Problem set 2, Task 1
Travelling waves
Computational Biology
FFR110/FIM740

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a.)

$$\frac{\partial n}{\partial t} = rn\left(1 - \frac{n}{K}\right) - \frac{An}{1 + \frac{n}{B}} + D\frac{\partial^2 n}{\partial x^2}$$

We change to dimensionless time $\tau = At$, position $\xi = x\sqrt{A/D}$ and population $u(\xi, \tau) = n(x, t)/B$. This gives us the following equations;

$$\begin{aligned}
\frac{\partial u}{\partial \tau} &= \frac{1}{AB}\frac{\partial n}{\partial t} \\
&= \frac{1}{AB}\left(rn\left(1 - \frac{n}{K}\right) - \frac{An}{1 + n\frac{n}{B}} + D\frac{\partial^2 n}{\partial x^2}\right) \\
&= \frac{1}{AB}\left(r(uB)\left(1 - \frac{(uB)}{K}\right) - \frac{A(uB)}{1 + \frac{(uB)}{B}} + D\frac{\partial^2 (uB)}{\partial(\xi/\sqrt{A/D})^2}\right) \\
&= \frac{1}{AB}\left(Bru - \frac{B^2ru^2}{K} - \frac{ABu}{1+u} + \frac{ABD}{D}\frac{\partial^2 u}{\partial \xi^2}\right) \\
&= \frac{Bru}{AB} - \frac{B^2ru^2}{ABK} - \frac{ABu}{(1+u)AB} + \frac{ABD}{ABD}\frac{\partial^2 u}{\partial \xi^2} \\
&= \frac{ru}{A} - \frac{Bru^2}{AK} - \frac{u}{1+u} + \frac{\partial^2 u}{\partial \xi^2} \qquad (1)
\end{aligned}$$

Introducing the dimensionless parameters $\rho = r/A$ and $q = K/B$ in Equation 1 gives us the following;

$$\frac{\partial u}{\partial \tau} = \rho u - \frac{\rho u^2}{q} - \frac{u}{1+u} + \frac{\partial^2 u}{\partial \xi^2} \qquad (2)$$

Now, to find the steady states, we need to find when $\frac{\partial u}{\partial \tau}$. by rewriting Equation 2, omitting the diffusion, we get;

$$0 = u\left(\rho - \frac{\rho u}{q} - \frac{1}{1+u}\right) \Rightarrow \tag{3}$$

$$u_1^* = 0 \tag{4}$$

The first part in Equation 3 gave us the first steady state. Now we need to solve for the second part to get the remaining steady state points;

$$\begin{aligned}
0 &= \rho - \frac{\rho u}{q} - \frac{1}{1+u} \\
&= \frac{\rho(1+u)}{(1+u)} - \frac{\rho u(1+u)}{q(1+u)} - \frac{1}{1+u} \\
&= \frac{\rho q(1+u)}{q(1+u)} - \frac{\rho u(1+u)}{q(1+u)} - \frac{q}{q(1+u)} \\
&= \frac{\rho q(1+u) - \rho u(1+u) - q}{q(1+u)} \\
&= \frac{(\rho q - q) + (q\rho - \rho)u + (-\rho)u^2}{q(1+u)}
\end{aligned} \tag{5}$$

Now we use the quadratic formula on the numerator from Equation 5, giving us the following;

$$\begin{aligned}
u_{1,2}^* &= \frac{-(q\rho - \rho) \pm \sqrt{(q\rho - \rho)^2 - 4(-\rho)(\rho q - q)}}{2(-\rho)} \\
&= -\frac{\rho - q\rho \pm \sqrt{(q\rho - \rho)^2 + 4\rho(\rho q - q)}}{2\rho}
\end{aligned}$$

$$u_1^* = \frac{-\rho + q\rho + \sqrt{(q\rho - \rho)^2 + 4\rho(\rho q - q)}}{2\rho} \tag{6}$$

$$u_2^* = \frac{-\rho + q\rho - \sqrt{(q\rho - \rho)^2 + 4\rho(\rho q - q)}}{2\rho} \tag{7}$$

We now have the steady state points for the dimensionless system in the form of ρ and q.

