Опр

$$z = x + iy$$

$$f(z) = u(x,y) + iv(x,y)$$

$$f - \mathbb{C} \text{ дифф } \Leftrightarrow u, v - \mathbb{R} \text{ - дифф}$$

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases} \quad \text{- усл. К-Р}$$

$$f'(z) = u'_x + iv'_x = v'_y - iu'_y = u'_x - iu'_y = v'_y + iv'_x$$

$$u = \operatorname{Re} f \quad (\mathbb{C} \text{ дифф})$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$v = \operatorname{Im} f \quad (\mathbb{C} \text{ дифф})$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Задача (1)

$$u(x,y) = xy - \frac{y}{x^2+y^2} \qquad v = ? \quad f(z) = ?$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\left(xy - \frac{y}{x^2+y^2}\right)'_y =$$

$$= -x + \left(\frac{y}{x^2+y^2}\right)'_y = -x + \frac{1\cdot(x^2+y^2) - y\cdot 2y}{(x^2+y^2)^2} = -x + \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = \left(xy - \frac{y}{x^2+y^2}\right)'_x =$$

$$= y - y\cdot\left(\frac{1}{x^2+y^2}\right)'_x = y + \frac{y\cdot 2x}{(x^2+y^2)^2 + +}$$

$$\begin{cases} \frac{\partial v}{\partial x} = -x + \frac{x^2-y^2}{(x^2+y^2)^2} \\ \frac{\partial v}{\partial y} = y + \frac{2xy}{(x^2+y^2)^2} \end{cases}$$

$$v = \int \frac{\partial v}{\partial y} dy = \int \left(y + \frac{2xy}{(x^2+y^2)^2}\right) dy =$$

$$= \frac{y^2}{2} + x \cdot \int \frac{2ydy}{(x^2+y^2)^2} = \frac{y^2}{2} + x \int \frac{dt}{t^2} =$$

$$t = x^2 + y^2$$

$$dt = 2ydy$$

$$= \frac{y^2}{2} + x\left(-\frac{1}{t}\right) + C$$

$$v = \frac{y^2}{2} - \frac{x}{x^2+y^2} + C \qquad C\text{- const отн } y \Rightarrow C = C(x)$$

$$\frac{\partial v}{\partial x} = 0 - \left(\frac{x}{x^2+y^2}\right)'_x + C'(x) =$$

$$= \frac{1\cdot(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} + C'(x) = -\frac{-x^2+y^2}{(x^2+y^2)^2} + C'(x)$$

$$\Rightarrow \frac{x^2-y^2}{(x^2+y^2)^2} + C'(x) = -x + \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\Rightarrow C'(x) = -x$$

$$C(x) = \int(-x)dx = -\frac{x^2}{2} + C_0 \qquad C_0 \text{ - const не зависит от } x, y$$

$$v = \frac{y^2}{2} - \frac{x}{x^2+y^2} - \frac{x^2}{2} + C_0$$

$$f(z) = u + iv = \left(xy - \frac{y}{x^2+y^2}\right) + i\left(\frac{y^2-x^2}{2} - \frac{x}{x^2+y^2}\right) + iC_0$$

Выразить через z

$$z = x + iy$$

$$\overline{z} = x - iy$$

$$x = \frac{z+\overline{z}}{2}$$

$$y = \frac{z-\overline{z}}{2i}$$

$$x^2 + y^2 = |z|^2 = z\cdot\overline{z}$$

$$= \frac{z+\overline{z}}{2}\cdot\frac{z\overline{z}}{2i} - \frac{\frac{z-\overline{z}}{2i}}{z\overline{z}} + i\left(\frac{\frac{(z-\overline{z})^2}{4i^2} - \frac{(z+\overline{z})^2}{4}}{2} - \frac{\frac{z+\overline{z}}{2}}{z\overline{z}}\right)$$

$$= \frac{z^2 - z^{-2}}{4i} - \frac{1}{2i\overline{z}} + \frac{1}{2iz} + i\left(\frac{-\frac{(z-\overline{z})^2}{4} - \frac{(z+\overline{z})^2}{4}}{2}\right) - \frac{1}{2\overline{z}} - \frac{1}{2z} =$$

$$= \frac{z^2 - \overline{z}^2}{4i} + \frac{i}{2\overline{z}} - \frac{i}{2z} + i\frac{-2z^2 - 2\overline{z}^2}{8} - \frac{i}{2\overline{z}} - \frac{i}{2z} =$$

$$= -\frac{iz^2}{4} + \frac{i\overline{z}^2}{4} - \frac{i}{z} - \frac{i\overline{z}^2}{4} = -\frac{iz^2}{2} - \frac{i}{z} + iC_0 = f(z)$$

