Machine Learning

PROBABILITY AND RANDOM VARIABLES

(background)

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Probability

Probability

- ❑ An event is something that happens
- ❑ A random event has an uncertain outcome
- ❑ The probability of an event measures how likely it is to occur

Example

- I've written a student's name of paper. What is that name?
- Event: E="George is written on the paper"
- Probability: $P(\mathrm{E})$ measures how likely it is that "George" is written on the paper
- Probability is a measurement tool
- Mathematical language for quantifying uncertainty

Sigma-Algebra

- Def: Sample space S is the set of all possible outcomes of an experiment
    - Ex: All student names $S = \{x_1, x_2, \dots, x_N\}$ (x_n denotes a name)
- Def: An outcome is an element in S, e.g., x_3
- Def: An event E is any subset of S
    - Ex: $\{x_1\}$, student with name x_1
    - Ex: Also, $\{x_1, x_4\}$, students with names x_1 and x_4

   Outcome x_3 and event $\{x_3\}$ are different, the latter is a set
- Def: A sigma-algebra $\mathcal{F}$ is a collection of events $E \subseteq S$ such that

   (i) The empty set $\emptyset$ belongs to $\mathcal{F}$: $\emptyset \in \mathcal{F}$

   (ii) Closed under complement: If $E \in \mathcal{F}$, then $E^c = S \setminus E \in \mathcal{F}$

   (iii) Closed under countable unions: If $E_1, E_2, \dots, E_N \in \mathcal{F}$, then $\cup_{i=1}^{N} E_i \in \mathcal{F}$

Sigma-Algebra (cont'd)

- Example
  - No student and all student names, i.e., $\mathcal{F}_0 := \{\emptyset, S\}$
- Example
  - Empty set, student names with more vowels, student names with more consonants, all student names:

    $\mathcal{F}_1 := \{\emptyset, \text{mV}, \text{mC}, S\}$
- Example
  - $\mathcal{F}_2$ including the empty set $\emptyset$ plus
  - All events (sets) with one student name $\{x_1\}, \dots, \{x_N\}$ plus
  - All events with two student names $\{x_1, x_2\}, \{x_1, x_3\}, \dots, \{x_1, x_N\}, \{x_2, x_3\}, \dots, \{x_2, x_N\}, \dots, \{x_{N-1}, x_N\}$ plus
  - All events with three, four, ..., N student names

    $\mathcal{F}_2$ is known as the power set of S, denoted 2^S

Axioms of Probability

- Define a function $P(E)$ from a sigma-algebra $\mathcal{F}$ to the real numbers
- $P(E)$ qualifies as a probability if

  A1) Non-negativity: $P(E) \geq 0$

  A2) Probability of universe: $P(S) = 1$

  A3) Additivity: Given sequence of disjoint events $E_1, E_2, \dots$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

  Disjoint (mutually exclusive) events means $E_i \cap E_j = \emptyset, i \neq j$

  Union of countably infinite many disjoint events

- Triplet $(S, \mathcal{F}, P(\cdot))$ is called a probability space.

Consequences of the Axioms

From axioms A1)-A3)

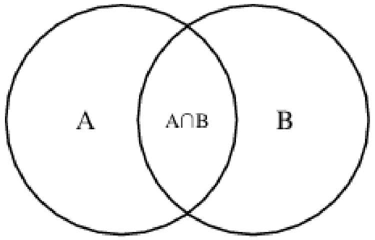
$\Rightarrow$ Impossible event: $P(\emptyset) = 0$

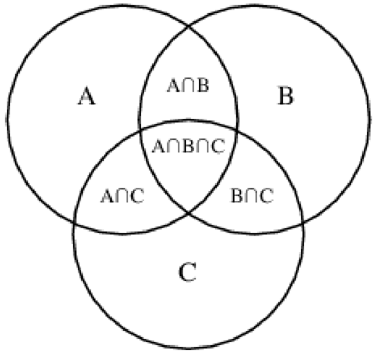
$\Rightarrow$ Monotonicity: $E_1 \subset E_2 \Rightarrow P(E_1) \leq P(E_2)$

$\Rightarrow$ Range: $0 \leq P(E) \leq 1$

$\Rightarrow$ Complement: $P(E^c) = 1 - P(E)$

$\Rightarrow$ Union (inclusion-exclusion): For any events $E_1, \ldots, E_N$





$$P\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{k=1}^{N} \left((-1)^{k-1} \sum_{\substack{I \subseteq \{1,\ldots,N\} \\ |I|=k}} P(\cap_{i \in I} E_I) \right)$$

Special case, $N = 2$: $P(E_1 \cup E_2) = \underbrace{+P(E_1) + P(E_2)}_{k=1} + \underbrace{(-P(E_1 \cap E_2))}_{k=2}$

Conditional Probability

- Consider events E and F, and suppose we know F occurred

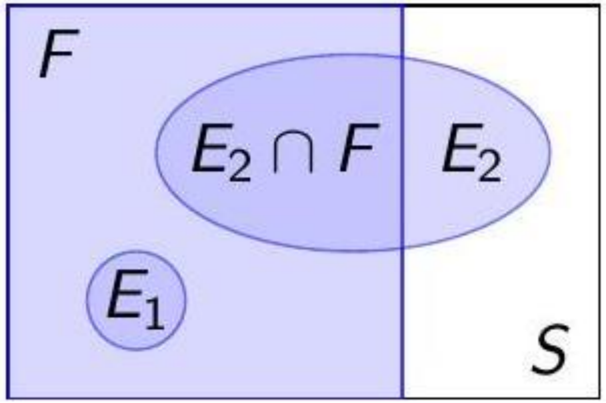
Q: What does this information imply about the probability of E ?

- Def: Conditional probability of E given F is (need $P(F) > 0$)

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

In general, $P(E \mid F) \neq P(F \mid E)$

- Renormalize probabilities to the set F
    - Discard a piece of S
    - May discard a piece of E as well
- For given F with $P(F) > 0$, $P(\cdot \mid F)$ satisfies the axioms of probability



Conditional Probability (cont'd)

- The number I wrote has more consonants. What is the probability of name x_n ?
- Assume the names with more consonants $F = \{x_1, \dots, x_M\} \Rightarrow P(F) = \frac{M}{N}$
- If name x_n has more consonants, $x_n \in F$ and we have for event $E = \{x_n\}$

$$P(E \cap F) = P(\{x_n\}) = \frac{1}{N}$$

Conditional probability is as you would expect

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{1/N}{M/N} = \frac{1}{M}$$

- If name has more vowels, $x_n \notin F$, then $P(E \cap F) = P(\emptyset) = 0$
- $\Rightarrow$ As you would expect, then $P(E \mid F) = 0$

Law of Total Probability

- Consider event E and events F and F^c

  F and F^c form a partition of the space $S (F \cup F^c = S, F \cap F^c = \emptyset)$

- Because $F \cup F^c = S$ cover space S, can write the set E as

$$E = E \cap S = E \cap [F \cup F^c] = [E \cap F] \cup [E \cap F^c]$$

- Because $F \cap F^c = \emptyset$ are disjoint, so is $[E \cap F] \cap [E \cap F^c] = \emptyset$

$$\Rightarrow P(E) = P([E \cap F] \cup [E \cap F^c]) = P(E \cap F) + P(E \cap F^c)$$

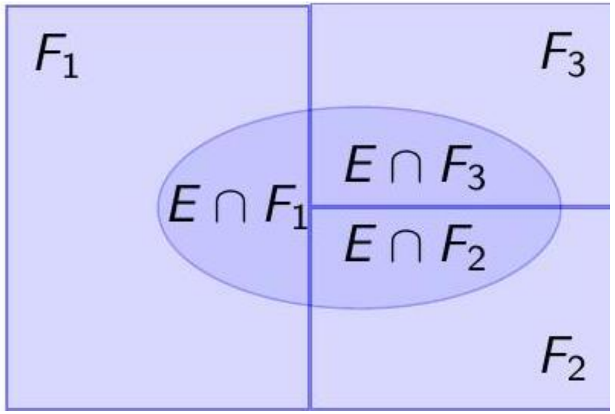
- Use definition of conditional probability

$$P(E) = P(E \mid F)P(F) + P(E \mid F^c)P(F^c)$$

- Translate conditional information $P(E \mid F)$ and $P(E \mid F^c)$

  $\Rightarrow$ Into unconditional information $P(E)$

Law of Total Probability (cont'd)



- In general, consider (possibly infinite) partition $F_i, i = 1,2, \ldots$ of S
- Sets are disjoint $\Rightarrow F_i \cap F_j = \emptyset$ for $i \neq j$
- Sets cover the space $\Rightarrow \cup_{i=1}^{\infty} F_i = S$
- As before, because $\cup_{i=1}^{\infty} F_i = S$ cover the space, can write set E as

$$E = E \cap S = E \cap \left[\bigcup_{i=1}^{\infty} F_i\right] = \bigcup_{i=1}^{\infty} [E \cap F_i]$$

- Because $F_i \cap F_j = \emptyset$ are disjoint, so is $[E \cap F_i] \cap [E \cap F_j] = \emptyset$. Thus

$$P(E) = P\left(\bigcup_{i=1}^{\infty} [E \cap F_i]\right) = \sum_{i=1}^{\infty} P(E \cap F_i) = \sum_{i=1}^{\infty} P(E \mid F_i) P(F_i)$$

Law of Total Probability (cont'd)

- Consider a probability class in some university
  - Seniors get an A with probability (w.p.) 0.9, juniors w.p. 0.8
  - An exchange student is a senior w.p. 0.7, and a junior w.p. 0.3

Q: What is the probability of the exchange student scoring an A?

