Problem 1: Time delayed model with allee effect

Collaborators

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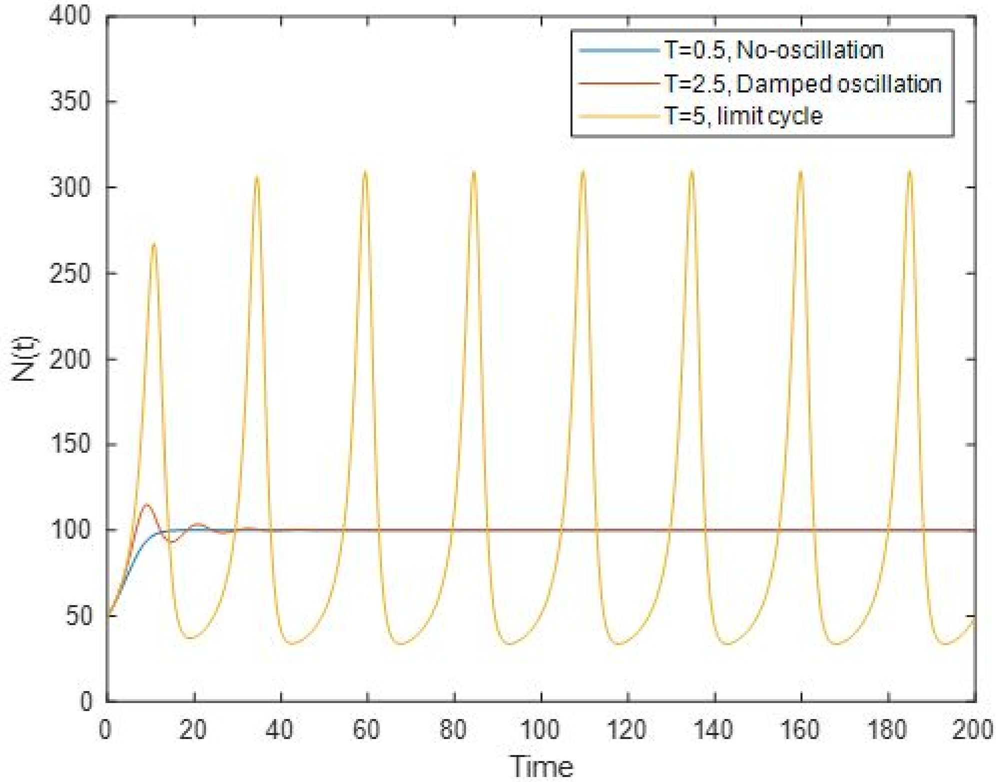
a) Example of dynamics of no-oscillation, damped oscillations and stable oscillation

