A priori data-driven robustness guarantees on strategic deviations from generalised Nash equilibria

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Abstract

In this paper we focus on noncooperative games with uncertain constraints coupling the agents' decisions. We consider a setting where bounded deviations of agents' decisions from the equilibrium are possible, and uncertain constraints are inferred from data. Building upon recent advances in the so called scenario approach, we propose a randomised algorithm that returns a nominal equilibrium such that a *pre-specified* bound on the probability of violation for yet unseen constraints is satisfied for an entire region of admissible deviations surrounding it—thus supporting neighbourhoods of equilibria with probabilistic feasibility certificates. For the case in which the game admits a potential function, whose minimum coincides with the social welfare optimum of the population, the proposed algorithmic scheme opens the road to achieve a trade-off between the guaranteed feasibility levels of the region surrounding the nominal equilibrium, and its system-level efficiency. Detailed numerical simulations corroborate our theoretical results.

Key words: Generalised equilibrium problem; Uncertain games; Randomized methods; Stochastic control and game theory; Multi-agent systems.

1 Introduction

The study of noncooperative games plays a significant role in a panoply of applications ranging from smartgrids [41] to communication [43] and social networks [2]. In these setups, agents can be modelled as self-interested entities that interact with each other and make decisions based on possibly misaligned individual criteria, while being subject to constraints (local or global) that restrict the domain of their choices. Even though a variety of systems can be analysed by means of deterministic game-theoretic tools [23,35,43], in many applications the decision making procedure is affected by uncertainty. A number of results in the literature have explicitly addressed uncertainty in a noncooperative setting, on the basis of specific assumptions on the probability distribution [14, 44] and/or the geometry of the sample space of the uncertainty [3, 25]; however, such assumptions can restrict the applicability of these methods.

To circumvent these limitations, recent developments adopt a data-driven perspective, focusing on the connection of game theory with the so called scenario approach [8]. This is based on the idea that an optimisation problem with constraints parametrised by an uncertain parameter—with fixed but possibly unknown support set and probability distribution—can be approximated by drawing samples of that parameter and solving the problem subject to the constraints generated by those samples only; this approximation is known as the *scenario program*. Standard results in the scenario approach [7, 10, 12] provide certificates on the probability that a new yet unseen constraint will violate the randomised solution obtained by solving the scenario program.

While the aforementioned results apply to uncertain *convex* optimisation problems, the works [11] and [13] paved the way towards the provision of data-driven robustness guarantees to solutions of more general nonconvex problems. In [21, 22, 34], these theoretical ad-

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vancements were leveraged for the first time in a game-theoretic context, for the formulation of distribution-free probabilistic feasibility guarantees for randomised Nash equilibria. These works provide guarantees for one specific equilibrium point (often assumed to be unique); this was extended in [36, 37], by providing *a posteriori* feasibility guarantees for the entire domain. Besides the game-theoretic context, alternative methodologies for set-oriented probabilistic feasibility guarantees have been proposed in the seminal works [5,15], which a priori characterise probabilistic feasibility regions constructed out of sampled constraints using statistical learning theoretic results. More recently, the so called probabilistic scaling [4, 31] has been proposed to obtain a posteriori guarantees on the probability that a polytope generated out of samples is a subset of some chance-constrained feasibility region. Following an approach similar to [36], the works [16, 17] deliver tighter probabilistic feasibility guarantees by focusing on variational-inequality (VI) solution sets.

The results above follow a standard approach in the game-theoretic literature, where a strict behavioural assumption—the so called *rationality*—is imposed on the players' decision making. Namely, the players are viewed as rational agents wishing to maximize their profit (expressed by some given cost function). However, studies have shown that this is unrealistic in practice [9, 27, 38, 45] and that agents usually exhibit a *boundedly rational* behaviour [39], i.e., their decisions can deviate from rationality due to individual biases, behavioural inertia, restricted computational power/time, etc. The consequences of this become relevant in engineering applications, as the human role in technical systems evolves beyond mere users and consumers to active agents, operators, decision-makers and enablers of efficient, resilient and sustainable infrastructures [30].

To bridge this gap between real-world applications and the cognate literature, here we study games with uncertain constraints, where deviations from a *nominal* equilibrium are explicitly considered. We follow a randomised approach to approximate the coupling constraints by means of data. In this more general setting, where deviations are considered, providing guarantees for a single solution is devoid of any meaning: indeed, repetition of the game might lead to a different solution in a neighbourhood around the nominal equilibrium, irrespective of the employed dataset. Technically speaking, this renders the identification of the data samples that support the solution (cf. sample compression [32]) a challenging task. Focussing on the class of generalised Nash equilibrium seeking problems [18], we contribute to the provision of data-driven robustness guarantees for the collection of possible deviations from the equilibrium as follows:

(1) Adopting a scenario-theoretic paradigm, we establish a methodology for the provision of *a posteriori* probabilistic feasibility guarantees for a domain around the randomised equilibrium of the game under study.
(2) We design a data-driven equilibrium-seeking algorithm that converges to a solution that meets an *a priori* defined level of probabilistic feasibility for a fixed admissible region surrounding this equilibrium. This can model possible deviations from a nominal equilibrium that the designer wishes to take into account when incentivising a certain operation profile. The strength of the provided feasibility guarantees depends on a prespecified quantity, which in turn can affect the location of the nominal equilibrium and the volume of the region for which these probabilistic guarantees hold. Furthermore, when the game under study admits a potential function—whose minimum coincides with some social welfare optimum—our methodology provides a new perspective for trading off the probabilistic feasibility of the region surrounding the nominal equilibrium and its system-level efficiency.

The rest of the paper is organized as follows. In Section 2 we provide fundamentals of game theory and the scenario approach which will be used as main ingredients for the subsequent developments. In Section 3.1 we show how the feasibility guarantees for a region around the game solution can be a posteriori quantified. In Section 3.2 we propose a data driven algorithm and prove its convergence to an equilibrium such that the considered neighbourhood of strategic deviations can satisfy prespecified probabilistic feasibility requirements. An illustrative example in Section 4 corroborates our theoretical analysis. Section 5 concludes the paper and presents future research directions. To streamline the presentation of our results, some proofs are deferred to the Appendix.

2 Preliminaries

Notation: All vectors are column unless otherwise indicated. We denote by $\mathbb{R}^n_+$ the nonnegative orthant in $\mathbb{R}^n$. When a matrix A is positive definite we write $A \succ 0$; similarly, positive semi-definitess is denoted as $A \succeq 0$. Note that our definition of (semi-)definiteness does not require the matrix to be symmetric. We denote by $\mathbf{0}_{q\times r}$ a $q\times r$ matrix full of zeros, by I_r the $r\times r$ identity matrix, and by $\mathbf{1}_r$ the vector of r ones; dimensions can be omitted when clear from the context. e_q is the unit vector whose q-th element is 1 and all other elements are 0, $\|\cdot\|_p$ the p-norm operator, and $(\cdot)_r$ denotes the r-th component of its vector argument. $\mathbb{B}_p(x,\rho) = \{y \in \mathbb{R}^d : \|y-x\|_p < \rho\}$ is the open p-normed ball centred at x with radius ρ; when p is omitted, any choice of norm is valid. For a set S, $|S|$ denotes its cardinality, while 2^S denotes its power set, i.e., the collection of all subsets of S. Finally, given $D \succ 0$, $\mathrm{proj}_{K,D}[x] := \arg\min_{y\in K}(y-x)^\intercal D(y-x)$ is the skewed projection of x onto the set K.

2.1 Games with uncertain constraints

We consider a population of agents with index set $\mathcal{N} = \{1, \dots, N\}$. The decision vector x_i of each agent $i \in \mathcal{N}$ takes value in the set $X_i \subseteq \mathbb{R}^n$, while $x = (x_i)_{i=1}^N \in X = \prod_{i=1}^N X_i \subseteq \mathbb{R}^{nN}$ is the global decision vector that is formed by concatenating the decisions of the entire population. The vector $x_{-i} \in \mathbb{R}^{n(N-1)}$ comprises all agents' decisions except for those of agent i. In our setup, the cost incurred by agent $i \in \mathcal{N}$ is expressed by a real-valued function $J_i(x_i, x_{-i})$ that depends on local decisions as well as on the decisions from other agents $j \in \mathcal{N} \setminus \{i\}$. In the following, with a slight abuse of notation, we can exchange x for (x_i, x_{-i}) to single out agent i's decision from the global decision vector. Furthermore, we consider *uncertain* constraints coupling the agents' decisions. These can be expressed in the form [1]

$$C_\delta = \{x \in X : g(x, \delta) \leq 0\},\ \delta \in \Delta, \tag{1}$$

where $g : \mathbb{R}^{nN} \times \Delta \to \mathbb{R}$ depends on some uncertain parameter δ taking values in a support set Δ according to a probability measure $\mathbb{P}$.

Feasible collective decisions under this setup can be found by letting every agent $i \in \mathcal{N}$ solve the following optimization program, where the decisions x_{-i} of all other agents are given,

$$\left.\begin{aligned} G:\ \min_{x_i \in X_i}\ & J_i(x_i, x_{-i}) \\ \text{subject to } & x_i \in \bigcap_{\delta \in \Delta} C_\delta^i(x_{-i}) \end{aligned}\right\} \forall i \in \mathcal{N}, \tag{2}$$

where $C_\delta^i(x_{-i}) = \{x_i \in X_i : g(x_i, x_{-i}, \delta) \leq 0\}$ denotes the projection of the coupling constraint on X_i for fixed x_{-i} and uncertain realization $\delta \in \Delta$. The collection of coupled optimization programs in (2) for all $i \in \mathcal{N}$ constitutes an *uncertain noncooperative game*; we denote it as G.

Note that G follows a worst-case paradigm, taking into account all possible coupling constraints that can be realised by variations of the uncertain parameter $\delta \in \Delta$. This can render the solutions of G rather conservative. Furthermore, it is in general not possible to compute a solution for G without an accurate knowledge of and/or additional assumptions on the support set Δ and the probability distribution $\mathbb{P}$. To circumvent these limitations, we follow a data-driven paradigm and approximate G by means of a finite number of samples drawn from Δ, namely the K-multisample $\boldsymbol{\delta}_K = (\delta_1, \dots, \delta_K) \in \Delta^K$. In the remainder of this document, we hold on to the standing assumption that these samples are independent and identically distributed (i.i.d.). Apart from this, no other knowledge on the support set Δ and the probability distribution $\mathbb{P}$ of the uncertain parameter is required. Then, for given multi-sample $\boldsymbol{\delta}_K$ and other agents' decisions x_{-i}, agent $i \in \mathcal{N}$ solves the optimization program

$$\left.\begin{aligned} G_K:\ \min_{x_i \in X_i}\ & J_i(x_i, x_{-i}) \\ \text{subject to } & x_i \in \bigcap_{k=1}^K C_{\delta_k}^i(x_{-i}) \end{aligned}\right\} \forall i \in \mathcal{N}. \tag{3}$$

Instead of considering all possible uncertainty realizations $\delta \in \Delta$ as in (2), we let the data encoded in $\boldsymbol{\delta}_K$ lead agents to their decision by solving (3). We refer to the collection of coupled optimization programs in (3) as the *scenario game* G_K, where the subscript K implies dependence on the drawn multi-sample $\boldsymbol{\delta}_K$. Under standard assumptions, a solution to the scenario game G_K exists and the problem is, in contrast to G, tractable using state-of-the-art equilibrium seeking algorithms.

