Magnetic propeties of a two dimentional electron gas strongly coupled to lights

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1 Schrödinger problem for Landau levels in dressed 2DEG

Our analysis is consider on 2 dimentional electronic gas which has distrubuted in (x, y) plane in configuration space. We are going to examine the properties of 2DEG with stationary magnetic field

$$\mathbf{B} = (0, 0, B)^T \tag{1.1}$$

which directed on z axis and a linearly y-polarized strong electomagnetic wave (dressing field) with electric field given by

$$\mathbf{E} = (0, E\sin(\omega t), 0)^T \tag{1.2}$$

which also propagate in z direction. Here B and E represent the amplitude of the stationary magnetic field and electric field of dressing field.

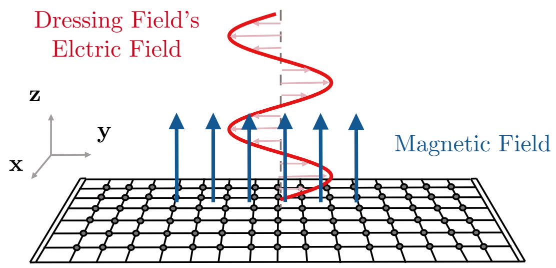

Figure 1: Stationary magnetic filed (blue color) and Strong EM wave (red color) applied to the 2DEG.

Using Landau gauge for the stationary magnetic field we can represent it using vector potential as

$$\mathbf{A}_s = (-By, 0, 0)^T \tag{1.3}$$

and choosing Coulomb gauge the dressing field can be present as the following vector potential

$$\mathbf{A}_d(t) = (0, [E/\omega]\cos(\omega t), 0)^T. \tag{1.4}$$

Now the Hamiltonian of an electron in 2DEG can be reads as

$$\hat{H}_e(t) = \frac{1}{2m_e}\Big[\hat{\mathbf{p}} - e\big(\mathbf{A}_s + \mathbf{A}_d(t)\big)\Big]^2 \tag{1.5}$$

where m_e is the effective mass of the electron and e is the magnitude (without considering the sign of the charge) of the electron charge. This can be simplified to

$$\hat{H}_e(t) = \frac{1}{2m_e}\Big[(\hat{p}_x + eBy)\mathbf{e}_x + (\hat{p}_y - \frac{eE}{\omega}\cos(\omega t))\mathbf{e}_y\Big]^2 \tag{1.6}$$

where $\mathbf{e}_x$ and $\mathbf{e}_y$ are unit vectors along x and y directions respectively. Moreover,

$$\hat{H}_e(t) = \frac{1}{2m_e}\Big[(\hat{p}_x + eBy)^2 + (\hat{p}_y - \frac{eE}{\omega}\cos(\omega t))^2\Big] \tag{1.7}$$

Since $[\hat{H}_e(t), \hat{p}_x] = 0$ both operators share same eigenvalue and eigen functions which are free electron wave functions. Therefore we can modify the Hamiltonian as follows

$$\hat{H}_e(t) = \frac{1}{2m_e}\Big[(p_x + eBy)^2 + (\hat{p}_y - \frac{eE}{\omega}\cos(\omega t))^2\Big]. \tag{1.8}$$

Using momentum operator definition

$$\hat{p}_y = -i\hbar\frac{\partial}{\partial y} \tag{1.9}$$

we can modify Eq. (1.8) as

$$\begin{aligned}\hat{H}_e(t) &= \frac{1}{2m_e}\Big[(p_x + eBy)^2 + \Big(-i\hbar\frac{\partial}{\partial y} - \frac{eE}{\omega}\cos(\omega t)\Big)^2\Big] \\ &= \frac{1}{2m_e}\Big[(p_x + eBy)^2 + \Big(i\hbar\frac{\partial}{\partial y} + \frac{eE}{\omega}\cos(\omega t)\Big)^2\Big].\end{aligned} \tag{1.10}$$

Define the *center of the cyclotron orbit* along y axis as

$$y_0 \equiv \frac{-p_x}{eB} \tag{1.11}$$

and the *cyclotron frequency* as

$$\omega_0 \equiv \frac{eB}{m_e}. \tag{1.12}$$

Then the Hamiltonian will leads to

$$\hat{H}_e(t) = \frac{m_e\omega_0^2}{2}(y - y_0)^2 + \frac{1}{2m_e}\Big(i\hbar\frac{\partial}{\partial y} + \frac{eE}{\omega}\cos(\omega t)\Big)^2 \tag{1.13}$$

$$\begin{aligned}\hat{H}_e(t) = \frac{m_e\omega_0^2}{2}(y - y_0)^2 + \frac{1}{2m_e}\Big(&-\hbar^2\frac{\partial^2}{\partial y^2} + i\hbar\frac{\partial}{\partial y}\Big[\frac{eE}{\omega}\cos(\omega t)\Big] \\ &+ \frac{i\hbar eE}{\omega}\cos(\omega t)\frac{\partial}{\partial y} + \frac{e^2E^2}{\omega^2}\cos^2(\omega t)\Big)\end{aligned} \tag{1.14}$$

$$\hat{H}_e(t) = \frac{m_e\omega_0^2}{2}(y - y_0)^2 + \frac{1}{2m_e}\Big(-\hbar^2\frac{\partial^2}{\partial y^2} + \frac{2i\hbar eE}{\omega}\cos(\omega t)\frac{\partial}{\partial y} + \frac{e^2E^2}{\omega^2}\cos^2(\omega t)\Big). \tag{1.15}$$

Let

$$(y - y_0) \rightarrow y \tag{1.16}$$

and then this becomes

$$\hat{H}_e(t) = \frac{m_e\omega_0^2}{2}y^2 + \frac{1}{2m_e}\Big(-\hbar^2\frac{\partial^2}{\partial y^2} + \frac{2i\hbar eE}{\omega}\cos(\omega t)\frac{\partial}{\partial y} + \frac{e^2E^2}{\omega^2}\cos^2(\omega t)\Big). \tag{1.17}$$

Now assume that the solution for the time-dependent schrödinger equation

$$i\hbar\frac{\mathrm{d}\psi}{\mathrm{d}t} = \hat{H}_e(t)\psi \tag{1.18}$$

can be represent by the following form

$$\psi(\mathbf{r}, t) = \frac{1}{\sqrt{L_x}}\exp\left(\frac{ip_x x}{\hbar} + \frac{ieE(y - y_0)}{\hbar\omega}\cos(\omega t)\right)\phi(y - y_0, t). \tag{1.19}$$

Using the same subtution from Eq. (1.16) this becomes

$$\psi(x, y, t) = \frac{1}{\sqrt{L_x}}\exp\left(\frac{ip_x x}{\hbar} + \frac{ieEy}{\hbar\omega}\cos(\omega t)\right)\phi(y, t). \tag{1.20}$$

Defining

$$\varphi(x, y, t) \equiv \frac{1}{\sqrt{L_x}}\exp\left(\frac{ip_x x}{\hbar} + \frac{ieEy}{\hbar\omega}\cos(\omega t)\right) \tag{1.21}$$

we can simply the the Eq. (1.20) as

$$\psi(x,y,t)=\varphi(x,y,t)\phi(y,t). \tag{1.22}$$

Let's subtitue Eq. (1.20) and Eq. (1.17) into Eq. (1.18) and we can observe that

$$\begin{aligned}\text{L.H.S}=i\hbar\frac{\mathrm{d}\psi}{\mathrm{d}t}&=i\hbar\Big(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\phi+\frac{\mathrm{d}\phi}{\mathrm{d}t}\varphi\Big)=i\hbar\Big(\Big[\frac{-ieEy}{\hbar}\sin(\omega t)\Big]\varphi\phi+\varphi\frac{\mathrm{d}\phi}{\mathrm{d}t}\Big)\\&=\big[eEy\sin(\omega t)\big]\varphi\phi+i\hbar\varphi\frac{\mathrm{d}\phi}{\mathrm{d}t}\end{aligned} \tag{1.23}$$

and

$$\begin{aligned}\text{R.H.S}&=\hat{H}_e(t)\psi\\&=\Big[\frac{m_e\omega_0^2}{2}y^2+\frac{1}{2m_e}\Big(-\hbar^2\frac{\partial^2}{\partial y^2}+\frac{2i\hbar eE}{\omega}\cos(\omega t)\frac{\partial}{\partial y}+\frac{e^2E^2}{\omega^2}\cos^2(\omega t)\Big)\Big]\varphi\phi\end{aligned} \tag{1.24}$$

