# Problem Set 3

Effects of stochasticity in deterministic system.

1st the deterministic

2nd The corresponding stochastic model

3rd Efficient Simulation

4th Simulation of distributions at different times.

a) 1 Note that the eq$^n$ 1 is 1D.

$I + S = N$ = constant

eliminate $S$, find an eq$^n$ for $I$.

2 $$\frac{dI}{dt} = f(\alpha, \beta, N, I) = 0$$

find the steady state. $I^*$

in order for the infection to sustain ad-infinitum

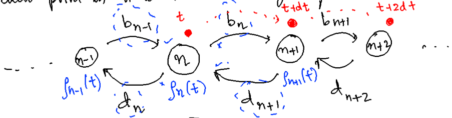
$\boxed{I^* \text{ should also be stable}}$ $\longrightarrow$ conditions on $\alpha, \beta$.

b) Stochastic model.

assumption (same as (a)) : the total population $N$: constant.

has the form of a Markov chain.

Idea: at each point in time the number of infectives : $n$ changes.



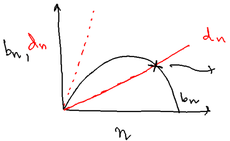

$b_n = \alpha n \left(1 - \dfrac{n}{N}\right)$ [ this is a rate, i.e its the change ~~prob~~ per unit time

$d_n = \beta n$

$$\boxed{\rho_n(t+dt) = \rho_n(t) + dt\left[\left(b_{n-1}\rho_{n-1}(t) + d_{n+1}\rho_{n+1}(t)\right) - \left(b_n \rho_n(t) + d_n \rho_n(t)\right)\right].}$$

$\alpha = \cdot$

$\beta = \cdots$

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- $\text{Prob}_n(t+dt) = \text{Prob}_n(t) + \text{Prob flux in}_t - \text{Prob flux out}_t$.

$$\rho_n(t+dt) = \rho_n(t) + dt\Big[\{b_{n-1}\rho_{n-1}(t) + d_{n+1}\rho_{n+1}(t)\} - \{b_n + d_n\}\rho_n(t)\Big].$$

$$\Rightarrow \frac{d\rho_n(t)}{dt} = b_{n-1}\rho_{n-1}(t) + d_{n+1}\rho_{n+1}(t) - \{b_n + d_n\}\rho_n(t)$$

- Derive the deterministic model from the Master eq$^n$ in the limit $N \to \infty$

$$\frac{d\rho_n(t)}{dt} = \alpha(n-1)\left(1-\frac{(n-1)}{N}\right)\rho_{n-1}(t) + \beta(n+1)\rho_{n+1}(t)$$
$$- \left(\alpha n\left(1-\frac{n}{N}\right) + \beta n\right)\rho_n(t)$$

$$I := \langle n \rangle = \sum_{n=0}^{\infty} n\,\rho_n(t)$$

$$I^2 := \langle n^2 \rangle$$

multiply throughout by $n$, sum $\sum_{n=0}^{\infty}$, and reproduce eq$^n$ from Lecture Notes.

quasi steady state: unstable state with a "very long" lifetime.

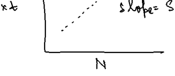
Need: an effective FOM in the large N limit.

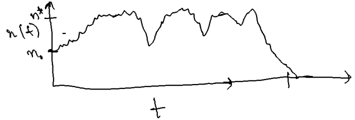
Pg 36 Lecture Notes: $T_{ext} \sim e^{N S(0)}$

$$S(0) = -\int_{1-1/R_0}^{0} d\bar{I} \, \log \left( R_0 (1-\bar{I}) \right]$$

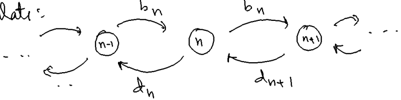
$R_0$ related to $\alpha$ & $\beta$

$$= \log R_0 - 1 - \frac{1}{R_0}$$






— Simulate:



1. Start at $n$: initialization

2. at time step 1 (which occurs after time $dt$) do:

$n \to n+1$ with $\text{Prob}(b_n\, dt)$

or

$n \to n-1$ with $\text{Prob}(d_n\, dt)$

end.

3. Repeat for many time steps.

the algo is accurate only when $b_n dt \ll 1$ & $d_n dt \ll 1$

Note: $dt \ll$ all other timescales

- Since $b_n, d_n$ are rates

$\frac{1}{b_n}, \frac{1}{d_n}$ are timescales

need to choose $dt$ small enough so that $b_n dt \ll 1$, $d_n dt \ll 1$.

choose $dt$ so that $b_n dt,\ d_n dt \ll 1$

order of $\sim 0.01$

just for concreteness, say $b_n dt \sim 0.02$

$d_n dt \sim 0.01$

do $n \to n+1$ with Prob $(0.02)$

or $n \to n-1$ with Prob $(0.01)$

$n \to n$ with Prob $(0.97)$.

most of the time the population remains constant in this algorithm.

end

Resolution: Gillespie Algorithm:

c). Check that the Gillespie Algorithm works.

example ① set $n = n^*$ (something reasonable). $b_n = 0.1$, $d_n = 0.2$.

