

Magnetic propeties of a two dimentional electron gas strongly coupled to lights

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1 Schrödinger problem for Landau levels in dressed 2DEG

Our analysis start with considering 2 dimentional free electronic gas which has been distrubuted in confined (x, y) plane in configuration space.

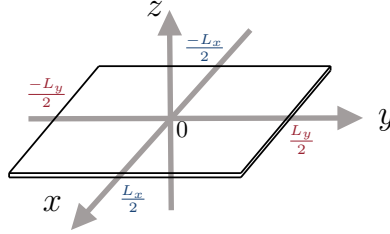


Figure 1: Confined 2DEG in configuration space with the size of $A = L_x L_y$.

We are going to examine the properties of 2DEG with stationary magnetic field

$$\mathbf{B} = (0, 0, B)^T \quad (1.1)$$

which directed on z axis and a linearly y -polarized strong electromagnetic wave (dressing field) with electric field given by

$$\mathbf{E} = (0, E \sin(\omega t), 0)^T \quad (1.2)$$

which also propagate in z direction. Here B and E represent the amplitude of the stationary magnetic field and electric field of dressing field.

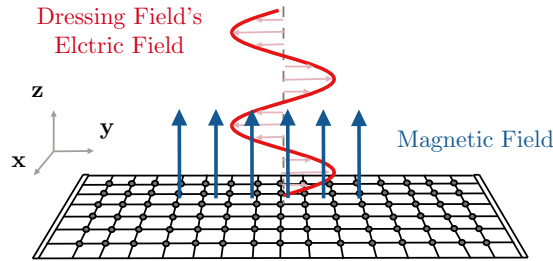


Figure 2: Stationary magnetic filed (blue color) and Strong EM wave (red color) applied to the 2DEG.

Using Landau gauge for the stationary magnetic field we can represent it using vector potential as

$$\mathbf{A}_s = (-By, 0, 0)^T \quad (1.3)$$

and choosing Coulomb gauge the dressing field can be present as the following vector potential

$$\mathbf{A}_d(t) = (0, [E/\omega] \cos(\omega t), 0)^T. \quad (1.4)$$

Now the Hamiltonian of an electron in 2DEG can be reads as

$$\hat{H}_e(t) = \frac{1}{2m_e} \left[\hat{\mathbf{p}} - e(\mathbf{A}_s + \mathbf{A}_d(t)) \right]^2 \quad (1.5)$$

where m_e is the effective mass of the electron and e is the magnitude (without considering the sign of the charge) of the electron charge. This can be simplified to

$$\hat{H}_e(t) = \frac{1}{2m_e} \left[(\hat{p}_x + eBy)\mathbf{e}_x + (\hat{p}_y - \frac{eE}{\omega} \cos(\omega t))\mathbf{e}_y \right]^2 \quad (1.6)$$

where \mathbf{e}_x and \mathbf{e}_y are unit vectors along x and y directions respectively. Moreover,

$$\hat{H}_e(t) = \frac{1}{2m_e} \left[(\hat{p}_x + eBy)^2 + (\hat{p}_y - \frac{eE}{\omega} \cos(\omega t))^2 \right] \quad (1.7)$$

Since $[\hat{H}_e(t), \hat{p}_x] = 0$ both operators share same (simultaneous) eigen functions which are free electron wave functions ($\frac{1}{\sqrt{L_x}} \exp(\frac{ip_x x}{\hbar})$). Therefore we can modify the Hamiltonian as follows

$$\hat{H}_e(t) = \frac{1}{2m_e} \left[(p_x + eBy)^2 + (\hat{p}_y - \frac{eE}{\omega} \cos(\omega t))^2 \right]. \quad (1.8)$$

Using momentum operator definition

$$\hat{p}_y = -i\hbar \frac{\partial}{\partial y} \quad (1.9)$$

we can modify Eq. (1.8) as

$$\begin{aligned} \hat{H}_e(t) &= \frac{1}{2m_e} \left[(p_x + eBy)^2 + \left(-i\hbar \frac{\partial}{\partial y} - \frac{eE}{\omega} \cos(\omega t) \right)^2 \right] \\ &= \frac{1}{2m_e} \left[(p_x + eBy)^2 + \left(i\hbar \frac{\partial}{\partial y} + \frac{eE}{\omega} \cos(\omega t) \right)^2 \right]. \end{aligned} \quad (1.10)$$

Define the *center of the cyclotron orbit* along y axis as

$$y_0 \equiv \frac{-p_x}{eB} \quad (1.11)$$

and the *cyclotron frequency* as

$$\omega_0 \equiv \frac{eB}{m_e}. \quad (1.12)$$

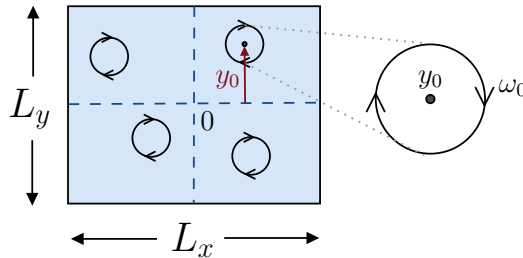


Figure 3: Paramters of the cyclotron orbits in the classical interpretation.

Then the Hamiltonian will leads to

$$\hat{H}_e(t) = \frac{m_e \omega_0^2}{2} (y - y_0)^2 + \frac{1}{2m_e} \left(i\hbar \frac{\partial}{\partial y} + \frac{eE}{\omega} \cos(\omega t) \right)^2 \quad (1.13)$$

$$\begin{aligned}\hat{H}_e(t) = \frac{m_e\omega_0^2}{2}(y-y_0)^2 + \frac{1}{2m_e} \left(-\hbar^2 \frac{\partial^2}{\partial y^2} + i\hbar \frac{\partial}{\partial y} \left[\frac{eE}{\omega} \cos(\omega t) \right] \right. \\ \left. + \frac{i\hbar eE}{\omega} \cos(\omega t) \frac{\partial}{\partial y} + \frac{e^2 E^2}{\omega^2} \cos^2(\omega t) \right)\end{aligned}\quad (1.14)$$

$$\hat{H}_e(t) = \frac{m_e\omega_0^2}{2}(y-y_0)^2 + \frac{1}{2m_e} \left(-\hbar^2 \frac{\partial^2}{\partial y^2} + \frac{2i\hbar eE}{\omega} \cos(\omega t) \frac{\partial}{\partial y} + \frac{e^2 E^2}{\omega^2} \cos^2(\omega t) \right). \quad (1.15)$$

Let

$$\tilde{y} = (y - y_0) \longrightarrow dy = d\tilde{y} \quad (1.16)$$

and then this becomes

$$\hat{H}_e(t) = \frac{m_e\omega_0^2}{2}\tilde{y}^2 + \frac{1}{2m_e} \left(-\hbar^2 \frac{\partial^2}{\partial \tilde{y}^2} + \frac{2i\hbar eE}{\omega} \cos(\omega t) \frac{\partial}{\partial \tilde{y}} + \frac{e^2 E^2}{\omega^2} \cos^2(\omega t) \right). \quad (1.17)$$

Now assume that the solution for the time-dependent schrödinger equation

$$i\hbar \frac{d\psi}{dt} = \hat{H}_e(t)\psi \quad (1.18)$$

can be represent by the following form

$$\psi(\mathbf{r}, t) = \frac{1}{\sqrt{L_x}} \exp \left(\frac{ip_x x}{\hbar} + \frac{ieE(y-y_0)}{\hbar\omega} \cos(\omega t) \right) \phi(y-y_0, t). \quad (1.19)$$

Using the same substitution from Eq. (1.16) this becomes

$$\psi(x, \tilde{y}, t) = \frac{1}{\sqrt{L_x}} \exp \left(\frac{ip_x x}{\hbar} + \frac{ieE\tilde{y}}{\hbar\omega} \cos(\omega t) \right) \phi(\tilde{y}, t). \quad (1.20)$$

Defining

$$\varphi(x, \tilde{y}, t) \equiv \frac{1}{\sqrt{L_x}} \exp \left(\frac{ip_x x}{\hbar} + \frac{ieE\tilde{y}}{\hbar\omega} \cos(\omega t) \right) \quad (1.21)$$

we can simplify the the Eq. (1.20) as

$$\psi(x, \tilde{y}, t) = \varphi(x, \tilde{y}, t) \phi(\tilde{y}, t). \quad (1.22)$$

Let's substitute Eq. (1.20) and Eq. (1.17) into Eq. (1.18) and we can observe that

$$\begin{aligned}\text{L.H.S} = i\hbar \frac{d\psi}{dt} &= i\hbar \left(\frac{d\varphi}{dt} \phi + \varphi \frac{d\phi}{dt} \right) = i\hbar \left(\left[\frac{-ieE\tilde{y}}{\hbar} \sin(\omega t) \right] \varphi \phi + \varphi \frac{d\phi}{dt} \right) \\ &= [eE\tilde{y} \sin(\omega t)] \varphi \phi + i\hbar \varphi \frac{d\phi}{dt}\end{aligned}\quad (1.23)$$

and

$$\begin{aligned}\text{R.H.S} = \hat{H}_e(t)\psi \\ = \left[\frac{m_e\omega_0^2}{2}\tilde{y}^2 + \frac{1}{2m_e} \left(-\hbar^2 \frac{\partial^2}{\partial \tilde{y}^2} + \frac{2i\hbar eE}{\omega} \cos(\omega t) \frac{\partial}{\partial \tilde{y}} + \frac{e^2 E^2}{\omega^2} \cos^2(\omega t) \right) \right] \varphi \phi\end{aligned}\quad (1.24)$$

where we will calculate this part by part as follows:

$$\begin{aligned}\frac{-\hbar^2}{2m_e} \frac{\partial^2}{\partial \tilde{y}^2} (\varphi \phi) &= \frac{-\hbar^2}{2m_e} \frac{\partial}{\partial \tilde{y}} \left[\left(\frac{ieE}{\hbar\omega} \cos(\omega t) \right) \varphi \phi + \varphi \frac{\partial \phi}{\partial \tilde{y}} \right] \\ &= \frac{-\hbar^2}{2m_e} \left[\left(\frac{ieE}{\hbar\omega} \cos(\omega t) \right)^2 \varphi \phi + \left(\frac{ieE}{\hbar\omega} \cos(\omega t) \right) \varphi \frac{\partial \phi}{\partial \tilde{y}} + \left(\frac{ieE}{\hbar\omega} \cos(\omega t) \right) \varphi \frac{\partial \phi}{\partial \tilde{y}} + \varphi \frac{\partial^2 \phi}{\partial \tilde{y}^2} \right] \\ &= \left(\frac{e^2 E^2}{2m_e \omega^2} \cos^2(\omega t) \right) \varphi \phi - \left(\frac{ieE\hbar}{m_e \omega} \cos(\omega t) \right) \varphi \frac{\partial \phi}{\partial \tilde{y}} - \frac{\hbar^2}{2m_e} \varphi \frac{\partial^2 \phi}{\partial \tilde{y}^2}\end{aligned}\quad (1.25)$$

and

$$\begin{aligned} \frac{2i\hbar eE}{2m_e\omega} \cos(\omega t) \frac{\partial}{\partial \tilde{y}} (\varphi\phi) &= \frac{i\hbar eE}{m_e\omega} \cos(\omega t) \left[\left(\frac{ieE}{\hbar\omega} \cos(\omega t) \right) \varphi\phi + \varphi \frac{\partial \phi}{\partial \tilde{y}} \right] \\ &= \left(\frac{-e^2 E^2}{m_e\omega^2} \cos(\omega t) \right) \varphi\phi + \frac{i\hbar eE}{m_e\omega} \cos(\omega t) \varphi \frac{\partial \phi}{\partial \tilde{y}}. \end{aligned} \quad (1.26)$$

Therefore we can derive that

$$\text{R.H.S} = \left[\frac{m_e\omega_0^2}{2} \tilde{y}^2 \varphi\phi - \frac{\hbar^2}{2m_e} \varphi \frac{\partial^2 \phi}{\partial \tilde{y}^2} \right]. \quad (1.27)$$

To satisfy the condition L.H.S=R.H.S we need to find a function $\phi(\tilde{y}, t)$ such that

$$[eE\tilde{y} \sin(\omega t)] \varphi\phi + i\hbar \varphi \frac{d\phi}{dt} = \left[\frac{m_e\omega_0^2}{2} \tilde{y}^2 \varphi\phi - \frac{\hbar^2}{2m_e} \varphi \frac{\partial^2 \phi}{\partial \tilde{y}^2} \right] \quad (1.28)$$

by removing φ this can be simplified as

$$\left[\frac{m_e\omega_0^2}{2} \tilde{y}^2 - eE\tilde{y} \sin(\omega t) - \frac{\hbar^2}{2m_e} \frac{\partial^2}{\partial \tilde{y}^2} - i\hbar \frac{d}{dt} \right] \phi(\tilde{y}, t) = 0. \quad (1.29)$$

If we turn off the external dressing field, this equation leads to simple harmonic oscillator Hamiltonian as follows

$$\left[\frac{m_e\omega_0^2}{2} \tilde{y}^2 - \frac{\hbar^2}{2m_e} \frac{\partial^2}{\partial \tilde{y}^2} - i\hbar \frac{d}{dt} \right] \phi(\tilde{y}, t) = 0 \quad (1.30)$$

$$i\hbar \frac{d\phi(\tilde{y}, t)}{dt} = \left[\frac{\hat{p}_{\tilde{y}}^2}{2m_e} + \frac{1}{2} m_e\omega_0^2 \tilde{y}^2 \right] \phi(\tilde{y}, t). \quad (1.31)$$

Therefore we can identify the $S(t) \equiv eE \sin(\omega t)$ part as a external force act on the harmonic oscillator and we can solve this as a forced harmonic oscillator in \tilde{y} axis.

$$i\hbar \frac{d\phi(\tilde{y}, t)}{dt} = \left[-\frac{\hbar^2}{2m_e} \frac{\partial^2}{\partial \tilde{y}^2} + \frac{1}{2} m_e\omega_0^2 \tilde{y}^2 - \tilde{y}S(t) \right] \phi(\tilde{y}, t). \quad (1.32)$$

This system can be exactly solvable and we can solve this equation using the methods explained by Husimi [*Ref:1] as follows.