- Let $A =$ "exchange student gets an A," S denote senior, and J junior
- Use the law of total probability

$$P(A) = P(A \mid S)P(S) + P(A \mid J)P(J)$$

$$= 0.9 \times 0.7 + 0.8 \times 0.3 = 0.87$$

Bayes' Rule

- From the definition of conditional probability

$$P(E \mid F)P(F) = P(E \cap F)$$

- Likewise, for F conditioned on E we have

$$P(F \mid E)P(E) = P(F \cap E)$$

- Quantities above are equal, giving Bayes' rule

$$P(E \mid F) = \frac{P(F \mid E)P(E)}{P(F)}$$

- Bayes' rule allows time reversion. If F (future) comes after E (past),
  - $P(E \mid F)$, probability of past (E) having seen the future (F)
  - $P(F \mid E)$, probability of future (F) having seen past (E)
- Models often describe future | past. Interest is often in past | future

Bayes' Rule (cont'd)

- Consider the following partition of my email
  - $E_1 =$ "spam" w.p. $P(E_1) = 0.7$
  - $E_2 =$ "low priority" w.p. $P(E_2) = 0.2$
  - $E_3 =$ "high priority" w.p. $P(E_3) = 0.1$
- Let $F =$ "an email contains the word free"

  From experience know $P(F \mid E_1) = 0.9, P(F \mid E_2) = P(F \mid E_3) = 0.01$

- I got an email containing "free". What is the probability that it is spam?
- Apply Bayes' rule
- $P(E_1 \mid F) = \frac{P(F|E_1)P(E_1)}{P(F)} = \frac{P(F|E_1)P(E_1)}{\sum_{i=1}^{3} P(F|E_i)P(E_i)} = 0.995$

$\Rightarrow$ Law of total probability very useful when applying Bayes' rule

Independence

- Def: Events E and F are independent if $P(E \cap F) = P(E)P(F)$
    - Events that are not independent are dependent
- According to definition of conditional probability

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)P(F)}{P(F)} = P(E)$$

- Intuitive, knowing F does not alter our perception of E
    - F bears no information about E
    - The symmetric is also true $P(F \mid E) = P(F)$
    - Whether E and F are independent relies strongly on $P(\cdot)$
- Avoid confusing with disjoint events, meaning $E \cap F = \emptyset$

Q: Can disjoint events with $P(E) > 0, P(F) > 0$ be independent? ***No***

Independence (cont'd)

- Wrote one name, asked a friend to write another (possibly the same)
- Probability space $(S, \mathcal{F}, P(\cdot))$ for this experiment
  - S is the set of all pairs of names $[x_n(1), x_n(2)]$, $|S| = N^2$
  - Sigma-algebra is (cartesian product) power set $\mathcal{F} = 2^S$
  - Define $P(E) = \frac{|E|}{|S|}$ as the uniform probability distribution
- Consider the events E_1 = 'I wrote x_1' and E_2 = 'My friend wrote x_2'

Q: Are they independent? Yes, since

$$P(E_1 \cap E_2) = P(\{(x_1, x_2)\}) = \frac{|\{(x_1, x_2)\}|}{|S|} = \frac{1}{N^2} = P(E_1)P(E_2)$$

- Dependent events: $E_1 =' I$ wrote x_1 ' and E_3 = 'Both names have more consonants'

Independence for >2 Events

- Def: Events $E_i, i = 1,2, \ldots$ are called mutually independent if
- $P(\bigcap_{i \in I} E_i) = \prod_{i \in I} P(E_i)$
- for every finite subset I of at least two integers
- Ex: Events E_1, E_2, and E_3 are mutually independent if all the following hold
    - $P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2)P(E_3)$
    - $P(E_1 \cap E_2) = P(E_1)P(E_2)$
    - $P(E_1 \cap E_3) = P(E_1)P(E_3)$
    - $P(E_2 \cap E_3) = P(E_2)P(E_3)$
- If $P\left(E_i \cap E_j\right) = P(E_i)P\left(E_j\right)$ for all (i, j), the E_i are pairwise independent

- Mutual independence → pairwise independence. **Not the other way**

Random Variable

Random Variable

- Def: RV $X(s)$ is a function that assigns a value to an outcome $s \in S$
- Think of RVs as measurements associated with an experiment

Example

- Throw a ball inside a $1m \times 1m$ square. Interested in ball position
- Uncertain outcome is the place $s \in [0,1]^2$ where the ball falls
- Random variables are $X(s)$ and $Y(s)$ position coordinates
- RV probabilities inferred from probabilities of underlying outcomes

$$P(X(s) = x) = P(\{s \in S: X(s) = x\})$$
$$P(X(s) \in (-\infty, x]) = P(\{s \in S: X(s) \in (-\infty, x]\})$$

- $X(s)$ is the random variable and x a particular value of $X(s)$

Random Variable – Example 1

- Throw coin for head (H) or tails (T). Coin is fair $P(H) = 1/2$, $P(T) = 1/2$. Pay \$1 for H, charge \$1 for T. Earnings?
- Possible outcomes are H and T
- To measure earnings, define RV X with values

$$X(H) = 1,\ X(T) = -1$$

- Probabilities of the RV are

$$P(X = 1) = P(H) = 1/2$$
$$P(X = -1) = P(T) = 1/2$$

Also have $P(X = x) = 0$ for all other $x \neq \pm 1$

Random Variable – Example 2

- Throw 2 coins. Pay \$1 for each H, charge \$1 for each T. Earnings?
- Now the possible outcomes are HH, HT, TH, and TT
- To measure earnings, define RV Y with values

$$Y(HH) = 2,\ Y(HT) = 0,\ Y(TH) = 0,\ Y(TT) = -2$$

- Probabilities of the RV are

$$\begin{gathered} P(Y = 2) = P(HH) = 1/4, \\ P(Y = 0) = P(HT) + P(TH) = 1/2, \\ P(Y = -2) = P(TT) \ = 1/4 \end{gathered}$$

- RVs are easier to manipulate than events
- Let $s_1 \in \{H, T\}$ be outcome of coin 1 and $s_2 \in \{H, T\}$ of coin 2. Can relate Y and X s as

$$Y(s_1, s_2) = X_1(s_1) + X_2(s_2)$$

Random Variable – Example 2 (cont'd)

- Throw N coins. Earnings?
- Enumeration becomes cumbersome
- Alternatively, let $s_n \in \{H, T\}$ be outcome of n-th toss and define:

$$Y(s_1, s_2, \dots, s_N) = \sum_{n=1}^{N} X_n(s_n)$$

- Will usually abuse notation and write $Y = \sum_{n=1}^{N} X_n$

Random Variable – Example 3

- Throw a coin until landing heads for the first time. $P(H) = p$
- Number of throws until the first head?
- Outcomes are H, TH, TTH, TTTH, … Note that $|S| = \infty$
  - Stop tossing after first H (thus THT not a possible outcome)
- Let N be a RV counting the number of throws
  - $N = n$ if we land T in the first $n-1$ throws and H in the n-th

$$P(N = 1) = P(H) = p$$

$$P(N = 2) = P(TH) = (1-p)p$$

$$\dots$$

$$P(N = n) = P(\underbrace{TT \dots T}_{n-1 \text{ tails}}\ H) = (1-p)^{n-1}p$$

Random Variable – Example 3 (cont'd)

- From A2) we should have $P(S) = \sum_{n=1}^{\infty} P(N = n) = 1$
- Holds because $\sum_{n=1}^{\infty} (1-p)^{n-1}$ is a geometric series

$$\sum_{n=1}^{\infty} (1-p)^{n-1} = 1 + (1-p) + (1-p)^2 + \cdots = \frac{1}{1-(1-p)} = \frac{1}{p}$$

- Plug the sum of the geometric series in the expression for $P(S)$

$$\sum_{n=1}^{\infty} P(N = n) = p \sum_{n=1}^{\infty} (1-p)^{n-1} = p \times \frac{1}{p} = 1 \checkmark$$

Indicator Function

- The indicator function of an event is a random variable
- Let $s \in S$ be an outcome, and $E \subset S$ be an event

$$\mathbb{I}\{E\}(s) = \begin{cases} 1, & \text{if } s \in E \\ 0, & \text{if } s \notin E \end{cases}$$

  - Indicates that outcome s belongs to set E, by taking value 1

Example

- Number of throws N until first H. Interested on N exceeding N_0
  - Event is $\{N: N > N_0\}$. Possible outcomes are $N = 1,2, \ldots$
  - Denote indicator function as $\mathbb{I}_{N_0} = \mathbb{I}\{N: N > N_0\}$
- Probability $P\left(\mathbb{I}_{N_0} = 1\right) = P(N > N_0) = (1-p)^{N_0}$
  - For N to exceed N_0 need N_0 consecutive tails
  - Doesn't matter what happens afterwards

Discrete RVs

- **Discrete RVs: Take discrete values**
- Continuous RVs: Take continuous values