Задача (2)

$$v = y\cos y \operatorname{ch} x + x \operatorname{sh} x \sin y \qquad u, f(z) =?$$

$$u'_x = v'_y = \cos y \operatorname{ch} x - \sin y y \operatorname{ch} x + \cos y x \operatorname{sh} x$$

$$u'_y = -v'_x = -(y\cos y \operatorname{sh} x + \operatorname{sh} x \sin y + x \operatorname{ch} x \sin y)$$

$$u = \int (\cos y - y\sin y)\operatorname{ch} x dx + \int \cos y \cdot (x \operatorname{sh} x) dx = (\cos y - y \sin y)\operatorname{sh} x +$$

$$y = const$$

$$\int x \operatorname{sh} x dx = x \operatorname{ch} x - \int \operatorname{ch} x 1 dx = x \operatorname{ch} x - \operatorname{sh} x$$

$$+ \cos y(x \operatorname{ch} x - \operatorname{sh} x) + C(y)$$

$$u'_y = (-\sin y - (\sin y + y\cos y))\operatorname{sh} x - \sin y(x \operatorname{ch} x - \operatorname{sh} x) + C'(y) =$$

$$= -y\cos y \operatorname{sh} x - \sin y(\operatorname{sh} x + x \operatorname{ch} x)$$

$$C'(y) = 0 \qquad C(y) = C_0$$

$$u = (\cos y - y\sin y)\operatorname{sh} x + \cos y(x \operatorname{ch} x - \operatorname{sh} x) + C$$

$$f(z) = u + iv = -y\sin y \operatorname{sh} x + x \operatorname{ch} x \cos y + i(y\cos y x + x \operatorname{sh} x \sin y)$$

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}$$

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}$$

$$\operatorname{sh} x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{ch} x = \frac{e^x + e^{-x}}{2}$$

$$-y\frac{e^{iy} - e^{-iy}}{2i} \cdot \frac{e^x - e^{-x}}{2} = x\frac{e^x + e^{-x}}{2} \cdot \frac{e^{iy} + e^{-iy}}{2} + iy\frac{e^{iy} + e^{-iy}}{2} \cdot \frac{e^x + e^{-x}}{2} + ix\frac{e^x - e^{-x}}{2} \cdot e$$

$$-y\frac{e^{x+iy} - e^{x-iy} - e^{-x+iy} + e^{-x-iy}}{4i} + x\frac{e^{x+iy} + e^{-x+iy} + e^{x-iy} + e^{-x-iy}}{4} +$$

$$+iy\frac{e^{x+iy}+e^{-x+iy}+e^{x-iy}+e^{-x-iy}}{4}+ix\frac{e^{x+iy}-e^{-x+iy}-e^{x+iy}+e^{-x-iy}}{4i}=$$

$$=(-y+ix)\frac{e^{z}+e^{-z}-e^{\overline{z}}-e^{-\overline{z}}}{4i}+(x+iy)\frac{e^{z}+e^{-z}+e^{\overline{z}}+e^{-\overline{z}}}{4}=$$

$$=(+iy+x)\frac{e^{z}+e^{-z}-e^{\overline{z}}-e^{-\overline{z}}}{4}+(x+iy)\frac{e^{z}+e^{-z}+e^{\overline{z}}+e^{-\overline{z}}}{4}=$$

$$=\frac{e^{z}+e^{-z}}{4}2z=z\operatorname{ch}z$$

Задача (3)

$$z=x+iy$$

$$z=r(\cos\varphi+i\sin\varphi)\qquad f(z)=u(r,\varphi)+iv(r,\varphi)$$

Составить уравнения К-Р для $u'_r,\ u'_\varphi,\ v'_r,\ v'_\varphi$

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

Замена переменных

$$x=r\cos\varphi$$

$$y=r\sin\varphi$$

две новые функции

$$u=w$$

$$v=w$$

$$x'_r=\cos\varphi$$

$$x'_\varphi=-r\sin\varphi$$

$$y'_r=\sin\varphi$$

$$y'_\varphi=r\cos\varphi$$

$$u(x,y)=u(x(r,\varphi),y(r,\varphi))$$

$$\begin{cases}u'_x x'_r+u'_y y'_r=u'_r\\ u'_x x'_\varphi+u'_y y'_\varphi=u'_\varphi\end{cases}$$

$$\begin{cases} u'_x \cos\varphi + u'_y \sin\varphi = u'_r & \cdot r\cos\varphi \\ u'_x \cdot (-r\sin\varphi) + u'_y r\cos\varphi = u'_\varphi & (-\sin\varphi) \end{cases}$$

