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Abstract. We study a forest-fire model recently introduced in a paper by Drossel and Schwabl, by means of simulations with high statistics and very close to the critical point. In this way we can correct some statements made in that paper. In particular, we find that the critical exponents are not the 'classical' ones proposed by Drossel and Schwabl. In spite of scaling laws with anomalous exponents, the typical states of the system are not 'critical' in the sense of being marginally stable locally.

Some time ago, Bak *et al* [1] introduced a forest-fire model which they claimed showed self-organized criticality [2]. This model contained three types of trees on a regular lattice: burning ones, burnt ones (ashes) and green ones. Burning trees burn for exactly one time unit, until they turn into ash. During this time step, they ignite all neighbouring green trees. The only free control parameter in this model is the probability p for a new green tree to grow out of ash. It is assumed that this growth is a random (Poissonian) process with rate p, and the growth itself also takes exactly one time step. The claim in [1] was that criticality shows up in the limit $p \to 0$, i.e. when the mean time for a new tree to grow is much larger than its burning time. Most of these simulations were done for two dimensions of space, and we shall also limit ourselves to two dimensions in the present paper.

This claim was mainly based on simulations of small systems with fairly large values of p and small statistics. It was refuted by Grassberger and Kantz [3] who obtained a different scenario for the limit $p \to 0$, by much larger simulations with much smaller values of p. According to them, the fire propagates essentially in regular fronts which proceed with finite velocity and burn down a finite fraction of trees. The distance between successive fronts is proportional to $1/p$. The intuitive reason for this behaviour is that fires survive (on sufficiently large lattices and for small p) typically in only a very few regions. If a region has been free of fires for a sufficiently long time so that the density of newly grown trees is already sufficient for a new front to propagate, there is simply no burning tree in the vicinity to ignite the region. Thus the density will have grown to a supercritical value before the next fire front reaches it. This was verified in [4].

However, it was also shown in [3] that this picture is somewhat too simplistic. Straight fronts break up repeatedly (even for the smallest values of p which could be studied), leading to an unstable pattern of spirals which evolves on a time scale much greater than $1/p$. Moreover, the possibility of an increasing 'softening' of the fronts in the limit $p \to 0$ could not be eliminated rigorously (though no hint for it was seen, and it was suggested to be very unlikely). It was suggested that in this alternative scenario propagation would be essentially percolation-like. The fire would propagate into regions which were populated with essentially random patterns of trees with a density which is just critical for propagation.

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If the tree patterns were exactly random, this argument would be exact. One result of the present paper is that it is not quite correct. Even if we modify the model so that it does become critical (in a sense to be specified below), patterns of trees are correlated with strong long-range correlations, and the spreading of fires is not in the same universality class as ordinary percolative spreading.

Such a modification is obtained by adding a new mechanism ('lightning') by which a green tree can catch fire without a burning neighbour. Such models have been recently proposed by Chen *et al* [5] and by Drossel and Schwabl [6]. In [5], the chance for spontaneously catching fire depends on the age of the tree (indeed, the process is formally deterministic). This makes it very cumbersome to simulate efficiently, and the conclusions drawn by the authors do not seem well justified [7]. I thus do not want to discuss it any further. On the other hand, the model in [6] is very elegant and allows for very efficient simulations.

In the Drossel–Schwabl model, lightning is also modelled as a Poisson process, with rate $f = p/\theta$, with $\theta \gg 1$: each green tree has a chance f to ignite spontaneously, independently of its age. Critical behaviour is observed in the double limit $f \ll p \ll 1$. But we also want f not to be too small either. Notice that the average size $\bar{s}$ of a fire (i.e. the number of trees destroyed by a single lightning) is proportional to θ. We want to consider the case where this is sufficiently small (for fixed $p \neq 0$), so that the growth of new trees can be neglected during the fire initiated by a typical lightning. Since we find the typical time for a fire to be $\bar{T} \propto \theta$ (in two dimensions; see below), we see that the requirement $\bar{T} \ll 1/p$ implies $f \gg p^2$ or $p \ll 1/\theta$. The advantage of this limit is that we can consider each fire as an isolated event: fires triggered by different lightnings do not overlap in spacetime, so the clusters destroyed by individual lightnings are well defined objects [6]. Simulation is then done in the limit $p = 0$, $\theta \gg 1$. It proceeds simply by burning the entire cluster of sites connected to a lightened site and attempting to grow θ new trees between any two fires.

