Условные экстремумы   11.10.19

$$u = f(x_1, ..., x_n) \text{ при усл } \begin{cases} \Phi_1(x_1, ..., x_n) = 0 \\ \vdots \\ \Phi_m(x_1, ..., x_n) = 0 \end{cases} \qquad m < n$$

1. Точка недифф-ти f или Φ_i
2. $\operatorname{rk} \Phi' < m$
3. $\mathscr{L} = f(x_1, ..., x_n) - \lambda_1 \Phi_1(x_1, ..., x_n) - \lambda_2 \Phi_2(x_1, ..., x_n) - ... - \lambda_m \Phi_m(x_1, ..., x_n)$

$$\Phi' = \begin{pmatrix} \frac{\partial \Phi_1}{\partial x_1} & \cdots & \frac{\partial \Phi_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \Phi_m}{\partial x_1} & \cdots & \frac{\partial \Phi_m}{\partial x_n} \end{pmatrix} \qquad m \times n$$

Точка экстремума удовлетворяет системе уравнений:

$$\begin{cases} \frac{\partial \mathscr{L}}{\partial x_1} = 0 \\ \vdots \\ \frac{\partial \mathscr{L}}{\partial x_n} = 0 \\ \Phi_1(x_1, ..., x_n) = 0 \\ \vdots \\ \Phi_m(x_1, ..., x_n) = 0 \end{cases} \qquad m + n \text{ уравнений}$$

$m + n$ неизвестных $x_1, ..., x_n, \lambda_1, ..., \lambda_m$

Задача (1)

$$f(x, y) = \frac{x}{a} + \frac{y}{b} \qquad a, b > 0 \text{ при усл. } x^2 + y^2 = 1 \Leftrightarrow \underbrace{x^2 + y^2 - 1 = 0}_{\Phi} \qquad M$$

$\Phi' = \begin{pmatrix} 2x & 2y \end{pmatrix}$ 1 ур-е $\Rightarrow$ 1 строка в матрице

$\operatorname{rk} \Phi' < 1 \Rightarrow \operatorname{rk} \Phi' = 0 \quad \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \quad \notin M$

$\forall (x, y) \in M \qquad \operatorname{rk} \Phi' = 1$

$\mathscr{L} = \dfrac{x}{a} + \dfrac{y}{b} - \lambda(x^2 + y^2 - 1)$

$$\begin{cases} \mathscr{L}'_x = \frac{1}{a} - 2\lambda \cdot x = 0 \Rightarrow \lambda \neq 0 & x = \frac{1}{2a\lambda} \\ \mathscr{L}'_y = \frac{1}{b} - a\lambda \cdot y = 0 \Rightarrow & y = \frac{1}{2b\lambda} \\ x^2 + y^2 = 1 \end{cases}$$

$$\frac{1}{4a^2\lambda^2} + \frac{1}{4b^2\lambda^2} = 1$$

$$\frac{b^2 + a^2}{4a^2b^2\lambda^2} = 1$$

$$a^2 + b^2 = 4a^2b^2\lambda^2$$

$$\lambda = \pm\frac{\sqrt{a^2 + b^2}}{2ab}$$

$$1. \begin{cases} x = \frac{1 \cdot 2ab}{2a\sqrt{a^2+b^2}} = \frac{b}{\sqrt{a^2+b^2}} \\ y = \frac{a}{\sqrt{a^2+b^2}} \\ \lambda = +\frac{\sqrt{a^2+b^2}}{2ab} \end{cases}$$

$$2. \begin{cases} x = -\frac{b}{\sqrt{a^2+b^2}} \\ y = -\frac{a}{\sqrt{a^2+b^2}} \\ \lambda = -\frac{\sqrt{a^2+b^2}}{2ab} \end{cases}$$

Выясним, что будет в этих точках

$$\mathscr{L}''_{x^2} = -2\lambda$$

$$\mathscr{L}''_{xy} = 0$$

$$\mathscr{L}''_{y^2} = -2\lambda$$

$$\begin{pmatrix} -2\lambda & 0 \\ 0 & -2\lambda \end{pmatrix} \qquad \Delta_1 = 2\lambda \quad \Delta_2 = 4\lambda^2$$

$$\begin{array}{llll} \text{для} & 1. & - \quad + & \max \\ & 2. & + \quad + & \min \end{array}$$