$$u'_x \cdot r(\cos^2\varphi + \sin^2\varphi) = u'_r r\cos\varphi - u'_\varphi \sin\varphi$$

$$u'_x = u'_r \cos\varphi - \frac{u'_\varphi \sin\varphi}{r}$$

$$\begin{cases} u'_x \cos\varphi + u'_y \sin\varphi = u'_r & \cdot r\sin\varphi \\ u'_x \cdot (-r\sin\varphi) + u'_y r\cos\varphi = u'_\varphi & \cdot\cos\varphi \end{cases}$$

$$u'_y r(\sin^2\varphi + \cos^2\varphi) = u'_r r\sin\varphi + u'_\varphi \cos\varphi$$

$$u'_y = u'_r \sin\varphi + \frac{u'_\varphi \cos\varphi}{r}$$

$$\begin{cases} u'_x = u'_r \cos\varphi - \frac{u'_\varphi \sin\varphi}{r} \\ u'_y = u'_r \sin\varphi + \frac{u'_\varphi \cos\varphi}{r} \end{cases}$$

$$\begin{cases} v'_x = v'_r \cos\varphi - \frac{v'_\varphi \sin\varphi}{r} \\ v'_y = v'_r \sin\varphi + \frac{v'_\varphi \cos\varphi}{r} \end{cases}$$

$$\begin{cases} u'_r \cos\varphi - \frac{u'_\varphi \sin\varphi}{r} = v'_r \sin\varphi + \frac{v'_\varphi \cos\varphi}{r} & \cdot cos\varphi \\ u'_r \sin\varphi + \frac{u'_\varphi \cos\varphi}{r} = -v'_r \cos\varphi + \frac{v'_\varphi \sin\varphi}{r} & \cdot\sin\varphi \end{cases}$$

$$u'_r(\cos^2\varphi + \sin^2\varphi) = v'_\varphi = v'_\varphi \frac{\cos^2\varphi + \sin^2\varphi}{r}$$

$$\frac{u'_\varphi}{r}(\cos^2 + \sin^2) = v'_r(-\sin^2 - \cos^2)$$

$$\begin{cases} u'_r = \frac{v'_\varphi}{r} \\ \frac{u'_\varphi}{r} = -v'_r \end{cases} \qquad \text{Уравнение К-Р}$$

<u>Задача</u> (4)

$$f(z) = R(x,y)(\cos\Phi(x,y) + i\sin\Phi(x,y))$$

$$u = R\cos\Phi$$

$$v = R\sin\Phi$$

нез. перем. те же

$$R'_x,\ R'_y,\ \Phi'_x,\ \Phi'_y$$

$$u'_x = R'_x \cos\Phi + R(-\sin\Phi)\Phi'_x$$

$$u'_y = R'_y \cos\Phi + R(-\sin\Phi)\Phi'_y$$
$$v'_x = R'_x \sin\Phi - R\cos\Phi\Phi'_x$$
$$v'_y = R'_y \sin\Phi - R\cos\Phi\Phi'_y$$

$$\begin{cases} R'_x\cos\Phi - R\sin\Phi\Phi'_x = R'_y\sin\Phi - R\cos\Phi\Phi'_y \\ R'_y\cos\Phi - R\Phi\Phi'_y = -R'_x\sin\Phi + R\cos\Phi\Phi'_x \end{cases}$$

$$R'_x\sin\Phi - R\cos\Phi\Phi'_x = -R'_y\cos\Phi + R\sin\Phi\Phi'_y$$
$$R'_x = R\Phi'_y$$
$$R'_y = -R\Phi'_x$$

<u>**Задача**</u> **(4)**

$$f(z) = R(r,\varphi)(\cos\Phi(r,\varphi) + i\sin\Phi(r,\varphi))$$
$$R'_r,\ R'_\varphi,\ \Phi'_r,\ \Phi'_\varphi$$

