Dynamical Systems TIF155/FIM770
Konstantinos Zakkas
Problem Set 1

1.4 Two-dimensional linear system

a)

In[∘]:=
```
M = {{σ + 1, 3}, {-2, σ - 1}}
X[t_] = {x[t], y[t]}
dynSystem = X'[t] == M.X[t]
```

Out[∘]= $\{\{1+\sigma, 3\}, \{-2, -1+\sigma\}\}$

Out[∘]= $\{x[t], y[t]\}$

Out[∘]= $\{x'[t], y'[t]\} == \{(1+\sigma)\, x[t] + 3\, y[t], -2\, x[t] + (-1+\sigma)\, y[t]\}$

In[∘]:=
```
Eigenvalues[M]
```

Out[∘]= $\{-i\sqrt{5}+\sigma, i\sqrt{5}+\sigma\}$

b)

In[∘]:=
```
sol = DSolve[dynSystem, {x, y}, t]
```

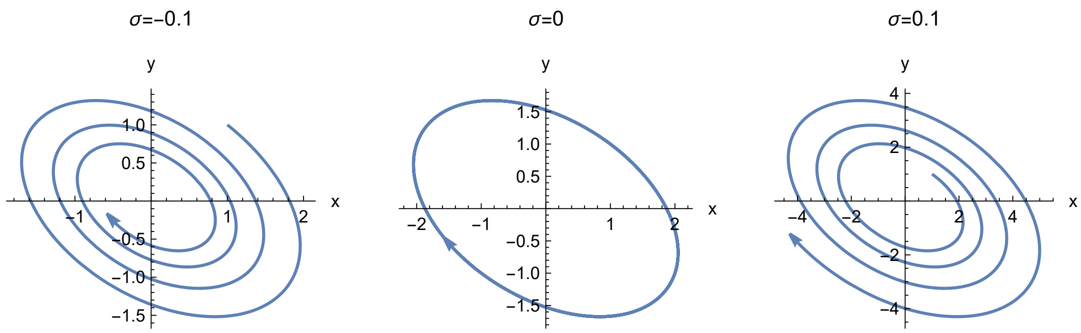
Out[∘]= $$\left\{\left\{x \to \text{Function}\left[\{t\}, \frac{3\, e^{t\sigma} c_2 \text{Sin}\left[\sqrt{5}\, t\right]}{\sqrt{5}} + \frac{1}{5} e^{t\sigma} c_1 \left(5\,\text{Cos}\left[\sqrt{5}\, t\right] + \sqrt{5}\,\text{Sin}\left[\sqrt{5}\, t\right]\right)\right],\right.\right.$$
$$\left.\left. y \to \text{Function}\left[\{t\}, -\frac{2\, e^{t\sigma} c_1 \text{Sin}\left[\sqrt{5}\, t\right]}{\sqrt{5}} + \frac{1}{5} e^{t\sigma} c_2 \left(5\,\text{Cos}\left[\sqrt{5}\, t\right] - \sqrt{5}\,\text{Sin}\left[\sqrt{5}\, t\right]\right)\right]\right\}\right\}$$

In[∘]:=
```
X[t_, x0_, y0_, σ_] = {x[t], y[t]} /. sol /. {C[1] → u, C[2] → v}
```

Out[∘]= $$\left\{\left\{\frac{3\, e^{t\sigma} v\, \text{Sin}\left[\sqrt{5}\, t\right]}{\sqrt{5}} + \frac{1}{5} e^{t\sigma} u \left(5\,\text{Cos}\left[\sqrt{5}\, t\right] + \sqrt{5}\,\text{Sin}\left[\sqrt{5}\, t\right]\right),\right.\right.$$
$$\left.\left. -\frac{2\, e^{t\sigma} u\, \text{Sin}\left[\sqrt{5}\, t\right]}{\sqrt{5}} + \frac{1}{5} e^{t\sigma} v \left(5\,\text{Cos}\left[\sqrt{5}\, t\right] - \sqrt{5}\,\text{Sin}\left[\sqrt{5}\, t\right]\right)\right\}\right\}$$

c)