Задача (2)

$$u = x^2 + y^2 + z^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad a > b > c > 0$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

$$\Phi' = \begin{pmatrix} \frac{2x}{a^2} & \frac{2y}{b^2} & \frac{2z}{c^2} \end{pmatrix}$$

$$\operatorname{rk}\Phi' = 0 \Rightarrow \quad x = y = z = 0 \qquad (0,0,0) \not\in M$$

$$\mathscr{L} = x^2 + y^2 + z^2 - \lambda(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1)$$

$$\begin{cases} \mathscr{L}'_x = 2x - \frac{2x\lambda}{a^2} = 0 \Rightarrow x(1 - \frac{\lambda}{a^2}) = 0 \\ \mathscr{L}'_y = 2y - \frac{2y\lambda}{b^2} = 0 \Rightarrow y(1 - \frac{\lambda}{b^2}) = 0 \\ \mathscr{L}'_z = 2z - \frac{2z\lambda}{c^2} = 0 \Rightarrow z(1 - \frac{\lambda}{c^2}) = 0 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \end{cases}$$

$$1 - \frac{\lambda}{b^2} = 0 \Rightarrow \lambda = b^2$$

$$x = z = 0 \qquad y \pm b$$

$$1 - \frac{\lambda}{c^2} = 0 \Rightarrow \lambda = c^2$$

$$x = y = 0 \qquad z = \pm c$$

$$1 - \frac{\lambda}{a^2} = 0$$

$$\lambda = a^2 \qquad 1 - \frac{a^2}{b^2} \neq 0$$

$$\Rightarrow y = 0$$

$$1 - \frac{a^2}{c^2} \neq 0 \Rightarrow z = 0$$

$$\Rightarrow \frac{x^2}{a^2} = 1$$

$$\begin{cases} x = \pm a \\ y = 0 \\ z = 0 \\ \lambda = a^2 \end{cases}$$

6 решений $\begin{pmatrix} \pm a & 0 & 0 & a^2 \end{pmatrix} \quad \begin{pmatrix} 0 & \pm b & 0 & b^2 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \pm c & c^2 \end{pmatrix}$

$$\begin{pmatrix} 2 - \frac{2\lambda}{a^2} & 0 & 0 \\ 0 & 2 - \frac{2\lambda}{b^2} & 0 \\ 0 & 0 & 2 - \frac{2\lambda}{c^2} \end{pmatrix}$$

$$\Delta_1 = 2 - \frac{2\lambda}{a^2} = 2(1 - \frac{\lambda}{a^2})$$

$$\Delta_2 = 4(1 - \frac{\lambda}{a^2})(1 - \frac{\lambda}{b^2})$$

$$\Delta_3 = 8(1 - \frac{\lambda}{a^2})(1 - \frac{\lambda}{b^2})(1 - \frac{\lambda}{c^2})$$

1. $\lambda = a^2 \qquad 0,\ 0,\ 0$
2. $\lambda = b^2 \qquad 1 - \frac{b^2}{a^2} > 0,\ 0,\ 0$
3. $\lambda = c^2 \qquad 1 - \frac{c^2}{a^2} > 0,\ (1 - \frac{c^2}{a^2})(1 - \frac{c^2}{b^2}) > 0,\ 0$

Но у нас 2 независимые переменные

$$d^2\mathscr{L} = 2(1 - \frac{\lambda^2}{a^2})(dx)^2 + 2(1 - \frac{\lambda}{b^2})(dy)^2 + 2(1 - \frac{\lambda}{c^2})(dz)^2$$

$$\frac{2x}{a^2}dx + \frac{2y}{b^2}dy + \frac{2z}{c^2}dz = 0$$

- линейная однородная система относительно диф-лов

dx, dy, dz - зависимы между собой

В точке $\left(\pm a,\ \ 0,\ \ 0,\ \ a^2\right)$ - максимум

$$\frac{\pm 2a}{a^2}dx = 0 \Rightarrow dx \equiv 0$$

$$d^2\mathscr{L} = 2(1 - \frac{a^2}{b^2})(dy)^2 + 2(1 - \frac{a^2}{c^2})(dz)^2$$

$$\begin{pmatrix} 2(1 - \frac{a^2}{b^2}) & 0 \\ 0 & 2(1 - \frac{a^2}{c^2}) \end{pmatrix} \qquad \begin{array}{lc} \Delta_1 = & 2(1 - \frac{a^2}{b^2}) < 0 \\ \Delta_2 = & 4(1 - \frac{a^2}{b^2})(1 - \frac{a^2}{c^2}) > 0 \\ - \quad + & \underline{\text{максимум}} \end{array}$$

В точке $\left(0,\ \ \pm b,\ \ 0,\ \ b^2\right)$ нет экстремума

$$\pm\frac{2b}{b^2}dy = 0 \Rightarrow dy = 0$$

$$d^2\mathscr{L} = 2(1 - \frac{b^2}{a^2})(dx)^2 + 2(1 - \frac{b^2}{c^2})(dz)^2$$

$$\begin{pmatrix} 2(1 - \frac{b^2}{a^2}) & 0 \\ 0 & 2(1 - \frac{b^2}{c^2}) \end{pmatrix} \qquad \begin{array}{lc} \Delta_1 = & 2(1 - \frac{b^2}{a^2}) > 0 \\ \Delta_2 = & 4(1 - \frac{b^2}{a^2})(1 - \frac{b^2}{c^2}) < 0 \\ + \quad - & \underline{\text{нет экстремума}} \end{array}$$