where we will can calculate this part by part as follows:

$$\begin{aligned}\frac{-\hbar^2}{2m_e}\frac{\partial^2}{\partial y^2}(\varphi\phi)&=\frac{-\hbar^2}{2m_e}\frac{\partial}{\partial y}\Big[\Big(\frac{ieE}{\hbar\omega}\cos(\omega)t\Big)\varphi\phi+\varphi\frac{\partial\phi}{\partial y}\Big]\\&=\frac{-\hbar^2}{2m_e}\Big[\Big(\frac{ieE}{\hbar\omega}\cos(\omega)t\Big)^2\varphi\phi+\Big(\frac{ieE}{\hbar\omega}\cos(\omega)t\Big)\varphi\frac{\partial\phi}{\partial y}+\Big(\frac{ieE}{\hbar\omega}\cos(\omega)t\Big)\varphi\frac{\partial\phi}{\partial y}+\varphi\frac{\partial^2\phi}{\partial y^2}\Big]\\&=\Big(\frac{e^2E^2}{2m_e\omega^2}\cos^2(\omega)t\Big)\varphi\phi-\Big(\frac{ieE\hbar}{m_e\omega}\cos(\omega)t\Big)\varphi\frac{\partial\phi}{\partial y}-\frac{\hbar^2}{2m_e}\varphi\frac{\partial^2\phi}{\partial y^2}\end{aligned} \tag{1.25}$$

and

$$\begin{aligned}\frac{2i\hbar eE}{2m_e\omega}\cos(\omega t)\frac{\partial}{\partial y}(\varphi\phi)&=\frac{i\hbar eE}{m_e\omega}\cos(\omega t)\Big[\Big(\frac{ieE}{\hbar\omega}\cos(\omega)t\Big)\varphi\phi+\varphi\frac{\partial\phi}{\partial y}\Big]\\&=\Big(\frac{-e^2E^2}{m_e\omega^2}\cos(\omega)t\Big)\varphi\phi+\frac{i\hbar eE}{m_e\omega}\cos(\omega t)\varphi\frac{\partial\phi}{\partial y}.\end{aligned} \tag{1.26}$$

Therefore we can derive that

$$\text{R.H.S}=\Big[\frac{m_e\omega_0^2}{2}y^2-\frac{\hbar^2}{2m_e}\varphi\frac{\partial^2\phi}{\partial y^2}\Big]\varphi\phi. \tag{1.27}$$

To satisfy the condition L.H.S=R.H.S we need to find a function $\phi(y,t)$ such that

$$\big[eEy\sin(\omega t)\big]\varphi\phi+i\hbar\varphi\frac{\mathrm{d}\phi}{\mathrm{d}t}=\Big[\frac{m_e\omega_0^2}{2}y^2-\frac{\hbar^2}{2m_e}\varphi\frac{\partial^2\phi}{\partial y^2}\Big]\varphi\phi \tag{1.28}$$

which can be simplyfied as

$$\Big[\frac{m_e\omega_0^2}{2}y^2-eEy\sin(\omega t)-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y^2}-i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\Big]\phi(y,t)=0. \tag{1.29}$$

If we turn off the external dressing field, this equation leads to simple harmonic oscillator Hamiltonian as follows

$$\Big[\frac{m_e\omega_0^2}{2}y^2-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y^2}-i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\Big]\phi(y,t)=0 \tag{1.30}$$

$$i\hbar\frac{\mathrm{d}\phi(y,t)}{\mathrm{d}t}=\Big[\frac{\hat{p}_y^2}{2m_e}+\frac{1}{2}m_e\omega_0^2y^2\Big]\phi(y,t). \tag{1.31}$$

Therefore we can identify the $S(t)\equiv eE\sin(\omega t)$ part as a external force act on the harmonic oscillator and we can solve this as a forced harmonic oscillator in y axis.

$$i\hbar\frac{\mathrm{d}\phi(y,t)}{\mathrm{d}t}=\Big[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y^2}+\frac{1}{2}m_e\omega_0^2y^2-yS(t)\Big]\phi(y,t). \tag{1.32}$$

This system can be extacly solvable and we can solve this equation using the methods explained by Husimi [*1] as follows.
First we can introduce the time dependent shifted corrdinte as

$$y \to y' = y - \zeta(t) \quad \Rightarrow \quad y = y' + \zeta(t) \tag{1.33}$$

and this implies that

$$\frac{\mathrm{d}\phi(y',t)}{\mathrm{d}t} = \frac{\partial\phi(y',t)}{\partial t} + \frac{\partial\phi(y',t)}{\partial y'}\frac{\partial y'}{\partial t} = \frac{\partial\phi(y',t)}{\partial t} - \dot{\zeta}(t)\frac{\partial\phi(y',t)}{\partial y'} \tag{1.34}$$

where $\dot{\zeta}(t) = \frac{\partial\zeta(t)}{\partial t}$. Therefore, Eq. (1.32) will be modified to

$$i\hbar\frac{\partial\phi(y',t)}{\partial t} = \Big[i\hbar\dot{\zeta}\frac{\partial}{\partial y'} - \frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y'^2} + \frac{1}{2}m_e\omega_0^2(y'+\zeta)^2 - (y'+\zeta)S(t)\Big]\phi(y',t). \tag{1.35}$$

Let's tranform the wave function using following unitary trasnform

$$\phi(y',t) = \exp\Bigg(\frac{im_e\dot{\zeta}y'}{\hbar}\Bigg)\varphi(y',t) \tag{1.36}$$

and subtitte this into the Eq. (1.35) and we will get the following

$$\text{R.H.S} = \Big[i\hbar\frac{\partial}{\partial t} - i\hbar\Big(\frac{im_e\ddot{\zeta}y'}{\hbar}\Big)\Big]\exp\Bigg(\frac{-im_e\dot{\zeta}y'}{\hbar}\Bigg)\varphi(y',t) \tag{1.37}$$

and

$$\begin{aligned}\text{L.H.S} = \Big[&i\hbar\dot{\zeta}\Big(\frac{im_e\dot{\zeta}}{\hbar}\Big) + i\hbar\dot{\zeta}\frac{\partial}{\partial y'} \\ &- \frac{\hbar^2}{2m_e}\Big[\Big(\frac{im_e\dot{\zeta}}{\hbar}\Big)^2 + \Big(\frac{2im_e\dot{\zeta}}{\hbar}\Big)\frac{\partial}{\partial y'} + \frac{\partial^2}{\partial y'^2}\Big] \\ &+ \frac{1}{2}m_e\omega_0^2y'^2 + \frac{1}{2}m_e\omega_0^2\zeta^2 + m_e\omega_0^2y'\zeta \\ &- y'S(t) - \zeta S(t)\Big]\exp\Bigg(\frac{-im_e\dot{\zeta}y'}{\hbar}\Bigg)\varphi(y',t).\end{aligned} \tag{1.38}$$

Combining these two we get derive that

$$\begin{aligned}i\hbar\frac{\partial\varphi(y',t)}{\partial t} = \Big[&-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y'^2} + \frac{1}{2}m_e\omega_0^2y'^2 + \Big[m_e\ddot{\zeta} + m_e\omega_0^2\zeta - S(t)\Big]y' \\ &+ \Big[-\frac{1}{2}m_e\dot{\zeta}^2 + \frac{1}{2}m_e\omega_0^2\zeta^2 - \zeta S(t)\Big]\Big]\varphi(y',t).\end{aligned} \tag{1.39}$$

Then we can restrict our $\zeta(t)$ function such that

$$m_e\ddot{\zeta} + m_e\omega_0^2\zeta = S(t) \tag{1.40}$$

and that leads to

$$i\hbar\frac{\partial\varphi(y',t)}{\partial t} = \Big[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y'^2} + \frac{1}{2}m_e\omega_0^2y'^2 - L(\zeta,\dot{\zeta},t)\Big]\varphi(y',t) \tag{1.41}$$

where

$$L(\zeta,\dot{\zeta},t) \equiv \frac{1}{2}m_e\dot{\zeta}^2 - \frac{1}{2}m_e\omega_0^2\zeta^2 + \zeta S(t) \tag{1.42}$$

is the largrangian of a driven oscillator.