2.2 Variational inequalities and game equilibria

Notably—under certain assumptions detailed next—solutions to the game G_K can be retrieved as solutions to a variational inequality (VI), for specific choices of the mapping $F : X \to \mathbb{R}^{nN}$ [18, Thm 3.9]:

$$\begin{aligned} \mathrm{VI}_K : \text{Find } & x^* \in \Pi_K \text{ such that} \\ & (x - x^*)^\intercal F(x^*) \geq 0 \text{ for any } x \in \Pi_K, \end{aligned} \tag{4}$$

where $\Pi_K := X \cap \bigcap_{k=1}^K C_{\delta_k}$ denotes the problem domain. A classic game solution concept, which encounters wide application in the literature, is the Nash equilibrium (NE) [33]. At a NE, no agent can decrease their cost by unilaterally changing their decision. Formally, this can be stated as follows.

Definition 1 *A point $x^* = (x_i^*, x_{-i}^*) \in \Pi_K$ is called a generalised Nash equilibrium (GNE) of G_K if, for all $i \in \mathcal{N}$,*

$$J_i\left(x_i^*, x_{-i}^*\right) \leq J_i(y_i, x_{-i}^*),\ \forall y_i \in X_i \cap \bigcap_{k=1}^K C_{\delta_k}^i(x_{-i}^*).$$

For our analysis, we rely on the following conditions:

Assumption 1 *For all $i \in \mathcal{N}$, $J_i(x_i, x_{-i})$ is convex and continuously differentiable in x_i for any fixed x_{-i}.*

Assumption 2 *(1) For any multi-sample $\boldsymbol{\delta}_K \in \Delta^K$, the domain Π_K is non-empty.*
(2) The set $X = \prod_{i=1}^N X_i$ is compact, polytopic and convex.

[1] This formulation can describe deterministic and/or local constraints as special cases.

(3) For any $\delta \in \Delta$, g is an affine function of the form $g(x,\delta) = a(\delta)^\intercal x - b(\delta)$, where $a : \Delta \to \mathbb{R}^{nN}$ and $b : \Delta \to \mathbb{R}$.

Under these assumptions, we can determine a GNE as in Definition 1 by solving (4) with

$$F(x) = F_{\mathrm{NE}}(x) := \begin{bmatrix} \nabla_{x_1} J_1(x_1, x_{-1}) \\ \vdots \\ \nabla_{x_N} J_N(x_N, x_{-N}) \end{bmatrix}. \qquad (5)$$

A class of problems of common interest can be modelled by the so called *aggregative* games [1,26,28], where the cost incurred by agents depends on some aggregate measure—typically the average—of the decision of the entire population. Formally, such a cost can be expressed in (3) by the real-valued function $J_i(x_i, \sigma(x))$, where the aggregate $\sigma : \mathbb{R}^{nN} \to \mathbb{R}^n$ is defined as the mapping $x \mapsto \frac{1}{N}\sum_{i=1}^N x_i$. A solution frequently linked to this class of games is the Wardrop equilibrium (WE), a concept akin to the NE but specifically defined in the context of transportation networks [6]. The variational WEs of G_K can be expressed by using $F(x) = F_{\mathrm{WE}}(x) := [\nabla_{x_i} J_i(x_i, z)_{|z=\sigma(x)}]_{i\in\mathcal{N}}$; notice that in this case the second argument of J_i is fixed and set to $\sigma(x)$, consistently with the notion of WE where agents neglect the impact of their decision on others.

We restrict the considered class of variational mappings as follows:

Assumption 3 *The mapping F is*

(1) α-strongly monotone, i.e., $(x-y)^\intercal(F(x)-F(y)) \geq \alpha\|x-y\|^2$ for any $x, y \in X$,
(2) L_F-Lipschitz continuous., i.e., $\|F(x) - F(y)\| \leq L_F\|x-y\|$ for any $x, y \in X$.

Assumptions 1 and 3 are standard in the game-theoretic literature [19, 43]. Assumption 2 is relatively mild; the affine form of the constraints is exploited in the proposed algorithm (see Section 3) for the convergence to an equilibrium bearing the desired robustness properties.

We point out that in general only a subset of solutions to G_K can be retrieved through (4): these are referred to as *variational equilibria*, and enjoy favourable properties over nonvariational ones, as with the former the coupling constraints' burden is equally split among agents [24,29]. The following lemma, adapted from [19, Thm. 2.3.3], formalises the connection between the solutions to VI_K and the GNEs (or GWEs) of G_K.

Lemma 1 *Under Assumptions 1, 2 and 3, VI_K has a unique solution that is also an equilibrium of G_K.*

For the considered class of VIs, several algorithms from the literature can be employed to obtain a variational equilibrium of G_K; see, e.g., [18, 35]. We remark that, even if not explicitly shown for ease of notation, any solution x^* to G_K is itself a function of the drawn multisample $\boldsymbol{\delta}_K \in \Delta^K$. Probabilistic feasibility guarantees for the unique solution of VI_K can then be provided both in an *a priori* and *a posteriori* fashion by resorting to the results in [21,22,34]. However, these results are tailored to the provision of probabilistic feasibility guarantees for a single point (namely the solution of a VI): any strategic deviation from the equilibrium is not covered by such guarantees. We cover this issue in Section 3. First, we provide some background on the scenario approach.

2.3 Basic concepts in the scenario approach

A fundamental notion in the scenario approach is the *probability of violation* of an uncertain constraint.

Definition 2 *(1) The probability of violation $\mathcal{V} : \mathbb{R}^{nN} \to [0,1]$ of a point $x \in \Pi_K$ is defined as the probability that a new yet unseen sample $\delta \in \Delta$ will give rise to a constraint C_δ (as defined in (1)) such that $x \notin C_\delta$, i.e.,*

$$\mathcal{V}(x) := \mathbb{P}\{\delta \in \Delta : x \notin C_\delta\}.$$

(2) The probability of violation $\mathbb{V} : 2^{\mathbb{R}^{nN}} \to [0,1]$ of a set $S \subseteq \Pi_K$ is defined as the worst-case $\mathcal{V}$ among all the points in S, i.e.,

$$\mathbb{V}(S) = \sup_{x\in S} \mathbb{P}\{\delta \in \Delta : x \notin C_\delta\}.$$

Formally, a data-driven decision-making process can be characterized by a mapping—the *algorithm*—that takes as input the data encoded by the samples and returns a set of decisions.

Definition 3 *An algorithm is a function $\mathscr{A} : \Delta^l \to 2^{\mathbb{R}^{nN}} \times \mathbb{R}^{nN}$ that takes as input an l-multisample and returns the pair (S_l^*, x^*), namely, a solution set S_l^* and a point $x^* \in S_l$.*

In the following, we interpret the above definition as context-dependent, in that the size l of the input multisample is admitted to vary—all else remaining fixed for a given algorithm $\mathscr{A}$. A key notion, strongly linked to that of algorithm, is the *minimal compression set* [32]. This concept springs from the observation that typically only a subset of the sampled data is relevant to a decision or set of decisions, and all other samples are redundant.

Definition 4 (Compression set) *Consider an algorithm $\mathscr{A}$ as in Definition 3. A subset of samples $I \subseteq \boldsymbol{\delta}_K$*

is called a compression *for* $\mathscr{A}(\boldsymbol{\delta}_K)$ *if* $\mathscr{A}(I) = \mathscr{A}(\boldsymbol{\delta}_K)$.[2] *As multiple subset of samples can exist that fulfil this property, the ones with the minimal cardinality are called* minimal *compression sets.*

If we feed the algorithm with the set of samples corresponding to a compression, then the same decision (in our case a set and a point in the set) will be returned as when we feed the algorithm with the entire multi-sample. The compression set is related to the notion of support samples that is presented below.

Definition 5 (Support sample) *A sample* $\delta_i \in \boldsymbol{\delta}_K$ *is a* support sample *if its removal changes the decision returned by* $\mathscr{A}$*, i.e., if* $\mathscr{A}(\boldsymbol{\delta}_K) \neq \mathscr{A}(\boldsymbol{\delta}_K \setminus \delta_i)$.

Notice that since the output of $\mathscr{A}$ is a pair, it follows directly from the previous definition that if a support sample is removed, at least one of the two elements of the pair will have to change for the algorithm's output to change as well. A sample whose removal changes the x^* component of $\mathscr{A}$ is called a support sample for x^*. We link the two previous concepts by imposing the following *non-degeneracy* assumption.

Assumption 4 *With* $\mathbb{P}^K$*-probability 1, if* I^* *is the set of support samples of* $(S_K^*, x^*) = \mathscr{A}(\boldsymbol{\delta}_K)$*, then* $\mathscr{A}(\boldsymbol{\delta}_K) = \mathscr{A}(I^*)$.

In other words, non-degeneracy implies that the solution returned by algorithm $\mathscr{A}$ when fed with all the drawn samples is the same as the one obtained by using only the support samples of the problem. Under such assumption, the set of support samples and the minimal compression set coincide—and the latter is unique. This avoids degenerate cases where there is an accumulation of constraints active at the solution.

3 Probabilistic feasibility of sets around equilibria

3.1 A first a posteriori result

Returning to the scenario game G_K in (3), we now consider a more general setup where agents are allowed to deviate from x^* following, e.g., unmodelled changes in their cost functions; while we suppose that these deviations are feasible with respect to the local constraints, we want to study the feasibility as regards the coupling constraints obtained through sampling. Specifically, the region in which agents' strategies can deviate from the nominal equilibrium is assumed to lie within

[2] With some abuse of notation, in the remainder the symbol $\boldsymbol{\delta}_K$ is interpreted as either the i.i.d. sample vector $\boldsymbol{\delta}_K \in \Delta^K$, or the set comprising its components, i.e., $\boldsymbol{\delta}_K = \{\delta_1, \dots, \delta_K\} \subseteq \Delta$, depending on the context.

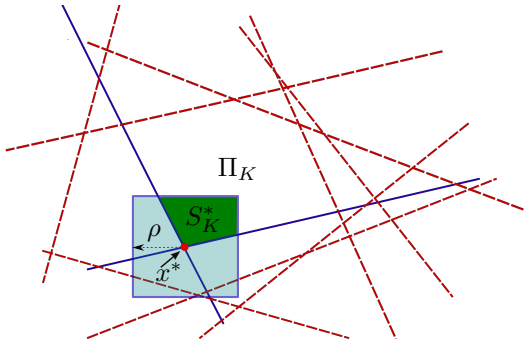


Fig. 1. Illustration of region S_K (in dark green) as the intersection of the set of deviations $\mathbb{B}_\infty(x^*, \rho)$ around the unique equilibrium x^* (red dot) with the domain Π_K.

a predefined open ball $\mathbb{B}(x^*, \rho)$, where $\rho > 0$ is a fixed radius that denotes the maximum possible distance of agents' deviations from x^*; the latter is assumed to be unique as per Lemma 1. As such, the region of interest is $S_K^* = \Pi_K \cap \mathbb{B}(x^*, \rho)$.

This is pictorially illustrated in Figure 1 using the ∞-norm (note that any other norm could have been used instead): an algorithm $\mathscr{A}$ (see Sec. 2.3) takes as input a multi-sample $\boldsymbol{\delta}_K$ and returns the region S_K^* around the solution $x^* \in \mathbb{R}^2$ of a game with two players whose decisions are defined as scalar quantities. We assume in this case that Π_K is shaped exclusively by sampled coupling constraints: only two constraints (in solid blue) are of support for the set, as the removal of any of the constraints depicted by dashed red lines will not change S_K^*. However, depending on the given multi-sample $\boldsymbol{\delta}_K$ and the location of the resulting equilibrium, more constraints could intersect the surrounding region $\mathbb{B}(x^*, \rho)$, thus increasing the cardinality of the support sample set.

