Magnetic propeties of a two dimentional electron gas strongly coupled to lights

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1 Floquet Fermi Goldern Rule

In this section we are going to derive the Floquet Fermi goldern rule for above derived quantum Floquet states using $t - t'$ formalism.

The Floquet states (**??**) fullfills the $t - t'$ Schrödinger equation [*Ref:myReport] as follows

$$i\hbar\frac{\partial}{\partial t}\left|\psi_\alpha(t,t')\right\rangle = H_F(t')\left|\psi_\alpha(t,t')\right\rangle \tag{1.1}$$

where Floquet Hamiltonian given by

$$H_F(t') \equiv H_e(t) - i\hbar\frac{\mathrm{d}}{\mathrm{d}t} \tag{1.2}$$

and

$$\left|\psi_\alpha(t,t')\right\rangle = \exp\left(-\frac{i}{\hbar}\varepsilon_\alpha t\right)\left|\phi_\alpha(t')\right\rangle \tag{1.3}$$

Now for the Eq. (1.1) corresponding time evolution operator satisfy the Schrödinger equation

$$U_0(t,t_0;t') = \exp\left(-\frac{i}{\hbar}H_F(t')[t-t_0]\right) \tag{1.4}$$

Consider a time-independent total perturbation $V(\mathbf{r})$ switched on at the reference time $t = t_0$, then Schrödinger equation becomes

$$i\hbar\frac{\partial}{\partial t}\left|\Psi_\alpha(t,t')\right\rangle = [H_F(t') + V(\mathbf{r})]\left|\Psi_\alpha(t,t')\right\rangle \tag{1.5}$$

and when $t \leq t_0$ both solutions of the Schrödinger equation coincide

$$\left|\psi_\alpha(t,t')\right\rangle = \left|\Psi_\alpha(t,t')\right\rangle \quad \text{when} \quad t \leq t_0 \tag{1.6}$$

Now, we can introduce the interaction picture representation of the $t - t'$ Floquet state as

$$\left|\Psi_\alpha(t,t')\right\rangle_I = U_0^\dagger(t,t_0;t')\left|\Psi_\alpha(t,t')\right\rangle \tag{1.7}$$

and the perturbation in the interaction picture will be

$$V_I(\mathbf{r}) = U_0^\dagger(t,t_0;t')V(\mathbf{r})U_0(t,t_0;t') = V(\mathbf{r}). \tag{1.8}$$

This leads to the Schrödinger eqution in the interction picture

$$i\hbar\frac{\partial}{\partial t}\left|\Psi_\alpha(t,t')\right\rangle_I = V_I(\mathbf{r})\left|\Psi_\alpha(t,t')\right\rangle_I \tag{1.9}$$

with the recursive solution

$$|\Psi_\alpha(t,t')\rangle_I = |\Psi_\alpha(t_0,t')\rangle_I + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\; V_I(\mathbf{r})\,|\Psi_\alpha(t_1,t')\rangle_I \tag{1.10}$$

Iterating the solution only upto first order (Born approximation) this leads to

$$|\Psi_\alpha(t,t')\rangle_I \approx |\psi_\alpha(t_0,t')\rangle + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\; V_I(\mathbf{r})\,|\psi_\alpha(t_0,t')\rangle \tag{1.11}$$

and multiply it by $\langle\psi_\beta(t_0,t')|$ and we will get

$$\langle\psi_\beta(t_0,t')|\Psi_\alpha(t,t')\rangle_I = \langle\psi_\beta(t_0,t')|\psi_\alpha(t_0,t')\rangle + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\;\langle\psi_\beta(t_0,t')|\,V_I(\mathbf{r})\,|\psi_\alpha(t_0,t')\rangle\,. \tag{1.12}$$

Then introdusing unitory operator U_0 we can re-write this as

$$\begin{aligned}\langle\psi_\beta(t_0,t')|U_0^\dagger(t,t_0;t')|\Psi_\alpha(t,t')\rangle = {} & \langle\psi_\beta(t_0,t')|U_0^\dagger(t,t_0;t')U_0(t,t_0;t')|\psi_\alpha(t_0,t')\rangle \\ & + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\;\langle\psi_\beta(t_0,t')|\,U_0^\dagger(t_1,t_0;t')V(\mathbf{r})U_0(t_1,t_0;t')\,|\psi_\alpha(t_0,t')\rangle\end{aligned} \tag{1.13}$$

and this can be simplied as

$$\langle\psi_\beta(t,t')|\Psi_\alpha(t,t')\rangle = \langle\psi_\beta(t,t')|\psi_\alpha(t,t')\rangle + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\;\langle\psi_\beta(t_1,t')|\,V(\mathbf{r})\,|\psi_\alpha(t_1,t')\rangle\,. \tag{1.14}$$

Since our $t-t'$ Floquet states are orthonormal [*Ref:myReport- t-t' formalism] we can derive that

$$\langle\psi_\beta(t,t')|\Psi_\alpha(t,t')\rangle = \delta_{\alpha\beta}\exp(i\omega[t'-t]) + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\;\langle\psi_\beta(t_1,t')|\,V(\mathbf{r})\,|\psi_\alpha(t_1,t')\rangle\,. \tag{1.15}$$

Now, set $t_0 = 0$ and for a case $\alpha \neq \beta$ where we can represent $\alpha = (n_\alpha, m_\alpha)$ and $\beta = (n_\beta, m_\beta)$ and this will simplied to

$$\langle\psi_\beta(t,t')|\Psi_\alpha(t,t')\rangle = -\frac{i}{\hbar}\int_{0}^{t} dt_1\;\langle\psi_\beta(t_1,t')|\,V(\mathbf{r})\,|\psi_\alpha(t_1,t')\rangle\,. \tag{1.16}$$

In addition, since our Floquet states create a basis for composite space we can represent any solution using our Floquet states

$$|\Psi_\alpha(t,t')\rangle = \sum_\beta a_{\alpha\beta}(t,t')\,|\psi_\beta(t,t')\rangle\,. \tag{1.17}$$

Therefore we can derive a equation for this *scattering amplitude* as

$$a_{\alpha\beta}(t,t') = \langle\psi_\beta(t,t')|\Psi_\alpha(t,t')\rangle = -\frac{i}{\hbar}\int_{0}^{t} dt_1\;\langle\psi_\beta(t_1,t')|\,V(\mathbf{r})\,|\psi_\alpha(t_1,t')\rangle\,. \tag{1.18}$$

Now lets assume a scattering event from a $t-t'$ Floquet state $|\psi_\beta(t,t')\rangle$ into another $t-t'$ Floquet state $|\Psi_\alpha(t,t')\rangle$ with constant quansienergy ε given as follows

$$|\Psi_\alpha(t,t')\rangle = \exp\left(-\frac{i}{\hbar}\varepsilon t\right)|\Phi_\alpha(t')\rangle \tag{1.19}$$

Now consider a scattering event

$$\psi_\beta(\mathbf{k}',t,t') = \exp\left(-\frac{i}{\hbar}\varepsilon_\beta t\right)\phi_\beta(\mathbf{k}',t') \longrightarrow \Psi_\alpha(\mathbf{k},t,t') = \exp\left(-\frac{i}{\hbar}\varepsilon t\right)\Phi_\alpha(\mathbf{k},t') \tag{1.20}$$

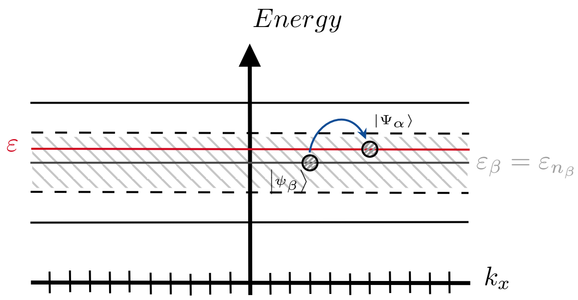

Figure 1: Scattering from $|\psi_\beta(t,t')\rangle$ to constant energy state $|\Psi_\alpha(t,t')\rangle$ due to scattering potential created by impurities.

