Project 1

CS 332, Fall 2025

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8 October, 2025

Outline

1. Part 1
   - 1. winning probability and expected utility with your bids
   - 2. the optimal bids
   - 3. Better bid strategy
2. Part 2
   - 1. optimal data-driven bid strategy
   - 2. The theoretically optimal bid in a two-player First-price auction
3. Usage of AI

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3. Usage of AI

Part 1

In Part 1,

- **Methods:** Exact Estimation and Monte Carlo Estimation.
- **Results:** Koshi's bid strategy($b = v - 1$) did not work well, while Ben's bid strategy(almost $b = \frac{v}{2}$) worked pretty well.
- **Takeaways:**: Monte Carlo approximately becomes exact as the sample size increases. The better bid strategy is to bid conservatively.

Model

- **Setting:** single-item, first-price, two bidders.
- **Private value:** $v \in \{10, 20, \ldots, 100\}$. You bid b.
- **Payoff:** if win, $u = v - b$; else $u = 0$.
- **Winning probability:**

$$P_{\text{win}}(b) = P(\text{opp} < b) + \tfrac{1}{2}P(\text{opp} = b).$$

- **Objective:**

$$EU(v, b) = (v - b)\, P_{\text{win}}(b), \qquad b^*(v) = \arg\max_b EU(v, b).$$

- EU = Expected Utility, MC = Monte Carlo, opt = optimal.

1-1. Calculation methods

- **Empirical analysis:**
  pick $V \sim \mathrm{Unif}\{10, \ldots, 100\}$; sample opponent bid set $\{b_i\}_{i=1}^{n}$ for that V.
- **Exact estimate:**

$$\widehat{P}_{\mathrm{win}}(b) \;=\; \frac{\#\{b_i < b\} \;+\; \frac{1}{2}\#\{b_i = b\}}{n}$$

$$\mathrm{EU}_{\mathrm{exact}}(v, b) \;=\; (v - b)\,\widehat{P}_{\mathrm{win}}(b)$$

- **Monte Carlo estimator:**
  draw $B_{\mathsf{opp}}^{(t)}$ from empirical model for $t = 1, \ldots, T$,

$$\widehat{EU}_{\mathsf{MC}}(v, b) = \frac{1}{T}\sum_{t}\Big(1\{b > B_{\mathsf{opp}}^{(t)}\} + \tfrac{1}{2}1\{b = B_{\mathsf{opp}}^{(t)}\}\Big)(v - b).$$

  - use $T = 20{,}000$ for stable MC estimates.

1.Calculate your winning probability and expected utility with your bids submitted in Ex 1.2 for each of your values.

value	my bid	win prob	EU exact	EU MC	opt bid	opt EU	regret
10	9	0.091	0.091	0.088	5	0.334	0.243
20	19	0.252	0.252	0.250	11	1.740	1.487
30	29	0.418	0.418	0.417	11	3.694	3.276
40	39	0.577	0.577	0.580	21	6.529	5.952
50	49	0.716	0.716	0.718	21	9.984	9.268
60	59	0.805	0.805	0.806	31	14.713	13.908
70	69	0.855	0.855	0.856	31	19.804	18.949
80	79	0.905	0.905	0.904	41	25.462	24.557
90	89	0.950	0.950	0.951	41	32.007	31.057
100	99	0.991	0.991	0.991	51	38.675	37.685

Koshi took a strategy in which he always bid $b = v - 1$.
Average Regret : **14.63**

1-1. Ben's Case

2. Calculate your winning probability and expected utility with your bids submitted in Ex 1.2 for each of your values.

value	my bid	win prob	EU exact	EU MC	opt bid	opt EU	regret
10	9	0.091	0.091	0.088	5	0.334	0.243
20	15	0.216	1.080	1.075	11	1.740	0.660
30	20	0.291	2.909	2.865	11	3.694	0.785
40	25	0.375	5.625	5.569	21	6.529	0.904
50	30	0.455	9.091	9.089	21	9.984	0.893
60	30	0.455	13.636	13.633	31	14.713	1.076
70	35	0.532	18.614	18.648	31	19.804	1.190
80	40	0.607	24.273	24.406	41	25.462	1.189
90	45	0.677	30.477	30.651	41	32.007	1.530
100	50	0.745	37.273	37.403	51	38.675	1.403

Ben took a strategy in which he always bid almost $b = \frac{v}{2}$.
Average Regret : **0.98**

