Dynamical Systems TIF155/FIM770
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Problem set 2

2.3 Hopf bifurcation

c)

In[ ]:=
```
f1[x_, y_] = -x^3;
g1[x_, y_] = 2 y^3;
f2[x_, y_] = -x^2;
g2[x_, y_] = 2 x^2;
```

In[ ]:=
```
ω1 = 3;
ω2 = -1;
```

For system (1)

In[ ]:= `fxxx = D[f1[x, y], {x, 3}]`

Out[ ]= -6

In[ ]:= `fxyy = D[f1[x, y], {x, 1}, {y, 2}]`

Out[ ]= 0

In[ ]:= `gxxy = D[g1[x, y], {x, 2}, {y, 1}]`

Out[ ]= 0

In[ ]:= `gyyy = D[g1[x, y], {y, 3}]`

Out[ ]= 12

In[ ]:= `fxy = D[D[f1[x, y], {x}], {y}]`

Out[ ]= 0

In[ ]:= `fxx = D[f1[x, y], {x, 2}]`

Out[ ]= $-6\,x$

In[ ]:= `fyy = D[f1[x, y], {y, 2}]`

Out[ ]= 0

In[ ]:= `gxy = D[D[g1[x, y], {x}], {y}]`

Out[ ]= 0

In[ ]:= `gxx = D[g1[x, y], {x, 2}]`

Out[ ]= 0

In[ ]:=
```
gyy = D[g1[x, y], {y, 2}]
```
Out[ ]= 12 y

In[ ]:=
```
α1 = fxxx + fxyy + gxxy + gyyy +
   1 / ω1 (fxy (fxx + fyy) - gxy (gxx + gyy) - fxx * gxx + fyy * gyy) - 16 α
```
Out[ ]= $6 - 16\,\alpha$

In[ ]:=
```
Solve[α1 == 0]
```
Out[ ]= $\left\{\left\{\alpha \to \frac{3}{8}\right\}\right\}$

α > 0 so the bifurcation is subcritical

<u>For system</u> (<u>2</u>)

In[ ]:=
```
fxxx = D[f2[x, y], {x, 3}]
```
Out[ ]= 0

In[ ]:=
```
fxyy = D[f2[x, y], {x}, {y, 3}]
```
Out[ ]= 0

In[ ]:=
```
gxxy = D[g2[x, y], {x, 2}, {y}]
```
Out[ ]= 0

In[ ]:=
```
gyyy = D[g2[x, y], {y, 3}]
```
Out[ ]= 0

In[ ]:=
```
fxy = D[f2[x, y], {x}, {y}]
```
Out[ ]= 0

In[ ]:=
```
fxx = D[f2[x, y], {x, 2}]
```
Out[ ]= -2

In[ ]:=
```
fyy = D[f2[x, y], {y, 2}]
```
Out[ ]= 0

In[ ]:=
```
gxy = D[g2[x, y], {x}, {y}]
```
Out[ ]= 0

In[ ]:=
```
gxx = D[g2[x, y], {x, 2}]
```
Out[ ]= 4

In[ ]:=
```
gyy = D[g2[x, y], {y, 2}]
```
Out[ ]= 0

In[•]:=
```
α2 = fxxx + fxyy + gxxy + gyyy +
  1 / ω2 (fxy (fxx + fyy) - gxy (gxx + gyy) - fxx * gxx + fyy * gyy) - 16 α
```
Out[•]= $-8 - 16\,\alpha$

In[•]:=
```
Solve[α2 == 0]
```
$\left\{\left\{\alpha \to -\frac{1}{2}\right\}\right\}$

