0.1 12.09.2019

0.1.1 Некоторые особенные примеры

Пример

$\lim\limits_{\substack{x\to 0\\ y\to 1}}(1+x)^{\frac{1}{x+x^2y}} = \lim\limits_{\substack{x\to 0\\ y\to 1}}((1+x)^{\frac{1}{x}})^{\frac{1}{1+xy}} = e$

Пример

$f(x,y) = \begin{cases} \frac{x^3-xy^2}{x^2+y^2} & , x^2+2^2 \neq 0 \\ a & , else \end{cases}$

1) $a =?$, т.ч. f - непр

2) $a =?$, f - непрю на прямых, проходящих через 0

Решение

1) $a = \lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x^3-xy^2}{x^2+y^2} = \lim\limits_{\substack{x\to 0\\ y\to 0}} x\frac{x^2-y^2}{x^2+y^2} = 0$

Замечание

$x^n y^m \leqslant (\sqrt{x^2+y^2})^{n+m}$ и $|x| \leqslant \sqrt{x^2+y^2}$

0.1.2 Частные производные. Определения

$f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$, $P_0 = (x_0, y_0, z_0)$

Опр

f - диф. в точке P_0, если $\exists A, B, C \in \mathbb{R}$, т.ч.

$f(x_0,+\delta x, y_0+\delta y, z+\delta z = f(x_0,y_0,z_0)+A\delta x+B\delta y+C\delta z+\overline{o}(\sqrt{(\delta x)^2+(\delta y)^2+(\delta z)^2})$

Пусть $h = (\delta x, \delta y, \delta z)^T$

$f(P_0+h) = f(P_0) = \begin{pmatrix} A \\ B \\ C \end{pmatrix}^T h + \overline{o}(|h|)$

$df(x,y,z) = Adx + Bdy + Cdz$

Дифференциал сопоставляет $(dx, dy, dz) \to Adx + Bdy + Cdz$

Опр

Частной произв. по перем. x в т. (x_0, y_0, z_0) называется предел (если $\exists$)

$$\lim_{t\to 0} \frac{f(x_0+t, y_0, t_0) - f(x_0, y_0, z_0)}{t} = \frac{\partial f}{\partial x}(x_0, y_0, z_0) = f'_x(x_0, y_0, z_0)$$

0.1.3 Частные производные. Примеры

Утв

f - дифф. $\Rightarrow \exists$ част. пр. и $A = \frac{\partial f}{\partial x}(x_0, y_0, z_0)$, $B = \frac{\partial f}{\partial x}$, $C = \frac{\partial f}{\partial x}$

Производные старшего порядка

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x})$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} \neq \text{(не всегда)}\ \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial y \partial x}$$

Частные производные сложной функции

$$w = f(x, y, z),\ \mathbb{R}^2 \to \mathbb{R}^3.\ (u, v) \to (\varphi(u, v), \psi(u, v), \chi(u, v))$$

$$w = f(\varphi(u, v), \psi(u, v), \chi(u, v))$$

$$\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x}, \frac{\partial \varphi}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial \psi}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial \chi}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial \varphi}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial \psi}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial \chi}{\partial v}$$

$$\begin{pmatrix} \frac{\partial w}{\partial u} \\ \frac{\partial w}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \psi}{\partial u} & \frac{\partial \chi}{\partial u} \\ \frac{\partial \varphi}{\partial v} & \frac{\partial \psi}{\partial v} & \frac{\partial \chi}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

Пример

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial f}{\partial x}\frac{\partial^2 \varphi}{\partial u \partial v} + (\frac{\partial^2 f}{\partial x^2}\frac{\partial u}{\partial v} + \frac{\partial^2 f}{\partial x \partial y}\frac{\partial \psi}{\partial v} + \frac{\partial^2 f}{\partial x \partial z}\frac{\partial \chi}{\partial v})\frac{\partial \varphi}{\partial u} + ...$$

Пример

$$F = f(x, xy, xyz) = f(u, v, w)$$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u}1 + \frac{\partial f}{\partial v}y + \frac{\partial f}{\partial w}yz$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + y\frac{\partial}{\partial v} + uz\frac{\partial}{\partial w}$$

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial u}) + \frac{\partial}{\partial x}(\frac{\partial f}{\partial v})y + \frac{\partial f}{\partial v}\underbrace{\frac{\partial}{\partial x}(y)}_{=0} + \frac{\partial}{\partial x}(\frac{\partial f}{\partial w})yz + \frac{\partial f}{\partial w}\frac{\partial}{\partial x}(yz)$$

$$\begin{aligned}\frac{\partial}{\partial x}(\frac{\partial f}{\partial u}) &= \frac{\partial^2 f}{\partial u^2} + y\frac{\partial^2 f}{\partial u \partial v} + yz\frac{\partial^2 f}{\partial u \partial w} = \\ &= \frac{\partial^2 f}{\partial u^2} + y\frac{\partial^2 f}{\partial v^2} + (yz)^2\frac{\partial^2 f}{\partial w^2} + zy\frac{\partial^2 f}{\partial u \partial v} + 2y^2 z\frac{\partial^2 f}{\partial v \partial w} + 2yz\frac{\partial^2 f}{\partial u \partial w}\end{aligned}$$

<u>Пример</u>

Дано $u = x^y$, найти $\dfrac{\partial^2 u}{\partial x^2}$, $\dfrac{\partial^2 u}{\partial y^2}$, $\dfrac{\partial^2 u}{\partial x \partial y}$

$$\frac{\partial u}{\partial x} = yx^{y-1}, \quad \frac{d^2 u}{\partial x^2} = y(y-1)x^{y-2}$$

$$\frac{\partial u}{\partial y} = \ln(x)x^y, \quad \frac{\partial^2 y}{\partial y^2} = \ln^2(x)x^y$$

$$\frac{\partial^2 u}{\partial x \partial y} = x^{y-1} + y\ln(x)x^{y-1}$$