We can quantify the number of samples that support S_K^* in an *a posteriori* fashion as established in Theorem 1. To this end, for a fixed confidence $\beta \in (0, 1)$, let the violation level be defined as a function $\epsilon : \{0, \dots, K\} \to [0, 1]$ satisfying

$$\epsilon(K) = 1 \text{ and } \sum_{i=0}^{K-1} \binom{K}{i} (1-\epsilon)^{K-i} = \beta, \qquad (6)$$

where K is the size of the multisample; see, e.g., [13, Eq. (7)].

Theorem 1 *Under Assumptions 1–4, consider some algorithm* $\mathscr{A}$ *returning a pair* (S_K^*, x^*) *where* S_K^* *is parametrised by* x^**. Fix a confidence parameter* $\beta \in (0, 1)$ *and a violation level* $\epsilon : \{0, \dots, K\} \to [0, 1]$ *that satisfies (6). We have that*

$$\mathbb{P}^K \left\{ \boldsymbol{\delta}_K \in \Delta^K : \mathbb{V}(S_K^*) > \epsilon(s^* + M) \right\} \leq \beta,$$

where s^* *is the number of samples that support the equi-*

librium x^* and M the number of facets of Π_K that intersect S_K^*.

Proof: Let (S_K^*, x^*) be the solution returned by $\mathscr{A}$ for some given $\boldsymbol{\delta}_K$, according to Definition 3. We aim at determining a compression set for $\mathscr{A}(\boldsymbol{\delta}_K)$, as this would in turn allow us to obtain the theorem's conclusion by means of Theorem 2 in [37]. Under Assumption 4, this is equivalent to determining an upper bound on the number of support samples for $\mathscr{A}(\boldsymbol{\delta}_K)$ (see Definition 5). Since by construction S_K^* is parametrised by x^*, the support samples of $\mathscr{A}(\boldsymbol{\delta}_K)$ are given by the union of (i) the samples that support the equilibrium x^*, and (ii) the samples that do not support x^*, but whose removal can still lead to a change of the region S_K^*. In the first case, removing a support sample for x^* would move the entire region (e.g., by its centre, as in Fig. 1); we denote the number of such samples as s^*. On the other hand, the removal of a sample corresponding to the second case yields a larger region S_K^*, leaving x^* unaffected; the number of such samples can be upper bounded by the M facets of Π_K that intersect S_K^*. Hence, the number of samples that form a compression set for $\mathscr{A}(\boldsymbol{\delta}_K)$ is bounded by $s^* + M$. Existence of a compression set I with a bound on its cardinality is sufficient for the application of Theorem 2 in [37]. The fact that for the minimal compression set $|I^*| \leq |I| \leq s^* + M$ always holds leads then to the statement of this theorem. ■

It is important to stress that the application of Theorem 1 is agnostic on the choice of the equilibrium seeking algorithm. To use the result of Theorem 1, one needs to quantify (an upper bound of) the number of samples s^* that support the randomised equilibrium x^* and (an upper bound of) the number M of coupling constraints that correspond to facets of S_K^*. While $s^* \leq nN$ under Assumptions 1–3,[3] an upper bound for M can in general only be achieved *a posteriori*, i.e., once $\boldsymbol{\delta}_K$ is sampled. Then an important question naturally arises: *When we allow for bounded deviations about some equilibrium, is it possible to devise an algorithm that converges to a solution such that an a priori defined robustness for all points in the region of admissible deviations is achieved?* We aim at answering this question next.

3.2 A priori probabilistic certificates

Consider the scenario game G_K and suppose that bounded deviations from the solution are allowed. We model such deviations as a ball of radius ρ around the equilibrium, as in Section 3.1. In contrast to the a posteriori nature of the result therein, our goal here is to achieve an a priori bound. Namely, we aim at establishing the main statement of Theorem 1 with a *prespecified* violation level, which does not depend on the given multi-sample $\boldsymbol{\delta}_K$. In other words, we seek a statement—holding with known confidence—of the form $\mathbb{V}(\Pi_K \cap \mathbb{B}(x^*, \rho)) < \bar{\epsilon}$, with $\bar{\epsilon} \in (0, 1)$ a priori fixed.

To achieve this, we build upon the previous conclusions, which expose a link between the probability of constraint violation and the number M of uncertainty samples constituting facets of Π_K that $\mathbb{B}(x^*, \rho)$ intersects. In particular, a monotonic relationship follows from (6): the smaller the number M of facets of the uncertain domain intersected by $\mathbb{B}(x^*, \rho)$, the better, i.e., less conservative, the theoretical feasibility guarantees on constraint violation for the strategies belonging to the feasible region S_K^* surrounding the equilibrium. Moreover, in certain cases (as illustrated next) a smaller value of M is associated with a larger region for which the guarantees of Theorem 1 hold—due to a smaller portion of $\mathbb{B}(x^*, \rho)$ being cut off by the intersection with Π_K. This motivates us to study the role of M as a modulating parameter for the robustness of the feasibility certificates offered for the region S_K^*, as well as the extent of deviation from the nominal equilibrium covered by such certificates. These concepts are leveraged in the algorithm proposed next.

3.2.1 GNE-seeking algorithm with a priori robustness guarantees

We consider an iterative scheme to determine a solution of VI_K in (4). In particular, since we address a NE problem characterised by coupling constraints, we build our Algorithm 1 upon a primal-dual scheme, where constraint satisfaction is achieved by the use of Lagrange multipliers. As deterministic constraints do not play a role in the evaluation of the robustness guarantees, suppose for ease of exposition that Π_K only comprises uncertain coupling constraints. Let $A \in \mathbb{R}^{m \times nN}$ and $b \in \mathbb{R}^m$ such that

$$\Pi_K = \{x \in X : Ax \leq b\}, \tag{7a}$$
$$\|a_\ell\|_2 = 1,\ \ell = 1, \ldots, m, \tag{7b}$$

where $a_\ell{}^\intercal$ denotes the ℓ-th row of A. Eq. (7) is the irredundant H-representation of the polytopic feasibility region Π_K defined in (4), where the rows of matrix A are unit vectors. Property (7b) is key to the second statement in Lemma 2. It entails no loss of generality, since for any A, b forming an equivalent H-representation of Π_K, (7) can be obtained by normalising each row of A and the corresponding component of b by the row-vector norm. Thus, the pair (A, b) encodes the set of randomised coupling constraints that constitute facets of Π_K. Formally, $A : \Delta^K \to \mathbb{R}^{m \times nN}$ and $b : \Delta^K \to \mathbb{R}^m$ are mappings from the K-multisample to the space of real $m \times nN$ matrices and m-dimensional vectors, respectively. Although we do not reflect this in the notation for simplicity, we will consider this to be implicit in the remainder of the paper.

[3] By arguments similar to those in [37] it can be shown that a tighter bound $s^* \leq n$ holds for the game G_K in case coupling constraints only concern the aggregate variable; see also [42].

Algorithm 1 A priori robust GNE seeking algorithm

Require: $y^{(0)}, \rho \in \mathbb{R}_+, \boldsymbol{\delta}_K \in \Delta^K, M \leq m, \xi \geq 0$
1: $\kappa \leftarrow 0$
2: **repeat**
3: $\quad y^{(\kappa+1)} \leftarrow \text{proj}_{X\times\mathcal{M},D}\left[y^{(\kappa)} - D^{-1}T(y^{(\kappa)}, \rho, M)\right]$
4: $\quad \kappa \leftarrow \kappa + 1$
5: **until** $\|y^{(\kappa+1)} - y^{(\kappa)}\| \leq \xi$
6: $y^* \leftarrow y^{(\kappa+1)}$
7: **return** y^* and $\Pi_K \cap \mathbb{B}(x^*, \rho)$

Now, to understand the mechanism underlying Algorithm 1, first note that tightening an affine constraint by a distance corresponding to the radius ρ is equivalent to preventing the original constraint from intersecting $\mathbb{B}(x^*, \rho)$. Based on this and the idea illustrated in previous paragraphs, we leverage the information carried by Lagrange multipliers to control the intersection between Π_K and $\mathbb{B}(x^{(\kappa)}, \rho)$ while the algorithm is running. In particular, all but the coupling constraints that correspond to the M largest multipliers are tightened in Algorithm 1 by a distance ρ; as a result, the number of facets of Π_K intersecting $\mathbb{B}(x^*, \rho)$ is at most M. This enables us to obtain an *a priori* estimate on the number of support samples, that in turn allows us to provide a priori bounds on $\mathbb{V}(\Pi_K \cap \mathbb{B}(x^*, \rho))$ (see Theorem 3).

It is worth pointing out that the constraints corresponding to the M largest multipliers contribute the most in reducing the nominal cost if x^* is allowed to lie at their boundaries. It can therefore be understood that while smaller values for M can result in a more robust—and possibly larger—region S_K^*, they can also move the location of the nominal equilibrium x^* to a less efficient point towards the interior of Π_K. As we will demonstrate numerically in the sequel, this is indeed the case with *potential* games [20].

We build upon the class of asymmetric projection algorithms (APA) [19, Ch. 12] to seek a solution of the game G_K. This involves an iterative scheme, whose trajectory is characterised by the dynamics in line 3 of Algorithm 1, where $y := (x^\intercal, \mu^\intercal)^\intercal \in \mathbb{R}^{nN+m}$ is the concatenation of the global decision vector x and the Lagrange multipliers $\mu = (\mu_\ell)_{\ell=1}^m \in \mathcal{M} \subseteq \mathbb{R}_+^m$, $D \succ 0$ is the so called *asymmetric projection* matrix, and the mapping $T : \mathbb{R}^{nN} \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{N} \to \mathbb{R}^{nN+m}$ is given by

$$T(y, \rho, M) = \begin{bmatrix} F(x) + A^\intercal \mu \\ -\left(Ax - b + Q(\mu, M)\boldsymbol{\rho}\right) \end{bmatrix}, \tag{8}$$

where $\boldsymbol{\rho} = c\rho \mathbf{1}_m$, and c, ρ, M are fixed during the execution of the algorithm. Note that T follows from the primal-dual conditions of the game solution; see [18, Sec. 4.2], [19, Sec. 1.4.1]. In (8), F is defined as in Section 2.2, and A, b are as in (7). Moreover, Algorithm 1 relies on the mapping $Q : \mathbb{R}_+^m \times \mathbb{N} \to \{0,1\}^{m\times m}$ which allows convergence to possibly different nominal solutions x^*, according to the specified parameters M and ρ. Q—formally introduced in the next subsection—allows to perform constraint tightening along the trajectory of the algorithm, following the concepts illustrated above.

As illustrated in Figure 2, $\mathcal{M}$ is the union of a finite number of disjoint convex sets, hence the projection in line 3 of Algorithm 1 is computationally viable (e.g., projecting on each convex subset of the union and then setting $y^{(\kappa+1)}$ to be the solution among these projections that results in the minimum distance from $y^{(\kappa)} - D^{-1}T(y^{(\kappa)}, \rho, M)$). Finally, it should be noted that Algorithm 1 could be performed in a distributed manner, although this is outside the scope of the present work.

3.2.2 Constraint tightening via mapping Q

We define the mapping Q as

$$Q(\mu, M) = P^\intercal(\mu) R(M), \tag{9}$$

where

- $P : \mathbb{R}^m \to \{0,1\}^{m\times m}$ returns a permutation matrix such that $P(\mu)\mu$ is the vector composed by the elements of μ arranged in decreasing order.
- $R : \mathbb{N} \to \{0,1\}^{m\times m}$ takes as input the number of coupling constraints $M \leq m$ we allow $\mathbb{B}(x^*, \rho)$ to intersect with and returns as output the matrix

$$R(M) = \left[\begin{array}{c|c} \mathbf{0}_{m\times M} & \begin{matrix} 0_{M\times m-M} \\ I_{m-M} \end{matrix} \end{array}\right]. \tag{10}$$

Compatibly with the definition of $P(\cdot)$, $R(M)P(\mu)\boldsymbol{\rho} = (\mathbf{0}_M{}^\intercal \; c\rho \mathbf{1}_{m-M}{}^\intercal)^\intercal = R(M)\boldsymbol{\rho}$, where the last equality holds since all components of $\boldsymbol{\rho}$ are equal.