α < 0 so the bifurcations is supercritical

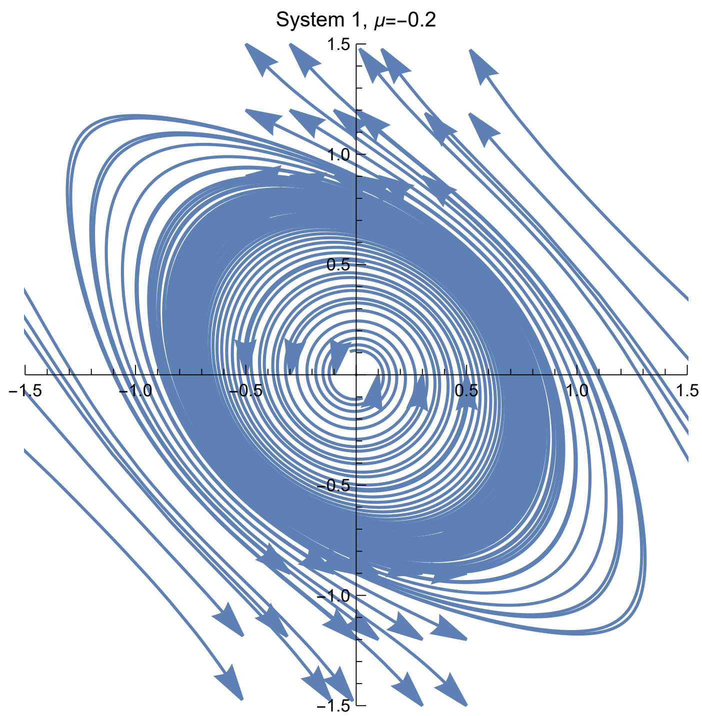
d)

```
(*System 1*)
```

In[1]:=
```
xDot[x_, y_, μ_] := μ x - 3 y - x^3;
yDot[x_, y_, μ_] := 3 x + μ y + 2 y^3;
```

In[•]:=
```
μ = -0.2;
t0 = 0;
tMax = -10;

sol = Table[NDSolve[{x'[t] == xDot[x[t], y[t], μ],
     y'[t] == yDot[x[t], y[t], μ], x[0] == x0, y[0] == y0}, {x[t], y[t]}, {t, t0, tMax}], {x0,
    {-0.5, -0.3, -0.1, 0, 0.1, 0.3, 0.5}}, {y0, {-1.5, -1.2, -0.9, 0, 0.9, 1.2, 1.5}}];
Length[sol]
p1 = ParametricPlot[{x[t], y[t]} /. sol, {t, t0, tMax},
   PlotRange → {{-1.5, 1.5}, {-1.5, 1.5}}, PlotLabel → StringForm["System 1, μ=``", μ]] /.
  Line[x_] :→ {Arrowheads[{0.05}], Arrow[x]}
```

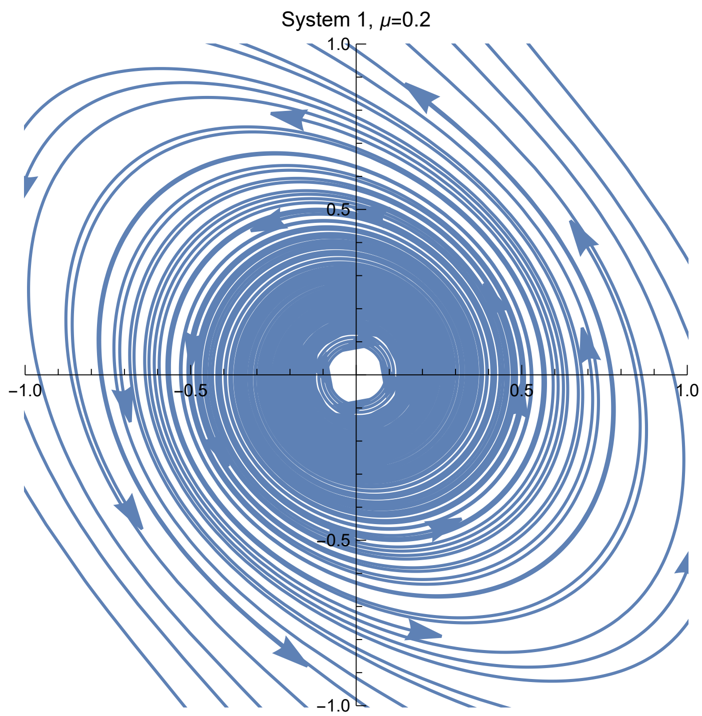
Out[•]=


```
μ = 0.2;
t0 = 0;
tMax = 7.7;

sol = Table[NDSolve[{x'[t] == xDot[x[t], y[t], μ],
      y'[t] == yDot[x[t], y[t], μ], x[0] == x0, y[0] == y0}, {x[t], y[t]}, {t, t0, tMax}],
   {x0, {-0.12, -0.08, -0.04, 0.04, 0.08, 0.12}},
   {y0, {-0.12, -0.08, -0.04, 0.04, 0.08, 0.12}}];

p2 = ParametricPlot[{x[t], y[t]} /. sol, {t, t0, tMax},
   PlotRange → {{-1, 1}, {-1, 1}}, PlotLabel → StringForm["System 1, μ=``", μ]] /.
  Line[x_] :→ {Arrowheads[{0.05}], Arrow[x]}
```

