Magnetic propeties of a two dimentional electron gas strongly coupled to lights

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1 Schrödinger problem for Landau levels in dressed 2DEG

Our analysis start with considering 2 dimentional free electronic gas which has been distrubuted in confined (x, y) plane in configuration space.

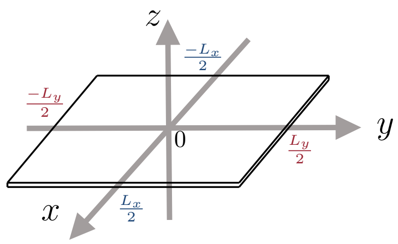

Figure 1: Confined 2DEG in configuration space with the size of $A = L_x L_y$.

We are going to examine the properties of 2DEG with stationary magnetic field

$$\mathbf{B} = (0, 0, B)^T \tag{1.1}$$

which directed on z axis and a linearly y-polarized strong electomagnetic wave (dressing field) with electric field given by

$$\mathbf{E} = (0, E\sin(\omega t), 0)^T \tag{1.2}$$

which also propagate in z direction. Here B and E represent the amplitude of the stationary magnetic field and electric field of dressing field.

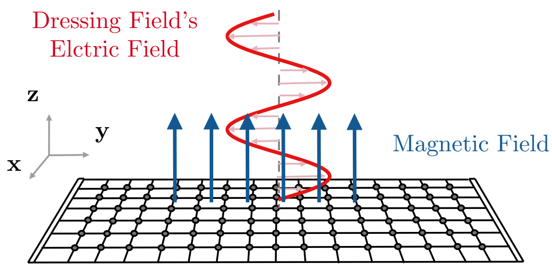

Figure 2: Stationary magnetic filed (blue color) and Strong EM wave (red color) applied to the 2DEG.

Using Landau gauge for the stationary magnetic field we can represent it using vector potential as

$$\mathbf{A}_s = (-By, 0, 0)^T \tag{1.3}$$

and choosing Coulomb gauge the dressing field can be present as the following vector potential

$$\mathbf{A}_d(t) = (0, [E/\omega]\cos(\omega t), 0)^T. \tag{1.4}$$

Now the Hamiltonian of an electron in 2DEG can be reads as

$$\hat{H}_e(t) = \frac{1}{2m_e}\Big[\hat{\mathbf{p}} - e\big(\mathbf{A}_s + \mathbf{A}_d(t)\big)\Big]^2 \tag{1.5}$$

where m_e is the effective mass of the electron and e is the magnitude (without considering the sign of the charge) of the electron charge. This can be simplified to

$$\hat{H}_e(t) = \frac{1}{2m_e}\Big[(\hat{p}_x + eBy)\mathbf{e}_x + (\hat{p}_y - \frac{eE}{\omega}\cos(\omega t))\mathbf{e}_y\Big]^2 \tag{1.6}$$

where $\mathbf{e}_x$ and $\mathbf{e}_y$ are unit vectors along x and y directions respectively. Moreover,

$$\hat{H}_e(t) = \frac{1}{2m_e}\Big[(\hat{p}_x + eBy)^2 + (\hat{p}_y - \frac{eE}{\omega}\cos(\omega t))^2\Big] \tag{1.7}$$

Since $[\hat{H}_e(t), \hat{p}_x] = 0$ both operators share same (simultaneous) eigen functions which are free electron wave functions $(\frac{1}{\sqrt{L_x}}\exp(\frac{ip_x x}{\hbar}))$. Therefore we can modify the Hamiltonian as follows

$$\hat{H}_e(t) = \frac{1}{2m_e}\Big[(p_x + eBy)^2 + (\hat{p}_y - \frac{eE}{\omega}\cos(\omega t))^2\Big]. \tag{1.8}$$

Using momentum operator definition

$$\hat{p}_y = -i\hbar\frac{\partial}{\partial y} \tag{1.9}$$

we can modify Eq. (1.8) as

$$\begin{aligned}\hat{H}_e(t) &= \frac{1}{2m_e}\Big[(p_x + eBy)^2 + \Big(-i\hbar\frac{\partial}{\partial y} - \frac{eE}{\omega}\cos(\omega t)\Big)^2\Big] \\ &= \frac{1}{2m_e}\Big[(p_x + eBy)^2 + \Big(i\hbar\frac{\partial}{\partial y} + \frac{eE}{\omega}\cos(\omega t)\Big)^2\Big].\end{aligned} \tag{1.10}$$

Define the *center of the cyclotron orbit* along y axis as

$$y_0 \equiv \frac{-p_x}{eB} \tag{1.11}$$

and the *cyclotron frequency* as

$$\omega_0 \equiv \frac{eB}{m_e}. \tag{1.12}$$

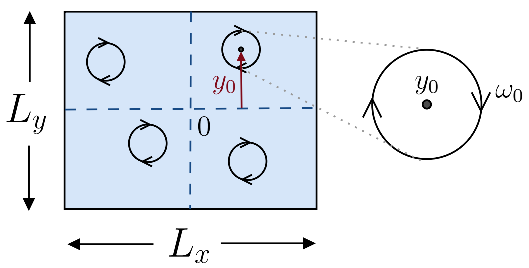

Figure 3: Paramters of the cyclotron orbits in the classical interpretation.

Then the Hamiltonian will leads to

$$\hat{H}_e(t) = \frac{m_e\omega_0^2}{2}(y - y_0)^2 + \frac{1}{2m_e}\Big(i\hbar\frac{\partial}{\partial y} + \frac{eE}{\omega}\cos(\omega t)\Big)^2 \tag{1.13}$$

$$\hat{H}_e(t) = \frac{m_e\omega_0^2}{2}(y-y_0)^2 + \frac{1}{2m_e}\Big(-\hbar^2\frac{\partial^2}{\partial y^2} + i\hbar\frac{\partial}{\partial y}\Big[\frac{eE}{\omega}\cos(\omega t)\Big] + \frac{i\hbar eE}{\omega}\cos(\omega t)\frac{\partial}{\partial y} + \frac{e^2E^2}{\omega^2}\cos^2(\omega t)\Big) \tag{1.14}$$

$$\hat{H}_e(t) = \frac{m_e\omega_0^2}{2}(y-y_0)^2 + \frac{1}{2m_e}\Big(-\hbar^2\frac{\partial^2}{\partial y^2} + \frac{2i\hbar eE}{\omega}\cos(\omega t)\frac{\partial}{\partial y} + \frac{e^2E^2}{\omega^2}\cos^2(\omega t)\Big). \tag{1.15}$$

Let

$$\tilde{y} = (y-y_0) \longrightarrow dy = d\tilde{y} \tag{1.16}$$

and then this becomes

$$\hat{H}_e(t) = \frac{m_e\omega_0^2}{2}\tilde{y}^2 + \frac{1}{2m_e}\Big(-\hbar^2\frac{\partial^2}{\partial \tilde{y}^2} + \frac{2i\hbar eE}{\omega}\cos(\omega t)\frac{\partial}{\partial \tilde{y}} + \frac{e^2E^2}{\omega^2}\cos^2(\omega t)\Big). \tag{1.17}$$

Now assume that the solution for the time-dependent schrödinger equation

$$i\hbar\frac{\mathrm{d}\psi}{\mathrm{d}t} = \hat{H}_e(t)\psi \tag{1.18}$$

can be represent by the following form

$$\psi(\mathbf{r},t) = \frac{1}{\sqrt{L_x}}\exp\left(\frac{ip_x x}{\hbar} + \frac{ieE(y-y_0)}{\hbar\omega}\cos(\omega t)\right)\phi(y-y_0,t). \tag{1.19}$$

Using the same subtution from Eq. (1.16) this becomes

$$\psi(x,\tilde{y},t) = \frac{1}{\sqrt{L_x}}\exp\left(\frac{ip_x x}{\hbar} + \frac{ieE\tilde{y}}{\hbar\omega}\cos(\omega t)\right)\phi(\tilde{y},t). \tag{1.20}$$

Defining

$$\varphi(x,\tilde{y},t) \equiv \frac{1}{\sqrt{L_x}}\exp\left(\frac{ip_x x}{\hbar} + \frac{ieE\tilde{y}}{\hbar\omega}\cos(\omega t)\right) \tag{1.21}$$

we can simply the the Eq. (1.20) as

$$\psi(x,\tilde{y},t) = \varphi(x,\tilde{y},t)\phi(\tilde{y},t). \tag{1.22}$$

Let's subtitue Eq. (1.20) and Eq. (1.17) into Eq. (1.18) and we can observe that

$$\begin{aligned} \text{L.H.S} = i\hbar\frac{\mathrm{d}\psi}{\mathrm{d}t} &= i\hbar\Big(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\phi + \frac{\mathrm{d}\phi}{\mathrm{d}t}\varphi\Big) = i\hbar\Big(\Big[\frac{-ieE\tilde{y}}{\hbar}\sin(\omega t)\Big]\varphi\phi + \varphi\frac{\mathrm{d}\phi}{\mathrm{d}t}\Big) \\ &= \big[eE\tilde{y}\sin(\omega t)\big]\varphi\phi + i\hbar\varphi\frac{\mathrm{d}\phi}{\mathrm{d}t} \end{aligned} \tag{1.23}$$

and

$$\begin{aligned} \text{R.H.S} &= \hat{H}_e(t)\psi \\ &= \Big[\frac{m_e\omega_0^2}{2}\tilde{y}^2 + \frac{1}{2m_e}\Big(-\hbar^2\frac{\partial^2}{\partial \tilde{y}^2} + \frac{2i\hbar eE}{\omega}\cos(\omega t)\frac{\partial}{\partial \tilde{y}} + \frac{e^2E^2}{\omega^2}\cos^2(\omega t)\Big)\Big]\varphi\phi \end{aligned} \tag{1.24}$$

where we will can calculate this part by part as follows:

$$\begin{aligned} \frac{-\hbar^2}{2m_e}\frac{\partial^2}{\partial \tilde{y}^2}(\varphi\phi) &= \frac{-\hbar^2}{2m_e}\frac{\partial}{\partial \tilde{y}}\Big[\Big(\frac{ieE}{\hbar\omega}\cos(\omega)t\Big)\varphi\phi + \varphi\frac{\partial\phi}{\partial\tilde{y}}\Big] \\ &= \frac{-\hbar^2}{2m_e}\Big[\Big(\frac{ieE}{\hbar\omega}\cos(\omega)t\Big)^2\varphi\phi + \Big(\frac{ieE}{\hbar\omega}\cos(\omega)t\Big)\varphi\frac{\partial\phi}{\partial\tilde{y}} + \Big(\frac{ieE}{\hbar\omega}\cos(\omega)t\Big)\varphi\frac{\partial\phi}{\partial\tilde{y}} + \varphi\frac{\partial^2\phi}{\partial\tilde{y}^2}\Big] \\ &= \Big(\frac{e^2E^2}{2m_e\omega^2}\cos^2(\omega)t\Big)\varphi\phi - \Big(\frac{ieE\hbar}{m_e\omega}\cos(\omega)t\Big)\varphi\frac{\partial\phi}{\partial\tilde{y}} - \frac{\hbar^2}{2m_e}\varphi\frac{\partial^2\phi}{\partial\tilde{y}^2} \end{aligned} \tag{1.25}$$

and

$$\frac{2i\hbar eE}{2m_e\omega}\cos(\omega t)\frac{\partial}{\partial\tilde{y}}(\varphi\phi) = \frac{i\hbar eE}{m_e\omega}\cos(\omega t)\Big[\Big(\frac{ieE}{\hbar\omega}\cos(\omega)t\Big)\varphi\phi + \varphi\frac{\partial\phi}{\partial\tilde{y}}\Big] \\ = \Big(\frac{-e^2E^2}{m_e\omega^2}\cos(\omega)t\Big)\varphi\phi + \frac{i\hbar eE}{m_e\omega}\cos(\omega t)\varphi\frac{\partial\phi}{\partial\tilde{y}}. \tag{1.26}$$

Therefore we can derive that

$$\text{R.H.S} = \Big[\frac{m_e\omega_0^2}{2}\tilde{y}^2\varphi\phi - \frac{\hbar^2}{2m_e}\varphi\frac{\partial^2\phi}{\partial\tilde{y}^2}\Big]. \tag{1.27}$$

To satisfy the condition L.H.S=R.H.S we need to find a function $\phi(\tilde{y}, t)$ such that

$$\big[eE\tilde{y}\sin(\omega t)\big]\varphi\phi + i\hbar\varphi\frac{\mathrm{d}\phi}{\mathrm{d}t} = \Big[\frac{m_e\omega_0^2}{2}\tilde{y}^2\varphi\phi - \frac{\hbar^2}{2m_e}\varphi\frac{\partial^2\phi}{\partial\tilde{y}^2}\Big] \tag{1.28}$$

by removing φ this can be simplyfied as

$$\Big[\frac{m_e\omega_0^2}{2}\tilde{y}^2 - eE\tilde{y}\sin(\omega t) - \frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial\tilde{y}^2} - i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\Big]\phi(\tilde{y}, t) = 0. \tag{1.29}$$

If we turn off the external dressing field, this equation leads to simple harmonic oscillator Hamiltonian as follows

$$\Big[\frac{m_e\omega_0^2}{2}\tilde{y}^2 - \frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial\tilde{y}^2} - i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\Big]\phi(\tilde{y}, t) = 0 \tag{1.30}$$

$$i\hbar\frac{\mathrm{d}\phi(\tilde{y}, t)}{\mathrm{d}t} = \Big[\frac{\hat{p}_{\tilde{y}}^2}{2m_e} + \frac{1}{2}m_e\omega_0^2\tilde{y}^2\Big]\phi(\tilde{y}, t). \tag{1.31}$$

Therefore we can identify the $S(t) \equiv eE\sin(\omega t)$ part as a external force act on the harmonic oscillator and we can solve this as a forced harmonic oscillator in $\tilde{y}$ axis.

$$i\hbar\frac{\mathrm{d}\phi(\tilde{y}, t)}{\mathrm{d}t} = \Big[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial\tilde{y}^2} + \frac{1}{2}m_e\omega_0^2\tilde{y}^2 - \tilde{y}S(t)]\Big]\phi(\tilde{y}, t). \tag{1.32}$$

This system can be extacly solvable and we can solve this equation using the methods explained by Husimi [*Ref:1] as follows.
First we can introduce the time dependent shifted corrdinte as

$$\tilde{y} \to y' = \tilde{y} - \zeta(t) \quad \Rightarrow \quad \tilde{y} = y' + \zeta(t) \tag{1.33}$$

and this implies that

$$\frac{\mathrm{d}\phi(y', t)}{\mathrm{d}t} = \frac{\partial\phi(y', t)}{\partial t} + \frac{\partial\phi(y', t)}{\partial y'}\frac{\partial y'}{\partial t} = \frac{\partial\phi(y', t)}{\partial t} - \dot{\zeta}(t)\frac{\partial\phi(y', t)}{\partial y'} \tag{1.34}$$

where $\dot{\zeta}(t) = \frac{\partial\zeta(t)}{\partial t}$. Therefore, Eq. (1.32) will be modified to

$$i\hbar\frac{\partial\phi(y', t)}{\partial t} = \Big[i\hbar\dot{\zeta}\frac{\partial}{\partial y'} - \frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y'^2} + \frac{1}{2}m_e\omega_0^2(y' + \zeta)^2 - (y' + \zeta)S(t)\Big]\phi(y', t). \tag{1.35}$$