It can be seen from (8) and the above definition that $Q(\cdot, M)$ allows to tighten the constraints corresponding to the smallest $m - M$ multipliers. In particular, following the discussion in Section 3.2.1, we consider tightening these constraints by an amount equal to the radius of the sphere that circumscribes $\mathbb{B}(x^*, \rho)$, for any choice of norm. This amount is equal to $\boldsymbol{\rho}_\ell = c\rho\|a_\ell\|_2 = c\rho$, where the last equality is due to (7b); $c = 1$ if $\mathbb{B}(\cdot, \rho)$ is expressed by a p-norm with $p \leq 2$, $c = \sqrt{n}$ otherwise. Conversely, at most M constraints can intersect $\mathbb{B}(x^*, \rho)$ upon convergence of the algorithm. Let $\mathcal{L}(M) \subseteq \{1, \ldots, m\}$ contain the indices of the M largest multipliers. Then, $\ell \in \mathcal{L}(M) \Leftrightarrow (Q(\mu, M)\boldsymbol{\rho})_\ell = 0$, and the second block row of T in (8) expresses

$$\begin{cases} a_\ell{}^\intercal x \leq b_\ell & \text{if } (Q(\mu, M)\boldsymbol{\rho})_\ell = 0, \\ a_\ell{}^\intercal x \leq b_\ell - c\rho & \text{if } (Q(\mu, M)\boldsymbol{\rho})_\ell = c\rho. \end{cases} \tag{11}$$

The following example further clarifies this concept.

Illustrative example: Suppose Π_K is composed of 3 uncertain coupling constraints and that we allow the region of strategic deviations $\mathbb{B}(\cdot,\rho)$ to intersect at most $M = 1$ of them. It follows from (10) that $R(M) = \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$. Now, suppose that at some iteration κ of Algorithm 1 the multiplier vector is $\mu^{(\kappa)} = (\mu_\ell^{(\kappa)})_{\ell=1}^3$ with

$$\mu_2^{(\kappa)} > \mu_1^{(\kappa)} > \mu_3^{(\kappa)}. \tag{12}$$

Then, $P(\mu^{(\kappa)}) = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$ is a permutation matrix such that $P(\mu^{(\kappa)})\mu^{(\kappa)} = (\mu_2^{(\kappa)}\ \mu_1^{(\kappa)}\ \mu_3^{(\kappa)})^\intercal$. Finally, we have $Q(\mu^{(\kappa)}, M)\boldsymbol{\rho} = P^\intercal(\mu^{(\kappa)})R(M)\boldsymbol{\rho} = (c\rho\ 0\ c\rho)^\intercal$, where $P^\intercal(\cdot)$ applies the correct ordering to the vector $R(M)\boldsymbol{\rho}$. Therefore, in case (12) continues to hold for all $j \geq \kappa$, the region $\mathbb{B}(x^{(j)},\rho)$ will intersect the constraint associated to the largest multiplier from some iteration $\hat{j} \geq \kappa$ until convergence.

3.3 Convergence analysis and main result

In this section we show convergence for Algorithm 1. Before proceeding, we provide details on the set $\mathcal{M} \subseteq \mathbb{R}_+^m$ where the Lagrange multipliers updates are projected (line 3, Algorithm 1). Let $\mathcal{Z} := [\zeta, +\infty) \cup \{0\}$, for some small $\zeta > 0$, i.e., $\mathcal{Z} \subset \mathbb{R}$ contains all nonnegative scalars which take value greater than ζ when nonzero.

Assumption 5 *$\mathcal{M}$ is compact and admits the form*

$$\mathcal{M} := \{\mu \in \mathbb{R}^m : (P(\mu)\mu)_{\ell+1} < (P(\mu)\mu)_\ell - \zeta, \\ \forall \ell = 1,\dots,m-1\} \cap \mathcal{Z}^m. \tag{13}$$

Recalling that $P(\mu)\mu$ rearranges the multipliers in descending order, the set $\mathcal{M}$ contains all vectors where the difference between every pair of strictly positive components—and the distance of the smallest of these from zero—exceeds ζ. We note that (13) can be expressed as the disjoint union of $q = m! + m + 1$ convex subsets of $\mathbb{R}_+^m$, each of which we denote in the following as $\mathcal{M}_j$, i.e., $\mathcal{M} = \bigcup_{j=1}^q \mathcal{M}_j$. Figure 2 provides an illustration of this set for the case $m = 3$.

Restricting the Lagrange multipliers to $\mathcal{M}$ facilitates convergence of the algorithm in cases where the VI$(X \times \mathcal{M}, T)$ has multiple solutions. In particular, it (i) alleviates discontinuity issues of the mapping T, and (ii) endows the latter with desired nonexpansiveness properties (see proof of Lemma 3). In the numerical implementation of the algorithm, ensuring $\mu \in \mathcal{M}$ can possibly introduce small perturbations in the multipliers—compared to standard formulations where $\mu \in \mathbb{R}_+^m$—which in turn could produce a slight violation of the constraints (this can be controlled through the magnitude of ζ). We further note that assuming compactness of $\mathcal{M}$ is only required for the proof of Theorem 2 below; the assumption can be numerically satisfied by intersecting (13) with an arbitrarily large compact set. While necessary for a formal statement of convergence, we note that in many practical instances, including our numerical study, the desired solution can be attained without the need of imposing such an assumption on $\mathcal{M}$.

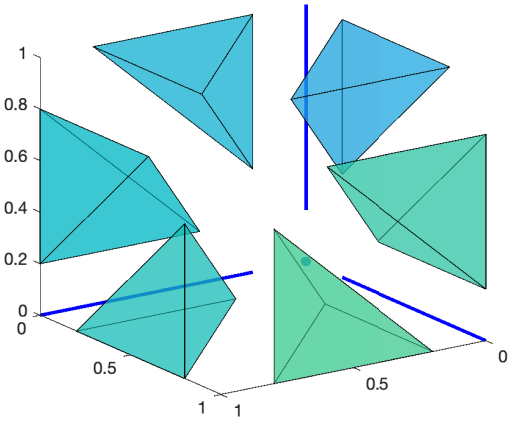


Fig. 2. Illustration of the domain $\mathcal{M}$ of the Lagrange multipliers associated to the coupling constraints, for the case $\zeta = 0.2$ and $m = 3$. This results in $q = 10$ convex subsets, including the origin and a portion of the axes.

We are now ready to introduce the following lemmas as an intermediate step towards proving convergence of Algorithm 1.

Lemma 2 *Define T as in (8)–(9), where A, b satisfy (7). Then, for any $\mu, \mu' \in \mathcal{M}$, $\mu \neq \mu'$, there exists an integer $0 \leq h \leq M$ such that*

$$(\mu - \mu')^\intercal (Q(\mu, M) - Q(\mu', M))\boldsymbol{\rho} \leq -h\zeta c\rho. \tag{14}$$

The construction of h can be found in the proof of Lemma 2; nonetheless, its exact value does not play a role in the subsequent analysis.

Lemma 3 *Consider T as in (8)–(9), where A, b satisfy (7) and $\mathcal{M} = \bigcup_{j=1}^q \mathcal{M}_j$ as in (13). For each $j = 1,\dots,q$, let VI$(X \times \mathcal{M}_j, T)$ denote the VI problem defined by the map T restricted to the subdomain $X \times \mathcal{M}_j$. Under Assumptions 1–3 the following holds:*

(1) T is continuous on $X \times \mathcal{M}$.
(2) Let

$$D = \begin{bmatrix} \frac{1}{\tau} I_{nN} & 0 \\ -2A & \frac{1}{\tau} I_m \end{bmatrix}, \tag{15}$$

and set $\tau > 0$ *such that*

$$\tau < \min\left\{ \frac{-L_F^2+\sqrt{L_F^4+4\alpha^2\|A\|^2}}{2\alpha\|A\|^2}, \right. \\ \left. \frac{-\rho(1+\|A\|^2)+\sqrt{\rho^2(1+\|A\|^2)^2+16\zeta^2\|A\|^2}}{4\zeta\|A\|^2} \right\}. \tag{16}$$

Then, for any $j = 1, \dots, q$*, Algorithm 1 converges to a solution of VI*$(X \times \mathcal{M}_j, T)$*, when the gradient step in line 3 is projected on the corresponding subdomain, for any* $y^{(0)} \in X \times \mathcal{M}_j$*.*

The second part of Lemma 3 provides an admissible range of values for τ such that Algorithm 1 converges to a solution of $\mathrm{VI}(X \times \mathcal{M}_j, T)$ if at each iteration the projection in line 3 is performed on the (convex) subdomain $\mathcal{M}_j \subset \mathcal{M}$, $j \in \{1, \dots, q\}$. However, we are interested in establishing convergence on the entire domain $\mathcal{M}$, so at each iteration the projected solution might belong to a different subdomain $\mathcal{M}_j$, $j \in \{1, \dots, q\}$. This does not trivially follow from the second part of Lemma 3; therefore, we capitalize on Lemmas 2 and 3 to establish an additional condition on τ such that Algorithm 1 retrieves a solution of $\mathrm{VI}(X \times \mathcal{M}, T)$. This is formally established in the following theorem.

Theorem 2 *Consider Assumptions 1, 2, 3, and 5. Fixed* $0 \leq M \leq m$*, assume the domain* Π_K *is nonempty for any of the* $\binom{m}{M}$ *combinations of constraints tightened as in* (11)*. Let* $D \succ 0$ *be defined as in* (15)*, where* τ *satisfies* (16) *and*

$$\tau < \frac{-(\bar{C}+\bar{R})+\sqrt{(\bar{C}+\bar{R})^2+2\zeta\bar{R}}}{2\bar{R}}, \tag{17}$$

where $\bar{R} = \max\left\{ \sup_{x\in X} \sup_{\mu\in\mathcal{M}} \|2A(F(x)+A^\intercal\mu)\|, \sup_{x\in X} \|Ax-b\| \right\}$ *and* $\bar{C} = c\rho\sqrt{m-M}$*.*

Then Algorithm 1 converges to a solution of VI$(X \times \mathcal{M})$ *for any given multi-sample* $\boldsymbol{\delta}_K \in \Delta^K$ *and initial condition* $y^{(0)} \in X \times \mathcal{M}$*.*

Note that as $\mu^{(\kappa)} \to \mu^*$, we have $Q(\mu^{(\kappa)}) \to Q(\mu^*) =: Q^*$. Then, the solution returned by Algorithm 1 is the equilibrium of a variant of G_K with $m - M$ tightened constraints—as it follows from (11) with $Q(\mu)$ replaced by Q^*. The next result accompanies the region $S_K^* = \Pi_K \cap \mathbb{B}(x^*, \rho)$ of strategic deviations from the equilibrium x^* with *a priori* probabilistic feasibility guarantees that can be tuned by means of M. It should be noted that Theorem 2 establishes that there exists a choice of τ to guarantee convergence of Algorithm 1. The exact admissible range for τ—albeit explicit via (16) and (17)—can be difficult to quantify due to $\bar{R}$. Numerical evidence suggests that selecting a small enough value is sufficient to achieve convergence.