In[ ]:=



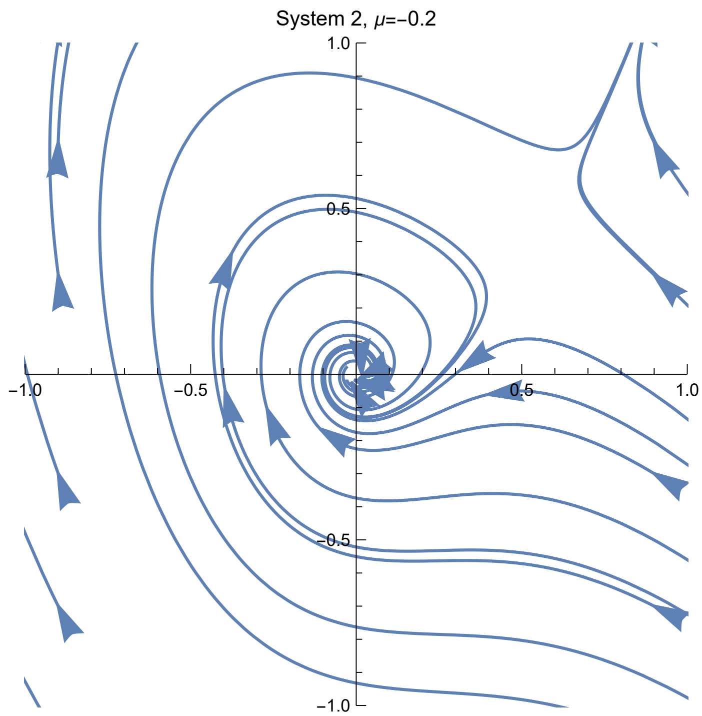
In[3]:=

```
(*System 2*)
μ =.
xDot[x_, y_, μ_] := μ x + y - x^2;
yDot[x_, y_, μ_] := -x + μ y + 2 x^2;
```

In[ ]:=
```
μ = -0.2;
t0 = 0;
tMax = 15;

sol = Table[NDSolve[{x'[t] == xDot[x[t], y[t], μ],
      y'[t] == yDot[x[t], y[t], μ], x[0] == x0, y[0] == y0}, {x[t], y[t]}, {t, t0, tMax}],
    {x0, {-1.5, -1.2, -0.9, 0.9, 1.2, 1.5}}, {y0, {-1.1, -0.7, -0.3, 0.3, 0.7, 1.1}}];
p3 = ParametricPlot[{x[t], y[t]} /. sol, {t, t0, tMax},
    PlotRange → {{-1, 1}, {-1, 1}}, PlotLabel → StringForm["System 2, μ=``", μ]] /.
   Line[x_] :> {Arrowheads[{{0.05}, {0.05, 0.5}, {0.05}}], Arrow[x]}
```

Out[ ]=



In[6]:=
```
μ = 0.2;
t0 = -15;
tMax = 15;

sol = Table[NDSolve[{x'[t] == xDot[x[t], y[t], μ],
      y'[t] == yDot[x[t], y[t], μ], x[0] == x0, y[0] == y0}, {x[t], y[t]}, {t, t0, tMax}],
    {x0, {-0.9, -0.5, -0.1, 0.1, 0.5, 0.9}}, {y0, {-0.9, -0.5, -0.1, 0.1, 0.5, 0.9}}];
p4 = ParametricPlot[{x[t], y[t]} /. sol, {t, t0, tMax}, PlotRange → {{-1, 1}, {-1, 1}},
     PlotStyle → {Red, Thick}, PlotLabel → StringForm["System 2, μ=``", μ]] /.
    Line[x_] :→ {Arrowheads[{{0.05}}], Arrow[x]};
stream4 = StreamPlot[{xDot[x, y, μ], yDot[x, y, μ]}, {x, -1, 1},
    {y, -1, 1}, StreamStyle → Blue, StreamColorFunction → None];
Show[{p4, stream4}]
```

Out[12]=