It is claimed in [6] that the critical exponents of this model are essentially 'classical'. This means, in particular, that the clusters are compact, i.e. the average radius for clusters of fixed mass (and fixed θ as well!) scales as

$$R(s) \sim s^{1/\mu} \tag{1}$$

with $\mu = d$ for $1 \ll s \ll s_{\max}$, where $s_{\max}$ is an effective cut-off. Similarly, the average cluster radius scales as

$$\bar{R} \sim \bar{s}^{\nu} \sim \theta^{\nu} \tag{2}$$

with $\nu = 1/d$ in d dimensions (in [6], a slightly different definition was used, with $\bar{R}$ replaced by $R(\bar{s})$).

The distribution of cluster (i.e. fire) size is denoted, for historic reasons, by $sN(s)$. It is assumed to scale as

$$sN(s) \sim s^{1-\tau} \tag{3}$$

for $s \ll s_{\max}$, leading to

$$P(s) \equiv \text{prob}\{\text{size} \geqslant s\} = \frac{1}{\sum_{s'=1}^{\infty} s'N(s')} \sum_{s'=s}^{\infty} s'N(s') \sim s^{\tau-2}. \tag{4}$$

A priori, s_{max} could scale with some power λ of $\bar{s}$,

$$s_{max} \sim \bar{s}^{\lambda}. \tag{5}$$

Assuming $s_{max} \sim \bar{s} \sim \theta$ (i.e. $\lambda = 1$), Drossel and Schwabl arrive at $\tau = 2$. For the average time of a fire, these predictions would give $\bar{T} \sim \sqrt{\theta}$. For $s \gtrsim s_{max}$, they finally assumed an ordinary scaling behaviour

$$s^{\tau} N(s) \approx C(s/s_{max}) \qquad s^{-1/\mu} R(s) \approx \tilde{C}(s/s_{max}) \tag{6}$$

with monotonic scaling functions $C(x)$ and $\tilde{C}(x)$ which are unity at $x = 1$ and vanish for $x \to \infty$.

These conjectures were supported in [6] by simulations with $\theta = 70$ and $\theta = 200$. These small values of θ allowed them to present parts of $N(s)$ and $R(s)$ for $s \lesssim 2000$.

In the following we shall present simulations much closer to the critical point, with $\theta = 125, 250, \ldots, 4000$. The lattice sizes were between 1024×1024 (for $\theta \leqslant 250$) and 8192×8192 for $\theta = 4000$. The number of lightnings used for averaging were between 2×10^6 (for $\theta = 4000$) and 7×10^6 (for $\theta = 1000$). In each case, a sufficient number of lightnings at the beginning were discarded, in order to reach the critical state. The burning was implemented with the recursive depth-first algorithm of [8]. The total CPU time spent on these simulations was approximately 170 h on a DECstation 2100 (approximately 12 MIPs).

While our simulations confirm, in some sense, the criticality of the model, they do not agree with the predicted exponents and with the scaling ansatz (6). Let us discuss them in detail.

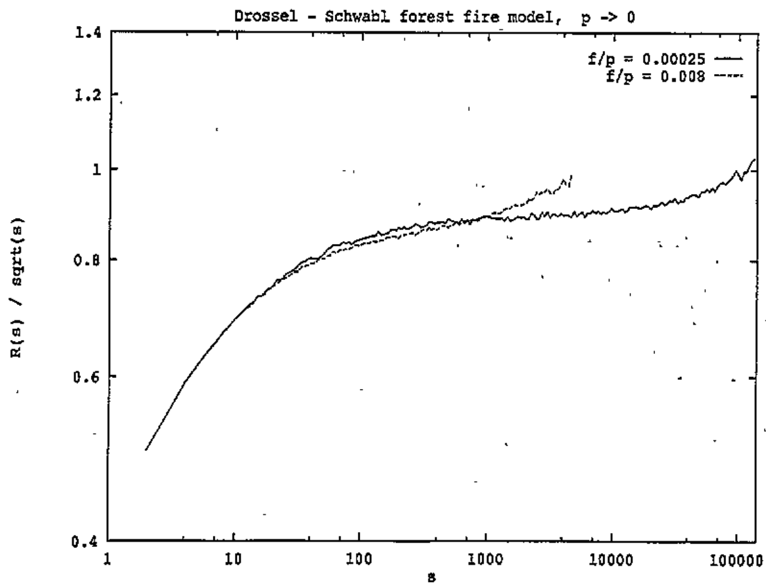

Figure 1. Log–log plot of the RMS radius of fire clusters with fixed size s. For larger sensitivity of the plot, $R(s)$ itself is not plotted, but $R(s)/\sqrt{s}$. Also, data in small intervals of s (of 5% width) have been lumped together in order to suppress statistical fluctuations. The broken curve is for $\theta \equiv p/f = 125$, the continuous curve for $\theta = 4000$.

