y - \gamma) \end{aligned} \quad (3.34)$$

and this can be re-write substituting μ and γ values as follows

$$\begin{aligned} \Phi_\alpha(k_x, k_y, t) = & \exp \left(\frac{-i}{\hbar} \left[\frac{eE\omega_0^2 \cos(\omega t)}{\omega(\omega_0^2 - \omega^2)} \right] y_0 \right) \exp \left(\frac{-i}{\hbar} \left[\frac{(eE)^2 (\omega_0^2 + \omega^2)}{8\omega m_e (\omega_0^2 - \omega^2)^2} \right] \sin(2\omega t) \right) \delta \left(k_x - \frac{p_x}{\hbar} \right) \\ & \times \exp \left(-ik_y \frac{eE \sin(\omega t)}{m_e (\omega_0^2 - \omega^2)} \right) \exp \left(\frac{i}{\hbar} \left[\frac{eE\omega_0^2 \cos(\omega t)}{\omega(\omega_0^2 - \omega^2)} \right] \frac{eE \sin(\omega t)}{m_e (\omega_0^2 - \omega^2)} \right) \exp(-ik_y y_0) \\ & \times \exp \left(i \frac{1}{\hbar} \left[\frac{eE\omega_0^2 \cos(\omega t)}{\omega(\omega_0^2 - \omega^2)} \right] y_0 \right) \frac{1}{\sqrt{L_x}} \tilde{\chi}_\alpha(k_y - \gamma) \end{aligned} \quad (3.35)$$

and

$$\begin{aligned} \Phi_\alpha(k_x, k_y, t) = & \exp \left(\frac{i}{\hbar} \left[\frac{(eE)^2 (3\omega_0^2 - \omega^2)}{8\omega m_e (\omega_0^2 - \omega^2)^2} \right] \sin(2\omega t) \right) \delta \left(k_x - \frac{p_x}{\hbar} \right) \\ & \times \exp \left(-ik_y \left[\frac{eE \sin(\omega t)}{m_e (\omega_0^2 - \omega^2)} + y_0 \right] \right) \frac{1}{\sqrt{L_x}} \tilde{\chi}_\alpha(k_y - \gamma) \end{aligned} \quad (3.36)$$

xxx