Problem2: Discrete growth models

Collaborators

Karthik Upendra
Konstantinos Zakkas

a) Analytically determine the non-negative steady states of the model

Given,

$$N_{\tau+1} = \frac{(r+1)N_\tau}{1+(\frac{N_\tau}{k})^b} = f(N_\tau) \qquad ....(1)$$

Fixed points occur at $N_{\tau+1} = N_\tau$. Substituting this in the above equation we get

$$N_\tau = \frac{(r+1)N_\tau}{1+(\frac{N_\tau}{k})^b}$$

$$1 + \left(\frac{N_\tau}{k}\right)^b = r + 1$$

Solving it using Mathematica, we get

$N_1^* = 0$ and $N_2^* = k\, r^{\frac{1}{b}}$

b) Perform linear stability analysis and discuss how the linear stability of the non-negative steady state depends on the parameter.

Equation (1) can be written as

$$N_{\tau+1} = f(N_\tau) \qquad ....(2)$$

Add a small perturbation ν_τ to the fixed point N^* .

$$N_\tau = N^* + \nu_\tau \qquad ....(3)$$

Substitute (3) in (2)

$$N^* + \nu_{\tau+1} = f(N^* + \nu_\tau)$$

Using Taylor series expansion around the fixed point

$$N^* + \nu_{\tau+1} = f(N^*) + f'(N^*)\nu_\tau + O(2)$$

According to fixed point definition, $f(N^*) = N^*$, and ignoring the second order term, we get

$$\nu_{\tau+1} = f'(N^*)\nu_\tau \qquad ....(4)$$

Let $f'(N^*) = \lambda$ and Solving this further we get

$$\nu_\tau = \lambda^\tau \nu_0$$

Therefore the perturbations vanishes for $|\lambda| < 1$, whereas the perturbations grows for $|\lambda| > 1$

From equation (1) we have

$$f(N_\tau) = \frac{(r+1)N_\tau}{1+(\frac{N_\tau}{k})^b}$$

Using Mathematica to differentiate it, we get

$$f'(N_\tau) = \frac{1+r-(-1+b)(1+r)(\frac{N_\tau}{k})^b}{\left(1+(\frac{N_\tau}{k})^b\right)^2}$$

Evaluating at the fixed points N_1^* and N_2^* , we get

$$f'(N_\tau)|_{N_1^*} = 1 + r$$

$f'(N_\tau)|_{N_2^*} = 1 - \frac{b\,r}{1+r}$

- Fixed point N_1^* is unstable because $f'(N_\tau)|_{N_1^*} > 1,\ \forall r > 0$
- Fixed point $N^*{}_2$ is stable if