1-2. Optimal-Bids

- Calculate the optimal bids values by grid search.
  - the answer is

value	b_opt_exact	util_opt_exact	b_opt_mc	util_opt_mc
10	5.1	0.3341	1.6	0.3507
20	11.1	1.7395	11.1	1.8067
30	11.1	3.6941	11.1	3.8367
40	21.1	6.5291	21.6	6.6608
50	21.1	9.9836	21.6	10.2808
60	31.1	14.7127	31.3	14.9455
70	31.1	19.8036	31.3	20.1530
80	41.1	25.4618	41.0	25.8814
90	41.1	32.0073	41.0	32.5176
100	51.1	38.6755	41.0	39.1539

1-2. Example

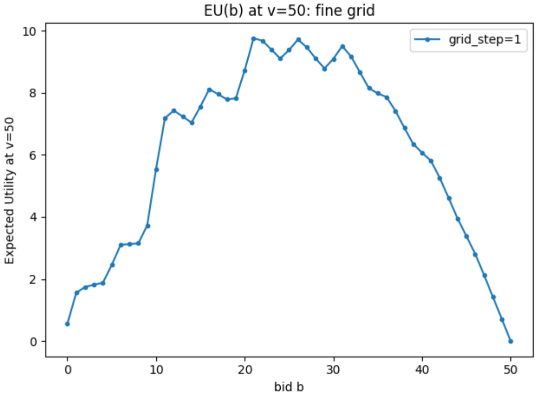
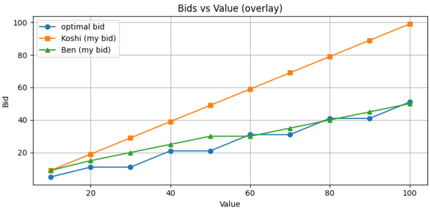


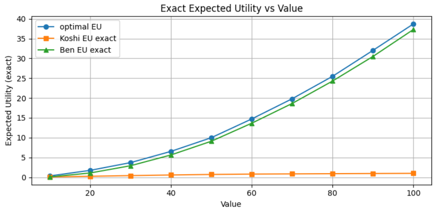
Figure: Change of Expected Utility when my value is 50

According to Figure, we concluded that the optimal bid is around 20 or 30 when the value is 50

- Compare the utility you obtained to the optimal utility you could have obtained. Can you conclude anything about a good strategy in this auction?
    - from Koshi's case: we observe that it is not recommended to bid too close to your valuation.
    - From Ben's case: he bids conservatively, yet his bids are close to the optimal bids and show very good expected-utility performance.

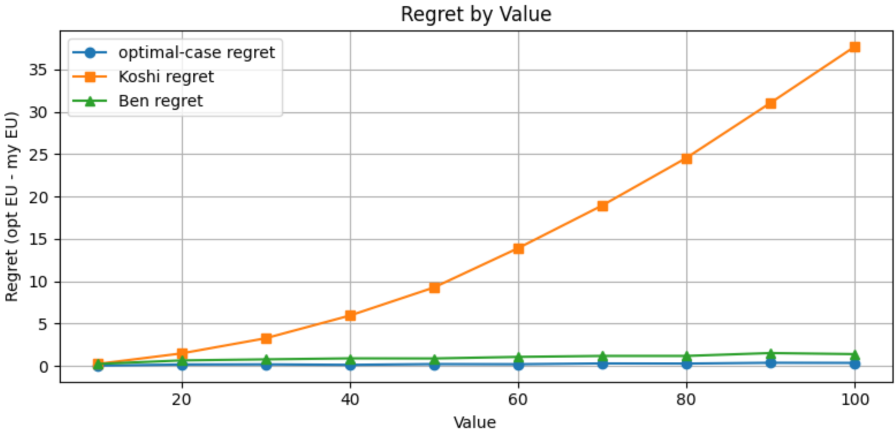


(a) Bid vs. value



(b) Expected utility vs. value

1-3. Appendix: Regret



(c) Regret

Outline

Part 2

In Part 2, we consider two things;

- 1. Optimal data-driven bid strategy (stated in CANVAS)
- 2. The theoretically optimal bid in a two-player First-price auction

2-1. Summary

Method:

- We then analyze how the empirical estimates converge to the true distribution as the sample size increases.

Results:

- The optimal bidding algorithm is obtained by choosing the bid function, and F(b) can be estimated through data.

$$b^*(v) = \arg\max_b (v - b)F(b).$$

- As the number of bid samples increases, the variance of the estimated expected utility decreases, leading to more accurate estimation of the opponent's bidding strategy and convergence of $\hat{b}^*(v)$ to $b^*(v)$.