Here we need to undestand a state of this considering system only be represented by two independent quantum numbers which are n energy eigen states and m quantum number which represents the qunatized momentum in x direction values. Lets calculate the scattering amplitutde of the above mentioned scattering scenario using the equation derived in (1.18).

$$\begin{aligned} a_{\alpha\beta}(\mathbf{k}',\mathbf{k},t,t') &= -\frac{i}{\hbar}\int_0^t dt_1 \ \langle\psi_{\beta,\mathbf{k}'}(t_1,t')|\, V(\mathbf{r})\,|\psi_{\alpha,\mathbf{k}}(t_1,t')\rangle \\ &= -\frac{i}{\hbar}\int_0^t dt_1 \ e^{\frac{i}{\hbar}(\varepsilon_\beta-\varepsilon)t_1}\,\langle\phi_{\beta,\mathbf{k}'}(t')|\, V(\mathbf{r})\,|\phi_{\alpha,\mathbf{k}}(t')\rangle \end{aligned} \tag{1.21}$$

Next assuimg this scenario for long time $t\to\infty$ we can turn this integral into a delta distrubution as follows

$$\begin{aligned} a_{\alpha\beta}(\mathbf{k}',\mathbf{k},t,t') &= -\frac{i}{\hbar}\lim_{t\to\infty}\left[\int_{-t/2}^{t/2} dt_1 \ e^{\frac{i}{\hbar}(\varepsilon_\beta-\varepsilon)t_1}\,\langle\phi_{\beta,\mathbf{k}'}(t')|\, V(\mathbf{r})\,|\phi_{\alpha,\mathbf{k}}(t')\rangle\right] \\ &= -2\pi i\delta(\varepsilon_\beta-\varepsilon)\,\langle\phi_{\beta,\mathbf{k}'}(t')|\, V(\mathbf{r})\,|\phi_{\alpha,\mathbf{k}}(t')\rangle \end{aligned} \tag{1.22}$$

Now lets consider about the inner product of the above derivation. Using completeness properties we can write that as follows

$$\begin{aligned} Q &\equiv \langle\phi_{\beta,\mathbf{k}'}(t')|\, V(\mathbf{r})\,|\phi_{\alpha,\mathbf{k}}(t')\rangle \\ &= \sum_{\mathbf{k}}\sum_{\mathbf{k}'}\langle\phi_{\beta,\mathbf{k}'}(t')|\mathbf{k}'\rangle\ \langle\mathbf{k}'|V(\mathbf{r})|\mathbf{k}\rangle\ \langle\mathbf{k}|\phi_{\alpha,\mathbf{k}}(t')\rangle \end{aligned} \tag{1.23}$$

and seperating x and y directional momentums we can modify this as follows (Assuming $L_y\to\infty$) and then using $\frac{1}{L_y}\sum_{k_y} = \frac{1}{2\pi}\int k_y$

$$\begin{aligned} Q &\equiv \langle\phi_{\beta,\mathbf{k}'}(t')|\, V(\mathbf{r})\,|\phi_{\alpha,\mathbf{k}}(t')\rangle \\ &= \frac{L_y^2}{4\pi^2}\sum_{k_x}\sum_{k'_x}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_y dk'_y\ \phi_\beta(\mathbf{k}',t')\,\langle\mathbf{k}'|V(\mathbf{r})|\mathbf{k}\rangle\,\phi_\alpha(\mathbf{k},t'). \end{aligned} \tag{1.24}$$

For a random white scattering potential we can represent the inner product of scattering potential with momentum as a constant value as

$$V_{\mathbf{k}',\mathbf{k}} \equiv \langle\mathbf{k}'|V(\mathbf{r})|\mathbf{k}\rangle\,. \tag{1.25}$$

In this study, the perturbation potential is assumed to be formed by an ensemble of randomly distributed impurities, since randomimpurities iin a disorded metal is a better approximation for experimental results.

Consider N_{imp} identical impurities positioned at the randomly distributed but fixed potions $\mathbf{r}_i$. The elastic scattering potential $V(\mathbf{r})$ is then given by the sum over uncorrelated single impurity potentials $\upsilon(\mathbf{r})$

$$V(\mathbf{r}) \equiv \sum_{i=1}^{N_{imp}}\upsilon(\mathbf{r}-\mathbf{r}_i). \tag{1.26}$$

Now assume that the perturbation $V(\mathbf{r})$ is a Gaussian random potential where one can choose the zero of enerrgy such that the potential is zero on average. This model characterized by [*Ref: e.Akkermans G. Montambaux]

$$\langle \upsilon(\mathbf{r})\rangle_{imp} = 0 \tag{1.27}$$

$$\langle \upsilon(\mathbf{r})\upsilon(\mathbf{r}')\rangle_{imp} = \Upsilon(\mathbf{r}-\mathbf{r}') \tag{1.28}$$

where $\langle\cdot\rangle_{imp}$ denoted the average over realizations of the impurity disorder. In addition, this model assume that $\upsilon(\mathbf{r}-\mathbf{r}')$ only dependes on the position difference $|\mathbf{r}-\mathbf{r}'|$ and it decays with a characteristic leangth r_c. Since the study considers the case where the waveleagth of radiation or scattering electrons is much faster than r_c, it is good approximation to make two-point correlation function to be

$$\langle \upsilon(\mathbf{r})\upsilon(\mathbf{r}')\rangle_{imp} = \Upsilon^2_{imp}\delta(\mathbf{r}-\mathbf{r}') \tag{1.29}$$

and a random potential $V(\mathbf{r})$ with this property is called white noise [*Ref: e.Akkermans G. Montambaux]. Then we can choose approximately total scattering potential as

$$V(\mathbf{r}) = \sum_{i=1}^{N_{imp}} \Upsilon_{imp}\delta(\mathbf{r}-\mathbf{r}_i). \tag{1.30}$$

Now we can calculate the Eq. (1.25) using this assumption as follows

$$\begin{aligned}
V_{\mathbf{k}',\mathbf{k}} =& \langle \mathbf{k}'|V(\mathbf{r})|\mathbf{k}\rangle \\
=& \left\langle \mathbf{k}' \middle| \sum_{i=1}^{N_{imp}} \Upsilon_{imp}\delta(\mathbf{r}-\mathbf{r}_i) \middle| \mathbf{k} \right\rangle \\
=& \left\langle \mathbf{k}' \middle| \sum_{i=1}^{N_{imp}} \Upsilon_{imp}\delta(x-x_i)\delta(y-y_i) \middle| \mathbf{k} \right\rangle \\
=& \left\langle k'_x \middle| \sum_{x_i=1}^{N_{imp}} \Upsilon_{imp}\delta(x-x_i) \middle| k_x \right\rangle \sum_{y_i=1}^{N_{imp}} \int_{-\infty}^{\infty} dy\ e^{ik'_y y}\delta(y-y_i)e^{-ik_y y} \\
=& \left\langle k'_x \middle| \sum_{x_i=1}^{N_{imp}} \Upsilon_{imp}\delta(x-x_i) \middle| k_x \right\rangle \sum_{y_i=1}^{N_{imp}} e^{i\left(k'_y-k_y\right)y_i}
\end{aligned} \tag{1.31}$$

Assuming the total umber of scatterers N_{imp} is macroscopically large we can achieve following expression

$$\begin{aligned}
V_{\mathbf{k}',\mathbf{k}} =& \left\langle k'_x \middle| \sum_{x_i=1}^{N_{imp}} \Upsilon_{imp}\delta(x-x_i) \middle| k_x \right\rangle \frac{N_{imp}}{L_y} \int_{-\infty}^{\infty} dy_i\ e^{i\left(k'_y-k_y\right)y_i} \\
=& \frac{N_{imp}}{L_y} V_{k'_x,k_x}\delta(k'_y-k_y)
\end{aligned} \tag{1.32}$$

where

$$V_{k'_x,k_x} \equiv \left\langle k'_x \middle| \sum_{x_i=1}^{N_{imp}} \Upsilon_{imp}\delta(x-x_i) \middle| k_x \right\rangle \tag{1.33}$$