Let's tranform the wave function using following unitary trasnform

$$\phi(y', t) = \exp\left(\frac{im_e\dot{\zeta}y'}{\hbar}\right)\varphi(y', t) \tag{1.36}$$

and subtitte this into the Eq. (1.35) and we will get the following

$$\text{L.H.S} = \Big[i\hbar\frac{\partial}{\partial t} - i\hbar\Big(\frac{im_e\ddot{\zeta}y'}{\hbar}\Big)\Big]\exp\left(\frac{-im_e\dot{\zeta}y'}{\hbar}\right)\varphi(y', t) \tag{1.37}$$

and

$$\begin{aligned}\text{R.H.S} = \Big[& i\hbar\dot{\zeta}\Big(\frac{im_e\dot{\zeta}}{\hbar}\Big) + i\hbar\dot{\zeta}\frac{\partial}{\partial y'} \\ & - \frac{\hbar^2}{2m_e}\Big[\Big(\frac{im_e\dot{\zeta}}{\hbar}\Big)^2 + \Big(\frac{2im_e\dot{\zeta}}{\hbar}\Big)\frac{\partial}{\partial y'} + \frac{\partial^2}{\partial y'^2}\Big] \\ & + \frac{1}{2}m_e\omega_0^2 y'^2 + \frac{1}{2}m_e\omega_0^2\zeta^2 + m_e\omega_0^2 y'\zeta \\ & - y'S(t) - \zeta S(t)\Big]\exp\left(\frac{-im_e\dot{\zeta}y'}{\hbar}\right)\varphi(y',t).\end{aligned} \tag{1.38}$$

Combining these two and removing exponential terms we can derive that

$$\begin{aligned}i\hbar\frac{\partial\varphi(y',t)}{\partial t} = \Big[& -\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y'^2} + \frac{1}{2}m_e\omega_0^2 y'^2 + \Big[m_e\ddot{\zeta} + m_e\omega_0^2\zeta - S(t)\Big]y' \\ & + \Big[-\frac{1}{2}m_e\dot{\zeta}^2 + \frac{1}{2}m_e\omega_0^2\zeta^2 - \zeta S(t)\Big]\Big]\varphi(y',t).\end{aligned} \tag{1.39}$$

Then we can restrict our $\zeta(t)$ function such that

$$m_e\ddot{\zeta} + m_e\omega_0^2\zeta = S(t) \tag{1.40}$$

and that leads to

$$i\hbar\frac{\partial\varphi(y',t)}{\partial t} = \Big[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y'^2} + \frac{1}{2}m_e\omega_0^2 y'^2 - L(\zeta,\dot{\zeta},t)\Big]\varphi(y',t) \tag{1.41}$$

where

$$L(\zeta,\dot{\zeta},t) \equiv \frac{1}{2}m_e\dot{\zeta}^2 - \frac{1}{2}m_e\omega_0^2\zeta^2 + \zeta S(t) \tag{1.42}$$

is the largrangian of a classical driven oscillator.

Now introduce new unitary transormation for the wavefunction as follows

$$\varphi(y',t) = \exp\Big(\frac{i}{\hbar}\int_0^t dt' L(\zeta,\dot{\zeta},t')\Big)\chi(y',t) \tag{1.43}$$

and subtite this into the Eq. (1.41) and gets

$$\begin{aligned}& i\hbar\Big[\exp\Big(\frac{i}{\hbar}\int_0^t dt' L(\zeta,\dot{\zeta},t')\Big)\frac{\partial}{\partial t} + i\hbar L(\zeta,\dot{\zeta},t)\exp\Big(\frac{i}{\hbar}\int_0^t dt' L(\zeta,\dot{\zeta},t')\Big)\Big]\chi(y',t) \\ & = \Big[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y'^2} + \frac{1}{2}m_e\omega_0^2 y'^2 - L(\zeta,\dot{\zeta},t)\Big]\exp\Big(\frac{i}{\hbar}\int_0^t dt' L(\zeta,\dot{\zeta},t')\Big)\chi(y',t)\end{aligned} \tag{1.44}$$

removing exponential terms finally we can derive that

$$i\hbar\frac{\partial}{\partial t}\chi(y',t) = \Big[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y'^2} + \frac{1}{2}m_e\omega_0^2 y'^2\Big]\chi(y',t). \tag{1.45}$$

This is the well known Schrodinger equation of a stationary quantum harmonic oscillator. In terms of the eigenvalues

$$E_n = \hbar\omega_0\big(n + \frac{1}{2}\big) \tag{1.46}$$

of well-known harmonic eigenfucntions (using Gauss-Hermite functions ϑ)

$$\chi_n(x) \equiv \sqrt{\kappa}\vartheta(\kappa x) \quad \text{where} \quad \vartheta(x) = \frac{1}{\sqrt{2^n n!}}e^{-x^2/2}\mathcal{H}_n(x) \quad \text{with} \quad \kappa = \sqrt{\frac{m_e\omega_0}{\hbar}} \tag{1.47}$$

being propositional to the Hermite functions $\mathcal{H}_n$, the solutions of Eq. (1.32) can be represent as

$$\phi_n(\tilde{y},t) = \chi_n(\tilde{y}-\zeta(t))\exp\Big(\frac{i}{\hbar}\Big[-E_n t + m_e\dot{\zeta}(t)\big(\tilde{y}-\zeta(t)\big) + \int_0^t dt' L(\zeta,\dot{\zeta},t')\Big]\Big) \tag{1.48}$$

The set $\{\chi_n(x)\}$ forms a complete set and thus any general solution $\phi_(\tilde{y},t)$ can be expaned in terms of the solutions in Eq. (1.48).

Next we consider special case where we assumed

$$S(t) = eE\sin(\omega t) \tag{1.49}$$

and one can derive the Eq. (1.40) for $\zeta(t)$

$$m_e\ddot{\zeta} + m_e\omega_0^2\zeta = eE\sin(\omega t) \tag{1.50}$$

and using Green function method the solution can be write as

$$\zeta(t) = \frac{eE}{m_e(\omega_0^2-\omega^2)}\sin(\omega t). \tag{1.51}$$

form this solutions we are able to derive the final solutions $\alpha = (n,m)$ where $n \in \mathbb{Z}_0^+ and m \in \mathbb{Z}$ are two quantum numbers that describe the state of the electron, can be present as

$$\begin{aligned}\psi_\alpha(x,\tilde{y},t) =& \frac{1}{\sqrt{L_x}}\chi_n\big(\tilde{y}-\zeta(t)\big)\\ &\times\exp\Big(\frac{i}{\hbar}\Big[-E_n t + p_x x + \frac{eE\tilde{y}}{\omega}\cos(\omega t) + m_e\dot{\zeta}(t)\big[\tilde{y}-\zeta(t)\big] + \int_0^t dt' L(\zeta,\dot{\zeta},t')\Big]\Big)\end{aligned} \tag{1.52}$$

and the exponential phase shifts represent the effect done by the stationary magnetic field and strong dressing field. In here p_x is qunatized with the quantum number m due to the spacial confinemnet in x direction.

$$p_x = m\frac{2\pi\hbar}{L_x} \quad , \quad m = 0, \pm1, \pm2, ... \tag{1.53}$$

Therefore we can assume that the magnetitranport properties of 2DEG will be renormalized by the magnetic field as well as the dressing field. ■

2 Scattering theory

Since in a real metal there would be many scatters that can be behave as obstacles for electron that have free wave functions. Therefore we need to calculate them to analyse the real behaviour of the electrons.
Then the wave function of the electron in a real matel $\Psi(\mathbf{r},t)$ should satisfy the following time-dependet Schrodinger equation

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = [H_e(t) + U(\mathbf{r})]\Psi(\mathbf{r},t) \tag{2.1}$$

where $U(\mathbf{r})$ is the total scattering potential. We have represented the all scatters using this potential. Since the solutions (1.52) are create a complete orthonormal basis we can represent this wave function using those as follows

$$\Psi(\mathbf{r},t) = \sum_j a_j(t)\,|\psi_j(t)\rangle \tag{2.2}$$

where the difference inidces j corresponding to the different sets of all quantum numbers p_x and n

$$j \to (m,n) \quad \text{where} \quad m,n = 0,1,2,... \tag{2.3}$$

with m is defined for quantized momentum in x direction

$$p_x = m\frac{2\pi\hbar}{L_x} \tag{2.4}$$

Now we can use the conventional pertubation theory to calculate scattering process of electron at a state $|\psi_j\rangle$ to a state $|\psi'_j\rangle$. For that assume an electron be in the j state at the time $t = 0$ and corresponding $a'_j(0) = \delta_{j,j'}$.
First subtitute a general electron state $\Psi(\mathbf{r},t)$ at time t as the incoming electron to the Schrodinger equation given in Eq. (2.1)

$$i\hbar\frac{\partial}{\partial t}\sum_j a_j(t)\,|\psi_j(t)\rangle = [H_e(t) + U(\mathbf{r})]\sum_j a_j(t)\,|\psi_j(t)\rangle \tag{2.5}$$

$$i\hbar\sum_j \dot{a_j}(t)\,|\psi_j(t)\rangle + a_j(t)\frac{\partial}{\partial t}\,|\psi_j(t)\rangle = [H_e(t) + U(\mathbf{r})]\sum_j a_j(t)\,|\psi_j(t)\rangle \tag{2.6}$$

since all the $|\psi(t)\rangle$ staistfy the Schrodinger equation (1.18)

$$i\hbar\sum_j \dot{a_j}(t)\,|\psi_j(t)\rangle = \sum_j U(\mathbf{r})a_j(t)\,|\psi_j(t)\rangle\,. \tag{2.7}$$

Then take inner product with state with the state $|\psi_{j'}(t)\rangle$

$$i\hbar\sum_j \dot{a_j}(t)\,\langle\psi_{j'}(t)|\psi_j(t)\rangle = \sum_j a_j(t)\,\langle\psi_{j'}(t)|\,U(\mathbf{r})\,|\psi_j(t)\rangle \tag{2.8}$$

But using the *Born approximation* we can assume that this incoming wave have the initial state of the electron at $t = 0$ and therefore this equation will modified to

$$i\hbar\sum_j \dot{a_j}(t)\,\langle\psi_{j'}(t)|\psi_j(t)\rangle = \langle\psi_{j'}(t)|\,U(\mathbf{r})\,|\psi_j(t)\rangle \tag{2.9}$$

due to orthonormality this becomes

$$i\hbar\dot{a_{j'}}(t) = \langle\psi_{j'}(t)|\,U(\mathbf{r})\,|\psi_j(t)\rangle \tag{2.10}$$

and finally this leads to first order pertubation theory for Sscattering as follows

$$\dot{a_{j'}}(t) = -\frac{i}{\hbar}\,\langle\psi_{j'}(t)|\,U(\mathbf{r})\,|\psi_j(t)\rangle \tag{2.11}$$

where

$$a_{j'}(t) = -\frac{i}{\hbar}\int_0^t dt' \int_S d\mathbf{r}\ \psi^*_{j'}(\mathbf{r},t')U(\mathbf{r})\psi_j(\mathbf{r},t') \tag{2.12}$$

where the integration should be performed over the 2DEG area $S = L_x L_y$. Then we can calculate this using the eqution we derived in (1.52) as follows

$$\begin{aligned}
a_{j'}(t) = &-\frac{i}{\hbar}\int_0^t dt' \int_S d\mathbf{r}\ \Bigg[\frac{1}{\sqrt{L_x}}\chi^*_{n'}\big(y - {y'}_0 - \zeta(t)\big) \\
&\times \exp\Bigg(\frac{i}{\hbar}\Bigg[E_{n'}t' - m'\frac{2\pi\hbar x}{L_x} - \frac{eE(y-{y'}_0)}{\omega}\cos(\omega t') - m_e\dot{\zeta}(t)\big[y - {y'}_0 - \zeta(t')\big] - \int_0^{t'} dt' L(\zeta,\dot{\zeta},t")\Bigg]\Bigg) \\
&\times U(\mathbf{r}) \\
&\times \frac{1}{\sqrt{L_x}}\chi_n\big(y - y_0 - \zeta(t')\big) \\
&\times \exp\Bigg(\frac{i}{\hbar}\Bigg[-E_n t' + m\frac{2\pi\hbar x}{L_x} - \frac{eE(y-y_0)}{\omega}\cos(\omega t') - m_e\dot{\zeta}(t')\big[y - y_0 - \zeta(t')\big] - \int_0^{t'} d\tilde{t}L(\zeta,\dot{\zeta},\tilde{t})\Bigg]\Bigg)\Bigg]
\end{aligned} \tag{2.13}$$

then this will be simplified to

$$\begin{aligned}
a_{j'}(t) = &-\frac{i}{\hbar}\int_0^t dt' \int_S d\mathbf{r}\ \Bigg[\frac{1}{\sqrt{L_x}}\chi^*_{n'}\big(y - {y'}_0 - \zeta(t')\big)U(\mathbf{r})\frac{1}{\sqrt{L_x}}\chi_n\big(y - y_0 - \zeta(t')\big) \\
&\times \exp\Bigg(\frac{i}{\hbar}\Bigg[E_{n'}t' - m'\frac{2\pi\hbar x}{L_x} - \frac{eE(y-{y'}_0)}{\omega}\cos(\omega t') - m_e\dot{\zeta}(t')\big[y - {y'}_0 - \zeta(t')\big] - \int_0^{t'} d\tilde{t}L(\zeta,\dot{\zeta},\tilde{t})\Bigg]\Bigg) \\
&\times \exp\Bigg(\frac{i}{\hbar}\Bigg[-E_n t' + m\frac{2\pi\hbar x}{L_x} + \frac{eE(y-y_0)}{\omega}\cos(\omega t') + m_e\dot{\zeta}(t')\big[y - y_0 - \zeta(t')\big] + \int_0^{t'} d\tilde{t}L(\zeta,\dot{\zeta},\tilde{t})\Bigg]\Bigg)\Bigg]
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
a_{j'}(t) = &-\frac{i}{\hbar}\int_0^t dt' \int_S d\mathbf{r}\ \Bigg[\frac{1}{\sqrt{L_x}}\chi^*_{n'}\big(y - {y'}_0 - \zeta(t')\big)U(\mathbf{r})\frac{1}{\sqrt{L_x}}\chi_n\big(y - y_0 - \zeta(t')\big)\exp\Bigg(\frac{2\pi i(m-m')\hbar x}{L_x}\Bigg) \\
&\times \exp\Bigg(\frac{i}{\hbar}\Bigg[E_{n'}t' + \frac{eE{y'}_0}{\omega}\cos(\omega t') + m_e\dot{\zeta}(t'){y'}_0\Bigg]\Bigg)\exp\Bigg(\frac{i}{\hbar}\Bigg[-E_n t' - \frac{eEy_0}{\omega}\cos(\omega t') - m_e\dot{\zeta}(t)y_0\Bigg]\Bigg)\Bigg].
\end{aligned} \tag{2.15}$$