Takeaway.

- More bid samples lead to a smoother, more accurate $\hat{F}$
- $\Rightarrow$ obtain bid strategy that converge to the true optimal policy.
- This analysis can be applied to arbitral type of distribution.

2-1. Setup

Setting.

- Same auction environment as in Part 1.
- Algorithm chooses a bid function $b(v)$ that maximizes expected utility.

Formulation of bid algorithm problem.

- Creating the algorithm which chooses a bid function $b(v)$ to maximize expected utility(in other words, choose the empirical optimal strategy):

$$b^*(v) = \arg\max_b (v - b)\ \Pr(B_{\text{opp}} \leq b)$$

- In our setting, value is parameterized discretely, so our goal is to find the optimal bid b^* for each value $v(= 10, 20, \ldots, 100)$.
- To avoid notational confusion, this presentation does not include analysis of cases in which items are randomly assigned between the two players when tie.
- Note that the discussion still holds even when tie cases are included

Win Probability. $\Pr(B_{\textbf{opp}} \leq b)$
Your win probability is the probability that the opponent's bid B_{opp} is below your bid b:

$$\Pr(B_{\text{opp}} \leq b).$$

This probability comes from two components:

- the distribution of the opponent's value V_{opp}, and
- the opponent's bidding function $b_{\text{opp}}(V_{\text{opp}})$

2-1. Method

What we estimate is win probability

- For each v_i'(drawn from uniform distribution), we observe n bids $B_{i,1}, \ldots, B_{i,n}$ from the dataset (`bid_data.csv`, and in ur experiment, n=22).
- Per value CDF:
$$\hat{G}_{v_i'}(b) = \frac{1}{n} \sum_{j=1}^{n} 1\{B_{i,j} \leq b\}.$$
- For each valuation level v_i', we observe n bids
$$\{B_{i,1}, \ldots, B_{i,n}\} \text{ i.i.d.} \sim G_{v_i'},$$
where $G_{v_i'}(b) = \Pr(B_{\text{opp}} \leq b \mid V_{\text{opp}} = v_i')$ is the conditional bid distribution.
- Aggregate CDF:
$$\hat{F}(b) = \frac{1}{10} \sum_{i=1}^{10} \hat{G}_{v_i'}(b).$$
- Therefore, our bid algorithm problem can be formulated as follows.
$$\hat{b}^*(v) \in \arg\max_b (v - b)\, \hat{F}(b).$$

2-1. Statistical Method

Estimator (per value)

$$\hat{G}_{v_i'}(b) = \frac{1}{n}\sum_{j=1}^{n} 1\{B_{i,j} \le b\},$$

$$\mathbb{E}[\hat{G}_{v_i'}(b)] = G_{v_i'}(b),$$

$$\mathrm{Var}[\hat{G}_{v_i'}(b)] = \frac{G_{v_i'}(b)\big(1 - G_{v_i'}(b)\big)}{n}.$$

Aggregated win CDF

$$F(b) = \frac{1}{10}\sum_{i=1}^{10} G_{v_i'}(b), \quad \hat{F}(b) = \frac{1}{10}\sum_{i=1}^{10} \hat{G}_{v_i'}(b).$$

Hence,

$$\mathbb{E}[\hat{F}(b)] = F(b), \quad \mathrm{Var}[\hat{F}(b)] = O\left(\frac{1}{n}\right).$$

2-1. Results (in theory)

Implication.
As n increases, the variance of the estimator decreases while its expected value remains unchanged. Therefore, our estimate of the opponent's bidding strategy becomes more precise, and the optimal bid converges to the true one:

$$\hat{b}^*(v) = \arg\max_b (v - b)\,\hat{F}(b) \;\longrightarrow\; b^*(v) = \arg\max_b (v - b)\,F(b).$$

- Here are the examples of Per value CDF.
- These figures shows culumulatice distribution functions given V. Vertical axis shows the amount of bid and horizontal axis shows that its culumulative density.