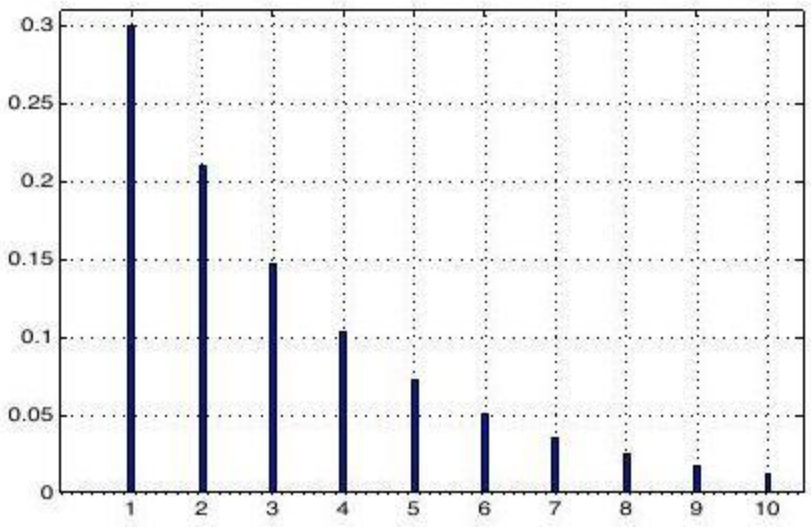
Probability Mass

- ❑ Discrete RV takes on, at most, a countable number of values
- ❑ Probability mass function (pmf) $p_X(x) = P(X = x)$
  - If RV is clear from context, just write $p_X(x) = p(x)$
- ❑ If X supported in $\{x_1, x_2, \dots\}$, pmf satisfies

  (i) $p(x_i) > 0$ for $i = 1,2, \dots$

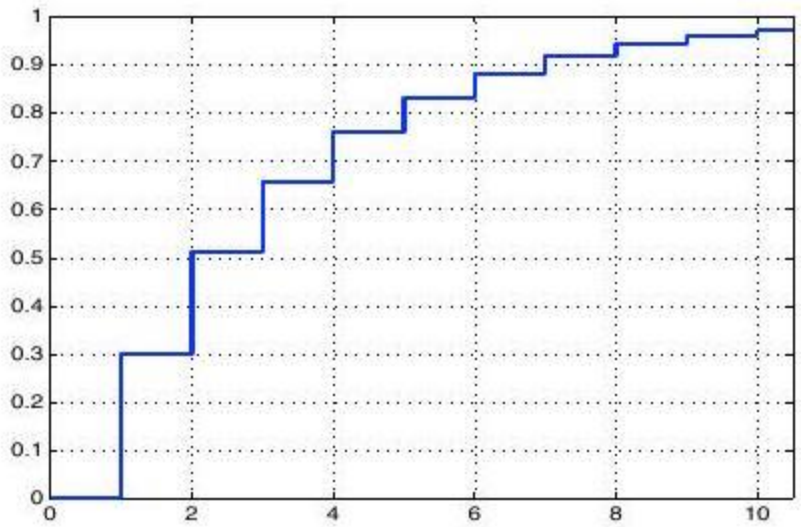
  (ii) $p(x) = 0$ for all other $x \neq x_i$

  (iii) $\sum_{i=1}^{\infty} p(x_i) = 1$
- ❑ Pmf for "throw to first heads" $(p = 0.3)$



Cumulative Distribution

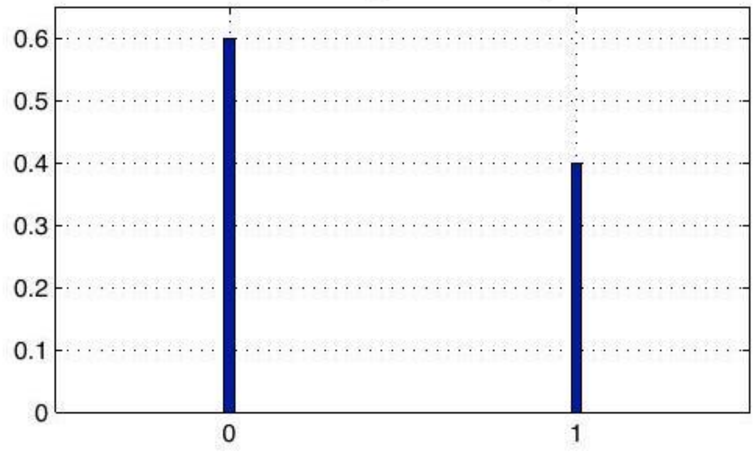
- Cumulative distribution function (cdf)
- $F_X(x) = P(X \leq x) = \sum_{i:x_i \leq x} p(x_i)$
  - Staircase function with jumps at x_i
- Cdf for "throw to first heads" $(p = 0.3)$



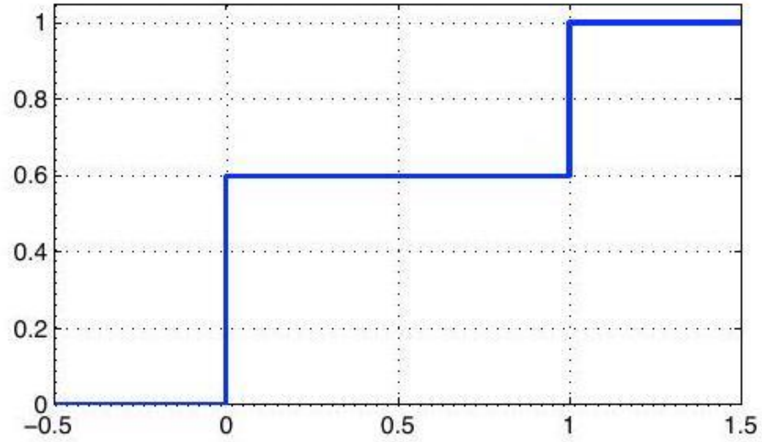
Bernoulli

- A trial/experiment/bet can succeed w.p. p or fail w.p. $q := 1 - p$
    - Ex: coin throws, any indication of an event
- Bernoulli X can be 0 or 1 . Pmf is $p(x) = p^x q^{1-x}$
- cdf:
- $F(x) = \begin{cases} 0, & x < 0 \\ q, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$

- pmf ($p = 0.4$)



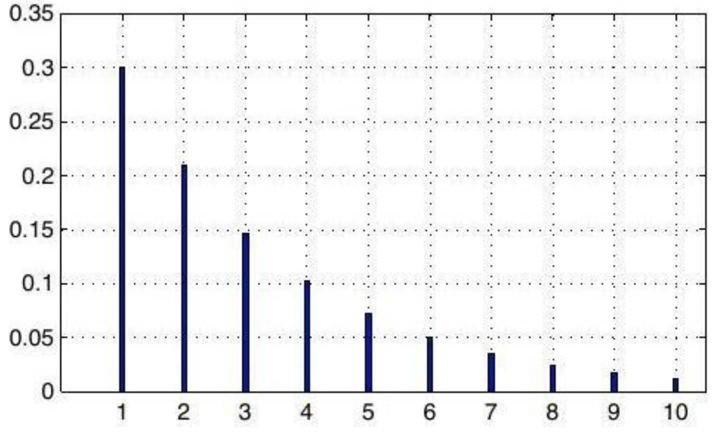
- cdf ($p = 0.4$)



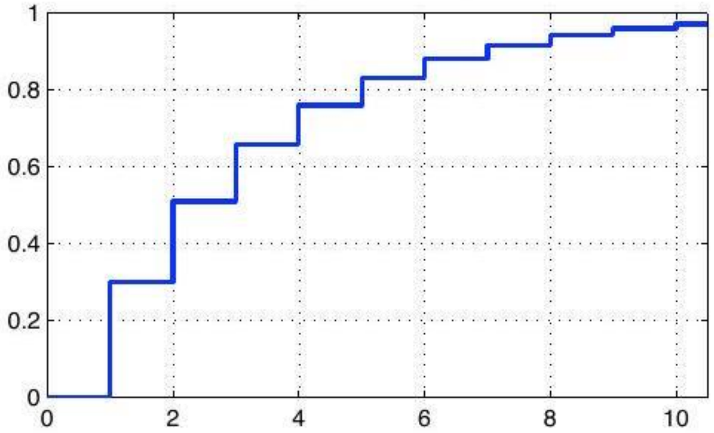
Geometric

- Count number of Bernoulli trials needed to register first success
  - Trials succeed w.p. p and are independent
- Number of trials X until success is geometric with parameter p
- Pmf:
- $p(x) = p(1-p)^{x-1}$
  - One success after $x-1$ failures, trials are independent
- Cdf:
- $F(x) = 1 - (1-p)^x$
  - Recall $\mathrm{P}(X > x) = (1-p)^x$; or just sum the geometric series

- pmf ($p = 0.3$)



- cdf ($p = 0.3$)