First we can introduce the time dependent shifted coordinate as

$$\tilde{y} \rightarrow y' = \tilde{y} - \zeta(t) \quad \Rightarrow \quad \tilde{y} = y' + \zeta(t) \quad (1.33)$$

and this implies that

$$\frac{d\phi(y', t)}{dt} = \frac{\partial \phi(y', t)}{\partial t} + \frac{\partial \phi(y', t)}{\partial y'} \frac{\partial y'}{\partial t} = \frac{\partial \phi(y', t)}{\partial t} - \dot{\zeta}(t) \frac{\partial \phi(y', t)}{\partial y'} \quad (1.34)$$

where $\dot{\zeta}(t) = \frac{\partial \zeta(t)}{\partial t}$. Therefore, Eq. (1.32) will be modified to

$$i\hbar \frac{\partial \phi(y', t)}{\partial t} = \left[i\hbar \dot{\zeta} \frac{\partial}{\partial y'} - \frac{\hbar^2}{2m_e} \frac{\partial^2}{\partial y'^2} + \frac{1}{2} m_e\omega_0^2 (y' + \zeta)^2 - (y' + \zeta)S(t) \right] \phi(y', t). \quad (1.35)$$

Let's transform the wave function using following unitary transform

$$\phi(y', t) = \exp\left(\frac{im_e\dot{\zeta}y'}{\hbar}\right) \varphi(y', t) \quad (1.36)$$

and substitute this into the Eq. (1.35) and we will get the following

$$\text{L.H.S} = \left[i\hbar \frac{\partial}{\partial t} - i\hbar \left(\frac{im_e\dot{\zeta}y'}{\hbar} \right) \right] \exp\left(\frac{-im_e\dot{\zeta}y'}{\hbar}\right) \varphi(y', t) \quad (1.37)$$

and

$$\begin{aligned}
\text{R.H.S} = & \left[i\hbar\dot{\zeta}\left(\frac{im_e\dot{\zeta}}{\hbar}\right) + i\hbar\dot{\zeta}\frac{\partial}{\partial y'} \right. \\
& - \frac{\hbar^2}{2m_e}\left[\left(\frac{im_e\dot{\zeta}}{\hbar}\right)^2 + \left(\frac{2im_e\dot{\zeta}}{\hbar}\right)\frac{\partial}{\partial y'} + \frac{\partial^2}{\partial y'^2}\right] \\
& + \frac{1}{2}m_e\omega_0^2 y'^2 + \frac{1}{2}m_e\omega_0^2\zeta^2 + m_e\omega_0^2 y'\zeta \\
& \left. - y'S(t) - \zeta S(t) \right] \exp\left(\frac{-im_e\dot{\zeta}y'}{\hbar}\right) \varphi(y', t).
\end{aligned} \tag{1.38}$$

Combining these two and removing exponential terms we can derive that

$$\begin{aligned}
i\hbar\frac{\partial\varphi(y', t)}{\partial t} = & \left[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y'^2} + \frac{1}{2}m_e\omega_0^2 y'^2 + [m_e\ddot{\zeta} + m_e\omega_0^2\zeta - S(t)]y' \right. \\
& \left. + \left[-\frac{1}{2}m_e\dot{\zeta}^2 + \frac{1}{2}m_e\omega_0^2\zeta^2 - \zeta S(t) \right] \right] \varphi(y', t).
\end{aligned} \tag{1.39}$$

Then we can restrict our $\zeta(t)$ function such that

$$m_e\ddot{\zeta} + m_e\omega_0^2\zeta = S(t) \tag{1.40}$$

and that leads to

$$i\hbar\frac{\partial\varphi(y', t)}{\partial t} = \left[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y'^2} + \frac{1}{2}m_e\omega_0^2 y'^2 - L(\zeta, \dot{\zeta}, t) \right] \varphi(y', t) \tag{1.41}$$

where

$$L(\zeta, \dot{\zeta}, t) \equiv \frac{1}{2}m_e\dot{\zeta}^2 - \frac{1}{2}m_e\omega_0^2\zeta^2 + \zeta S(t) \tag{1.42}$$

is the largrangian of a classical driven oscillator.

Now introduce new unitary transormation for the wavefunction as follows

$$\varphi(y', t) = \exp\left(\frac{i}{\hbar}\int_0^t dt' L(\zeta, \dot{\zeta}, t')\right) \chi(y', t) \tag{1.43}$$

and subtitte this into the Eq. (1.41) and gets

$$\begin{aligned}
i\hbar\left[\exp\left(\frac{i}{\hbar}\int_0^t dt' L(\zeta, \dot{\zeta}, t')\right)\frac{\partial}{\partial t} + i\hbar L(\zeta, \dot{\zeta}, t)\exp\left(\frac{i}{\hbar}\int_0^t dt' L(\zeta, \dot{\zeta}, t')\right)\right] \chi(y', t) \\
= \left[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y'^2} + \frac{1}{2}m_e\omega_0^2 y'^2 - L(\zeta, \dot{\zeta}, t) \right] \exp\left(\frac{i}{\hbar}\int_0^t dt' L(\zeta, \dot{\zeta}, t')\right) \chi(y', t)
\end{aligned} \tag{1.44}$$

removing exponential terms finally we can derive that

$$i\hbar\frac{\partial}{\partial t}\chi(y', t) = \left[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y'^2} + \frac{1}{2}m_e\omega_0^2 y'^2 \right] \chi(y', t). \tag{1.45}$$

This is the well known Schrodinger equation of a stationary quantum harmonic oscillator. In terms of the eigenvalues

$$E_n = \hbar\omega_0\left(n + \frac{1}{2}\right) \tag{1.46}$$

of well-known harmonic eigenfucntions (using Gauss-Hermite functions ϑ)

$$\chi_n(x) \equiv \sqrt{\kappa}\vartheta(\kappa x) \quad \text{where} \quad \vartheta(x) = \frac{1}{\sqrt{2^n n!}} e^{-x^2/2} \mathcal{H}_n(x) \quad \text{with} \quad \kappa = \sqrt{\frac{m_e\omega_0}{\hbar}} \tag{1.47}$$

being propositional to the Hermite functions \mathcal{H}_n , the solutions of Eq. (1.32) can be represent as

$$\phi_n(\tilde{y}, t) = \chi_n(\tilde{y} - \zeta(t)) \exp\left(\frac{i}{\hbar} \left[-E_n t + m_e \zeta(\dot{t})(\tilde{y} - \zeta(t)) + \int_0^t dt' L(\zeta, \dot{\zeta}, t') \right]\right) \quad (1.48)$$

The set $\{\chi_n(x)\}$ forms a complete set and thus any general solution $\phi(\tilde{y}, t)$ can be expanded in terms of the solutions in Eq. (1.48).

Next we consider special case where we assumed

$$S(t) = eE \sin(\omega t) \quad (1.49)$$

and one can derive the Eq. (1.40) for $\zeta(t)$

$$m_e \ddot{\zeta} + m_e \omega_0^2 \zeta = eE \sin(\omega t) \quad (1.50)$$

and using Green function method the solution can be write as

$$\zeta(t) = \frac{eE}{m_e(\omega_0^2 - \omega^2)} \sin(\omega t). \quad (1.51)$$

form this solutions we are able to derive the final solutions $\alpha = (n, m)$ where $n \in \mathbb{Z}_0^+$ and $m \in \mathbb{Z}$ are two quantum numbers that describe the state of the electron, can be present as

$$\begin{aligned} \psi_\alpha(x, \tilde{y}, t) = & \frac{1}{\sqrt{L_x}} \chi_n(\tilde{y} - \zeta(t)) \\ & \times \exp\left(\frac{i}{\hbar} \left[-E_n t + p_x x + \frac{eE\tilde{y}}{\omega} \cos(\omega t) + m_e \zeta(\dot{t})[\tilde{y} - \zeta(t)] + \int_0^t dt' L(\zeta, \dot{\zeta}, t') \right]\right) \end{aligned} \quad (1.52)$$

and the exponential phase shifts represent the effect done by the stationary magnetic field and strong dressing field. In here p_x is qunatized with the quantum number m due to the spacial confinemet in x direction.

$$p_x = m \frac{2\pi\hbar}{L_x}, \quad m = 0, \pm 1, \pm 2, \dots \quad (1.53)$$

Therefore we can assume that the magnetitransport properties of 2DEG will be renormalized by the magnetic field as well as the dressing field. ■

2 Scattering theory

Since in a real metal there would be many scatters that can behave as obstacles for electron that have free wave functions. Therefore we need to calculate them to analyse the real behaviour of the electrons.

Then the wave function of the electron in a real metal $\Psi(\mathbf{r}, t)$ should satisfy the following time-dependent Schrodinger equation

$$i\hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = [H_e(t) + U(\mathbf{r})]\Psi(\mathbf{r}, t) \quad (2.1)$$

where $U(\mathbf{r})$ is the total scattering potential. We have represented the all scatters using this potential. Since the solutions (1.52) create a complete orthonormal basis we can represent this wave function using those as follows

$$\Psi(\mathbf{r}, t) = \sum_j a_j(t) |\psi_j(t)\rangle \quad (2.2)$$

where the difference indices j corresponding to the different sets of all quantum numbers p_x and n

$$j \rightarrow (m, n) \quad \text{where} \quad m, n = 0, 1, 2, \dots \quad (2.3)$$

with m is defined for quantized momentum in x direction

$$p_x = m \frac{2\pi\hbar}{L_x} \quad (2.4)$$

Now we can use the conventional perturbation theory to calculate scattering process of electron at a state $|\psi_j\rangle$ to a state $|\psi_{j'}\rangle$. For that assume an electron be in the j state at the time $t = 0$ and corresponding $a'_j(0) = \delta_{j,j'}$.

