Problem3: A route to chaos

Collaborators

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a) Plot of η_τ Vs R

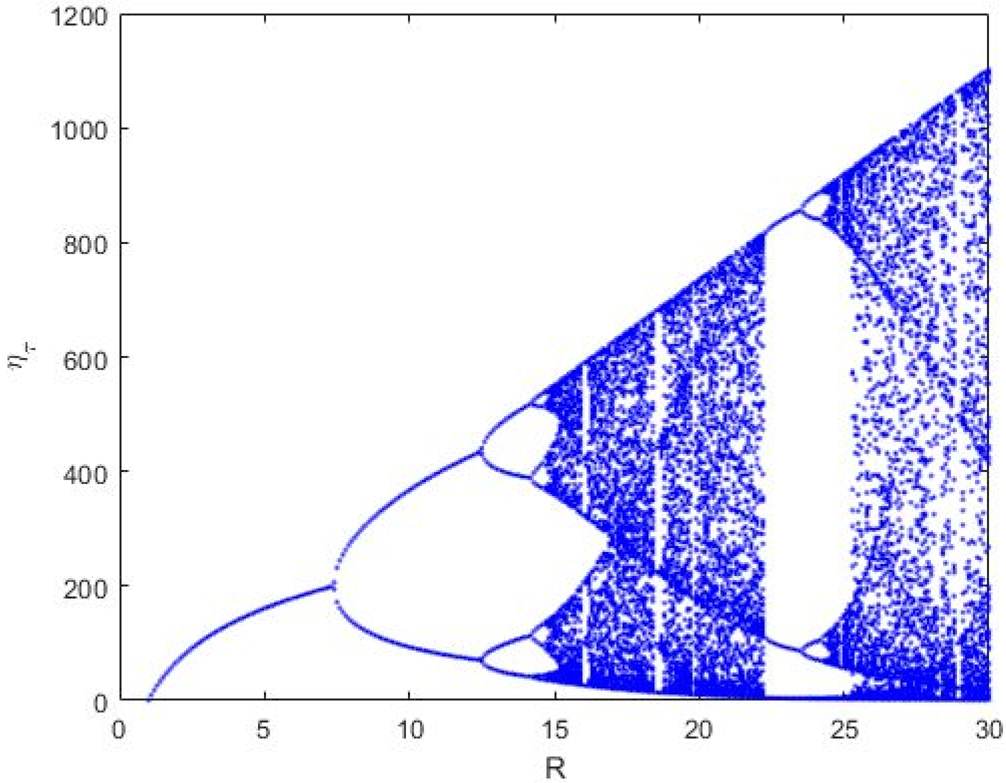
The bifurcation diagram for $1 \le R \le 30$ is shown below. As the parameter R is varied, the system reaches new equilibrium. Between $1 < R < e^2$, only one fixed point exists.

As R is increased further, the fixed point undergoes pitchfork bifurcation producing 2 more fixed points. This continues and a cascade of period doubling bifurcation takes place as R is increased. This cascade of first-period doubling is followed by intermittency and chaos and then a new window appears. (Example of first-period doubling bifurcation (or pitchfork bifurcation) is listed below)

- Period 2 cycle starts at $R = e^2$ (please read further for calculation
- Period 4 cycle starts at $R \approx 12.5$ (please read further for calculations)