Theorem 3 *Consider Assumptions 1–5. Let* x^* *and* $S_K^* = \Pi_K \cap \mathbb{B}(x^*, \rho)$ *be returned by Algorithm 1; fix* $\bar{\epsilon} \in (0, 1)$ *and* M*. We then have that*

$$\mathbb{P}^K\left\{ \boldsymbol{\delta}_K \in \Delta^K : \mathbb{V}(S_K^*) \leq \bar{\epsilon} \right\} \\ \geq 1 - \sum_{i=0}^{nN+M-1} \binom{K}{i} \bar{\epsilon}^i (1-\bar{\epsilon})^{K-i}. \tag{18}$$

By Definition 2, Theorem 3 guarantees that for any point in S_K^*, the probability of constraint violation is bounded by $\bar{\epsilon}$, with confidence at least $1 - \sum_{i=0}^{nN+M-1} \binom{K}{i} \bar{\epsilon}^i (1-\bar{\epsilon})^{K-i}$. The dependence of this term on M gives us an additional degree of freedom in trading the robustness of the solution for its associated probabilistic confidence. The choice of M can also have an effect on the size of S_K^*, as well as on the location of x^*, thus resulting in a trade-off between performance and robustness.

For the case in which the coupling constraints concern exclusively the aggregate variable, it can be shown that the support rank for all points in the region S_K^* is upper-bounded by $n + M - 1$. This allows to state (18) with a much higher confidence of $1 - \sum_{i=0}^{n+M-1} \binom{K}{i} \bar{\epsilon}^i (1-\bar{\epsilon})^{K-i}$; for details, we refer the reader to [37] and [42].

4 Numerical example

Consider a game with N agents whose decisions are subject to deterministic local constraints and uncertain coupling constraints on the aggregate decision:

$$\left.\begin{array}{l} \min\limits_{x_i \in X_i} \; x_i{}^\intercal (C\sigma(x) + d) \\ \text{subject to } \; \underline{b}_{\delta_k} \leq \sigma(x) \leq \bar{b}_{\delta_k}, \\ \qquad\qquad\qquad\qquad k = 1, \dots, K \end{array}\right\} \; \forall i \in \mathcal{N}, \tag{19}$$

where $C \succ \alpha I_n$, for some $\alpha > 0$, and $d \in \mathbb{R}^n$. We impose no knowledge of Δ and $\mathbb{P}$; we rely instead on a scenario-based approximation of the game, whereby each sample $\delta_k \in \boldsymbol{\delta}_K$ gives rise to $\underline{b}_{\delta_k}, \bar{b}_{\delta_k}$. Eq. (19) is an *aggregative* game in the form of (3).

In this instance, we assume each agent's action has negligible effect on the aggregate, and accordingly consider a WE-seeking problem. Following the definition of F_{WE} (Sec. 2.2), we get $F(x) = F_{\mathrm{WE}} = [C\sigma(x) + c]_{i\in\mathcal{N}}$. It can be verified that F is Lipschitz continuous and strongly monotone with respect to σ: by [19, Thm. 2.3.3], (19) admits a unique aggregate equilibrium $\sigma^* = \sigma(x^*)$. [4]

[4] We note that this case slightly transcends the conditions in Theorem 2, as F does not comply with Assumption 3-(1). Convergence of Algorithm 1 (following from the nonexpan-

We employ Algorithm 1 to seek a WE x^* such that, by fixing M, a prespecified theoretical violation level is guaranteed for the set $\Pi_K \cap \mathbb{B}(x^*, \rho)$. Due to uniqueness of σ^*, all sets $\mathbb{B}(x^*, \rho)$—parametrised by any x^* solving (19)—are projected into the unique ball $\mathbb{B}(\sigma^*, \rho/N)$ in the aggregate space. Also note that by definition of σ, at most n non-redundant samples will contribute to define the domain $\Pi_K^\sigma := \frac{1}{N}(\mathbf{1}_N{}^\intercal \otimes I_n)X \cap \left(\bigcap_{k=1}^K C_{\delta_k}\right)$ in (19). For the derivation of the robustness guarantees, we can thus restrict our attention to $S_K^* = \Pi_K^\sigma \cap \mathbb{B}(\sigma^*, \rho/N) \subseteq \mathbb{R}^n$. As remarked at the end of Section 3.3, we can apply (18) from Theorem 3 with tighter confidence, since in this case the support rank for all points belonging to S_K^* is bounded by $n+M-1$. For the case $n = 2, N = 50$, and for different choices of M, Figure 3 depicts the iterates $\{\sigma(x^{(\kappa)})\}$, $\kappa = 1, 2, \ldots$, generated by our algorithm, i.e., the projection of the algorithm iterates on the space Π_K^σ. It can be observed how the region S_K^* changes as the value of M is modified.

It is worth noting that in this case $F(x)$ is *integrable*—this can be inferred by [19, Thm. 1.3.1] since the Jacobian of the game is symmetric, i.e., $\nabla_x F(x) = \nabla_x F(x)^\intercal$. Therefore, a WE x^* can also be obtained by solving

$$\begin{aligned} &\min_{x \in X} \ \sigma(x)^\intercal C \sigma(x) + d^\intercal \sigma(x) \\ &\text{subject to } \underline{b}_{\delta_k} \leq \sigma(x) \leq \overline{b}_{\delta_k}, \quad k = 1, \ldots, K. \end{aligned} \tag{20}$$

In other words, this game admits a *potential function* $E(x) := \sigma(x)^\intercal C \sigma(x) + d^\intercal \sigma(x)$, whose minimizers correspond to a WE. E can be interpreted as the total cost incurred by the population of agents, and its minimization leads to the optimum social welfare. The contour lines of E are depicted in Figure 3: since x^* minimises $E(\cdot)$, σ^* lies on the contour associated to the minimum value of E within the feasible domain. Lower values of M result in larger regions for which guarantees are provided. Figure 4 shows how the sequence $\{E(x^{(\kappa)})\}_{\kappa=1,2,\ldots}$ converges to the minimum *potential* within the possibly tightened feasibility region. It can be observed how in this case the efficiency of the equilibrium decreases as smaller values of M are chosen. The three panels in Figure 4 show the trade-off between system level efficiency and the guaranteed robustness levels. The lower the value of M, the lower the empirical constraints violation—corresponding to a better confidence bound in the right-hand side of (18).

5 Concluding remarks

This work proposes a data-driven equilibrium-seeking algorithm such that probabilistic feasibility guarantees are provided for a region surrounding a game equilibrium. These guarantees are a priori and the region that is accompanied with such a probabilistic certificate is tunable. For games that admit a potential function, the proposed scheme is shown to achieve a trade-off between cost and the level of probabilistic feasibility guarantees. In fact, our scheme returns the most efficient equilibrium such that the predefined guarantees are achieved. Proving this conjecture is left for future work.

6 Appendix

6.1 Proof of Lemma 2

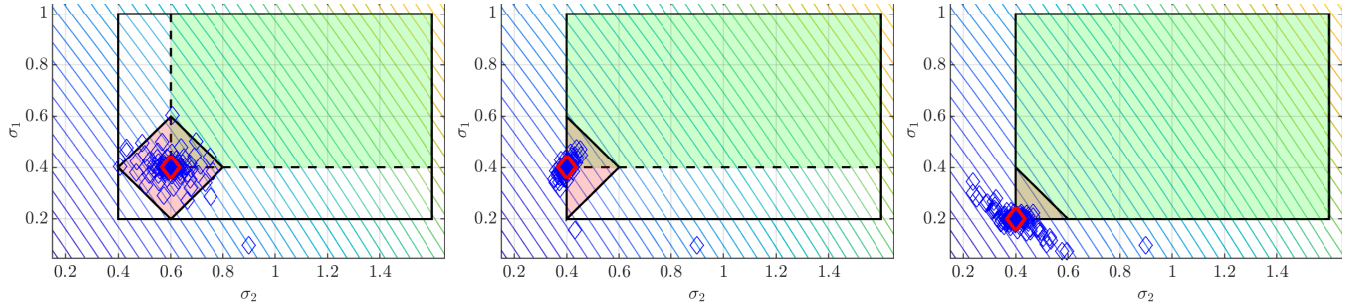
Let μ, z be arbitrary vectors in $\mathcal{M}$ and, as in the proof of Lemma 3, define $\vec{\mu}, \vec{z}$ as the vectors composed by rearranging the elements of μ, z in decreasing order. According to this arrangement, let $\mathcal{I}_\mu = \{i_1, i_2, \ldots, i_m\}$ be the ordered set of indices of μ, i.e., $i_k : \mu_{i_k} = \vec{\mu}_k$, $k = 1, \ldots, m$; as a result, i_1 and i_m will be the indices of the largest and smallest components of μ, respectively. Applying a similar definition to z, we denote the corresponding set $\mathcal{I}_z := \{j_1, j_2, \ldots, j_m\}$. Then, the first M indices in $\mathcal{I}_\mu$ and $\mathcal{I}_z$, denoted as $\mathcal{L}_\mu$ and $\mathcal{L}_z$, respectively, are relative to the constraints not tightened by the application of $Q(\cdot, M)$. In other words, for all $\ell \in \mathcal{L}_\mu$, $(Q(\mu, M)\boldsymbol{\rho})_\ell = 0$—and similarly for z. Vice versa, the complementary sets $\mathcal{L}_\mu^c = \mathcal{I}_\mu \setminus \mathcal{L}_\mu$ and $\mathcal{L}_z^c = \mathcal{I}_z \setminus \mathcal{L}_z$ are such that for all $\ell \in \mathcal{L}_\mu^c$, $(Q(\mu, M)\boldsymbol{\rho})_\ell = c\rho$, and for all $\ell \in \mathcal{L}_z^c$, $(Q(z, M)\boldsymbol{\rho})_\ell = c\rho$. Let $q = [Q(\mu, M) - Q(z, M)]\boldsymbol{\rho}$. We distinguish between the following cases:

(1) $\ell \in \mathcal{L}_\mu^c \cap \mathcal{L}_z$: we have $(Q(\mu, M)\boldsymbol{\rho})_\ell = c\rho$ since $\ell \in \mathcal{L}_\mu^c$, while $(Q(z, M)\boldsymbol{\rho})_\ell = 0$ as $\ell \in \mathcal{L}_z$. Then, $q_\ell = c\rho$.
(2) $\ell \in \mathcal{L}_\mu \cap \mathcal{L}_z^c$: from $\ell \in \mathcal{L}_z^c$ we have $(Q(z, M)\boldsymbol{\rho})_\ell = c\rho$. On the other hand, since $\ell \in \mathcal{L}_\mu$, $(Q(\mu, M)\boldsymbol{\rho})_\ell = 0$. This results in $q_\ell = -c\rho$.
(3) $\ell \in (\mathcal{L}_\mu \cap \mathcal{L}_z) \cup (\mathcal{L}_\mu^c \cap \mathcal{L}_z^c)$. If $\ell \in \mathcal{L}_\mu \cap \mathcal{L}_z$ then $(Q(\cdot, M)\boldsymbol{\rho})_\ell = 0$ for both μ and z. Therefore, $q_\ell = 0$. Conversely, if $\ell \in \mathcal{L}_\mu^c \cap \mathcal{L}_z^c$, then $(Q(\cdot, M)\boldsymbol{\rho})_\ell = c\rho$ for both μ and z, which results again in $q_\ell = 0$.