Therefore, using the Eq. (**??**), the Eq. (1.24) modified to (we can change varable $t' \to t$)

$$\begin{aligned}
Q =& \sum_{k_x}\sum_{k'_x} \frac{N_{imp}L_y V_{k'_x,k_x}}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_y dk'_y\ \delta(k'_y-k_y) \\
&\times \sqrt{L_x}\exp(-ib\sin(2\omega t))\exp(ik'_y[d\sin(\omega t)+y'_0])\tilde{\chi}_{n_\beta}(k'_y-g\cos(\omega t)) \\
&\times \sqrt{L_x}\exp(ib\sin(2\omega t))\exp(-ik_y[d\sin(\omega t)+y_0])\tilde{\chi}_{n_\alpha}(k_y-g\cos(\omega t))
\end{aligned} \tag{1.34}$$

and we can simplify this as

$$Q = \sum_{k_x}\sum_{k'_x} \frac{N_{imp}L_xL_yV_{k'_x,k_x}}{4\pi^2}\int_{-\infty}^{\infty} dk_y \\ \times \exp(ik_y y'_0)\tilde{\chi}_{n_\beta}(k_y - g\cos(\omega t))\exp(-ik_y y_0)\tilde{\chi}_{n_\alpha}(k_y - g\cos(\omega t)) \tag{1.35}$$

and this can re-write as

$$Q = \sum_{k_x}\sum_{k'_x} \frac{N_{imp}L_xL_yV_{k'_x,k_x}}{4\pi^2} I \tag{1.36}$$

where

$$I \equiv \int_{-\infty}^{\infty} dk_y\ \tilde{\chi}_{n_\beta}(k_y - g\cos(\omega t))\tilde{\chi}_{n_\alpha}(k_y - g\cos(\omega t))\exp(-ik_y[y_0 - y'_0]). \tag{1.37}$$

To avoid the energy tramision from external high-frequency field and electrons in the system, the applied radiation should be purely dressing field. Therefore, the only effect of the dressing field on 2DEG is the renormalization of the probability of elastic electron scattering within same Landau level ($n_\alpha = n_\beta$). Therefore Eq. (1.37) can be modified to

$$I \equiv \int_{-\infty}^{\infty} dk_y\ \tilde{\chi}^2_{n_\beta}(k_y - g\cos(\omega t))\exp(-ik_y[y_0 - y'_0]). \tag{1.38}$$

Lets consider about this integral and we can calculate it as using the following subtitution. Let

$$k_y - g\cos(\omega t) = \bar{k}_y \longrightarrow dk_y = d\bar{k}_y \tag{1.39}$$

and this leads to

$$I \equiv 2\pi \times \frac{1}{2\pi}\int_{-\infty}^{\infty} d\bar{k}_y\ \tilde{\chi}^2_{n_\alpha}(\bar{k}_y)\exp\big(-i\big(\bar{k}_y + g\cos(\omega t)\big)(y_0 - y'_0)\big). \tag{1.40}$$

Using Fourier transform of Gauss-Hermite functions and convolution theorem we can write this as

$$I \equiv 2\pi\exp(g[y'_0 - y_0]\cos(\omega t))\int_{-\infty}^{\infty} dy\ \chi_{n_\beta}(y)\chi_{n_\beta}(y_0 - y'_0 - y). \tag{1.41}$$

Therefore the scatterng amplitude (1.22) will modified to

$$a_{\alpha\beta}(k'_x, k_x, t) = -2\pi i\delta(\varepsilon_\beta - \varepsilon)\sum_{k_x}\sum_{k'_x} \frac{N_{imp}L_xL_yV_{k'_x,k_x}}{4\pi^2} I \tag{1.42}$$

Consideirng qunatized momentum given in x direction derived in Eq. (**??**), we can identify the non-zero values for scattering amplitude using following conditions

$$k'_x = \frac{p_{x_\beta}}{\hbar} = m'\frac{2\pi}{L_x} \quad \text{and} \quad k_x = \frac{p_{x_\alpha}}{\hbar} = m\frac{2\pi}{L_x}. \tag{1.43}$$

Then we can simplfied scattering amplitude for given k'_x and k_x as

$$a_{\alpha\beta}(k'_x, k_x, t) = \delta(\varepsilon_\beta - \varepsilon)\left[\frac{-iN_{imp}L_xL_yV_{k'_x,k_x}}{2\pi}\right]\exp(g[y'_0 - y_0]\cos(\omega t)) \\ \times \int_{-\infty}^{\infty} dy\ \chi_{n_\beta}(y)\chi_{n_\beta}(y_0 - y'_0 - y) \tag{1.44}$$

Since this scattering amplitude is time-periodic we can write this as a Fourier series expansion

$$a_{\alpha\beta}(k'_x, k_x, t) = \sum_{l=-\infty}^{\infty} a^l_{\alpha\beta}(k'_x, k_x)e^{-il\omega t}. \tag{1.45}$$

In addition, using Jacobi-Anger expansion

$$e^{iz\cos(\theta)} = \sum_{l=-\infty}^{\infty} i^l J_l(z)e^{-il\theta} \tag{1.46}$$

we can re-write the Eq.(1.44) as folllows

$$
\begin{aligned}
a_{\alpha\beta}(k'_x, k_x, t) = \delta(\varepsilon_\beta - \varepsilon)\left[\frac{-iN_{imp}L_xL_yV_{k'_x,k_x}}{2\pi}\right]\sum_{l=-\infty}^{\infty} i^l J_l(g[y'_0 - y_0])e^{-il\omega t} \\
\times \int_{-\infty}^{\infty} dy\ \chi_{n_\beta}(y)\chi_{n_\beta}(y_0 - y'_0 - y)
\end{aligned}
\tag{1.47}
$$

$$
\begin{aligned}
a_{\alpha\beta}(k'_x, k_x, t) = \sum_{l=-\infty}^{\infty} \delta(\varepsilon_\beta - \varepsilon)\left[\frac{-i^{l+1}N_{imp}L_xL_yV_{k'_x,k_x}}{2\pi}\right] J_l(g[y'_0 - y_0]) \\
\times \int_{-\infty}^{\infty} dy\ \chi_{n_\beta}(y)\chi_{n_\beta}(y_0 - y'_0 - y)e^{-il\omega t}
\end{aligned}
\tag{1.48}
$$

Then we can identified the Fourier series component as

$$
a^l_{\alpha\beta}(k'_x, k_x) = \delta(\varepsilon_\beta - \varepsilon)\left[\frac{-i^{l+1}N_{imp}L_xL_yV_{k'_x,k_x}}{2\pi}\right] J_l(g[y'_0 - y_0]) \int_{-\infty}^{\infty} dy\ \chi_{n_\beta}(y)\chi_{n_\beta}(y_0 - y'_0 - y)
\tag{1.49}
$$

Now one can introduce the definition of the *transition probability matrix* as

$$
(A_{\alpha\beta}(k'_x, k_x))_{l,l'} \equiv a^l_{\alpha\beta}(k'_x, k_x)\left[a^{l'}_{\alpha\beta}(k'_x, k_x)\right]^*
\tag{1.50}
$$

and this becomes

$$
\begin{aligned}
(A_{\alpha\beta}(k'_x, k_x))_{l,l'} = \left[\frac{N^2_{imp}A^2|V_{k'_x,k_x}|^2}{4\pi^2}\right] J_l(g[y'_0 - y_0])J_{l'}(g[y'_0 - y_0])\delta^2(\varepsilon_\beta - \varepsilon) \\
\times \int_{-\infty}^{\infty} dy\ \chi_{n_\beta}(y)\chi_{n_\beta}(y_0 - y'_0 - y)\int_{-\infty}^{\infty} d\bar{y}\ \chi_{n_\beta}(\bar{y})\chi_{n_\beta}(y_0 - y'_0 - \bar{y}).
\end{aligned}
\tag{1.51}
$$

We can reduce these intragal into one variable and derive

$$
\begin{aligned}
(A_{\alpha\beta}(k'_x, k_x))_{l,l'} = \left[\frac{N^2_{imp}A^2|V_{k'_x,k_x}|^2}{4\pi^2}\right] J_l(g[y'_0 - y_0])J_{l'}(g[y'_0 - y_0])\delta^2(\varepsilon_\beta - \varepsilon) \\
\times \left|\int_{-\infty}^{\infty} dy\ \chi_{n_\beta}(y)\chi_{n_\beta}(y_0 - y'_0 - y)\right|^2.
\end{aligned}
\tag{1.52}
$$

Then desribing the square of the delta distribution using following procedure

$$
\delta^2(\varepsilon) = \delta(\varepsilon)\delta(0) = \frac{\delta(\varepsilon)}{2\pi\hbar}\int_{-t/2}^{t/2} e^{i0\times t'/\hbar}dt' \ = \frac{\delta(\varepsilon)t}{2\pi\hbar}
\tag{1.53}
$$

one can modify our derivation in Eq. (1.51) as

$$
\begin{aligned}
(A_{\alpha\beta}(k'_x, k_x))_{l,l'} = \left[\frac{N^2_{imp}A^2|V_{k'_x,k_x}|^2}{4\pi^2}\right] J_l(g[y'_0 - y_0])J_{l'}(g[y'_0 - y_0])\delta(\varepsilon_\beta - \varepsilon)\frac{t}{2\pi\hbar} \\
\times \left|\int_{-\infty}^{\infty} dy\ \chi_{n_\beta}(y)\chi_{n_\beta}(y_0 - y'_0 - y)\right|^2.
\end{aligned}
\tag{1.54}
$$