In figure 1 we show $R(s)/\sqrt{s}$. According to equation (1) with $\mu = 2$, we expect $R(s)/\sqrt{s} \approx$ constant. We see strong deviations from this in figure 1, but they seem to be corrections to scaling due to the finiteness of θ. This is strongly suggested by the comparison of results for $\theta = 4000$ with $\theta = 125$. We thus see that equation (1) is indeed correct, as well as the prediction $\mu = 2$.

As we have said above, this would also suggest that $\bar{R} \sim \theta^{1/2}$. But as seen in figure 2, this is not the case. Instead we find perfect scaling behaviour with

$$\nu = 0.584 \pm 0.01. \tag{7}$$

Notice that $\bar{R}$ is dominated by contributions from the very large clusters. The last result tells us that either the very large clusters have much larger $R(s)$ (for which we did indeed see some evidence, but not nearly enough to explain equation (7)) or that $s_{\max}$ increases more rapidly than $\bar{s}$.

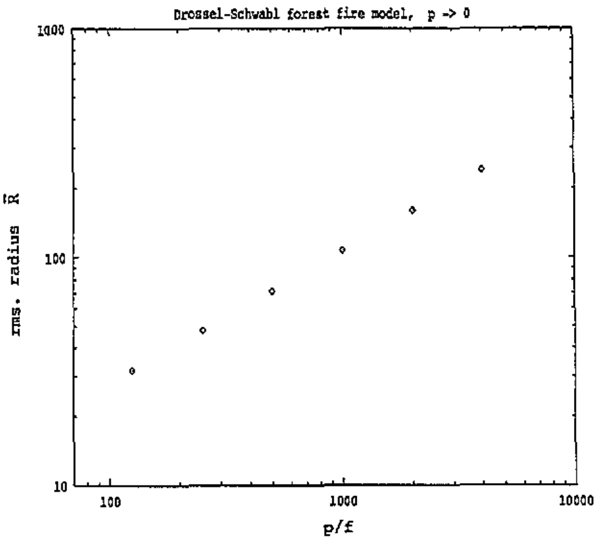

Figure 2. Log–log plot of the grand average radii $\bar{R}$ of fire clusters against θ. Statistical errors are much smaller than the size of the symbols.

Before we present direct evidence for the latter, we show in figure 3 the average fire lifetime $\bar{T}$ against θ. In contrast to figure 2, we now see very important deviations from scaling behaviour (in figures 2 and 3, the statistical errors are much smaller than the sizes of the symbols). While we would get only slight disagreement with the predicted behaviour $\bar{T} \sim \theta^{1/2}$ for very small θ, this prediction is very wrong in the critical limit $\theta \to \infty$. For our largest values of θ we get $\bar{T} \propto \theta^{0.87\pm0.03}$. Our best extrapolation to $\theta \to \infty$ is close to $\bar{T} \sim \theta$, as mentioned above.

Next, we present the cluster distribution $sN(s)$ for three values of θ in figure 4. We see again very strong corrections to scaling. In particular, the slope of $sN(s)$ against s (on a log–log plot) is not monotonic for large values of θ, contradicting the assumed monotonicity of $C(x)$. For small values of θ, we again find reasonable agreement with the prediction $sN(s) \sim 1/s$, but not for large θ. For $\theta = 4000$, the scaling region seems to be roughly $0 < s < 400$. A fit in that region would give $\tau = 2.19 \pm 0.04$, excluding the proposed value $\tau = 2$.

Clearly the bump in $N(s)$ for large s represents those clusters which would have had even larger s in the scaling limit $\theta \to \infty$, and which are shifted to $s \lesssim s_{\max}$ by the finiteness of θ.