$\frac{1}{b_n} = 10$, $\frac{1}{d_n} = 5$

time step 1

② do:

$n \to n+1$ with Prob $(b_n dt = 10^{-3})$ $dt = 0.01$

or

$n \to n-1$ with Prob $(d_n dt = \cancel{5\times10^{-2}}\ 2\times10^{-3})$

or $n \to n$ with Prob $(1-(3\cdot10^{-3}))$.

end

③ if $n \neq n^*$ $\boxed{\text{record time step}}$. and the event $\begin{bmatrix}\text{birth}\\\text{death}\end{bmatrix}$

if $n = n^*$ then GOTO step 2 and repeat.

birth $t_b^{(1)}\ t_b^{(2)}\ t_b^{(3)} \dots$

Plot a histogram of $t_b$


death $t_d^{(1)}\ t_d^{(2)}\ t_d^{(3)} \dots$

histogram of $t_d$


$$P(t_b) \sim b_n e^{-b_n t_b}$$

Conclusion: times until birth / death event happens is distributed exponentially with parameters $b_n$, $d_n$ respecti

Gillespie:

DO: $R_b$ sampled from exp distribution with para $= b_n$

(how to sample from Exp)

(Matlab has one built in)

$R_d$ sampled from exp dist with para $= d_n$

if $R_b < R_d$

$t \to t + R_b$

$n \to n+1$

else $t \to t + R_d$; $n \to n-1$ end

if $R_b < R_d$

$t \to t + R_b$

$n \to n+1$

else

$t \to t + R_d$

$n \to n-1$

end.

d).

$$T_{ext} \sim e^{N\left\{\log r_0 + 1 - \frac{1}{r_0}\right\}}$$

Dont know proportionality constant

choose 3 time $\quad t_1 < T_{ext}$

$t_2 \sim T_{ext}$

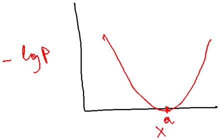
$t_3 > T_{ext}$

Plot $-\log P(n_{t_1})$, $-\log P(n_{t_2})$, $-\log P(n_{t_3})$.

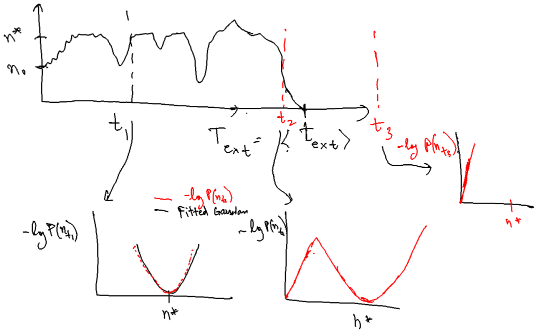
if Gaussian dist

$P_{(x)} \sim e^{-\frac{(x-a)^2}{2\sigma^2}}$

$-\log P_{(x)} \sim \frac{(x-a)^2}{2\sigma^2}$




1) $T_{ext}$ : start with some population $n_0$
run many trajectories until extinction $(n=0)$
& average the individual $t_{ext}$ to find $T_{ext}$.



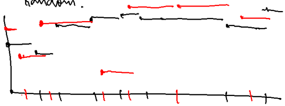
n*
n₀
t₁
t₂
t₃
T_ext = ⟨t_ext⟩
-log P(n_t3)
-log P(n_t2)
Fitted Gaussian
-log P(n_t1)
-log P(n_t2)
n*
n*
n*

For the fitted gaussian

Compare to theory!

(In lecture notes there is an expression for the distribution).

- When using the Gillespie algo: increments are random.



Physically, $S_0$ is the "optimal action".

$$S_0(I) = -\int_{1-\frac{1}{\eta_0}}^{I} \log\left(s_0(1-y)\right)\, dy .$$

$\boxed{2.}$ Population Genetics by John Gillespie.

lecture notes: $P(S_n = j) = \dfrac{(u\, T_c)^j \, e^{-u T_c}}{j!}$

where $T_c = \sum_{j=2}^{n} j\, T_j$.

AND $P(T_j) = \lambda_j \, e^{-\lambda_j T_j}$ ; $\lambda_j = \dfrac{\binom{j}{2}}{N}$

$P(S_n = 0) = \dfrac{1}{1} \cdot \langle e^{-u T_c} \rangle_{T_c}$

realise that all $T_j$ are indefendant

$$\left\langle e^{-\mu T_c} \right\rangle_{T_c} = \int P(T_j)\, dT_j \; e^{-\mu T_c}$$

$$= \prod_{j=2}^{n} \int_0^\infty P(T_j)\, dT_j \; e^{-\mu \{ T_1 + 2T_2 + 3T_3 \cdot + nT_n \}}$$

$$= \prod_{j=2}^{n} \int_0^\infty P(T_j)\, dT_j \; e^{-\mu j T_j}$$

$$\boxed{2N\mu = \theta}$$

repeat with right parameters for (b).