Lebesgue's Integral

A Historical Exploration into the Development of Modern Integration Theory

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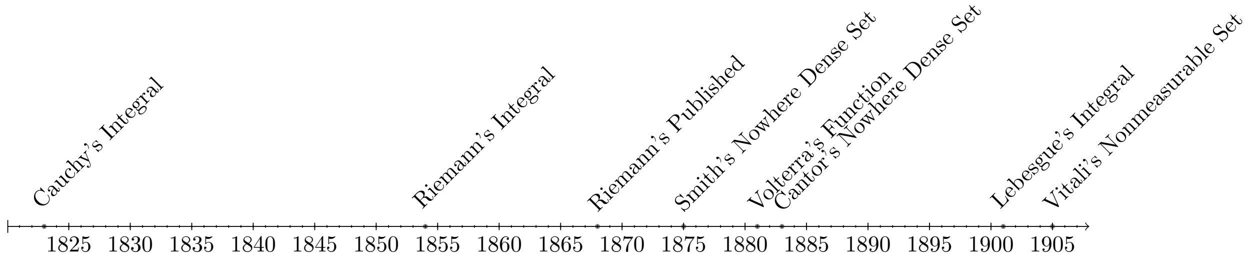
Cauchy's Integral
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1 Introduction

> *Mathematicians have often been the most curious of scholarly creatures in that they have seldom been curious about the history of their own subject.*
> -MICHAEL BERNKOPF

Students learning mathematics in the 20^{th} and 21^{th} century have a near universal experience. They begin with simple number sense; they then move to algebra and geometry; in their freshman year at college, they study computational calculus and linear algebra; and if they choose to move on, they are finally introduced to proof writing and rigorous theoretical mathematics.

While this approach to mathematical education is efficient and streamlined, it leaves out important context. Students are presented with polished and elegant theorems that connect to each other in beautiful and unexpected ways. While this is useful for clarity, it masks the reality that these discoveries involved life times of work. The development of new mathematics is challenging, frustrating, time consuming, and exciting. By hiding this from students we risk losing their interest.

As a student in mathematics, I would often think to myself, "how could anyone possibly come up with this?" However, after spending time studying mathematical history one learns about the amount effort that mathematicians put into their brilliant results. Additionally, spending time looking into the history provides the student with important context into why mathematicians were interested in these subjects in the first place. Often, in modern math classes, theorems and definitions are presented to the student with little context about their importance to greater field of mathematics. This varies from class-to-class and professor-to-professor, but the general trend is evident.

In this paper, I seek to provide some of that clarity and historical context behind the modern theory of integration. Today, many freshman calculus students and even the sophomore advanced calculus students take the theory of integration for gradated. It is easy to get swept up in integration techniques while losing the greater picture. I hope that the reader will find the subject of this paper interesting and be inspired by the work and care that went into the development of the Lebesgue integral–while paying homage to the mathematical giants that contributed to its invention.

2 General Historical Context

Many have the notion that mathematical thinking was always on solid ground. However, this is far from the reality. Until the 19^{th} century, mathematical thought was disorganized and decentralized, containing a wide variety of ideas, notations, paradoxes, and other inconsistencies that concerned many mathematicians and logicians of the time [Kli]. Motivated by this concern, mathematicians of the 19^{th} century developed the more organized and logically consistent mathematics that we recognize today.

In particular, the field of mathematical analysis went through major growing pains in the 1800s. In a letter written in 1826 mathematician N.H. Abel complained about the,

> tremendous obscurity on which one unquestionably finds in analysis. It lacks so completely all plan and system that it is peculiar that so many men could have studied it. The worst of it is, it has never been treated stringently.

At this time, analysis was recognized as an important field, but it was often criticised as inaccessible due to its lack of structure. In order to remedy this fundamental disorder, mathematicians of the time tirelessly developed more rigours definitions and methods for studying analysis. Specifically, there was a desire to move away from purely geometric understandings of analysis towards stricter, arithmetic definitions [Kli].

One area of analysis that went through major development during this time period was the study of integration theory. Contributions from mathematical heavyweights such as Newton, Cauchy, Riemann, Volterra, and Lebesgue gave birth to modern integration theory. It took more than a century of work to put integration on solid ground. As we will see the study of integration has undergone as series of changes in order to make it more rigorous.

3 The State of the Integral Before Lebesgue

It is difficult to discuss the Lebesgue integral without a strong introduction into the development of integration theory. In fact, the invention of the Lebesgue integral was a direct response to the limitations of early conceptions of the integral. At this time, it was common for mathematicians to directly respond to other mathematicians in their works, and as such, illustrates the inherit collaborative and compounding nature of mathematical research. As we go through the development of these ideas, it is useful to look the evolution of integration as a conversation between many mathematicians over hundreds of years.

3.1 Early Ideas: Newton and Leibniz

The foundations of integration theory are quite old and can even be traced back thousands of years to Greek, Arabic, and Chinese mathematicians. However, the modern integral has more direct roots in the development of calculus in the late 17^{th} and early 18^{th} centuries beginning with Isaac Newton (1643-1727) and Gottfried Leibniz (1646-1716). Newton and Leibniz are both considered to be the fathers of calculus, and by extension, modern integration theory.

Interestingly, while they both studied integration, Newton focused on integration as a reversal of differentiation while Leibniz focused on differentiation as a reversal of integration. Both, of course, were correct as demonstrated by the fundamental theorem of calculus. When it came to actually defining the integral, Leibniz insisted that the integral be defined as an infinite sum of rectangles/cylinders to approximate area; whereas Newton preferred the idea of defining the integral as the inverse of differentiation (often referred to as an anti-derivative) [Kli]. However, while their approach to conceptualizing integration differed, Leibniz and Newton shared similar methods of computing integrals. Both understood that one could compute the integral of a function on a interval by measuring the area under the curve.

In his text *Mathematical Principals of Natural Philosophy*[1], Newton introduces the idea of rather than calculating the area directly, we should try to estimate the area under a curve using left hand (lower) and right hand (upper) rectangles depicted blow.

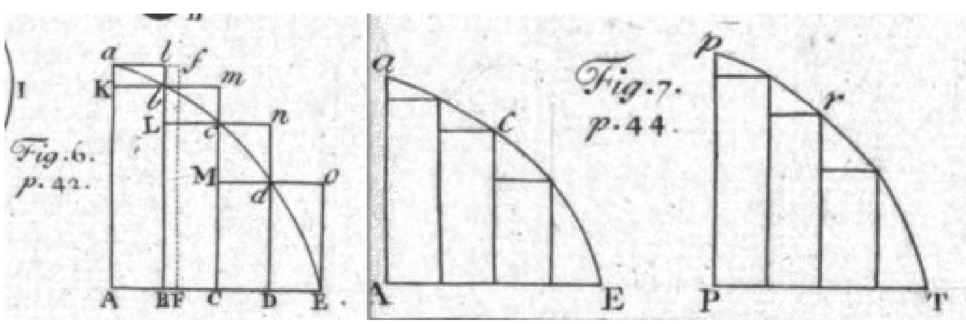

Figure 1: Newton's Motivating Figures for Lemma II
Philosophiæ Naturalis Principia Mathematica, CC0

Newton observed that as he decreased the width of the rectangles, thus increasing the quantity, he could get a better and better estimate of the true area. He summarized this idea in a theorem he called Lemma II.

A Note About Reading Historical Math Texts: Before going over this theorem it is important to note the difficulty of historical mathematical texts. In many cases, the difficulty of these texts does not always originate from the mathematical content. Due to the lack of standardization of notation and antiquated writing styles, older math texts can be difficult to parse. [Bou10]. Newton in particular was known for using obscure and difficult notation which made his work hard to interpret despite its brilliance.

1. In Old English, the symbol we use for integration today ($\int$) was analogous to the letter S. For example, where we would write Sum, Newton would write $\int$um.
2. Another convention in older texts is the symbol *&c* which means etc. or et cetera. In fact, *et* is Latin for and, so the use of the ampersand is quite intuitive.

[1] Philosophiæ Naturalis Principia Mathematica

3. Finally, Newton, among other mathematicians, frequently used the Latin phrase *ad infinitum* or just *infinitum* which means to infinity or infinitely. This phrase was invoked often to represent the idea of a limit or the repetition of some process to infinity.

We will see these conventions appear again when we look at the work of other early mathematical work.

Lemma 1. *[New, Lemma II] If any figure AacE (Pl.I.Fig6.) terminated by the right lines A a, AE, and the curve AcE, there be in∫scrib'd any number of parallelograms A b, B c, C d, &c. comprehended under equal b∫es A B,B C, C D, &c. and the ∫ides B b, C c, D d, &c. parallel to one ∫ide A a of the figure; and the parallelogram a K b l, b L c m, c M d n, &c are completed. Then if the breadth of tho∫e parallelograms be ∫uppos'd to be dimini∫ed, and their number to be augmented in infinitum: I ∫ay that the ultimate ratio's which the in∫crib'd figure A K b L c M d D, the circum∫scrib'd figure A a l b c n do E, and curvilinear figure A a b c d E, will have to one another, are ratio's of equality.*

In more modern language, Newton is saying that as the number of rectangles increases and their width decreases, the ratio between their sums of the upper and lower sums will get closer to 1 (or in other words produce an equal sum). Already in these early conceptions of the integral, we see glimpses of more rigorous definitions of the integral introduced first by Cauchy, improved by Riemann, and further expanded upon by Lebesgue.

3.2 Cauchy's Integral

Augustin-Louis Cauchy (1789-1857) was a prolific mathematician of the 19^{th} century. Aside from this many specific mathematical contributions, Cauchy's goal was to develop a rigours analysis, and of course, this included the integral. The ideas of Newton and Leibniz provided a foundation for integration, but Cauchy understood that their early conceptions still lacked rigor.

In his text *Summary of some Lessons on the Infinitesimal Calculus*[2], Cauchy introduces what he calls the definite integral of a function $f(x)$. His approach is similar to Leibniz and Newton in that he opts to partition the function's domain and estimate the area with an infinite sum of rectangles. He writes,

Si l'on divise $X - x_0$ en élémens infiniment petits $x_1 - x_0, x_2 - x_1, \cdots X - x_{n-1}$, la somme

$$S = (x_1 - x_0 f)(x_0) + (x_1 - x_2)f(x_1) + \cdots + (X - x_{n-1})f(x_{n-1})$$

convergera vers une limite représentée par l'intégrale définie

$$\int_{x_0}^{X} f(x)dx.$$

If we divide $X - x_0$ into infinitely small elements, $x_1 - x_0, x_2 - x_1, \ldots, X - x_{n-1}$, the sum

$$S = (x_1 - x_0)f(x_0) + (x_2 - x_1)f(x_1) + \ldots + (X - x_{n-1})f(x_{n-1})$$

will converge towards a limit represented by the definite integral

$$\int_{x_0}^{X} f(x)dx.$$

In this first conception, Cauchy only looks at left hand (lower) rectangles in his construction. Additionally, he does not specify that each rectangle have an equal base. In other words $x_i - x_{i-1}$ need not equal $x_j - x_{j-1}$ [Gil15]. It is important to note that, Cauchy only considered functions that were continuous on $[x_0, X]$. More exotic discontinuous functions were of less interest at this time (if even fully understood). Additionally, without comment, he implies that the length of the largest subinterval of the domain approaches 0 [Kli]. Many early math texts often assumed the reader would figure out key details or took them as obvious.