Using $\rho = 0.5$ and $q = 8$ in Equation 4, 6 and 7, we get the following numerical values for our steady state points;

$$
\begin{aligned}
u_0^* &= 0 \qquad (8)\\
u_1^* &= \frac{-\rho + q\rho + \sqrt{(q\rho-\rho)^2 + 4\rho(\rho q - q)}}{2\rho}\\
&= \frac{-0.5 + 8*0.5 + \sqrt{(8*0.5-0.5)^2 + 4*0.5(0.5*8-8)}}{2*0.5}\\
&= \frac{-0.5 + 4 + \sqrt{(4-0.5)^2 + 2(4-8)}}{1}\\
&= 3.5 + \sqrt{(3.5)^2 - 8} = 3.5 + \sqrt{12.25 - 8} \approx 3.5 + 2.06155\\
&\approx 5.56155 \qquad (9)\\
u_2^* &= \frac{-\rho + q\rho - \sqrt{(q\rho-\rho)^2 + 4\rho(\rho q - q)}}{2\rho}\\
&= \frac{-0.5 + 8*0.5 - \sqrt{(8*0.5-0.5)^2 + 4*0.5(0.5*8-8)}}{2*0.5}\\
&= \frac{-0.5 + 4 - \sqrt{(4-0.5)^2 + 2(4-8)}}{1}\\
&= 3.5 - \sqrt{(3.5)^2 - 8} = 3.5 - \sqrt{12.25 - 8} \approx 3.5 - 2.06155\\
&\approx 1.43845 \qquad (10)
\end{aligned}
$$

b.)

To later classify the traveling waves fixed points using our estimated velocity (c), we do the following ansatz

$$u(\xi,\tau) = n(\xi - c\tau) = n(z)$$
$$\frac{\partial u}{\partial \tau} = -c\frac{\partial n}{\partial z}$$
$$\frac{\partial u}{\partial \xi} = \frac{\partial n}{\partial z}$$

we then intrude the help variable v giving;

$$\frac{\partial n}{\partial z} = v$$
$$\frac{\partial v}{\partial z} = \frac{-cv - f}{D}$$

Where $f = \rho u - \frac{\rho u^2}{q} - \frac{u}{1+u} = \frac{u}{2} - \frac{u^2}{16} - \frac{u}{1+u}$. To classify the fixed points we use the Jacobian

$$\mathbb{J} = \begin{pmatrix} 0 & 1 \\ \frac{-f'(u^*)}{D} & -\frac{c}{D} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} + \frac{u}{8} + \frac{1}{(1+u)^2} & -c \end{pmatrix} \tag{11}$$

Giving eigenvalues;

$$\lambda_{\pm} = \frac{1}{2}(-c \pm \sqrt{c^2 - 2 + \frac{u}{2} + \frac{4}{(1+u)^2}} \tag{12}$$

b.) (i)

Describe how the total population expands over the habitat. Do you find a travelling wave?
We find a travelling wave, moving between u_1^* and u_0^* values, moving forward in the ξ plain, increasing the total population.

If you find a travelling wave, estimate its velocity c
Using a visual inspection of the two different τ values in Figure 2, we find $c \approx 0.1775$ (at u=1.00, $\xi_{\tau=100} \approx 39.43$, $\xi_{\tau=120} \approx 42.98$, $c \approx (\xi_{\tau=120} - \xi_{\tau=100})/(\tau_{\tau=100} - \tau_{\tau=120})$).

Make a plot with two panels
See Figure 2.