The sets $\mathcal{L}_\mu^c \cap \mathcal{L}_z$, $\mathcal{L}_\mu \cap \mathcal{L}_z^c$, $(\mathcal{L}_\mu \cap \mathcal{L}_z) \cup (\mathcal{L}_\mu^c \cap \mathcal{L}_z^c)$ are pairwise disjoint and exhaust the set $\{1, \ldots, m\}$. Hence we can write

$$\begin{aligned} U &= (\mu - z)^\intercal (Q(\mu, M) - Q(z, M))\boldsymbol{\rho} = \sum_{\ell=1}^m (\mu_\ell - z_\ell) q_\ell \\ &= \sum_{i \in \mathcal{L}_\mu^c \cap \mathcal{L}_z} (\mu_i - z_i) c\rho + \sum_{j \in \mathcal{L}_\mu \cap \mathcal{L}_z^c} (\mu_j - z_j) \cdot (-c\rho) \\ &= \underbrace{\Big(\sum_{i \in \mathcal{L}_\mu^c \cap \mathcal{L}_z} \mu_i - \sum_{j \in \mathcal{L}_\mu \cap \mathcal{L}_z^c} \mu_j \Big)}_{=:U_1} + \underbrace{\sum_{j \in \mathcal{L}_\mu \cap \mathcal{L}_z^c} z_j - \sum_{i \in \mathcal{L}_\mu^c \cap \mathcal{L}_z} z_i}_{=:U_2} \Big) c\rho, \end{aligned} \tag{21}$$

siveness of T on each subdomain $\mathcal{M}_j$) can still be ensured on this example due to the affine structure of F; cf. [19, Sec. 12.5.1].



(a) $M = 0$ (b) $M = 1$ (c) $M \geq 2$

Fig. 3. Iterates generated by Algorithm 1 (blue diamonds) for different choices of M. In this numerical instance, $N = 50$, $\rho = 10$, and $X_i := \{x_i \in \mathbb{R}^n : x_i \in [\underline{x}_i, \overline{x}_i]\}$, with $\underline{x}_i = [0\ 0]^\intercal$, $\overline{x}_i = [3.5\ 3.5]^\intercal$. The randomly generated coupling constraints form the rectangular feasibility region Π_K^q (delineated by the solid black line). The red-shaded region represents the intersection between the latter and the ball $\mathbb{B}_1(\sigma^*, \rho/N)$ around the aggregate equilibrium σ^* (red diamond marker). In this instance, its volume increases as larger values for M are chosen. The value associated to the contour lines of the potential function E decreases from top-right to bottom-left, showing that σ^* is the unique minimiser in the admissible region (shaded in green) after constraint tightening is performed by the algorithm (see Sec. 3.2.2).

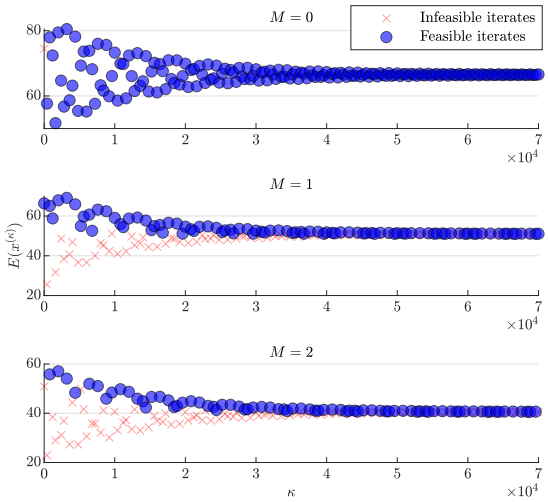


Fig. 4. Potential function $E(x^{(\kappa)})$ evaluated along the iterations of Algorithm 1. Lower values of M yield better confidence on the theoretical robustness certificates for the considered region (see Thm. 3), which results in a lower empirical probability of constraint violation. On the other hand, the system-level efficiency of the equilibrium increases for higher values of M.

Now, notice that for any $i \in \mathcal{L}_\mu^c \cap \mathcal{L}_z \subseteq \mathcal{L}_\mu^c$ and $j \in \mathcal{L}_\mu \cap \mathcal{L}_z^c \subseteq \mathcal{L}_\mu$, we have by definition of $\mathcal{L}_\mu$ and $\mathcal{L}_\mu^c$ that $\mu_i \leq \mu_j$ (which by (13) only holds with equality if $\mu_i = \mu_j = 0$). With analogous reasoning, we have $z_i \geq z_j$ for any $i \in \mathcal{L}_\mu^c \cap \mathcal{L}_z \subseteq \mathcal{L}_z$ and $j \in \mathcal{L}_\mu \cap \mathcal{L}_z^c \subseteq \mathcal{L}_z^c$. Let h_1 be the cardinality of the set $\mathcal{L}_\mu^c \cap \mathcal{L}_z$, and h_2 that of $\mathcal{L}_\mu \cap \mathcal{L}_z^c$. Then,

$$h_1 = |\mathcal{L}_\mu^c \cap \mathcal{L}_z| \overset{(a)}{=} |\mathcal{L}_z \setminus \mathcal{L}_\mu| = |\mathcal{L}_z| - |\mathcal{L}_\mu \cap \mathcal{L}_z|$$
$$\overset{(b)}{=} |\mathcal{L}_\mu| - |\mathcal{L}_z \cap \mathcal{L}_\mu| = |\mathcal{L}_\mu \setminus \mathcal{L}_z| = |\mathcal{L}_\mu \cap \mathcal{L}_z^c| = h_2,$$

where (a) holds since $\mathcal{L}_\mu, \mathcal{L}_z \subseteq \{1, \dots, M\}$, and (b) follows from $|\mathcal{L}_\mu| = |\mathcal{L}_z| = M$. Therefore $h_1 = h_2 =: h$ and $0 \leq h \leq M$, which implies $U_1 \leq 0$ and $U_2 \leq 0$ in (21). We can observe that $U_1 < 0$ and $U_2 < 0$ if $\mathcal{L}_\mu \cap \mathcal{L}_z^c$ and $\mathcal{L}_\mu^c \cap \mathcal{L}_z$ are nonempty and the corresponding components of μ and z are nonzero. In such a case $h \geq 1$ and we can write

$$U_1 = \sum_{i \in \mathcal{L}_\mu^c \cap \mathcal{L}_z} \mu_i - \sum_{j \in \mathcal{L}_\mu \cap \mathcal{L}_z^c} \mu_j \leq -h\zeta, \tag{22}$$

where the inequality follows from (13) and the above discussion. A similar reasoning holds for U_2. Lastly, note that if $\mu \neq z$ and $h \geq 1$, then at least one of $U_1 \leq -h\zeta$ and $U_2 \leq -h\zeta$ will hold. By (21), we can thus conclude $U \leq -h\zeta c\rho$ for any $\mu, z \in \mathcal{M}$, $\mu \neq z$. ■

6.2 *Proof of Lemma 3*

Part (1): To prove that the mapping T is continuous on its domain, we first notice that T is by construction continuous on $X \times \mathcal{M}$ when the operator $Q(\cdot, M)$ is continuous on $\mathcal{M}$ (as the parameter M is fixed). Therefore, it is sufficient to show that for any $\mu, z \in \mathcal{M}$ and any $\eta > 0$, there exists $\delta > 0$ such that

$$\|\mu - z\| < \delta \ \Rightarrow\ \|Q(\mu, M) - Q(z, M)\|\|\boldsymbol{\rho}\| < \eta, \tag{23}$$

where $\boldsymbol{\rho} = c\rho \mathbf{1}_m \neq \mathbf{0}$. To this end, consider any $\mu, z \in \mathcal{M}$ such that $\|\mu - z\| < \frac{\zeta}{2}$, with ζ as defined in (13). [5] Let $\vec{\mu}$ and $\vec{z}$ denote the vectors μ and z sorted in decreasing order; thus, $\vec{\mu}_\ell$ is the ℓ-th largest element of μ (and similarly for z). For any given ℓ, let $i : \mu_i = \vec{\mu}_\ell$, $j : z_j = \vec{z}_\ell$, and $\bar{\ell} := \min_{\{1,\dots,m\}} \ell : i \neq j$. In words, $\bar{\ell}$ is the smallest index for which the ℓ-th largest elements of μ and z do

[5] The proof of this part also holds for $\mu \in \mathbb{R}_+^m \supset \mathcal{M}$.

not appear at the same row of their respective vectors. We then let $\mathcal{I}$ be the set of indices for which the ordering of the elements of μ and z agrees, i.e., for all $k \in \mathcal{I}$, there exists $\ell < \bar{\ell}$ such that $i = j = k$, with $i : \mu_i = \vec{\mu}_\ell$ and $j : z_j = \vec{z}_\ell$.

We prove our statement by contradiction. Suppose there exists $i, j \notin \mathcal{I}$ such that $i : \mu_i = \vec{\mu}_\ell$ and $j : z_j = \vec{z}_\ell$ for some $\ell > \bar{\ell}$, where $\mu_i < \mu_j$ and $z_i > z_j$. First, we note that such an instance exists by hypothesis, as otherwise the only possible case is where $i = j$, which contradicts $i, j \notin \mathcal{I}$ and implies $Q(\mu, M) = Q(z, M)$. Since $z \in \mathcal{M}$, it further holds $z_j < z_i - \zeta$, which by $\|\mu_i - z_i\| \leq \|\mu - z\| < \frac{\zeta}{2}$ implies

$$z_j < \mu_i + \frac{\zeta}{2} - \zeta. \tag{24}$$

We bound (24) from below by noting $z_j > \mu_j - \frac{\zeta}{2}$, which holds since $\|\mu_j - z_j\| < \frac{\zeta}{2}$, obtaining

$$\mu_j - \frac{\zeta}{2} < \mu_i + \frac{\zeta}{2} - \zeta,$$

or equivalently $\mu_j < \mu_i$, which contradicts our hypothesis. Hence the elements of any pair of vectors $\mu, z \in \mathcal{M}$ such that $\|\mu - z\| < \frac{\zeta}{2}$ must follow the same ordering. By definition of $P(\cdot)$, this implies $P(\mu) = P(z)$ and, in turn, $\|Q(\mu, M) - Q(z, M)\| = 0$. This validates (23) with $\delta = \frac{\zeta}{2}$ and any $\eta > 0$, establishing the continuity of $Q(\cdot, M)$ on $\mathcal{M}$ and concluding the proof of the first part.

Part (2): We show here that the mapping T fulfils certain nonexpansiveness properties required for the convergence of Algorithm 1, for compatible choices of τ. In particular, we provide here a sufficient condition for which the iteration

$$y^{(\kappa+1)} = \mathrm{proj}_{X \times \mathcal{M}_j, D} \left[y^{(\kappa)} - D^{-1} T(y^{(\kappa)}, \rho, M) \right], \tag{25}$$

converges to a solution of $\mathrm{VI}(X \times \mathcal{M}_j, T)$, where $j \in \{1, \ldots, q\}$ is fixed, for any $y^{(0)} \in X \times \mathcal{M}_j$. Notice that in (25) the skew projection is performed on the convex subdomain $X \times \mathcal{M}_j$. Then (25) is the solution of the $\mathrm{VI}(X \times \mathcal{M}_j, T_D^{(\kappa)})$ (see [19, Sec. 12.5.1]), where $T_D^{(\kappa)}(y) := T(y^{(\kappa)}, \rho, M) + D(y - y^{(\kappa)})$ is strongly monotone due to $D \succ 0$ and $(T(y, \rho, M) - T(y', \rho, M))^\intercal (y - y') \geq 0$, for all $y, y' \in X \times \mathcal{M}$, which in turn follows from Assumption 3 and Lemma 2. The fixed-point iteration (25) is an instance of the forward-backward splitting method: we thus resort to standard results in the literature to prove its convergence. Following the notation in [19, Sec. 12.5.1], let $\tilde{D} := D_s^{-1/2}(D - D_s)D_s^{-1/2}$, where $D_s := \frac{D + D^\intercal}{2}$. Also, let $\mathcal{U}_j := \{D_s^{1/2} y : y \in X \times \mathcal{M}_j\}$, $\mathcal{U} = \bigcup_{j=1}^q \mathcal{U}_j$, and $\tilde{T}(w) := D_s^{-1/2} T(D_s^{-1/2} w, \rho, M)$, for all $w \in \mathcal{U}$. To ease notation, we drop the dependence of $\tilde{T}$ and $\tilde{T}_D$ on ρ, M, as they remain fixed throughout the proof. According to [19, Thm. 12.5.2] (see also [46, Sec. 4.3]), to ensure convergence of (25) to a solution of the $\mathrm{VI}(X \times \mathcal{M}_j, T)$ it is sufficient to show that $\tilde{T}_D = \tilde{T} - \tilde{D}$ is β-cocoercive on $\mathcal{U}_j$, i.e.,

