

B15 Linear Dynamic Systems and Optimal Control

Example Paper 1

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Questions

1. Consider the amplifier of Figure 1, where $v_i(t)$ is the input voltage and $v_o(t)$ the output one. Denote by $v_{C_1}(t)$ and $v_{C_2}(t)$ the voltage across the capacitor C_1 and C_2 , respectively, and assume that the amplifier is ideal.

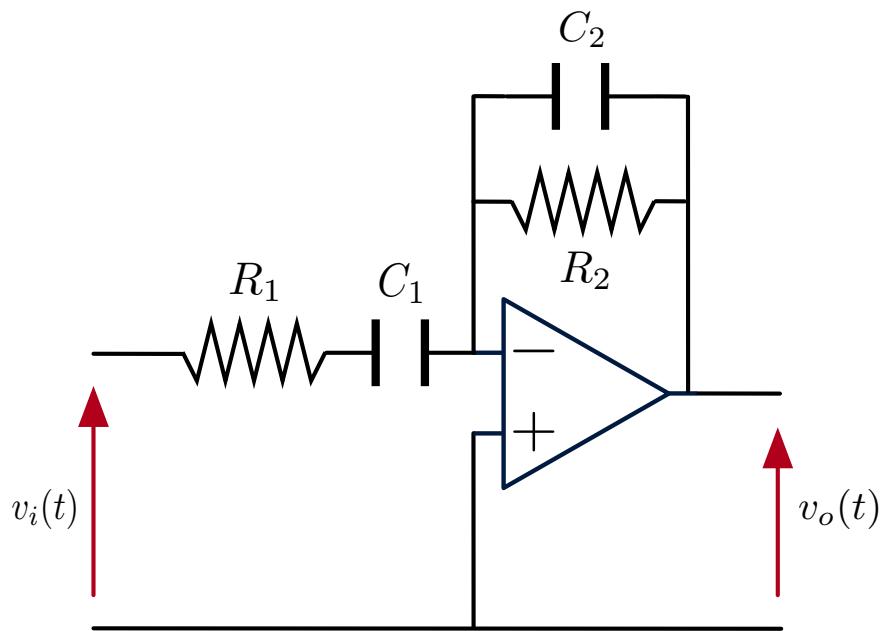


Figure 1: Amplifier circuit.

- (a) Derive the ordinary differential equations (ODEs) that capture the evolution of $v_{C_1}(t)$ and $v_{C_2}(t)$.
- (b) Provide a state space description of the amplifier circuit. What is the order of the resulting system?
- (c) Is the resulting system autonomous? Is it linear? Justify your answers.

2. Consider the following dynamical system

$$\ddot{z}(t) = 1 - \frac{1}{(z(t) + z^*)^2} u(t),$$

where $z^* > 0$ is a fixed parameter.

- (a) Find the constant input $u(t) = u^*$ for all $t \geq 0$ that renders 0 an equilibrium of the system, i.e., starting at $z(t) = 0$ the system does not move.
- (b) Use $x(t) = [z(t) \ \dot{z}(t)]^\top$ and $y(t) = z(t)$ as the state vector and system output, respectively. Write the given dynamical system in state space form. Is the resulting system linear?
- (c) Linearize the system around $x^* = [0 \ 0]^\top$ and the value for u^* computed in part (a).

3. Recall that the state transition matrix for linear time invariant (LTI) systems with starting time zero is given by

$$\Phi(t) = e^{At} = I + At + \frac{A^2 t^2}{2!} + \dots + \frac{A^k t^k}{k!} + \dots$$

Show that it satisfies the following properties:

- (a) $\Phi(0) = I$.
- (b) $\frac{d}{dt}\Phi(t) = A\Phi(t)$.
- (c) $\Phi(t)\Phi(-t) = \Phi(-t)\Phi(t) = I$.
What is the role of $\Phi(-t)$ in this case?
- (d) For any $t_1, t_2 \in \mathbb{R}$, $\Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2)$.

4. For each case below comment on whether matrix A is diagonalizable, and determine the matrix exponential e^{At} .

- (a) $A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$, where $\omega \neq 0$.

(b) $A = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix}$, where $\sigma \neq 0$.

(c) $A = \begin{bmatrix} \lambda_1 & \lambda_2 - \lambda_1 \\ 0 & \lambda_2 \end{bmatrix}$, where $\lambda_1, \lambda_2 \neq 0$.

(d) $A = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

5. The so called Wien oscillator is the circuit shown in Figure 2 with $k > 1$. Denote by $v_{C_1}(t)$ and $v_{C_2}(t)$ the voltage across the capacitor C_1 and C_2 , respectively, and assume that the amplifier is ideal.