Later, Cauchy gave a more precise definition his integral where he allowed the height of the rectangles to vary. Instead of fixing the height of each rectangle to strictly be the image the left endpoint of each subinterval, he assigns a new variable ζ_i to be some random element from each subinterval and allows its image be the height of the rectangle [Cau23]. In more modern notation, we define Cauchy's definite integral

[2] Résumé des Leçons sur Le Calcul Infinitésimal

as follows:

$$\int_{x_0}^{X} f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\zeta_i)(x_i - x_{i-1}).$$

This construction allowed Cauchy to define the integral arithmetically as opposed the more loose geometric definitions that proceeded it.

3.3 Riemann's Integral

Georg Frierdrich Bernhard Riemann (1826-1866) was one of the most famous mathematicians from the 19^{th} century for his contributions to analysis and number theory. Interestingly, even though Riemann's integral is one of the most ubiquitous definitions of integration, it was more-or-less a footnote in his 1854 paper, *On the representability of a function by trigonometric series*[3]. Riemann developed his integral to be able to calculate coefficients in infinite trigonometric series. Riemann needed a definition of integration that could handle functions with many discontinuities and developed it based on Cauchy's work earlier that century. In fact, Riemann's integral is equivalent to Cuachy's, but his is far more rigorously defined.

We will proceed with a modern definition of Riemann's integral in order to more easily demonstrate its strengths and limitations. We will begin by setting up some standard definitions.

Definition 1. A partition of an interval $[a, b]$ is defined as a finite sequence of real numbers such that:

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b. \tag{1}$$

Definition 2. We define the length (or difference) of a subinterval of a partition as $\forall i \in \{1, 2, \ldots, n\}$:

$$\Delta x_i = x_i - x_{i-1}. \tag{2}$$

Definition 3. We define the mesh (or norm) of a partition $a = x_0 < x_1 < \ldots < x_n = b$ of the interval $[a, b]$ as:

$$\max \Delta x \coloneqq \max\{|\Delta x_i| : i = 1, \ldots, n\}. \tag{3}$$

Definition 4. Let $a = x_0 < \ldots < x_n = b$ be a partition of the interval $[a, b]$. We say $t_0, \ldots, t_{n-1}$ is a tagged partition if $\forall i \in \{1, 2, \ldots., n\}$:

$$x_i \le t_i \le x_{i+1}. \tag{4}$$

The following definition is commonly referred to as the Riemann integral.

Definition 5. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function defined on the interval $[a, b]$, and let x be a partition of $[a, b]$. Choose points t_i in $[x_{i-1}, x_i]$. We define the Riemann integral, if it exists, of f from a to b as:

$$\int_a^b f(x)dx = \lim_{\max \Delta x \to 0} \sum_{i=1}^{n} f(t_i)(\Delta x_i). \tag{5}$$

From this definition, Riemann also gives us a criteria for integrability. In his original paper, Riemann defines upper and lower sums as follows. First, for any of the subintervals in the partition, he lets $M_i \coloneqq \max f([x_{i-1}, x_i])$ and $m_i \coloneqq \min f([x_{i-1}, x_i])$. We then have that the upper and lower sums respectively:

$$S \coloneqq \sum_{i=1}^{n} M_i \Delta x_i, s \coloneqq \sum_{i=1}^{n} m_i \Delta x_i.$$

He defines $D_i \coloneqq M_i - m_i$ (this will always be positive by construction). He states that the integral of a function $f(x)$ exists if and only if the following condition holds:

$$\lim_{\max \Delta x \to 0} \sum_{i=1}^{n} D_i \Delta x_i = 0.$$

In other words, as the mesh of the partition tends towards 0, the upper and lower sums will become closer and closer in size eventually cancelling each other out. If the area of the rectangles do not get closer together in area, then we cannot take the integral.

[3] Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe

3.4 Limitations of Riemann's Integral

For most students, especially first year calculus students, the Riemann integral takes center stage as a highly intuitive and natural approach to integration, but we will soon see that although the Riemann integral is an incredibly powerful tool, it has many shortcomings which prove problematic for a generalized theory of integration.

3.4.1 Non-Riemann Integrable Functions

The first limitation of the Riemann integral is the existence of bounded functions whose Riemann integral is not defined. We begin with a standard example to demonstrate this. Let us define an indicator function on the rational numbers, $\mathbb{Q}$. We define $1_{\mathbb{Q}} : \mathbb{R} \to \{0, 1\}$ as,

$$1_{\mathbb{Q}} = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

This function was first introduced by Johann Peter Gustav Lejeune Dirichlet (1805-1859) in 1829, and while it feels quite artificial, it serves as a useful counterexample. By the density of the irrationals in $\mathbb{R}$, it is clear that this function is highly discontinuous. We will see that this "amount" of discontinuities is enough to make this simple function non-Riemann integrable. Let's show this using Riemann's criteria.

Proposition 1. $1_{\mathbb{Q}} : [0, 1] \to \{0, 1\}$ *is **not** Riemann integrable.*

Proof. Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be a partition of $[0, 1]$ such that length of each subinterval of our partition is of equal size:

$$\Delta x = \frac{1-0}{n}.$$

It suffices to show that $\lim_{\max \Delta \to 0} \sum_{i=1}^{n} D_i \Delta x_i \neq 0$. We observe that $\mathbb{Q} \cap [0, 1]$ and $(\mathbb{Q} \setminus \mathbb{R}) \cap [0, 1]$ are both dense in $[0, 1]$. Thus every subinterval in our partition will contain elements both from $\mathbb{Q} \cap [0, 1]$ and $(\mathbb{Q} \setminus \mathbb{R}) \cap [0, 1]$. This gives us that $\forall i, M_i = \max 1_{\mathbb{Q}}([x_{i-1}, x_i]) = 1$ and that $m_i = \min 1_{\mathbb{Q}}([x_{i-1}, x_i]) = 0$. Thus we have $\forall i, D_i = M_i - m_i = 1$. Taking our sum gives us:

$$\begin{aligned} \lim_{\max \Delta x \to 0} \sum_{i=1}^{n} D_i \Delta x_i &= \lim_{n \to \infty} \sum_{i=1}^{n} \frac{i}{n} \\ &= \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} i \\ &= \lim_{n \to \infty} \frac{n(n+1)}{2n} \\ &= \lim_{n \to \infty} \frac{n+1}{2}. \end{aligned}$$

Thus our series diverges. ■

3.4.2 The Limit Problem

The second major issue with the Riemann integral is often referred to as the limit problem [Loy]. Suppose we have a sequence of Riemann integrable bounded functions $f_n : [a, b] \to \mathbb{R}$ that converge to some bounded function $f : [a, b] \to \mathbb{R}$ does the following property always hold:

$$\int_a^b f(x)dx = \lim_{n \to \infty} \int_a^b f_n(x)dx?$$

We might expect this property to hold, but it does not. There are cases where the limit of a sequence of Riemann integrable functions is not a Riemann integrable function! In 1898, french mathematician

René-Louis Baire (1874-1932) introduced a fairly innocent sequence of functions. Suppose that we define $\forall n \in \mathbb{N}, f_n : [0,1] \to \{0,1\}$ as:

$$f_n(x) = \begin{cases} 1 & \text{if } x = p/q \text{ is rational in lowest terms with } q \leq n \\ 0 & \text{otherwise} \end{cases}.$$

Let us look at an example. Suppose $n = 4$, this gives us the following function:

$$f_4(x) = \begin{cases} 1 & \text{if } x \in \{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1\} \\ 0 & \text{otherwise} \end{cases}.$$

We also have that given an $n \in \mathbb{N}$, f_n is Riemann integrable. Let's use f_2 as an example. We have finite discontinuities at $\{0, \frac{1}{2}, 1\}$. Taking improper Riemann integrals gives us:

$$\begin{aligned} \int_0^1 f_2(x)dx &= \lim_{t \to 0^+} \int_t^{\frac{1}{4}} f_2(x)dx + \lim_{t \to \frac{1}{2}^-} \int_{\frac{1}{4}}^t f_2(x)dx + \lim_{t \to \frac{1}{2}^+} \int_t^{\frac{3}{4}} f_2(x)dx + \lim_{t \to 1^-} \int_{\frac{3}{4}}^t f_2(x)dx \\ &= 0 + 0 + 0 + 0 = 0. \end{aligned}$$

We see that this generalizes given any value $n \in \mathbb{N}$, f_n will have a Riemann integral of value 0. Thus every f_n in our sequence is Riemann integrable. However, observe what happens when we take the limit of f_n. $\forall x \in [0,1]$, we have:

$$\lim_{n \to \infty} f_n(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases} = 1_{\mathbb{Q}}(x).$$

Therefore there is a possibility that a sequence of Riemann integrable functions can converge to a nonintegrable function.

3.4.3 The Anti-Derivative Problem

There is another issue with the Riemann integral relating to the fundamental theorem of calculus (FTC). For some time, the following statement was taken as fact.

Let $f : [a,b] \to \mathbb{R}$ be a bounded function on $[a,b]$ and F be the anti-derivative of f such that $\forall x \in [a,b], F'(x) = f(x)$. Then the following equality holds.

$$\int_a^b f(x)dx = F(b) - F(a).$$

However, if the integral in the above equality is the Riemann integral, this equality does not always hold. That is, we can show that there exists a bounded, continuous, real-valued function F on $[a,b]$ whose derivative f exists, yet $\int_a^b f(x)$ is not defined in terms of Riemann. Such a function was described by Italian mathematician Vito Volterra (1860-1940) in 1881, but it requires some more theory to set up which will will discuss in a later section. For now, know that this is a limitation of the Riemann integral.

3.5 Summary

From the 16^{th} to 19^{th} century, the mathematical community developed a more rigorous definition of the definite integral based on the properties of infinite sums. These integrals all relied on partitioning the domain of a function. They were able to integrate both continuous functions and functions with jump discontinuities. However, this definition of the integral had some severe limitations that motivated future development of the theory of integration and raised the following questions:

1. Can we develop a more inclusive integral to handle highly discontinuous functions?
2. How discontinuous can a function be before it can no longer be Riemann integrable?
3. Can we fix fundamental theorem of calculus?

French mathematician Henri Lebesgue (1875-1941) saw these issues and came up with a novel solution that changed the theory of integration in a revolutionary way. In his ground breaking paper *On a generalization of the definite integral*, [4] Lebesgue writes, "Riemann defined the integral of certain discontinuous functions, but all derivatives are not integrable in the sense of Riemann. Research into the the problem of anti-derivatives is thus not solved by integration, and one can desire a definition of the integral...allowing one to solve the problem of anti-derivatives [Lebb]." It is important to keep in mind that this was Lebesgue's primary motivator when exploring integration. Like Newton, his approach was to treat integral as an inverse of differentiation.

4 Measure Theory

In order to discuss Lebesgue's integral, we need to set up some important tools–particularly a general theory of measure on sets. Measure theory is the foundation of Lebesgue's theory of integration, and it requires much discussion. Before jumping into the theory of measure in $\mathbb{R}$ we briefly need to set up some new notation.