First substitute a general electron state $\Psi(\mathbf{r}, t)$ at time t as the incoming electron to the Schrodinger equation given in Eq. (2.1)

$$i\hbar \frac{\partial}{\partial t} \sum_j a_j(t) |\psi_j(t)\rangle = [H_e(t) + U(\mathbf{r})] \sum_j a_j(t) |\psi_j(t)\rangle \quad (2.5)$$

$$i\hbar \sum_j \dot{a}_j(t) |\psi_j(t)\rangle + a_j(t) \frac{\partial}{\partial t} |\psi_j(t)\rangle = [H_e(t) + U(\mathbf{r})] \sum_j a_j(t) |\psi_j(t)\rangle \quad (2.6)$$

since all the $|\psi(t)\rangle$ satisfy the Schrodinger equation (1.18)

$$i\hbar \sum_j \dot{a}_j(t) |\psi_j(t)\rangle = \sum_j U(\mathbf{r}) a_j(t) |\psi_j(t)\rangle. \quad (2.7)$$

Then take inner product with state with the state $|\psi_{j'}(t)\rangle$

$$i\hbar \sum_j \dot{a}_j(t) \langle \psi_{j'}(t) | \psi_j(t) \rangle = \sum_j a_j(t) \langle \psi_{j'}(t) | U(\mathbf{r}) | \psi_j(t) \rangle \quad (2.8)$$

But using the *Born approximation* we can assume that this incoming wave have the initial state of the electron at $t = 0$ and therefore this equation will modified to

$$i\hbar \sum_j \dot{a}_j(t) \langle \psi_{j'}(t) | \psi_j(t) \rangle = \langle \psi_{j'}(t) | U(\mathbf{r}) | \psi_j(t) \rangle \quad (2.9)$$

due to orthonormality this becomes

$$i\hbar \dot{a}_{j'}(t) = \langle \psi_{j'}(t) | U(\mathbf{r}) | \psi_j(t) \rangle \quad (2.10)$$

and finally this leads to first order perturbation theory for Scattering as follows

$$a_{j'}(t) = -\frac{i}{\hbar} \langle \psi_{j'}(t) | U(\mathbf{r}) | \psi_j(t) \rangle \quad (2.11)$$

where

$$a_{j'}(t) = -\frac{i}{\hbar} \int_0^t dt' \int_S d\mathbf{r} \psi_{j'}^*(\mathbf{r}, t') U(\mathbf{r}) \psi_j(\mathbf{r}, t') \quad (2.12)$$

where the integration should be performed over the 2DEG area $S = L_x L_y$. Then we can calculate this using the equation we derived in (1.52) as follows

$$\begin{aligned} a_{j'}(t) = & -\frac{i}{\hbar} \int_0^t dt' \int_S d\mathbf{r} \left[\frac{1}{\sqrt{L_x}} \chi_{n'}^*(y - y'_0 - \zeta(t)) \right. \\ & \times \exp \left(\frac{i}{\hbar} \left[E_{n'} t' - m' \frac{2\pi \hbar x}{L_x} - \frac{eE(y - y'_0)}{\omega} \cos(\omega t') - m_e \dot{\zeta}(t) [y - y'_0 - \zeta(t')] - \int_0^{t'} dt'' L(\zeta, \dot{\zeta}, t'') \right] \right) \\ & \times U(\mathbf{r}) \\ & \times \frac{1}{\sqrt{L_x}} \chi_n(y - y_0 - \zeta(t')) \\ & \times \exp \left(\frac{i}{\hbar} \left[-E_n t' + m \frac{2\pi \hbar x}{L_x} - \frac{eE(y - y_0)}{\omega} \cos(\omega t') - m_e \dot{\zeta}(t') [y - y_0 - \zeta(t')] - \int_0^{t'} dt'' \tilde{L}(\zeta, \dot{\zeta}, \tilde{t}) \right] \right) \Big] \end{aligned} \quad (2.13)$$

then this will be simplified to

$$\begin{aligned} a_{j'}(t) = & -\frac{i}{\hbar} \int_0^t dt' \int_S d\mathbf{r} \left[\frac{1}{\sqrt{L_x}} \chi_{n'}^*(y - y'_0 - \zeta(t')) U(\mathbf{r}) \frac{1}{\sqrt{L_x}} \chi_n(y - y_0 - \zeta(t')) \right. \\ & \times \exp \left(\frac{i}{\hbar} \left[E_{n'} t' - m' \frac{2\pi \hbar x}{L_x} - \frac{eE(y - y'_0)}{\omega} \cos(\omega t') - m_e \dot{\zeta}(t') [y - y'_0 - \zeta(t')] - \int_0^{t'} dt'' \tilde{L}(\zeta, \dot{\zeta}, \tilde{t}) \right] \right) \\ & \times \exp \left(\frac{i}{\hbar} \left[-E_n t' + m \frac{2\pi \hbar x}{L_x} + \frac{eE(y - y_0)}{\omega} \cos(\omega t') + m_e \dot{\zeta}(t') [y - y_0 - \zeta(t')] + \int_0^{t'} dt'' \tilde{L}(\zeta, \dot{\zeta}, \tilde{t}) \right] \right) \Big] \end{aligned} \quad (2.14)$$

$$\begin{aligned} a_{j'}(t) = & -\frac{i}{\hbar} \int_0^t dt' \int_S d\mathbf{r} \left[\frac{1}{\sqrt{L_x}} \chi_{n'}^*(y - y'_0 - \zeta(t')) U(\mathbf{r}) \frac{1}{\sqrt{L_x}} \chi_n(y - y_0 - \zeta(t')) \exp \left(\frac{2\pi i(m - m') \hbar x}{L_x} \right) \right. \\ & \times \exp \left(\frac{i}{\hbar} \left[E_{n'} t' + \frac{eE y'_0}{\omega} \cos(\omega t') + m_e \dot{\zeta}(t') y'_0 \right] \right) \exp \left(\frac{i}{\hbar} \left[-E_n t' - \frac{eE y_0}{\omega} \cos(\omega t') - m_e \dot{\zeta}(t) y_0 \right] \right) \Big]. \end{aligned} \quad (2.15)$$

The time dependence of the $\chi_{n'}(y)$ can neglect since it is integrate over all the values of the y and we can write this as

$$\begin{aligned} a_{j'}(t) = & -\frac{i}{\hbar} \int_S d\mathbf{r} \frac{1}{\sqrt{L_x}} \chi_{n'}^*(y - y'_0 - \zeta(t')) U(\mathbf{r}) \frac{1}{\sqrt{L_x}} \chi_n(y - y_0 - \zeta(t')) \exp \left(\frac{2\pi i(m - m') \hbar x}{L_x} \right) \\ & \times \int_0^t dt' \left[\exp \left(\frac{i}{\hbar} \left[(E_{n'} - E_n) t' + \frac{eE(y'_0 - y_0) \omega_0^2}{\omega(\omega_0^2 - \omega^2)} \cos(\omega t') \right] \right) \right]. \end{aligned} \quad (2.16)$$

Using Jacobi-Anger expansion

$$e^{iz \cos(\theta)} = \sum_{l=-\infty}^{\infty} i^l J_l(z) e^{in\theta} \quad (2.17)$$

above equation can be modified as

$$a_{j'}(t) = -\frac{i}{\hbar} U_{j'j} \int_0^t dt' \left[\sum_{l=-\infty}^{\infty} i^l J_l \left[\frac{eE(y'_0 - y_0) \omega_0^2}{\hbar \omega(\omega_0^2 - \omega^2)} \right] \exp \left(\frac{i}{\hbar} (E_{n'} - E_n + l \hbar \omega) t' \right) \right] \quad (2.18)$$

where

$$U_{j'j} \equiv \langle \Phi_{j'}(\mathbf{r}) | U(\mathbf{r}) | \Phi_j(\mathbf{r}) \rangle \quad (2.19)$$

with bare electron eigen states (without dressing field)

$$\Phi_j(\mathbf{r}) = \frac{1}{\sqrt{L_x}} \exp\left(\frac{2\pi i m \hbar x}{L_x}\right) \chi_n(y). \quad (2.20)$$

Considering time evaluation from negative values we can write the same expression as follows

$$a_{j'}(t) = -\frac{i}{\hbar} U_{j'j} \int_{-t/2}^{t/2} dt' \left[\sum_{l=-\infty}^{\infty} i^l J_l \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \exp\left(\frac{i}{\hbar}(E_{n'} - E_n + l\hbar\omega)t'\right) \right]. \quad (2.21)$$

To calculate scattering probability we can use this scattering amplitude's square value

$$\begin{aligned} |a_{j'}(t)|^2 &= \frac{|U_{j'j}|^2}{\hbar^2} \int_{-t/2}^{t/2} dt' \left[\sum_{l=-\infty}^{\infty} -i^l J_l \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \exp\left(\frac{-i}{\hbar}(E_{n'} - E_n + l\hbar\omega)t'\right) \right] \\ &\quad \times \int_{-t/2}^{t/2} dt'' \left[\sum_{k=-\infty}^{\infty} i^k J_k \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \exp\left(\frac{i}{\hbar}(E_{n'} - E_n + k\hbar\omega)t''\right) \right] \end{aligned} \quad (2.22)$$

Considering long time $t \rightarrow \infty$ we can make the integral into a delta function as follows

$$\begin{aligned} |a_{j'}(t)|^2 &= 4\pi^2 |U_{j'j}|^2 \left[\sum_{l=-\infty}^{\infty} -i^l J_l \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \delta(-E_{n'} + E_n - l\hbar\omega) \right] \\ &\quad \times \left[\sum_{k=-\infty}^{\infty} i^k J_k \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \delta(E_{n'} - E_n + k\hbar\omega) \right] \end{aligned} \quad (2.23)$$

and this implies $l = k$ and this leads to

$$|a_{j'}(t)|^2 = 4\pi^2 |U_{j'j}|^2 \left[\sum_{l=-\infty}^{\infty} J_l^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \delta^2(E_{n'} - E_n + l\hbar\omega) \right]. \quad (2.24)$$

Then using the famous the square δ function transformation method

$$\delta^2(\epsilon) = \delta(\epsilon) \delta^2(0) \lim_{t \rightarrow \infty} \int_{-t/2}^{t/2} e^{i0 \times t' / \hbar} dt' = \frac{\delta(\epsilon)t}{2\pi\hbar} \quad (2.25)$$

we can calculate the probability of electron scattering between states j and j' per unit time as

$$\mathcal{W}_{j'j} \equiv \frac{d|a_{j'}(t)|^2}{dt} = |U_{j'j}|^2 \sum_{l=-\infty}^{\infty} J_l^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \times \frac{2\pi}{\hbar} \delta(E_{n'} - E_n + l\hbar\omega) \quad (2.26)$$

To avoid the energy exchange between a high-frequency field and electrons, the field should be purely dressing. We can achieve that by using the field with off-resonant and high frequency. Therefore, the only effect of the dressing field on 2DEG is the renormalization of the probability of elastic electron scattering within the same Landau level ($E_{n'} = E_n$), which is described by the term with $l = 0$ in the Eq. (2.26) leads to

$$\mathcal{W}_{j'j} = J_0^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \mathcal{W}_{j'j}^{(0)} \quad (2.27)$$

where

$$\mathcal{W}_{j'j}^{(0)} = \frac{2\pi}{\hbar} |U_{j'j}|^2 \delta(E_{n'} - E_n) \quad (2.28)$$

is the probability of scattering of a *bare electron*. It is important to notice that the Bessel function factor depends on both the dressing field and stationary magnetic field. This factor is responsible for all the effects discussed in this article.

One can define the lifetime of the dressed electron at the Landau level τ is renormalized by the Bessel function as below

$$\frac{1}{\tau} \equiv \sum_{j'} \mathcal{W}_{j'j} = \sum_{j'} J_0^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \mathcal{W}_{j'j}^{(0)} \quad (2.29)$$

where we have consider all possibilities that electron can jump to the state j' . Then rewrite the delat function as follows

$$\delta(\epsilon) = \frac{1}{\pi} \lim_{\Gamma \rightarrow 0} \frac{\Gamma}{\Gamma^2 + \epsilon^2} \quad (2.30)$$

where in this study we can assume that the paramater $\Gamma \equiv \hbar/\tau$ as scattering induced broading of the Landau level. But for the elestic scatteing within the same Landau level, we can write the δ function as

$$\delta(E_{n'} - E_n) \approx \frac{1}{\pi\Gamma}. \quad (2.31)$$

Therefore Eq. (2.29) will change to

$$\frac{1}{\tau} = \sum_{j'} J_0^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \times \frac{2\pi}{\hbar} |U_{j'j}|^2 \times \frac{1}{\pi\Gamma} \quad (2.32)$$

$$\frac{1}{\tau} = \sum_{j'} J_0^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \times \frac{2}{\hbar} |U_{j'j}|^2 \times \frac{\tau}{\hbar} \quad (2.33)$$

and finally this can be modified to

$$\frac{1}{\tau} = \left[\frac{2}{\hbar^2} \sum_{j'} J_0^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] |U_{j'j}|^2 \right]^{1/2} \quad (2.34)$$

where the summation is performed over electron states j' within the same Landau level. Now lets specify more on the scattering potential where we can model them as randomly distributed delta fucntions as follows

$$U(\mathbf{r}) \equiv \sum_{i=1}^{N_s} U_0 \delta(\mathbf{r} - \mathbf{r}_i) \quad (2.35)$$