$$u(r,\varphi) = R(r,\varphi)\cos\Phi(r,\varphi)$$
$$v(r,\varphi) = R(r,\varphi)\sin\Phi(r,\varphi)$$
$$u'_r = R'_r\cos\Phi - R\sin\Phi\cdot\Phi'_r = \frac{1}{r}(R'_\varphi\sin\Phi + R\cos\Phi\cdot\Phi'_\varphi) = \frac{1}{r}v'_\varphi \qquad \cdot\cos\Phi$$
$$v'_r = R'_r\sin\Phi + R\cos\Phi\cdot\Phi'_r = -\frac{1}{r}(R'_\varphi\cos\Phi - R\sin\Phi\cdot\Phi'_\varphi) = -\frac{1}{r}u'_\varphi \qquad \cdot\sin\Phi$$
$$R'_r = \frac{R}{r}\Phi'_\varphi$$
$$u'_r = R'_r\cos\Phi - R\sin\Phi\cdot\Phi'_r = \frac{1}{r}(R'_\varphi\sin\Phi + R\cos\Phi\cdot\Phi'_\varphi) = \frac{1}{r}v'_\varphi \qquad \cdot -\sin\Phi$$
$$v'_r = R'_r\sin\Phi + R\cos\Phi\cdot\Phi'_r = -\frac{1}{r}(R'_\varphi\cos\Phi - R\sin\Phi\cdot\Phi'_\varphi) = -\frac{1}{r}u'_\varphi \qquad \cdot\cos\Phi$$
$$R\cdot\Phi'_r = -\frac{1}{r}R'_\varphi$$
$$\begin{cases} R'_r = \frac{R}{r}\Phi'_\varphi \\ R'_\varphi = -Rr\Phi'_r \end{cases}$$

Задача (5)

$$\Phi(r,\varphi)=3\varphi \qquad R, f(z)=?$$

$$\Phi'_r=0$$

$$\Phi'_\varphi=3$$

$$R'_r=3\frac{R}{r} \qquad \frac{R'_r}{R}=\frac{3}{r}$$

$$\frac{1}{R}R'=(\ln R)'=\frac{3}{r}$$

$$\ln R=3\ln r+c$$

$$R=e^{3\ln r+c}=r^3\cdot c$$

$$f(z)=R(\cos\Phi+i\sin\Phi)=c_1r^3(\cos 3\varphi+i\sin 3\varphi)=c_1z^3$$

Опр Степенной ряд

$$f(z)=\sum_{k=0}^{\infty}a_n(z-a)^n$$

$$a_n=\frac{f^{(n)}(a)}{n!}$$

Формула Коши-Адамара

$$R_{\text{сх}}=\frac{1}{\overline{\lim}\sqrt[n]{|a_n|}}$$

$$\overline{\lim}\sqrt[n]{|a_n|}=0\Rightarrow R_{\text{сх}}=+\infty$$

$$\overline{\lim}\sqrt[n]{|a_n|}=+\infty\Rightarrow R_{\text{сх}}=0$$

Ряд сх абс при $|z-a|<R$
Ряд расх при $|z-a|>$ R
R - расст. от центра (от точки a) до ближайшей особой точки

$$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!} \qquad R=+\infty$$

$$\sin z=\sum_{n=0}^{\infty}\frac{(-1)^{n-1}z^{2n-1}}{(2n-1)!}=z-\frac{z^3}{3!}+\frac{z^5}{5!}+...$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + ...$$

$$(1+z)^\alpha = 1 + \sum_{n+1}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)...(\alpha-(n-1))}{n!} z^n$$

$$\alpha = const$$

Берется главное значение лог-ма
Если, $\alpha = 0, 1, 2, ...$ то $R = +\infty$
Если $\alpha \neq 0, 1, 2$, то $R = 1$, т.к. ближ особоая точка -1

$$\frac{1}{1+z} = 1 + \sum_{n+1}^{\infty} \frac{(-1)(-2)...(-n)}{n!} z^n = 1 + \sum_{n=1}^{\infty} (-1)^n z^n = \sum_{n=0}^{\infty} (-1)^n z^n$$

$$\frac{1}{(1+z)^2} = 1 + \sum_{n=1}^{\infty} \frac{(-2)(-3)...(-n-1)}{n!} z^n = \sum_{n=0}^{\infty} (-1)^n (n+1) z^n = 1 - 2z + 3z^2 ...$$

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n} =$$

Как умножать два ряда?

$$(\sum_{m=0}^{\infty} a_m z^m)(\sum_{n=0}^{\infty} b_n z^n) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_m b_n z^{m+n}$$

Если оба абсолютно сходятся, то и произв. абс . сходится

$$\exists\ m + n = k \quad (\text{фикс}), \quad k \geqslant 0$$

$$n = k - m$$

$$m \geqslant 0 \qquad n = k - m \geqslant 0 \quad \Rightarrow m \leqslant k$$

$$= \sum_{k=0}^{\infty} (\sum_{m+n=k} a_m b_n z^{m+n}) = \sum_{k=0}^{\infty} (\sum_{m=0}^{k} a_m b_{k-m}) z^k$$