Definition 6. We define the extended real numbers as the following set:

$$\bar{\mathbb{R}} \coloneqq \mathbb{R} \cup \{-\infty, \infty\}. \tag{6}$$

In the extended real numbers, we treat $\pm\infty$ as a valid mathematical object that sits on the line of extended real numbers. Keeping this concept in mind, in the most abstract sense, a measure is a mapping from a given set to some value in the extended real number line. The notation of the extended reals allows a set to have measure ∞ or $-\infty$.

$$m : \mathcal{M} \to \bar{\mathbb{R}}.$$

4.1 Measure in $\mathbb{R}$

While most modern applications measure theory are concerned with more abstract spaces, historically, Lebesgue and other measure theorists began by investigating measure on line and the plane. Since we are approaching Lebesgue's theory of integration from a historical perspective, we will begin on the line.

In his 1902 paper titled *Integral, Length, Area*[5] [Leba], Lebesgue outlined the following conditions for his measure. He wrote,

> Nous nous proposons d'attacher à chaque ensemble borné un nombre positif ou nul que nous appellerons sa mesure et satisfaisant aux conditions suivantes:
>
> 1. Il existe des ensembles dont la mesure n'est pas nulle.
> 2. Deux ensembles égaux ont même mesure.
> 3. La mesure de la somme d'un nombre fini out d'une infinité dénombrable d'ensembles, sans points communs, deux à deux, est la somme des mesures de ces ensembles.

> We propose to attach a positive number or zero to each bounded set that we call its measure and satisfies the following conditions:
>
> 1. There exists sets that have a non-zero measure.
> 2. Two sets that are equal have the same measure.
> 3. The measure of the sum of a finite or a denumerable [countablely] infinite number of sets without points in common, pairwise disjoint, is the sum of measures of these sets.

In more modern notation, we define such a measure as the *Lebesgue measure*. Let us introduce a measure and see if it satisfies Lebesgue's criteria. We will by defining the length of an interval.

Definition 7. Let $I \subset \mathbb{R}$ be an interval with endpoints $a \leq b$. We define the length of I as:

$$|I| \coloneqq b - a. \tag{7}$$

[4] Sur une généralisation de l'intérale définie

[5] Intégrale, Longueur, Aire

From this simple definition we elect the following measure as a candidate to satisfy the criteria for Lebesgue's measure.

Definition 8. Let $A \subseteq \mathbb{R}$. We define the *outer measure* of A as:

$$m^*(A) \coloneqq \inf \left\{ \sum_{k=1}^{\infty} |I_k| : \ (I_k)_{k=1}^{\infty} \textit{ is a sequence of open intervals with } A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}. \tag{8}$$

Before moving on, we will outline some important properties of the outer measure so that we can become more comfortable using it. First we will show that the outer measure of a closed interval is just its length.

Theorem 1. *Given any closed and bounded interval $A \subset \mathbb{R}$ with endpoints $a \leq b$, then $m^*(A) = |A|$.*

Proof. It suffices to show that $m^*(A) \leq |A|$ and $m^*(A) \geq |A|$.

1. We want to show $m^*(A) \leq |A|$. Let $\epsilon > 0$ be given. Define an open covering of A as $\forall k \in \mathbb{N}$:

$$I_k = \begin{cases} (a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2}) & k = 1 \\ \emptyset & k > 1. \end{cases}$$

   Taking the outer measure of $\bigcup_{k=1}^{\infty}$ gives us:

$$m^*(A) \leq \left(b + \frac{\epsilon}{2}\right) - \left(a - \frac{\epsilon}{2}\right) = b - a + \epsilon.$$

   Therefore we have $m^*(A) \leq b - a = |A|$ as required.

2. We want to show that $m^*(A) \geq |A|$. Let a be the left endpoint of A and b be the right such that $a \leq b$. Let $(I_k)_{k=1}^{\infty}$ be some open cover of A. Since A is an interval, then it is compact. Thus, we can choose a finite subcover of A. Let us enumerate our finite covering $I_1, I_2, \cdots, I_N$ such that $a \in I_1$ and $b \in I_N$ (provided that $b \neq a$ otherwise, $b \in I_1$). This gives us:

$$a = a_1 < a_2 < \cdots < a_{N+1} = b, (a_j, a_{j+1}) \subseteq I_j.$$

   Taking the infimum over all possible finite subcovers gives us:

$$|A| = b - a = a_{N+1} - a_1 = \sum_{j=1}^{N} a_{j+1} - a_j \leq \sum_{j=1}^{N} |I_j| \leq \sum_{j=1}^{\infty} |I_j|.$$

   Therefore $m^*(A) \geq |A|$.

■

The proof for open and half open intervals is similar which yields the following theorem.

Theorem 2. *Let $I \subseteq \mathbb{R}$ be an interval, then $m^*(I) = |I|$.*

Additionally, this theorem shows that this outer measure satisfies Lebesgue's first criteria for a measure. Take the unit interval $m^*([0, 1]) = 1$. Thus, there exists a set without measure 0. Additionally, because of uniqueness of infimum, two equal sets will have the same outer measure satisfying the second condition. Next, we will show that all countable sets have a measure of 0.

Theorem 3. *Let A be a countable set, then $m^*(A) = 0$.*

Proof. It suffices to show that $\forall \epsilon > 0, m^*(A) \leq \epsilon$. Let $\epsilon > 0$ be given, enumerate the elements of A as $(x_k)_{k=1}^{\infty}$, and let I_k be an open interval of length $\epsilon / 2^k$ that contains x_k. Then we have that the collection

$\{I_k\}$ covers A. This gives us:

$$\begin{aligned} m^*(A) \leq \sum_{k=1}^{\infty} |I_k| &= \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} \\ &= \sum_{k=1}^{\infty} \epsilon \left(\frac{1}{2}\right)^k \\ &= \epsilon\left(\frac{1/2}{1-1/2}\right) \\ &= \epsilon. \end{aligned}$$

■

These theorems yield the following properties of the outer measure.

Theorem 4. *The following properties of the outer measure hold.*

1. $0 \leq m^*(A) \leq +\infty$;
2. $A \subseteq B \implies m^*(A) \leq m^*(B)$;
3. $A \subseteq \bigcup_{k=1}^{\infty} A_k \implies m^*(A) \leq \sum_{k=1}^{\infty} m^*(A_k)$;
4. *Let A be an interval, then $m^*(A) = |A|$;*
5. $m^*(A+h) = m^*(A)$.

4.2 Measureability in $\mathbb{R}$

As we have been working with the outer measure, the reader will note that we have demonstrated the outer measure satisfies Lebesgue's first two criteria for a measure, but we have have thus far been silent on the third criteria. Rewriting Lebesgue's third requirement in modern notation, we would expect that given any countable collection of pairwise disjoint sets $\{C_k\}$,

$$m^*(C_0 \sqcup C_1 \sqcup \ldots \sqcup C_k \sqcup \ldots) = m^*(C_0) + m^*(C_1) + \ldots + m^*(C_k) + \ldots$$

This only seems natural. The next question is: does this work for every subset of the real line? As surprising as it is, the answer is no. Will see in a later section that Italian mathematician Giuseppe Vitali (1875-1932) demonstrated that, in fact, not all sets of the real line satisfy this property. If we try to measure a set that does *not* have this property, then the measure of such a set would be of little use to us. In order to account for the fact that not all sets conform to this criteria, we simply restrict the sets that we that we can measure.

This was no surprise to Lebesgue. Returning to his work *Integral, Length, and Area*[Leba], he commented heavily on the possibility of nonmeasurable sets. He referred to this issue generally as, *the problem of measure*. In fact, Lebesgue outlined a criteria of measureability that he ad adapted from the work of mathematician Émile Borel (1871-1956) who also made major contributions to measure theory. Lebesgue states,

> Nous ne résoudrons ce *problème de la mesure* que pour les ensembles que nous appellerons mesurables...Nous appellerons *ensembles mesurables ceux dont les mesures extieure et intérieure sont égales*, la valeur commune de ces deux nombres sera la mesure de l'ensemble, problème de la mesure est possible.

> We will [only] solve this *problem of measure* for sets that we call measurable...We will call *measurable sets those whose exterior measure and interior measure are equal.* The value in common between these two numbers will be the measure of the set, if the problem of measure is possible.

Today, we still use this concept of measureability in a more concise form. The following is the modern definition of measureability.

Definition 9. Let $A \subseteq \mathbb{R}$. A is said to be measurable if, $\forall E \subseteq \mathbb{R}$,

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A^c). \tag{9}$$

Abiding by this restriction, it is useful to define a superset that contains all the sets which Lebesgue deemed measurable.

Definition 10. We denote $\mathcal{M}$ as the set of all measurable sets.

The next theorem shows that, because of the subadditivity of the outer measure, we can obtain a lighter condition for measureability.

Theorem 5. *Let* $A \subseteq \mathbb{R}, \forall E \subset \mathbb{R}, m^*(E) \geq m^*(E \cap A) + m^*(E \cap A^c) \implies A \in \mathcal{M}$

Proof. Let $E \subseteq \mathbb{R}$ be given. Observe that $E = (E \cap A) \cup (E \cap A^c)$. By subadditivity we have:

$$m^*(E) = m^*((E \cap A) \cup (E \cap A^c)) \leq m^*(E \cap A) + m^*(E \cap A^c).$$

Thus by definition of measurability, it suffices to prove show that $m^*(E) \geq m^*(E \cap A) + m^*(E \cap A^c)$. ■

The following theorem shows that all sets of measure 0 are measurable.

Theorem 6. *Let* $A \subseteq \mathbb{R}, m^*(A) = 0 \implies A \in \mathcal{M}$.

Proof. Let $E \subseteq \mathbb{R}$ be given. We observe that $E \cap A \subseteq A$, thus we have that:

$$0 \leq m^*(E \cap A) \leq m^*(A) = 0.$$

We also have that $A^c \cap E \subseteq E$ which gives us $m^*(A^c \cap E) \leq m^*(E)$. Thus we have:

$$m^*(E \cap A) + m^*(E \cap A^c) \leq m^*(E).$$

■

An additional property is that, if a set is measurable, then its complement is also measurable. This follows directly from the definition of measureability.

Theorem 7. $A \in \mathcal{M} \implies A^c \in \mathcal{M}$.

Proof. Let E be given. Since $A \in \mathcal{M}$, we have:

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A^c) = m^*(E \cap A^c) + m^*(E \cap A). \tag{10}$$

■

The next theorem will prove that if a collection of disjoint sets are each measurable, then the outer measure will satisfy Lebesgue's third condition for measure. First we will need to prove a lemma.