The time dependence of the $chi_n(y)$ can neglect since it is integrate over all the values of the y and we can write this as

$$\begin{aligned}
a_{j'}(t) = &-\frac{i}{\hbar}\int_S d\mathbf{r}\ \frac{1}{\sqrt{L_x}}\chi^*_{n'}\big(y - {y'}_0 - \zeta(t')\big)U(\mathbf{r})\frac{1}{\sqrt{L_x}}\chi_n\big(y - y_0 - \zeta(t')\big)\exp\Bigg(\frac{2\pi i(m-m')\hbar x}{L_x}\Bigg) \\
&\times \int_0^t dt'\ \Bigg[\exp\Bigg(\frac{i}{\hbar}\Bigg[(E_{n'} - E_n)t' + \frac{eE({y'}_0 - y_0)\omega_0^2}{\omega(\omega_0^2 - \omega^2)}\cos(\omega t')\Bigg]\Bigg)\Bigg].
\end{aligned} \tag{2.16}$$

Using Jacobi-Anger expansion

$$e^{iz\cos(\theta)} = \sum_{l=-\infty}^{\infty} i^l J_j(z)e^{in\theta} \tag{2.17}$$

above eqution can be modified as

$$a_{j'}(t) = -\frac{i}{\hbar}U_{j'j}\int_0^t dt'\ \Bigg[\sum_{l=-\infty}^{\infty} i^l J_l\Bigg[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)}\Bigg]\exp\Bigg(\frac{i}{\hbar}(E_{n'} - E_n + l\hbar\omega)t'\Bigg)\Bigg] \tag{2.18}$$

where

$$Uj'j \equiv \langle\Phi_{j'}(\mathbf{r})|U(\mathbf{r})|\Phi_j(\mathbf{r})\rangle \tag{2.19}$$

with bare electron eigen states (without dressing field)

$$\Phi_j(\mathbf{r}) = \frac{1}{\sqrt{L_x}} \exp\left(\frac{2\pi i m\hbar x}{L_x}\right)\chi_n(y). \tag{2.20}$$

Considering time evalution from negative values we can write the same expression as follows

$$a_{j'}(t) = -\frac{i}{\hbar}U_{j'j}\int_{-t/2}^{t/2} dt' \left[\sum_{l=-\infty}^{\infty} i^l J_l\left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2-\omega^2)}\right]\exp\left(\frac{i}{\hbar}(E_{n'} - E_n + l\hbar\omega)t'\right)\right]. \tag{2.21}$$

To calculate scattering probability we can use this scattering amplitude's squre value

$$\begin{aligned}|a_{j'}(t)|^2 = \frac{|U_{j'j}|^2}{\hbar^2}\int_{-t/2}^{t/2} dt' &\left[\sum_{l=-\infty}^{\infty} -i^l J_l\left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2-\omega^2)}\right]\exp\left(\frac{-i}{\hbar}(E_{n'} - E_n + l\hbar\omega)t'\right)\right] \\ \times \int_{-t/2}^{t/2} dt'' &\left[\sum_{k=-\infty}^{\infty} i^k J_k\left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2-\omega^2)}\right]\exp\left(\frac{i}{\hbar}(E_{n'} - E_n + k\hbar\omega)t''\right)\right]\end{aligned} \tag{2.22}$$

Considering long time $t \to \infty$ we can make the integral into a delta function as follows

$$\begin{aligned}|a_{j'}(t)|^2 = 4\pi^2|U_{j'j}|^2 &\left[\sum_{l=-\infty}^{\infty} -i^l J_l\left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2-\omega^2)}\right]\delta(-E_{n'} + E_n - l\hbar\omega)\right] \\ \times &\left[\sum_{k=-\infty}^{\infty} i^k J_k\left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2-\omega^2)}\right]\delta(E_{n'} - E_n + k\hbar\omega)\right]\end{aligned} \tag{2.23}$$

and this implies $l = k$ and this leads to

$$|a_{j'}(t)|^2 = 4\pi^2|U_{j'j}|^2\left[\sum_{l=-\infty}^{\infty} J_l^2\left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2-\omega^2)}\right]\delta^2(E_{n'} - E_n + l\hbar\omega). \tag{2.24}$$

Then using the famous the square δ function transormation method

$$\delta^2(\epsilon) = \delta(\epsilon)\delta^2(0)\lim_{t\to\infty}\int_{-t/2}^{t/2} e^{i0\times t'/\hbar}dt' = \frac{\delta(\epsilon)t}{2\pi\hbar} \tag{2.25}$$

we can calculatre the probability of electron scattering between states j and j' per unit time as

$$\mathcal{W}_{j'j} \equiv \frac{\mathrm{d}|a_{j'}(t)|^2}{\mathrm{d}t} = |U_{j'j}|^2\sum_{l=-\infty}^{\infty} J_l^2\left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2-\omega^2)}\right] \times \frac{2\pi}{\hbar}\delta(E_{n'} - E_n + l\hbar\omega) \tag{2.26}$$

To avoid thee energy echange betwen a high-frequency field and electrons, the field should be purely dressing. We can achieve that by using the field with off-resonant and high frequency. Therefore, the only effect of the dressing field on 2DEG is the renormalization of the probability of elastic electron scattering within the same Landau level ($E_{n'} = E_n$), which described by the term with $l = 0$ the Eq. (2.26) leads to

$$\mathcal{W}_{j'j} = J_0^2\left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2-\omega^2)}\right]\mathcal{W}_{j'j}^{(0)} \tag{2.27}$$

where

$$\mathcal{W}_{j'j}^{(0)} = \frac{2\pi}{\hbar}|U_{j'j}|^2\delta(E_{n'} - E_n) \tag{2.28}$$

is the probability of scattering of a *bare electron*. It is important to notice that the Bessel function factor depend on both the dressing field and stationaty magnetic field. This factor is responcible for all the effcts discussed in thisarticle.

One can define the lifetime of the dressed electron at the Landau level τ is renormalized by the Bessel function as below

$$\frac{1}{\tau} \equiv \sum_{j'} \mathcal{W}_{j'j} = \sum_{j'} J_0^2 \left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \mathcal{W}_{j'j}^{(0)} \tag{2.29}$$

where we have consider all posibilities that electron can jump to the state j'. Then rewrite the delat function as follows

$$\delta(\epsilon) = \frac{1}{\pi} \lim_{\Gamma \to 0} \frac{\Gamma}{\Gamma^2 + \epsilon^2} \tag{2.30}$$

where in this study we can assume that the paramater $\Gamma \equiv \hbar/\tau$ as scattering induced broading of the Landau level. But for the elestic scatteing within the same Landau level, we can write the δ function as

$$\delta(E_{n'} - E_n) \approx \frac{1}{\pi\Gamma}. \tag{2.31}$$

Therefore Eq. (2.29) will change to

$$\frac{1}{\tau} = \sum_{j'} J_0^2 \left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \times \frac{2\pi}{\hbar} |U_{j'j}|^2 \times \frac{1}{\pi\Gamma} \tag{2.32}$$

$$\frac{1}{\tau} = \sum_{j'} J_0^2 \left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \times \frac{2}{\hbar} |U_{j'j}|^2 \times \frac{\tau}{\hbar} \tag{2.33}$$

and finally this can be modified to

$$\frac{1}{\tau} = \left[\frac{2}{\hbar^2} \sum_{j'} J_0^2 \left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] |U_{j'j}|^2 \right]^{1/2} \tag{2.34}$$

where the summation is performed over electron states j' within the same Landau level.
Now lets specify more on the scattering potential where we can model them as randomly distributed delta fucntions as follows

$$U(\mathbf{r}) \equiv \sum_{i=1}^{N_s} U_0 \delta(\mathbf{r} - \mathbf{r}_i) \tag{2.35}$$

where N_s is the total number of scatters in the considering matel. Now we can calculate $|U_{j'j}|^2$ as follows

$$\begin{aligned} |U_{j'j}|^2 = & \sum_{i=1}^{N_s} \frac{1}{L_x^2} \int\int dx_1 dy_1 \exp\left(\frac{-{p'}_x x_1}{\hbar} \right) \chi_n^*(y_1 - {y'}_0) U_0 \delta(x_1 - x_i) \delta(y_1 - y_i) \exp\left(\frac{p_x x_1}{\hbar} \right) \chi_n(y_1 - y_0) \\ & \times \int\int dx_2 dy_2 \exp\left(\frac{{p'}_x x_2}{\hbar} \right) \chi_n(y_2 - {y'}_0) U_0 \delta(x_2 - x_i) \delta(y_2 - y_i) \exp\left(\frac{-p_x x_2}{\hbar} \right) \chi_n^*(y_2 - y_0) \end{aligned} \tag{2.36}$$

and considering only non-zero values for x_1 and x_2 integrals we can re-write this as

$$\begin{aligned} |U_{j'j}|^2 = & \sum_{i=1}^{N_s} \frac{U_0^2}{L_x^2} \int dy_1 \exp\left(\frac{-{p'}_x x_i}{\hbar} \right) \chi_n^*(y_1 - {y'}_0) \delta(y_1 - y_i) \exp\left(\frac{p_x x_i}{\hbar} \right) \chi_n(y_1 - y_0) \\ & \times \int dy_2 \exp\left(\frac{{p'}_x x_i}{\hbar} \right) \chi_n(y_2 - {y'}_0) \delta(y_2 - y_i) \exp\left(\frac{-p_x x_i}{\hbar} \right) \chi_n^*(y_2 - y_0) \end{aligned} \tag{2.37}$$

and this will be simplified to

$$\begin{aligned} |U_{j'j}|^2 = & \sum_{i=1}^{N_s} \frac{U_0^2}{L_x^2} \int dy_1 \chi_n^*(y_1 - {y'}_0) \delta(y_1 - y_i) \chi_n(y_1 - y_0) \\ & \times \int dy_2 \chi_n(y_2 - {y'}_0) \delta(y_2 - y_i) \chi_n^*(y_2 - y_0). \end{aligned} \tag{2.38}$$

Again considering only non-zero values for y_1 and y_2 integrals we can re-write this as

$$|U_{j'j}|^2 = \sum_{i=1}^{N_s} \frac{U_0^2}{L_x^2} \chi_n^*(y_i - {y'}_0)\chi_n(y_i - y_0)\chi_n(y_i - {y'}_0)\chi_n^*(y_i - y_0). \tag{2.39}$$

$$|U_{j'j}|^2 = \frac{U_0^2}{L_x^2} \sum_{i=1}^{N_s} \chi_n^2(y_i - {y'}_0)\chi_n^2(y_i - y_0). \tag{2.40}$$

Now subtituting this derivation into the Eq. (2.34) we will get

$$\frac{1}{\tau} = \left[\frac{2U_0^2}{\hbar^2 L_x^2} \sum_{{y'}_0} J_0^2 \left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)}\right] \sum_{i=1}^{N_s} \chi_n^2(y_i - {y'}_0)\chi_n^2(y_i - y_0)\right]^{1/2} \tag{2.41}$$

where j' reduced to ${p'}_x$ (since $n' = n$)and we can represent it by ${y'}_0$. Then this will modified to

$$\frac{1}{\tau} = \left[\frac{2U_0^2}{\hbar^2 L_x^2} \sum_{{y'}_0} \sum_{i=1}^{N_s} J_0^2 \left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)}\right] \chi_n^2(y_i - {y'}_0)\chi_n^2(y_i - y_0)\right]^{1/2}. \tag{2.42}$$

Now considering large size of sample and a macroscopically large N_s scatters we can promate the summation to integrations as follows

$$\frac{1}{\tau} = \left[\frac{2U_0^2}{\hbar^2 L_x^2} \frac{eBL_x}{2\pi\hbar} \int d{y'}_0 \frac{N_s}{L_x} \int dy_i J_0^2 \left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)}\right] \chi_n^2(y_i - {y'}_0)\chi_n^2(y_i - y_0)\right]^{1/2}. \tag{2.43}$$

Assuming $L_x = L_y$ we can define the area of the 2D material as

$$S \equiv L_x L_x = L_x L_y \tag{2.44}$$

and then we can re-write the above as

$$\frac{1}{\tau} = \left[\frac{eBN_sU_0^2}{\pi\hbar^3 S} \int d{y'}_0 \int dy_i J_0^2 \left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)}\right] \chi_n^2(y_i - {y'}_0)\chi_n^2(y_i - y_0)\right]^{1/2}. \tag{2.45}$$

Define the *density of scatters* per unit area of 2DEG

$$n_s \equiv \frac{N_s}{S} \tag{2.46}$$

and the *magnetic length* as

$$l_0 \equiv \sqrt{\frac{\hbar}{eB}}. \tag{2.47}$$

Now our Eq. (2.45) leads to

$$\frac{1}{\tau} = \sqrt{\frac{n_s U_0^2}{\pi l_0^2 \hbar^2}} \left[\int d{y'}_0 \int dy_i J_0^2 \left[\frac{eE({y'}_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)}\right] \chi_n^2(y_i - {y'}_0)\chi_n^2(y_i - y_0)\right]^{1/2} \tag{2.48}$$

and now define new dummy variables as follows (since y_0 is a paramter)

$$({y'}_0 - y_0) \to y \quad \text{and} \quad (y_i - {y'}_0) \to y' \tag{2.49}$$

and finally we will get the eqution for the dressed electron lifetime at the nth Landau level as

$$\frac{1}{\tau} = \sqrt{\frac{n_s U_0^2}{\pi l_0^2 \hbar^2}} \left[\int\int dy dy' \; J_0^2 \left[\frac{eEy\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)}\right] \chi_n^2(y')\chi_n^2(y + y')\right]^{1/2} \tag{2.50}$$

■

3 Floquet theory

Since we describe the lifetime of an electron in certain Landau level using conventianal perturbation theeory, now we can apply the Floquet theory to identify the difference of these methods.
First we need to identify the *quasienergies* and periodic *Floquet modes* for derived wavefunctions (1.52) for a 2DEG system with both stationary magnetic field and strong dressing filed.
Let's consider the following paramter which is lineraly increasing in time

$$\Delta_E t \equiv \frac{t}{T}\int_0^T dt'\ L(\zeta,\dot{\zeta},t') \tag{3.1}$$

where we can calculate this using Eq. (1.42) and (1.51) as follows

$$\begin{aligned}\Delta_E t = \frac{t}{T}\int_0^T dt'\ \frac{1}{2}m_e\frac{(eE\omega)^2}{m_e^2(\omega_0^2-\omega^2)^2}\cos^2(\omega t') - \frac{1}{2}m_e\omega_0^2\frac{(eE)^2}{m_e^2(\omega_0^2-\omega^2)^2}\sin^2(\omega t') \\ + \frac{eE}{m_e(\omega_0^2-\omega^2)}\sin(\omega t')eE\sin(\omega t')\end{aligned} \tag{3.2}$$

$$\begin{aligned}\Delta_E t = \frac{t\omega}{2\pi}\times\frac{(eE)^2}{2m_e(\omega_0^2-\omega^2)^2}\Bigg[\omega^2\int_0^T dt'\ \cos^2(\omega t') - \omega_0^2\int_0^T dt'\ \sin^2(\omega t') \\ + 2(\omega_0^2-\omega^2)\int_0^T dt'\ \sin^2(\omega t')\Bigg]\end{aligned} \tag{3.3}$$

$$\Delta_E t = \frac{t\omega}{2\pi}\times\frac{(eE)^2}{2m_e(\omega_0^2-\omega^2)^2}\left[\omega^2\frac{\pi}{\omega} - \omega_0^2\frac{\pi}{\omega} + 2(\omega_0^2-\omega^2)\frac{\pi}{\omega}\right] \tag{3.4}$$

$$\Delta_E t = \frac{t\omega}{2}\times\frac{(eE)^2}{2m_e(\omega_0^2-\omega^2)^2}(\omega_0^2-\omega^2) = \frac{(eE)^2}{4m_e(\omega_0^2-\omega^2)}t \tag{3.5}$$