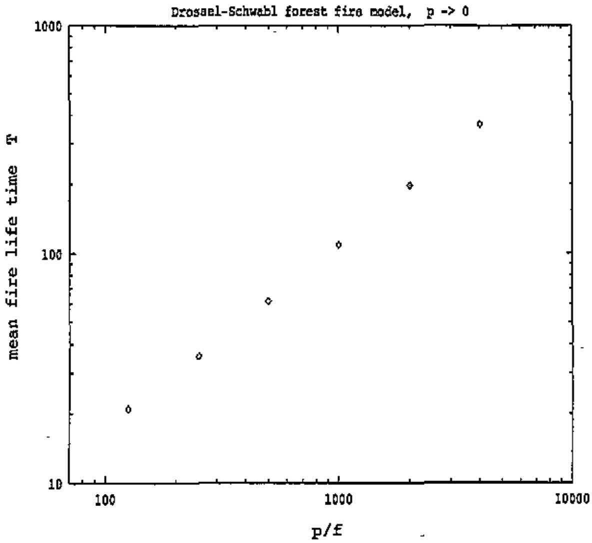

Figure 3. Same as figure 2, but for the average lifetimes of fires.

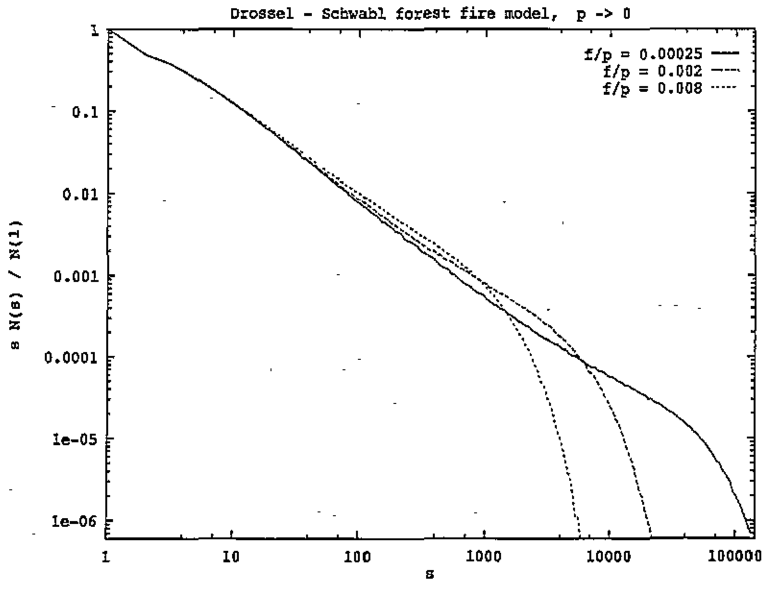

Figure 4. Log–log plot of the distribution of cluster sizes, for three different values of θ. The same binning has been performed as in figure 1, in order to reduce fluctuations.

We thus expect a faster approach to scaling if we consider not the differential distribution $N(s)$ but the integrated distribution $P(s)$. This is shown, for all six values of θ, in figure 5. From this figure we can read several results. First of all, there is indeed a very nice scaling region. For $\theta = 4000$ it ranges from $s = 8$ to $s = 1000$. The slope in this region seems to decrease slightly as we approach the critical point. Taking the uncertainty implied by this decrease into account, we find

$$\tau = 2.15 \pm 0.02. \qquad (8)$$

The second result directly read from figure 5 is that $s_{\max}$ increases faster than θ. More precisely, we find

$$\lambda = 1.08 \pm 0.02. \tag{9}$$

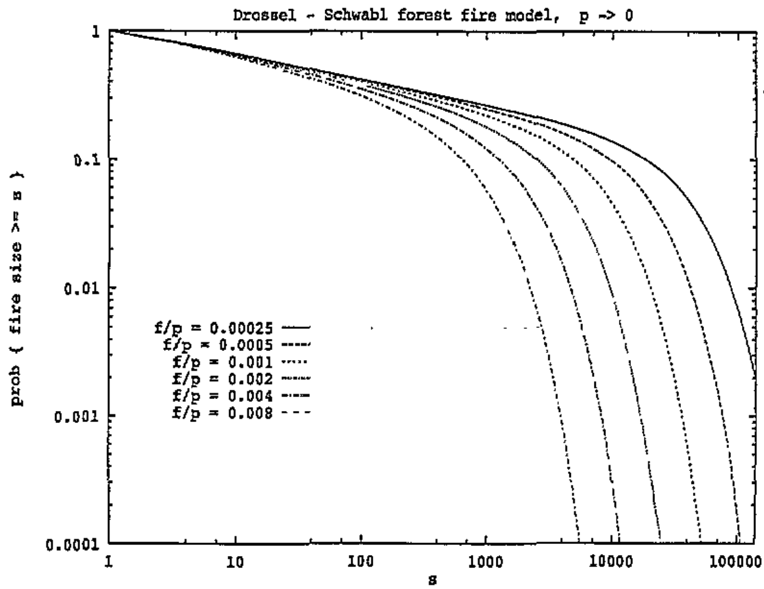

Figure 5. Log–log plot of the integrated distribution $P(s)$ of cluster sizes, for all six values of θ between 125 and 4000.