Now introduce new unitary transormation for the wavefunction as follows

$$\varphi(y',t) = \exp\biggl(\frac{i}{\hbar}\int_0^t dt' L(\zeta,\dot{\zeta},t')\biggr)\chi(y',t) \tag{1.43}$$

and subtite this into the Eq. (1.41) and gets

$$\begin{aligned} i\hbar\biggl[\exp\biggl(\frac{i}{\hbar}\int_0^t dt' L(\zeta,\dot{\zeta},t')\biggr)\frac{\partial}{\partial t} + i\hbar L(\zeta,\dot{\zeta},t)\exp\biggl(\frac{i}{\hbar}\int_0^t dt' L(\zeta,\dot{\zeta},t')\biggr)\biggr]\chi(y',t) \\ = \biggl[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y'^2} + \frac{1}{2}m_e\omega_0^2 y'^2 - L(\zeta,\dot{\zeta},t)\biggr]\exp\biggl(\frac{i}{\hbar}\int_0^t dt' L(\zeta,\dot{\zeta},t')\biggr)\chi(y',t) \end{aligned} \tag{1.44}$$

and finally we can derive that

$$i\hbar\frac{\partial}{\partial t}\chi(y',t) = \biggl[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y'^2} + \frac{1}{2}m_e\omega_0^2 y'^2\biggr]\chi(y',t). \tag{1.45}$$

This is the well known Schrodinger equation of a stationary harmonic oscillator. In terms of the eigenvalues

$$E_n = \hbar\omega_0(n+\frac{1}{2}) \tag{1.46}$$

of well-known harmonic eigenfucntions

$$\chi_n(y') = \frac{1}{\sqrt{2^n n!}}\cdot\biggl(\frac{m_e\omega_0}{\pi\hbar}\biggr)^{1/4}\cdot e^{-\frac{m_e\omega_0 y'^2}{2\hbar}}\cdot\mathcal{H}_n\biggl(\sqrt{\frac{m_e\omega_0}{\hbar}}y'\biggr) \tag{1.47}$$

being propositional to the Hermite functions $\mathcal{H}_n$, the solutions of Eq. (1.32) can be represent as

$$\phi_n(y,t) = \chi_n(y-\zeta(t))\exp\biggl(\frac{i}{\hbar}\biggl[-E_n t + m_e\dot{\zeta}(t)\bigl(y-\zeta(t)\bigr) + \int_0^t dt' L(\zeta,\dot{\zeta},t')\biggr]\biggr) \tag{1.48}$$

The set $\chi(y)$ forms a complete set and thus ay general solution $\phi_(y,t)$ can be expaned in terms of the solutions in Eq. (1.48).
Next we consider special case where we assumed

$$S(t) = eE\sin(\omega t) \tag{1.49}$$

and one can derive the Eq. (1.40) for $\zeta(t)$

$$m_e\ddot{\zeta} + m_e\omega_0^2\zeta = eE\sin(\omega t) \tag{1.50}$$

and using Green function method the solution can be write as

$$\zeta(t) = \frac{eE}{m_e(\omega_0^2-\omega^2)}\sin(\omega t). \tag{1.51}$$

form this solutions we are able to derive the final solutions $\alpha = (n,m)$ where $n,m \in \mathbb{Z}^+$ are two quantum numbers that describe the state of the electron, can be present as

$$\begin{aligned} \psi_\alpha(x,y,t) =& \frac{1}{\sqrt{L_x}}\chi_n\bigl(y-\zeta(t)\bigr) \\ &\times\exp\biggl(\frac{i}{\hbar}\biggl[-E_n t + p_x x + \frac{eEy}{\omega}\cos(\omega t) + m_e\dot{\zeta}(t)\bigl[y-\zeta(t)\bigr] + \int_0^t dt' L(\zeta,\dot{\zeta},t')\biggr]\biggr) \end{aligned} \tag{1.52}$$

and the exponential phase shifts represent the effect done by the stationary magnetic field and strong dressing field.Therefore we can assume that the magnetitranport properties of 2DEG will be renormalized by the magnetic field as well as the dressing field. ■

2 Scattering theory

Since in a real metal there would be many scatters that can be behave as obstacles for electron that have free wave functions. Therefore we need to calculate them to analyse the real behaviour of the electrons.
Then the wave function of the electron in a real matel $\Psi(\mathbf{r},t)$ should satisfy the following time-dependet Schrodinger equation

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}=[H_e(t)+U(\mathbf{r})]\Psi(\mathbf{r},t) \tag{2.1}$$

where $U(\mathbf{r})$ is the total scattering potential. We have represented the all scatters using this potential. Since the solutions (1.52) are create a complete orthonormal basis we can represent this wave function using those as follows

$$\Psi(\mathbf{r},t)=\sum_j a_j(t)\left|\psi_j(t)\right\rangle \tag{2.2}$$

where the difference inidces j corresponding to the different sets of all quantum numbers p_x and n

$$j\rightarrow(m,n) \quad \text{where} \quad m,n=0,1,2,... \tag{2.3}$$

with m is defined for quantized momentum in x direction

$$p_x=m\frac{2\pi\hbar}{L_x} \tag{2.4}$$

Now we can use the conventional pertubation theory to calculate scattering process of electron at a state $\left|\psi_j\right\rangle$ to a state $\left|\psi_j'\right\rangle$. For that assume an electron be in the j state at the time $t=0$ and corresponding $a_j'(0)=\delta_{j,j'}$.
First subtitute a general electron state $\Psi(\mathbf{r},t)$ at time t as the incoming electron to the Schrodinger equation given in Eq. (2.1)

$$i\hbar\frac{\partial}{\partial t}\sum_j a_j(t)\left|\psi_j(t)\right\rangle=[H_e(t)+U(\mathbf{r})]\sum_j a_j(t)\left|\psi_j(t)\right\rangle \tag{2.5}$$

$$i\hbar\sum_j \dot{a_j}(t)\left|\psi_j(t)\right\rangle+a_j(t)\frac{\partial}{\partial t}\left|\psi_j(t)\right\rangle=[H_e(t)+U(\mathbf{r})]\sum_j a_j(t)\left|\psi_j(t)\right\rangle \tag{2.6}$$

since all the $\left|\psi(t)\right\rangle$ staistfy the Schrodinger equation (1.18)

$$i\hbar\sum_j \dot{a_j}(t)\left|\psi_j(t)\right\rangle=\sum_j U(\mathbf{r})a_j(t)\left|\psi_j(t)\right\rangle. \tag{2.7}$$

Then take inner product with state with the state $\left|\psi_{j'}(t)\right\rangle$

$$i\hbar\sum_j \dot{a_j}(t)\left\langle\psi_{j'}(t)|\psi_j(t)\right\rangle=\sum_j a_j(t)\left\langle\psi_{j'}(t)\right|U(\mathbf{r})\left|\psi_j(t)\right\rangle \tag{2.8}$$

But using the *Born approximation* we can assume that this incoming wave have the initial state of the electron at $t=0$ and therefore this equation will modified to

$$i\hbar\sum_j \dot{a_j}(t)\left\langle\psi_{j'}(t)|\psi_j(t)\right\rangle=\left\langle\psi_{j'}(t)\right|U(\mathbf{r})\left|\psi_j(t)\right\rangle \tag{2.9}$$

due to orthonormality this becomes

$$i\hbar\dot{a_{j'}}(t)=\left\langle\psi_{j'}(t)\right|U(\mathbf{r})\left|\psi_j(t)\right\rangle \tag{2.10}$$

and finally this leads to first order pertubation theory for Sscattering as follows

$$\dot{a_{j'}}(t)=-\frac{i}{\hbar}\left\langle\psi_{j'}(t)\right|U(\mathbf{r})\left|\psi_j(t)\right\rangle \tag{2.11}$$

where

$$a_{j'}(t) = -\frac{i}{\hbar}\int_0^t dt' \int_S d\mathbf{r}\ \psi^*_{j'}(\mathbf{r},t')U(\mathbf{r})\psi_j(\mathbf{r},t') \tag{2.12}$$

where the integration should be performed over the 2DEG area $S = L_x L_y$. Then we can calculate this using the eqution we derived in (1.52) as follows