Since this is the continuous increasing part of the Laggrangian integral in Eq. (1.52) we can make this as 2ω periodic function as follows

$$\Lambda \equiv \int_0^t dt'\ L(\zeta,\dot{\zeta},t') - \frac{t}{T}\int_0^T dt'\ L(\zeta,\dot{\zeta},t') \tag{3.6}$$

which can be proved as follows. First consider the first term of the Λ

$$\begin{aligned}\int_0^t dt'\ L(\zeta,\dot{\zeta},t') = \frac{(eE)^2}{2m_e(\omega_0^2-\omega^2)^2}\Bigg[\omega^2\int_0^t dt'\ \cos^2(\omega t') - \omega_0^2\int_0^t dt'\ \sin^2(\omega t') \\ + 2(\omega_0^2-\omega^2)\int_0^t dt'\ \sin^2(\omega t')\Bigg]\end{aligned} \tag{3.7}$$

$$\begin{aligned}\int_0^t dt'\ L(\zeta,\dot{\zeta},t') = \frac{(eE)^2}{2m_e(\omega_0^2-\omega^2)^2}\Bigg[\omega^2\left[\frac{t}{2}+\frac{\sin(2\omega t)}{4\omega}\right] - \omega_0^2\left[\frac{t}{2}-\frac{\sin(2\omega t)}{4\omega}\right] \\ + 2(\omega_0^2-\omega^2)\left[\frac{t}{2}-\frac{\sin(2\omega t)}{4\omega}\right]\Bigg]\end{aligned} \tag{3.8}$$

$$\begin{aligned}\int_0^t dt'\ L(\zeta,\dot{\zeta},t') = \frac{(eE)^2}{2m_e(\omega_0^2-\omega^2)^2}\Bigg[\frac{t}{2}\left[\omega^2-\omega_0^2+2\omega_0^2-2\omega^2\right] \\ + \frac{\sin(2\omega t)}{4\omega}\left[\omega^2+\omega_0^2-2\omega_0^2+2\omega^2\right]\Bigg]\end{aligned} \tag{3.9}$$

$$\int_0^t dt'\ L(\zeta,\dot{\zeta},t') = \frac{(eE)^2}{4m_e(\omega_0^2-\omega^2)}t + \frac{(eE)^2(3\omega^2-\omega_0^2)}{8m_e\omega(\omega_0^2-\omega^2)^2}\sin(2\omega t) \tag{3.10}$$

then using Eq.(3.5) we can write this as

$$\int_0^t dt'\ L(\zeta,\dot{\zeta},t') = \Delta_E t + \frac{(eE)^2\big(3\omega^2-\omega_0^2\big)}{8m_e\omega(\omega_0^2-\omega^2)^2}\sin(2\omega t). \tag{3.11}$$

Now we can express

$$\Lambda = \Delta_E t + \frac{(eE)^2\big(3\omega^2-\omega_0^2\big)}{8m_e\omega(\omega_0^2-\omega^2)^2}\sin(2\omega t) - \Delta_E t = \frac{(eE)^2\big(3\omega^2-\omega_0^2\big)}{8m_e\omega(\omega_0^2-\omega^2)^2}\sin(2\omega t) \tag{3.12}$$

which is a periodic function in time with 2ω frequency.
Now using this parmaters we can factorize the wavefunction (1.52) as linearly time dependend part and periodic time dependend part as follows

$$\begin{aligned}\psi_\alpha(x,y,t) =& \exp\left(\frac{i}{\hbar}[-E_n t + \Delta_E t]\right)\frac{1}{\sqrt{L_x}}\chi_n\big(y-\zeta(t)\big)\\ &\times\exp\left(\frac{i}{\hbar}\left[p_x x + \frac{eEy}{\omega}\cos(\omega t) + m_e\zeta(\dot{t})\big[y-\zeta(t)\big] + \int_0^t dt' L(\zeta,\dot{\zeta},t') - \Delta_E t\right]\right)\end{aligned} \tag{3.13}$$

where we can identify(let $\alpha \to (n,m)$) the *quasienergies* as

$$\varepsilon_\alpha \equiv \varepsilon_n = \hbar\omega_0\left(n+\frac{1}{2}\right) - \Delta_E \quad \text{where} \quad n = 0,1,2,\ldots \quad \text{for any given } m \tag{3.14}$$

which is only depend on one quantum number (n) and *Floquet modes* as

$$\phi_\alpha(x,\tilde{y},t) \equiv \frac{1}{\sqrt{L_x}}\chi_n\big(\tilde{y}-\zeta(t)\big)\exp\left(\frac{i}{\hbar}\left[p_x x + \frac{eE\tilde{y}}{\omega}\cos(\omega t) + m_e\dot{\zeta}(t)\big[\tilde{y}-\zeta(t)\big] + \Lambda\right]\right) \tag{3.15}$$

with

$$\zeta(t) = \frac{eE}{m_e(\omega_0^2-\omega^2)}\sin(\omega t) \quad \text{and} \quad \dot{\zeta}(t) = \frac{eE\omega}{m_e(\omega_0^2-\omega^2)}\cos(\omega t) \tag{3.16}$$

where *Floquet modes* are time-periodic functions that also create a complete orthonormal set. ■

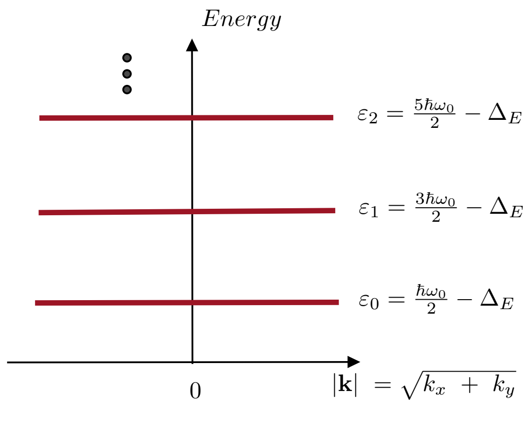

Figure 4: Quasienergies for each Landau levels against magnitude of momentum.

Therefore using Floquet theory, the solutions (Floquet states) for the periodic Hamiltonian (1.5) can be written in position space as

$$\psi_\alpha(x,\tilde{y},t) = \exp\left(-\frac{i}{\hbar}\varepsilon_\alpha t\right)\phi_\alpha(x,\tilde{y},t) \tag{3.17}$$

where

$$\varepsilon_\alpha \equiv \left(\frac{eB\hbar}{m_e}\right)\left(n+\frac{1}{2}\right) - \frac{(eE)^2}{4m_e(\omega_0^2-\omega^2)} \quad \text{where} \quad n = 0,1,2,\ldots \tag{3.18}$$

and

$$\begin{aligned}\phi_\alpha(x,\tilde{y},t) \equiv& \frac{1}{\sqrt{L_x}}\chi_n\left(\tilde{y} - \frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}\right)\\ &\times \exp\left(\frac{i}{\hbar}\left[p_x x + \frac{eE\tilde{y}}{\omega}\cos(\omega t) + \frac{eE\omega\tilde{y}}{(\omega_0^2-\omega^2)}\cos(\omega t)\right]\right)\\ &\times \exp\left(\frac{i}{\hbar}\left[-\frac{(eE)^2\omega}{2m_e(\omega^2-\omega_0^2)^2}\sin(2\omega t) + \frac{(eE)^2\left(3\omega_0^2-\omega^2\right)}{8m_e\omega(\omega_0^2-\omega^2)^2}\sin(2\omega t)\right]\right)\end{aligned} \tag{3.19}$$

Now we can write this by more simplying and considering spacial dependencies and using previous subtituting done in Eq. (1.16) and now χ function depend on both quantum numbers because y_0 gives the p_x dependence and we can present as

$$\begin{aligned}\phi_\alpha(x,y,t) \equiv \frac{1}{\sqrt{L_x}}\chi_n\left(y - y_0 - \frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}\right)\exp\left(\frac{ip_x}{\hbar}x\right)\exp\left(\frac{i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right](y-y_0)\right)\\ \times\exp\left(\frac{-i}{\hbar}\left[\frac{(eE)^2\left(\omega_0^2+\omega^2\right)}{8\omega m_e(\omega_0^2-\omega^2)^2}\right]\sin(2\omega t)\right)\end{aligned} \tag{3.20}$$

Now we can transform this solution in spacial variable into the momentum space using Fourier trasform over the considering confined space $A = L_x L_y$.

$$\begin{aligned}\phi_\alpha(k_x,k_y,t) =& \int_{-L_y/2}^{L_y/2} dy\ \exp(-ik_y y)\left[\frac{1}{\sqrt{L_x}}\chi_n\left(y - y_0 - \frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}\right)\exp\left(\frac{i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y\right)\right]\\ &\times\int_{-L_x/2}^{L_x/2} dx\ \exp(-ik_x x)\left[\exp\left(\frac{ip_x}{\hbar}x\right)\right]\\ &\times\exp\left(\frac{-i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y_0\right)\times\exp\left(\frac{-i}{\hbar}\left[\frac{(eE)^2\left(\omega_0^2+\omega^2\right)}{8\omega m_e(\omega_0^2-\omega^2)^2}\right]\sin(2\omega t)\right)\end{aligned} \tag{3.21}$$

Then this can be re-write as follows

$$\phi_\alpha(k_x,k_y,t) = \exp\left(\frac{-i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y_0\right)\exp\left(\frac{-i}{\hbar}\left[\frac{(eE)^2\left(\omega_0^2+\omega^2\right)}{8\omega m_e(\omega_0^2-\omega^2)^2}\right]\sin(2\omega t)\right)\Theta_\alpha(k_y,t)\delta_{k_x,\frac{p_x}{\hbar}} \tag{3.22}$$

where we used

$$\int_{L_x} dx\ \exp\left(-ik_x x + \frac{ip_x}{\hbar}x\right) = L_x\delta_{k_x,\frac{p_x}{\hbar}} \tag{3.23}$$

and

$$\Theta_\alpha(k_y,t) \equiv \int_{-L_y/2}^{L_y/2} dy\ \exp(-ik_y y)\left[\sqrt{L_x}\chi_n\left(y - y_0 - \frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}\right)\exp\left(\frac{i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y\right)\right] \tag{3.24}$$

and this can be simplied as

$$\Theta_\alpha(k_y,t) = \sqrt{L_x}\int_{-L_y/2}^{L_y/2} dy\ \chi_n\left(y - y_0 - \frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}\right)\exp\left(-ik_y y + \frac{i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y\right). \tag{3.25}$$

Then by defining

$$\mu(t) \equiv \frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)} + y_0 \tag{3.26}$$

and

$$\gamma(t) \equiv \frac{eE\omega_0^2\cos(\omega t)}{\hbar\omega(\omega_0^2-\omega^2)} \tag{3.27}$$

we can re-write this by neglecting time dependencies as

$$\Theta_\alpha(k_y,t) = \sqrt{L_x}\int_{-\infty}^{\infty} dy\, \chi_n(y-\mu)\exp(-i(k_y-\gamma)y). \tag{3.28}$$

We can subtitute following variables

$$k_y{}' = k_y - \gamma \quad \text{and} \quad y' = y-\mu \tag{3.29}$$

and for $L_y \to \infty$ this leads to

$$\Theta_\alpha(k_y{}',t) = \sqrt{L_x}e^{-ik_y{}'\mu}\int_{-\infty}^{\infty} dy'\, \chi_n(y')\exp\big(-ik_y{}'y'\big) = \sqrt{L_x}e^{-ik_y{}'\mu}\sqrt{\kappa}\int_{-\infty}^{\infty} dy'\, \vartheta_n(\kappa y')\exp\big(-ik_y{}'y'\big) \tag{3.30}$$

We know that $\{\chi_\alpha\}$ are well-known harmonic eigenfunctions(with Gauss-Hermite functions) as given in the Eq. (1.47). However, the equation in (3.30) represents the Fourier transform of the these Gauss-Hermite functions. Due to the symmetric condition [*Ref:E.Celeghini] the Fourier transform of these functions can be represent as

$$\mathcal{FT}[\vartheta_n(\kappa x), x, k] = \frac{i^n}{|\kappa|}\vartheta_n(k/\kappa) \tag{3.31}$$

Therefore

$$\Theta_\alpha(k_y{}',t) = \sqrt{L_x}e^{-ik_y{}'\mu} \times \frac{i^n}{\sqrt{\kappa}}\vartheta_n\left(\frac{k_y{}'}{\kappa}\right) = \sqrt{L_x}e^{-ik_y{}'\mu}\tilde{\chi}_n(k_y{}') \tag{3.32}$$

where

$$\tilde{\chi}_n(k) = \frac{i^n}{\sqrt{2^n n!\sqrt{\pi}}}\left(\frac{1}{\kappa}\right)^{1/2} e^{-\frac{k^2}{2\kappa^2}}\mathcal{H}_\alpha\left(\frac{k}{\kappa}\right). \tag{3.33}$$

Using Eq. (3.32) and Eq. (3.22) we can derive that

$$\begin{aligned}\phi_\alpha(k_y,t) = \exp\left(\frac{-i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y_0\right)\exp\left(\frac{-i}{\hbar}\left[\frac{(eE)^2\big(\omega_0^2+\omega^2\big)}{8\omega m_e(\omega_0^2-\omega^2)^2}\right]\sin(2\omega t)\right)\\ \times\sqrt{L_x}e^{-i(k_y-\gamma)\mu}\tilde{\chi}_n(k_y-\gamma)\end{aligned} \tag{3.34}$$

where we included the k_x dependence into α quantum number using m value and this can be re-write subtituting μ and γ values as follows

$$\begin{aligned}\phi_\alpha(k_y,t) = \sqrt{L_x}\exp\left(\frac{-i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y_0\right)\exp\left(\frac{-i}{\hbar}\left[\frac{(eE)^2\big(\omega_0^2+\omega^2\big)}{8\omega m_e(\omega_0^2-\omega^2)^2}\right]\sin(2\omega t)\right)\\ \times\exp\left(-ik_y\frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}\right)\exp\left(\frac{i}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]\frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}\right)\exp(-ik_y y_0)\\ \times\exp\left(i\frac{1}{\hbar}\left[\frac{eE\omega_0^2\cos(\omega t)}{\omega(\omega_0^2-\omega^2)}\right]y_0\right)\tilde{\chi}_n(k_y-\gamma)\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}\phi_\alpha(k_y,t) = \sqrt{L_x}\exp\left(\frac{i}{\hbar}\left[\frac{(eE)^2\big(3\omega_0^2-\omega^2\big)}{8\omega m_e(\omega_0^2-\omega^2)^2}\right]\sin(2\omega t)\right)\\ \times\exp\left(-ik_y\left[\frac{eE\sin(\omega t)}{m_e(\omega_0^2-\omega^2)}+y_0\right]\right)\tilde{\chi}_n(k_y-\gamma).\end{aligned} \tag{3.36}$$

For notation convinient we can introduce few constant as follows

$$b \equiv \frac{(eE)^2\big(3\omega_0^2-\omega^2\big)}{8\hbar\omega m_e(\omega_0^2-\omega^2)^2} \tag{3.37}$$

and

$$d \equiv \frac{eE}{m_e(\omega_0^2-\omega^2)} \tag{3.38}$$

with

$$g \equiv \frac{eE\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)}. \tag{3.39}$$

Therefore we can write Eq. (3.36) as

$$\phi_\alpha(k_y, t) = \sqrt{L_x} e^{ib\sin(2\omega t)} e^{-ik_y[d\sin(\omega t)+y_0]} \tilde{\chi}_n(k_y - g\cos(\omega t)). \tag{3.40}$$

4 Floquet Fermi Goldern Rule

In this section we are going to derive the Floquet Fermi goldern rule for above derived quantum Floquet states using $t - t'$ formalism.