Third, we see that the scaling ansatz (6) cannot be correct. It would have led to a similar scaling ansatz

$$\log P(s) \approx (\tau - 2)\log s + F(s/s_{\max}). \tag{10}$$

This would mean that the curves in figure 5 could be brought to coincide by simple shifts. By superimposing copies of this figure one sees that this is not the case. Instead, one finds that the bends in these curves become increasingly sharper with increasing θ, suggesting that the scaling limit of these curves is a straight line with a sharp cutoff at $s_{\max}$. This would mean that the broad bumps seen in figure 4 would tend towards delta peaks for $\theta \to \infty$. Indeed, these bumps do become narrower with increasing θ, but this is less clear than in figure 5 due to the larger statistical fluctuations.

Finally, we studied directly the distributions of green trees. Figure 6 shows a typical snapshot of part of a large lattice (2048×2048) at $\theta = 2000$. Green trees are black pixels, ashes are white. Essentially we see in this figure a collection of patches with different degrees of uniform greyness. The interpretation of this pattern is clear: bright patches are those regions which have been burnt recently, while dark patches are regions where so many trees have grown since the last fire that the next lightning will induce a fire with great probability. The compact shapes of these patches agree with our findings that $R(s) \sim \sqrt{s}$. The boundaries of the patches seem fractal, though it is not easy to determine their fractal dimensions directly.

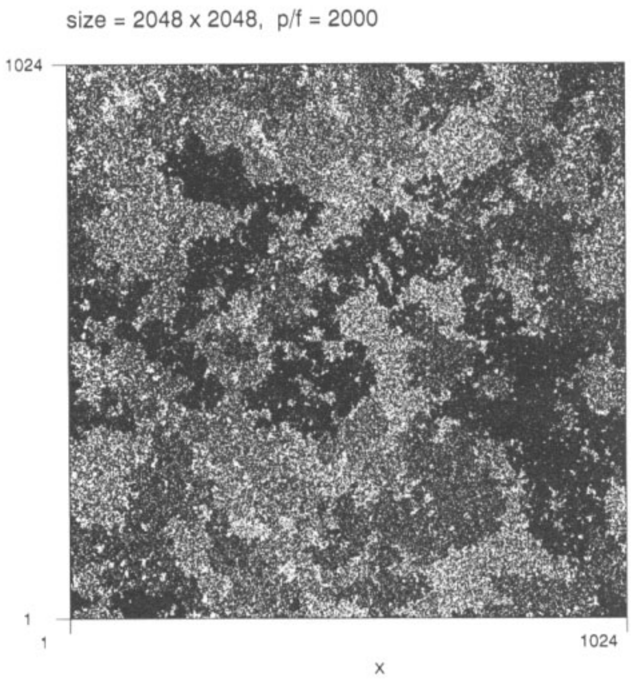

Figure 6. Typical pattern of green trees (black dots) on one quarter of a lattice of size 2048×2048. The inverse lightning rate was $\theta = 2000$, and approximately 140 000 lightnings had already struck the lattice.

We conjecture that this scenario becomes exact in the limit $\theta \to \infty$. In this limit, the fluctuations of greyness within a patch should be negligible on the scale of $R(s) \sim \sqrt{\theta}$, and the boundaries are sharp on this length scale.

An indirect argument that the boundaries are indeed *not fractal* is provided by the dependence of the average density $\bar{\rho}$ on θ. In figure 7 we show $\bar{\rho}$ against $1/\sqrt{\theta}$. We see a perfect straight line with slope -1 (statistical errors are of the size of the symbols), indicating that

$$\bar{\rho} = 0.4075 - 1/\sqrt{\theta}. \tag{11}$$

Since the deviation of $\bar{\rho}$ from its asymptotic value should be due to the boundaries between patches, this suggests that the length of these boundaries scales as their radii, i.e. that the patches have non-fractal boundaries. Clearly more work has to be done to support this surprising prediction.