  and so on

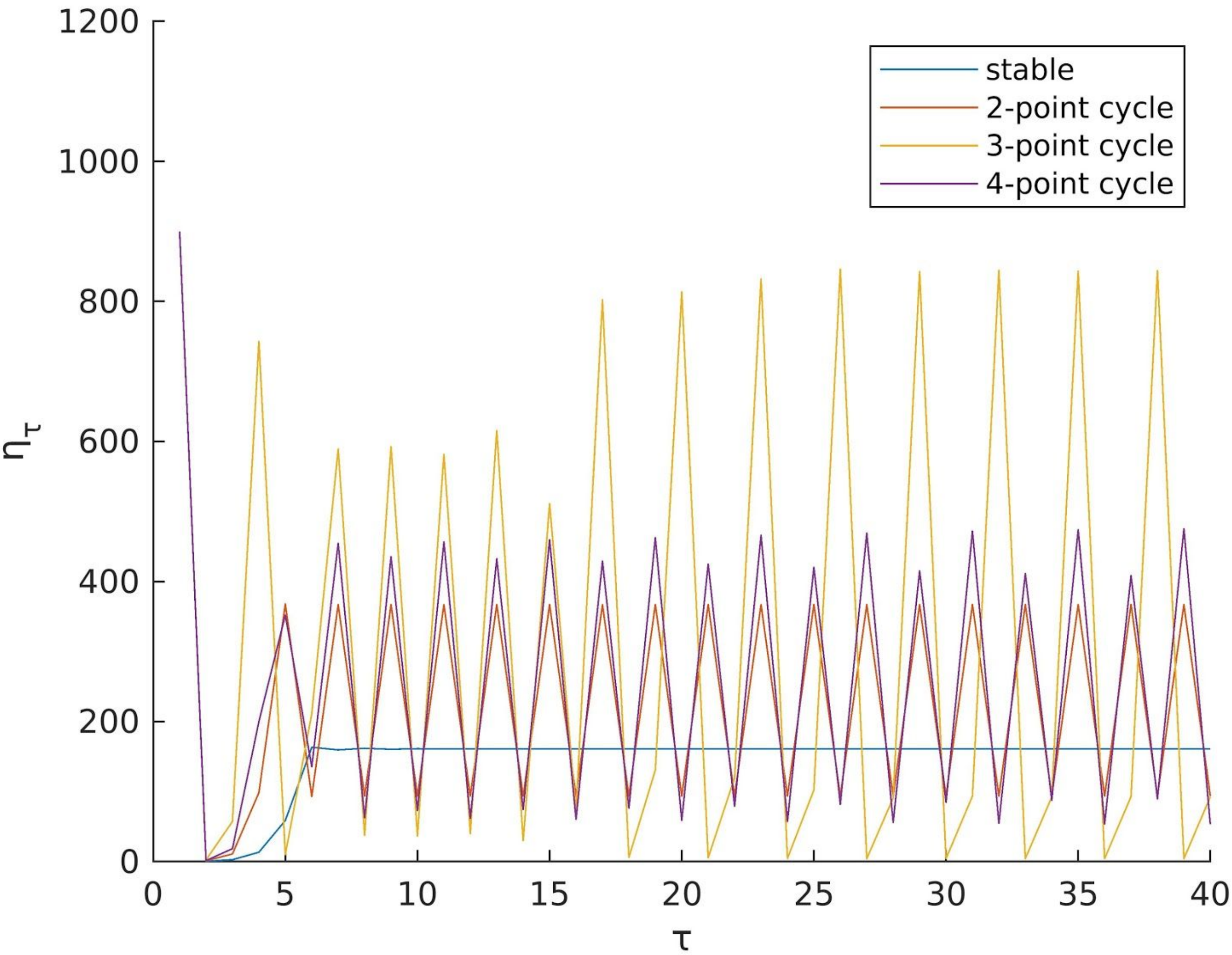
The beginning of the new window is usually created due to tangent bifurcation. This new window will now consist of new period doubling bifurcation (May be 5-period or 3 period window or ..). This behaviour continues until the end.