Does the wave connect between two fixed points in the phase plane? If so, classify these fixed points using your numerically estimated velocity c.
In Figure 2, it can be seen that the wave connects between u_0^* and u_1^*. Using Equation (12) for the the first fixed point $(u_0^*, v^*) = (0, 0)$ and for the estimated velocity $c = 0.1775$ we get

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}(-0.1775 + \sqrt{(0.1775)^2 - 2 + \frac{0}{2} + \frac{4}{(1+0)^2}}) \\
&= \frac{1}{2}(-0.1775 + \sqrt{2.0315}) \\
&= \frac{1}{2}(-0.1775 + 1.4253) \\
&= \frac{1.2478}{2} \\
&= 0.6239
\end{aligned} \tag{13}$$

$$\begin{aligned}
\lambda_2 &= \frac{1}{2}(-0.1775 - \sqrt{(0.1775)^2 - 2 + \frac{0}{2} + \frac{4}{(1+0)^2}}) \\
&= \frac{1}{2}(-0.1775 - \sqrt{2.0315}) \\
&= \frac{1}{2}(-0.1775 - 1.4253) \\
&= \frac{-1.6028}{2} \\
&= -0.8014
\end{aligned} \tag{14}$$

Thus since $\lambda_2 < 0 < \lambda_1$ the fixed point $(u_0^*, v^*) = (0, 0)$ is an unstable saddle point. In the same way for the second fixed point $(u_1^*, v^*) = (5.56155, 0)$ we have

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2}(-0.1775 + \sqrt{(0.1775)^2 - 2 + \frac{5.56155}{2} + \frac{4}{(1+5.56155)^2}}) \\
&= \frac{1}{2}(-0.1775 + \sqrt{0.9051}) \\
&= \frac{1}{2}(-0.1775 + 0.9513) \\
&= \frac{0.7738}{2} \\
&= 0.3869
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2}(-0.1775 - \sqrt{(0.1775)^2 - 2 + \frac{5.56155}{2} + \frac{4}{(1+5.56155)^2}}) \\
&= \frac{1}{2}(-0.1775 - \sqrt{0.9051}) \\
&= \frac{1}{2}(-0.1775 - 0.9513) \\
&= -\frac{1.1288}{2} \\
&= -0.5644
\end{aligned} \tag{16}
$$

Thus since $\lambda_2 < 0 < \lambda_1$ the fixed point $(u_1^*, v^*) = (5.56155, 0)$ is an unstable saddle point. Finally for the last fixed point $(u_2^*, v^*) = (1.43845, 0)$ we get

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2}(-0.17750 + \sqrt{(0.17750)^2 - 2 + \frac{1.43845}{2} + \frac{1}{(1+1.43845)^2}}) \\
&= \frac{1}{2}(-0.17750 + \sqrt{-0.57655}) \\
&= \frac{1}{2}(-0.17750 + 0.75931i) \\
&= -\frac{0.17750}{2} + \frac{0.75931}{2}i \\
&= -0.08875 + 0.37965i
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
\lambda_2 &= \frac{1}{2}(-0.17750 - \sqrt{(0.17750)^2 - 2 + \frac{1.43845}{2} + \frac{1}{(1+1.43845)^2}}) \\
&= \frac{1}{2}(-0.17750 - \sqrt{-0.57655}) \\
&= \frac{1}{2}(-0.17750 - 0.75931i) \\
&= -\frac{0.17750}{2} - \frac{0.75931}{2}i \\
&= -0.08875 - 0.37965i
\end{aligned} \tag{18}
$$

The eigenvalues have negative real part so the fixed point $(u_2^*, v^*) = (1.43845, 0)$ is a stable spiral.

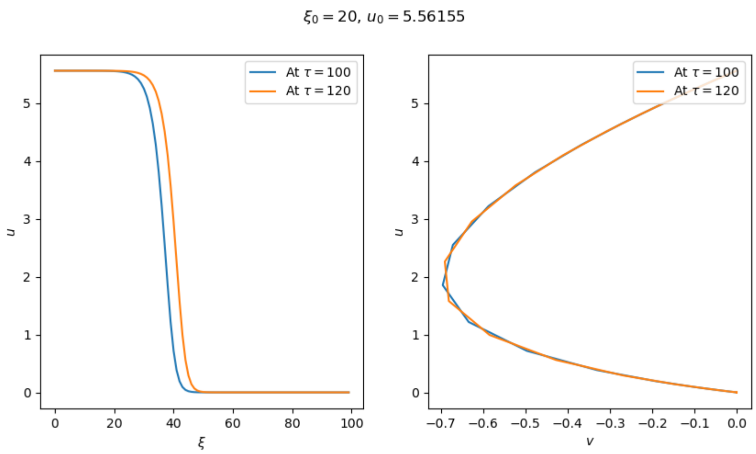


Figure 1: System plots for parameters in b.) (i)

b.) (ii)

Describe how the total population expands over the habitat. Do you find a travelling wave?
We find a travelling wave, moving between u_2^* and u_0^*, moving backwards in the ξ plain, decreasing the population.

