

# B15 Linear Dynamic Systems and Optimal Control

## Example Paper 2

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### Questions

1. Consider an LTI system whose state evolves according to

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_1(t) - 2x_2(t) + u(t).\end{aligned}$$

- (a) Verify that it is controllable.
- (b) Design a state feedback controller that places the eigenvalues of the closed loop system at  $-2$  and  $-4$ .

2. Consider the following LTI system:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ &= \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} u(t), \\ y(t) &= Cx(t) = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix} x(t).\end{aligned}$$

- (a) Verify that the system is controllable.
- (b) Determine  $K$  such that the state feedback  $u(t) = Kx(t)$  results in a closed loop system with three eigenvalues at  $-2$ .

3. Consider the LTI system of Question 2.

- (a) Verify that the system is observable.

- (b) If  $\hat{x}(t)$  denotes the state estimated by means of a linear state observer with gain matrix  $L$ , determine the dynamics of the estimation error  $e(t) = x(t) - \hat{x}(t)$ .
- (c) Determine  $L$  such that the dynamics of the estimation error have three eigenvalues at  $-3$ .

4. Consider the transfer function

$$G(s) = \frac{\omega_0^2}{s^2 + \omega_0 s + \omega_0^2}.$$

- (a) Determine the poles of  $G(s)$  and specify their damping ratio.
- (b) Determine a realization  $(A, B, C, D)$  of  $G(s)$ .
- (c) Compute the gains of a state feedback controller as a function of  $\omega_0$  so that the closed loop system has a complex conjugate pair of eigenvalues with damping ratio  $\frac{1}{\sqrt{2}}$ .
- (d) What was the purpose of this controller?

*Hint:* Recall that the general description of a complex conjugate pole (eigenvalue) pair is given by  $-\zeta\omega_0 \pm j\omega_0\sqrt{1 - \zeta^2}$ , where  $\zeta$  denotes the damping ratio.

5. Consider the transfer function  $G(s)$  of Question 4, and the realization computed in part (b). Compute the gains of a linear state observer as a function of  $\omega_0$  so that the estimation error dynamics are 10 times faster (with the same damping ratio) than the dynamics of the closed loop system computed in Question 4.

6. Let  $T$  be a given time horizon length, and consider the following finite

horizon optimal control problem:

$$\begin{aligned} & \text{minimize } \int_0^T u(t)^2 dt + x(T)^2 \\ & \text{subject to } \dot{x}(t) = x(t) + u(t), \text{ for all } t \in [0, T], \\ & \quad x(0) = 1. \end{aligned}$$

- (a) Determine matrices  $A$  and  $B$  corresponding to the state space description of the system's dynamics. Determine matrices  $Q$ ,  $R$  and  $Q_T$  so that the cost criterion can be written in the form

$$\int_0^T (x(t)^\top Q x(t) + u(t)^\top R u(t)) dt + x(T)^\top Q_T x(T).$$

- (b) State and solve the Riccati differential equation associated with this finite horizon linear quadratic regulation (LQR) problem.
- (c) Compute the optimal LQR controller and the associated optimal cost.

7. Consider the following infinite horizon optimal control problem:

$$\begin{aligned} & \text{minimize } \frac{1}{2} \int_0^\infty (x_1(t)^2 + \frac{1}{8} u(t)^2) dt \\ & \text{subject to } \dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = -x_1(t) + u(t), \text{ for all } t, \\ & \quad x_1(0), x_2(0) : \text{ given.} \end{aligned}$$

Let  $y(t) = x_1(t)$  denote the output of the underlying LTI system.

- (a) Determine matrices  $A$ ,  $B$  and  $C$  corresponding to the state space description of the system's dynamics. Determine matrices  $Q$ ,  $R$  and  $Q_T$  so that the cost criterion can be written in the form

$$\int_0^\infty (x(t)^\top Q x(t) + u(t)^\top R u(t)) dt.$$

- (b) State and solve the algebraic Riccati equation associated with this infinite horizon linear quadratic regulation (LQR) problem.
- (c) Does the algebraic Riccati equation admit a unique positive semidefinite solution? If yes, justify whether this is anticipated.

- (d) Compute the optimal LQR controller and the associated optimal cost.
8. Consider the following LQR problem with  $\mu > 0$ :

$$\begin{aligned} &\text{minimize } \int_0^\infty (\mu^2 x(t)^2 + u(t)^2) dt \\ &\text{subject to } \dot{x}(t) = u(t), \text{ for all } t, \\ &x(0) = x_0. \end{aligned}$$

Let  $y(t) = x(t)$  denote the output of the underlying LTI system.

- (a) Compute the optimal LQR controller.
- (b) Comment on the effect of the choice of  $\mu$  on the behaviour of the closed loop system state  $x(t)$ .

*Hint:* For part (a) adapt the solution of the Riccati equation computed in Examples 18 & 19 in the notes to account for the presence of parameter  $\mu$  in the cost function.

9. **OPTIONAL:** Consider an (open loop) LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

with  $n$  states, a single input and a single output. Assume that a state feedback controller  $u(t) = Kx(t) + r(t)$  is designed with  $K \in \mathbb{R}^{1 \times n}$ . The closed loop system is then given by

$$\begin{aligned} \dot{x}(t) &= (A + BK)x(t) + Br(t), \\ y(t) &= (C + DK)x(t) + Dr(t). \end{aligned}$$

- (a) Show that if the open loop system is controllable, then the closed loop system is controllable as well.
- (b) Use your answer in Question 1 to construct a counterexample of an open loop system that is observable, while the closed loop is not.

*Note:* The condition of part (a) is in fact an “if and only if” one.