**Boolean Functions (Expressions)**

It is useful to know how many different Boolean functions can be constructed on a set of Boolean variables.

When there are no variables, there are two expressions

$$\texttt{False} = 0 \qquad \text{and} \qquad \texttt{True} = 1$$

**Boolean Functions of One Variable**

For one variable $p$, four functions can be constructed.

Recall, a function maps each input value of a variable to one and only one output value.

1. The $\texttt{False}(p)$ function maps each value of $p$ to $0$ ($\texttt{False}$).

2. The identity$(p)$ function maps each value of $p$ to the identical value.

3. The flip$(p)$ function maps $\texttt{False}$ to $\texttt{True}$ and $\texttt{True}$ to $\texttt{False}$.
4. The $\texttt{True}(p)$ function maps each value of $p$ to $1$ ($\texttt{True}$).

**Boolean Functions of One Variable**

For one variable $p$, $4 = 2^{2^1}$ functions can be constructed.

This information can be collected into a table.

| Input | Functions | | | |
|---|---|---|---|---|
| $p$ | False | $p$ | $\neg p$ | True |
| 0 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 |

**Boolean Functions of Two Variables**

For two variables P and Q, 16 Boolean functions can be constructed.

One way to see there are 16 two variables Boolean functions $f(P, Q)$ is to list the 4 truth assignments for P and Q.

| P | Q | $f(P, Q)$ |
|---|---|---|
| 0 | 0 | $a$ |
| 0 | 1 | $b$ |
| 1 | 0 | $c$ |
| 1 | 1 | $d$ |

The values $a, b, c, d$ of the function $f$ can be either a 0 or a 1. Since there are 2 choices for 4 values, there are a total of $16 = 2^4 = 2^{2^2}$ different functions.

**Boolean Functions of Two Variables**

Several of the two variable functions are important. The table below describes the AND, NAND, OR, and NOR functions.

| Input | | Functions | | | |
|---|---|---|---|---|---|
| | | AND | NAND | OR | NOR |
| P | Q | $P \wedge Q$ | $\neg(P \wedge Q)$ | $P \vee Q$ | $\neg(P \vee Q)$ |
| 0 | 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 |
| 1 | 1 | 1 | 0 | 1 | 0 |
| | | 1 | 14 | 7 | 8 |

**Boolean Functions of Two Variables**

The table below describes the EQUIVALENT, EXCLUSIVE OR, P IMPLIES Q, and NOT (P IMPLIES Q) functions.

| Input | | Functions | | | |
|---|---|---|---|---|---|
| | | EQUIV | XOR | PIMPLIESQ | NOT (P IMPLIES Q) |
| P | Q | $P \equiv Q$ | $P \oplus Q$ | $P \rightarrow Q$ | $\neg(P \rightarrow Q)$ |
| 0 | 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 | 1 |
| 1 | 1 | 1 | 0 | 1 | 0 |
| | | 9 | 6 | 13 | 2 |

**Boolean Functions of Two Variables**

The table below describes the `False`, `True`, IDENTITY(P), and FLIP(P) functions.

| Input | | Functions | | | |
|---|---|---|---|---|---|
| | | `False` | `True` | IDENTITY(P) | FLIP(P) |
| P | Q | $P \wedge \neg P$ | $P \vee \neg P$ | P | $\neg P$ |
| 0 | 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 |
| 1 | 1 | 0 | 1 | 1 | 0 |
| | | 0 | 15 | 3 | 12 |

**Boolean Functions of Two Variables**

The table below describes the IDENTITY(Q), FLIP(Q) functions, Q IMPLIES P, and NOT(Q IMPLIES P)

| Input | | Functions | | | |
|---|---|---|---|---|---|
| | | ID(Q) | FLIP(Q) | QIMPLIESP | NOT(Q IMPLIES P) |
| P | Q | Q | $\neg Q$ | $Q \rightarrow P$ | $\neg(Q \rightarrow P)$ |
| 0 | 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 |
| 1 | 1 | 1 | 0 | 1 | 0 |
| | | 5 | 10 | 11 | 4 |

**Boolean Functions of $n$ Variables**

Given $n$ Boolean variables, how many different Boolean functions can be made?

| Number of Variables | Number of Boolean Functions |
|---|---|
| 0 | $2 = 2^1 = 2^{2^0}$ |
| 1 | $4 = 2^2 = 2^{2^1}$ |
| 2 | $16 = 2^4 = 2^{2^2}$ |
| 3 | $256 = 2^8 = 2^{2^3}$ |
| 4 | $65,536 = 2^{16} = 2^{2^4}$ |
| $n$ | $2^{2^n}$ |

**Theorem 1.** *There are $2^{2^n}$ different Boolean functions on n Boolean variables.*