b) Population dynamics η_τ Vs τ

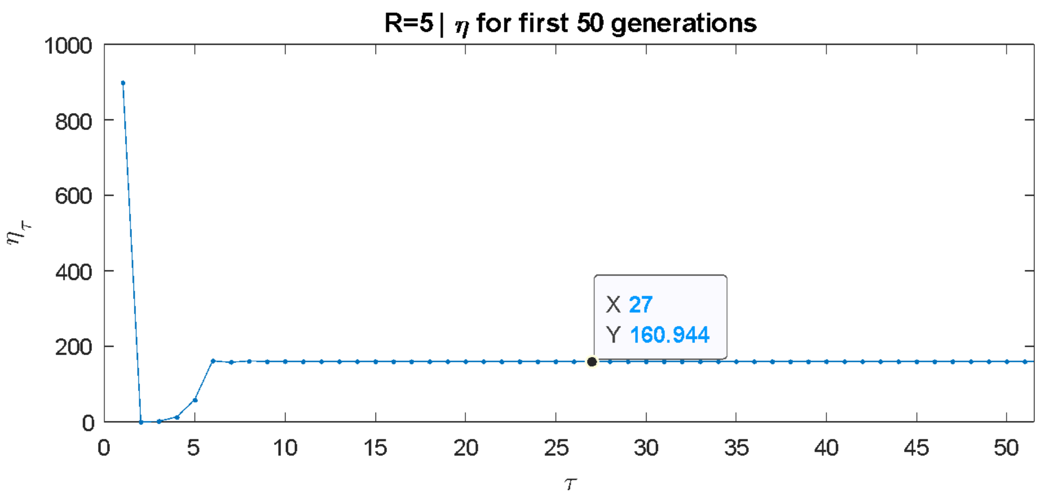
The population dynamics showing the stable equilibrium, 2-point cycle, 3-point cycle and 4-point cycle is shown below. (Please read further for representative values).

In all the graphs, we see an initial transient where the population falls to zero. This is because at high initial population value, the probability of offspring survival to maturity $e^{-\alpha n_\tau}$ drops to 0. After the initial transient, the probability of offspring survival increases, and the system eventually reaches a steady state population



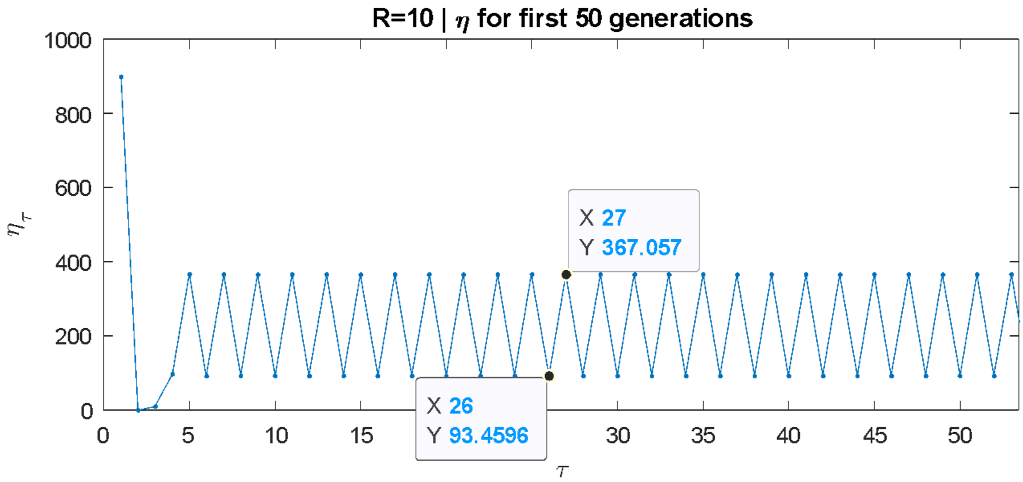
1. Stable equilibrium

$R = 5$ is a representative value for a system exhibiting stable equilibrium. After the initial transient, the probability of offspring survival increases, and the system eventually reaches a steady state population of approximately 160.