$-1 < 1 - \frac{br}{1+r} < 1$

$-2 < -\frac{br}{1+r} < 0$

$0 < \frac{br}{1+r} < 2$

$0 < br < 2 + 2r$

$0 < b < 2 + \frac{2}{r}$

It is given that $b \geq 1$ so

$1 \leq b < 2 + \frac{2}{r},\ \forall r > 0$

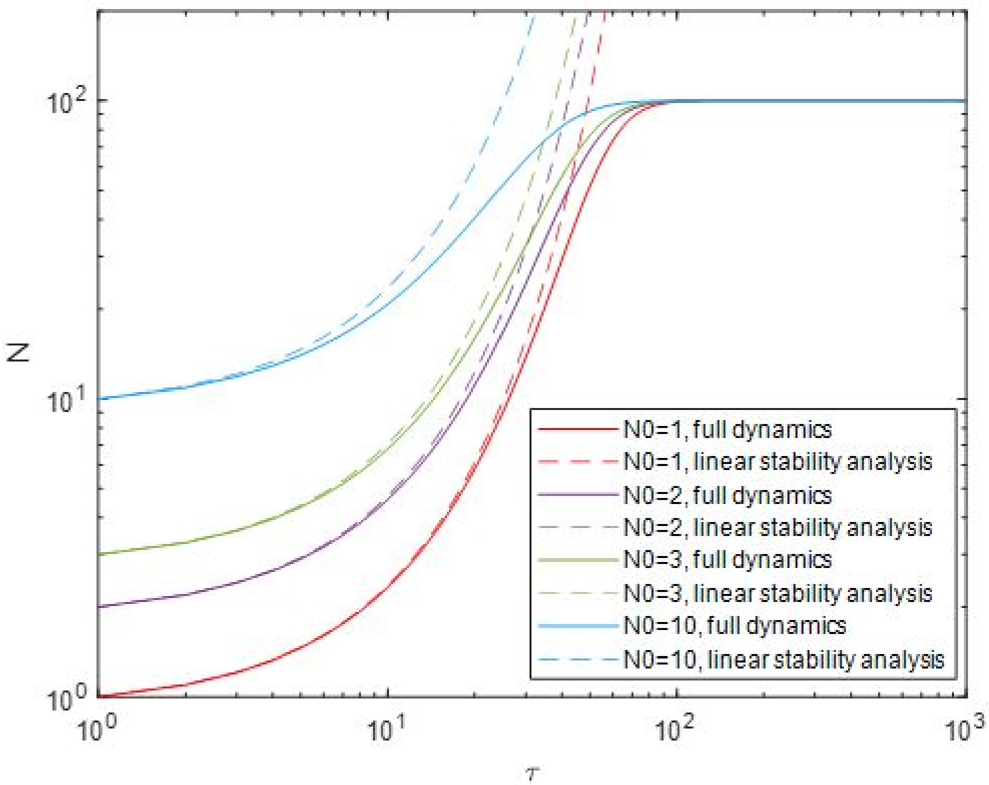
c) Analytically find all the conditions of the parameter where a bifurcation from a stable steady state to an unstable steady state occurs.

The second fixed point undergoes bifurcation when it passes $f'(N_\tau)|_{N_2^*} = -1$. Therefore,

$$1 - \frac{b\,r}{1+r} = -1$$

$$b = \frac{2(1+r)}{r}$$

d) Comparison of the actual dynamics versus linear stability analysis around unstable fixed point



e) Discussion of the graph from part (d)

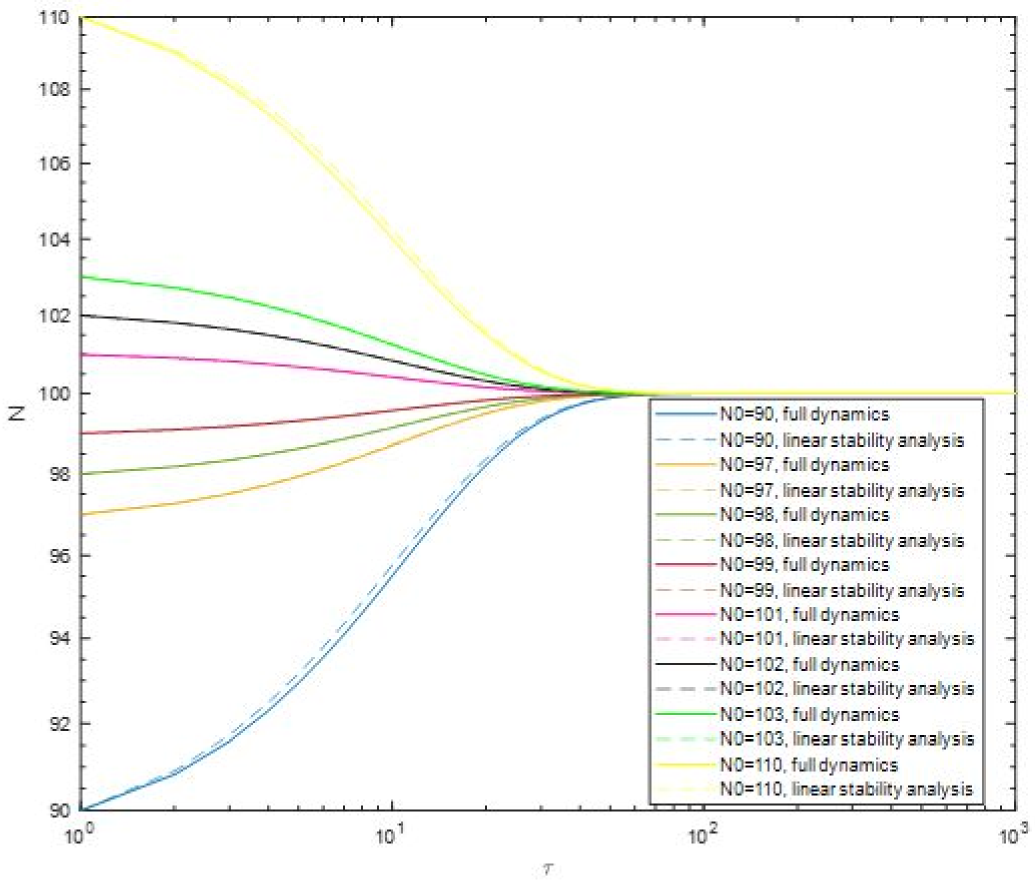
The dynamics of the system for different initial conditions are shown in the figure in part d) as solid lines. Whereas, the dashed line shows the linear stability analysis performed around the unstable fixed point $N_1^* = 0$. The linear stability analysis is done in part b)