Then performing thetime derivation of each matrix element yeild the *transition amplitude matrix* as folllows

$$
\begin{aligned}
\Gamma^{ll'}_{\alpha\beta}(k'_x, k_x) &\equiv \frac{d(A_{\alpha\beta}(k'_x, k_x))_{l,l'}}{dt} \\
&= \Lambda|V_{k'_x,k_x}|^2\delta(\varepsilon_\beta - \varepsilon)J_l(g[y'_0 - y_0])J_{l'}(g[y'_0 - y_0])\left|\int_{-\infty}^{\infty} dy\ \chi_{n_\beta}(y)\chi_{n_\beta}(y_0 - y'_0 - y)\right|^2
\end{aligned}
\tag{1.55}
$$

where

$$\Lambda \equiv \frac{N_{imp}^2 A^2}{8\pi^3\hbar} \tag{1.56}$$

Now using defintion of y_0 given in Eq. (**??**) we can write that

$$y_0 - {y'}_0 = -\frac{p_{x_\alpha}}{eB} + \frac{p_{x_\beta}}{eB} = \frac{\hbar {k'}_x}{eB} - \frac{\hbar k_x}{eB} = \frac{\hbar}{eB}[{k'}_x - k_x] \tag{1.57}$$

and this leads Eq. (1.56) to

$$\Gamma_{\alpha\beta}^{ll'}({k'}_x, k_x) = \Lambda |V_{{k'}_x,k_x}|^2 \delta(\varepsilon_\beta - \varepsilon) J_l\left(\frac{g\hbar}{eB}[k_x - {k'}_x]\right) J_{l'}\left(\frac{g\hbar}{eB}[k_x - {k'}_x]\right) \\ \times \left| \int_{-\infty}^{\infty} dy\, \chi_{n_\beta}(y) \chi_{n_\beta}\left(\frac{\hbar}{eB}[{k'}_x - k_x] - y\right)\right|^2 \tag{1.58}$$

An impurity average of white noise potential allows to identify $\left\langle |V_{{k'}_x,k_x}|^2 \right\rangle = V_{imp}$ and the inverse scattering time matrix is the sum over all momentum over the transition probability matrix

$$\left(\frac{1}{\tau(\varepsilon, k_x)}\right)_{\alpha\beta}^{ll'} \equiv \frac{1}{L_x} \sum_{{k'}_x} \langle \Gamma_{\alpha\beta}^{ll'}({k'}_x, k_x)\rangle_{imp} \tag{1.59}$$

and this implies

$$\Gamma_{\alpha\beta}^{ll'}({k'}_x, k_x) = \frac{\Lambda V_{imp}}{L_x} \sum_{{k'}_x} \delta(\varepsilon_\beta - \varepsilon) J_l\left(\frac{g\hbar}{eB}[k_x - {k'}_x]\right) J_{l'}\left(\frac{g\hbar}{eB}[k_x - {k'}_x]\right) \\ \times \left| \int_{-\infty}^{\infty} dy\, \chi_{n_\beta}(y) \chi_{n_\beta}\left(\frac{\hbar}{eB}[{k'}_x - k_x] - y\right)\right|^2 \tag{1.60}$$

For the 1-dimentional case introduce the momentum continuum limit as folllows

$$\frac{1}{L_x} \sum_{{k'}_x} \longrightarrow \frac{1}{2\pi} \int d{k'}_x \tag{1.61}$$

and this leads to

$$\left(\frac{1}{\tau(\varepsilon, k_x)}\right)_{\alpha\beta}^{ll'} = \frac{\Lambda V_{imp}}{2\pi} \delta(\varepsilon_\beta - \varepsilon) \int_{-\infty}^{\infty} d{k'}_x J_l\left(\frac{g\hbar}{eB}[k_x - {k'}_x]\right) J_{l'}\left(\frac{g\hbar}{eB}[k_x - {k'}_x]\right) \\ \times \left| \int_{-\infty}^{\infty} dy\, \chi_{n_\beta}(y) \chi_{n_\beta}\left(\frac{\hbar}{eB}[{k'}_x - k_x] - y\right)\right|^2 \tag{1.62}$$

Using following subtitution

$$y = \frac{\hbar \bar{k}}{eB} \longrightarrow dy = \frac{\hbar}{eB} d\bar{k} \tag{1.63}$$

we can modify above derivation as

$$\left(\frac{1}{\tau(\varepsilon, k_x)}\right)_{\alpha\beta}^{ll'} = \frac{\Lambda V_{imp}}{2\pi} \delta(\varepsilon_\beta - \varepsilon) \int_{-\infty}^{\infty} d{k'}_x J_l\left(\frac{g\hbar}{eB}[k_x - {k'}_x]\right) J_{l'}\left(\frac{g\hbar}{eB}[k_x - {k'}_x]\right) \\ \times \left(\frac{\hbar}{eB}\right)^2 \left| \int_{-\infty}^{\infty} d\bar{k}\, \chi_{n_\beta}\left(\frac{\hbar}{eB}\bar{k}\right) \chi_{n_\beta}\left(\frac{\hbar}{eB}\left[{k'}_x - k_x - \bar{k}\right]\right)\right|^2, \tag{1.64}$$

and finally we can derive our expression for the *inverse scattering time matrix* for Nth Landau level (let $n_\alpha = n_\beta = N$)

$$\left(\frac{1}{\tau(\varepsilon, k_x)}\right)_{N}^{ll'} = \frac{N_{imp}^2 A^2 \hbar V_{imp}}{16\pi^4 (eB)^2} \delta(\varepsilon - \varepsilon_N) \int_{-\infty}^{\infty} d{k'}_x J_l\left(\frac{g\hbar}{eB}[k_x - {k'}_x]\right) J_{l'}\left(\frac{g\hbar}{eB}[k_x - {k'}_x]\right) \\ \times \left| \int_{-\infty}^{\infty} d\bar{k}\, \chi_N\left(\frac{\hbar}{eB}\bar{k}\right) \chi_N\left(\frac{\hbar}{eB}\left[{k'}_x - k_x - \bar{k}\right]\right)\right|^2. \tag{1.65}$$

■

2 Inverse Scattering Time Analysis

We hace derived the inverse scattering time matrix element from previous section as follows

$$\left(\frac{1}{\tau(\varepsilon,k_x)}\right)_N^{ll'} = \frac{N_{imp}^2 A^2 \hbar V_{imp}}{16\pi^4 (eB)^2}\delta(\varepsilon-\varepsilon_N)\int_{-\infty}^{\infty} dk'_x J_l\left(\frac{g\hbar}{eB}[k_x-k'_x]\right) J_{l'}\left(\frac{g\hbar}{eB}[k_x-k'_x]\right) \times \left|\int_{-\infty}^{\infty} d\bar{k}\,\chi_N\left(\frac{\hbar}{eB}\bar{k}\right)\chi_N\left(\frac{\hbar}{eB}\left[k'_x-k_x-\bar{k}\right]\right)\right|^2. \tag{2.1}$$

The disorder in the system is not supposed to change the eigenenergies of the bare system, hence all off-diogonal elements of the self-energy were neglected. Thesefore we can consider only the central diagonal element ($l = l' = 0$) of the inverse scattering time matrix which has the largest contribution

$$\left(\frac{1}{\tau(\varepsilon,k_x)}\right)_N^{00} = \frac{N_{imp}^2 A^2 \hbar V_{imp}}{16\pi^4 (eB)^2}\delta(\varepsilon-\varepsilon_N)\int_{-\infty}^{\infty} dk'_x\; J_l^2\left(\frac{g\hbar}{eB}[k_x-k'_x]\right) \times \left|\int_{-\infty}^{\infty} d\bar{k}\,\chi_N\left(\frac{\hbar}{eB}\bar{k}\right)\chi_N\left(\frac{\hbar}{eB}\left[k'_x-k_x-\bar{k}\right]\right)\right|^2. \tag{2.2}$$