If you find a travelling wave, estimate its velocity c
Using a visual inspection of the two different τ values in Figure 3, we find $c \approx -0.6075$ (at u=0.20, $\xi_{\tau=10} \approx 48.21$, $\xi_{\tau=30} \approx 36.06$, $c \approx (\xi_{\tau=30} - \xi_{\tau=10})/(\tau_{\tau=10} - \tau_{\tau=30})$).

Make a plot with two panels
See Figure 3.

Does the wave connect between two fixed points in the phase plane? If so, classify these fixed points using your numerically estimated velocity c.
In Figure 3, it can be seen that the wave connects between u_0^* and u_2^*. Using Equation (12) for the the first fixed point $(u_0^*, v^*) = (0, 0)$ and for the estimated velocity $c = -0.6075$ we get

$$\begin{aligned}\lambda_1 &= \frac{1}{2}(-(-0.6075) + \sqrt{(-0.6075)^2 - 2 + \frac{0}{2} + \frac{4}{(1+0)^2}})\\ &= \frac{1}{2}(0.6075 + \sqrt{2.3690})\\ &= \frac{1}{2}(0.6075 + 1.5391)\\ &= \frac{2.1466}{2}\\ &= 1.0733\end{aligned} \tag{19}$$

$$\begin{aligned}\lambda_2 &= \frac{1}{2}(-(-0.6075) - \sqrt{(-0.6075)^2 - 2 + \frac{0}{2} + \frac{4}{(1+0)^2}})\\ &= \frac{1}{2}(0.6075 - \sqrt{2.3690})\\ &= \frac{1}{2}(0.6075 - 1.5391)\\ &= \frac{-0.9316}{2}\\ &= -0.4658\end{aligned} \tag{20}$$

Thus since $\lambda_2 < 0 < \lambda_1$ the fixed point $(u_0^*, v^*) = (0, 0)$ is an unstable saddle point. In the same way for the second fixed point $(u_1^*, v^*) = (5.56155, 0)$ we have

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}(-(-0.6075) + \sqrt{(0.6075)^2 - 2 + \frac{5.56155}{2} + \frac{4}{(1+5.56155)^2}})\\
&= \frac{1}{2}(0.6075 + \sqrt{1.2427})\\
&= \frac{1}{2}(0.6075 + 1.1147)\\
&= \frac{1.7222}{2}\\
&= 0.8611
\end{aligned} \tag{21}$$

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}(-(-0.6075) - \sqrt{(0.6075)^2 - 2 + \frac{5.56155}{2} + \frac{4}{(1+5.56155)^2}})\\
&= \frac{1}{2}(0.6075 - \sqrt{1.2427})\\
&= \frac{1}{2}(0.6075 - 1.1147)\\
&= -\frac{0.5072}{2}\\
&= -0.2536
\end{aligned} \tag{22}$$

Thus since $\lambda_2 < 0 < \lambda_1$ the fixed point $(u_1^*, v^*) = (5.56155, 0)$ is an unstable saddle point. Finally for the last fixed point $(u_2^*, v^*) = (1.43845, 0)$ we get

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}(-(-0.6075)0 + \sqrt{(0.6075)^2 - 2 + \frac{1.43845}{2} + \frac{1}{(1+1.43845)^2}})\\
&= \frac{1}{2}(0.6075 + \sqrt{-0.2390})\\
&= \frac{1}{2}(0.6075 + 0.4888i)\\
&= \frac{0.6075}{2} + \frac{0.4888}{2}i\\
&= 0.3037 + 0.2444i
\end{aligned} \tag{23}$$

$$\begin{aligned}
\lambda_2 &= \frac{1}{2}(-(-0.6075)0 - \sqrt{(0.6075)^2 - 2 + \frac{1.43845}{2} + \frac{1}{(1+1.43845)^2}})\\
&= \frac{1}{2}(0.6075 - \sqrt{-0.2390})\\
&= \frac{1}{2}(0.6075 - 0.4888i)\\
&= \frac{0.6075}{2} - \frac{0.4888}{2}i\\
&= 0.3037 - 0.2444i
\end{aligned} \tag{24}$$

The eigenvalues have positive real part so the fixed point $(u_2^*, v^*) = (1.43845, 0)$ is an unstable spiral.