The full dynamics of the system settles down to the only available stable fixed point $N_2^* = k\, r^{\frac{1}{b}}$. The equilibrium is independent of the initial conditions.

The linear stability analysis approximates the dynamics around the fixed point quite well, but not good away from it. In this case, linear stability analysis shows that the perturbations grow to infinity as generations tend to infinity. This confirms our claims made in part b that the origin is unstable. The initial conditions do not affect the equilibrium state .

Note: Linear stability analysis works well only around the fixed point in question. It doesn't work well far away from it.

f) Comparison of the actual dynamics versus linear stability analysis around stable fixed point



The dynamics of the system for different initial conditions are shown in the above figure as solid lines. Whereas, the dashed line shows the linear stability analysis performed around the stable fixed point $N_2^* = k\, r^{\frac{1}{b}}$. The linear stability analysis is done in part b)

The linear stability analysis approximates the dynamics quite well near the fixed point. In this case, linear stability analysis shows that the perturbations vanish as generations tend to infinity. This confirms our claims made in part b. The initial conditions do not affect the equilibrium state.

Appendix

Code for part a

Solve[N_τ==((r+1)N_τ)/(1+ (Subscript[N, τ]/k)^b),N_τ]

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

{{N_τ->0},{N_τ->k }}

Code for part b

D[((r+1) N_τ)/(1+(Subscript[N, τ]/k)^b),N_τ]//FullSimplify

(1+r-(-1+b) (1+r) (Subscript[N, τ]/k)^b)/(1+(Subscript[N, τ]/k)^b)^2

Assuming[b>=1,FullSimplify[(1+r-(-1+b) (1+r) (Subscript[N, τ]/k)^b)/(1+(Subscript[N, τ]/k)^b)^2/.N_τ->0]]

1+r

Assuming[b>=1,FullSimplify[(1+r-(-1+b) (1+r) (Subscript[N, τ]/k)^b)/(1+(Subscript[N, τ]/k)^b)^2/.N_τ->k]]

1-(b r)/(1+r)

Code for part c

Solve[1-(b r)/(1+r)==-1,b]

{{b->(2 (1+r))/r}}

b->(2 (1+r))/r/.r->10

b->11/5

Code for part d

```
close all; clear all;clc;
k = 1000;
r = 0.1;
b = 1;
generation=1000;

%%
NInitial=[1 2 3 10];

for i=1:length(NInitial)
    N=zeros(generation,1);
    v=zeros(generation,1);
    N(1)=NInitial(i);
    v(1)=NInitial(i);
  for j=1:generation
        N(j+1)=((r+1)*N(j))/(1+(N(j)/k)^b);
        v(j+1)= (1+r)*v(j);
  end
  % figure
    loglog(N);
    hold on;
    loglog(v);
```

```
    ylim([1 200]);
end
```

Code for part f

```
close all; clear all;clc;
k = 1000;
r = 0.1;
b = 1;
generation=1000;

%%
N2Star=k*r^(1/b);
vinitial=[-10 -3 -2 -1 1 2 3 10];
NInitial=[N2Star+vinitial];

for i=1:length(NInitial)
    N=zeros(generation,1);
    N(1)=NInitial(i);
    NP=zeros(generation,1);
    v=zeros(generation,1);
    v(1)=vinitial(i);
    for j=1:generation
        N(j+1)=((r+1)*N(j))/(1+(N(j)/k)^b);
        %NP(j+1)=(((r+1)*NP(j))/(1+(NP(j)/k)^b))+v(j);
        NP(j)=N2Star+v(j);
        v(j+1)=((1-((b*r)/(1+r)))*v(j));
    end
    loglog(N);
    hold on;
    loglog(NP,'--')

end
```