In[ ]:=
```
p1 = ParametricPlot[X[t, 1, 1, -0.1], {t, 0, 10},
     AxesLabel → {"x", "y"}, PlotLabel → "σ=-0.1"] /. Line → Arrow;
p2 = ParametricPlot[X[t, 1, 1, 0], {t, 0, 10}, AxesLabel → {"x", "y"},
     PlotLabel → "σ=0"] /. Line → Arrow;
p3 = ParametricPlot[X[t, 1, 1, 0.1], {t, 0, 10}, AxesLabel → {"x", "y"},
     PlotLabel → "σ=0.1"] /. Line → Arrow;
plot = {p1, p2, p3};
Show[GraphicsRow[plot]]
```



d)

In[ ]:=
```
Solve [X[0, u, v, 0] == X[t, u, v, 0], t]
```

Out[ ]= $\left\{\left\{t \to \boxed{\frac{2\pi\,\mathbb{c}_1}{\sqrt{5}} \text{ if } \mathbb{c}_1 \in \mathbb{Z}}\right\}\right\}$

In[ ]:=
```
X[t_, x0_, y0_] = X[t, u, v, 0] // Simplify
```

Out[ ]= $\left\{\left\{u \operatorname{Cos}\left[\sqrt{5}\,t\right] + \frac{(u+3v)\operatorname{Sin}\left[\sqrt{5}\,t\right]}{\sqrt{5}},\ v \operatorname{Cos}\left[\sqrt{5}\,t\right] - \frac{(2u+v)\operatorname{Sin}\left[\sqrt{5}\,t\right]}{\sqrt{5}}\right\}\right\}$

In[ ]:=
```
L[t_] = X[t, x0, y0]⟦1⟧⟦1⟧^2 + X[t, x0, y0]⟦1⟧⟦2⟧^2
```

Out[ ]= $\left(v \operatorname{Cos}\left[\sqrt{5}\,t\right] - \frac{(2u+v)\operatorname{Sin}\left[\sqrt{5}\,t\right]}{\sqrt{5}}\right)^2 + \left(u \operatorname{Cos}\left[\sqrt{5}\,t\right] + \frac{(u+3v)\operatorname{Sin}\left[\sqrt{5}\,t\right]}{\sqrt{5}}\right)^2$

In[ ]:=
```
L[t_] = L[t] // Simplify
```

Out[ ]= $\left(v \operatorname{Cos}\left[\sqrt{5}\,t\right] - \frac{(2u+v)\operatorname{Sin}\left[\sqrt{5}\,t\right]}{\sqrt{5}}\right)^2 + \left(u \operatorname{Cos}\left[\sqrt{5}\,t\right] + \frac{(u+3v)\operatorname{Sin}\left[\sqrt{5}\,t\right]}{\sqrt{5}}\right)^2$

In[•]:= `dL[t_] = D[L[t], t]`

Out[•]=
$$2\left(-\left((2u+v)\cos\left[\sqrt{5}\,t\right]\right)-\sqrt{5}\,v\sin\left[\sqrt{5}\,t\right]\right)\left(v\cos\left[\sqrt{5}\,t\right]-\frac{(2u+v)\sin\left[\sqrt{5}\,t\right]}{\sqrt{5}}\right)+2\left((u+3v)\cos\left[\sqrt{5}\,t\right]-\sqrt{5}\,u\sin\left[\sqrt{5}\,t\right]\right)\left(u\cos\left[\sqrt{5}\,t\right]+\frac{(u+3v)\sin\left[\sqrt{5}\,t\right]}{\sqrt{5}}\right)$$

In[•]:= `dL[t_] = dL[t] // Simplify`

Out[•]=
$$2\left(u^2+u\,v-v^2\right)\cos\left[2\sqrt{5}\,t\right]+\sqrt{5}\,v\,(2u+v)\sin\left[2\sqrt{5}\,t\right]$$