The Floquet states (3.17) fullfills the $t - t'$ Schrödinger equation [*Ref:myReport] as follows

$$i\hbar \frac{\partial}{\partial t} |\psi_\alpha(t,t')\rangle = H_F(t') |\psi_\alpha(t,t')\rangle \tag{4.1}$$

where Floquet Hamiltonian given by

$$H_F(t') \equiv H_e(t) - i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \tag{4.2}$$

and

$$|\psi_\alpha(t,t')\rangle = \exp\left(-\frac{i}{\hbar}\varepsilon_\alpha t\right) |\phi_\alpha(t')\rangle \tag{4.3}$$

Now for the Eq. (4.1) corresponding time evolution operator satisfy the Schrödinger equation

$$U_0(t,t_0;t') = \exp\left(-\frac{i}{\hbar} H_F(t')[t - t_0]\right) \tag{4.4}$$

Consider a time-independent total perturbation $V(\mathbf{r})$ switched on at the reference time $t = t_0$, then Schrödinger equation becomes

$$i\hbar \frac{\partial}{\partial t} |\Psi_\alpha(t,t')\rangle = [H_F(t') + V(\mathbf{r})] |\Psi_\alpha(t,t')\rangle \tag{4.5}$$

and when $t \leq t_0$ both solutions of the Schrödinger equation coincide

$$|\psi_\alpha(t,t')\rangle = |\Psi_\alpha(t,t')\rangle \quad \text{when} \quad t \leq t_0 \tag{4.6}$$

Now, we can introduce the interaction picture representation of the $t - t'$ Floquet state as

$$|\Psi_\alpha(t,t')\rangle_I = U_0^\dagger(t,t_0;t') |\Psi_\alpha(t,t')\rangle \tag{4.7}$$

and the perturbation in the interaction picture will be

$$V_I(\mathbf{r}) = U_0^\dagger(t,t_0;t') V(\mathbf{r}) U_0(t,t_0;t') = V(\mathbf{r}). \tag{4.8}$$

This leads to the Schrödinger eqution in the interction picture

$$i\hbar \frac{\partial}{\partial t} |\Psi_\alpha(t,t')\rangle_I = V_I(\mathbf{r}) |\Psi_\alpha(t,t')\rangle_I \tag{4.9}$$

with the recursive solution

$$|\Psi_\alpha(t,t')\rangle_I = |\Psi_\alpha(t_0,t')\rangle_I + \frac{1}{i\hbar} \int_{t_0}^{t} dt_1 \; V_I(\mathbf{r}) |\Psi_\alpha(t_1,t')\rangle_I \tag{4.10}$$

Iterating the solution only upto first order (Born approximation) this leads to

$$|\Psi_\alpha(t,t')\rangle_I \approx |\psi_\alpha(t_0,t')\rangle + \frac{1}{i\hbar} \int_{t_0}^{t} dt_1 \; V_I(\mathbf{r}) |\psi_\alpha(t_0,t')\rangle \tag{4.11}$$

and multiply it by $\langle\psi_\beta(t_0,t')|$ and we will get

$$\langle\psi_\beta(t_0,t')|\Psi_\alpha(t,t')\rangle_I = \langle\psi_\beta(t_0,t')|\psi_\alpha(t_0,t')\rangle + \frac{1}{i\hbar} \int_{t_0}^{t} dt_1 \; \langle\psi_\beta(t_0,t')| V_I(\mathbf{r}) |\psi_\alpha(t_0,t')\rangle \,. \tag{4.12}$$

Then introdusing unitory operator U_0 we can re-write this as

$$\langle\psi_\beta(t_0,t')|U_0^\dagger(t,t_0;t')|\Psi_\alpha(t,t')\rangle = \langle\psi_\beta(t_0,t')|U_0^\dagger(t,t_0;t')U_0(t,t_0;t')|\psi_\alpha(t_0,t')\rangle + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\ \langle\psi_\beta(t_0,t')|\,U_0^\dagger(t_1,t_0;t')V(\mathbf{r})U_0(t_1,t_0;t')\,|\psi_\alpha(t_0,t')\rangle \tag{4.13}$$

and this can be simplied as

$$\langle\psi_\beta(t,t')|\Psi_\alpha(t,t')\rangle = \langle\psi_\beta(t,t')|\psi_\alpha(t,t')\rangle + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\ \langle\psi_\beta(t_1,t')|\,V(\mathbf{r})\,|\psi_\alpha(t_1,t')\rangle\,. \tag{4.14}$$

Since our $t-t'$ Floquet states are orthonormal [*Ref:myReport- t-t' formalism] we can derive that

$$\langle\psi_\beta(t,t')|\Psi_\alpha(t,t')\rangle = \delta_{\alpha\beta}\exp(i\omega[t'-t]) + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\ \langle\psi_\beta(t_1,t')|\,V(\mathbf{r})\,|\psi_\alpha(t_1,t')\rangle\,. \tag{4.15}$$

Now, set $t_0 = 0$ and for a case $\alpha \neq \beta$ where we can represent $\alpha = (n_\alpha, m_\alpha)$ and $\beta = (n_\beta, m_\beta)$ and this will simplied to

$$\langle\psi_\beta(t,t')|\Psi_\alpha(t,t')\rangle = -\frac{i}{\hbar}\int_{0}^{t} dt_1\ \langle\psi_\beta(t_1,t')|\,V(\mathbf{r})\,|\psi_\alpha(t_1,t')\rangle\,. \tag{4.16}$$

In addition, since our Floquet states create a basis for composite space we can represent any solution using our Floquet states

$$|\Psi_\alpha(t,t')\rangle = \sum_\beta a_{\alpha\beta}(t,t')\,|\psi_\beta(t,t')\rangle\,. \tag{4.17}$$

Therefore we can derive a equation for this *scattering amplitude* as

$$a_{\alpha\beta}(t,t') = \langle\psi_\beta(t,t')|\Psi_\alpha(t,t')\rangle = -\frac{i}{\hbar}\int_{0}^{t} dt_1\ \langle\psi_\beta(t_1,t')|\,V(\mathbf{r})\,|\psi_\alpha(t_1,t')\rangle\,. \tag{4.18}$$

Now lets assume a scattering event from a $t-t'$ Floquet state $|\psi_\beta(t,t')\rangle$ into another $t-t'$ Floquet state $|\Psi_\alpha(t,t')\rangle$ with constant quansienergy ε given as follows

$$|\Psi_\alpha(t,t')\rangle = \exp\left(-\frac{i}{\hbar}\varepsilon t\right)|\Phi_\alpha(t')\rangle \tag{4.19}$$

Now consider a scattering event

$$\psi_\beta(\mathbf{k}',t,t') = \exp\left(-\frac{i}{\hbar}\varepsilon_\beta t\right)\phi_\beta(\mathbf{k}',t') \longrightarrow \Psi_\alpha(\mathbf{k},t,t') = \exp\left(-\frac{i}{\hbar}\varepsilon t\right)\Phi_\alpha(\mathbf{k},t') \tag{4.20}$$

Here we need to undestand a state of this considering system only be represented by two independent quantum numbers which are n energy eigen states and m quantum number which represents the qunatized momentum in x direction values. Lets calculate the scattering amplitutde of the above mentioned scattering scenario using the equation derived in (4.18).

$$\begin{aligned} a_{\alpha\beta}(\mathbf{k}',\mathbf{k},t,t') &= -\frac{i}{\hbar}\int_{0}^{t} dt_1\ \langle\psi_{\beta,\mathbf{k}'}(t_1,t')|\,V(\mathbf{r})\,|\psi_{\alpha,\mathbf{k}}(t_1,t')\rangle \\ &= -\frac{i}{\hbar}\int_{0}^{t} dt_1\ e^{\frac{i}{\hbar}(\varepsilon_\beta-\varepsilon)t_1}\,\langle\phi_{\beta,\mathbf{k}'}(t')|\,V(\mathbf{r})\,|\phi_{\alpha,\mathbf{k}}(t')\rangle \end{aligned} \tag{4.21}$$

Next assuimg this scenario for long time $t \to \infty$ we can turn this integral into a delta distrubution as follows

$$\begin{aligned} a_{\alpha\beta}(\mathbf{k}',\mathbf{k},t,t') &= -\frac{i}{\hbar}\lim_{t\to\infty}\left[\int_{-t/2}^{t/2} dt_1\ e^{\frac{i}{\hbar}(\varepsilon_\beta-\varepsilon)t_1}\,\langle\phi_{\beta,\mathbf{k}'}(t')|\,V(\mathbf{r})\,|\phi_{\alpha,\mathbf{k}}(t')\rangle\right] \\ &= -2\pi i\delta(\varepsilon_\beta-\varepsilon)\,\langle\phi_{\beta,\mathbf{k}'}(t')|\,V(\mathbf{r})\,|\phi_{\alpha,\mathbf{k}}(t')\rangle \end{aligned} \tag{4.22}$$

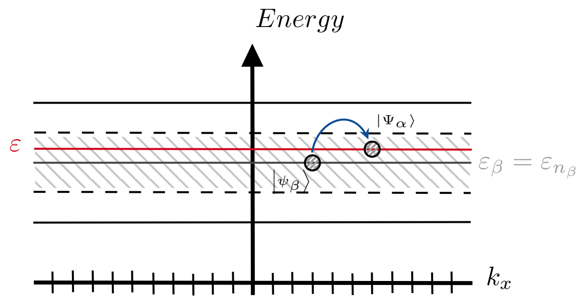

Figure 5: Scattering from $|\psi_\beta(t,t')\rangle$ to constant energy state $|\Psi_\alpha(t,t')\rangle$ due to scattering potential created by impurities.

Now lets consider about the inner product of the above derivation. Using completeness properties we can write that as follows

$$\begin{aligned} Q &\equiv \langle\phi_{\beta,\mathbf{k}'}(t')|\, V(\mathbf{r})\, |\phi_{\alpha,\mathbf{k}}(t')\rangle \\ &= \sum_{\mathbf{k}}\sum_{\mathbf{k}'} \langle\phi_{\beta,\mathbf{k}'}(t')|\mathbf{k}'\rangle \, \langle\mathbf{k}'|V(\mathbf{r})|\mathbf{k}\rangle \, \langle\mathbf{k}|\phi_{\alpha,\mathbf{k}}(t')\rangle \end{aligned} \tag{4.23}$$

and seperating x and y directional momentums we can modify this as follows (Assuming $L_y \to \infty$) and then using $\frac{1}{L_y}\sum_{k_y} = \frac{1}{2\pi}\int k_y$

$$\begin{aligned} Q &\equiv \langle\phi_{\beta,\mathbf{k}'}(t')|\, V(\mathbf{r})\, |\phi_{\alpha,\mathbf{k}}(t')\rangle \\ &= \frac{L_y^2}{4\pi^2}\sum_{k_x}\sum_{k'_x}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_y dk'_y \; \phi_\beta(\mathbf{k}',t')\, \langle\mathbf{k}'|V(\mathbf{r})|\mathbf{k}\rangle \, \phi_\alpha(\mathbf{k},t'). \end{aligned} \tag{4.24}$$

For a random white scattering potential we can represent the inner product of scattering potential with momentum as a constant value as

$$V_{\mathbf{k}',\mathbf{k}} \equiv \langle\mathbf{k}'|V(\mathbf{r})|\mathbf{k}\rangle\,. \tag{4.25}$$

In this study, the perturbation potential is assumed to be formed by an ensemble of randomly distributed impurities, since randomimpurities iin a disorded metal is a better approximation for experimental results.
Consider N_{imp} identical impurities positioned at the randomly distributed but fixed potions $\mathbf{r}_i$. The elastic scattering potential $V(\mathbf{r})$ is then given by the sum over uncorrelated single impurity potentials $\upsilon(\mathbf{r})$

$$V(\mathbf{r}) \equiv \sum_{i=1}^{N_{imp}} \upsilon(\mathbf{r}-\mathbf{r}_i). \tag{4.26}$$

Now assume that the perturbation $V(\mathbf{r})$ is a Gaussian random potential where one can choose the zero of enerrgy such that the potential is zero on average. This model characterized by [*Ref: e.Akkermans G. Montambaux]

$$\langle\upsilon(\mathbf{r})\rangle_{imp} = 0 \tag{4.27}$$

$$\langle\upsilon(\mathbf{r})\upsilon(\mathbf{r}')\rangle_{imp} = \Upsilon(\mathbf{r}-\mathbf{r}') \tag{4.28}$$

where $\langle\cdot\rangle_{imp}$ denoted the average over realizations of the impurity disorder. In addition, this model assume that $\upsilon(\mathbf{r}-\mathbf{r}')$ only dependes on the position difference $|\mathbf{r}-\mathbf{r}'|$ and it decays with a characteristic leangth r_c. Since the study considers the case where the waveleagth of radiation or scattering electrons is much faster than r_c, it is good approximation to make two-point correlation function to be

$$\langle\upsilon(\mathbf{r})\upsilon(\mathbf{r}')\rangle_{imp} = \Upsilon^2_{imp}\delta(\mathbf{r}-\mathbf{r}') \tag{4.29}$$

and a random potential $V(\mathbf{r})$ with this property is called white noise [*Ref: e.Akkermans G. Montambaux]. Then we can choose approximately total scattering potential as

$$V(\mathbf{r}) = \sum_{i=1}^{N_{imp}} \Upsilon_{imp}\delta(\mathbf{r}-\mathbf{r}_i). \tag{4.30}$$