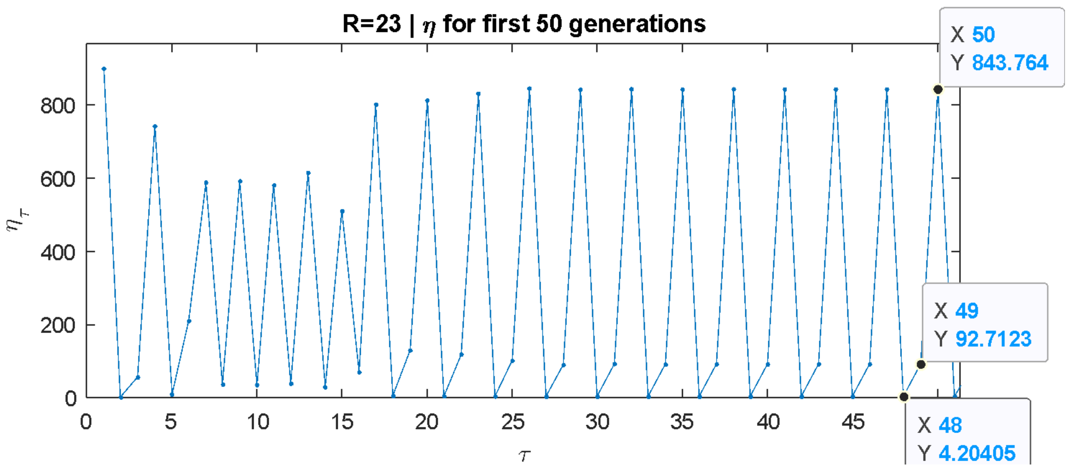
2. 2-point cycle

After the initial transient, the system oscillates around the fixed point ($\eta_\tau^* \approx 230$) exhibiting a 2-point cycle. That is the system repeats the same value every two generations, i.e. $\eta_{\tau+2} = \eta_\tau$. The oscillation around the fixed point is due to the discrete system.



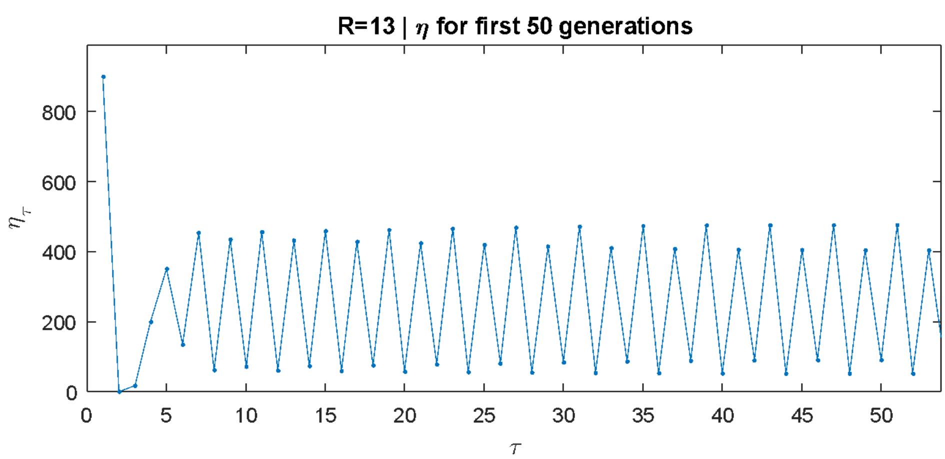
3. 3-point cycle

After the initial transient, the system oscillates around the fixed point ($\eta_\tau^* \approx 313$) exhibiting a 3-point cycle. That is the system repeats the same value every three generation, i.e. $\eta_{\tau+3} = \eta_\tau$. The oscillation around the fixed point is due to the discrete system.



4. 4-point cycle

After the initial transient, the system oscillates around the fixed point ($\eta_\tau^* \approx 256$) exhibiting a 4-point cycle. That is the system repeats the same value every four generations, i.e. $\eta_{\tau+4} = \eta_\tau$



c) Period 2 and period 4 bifurcation

The fixed point of the system

$\eta_{\tau+1} = R\,\eta_\tau e^{-\alpha\eta_\tau}$

could be found analytically using Mathematica. They are $\eta_1^* = 0$ and $\eta_2^* = -100\log\frac{1}{R}$

<u>Stability analysis around the fixed point</u>

Differentiating the given equation with respect to η_τ, we get

$f'(\eta_\tau) = e^{-0.01\,\eta_\tau} R\,(1 - 0.01\eta_\tau)$

Substituting for the first fixed point $\eta_1^* = 0$, we get

$f'(\eta_1^*) = R$. Therefore, the fixed point is unstable for $1 < R \leq 30$

Substituting for the second fixed point $\eta_2^* = -100\log\frac{1}{R}$, we get

$f'(\eta_2^*) = 1 + Log(\frac{1}{R})$. Therefore, the second fixed point is stable for $1 < R < e^2$