$$(\tilde{T}_D(v) - \tilde{T}_D(w))^\intercal (v - w) \geq \beta \|\tilde{T}_D(v) - \tilde{T}_D(w)\|^2, \tag{26}$$

for some $\beta > \frac{1}{2}$ and all $v, w \in \mathcal{U}_j$, $j \in \{1, \ldots, q\}$. In fact, we will go a step forward and demonstrate here that $\tilde{T}_D$ is co-coercive on $\mathcal{U}$ with $\beta > \frac{1}{2}$. Due to the particular saddle problem structure of the mapping in (8), we adopt the procedure in [19, Prop. 12.5.4] and define D as in (15) (see also [35]). It then follows from the above definitions that $\tilde{T}_D(w)$, for any $w \in \mathcal{U}$, reduces to

$$\tilde{T}_D(D_s^{1/2} y) = D_s^{-1/2} \begin{bmatrix} F(x) \\ b - Q(\mu, M)\boldsymbol{\rho} \end{bmatrix}, \ \forall y \in X \times \mathcal{M}, \tag{27}$$

which can be easily seen by rewriting (8) as

$$T(y, \rho, M) = \begin{bmatrix} F(x) \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & A^\intercal \\ -A & 0 \end{bmatrix}}_{D - D_s} y + \begin{bmatrix} 0 \\ b - Q(\mu, M)\boldsymbol{\rho} \end{bmatrix}.$$

Define $W := (D_s^{-1/2})^\intercal D_s^{-1/2} = D_s^{-1}$, and let $\vec{Q}(\cdot)$ be a shorthand for $Q(\cdot, M)\boldsymbol{\rho}$ (as M is a fixed parameter). Then, for any $w_a, w_b \in \mathcal{U}$, we can expand (26) by using (27), obtaining

$$\begin{aligned}
&(w_a - w_b)^\intercal (\tilde{T}_D(w_a) - \tilde{T}_D(w_b)) - \beta \|\tilde{T}_D(w_a) - \tilde{T}_D(w_b)\|^2 \\
&= (D_s^{-1/2} w_a - D_s^{-1/2} w_b)^\intercal \begin{bmatrix} F(x_a) - F(x_b) \\ \vec{Q}(\mu_b) - \vec{Q}(\mu_a) \end{bmatrix} \\
&\qquad - \beta \left\| D_s^{-1/2} \begin{bmatrix} F(v_a) - F(v_b) \\ \vec{Q}(\mu_b) - \vec{Q}(\mu_a) \end{bmatrix} \right\|^2 \\
&= \begin{bmatrix} x_a - x_b \\ \mu_a - \mu_b \end{bmatrix}^\intercal \begin{bmatrix} F(x_a) - F(x_b) \\ \vec{Q}(\mu_b) - \vec{Q}(\mu_a) \end{bmatrix} \\
&\quad - \beta \begin{bmatrix} F(x_a) - F(x_b) \\ \vec{Q}(\mu_b) - \vec{Q}(\mu_a) \end{bmatrix}^\intercal W \begin{bmatrix} F(x_a) - F(x_b) \\ \vec{Q}(\mu_b) - \vec{Q}(\mu_a) \end{bmatrix},
\end{aligned} \tag{28}$$

for all $y_a, y_b \in X \times \mathcal{M}$, where the last equality follows from the definition of $\mathcal{U}_j$ and by expanding the norm. Matrix W can be written as $W = \left[\begin{smallmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{smallmatrix}\right]$, where

$W_{11} \in \mathbb{R}^{nN\times nN}$, $W_{12} \in \mathbb{R}^{nN\times m}$, $W_{33} \in \mathbb{R}^{m\times m}$ are:

$$\begin{aligned}
W_{11} &= \tau(I_n - \tau^2 A^\intercal A)^{-1},\\
W_{12} &= W_{21}{}^\intercal = \tau^2(I_n - \tau^2 A^\intercal A)^{-1}A^\intercal,\\
W_{22} &= \tau I_m + \tau^3 A(I_n - \tau^2 A^\intercal A)^{-1}A^\intercal.
\end{aligned}$$

Expanding the inner product in (28) with respect to the matrix blocks $W_{11}, W_{12}, W_{21}, W_{33}$ we obtain

$$\begin{aligned}
&\beta(F(x_a) - F(x_b))^\intercal\big[\tfrac{1}{\beta}(x_a - x_b)\\
&\quad - W_{11}(F(x_a) - F(x_b)) - 2W_{12}(\vec{Q}(\mu_b) - \vec{Q}(\mu_1))\big]\\
&\quad + \beta(\vec{Q}(\mu_b) - \vec{Q}(\mu_a))^\intercal\big[\tfrac{1}{\beta}(\mu_a - \mu_b)\\
&\qquad\qquad - W_{22}(\vec{Q}(\mu_b) - \vec{Q}(\mu_a))\big]\\
&= (F(x_a) - F(x_b))^\intercal(x_a - x_b)\\
&\quad - \beta(F(x_a) - F(x_b))^\intercal W_{11}(F(x_a) - F(x_b))\\
&\quad - 2\beta(F(x_a) - F(x_b))^\intercal W_{12}(\vec{Q}(\mu_b) - \vec{Q}(\mu_a))\\
&\quad + (\vec{Q}(\mu_b) - \vec{Q}(\mu_a))^\intercal(\mu_a - \mu_b)\\
&\quad - \beta(\vec{Q}(\mu_b) - \vec{Q}(\mu_a))^\intercal W_{22}(\vec{Q}(\mu_b) - \vec{Q}(\mu_a)).
\end{aligned}$$

Setting $p_\tau := (I - \tau^2 A^\intercal A)^{-1/2}(F(x_a) - F(x_b))$ and $q_\tau := \tau(I - \tau^2 A^\intercal A)^{-1/2}A^\intercal(\vec{Q}(\mu_b) - \vec{Q}(\mu_a))$ above we obtain

$$\begin{aligned}
&(F(x_a) - F(x_b))^\intercal(x_a - x_b)\\
&\quad + (\vec{Q}(\mu_b) - \vec{Q}(\mu_a))^\intercal(\mu_a - \mu_b)\\
&\quad - \beta\tau(\vec{Q}(\mu_b) - \vec{Q}(\mu_a))^\intercal(\vec{Q}(\mu_b) - \vec{Q}(\mu_a))\\
&\quad - \beta\tau(p_\tau + q_\tau)^\intercal(p_\tau + q_\tau)\\
&\quad \geq \alpha\|x_a - x_b\|^2 + 2h\zeta c\rho\\
&\qquad\qquad - 2\beta\tau h(c\rho)^2 - 2\beta\tau(p_\tau{}^\intercal p_\tau + q_\tau{}^\intercal q_\tau),
\end{aligned} \tag{29}$$

where for the last inequality we used, in order, (i) strong monotonicity of F (cf. Assumption 3), (ii) Lemma 2, (iii) $\|\vec{Q}(\mu_b) - \vec{Q}(\mu_a)\|^2 \leq 2h(c\rho)^2$—which follows from the same arguments used in the proof of Lemma 2—and (iv) $(p_\tau + q_\tau)^\intercal(p_\tau + q_\tau) \leq 2(p_\tau{}^\intercal p_\tau + q_\tau{}^\intercal q_\tau)$. Expanding the term containing p_τ, q_τ in (29) we get

$$\begin{aligned}
&\alpha\|x_a - x_b\|^2 + 2h\zeta c\rho - 2\beta\tau h(c\rho)^2\\
&- 2\beta\tau(F(x_a) - F(x_b))^\intercal(I_n - \tau^2 A^\intercal A)^{-1}(F(x_a) - F(x_b))\\
&- 2\beta\tau^3(\vec{Q}(\mu_b) - \vec{Q}(\mu_a))^\intercal\\
&\qquad\qquad \cdot A(I_n - \tau^2 A^\intercal A)^{-1}A^\intercal(\vec{Q}(\mu_b) - \vec{Q}(\mu_a))\\
&\overset{(a)}{\geq} \alpha\|x_a - x_b\|^2 + 2h\zeta c\rho - 2\beta\tau h(c\rho)^2\\
&\quad - 2\beta\tau\|F(x_a) - F(x_b)\|^2 \cdot \|(I_n - \tau^2 A^\intercal A)^{-1}\|\\
&\quad - 2\beta\tau^3\|\vec{Q}(\mu_b) - \vec{Q}(\mu_a)\|^2 \cdot \|(I_n - \tau^2 A^\intercal A)^{-1}\| \cdot \|A\|^2\\
&\overset{(b)}{\geq} (\alpha - 2\beta\tau L_F^2\|(I_n - \tau^2 A^\intercal A)^{-1}\|)\|x_a - x_b\|^2\\
&\quad + 2h\zeta c\rho - 2\beta\tau h(c\rho)^2\bigg(1 + \frac{2\tau^2}{1 - \tau^2\|A\|^2}\|A\|^2\bigg),
\end{aligned} \tag{30}$$

where (a) is obtained by applying the Cauchy-Schwarz inequality, and in (b) we use the Lipschitz continuity of F (cf. Assum. 3), $\|\vec{Q}(\mu_b) - \vec{Q}(\mu_a)\|^2 \leq 2h(c\rho)^2$, and the triangle inequality. Now, notice that for the last term in (30),

$$\begin{aligned}
&2\beta\tau h(c\rho)^2\bigg(1 + \frac{2\tau^2}{1 - \tau^2\|A\|^2}\|A\|^2\bigg)\\
&= 2\beta\tau h(c\rho)^2\frac{1 + \tau^2\|A\|^2}{1 - \tau^2\|A\|^2} \leq 2\beta\tau h(c\rho)^2\frac{1 + \|A\|^2}{1 - \tau^2\|A\|^2},
\end{aligned} \tag{31}$$

holds for any choice of $\tau \in \big(0, \max\big\{\frac{1}{\|A\|}, 1\big\}\big)$. Recall that by invoking [19, Thm. 12.5.2], our objective is to show that (26) holds for some $\tau > 0$ and $\beta > \frac{1}{2}$. Then, by inspecting (30) and using (31), to achieve this it is sufficient to guarantee

$$\begin{aligned}
&\alpha - \tau L_F^2\|(I_n - \tau^2 A^\intercal A)^{-1}\| > 0,\\
&2h\zeta c\rho - \tau h(c\rho)^2\frac{1 + \|A\|^2}{1 - \tau^2\|A\|^2} > 0, \text{ if } 1 \leq h \leq M.
\end{aligned}$$

Solving the quadratic expressions above with respect to τ results in the admissible range of values in (16) (these are also satisfying $\tau \in \big(0, \max\big\{\frac{1}{\|A\|}, 1\big\}\big)$, required for (31) to hold). Therefore, for any τ satisfying this condition, $\tilde{T}_D$ is co-coercive with $\beta > \frac{1}{2}$ on the entire domain $\mathcal{U}$, which in turn implies that co-coercivity of $\tilde{T}_D$ holds on each subdomain $\mathcal{U}_j$, $j = 1, \ldots, q$, with the same modulus. By [19, Thm. 12.5.2], this is sufficient to guarantee the convergence of (25) to a solution of the VI$(X \times \mathcal{M}_j, T)$, thus concluding the proof. ■