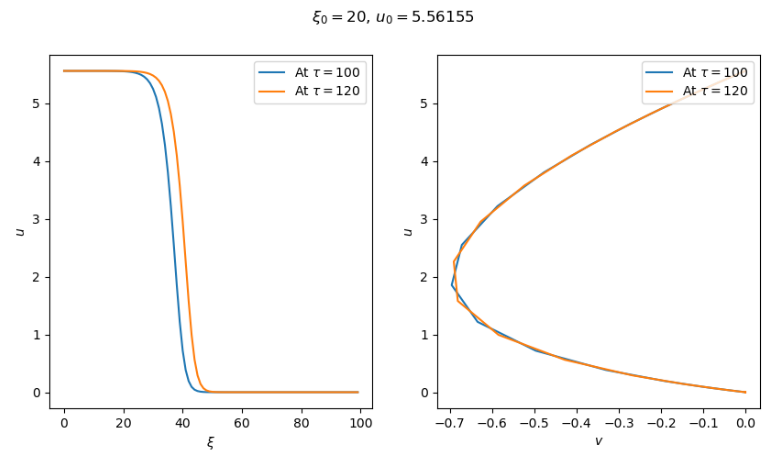


Figure 2: System plots for parameters in b.) (i)

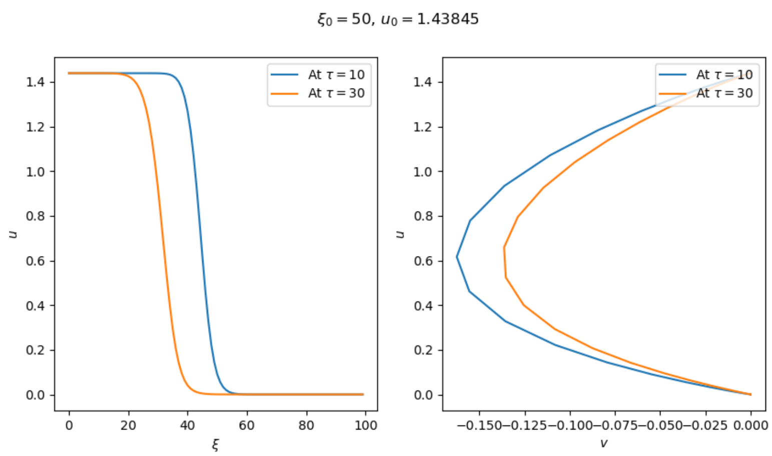


Figure 3: System plots for parameters in b.) (ii)

b.) (iii)

Describe how the total population expands over the habitat. Do you find a travelling wave?
We find a travelling wave, moving between u_1^* and u_0^* values, moving forward in the ξ plain, increasing the total population. This one have a initial transient, as we don't start at the maximum value of each ξ point, it will move from ξ_0 to ξ_1^* at the higher ξ_0 initial values.

If you find a travelling wave, estimate its velocity c
Using a visual inspection of the two different τ values in Figure 4, we find $c \approx 0.180$ (at u=1.00, $\xi_{\tau=100} \approx 56.7$, $\xi_{\tau=120} \approx 60.35$, $c \approx (\xi_{\tau=120} - \xi_{\tau=100})/(\tau_{\tau=100} - \tau_{\tau=120})$).

Make a plot with two panels
See Figure 4.