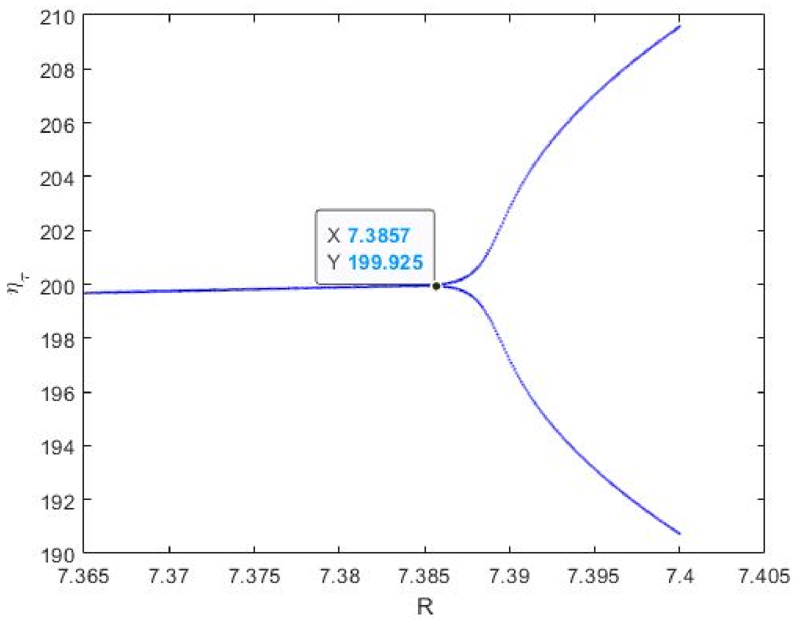
The second fixed point undergoes a pitchfork bifurcation at

$f'(\eta_2^*) = 1 + Log\left(\frac{1}{R}\right) =- 1$

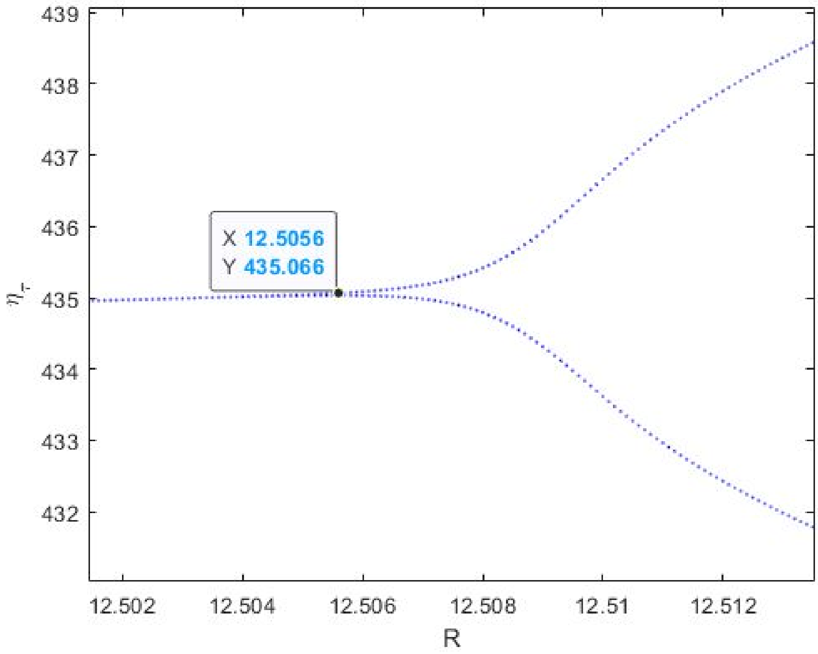
$R = e^2$.

Therefore, the stable 2-point cycle is born at $R = e^2$

Numerical investigation with 10000 generation and step in R of 10^-4 gives $R_1 \approx 7.3857$