<u>Задача</u> (6)

$$f(z) = e^z \sin z = \left(\sum_{n=0}^{\infty} \frac{z^m}{m!}\right) \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!}\right) =$$

$$= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{m!(2n-1)!} z^{m+2n-1}$$

$$m + 2n - 1 = k$$

$$m = k + 1 - 2n$$

$$n \geqslant 1$$

$$m \geqslant 0 \Rightarrow k + 1 - 2n \geqslant 0$$

$$n \leqslant \frac{k+1}{2} \qquad n \leqslant \left[\frac{k+1}{2}\right] \qquad n \text{ - целые}$$

$$= \sum_{k=1}^{\infty} \left(\sum_{m+2n-1=k} \frac{(-1)^{n-1}}{m!(2n-1)!} z^k \right) = \sum_{k=1}^{\infty} \left(\sum_{n=1}^{[\frac{k+1}{2}]} \frac{(-1)^{n-1}}{(k+1-2n)!(2n-1)!} \right) z^k$$

Другой способ

$$e^z \sin z = e^z \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{z+iz} - e^{z-iz}}{2i} = \frac{1}{2i}(e^{z(1+i)} - e^{z(1-i)}) =$$

$$= \frac{1}{2i} \left(\sum_{n=0}^{\infty} \frac{z^n (1+i)^n}{n!} - \sum_{n=0}^{\infty} \frac{z^n (1-i)^n}{n!} \right) =$$

$$= \sum_{n=0}^{\infty} \frac{(1+i)^n - (1-i)^n}{2in!} z^n = \sum_{n=0}^{\infty} \frac{(\sqrt{2})^n ((\cos\frac{n\pi}{4} + i \sin\frac{n\pi}{4}) - (\cos(\frac{n\pi}{4} + i\sin\frac{n\pi}{4})))}{2in!} =$$

$$\sum_{n=0}^{\infty} \frac{2^{n/2} 2i \sin\frac{n\pi}{4}}{2in!} z^n = \sum_{n=0}^{\infty} \frac{2^{n/2} \sin\frac{n\pi}{4}}{n!} z^n$$

0.1 Самый общий способ разложения

Опр

$$\frac{A}{(z-a)^k} = \frac{A}{(-a)^k (1 - \frac{z}{a})^k} = \frac{A}{(-a)^k} (1 - \frac{z}{a})^{-k}$$

$$\frac{P(z)}{Q(z)}$$

разложить в простейшие над $\mathbb{C}$ (только линейные)

Задача (7)

$$\frac{1}{z^2 + 2z + 2} \text{ - разложить по степеням } z \text{ и найти радиус сходимости}$$

Ищем корни

$$D = 4 - 4 \cdot 2 = -4 \qquad \sqrt{D} = \pm 2i$$

$$z_{1,2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$\frac{1}{z^2 + 2z + 2} = \frac{1}{(z+1-i)(z+1+i)} = \frac{A}{(z+1-i)} + \frac{B}{z+1+i}$$

$$1 = Az + A + iA + Bz + B - iB$$

$$(A + B + iA - iB) = 0$$

$$0 = A + B$$

$$A = -B$$

$$B = -\frac{i}{2}$$

$$A = \frac{i}{2}$$

$$\frac{1}{z+1+i} = \frac{1}{(1+i)(1+\frac{z}{1+i})} = \frac{1}{1+i}(1+\frac{z}{1+i})^{-1} = \frac{1}{1+i}\sum_{n=0}^{\infty}(-1)^n(\frac{z}{1+i})^n =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(1+i)^{n+1}} z^n$$

$$\frac{1}{z+1-i} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(1-i)^{n+1}} z^n$$

$$\sum_{n=0}^{\infty}(\frac{(-1)^n}{(1+i)^{n+1}} - \frac{(-1)^n}{(i-i)^{n_1}})z^n$$

$$(1+i)^{-(n+1)} = (\sqrt{2})^{-(n+1)} \left(\cos\frac{-(n+1)\pi}{4} + i\sin\frac{-(n+1)\pi}{4}\right)$$

$$(1-i)^{-(n+1)} = (\sqrt{2})^{-(n+1)} \left(\cos\frac{(n+1)\pi}{4} + i\sin\frac{(n+1)\pi}{4}\right)$$

$$\frac{1}{(1+i)^{n+1}} - \frac{1}{(1-i)^{n+1}} = (\sqrt{2})^{n+1}(-2\sin\frac{(n+1)\pi}{4})$$

$$R = \sqrt{2}$$

Дз: Закончить 7 довести до вещ. вида
Разложить 166 168 177 179 Волковыский
На разл. в ряды 457 464 472 475