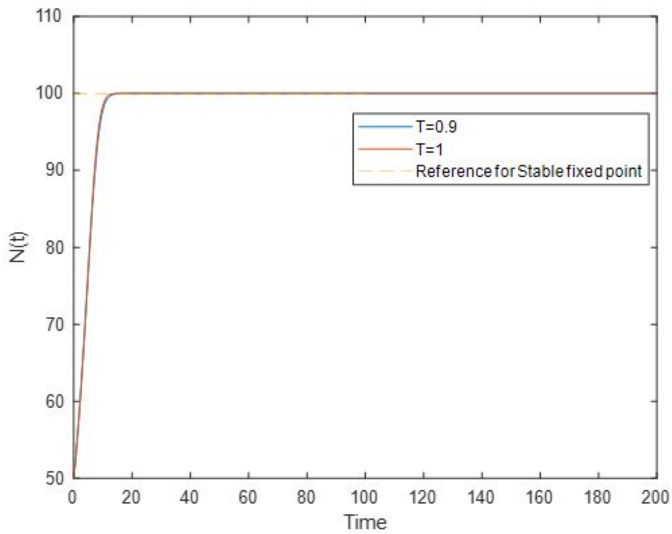
Example of dynamics for

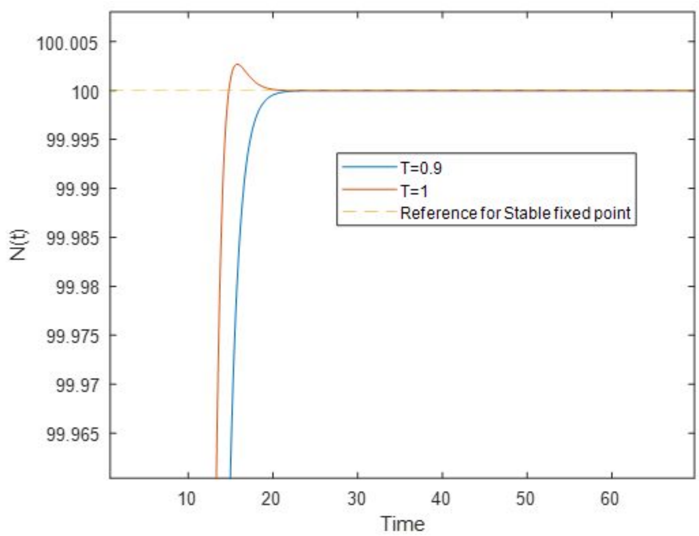
- $T = 0.5$, no-oscillation
- $T = 2.5$, damped oscillation
- $T = 5.0$, Limit cycle



b) At which T does the dynamics starts exhibiting damped oscillation.

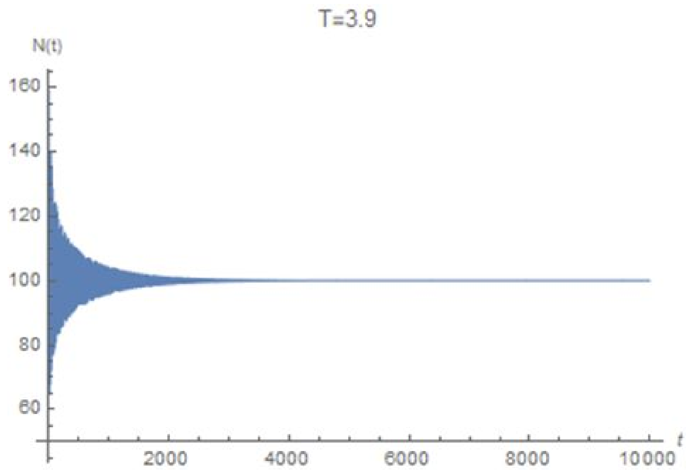
At around T=1.0, the dynamics starts exhibiting damped oscillation. The dynamics starts to overshoot and then settles down to equilibrium $N^* = K = 100$, as shown in the figures below. The second figure shows the detailed view of the overshoot.

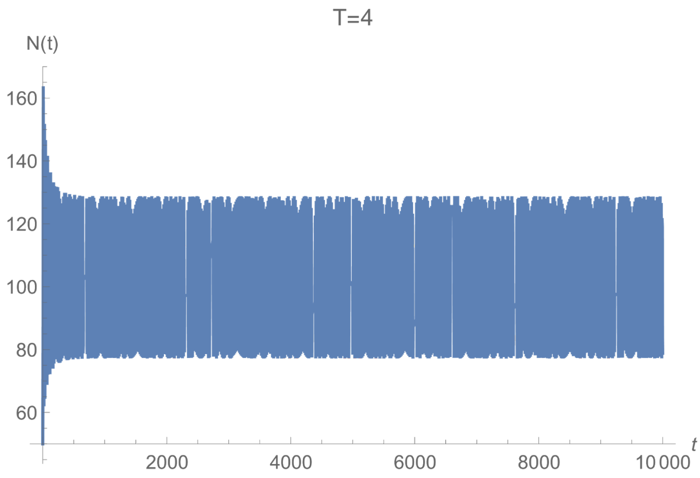




c) Approximate at which T the Hopf bifurcation occurs.

At T=4.0, the dynamics changes to an unstable state. Therefore, at approximately T=4.0, Hopf bifurcation occurs.





d) Analytically compute T_H

Given

$$\frac{dN(t)}{dt} = r\,N(t)\left(1 - \frac{N(t-T)}{K}\right)\left(\frac{N(t)}{A} - 1\right) \qquad \text{.......................... (1)}$$

Add a small perturbation $e^{\lambda t}$ around the fixed point $N^* = K$

Therefore, let $N(t) = K + e^{\lambda t}$

Substituting this in equation (1), we get

$$\frac{d(K+e^{\lambda t})}{dt} = r\,(K + e^{\lambda t})\left(1 - \frac{K+e^{\lambda(t-T)}}{K}\right)\left(\frac{K+e^{\lambda t}}{A} - 1\right)$$

$$\lambda e^{\lambda t} = r\,(K + e^{\lambda t})\left(1 - \frac{K+e^{\lambda(t-T)}}{K}\right)\left(\frac{K+e^{\lambda t}}{A} - 1\right)$$

$$\lambda = \frac{1}{e^{\lambda t}}\left[r\,(K + e^{\lambda t})\left(1 - \frac{K+e^{\lambda(t-T)}}{K}\right)\left(\frac{K+e^{\lambda t}}{A} - 1\right)\right]$$