where N_s is the total number of scatters in the considering matel. Now we can calculate $|U_{j'j}|^2$ as follows

$$\begin{aligned} |U_{j'j}|^2 &= \sum_{i=1}^{N_s} \frac{1}{L_x^2} \int \int dx_1 dy_1 \exp\left(\frac{-p'_x x_1}{\hbar}\right) \chi_n^*(y_1 - y'_0) U_0 \delta(x_1 - x_i) \delta(y_1 - y_i) \exp\left(\frac{p_x x_1}{\hbar}\right) \chi_n(y_1 - y_0) \\ &\quad \times \int \int dx_2 dy_2 \exp\left(\frac{p'_x x_2}{\hbar}\right) \chi_n(y_2 - y'_0) U_0 \delta(x_2 - x_i) \delta(y_2 - y_i) \exp\left(\frac{-p_x x_2}{\hbar}\right) \chi_n^*(y_2 - y_0) \end{aligned} \quad (2.36)$$

and considering only non-zero values for x_1 and x_2 integrals we can re-write this as

$$\begin{aligned} |U_{j'j}|^2 &= \sum_{i=1}^{N_s} \frac{U_0^2}{L_x^2} \int dy_1 \exp\left(\frac{-p'_x x_i}{\hbar}\right) \chi_n^*(y_1 - y'_0) \delta(y_1 - y_i) \exp\left(\frac{p_x x_i}{\hbar}\right) \chi_n(y_1 - y_0) \\ &\quad \times \int dy_2 \exp\left(\frac{p'_x x_i}{\hbar}\right) \chi_n(y_2 - y'_0) \delta(y_2 - y_i) \exp\left(\frac{-p_x x_i}{\hbar}\right) \chi_n^*(y_2 - y_0) \end{aligned} \quad (2.37)$$

and this will be simplified to

$$\begin{aligned} |U_{j'j}|^2 &= \sum_{i=1}^{N_s} \frac{U_0^2}{L_x^2} \int dy_1 \chi_n^*(y_1 - y'_0) \delta(y_1 - y_i) \chi_n(y_1 - y_0) \\ &\quad \times \int dy_2 \chi_n(y_2 - y'_0) \delta(y_2 - y_i) \chi_n^*(y_2 - y_0). \end{aligned} \quad (2.38)$$

Again considering only non-zero values for y_1 and y_2 integrals we can re-write this as

$$|U_{j'j}|^2 = \sum_{i=1}^{N_s} \frac{U_0^2}{L_x^2} \chi_n^*(y_i - y'_0) \chi_n(y_i - y_0) \chi_n(y_i - y'_0) \chi_n^*(y_i - y_0). \quad (2.39)$$

$$|U_{j'j}|^2 = \frac{U_0^2}{L_x^2} \sum_{i=1}^{N_s} \chi_n^2(y_i - y'_0) \chi_n^2(y_i - y_0). \quad (2.40)$$

Now substituting this derivation into the Eq. (2.34) we will get

$$\frac{1}{\tau} = \left[\frac{2U_0^2}{\hbar^2 L_x^2} \sum_{y'_0} J_0^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \sum_{i=1}^{N_s} \chi_n^2(y_i - y'_0) \chi_n^2(y_i - y_0) \right]^{1/2} \quad (2.41)$$

where j' reduced to p'_x (since $n' = n$) and we can represent it by y'_0 . Then this will modified to

$$\frac{1}{\tau} = \left[\frac{2U_0^2}{\hbar^2 L_x^2} \sum_{y'_0} \sum_{i=1}^{N_s} J_0^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \chi_n^2(y_i - y'_0) \chi_n^2(y_i - y_0) \right]^{1/2}. \quad (2.42)$$

Now considering large size of sample and a macroscopically large N_s scatters we can promote the summation to integrations as follows

$$\frac{1}{\tau} = \left[\frac{2U_0^2}{\hbar^2 L_x^2} \frac{eBL_x}{2\pi\hbar} \int dy'_0 \frac{N_s}{L_x} \int dy_i J_0^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \chi_n^2(y_i - y'_0) \chi_n^2(y_i - y_0) \right]^{1/2}. \quad (2.43)$$

Assuming $L_x = L_y$ we can define the area of the 2D material as

$$S \equiv L_x L_y = L_x L_y \quad (2.44)$$

and then we can re-write the above as

$$\frac{1}{\tau} = \left[\frac{eBN_s U_0^2}{\pi\hbar^3 S} \int dy'_0 \int dy_i J_0^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \chi_n^2(y_i - y'_0) \chi_n^2(y_i - y_0) \right]^{1/2}. \quad (2.45)$$

Define the *density of scatters* per unit area of 2DEG

$$n_s \equiv \frac{N_s}{S} \quad (2.46)$$

and the *magnetic length* as

$$l_0 \equiv \sqrt{\frac{\hbar}{eB}}. \quad (2.47)$$

Now our Eq. (2.45) leads to

$$\frac{1}{\tau} = \sqrt{\frac{n_s U_0^2}{\pi l_0^2 \hbar^2}} \left[\int dy'_0 \int dy_i J_0^2 \left[\frac{eE(y'_0 - y_0)\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \chi_n^2(y_i - y'_0) \chi_n^2(y_i - y_0) \right]^{1/2} \quad (2.48)$$

and now define new dummy variables as follows (since y_0 is a paramter)

$$(y'_0 - y_0) \rightarrow y \quad \text{and} \quad (y_i - y'_0) \rightarrow y' \quad (2.49)$$

and finally we will get the equation for the dressed electron lifetime at the n th Landau level as

$$\frac{1}{\tau} = \sqrt{\frac{n_s U_0^2}{\pi l_0^2 \hbar^2}} \left[\int \int dy dy' J_0^2 \left[\frac{eE y \omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \right] \chi_n^2(y') \chi_n^2(y + y') \right]^{1/2} \quad (2.50)$$

■

3 Floquet theory

Since we describe the lifetime of an electron in certain Landau level using conventional perturbation theory, now we can apply the Floquet theory to identify the difference of these methods. First we need to identify the *quasienergies* and periodic *Floquet modes* for derived wavefunctions (1.52) for a 2DEG system with both stationary magnetic field and strong dressing filed. Let's consider the following paramter which is lineraly increasing in time

$$\Delta_E t \equiv \frac{t}{T} \int_0^T dt' L(\zeta, \dot{\zeta}, t') \quad (3.1)$$

where we can calculate this using Eq. (1.42) and (1.51) as follows

$$\begin{aligned} \Delta_E t = \frac{t}{T} \int_0^T dt' \frac{1}{2} m_e \frac{(eE\omega)^2}{m_e^2(\omega_0^2 - \omega^2)^2} \cos^2(\omega t') - \frac{1}{2} m_e \omega_0^2 \frac{(eE)^2}{m_e^2(\omega_0^2 - \omega^2)^2} \sin^2(\omega t') \\ + \frac{eE}{m_e(\omega_0^2 - \omega^2)} \sin(\omega t') eE \sin(\omega t') \end{aligned} \quad (3.2)$$

$$\begin{aligned} \Delta_E t = \frac{t\omega}{2\pi} \times \frac{(eE)^2}{2m_e(\omega_0^2 - \omega^2)^2} \left[\omega^2 \int_0^T dt' \cos^2(\omega t') - \omega_0^2 \int_0^T dt' \sin^2(\omega t') \right. \\ \left. + 2(\omega_0^2 - \omega^2) \int_0^T dt' \sin^2(\omega t') \right] \end{aligned} \quad (3.3)$$

$$\Delta_E t = \frac{t\omega}{2\pi} \times \frac{(eE)^2}{2m_e(\omega_0^2 - \omega^2)^2} \left[\omega^2 \frac{\pi}{\omega} - \omega_0^2 \frac{\pi}{\omega} + 2(\omega_0^2 - \omega^2) \frac{\pi}{\omega} \right] \quad (3.4)$$

$$\Delta_E t = \frac{t\omega}{2} \times \frac{(eE)^2}{2m_e(\omega_0^2 - \omega^2)^2} (\omega_0^2 - \omega^2) = \frac{(eE)^2}{4m_e(\omega_0^2 - \omega^2)} t \quad (3.5)$$

Since this is the continuous increasing part of the Laggrangian integral in Eq. (1.52) we can make this as 2ω periodic function as follows

$$\Lambda \equiv \int_0^t dt' L(\zeta, \dot{\zeta}, t') - \frac{t}{T} \int_0^T dt' L(\zeta, \dot{\zeta}, t') \quad (3.6)$$

which can be proved as follows. First consider the first term of the Λ

$$\begin{aligned} \int_0^t dt' L(\zeta, \dot{\zeta}, t') = \frac{(eE)^2}{2m_e(\omega_0^2 - \omega^2)^2} \left[\omega^2 \int_0^t dt' \cos^2(\omega t') - \omega_0^2 \int_0^t dt' \sin^2(\omega t') \right. \\ \left. + 2(\omega_0^2 - \omega^2) \int_0^t dt' \sin^2(\omega t') \right] \end{aligned} \quad (3.7)$$

$$\begin{aligned} \int_0^t dt' L(\zeta, \dot{\zeta}, t') = \frac{(eE)^2}{2m_e(\omega_0^2 - \omega^2)^2} \left[\omega^2 \left[\frac{t}{2} + \frac{\sin(2\omega t)}{4\omega} \right] - \omega_0^2 \left[\frac{t}{2} - \frac{\sin(2\omega t)}{4\omega} \right] \right. \\ \left. + 2(\omega_0^2 - \omega^2) \left[\frac{t}{2} - \frac{\sin(2\omega t)}{4\omega} \right] \right] \end{aligned} \quad (3.8)$$

$$\begin{aligned} \int_0^t dt' L(\zeta, \dot{\zeta}, t') = \frac{(eE)^2}{2m_e(\omega_0^2 - \omega^2)^2} \left[\frac{t}{2} [\omega^2 - \omega_0^2 + 2\omega_0^2 - 2\omega^2] \right. \\ \left. + \frac{\sin(2\omega t)}{4\omega} [\omega^2 + \omega_0^2 - 2\omega_0^2 + 2\omega^2] \right] \end{aligned} \quad (3.9)$$

$$\int_0^t dt' L(\zeta, \dot{\zeta}, t') = \frac{(eE)^2}{4m_e(\omega_0^2 - \omega^2)^2} t + \frac{(eE)^2 (3\omega^2 - \omega_0^2)}{8m_e(\omega_0^2 - \omega^2)^2} \sin(2\omega t) \quad (3.10)$$

then using Eq.(3.5) we can write this as

$$\int_0^t dt' L(\zeta, \dot{\zeta}, t') = \Delta_E t + \frac{(eE)^2(3\omega^2 - \omega_0^2)}{8m_e\omega(\omega_0^2 - \omega^2)^2} \sin(2\omega t). \quad (3.11)$$

Now we can express

$$\Lambda = \Delta_E t + \frac{(eE)^2(3\omega^2 - \omega_0^2)}{8m_e\omega(\omega_0^2 - \omega^2)^2} \sin(2\omega t) - \Delta_E t = \frac{(eE)^2(3\omega^2 - \omega_0^2)}{8m_e\omega(\omega_0^2 - \omega^2)^2} \sin(2\omega t) \quad (3.12)$$

which is a periodic function in time with 2ω frequency.