Lemma 2. *Let* $\{A_N\}$ *be a countable collection of pairwise disjoint and measurable sets. Let* $E \subset \mathbb{R}$. *Then the following holds:*

$$m^*\left(E \cap \bigcup_{k=1}^{N} A_k\right) = \sum_{k=1}^{N} m^*(E \cap A_k). \tag{11}$$

Proof. The base case $N = 1$ is trivial. For $1 \leq n \leq N$, let $B_n = \bigcup_{k=1}^{n} A_k$. By induction, assume that,

$$m^*(E \cap B_n) = \sum_{k=1}^{n} m^*(E \cap A_k).$$

Applying our induction hypothesis gives the following,

$$\begin{aligned} m^*(E \cap B_{n+1}) &= m^*(E \cap A_{n+1}^c \cap B_{n+1}) + m^*(E \cap A_{n+1} \cap B_{n+1}) \\ &= m^*(E \cap B_n) + m^*(E \cap A_{n+1}) \\ &= \sum_{k=1}^{n} m^*(E \cap A_k) + m^*(E \cap A_{n+1}). \end{aligned}$$

■

Theorem 8. *Let $(A_k)_{k=1}^{\infty}$ be a sequence of pairwise disjoint and measurable sets, then $\bigcup_{k=1}^{\infty} A_k$ is measurable and*

$$m^*\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} m^*(A_k). \tag{12}$$

Proof. We want to show that $\bigcup_{k=1}^{\infty} A_k$ is measurable and $m^*\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} m^*(A_k)$.

1. Let $E \subset \mathbb{R}$ be given. Let $B_n = \cup_{k=1}^{n} A_k$ and $B = \cup_{k=1}^{\infty}$. Since B_n is measurable, by lemma 2,

$$m^*(E) = m^*(E \cap B_n) + m^*(E \cap B_n^c) = \left(\sum_{n=1}^{n} m^*(E \cap A_n)\right) + m^*(E \cap B_n^c).$$

   Since $B^c \subseteq B_n^c$, we can take the limit as $n \to \infty$ giving us,

$$m^*(E) \geq \left(\sum_{k=1}^{\infty} m^*(E \cap A_k)\right) + m^*(E \cap B^c) \geq m^*(E \cap B) + m^*(E \cap B^c).$$

   Therefore $B = \bigcup_{n=1}^{\infty} A_n$ is measurable. We also have that,

$$m^*(E) = m^*(E \cap B) + m^*(E \cap B^c) \leq \left(\sum_{n=1}^{\infty} m^*(E \cap A_n)\right) + m^*(E \cap B^c)$$

2. Now, we want to show that this sum is additive. From above, we have the following equality,

$$m^*(E) = \sum_{k=1}^{\infty} m^*(E \cap A_k) + m^*\left(E \cap \left(\bigcup_{k}^{\infty} A_k\right)^c\right)$$

   Since E was taken to be arbitrary, let $E = \bigcup_{k}^{\infty} A_k$. This gives us,

$$m^*\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} m^*(A_k).$$

■

From this, it follows the infinite intersection of a sequence of measurable sets is also measurable. This fact will prove useful in later sections.

Theorem 9. *Let $(A_k)_{k=1}^{\infty}$ be a sequence of pairwise disjoint measurable sets. Then $\bigcap_{k=1}^{\infty} A_k$ is also measurable.*

Proof. We have:

$$\bigcap_{k=1}^{\infty} A_k = \left(\bigcup_{k=1}^{\infty} A_k^c\right)^c.$$

By theorem 7, since each A_k is measurable, then A_k^c is also measurable, so applying theorem 8 we have that $\bigcup_{k=1}^{\infty} A_k^c$ is measurable. Finally, we apply theorem 7 again, and conclude $\bigcap_{k=1}^{\infty} A_k$ is measurable. ■

Now that we have shown that the outer measure satisfies Lebesgue's three criteria under the constraint of measureability we can finally define the Lebesgue measure.

Definition 11. Let $A \subseteq \mathbb{R}$ be measurable, then its *Lebesgue measure* is:

$$m(A) \coloneqq m^*(A). \tag{13}$$

At this point, it should be clear why the Lebesgue measure is a powerful tool. Many of the basic sets that we would expect to be measurable are. However, we still have the question of whether or not nonmeasurable sets exist.

4.3 A Nonmeasurable Set

The problem of measuring sets of points of a straight line is insurmountable.
- GIUSEPPE VITALI

As alluded to earlier, nonmeasurable sets exist. The existence of such sets were of great concern to mathematicians at the turn of the 20^{th} century as it called into question the utility of measure. In this next section, we will give an example of a nonmeasurable set in addition to discussing the theoretical considerations given its existence.

4.3.1 A Nonmeasurable Construction in $\mathbb{R}$

The first mathematician to construct a Lebesgue nonmeasurable set was Giuseppe Vitali in 1905 [Vit05]. Vitali's construction relied on a clever use of the Axiom of Choice. This move is subtle, but it is the crux of the argument. We will go over an example presented by Frank Burk in his book *Lebesgue Measure and Integration: An Introduction* [Bur], as it provides far more detail than Vitali's original paper.

We begin by setting up an equivalence relation such that for any $x, y \in \mathbb{R}, x \sim y$ if and only if the difference $x - y$ is rational. These give rise to the following equivalence classes:

$$E_x = \{y \in \mathbb{R} : x - y \in \mathbb{Q}\}.$$

These equivalence classes have the following properties:

1. $\mathbb{R} = \cup E_x$.
2. $x_1 \sim x_2 \implies E_{x_1} = E_{x_2}$.
3. $E_{x_1} \cap E_{x_2} \neq \emptyset \implies E_{x_1} = E_{x_2}$.

Based on these properties we observe that we have created a valid partition of $\mathbb{R}$. Using this partition, we will create a nonmeasurable set of $\mathbb{R}$ contained in the open interval $(-1, 1)$. We begin by creating equivalence classes of the elements in $(-1, 1)$ such that

$$(-1, 1) \cap E_x = \{y \in (-1, 1) : x - y \in \mathbb{Q}\}.$$

We observe the following about the collection $\{E_x\}$:

1. Each E_x is countable;
2. The entire collection of distinct sets E_x is uncountable.

Now, let us *choose* one element from each E_x such and place it in a new set $\mathcal{V}$. This step is subtle, but we have just invoked the Axiom of Choice to construct this set. This will inevitably lead to the nonmeasurability of this set.

Proposition 2. *$\mathcal{V}$ is nonmeasurable.*

Proof. Assume towards contradiction that $\mathcal{V}$ is measurable. Observe the following about $\mathcal{V}$:

1. $\mathcal{V}$ is uncountable.
2. $\forall x \in (-1, 1), \mathcal{V} \cap E_x$ is a singleton.

Next, let us enumerate the rational numbers in the open interval $(-2, 2)$. Let us call this enumeration $(q_k)_{k=1}^{\infty}$. We define:

$$\mathcal{V} + q_i = \{x + q_n : x \in \mathcal{V}, q_i \in (q_k)_{k=1}^{\infty}\}.$$

We then observe the following containments:

- $(-1, 1) \subseteq \cup \mathcal{V} + q_k$;
- $\mathcal{V} + q_i \subseteq (-3, 3)$.

Thus we have that the collection of sets $\{\mathcal{V}+q_n\}$ creates a countable and disjoint cover of $(-1,1)$. Thus we have:

$$(-1,1) \subset \bigcup_{k=1}^{\infty}(\mathcal{V}+q_k) \subset (-3,3).$$

The monotonicity of the outer measure measure gives us:

$$m(-1,1) \leq m\left(\bigcup_{k=1}^{\infty}(\mathcal{V}+q_k)\right) \leq m(-3,3).$$

The countable additivity of the outer measure gives us:

$$m(-1,1) \leq \sum_{k=1}^{\infty} m(\mathcal{V}+q_k) \leq m(-3,3).$$

Since each $\{\mathcal{V}+q_k\}$ is a translate of $\mathcal{V}$, then by (5) of theorem 4, we have that each $m(\mathcal{V}+q_k) = m(\mathcal{V})$. Substitution gives us:

$$m(-1,1) \leq \sum_{k=1}^{\infty} m(\mathcal{V}) \leq m(-3,3).$$

Evaluating $m((-1,1)) = 2$ and $m((-3,3)) = 6$. Since $\sum_{k=1}^{\infty} m(\mathcal{V}) \leq 6$, then $m(\mathcal{V}) = 0$ because a infinite sum of a constant diverges to ∞. Since $2 \leq \sum_{k=1}^{\infty} m(\mathcal{V})$, then $m(\mathcal{V}) > 0$. Thus we have a contradiction, and $\mathcal{V}$ is nonmeasurable. ■

4.3.2 An Analogy About Sawdust

Now that we have seen that there exists a nonmeasuarable set, this raises a natural question: if there exists sets that are not measurable by the outer measure, then why not develop a better measure? This is a fair question because having a theory of measurement that cannot handle every set is concerning to some. Historically, many mathematicians have pushed against measure theory because of the existence of such sets. This, of course, greatly over looks the utility of measure which we will explore in later sections. It is important to note that a measurement is only as useful as its utility. The fact is, it is rare that we encounter unmeasurable sets. It is not impossible, as we have demonstrated above, but one can see that they are not obvious.

To drive this point home, I will reference an analogy from my capstone mentor. He said, "you would use a ruler to measure a board, but it would be impractical to use a ruler to measure sawdust." Measure, like a ruler, is a tool that has practicality and great use in many circumstances even if it has constraints. Additionally, this does not mean we cannot "measure" sawdust, we simply need a different *way* to measure it–take for example weight or volume. The same can be said for these nonmeasurable sets; their existence does not threaten the utility of measure theory. Instead, they demonstrate a context in which this particular measure would be a poor tool.

The fact is, the outer measure is a *very* practical measure. The nonmeasurable sets in terms of Lebesgue are relatively rare and often only of interest when demonstrating that they are not measurable. Additionally, the outer measure is a fairly intuitive measure. As we have seen, when working with intervals it takes their lengths; it assigns a measure of 0 to countable sets; it handles both simple and complex sets with relative ease; and it generalizes quite nicely. Fundamentally, the outer measure is a tool, and like all tools, it is incumbent on the user to know when it is appropriate to use it.

As a final word on this subject, we mentioned during the construction that this example relies on a clever use of the Axiom of Choice. A natural question might be: can we construct a nonmeasurable set without relying on the Axiom of Choice? As it turns out, it is not possible. In his 1970 paper *A Model of Set-Theory in Which Every Set of Reals is Lebesgue Measurable*, mathematician Robert M. Solovay demonstrates that the only way to construct a nonmeasurable set in terms of Lebesgue is by use of the Axiom of Choice [Sol70].

5 A Lengthy Aside on Smith, Cantor, and Volterra

Now that we have a notion of measure and measureability in $\mathbb{R}$, we will return to the Riemann integral to develop a better understanding of why Lebesgue was so interested in the anti-derivative problem.

After the introduction of the Riemann integral to the greater mathematical community in 1868, the following decade saw mathematicians become increasingly interested in investigating the, "conditions under which a function could be integrated [Fle94]." Specifically, mathematicians spent a great deal of time constructing functions with infinite discontinuities that were still integrable in the sense of Riemann [Kli]. This work gave rise to a famous theorem called the Lebesgue Criterion for Riemann integrability which gives us precise conditions under which a function can be integrated using Riemann's definition. The next theorem is the Lebesgue Criterion for Riemann integration.

Theorem 10 (Lebesgue Criterion for Riemann Integration)**.** *Let $f : [a, b] \to \mathbb{R}$, and let $D(f) \coloneqq \{x \in [a, b] : f \text{ is discontinuous at } x\}$. We say that f is Riemann integrable if and only if f is bounded and $D(f)$ has Lebesgue measure 0.*

Before moving on, let's look at some examples to illustrate the necessity of the hypotheses. Let us first return to the Dirichlet function. Clearly $1_{\mathbb{Q}}$ is a bounded function, but from an earlier section, we know that it is not Riemann integrable, so what is its domain of discontinuity? We claim that $1_{\mathbb{Q}}$ is everywhere discontinuous.