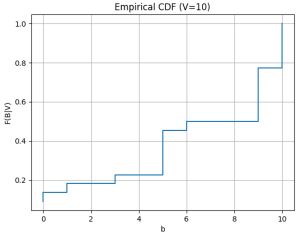


Figure: V=10

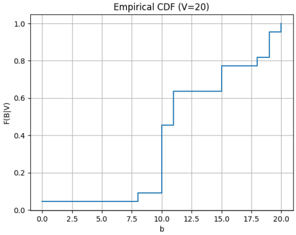


Figure: V=20

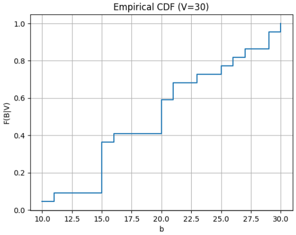


Figure: V=30

2-1. Results (in empirical)

- Here are the examples of Aggregated CDF.
- We assume a uniform distribution, so treat all bids in samples equally.
- Therefore, the aggregated CDF is equivalent to constructing a cumulative distribution that gives equal weight to all observed samples (in our experiment, 22×10 samples in total)
- Vertical axis shows the amount of bid and horizontal axis shows that its culumulative density.

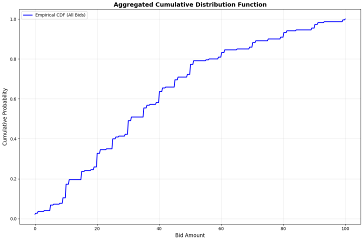


Figure: Aggregated CDF

2-1. Appendix: Distribution(V=10,20,30)

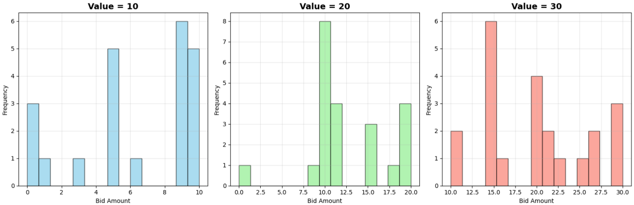


Figure: Discrete Distribution

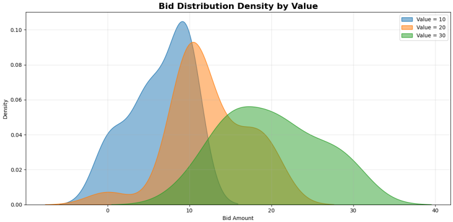


Figure: Continuous Distribution

2-2. Observation of the optimal bid

- **First-price Auction:** : It seems like the optimal bid is linear in value, but its coefficient is half not 1(truthful).
- From this, we wondered whether this bidding strategy would be optimal in this particular setting.

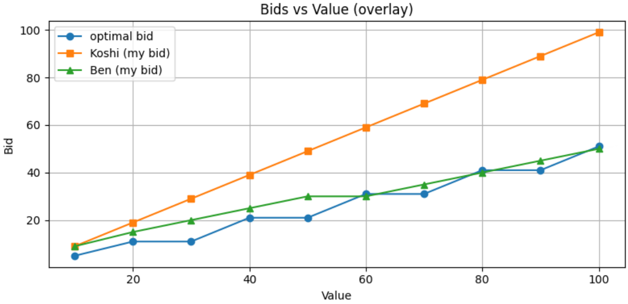


Figure: Comparing bid strategy

2-2. Bid Strategy in First-Price Auction

Setup:

- First-price Auction
- Two bidders, values drawn from $U[0, 1]$ i.i.d.
- The valuation takes continuous values for simplicity, instead of discrete.

Method:

- I made certain assumptions and,
- based on the best response functions and first-order conditions, derived the Nash equilibrium through simple algebraic manipulation.

Results:

- Assumption: opponent's bid strategy is deterministic and linear in his value.
- Best bid strategy is $b(v) = \frac{v}{2}$.
- From our empirical research so far, this strategy is robust.

2-2. Calculation process

- I assume that opponent's bid strategy is deterministic and linear in his value($b(x) = \alpha x$).
- Optimal bid problem is formulated by following formula;

$$b^* = \arg\max_b (v - b) \Pr(b \leq b_{opp})$$

- Considering Best respond to a opponent,

$$\Pr(\text{win at bid } b) = \Pr(\alpha X < b) = \tfrac{b}{\alpha}$$

- so,

$$b^* = \arg\max_b (v - b) \tfrac{b}{\alpha} \quad (= U).$$

- First-Order Condition in b:

$$\partial U / \partial b = \tfrac{1}{\alpha}(v - 2b) = 0 \;\Rightarrow\; b^*(v) = \tfrac{1}{2} v.$$

- By symmetry,

$$b(v) = \alpha v = b^*(v) \;\Rightarrow\; \alpha = \frac{1}{2}.$$

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Usage of AI

AI was used for coding; final review and responsibility by the authors.