C21 Nonlinear Systems
Example Paper

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Problems

1. (a) Consider the following autonomous, nonlinear system

$$\dot{x}(t) = -x^3(t) + \sin^4 x(t).$$

Determine the equilibrium points of this system.

Hint: Consider the points for which $x = \sin x$.

(b) Consider the following differential equation

$$\ddot{x}(t) + (x(t) - 1)^2 \dot{x}^5(t) + x^2(t) = \sin\left(\frac{\pi}{2} x(t)\right).$$

Write the system in state space form, using $(x_1(t), x_2(t)) = (x(t), \dot{x}(t))$ as the state vector. Deduce that $\dot{x}(t) = 0$ at an equilibrium point, and hence determine the values of $x(t)$ at equilibrium.

2. The rotational motion of a drifting spacecraft is described by the dynamics

$$\dot{\omega}_x = a\omega_y\omega_z, \qquad \dot{\omega}_y = -b\omega_x\omega_z, \qquad \dot{\omega}_z = c\omega_x\omega_y,$$

where $\omega_x, \omega_y, \omega_z$ are angular velocities measured in a coordinate frame attached to the spacecraft (see Figure 1); their dependency on time is not shown explicitly to ease notation. Parameters a, b, c are positive constants.

(a) Determine the equilibrium points of this system.

(b) Show that the equilibrium $(\omega_x^\star, \omega_y^\star, \omega_z^\star) = (0, 0, 0)$, corresponding to zero rotation, is stable.

Hint: Employ Lyapunov's direct method using a candidate Lyapunov function $V(\omega_x, \omega_y, \omega_z) = p\omega_x^2 + q\omega_y^2 + r\omega_z^2$, where the coefficients p, q and r are all positive, and satisy $ap - bq + cr = 0$.

(c) Consider any $\omega_0 > 0$. Verify that the function

$$V(\omega_x, \omega_y, \omega_z) = c\omega_y^2 + b\omega_z^2 + \left(2ac\omega_y^2 + ab\omega_z^2 + bc(\omega_x^2 - \omega_0^2)\right)^2,$$

satisfies $\dot{V}(\omega_x, \omega_y, \omega_z) = 0$ along the trajectories of the system. Using Lypunov's direct method with this candidate Lyapunov function, investigate the stability properties of any non-zero equilibrium point of the form $(\omega_x^\star, \omega_y^\star, \omega_z^\star) = (\pm\omega_0, 0, 0)$.

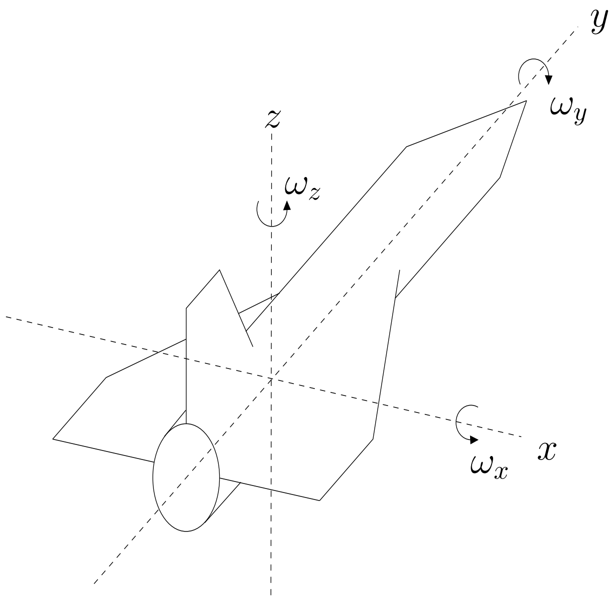

Figure 1: Pictorial illustration of the rotating spacecraft of Problem 2.