Proposition 3. *$1_{\mathbb{Q}} : [0, 1] \to \{0, 1\}$ is nowhere continuous (everywhere discontinuous) on $[0, 1]$.*

Proof. Let $x \in [0, 1]$ be given. We have exactly two cases (1) $x \in \mathbb{Q} \cap [0, 1]$ or (2) $x \in [0, 1] \setminus \mathbb{Q}$.

1. Assume that x is rational. Choose $\epsilon = \frac{1}{2}$. Let $\delta > 0$ be given. By density of the irrationals in $\mathbb{R}$, choose an irrational number $y \in [0, 1]$ such that $|x - y| < \delta$. We have:

$$|1_{\mathbb{Q}}(x) - 1_{\mathbb{Q}}(y)| = |1 - 0| = 1 > \frac{1}{2}.$$

2. Assume that x is irrational. Again, choose $\epsilon = \frac{1}{2}$. Let $\delta > 0$ be given. By density of the rationals in $\mathbb{R}$, choose a rational number $y \in [0, 1]$ such that $|x - y| < \delta$. We have:

$$|1_{\mathbb{Q}}(x) - 1_{\mathbb{Q}}(y)| = |0 - 1| = 1 > \frac{1}{2}.$$

■

This fact tells us the set of discontinuities of $1_{\mathbb{Q}}$ on $[0, 1]$ is $[0, 1]$, and since $[0, 1]$ is an interval, then $m([0, 1]) = 1 > 0$. Thus we cannot simply have a bounded function.

The other hypothesis states that Riemann integral must be taken on a bounded function. This was mentioned in the definition of the Riemann integral, but we have not explored it in great detail. Here is another question, what would happen if we tried to take the integral of the following function on the interval $[0, 1]$?

$$h(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}. \tag{14}$$

First, it is clear that on $[0, 1]$ that $D(h) = \{0\}$ and as a result has measure 0. Further more, it is obvious that h is not bounded. We see that it approaches ∞ as $x \to 0^-$. Thus we only have one hypothesis satisfied. We will show that this is not a sufficient condition for Riemann integrability.

Let us begin in our usual fashion by creating a partition of $[0, 1]$ such that $0 = x_1 < x_2 < \cdots < x_n = 1$. Suppose now that we want to find the height of the right hand rectangles on this partition, we have that $\sup(h([0, x_1])) = \infty$. Clearly we cannot interpret such a height. Thus we must have a bounded function.

Now that we have briefly explored the necessity of the hypotheses in Lebesgue's criteria, we will look at some historical motivating examples that led Lebesgue to become interested in the anti-derivative problem mentioned in section 3.4.3.

5.1 Cantor Sets

Historically, our conversation begins with Cantor sets. Cantor sets are quite famous and appear in many fields of mathematics in novel and unexpected ways. However, their discovery was directly related to integration theory. While Cantor sets are most commonly associated with the mathematician Georg Cantor (1845-1918) who wrote about them in 1883, they were actually introduced in 1875 by Irish, Oxford professor, Henry John Stephen Smith (1826-1883) in his paper titled *On the Integration of Discontinuous Functions*, and to this date, is their earliest known mention. While Cantor's paper is much more famous, Smith's paper introduced a important idea in the development of understanding integrable functions [Sco19]. At the time, Smith was relatively unknown to the greater mathematical community, and his discoveries were not recognized until much later. In his work, Smith was interested in constructing sets whose points he described as lying in *loose order*. He wrote,

> A system of points is said to *fill completely* a given interval, when any segment of the interval being taken, however small, one point at least of the system lies on that segment. . . points are in close order on any segment when the completely fill it, and in *loose order*, and in *loose order* when they do not completely fill it or any pare of it, however small [Smi75].

The observant reader will note that when Smith talks about points who *fill and interval completely* he is discussing the concept of density. Smith's concept of a system of *loose order* order points would be describe today as a *nowhere dense set*.

The sets we have looked at up until this point have been fairly straight forward and easy to visualize (aside from $\mathcal{V}$), but the measure of a nowhere dense set like the Cantor Set is not immediately obvious. While Smith was the first person to come up with a generalized Cantor set, Cantor's middle thirds set serves as a fairly straight forward introduction into nowhere dense sets. In his famous paper titled *On infinite linear manifolds of points, Part 5*, Cantor defined his set by beginning with the closed unit interval $[0, 1]$ which we will denote C_0. Then he removes the middle third, and labels the union of the remaining closed intervals C_1:

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Then, he continues this process for each smaller interval, so for example, to go from C_1 to C_2, we remove the middle thirds of the two intervals that make up C_1.

$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

As we continue this process to infinity, we observe the following relationship:

$$C_1 \supset C_2 \supset C_3 \supset \ldots \supset C_k \supset \cdots$$

After constructing this sequence of subsets of the unit interval, Cantor then defines his set as their infinite intersection,

$$\mathcal{C} \coloneqq \bigcap_{k=1}^{\infty} C_k. \tag{15}$$

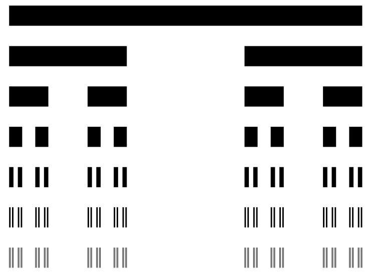

Figure 2: Visualization of the middle thirds Cantor Set's Construction

With such a simple construction we would expect a set like this to be easy to understand, yet only after a few iterations, we see how complex it becomes. Naturally, since we are discussing measure theory, we might ask two questions:

1. Is such a set measurable?
2. If so, what is its measure?

Fortunately, the Cantor set is both measurable and we are able to calculate it [ES05]. Before taking its measure, let's discuss some nice properties about the Cantor set. Because $\mathcal{C} \subset [0, 1]$, $\mathcal{C}$ is trivially bounded. Also, because $\mathcal{C}$ is the infinite intersection of closed sets, then $\mathcal{C}$ is also closed. Finally, because $\mathcal{C}$ is both closed and bounded, by Heine-Borel, $\mathcal{C}$ is compact.

Proposition 4. *$m^*(\mathcal{C}) = 0$ and $\mathcal{C} \in \mathcal{M}$.*

Proof. By theorem 6, it suffices to show $m(\mathcal{C}) = 0$. Let us look at the length of any given C_k. Since C_k is a disjoint union of closed 2^k intervals, it follows that $m^*(C_k) = (2/3)^k$. Thus if we take the intersection of these sets, their Lebesgue measure will be the k^{th} iteration. Therefore our outer measure will be the limit of the following sequence:

$$\lim_{k\to\infty} (2/3)^k = 0.$$

∎

Finally, using these properties, we will show that $\mathcal{C}$ is nowhere dense in $[0, 1]$.

Proposition 5. *$\mathcal{C}$ is nowhere dense in $[0, 1]$.*

Proof. We want to show that $\text{int}(\text{cl}\mathcal{C}) = \emptyset$. Because $\mathcal{C}$ is closed, then $\text{cl}\mathcal{C} = \mathcal{C}$. Thus it suffices to show that $\text{int}(\mathcal{C}) = \emptyset$. Assume towards contradiction that $\text{int}(\mathcal{C}) \neq \emptyset$. Let $x \in \text{int}(\mathcal{C})$ such that for some $\epsilon > 0$, $B_\epsilon(x) \subset \mathcal{C}$. We have $m(B_\epsilon(x)) = 2\epsilon > 0$, but $m(\mathcal{C}) = 0$. Contradiction. ∎

5.2 A Nowhere Dense Set with Positive Outer Measure

Since the middle thirds Cantor set is nowhere dense in $[0, 1]$, it makes its measure of 0 feel fairly natural. We may even feel temped to say that all nowhere dense sets have measure 0. In fact, this was the assumption of early mathematical work on the subject of nowhere dense sets. Mathematician Hermann Hankel (1839-1873) claimed that, "all nowhere dense subsets of the real line could be enclosed in intervals of arbitrarily small total length (i.e. had zero outer content) [Fle94]." Hankel's assumption was incorrect. Smith's paper is primarily a response to Hankel's flawed claim where Smith introduces a rather interesting set that has this nowhere dense property but also has positive outer measure. Tragically, Hankel was never able to read Smith's paper as he passed away at the young age of 34, and in his paper, Smith takes the time to briefly eulogize Hankel commenting that his death was,"a great loss to the mathematical community [Smi75]."

In order to counter Hankel, Smith outlines a general construction for nowhere dense sets. He states,

> Let m be any given integral number [integer] greater than 2. Divide the interval from 0 to 1 into m equal parts; and exempt the last segment from any subsequent division. Divide each of the remaining $m-1$ segments into m equal parts; and exempt the last segments from any subsequent subdivision. If this operation be continued *ad infinitum* [infintly],we shall obtain an infinite number of points of division P upon the line from 0 to 1. These points lie in loose order [nowhere dense] [Smi75].

Today, we recognize this as a generalization of Cantor sets, where the middle thirds Cantor set is the special case of $m = 3$. It should also be noted that in this general construction, Smith chooses to remove the last segment as opposed to the middle segment.

Later in his paper, Smith discussed another set that was similar in nature to the Cantor set. Starting with a similar premise, Smith started with a closed interval $[0, 1]$ and began dividing it. However, instead of dividing the intervals at each step by a constant factor of m, he decided to divide each interval by a factor of m^k. Thus, at every step, the proportion that he ends up removing decreases with every step. Let's take a look at the middle-thirds Cantor set, but instead of removing a constant $m = 3$ for every step, we divide every interval by a factor of of 4^k. Again, we begin with the unit interval $F_0 = [0, 1]$. We then remove the middle fourth.

$$F_1 = \left[0, \frac{3}{8}\right] \cup \left[\frac{5}{8}, 1\right].$$

Repeating this process, we get:

$$F_2 = \left[0, \frac{5}{32}\right] \cup \left[\frac{7}{32}, \frac{3}{8}\right] \cup \left[\frac{5}{8}, \frac{25}{32}\right] \cup \left[\frac{27}{32}, 1\right].$$

Like the middle-thirds Cantor set, we continue this process to infinity and finally take the infinite intersection:

$$\mathcal{F} := \bigcap_{k=1}^{\infty} F_k. \tag{16}$$

Up until this point, the construction feels fairly familiar, but this set has a fundamental difference from $\mathcal{C}$–its outer measure is non-zero. Before taking its measure, we observe that, $\mathcal{F}$ is a bounded set whose construction is infinite intersection of disjoint intervals, thus by theorem 9, $\mathcal{F}$ is measurable.

To compute its measure, we will take a different approach from $\mathcal{C}$. We observe that at each step, our length (measure) decreases, but the amount that it decreases varies at each step. On the first step, we split our interval into two intervals, and our length decreases by $\frac{1}{4}$ from 1 to $\frac{3}{4}$. On the second step, we now have 4 intervals, length decreases by $\frac{2}{16}$ from $\frac{3}{4}$ to $\frac{5}{8}$. Continuing this process, we represent it with the following sum:

$$\begin{aligned} 1 - \sum_{k=0}^{\infty} \frac{2^k}{4^{k+1}} &= 1 - \frac{1}{4} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \\ &= 1 - \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

We have alluded to the fact that this set is nowhere dense, but it is not immediately obvious that this is the case. Because this set has positive Lebesgue measure, then we will take a slightly different approach than we did with $\mathcal{C}$ [Mer15].