Now we can introduce a new parameter with physical meaning of scatterin-induced broading of the Landau level as follows

$$\Gamma_N^{00}(\varepsilon,k_x) \equiv \hbar\left(\frac{1}{\tau(\varepsilon,k_x)}\right)_N^{00} \tag{2.3}$$

and this modify our previous expressing as

$$\Gamma_N^{00}(\varepsilon,k_x) = \frac{N_{imp}^2 A^2 \hbar^2 V_{imp}}{16\pi^4 (eB)^2}\delta(\varepsilon-\varepsilon_N)\int_{-\infty}^{\infty} dk'_x\; J_0^2\left(\frac{g\hbar}{eB}[k_x-k'_x]\right) \times \left|\int_{-\infty}^{\infty} d\bar{k}\,\chi_N\left(\frac{\hbar}{eB}\bar{k}\right)\chi_N\left(\frac{\hbar}{eB}\left[k'_x-k_x-\bar{k}\right]\right)\right|^2. \tag{2.4}$$

In addition, for the case of elastic scattering within the same Landau level, one can present the delta distribution of the energy using the same physical interpretation as follows

$$\delta(\varepsilon-\varepsilon_N) \approx \frac{1}{\pi\Gamma_N^{00}(\varepsilon,k_x)} \tag{2.5}$$

and this leads to

$$\left[\Gamma_N^{00}(\varepsilon,k_x)\right]^2 = \frac{N_{imp}^2 A^2 \hbar^2 V_{imp}}{16\pi^5 (eB)^2}\int_{-\infty}^{\infty} dk'_x\; J_0^2\left(\frac{g\hbar}{eB}[k_x-k'_x]\right)\left|\int_{-\infty}^{\infty} d\bar{k}\,\chi_N\left(\frac{\hbar}{eB}\bar{k}\right)\chi_N\left(\frac{\hbar}{eB}\left[k'_x-k_x-\bar{k}\right]\right)\right|^2. \tag{2.6}$$

and

$$\Gamma_N^{00}(\varepsilon,k_x) = \left[\frac{N_{imp}^2 A^2 \hbar^2 V_{imp}}{16\pi^5 (eB)^2}\int_{-\infty}^{\infty} dk'_x\; J_0^2\left(\frac{g\hbar}{eB}[k_x-k'_x]\right)\left|\int_{-\infty}^{\infty} d\bar{k}\,\chi_N\left(\frac{\hbar}{eB}\bar{k}\right)\chi_N\left(\frac{\hbar}{eB}\left[k'_x-k_x-\bar{k}\right]\right)\right|^2\right]^{-1/2}. \tag{2.7}$$

This can be write in more compact form as follows

$$\Gamma_N^{00}(\varepsilon,k_x) = \left[\frac{N_{imp}^2 A^2 \hbar^2 V_{imp}}{16\pi^5 (eB)^2}\int_{-\infty}^{\infty} dk'_x\; J_0^2(g\sigma[k_x-k'_x])\left|\int_{-\infty}^{\infty} d\bar{k}\,\chi_N(\sigma\bar{k})\chi_N\left(\sigma\left[k'_x-k_x-\bar{k}\right]\right)\right|^2\right]^{-1/2} \tag{2.8}$$

where

$$g = \frac{eE\omega_0^2}{\hbar\omega(\omega_0^2-\omega^2)} \qquad \sigma = \frac{\hbar}{eB} \tag{2.9}$$

and

$$\chi_N(x) = \frac{\sqrt{\kappa}}{\sqrt{2^N N!\sqrt{\pi}}} \exp\left(-\frac{\kappa^2 x^2}{2}\right)\mathcal{H}_N(\kappa x) \quad \text{with} \quad \kappa \equiv \sqrt{\frac{m_e\omega_0}{\hbar}}. \tag{2.10}$$

Using above definition we can identify Gauss-Hermite functions ($\tilde{\chi}_N$) and this can re-write as

$$\Gamma_N^{00}(\varepsilon, k_x) = \left[\frac{N_{imp}^2 A^2 \hbar^2 V_{imp}\kappa^4}{16\pi^5 (eB)^2} \int_{-\infty}^{\infty} dk'_x \; J_0^2(g\sigma[k_x - k'_x]) \left| \int_{-\infty}^{\infty} d\bar{k} \; \tilde{\chi}_N\big(\sigma\kappa\bar{k}\big)\tilde{\chi}_N\big(\sigma\kappa\big[k'_x - k_x - \bar{k}\big]\big)\right|^2\right]^{-1/2} \tag{2.11}$$

where

$$\tilde{\chi}_N(x) = \frac{1}{\sqrt{2^N N!\sqrt{\pi}}} \exp\left(-\frac{x^2}{2}\right)\mathcal{H}_N(x) \tag{2.12}$$

and this will be simplified to

$$\Gamma_N^{00}(\varepsilon, k_x) = \eta\left[\int_{-\infty}^{\infty} dk_1 \; J_0^2(\lambda_1[k_x - k_1]) \left| \int_{-\infty}^{\infty} dk_2 \; \tilde{\chi}_N(\lambda_2 k_2)\tilde{\chi}_N(\lambda_2[k_1 - k_2 - k_x])\right|^2\right]^{-1/2} \tag{2.13}$$

where

$$\eta = \left[\frac{N_{imp}^2 A^2 V_{imp}}{16\pi^5}\right]^{1/2} \quad , \quad \lambda_1 = g\sigma \quad , \quad \lambda_2 = \sigma\kappa. \tag{2.14}$$

Now we can analyze the behaviour of the normalized N-th Landau level for broading as follows

$$\Lambda_N(k_x) \equiv \frac{(1/\tau)_N^{00}}{(1/\tau)_0^{00}\big|_{E=0}} = \frac{\Gamma_N^{00}(\varepsilon, k_x)}{\Gamma_0^{00}(\varepsilon, k_x)\big|_{E=0}} \tag{2.15}$$

and this will be

$$\Lambda_N(k_x) = \left[\frac{\int_{-\infty}^{\infty} dk_1 \; J_0^2(\lambda_1[k_x - k_1]) \left| \int_{-\infty}^{\infty} dk_2 \; \tilde{\chi}_N(\lambda_2 k_2)\tilde{\chi}_N(\lambda_2[k_1 - k_2 - k_x])\right|^2}{\int_{-\infty}^{\infty} dk_1 \left| \int_{-\infty}^{\infty} dk_2 \; \tilde{\chi}_0(\lambda_2 k_2)\tilde{\chi}_0(\lambda_2[k_1 - k_2 - k_x])\right|^2}\right]^{1/2}. \tag{2.16}$$

Lets calculate these constants for GaAs-based quantum well with following given physical constants and system external paramters.

Physical constant name	**Symbol**	**Value in SI-units**
Electron charge	e	1.602×10^{-19} C
Electron mass	m	9.109×10^{-31} kg
Reduced Planck's constant	$\hbar$	1.054×10^{-34} kgm^{2}s^{-1}
Speed of light	c	2.998×10^{8} ms^{-1}
Vacuum permittivity	ε_0	8.854×10^{-12} C^{2}s^{2}kg^{-1}m^{-3}

Table 1: Physical constant values in SI-units

External paramter name	**Symbol**	**Value in SI-units**
Average intensity	I	$\tilde{I} \times 100$ W/cm^2 $= \tilde{I} \times 10^6$ W/m^2
Magnetic field	B	1.2 T
Driving frequency	ω	2×10^{12} rads^{-1}
Effective mass	m_e	$0.071 \times m = 6.467 \times 10^{-32}$ kg

Table 2: System external paramter values. ($\tilde{I}$ is a dimentionless value.)