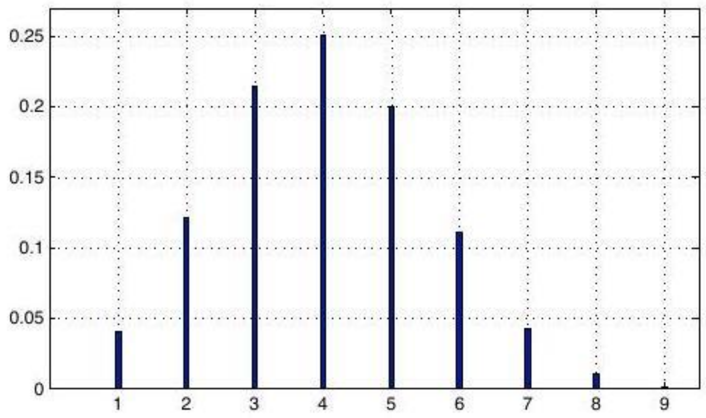


Binomial

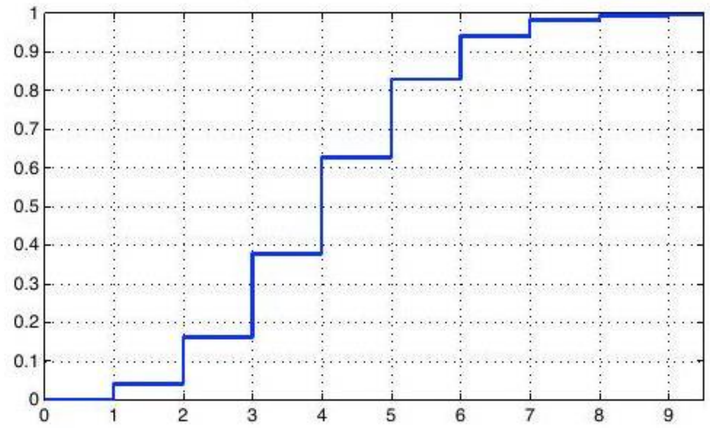
- ❑ Count number of successes X in n Bernoulli trials
  - Trials succeed w.p. p and are independent

- ❑ Number of successes X is binomial with parameters (n, p).
- ❑ Pmf is:
- ❑ $p(x) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}$
  - $X = x$ for x successes (p^x) and $n - x$ failures $((1-p)^{n-x})$
  - $\binom{n}{x}$ ways of drawing x successes and $n - x$ failures

❑ pmf$(n = 9, p = 0.4)$



❑ cdf$(n = 9, p = 0.4)$



Binomial (cont'd)

- Let $Y_i, i = 1, \dots n$ be Bernoulli RV s with parameter p
  - Y_i associated with independent events
- Can write binomial X with parameters (n, p) as $\Rightarrow X = \sum_{i=1}^{n} Y_i$

Example

- Consider binomials Y and Z with parameters (n_Y, p) and (n_Z, p)

Q: Probability distribution of $X = Y + Z$?

- Write $Y = \sum_{i=1}^{n_Y} Y_i$ and $Z = \sum_{i=1}^{n_Z} Z_i$, thus

$$X = \sum_{i=1}^{n_Y} Y_i + \sum_{i=1}^{n_Z} Z_i$$

  - $\Rightarrow X$ is binomial with parameter $(n_Y + n_Z, p)$

Poisson

- Counts of rare events (radioactive decay, packet arrivals, accidents)
- Usually modeled as Poisson with parameter λ and
- Pmf: $p(x) = e^{-\lambda}\frac{\lambda^x}{x!}$

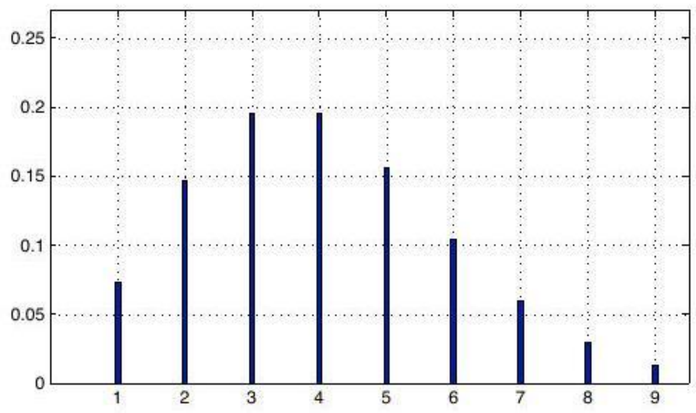
Q: Is this a properly defined pmf? Yes

- Taylor's expansion of $e^x = 1 + x + x^2/2 + \cdots + x^i/i! + \cdots$

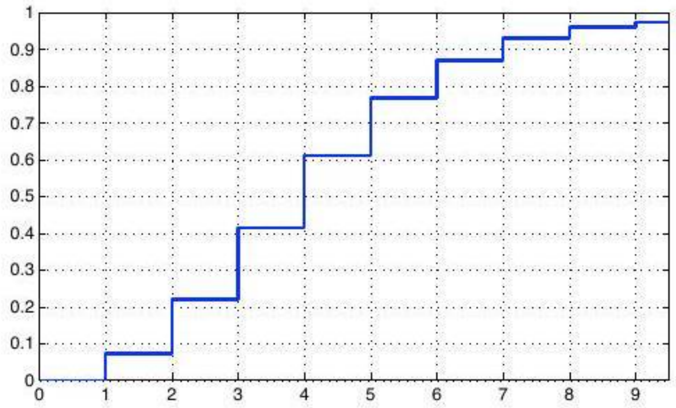
Then

- $P(S) = \sum_{i=0}^{\infty} p(i) = e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} = e^{-\lambda}e^{\lambda} = 1\checkmark$

pmf($\lambda = 4$)

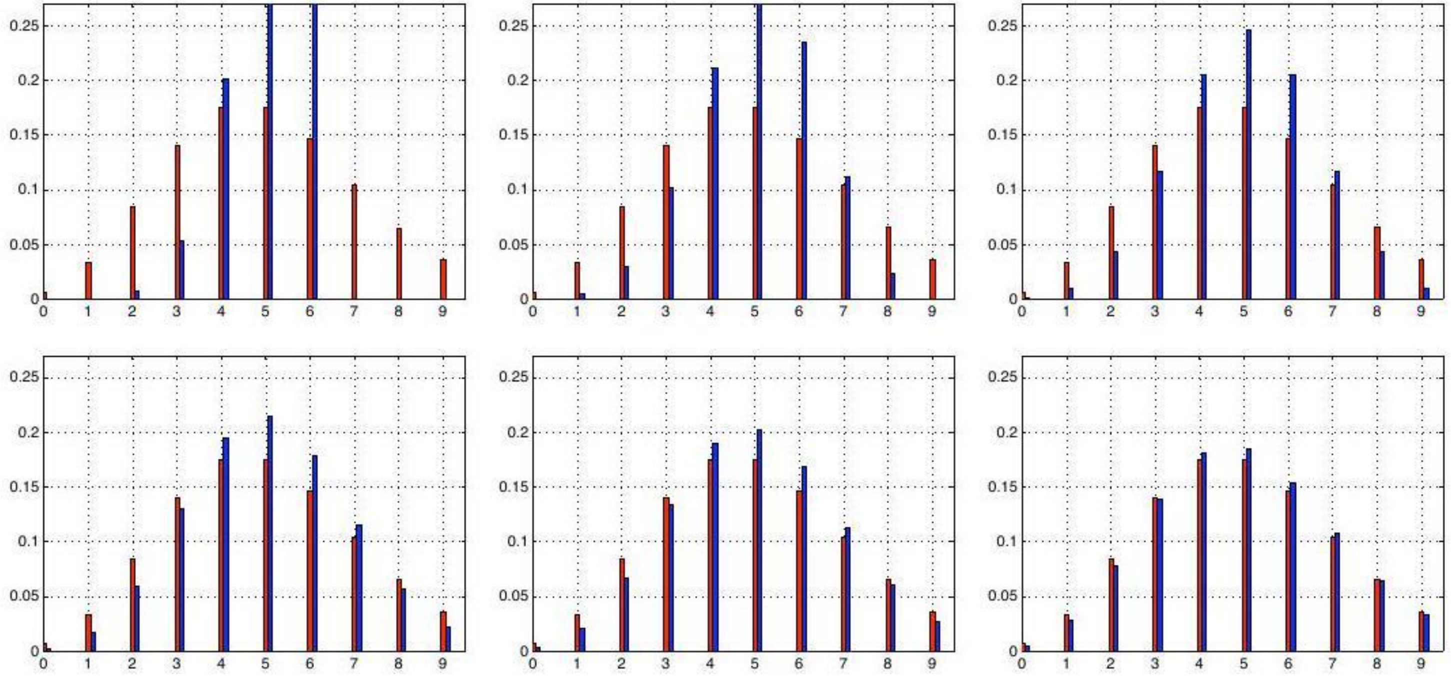


cdf($\lambda = 4$)



Poisson & Binomial

- X is binomial with parameters (n, p)
- Let $n \to \infty$ while maintaining a constant product $np = \lambda$
    - If we just let $n \to \infty$ number of successes diverges. Boring
- Compare with Poisson distribution with parameter λ
    - $\lambda = 5, n = 6,8,10,15,20,50$



Poisson & Binomial (cont'd)

- This is, in fact, the motivation for the definition of a Poisson RV
- Substituting $p = \lambda/n$ in the pmf of a binomial RV

$$p_n(x) = \frac{n!}{(n-x)!\,x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} = \frac{n(n-1)\ldots(n-x+1)}{n^x}\frac{\lambda^x}{x!}\frac{(1-\lambda/n)^n}{(1-\lambda/n)^x}$$

  - Used factorials' defs., $(1-\lambda/n)^{n-x} = \frac{(1-\lambda/n)^n}{(1-\lambda/n)^x}$, and reordered terms

- In the limit, red term is $\lim_{n\to\infty}(1-\lambda/n)^n = e^{-\lambda}$
- Black and blue terms converge to 1 . From both observations

$$\lim_{n\to\infty} p_n(x) = 1\frac{\lambda^x}{x!}\frac{e^{-\lambda}}{1} = e^{-\lambda}\frac{\lambda^x}{x!}$$

- Limit is the pmf of a Poisson RV

Discrete RVs - Remarks

- Binomial distribution is motivated by counting successes
- The Poisson is an approximation for large number of trials n
- Poisson distribution is more tractable (compare pmfs)
- Sometimes called "law of rare events"
- Individual events (successes) happen with small probability $p = \lambda/n$
- Aggregate event (number of successes), though, need not be rare
- Notice that all four RVs seen so far are related to "coin tosses"

Probability Space in D-RVs?