6.3 Proof of Theorem 2

Fix any τ satisfying the conditions of Lemma 3 and (17). The sequence of iterates $\{y^{(\kappa)}\}_{\kappa=1,2,\ldots}$ (where $y^{(\kappa)} = (x^{(\kappa)}, \mu^{(\kappa)})$) generated by Algorithm 1 lives in a bounded set since X and $\mathcal{M}$ are assumed to be compact (see Assumption 5). As such there exist convergent subsequences, or in other words, the set

$$\begin{aligned}
\Omega := \Big\{\bar{y} = (\bar{x}, \bar{\mu})\colon\ &\exists \text{ subsequence } \{\kappa_i\}_{i\in\mathbb{N}}\\
&\text{such that } \lim_{i\to\infty}\kappa_i = \infty,\ \lim_{i\to\infty} y^{(\kappa_i)} = \bar{y}\Big\},
\end{aligned} \tag{32}$$

containing the limit points of $\{y^{(\kappa)}\}$ is non-empty; see, e.g., [40, p. 48]. In particular, we will show that Ω is a singleton for any τ satisfying (16)–(17), which implies that the iterates generated by Algorithm 1 have a unique limit point, hence they converge. To achieve this, we assume for the sake of contradiction that there exist two cluster points $\bar{y}_1, \bar{y}_2 \in \Omega$, where $\bar{y}_1 = (\bar{x}_1, \bar{\mu}_1)$ and $\bar{y}_2 = (\bar{x}_2, \bar{\mu}_2)$. Moreover, we assume that $\bar{\mu}_1 \in \mathcal{M}_i$, and $\bar{\mu}_2 \in \mathcal{M}_j$, with

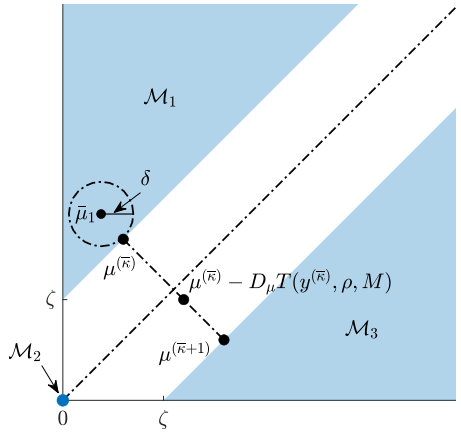


Fig. 5. Domain $\mathcal{M}$ of the Lagrange multipliers associated to the coupling constraints, for the case $m = 2$. Notice the minimum distance ζ between any two subdomains of $\mathcal{M}$ involves the origin as one of the subdomains.

$i \neq j$. Note that if this were not the case, then we would be in a trivial case where $\bar{y}_1 = \bar{y}_2$, due to co-coercivity of T (see Lemma 3)—by which Algorithm 1 converges to a unique solution when restricted to any convex subdomain $X \times \mathcal{M}_j$, $j = 1, \ldots, q$. To ease the notation in the remainder of the proof, we assume without loss of generality that $\bar{\mu}_1 \in \mathcal{M}_1$, $\bar{\mu}_2 \in \mathcal{M}_2$ (see Fig. 5). By (32) there exist an infinite-length subsequence $\{\kappa_i\}_{i\in\mathbb{N}}$ of the iterates generated by Algorithm 1 whose elements get arbitrarily close to $\bar{\mu}_1$ while staying in $\mathcal{M}_1$ where this cluster point belongs (similarly for $\bar{\mu}_2$). We then have that for any $\delta > 0$, there exists $\tilde{\kappa}$ such that for all $\kappa_i \geq \tilde{\kappa}$, $\|y^{(\kappa_i)} - \bar{y}_1\| \leq \delta$; this implies $\|x^{(\kappa_i)} - \bar{x}_1\| \leq \delta$ and $\|\mu^{(\kappa_i)} - \bar{\mu}_1\| \leq \delta$.

Due to our contradiction hypothesis (recall that $\{\kappa_i\}_{i\in\mathbb{N}}$ is a subsequence), the sequence of iterates generated by Algorithm 1 would be leaving $\mathcal{M}_1$ towards $\mathcal{M}_2$ infinitely often. Denote then by $\overline{\kappa} > \tilde{\kappa}$ the smallest index of the subsequence such that $\mu^{(\overline{\kappa})} \in \mathcal{M}_1$, but $\mu^{(\overline{\kappa}+1)} \in \mathcal{M}_2$, i.e., after the $\overline{\kappa}$-th iterate the original sequence would jump to $\mathcal{M}_2$ for the first time after $\tilde{\kappa}$.

For this jump to occur, the unprojected solution for the Lagrange multipliers must be "closer" to $\mathcal{M}_2$ than to any other sub-domain of $\mathcal{M}$. To see this more formally, let D_μ^{-1} denote the lower block-row of $D^{-1} = \begin{bmatrix} \tau I_{nN} & 0 \\ 2A\tau^2 & \tau I_m \end{bmatrix}$, corresponding to the Lagrange multiplier update in line 3 of Algorithm 1. By definition of $\mathcal{M}$, such a jump requires the distance between the unprojected gradient step at $\overline{\kappa}+1$ and $\mu^{(\overline{\kappa})}$ to satisfy

$$\|\mu^{(\overline{\kappa})} - D_\mu^{-1} T(y^{(\overline{\kappa})}, \rho, M) - \mu^{(\overline{\kappa})}\| > \zeta/2, \qquad (33)$$

where ζ is the worst-case (minimum) distance between any two subdomains of $\mathcal{M}$—corresponding to the case where one of these is the origin (in this case $\mathcal{M}_2$). Figure 5 provides a pictorial illustration of this construction. However, we have that

$$\begin{aligned}
&\|\mu^{(\overline{\kappa})} - D_\mu^{-1} T(y^{(\overline{\kappa})}, \rho, M) - \mu^{(\overline{\kappa})}\| \\
&\quad = \tau \| - 2\tau A(F(x^{(\overline{\kappa})}) + A^\intercal \mu^{(\overline{\kappa})}) \\
&\qquad + Ax^{(\overline{\kappa})} - b + Q(\mu^{(\overline{\kappa})}, M)\boldsymbol{\rho}\| \\
&\quad = \tau \| - 2\tau A(F(x^{(\overline{\kappa})}) - F(\bar{x}_1) + A^\intercal(\mu^{(\overline{\kappa})} - \bar{\mu}_1)) \\
&\qquad + A(x^{(\overline{\kappa})} - \bar{x}_1) + Q(\mu^{(\overline{\kappa})}, M)\boldsymbol{\rho} \\
&\qquad - 2\tau A(F(\bar{x}_1) + A^\intercal \bar{\mu}_1) + (A\bar{x}_1 - b)\| \\
&\quad \leq \tau^2 \|2A(F(\bar{x}_1) + A^\intercal \bar{\mu}_1)\| + \tau \|A\bar{x}_1 - b\| \\
&\qquad + \tau \|Q(\mu^{(\overline{\kappa})}, M)\boldsymbol{\rho}\| + \tau \|A\| \|x^{(\overline{\kappa})} - \bar{x}_1\| \\
&\qquad + 2\tau^2 \big(\|A(F(x^{(\overline{\kappa})}) - F(\bar{x}_1))\| + \|AA^\intercal(\mu^{(\overline{\kappa})} - \bar{\mu}_1)\| \big) \\
&\quad \leq (\tau^2 + \tau)\bar{R} + \tau c\rho\sqrt{m - M} \\
&\qquad + \tau\delta\big(2\tau(L_F\|A\| + \|AA^\intercal\|) + \|A\|\big) \qquad (34)
\end{aligned}$$

where the first equality follows from the definition of D_μ^{-1} and T, and the second one by adding and subtracting $F(\bar{x}_1)$, $A^\intercal\bar{\mu}_1$ and $A\bar{x}_1$. The first inequality is due to the triangle inequality, while the last one follows from the previous one by upper-bounding (i) the first two terms using the definition of $\bar{R}$; (ii) $\|Q(\mu^{(\overline{\kappa})}, M)\boldsymbol{\rho}\|$ by $c\rho\sqrt{m - M}$ based on its definition; and (iii) the last three terms using $\|F(x^{(\overline{\kappa})}) - F(\bar{x}_1)\| \leq L_F\|x^{(\overline{\kappa})} - \bar{x}_1\|$ by Assumption 3, and $\|x^{(\overline{\kappa})} - \bar{x}_1\| \leq \delta$, $\|\mu^{(\overline{\kappa})} - \bar{\mu}_1\| \leq \delta$.

By (34), and choosing τ as in (17), we have that

$$\|\mu^{(\overline{\kappa})} - D_\mu^{-1} T(y^{(\overline{\kappa})}, \rho, M) - \mu^{(\overline{\kappa})}\| < \frac{\zeta}{2} + \bar{K}\delta \qquad (35)$$

where $\bar{K}$ is a constant, emanating from the coefficient of δ in (34) when substituting for τ the upper-bound in (17). Since δ is arbitrary, taking $\limsup_{\delta\to 0}$ in (35) establishes a contradiction with (33). The latter implies then that $\bar{\mu}_2$ must belong to the same subdomain $\mathcal{M}_1$ with $\bar{\mu}_1$, i.e., all cluster points should be in the same subdomain of $\mathcal{M}$. However, by Lemma 3, due to co-coercivity of T on each subdomain $X \times \mathcal{M}_j, j = 1, \ldots, q$, we can only have one cluster point, i.e., Ω is a singleton, implying that Algorithm 1 converges, thus concluding the proof. ■

6.4 Proof of Theorem 3

The elements of the minimal compression set I of Algorithm 1 can belong to one or both of the following subsets:

(1) The subset I_1 of samples that support the solution x^*. Note that since Algorithm 1 converges to the point (x^*, μ^*) for a fixed choice of M, $Q(\mu^*, M)$ will be a fixed quantity. As such, Algorithm 1 will

converge to a solution of

$$\begin{aligned} &\text{Find } x^* \in \widehat{\Pi}_K \text{ such that} \\ &F(x^*)^\intercal (x - x^*) \geq 0 \text{ for any } x \in \widehat{\Pi}_K, \end{aligned} \tag{36}$$

where $\widehat{\Pi}_K$ denotes the polytope obtained from Π_K by tightening at most M coupling constraints, as dictated by (11) with $Q(\mu^*, M)$. The constraints in (36) are equivalent to $F(x^*)^\intercal x \geq F(x^*)^\intercal x^*$ for all $x \in \widehat{\Pi}_K$. Then, x^* is the minimiser of

$$\begin{aligned} &\min_{x \in \mathbb{R}^{nN}} F(x^*)^\intercal x \\ &\text{subject to } x \in \widehat{\Pi}_K, \end{aligned} \tag{37}$$

which is unique due to Lemma 1. Since the cost function is linear in x and $\widehat{\Pi}_K$ is convex by Assumption 2, we obtain a scenario program as in [10]. Applying [10, Prop. 1] to (37), we have that $|I_1| \leq nN$, i.e., the number of support samples of x^* is bounded by the dimension of the decision vector nN.

(2) The subset I_2 of samples whose corresponding coupling constraints intersect $\mathbb{B}(x^*, \rho)$. By construction of Algorithm 1 we have that $|I_2| \leq M$.

As such, we have that $I = I_1 \cup I_2$ is a compression set with cardinality $|I| = |I_1 \cup I_2| \leq |I_1| + |I_2| \leq nN + M$. Then, from Corollary 2 in [32] it follows that

$$\begin{aligned} &\mathbb{P}^K \Big\{ \delta_K \in \Delta^K : \ \mathbb{V}(S_K^*) > \bar{\epsilon} \Big\} \\ &\qquad \leq \sum_{i=0}^{nN+M-1} \binom{K}{i} \bar{\epsilon}^i (1 - \bar{\epsilon})^{K-i}, \end{aligned} \tag{38}$$

which concludes the proof. ■

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