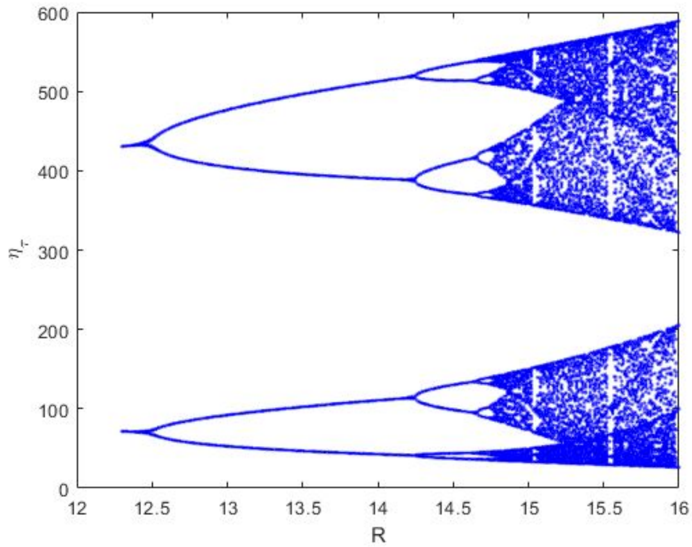


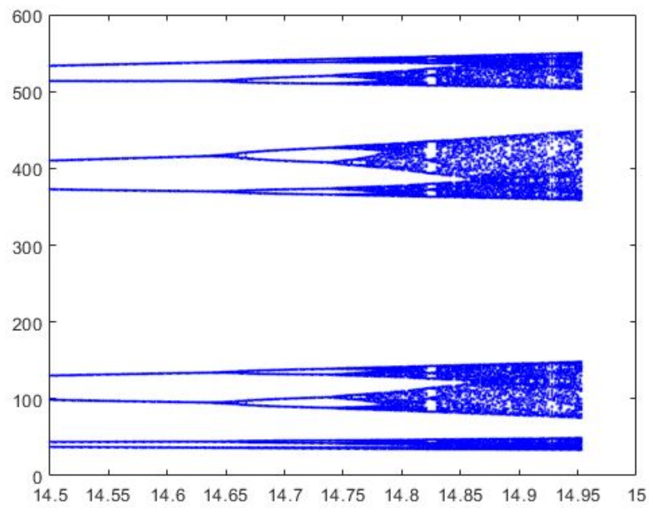
The start of a 4-point cycle is harder to find analytically, therefore a numerical approach with 10000 generation and step in R of 10^-4 gives $R_2 \approx 12.5056$. (Note: only one branch is shown below in the picture for clarity. This is because bifurcation occurs simultaneously)