- Random variables are mappings $X(s): S \mapsto \mathbb{R}$
  - The underlying probability space often "disappears"
  - This is for notational convenience, but it's still there

Example

- Let's construct a probability space for a Bernoulli RV
- Let $S = [0,1], \mathcal{F}$ the Borel sigma-field and $\mathrm{P}([a, b]) = b - a, a \leq b$
- Fix a parameter $p \in [0,1]$ and define

$$X(s) = \begin{cases} 1, & s \leq p, \\ 0, & s > p \end{cases}$$

$$\Rightarrow \mathrm{P}(X = 1) = \mathrm{P}(s \leq p) = \mathrm{P}([0, p]) = p \text{ and } \mathrm{P}(X = 0) = 1 - p$$

- Can do a similar construction for all distributions consider so far

Continuous RVs

- Discrete RVs: Take discrete values
- **Continuous RVs: Take continuous values**

Probability Density Function

- Possible values for continuous RV X form a dense subset $\mathcal{X} \subseteq \mathbb{R}$
  - Uncountably infinite number of possible values
- Probability density function (pdf) $f_X(x) \geq 0$ is such that for any subset $\mathcal{X} \subseteq \mathbb{R}$

$$P(X \in \mathcal{X}) = \int_{\mathcal{X}} f_X(x)dx$$

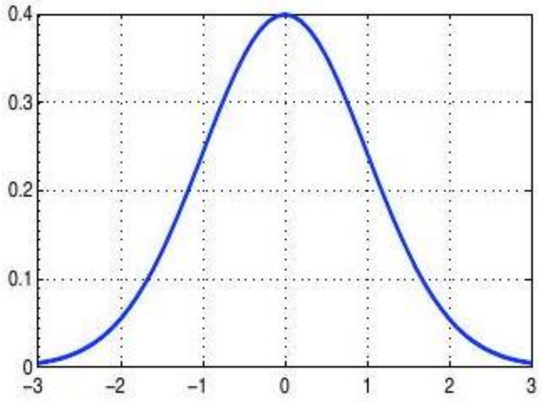
  - Will have $P(X = x) = 0$ for all $x \in \mathcal{X}$

- cdf defined as before and related to the pdf

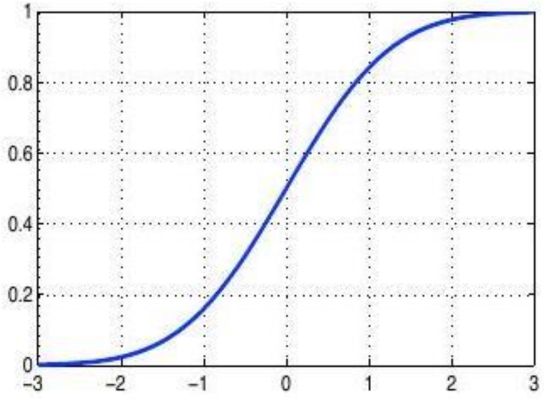
$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(u)du$$

- $\Rightarrow P(X \leq \infty) = F_X(\infty) = \lim_{x\to\infty} F_X(x) = 1$

- Normal pdf



- Normal cdf



More on CDFs and PDFs

- When the set $\mathcal{X} = [a, b]$ is an interval of $\mathbb{R}$

$$\mathrm{P}(X \in [a, b]) = \mathrm{P}(X \leq b) - \mathrm{P}(X \leq a) = F_X(b) - F_X(a)$$

- In terms of the pdf it can be written as

$$P(X \in [a, b]) = \int_a^b f_X(x)dx$$

- For small interval $[x_0, x_0 + \delta x]$, in particular

$$\mathrm{P}(X \in [x_0, x_0 + \delta x]) = \int_{x_0}^{x_0+\delta x} f_X(x)dx \approx f_X(x_0)\delta x$$

    - Probability is the "area under the pdf" (thus "density")

- Another relationship between pdf and cdf is $\Rightarrow \frac{\partial F_X(x)}{\partial x} = f_X(x)$

    - Fundamental theorem of calculus ("derivative inverse of integral")

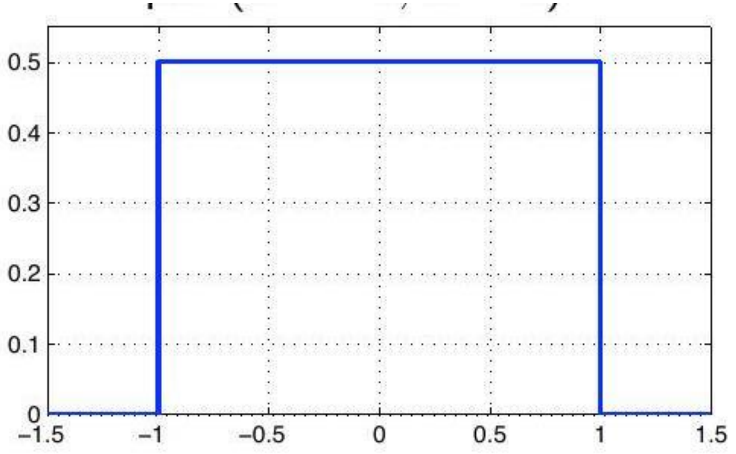
Uniform

- Model problems with equal probability of landing on an interval $[a, b]$
- Pdf of uniform RV is $f(x) = 0$ outside the interval $[a, b]$ and

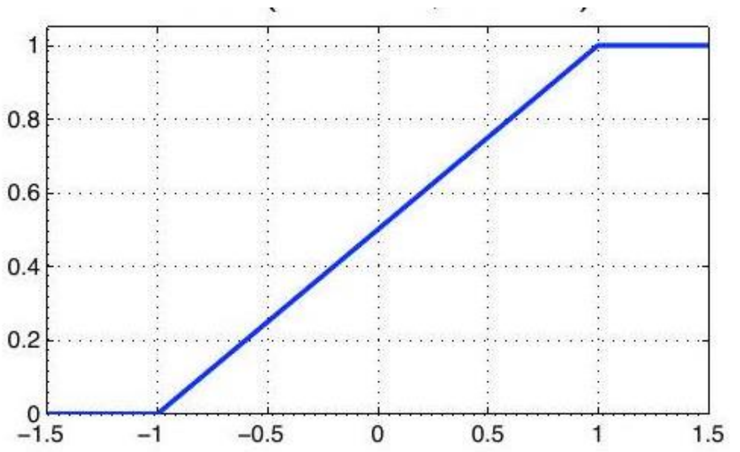
$$f(x) = \frac{1}{b-a}, \quad \text{for } a \leq x \leq b$$

- Cdf is $F(x) = (x - a)/(b - a)$ in the interval $[a, b]$ (0 before, 1 after)
- Prob. of interval $[\alpha, \beta] \subseteq [a, b]$ is $\int_{\alpha}^{\beta} f(x)dx = (\beta - \alpha)/(b - a)$
- Depends on interval's width $\beta - \alpha$ only, not on its position

pdf$(a = -1, b = 1)$



cdf$(a = -1, b = 1)$



Exponential

- Model duration of phone calls, lifetime of electronic components

- Pdf of exponential RV is

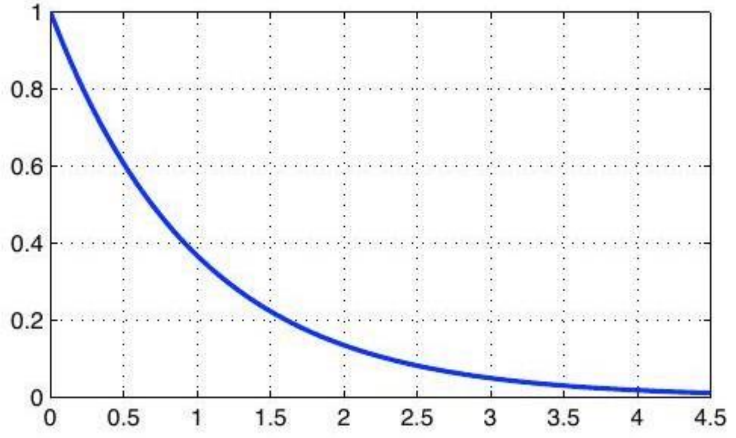
$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