In[•]:= `tVal = Solve[dL[t] == 0, t]`

Out[•]=
$$\left\{\left\{t\to\boxed{\frac{1}{2\sqrt{5}}\left(\mathrm{ArcTan}\left[-\frac{\sqrt{5}\,v\,(2u+v)}{\sqrt{4u^4+8u^3v+16u^2v^2+12uv^3+9v^4}},\ \frac{1}{2u+v}\,2\left(\frac{2u^3}{\sqrt{\left(2u^2+2uv+3v^2\right)^2}}+\frac{3u^2v}{\sqrt{\left(2u^2+2uv+3v^2\right)^2}}-\frac{uv^2}{\sqrt{\left(2u^2+2uv+3v^2\right)^2}}-\frac{v^3}{\sqrt{\left(2u^2+2uv+3v^2\right)^2}}\right)\right]+2\pi c_1\right)\ \text{if}\ c_1\in\mathbb{Z}}\right\},\right.$$

$$\left.\left\{t\to\boxed{\frac{1}{2\sqrt{5}}\left(\mathrm{ArcTan}\left[\frac{\sqrt{5}\,v\,(2u+v)}{\sqrt{4u^4+8u^3v+16u^2v^2+12uv^3+9v^4}},\ \frac{1}{2u+v}\,2\left(-\frac{2u^3}{\sqrt{\left(2u^2+2uv+3v^2\right)^2}}-\frac{3u^2v}{\sqrt{\left(2u^2+2uv+3v^2\right)^2}}+\frac{uv^2}{\sqrt{\left(2u^2+2uv+3v^2\right)^2}}+\frac{v^3}{\sqrt{\left(2u^2+2uv+3v^2\right)^2}}\right)\right]+2\pi c_1\right)\ \text{if}\ c_1\in\mathbb{Z}}\right\}\right\}$$

In[•]:= `tVal = tVal // Simplify`

Out[•]=
$$\left\{\left\{t\to\boxed{\frac{\mathrm{ArcTan}\left[-\frac{\sqrt{5}\,v\,(2u+v)}{\sqrt{\left(2u^2+2uv+3v^2\right)^2}},\ \frac{2\left(u^2+uv-v^2\right)}{\sqrt{\left(2u^2+2uv+3v^2\right)^2}}\right]+2\pi c_1}{2\sqrt{5}}\ \text{if}\ c_1\in\mathbb{Z}}\right\},\right.$$

$$\left.\left\{t\to\boxed{\frac{\mathrm{ArcTan}\left[\frac{\sqrt{5}\,v\,(2u+v)}{\sqrt{\left(2u^2+2uv+3v^2\right)^2}},\ -\frac{2\left(u^2+uv-v^2\right)}{\sqrt{\left(2u^2+2uv+3v^2\right)^2}}\right]+2\pi c_1}{2\sqrt{5}}\ \text{if}\ c_1\in\mathbb{Z}}\right\}\right\}$$

In[•]:=
```
t1 = t /. tVal[[1]]
```

Out[•]=
$$\frac{\text{ArcTan}\left[-\frac{\sqrt{5}\ v\ (2\ u+v)}{\sqrt{\left(2\ u^2+2\ u\ v+3\ v^2\right)^2}},\ \frac{2\left(u^2+u\ v-v^2\right)}{\sqrt{\left(2\ u^2+2\ u\ v+3\ v^2\right)^2}}\right]+2\ \pi\ c_1}{2\ \sqrt{5}}\quad \text{if}\ c_1\in\mathbb{Z}$$

In[•]:=
```
t2 = t /. tVal[[2]]
```

Out[•]=
$$\frac{\text{ArcTan}\left[\frac{\sqrt{5}\ v\ (2\ u+v)}{\sqrt{\left(2\ u^2+2\ u\ v+3\ v^2\right)^2}},\ -\frac{2\left(u^2+u\ v-v^2\right)}{\sqrt{\left(2\ u^2+2\ u\ v+3\ v^2\right)^2}}\right]+2\ \pi\ c_1}{2\ \sqrt{5}}\quad \text{if}\ c_1\in\mathbb{Z}$$