Now using this parameters we can factorize the wavefunction (1.52) as linearly time dependent part and periodic time dependent part as follows

$$\begin{aligned} \psi_\alpha(x, y, t) = & \exp\left(\frac{i}{\hbar}[-E_n t + \Delta_E t]\right) \frac{1}{\sqrt{L_x}} \chi_n(y - \zeta(t)) \\ & \times \exp\left(\frac{i}{\hbar}\left[p_x x + \frac{eE y}{\omega} \cos(\omega t) + m_e \dot{\zeta}(t)[y - \zeta(t)] + \int_0^t dt' L(\zeta, \dot{\zeta}, t') - \Delta_E t\right]\right) \end{aligned} \quad (3.13)$$

where we can identify (let $\alpha \rightarrow (n, m)$) the *quasienergies* as

$$\varepsilon_\alpha \equiv \varepsilon_n = \hbar\omega_0\left(n + \frac{1}{2}\right) - \Delta_E \quad \text{where } n = 0, 1, 2, \dots \quad \text{for any given } m \quad (3.14)$$

which is only depend on one quantum number (n) and *Floquet modes* as

$$\phi_\alpha(x, \tilde{y}, t) \equiv \frac{1}{\sqrt{L_x}} \chi_n(\tilde{y} - \zeta(t)) \exp\left(\frac{i}{\hbar}\left[p_x x + \frac{eE \tilde{y}}{\omega} \cos(\omega t) + m_e \dot{\zeta}(t)[\tilde{y} - \zeta(t)] + \Lambda\right]\right) \quad (3.15)$$

with

$$\zeta(t) = \frac{eE}{m_e(\omega_0^2 - \omega^2)} \sin(\omega t) \quad \text{and} \quad \dot{\zeta}(t) = \frac{eE\omega}{m_e(\omega_0^2 - \omega^2)} \cos(\omega t) \quad (3.16)$$

where *Floquet modes* are time-periodic functions that also create a complete orthonormal set. ■

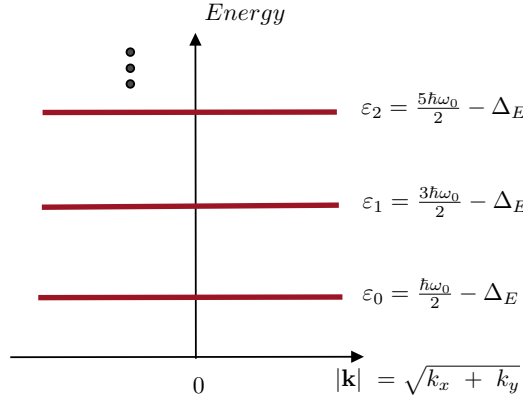


Figure 4: Quasienergies for each Landau levels against magnitude of momentum.

Therefore using Floquet theory, the solutions (Floquet states) for the periodic Hamiltonian (1.5) can be written in position space as

$$\psi_\alpha(x, \tilde{y}, t) = \exp\left(-\frac{i}{\hbar}\varepsilon_\alpha t\right) \phi_\alpha(x, \tilde{y}, t) \quad (3.17)$$

where

$$\varepsilon_\alpha \equiv \left(\frac{eB\hbar}{m_e}\right)\left(n + \frac{1}{2}\right) - \frac{(eE)^2}{4m_e(\omega_0^2 - \omega^2)} \quad \text{where } n = 0, 1, 2, \dots \quad (3.18)$$

and

$$\begin{aligned}\phi_\alpha(x, \tilde{y}, t) &\equiv \frac{1}{\sqrt{L_x}} \chi_n \left(\tilde{y} - \frac{eE \sin(\omega t)}{m_e(\omega_0^2 - \omega^2)} \right) \\ &\times \exp \left(\frac{i}{\hbar} \left[p_x x + \frac{eE \tilde{y}}{\omega} \cos(\omega t) + \frac{eE \omega \tilde{y}}{(\omega_0^2 - \omega^2)} \cos(\omega t) \right] \right) \\ &\times \exp \left(\frac{i}{\hbar} \left[-\frac{(eE)^2 \omega}{2m_e(\omega^2 - \omega_0^2)^2} \sin(2\omega t) + \frac{(eE)^2 (3\omega_0^2 - \omega^2)}{8m_e \omega (\omega_0^2 - \omega^2)^2} \sin(2\omega t) \right] \right)\end{aligned}\quad (3.19)$$

Now we can write this by more simplifying and considering spacial dependencies and using previous substituting done in Eq. (1.16) and now χ function depend on both quantum numbers because y_0 gives the p_x dependence and we can present as

$$\begin{aligned}\phi_\alpha(x, y, t) &\equiv \frac{1}{\sqrt{L_x}} \chi_n \left(y - y_0 - \frac{eE \sin(\omega t)}{m_e(\omega_0^2 - \omega^2)} \right) \exp \left(\frac{ip_x}{\hbar} x \right) \exp \left(\frac{i}{\hbar} \left[\frac{eE \omega_0^2 \cos(\omega t)}{\omega(\omega_0^2 - \omega^2)} \right] (y - y_0) \right) \\ &\times \exp \left(\frac{-i}{\hbar} \left[\frac{(eE)^2 (\omega_0^2 + \omega^2)}{8\omega m_e (\omega_0^2 - \omega^2)^2} \right] \sin(2\omega t) \right)\end{aligned}\quad (3.20)$$

Now we can transform this solution in spacial variable into the momentum space using Fourier transform over the considering confined space $A = L_x L_y$.

$$\begin{aligned}\phi_\alpha(k_x, k_y, t) &= \int_{-L_y/2}^{L_y/2} dy \exp(-ik_y y) \left[\frac{1}{\sqrt{L_x}} \chi_n \left(y - y_0 - \frac{eE \sin(\omega t)}{m_e(\omega_0^2 - \omega^2)} \right) \exp \left(\frac{i}{\hbar} \left[\frac{eE \omega_0^2 \cos(\omega t)}{\omega(\omega_0^2 - \omega^2)} \right] y \right) \right] \\ &\times \int_{-L_x/2}^{L_x/2} dx \exp(-ik_x x) \left[\exp \left(\frac{ip_x}{\hbar} x \right) \right] \\ &\times \exp \left(\frac{-i}{\hbar} \left[\frac{eE \omega_0^2 \cos(\omega t)}{\omega(\omega_0^2 - \omega^2)} \right] y_0 \right) \times \exp \left(\frac{-i}{\hbar} \left[\frac{(eE)^2 (\omega_0^2 + \omega^2)}{8\omega m_e (\omega_0^2 - \omega^2)^2} \right] \sin(2\omega t) \right)\end{aligned}\quad (3.21)$$

Then this can be re-write as follows

$$\phi_\alpha(k_x, k_y, t) = \exp \left(\frac{-i}{\hbar} \left[\frac{eE \omega_0^2 \cos(\omega t)}{\omega(\omega_0^2 - \omega^2)} \right] y_0 \right) \exp \left(\frac{-i}{\hbar} \left[\frac{(eE)^2 (\omega_0^2 + \omega^2)}{8\omega m_e (\omega_0^2 - \omega^2)^2} \right] \sin(2\omega t) \right) \Theta_\alpha(k_y, t) \delta_{k_x, \frac{p_x}{\hbar}} \quad (3.22)$$

where we used

$$\int_{L_x} dx \exp \left(-ik_x x + \frac{ip_x}{\hbar} x \right) = L_x \delta_{k_x, \frac{p_x}{\hbar}} \quad (3.23)$$

and

$$\Theta_\alpha(k_y, t) \equiv \int_{-L_y/2}^{L_y/2} dy \exp(-ik_y y) \left[\sqrt{L_x} \chi_n \left(y - y_0 - \frac{eE \sin(\omega t)}{m_e(\omega_0^2 - \omega^2)} \right) \exp \left(\frac{i}{\hbar} \left[\frac{eE \omega_0^2 \cos(\omega t)}{\omega(\omega_0^2 - \omega^2)} \right] y \right) \right] \quad (3.24)$$

and this can be simplified as

$$\Theta_\alpha(k_y, t) = \sqrt{L_x} \int_{-L_y/2}^{L_y/2} dy \chi_n \left(y - y_0 - \frac{eE \sin(\omega t)}{m_e(\omega_0^2 - \omega^2)} \right) \exp \left(-ik_y y + \frac{i}{\hbar} \left[\frac{eE \omega_0^2 \cos(\omega t)}{\omega(\omega_0^2 - \omega^2)} \right] y \right). \quad (3.25)$$

Then by defining

$$\mu(t) \equiv \frac{eE \sin(\omega t)}{m_e(\omega_0^2 - \omega^2)} + y_0 \quad (3.26)$$

and

$$\gamma(t) \equiv \frac{eE \omega_0^2 \cos(\omega t)}{\hbar \omega (\omega_0^2 - \omega^2)} \quad (3.27)$$

we can re-write this by neglecting time dependencies as

$$\Theta_\alpha(k_y, t) = \sqrt{L_x} \int_{-\infty}^{\infty} dy \chi_n(y - \mu) \exp(-i(k_y - \gamma)y). \quad (3.28)$$

We can substitute following variables

$$k_y' = k_y - \gamma \quad \text{and} \quad y' = y - \mu \quad (3.29)$$

and for $L_y \rightarrow \infty$ this leads to

$$\Theta_\alpha(k_y', t) = \sqrt{L_x} e^{-ik_y' \mu} \int_{-\infty}^{\infty} dy' \chi_n(y') \exp(-ik_y' y') = \sqrt{L_x} e^{-ik_y' \mu} \sqrt{\kappa} \int_{-\infty}^{\infty} dy' \vartheta_n(\kappa y') \exp(-ik_y' y') \quad (3.30)$$

We know that $\{\chi_\alpha\}$ are well-known harmonic eigenfunctions (with Gauss-Hermite functions) as given in the Eq. (1.47). However, the equation in (3.30) represents the Fourier transform of these Gauss-Hermite functions. Due to the symmetric condition [*Ref:E.Celeghini] the Fourier transform of these functions can be represent as

$$\mathcal{FT}[\vartheta_n(\kappa x), x, k] = \frac{i^n}{|\kappa|} \vartheta_n(k/\kappa) \quad (3.31)$$

Therefore

$$\Theta_\alpha(k_y', t) = \sqrt{L_x} e^{-ik_y' \mu} \times \frac{i^n}{\sqrt{\kappa}} \vartheta_n\left(\frac{k_y'}{\kappa}\right) = \sqrt{L_x} e^{-ik_y' \mu} \tilde{\chi}_n(k_y') \quad (3.32)$$

where

$$\tilde{\chi}_n(k) = \frac{i^n}{\sqrt{2^n n! \sqrt{\pi}}} \left(\frac{1}{\kappa}\right)^{1/2} e^{-\frac{k^2}{2\kappa^2}} \mathcal{H}_n\left(\frac{k}{\kappa}\right). \quad (3.33)$$

Using Eq. (3.32) and Eq. (3.22) we can derive that

$$\begin{aligned} \phi_\alpha(k_y, t) = \exp\left(\frac{-i}{\hbar} \left[\frac{eE\omega_0^2 \cos(\omega t)}{\omega(\omega_0^2 - \omega^2)} \right] y_0\right) \exp\left(\frac{-i}{\hbar} \left[\frac{(eE)^2(\omega_0^2 + \omega^2)}{8\omega m_e(\omega_0^2 - \omega^2)^2} \right] \sin(2\omega t)\right) \\ \times \sqrt{L_x} e^{-i(k_y - \gamma)\mu} \tilde{\chi}_n(k_y - \gamma) \end{aligned} \quad (3.34)$$

where we included the k_x dependence into α quantum number using m value and this can be re-write substituting μ and γ values as follows

$$\begin{aligned} \phi_\alpha(k_y, t) = \sqrt{L_x} \exp\left(\frac{-i}{\hbar} \left[\frac{eE\omega_0^2 \cos(\omega t)}{\omega(\omega_0^2 - \omega^2)} \right] y_0\right) \exp\left(\frac{-i}{\hbar} \left[\frac{(eE)^2(\omega_0^2 + \omega^2)}{8\omega m_e(\omega_0^2 - \omega^2)^2} \right] \sin(2\omega t)\right) \\ \times \exp\left(-ik_y \frac{eE \sin(\omega t)}{m_e(\omega_0^2 - \omega^2)}\right) \exp\left(\frac{i}{\hbar} \left[\frac{eE\omega_0^2 \cos(\omega t)}{\omega(\omega_0^2 - \omega^2)} \right] \frac{eE \sin(\omega t)}{m_e(\omega_0^2 - \omega^2)}\right) \exp(-ik_y y_0) \\ \times \exp\left(i \frac{1}{\hbar} \left[\frac{eE\omega_0^2 \cos(\omega t)}{\omega(\omega_0^2 - \omega^2)} \right] y_0\right) \tilde{\chi}_n(k_y - \gamma) \end{aligned} \quad (3.35)$$

and

$$\begin{aligned} \phi_\alpha(k_y, t) = \sqrt{L_x} \exp\left(\frac{i}{\hbar} \left[\frac{(eE)^2(3\omega_0^2 - \omega^2)}{8\omega m_e(\omega_0^2 - \omega^2)^2} \right] \sin(2\omega t)\right) \\ \times \exp\left(-ik_y \left[\frac{eE \sin(\omega t)}{m_e(\omega_0^2 - \omega^2)} + y_0 \right]\right) \tilde{\chi}_n(k_y - \gamma). \end{aligned} \quad (3.36)$$

For notation convinient we can introduce few constant as follows

$$b \equiv \frac{(eE)^2(3\omega_0^2 - \omega^2)}{8\hbar\omega m_e(\omega_0^2 - \omega^2)^2} \quad (3.37)$$

and

$$d \equiv \frac{eE}{m_e(\omega_0^2 - \omega^2)} \quad (3.38)$$

with

$$g \equiv \frac{eE\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)}. \quad (3.39)$$

Therefore we can write Eq. (3.36) as

$$\phi_\alpha(k_y, t) = \sqrt{L_x} e^{ib \sin(2\omega t)} e^{-ik_y[d \sin(\omega t) + y_0]} \tilde{\chi}_n(k_y - g \cos(\omega t)). \quad (3.40)$$

4 Floquet Fermi Goldern Rule

In this section we are going to derive the Floquet Fermi goldern rule for above derived quantum Floquet states using $t - t'$ formalism.