Proposition 6. *$\mathcal{F}$ is nowhere dense in* $[0, 1]$.

Proof. Because $\mathcal{F}$ is closed, it suffices to show that $\text{int}\mathcal{F} = \emptyset$. First, we claim that $\mathcal{F}$ contains no intervals. Let $x, y \in \mathcal{F}$ such that $x < y$. Then for some $k \in \mathbb{N}$, we have:

$$\ell_k = \frac{2^{k+1}}{2^{2k+1}} < y - x.$$

Since $\ell_k < y - x$, then they will not be in the same subinterval of F_k. thus we can choose some $z \notin \mathcal{F}$ such that $x < z < y$. Since x, y, z were taken to be arbitrary, then no interval exists in $\mathcal{F}$.

Assume towards contradiction that $x \in \text{int}\mathcal{F}$, then there exists some $\epsilon > 0$ such that $B_\epsilon(x) \subset \mathcal{F}$, but we showed that $\mathcal{F}$ contains no intervals. Contradiction. ■

Thus we have shown that $\mathcal{F}$ is indeed nowhere dense. Sets of this type of sets are called Smith-Volterra-Cantor sets, but it is more common to hear them lovingly called "Fat Cantor Sets." In fact, there is nothing special about this set having an Lebesgue measure of $\frac{1}{2}$. Using this general approach, we can construct nowhere dense sets of varying positive Lebesgue measure less than 1.

In fact, Smith's astounding discovery not only challenged Hankel's view on nowhere dense sets, but it further challenged his understanding of Riemann integrable functions. Because he believed that all nowhere dense sets had no positive outer content, Hankel developed his own criteria for Riemann integrability. He claimed that a function is Riemann integrable if and only if it is *pointwise discontinuous*. The notion of pointwise discontinuity is now an antiquated term, but we say that a function f is "*pointwise discontinuous* if it has infinitely many points of discontinuity, yet is continuous on a dense set [JCPC15]." In fact, in the next section we will see how the existence of Fat Cantor Sets challenged mathematicians understanding of the FTC. After, his discovery, Smith wrote, "the result obtained in the last example deserves attention, because it is opposed to a theory of discontinuous functions, which has received the sanction of an eminent geometer, Dr. Hermann Hankel [Smi75]."

5.3 Volterra's Function

> *In some cases, it can happen that the ordinary definition of the integral is not included in that of Riemann.*
> - VITO VOLTERRA

In section 3.4 we noted that the if the Riemann integral is used in the fundamental theorem of calculus (FTC), then the FTC might not hold for all functions. In fact, the most famous counter example, is directly related to the Fat Cantor set. In 1881, Italian mathematician Vito Volterra (1860-1940) came up with a continuous function with a bounded yet non-Riemann integrable derivative. More specifically, he constructed a function $V : [0, 1] \to \mathbb{R}$ that makes the following equation meaningless in context of Riemann's integral:

$$\int_a^b V'(x)dx = V(b) - V(a).$$

The existence of such functions influenced Lebesgue to develop a more inclusive integral, as Volterra's function explicitly demonstrated the limitations of Riemann's integral.We will use an example from Juan Ponce-Campuzano and Miguel Maldonado-Aguilar which models itself off of Volterra's original construction [JCPC15].

Volterra used a Fat Cantor set as a basis for his function. We will use our construction $\mathcal{F}$ as defined in section 5.2. He begins his construction by defining a function $f_{a,b} : [a, b] \to \mathbb{R}$ that behaves like $f(x) = x^2 \cdot \sin(\frac{1}{x})$ on every interval. Before defining f, we will define the values x_1 and x_2. $\forall a < b \in \mathbb{R}$,

- Let x_1 be largest number less than or equal to $\frac{a+b}{2}$ such that $(x - a)^2 \sin(\frac{1}{x-a})$ has a maximum;
- Let x_2 be the smallest number greater than or equal to $\frac{a+b}{2}$ such that $(b-x)^2 \sin(\frac{1}{b-x})$ has a maximum;

we define $f_{a,b}$ as $\forall x \in \mathbb{R}$:

$$f_{a,b}(x) = \begin{cases} 0 & \text{if } x = a \text{ or } x = b \\ (x-a)^2 \sin(\frac{1}{x-a}) & \text{if } a < x \leq x_1 \\ (b-x)^2 \sin(\frac{1}{b-x}) & \text{if } x_2 \leq x < b \\ (x-x_1)^2 \sin(\frac{1}{x_1-a}) & \text{if } x_1 \leq x \leq x_2 \end{cases}. \tag{17}$$

The following figure is a representation of $f_{a,b}$. As can be seen, this function infinitely oscillates towards the endpoints a and b.

Figure 3: Representation of $f_{0.5,0.8}$

Let us focus on the endpoint a to gain some intuition about this function. It is obvious that this function is continuous at every point on $[a, b]$, but we will focus on the continuity at a. We see that $\forall x \in [a, b], |f_{a,b}(x)| < (x - a)^2$. Thus by the squeeze theorem, we have that

$$\lim_{x \to a^+} (x - a)^2 = 0 \implies \lim_{x \to a^+} f_{a,b}(x) = 0.$$

A similar argument holds for b with the bound $\forall x \in [a, b], (b - x)^2$.

It is also important to observe that near the endpoints a and b, the derivative of this function behaves much like the derivative of $x^2 \sin(\frac{1}{x})$. For example, near a we have,

$$f'_{a,b}(x) = 2(x - a) \sin\left(\frac{1}{x - a}\right) - \cos\left(\frac{1}{x - a}\right).$$

Thus we have that there exists an epsilon neighborhood about a such that $f'_{a,b}$ oscillates infinitely between an upper bound of 1 and a lower bound of -1. This implies we have an essential discontinuity at a. A similar argument can be made for the endpoint b. In fact, the derivative exists on the entire open interval (a, b).

Finally, we observe that the derivative of this function on any interval is bounded as follows:

$$|f'_{a,b}(x)| < 2(b - a) + 1.$$

Now we are ready to define Volterra's function. We define Volterra's function $V : [0, 1] \to \mathbb{R}$ as follows,

$$V(x) = \begin{cases} f_{a,b}(x), & \text{if } x \in (a, b) \text{ for some interval } (a, b) \subset [0, 1] \setminus \mathcal{F} \\ 0, & \text{if } x \in \mathcal{F} \end{cases}. \tag{18}$$

This next part is important. As we showed in the proof of proposition 6, there no subinterval of $[0, 1]$ is a subset of $\mathcal{F}$. Thus, for every interval $(a, b) \subset [0, 1] \setminus \mathcal{F}$, the endpoints a, b are members of $\mathcal{F}$. We have that V is continuous on $[0, 1]$.

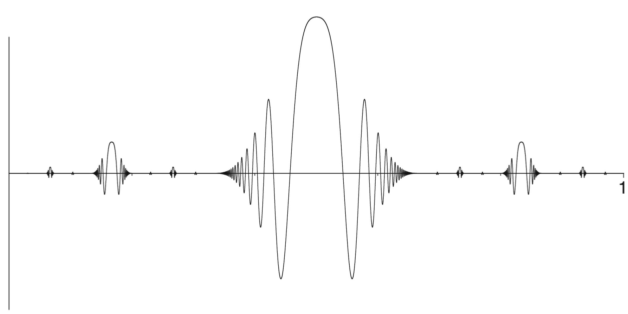

Figure 4: Representation of V on $[0, 1]$
Image Courtesy of Professor Dave Richeson

We also have that V is differentiable on $[0, 1]$. It is obvious that $x \notin \mathcal{F}$ since $f_{a,b}$ is differentiable on any interval it is defined on. However, we need to show that it is differentiable for all points in $\mathcal{F}$. We claim that for all elements in $\mathcal{F}$, the derivative of V is 0.

Proposition 7. $\forall x \in \mathcal{F}, V'(x) = 0$.

Proof. We want to show that $\forall \epsilon > 0, \forall t \in [0,1], \forall x \in \mathcal{F}, |t-x| < \epsilon \implies \frac{V(t)-V(x)}{t-x} = 0$. Let $\epsilon > 0, t \in [0,1]$, and $x \in \mathcal{F}$ be given. Suppose that $|x-t| < \epsilon$. We have two cases.

1. Suppose $t \in \mathcal{F}$. Then we have the following,

$$\frac{V(t)-V(x)}{t-x} = \frac{0}{t-x} = 0.$$

2. Suppose $t \in \mathcal{F}$. Then by definition, we have that $t \in (a,b) \subset [0,1] \setminus \mathcal{F}$. Without loss of generality, suppose that a is the endpoint nearest to x. Then by definition of Volterra's function, we have,

$$\frac{V(t)-V(x)}{t-x} = \frac{f(t)-0}{t-x} \leq \frac{f(t)}{t-a} < \frac{|t-a|^2}{|t-a|} = |t-a| < \epsilon.$$

■

Since the derivative at every point of the Volterra function has an oscillation similar to $\cos(\frac{1}{x})$ approaching from either the left, right, or both, then the derivative at that point will be discontinuous. In other words, the domain of discontinuity $D(V') = \mathcal{F}$. As discussed earlier, $m(\mathcal{F}) > 0$, thus by Lebesgue's criteria for Riemann integration, $\int_a^b V'(x)dx$ is not defined.

6 Lebesgue's Integral

> *I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral. But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral.*
> - HENRI LEBESGUE

Finally, we arrive at Lebesgue's theory of integration. In his foundational paper, Lebesgue introduces his integral with a simple yet brilliant construction. Lebesgue begins,

> To define the integral of an increasing continuous function,
>
> $$y(x)(a \leq x \leq b)$$
>
> We divide the interval (a,b) into subintervals and sums the quantities obtained by multiplying the length of each subinterval by one of the values of y when x is in the subinterval by one of the values. If x is in the interval (a_i, a_{i+1}), y varies between certain limits m_i and m_{i+1}, x is between a_i and $a_{i+1} \cdots$ Let the function y range between m and M. Consider the situation
>
> $$m = m_0 < m_1 < m_2 < \ldots < m_{p-1} < M = m_p$$
>
> $y = m$ when x belongs to the set E_0; m_{i-1} when x belongs to the set E_i[6]. We will define the measures λ_0, λ_i of these sets. Let us consider one or the other of the two sums
>
> $$m_0\lambda_0 + \sum m_i\lambda_i; m_0\lambda_0 + \sum m_{i-1}\lambda_i;$$
>
> *If, when the maximum difference between two consecutive m_i tends to zero, these sums tend to the same limit independent of the chosen m_i, this limit will be, by definition, the integral of y, which will be called integrable.*

[6]Translator's Note: Without comment, he is defining $E_0 = y^{-1}(m) = \{x \in [a,b] : y(m) = m\}$, and each $E_i = y^{-1}(m_{i-1}, m_i]$.