Now we can calculate the Eq. (4.25) using this assumption as follows

$$\begin{aligned} V_{\mathbf{k}',\mathbf{k}} &= \langle \mathbf{k}'|V(\mathbf{r})|\mathbf{k}\rangle \\ &= \left\langle \mathbf{k}' \middle| \sum_{i=1}^{N_{imp}} \Upsilon_{imp}\delta(\mathbf{r}-\mathbf{r}_i) \middle| \mathbf{k} \right\rangle \\ &= \left\langle \mathbf{k}' \middle| \sum_{i=1}^{N_{imp}} \Upsilon_{imp}\delta(x-x_i)\delta(y-y_i) \middle| \mathbf{k} \right\rangle \\ &= \left\langle k'_x \middle| \sum_{x_i=1}^{N_{imp}} \Upsilon_{imp}\delta(x-x_i) \middle| k_x \right\rangle \sum_{y_i=1}^{N_{imp}} \int_{-\infty}^{\infty} dy\; e^{ik'_y y}\delta(y-y_i)e^{-ik_y y} \\ &= \left\langle k'_x \middle| \sum_{x_i=1}^{N_{imp}} \Upsilon_{imp}\delta(x-x_i) \middle| k_x \right\rangle \sum_{y_i=1}^{N_{imp}} e^{i\left(k'_y-k_y\right)y_i} \end{aligned} \tag{4.31}$$

Assuming the total umber of scatterers N_{imp} is macroscopically large we can achieve following expression

$$\begin{aligned} V_{\mathbf{k}',\mathbf{k}} &= \left\langle k'_x \middle| \sum_{x_i=1}^{N_{imp}} \Upsilon_{imp}\delta(x-x_i) \middle| k_x \right\rangle \frac{N_{imp}}{L_y} \int_{-\infty}^{\infty} dy_i\; e^{i\left(k'_y-k_y\right)y_i} \\ &= \frac{N_{imp}}{L_y} V_{k'_x,k_x}\delta(k'_y-k_y) \end{aligned} \tag{4.32}$$

where

$$V_{k'_x,k_x} \equiv \left\langle k'_x \middle| \sum_{x_i=1}^{N_{imp}} \Upsilon_{imp}\delta(x-x_i) \middle| k_x \right\rangle \tag{4.33}$$

Therefore, using the Eq. (3.36), the Eq. (4.24) modified to (we can change varable $t' \to t$)

$$\begin{aligned} Q = \sum_{k_x}\sum_{k'_x} &\frac{N_{imp}L_y V_{k'_x,k_x}}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_y dk'_y\; \delta(k'_y-k_y) \\ &\times \sqrt{L_x}\exp(-ib\sin(2\omega t))\exp(ik'_y[d\sin(\omega t)+y'_0])\tilde{\chi}_{n_\beta}(k'_y-g\cos(\omega t)) \\ &\times \sqrt{L_x}\exp(ib\sin(2\omega t))\exp(-ik_y[d\sin(\omega t)+y_0])\tilde{\chi}_{n_\alpha}(k_y-g\cos(\omega t)) \end{aligned} \tag{4.34}$$

and we can simplify this as

$$\begin{aligned} Q = \sum_{k_x}\sum_{k'_x} &\frac{N_{imp}L_xL_y V_{k'_x,k_x}}{4\pi^2} \int_{-\infty}^{\infty} dk_y \\ &\times \exp(ik_y y'_0)\tilde{\chi}_{n_\beta}(k_y-g\cos(\omega t))\exp(-ik_y y_0)\tilde{\chi}_{n_\alpha}(k_y-g\cos(\omega t)) \end{aligned} \tag{4.35}$$

and this can re-write as

$$Q = \sum_{k_x}\sum_{k'_x} \frac{N_{imp}L_xL_y V_{k'_x,k_x}}{4\pi^2} I \tag{4.36}$$

where

$$I \equiv \int_{-\infty}^{\infty} dk_y\; \tilde{\chi}_{n_\beta}(k_y-g\cos(\omega t))\tilde{\chi}_{n_\alpha}(k_y-g\cos(\omega t))\exp(-ik_y[y_0-y'_0]). \tag{4.37}$$

To avoid the energy tramision from external high-frequency field and electrons in the system, the applied radiation should be purely dressing field. Therefore, the only effect of the dressing field on 2DEG is the renormalization of the probability of elastic electron scattering within same Landau level ($n_\alpha = n_\beta$). Therefore Eq. (4.37) can be modified to

$$I \equiv \int_{-\infty}^{\infty} dk_y \; \tilde{\chi}^2_{n_\beta}(k_y - g\cos(\omega t)) \exp(-ik_y[y_0 - {y'}_0]). \tag{4.38}$$

Lets consider about this integral and we can calculate it as using the following subtitution. Let

$$k_y - g\cos(\omega t) = \bar{k}_y \longrightarrow dk_y = d\bar{k}_y \tag{4.39}$$

and this leads to

$$I \equiv 2\pi \times \frac{1}{2\pi}\int_{-\infty}^{\infty} d\bar{k}_y \; \tilde{\chi}^2_{n_\alpha}\big(\bar{k}_y\big) \exp\big(-i\big(\bar{k}_y + g\cos(\omega t)\big)(y_0 - {y'}_0)\big). \tag{4.40}$$

Using Fourier transform of Gauss-Hermite functions and convolution theorem we can write this as

$$I \equiv 2\pi \exp(g[{y'}_0 - y_0]\cos(\omega t)) \int_{-\infty}^{\infty} dy \; \chi_{n_\beta}(y)\chi_{n_\beta}(y_0 - {y'}_0 - y). \tag{4.41}$$

Therefore the scatterng amplitude (4.22) will modified to

$$a_{\alpha\beta}({k'}_x, k_x, t) = -2\pi i\delta(\varepsilon_\beta - \varepsilon)\sum_{k_x}\sum_{{k'}_x}\frac{N_{imp}L_xL_yV_{{k'}_x,k_x}}{4\pi^2}I \tag{4.42}$$

Consideirng qunatized momentum given in x direction derived in Eq. (2.4), we can identify the non-zero values for scattering amplitude using following conditions

$${k'}_x = \frac{p_{x_\beta}}{\hbar} = m'\frac{2\pi}{L_x} \quad \text{and} \quad k_x = \frac{p_{x_\alpha}}{\hbar} = m\frac{2\pi}{L_x}. \tag{4.43}$$

Then we can simplfied scattering amplitude for given ${k'}_x$ and k_x as

$$\begin{aligned} a_{\alpha\beta}({k'}_x, k_x, t) = \delta(\varepsilon_\beta - \varepsilon)\left[\frac{-iN_{imp}L_xL_yV_{{k'}_x,k_x}}{2\pi}\right] &\exp(g[{y'}_0 - y_0]\cos(\omega t)) \\ &\times \int_{-\infty}^{\infty} dy \; \chi_{n_\beta}(y)\chi_{n_\beta}(y_0 - {y'}_0 - y) \end{aligned} \tag{4.44}$$

Since this scattering amplitude is time-periodic we can write this as a Fourier series expansion

$$a_{\alpha\beta}({k'}_x, k_x, t) = \sum_{l=-\infty}^{\infty} a^l_{\alpha\beta}({k'}_x, k_x)e^{-il\omega t}. \tag{4.45}$$

In addition, using Jacobi-Anger expansion

$$e^{iz\cos(\theta)} = \sum_{l=-\infty}^{\infty} i^l J_l(z)e^{-il\theta} \tag{4.46}$$

we can re-write the Eq.(4.44) as folllows

$$\begin{aligned} a_{\alpha\beta}({k'}_x, k_x, t) = \delta(\varepsilon_\beta - \varepsilon)\left[\frac{-iN_{imp}L_xL_yV_{{k'}_x,k_x}}{2\pi}\right] &\sum_{l=-\infty}^{\infty} i^l J_l(g[{y'}_0 - y_0])e^{-il\omega t} \\ &\times \int_{-\infty}^{\infty} dy \; \chi_{n_\beta}(y)\chi_{n_\beta}(y_0 - {y'}_0 - y) \end{aligned} \tag{4.47}$$

$$\begin{aligned} a_{\alpha\beta}({k'}_x, k_x, t) = \sum_{l=-\infty}^{\infty} \delta(\varepsilon_\beta - \varepsilon)\left[\frac{-i^{l+1}N_{imp}L_xL_yV_{{k'}_x,k_x}}{2\pi}\right] &J_l(g[{y'}_0 - y_0]) \\ \times \int_{-\infty}^{\infty} dy \; \chi_{n_\beta}(y)\chi_{n_\beta}(y_0 - {y'}_0 - y)&e^{-il\omega t} \end{aligned} \tag{4.48}$$

Then we can identified the Fourier series component as

$$a^l_{\alpha\beta}(k'_x,k_x) = \delta(\varepsilon_\beta-\varepsilon)\left[\frac{-i^{l+1}N_{imp}L_xL_yV_{k'_x,k_x}}{2\pi}\right]J_l(g[y'_0-y_0])\int_{-\infty}^{\infty}dy\ \chi_{n_\beta}(y)\chi_{n_\beta}(y_0-y'_0-y) \tag{4.49}$$

Now one can introduce the definition of the *transition probability matrix* as

$$(A_{\alpha\beta}(k'_x,k_x))_{l,l'} \equiv a^l_{\alpha\beta}(k'_x,k_x)\left[a^{l'}_{\alpha\beta}(k'_x,k_x)\right]^* \tag{4.50}$$

and this becomes

$$\begin{aligned}(A_{\alpha\beta}(k'_x,k_x))_{l,l'} = &\left[\frac{N^2_{imp}A^2|V_{k'_x,k_x}|^2}{4\pi^2}\right]J_l(g[y'_0-y_0])J_{l'}(g[y'_0-y_0])\delta^2(\varepsilon_\beta-\varepsilon)\\ &\times\int_{-\infty}^{\infty}dy\ \chi_{n_\beta}(y)\chi_{n_\beta}(y_0-y'_0-y)\int_{-\infty}^{\infty}d\bar{y}\ \chi_{n_\beta}(\bar{y})\chi_{n_\beta}(y_0-y'_0-\bar{y}).\end{aligned} \tag{4.51}$$

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We can reduce these intragal into one variable and derive

$$\begin{aligned}(A_{\alpha\beta}(k'_x,k_x))_{l,l'} = &\left[\frac{N^2_{imp}A^2|V_{k'_x,k_x}|^2}{4\pi^2}\right]J_l(g[y'_0-y_0])J_{l'}(g[y'_0-y_0])\delta^2(\varepsilon_\beta-\varepsilon)\\ &\times\left|\int_{-\infty}^{\infty}dy\ \chi_{n_\beta}(y)\chi_{n_\beta}(y_0-y'_0-y)\right|^2.\end{aligned} \tag{4.52}$$

Then desribing the square of the delta distribution using following procedure

$$\delta^2(\varepsilon) = \delta(\varepsilon)\delta(0) = \frac{\delta(\varepsilon)}{2\pi\hbar}\int_{-t/2}^{t/2}e^{i0\times t'/\hbar}dt' = \frac{\delta(\varepsilon)t}{2\pi\hbar} \tag{4.53}$$

one can modify our derivation in Eq. (4.51) as

$$\begin{aligned}(A_{\alpha\beta}(k'_x,k_x))_{l,l'} = &\left[\frac{N^2_{imp}A^2|V_{k'_x,k_x}|^2}{4\pi^2}\right]J_l(g[y'_0-y_0])J_{l'}(g[y'_0-y_0])\delta(\varepsilon_\beta-\varepsilon)\frac{t}{2\pi\hbar}\\ &\times\left|\int_{-\infty}^{\infty}dy\ \chi_{n_\beta}(y)\chi_{n_\beta}(y_0-y'_0-y)\right|^2.\end{aligned} \tag{4.54}$$

Then performing thetime derivation of each matrix element yeild the *transition amplitude matrix* as folllows

$$\begin{aligned}\Gamma^{ll'}_{\alpha\beta}(k'_x,k_x) &\equiv \frac{d(A_{\alpha\beta}(k'_x,k_x))_{l,l'}}{dt}\\ &= \Lambda|V_{k'_x,k_x}|^2\delta(\varepsilon_\beta-\varepsilon)J_l(g[y'_0-y_0])J_{l'}(g[y'_0-y_0])\left|\int_{-\infty}^{\infty}dy\ \chi_{n_\beta}(y)\chi_{n_\beta}(y_0-y'_0-y)\right|^2\end{aligned} \tag{4.55}$$

where

$$\Lambda \equiv \frac{N^2_{imp}A^2}{8\pi^3\hbar} \tag{4.56}$$

Now using defintion of y_0 given in Eq. (1.11) we can write that

$$y_0-y'_0 = -\frac{p_{x_\alpha}}{eB}+\frac{p_{x_\beta}}{eB} = \frac{\hbar k'_x}{eB}-\frac{\hbar k_x}{eB} = \frac{\hbar}{eB}[k'_x-k_x] \tag{4.57}$$

and this leads Eq. (4.56) to

$$\begin{aligned}\Gamma^{ll'}_{\alpha\beta}(k'_x,k_x) = \Lambda|V_{k'_x,k_x}|^2\delta(\varepsilon_\beta-\varepsilon)&J_l\left(\frac{g\hbar}{eB}[k_x-k'_x]\right)J_{l'}\left(\frac{g\hbar}{eB}[k_x-k'_x]\right)\\ &\times\left|\int_{-\infty}^{\infty}dy\ \chi_{n_\beta}(y)\chi_{n_\beta}\left(\frac{\hbar}{eB}[k'_x-k_x]-y\right)\right|^2\end{aligned} \tag{4.58}$$

An impurity average of white noise potential allows to identify $\left\langle |V_{k'_x,k_x}|^2 \right\rangle = V_{imp}$ and the inverse scattering time matrix is the sum over all momentum over the transition probability matrix

$$\left(\frac{1}{\tau(\varepsilon,k_x)}\right)^{ll'}_{\alpha\beta} \equiv \frac{1}{L_x}\sum_{k'_x} \langle \Gamma^{ll'}_{\alpha\beta}(k'_x,k_x)\rangle_{imp} \tag{4.59}$$

and this implies

$$\begin{aligned}\Gamma^{ll'}_{\alpha\beta}(k'_x,k_x) = \frac{\Lambda V_{imp}}{L_x}\sum_{k'_x}\delta(\varepsilon_\beta-\varepsilon)J_l\left(\frac{g\hbar}{eB}[k_x-k'_x]\right)J_{l'}\left(\frac{g\hbar}{eB}[k_x-k'_x]\right)\\ \times\left|\int_{-\infty}^{\infty} dy\ \chi_{n_\beta}(y)\chi_{n_\beta}\left(\frac{\hbar}{eB}[k'_x-k_x]-y\right)\right|^2\end{aligned} \tag{4.60}$$