В точке $\left(0,\ \ 0,\ \ \pm c,\ \ c^2\right)$ - минимум

$$\pm\frac{2c}{c^2}dz = 0 \Rightarrow dz = 0$$

$$d^2\mathscr{L} = 2(1 - \frac{c^2}{a^2})(dx)^2 + 2(1 - \frac{c^2}{b^2})(dy)^2$$

$$\begin{pmatrix} 2(1 - \frac{c^2}{a^2}) & 0 \\ 0 & 2(1 - \frac{c^2}{b^2}) \end{pmatrix} \qquad \begin{array}{lc} \Delta_1 = & 2(1 - \frac{c^2}{a^2}) > 0 \\ \Delta_2 = & 4(1 - \frac{c^2}{a^2})(1 - \frac{c^2}{b^2}) > 0 \\ + \quad + & \underline{\text{минимум}} \end{array}$$

Задача (3)

$$u = xy + yz$$

$$\begin{cases} x^2 + y^2 = 2 \\ y + z = 2 \end{cases} \qquad \begin{cases} x^2 + y^2 - 2 = 0 \\ y + z - 2 = 0 \end{cases} \qquad M$$

$$\Phi' = \begin{pmatrix} 2x & 2y & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \operatorname{rk} \Phi' < 2$$

$$\Delta_1 = \begin{vmatrix} 2x & 2y \\ 0 & 1 \end{vmatrix} = 2x$$

$$\Delta_2 = \begin{vmatrix} 2x & 0 \\ 0 & 1 \end{vmatrix} = 2x$$

$$\Delta_3 = \begin{vmatrix} 2y & 0 \\ 1 & 1 \end{vmatrix} = 2y$$

$$\Delta_1 = \Delta_2 = \Delta_3 = 0 \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \qquad \text{противоречие с } x^2 + y^2 = 2$$

$$\forall (x, y) \in M \qquad \operatorname{rk} \Phi' = 2$$

$$\mathscr{L} = xy + yz - \lambda_1(x^2 + y^2 - 2) - \lambda_2(y + z - 2)$$

$$\begin{cases} \mathscr{L}'_x = y - 2\lambda_1 x & = 0 \\ \mathscr{L}'_y = x + z - 2\lambda_1 y - \lambda_2 & = 0 \\ \mathscr{L}'_z = y - \lambda_2 & = 0 \\ x^2 + y^2 & = 2 \\ y + z & = 2 \end{cases}$$

$$\Rightarrow x \neq 0 \quad \lambda_1 = \frac{y}{2x}$$

$x = 0 \Rightarrow y = 0$ - противореч с $x^2 + y^2 = 2$

$$\Leftrightarrow \begin{cases} \lambda_1 = \frac{y}{2x} \\ \lambda_2 = y \\ x + z - \frac{y^2}{x} - y = 0 \\ x^2 + y^2 = z \\ z = 2 - y \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = \frac{y}{2x} \\ \lambda_2 = y \\ x + 2 - y - \frac{y^2}{x} - y = 0 \\ x^2 + y^2 = 2 \\ z = 2 - y \end{cases} \overset{x \neq 0}{\Leftrightarrow}$$

$$\Leftrightarrow \begin{cases} \lambda_1 = \frac{y}{2x} \\ \lambda_2 = y \\ x^2 + 2(1-y)x - y^2 = 0 \\ x^2 = 2 - y^2 \\ z = 2 - y \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = \frac{y}{2x} \\ \lambda_2 = y \\ 2 - 2y^2 + 2(1-y)x = 0 \\ x^2 = 2 - y^2 \\ z = 2 - y \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \lambda_1 = \frac{y}{2x} \\ \lambda_2 = y \\ (1-y)(1+y) + x(1-y) = 0 \\ x^2 = 2 - y^2 \\ z = 2 - y \end{cases}$$

$$(1-y)(1+y+x) = 0$$

1. $y = 1$

$$x^2 = 1 \Rightarrow x = \pm 1$$

$$z = 2 - y = 1$$

$$\lambda_2 = 1$$

$$\lambda_1 = \pm\frac{1}{2}$$

$$(1) \begin{cases} x = 1 \\ y = 1 \\ z = 1 \\ \lambda_1 = \frac{1}{2} \\ \lambda_2 = 1 \end{cases} \qquad (2) \begin{cases} x = -1 \\ y = 1 \\ z = 1 \\ \lambda_1 = -\frac{1}{2} \\ \lambda_2 = 1 \end{cases}$$

2. $1 + y + x = 0$

$$x = -1 - y$$

$$(-1-y)^2 = 2 - y^2 \qquad y = \frac{-1 \pm \sqrt{3}}{2}$$

$$y^2 + 2y + 1 = 2 - y^2 \qquad z = 1 - \frac{-1 \pm \sqrt{3}}{2} = \frac{-2 + 1 \mp \sqrt{3}}{2}$$

$$2y^2 + 2y - 1 = 0$$