$$\begin{aligned}
a_{j'}(t) = &-\frac{i}{\hbar}\int_0^t dt' \int_S d\mathbf{r}\ \Bigg[\frac{1}{\sqrt{L_x}}\chi^*_{n'}\big(y - {y'}_0 - \zeta(t)\big) \\
&\times \exp\Bigg(\frac{i}{\hbar}\Bigg[E_{n'}t' - m'\frac{2\pi\hbar x}{L_x} - \frac{eE(y-{y'}_0)}{\omega}\cos(\omega t') - m_e\dot{\zeta}(t)\big[y - {y'}_0 - \zeta(t')\big] - \int_0^{t'} dt' L(\zeta,\dot{\zeta},t")\Bigg]\Bigg) \\
&\times U(\mathbf{r}) \\
&\times \frac{1}{\sqrt{L_x}}\chi_n\big(y - y_0 - \zeta(t')\big) \\
&\times \exp\Bigg(\frac{i}{\hbar}\Bigg[-E_n t' + m\frac{2\pi\hbar x}{L_x} - \frac{eE(y-y_0)}{\omega}\cos(\omega t') - m_e\dot{\zeta}(t')\big[y - y_0 - \zeta(t')\big] - \int_0^{t'} d\tilde{t}L(\zeta,\dot{\zeta},\tilde{t})\Bigg]\Bigg)\Bigg]
\end{aligned} \tag{2.13}$$

then this will be simplified to

$$\begin{aligned}
a_{j'}(t) = &-\frac{i}{\hbar}\int_0^t dt' \int_S d\mathbf{r}\ \Bigg[\frac{1}{\sqrt{L_x}}\chi^*_{n'}\big(y - {y'}_0 - \zeta(t')\big)U(\mathbf{r})\frac{1}{\sqrt{L_x}}\chi_n\big(y - y_0 - \zeta(t')\big) \\
&\times \exp\Bigg(\frac{i}{\hbar}\Bigg[E_{n'}t' - m'\frac{2\pi\hbar x}{L_x} - \frac{eE(y-{y'}_0)}{\omega}\cos(\omega t') - m_e\dot{\zeta}(t')\big[y - {y'}_0 - \zeta(t')\big] - \int_0^{t'} d\tilde{t}L(\zeta,\dot{\zeta},\tilde{t})\Bigg]\Bigg) \\
&\times \exp\Bigg(\frac{i}{\hbar}\Bigg[-E_n t' + m\frac{2\pi\hbar x}{L_x} + \frac{eE(y-y_0)}{\omega}\cos(\omega t') + m_e\dot{\zeta}(t')\big[y - y_0 - \zeta(t')\big] + \int_0^{t'} d\tilde{t}L(\zeta,\dot{\zeta},\tilde{t})\Bigg]\Bigg)\Bigg]
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
a_{j'}(t) = &-\frac{i}{\hbar}\int_0^t dt' \int_S d\mathbf{r}\ \Bigg[\frac{1}{\sqrt{L_x}}\chi^*_{n'}\big(y - {y'}_0 - \zeta(t')\big)U(\mathbf{r})\frac{1}{\sqrt{L_x}}\chi_n\big(y - y_0 - \zeta(t')\big)\exp\Bigg(\frac{2\pi i(m-m')\hbar x}{L_x}\Bigg) \\
&\times \exp\Bigg(\frac{i}{\hbar}\Bigg[E_{n'}t' + \frac{eE{y'}_0}{\omega}\cos(\omega t') + m_e\dot{\zeta}(t'){y'}_0\Bigg]\Bigg)\exp\Bigg(\frac{i}{\hbar}\Bigg[-E_n t' - \frac{eEy_0}{\omega}\cos(\omega t') - m_e\dot{\zeta}(t)y_0\Bigg]\Bigg)\Bigg].
\end{aligned} \tag{2.15}$$

The time dependence of the $chi_n(y)$ can neglect since it is integrate over all the values of the y and we can write this as

$$\begin{aligned}
a_{j'}(t) = &-\frac{i}{\hbar}\int_S d\mathbf{r}\ \frac{1}{\sqrt{L_x}}\chi^*_{n'}\big(y - {y'}_0 - \zeta(t')\big)U(\mathbf{r})\frac{1}{\sqrt{L_x}}\chi_n\big(y - y_0 - \zeta(t')\big)\exp\Bigg(\frac{2\pi i(m-m')\hbar x}{L_x}\Bigg) \\
&\times \int_0^t dt' \Bigg[\exp\Bigg(\frac{i}{\hbar}\Bigg[(E_{n'} - E_n)t' + \frac{eE({y'}_0 - y_0)\omega_0^2}{\omega(\omega_0^2 - \omega^2)}\cos(\omega t')\Bigg]\Bigg)\Bigg].
\end{aligned} \tag{2.16}$$

Using Jacobi-Anger expansion

$$e^{iz\cos(\theta)} = \sum_{l=-\infty}^{\infty} i^l J_j(z)e^{in\theta} \tag{2.17}$$

above eqution can be modified as

$$a_{j'}(t) = -\frac{i}{\hbar}U_{j'j}\int_0^t dt' \Bigg[\sum_{l=-\infty}^{\infty} i^l J_l\Bigg[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)}\Bigg]\exp\Bigg(\frac{i}{\hbar}(E_{n'} - E_n + l\hbar\omega)t'\Bigg)\Bigg] \tag{2.18}$$

where

$$Uj'j \equiv \langle\Phi_{j'}(\mathbf{r})|U(\mathbf{r})|\Phi_j(\mathbf{r})\rangle \tag{2.19}$$

with bare electron eigen states (without dressing field)

$$\Phi_j(\mathbf{r}) = \frac{1}{\sqrt{L_x}} \exp\left(\frac{2\pi i m\hbar x}{L_x}\right)\chi_n(y). \tag{2.20}$$

Considering time evalution from negative values we can write the same expression as follows

$$a_{j'}(t) = -\frac{i}{\hbar}U_{j'j}\int_{-t/2}^{t/2} dt' \left[\sum_{l=-\infty}^{\infty} i^l J_l\left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2-\omega^2)}\right]\exp\left(\frac{i}{\hbar}(E_{n'} - E_n + l\hbar\omega)t'\right)\right]. \tag{2.21}$$

To calculate scattering probability we can use this scattering amplitude's squre value

$$\begin{aligned}|a_{j'}(t)|^2 = \frac{|U_{j'j}|^2}{\hbar^2}\int_{-t/2}^{t/2} dt' &\left[\sum_{l=-\infty}^{\infty} -i^l J_l\left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2-\omega^2)}\right]\exp\left(\frac{-i}{\hbar}(E_{n'} - E_n + l\hbar\omega)t'\right)\right]\\ \times\int_{-t/2}^{t/2} dt'' &\left[\sum_{k=-\infty}^{\infty} i^k J_k\left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2-\omega^2)}\right]\exp\left(\frac{i}{\hbar}(E_{n'} - E_n + k\hbar\omega)t''\right)\right]\end{aligned} \tag{2.22}$$

Considering long time $t \to \infty$ we can make the integral into a delta function as follows

$$\begin{aligned}|a_{j'}(t)|^2 = 4\pi^2|U_{j'j}|^2&\left[\sum_{l=-\infty}^{\infty} -i^l J_l\left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2-\omega^2)}\right]\delta(-E_{n'} + E_n - l\hbar\omega)\right]\\ \times&\left[\sum_{k=-\infty}^{\infty} i^k J_k\left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2-\omega^2)}\right]\delta(E_{n'} - E_n + k\hbar\omega)\right]\end{aligned} \tag{2.23}$$

and this implies $l = k$ and this leads to

$$|a_{j'}(t)|^2 = 4\pi^2|U_{j'j}|^2\left[\sum_{l=-\infty}^{\infty} J_l^2\left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2-\omega^2)}\right]\delta^2(E_{n'} - E_n + l\hbar\omega). \tag{2.24}$$