- As parameter λ increases, "height" increases and "width" decreases

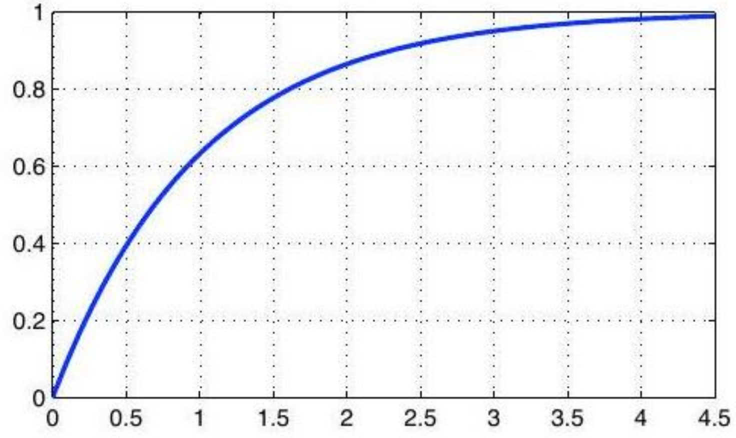
- Cdf obtained by integrating pdf

$$F(x) = \int_{-\infty}^{x} f(u)du = \int_{0}^{x} \lambda e^{-\lambda u} du = -e^{-\lambda u}\Big|_0^x = 1 - e^{-\lambda x}$$

pdf($\lambda = 1$)



cdf($\lambda = 1$)



Normal / Gaussian

- Model randomness arising from large number of random effects

- Pdf of normal RV is

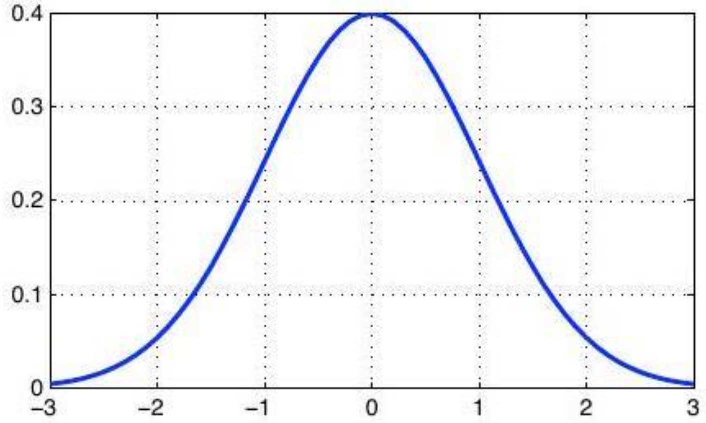
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

- μ is the mean (center), σ^2 is the variance (width)
- 0.68 prob. between $\mu \pm \sigma$, 0.997 prob. in $\mu \pm 3\sigma$

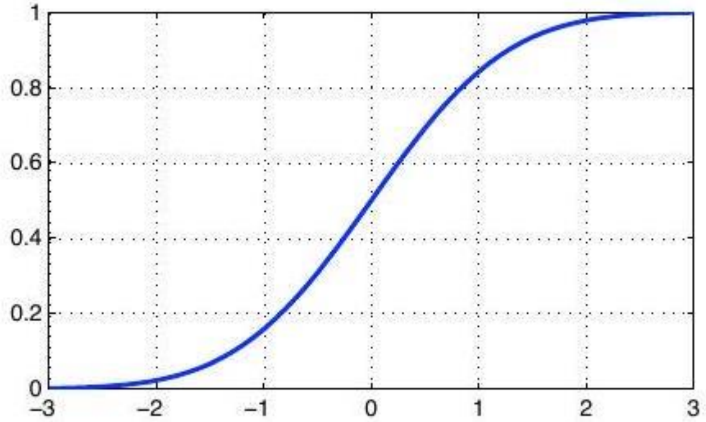
- Standard normal RV has $\mu = 0$ and $\sigma^2 = 1$

- Cdf $F(x)$ cannot be expressed in terms of elementary functions

pdf$(\mu = 0, \sigma = 1)$



cdf$(\mu = 0, \sigma = 1)$



Expected Values

- We are asked to summarize information about a RV in a single value. What should this value be?
- If we are allowed a description with a few values. What should they be?

- Expected (mean) values are convenient answers to these questions
- Beware: Expectations are condensed descriptions
  - They overlook some aspects of the random phenomenon
  - Whole story told by the probability distribution (cdf)

Expected Values for D-RVs

- Discrete RV X taking on values $x_i, i = 1,2, \ldots$ with pmf $p(x)$
- Def: The expected value of the discrete RV X is

$$\mathbb{E}[X] := \sum_{i=1}^{\infty} x_i p(x_i) = \sum_{x:p(x)>0} x p(x)$$

- Weighted average of possible values x_i. Probabilities are weights.
- Common average if RV takes values $x_i, i = 1, \ldots, N$ equiprobably

$$\mathbb{E}[X] = \sum_{i=1}^{N} x_i p(x_i) = \sum_{i=1}^{N} x_i \frac{1}{N} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

Exp. Val. of Bernoulli and Geometric RVs

- Ex: For a Bernoulli RV $p(x) = p^x q^{1-x}$, for $x \in \{0,1\}$

$$\mathbb{E}[X] = 1 \times p + 0 \times q = p$$

- Ex: For a geometric RV$p(x) = p(1-p)^{x-1} = pq^{x-1}$, for $x \geq 1$
- Note that $\partial q^x / \partial q = xq^{x-1}$ and that derivatives are linear operators

$$\mathbb{E}[X] = \sum_{x=1}^{\infty} xpq^{x-1} = p \sum_{x=1}^{\infty} \frac{\partial q^x}{\partial q} = p \frac{\partial}{\partial q}\left(\sum_{x=1}^{\infty} q^x\right)$$

- Sum inside derivative is geometric. Sums to $q/(1-q)$, thus

$$\mathbb{E}[X] = p \frac{\partial}{\partial q}\left(\frac{q}{1-q}\right) = \frac{p}{(1-q)^2} = \frac{1}{p}$$

- Time to first success is inverse of success probability. Reasonable.

Exp. Val. of Poisson RV

- Ex: For a Poisson RV $p(x) = e^{-\lambda}(\lambda^x/x!)$, for $x \geq 0$
- First summand in definition is 0 , pull λ out, and use $\frac{x}{x!} = \frac{1}{(x-1)!}$

$$\mathbb{E}[X] = \sum_{x=0}^{\infty} x e^{-\lambda} \frac{\lambda^x}{x!} = \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

- Sum is Taylor's expansion of $e^{\lambda} = 1 + \lambda + \lambda^2/2! + \cdots + \lambda^x/x!$

$$\mathbb{E}[X] = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

- Poisson is limit of binomial for large number of trials n, with $\lambda = np$
  - Counts number of successes in n trials that succeed w.p. p
- Expected number of successes is $\lambda = np$
  - Number of trials × probability of individual success. Reasonable.

Expected Values for C-RVs

- Continuous RV X taking values on $\mathbb{R}$ with pdf $f(x)$
- Def: The expected value of the continuous RV X is

$$\mathbb{E}[X] := \int_{-\infty}^{\infty} x f(x) dx$$

  - Compare with $\mathbb{E}[X] := \sum_{x:p(x)>0} x p(x)$ in the discrete RV case
- Note that the integral or sum are assumed to be well defined
  - Otherwise, we say the expectation does not exist

Exp. Val. of Normal RV

- Ex: For a normal RV add and subtract μ, separate integrals

$$\mathbb{E}[X] = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x+\mu-\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \mu \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

- First integral is 1 because it integrates a pdf in all $\mathbb{R}$
- Second integral is 0 by symmetry. Both observations yield

$$\mathbb{E}[X] = \mu$$

- The mean of a RV with a symmetric pdf is the point of symmetry.

Exp. Val. of Uniform and Exponential RVs

- Ex: For a uniform RV $f(x) = 1/(b-a)$, for $a \leq x \leq b$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_{a}^{b} \frac{x}{b-a} dx = \frac{b^2 - a^2}{2(b-a)} = \frac{(a+b)}{2}$$

- Makes sense, since pdf is symmetric around midpoint $(a+b)/2$
- Ex: For an exponential RV (non symmetric) integrate by parts

$$\mathbb{E}[X] = \int_{0}^{\infty} x \lambda e^{-\lambda x} dx = -x e^{-\lambda x}\Big|_{0}^{\infty} + \int_{0}^{\infty} e^{-\lambda x} dx = -x e^{-\lambda x}\Big|_{0}^{\infty} - \frac{e^{-\lambda x}}{\lambda}\Bigg|_{0}^{\infty} = \frac{1}{\lambda}$$

Expected Value of a Function of a RV

- ❑ Consider a function $g(X)$ of a RV X. Expected value of $g(X)$?
- ❑ g(X) is also a RV, then it also has a pmf $p_{g(X)}(g(x))$

$$\mathbb{E}[g(X)] = \sum_{g(x):p_{g(X)}(g(x))>0} g(x)p_{g(X)}\big(g(x)\big)$$

Requires calculating the pmf of $g(X)$. There is a simpler way.