Does the wave connect between two fixed points in the phase plane? If so, classify these fixed points using your numerically estimated velocity c.
In Figure 4, it can be seen that the wave connects between u_0^* and u_1^*. Using Equation (12) for the the first fixed point $(u_0^*, v^*) = (0, 0)$ and for the estimated velocity $c = 0.180$ we get

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2}(-0.180 + \sqrt{(0.180)^2 - 2 + \frac{0}{2} + \frac{4}{(1+0)^2}})\\
&= \frac{1}{2}(-0.180 + \sqrt{2.0324})\\
&= \frac{1}{2}(-0.180 + 1.4256)\\
&= \frac{1.2456}{2}\\
&= 0.6228
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
\lambda_2 &= \frac{1}{2}(-0.180 - \sqrt{(0.180)^2 - 2 + \frac{0}{2} + \frac{4}{(1+0)^2}})\\
&= \frac{1}{2}(-0.180 - \sqrt{2.0324})\\
&= \frac{1}{2}(-0.180 - 1.4256)\\
&= \frac{-1.6056}{2}\\
&= -0.8028
\end{aligned} \tag{26}
$$

Thus since $\lambda_2 < 0 < \lambda_1$ the fixed point $(u_0^*, v^*) = (0, 0)$ is an unstable saddle point. In the same way for the second fixed point $(u_1^*, v^*) = (5.56155, 0)$ we have

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}(-0.180 + \sqrt{(0.180)^2 - 2 + \frac{5.56155}{2} + \frac{4}{(1+5.56155)^2}}) \\
&= \frac{1}{2}(-0.180 + \sqrt{0.906}) \\
&= \frac{1}{2}(-0.180 + 0.9518) \\
&= \frac{0.7718}{2} \\
&= 0.3859
\end{aligned} \tag{27}$$

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}(-0.180 - \sqrt{(0.180)^2 - 2 + \frac{5.56155}{2} + \frac{4}{(1+5.56155)^2}}) \\
&= \frac{1}{2}(-0.180 - \sqrt{0.906}) \\
&= \frac{1}{2}(-0.180 - 0.9518) \\
&= -\frac{1.1318}{2} \\
&= -0.5659
\end{aligned} \tag{28}$$

Thus since $\lambda_2 < 0 < \lambda_1$ the fixed point $(u_1^*, v^*) = (5.56155, 0)$ is an unstable saddle point. Finally for the last fixed point $(u_2^*, v^*) = (1.43845, 0)$ we get

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}(-0.180 + \sqrt{(0.180)^2 - 2 + \frac{1.43845}{2} + \frac{1}{(1+1.43845)^2}}) \\
&= \frac{1}{2}(-0.180 + \sqrt{-0.57565}) \\
&= \frac{1}{2}(-0.180 + 0.75868i) \\
&= -\frac{0.180}{2} + \frac{0.75868}{2}i \\
&= -0.09 + 0.37934i
\end{aligned} \tag{29}$$

$$
\begin{aligned}
\lambda_2 &= \frac{1}{2}(-0.180 - \sqrt{(0.180)^2 - 2 + \frac{1.43845}{2} + \frac{1}{(1+1.43845)^2}})\\
&= \frac{1}{2}(-0.180 - \sqrt{-0.57565})\\
&= \frac{1}{2}(-0.180 - 0.75868i)\\
&= -\frac{0.180}{2} - \frac{0.75868}{2}i\\
&= -0.09 - 0.37934i
\end{aligned} \tag{30}
$$

The eigenvalues have negative real part so the fixed point $(u_2^*, v^*) = (1.43845, 0)$ is a stable spiral.