The Floquet states (3.17) fullfills the $t - t'$ Schrödinger equation [*Ref:myReport] as follows

$$i\hbar \frac{\partial}{\partial t} |\psi_\alpha(t, t')\rangle = H_F(t') |\psi_\alpha(t, t')\rangle \quad (4.1)$$

where Floquet Hamiltonian given by

$$H_F(t') \equiv H_e(t) - i\hbar \frac{d}{dt} \quad (4.2)$$

and

$$|\psi_\alpha(t, t')\rangle = \exp\left(-\frac{i}{\hbar} \varepsilon_\alpha t\right) |\phi_\alpha(t')\rangle \quad (4.3)$$

Now for the Eq. (4.1) corresponding time evolution operator satisfy the Schrödinger equation

$$U_0(t, t_0; t') = \exp\left(-\frac{i}{\hbar} H_F(t') [t - t_0]\right) \quad (4.4)$$

Consider a time-independent total perturbation $V(\mathbf{r})$ switched on at the reference time $t = t_0$, then Schrödinger equation becomes

$$i\hbar \frac{\partial}{\partial t} |\Psi_\alpha(t, t')\rangle = [H_F(t') + V(\mathbf{r})] |\Psi_\alpha(t, t')\rangle \quad (4.5)$$

and when $t \leq t_0$ both solutions of the Schrödinger equation coincide

$$|\psi_\alpha(t, t')\rangle = |\Psi_\alpha(t, t')\rangle \quad \text{when } t \leq t_0 \quad (4.6)$$

Now, we can introduce the interaction picture representation of the $t - t'$ Floquet state as

$$|\Psi_\alpha(t, t')\rangle_I = U_0^\dagger(t, t_0; t') |\Psi_\alpha(t, t')\rangle \quad (4.7)$$

and the perturbation in the interaction picture will be

$$V_I(\mathbf{r}) = U_0^\dagger(t, t_0; t') V(\mathbf{r}) U_0(t, t_0; t') = V(\mathbf{r}). \quad (4.8)$$

This leads to the Schrödinger equation in the interaction picture

$$i\hbar \frac{\partial}{\partial t} |\Psi_\alpha(t, t')\rangle_I = V_I(\mathbf{r}) |\Psi_\alpha(t, t')\rangle_I \quad (4.9)$$

with the recursive solution

$$|\Psi_\alpha(t, t')\rangle_I = |\Psi_\alpha(t_0, t')\rangle_I + \frac{1}{i\hbar} \int_{t_0}^t dt_1 V_I(\mathbf{r}) |\Psi_\alpha(t_1, t')\rangle_I \quad (4.10)$$

Iterating the solution only upto first order (Born approximation) this leads to

$$|\Psi_\alpha(t, t')\rangle_I \approx |\psi_\alpha(t_0, t')\rangle + \frac{1}{i\hbar} \int_{t_0}^t dt_1 V_I(\mathbf{r}) |\psi_\alpha(t_0, t')\rangle \quad (4.11)$$

and multiply it by $\langle \psi_\beta(t_0, t') |$ and we will get

$$\langle \psi_\beta(t_0, t') | \Psi_\alpha(t, t') \rangle_I = \langle \psi_\beta(t_0, t') | \psi_\alpha(t_0, t') \rangle + \frac{1}{i\hbar} \int_{t_0}^t dt_1 \langle \psi_\beta(t_0, t') | V_I(\mathbf{r}) | \psi_\alpha(t_0, t') \rangle. \quad (4.12)$$

Then introducing unitary operator U_0 we can re-write this as

$$\begin{aligned} \langle \psi_\beta(t_0, t') | U_0^\dagger(t, t_0; t') | \Psi_\alpha(t, t') \rangle &= \langle \psi_\beta(t_0, t') | U_0^\dagger(t, t_0; t') U_0(t, t_0; t') | \psi_\alpha(t_0, t') \rangle \\ &+ \frac{1}{i\hbar} \int_{t_0}^t dt_1 \langle \psi_\beta(t_0, t') | U_0^\dagger(t_1, t_0; t') V(\mathbf{r}) U_0(t_1, t_0; t') | \psi_\alpha(t_0, t') \rangle \end{aligned} \quad (4.13)$$

and this can be simplified as

$$\langle \psi_\beta(t, t') | \Psi_\alpha(t, t') \rangle = \langle \psi_\beta(t, t') | \psi_\alpha(t, t') \rangle + \frac{1}{i\hbar} \int_{t_0}^t dt_1 \langle \psi_\beta(t_1, t') | V(\mathbf{r}) | \psi_\alpha(t_1, t') \rangle. \quad (4.14)$$

Since our $t - t'$ Floquet states are orthonormal [*Ref:myReport- t-t' formalism] we can derive that

$$\langle \psi_\beta(t, t') | \Psi_\alpha(t, t') \rangle = \delta_{\alpha\beta} \exp(i\omega[t' - t]) + \frac{1}{i\hbar} \int_{t_0}^t dt_1 \langle \psi_\beta(t_1, t') | V(\mathbf{r}) | \psi_\alpha(t_1, t') \rangle. \quad (4.15)$$

Now, set $t_0 = 0$ and for a case $\alpha \neq \beta$ where we can represent $\alpha = (n_\alpha, m_\alpha)$ and $\beta = (n_\beta, m_\beta)$ and this will simplified to

$$\langle \psi_\beta(t, t') | \Psi_\alpha(t, t') \rangle = -\frac{i}{\hbar} \int_0^t dt_1 \langle \psi_\beta(t_1, t') | V(\mathbf{r}) | \psi_\alpha(t_1, t') \rangle. \quad (4.16)$$

In addition, since our Floquet states create a basis for composite space we can represent any solution using our Floquet states

$$|\Psi_\alpha(t, t')\rangle = \sum_{\beta} a_{\alpha\beta}(t, t') |\psi_\beta(t, t')\rangle. \quad (4.17)$$

Therefore we can derive a equation for this *scattering amplitude* as

$$a_{\alpha\beta}(t, t') = \langle \psi_\beta(t, t') | \Psi_\alpha(t, t') \rangle = -\frac{i}{\hbar} \int_0^t dt_1 \langle \psi_\beta(t_1, t') | V(\mathbf{r}) | \psi_\alpha(t_1, t') \rangle. \quad (4.18)$$

Now lets assume a scattering event from a $t - t'$ Floquet state $|\psi_\beta(t, t')\rangle$ into another $t - t'$ Floquet state $|\Psi_\alpha(t, t')\rangle$ with constant quansienenergy ε given as follows

$$|\Psi_\alpha(t, t')\rangle = \exp\left(-\frac{i}{\hbar}\varepsilon t\right) |\Phi_\alpha(t')\rangle \quad (4.19)$$

Now consider a scattering event

$$\psi_\beta(\mathbf{k}', t, t') = \exp\left(-\frac{i}{\hbar}\varepsilon_\beta t\right) \phi_\beta(\mathbf{k}', t') \longrightarrow \Psi_\alpha(\mathbf{k}, t, t') = \exp\left(-\frac{i}{\hbar}\varepsilon t\right) \Phi_\alpha(\mathbf{k}, t') \quad (4.20)$$

Here we need to undestand a state of this considering system only be represented by two indepen-
dent quantum numbers which are n energy eigen states and m quantum number which represents
the qunatized momentum in x direction values. Lets calculate the scattering amplitudte of the
above mentioned scattering scenario using the equation derived in (4.18).

$$\begin{aligned} a_{\alpha\beta}(\mathbf{k}', \mathbf{k}, t, t') &= -\frac{i}{\hbar} \int_0^t dt_1 \langle \psi_{\beta, \mathbf{k}'}(t_1, t') | V(\mathbf{r}) | \psi_{\alpha, \mathbf{k}}(t_1, t') \rangle \\ &= -\frac{i}{\hbar} \int_0^t dt_1 e^{\frac{i}{\hbar}(\varepsilon_\beta - \varepsilon)t_1} \langle \phi_{\beta, \mathbf{k}'}(t') | V(\mathbf{r}) | \phi_{\alpha, \mathbf{k}}(t') \rangle \end{aligned} \quad (4.21)$$

Next assuimg this scenario for long time $t \rightarrow \infty$ we can turn this integral into a delta distrubution as follows

$$\begin{aligned} a_{\alpha\beta}(\mathbf{k}', \mathbf{k}, t, t') &= -\frac{i}{\hbar} \lim_{t \rightarrow \infty} \left[\int_{-t/2}^{t/2} dt_1 e^{\frac{i}{\hbar}(\varepsilon_\beta - \varepsilon)t_1} \langle \phi_{\beta, \mathbf{k}'}(t') | V(\mathbf{r}) | \phi_{\alpha, \mathbf{k}}(t') \rangle \right] \\ &= -2\pi i \delta(\varepsilon_\beta - \varepsilon) \langle \phi_{\beta, \mathbf{k}'}(t') | V(\mathbf{r}) | \phi_{\alpha, \mathbf{k}}(t') \rangle \end{aligned} \quad (4.22)$$

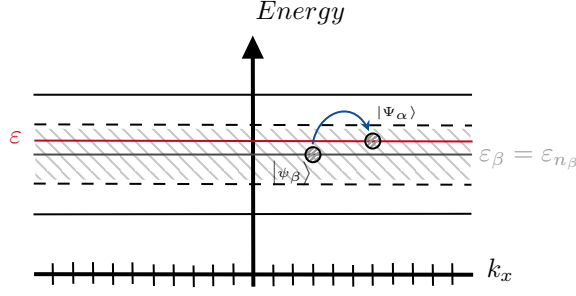


Figure 5: Scattering from $|\psi_\beta(t, t')\rangle$ to constant energy state $|\Psi_\alpha(t, t')\rangle$ due to scattering potential created by impurities.

Now let's consider about the inner product of the above derivation. Using completeness properties we can write that as follows

$$\begin{aligned} Q &\equiv \langle \phi_{\beta, \mathbf{k}'}(t') | V(\mathbf{r}) | \phi_{\alpha, \mathbf{k}}(t') \rangle \\ &= \sum_{\mathbf{k}} \sum_{\mathbf{k}'} \langle \phi_{\beta, \mathbf{k}'}(t') | \mathbf{k}' \rangle \langle \mathbf{k}' | V(\mathbf{r}) | \mathbf{k} \rangle \langle \mathbf{k} | \phi_{\alpha, \mathbf{k}}(t') \rangle \end{aligned} \quad (4.23)$$

and separating x and y directional momentums we can modify this as follows (Assuming $L_y \rightarrow \infty$) and then using $\frac{1}{L_y} \sum_{k_y} = \frac{1}{2\pi} \int k_y$

$$\begin{aligned} Q &\equiv \langle \phi_{\beta, \mathbf{k}'}(t') | V(\mathbf{r}) | \phi_{\alpha, \mathbf{k}}(t') \rangle \\ &= \frac{L_y^2}{4\pi^2} \sum_{k_x} \sum_{k'_x} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_y dk'_y \phi_{\beta}(\mathbf{k}', t') \langle \mathbf{k}' | V(\mathbf{r}) | \mathbf{k} \rangle \phi_{\alpha}(\mathbf{k}, t'). \end{aligned} \quad (4.24)$$

For a random white scattering potential we can represent the inner product of scattering potential with momentum as a constant value as

$$V_{\mathbf{k}', \mathbf{k}} \equiv \langle \mathbf{k}' | V(\mathbf{r}) | \mathbf{k} \rangle. \quad (4.25)$$

In this study, the perturbation potential is assumed to be formed by an ensemble of randomly distributed impurities, since random impurities in a disordered metal is a better approximation for experimental results.