In[∘]:=
```
R = Sqrt[L[t1] / L[t2]]
```

Out[∘]=

$$\sqrt{\frac{\left(v \operatorname{Cos}\left[\frac{1}{2}\left(\operatorname{ArcTan}\left[-\frac{\sqrt{5}\, v\, (2 u+v)}{\sqrt{\left(2 u^2+2 u v+3 v^2\right)^2}}, \frac{2\left(u^2+u v-v^2\right)}{\sqrt{\left(2 u^2+2 u v+3 v^2\right)^2}}\right]+2 \pi \mathbb{c}_1\right)\right]-\frac{1}{\sqrt{5}}(2 u+v) \operatorname{Sin}\left[\frac{1}{2}\left(\operatorname{ArcTan}\left[-\frac{\sqrt{5}\, v\, (2 u+v)}{\sqrt{\left(2 u^2+2 u v+3 v^2\right)^2}}, \frac{2\left(u^2+u v-v^2\right)}{\sqrt{\left(2 u^2+2 u v+3 v^2\right)^2}}\right]+2 \pi \mathbb{c}_1\right)\right]\right)^2+\left(u \operatorname{Cos}\left[\frac{1}{2}\left(\operatorname{ArcTan}\left[-\frac{\sqrt{5}\, v\, (2 u+v)}{\sqrt{\left(2 u^2+2 u v+3 v^2\right)^2}}, \frac{2\left(u^2+u v-v^2\right)}{\sqrt{\left(2 u^2+2 u v+3 v^2\right)^2}}\right]+2 \pi \mathbb{c}_1\right)\right]+\frac{1}{\sqrt{5}}(u+3 v) \operatorname{Sin}\left[\frac{1}{2}\left(\operatorname{ArcTan}\left[-\frac{\sqrt{5}\, v\, (2 u+v)}{\sqrt{\left(2 u^2+2 u v+3 v^2\right)^2}}, \frac{2\left(u^2+u v-v^2\right)}{\sqrt{\left(2 u^2+2 u v+3 v^2\right)^2}}\right]+2 \pi \mathbb{c}_1\right)\right]\right)^2}{\left(v \operatorname{Cos}\left[\frac{1}{2}\left(\operatorname{ArcTan}\left[\frac{\sqrt{5}\, v\, (2 u+v)}{\sqrt{\left(2 u^2+2 u v+3 v^2\right)^2}}, -\frac{2\left(u^2+u v-v^2\right)}{\sqrt{\left(2 u^2+2 u v+3 v^2\right)^2}}\right]+2 \pi \mathbb{c}_1\right)\right]-\frac{1}{\sqrt{5}}(2 u+v) \operatorname{Sin}\left[\frac{1}{2}\left(\operatorname{ArcTan}\left[\frac{\sqrt{5}\, v\, (2 u+v)}{\sqrt{\left(2 u^2+2 u v+3 v^2\right)^2}}, -\frac{2\left(u^2+u v-v^2\right)}{\sqrt{\left(2 u^2+2 u v+3 v^2\right)^2}}\right]+2 \pi \mathbb{c}_1\right)\right]\right)^2+\left(u \operatorname{Cos}\left[\frac{1}{2}\left(\operatorname{ArcTan}\left[\frac{\sqrt{5}\, v\, (2 u+v)}{\sqrt{\left(2 u^2+2 u v+3 v^2\right)^2}}, -\frac{2\left(u^2+u v-v^2\right)}{\sqrt{\left(2 u^2+2 u v+3 v^2\right)^2}}\right]+2 \pi \mathbb{c}_1\right)\right]+\frac{1}{\sqrt{5}}(u+3 v) \operatorname{Sin}\left[\frac{1}{2}\left(\operatorname{ArcTan}\left[\frac{\sqrt{5}\, v\, (2 u+v)}{\sqrt{\left(2 u^2+2 u v+3 v^2\right)^2}}, -\frac{2\left(u^2+u v-v^2\right)}{\sqrt{\left(2 u^2+2 u v+3 v^2\right)^2}}\right]+2 \pi \mathbb{c}_1\right)\right]\right)^2}} \quad \text{if } \mathbb{c}_1 \in \mathbb{Z}$$