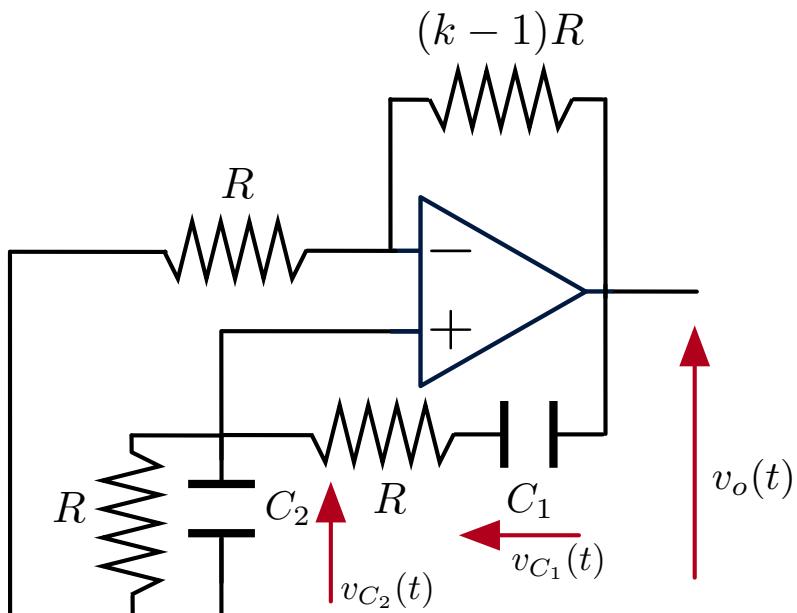


Figure 2: Wien oscillator.

- (a) Let $x(t) = \begin{bmatrix} v_{C_1}(t) \\ v_{C_2}(t) \end{bmatrix}$ denote the state vector, and $y(t) = v_o(t)$ the output of the Wien oscillator circuit (notice that there is no input).

Show that its state space representation is given by

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} -\frac{1}{RC_1} & \frac{1-k}{RC_1} \\ \frac{1}{RC_2} & \frac{k-2}{RC_2} \end{bmatrix} x(t), \\ y(t) &= \begin{bmatrix} 0 & k \end{bmatrix} x(t).\end{aligned}$$

- (b) If $C_1 = C_2 = C$, determine the range of values for the parameter k for which the resulting system is stable, asymptotically stable, or unstable.

6. Consider an LTI system whose state $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ evolves according to

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

where $u(t)$ is an external input.

- (a) Does there exist a control input u (as a function of time) such that we can drive the system state from $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $x(2\pi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$?
- (b) Consider the following piecewise constant input:

$$u(t) = \begin{cases} u_1 & \text{if } 0 \leq t < \frac{2\pi}{3}; \\ u_2 & \text{if } \frac{2\pi}{3} \leq t < \frac{4\pi}{3}; \\ u_3 & \text{if } \frac{4\pi}{3} \leq t \leq 2\pi. \end{cases}$$

Do there exist u_1 , u_2 and u_3 such that we can drive the system state from $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $x(2\pi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$?

Hint: Note that the system's "A" matrix is in the form of the one in Question 3(a).

7. (a) Consider a system whose state evolves according to

$$\begin{aligned}\dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= u(t),\end{aligned}$$

where $u(t)$ is a control input. Let $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. Do there exist u_1 and u_2 such that $u(t) = u_1 t + u_2$ can drive the system state from $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $x(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

(b) Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. If (A, B) is controllable, would (A^2, B) be controllable as well? Justify your answer.

8. Consider the following two systems:

$$\begin{aligned}\text{system } S: \quad &\dot{x}(t) = Ax(t) + Bu(t), \\ &y(t) = Cx(t) + Du(t),\end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$, and

$$\begin{aligned}\text{system } \hat{S}: \quad &\dot{\hat{x}}(t) = A^\top \hat{x}(t) + C^\top \hat{u}(t), \\ &\hat{y}(t) = B^\top \hat{x}(t) + D^\top \hat{u}(t),\end{aligned}$$

where $\hat{x}(t) \in \mathbb{R}^n$, $\hat{u}(t) \in \mathbb{R}^p$ and $\hat{y}(t) \in \mathbb{R}^m$.

(a) Show that S is controllable if and only if \hat{S} is observable.

(b) Show that S is observable if and only if \hat{S} is controllable.

9. Consider the transfer function

$$G(s) = \frac{1}{(s+1)(s+2)}.$$

(a) Is (A, B, C, D) with

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 0,$$

a realization of $G(s)$? Is the system with the matrices (A, B, C, D) above controllable and observable?

(b) Is (A, B, C, D) with

$$A = \begin{bmatrix} -6 & -11 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix}, \quad D = 0,$$

also a realization of $G(s)$? Is the system with the matrices (A, B, C, D) above controllable and observable?

(c) Comment on the effect that lack of controllability or observability may have on the realization of a transfer function.

10. **OPTIONAL:** Consider the controllability Gramian

$$W_c(t) = \int_0^t e^{A\tau} BB^\top e^{A^\top \tau} d\tau \in \mathbb{R}^{n \times n}.$$

Show that if it is invertible for a particular $\bar{t} > 0$, i.e., $W_c(\bar{t}) \succ 0$, then it is invertible for any $t > 0$, i.e., $W_c(t) \succ 0$, for all $t > 0$.

Note: The fact that $W_c(t) \succ 0$, for all $t \geq \bar{t}$ is easier to show compared to the case where $t < \bar{t}$.