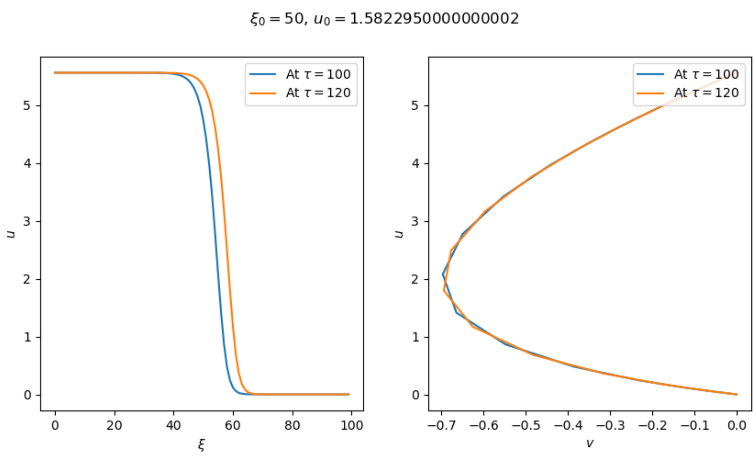
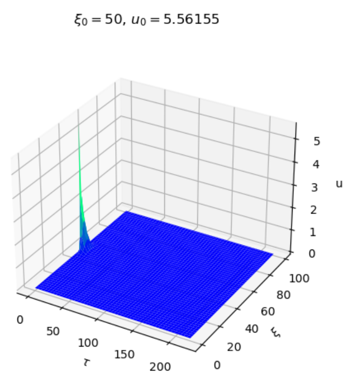


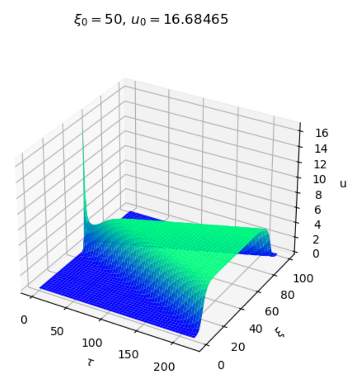
Figure 4: System plots for parameters in b.) (iii)

c.)

With case 1 ($u_0 = u_1^*$) will not give a traveling wave, as the population dies out very early (see Figure 5a). For the second case ($u_0 = 3 * u_1^*$) we get a double sided wave, where the population spread from the middle point, outwards to the edges (see Figure 5b).



(a) Case 1



(b) Case 2

Figure 5: 3d representations of the population (u) against τ and ξ for c)

