0.1 21.10.2019

0.1.1 Продолжаем делать примеры

<u>Пример</u> (3475)

$$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = z^2$$

$$x,\ y,\ z(x,y) \to u,\ v,\ w(u,v)$$

$$u = x, \quad v = \frac{1}{y} - \frac{1}{x}, \quad w = \frac{1}{z} - \frac{1}{x}$$

<u>Решение</u>

Выразим старые переменные через новые:

$$x = u, \quad y = \frac{u}{uv+1}, \quad z = \frac{u}{uw+1}$$

Можем составить тождество:

$$\frac{u}{uw+1} = z(x,\ y) = z(u,\ \frac{u}{uv+1})$$

Продифференцируем ЛЧ:

$$\Rightarrow \left(\frac{u}{uw+1}\right)'_u = \frac{(uw+1)-(w+uw'_u)u'}{(uw+1)^2} = \frac{1-uw'_uu'}{(uw+1)^2}$$

$$\Rightarrow \left(\frac{u}{uw+1}\right)'_v = \frac{-u^2w'_v}{(uw+1)^2}$$

Теперь продифференцируем ПЧ и составим систему:

$$\begin{cases} z\left(u,\ \dfrac{u}{uv+1}\right)'_u = \dfrac{\partial z}{\partial x}\cdot 1 + \dfrac{\partial z}{\partial y}\left(\dfrac{1(\cancel{uv}+1)-\cancel{vu}}{(uv+1)^2}\right) = \dfrac{1-uw'_uu'}{(uw+1)^2} \\[2ex] z\left(u,\ \dfrac{u}{uv+1}\right)'_v = \dfrac{\partial z}{\partial y}\left(\dfrac{-u^2}{(uv+1)^2}\right) = \dfrac{1-uw'_uu'}{(uw+1)^2} \end{cases}$$

Мы нашли то что хотели:

$$\frac{\partial z}{\partial y} = \frac{w'_v(uv+1)^2}{(uw+1)^2}$$

$$\frac{\partial z}{\partial x} = \frac{1-u^2w'_u}{(uw+1)^2} - \frac{w'_v(vu+1)^2}{(uw+1)^2}\frac{1}{(uv+1)^2}$$

Пример

$$\frac{\partial^2 z}{\partial x^2} + z\frac{\partial^2 z}{\partial z \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

$$u = x + y, \quad v = x - y, \quad w = xy - z$$

Решение

Составим тождество

$$xy - z = w(x + y,\ x - y) = w(u, v)$$

Дифференцируем по x:

$$\frac{\partial w}{\partial u_1} + \frac{\partial w}{\partial v} = y - z'_x$$

$$w'_x = (xy - z)'_x = y - z'_x$$

Дифференцируем по y:

$$\frac{\partial w}{\partial u}\underbrace{\frac{\partial u}{\partial y}}_{=1} + \frac{\partial w}{\partial v}\underbrace{\frac{\partial v}{\partial y}}_{=-1} = \frac{\partial w}{\partial u} - \frac{\partial w}{\partial v} = x - z'_y$$

$$w'_y = (xy - z)'_y = x - z'_y$$

$$z'_x = y - \underbrace{\frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}}_{w(u,v)=h(x+y,\ x-y)}$$

$$\frac{\partial^2 z}{\partial x^2} = \underbrace{\frac{\partial y}{\partial x}}_{=0} - \frac{\partial}{\partial x}\big(h(\underbrace{x+y}_{u},\ \underbrace{x-y}_{v})\big) = \frac{\partial h}{\partial u} + \frac{\partial h}{\partial v} = \underbrace{-\frac{\partial^2 w}{\partial u^2} - 2\frac{\partial^2 w}{\partial v \partial u} - \frac{\partial^2 w}{\partial v^2}}_{\text{из начальных уравнений}}$$

$$z'_y = x + \frac{\partial w}{\partial v} - \frac{\partial w}{\partial u}$$

$$\frac{\partial^2 z}{\partial y^2} = \underbrace{\frac{\partial x}{\partial y}}_{=0} + \frac{\partial}{\partial y}\big(h_1(x + y,\ x - y)\big) = \frac{\partial h_1}{\partial u} - \frac{\partial h_1}{\partial v} = 2\frac{\partial^2 w}{\partial v \partial u} - \frac{\partial^2 w}{\partial u^2} - \frac{\partial^2 w}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 1 - \frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2}$$