Theorem:

- ❑ Consider a function $g(X)$ of a discrete RV X with pmf $p_X(x)$. Then: $\mathbb{E}[g(X)] = \sum_{i=1}^{\infty} g(x_i)p_X(x_i)$
  - Weighted average of functional values. No need to find pmf of $g(X)$.
- ❑ Same can be proved for a continuous RV: $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x)dx$

Expected Value of Linear Transformation

- Consider a linear function (actually affine) $g(X) = aX + b$

$$\mathbb{E}[aX + b] = \sum_{i=1}^{\infty} (ax_i + b)p_X(x_i) = \sum_{i=1}^{\infty} ax_i p_X(x_i) + \sum_{i=1}^{\infty} b p_X(x_i)$$

$$= a\sum_{i=1}^{\infty} x_i p_X(x_i) + b\sum_{i=1}^{\infty} p_X(x_i) = a\mathbb{E}[X] + b \cdot 1$$

- Can interchange expectation with additive/multiplicative constants.

- $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$
  - Again, the same holds for a continuous RV

Expected Value of Indication Function

- Let X be a RV and $\mathcal{X}$ be a set

$$\mathbb{I}\{X \in \mathcal{X}\} = \begin{cases} 1, & \text{if } x \in \mathcal{X} \\ 0, & \text{if } x \notin \mathcal{X} \end{cases}$$

- Expected value of $\mathbb{I}\{X \in \mathcal{X}\}$ in the discrete case

$$\mathbb{E}[\mathbb{I}\{X \in \mathcal{X}\}] = \sum_{x:p_X(x)>0} \mathbb{I}\{x \in \mathcal{X}\} p_X(x) = \sum_{x \in \mathcal{X}} p_X(x) = \mathrm{P}(X \in \mathcal{X})$$

- Likewise in the continuous case

$$\mathbb{E}[\mathbb{I}\{X \in \mathcal{X}\}] = \int_{-\infty}^{\infty} \mathbb{I}\{x \in \mathcal{X}\} f_X(x) dx = \int_{x \in \mathcal{X}} f_X(x) dx = P(X \in \mathcal{X})$$

- Expected value of indicator RV = Probability of indicated event $\Rightarrow$ Recall $\mathbb{E}[X] = p$ for Bernoulli RV (it "indicates success")

Moments, Central Moments, Variance

- Def: The n-th moment ($n \geq 0$) of a RV is

$$\mathbb{E}[X^n] = \sum_{i=1}^{\infty} x_i^n p(x_i)$$

- Def: The n-th central moment corrects for the mean, that is

$$\mathbb{E}[(X - \mathbb{E}[X])^n] = \sum_{i=1}^{\infty} (x_i - \mathbb{E}[X])^n p(x_i)$$

- 0-th order moment is $\mathbb{E}[X^0] = 1$; 1-st moment is the mean $\mathbb{E}[X]$
- 2-nd central moment is the variance. Measures width of the pmf

$$\mathrm{var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}^2[X]$$

- Ex: For affine functions

$$\mathrm{var}[aX + b] = a^2 \mathrm{var}[X]$$

Variance of Bernoulli and Poisson RVs

- Ex: For a Bernoulli RV X with parameter p, $\mathbb{E}[X] = \mathbb{E}[X^2] = p$
- $\Rightarrow \mathrm{var}[X] = \mathbb{E}[X^2] - \mathbb{E}^2[X] = p - p^2 = p(1-p)$
- Ex: For Poisson RV Y with parameter λ, second moment is

$$\mathbb{E}[Y^2] = \sum_{y=0}^{\infty} y^2 e^{-\lambda} \frac{\lambda^y}{y!}$$

$$= \sum_{y=1}^{\infty} y \frac{e^{-\lambda}\lambda^y}{(y-1)!} = \sum_{y=1}^{\infty} (y-1) \frac{e^{-\lambda}\lambda^y}{(y-1)!} + \sum_{y=1}^{\infty} \frac{e^{-\lambda}\lambda^y}{(y-1)!}$$

$$= e^{-\lambda}\lambda^2 \sum_{y=2}^{\infty} \frac{\lambda^{y-2}}{(y-2)!} + e^{-\lambda}\lambda \sum_{y=1}^{\infty} \frac{\lambda^{y-}}{(y-1)!} = e^{-\lambda}\lambda^2 e^{\lambda} + e^{-\lambda}\lambda e^{\lambda} = \lambda^2 + \lambda$$

- $\mathrm{var}[Y] = \mathbb{E}[Y^2] - \mathbb{E}^2[Y] = \lambda^2 + \lambda - \lambda^2 = \lambda$

Moments of Uniform

- Ex: For a Uniform RV $X \in [-a, b]$, the n-th order moment is
- $\mathbb{E}\{X^n\} = \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}$
- For example:
  - $\mathbb{E}\{X^1\} = \frac{b^2 - a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{(b+a)}{2}$
  - $\mathbb{E}\{X^2\} = \frac{b^3 - a^3}{3(b-a)}$
  - $\mathbb{E}\{X^3\} = \frac{b^4 - a^4}{4(b-a)}$

Multiple Random Variables

Joint CDF

- Want to study problems with more than one RV. Say, e.g., X and Y
- Probability distributions of X and Y are not sufficient
  - Joint probability distribution (cdf) of (X, Y) defined as

$$F_{XY}(x, y) = \mathrm{P}(X \leq x, Y \leq y)$$

- If X, Y clear from context omit subindex to write $F_{XY}(x, y) = F(x, y)$
- Can recover $F_X(x)$ by considering all possible values of Y

$$F_X(x) = \mathrm{P}(X \leq x) = \mathrm{P}(X \leq x, Y \leq \infty) = F_{XY}(x, \infty)$$

- $F_X(x)$ and $F_Y(y) = F_{XY}(\infty, y)$ are called marginal cdfs

Joint PMF

- Consider discrete RVs X and Y
- X takes values in $\mathcal{X} := \{x_1, x_2, \dots\}$ and Y in $\mathcal{Y} := \{y_1, y_2, \dots\}$
- Joint pmf of (X, Y) defined as

$$p_{XY}(x, y) = P(X = x, Y = y)$$

- Possible values (x, y) are elements of the Cartesian product $\mathcal{X} \times \mathcal{Y}$
  - $(x_1, y_1), (x_1, y_2), \dots, (x_2, y_1), (x_2, y_2), \dots, (x_3, y_1), (x_3, y_2), \dots$
- Marginal pmf $p_X(x)$ obtained by summing over all values of Y

$$p_X(x) = \mathrm{P}(X = x) = \sum_{y \in \mathcal{Y}} \mathrm{P}(X = x, Y = y) = \sum_{y \in \mathcal{Y}} p_{XY}(x, y)$$

- Likewise, $p_Y(y) = \sum_{x \in \mathcal{X}} p_{XY}(x, y)$. Marginalize by summing

Joint PDF

- Consider continuous RVs X, Y. Arbitrary set $\mathcal{A} \in \mathbb{R}^2$
- Joint pdf is a function $f_{XY}(x, y): \mathbb{R}^2 \to \mathbb{R}^+$such that

$$\mathrm{P}((X, Y) \in \mathcal{A}) = \iint_{\mathcal{A}} f_{XY}(x, y) dx dy$$

- Marginalization. There are two ways of writing $\mathrm{P}(X \in \mathcal{X})$

$$\mathrm{P}(X \in \mathcal{X}) = \mathrm{P}(X \in \mathcal{X}, Y \in \mathbb{R}) = \int_{X \in \mathcal{X}} \int_{-\infty}^{+\infty} f_{XY}(x, y) dy dx$$

- Definition of $f_X(x) \Rightarrow \mathrm{P}(X \in \mathcal{X}) = \int_{X \in \mathcal{X}} f_X(x) dx$
- $f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y) dy,\ f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x, y) dx$

Joint PDF (cont'd)

- Consider two Bernoulli RVs$_1, B_2$, with the same parameter p
- Define $X = B_1$ and $Y = B_1 + B_2$
- The pmf of X is

$$p_X(0) = 1 - p,\; p_X(1) = p$$

- Likewise, the pmf of Y is

$$p_Y(0) = (1-p)^2,\; p_Y(1) = 2p(1-p),\; p_Y(2) = p^2$$

- The joint pmf of X and Y is

- $p_{XY}(0,0) = (1-p)^2$
- $p_{XY}(0,1) = p(1-p)$
- $p_{XY}(0,2) = 0$
- $p_{XY}(1,0) = 0$
- $p_{XY}(1,1) = p(1-p)$
- $p_{XY}(1,2) = p^2$

Random Vectors

- For convenience often arrange RVs in a vector
  - Prob. distribution of vector is joint distribution of its entries
- Consider, e.g., two RVs X and Y. Random vector is $\mathbf{X} = [X, Y]^T$
- If X and Y are discrete, vector variable $\mathbf{X}$ is discrete with pmf

$$p_{\mathbf{X}}(\mathbf{x}) = p_{\mathbf{X}}([x, y]^T) = p_{XY}(x, y)$$

- If X, Y continuous, $\mathbf{X}$ continuous with pdf

$$f_{\mathbf{X}}(\mathbf{x}) = f_{\mathbf{X}}([x, y]^T) = f_{XY}(x, y)$$

- Vector cdf is $\Rightarrow F_{\mathbf{X}}(\mathbf{x}) = F_{\mathbf{X}}([x, y]^T) = F_{XY}(x, y)$
- In general, can define n-dimensional RVS$\mathbf{X} := [X_1, X_2, \ldots, X_n]^T$
  - Just notation, definitions carry over from the $n = 2$ case