Consider N_{imp} identical impurities positioned at the randomly distributed but fixed positions \mathbf{r}_i . The elastic scattering potential $V(\mathbf{r})$ is then given by the sum over uncorrelated single impurity potentials $v(\mathbf{r})$

$$V(\mathbf{r}) \equiv \sum_{i=1}^{N_{imp}} v(\mathbf{r} - \mathbf{r}_i). \quad (4.26)$$

Now assume that the perturbation $V(\mathbf{r})$ is a Gaussian random potential where one can choose the zero of energy such that the potential is zero on average. This model characterized by [*Ref: e.Akkermans G. Montambaux]

$$\langle v(\mathbf{r}) \rangle_{imp} = 0 \quad (4.27)$$

$$\langle v(\mathbf{r}) v(\mathbf{r}') \rangle_{imp} = \Upsilon(\mathbf{r} - \mathbf{r}') \quad (4.28)$$

where $\langle \cdot \rangle_{imp}$ denoted the average over realizations of the impurity disorder. In addition, this model assume that $v(\mathbf{r} - \mathbf{r}')$ only depends on the position difference $|\mathbf{r} - \mathbf{r}'|$ and it decays with a characteristic length r_c . Since the study considers the case where the wavelength of radiation or scattering electrons is much faster than r_c , it is good approximation to make two-point correlation function to be

$$\langle v(\mathbf{r}) v(\mathbf{r}') \rangle_{imp} = \Upsilon_{imp}^2 \delta(\mathbf{r} - \mathbf{r}') \quad (4.29)$$

and a random potential $V(\mathbf{r})$ with this property is called white noise [*Ref: e.Akkermans G. Montambaux]. Then we can choose approximately total scattering potential as

$$V(\mathbf{r}) = \sum_{i=1}^{N_{imp}} \Upsilon_{imp} \delta(\mathbf{r} - \mathbf{r}_i). \quad (4.30)$$

Now we can calculate the Eq. (4.25) using this assumption as follows

$$\begin{aligned} V_{\mathbf{k}', \mathbf{k}} &= \langle \mathbf{k}' | V(\mathbf{r}) | \mathbf{k} \rangle \\ &= \left\langle \mathbf{k}' \left| \sum_{i=1}^{N_{imp}} \Upsilon_{imp} \delta(\mathbf{r} - \mathbf{r}_i) \right| \mathbf{k} \right\rangle \\ &= \left\langle \mathbf{k}' \left| \sum_{i=1}^{N_{imp}} \Upsilon_{imp} \delta(x - x_i) \delta(y - y_i) \right| \mathbf{k} \right\rangle \\ &= \left\langle k'_x \left| \sum_{x_i=1}^{N_{imp}} \Upsilon_{imp} \delta(x - x_i) \right| k_x \right\rangle \sum_{y_i=1}^{N_{imp}} \int_{-\infty}^{\infty} dy e^{ik'_y y} \delta(y - y_i) e^{-ik_y y} \\ &= \left\langle k'_x \left| \sum_{x_i=1}^{N_{imp}} \Upsilon_{imp} \delta(x - x_i) \right| k_x \right\rangle \sum_{y_i=1}^{N_{imp}} e^{i(k'_y - k_y) y_i} \end{aligned} \quad (4.31)$$

Assuming the total number of scatterers N_{imp} is macroscopically large we can achieve following expression

$$\begin{aligned} V_{\mathbf{k}', \mathbf{k}} &= \left\langle k'_x \left| \sum_{x_i=1}^{N_{imp}} \Upsilon_{imp} \delta(x - x_i) \right| k_x \right\rangle \frac{N_{imp}}{L_y} \int_{-\infty}^{\infty} dy e^{i(k'_y - k_y) y} \\ &= \frac{N_{imp}}{L_y} V_{k'_x, k_x} \delta(k'_y - k_y) \end{aligned} \quad (4.32)$$

where

$$V_{k'_x, k_x} \equiv \left\langle k'_x \left| \sum_{x_i=1}^{N_{imp}} \Upsilon_{imp} \delta(x - x_i) \right| k_x \right\rangle \quad (4.33)$$

Therefore, using the Eq. (3.36), the Eq. (4.24) modified to (we can change variable $t' \rightarrow t$)

$$\begin{aligned} Q &= \sum_{k_x} \sum_{k'_x} \frac{N_{imp} L_y V_{k'_x, k_x}}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_y dk'_y \delta(k'_y - k_y) \\ &\quad \times \sqrt{L_x} \exp(-ib \sin(2\omega t)) \exp(ik'_y [d \sin(\omega t) + y'_0]) \tilde{\chi}_{n_\beta}(k'_y - g \cos(\omega t)) \\ &\quad \times \sqrt{L_x} \exp(ib \sin(2\omega t)) \exp(-ik_y [d \sin(\omega t) + y_0]) \tilde{\chi}_{n_\alpha}(k_y - g \cos(\omega t)) \end{aligned} \quad (4.34)$$

and we can simplify this as

$$\begin{aligned} Q &= \sum_{k_x} \sum_{k'_x} \frac{N_{imp} L_x L_y V_{k'_x, k_x}}{4\pi^2} \int_{-\infty}^{\infty} dk_y \\ &\quad \times \exp(ik_y y'_0) \tilde{\chi}_{n_\beta}(k_y - g \cos(\omega t)) \exp(-ik_y y_0) \tilde{\chi}_{n_\alpha}(k_y - g \cos(\omega t)) \end{aligned} \quad (4.35)$$

and this can re-write as

$$Q = \sum_{k_x} \sum_{k'_x} \frac{N_{imp} L_x L_y V_{k'_x, k_x}}{4\pi^2} I \quad (4.36)$$

where

$$I \equiv \int_{-\infty}^{\infty} dk_y \tilde{\chi}_{n_\beta}(k_y - g \cos(\omega t)) \tilde{\chi}_{n_\alpha}(k_y - g \cos(\omega t)) \exp(-ik_y [y_0 - y'_0]). \quad (4.37)$$

To avoid the energy transmission from external high-frequency field and electrons in the system, the applied radiation should be purely dressing field. Therefore, the only effect of the dressing field on 2DEG is the renormalization of the probability of elastic electron scattering within same Landau level ($n_\alpha = n_\beta$). Therefore Eq. (4.37) can be modified to

$$I \equiv \int_{-\infty}^{\infty} dk_y \tilde{\chi}_{n_\beta}^2(k_y - g \cos(\omega t)) \exp(-ik_y[y_0 - y'_0]). \quad (4.38)$$

Lets consider about this integral and we can calculate it as using the following substitution. Let

$$k_y - g \cos(\omega t) = \bar{k}_y \longrightarrow dk_y = d\bar{k}_y \quad (4.39)$$

and this leads to

$$I \equiv 2\pi \times \frac{1}{2\pi} \int_{-\infty}^{\infty} d\bar{k}_y \tilde{\chi}_{n_\alpha}^2(\bar{k}_y) \exp(-i(\bar{k}_y + g \cos(\omega t))(y_0 - y'_0)). \quad (4.40)$$

Using Fourier transform of Gauss-Hermite functions and convolution theorem we can write this as

$$I \equiv 2\pi \exp(g[y'_0 - y_0] \cos(\omega t)) \int_{-\infty}^{\infty} dy \chi_{n_\beta}(y) \chi_{n_\beta}(y_0 - y'_0 - y). \quad (4.41)$$

Therefore the scattering amplitude (4.22) will modified to

$$a_{\alpha\beta}(k'_x, k_x, t) = -2\pi i \delta(\varepsilon_\beta - \varepsilon) \sum_{k_x} \sum_{k'_x} \frac{N_{imp} L_x L_y V_{k'_x, k_x}}{4\pi^2} I \quad (4.42)$$

Considerng quantized momentum given in x direction derived in Eq. (2.4), we can identify the non-zero values for scattering amplitude using following conditions

$$k'_x = \frac{p_{x_\beta}}{\hbar} = m' \frac{2\pi}{L_x} \quad \text{and} \quad k_x = \frac{p_{x_\alpha}}{\hbar} = m \frac{2\pi}{L_x}. \quad (4.43)$$

Then we can simplified scattering amplitude for given k'_x and k_x as

$$a_{\alpha\beta}(k'_x, k_x, t) = \delta(\varepsilon_\beta - \varepsilon) \left[\frac{-i N_{imp} L_x L_y V_{k'_x, k_x}}{2\pi} \right] \exp(g[y'_0 - y_0] \cos(\omega t)) \times \int_{-\infty}^{\infty} dy \chi_{n_\beta}(y) \chi_{n_\beta}(y_0 - y'_0 - y) \quad (4.44)$$

Since this scattering amplitude is time-periodic we can write this as a Fourier series expansion

$$a_{\alpha\beta}(k'_x, k_x, t) = \sum_{l=-\infty}^{\infty} a_{\alpha\beta}^l(k'_x, k_x) e^{-il\omega t}. \quad (4.45)$$

In addition, using Jacobi-Anger expansion

$$e^{iz \cos(\theta)} = \sum_{l=-\infty}^{\infty} i^l J_l(z) e^{-il\theta} \quad (4.46)$$

we can re-write the Eq.(4.44) as follows

$$a_{\alpha\beta}(k'_x, k_x, t) = \delta(\varepsilon_\beta - \varepsilon) \left[\frac{-i N_{imp} L_x L_y V_{k'_x, k_x}}{2\pi} \right] \sum_{l=-\infty}^{\infty} i^l J_l(g[y'_0 - y_0]) e^{-il\omega t} \times \int_{-\infty}^{\infty} dy \chi_{n_\beta}(y) \chi_{n_\beta}(y_0 - y'_0 - y) \quad (4.47)$$

$$a_{\alpha\beta}(k'_x, k_x, t) = \sum_{l=-\infty}^{\infty} \delta(\varepsilon_\beta - \varepsilon) \left[\frac{-i^{l+1} N_{imp} L_x L_y V_{k'_x, k_x}}{2\pi} \right] J_l(g[y'_0 - y_0]) \times \int_{-\infty}^{\infty} dy \chi_{n_\beta}(y) \chi_{n_\beta}(y_0 - y'_0 - y) e^{-il\omega t} \quad (4.48)$$

Then we can identified the Fourier series component as

$$a_{\alpha\beta}^l(k'_x, k_x) = \delta(\varepsilon_\beta - \varepsilon) \left[\frac{-i^{l+1} N_{imp} L_x L_y V_{k'_x, k_x}}{2\pi} \right] J_l(g[y'_0 - y_0]) \int_{-\infty}^{\infty} dy \chi_{n_\beta}(y) \chi_{n_\beta}(y_0 - y'_0 - y) \quad (4.49)$$

Now one can introduce the definition of the *transition probability matrix* as

$$(A_{\alpha\beta}(k'_x, k_x))_{l,l'} \equiv a_{\alpha\beta}^l(k'_x, k_x) \left[a_{\alpha\beta}^{l'}(k'_x, k_x) \right]^* \quad (4.50)$$

and this becomes

$$(A_{\alpha\beta}(k'_x, k_x))_{l,l'} = \left[\frac{N_{imp}^2 A^2 |V_{k'_x, k_x}|^2}{4\pi^2} \right] J_l(g[y'_0 - y_0]) J_{l'}(g[y'_0 - y_0]) \delta^2(\varepsilon_\beta - \varepsilon) \times \int_{-\infty}^{\infty} dy \chi_{n_\beta}(y) \chi_{n_\beta}(y_0 - y'_0 - y) \int_{-\infty}^{\infty} d\bar{y} \chi_{n_\beta}(\bar{y}) \chi_{n_\beta}(y_0 - y'_0 - \bar{y}). \quad (4.51)$$

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We can reduce these intragal into one variable and derive

$$(A_{\alpha\beta}(k'_x, k_x))_{l,l'} = \left[\frac{N_{imp}^2 A^2 |V_{k'_x, k_x}|^2}{4\pi^2} \right] J_l(g[y'_0 - y_0]) J_{l'}(g[y'_0 - y_0]) \delta^2(\varepsilon_\beta - \varepsilon) \times \left| \int_{-\infty}^{\infty} dy \chi_{n_\beta}(y) \chi_{n_\beta}(y_0 - y'_0 - y) \right|^2. \quad (4.52)$$

Then desribing the square of the delta distribution using following procedure

$$\delta^2(\varepsilon) = \delta(\varepsilon) \delta(0) = \frac{\delta(\varepsilon)}{2\pi\hbar} \int_{-t/2}^{t/2} e^{i0 \times t' / \hbar} dt' = \frac{\delta(\varepsilon)t}{2\pi\hbar} \quad (4.53)$$

one can modify our derivation in Eq. (4.51) as

$$(A_{\alpha\beta}(k'_x, k_x))_{l,l'} = \left[\frac{N_{imp}^2 A^2 |V_{k'_x, k_x}|^2}{4\pi^2} \right] J_l(g[y'_0 - y_0]) J_{l'}(g[y'_0 - y_0]) \delta(\varepsilon_\beta - \varepsilon) \frac{t}{2\pi\hbar} \times \left| \int_{-\infty}^{\infty} dy \chi_{n_\beta}(y) \chi_{n_\beta}(y_0 - y'_0 - y) \right|^2. \quad (4.54)$$