For the 1-dimentional case introduce the momentum continuum limit as folllows

$$\frac{1}{L_x}\sum_{k'_x} \longrightarrow \frac{1}{2\pi}\int dk'_x \tag{4.61}$$

and this leads to

$$\begin{aligned}\left(\frac{1}{\tau(\varepsilon,k_x)}\right)^{ll'}_{\alpha\beta} = \frac{\Lambda V_{imp}}{2\pi}\delta(\varepsilon_\beta-\varepsilon)\int_{-\infty}^{\infty} dk'_x J_l\left(\frac{g\hbar}{eB}[k_x-k'_x]\right)J_{l'}\left(\frac{g\hbar}{eB}[k_x-k'_x]\right)\\ \times\left|\int_{-\infty}^{\infty} dy\ \chi_{n_\beta}(y)\chi_{n_\beta}\left(\frac{\hbar}{eB}[k'_x-k_x]-y\right)\right|^2\end{aligned} \tag{4.62}$$

Using following subtitution

$$y = \frac{\hbar\bar{k}}{eB} \longrightarrow dy = \frac{\hbar}{eB}d\bar{k} \tag{4.63}$$

we can modify above derivation as

$$\begin{aligned}\left(\frac{1}{\tau(\varepsilon,k_x)}\right)^{ll'}_{\alpha\beta} = \frac{\Lambda V_{imp}}{2\pi}\delta(\varepsilon_\beta-\varepsilon)\int_{-\infty}^{\infty} dk'_x J_l\left(\frac{g\hbar}{eB}[k_x-k'_x]\right)J_{l'}\left(\frac{g\hbar}{eB}[k_x-k'_x]\right)\\ \times\left(\frac{\hbar}{eB}\right)^2\left|\int_{-\infty}^{\infty} d\bar{k}\ \chi_{n_\beta}\left(\frac{\hbar}{eB}\bar{k}\right)\chi_{n_\beta}\left(\frac{\hbar}{eB}\left[k'_x-k_x-\bar{k}\right]\right)\right|^2,\end{aligned} \tag{4.64}$$

and finally we can derive our expression for the *inverse scattering time matrix* for Nth Landau level (let $n_\alpha = n_\beta = N$)

$$\begin{aligned}\left(\frac{1}{\tau(\varepsilon,k_x)}\right)^{ll'}_{N} = \frac{N^2_{imp}A^2\hbar V_{imp}}{16\pi^4(eB)^2}\delta(\varepsilon-\varepsilon_N)\int_{-\infty}^{\infty} dk'_x J_l\left(\frac{g\hbar}{eB}[k_x-k'_x]\right)J_{l'}\left(\frac{g\hbar}{eB}[k_x-k'_x]\right)\\ \times\left|\int_{-\infty}^{\infty} d\bar{k}\ \chi_{N}\left(\frac{\hbar}{eB}\bar{k}\right)\chi_{N}\left(\frac{\hbar}{eB}\left[k'_x-k_x-\bar{k}\right]\right)\right|^2.\end{aligned} \tag{4.65}$$

5 Inverse Scattering Time Analysis

We hace derived the inverse scattering time matrix element from previous section as follows

$$\left(\frac{1}{\tau(\varepsilon,k_x)}\right)_N^{ll'} = \frac{N_{imp}^2 A^2 \hbar V_{imp}}{16\pi^4 (eB)^2}\delta(\varepsilon-\varepsilon_N)\int_{-\infty}^{\infty} dk'_x J_l\left(\frac{g\hbar}{eB}[k_x-k'_x]\right)J_{l'}\left(\frac{g\hbar}{eB}[k_x-k'_x]\right) \times \left|\int_{-\infty}^{\infty} d\bar{k}\,\chi_N\left(\frac{\hbar}{eB}\bar{k}\right)\chi_N\left(\frac{\hbar}{eB}\left[k'_x-k_x-\bar{k}\right]\right)\right|^2. \tag{5.1}$$

The disorder in the system is not supposed to change the eigenenergies of the bare system, hence all off0diogonal elements of the self-energy were neglected. Thesefore we can consider only the diagonal elements of the inverse scattering time matrix

$$\left(\frac{1}{\tau(\varepsilon,k_x)}\right)_N^{ll} = \frac{N_{imp}^2 A^2 \hbar V_{imp}}{16\pi^4 (eB)^2}\delta(\varepsilon-\varepsilon_N)\int_{-\infty}^{\infty} dk'_x\ J_l^2\left(\frac{g\hbar}{eB}[k_x-k'_x]\right) \times \left|\int_{-\infty}^{\infty} d\bar{k}\,\chi_N\left(\frac{\hbar}{eB}\bar{k}\right)\chi_N\left(\frac{\hbar}{eB}\left[k'_x-k_x-\bar{k}\right]\right)\right|^2. \tag{5.2}$$

Lets consider how this expression change when we have turn off the dressing field ($E=0$). Thesefore the inverse scattering time becomes valid for only $l=0$

$$\left.\left(\frac{1}{\tau(\varepsilon,k_x)}\right)_N^{00}\right|_{E=0} = \frac{N_{imp}^2 A^2 \hbar V_{imp}}{16\pi^4 (eB)^2}\delta(\varepsilon-\varepsilon_N)\int_{-\infty}^{\infty} dk'_x \left|\int_{-\infty}^{\infty} d\bar{k}\,\chi_N\left(\frac{\hbar}{eB}\bar{k}\right)\chi_N\left(\frac{\hbar}{eB}\left[k'_x-k_x-\bar{k}\right]\right)\right|^2. \tag{5.3}$$

Thesefore we can analyze the behaviour of the inverse scattering time with $l=0$ central element of the matrix.

$$\Lambda_{00} \equiv \frac{(1/\tau)_N^{00}}{(1/\tau)_N^{00}\big|_{E=0}} \tag{5.4}$$

and this will be

$$\Lambda_{00}(k_x) = \frac{\int_{-\infty}^{\infty} dk'_x\ J_l^2(g\gamma[k_x-k'_x])\left|\int_{-\infty}^{\infty} d\bar{k}\,\chi_N(\gamma\bar{k})\chi_N(\gamma\left[k'_x-k_x-\bar{k}\right])\right|^2}{\int_{-\infty}^{\infty} dk'_x\left|\int_{-\infty}^{\infty} d\bar{k}\,\chi_N(\gamma\bar{k})\chi_N(\gamma\left[k'_x-k_x-\bar{k}\right])\right|^2} \tag{5.5}$$

where

$$g = \frac{eE\omega_0^2}{\hbar\omega(\omega_0^2-\omega^2)} \qquad \gamma = \frac{\hbar}{eB} \tag{5.6}$$

and

$$\chi_N(y) = \frac{\sqrt{\kappa}}{\sqrt{2^N N!\sqrt{\pi}}}\exp\left(-\frac{\kappa^2 y^2}{2}\right)\mathcal{H}_N(\kappa y) \quad \text{with} \quad \kappa \equiv \sqrt{\frac{m_e\omega_0}{\hbar}}. \tag{5.7}$$

Lets calculate these constants for GaAs-based quantum well with following given system external paramters and physical constants.

External paramter name	**Symbol**	**Value in SI-units**
Average intensity	I	$200\ \mathrm{W/cm^2} = 2\times10^6\ \mathrm{W/m^2}$
Magnetic field	B	1.2 T
Driving frequency	ω	$2\times10^{12}\ \mathrm{rads^{-1}}$
Effective mass	m_e	$0.071\times m = 6.467\times10^{-32}$ kg

Table 1: System external paramter values

Physical constant name	**Symbol**	**Value in SI-units**
Electron charge	e	1.602×10^{-19} C
Electron mass	m	9.109×10^{-31} kg
Reduced Planck's constant	$\hbar$	1.054×10^{-34} kgm^{2}s^{-1}
Speed of light	c	2.998×10^{8} ms^{-1}
Vacuum permittivity	ε_0	8.854×10^{-12} C^{2}s^{2}kg^{-1}m^{-3}

Table 2: Physical constant values in SI-units

Therfore we can calculate following values

$$\omega_0 = \frac{eB}{m_e} = 2.97265 \times 10^{12}\ \mathrm{s}^{-1} \tag{5.8}$$

$$\gamma = \frac{\hbar}{eB} = 5.4851 \times 10^{-16}\ \mathrm{m}^2 \tag{5.9}$$

$$g = \frac{eE\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} = 5.38756 \times 10^{7}\ \mathrm{m}^{-1} \tag{5.10}$$

$$\kappa = \sqrt{\frac{m_e\omega_0}{\hbar}} = 4.2698 \times 10^{7}\ \mathrm{m}^{-1} \tag{5.11}$$

Since

$$g\gamma = 2.95513 \times 10^{-8}\ \mathrm{m} \quad \text{and} \quad \kappa\gamma = 2.34202 \times 10^{-8}\ \mathrm{m} \tag{5.12}$$

we can choose our integral dummy variables ${k'}_x$, $\bar{k}$ and momentum variable k_x are in one range as follows

$${k'}_x, \bar{k}, k_x \approx 10^{-8}\ \mathrm{m}^{-1} \tag{5.13}$$

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6 Floquet Conductivity in Landau Levels

Now we can derive conductivity expression for a given Landau level using Floquet conductivity expression derived in [*Ref:my report 2.488]. Before that let's consider the inverse scattering time matrix element in the previous section. From Eq. (5.3) we can express the Nth Landau level's inverse scattering time central element ($n = n' = 0$) as

$$\left(\frac{1}{\tau(\varepsilon,k_x)}\right)^{00}_{N} = \frac{N_{imp}^2 A^2 \hbar V_{imp}}{16\pi^4 (eB)^2}\delta(\varepsilon-\varepsilon_N)\int_{-\infty}^{\infty} dk'_x\; J_0^2\left(\frac{g\hbar}{eB}[k_x - k'_x]\right) \times \left|\int_{-\infty}^{\infty} d\bar{k}\;\chi_N\left(\frac{\hbar}{eB}\bar{k}\right)\chi_N\left(\frac{\hbar}{eB}\left[k'_x - k_x - \bar{k}\right]\right)\right|^2. \tag{6.1}$$

Now we can introduce a new parameter with physical meaning of scatterin-induced broading of the Landau level as follows

$$\Gamma_N^{00}(\varepsilon,k_x) \equiv \hbar\left(\frac{1}{\tau(\varepsilon,k_x)}\right)^{00}_{N} \tag{6.2}$$

and this modify our previous expressing as

$$\Gamma_N^{00}(\varepsilon,k_x) = \frac{N_{imp}^2 A^2 \hbar V_{imp}}{16\pi^4 (eB)^2}\delta(\varepsilon-\varepsilon_N)\int_{-\infty}^{\infty} dk'_x\; J_0^2\left(\frac{g\hbar}{eB}[k_x - k'_x]\right) \times \left|\int_{-\infty}^{\infty} d\bar{k}\;\chi_N\left(\frac{\hbar}{eB}\bar{k}\right)\chi_N\left(\frac{\hbar}{eB}\left[k'_x - k_x - \bar{k}\right]\right)\right|^2. \tag{6.3}$$

In addition, for the case of elastic scattering within the same Landau level, one can present the delta distribution of the energy using the same physical interpretation as follows

$$\delta(\varepsilon-\varepsilon_N) \approx \frac{1}{\pi\Gamma_N^{00}(\varepsilon,k_x)} \tag{6.4}$$

and this leads to

$$\left[\Gamma_N^{00}(\varepsilon,k_x)\right]^2 = \frac{N_{imp}^2 A^2 \hbar V_{imp}}{16\pi^5 (eB)^2}\int_{-\infty}^{\infty} dk'_x\; J_0^2\left(\frac{g\hbar}{eB}[k_x - k'_x]\right)\left|\int_{-\infty}^{\infty} d\bar{k}\;\chi_N\left(\frac{\hbar}{eB}\bar{k}\right)\chi_N\left(\frac{\hbar}{eB}\left[k'_x - k_x - \bar{k}\right]\right)\right|^2. \tag{6.5}$$

and

$$\Gamma_N^{00}(\varepsilon,k_x) = \left[\frac{N_{imp}^2 A^2 \hbar V_{imp}}{16\pi^5 (eB)^2}\int_{-\infty}^{\infty} dk'_x\; J_0^2\left(\frac{g\hbar}{eB}[k_x - k'_x]\right)\left|\int_{-\infty}^{\infty} d\bar{k}\;\chi_N\left(\frac{\hbar}{eB}\bar{k}\right)\chi_N\left(\frac{\hbar}{eB}\left[k'_x - k_x - \bar{k}\right]\right)\right|^2\right]^{-1/2}. \tag{6.6}$$

Using the numerical calculations we can see that the above integral is does not depend on the value of k_x and we can choose any values for k_x. Therefore applying $k_x = 0$ and letting $k'_x \to k_1, \bar{k} \to k_2$ we can modify our equation as

$$\Gamma_N^{00}(\varepsilon,k_x) = \left[\frac{N_{imp}^2 A^2 \hbar V_{imp}}{16\pi^5 (eB)^2}\int_{-\infty}^{\infty} dk_1\; J_0^2\left(\frac{g\hbar}{eB}k_1\right)\left|\int_{-\infty}^{\infty} dk_2\;\chi_N\left(\frac{\hbar}{eB}k_2\right)\chi_N\left(\frac{\hbar}{eB}[k_1 - k_2]\right)\right|^2\right]^{-1/2}. \tag{6.7}$$

Now we can compare the central element of energy level broading for each Landau level using normalized Landau energy broading(inverse scattering time) against applied dressing field's electric field's amplitude (E) as follows

$$\Lambda_N^{00} \equiv \frac{\Gamma_N^{00}(\varepsilon,k_x)}{\Gamma_N^{00}(\varepsilon,k_x)\big|_{E=0}}. \tag{6.8}$$

As you can see in the Fig. 6, when the applied dressing field's intensity is increasing the broading of the Landau energy level decreasing. The effect of this decreasing is depend on the considering Landau level and for higher the level lower the effect.

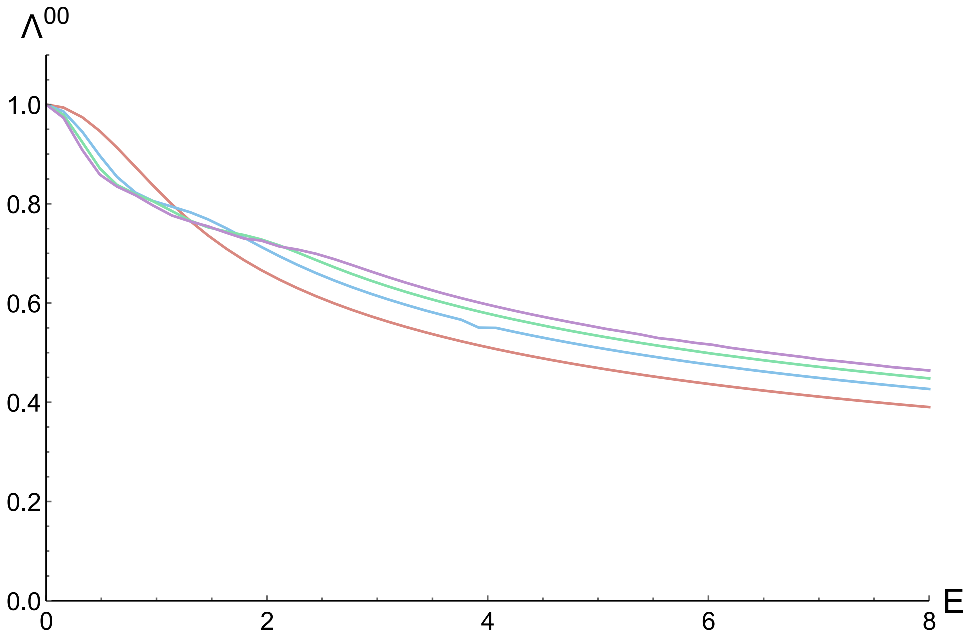

Figure 6: Normalized broading of Landau levels agaist dressing fileds amplitude. Red line represents $N = 0$, blue line represents $N = 1$, green line represents $N = 2$ and purplr color line represents $N = 3$ Landau levels.