In[∘]:=
```
R = Simplify[R /. C[1] → 0]
```

Out[∘]=

$$\sqrt{-\frac{2 \sqrt{5}\, u^2+2 \sqrt{5}\, u v+3 \sqrt{5}\, v^2+5 \sqrt{\left(2 u^2+2 u v+3 v^2\right)^2}}{2 \sqrt{5}\, u^2+2 \sqrt{5}\, u v+3 \sqrt{5}\, v^2-5 \sqrt{\left(2 u^2+2 u v+3 v^2\right)^2}}}$$

In[∘]:=
```
R = Simplify[R, {Element[u, Reals], Element[v, Reals]}]
```

Out[∘]= $\sqrt{\frac{1}{2}\times\left(3+\sqrt{5}\right)}$

In[∘]:=
```
N[R]
```

Out[∘]= 1.61803

In[∘]:=
```
u =.;
v =.;
direction = X[t1 /. C[1] → 0, u, v] // Simplify
u = 1;
v = 1;
dir =
 Normalize[direction[[1]]] // Simplify // TrigReduce // Expand // TrigExpand // PowerExpand //
  Simplify
N[
 dir]
```

Out[∘]= $$\left\{\left\{u\,\mathrm{Cos}\left[\frac{1}{2}\,\mathrm{ArcTan}\left[-\frac{\sqrt{5}\,v\,(2\,u+v)}{\sqrt{\left(2\,u^2+2\,u\,v+3\,v^2\right)^2}},\frac{2\left(u^2+u\,v-v^2\right)}{\sqrt{\left(2\,u^2+2\,u\,v+3\,v^2\right)^2}}\right]\right]+\frac{(u+3\,v)\,\mathrm{Sin}\left[\frac{1}{2}\,\mathrm{ArcTan}\left[-\frac{\sqrt{5}\,v\,(2\,u+v)}{\sqrt{\left(2\,u^2+2\,u\,v+3\,v^2\right)^2}},\frac{2\left(u^2+u\,v-v^2\right)}{\sqrt{\left(2\,u^2+2\,u\,v+3\,v^2\right)^2}}\right]\right]}{\sqrt{5}},\right.\right.$$
$$\left.\left.v\,\mathrm{Cos}\left[\frac{1}{2}\,\mathrm{ArcTan}\left[-\frac{\sqrt{5}\,v\,(2\,u+v)}{\sqrt{\left(2\,u^2+2\,u\,v+3\,v^2\right)^2}},\frac{2\left(u^2+u\,v-v^2\right)}{\sqrt{\left(2\,u^2+2\,u\,v+3\,v^2\right)^2}}\right]\right]-\frac{(2\,u+v)\,\mathrm{Sin}\left[\frac{1}{2}\,\mathrm{ArcTan}\left[-\frac{\sqrt{5}\,v\,(2\,u+v)}{\sqrt{\left(2\,u^2+2\,u\,v+3\,v^2\right)^2}},\frac{2\left(u^2+u\,v-v^2\right)}{\sqrt{\left(2\,u^2+2\,u\,v+3\,v^2\right)^2}}\right]\right]}{\sqrt{5}}\right\}\right\}$$

Out[∘]= $\left\{-\frac{1}{10}\times\left(-5+\sqrt{5}\right)\sqrt{5+2\sqrt{5}},-\frac{3+\sqrt{5}}{\sqrt{50+22\sqrt{5}}}\right\}$

Out[∘]= {0.850651, -0.525731}