B15 Linear Dynamic Systems and Optimal Control
Example Paper 2

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Note

Some problems are taken from teaching material that Prof. John Lygeros has been using at ETH Zurich; this is gratefully acknowledged. Any errors or typos should be referred to kostas.margellos@eng.ox.ac.uk

Questions

1. Consider an LTI system whose state evolves according to

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_1(t) - 2x_2(t) + u(t). \end{aligned}$$

   (a) Verify that it is controllable.

   (b) Design a state feedback controller that places the eigenvalues of the closed loop system at -2 and -4.

2. Consider the following LTI system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ &= \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} u(t), \\ y(t) &= Cx(t) = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix} x(t). \end{aligned}$$

   (a) Verify that the system is controllable.

   (b) Determine K such that the state feedback $u(t) = Kx(t)$ results in a closed loop system with three eigenvalues at -2.

3. Consider the LTI system of Question 2.

   (a) Verify that the system is observable.

   (b) If $\hat{x}(t)$ denotes the state estimated by means of a linear state observer with gain matrix L, determine the dynamics of the estimation error $e(t) = x(t) - \hat{x}(t)$.

   (c) Determine L such that the dynamics of the estimation error have three eigenvalues at -3.

4. Consider the transfer function

$$G(s) = \frac{\omega_0^2}{s^2 + \omega_0 s + \omega_0^2}.$$

   (a) Determine the poles of $G(s)$ and specify their damping ratio.

   (b) Determine a realization (A, B, C, D) of $G(s)$.

   (c) Compute the gains of a state feedback controller as a function of ω_0 so that the closed loop system has a complex conjugate pair of eigenvalues with damping ratio $\frac{1}{\sqrt{2}}$.

   (d) What was the purpose of this controller?

   Hint: Recall that the general description of a complex conjugate pole (eigenvalue) pair is given by $-\zeta\omega_0 \pm j\omega_0\sqrt{1-\zeta^2}$, where ζ denotes the damping ratio.

5. Consider the transfer function $G(s)$ of Question 4, and the realization computed in part (b). Compute the gains of a linear state observer as a function of ω_0 so that the estimation error dynamics are 10 times faster (with the same damping ratio) than the dynamics of the closed loop system computed in Question 4.

6. Let T be a given time horizon length, and consider the following finite

horizon optimal control problem:

$$\begin{aligned}\text{minimize } & \int_0^T u(t)^2 dt + x(T)^2 \\ \text{subject to } & \dot{x}(t) = x(t) + u(t), \text{ for all } t \in [0, T], \\ & x(0) = 1.\end{aligned}$$

(a) Determine matrices A and B corresponding to the state space description of the system's dynamics. Determine matrices Q, R and Q_T so that the cost criterion can be written in the form

$$\int_0^T \big(x(t)^\top Q x(t) + u(t)^\top R u(t)\big) dt + x(T)^\top Q_T x(T).$$

(b) State and solve the Riccati differential equation associated with this finite horizon linear quadratic regulation (LQR) problem.

(c) Compute the optimal LQR controller and the associated optimal cost.

7. Consider the following infinite horizon optimal control problem:

$$\begin{aligned}\text{minimize } & \frac{1}{2}\int_0^\infty \big(x_1(t)^2 + \frac{1}{8}u(t)^2\big) dt \\ \text{subject to } & \dot{x}_1(t) = x_2(t),\ \dot{x}_2(t) = -x_1(t) + u(t), \text{ for all } t, \\ & x_1(0), x_2(0): \text{ given.}\end{aligned}$$

Let $y(t) = x_1(t)$ denote the output of the underlying LTI system.

(a) Determine matrices A, B and C corresponding to the state space description of the system's dynamics. Determine matrices Q, R and Q_T so that the cost criterion can be written in the form

$$\int_0^\infty \big(x(t)^\top Q x(t) + u(t)^\top R u(t)\big) dt.$$

(b) State and solve the algebraic Riccati equation associated with this infinite horizon linear quadratic regulation (LQR) problem.

(c) Does the algebraic Riccati equation admit a unique positive semidefinite solution? If yes, justify whether this is anticipated.

(d) Compute the optimal LQR controller and the associated optimal cost.

8. Consider the following LQR problem with $\mu > 0$:

$$\text{minimize} \int_0^\infty \left(\mu^2 x(t)^2 + u(t)^2\right) dt$$
$$\text{subject to } \dot{x}(t) = u(t), \text{ for all } t,$$
$$x(0) = x_0.$$

Let $y(t) = x(t)$ denote the output of the underlying LTI system.

(a) Compute the optimal LQR controller.

(b) Comment on the effect of the choice of μ on the behaviour of the closed loop system state $x(t)$.

Hint: For part (a) adapt the solution of the Riccati equation computed in Examples 18 & 19 in the notes to account for the presence of parameter μ in the cost function.

9. OPTIONAL: Consider an (open loop) LTI system

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

with n states, a single input and a single output. Assume that a state feedback controller $u(t) = Kx(t) + r(t)$ is designed with $K \in \mathbb{R}^{1\times n}$. The closed loop system is then given by

$$\dot{x}(t) = (A + BK)x(t) + Br(t),$$
$$y(t) = (C + DK)x(t) + Dr(t).$$

(a) Show that if the open loop system is controllable, then the closed loop system is controllable as well.

(b) Use your answer in Question 1 to construct a counterexample of an open loop system that is observable, while the closed loop is not.

Note: The condition of part (a) is in fact an "if and only if" one.