Then we can use Floquet conductivity expression derived in [*Ref:my report 2.488] as follows

$$\lim_{\omega\to 0} \mathrm{Re}[\sigma^{xx}(0,\omega)] = \frac{-1}{4\pi\hbar A}\int_{\lambda-\hbar\Omega/2}^{\lambda+\hbar\Omega/2} d\varepsilon\Big(-\frac{\partial f}{\partial \varepsilon}\Big)\frac{1}{A}\sum_{\mathbf{k}} \times \sum_{s,s'=-\infty}^{\infty} j_s^x(\mathbf{k}) j_{s'}^x(\mathbf{k})\,\mathrm{tr}_s\left[\big(\mathbf{G}_0^r(\varepsilon;\mathbf{k})-\mathbf{G}_0^a(\varepsilon;\mathbf{k})\big)\odot_s\big(\mathbf{G}_0^r(\varepsilon;\mathbf{k})-\mathbf{G}_0^a(\varepsilon;\mathbf{k})\big)\right]. \tag{6.9}$$

However, in this case we are consider only x directional momentum as a quantum number to seperate different states and let $\lambda = \varepsilon_N$. In addition, with the current operator derivation IF we get component values only for $s = s' = 0$ our conductivity rquation will be modified to

$$\lim_{\omega\to 0} \mathrm{Re}[\sigma^{xx}(0,\omega)] = \frac{-1}{4\pi\hbar A}\int_{\varepsilon_N-\hbar\Omega/2}^{\varepsilon_N+\hbar\Omega/2} d\varepsilon\Big(-\frac{\partial f}{\partial \varepsilon}\Big)\frac{1}{L_x}\sum_{k_x}[j_0^x(k_x)]^2 \times \mathrm{tr}\left[\big(\mathbf{G}_0^r(\varepsilon;k_x)-\mathbf{G}_0^a(\varepsilon;k_x)\big)\big(\mathbf{G}_0^r(\varepsilon;k_x)-\mathbf{G}_0^a(\varepsilon;k_x)\big)\right]. \tag{6.10}$$

Now we can expand the above expressing using unitary transformation

$$(\mathbf{T})_\alpha^{nn'} \equiv |\phi_\alpha^{n+n'}\rangle \tag{6.11}$$

where

$$|\phi_\alpha(t)\rangle = \sum_{n=-\infty}^{\infty} e^{-in\omega t}\,|\phi_\alpha^n\rangle\,. \tag{6.12}$$

Therefore our conductivity equation becomes

$$\lim_{\omega\to 0} \mathrm{Re}[\sigma^{xx}(0,\omega)] = \frac{-1}{4\pi\hbar A}\int_{\varepsilon_N-\hbar\Omega/2}^{\varepsilon_N+\hbar\Omega/2} d\varepsilon\Big(-\frac{\partial f}{\partial \varepsilon}\Big)\frac{1}{L_x}\sum_{k_x}[j_0^x(k_x)]^2 \times \left[\mathbf{T}^\dagger(k_x)\big(\mathbf{G}_0^r(\varepsilon;k_x)-\mathbf{G}_0^a(\varepsilon;k_x)\big)\mathbf{T})(k_x)\right]_{00}^2. \tag{6.13}$$

As derived in [*Ref:my report 2.547] we can present this matrix multiplication result as follows

$$\begin{aligned}\big[\mathbf{T}^\dagger(k_x)\big(\mathbf{G}_0^r(\varepsilon;k_x)-\mathbf{G}_0^a(\varepsilon;k_x)\big)\mathbf{T})(k_x)\big]_{00}^2 &\approx \big[\mathbf{T}^\dagger(k_x)\big(\mathbf{G}_0^r(\varepsilon;k_x)\mathbf{G}_0^a(\varepsilon;k_x)\big)\mathbf{T})(k_x)\big]_{00} \\ &= \frac{-1}{\left(\frac{\varepsilon}{\hbar}-\frac{\varepsilon_N}{\hbar}\right)^2+\left(\frac{\Gamma_N^{00}(\varepsilon,k_x)}{2\hbar}\right)^2}\end{aligned} \tag{6.14}$$

and this can be more simplified as

$$\big[\mathbf{T}^\dagger(k_x)\big(\mathbf{G}_0^r(\varepsilon;k_x)-\mathbf{G}_0^a(\varepsilon;k_x)\big)\mathbf{T})(k_x)\big]_{00}^2 = \frac{-\hbar^2}{[\Gamma_N^{00}(\varepsilon,k_x)]^2}\left[1+\left(\frac{\varepsilon-\varepsilon_N}{\Gamma_N^{00}(\varepsilon,k_x)/2}\right)^2\right]^{-1}. \tag{6.15}$$

Since the squared termed in the square brackets goinf to zero with valid conditions we can use binomial approximation for the square bracket term and get the following derivation

$$\big[\mathbf{T}^\dagger(k_x)\big(\mathbf{G}_0^r(\varepsilon;k_x)-\mathbf{G}_0^a(\varepsilon;k_x)\big)\mathbf{T})(k_x)\big]_{00}^2 = \frac{-\hbar^2}{[\Gamma_N^{00}(\varepsilon,k_x)]^2}\left[1-4\left(\frac{\varepsilon-\varepsilon_N}{\Gamma_N^{00}(\varepsilon,k_x)}\right)^2\right] \tag{6.16}$$

Now the conductivity expression get modify as

$$\lim_{\omega\to 0}\mathrm{Re}[\sigma^{xx}(0,\omega)] = \frac{\hbar}{4\pi A L_x}\int_{\varepsilon_N-\hbar\Omega/2}^{\varepsilon_N+\hbar\Omega/2} d\varepsilon\left(-\frac{\partial f}{\partial\varepsilon}\right)\sum_{k_x}[j_0^x(k_x)]^2\frac{\hbar^2}{[\Gamma_N^{00}(\varepsilon,k_x)]^2}\left[1-4\left(\frac{\varepsilon-\varepsilon_N}{\Gamma_N^{00}(\varepsilon,k_x)}\right)^2\right] \tag{6.17}$$

Then assuming we are considering fermions in aero temperature for this scenario we can describe the particle distribution function using the Fermi-Dirac distribution in zero-tempurature.

$$-\frac{\partial f}{\partial\varepsilon} = \delta(\varepsilon_F-\varepsilon) \tag{6.18}$$

where ε_F represent the Fermi energy for the considered material.
With this approximation we can derive that

$$\lim_{\omega\to 0}\mathrm{Re}[\sigma^{xx}(0,\omega)] = \frac{\hbar}{4\pi A L_x}\int_{\varepsilon_N-\hbar\Omega/2}^{\varepsilon_N+\hbar\Omega/2} d\varepsilon\,\delta(\varepsilon_F-\varepsilon)\sum_{k_x}[j_0^x(k_x)]^2\frac{\hbar^2}{[\Gamma_N^{00}(\varepsilon,k_x)]^2}\left[1-4\left(\frac{\varepsilon-\varepsilon_N}{\Gamma_N^{00}(\varepsilon,k_x)}\right)^2\right] \tag{6.19}$$

and

$$\lim_{\omega\to 0}\mathrm{Re}[\sigma^{xx}(0,\omega)] = \frac{\hbar}{4\pi A L_x}\sum_{k_x}[j_0^x(k_x)]^2\frac{\hbar^2}{[\Gamma_N^{00}(\varepsilon_F,k_x)]^2}\left[1-4\left(\frac{\varepsilon_F-\varepsilon_N}{\Gamma_N^{00}(\varepsilon_F,k_x)}\right)^2\right] \tag{6.20}$$

■

7 Current Operator in Landau Levels

Now consider about the current density operator for Nth Landau level. Since we have already found the extact solution for our time depenedent Hamiltonian and we have derive them as Floquet states with quesi energies. Using second quantization notaion we can represent our Hamiltonian using creating and anhilation operators for Floquet states which are eigen function of this Hamiltonian.

Let's introduce our creation and anhilation operators in momentum space for the Nth Landau level states as follows

$$c^\dagger_{k_x}\left|0\right\rangle = \left|\psi_{N,k_x}\right\rangle \qquad c_{k_x}\left|\psi_{N,k_x}\right\rangle = \left|0\right\rangle . \tag{7.1}$$

Therefore with second quantization we can represent our Hamiltonian as

$$\hat{H}_N = \sum_{k_x} \varepsilon_N c^\dagger_{k_x} c_{k_x} \tag{7.2}$$

and particle density operator as

$$\hat{\rho}_N = \sum_{k'_x} c^\dagger_{k'_x} c_{k'_x} . \tag{7.3}$$

Now we can find the commutation relationship with each other as

$$\left[\hat{H}_N, \hat{\rho}_N\right] = \left[\sum_{k_x} \varepsilon_N c^\dagger_{k_x} c_{k_x}, \sum_{k'_x} c^\dagger_{k'_x} c_{k'_x}\right] \tag{7.4}$$

and this can be simplified using fermions commutation relationship and one can derive that

$$\left[\hat{H}_N, \hat{\rho}_N\right] = \sum_{k_x} \varepsilon_N c^\dagger_{k_x} c_{k_x} . \tag{7.5}$$

Now using *Liouville-Von Neumann equation* we can derive that

$$\frac{\partial \hat{\rho}_N}{\partial t} = -\frac{i}{\hbar}\left[\hat{H}_N, \hat{\rho}_N\right] = \sum_{k_x} -\frac{i}{\hbar} \varepsilon_N c^\dagger_{k_x} c_{k_x} \tag{7.6}$$

and using famous *continuty equation* we can make relationship with probability current density ($\mathbf{j}(\mathbf{r},t)$) operator as follows

$$\frac{\partial \hat{\rho}_N}{\partial t} = -\boldsymbol{\nabla} \cdot \hat{\mathbf{j}}(\mathbf{r},t). \tag{7.7}$$

However, we can assume that the current flow of this system only can be happed in x direction due to y direction restriction by magnetic leangth. Therefore we can re-write the above equation as follows

$$\frac{\partial \hat{\rho}_N}{\partial t} = -\frac{\partial \hat{jx}^x(\mathbf{r},t)}{\partial x} \tag{7.8}$$

and using this on Eq. (7.6) we can derive that

$$-\frac{\partial \hat{j}^x(\mathbf{r},t)}{\partial x} = \sum_{k_x} -\frac{i}{\hbar} \varepsilon_N c^\dagger_{k_x} c_{k_x} \tag{7.9}$$

and this leads to

$$\hat{j}^x(\mathbf{r},t) = \sum_{k_x} -\frac{iL_x}{\hbar} \varepsilon_N c^\dagger_{k_x} c_{k_x} . \tag{7.10}$$

Therefore we can identify the momentum space component of the current density operator as

$$-j^x(k_x,t) = -\frac{iL_x}{\hbar} \varepsilon_N \tag{7.11}$$

and find the time space fourier series components and this will vanish all the components expect $s = 0$ components because

$$\int dte^{-is\omega t} = 2\pi\delta_{s,0} \tag{7.12}$$

and then we can find the time fourier series components of current operator as follows

$$j^x_{s=0}(k_x, t) = \frac{i2\pi L_x}{\hbar}\varepsilon_N = i2\pi\omega_0 L_x\left(N + \frac{1}{2}\right) \tag{7.13}$$

and for electric current flow we can introfuce the electron's charge as $-e$ and this will be modified to

$$j^x_{s=0}(k_x, t) = -i2\pi e\omega_0 L_x\left(N + \frac{1}{2}\right) \tag{7.14}$$

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Now we can use this in our previously derived conductivity formula (6.20) and get

$$\lim_{\omega\to 0}\text{Re}[\sigma^{xx}(0,\omega)] = \frac{\hbar}{4\pi AL_x}\sum_{k_x}\left[-i2\pi e\omega_0 L_x\left(N+\frac{1}{2}\right)\right]^2 \frac{\hbar^2}{[\Gamma^{00}_N(\varepsilon_F, k_x)]^2}\left[1 - 4\left(\frac{\varepsilon_F - \varepsilon_N}{\Gamma^{00}_N(\varepsilon_F, k_x)}\right)^2\right] \tag{7.15}$$

and this leads to

$$\lim_{\omega\to 0}\text{Re}[\sigma^{xx}(0,\omega)] = \frac{\pi e^2\omega_0^2\hbar}{L_x}\sum_{k_x}\left(N+\frac{1}{2}\right)^2 \frac{\hbar^2}{[\Gamma^{00}_N(\varepsilon_F, k_x)]^2}\left[1 - 4\left(\frac{\varepsilon_F - \varepsilon_N}{\Gamma^{00}_N(\varepsilon_F, k_x)}\right)^2\right]. \tag{7.16}$$

However we know that $\Gamma^{00}_N(\varepsilon_F, k_x)$ is independent of k_x and we can get summation over availble all momentum through the summation. However by the condition that the center of the force of the oscillator y_0 must physically liw within the system $-L_y/2 < y_0 < L_y/2$, one can derive that

$$-\frac{m\omega_0 Ly}{2\hbar} \le k_x \le \frac{m\omega_0 Ly}{2\hbar} \tag{7.17}$$

Therefore the Eq. (7.16) can be simplified to

$$\lim_{\omega\to 0}\text{Re}[\sigma^{xx}(0,\omega)] = \frac{\pi e^2\omega_0^2\hbar}{L_x}\frac{m\omega_0 Ly}{\hbar}\left(N+\frac{1}{2}\right)^2 \frac{\hbar^2}{[\Gamma^{00}_N]^2}\left[1 - 4\left(\frac{\varepsilon_F - \varepsilon_N}{\Gamma^{00}_N}\right)^2\right] \tag{7.18}$$

and this will becomes

$$\lim_{\omega\to 0}\text{Re}[\sigma^{xx}(0,\omega)] = \pi e^2\omega_0 m\frac{(\hbar\omega_0)^2}{[\Gamma^{00}_N]^2}\left(N+\frac{1}{2}\right)^2\left[1 - 4\left(\frac{\varepsilon_F - \varepsilon_N}{\Gamma^{00}_N}\right)^2\right]. \tag{7.19}$$

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