Using Mathematica to solve the above equation we get,

$$\lambda = -0.00005\, e^{-\lambda T}\,(80 + e^{\lambda t})\,(100 + e^{\lambda t})$$

Ignoring the term of the order $O\left((e^{\lambda t})^2\right)$, we get

$$\lambda = -0.4\, e^{-\lambda T}$$

Assuming complex eigenvalues $\lambda = \mu + i\omega$

$$\mu + i\omega = -0.4\, e^{-(\mu + i\omega)T}$$

$$\mu + i\omega = -0.4\, e^{-\mu T} e^{-i\omega T}$$

$$\mu + i\omega = -0.4\, e^{-\mu T}(cos\omega T - isin\,\omega T)$$

Separating the real and imaginary terms we get,

$$\mu = -0.4\, e^{-\mu T} cos\omega T$$

$$\omega = 0.4\, e^{-\mu T} sin\,\omega T$$

Hopf bifurcation occurs when the real part is equal to 0. Therefore,

$$\omega T = \frac{\pi}{2}$$

$$\omega = 0.4\, sin\,\omega T$$

Therefore, substituting $\omega T = \frac{\pi}{2}$ in $\omega = 0.4\, sin\, \omega T$, we get

$\omega = 0.4$.

Substituting this in $\omega T = \frac{\pi}{2}$, we get

$T_H = 3.9269$

$T_H = 3.9269$ is very close to what we obtained in part c. The error between the two values is because of the discrete steps taken in part c.

Appendix

a) Show examples of the different dynamics obtained in this model: no oscillations, damped oscillations, and stable oscillations (limit cycle).

r=0.1;k=100;a=20;n_0=50;tend=300;

Manipulate[s=NDSolve[{n'[t]==r n[t] (1-n[t-T]/k) (n[t]/a-1),n[t/;t<=0]==n_0},n[t],{t,0,tend}],{T},{T,0,5,0.1}]

Manipulate[Plot[Evaluate[n[t]/.s],{t,0,tend},PlotRange->Full,PlotLabel->"T=4",AxesLabel->{t,"N(t)"}],{T, 0,5,ControlType->None}]

Manipulate[Plot[Evaluate[n[t]/.s],{t,0,tend},PlotLabel->"T=1.5",AxesLabel->{t,"N(t)"},PlotRange->{{0,100},{99,101}}],{T,0,5,ControlType->None}]

b) Analytically compute Th

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λ ==(1/$E^{\lambda t}$)(r (k+$E^{\lambda t}$)(1-(k+$E^{\lambda(t-T)}$)/k)((k+$E^{\lambda t}$)/a-1))//Simplify

λ==-(($E^{-T\lambda}$ ($E^{t\lambda}$+k) (-a+$E^{t\lambda}$+k) r)/(a k))

λ==-(($E^{-T\lambda}$ ($E^{t\lambda}$+k) (-a+$E^{t\lambda}$+k) r)/(a k))/.k->100/.r->0.1/.a->20//FullSimplify

λ==-0.00005 $E^{-T\lambda}$ (80. +$E^{t\lambda}$) (100. +$E^{t\lambda}$)

λ==-0.00005 $E^{-T\lambda}$ (8000 +180$E^{t\lambda}$+(E^(t λ))^2)

Neglecting the second order term we get

-0.00005 $E^{-T\lambda}$ 8000

-0.4 $E^{-T\lambda}$

Assuming complex eigenvalues, $\lambda=\mu+i\omega$

λ = -0.4 $E^{-T\lambda}$

$\mu + i\omega$ = -0.4 $E^{-T(\mu+i\omega)}$

$\mu + i\omega$ = -0.4 $E^{-\mu T}$ $E^{-i\omega T}$

$\mu + i\omega$ = -0.4 $E^{-\mu T}$ (Cos[wT]-iSin[wT])

μ = -0.4 $E^{-\mu T}$ Cos ωT

ω = 0.4 $E^{-\mu T}$ Sin ωT

Hopf bifurcation occurs when the real eigenvalue μ crosses 0. Substituting μ =0, we get

(π/2)=ωT. and ω=0.4. Then T=π/0.8

Pi/0.8

3.92699