Therfore we can calculate following values

$$\omega_0 = \frac{eB}{m_e} = 2.97265 \times 10^{12}\ \mathrm{s}^{-1} \tag{2.17}$$

$$\sigma = \frac{\hbar}{eB} = 5.4851 \times 10^{-16}\ \mathrm{m}^2 \tag{2.18}$$

$$E = \sqrt{\frac{2I}{c\varepsilon_0}} \tag{2.19}$$

$$g = \frac{eE\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} = \frac{e\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)}\sqrt{\frac{2I}{c\varepsilon_0}} = 3.80958 \times 10^7 \times \sqrt{\tilde{I}}\ \mathrm{m}^{-1} \tag{2.20}$$

$$\kappa = \sqrt{\frac{m_e\omega_0}{\hbar}} = 4.2698 \times 10^7\ \mathrm{m}^{-1} \tag{2.21}$$

Since

$$\lambda_1 = g\sigma = 2.08959 \times 10^{-8} \times \sqrt{\tilde{I}}\ \mathrm{m} \quad \text{and} \quad \lambda_2 = \kappa\sigma = 2.34203 \times 10^{-8}\ \mathrm{m} \tag{2.22}$$

we can choose our integral dummy variables k_1, k_2 and momentum variable k_x are in one range as follows

$$k_x, k_1, k_2 \approx 10^{-8}\ \mathrm{m}^{-1} \tag{2.23}$$

Using above values we can re-write the normalized energy broading of the N-th Landau level as

$$\Lambda_N(k_x) = \left[\frac{\int_{-\infty}^{\infty} dk_1\ J_0^2\Big(2.090\sqrt{\tilde{I}} \times [k_x - k_1]\Big)\left|\int_{-\infty}^{\infty} dk_2\ \tilde{\chi}_N(2.342 \times k_2)\tilde{\chi}_N(2.342 \times [k_1 - k_2 - k_x])\right|^2}{\int_{-\infty}^{\infty} dk_1 \left|\int_{-\infty}^{\infty} dk_2\ \tilde{\chi}_0(2.342 \times k_2)\tilde{\chi}_0(2.342 \times [k_1 - k_2 - k_x])\right|^2}\right]^{1/2}. \tag{2.24}$$

To check the variability of this expression with k_x value we check it with a constant intensity. Therefore let $\tilde{I} = 1$ and we can graph the $\Lambda_N(k_x)$ against k_x for different Landau levels (N) using following equation.

$$\Lambda_N(k_x) = \left[\frac{\int_{-\infty}^{\infty} dk_1\ J_0^2(2.090 \times [k_x - k_1])\left|\int_{-\infty}^{\infty} dk_2\ \tilde{\chi}_N(2.342 \times k_2)\tilde{\chi}_N(2.342 \times [k_1 - k_2 - k_x])\right|^2}{\int_{-\infty}^{\infty} dk_1 \left|\int_{-\infty}^{\infty} dk_2\ \tilde{\chi}_0(2.342 \times k_2)\tilde{\chi}_0(2.342 \times [k_1 - k_2 - k_x])\right|^2}\right]^{1/2}. \tag{2.25}$$

xx

3 Current Operator in Landau Levels

Now consider about the current density operator for Nth Landau level. Since we have already found the extact solution for our time depenedent Hamiltonian and we have identify them as Floquet states with quesi energies. From these solutions we can identify the *Floquet modes* as given in Eq. (**??**) and using quantum numbers we can represent those states as follows

$$|\phi_\alpha\rangle = |\phi_{n,m}\rangle \equiv |n, k_x\rangle \quad \text{where} \quad k_x = m\frac{2\pi}{L_x} \tag{3.1}$$

Using above complete set of eigenstates of Floquet Hamiltonian we can represent the single particle current operator's matrix element as

$$(\mathbf{j})_{nm,n'm'} = \langle n, k_x|\, \hat{\mathbf{j}}\, |n', k_x'\rangle \tag{3.2}$$

where particle current operator for this system will be

$$\hat{\mathbf{j}} = \frac{1}{m}\Big(\hat{\mathbf{P}} - e[\mathbf{A}_s + \mathbf{A}_d(t)]\Big). \tag{3.3}$$

Howver, we are only consider the transverse conductivity in x direction we can identify that x directional current operator as

$$\hat{j}_x = \frac{1}{m}\left(-i\hbar\frac{\partial}{\partial x} + eBy\right). \tag{3.4}$$

Now we can calculate the matrix elements of x directional current operator's matrix in Floquet mode basis as

$$(j_x)_{nm,n'm'} = \langle n, k_x|\, \hat{j}_x\, |n', k_x'\rangle = \langle n, k_x|\, \frac{1}{m}\left(-i\hbar\frac{\partial}{\partial x} + eBy\right) |n', k_x'\rangle \tag{3.5}$$

and we can evaluate these using Floquet modes derived in Eq.(**??**) as follows

$$\begin{aligned}(j_x)_{nm,n'm'} = &\int dx \int dy\, \frac{1}{\sqrt{L_x}}\chi_n\big(y - y_0 - \zeta(t)\big)\exp(-ik_x x) \\ &\times \frac{1}{m}\left(-i\hbar\frac{\partial}{\partial x} + eBy\right)\frac{1}{\sqrt{L_x}}\chi_{n'}\big(y - y_0 - \zeta(t)\big)\exp(ik_x' x)\end{aligned} \tag{3.6}$$

and this can be simplified as

$$\begin{aligned}(j_x)_{nm,n'm'} = &\frac{1}{mL_x}\int dx\, \exp(-i(k_x - k_x')x) \int dy\; \chi_n\big(y - y_0 - \zeta(t)\big) \\ &\times (\hbar k_x' + eBy)\chi_{n'}\big(y - y_0 - \zeta(t)\big)\end{aligned} \tag{3.7}$$

and

$$(j_x)_{nm,n'm'} = \frac{1}{m}\delta_{k_x,k_x'}\int dy\; (\hbar k_x' + eBy)\chi_n\big(y - y_0 - \zeta(t)\big)\chi_{n'}\big(y - y_0 - \zeta(t)\big). \tag{3.8}$$

Now let $y - y_0 - \zeta(t) = \bar{y}$ and we will get

$$(j_x)_{nm,n'm'} = \frac{1}{m}\delta_{k_x,k_x'}\int d\bar{y}\; (\hbar k_x' + eB\bar{y} + eBy_0 + eB\zeta(t))\chi_n(\bar{y})\chi_{n'}(\bar{y}). \tag{3.9}$$

using definition of y_0 given in Eq. (**??**) this will be modiofied to

$$(j_x)_{nm,n'm'} = \frac{1}{m}\delta_{k_x,k_x'}\int d\bar{y}\; (\hbar k_x' + eB\bar{y} - \hbar k_x' + eB\zeta(t))\chi_n(\bar{y})\chi_{n'}(\bar{y}) \tag{3.10}$$

and using integral indentities of Gauss-Hermite functions

$$\int dy\; \chi_n(y)\chi_{n'}(y) = \delta_{n',n} \tag{3.11}$$

$$\int dy\; y\chi_n(y)\chi_{n'}(y) = \left(\sqrt{\frac{n+1}{2}}\delta_{n',n+1} + \sqrt{\frac{n}{2}}\delta_{n',n-1}\right) \tag{3.12}$$

this becomes

$$(j_x)_{nm,n'm'} = \frac{1}{m}\delta_{k_x,k'_x} eB\left[\left(\sqrt{\frac{n+1}{2}}\delta_{n',n+1} + \sqrt{\frac{n}{2}}\delta_{n',n-1}\right) + \zeta(t)\delta_{n',n}\right] \tag{3.13}$$

Due to complexity we can only consider the constant contribution and we allows only the one-cycle averaged current flow and then we can derive the $s = 0$ components of the Fourier series as

$$(j^x_{s=0})_{nm,n'm'} = \frac{1}{T}\int_0^T dt\; \frac{1}{m}\delta_{k_x,k'_x} eB\left[\left(\sqrt{\frac{n+1}{2}}\delta_{n',n+1} + \sqrt{\frac{n}{2}}\delta_{n',n-1}\right) + \frac{eE}{m(\omega_0^2-\omega^2)}\sin(\omega t)\delta_{n',n}\right] \tag{3.14}$$

and this can be evaluate and get

$$(j^x_{s=0})_{nm,n'm'} = \frac{eB}{m}\delta_{k_x,k'_x}\left(\sqrt{\frac{n+1}{2}}\delta_{n',n+1} + \sqrt{\frac{n}{2}}\delta_{n',n-1}\right) \tag{3.15}$$

For electric current operator we can introduce the electron's charge and effective mass

$$(j^x_{s=0})_{nm,n'm'} = \frac{e^2B}{m_e}\delta_{k_x,k'_x}\left(\sqrt{\frac{n+1}{2}}\delta_{n',n+1} + \sqrt{\frac{n}{2}}\delta_{n',n-1}\right) \tag{3.16}$$

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4 Floquet-Drude Conductivity in Quantum Hall Systems