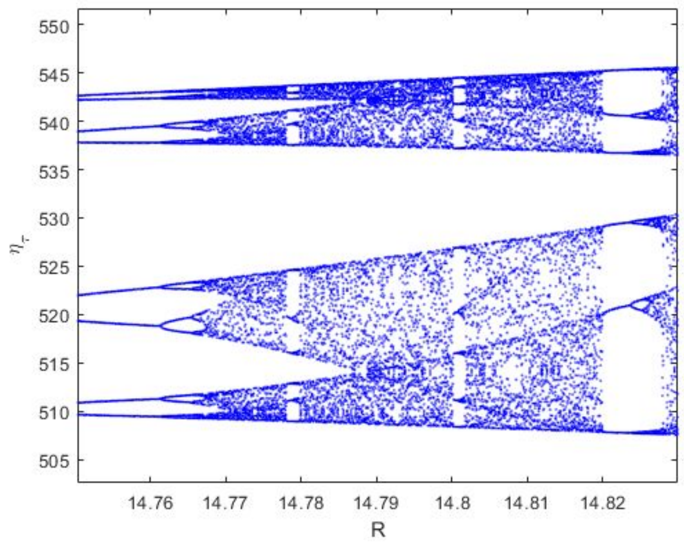


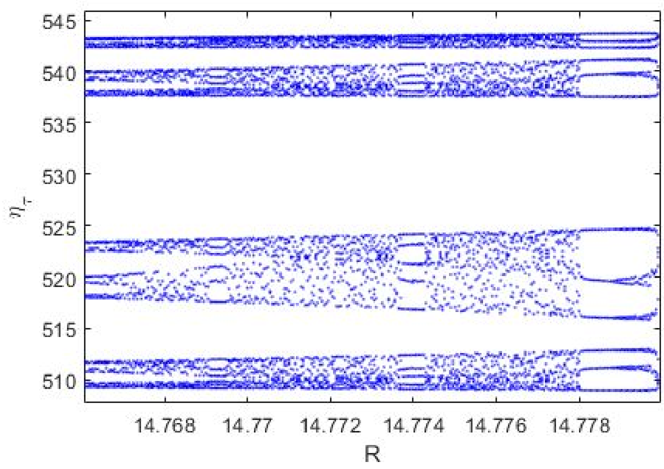
d) Rough estimate of R_∞

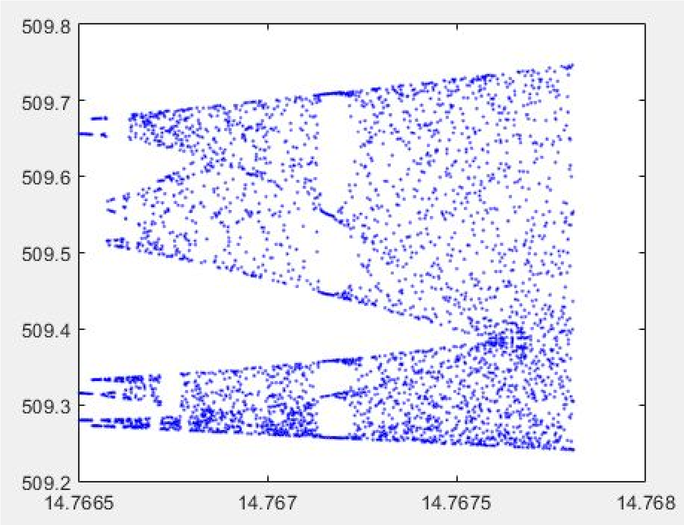
Several detailed views of the graphs are shown below. In each graph the range of x axis is narrowed down towards R_∞

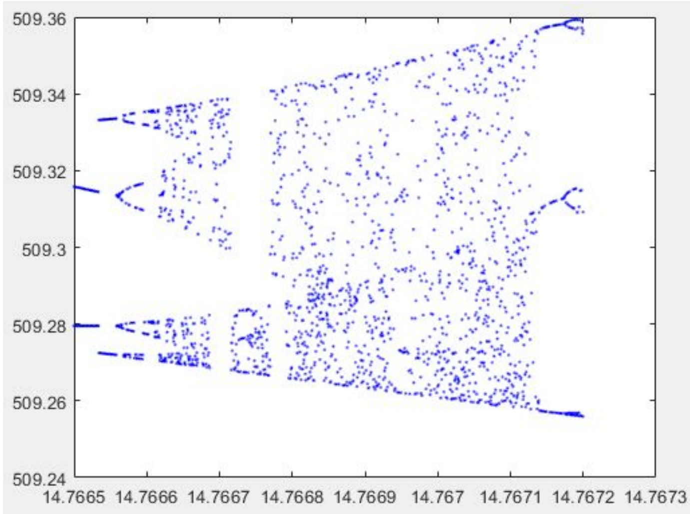












The final graph shows that a new window appears close to 14.7671. A new window suggests the birth of a new period doubling bifurcation. Just before this window first-period doubling comes to an end followed by chaos. Therefore the approximation of R_∞ is approximately 14.7668.

From the analysis above, a summary of rough approximation of first period doubling cascade is given below. (Note: the values are not the beginning of a 2-period cycle, but instead a representative R)

R	First period doubling
10	2-point cycle
13	4-point cycle
14.5	8- point cycle
14.7	16- point cycle
14.75	32- point cycle
14.764	64-point cycle
.	.
.	.
14.7668	Approximation of R_∞

Appendix

```
clear all; clc;close all

alpha=0.01;

etaInitial=900;

generations=1000000;

R=14.7665:0.0000001:14.7668;

% R=12.501:0.00001:12.505;

%%

eta=zeros(generations,1);

eta(1)=etaInitial;

for i=1:length(R)

      for j=1:generations-1

        eta(j+1)=R(i)*eta(j)*exp(-alpha*eta(j));

      end

    figure(1)

    temp=eta(end-100:end);

    temp(temp<500|temp>509.3)=[];

    plot(R(i),temp,'b.','MarkerSize',3)

    hold on;

%      figure(2)

%      plot(eta(end-100:end),'.-')

end

figure(1)

xlabel('R');ylabel('{\eta}_{\tau}');
```