Then using the famous the square δ function transormation method

$$\delta^2(\epsilon) = \delta(\epsilon)\delta^2(0)\lim_{t\to\infty}\int_{-t/2}^{t/2} e^{i0\times t'/\hbar}dt' = \frac{\delta(\epsilon)t}{2\pi\hbar} \tag{2.25}$$

we can calculatre the probability of electron scattering between states j and j' per unit time as

$$\mathcal{W}_{j'j} \equiv \frac{\mathrm{d}|a_{j'}(t)|^2}{\mathrm{d}t} = |U_{j'j}|^2\sum_{l=-\infty}^{\infty} J_l^2\left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2-\omega^2)}\right] \times \frac{2\pi}{\hbar}\delta(E_{n'} - E_n + l\hbar\omega) \tag{2.26}$$

To avoid thee energy echange betwen a high-frequency field and electrons, the field should be purely dressing. We can achieve that by using the field with off-resonant and high frequency. Therefore, the only effect of the dressing field on 2DEG is the renormalization of the probability of elastic electron scattering within the same Landau level ($E_{n'} = E_n$), which described by the term with $l = 0$ the Eq. (2.26) leads to

$$\mathcal{W}_{j'j} = J_0^2\left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2-\omega^2)}\right]\mathcal{W}^{(0)}_{j'j} \tag{2.27}$$

where

$$\mathcal{W}^{(0)}_{j'j} = \frac{2\pi}{\hbar}|U_{j'j}|^2\delta(E_{n'} - E_n) \tag{2.28}$$

is the probability of scattering of a *bare electron.* It is important to notice that the Bessel function factor depend on both the dressing field and stationaty magnetic field. This factor is responcible for all the effcts discussed in thisarticle.

One can define the lifetime of the dressed electron at the Landau level τ is renormalized by the Bessel function as below

$$\frac{1}{\tau} \equiv \sum_{j'} \mathcal{W}_{j'j} = \sum_{j'} J_0^2\left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)}\right]\mathcal{W}_{j'j}^{(0)} \tag{2.29}$$

where we have consider all posibilities that electron can jump to the state j'. Then rewrite the delat function as follows

$$\delta(\epsilon) = \frac{1}{\pi}\lim_{\Gamma\to 0}\frac{\Gamma}{\Gamma^2 + \epsilon^2} \tag{2.30}$$

where in this study we can assume that the paramater $\Gamma \equiv \hbar/\tau$ as scattering induced broading of the Landau level. But for the elestic scatteing within the same Landau level, we can write the δ function as

$$\delta(E_{n'} - E_n) \approx \frac{1}{\pi\Gamma}. \tag{2.31}$$

Therefore Eq. (2.29) will change to

$$\frac{1}{\tau} = \sum_{j'} J_0^2\left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)}\right] \times \frac{2\pi}{\hbar}|U_{j'j}|^2 \times \frac{1}{\pi\Gamma} \tag{2.32}$$

$$\frac{1}{\tau} = \sum_{j'} J_0^2\left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)}\right] \times \frac{2}{\hbar}|U_{j'j}|^2 \times \frac{\tau}{\hbar} \tag{2.33}$$

and finally this can be modified to

$$\frac{1}{\tau} = \left[\frac{2}{\hbar^2}\sum_{j'} J_0^2\left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)}\right]|U_{j'j}|^2\right]^{1/2} \tag{2.34}$$

where the summation is performed over electron states j' within the same Landau level.
Now lets specify more on the scattering potential where we can model them as randomly distributed delta fucntions as follows

$$U(\mathbf{r}) \equiv \sum_{i=1}^{N_s} U_0\delta(\mathbf{r} - \mathbf{r}_i) \tag{2.35}$$

where N_s is the total number of scatters in the considering matel. Now we can calculate $|U_{j'j}|^2$ as follows

$$\begin{aligned}|U_{j'j}|^2 =& \sum_{i=1}^{N_s}\frac{1}{L_x^2}\int\int dx_1dy_1 \exp\left(\frac{-{p'}_x x_1}{\hbar}\right)\chi_n^*(y_1 - {y'}_0)U_0\delta(x_1 - x_i)\delta(y_1 - y_i)\exp\left(\frac{p_x x_1}{\hbar}\right)\chi_n(y_1 - y_0)\\ &\times\int\int dx_2dy_2 \exp\left(\frac{{p'}_x x_2}{\hbar}\right)\chi_n(y_2 - {y'}_0)U_0\delta(x_2 - x_i)\delta(y_2 - y_i)\exp\left(\frac{-p_x x_2}{\hbar}\right)\chi_n^*(y_2 - y_0)\end{aligned} \tag{2.36}$$

and considering only non-zero values for x_1 and x_2 integrals we can re-write this as

$$\begin{aligned}|U_{j'j}|^2 =& \sum_{i=1}^{N_s}\frac{U_0^2}{L_x^2}\int dy_1 \exp\left(\frac{-{p'}_x x_i}{\hbar}\right)\chi_n^*(y_1 - {y'}_0)\delta(y_1 - y_i)\exp\left(\frac{p_x x_i}{\hbar}\right)\chi_n(y_1 - y_0)\\ &\times\int dy_2 \exp\left(\frac{{p'}_x x_i}{\hbar}\right)\chi_n(y_2 - {y'}_0)\delta(y_2 - y_i)\exp\left(\frac{-p_x x_i}{\hbar}\right)\chi_n^*(y_2 - y_0)\end{aligned} \tag{2.37}$$

and this will be simplified to

$$\begin{aligned}|U_{j'j}|^2 =& \sum_{i=1}^{N_s}\frac{U_0^2}{L_x^2}\int dy_1\chi_n^*(y_1 - {y'}_0)\delta(y_1 - y_i)\chi_n(y_1 - y_0)\\ &\times\int dy_2\chi_n(y_2 - {y'}_0)\delta(y_2 - y_i)\chi_n^*(y_2 - y_0).\end{aligned} \tag{2.38}$$

Again considering only non-zero values for y_1 and y_2 integrals we can re-write this as

$$|U_{j'j}|^2 = \sum_{i=1}^{N_s} \frac{U_0^2}{L_x^2} \chi_n^*(y_i - {y'}_0)\chi_n(y_i - y_0)\chi_n(y_i - {y'}_0)\chi_n^*(y_i - y_0). \tag{2.39}$$

$$|U_{j'j}|^2 = \frac{U_0^2}{L_x^2} \sum_{i=1}^{N_s} \chi_n^2(y_i - {y'}_0)\chi_n^2(y_i - y_0). \tag{2.40}$$

Now subtituting this derivation into the Eq. (2.34) we will get

$$\frac{1}{\tau} = \left[\frac{2U_0^2}{\hbar^2 L_x^2} \sum_{{y'}_0} J_0^2 \left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \sum_{i=1}^{N_s} \chi_n^2(y_i - {y'}_0)\chi_n^2(y_i - y_0) \right]^{1/2} \tag{2.41}$$

where j' reduced to ${p'}_x$ (since $n' = n$)and we can represent it by ${y'}_0$. Then this will modified to

$$\frac{1}{\tau} = \left[\frac{2U_0^2}{\hbar^2 L_x^2} \sum_{{y'}_0} \sum_{i=1}^{N_s} J_0^2 \left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \chi_n^2(y_i - {y'}_0)\chi_n^2(y_i - y_0) \right]^{1/2}. \tag{2.42}$$

Now considering large size of sample and a macroscopically large N_s scatters we can promate the summation to integrations as follows

$$\frac{1}{\tau} = \left[\frac{2U_0^2}{\hbar^2 L_x^2} \frac{eBL_x}{2\pi\hbar} \int d{y'}_0 \frac{N_s}{L_x} \int dy_i J_0^2 \left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \chi_n^2(y_i - {y'}_0)\chi_n^2(y_i - y_0) \right]^{1/2}. \tag{2.43}$$

Assuming $L_x = L_y$ we can define the area of the 2D material as

$$S \equiv L_x L_x = L_x L_y \tag{2.44}$$

and then we can re-write the above as

$$\frac{1}{\tau} = \left[\frac{eBN_sU_0^2}{\pi\hbar^3 S} \int d{y'}_0 \int dy_i J_0^2 \left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \chi_n^2(y_i - {y'}_0)\chi_n^2(y_i - y_0) \right]^{1/2}. \tag{2.45}$$