The general expression for the conductivity [*Ref: Martin Wackerl Thesis 1.250] with the disorder averaging can be represent as follows

$$\lim_{\omega\to 0}\mathrm{Re}[\sigma^{xx}(0,\omega)] = \frac{-1}{4\pi\hbar A}\int_{\lambda-\hbar\Omega/2}^{\lambda+\hbar\Omega/2} d\varepsilon\left(-\frac{\partial f}{\partial\varepsilon}\right) \times \mathrm{tr}\,[j_0^x(\mathbf{G}^r(\varepsilon)-\mathbf{G}^a(\varepsilon))j_0^x(\mathbf{G}_0^r(\varepsilon)-\mathbf{G}_0^a(\varepsilon))]. \tag{4.1}$$

where j_0^x and $\mathbf{G}^{r,a}(\varepsilon)$ are x directional current operator matrix and white noise disorder averaged Green function matrix respectvely defined against to the *Floquet modes* of the system. Here we have assumed that only $s=0$ Fourier component of the current operator is contributing to the conductivity.
Now this can be expand in off resonant regime ($\omega\tau_0 \gg 1$)using only central entry Fourier components ($l=l'=0$) of *Floquet modes* mentioned in Eq. (3.1) as

$$\lim_{\omega\to 0}\mathrm{Re}[\sigma^{xx}(0,\omega)] = \frac{-1}{4\pi\hbar A}\int_{\lambda-\hbar\Omega/2}^{\lambda+\hbar\Omega/2} d\varepsilon\left(-\frac{\partial f}{\partial\varepsilon}\right) \times \frac{1}{V_{k_x}}\sum_{k_x}\sum_{n}\langle n,k_x|j_0^x(\mathbf{G}^r(\varepsilon)-\mathbf{G}^a(\varepsilon))j_0^x(\mathbf{G}_0^r(\varepsilon)-\mathbf{G}_0^a(\varepsilon))|n,k_x\rangle \tag{4.2}$$

and one can evaluate these matrix elements as follows

$$\begin{aligned}\lim_{\omega\to 0}\mathrm{Re}[\sigma^{xx}(0,\omega)] = {}& \frac{-1}{4\pi\hbar A}\int_{\lambda-\hbar\Omega/2}^{\lambda+\hbar\Omega/2} d\varepsilon\left(-\frac{\partial f}{\partial\varepsilon}\right)\frac{1}{V_{k_x}}\sum_{k_x}\sum_{n}\frac{1}{{L_x}^3}\sum_{k_{x1},k_{x2},k_{x3}}\sum_{n_1,n_2,n_3} \\ & \times\ \langle n,k_x|j_0^x|n_1,k_{x1}\rangle\ \langle n_1,k_{x1}|(\mathbf{G}^r(\varepsilon)-\mathbf{G}^a(\varepsilon))|n_2,k_{x2}\rangle \\ & \times\ \langle n_2,k_{x2}|j_0^x|n_3,k_{x3}\rangle\ \langle n_3,k_{x3}|(\mathbf{G}^r(\varepsilon)-\mathbf{G}^a(\varepsilon))|n,k_x\rangle\end{aligned} \tag{4.3}$$

Since we can diagonalize the impurity averaged Green's function using unitary trasnformation ($\mathbf{T}=|n,k_x\rangle$) [*Ref: Martin Wackerl - Paper] and we can evaluate the matrix element of differece between retarded and advanced Green's function as follows [*Ref: My report 2.535]

$$\langle n_1,k_{x1}|\mathbf{T}^\dagger(\mathbf{G}^r(\varepsilon)-\mathbf{G}^a(\varepsilon))\mathbf{T}|n_2,k_{x2}\rangle = \left[\frac{2i\mathrm{Im}\big(\mathbf{T}^\dagger\sum^r\mathbf{T}\big)\delta_{n_1,n_2}\delta_{k_{x1},k_{x2}}}{\big(\frac{1}{\hbar}\varepsilon-\frac{1}{\hbar}\varepsilon_{n_1}\big)^2+[\mathrm{Im}(\mathbf{T}^\dagger\sum^r\mathbf{T})]^2}\right] \tag{4.4}$$

and

$$\langle n_3,k_{x3}|\mathbf{T}^\dagger(\mathbf{G}^r(\varepsilon)-\mathbf{G}^a(\varepsilon))\mathbf{T}|n,k_x\rangle = \left[\frac{2i\mathrm{Im}\big(\mathbf{T}^\dagger\sum^r\mathbf{T}\big)\delta_{n_3,n}\delta_{k_{x3},k_x}}{\big(\frac{1}{\hbar}\varepsilon-\frac{1}{\hbar}\varepsilon_{n}\big)^2+[\mathrm{Im}(\mathbf{T}^\dagger\sum^r\mathbf{T})]^2}\right] \tag{4.5}$$

Then appying the results we derived in previous section (4.21) we can calculate the conductivity

$$\begin{aligned}\lim_{\omega\to 0}\mathrm{Re}[\sigma^{xx}(0,\omega)] = {}& \frac{-1}{4\pi\hbar A}\int_{\lambda-\hbar\Omega/2}^{\lambda+\hbar\Omega/2} d\varepsilon\left(-\frac{\partial f}{\partial\varepsilon}\right)\frac{1}{V_{k_x}}\sum_{k_x}\sum_{n}\frac{1}{{V_{k_x}}^3}\sum_{k_{x1},k_{x2},k_{x3}}\sum_{n_1,n_2,n_3} \\ & \times\frac{e^2B}{m_e}\delta_{k_x,k_{x1}}\left(\sqrt{\frac{n+1}{2}}\delta_{n_1,n+1}+\sqrt{\frac{n}{2}}\delta_{n_1,n-1}\right)\left[\frac{2i\mathrm{Im}\big(\mathbf{T}^\dagger\sum^r\mathbf{T}\big)\delta_{n_1,n_2}\delta_{k_{x1},k_{x2}}}{\big(\frac{1}{\hbar}\varepsilon-\frac{1}{\hbar}\varepsilon_{n_1}\big)^2+[\mathrm{Im}(\mathbf{T}^\dagger\sum^r\mathbf{T})]^2}\right] \\ & \times\frac{e^2B}{m_e}\delta_{k_{x2},k_{x3}}\left(\sqrt{\frac{n_2+1}{2}}\delta_{n_3,n_2+1}+\sqrt{\frac{n_2}{2}}\delta_{n_3,n_2-1}\right)\left[\frac{2i\mathrm{Im}\big(\mathbf{T}^\dagger\sum^r\mathbf{T}\big)\delta_{n_3,n}\delta_{k_{x3},k_x}}{\big(\frac{1}{\hbar}\varepsilon-\frac{1}{\hbar}\varepsilon_{n}\big)^2+[\mathrm{Im}(\mathbf{T}^\dagger\sum^r\mathbf{T})]^2}\right]\end{aligned} \tag{4.6}$$

and this will be modified to

$$\lim_{\omega\to 0}\mathrm{Re}[\sigma^{xx}(0,\omega)] = \frac{-1}{4\pi\hbar A}\int_{\lambda-\hbar\Omega/2}^{\lambda+\hbar\Omega/2} d\varepsilon\Big(-\frac{\partial f}{\partial\varepsilon}\Big)\frac{1}{V_{k_x}}\sum_{k_x}\sum_{n}\sum_{n_1,n_2}$$
$$\times\frac{e^2B}{m_e}\left(\sqrt{\frac{n+1}{2}}\delta_{n_1,n+1}+\sqrt{\frac{n}{2}}\delta_{n_1,n-1}\right)\left[\frac{2i\mathrm{Im}\big(\mathbf{T}^\dagger\sum^r\mathbf{T}\big)\delta_{n_1,n_2}}{\big(\frac{1}{\hbar}\varepsilon-\frac{1}{\hbar}\varepsilon_{n_1}\big)^2+[\mathrm{Im}(\mathbf{T}^\dagger\sum^r\mathbf{T})]^2}\right]$$
$$\times\frac{e^2B}{m_e}\left(\sqrt{\frac{n_2+1}{2}}\delta_{n,n_2+1}+\sqrt{\frac{n_2}{2}}\delta_{n,n_2-1}\right)\left[\frac{2i\mathrm{Im}\big(\mathbf{T}^\dagger\sum^r\mathbf{T}\big)}{\big(\frac{1}{\hbar}\varepsilon-\frac{1}{\hbar}\varepsilon_{n}\big)^2+[\mathrm{Im}(\mathbf{T}^\dagger\sum^r\mathbf{T})]^2}\right] \tag{4.7}$$