Then performing thetime derivation of each matrix element yeild the *transition amplitude matrix* as follows

$$\Gamma_{\alpha\beta}^{ll'}(k'_x, k_x) \equiv \frac{d(A_{\alpha\beta}(k'_x, k_x))_{l,l'}}{dt} = \Lambda |V_{k'_x, k_x}|^2 \delta(\varepsilon_\beta - \varepsilon) J_l(g[y'_0 - y_0]) J_{l'}(g[y'_0 - y_0]) \left| \int_{-\infty}^{\infty} dy \chi_{n_\beta}(y) \chi_{n_\beta}(y_0 - y'_0 - y) \right|^2 \quad (4.55)$$

where

$$\Lambda \equiv \frac{N_{imp}^2 A^2}{8\pi^3 \hbar} \quad (4.56)$$

Now using defintion of y_0 given in Eq. (1.11) we can write that

$$y_0 - y'_0 = -\frac{p_{x_\alpha}}{eB} + \frac{p_{x_\beta}}{eB} = \frac{\hbar k'_x}{eB} - \frac{\hbar k_x}{eB} = \frac{\hbar}{eB} [k'_x - k_x] \quad (4.57)$$

and this leads Eq. (4.56) to

$$\Gamma_{\alpha\beta}^{ll'}(k'_x, k_x) = \Lambda |V_{k'_x, k_x}|^2 \delta(\varepsilon_\beta - \varepsilon) J_l \left(\frac{g\hbar}{eB} [k_x - k'_x] \right) J_{l'} \left(\frac{g\hbar}{eB} [k_x - k'_x] \right) \times \left| \int_{-\infty}^{\infty} dy \chi_{n_\beta}(y) \chi_{n_\beta} \left(\frac{\hbar}{eB} [k'_x - k_x] - y \right) \right|^2 \quad (4.58)$$

An impurity average of white noise potential allows to identify $\langle |V_{k'_x, k_x}|^2 \rangle = V_{imp}$ and the inverse scattering time matrix is the sum over all momentum over the transition probability matrix

$$\left(\frac{1}{\tau(\varepsilon, k_x)} \right)_{\alpha\beta}^{ll'} \equiv \frac{1}{L_x} \sum_{k'_x} \langle \Gamma_{\alpha\beta}^{ll'}(k'_x, k_x) \rangle_{imp} \quad (4.59)$$

and this implies

$$\begin{aligned} \Gamma_{\alpha\beta}^{ll'}(k'_x, k_x) &= \frac{\Lambda V_{imp}}{L_x} \sum_{k'_x} \delta(\varepsilon_\beta - \varepsilon) J_l \left(\frac{g\hbar}{eB} [k_x - k'_x] \right) J_{l'} \left(\frac{g\hbar}{eB} [k_x - k'_x] \right) \\ &\quad \times \left| \int_{-\infty}^{\infty} dy \chi_{n_\beta}(y) \chi_{n_\beta} \left(\frac{\hbar}{eB} [k'_x - k_x] - y \right) \right|^2 \end{aligned} \quad (4.60)$$

For the 1-dimentional case introduce the momentum continuum limit as follows

$$\frac{1}{L_x} \sum_{k'_x} \rightarrow \frac{1}{2\pi} \int dk'_x \quad (4.61)$$

and this leads to

$$\begin{aligned} \left(\frac{1}{\tau(\varepsilon, k_x)} \right)_{\alpha\beta}^{ll'} &= \frac{\Lambda V_{imp}}{2\pi} \delta(\varepsilon_\beta - \varepsilon) \int_{-\infty}^{\infty} dk'_x J_l \left(\frac{g\hbar}{eB} [k_x - k'_x] \right) J_{l'} \left(\frac{g\hbar}{eB} [k_x - k'_x] \right) \\ &\quad \times \left| \int_{-\infty}^{\infty} dy \chi_{n_\beta}(y) \chi_{n_\beta} \left(\frac{\hbar}{eB} [k'_x - k_x] - y \right) \right|^2 \end{aligned} \quad (4.62)$$

Using following substitution

$$y = \frac{\hbar \bar{k}}{eB} \rightarrow dy = \frac{\hbar}{eB} d\bar{k} \quad (4.63)$$

we can modify above derivation as

$$\begin{aligned} \left(\frac{1}{\tau(\varepsilon, k_x)} \right)_{\alpha\beta}^{ll'} &= \frac{\Lambda V_{imp}}{2\pi} \delta(\varepsilon_\beta - \varepsilon) \int_{-\infty}^{\infty} dk'_x J_l \left(\frac{g\hbar}{eB} [k_x - k'_x] \right) J_{l'} \left(\frac{g\hbar}{eB} [k_x - k'_x] \right) \\ &\quad \times \left(\frac{\hbar}{eB} \right)^2 \left| \int_{-\infty}^{\infty} d\bar{k} \chi_{n_\beta} \left(\frac{\hbar}{eB} \bar{k} \right) \chi_{n_\beta} \left(\frac{\hbar}{eB} [k'_x - k_x - \bar{k}] \right) \right|^2, \end{aligned} \quad (4.64)$$

and finally we can derive our expression for the *inverse scattering time matrix* for N th Landau level (let $n_\alpha = n_\beta = N$)

$$\begin{aligned} \left(\frac{1}{\tau(\varepsilon, k_x)} \right)_N^{ll'} &= \frac{N_{imp}^2 A^2 \hbar V_{imp}}{16\pi^4 (eB)^2} \delta(\varepsilon - \varepsilon_N) \int_{-\infty}^{\infty} dk'_x J_l \left(\frac{g\hbar}{eB} [k_x - k'_x] \right) J_{l'} \left(\frac{g\hbar}{eB} [k_x - k'_x] \right) \\ &\quad \times \left| \int_{-\infty}^{\infty} d\bar{k} \chi_N \left(\frac{\hbar}{eB} \bar{k} \right) \chi_N \left(\frac{\hbar}{eB} [k'_x - k_x - \bar{k}] \right) \right|^2. \end{aligned} \quad (4.65)$$

5 Inverse Scattering Time Analysis

We have derived the inverse scattering time matrix element from previous section as follows

$$\left(\frac{1}{\tau(\varepsilon, k_x)}\right)_N^{ll'} = \frac{N_{imp}^2 A^2 \hbar V_{imp}}{16\pi^4 (eB)^2} \delta(\varepsilon - \varepsilon_N) \int_{-\infty}^{\infty} dk'_x J_l \left(\frac{g\hbar}{eB} [k_x - k'_x] \right) J_{l'} \left(\frac{g\hbar}{eB} [k_x - k'_x] \right) \times \left| \int_{-\infty}^{\infty} d\bar{k} \chi_N \left(\frac{\hbar}{eB} \bar{k} \right) \chi_N \left(\frac{\hbar}{eB} [k'_x - k_x - \bar{k}] \right) \right|^2. \quad (5.1)$$

The disorder in the system is not supposed to change the eigenenergies of the bare system, hence all off-diagonal elements of the self-energy were neglected. Therefore we can consider only the diagonal elements of the inverse scattering time matrix

$$\left(\frac{1}{\tau(\varepsilon, k_x)}\right)_N^{ll} = \frac{N_{imp}^2 A^2 \hbar V_{imp}}{16\pi^4 (eB)^2} \delta(\varepsilon - \varepsilon_N) \int_{-\infty}^{\infty} dk'_x J_l^2 \left(\frac{g\hbar}{eB} [k_x - k'_x] \right) \times \left| \int_{-\infty}^{\infty} d\bar{k} \chi_N \left(\frac{\hbar}{eB} \bar{k} \right) \chi_N \left(\frac{\hbar}{eB} [k'_x - k_x - \bar{k}] \right) \right|^2. \quad (5.2)$$

Lets consider how this expression change when we have turn off the dressing field ($E = 0$). Therefore the inverse scattering time becomes valid for only $l = 0$

$$\left(\frac{1}{\tau(\varepsilon, k_x)}\right)_N^{00} \Big|_{E=0} = \frac{N_{imp}^2 A^2 \hbar V_{imp}}{16\pi^4 (eB)^2} \delta(\varepsilon - \varepsilon_N) \int_{-\infty}^{\infty} dk'_x \left| \int_{-\infty}^{\infty} d\bar{k} \chi_N \left(\frac{\hbar}{eB} \bar{k} \right) \chi_N \left(\frac{\hbar}{eB} [k'_x - k_x - \bar{k}] \right) \right|^2. \quad (5.3)$$

Therefore we can analyze the behaviour of the inverse scattering time with $l = 0$ central element of the matrix.

$$\Lambda_{00} \equiv \frac{(1/\tau)_N^{00}}{(1/\tau)_N^{00} \Big|_{E=0}} \quad (5.4)$$

and this will be

$$\Lambda_{00}(k_x) = \frac{\int_{-\infty}^{\infty} dk'_x J_l^2(g\gamma[k_x - k'_x]) \left| \int_{-\infty}^{\infty} d\bar{k} \chi_N(\gamma\bar{k}) \chi_N(\gamma[k'_x - k_x - \bar{k}]) \right|^2}{\int_{-\infty}^{\infty} dk'_x \left| \int_{-\infty}^{\infty} d\bar{k} \chi_N(\gamma\bar{k}) \chi_N(\gamma[k'_x - k_x - \bar{k}]) \right|^2} \quad (5.5)$$

where

$$g = \frac{eE\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} \quad \gamma = \frac{\hbar}{eB} \quad (5.6)$$

and

$$\chi_N(y) = \frac{\sqrt{\kappa}}{\sqrt{2^N N! \sqrt{\pi}}} \exp\left(-\frac{\kappa^2 y^2}{2}\right) \mathcal{H}_N(\kappa y) \quad \text{with} \quad \kappa \equiv \sqrt{\frac{m_e \omega_0}{\hbar}}. \quad (5.7)$$

Lets calculate these constants for GaAs-based quantum well with following given system external parameters and physical constants.

External paramter name	Symbol	Value in SI-units
Average intensity	I	$200 \text{ W/cm}^2 = 2 \times 10^6 \text{ W/m}^2$
Magnetic field	B	1.2 T
Driving frequency	ω	$2 \times 10^{12} \text{ rads}^{-1}$
Effective mass	m_e	$0.071 \times m = 6.467 \times 10^{-32} \text{ kg}$

Table 1: System external paramter values

Physical constant name	Symbol	Value in SI-units
Electron charge	e	$1.602 \times 10^{-19} \text{ C}$
Electron mass	m	$9.109 \times 10^{-31} \text{ kg}$
Reduced Planck's constant	\hbar	$1.054 \times 10^{-34} \text{ kgm}^2\text{s}^{-1}$
Speed of light	c	$2.998 \times 10^8 \text{ ms}^{-1}$
Vacuum permittivity	ε_0	$8.854 \times 10^{-12} \text{ C}^2\text{s}^2\text{kg}^{-1}\text{m}^{-3}$

Table 2: Physical constant values in SI-units

Therefore we can calculate following values

$$\omega_0 = \frac{eB}{m_e} = 2.97265 \times 10^{12} \text{ s}^{-1} \quad (5.8)$$

$$\gamma = \frac{\hbar}{eB} = 5.4851 \times 10^{-16} \text{ m}^2 \quad (5.9)$$

$$g = \frac{eE\omega_0^2}{\hbar\omega(\omega_0^2 - \omega^2)} = 5.38756 \times 10^7 \text{ m}^{-1} \quad (5.10)$$

$$\kappa = \sqrt{\frac{m_e\omega_0}{\hbar}} = 4.2698 \times 10^7 \text{ m}^{-1} \quad (5.11)$$

Since

$$g\gamma = 2.95513 \times 10^{-8} \text{ m} \quad \text{and} \quad \kappa\gamma = 2.34202 \times 10^{-8} \text{ m} \quad (5.12)$$

we can choose our integral dummy variables k'_x , \bar{k} and momentum variable k_x are in one range as follows

$$k'_x, \bar{k}, k_x \approx 10^{-8} \text{ m}^{-1} \quad (5.13)$$

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