Define the *density of scatters* per unit area of 2DEG

$$n_s \equiv \frac{N_s}{S} \tag{2.46}$$

and the *magnetic length* as

$$l_0 \equiv \sqrt{\frac{\hbar}{eB}}. \tag{2.47}$$

Now our Eq. (2.45) leads to

$$\frac{1}{\tau} = \sqrt{\frac{n_sU_0^2}{\pi l_0^2 \hbar^2}} \left[\int d{y'}_0 \int dy_i J_0^2 \left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \chi_n^2(y_i - {y'}_0)\chi_n^2(y_i - y_0) \right]^{1/2} \tag{2.48}$$

and now define new dummy variables as follows (since y_0 is a paramter)

$$({y'}_0 - y_0) \to y \quad \text{and} \quad (y_i - {y'}_0) \to y' \tag{2.49}$$

and finally we will get the eqution for the dressed electron lifetime at the nth Landau level as

$$\frac{1}{\tau} = \sqrt{\frac{n_sU_0^2}{\pi l_0^2 \hbar^2}} \left[\int\int dy dy' \; J_0^2 \left[\frac{eEy\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \chi_n^2(y')\chi_n^2(y + y') \right]^{1/2} \tag{2.50}$$

■

3 Floquet theory

Since we describe the lifetime of an electron in certain Landau level using conventianal perturbation theeory, now we can apply the Floquet theory to identify the difference of these methods.
First we need to identify the *quasienergies* and periodic *Floquet modes* for derived wavefunctions (1.52) for a 2DEG system with both stationary magnetic field and strong dressing filed.
Let's consider the following paramter which is lineraly increasing in time

$$\Delta_E t \equiv \frac{t}{T}\int_0^T dt'\ L(\zeta,\dot{\zeta},t') \tag{3.1}$$

where we can calculate this using Eq. (1.42) and (1.51) as follows

$$\begin{aligned}\Delta_E t = \frac{t}{T}\int_0^T dt'\ \frac{1}{2}m_e\frac{(eE\omega)^2}{m_e^2(\omega_0^2-\omega^2)^2}\cos^2(\omega t') - \frac{1}{2}m_e\omega_0^2\frac{(eE)^2}{m_e^2(\omega_0^2-\omega^2)^2}\sin^2(\omega t')\\ + \frac{eE}{m_e(\omega_0^2-\omega^2)}\sin(\omega t')eE\sin(\omega t')\end{aligned} \tag{3.2}$$

$$\begin{aligned}\Delta_E t = \frac{t\omega}{2\pi}\times\frac{(eE)^2}{2m_e(\omega_0^2-\omega^2)^2}\Bigg[\omega^2\int_0^T dt'\ \cos^2(\omega t') - \omega_0^2\int_0^T dt'\ \sin^2(\omega t')\\ + 2(\omega_0^2-\omega^2)\int_0^T dt'\ \sin^2(\omega t')\Bigg]\end{aligned} \tag{3.3}$$

$$\Delta_E t = \frac{t\omega}{2\pi}\times\frac{(eE)^2}{2m_e(\omega_0^2-\omega^2)^2}\left[\omega^2\frac{\pi}{\omega} - \omega_0^2\frac{\pi}{\omega} + 2(\omega_0^2-\omega^2)\frac{\pi}{\omega}\right] \tag{3.4}$$

$$\Delta_E t = \frac{t\omega}{2}\times\frac{(eE)^2}{2m_e(\omega_0^2-\omega^2)^2}(\omega_0^2-\omega^2) = \frac{(eE)^2}{4m_e(\omega_0^2-\omega^2)}t \tag{3.5}$$

Since this is the continuous increasing part of the Laggrangian integral in Eq. (1.52) we can make this as 2ω periodic function as follows

$$\Lambda \equiv \int_0^t dt'\ L(\zeta,\dot{\zeta},t') - \frac{t}{T}\int_0^T dt'\ L(\zeta,\dot{\zeta},t') \tag{3.6}$$

which can be proved as follows. First consider the first term of the Λ

$$\begin{aligned}\int_0^t dt'\ L(\zeta,\dot{\zeta},t') = \frac{(eE)^2}{2m_e(\omega_0^2-\omega^2)^2}\Bigg[\omega^2\int_0^t dt'\ \cos^2(\omega t') - \omega_0^2\int_0^t dt'\ \sin^2(\omega t')\\ + 2(\omega_0^2-\omega^2)\int_0^t dt'\ \sin^2(\omega t')\Bigg]\end{aligned} \tag{3.7}$$

$$\begin{aligned}\int_0^t dt'\ L(\zeta,\dot{\zeta},t') = \frac{(eE)^2}{2m_e(\omega_0^2-\omega^2)^2}\Bigg[\omega^2\left[\frac{t}{2}+\frac{\sin(2\omega t)}{4\omega}\right] - \omega_0^2\left[\frac{t}{2}-\frac{\sin(2\omega t)}{4\omega}\right]\\ + 2(\omega_0^2-\omega^2)\left[\frac{t}{2}-\frac{\sin(2\omega t)}{4\omega}\right]\Bigg]\end{aligned} \tag{3.8}$$

$$\begin{aligned}\int_0^t dt'\ L(\zeta,\dot{\zeta},t') = \frac{(eE)^2}{2m_e(\omega_0^2-\omega^2)^2}\Bigg[\frac{t}{2}\left[\omega^2-\omega_0^2+2\omega_0^2-2\omega^2\right]\\ + \frac{\sin(2\omega t)}{4\omega}\left[\omega^2+\omega_0^2-2\omega_0^2+2\omega^2\right]\Bigg]\end{aligned} \tag{3.9}$$

$$\int_0^t dt'\ L(\zeta,\dot{\zeta},t') = \frac{(eE)^2}{4m_e(\omega_0^2-\omega^2)}t + \frac{(eE)^2(3\omega^2-\omega_0^2)}{8m_e\omega(\omega_0^2-\omega^2)^2}\sin(2\omega t) \tag{3.10}$$

then using Eq.(3.5) we can write this as

$$\int_0^t dt'\ L(\zeta,\dot{\zeta},t') = \Delta_E t + \frac{(eE)^2\big(3\omega^2-\omega_0^2\big)}{8m_e\omega(\omega_0^2-\omega^2)^2}\sin(2\omega t). \tag{3.11}$$

Now we can express

$$\Lambda = \Delta_E t + \frac{(eE)^2\big(3\omega^2-\omega_0^2\big)}{8m_e\omega(\omega_0^2-\omega^2)^2}\sin(2\omega t) - \Delta_E t = \frac{(eE)^2\big(3\omega^2-\omega_0^2\big)}{8m_e\omega(\omega_0^2-\omega^2)^2}\sin(2\omega t) \tag{3.12}$$

which is a periodic function in time with 2ω frequency.
Now using this parmaters we can factorize the wavefunction (1.52) as linearly time dependend part and periodic time dependend part as follows

$$\begin{aligned}\psi_\alpha(x,y,t) =& \exp\left(\frac{i}{\hbar}[-E_n t+\Delta_E t]\right)\frac{1}{\sqrt{L_x}}\chi_n\big(y-\zeta(t)\big)\\ &\times\exp\left(\frac{i}{\hbar}\left[p_x x+\frac{eEy}{\omega}\cos(\omega t)+m_e\dot{\zeta(t)}\big[y-\zeta(t)\big]+\int_0^t dt' L(\zeta,\dot{\zeta},t')-\Delta_E t\right]\right)\end{aligned} \tag{3.13}$$

where we can identify(let $n\to\alpha$) the *quasienergies* as

$$\varepsilon_\alpha \equiv \hbar\omega_0\left(n+\frac{1}{2}\right)-\Delta_E \quad \text{where} \quad \alpha = 0,1,2,... \tag{3.14}$$

which is only depend on one quantum number (n) and *Floquet modes* as

$$\phi_\alpha(x,y,t) \equiv \frac{1}{\sqrt{L_x}}\chi_n\big(y-\zeta(t)\big)\exp\left(\frac{i}{\hbar}\left[p_x x+\frac{eEy}{\omega}\cos(\omega t)+m_e\dot{\zeta}(t)\big[y-\zeta(t)\big]+\Lambda\right]\right) \tag{3.15}$$

with

$$\zeta(t) = \frac{eE}{m_e(\omega_0^2-\omega^2)}\sin(\omega t) \quad \text{and} \quad \dot{\zeta}(t) = \frac{eE\omega}{m_e(\omega_0^2-\omega^2)}\cos(\omega t) \tag{3.16}$$

where *Floquet modes* are time-periodic functions that also create a complete orthonormal set. ■