and the only non-zero term would be

$$\lim_{\omega\to 0}\mathrm{Re}[\sigma^{xx}(0,\omega)] = \frac{-1}{4\pi\hbar A}\frac{e^4B^2}{{m_e}^2}\int_{\lambda-\hbar\Omega/2}^{\lambda+\hbar\Omega/2} d\varepsilon\Big(-\frac{\partial f}{\partial\varepsilon}\Big)\frac{1}{V_{k_x}}\sum_{k_x}\sum_{n}\left(\frac{n+1}{2}\right)$$
$$\times\left[\frac{2i\mathrm{Im}\big(\mathbf{T}^\dagger\sum^r\mathbf{T}\big)_{\varepsilon_{n+1}}}{\big(\frac{1}{\hbar}\varepsilon-\frac{1}{\hbar}\varepsilon_{n+1}\big)^2+\Big[\mathrm{Im}(\mathbf{T}^\dagger\sum^r\mathbf{T})_{\varepsilon_{n+1}}\Big]^2}\right]\left[\frac{2i\mathrm{Im}\big(\mathbf{T}^\dagger\sum^r\mathbf{T}\big)_{\varepsilon_{n}}}{\big(\frac{1}{\hbar}\varepsilon-\frac{1}{\hbar}\varepsilon_{n}\big)^2+\Big[\mathrm{Im}(\mathbf{T}^\dagger\sum^r\mathbf{T})_{\varepsilon_{n}}\Big]^2}\right] \tag{4.8}$$

■

Then using the following identity derived in [*Ref: My report 2.509]

$$\left(\frac{1}{\tau(\varepsilon,k_x)}\right)_{ll} = -2\mathrm{Im}\Big[\Big(\mathbf{T}^\dagger\sum^r\mathbf{T}\Big)_{\varepsilon}\Big]_{ll} \tag{4.9}$$

using central element of the inverse scattering time matrix we can modify our result as

$$\lim_{\omega\to 0}\mathrm{Re}[\sigma^{xx}(0,\omega)] = \frac{1}{4\pi\hbar A}\frac{e^4B^2}{{m_e}^2}\int_{\lambda-\hbar\Omega/2}^{\lambda+\hbar\Omega/2} d\varepsilon\Big(-\frac{\partial f}{\partial\varepsilon}\Big)\frac{1}{V_{k_x}}\sum_{k_x}\sum_{n}\left(\frac{n+1}{2}\right)$$
$$\times\left[\frac{\Big(\frac{1}{\tau(\varepsilon_{n+1},k_x)}\Big)}{\big(\frac{1}{\hbar}\varepsilon-\frac{1}{\hbar}\varepsilon_{n+1}\big)^2+\Big(\frac{1}{2\tau(\varepsilon_{n+1},k_x)}\Big)^2}\right]\left[\frac{\Big(\frac{1}{\tau(\varepsilon_{n},k_x)}\Big)}{\big(\frac{1}{\hbar}\varepsilon-\frac{1}{\hbar}\varepsilon_{n}\big)^2+\Big(\frac{1}{2\tau(\varepsilon_{n},k_x)}\Big)^2}\right] \tag{4.10}$$

We have identited that the inverse scattering time matrix's central element is not k_x dependent we can get the sum over all available momentum space in x direction. However by considering the condition that the center of the force of the oscillator y_0 must physically liw within the system $-L_y/2 < y_0 < L_y/2$, one can derive that

$$-\frac{m_e\omega_0 Ly}{2\hbar}\leq k_x\leq\frac{m_e\omega_0 Ly}{2\hbar} \tag{4.11}$$

and we can derive that

$$\frac{1}{V_{k_x}}\sum_{k_x} = \frac{m_e\omega_0 Ly}{\hbar V_{k_x}} = 1 \tag{4.12}$$

Thefore Eq. (4.10) modified to

$$\lim_{\omega\to 0}\mathrm{Re}[\sigma^{xx}(0,\omega)] = \frac{e^2\omega_0^2}{4\pi\hbar A}\int_{\lambda-\hbar\Omega/2}^{\lambda+\hbar\Omega/2} d\varepsilon\Big(-\frac{\partial f}{\partial\varepsilon}\Big)\sum_{n}\left(\frac{n+1}{2}\right)$$
$$\times\left[\frac{\Big(\frac{1}{\tau(\varepsilon_{n+1})}\Big)}{\big(\frac{1}{\hbar}\varepsilon-\frac{1}{\hbar}\varepsilon_{n+1}\big)^2+\Big(\frac{1}{2\tau(\varepsilon_{n+1})}\Big)^2}\right]\left[\frac{\Big(\frac{1}{\tau(\varepsilon_{n})}\Big)}{\big(\frac{1}{\hbar}\varepsilon-\frac{1}{\hbar}\varepsilon_{n}\big)^2+\Big(\frac{1}{2\tau(\varepsilon_{n})}\Big)^2}\right] \tag{4.13}$$

Then using Fermi-Dirac distribution as our partical distribution function (f) for this system

$$f(\varepsilon) = \frac{1}{[\exp(\varepsilon-\varepsilon_F)/k_BT]+1} \tag{4.14}$$

where k_B is Botlzmann constant, T is absolute tempurature and ε_F is Fermi energy of the system. Using above distribution, for extreamly low tempuratures we can appromixate that

$$-\frac{\partial f(\varepsilon)}{\partial \varepsilon} \approx \delta(\varepsilon - \varepsilon_F) \tag{4.15}$$

and this will mpre simplify our derivation of conductivity as

$$\lim_{\omega\to 0} \mathrm{Re}[\sigma^{xx}(0,\omega)] = \frac{e^2\omega_0^2}{4\pi\hbar A}\sum_n \left(\frac{n+1}{2}\right)\left[\frac{\left(\frac{1}{\tau(\varepsilon_{n+1})}\right)}{\left(\frac{1}{\hbar}\varepsilon_F - \frac{1}{\hbar}\varepsilon_{n+1}\right)^2 + \left(\frac{1}{2\tau(\varepsilon_{n+1})}\right)^2}\right]\left[\frac{\left(\frac{1}{\tau(\varepsilon_n)}\right)}{\left(\frac{1}{\hbar}\varepsilon_F - \frac{1}{\hbar}\varepsilon_n\right)^2 + \left(\frac{1}{2\tau(\varepsilon_n)}\right)^2}\right] \tag{4.16}$$

Now introduce a new paramter with a physical meaning od scattering-induced broading of the Landau level as follows

$$\Gamma_n \equiv \Gamma(\varepsilon_n) \equiv \left(\frac{\hbar}{2\tau(\varepsilon_n)}\right) \tag{4.17}$$

and then we can re-write Eq. (4.16) as follows

$$\lim_{\omega\to 0} \mathrm{Re}[\sigma^{xx}(0,\omega)] = \frac{e^2(\hbar\omega_0)^2}{\pi\hbar A}\sum_n \left(\frac{n+1}{2}\right)\left[\frac{\Gamma(\varepsilon_{n+1})}{(\varepsilon_F - \varepsilon_{n+1})^2 + \Gamma^2(\varepsilon_{n+1})}\right]\left[\frac{\Gamma(\varepsilon_n)}{(\varepsilon_F - \varepsilon_n)^2 + \Gamma^2(\varepsilon_n)}\right] \tag{4.18}$$

$$\lim_{\omega\to 0} \mathrm{Re}[\sigma^{xx}(0,\omega)] = \frac{e^2(\hbar\omega_0)^2}{\pi\hbar A}\sum_n \left(\frac{n+1}{2}\right)\left[\frac{\Gamma_{n+1}}{(\varepsilon_F - \varepsilon_{n+1})^2 + \Gamma_{n+1}^2}\right]\left[\frac{\Gamma_n}{(\varepsilon_F - \varepsilon_n)^2 + \Gamma_n^2}\right] \tag{4.19}$$

Now use new dimentionless paramters

$$X_F \equiv \frac{\varepsilon_F}{\hbar\omega_0} - \frac{1}{2} \tag{4.20}$$

and

$$\gamma_n \equiv \frac{\Gamma_n}{\hbar\omega_0}. \tag{4.21}$$

Therefore the Eq. (4.19) leads to

$$\lim_{\omega\to 0} \mathrm{Re}[\sigma^{xx}(0,\omega)] = \frac{e^2}{\hbar}\frac{1}{2\pi A}\sum_n (n+1)\left[\frac{\gamma_{n+1}}{(X_F - n - 1)^2 + \gamma_{n+1}^2}\right]\left[\frac{\gamma_n}{(X_F - n)^2 + \gamma_n^2}\right] \tag{4.22}$$

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