A lightning striking into such a patch will only have a chance to induce a large fire if its density ρ is above a threshold ρ_c, and it will leave a depleted region with density $\rho' < \rho_c$ which is a decreasing function of ρ. The large differences in greyness seen in figure 6 indicate that typically a lightning does not hit a region before ρ has grown well above ρ_c. Thus, like the original model of [1], the model is not 'critical' in the sense that locally the state is nearly everywhere far away from the instability threshold. As a consequence, the average density of green trees is not given by the threshold for site percolation [9], $\bar{\rho} < 0.5927$. It is very close to the value 0.39 conjectured in [6], but we believe that the difference is significant. The reason for the conjecture in [6] was that 0.39 is the minimum

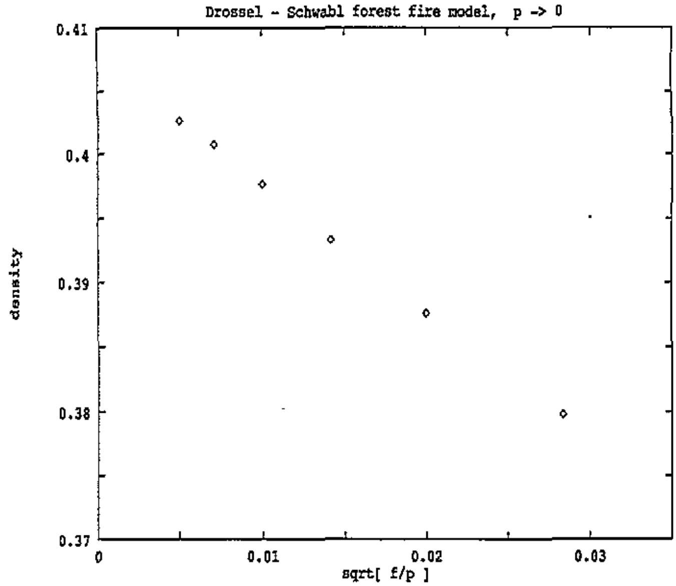

Figure 7. Average density of unburnt trees in the critical state, plotted against $1/\sqrt{\theta}$. The errors are typically of the size of the symbols.

possible density if we assume that the fire always finds completely *random* configurations of unburnt trees. But there is no reason why the configuration should be random (unless the last fire had left $\rho' = 0$ behind), and we see no strong rationale behind this argument.

In summary we have verified that the forest-fire model introduced in [6] is indeed critical in the sense that it leads to anomalous scaling laws. In this sense it is, moreover, in a new universality class. It is not critical in the sense that locally it is not close to the limit of stability. We have also shown that the scaling laws are much less trivial than proposed in [6]. While some variables show early scaling, others display very large corrections to scaling. This holds, in particular, for the cluster-size distribution. The latter is very reminiscent of the Abelian sand-pile model [2] where the cluster-size distribution also shows very late scaling with very large corrections [10, 11]. Moreover, the exponent τ in both models is nearly the same. In previous simulations of the sand-pile model, avalanches which started near the boundary were also included in the averages, although they clearly have the largest finite-size corrections. If such clusters are excluded, the cluster distribution $N(s)$ of the sand-pile model is very similar to that in figure 4. In that case it also seems that the most precise determination of τ does not use $N(s)$ but the integrated distribution $P(s)$.

But apart from this analogy, the present model and the sand-pile model are very different. In the sand-pile model a conservation law during 'stress' relaxation is responsible for the criticality†, in conjunction with a slow increase in the stress between the relaxation ('avalanche') events. In the present model, a doubly slow process is needed: a slow increase in *susceptibility* to relaxation, and an even slower process of events which actually trigger the relaxation processes. Due to the latter, a typical relaxation processes will discharge so much 'stress' that no local conservation is needed for avalanches to be large. Of course, the behaviour of the present forest-fire model is also very different from the model of [1, 3] quoted in the introduction where the doubly slow process triggering the relaxation is the

† At least in average, see [12]. For a different view based on simulations of a slightly different model on very small lattices, see [13].

propagation of fire fronts from very distant regions of the lattice.

In the case of sand-pile models, it is known that the behaviour is percolation-like in high ($\geqslant 6$) dimensions [2, 10]. We conjecture that the same will be true for the present model, but we have not made any attempt to study the model in dimensions greater than two.

Acknowledgments

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