Therefore the solutions (Floquet states) for the periodic Hamiltonian (1.5) can be writting in position space as

$$\psi_\alpha(x,y,t) = \exp\left(-\frac{i}{\hbar}\varepsilon_\alpha t\right)\phi_\alpha(x,y,t) \tag{3.17}$$

where

$$\varepsilon_\alpha \equiv \left(\frac{eB\hbar}{m_e}\right)\left(n+\frac{1}{2}\right)-\frac{(eE)^2}{4m_e(\omega_0^2-\omega^2)} \quad \text{where} \quad n = 0,1,2,... \tag{3.18}$$

and

$$\begin{aligned}\phi_\alpha(x,y,t) \equiv& \frac{1}{\sqrt{L_x}}\chi_n\left(y-\frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}\right)\\ &\times\exp\left(\frac{i}{\hbar}\left[p_x x+\frac{eEy}{\omega}\cos(\omega t)+\frac{eE\omega y}{(\omega_0^2-\omega^2)}\cos(\omega t)\right]\right)\\ &\times\exp\left(\frac{i}{\hbar}\left[-\frac{(eE)^2\omega}{2m_e(\omega^2-\omega_0^2)^2}\sin(2\omega t)+\frac{(eE)^2\big(3\omega_0^2-\omega^2\big)}{8m_e\omega(\omega_0^2-\omega^2)^2}\sin(2\omega t)\right]\right)\end{aligned} \tag{3.19}$$

Now we can write this by more simplying and considering spacial dependencies and using previous subtituting done in Eq. (1.16) and now χ function depend on both quantum numbers because y_0 gives the p_x dependence and we can present as

$$\begin{aligned}\phi_\alpha(x,y,t) \equiv& \frac{1}{\sqrt{L_x}}\chi_n\left(y-y_0-\frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}\right)\exp\left(\frac{ip_x}{\hbar}x\right)\exp\left(\frac{i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right](y-y_0)\right)\\ &\times\exp\left(\frac{-i}{\hbar}\left[\frac{(eE)^2(\omega_0^2+\omega^2)}{8\omega m_e(\omega_0^2-\omega^2)^2}\right]\sin(2\omega t)\right)\end{aligned} \tag{3.20}$$

Now we can transform this solution in spacial variable into the momentum space using Fourier trasform over the considering space.

$$\begin{aligned}\phi_\alpha(k_x,k_y,t) = &\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dy\ \exp(-ik_y y)\left[\frac{1}{\sqrt{L_x}}\chi_n\left(y-y_0-\frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}\right)\exp\left(\frac{i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y\right)\right]\\ &\times\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\ \exp(-ik_x x)\left[\exp\left(\frac{ip_x}{\hbar}x\right)\right]\\ &\times\exp\left(\frac{-i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y_0\right)\times\exp\left(\frac{-i}{\hbar}\left[\frac{(eE)^2(\omega_0^2+\omega^2)}{8\omega m_e(\omega_0^2-\omega^2)^2}\right]\sin(2\omega t)\right)\end{aligned} \tag{3.21}$$

Then this can be re-write as follows

$$\phi_\alpha(k_x,k_y,t) = \exp\left(\frac{-i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y_0\right)\exp\left(\frac{-i}{\hbar}\left[\frac{(eE)^2(\omega_0^2+\omega^2)}{8\omega m_e(\omega_0^2-\omega^2)^2}\right]\sin(2\omega t)\right)\delta\left(k_x-\frac{p_x}{\hbar}\right)\Theta_\alpha(k_y,t) \tag{3.22}$$

where we used

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\ \exp\left(-ik_x x+\frac{ip_x}{\hbar}x\right) = \sqrt{2\pi}\delta\left(k_x-\frac{p_x}{\hbar}\right) \tag{3.23}$$

and

$$\Theta_\alpha(k_y,t) \equiv \int_{-\infty}^{\infty} dy\ \exp(-ik_y y)\left[\frac{1}{\sqrt{L_x}}\chi_n\left(y-y_0-\frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}\right)\exp\left(\frac{i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y\right)\right] \tag{3.24}$$

and this can be simplied as

$$\Theta_\alpha(k_y,t) = \frac{1}{\sqrt{L_x}}\int_{-\infty}^{\infty} dy\ \chi_n\left(y-y_0-\frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}\right)\exp\left(-ik_y y+\frac{i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y\right). \tag{3.25}$$

Then by defining

$$\mu(t) \equiv \frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)} + y_0 \tag{3.26}$$

and

$$\gamma(t) \equiv \frac{eE\omega_0^2\cos(\omega t)}{\hbar\omega(\omega_0^2-\omega^2)} \tag{3.27}$$

we can re-write this by neglecting time dependencies as

$$\Theta_\alpha(k_y,t) = \frac{1}{\sqrt{L_x}}\int_{-\infty}^{\infty} dy\ \chi_n(y-\mu)\exp(-i(k_y-\gamma)y). \tag{3.28}$$

We can subtitute following variables

$${k_y}' = k_y-\gamma \quad \text{and} \quad y' = y-\mu \tag{3.29}$$

and this leads to

$$\Theta_\alpha({k_y}',t) = \frac{\sqrt{2}e^{-i{k_y}'\mu}}{\sqrt{L_x}}\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} dy'\ \chi_n(y')\exp\left(-i{k_y}'y'\right). \tag{3.30}$$

We know that $\{\chi_\alpha\}$ are well-known harmonic eigenfunctions(Gauss-Hermite functions) as given in the Eq. (1.47). However, the equation in (3.30) represents the Fourier transform of the these Gauss-Hermite functions. Due to the symmetric condition [*Ref:E.Celeghini] the Fourier transform of these functions can be represent as

$$\mathcal{FT}[\Theta_n(ax+b),x,k] = \frac{i^n}{|a|}\exp\left(-\frac{ibk}{a}\right)\Theta_n(k/a) \tag{3.31}$$

Therefore

$$\Theta_\alpha({k_y}',t) = \frac{\sqrt{2}e^{-i{k_y}'\mu}}{\sqrt{L_x}}\tilde{\chi}_n\left({k_y}'\right) \tag{3.32}$$

where

$$\tilde{\chi}_n(k_y{}') \equiv \frac{1}{\sqrt{2^n\alpha!}}\cdot\left(\frac{\hbar}{\pi m_e\omega_0}\right)^{1/4}\cdot e^{-\frac{\hbar}{2m_e\omega_0}(k_y{}')^2}\cdot\mathcal{H}_\alpha\left(\sqrt{\frac{\hbar}{m_e\omega_0}}k_y{}'\right) \tag{3.33}$$

Using Eq. (3.32) and Eq. (3.22) we can derive that

$$\begin{aligned}\phi_\alpha(k_x,k_y,t) = \exp\left(\frac{-i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y_0\right)\exp\left(\frac{-i}{\hbar}\left[\frac{(eE)^2(\omega_0^2+\omega^2)}{8\omega m_e(\omega_0^2-\omega^2)^2}\right]\sin(2\omega t)\right)\delta\left(k_x-\frac{p_x}{\hbar}\right)\\ \times\frac{i^n\sqrt{2}e^{-i(k_y-\gamma)\mu}}{\sqrt{L_x}}\tilde{\chi}_n(k_y-\gamma)\end{aligned} \tag{3.34}$$

and this can be re-write subtituting μ and γ values as follows

$$\begin{aligned}\phi_\alpha(k_x,k_y,t) = \exp\left(\frac{-i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y_0\right)\exp\left(\frac{-i}{\hbar}\left[\frac{(eE)^2(\omega_0^2+\omega^2)}{8\omega m_e(\omega_0^2-\omega^2)^2}\right]\sin(2\omega t)\right)\delta\left(k_x-\frac{p_x}{\hbar}\right)\\ \times\exp\left(-ik_y\frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}\right)\exp\left(\frac{i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]\frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}\right)\exp(-ik_yy_0)\\ \times\exp\left(i\frac{1}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y_0\right)\frac{i^n\sqrt{2}}{\sqrt{L_x}}\tilde{\chi}_n(k_y-\gamma)\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}\phi_\alpha(k_x,k_y,t) = \exp\left(\frac{i}{\hbar}\left[\frac{(eE)^2(3\omega_0^2-\omega^2)}{8\omega m_e(\omega_0^2-\omega^2)^2}\right]\sin(2\omega t)\right)\delta\left(k_x-\frac{p_x}{\hbar}\right)\\ \times\exp\left(-ik_y\left[\frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}+y_0\right]\right)\frac{i^n\sqrt{2}}{\sqrt{L_x}}\tilde{\chi}_n(k_y-\gamma)\end{aligned} \tag{3.36}$$

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4 Floquet Fermi Goldern Rule

In this section we are going to derive the Floquet Fermi goldern rule for above derived quantum Floquet states using $t - t'$ formalism.