Appendix

```python
#Course : FFR110/FIM740 Computational Biology
#Problem : Problem set 2, Task 1. Travelling waves
#Code :  Python 3.8.5

import numpy as np
import matplotlib.pyplot as plt
import math

tau_Max = 200
tau_StepSize = 0.1
tau_MaxSteps = int(tau_Max/tau_StepSize)

def WaveEqu(tau_Max=200, tau_StepSize = 0.1, xi_Max = 100, xi_0 = 20, u_0=5.56155, p=0.5, q=8, init_setting = 'b'):
    tau_MaxSteps = int(tau_Max/tau_StepSize)

    u = np.zeros((xi_Max,tau_MaxSteps))
    dudv = np.zeros((xi_Max,tau_MaxSteps))

    #Setting the u(xi,0) part
    for xi in range(xi_Max):
        if init_setting == 'c':
            u[xi,0]=u_0*np.exp(-((xi+1)-xi_0)**2)
        else:
            u[xi,0]=u_0/(1+np.exp((xi+1)-xi_0))

    for tau in range(tau_MaxSteps-1):

        for xi in range(xi_Max):

            du1_part1 = p*u[xi,tau]*(1-u[xi,tau]/q)

            du1_part2 = u[xi,tau]/(1+u[xi,tau])

            if xi == 0: #Edge case 1
                du2_xi = u[xi+1,tau]-u[xi,tau]
            elif xi == xi_Max-1:  #Edge case 2
                du2_xi = u[xi,tau]-u[xi-1,tau]
            else: #Normal case (central differentiation, h=1)
                du2_xi = (u[xi+1,tau]+u[xi-1,tau]-2*u[xi,tau])
                dudv[xi,tau+1] = u[xi,tau]-u[xi-1,tau]

            du_dtau = du1_part1-du1_part2 + du2_xi
            u[xi,tau+1] = u[xi,tau] + tau_StepSize*du_dtau

            #Cant have a negative population
            if u[xi,tau+1] <= 0:
                u[xi,tau+1] = 0

    return u, dudv

def PlotWave(xi_0, u_0, tau_Max, tau_Plot, tau_Plot_diff, plot_3d = True, init_setting = 'b'):

    tau_StepSize = 0.1
    tau_MaxSteps = int(tau_Max/tau_StepSize)
    tau_PlotPoint = int(tau_Plot/tau_StepSize)
    tau_Plot_diffPoint = int(tau_Plot_diff/tau_StepSize)

    xi_Max = 100

    x = range(tau_MaxSteps)
    y = range(xi_Max)
    X, Y = np.meshgrid(x, y)

    u, dudv = WaveEqu(xi_0 = xi_0, u_0=u_0, tau_Max=tau_Max, init_setting = init_setting)

    if plot_3d:
        fig = plt.figure()

        ax1 = fig.add_subplot(121, projection='3d')
        ax1.plot_surface(X*tau_StepSize, Y, u, cmap='winter')
        ax1.set_xlabel(r'$\tau$')
        ax1.set_ylabel(r'$\xi$')
        ax1.set_zlabel('u')

        ax2 = fig.add_subplot(122, projection='3d')
        ax2.plot_surface(X*tau_StepSize, Y, dudv, cmap='winter')
        ax2.set_xlabel(r'$\tau$')
        ax2.set_ylabel(r'$\xi$')
        ax2.set_zlabel('v_(du/dxi)')

        plt.show()

        fig = plt.figure(figsize=(5,5))
        fig.suptitle(r'$\xi_0=$' +str(xi_0) + ',_' +r'$u_0=$' +str(u_0))

        ax1 = fig.add_subplot(111, projection='3d')
        ax1.plot_surface(X*tau_StepSize, Y, u, cmap='winter')
        ax1.set_xlabel(r'$\tau$')
        ax1.set_ylabel(r'$\xi$')
        ax1.set_zlabel('u')
```

```
        plt.show()

    fig = plt.figure(figsize=(10,5))
    fig.suptitle(r'$\xi_0=$' +str(xi_0) + ', ' +r'$u_0=$' +str(u_0))
    ax1 = fig.add_subplot(121)
    ax2 = fig.add_subplot(122)

    ax1.plot(y, u[:,tau_PlotPoint], label='At '+ r'$\tau=$'+ str(tau_Plot))
    ax1.plot(y, u[:,tau_PlotPoint+tau_Plot_diffPoint], label='At '+ r'$\tau=$'+ str(tau_Plot+tau_Plot_diff))

    ax1.legend(loc="upper right")
    ax1.set_ylabel(r'$u$')
    ax1.set_xlabel(r'$\xi$')

    ax2.plot(dudv[:,tau_PlotPoint], u[:,tau_PlotPoint], label='At '+ r'$\tau=$'+ str(tau_Plot))
    ax2.plot(dudv[:,tau_PlotPoint+tau_Plot_diffPoint], u[:,tau_PlotPoint+tau_Plot_diffPoint], label='At '+ r'$\tau=$'+

    ax2.legend(loc="upper right")
    ax2.set_ylabel(r'$u$')
    ax2.set_xlabel(r'$v$')

    plt.show()


u_1=5.56155
u_2=1.43845

#For question 1b i
if True:
    xi_0=20
    tau_Max=220
    tau_Plot=100
    tau_Plot_diff = 20

    PlotWave(xi_0, u_1, tau_Max, tau_Plot, tau_Plot_diff, plot_3d=False)

#For question 1b ii
if True:
    xi_0=50
    tau_Max=220
    tau_Plot=10
    tau_Plot_diff = 20

    PlotWave(xi_0, u_2, tau_Max, tau_Plot, tau_Plot_diff, plot_3d=False)

#For question 1b iii
if True:
    xi_0=50
    tau_Max=220
    tau_Plot=100
    tau_Plot_diff = 20

    PlotWave(xi_0, 1.1*u_2, tau_Max, tau_Plot, tau_Plot_diff, plot_3d=False)

#For question 1c
if True:
    xi_0=50
    tau_Max=220
    tau_Plot=100
    tau_Plot_diff = 20

    PlotWave(xi_0, u_1, tau_Max, tau_Plot, tau_Plot_diff, plot_3d=True, init_setting = 'c')
    PlotWave(xi_0, 3*u_1, tau_Max, tau_Plot, tau_Plot_diff, plot_3d=True, init_setting = 'c')
```