6.1 Building Intuition

Lebesgue's integral is fairly simple to define, yet upon first read it is hard to grasp what he is actually doing. On a more intuitive level, we can understand Lebesgue's integral as a backwards approach to the Riemann integral–instead of partitioning the domain, we partition the range. However, by this, we implicitly partition the domain! Here's another way to think about it: when Riemann partitions the domain, he chooses what the width of each rectangle will be, and the function determines the height for each rectangle. When Lebesgue partitions the range, he decides the height of the rectangle, then finds where on the function's domain the function allows him to build such rectangles.

Let's look at an example to drive this point home. Suppose that we wanted to approximate the integral of the function $f(x) = 2 - x^2$ on the interval $[-\sqrt{2}, \sqrt{2}]$. How would we do this using Lebesgue's approach? Let's begin by partitioning the range into 4 parts $\mathbb{I}_f = [0, 2] = [0, \frac{1}{2}) \cup [\frac{1}{2}, 1) \cup [1, \frac{3}{2}) \cup [\frac{3}{2}, 2]$. Then we find the pre-image of partition. We have:

- $P_1 = f^{-1}([0, \frac{1}{2})) = [-\sqrt{2}, -\sqrt{3/2}) \cup (\sqrt{3/2}, \sqrt{2}]$;
- $P_2 = f^{-1}([\frac{1}{2}, 1)) = [-\sqrt{3/2}, -1) \cup (1, \sqrt{3/2}]$;
- $P_3 = f^{-1}([1, \frac{3}{2})) = [-1, -1/\sqrt{2}) \cup (1/\sqrt{2}, 1]$;
- $P_4 = f^{-1}([\frac{3}{2}, 2]) = [-1/\sqrt{2}, 1/\sqrt{2}]$.

Observe that $P_1 \cup P_2 \cup P_3 \cup P_4 = \mathbb{D}_f$! Now, like Riemann, we can approximate area by both upper and lower rectangles. Let the height of the upper rectangles be defined as $\sup(f(P_k))$ and the height of the lower rectangles be defined as $\inf(f(P_k))$. This gives us the following,

- $\inf(f(P_1)) = 0; \sup(f(P_1)) = \frac{1}{2}$.
- $\inf(f(P_2)) = \frac{1}{2}; \sup(f(P_2)) = 1$.
- $\inf(f(P_3)) = 1; \sup(f(P_3)) = \frac{3}{2}$.
- $\inf(f(P_4)) = \frac{3}{2}; \sup(f(P_4)) = 2$.

Here we finally realize why we spent so much time building up measure theory. We see that the pre-image of many of these intervals have breaks, so in order to measure the "width" of all the rectangles with a certain height, we simply take the Lebesgue measure of pre-image! Since each P_k is an interval or an union of interval, we calculate the measure of each as follows.

- $m(P_1) = 2(\sqrt{2} - \sqrt{3/2})$.
- $m(P_2) = 2(\sqrt{3/2} - 1)$.
- $m(P_3) = 2(1 - \frac{1}{\sqrt{2}})$.
- $m(P_4) = \sqrt{2}$.

This gives use the following upper and lower sums:

$$S = \frac{1}{2} \cdot m(P_1) + 1 \cdot m(P_2) + \frac{3}{2} \cdot m(P_3) + 2 \cdot m(P_4) \approx 4.352,$$

$$s = 0 \cdot m(P_1) + \frac{1}{2} \cdot m(P_2) + 1 \cdot m(P_3) + \frac{3}{2} \cdot m(P_4) \approx 2.931.$$

Relating this back to Lebesgue's definition, we can see that as lengths of the range's partitions tend towards 0, our approximation will approach the actual area. In fact, because this function is so simple, we can calculate its integral exactly. We have,

$$\int_{-\sqrt{2}}^{\sqrt{2}} 2 - x^2 dx = \left[2x - \frac{x^3}{3}\right]_{-\sqrt{2}}^{\sqrt{2}} = \left(2(\sqrt{2}) - \frac{\sqrt{2}^3}{3}\right) - \left(2(-\sqrt{2}) - \frac{(-\sqrt{2})^3}{3}\right) \approx 3.771.$$

Thus we can see that this Lebesgue approximation gives us a fairly decent bound on the actual value.

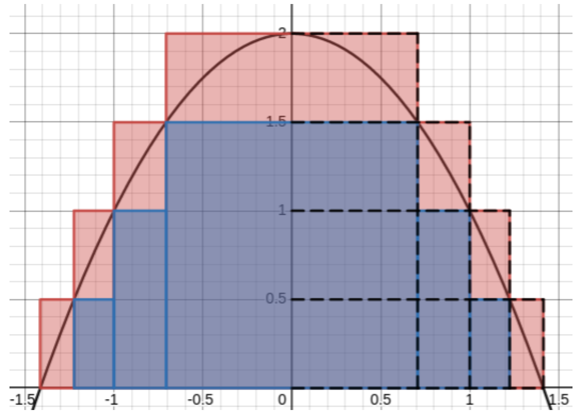

Figure 5: Lebesgue Approximation of $f(x) = 2 - x^2$ with 4 Range Partitions

After all of that, it is clear that performing a Riemann approximation would be much easier. However, we are not interested in Lebesgue's integral for its computational ease. It is the theoretical advantages we are after. In fact, if a function is Riemann integrable it is automatically Lebesgue integrable. This fact can ease computational headaches.

6.2 Integration of Simple Functions

In order to get a better grasp on Lebesgue integration, it is best to start simple. We will introduce a class of simple functions that are easy to integrate and build off them.

Definition 12. Let $A \subseteq \mathbb{R}$, we define the *indicator function* of A as $\forall x \in \mathbb{R}$,

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}. \tag{19}$$

Definition 13. Let $a_1, \cdots, a_k \in \mathbb{R}$ and the sets $A_1, \cdots, A_k \subset \mathbb{R}$ be disjoint measurable sets with finite measure. We define an *simple function* (SF) f as any function that can be written in the form,

$$f = \sum_{i=1}^{k} a_i 1_{A_i}. \tag{20}$$

Definition 14. The *Lebesgue Integral* of an SF is $f : [a, b] \to \mathbb{R}$,

$$\int_{\mathbb{R}} f(x)dx := \sum_{i=1}^{k} a_k m(A_k). \tag{21}$$

Now that we have a definition, let's return to our old friend $1_{\mathbb{Q}} : [0, 1] \to \{0, 1\}$. What is the Lebesgue integral of this function? Since $\mathbb{Q} \cap [0, 1]$ is countable, we have that $m(\mathbb{Q} \cap [0, 1]) = 0$, and since $1_{\mathbb{Q}}$ is an SF, we have that the integral of $1_{\mathbb{Q}}$ is,

$$\int_{[0,1]} 1_{\mathbb{Q}} = 1 \cdot m(\mathbb{Q} \cap [0, 1]) = 0.$$

We see that this simple application of the Lebesgue integral is able to deal with functions that caused serious problems for Riemann's integral. The next theorem outlines some of properties of SF.

Theorem 11. *Let f, g be SF's and $c \in \mathbb{R}$. The following are true:*

1. *cf is also an SF and*

$$c\int f = \int cf. \tag{22}$$

2. *$f+g$ is also an SF and*

$$\int f+g = \int f + \int g. \tag{23}$$

3. *$|f|$ is also an SF and*

$$\left|\int f\right| = \int |f|. \tag{24}$$

4. *$f \leq g \implies \int f \leq \int g$.*

6.3 Integration of Measurable Functions

Now that we have established a theory of integration for SF, we can use these facts to further generalize the Lebesgue integral to more general functions. Let us define how we can extend measure theory to functions. We define a non-negative measurable function as follows.

Definition 15. Let $E \subset \mathbb{R}$ be a measurable set and $f : E \to \bar{\mathbb{R}}$. We say f is a measurable function if $\forall \alpha \in \mathbb{R}, f^{-1}((\alpha, \infty])$ is measurable.

Theorem 12. *Let $E \subset \mathbb{R}$ be a measurable set and $f : E \to \bar{\mathbb{R}}$. Then the following are equivalent:*

1. *$\forall \alpha \in \mathbb{R}, f^{-1}([\alpha, \infty])$ is measurable.*
2. *$\forall \alpha \in \mathbb{R}, f^{-1}([-\infty, \alpha])$ is measurable.*

Definition 16. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative measurable function. We define the Lebesgue integral of f as,

$$\sup\left\{\int g : g \text{ is an SF such that } 0 \leq g \leq f\right\}. \tag{25}$$

In other words, for some non-negative measurable function, we are able to approximate its integral with SF's from below, and the least upper bound will be the integral of this function. Observe that $\int f = +\infty$ and f may be an SF as well without conflict.

Theorem 13. *Let f, g be non-negative measurable functions and $c \in \mathbb{R}^+$. The following properties hold:*

1.

$$\int cf = c \cdot \int f. \tag{26}$$

2.

$$\int (f+g) = \int f + \int g \tag{27}$$

The proof for additivity of the Lebesgue integral is non-trivial. We will need a squeeze theorem to help us along the way. This proof is taken from Richard Beals' Real Analysis with some adjustments for clarity. Let's first look at the lemma.

Lemma 3. *Let f be a bounded, measurable function and A be a set with finite measure. $\forall \epsilon > 0$, there exists SFs f_1 and f_2 such that $f_1 \leq f \leq f_2$ on A and $\forall x \in \mathbb{R}, f_1 - f_2 \leq \epsilon$.*

Proof. Let $\epsilon > 0$ be given. Choose an M such that $\forall x \in \mathbb{R}, |f(x)| \leq M$ and partition the interval $[-M, M]$ into disjoint intervals $I_1, I_2, \ldots, I_n$ such that given any $I_k, |I_k| \leq \epsilon$. Let $A_k = A \cap f^{-1}(I_k)$. Given any I_k, let a_k be its left end point and b_k be its right endpoint. Define f_1, f_2 as,

$$f_1 \coloneqq \sum_{k=1}^{n} a_k 1_{A_k}, f_2 \coloneqq \sum_{k=1}^{n} b_k 1_{A_k}.$$

1. We want to show that $\forall x \in A, f_1(x) \leq f(x) \leq f_2(x)$. Let $x \in A$ be given, choose I_k such that $x \in A_k$. Then we have that $f(x) \in I_k$. By choice of f_1 and f_2, we have that

$$f_1(x) = a_k \leq f(x) \leq b_k = f_2(x).$$

2. We want to show that $f_1 - f_2 \leq \epsilon$. We have that,

$$f_1 - f_2 = \sum_{k=1}^{n} a_k 1_{A_k} - \sum_{k=1}^{n} b_k 1_{A_k} = \sum_{k=1}^{n} (a_k - b_k) 1_{A_k} = \sum_{k=1}^{n} |I_k| 1_{A_k} \leq \sum_{k=1}^{n} \epsilon \cdot 1_{A_k}$$

Since each I_k is disjoint, then given an $x \in A_k = A \cap f^{-1}(I_k)$, x will be unique to that A_k, so only one of the indicator functions will evaluate to 1 otherwise each 1_{A_k} will evaluate to 0. This gives us $\forall x \in \mathbb{R}$,

$$\epsilon \cdot \sum_{k=1}^{n} 1_{A_k}(x) \leq \epsilon.$$

■

Theorem 14. *Let f, g be bounded, non-negative, and measurable functions, then we have $\int f + \int g = \int (g + f)$.*

Proof. Let f_1, g_1 both be SFs such that $0 \leq f_1 \leq f$ and $0 \leq g_1 \leq g$. This gives us,

$$\int f_1 + \int g_1 = \int (f_1 + g_1) \leq \int (f + g).$$

If we take the supremeum over all such possible SFs, we get the following

$$\int f + \int g \leq \int (f + g).$$

The reverse inequality is less trivial. Let h be a SF such that $0 \leq h \leq f + g$. Let A be the set where $h > 0$. We note that this set has finite measure and h is bounded. Thus we have $h \wedge g$ and $h \wedge g$ are both bounded. Using the lemma from above, choose SFs f_1, g_1 such that,

$$0 \leq f_1 \leq h \wedge f \leq f_1 + \epsilon, 0 \leq g_1 \leq h \wedge g \leq g_1 + \epsilon.$$

Because $h \leq f + g$, combining these inequalities gives us the following,

$$h \leq (h \wedge f) + (h \wedge g) \leq f_1 + g_1 + 2\epsilon \cdot 1_A.$$

Thus we can construct the following inequality,

$$\int h \leq \int f_1 + \int g_1 + 2\epsilon m(A) \leq \int f + \int g + 2\epsilon m(A).$$

Next, we take the infimum over $\epsilon > 0$, and then take the supremum over $h \leq f + g$ we get,

$$\int (f + g) \leq \int f + \int g.$$

■

6.4 Signed Measurable Functions

Now that we have a working definition for non-negative measurable functions, we need a way to handle negative measurable functions. We can do this by signing the functions.