The Floquet states (3.17) fullfills the $t - t'$ Schrödinger equation [*Ref:myReport] as follows

$$i\hbar \frac{\partial}{\partial t} |\psi_\alpha(t,t')\rangle = H_F(t') |\psi_\alpha(t,t')\rangle \tag{4.1}$$

where Floquet Hamiltonian given by

$$H_F(t') \equiv H_e(t) - i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \tag{4.2}$$

and

$$|\psi_\alpha(t,t')\rangle = \exp\left(-\frac{i}{\hbar}\varepsilon_\alpha t\right) |\phi_\alpha(t')\rangle \tag{4.3}$$

Now for the Eq. (4.1) corresponding time evolution operator satisfy the Schrödinger equation

$$U_0(t,t_0;t') = \exp\left(-\frac{i}{\hbar} H_F(t')[t-t_0]\right) \tag{4.4}$$

Consider a time-independent total perturbation $V(\mathbf{r})$ switched on at the reference time $t- = t_0$, then Schrödinger equation becomes

$$i\hbar \frac{\partial}{\partial t} |\Psi_\alpha(t,t')\rangle = [H_F(t') + V(\mathbf{r})] |\Psi_\alpha(t,t')\rangle \tag{4.5}$$

and when $t \leq t_0$ both solutions of the Schrödinger equation coincide

$$|\psi_\alpha(t,t')\rangle = |\Psi_\alpha(t,t')\rangle \quad \text{when} \quad t \leq t_0 \tag{4.6}$$

Now, we can introduce the interaction picture representation of the $t - t'$ Floquet state as

$$|\Psi_\alpha(t,t')\rangle_I = U_0^\dagger(t,t_0;t') |\Psi_\alpha(t,t')\rangle \tag{4.7}$$

and the perturbation in the interaction picture will be

$$V_I(\mathbf{r}) = U_0^\dagger(t,t_0;t') V(\mathbf{r}) U_0(t,t_0;t') = V(\mathbf{r}). \tag{4.8}$$

This leads to the Schrödinger eqution in the interction picture

$$i\hbar \frac{\partial}{\partial t} |\Psi_\alpha(t,t')\rangle_I = V_I(\mathbf{r}) |\Psi_\alpha(t,t')\rangle_I \tag{4.9}$$

with the recursive solution

$$|\Psi_\alpha(t,t')\rangle_I = |\Psi_\alpha(t_0,t')\rangle_I + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1 \; V_I(\mathbf{r}) |\Psi_\alpha(t_1,t')\rangle_I \tag{4.10}$$

Iterating the solution only upto first order (Born approximation) this leads to

$$|\Psi_\alpha(t,t')\rangle_I \approx |\psi_\alpha(t_0,t')\rangle + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1 \; V_I(\mathbf{r}) |\psi_\alpha(t_0,t')\rangle \tag{4.11}$$

and multiply it by $\langle\psi_\beta(t_0,t')|$ and we will get

$$\langle\psi_\beta(t_0,t')|\Psi_\alpha(t,t')\rangle_I = \langle\psi_\beta(t_0,t')|\psi_\alpha(t_0,t')\rangle + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1 \; \langle\psi_\beta(t_0,t')| V_I(\mathbf{r}) |\psi_\alpha(t_0,t')\rangle \,. \tag{4.12}$$

Then introdusing unitory operator U_0 we can re-write this as

$$\begin{aligned}\langle\psi_\beta(t_0,t')|U_0^\dagger(t,t_0;t')|\Psi_\alpha(t,t')\rangle = &\ \langle\psi_\beta(t_0,t')|U_0^\dagger(t,t_0;t')U_0(t,t_0;t')|\psi_\alpha(t_0,t')\rangle \\ &+\frac{1}{i\hbar}\int_{t_0}^{t} dt_1\ \langle\psi_\beta(t_0,t')|\,U_0^\dagger(t_1,t_0;t')V(\mathbf{r})U_0(t_1,t_0;t')\,|\psi_\alpha(t_0,t')\rangle\end{aligned} \tag{4.13}$$

and this can be simplied as

$$\langle\psi_\beta(t,t')|\Psi_\alpha(t,t')\rangle = \langle\psi_\beta(t,t')|\psi_\alpha(t,t')\rangle + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\ \langle\psi_\beta(t_1,t')|\,V(\mathbf{r})\,|\psi_\alpha(t_1,t')\rangle\,. \tag{4.14}$$

Since our $t-t'$ Floquet states are orthonormal [*Ref:myReport- t-t' formalism] we can derive that

$$\langle\psi_\beta(t,t')|\Psi_\alpha(t,t')\rangle = \delta_{\alpha\beta}\exp(i\omega[t'-t]) + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\ \langle\psi_\beta(t_1,t')|\,V(\mathbf{r})\,|\psi_\alpha(t_1,t')\rangle\,. \tag{4.15}$$

Now, set $t_0 = 0$ and for a case $\alpha \neq \beta$ this will simplied to

$$\langle\psi_\beta(t,t')|\Psi_\alpha(t,t')\rangle = -\frac{i}{\hbar}\int_{0}^{t} dt_1\ \langle\psi_\beta(t_1,t')|\,V(\mathbf{r})\,|\psi_\alpha(t_1,t')\rangle\,. \tag{4.16}$$

In addition, since our Floquet states create a basis for composite space we can represent any solution using our Floquet states

$$|\Psi_\alpha(t,t')\rangle = \sum_\beta a_{\alpha\beta}(t,t')\,|\psi_\beta(t,t')\rangle\,. \tag{4.17}$$

Therefore we can derive a equation for this *scattering amplitude* as

$$a_{\alpha\beta}(t,t') = \langle\psi_\beta(t,t')|\Psi_\alpha(t,t')\rangle = -\frac{i}{\hbar}\int_{0}^{t} dt_1\ \langle\psi_\beta(t_1,t')|\,V(\mathbf{r})\,|\psi_\alpha(t_1,t')\rangle\,. \tag{4.18}$$

Now lets assume a scattering event from a $t-t'$ Floquet state $|\psi_\beta(t,t')\rangle$ into another $t-t'$ Floquet state $|\Psi_\alpha(t,t')\rangle$ with constant quansienergy ε given as follows

$$|\Psi_\alpha(t,t')\rangle = \exp\left(-\frac{i}{\hbar}\varepsilon t\right)|\Phi_\alpha(t')\rangle \tag{4.19}$$

Now consider a scattering event

$$\psi_\beta(\mathbf{k}',t,t') = \exp\left(-\frac{i}{\hbar}\varepsilon_\beta t\right)\phi_\beta(\mathbf{k}',t') \longrightarrow \Psi_\alpha(\mathbf{k},t,t') = \exp\left(-\frac{i}{\hbar}\varepsilon t\right)\Phi_\alpha(\mathbf{k},t') \tag{4.20}$$

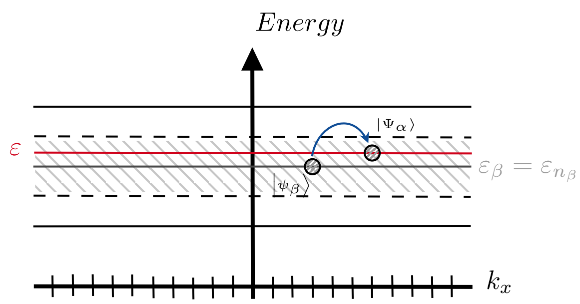

Figure 2: Scattering from $|\psi_\beta(t,t')\rangle$ to constant energy state $|\Psi_\alpha(t,t')\rangle$ due to scattering potential created by impurities.

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