3. Consider the autonomous, nonlinear system

$$\begin{aligned}\dot{x}_1(t) &= x_2(t),\\ \dot{x}_2(t) &= -x_2(t)(x_1(t) - 1)^2 - x_1(t)(x_1^2(t) - 1).\end{aligned}$$

(a) Show that the only equilibria of this system are $(0, 0)$, $(1, 0)$, $(-1, 0)$.

(b) Using Lyapunov's indirect method comment on the stability properties of these equilibria.

4. Consider the same system with Problem 3, and the candidate Lyapunov function

$$V(x_1, x_2) = \frac{1}{4}x_1^2(x_1^2 - 2) + \frac{1}{2}x_2^2 + \frac{1}{4}.$$

(a) Using Lyapunov's indirect method show that the equilibrium points $(1, 0)$ and $(-1, 0)$ are stable.

Hint: Show that these equilibria constitute local minima of $V(x_1, x_2)$: check the gradient of V and compute the Hessian at these points.

(b) Let $\overline{S} = \Big\{(0,0), (-1,0), (1,0)\Big\}$ be the set containing all equilibria. Consider a big enough $c > 0$ such that the level-set

$$S_c = \Big\{(x_1, x_2) : \; V(x_1, x_2) \leq c\Big\},$$

contains $\overline{S}$. Justify whether S_c is compact and invariant.

(c) Apply La Salle's invariance principle to show that state trajectories tend to $\overline{S}$ as $t \to \infty$.

5. (a) Consider the scalar system (assume solutions exist and are unique)

$$\dot{x}(t) = -b(x(t)),$$

where b is a continuous nonlinear function such that $xb(x) > 0$ for all $x \neq 0$.

Choose a quadratic Lyapunov function and, using Lyapunov's direct method, show that $x^\star = 0$ is a globally asymptotically stable equilibrium point.

(b) Consider a two-state system of the form (assume solutions exist and are unique)

$$\dot{x}_1(t) = x_2(t),$$
$$\dot{x}_2(t) = -b(x_2(t)) - c(x_1(t)),$$

where b and c are continuous nonlinear functions such that

$$x_2 b(x_2) > 0, \text{ for all } x_2 \neq 0,$$
$$x_1 c(x_1) > 0, \text{ for all } x_1 \neq 0.$$

Show that $(x_1^\star, x_2^\star) = (0,0)$ is the only equilibrium point. Consider the candidate Lyapunov function

$$V(x_1, x_2) = \frac{1}{2} x_2^2 + \int_0^{x_1} c(x) \, \mathrm{d}x.$$

Using La Salle's invariance principle show that $(x_1^\star, x_2^\star)$ is locally asymptotically stable.

6. (a) The nonlinear circuit in the left panel of Figure 2 is described by the equations:

$$\dot{x}_1 = \frac{x_2}{L(x_2)},$$
$$\dot{x}_2 = -\frac{x_1}{C(x_1)} - \frac{R_1 x_2}{L(x_2)} + e,$$

where x_1 is the charge on the capacitor and x_2 is the magnetic flux in the inductor. Notice that the capacitance C depends on x_1, and the inductance L on x_2, in a continuous but potentially nonlinear manner. The resistance R_1 is time invariant and positive. Moreover, $C(x_1) > 0$ for all x_1, $L(x_2) > 0$ for all x_2, and $R_1 > 0$. The dependency of x_1, x_2 and e on time is not shown explicitly to ease notation.

Consider the candidate storage function

$$V_1(x_1, x_2) = \int_0^{x_2} \frac{x}{L(x)}\,\mathrm{d}x \; + \int_0^{x_1} \frac{x}{C(x)}\,\mathrm{d}x.$$

Show that the system with input $u = e$ and output $y = \dot{x}_1$ is strictly passive.