Definition 17. Let f be a real-valued or extended-real-valued function defined on the set E such that $\forall x \in E$,

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{otherwise} \end{cases}. \tag{28}$$

$$f^-(x) = \begin{cases} -f(x) & \text{if } f(x) < 0 \\ 0 & \text{otherwise} \end{cases}. \tag{29}$$

It follows from the definition that these functions are measurable if f is measurable, and we have:

$$f = f^+ - f^-; |f| = f^+ + f^-. \tag{30}$$

Definition 18. The signed function f is said to be integrable if $\int |f| < +\infty$.

This concludes our introduction of the basic definition and properties of Lebesgue's integral. Finally, we will move on to its two main theoretical advantages over Riemann's integral.

6.5 Dominated Convergence Theorem

If we recall back to section 3.4.2 regarding the limit problem where we could define a sequence of bounded functions each of whom were be Riemann integrable, yet the limit was not integrable. This reality raised the question: under what conditions can we move the limit inside of the integral? Does Lebesgue's integral resolve this issue? The answer is somewhat.

First, we must dispense with the possibility that a sequence of measurable functions could converge to a nonmeasurable function. Fortunately, a convergent sequence of measurable functions will always converge pointwise to a measurable function. The following proof is again adapted from Richard Beals' Real Analysis.

Theorem 15. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions such that $\forall x \in \mathbb{R}, \lim_{n\to\infty} f_n(x) = f(x)$, then f is measurable.*

Proof. Let $\alpha, x \in \mathbb{R}$ be given. We want to show that $x \in \{z \in \mathbb{R} : f(z) > \alpha\}$ is measurable. Suppose that $x \in \{z \in \mathbb{R} : f(z) > \alpha\}$. Then we can say for some $m \in \mathbb{N}, f(x) > \alpha + \frac{1}{m}$. Since $f_n \to f$, there exists an $N \in \mathbb{N}$ such that $\forall n \geq N, f_n(x) > \alpha + \frac{1}{m}$. This yields the following containment,

$$\{z \in \mathbb{R} : f(z) > \alpha\} \subset \bigcup_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \left\{ z \in \mathbb{R} : f_n(z) > \alpha + \frac{1}{m} \right\}$$

Now suppose that x is contained in the right hand set. Then for some $m, N \in \mathbb{N}$, we have $\forall n \geq N, f_n(x) > \alpha + \frac{1}{m}$. Thus for the limit $f(x)$, we have,

$$f(x) \geq \alpha + \frac{1}{m} > \alpha.$$

Therefore we have set equality and $\{z \in \mathbb{R} : f(z) > \alpha\}$ is measurable. ■

This fact gives us the assurance that we will not converge to a function that we cannot integrate given a sequence of integrable functions. However, it is still not the case that we can always move the limit inside of the integral. The following theorem is called Lebesgue's Dominated Convergence Theorem (DCT), and is considered one of the greatest advantages of the Lebesgue integral over the Riemann integral.

Theorem 16 (Dominated Convergence Theorem)**.** *Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions such that $\forall x \in \mathbb{R}, \lim_{n\to\infty} f_n(x) = f(x)$. Assume that $\forall n \in \mathbb{N}$ there exists an integrable function g such that $\forall x \in \mathbb{R}, |f_n(x)| \leq g(x)$. Then,*

$$\lim_{n\to\infty} \int f_n = \int f. \tag{31}$$

This theorem gives us precise conditions under which we can move the limit inside the integral. Summarizing this theorem, it says: if we have a sequence of measurable functions that converge to a measurable function, and that sequence (including its limit) is *dominated* pointwise by some function g, then we can move the limit in. Of course, good mathematicians always ask, can we reduce our assumptions? Unfortunately, we cannot strengthen this theorem. Next, we will go over a simple example that illustrates why we cannot strengthen this theorem.

Let us define a simple sequence of functions $f_n : [0,1] \to \mathbb{R}$ such that $\forall n \in \mathbb{N}, \forall x \in \mathbb{R}$,

$$f_n(x) = n \cdot 1_{(0,\frac{1}{n})}(x).$$

We see as $n \to \infty$ that this function cannot be dominated. In fact suppose that could be.

Proof. Assume towards contradiction that there is some $g : \mathbb{R} \to \mathbb{R}$ such that $\forall n \in \mathbb{N}, \forall x \in \mathbb{R}, |f_n(x)| \leq g(x)$. Then, choose an $n \in \mathbb{N}$ such that $n > \sup(g[0,1])$, and choose $x \in (0, \frac{1}{n})$. We have:

$$|f_n(x)| = |n \cdot 1_{(0,\frac{1}{n})}(x)| = n > g(x).$$

Contradiction. ■

Thus we see that this function does not satisfy the dominated hypothesis in the DCT. The next natural question is what is the limit of this sequence? We claim that this sequence approaches the 0 function.

Proposition 8. $\forall x \in [0,1], \lim_{n\to\infty} f_n(x) = \mathbf{0}(x)$.

Proof. Let $x \in [0,1]$ be given. We have exactly two cases.

1. Assume that $x = 0$. Then $\forall n \in \mathbb{N}, f_n(x) = 0$.
2. Assume that $x \neq 0$. Let $\epsilon > 0, n \in \mathbb{N}$ be given. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < x$. Assume that $n \geq N$. Then we have,
$$|f_n(x) - 0| = |n \cdot 1_{(0,\frac{1}{n})}(x)| = 0 < \epsilon.$$

■

Now, let us evaluate both sides of the equality of the DCT and see if it holds. For the left hand side we have,

$$\lim_{n\to\infty} \int_{[0,1]} f_n(x)dx = \lim_{n\to\infty} n \cdot \frac{1}{n} = 1.$$

and for the right hand side we have,

$$\int_{[0,1]} \lim_{n\to\infty} f_n(x)dx = \int_{[0,1]} \mathbf{0}(x)dx = 0.$$

Therefore we see that it is not possible to move the limit inside of the integral, and we conclude that we cannot remove this condition. Even though we cannot remove this condition, because it is so mild, the DCT is an extremely powerful tool to have.

6.6 Lebesgue's Fundamental Theorem of Calculus

The final major issue that Lebesgue's integral addresses is the anti-derivative problem. After all, this was Lebesgue's primary goal. Lebesgue's integral gives us a precise condition for which functions satisfy the FTC. It involves a stricter notion of continuity called absolute continuity. We define it as follows.

Definition 19. Let $[a,b] \subset \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$, we say that f is *absolutely continuous* if $\forall \epsilon > 0$, $\exists \delta > 0$, such that when a sequence of pairwise disjoint subintervals (x_k, y_k) for $x_k < y_k \in [a,b]$ satisfies:

$$\sum_k (x_k - y_k) < \delta \implies \sum_k |f(x_k) - f(y_k)| < \epsilon. \tag{32}$$

The next theorem is Lebesgue's Fundamental Theorem of Calculus.

Theorem 17 (Lebesgue's Fundamental Theory of Calculus)**.** *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$, then,*

$$\int_{[a,b]} f'(x)dx = f(b) - f(a). \tag{33}$$

Unfortunately, deeper investigation into why such functions have this property is beyond the scope of this paper.[7] The key take away is that Lebesgue's theory of integration provides us with a clear context in which we can apply the FTC to a given function. Ultimately, Lebesgue's integral puts the theory of integration on firm theoretical ground and addresses many of the theoretical concerns prior to its introduction.

[7] For a detailed proof see [Pou12]

References

[Bou10] Murray Bourne. What did newton originally say about integration, 2010.

[Bur] Frank Burk. *Lebesgue measure and Integration: An Introduction*. John Wiley & Sons.

[Cau23] Augustin-Louis Cauchy. *Résumé des leçons sur le calcul infinitésimal*. 1823.

[ES05] Rami Shakarchi Elias Stein. *Real Analysis: Measure Theory, Integration, & Hilbert Spaces*. Princeton University Press, 2005.

[Fle94] Julian Fleron. A note on the history of the cantor set and cantor function. 1994.

[Gil15] D. C. Gillespie. The cauchy definition of a definite integral. *Annals of Mathematics*, 1915.

[JCPC15] Miguel Ángel Maldonado-Aguilar Juan Carlos Ponce-Campuzano. Vito volterra's construction of a nonconstant function with a bounded, non-riemann integrable derivative. *BSHM Bulletin: Journal of the British Society for the History of Mathematics*, 2015.

[Kli] Morris Kline. *Mathematical Thought Ancient To Modern Times*, volume 3. Oxford University Press.

[Leba] Heri Lebesgue. *Intégrale, Longueur, Aire*.

[Lebb] Heri Lebesgue. *Sur une généralisation de l'intégral définie*. Translated by Paul Loya and Emanuele Delucchi.

[Loy] Paul Loya. Lebesgue's remarkable theory of measureand integration with probability. https://www.scribd.com/document/192273466/Loyabook#.

[Mer15] Jean-Pierre Merx. The smith volterra cantor set. https://www.mathcounterexamples.net/the-smith-volterra-cantor-set/, 2015.

[New] Isaac Newton. *Mathematical Principals of Natural Philosophy*, volume 1.

[Pou12] Rodrigo López Pouso. A simple proof of the fundamental theorem of calculus for the lebesgue integral. https://arxiv.org/pdf/1203.1462.pdf, 2012.

[Sco19] Nicholas Scoville. The cantor set before cantor. 2019.

[Smi75] Henery John Stephen Smith. On the integration of discontinuous functions. 1875.

[Sol70] Robert M. Solovay. A model of set-theory in which every set of reals is lebesgue measurable. 1970.

[Vit05] Giuseppe Vitali. Sul problema della misura dei gruppi di punti di una retta. 1905. Translated by James R. Meyer.