Joint Expectations

- RVs X and Y and function $g(X, Y)$. Function $g(X, Y)$ also a RV
- Expected value of $g(X, Y)$ when X and Y discrete can be written as

$$\mathbb{E}[g(X, Y)] = \sum_{x,y:p_{XY}(x,y)>0} g(x, y) p_{XY}(x, y)$$

- When X and Y are continuous

$$\mathbb{E}[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{XY}(x, y) dx dy$$

  - Can have more than two RVs and use vector notation
- Ex: Linear transformation of a vector RV $\boldsymbol{x} \in \mathbb{R}^n$: $g(\boldsymbol{x}) = \mathbf{a}^T \boldsymbol{x}$

$$\mathbb{E}[\mathbf{a}^T \boldsymbol{x}] = \int_{\mathbb{R}^n} \mathbf{a}^T \mathbf{x} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

Mean of LC of RVs

- Start with LC of $N = 1$ RVs

$$\mathbb{E}[a_1X_1 + a_2X_2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(a_1w + a_2q)f_{X_1X_2}(q,w)dqdw$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_1q\, f_{X_1X_2}(q,w)dqdw + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_2wf_{X_1X_2}(q,w)dqdw$$

- Reordering and using marginal expressions:

$$\mathbb{E}[a_1X_1 + a_2X_2] = a_1\int_{-\infty}^{\infty} q\left(\int_{-\infty}^{\infty} f_{X_1X_2}(q,w)dw\right)dq + a_2\int_{-\infty}^{\infty} w\left(\int_{-\infty}^{\infty} f_{X_1X_2}(q,w)dq\right)dw$$

$$= a_1\int_{-\infty}^{\infty} q\cdot f_{X_1}(q)\cdot dq + a_2\int_{-\infty}^{\infty} w\cdot f_{X_2}(w)\cdot dw = a_1\mathbb{E}[X_1] + a_2\mathbb{E}[X_2]$$

Mean of LC of RVs (cont'd)

- In general, for N RVs $X_1, \dots, X_N$

$$\mathbb{E}\left[\sum_{n=1}^{N} a_n X_n\right] = \sum_{n=1}^{N} a_n \mathbb{E}[X_n]$$

- That is, $\mathbb{E}$ is a **linear operator** and as such can swap order with linear combinations.
- Practice to make sure that you can derive the general N case above.

Variance of LC of RVs

- For N RVs $X_1, \dots, X_N$

$$Var\left[\sum_{n=1}^{N} a_n X_n\right] = \mathbb{E}\left[\left(\sum_{n=1}^{N} a_n X_n - \mathbb{E}\left(\sum_{m=1}^{N} a_m X_m\right)\right)^2\right]$$

$$= \mathbb{E}\left[\left(\sum_{n=1}^{N} a_n \big(X_n - \mathbb{E}(X_n)\big)\right)^2\right] = \mathbb{E}\left[\sum_{n=1}^{N}\sum_{m=1}^{N} a_n a_M \big(X_n - \mathbb{E}(X_n)\big)\big(X_m - \mathbb{E}(X_m)\big)\right]$$

$$= \sum_{n=1}^{N}\sum_{m=1}^{N} a_n a_M \mathbb{E}\Big(\big(X_n - \mathbb{E}(X_n)\big)\big(X_m - \mathbb{E}(X_m)\big)\Big)$$

Independence of RVs

- Events E and F are independent if $\mathrm{P}(E \cap F) = \mathrm{P}(E)\mathrm{P}(F)$
- Def: RVs X and Y are independent if events $X \leq x$ and $Y \leq y$ are independent for all x and y, i.e.

$$\mathrm{P}(X \leq x, Y \leq y) = \mathrm{P}(X \leq x)\mathrm{P}(Y \leq y)$$

  - By definition, equivalent to $F_{XY}(x, y) = F_X(x)F_Y(y)$

- For discrete RV s equivalent to analogous relation between pmfs

$$p_{XY}(x, y) = p_X(x)p_Y(y)$$

- For continuous RVs the analogous is true for pdfs

$$f_{XY}(x, y) = f_X(x)f_Y(y)$$

- Independence $\Leftrightarrow$ Joint distribution factorizes into product of marginals

Covariance

- Def: The covariance of X and Y is (generalizes variance to pairs of RV)

$$\mathrm{cov}(X,Y) = \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

- If $\mathrm{cov}(X,Y) = 0$ variables X and Y are said to be uncorrelated
- If X, Y independent then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ and $\mathrm{cov}(X,Y) = 0$
  - Independence implies uncorrelated RV s
- Opposite is not true, may have $\mathrm{cov}(X,Y) = 0$ for dependent X, Y
- Ex: X uniform in $[-a, a]$ and $Y = X^2$
  - But uncorrelatedness implies independence if X, Y are normal
- If $\mathrm{cov}(X,Y) > 0$ then X and Y tend to move in the same direction
  - Positive correlation
- If $\mathrm{cov}(X,Y) < 0$ then X and Y tend to move in opposite directions
  - Negative correlation

Covariance – Example

- Let X be a zero-mean random signal and Z zero-mean noise
  - Signal X and noise Z are independent
- Consider received signals $Y_1 = X + Z$ and $Y_2 = -X + Z$

(1) Y_1 and X are positively correlated (X, Y_1 move in same direction)

$$\begin{aligned} \mathrm{cov}(X, Y_1) &= \mathbb{E}[XY_1] - \mathbb{E}[X]\mathbb{E}[Y_1] \\ &= \mathbb{E}[X(X+Z)] - \mathbb{E}[X]\mathbb{E}[X+Z] \end{aligned}$$

- Second term is $0 (\mathbb{E}[X] = 0)$. For first term independence of X, Z

$$\mathbb{E}[X(X+Z)] = \mathbb{E}[X^2] + \mathbb{E}[X]\mathbb{E}[Z] = \mathbb{E}[X^2]$$

- Combining observations $\Rightarrow \mathrm{cov}(X, Y_1) = \mathbb{E}[X^2] > 0$

Covariance – Example (cont'd)

(2) Y_2 and X are negatively correlated (X, Y_2 move opposite direction)

- ❑ Same computations $\Rightarrow \mathrm{cov}(X, Y_2) = -\mathbb{E}[X^2] < 0$

(3) Can also compute correlation between Y_1 and Y_2

$$\begin{aligned} \mathrm{cov}(Y_1, Y_2) \quad &= \mathbb{E}[(X+Z)(-X+Z)] - \mathbb{E}[(X+Z)]\mathbb{E}[(-X+Z)] \\ &= -\mathbb{E}[X^2] + \mathbb{E}[Z^2] \end{aligned}$$

- ❑ Negative correlation if $\mathbb{E}[X^2] > \mathbb{E}[Z^2]$ (small noise)
- ❑ Type equation here.Positive correlation if $\mathbb{E}[X^2] < \mathbb{E}[Z^2]$ (large noise)

- ❑ Correlation between X and Y_1 or X and Y_2 comes from causality
- ❑ Correlation between Y_1 and Y_2 does not. Latent variables X and Z
    - ❑ Correlation does not imply causation

Back to Variance of LC of RVs

- For N RVs $X_1, \dots, X_N$

$$Var\left[\sum_{n=1}^{N} a_n X_n\right] = \sum_{n=1}^{N}\sum_{m=1}^{N} a_n a_M \mathbb{E}\left(\left(X_n - \mathbb{E}(X_n)\right)\left(X_m - \mathbb{E}(X_m)\right)\right) = \sum_{n=1}^{N}\sum_{m=1}^{N} a_n a_M \mathrm{cov}(X_n, X_m)$$

- In the special case of independent RVs

$$Var\left[\sum_{n=1}^{N} a_n X_n\right] = \sum_{n=1}^{N} a_n^2 \mathrm{var}[X_n] + \sum_{m\neq n} a_n a_M \mathrm{cov}(X_n, X_m) = \sum_{n=1}^{N} a_n^2 \mathrm{var}[X_n]$$

Mean of Product of Independent RVs

- For independent RVs, X and Y, and arbitrary functions $g(X)$ and $h(Y)$:

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$$

- The expected value of the product is the product of the expected values
- Can show that $g(X)$ and $h(Y)$ are also independent. Intuitive

- Ex: Special case when $g(X) = X$ and $h(Y) = Y$ yields

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

- Expectation and product can be interchanged if RV s are independent
- Different from interchange with linear operations (always possible)

Mean of Product of Independent RVs - Proof

- Suppose X and Y continuous RV. Use definition of independence

$$\begin{aligned}\mathbb{E}[g(X)h(Y)] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_{XY}(x,y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_X(x)f_Y(y)dxdy\end{aligned}$$

- Integrand is product of a function of x and a function of y

$$\begin{aligned}\mathbb{E}[g(X)h(Y)] &= \int_{-\infty}^{\infty} g(x)f_X(x)dx \int_{-\infty}^{\infty} h(y)f_Y(y)dy \\ &= \mathbb{E}[g(X)]\mathbb{E}[h(Y)]\end{aligned}$$