(b) Consider now the circuit in the right panel of Figure 2. Denote by x_1, x_2 the charge on the capacitor and the flux in the inductor in the left-branch of the circuit, and by x_3, x_4, the respective quantities in the right-branch of the circuit. Let $x = (x_1, x_2, x_3, x_4)$. Assume that switch S is closed, and notice that by Kirchhoff's current law $\dot{x}_1 + \dot{x}_3 = i$ (where i is the current shown in the figure).

Find a function $V(x)$ such that for all x,

$$V(x) \geq 0, \text{ and } \dot{V}(x) = ie - \frac{R_1}{L^2(x_2)}x_2^2 - \frac{R_2}{L^2(x_4)}x_4^2.$$

What does this imply about the passivity properties of the system?

(c) Consider the same setting with part (b), and a function V with these properties. Assume that the switch S opens up. Using La Salle's invariance principle determine the set of states towards which the system converges as $t \to \infty$.

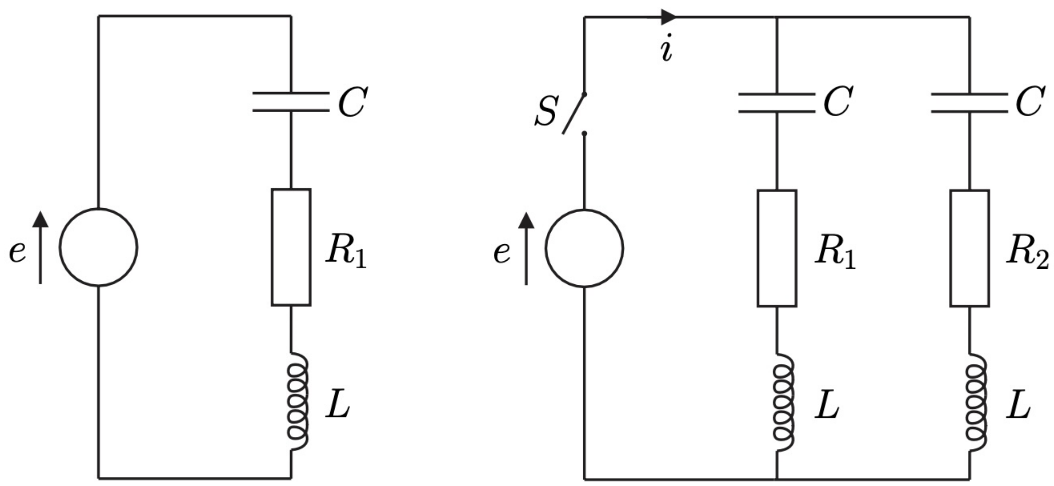

Figure 2: Left panel: Electric circuit of part (a); Right panel: Electric circuit of parts (b) and (c).

7. (a) Show that if there exist symmetric positive definite matrices P and Q satisfying

$$A^\top P + PA + 2\mu P = -Q, \text{ for some } \mu > 0,$$

then each eigenvalue $\lambda(A)$ of A satisfies $\mathrm{Re}\big[\lambda(A)\big] < -\mu$.

(b) Consider the transfer function

$$G(s) = \frac{1}{s^2 + s + 1}.$$

Is this transfer function strictly positive real? Justify your answer.

8. Consider a linear system with input u and output y. The system has a transfer function $G(s)$, with all its poles having negative real part. This system is to be controlled via feedback $u = -\phi(y)$, where ϕ is a static nonlinearity. For all $\omega \in \mathbb{R}$, $G(j\omega)$ lies within the bounds:

$$-1 < \mathrm{Re}\big[G(j\omega)\big] < 2, \qquad -2 < \mathrm{Im}\big[G(j\omega)\big] < 2.$$

(a) Show that the origin is an asymptotically stable equilibrium of the closed-loop for any function ϕ belonging to the sector $[0, 1]$ or $[-\frac{1}{3}, \frac{1}{2}]$.

(b) Does this imply that the origin will be an asymptotically stable equilibrium of the closed-loop system for all ϕ in the sector $[-\frac{1}{3}, 1]$? Justify your answer.