# Uniform error bounds of the ensemble transform Kalman filter for infinite-dimensional dynamics with multiplicative covariance inflation

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- Research interests:
  - Mathematics of Data Assimilation(DA)
  - Uncertainty Quantification (UQ) related to fluid dynamics
  - Numerical analysis (deterministic & stochastic)
  - Topological Data Analysis (TDA)

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# Data Assimilation(DA)

Seamless integration of observation data into numerical models.

Mainly applied to numerical weather prediction.

# Math of DA

Situation:

- Little mathematical explanation for nonlinear DA
  $\rightarrow$ Motivation
- Only few mathematicians working on in Japan
  $\rightarrow$ Need to expand
- DA provides many mathematical problems related to fields such as
  *PDE, dynamical system, inverse problem, optimization, probability, stochastic analysis, numerical analysis, computational science ...*

Goal:
Not only explain but also contribute to applied fields of DA

Today's talk:
Basic explanation of a Nonlinear DA algorithm in an ideal setting.

# Spaces

Only focus on a finite dimensional case in this talk.

- $\mathcal{X} = \mathbb{R}^{N_x}$: state space with a norm $\|\cdot\|$ and a inner product $\langle \cdot, \cdot \rangle$.
  $\mathcal{Y} = \mathbb{R}^{N_y} \subset \mathcal{X}$: observation space. ($N_x, N_y \in \mathbb{N}$, $N_x \geq N_y$.)
- In general, we can generalize $\mathcal{X}$ to a separable Hilbert space and the some results still hold by the similar way.
  See the paper [1].

# State space model

## Dynamical model ($\mathcal{F} \leftrightarrow \mathcal{M}$)

$\mathcal{F} : \mathbb{R}^{N_x} \to \mathbb{R}^{N_x}$,

$$\text{(Continuous)} \qquad \frac{d\boldsymbol{x}^\dagger}{dt} = \mathcal{F}(\boldsymbol{x}^\dagger), \quad \boldsymbol{x}^\dagger(0) = \boldsymbol{x}_0^\dagger \in \mathbb{R}^{N_x}. \tag{1}$$

Equivalently, time step $h > 0$, $\mathcal{M} : \mathbb{R}^{N_x} \to \mathbb{R}^{N_x}$,

$$\text{(Discrete)} \qquad \boldsymbol{x}_j^\dagger = \mathcal{M}(\boldsymbol{x}_{j-1}^\dagger), \quad j \in \mathbb{N}. \tag{2}$$

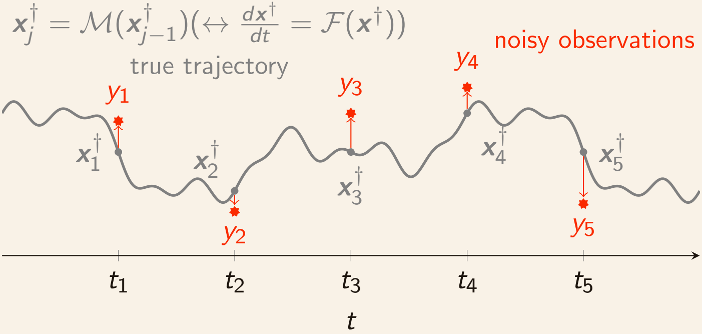
## Observation model

Observation period $h$,

$$y_j = H\boldsymbol{x}_j^\dagger + \xi_j, \quad \text{i.i.d. } \xi_j \sim N(0, R), \tag{3}$$

where $H \in \mathbb{R}^{N_y \times N_x}$ and $R \in \mathbb{R}^{N_y \times N_y}$ positive definite.

# State space model & Filtering problem




Uncertain initial state $x_0^\dagger \sim \mu_0 \to$ estimate $x_j^\dagger$.

**Filtering Problem**

Estimate $x_j^\dagger$ by probability distribution $\mathbb{P}(dx_j|Y_j)$ provided observations $Y_j = \{y_1, \ldots, y_j\}$.

# Data assimilation process

2 steps in one cycle: (i) forecast & (ii) analysis.

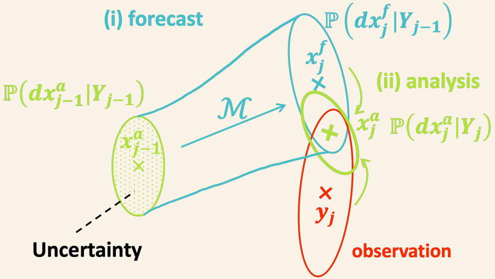
$\mathbb{P}(dx^a_{j-1}|Y_{j-1})$

$\mathcal{M} \Downarrow$ **(i) forecast**

$\mathbb{P}\left(dx^f_j|Y_{j-1}\right)$

$y_j \Downarrow$ **(ii) analysis**

$\mathbb{P}(dx^a_j|Y_j)$




**issue**

If $\mathcal{M}$ is nonlinear, it is difficult to compute propagation of uncertainty in (i) $\rightarrow$ employ EnKF.

## Notations: ensemble

Ensemble $\rightarrow$ a set of particle.
Preparing for ensemble approximation of a probability distribution $\mathbb{P}(d\boldsymbol{x}_j|Y_j)$ to estimate the true state.

- $X^f = [\boldsymbol{x}^{f(1)}, \ldots, \boldsymbol{x}^{f(m)}] \in \mathcal{X}^m \in \mathbb{R}^{N_x \times m}$: forecast ensemble,
  $X^a = [\boldsymbol{x}^{a(1)}, \ldots, \boldsymbol{x}^{a(m)}] \in \mathcal{X}^m \in \mathbb{R}^{N_x \times m}$: analysis ensemble ($m \in \mathbb{N}$).

- for $X^f$, mean: $\overline{\boldsymbol{x}}^f = \frac{1}{m}\sum_{n=1}^{m} \boldsymbol{x}^{f(n)} \in \mathbb{R}^{N_x}$,
  deviations: $dX^f = [\boldsymbol{x}^{f(n)} - \overline{\boldsymbol{x}}^f]_{n=1}^{m} \in \mathbb{R}^{N_x \times m}$,

  ensemble covariance matrix:
  $P^f = \mathrm{Cov}(X^f) := \frac{1}{m-1} dX^f {dX^f}^\top \in \mathbb{R}^{N_x \times N_x}$.

  (similarly for analysis ensemble $X^a$)

# Ensemble Kalman filter (EnKF)

## Ensemble approximation

$$X_j^a = [\boldsymbol{x}_j^{a(n)}]_{n=1}^m, \left( X_j^f = [\boldsymbol{x}_j^{f(n)}]_{n=1}^m \right) \in \mathcal{X}^m,$$

$$\mathbb{P}(d\boldsymbol{x}_j | Y_j) \approx \frac{1}{N} \sum_{n=1}^m \delta_{\boldsymbol{x}_j^{a(n)}}(dv_j), \quad \text{(similarly } \mathbb{P}(d\boldsymbol{x}_j | Y_{j-1})\text{)}.$$

Algorithm:

(i) $X_{j-1}^a \to X_j^f$: Evolve each member,

$$\boldsymbol{x}_j^{a(n)} = \mathcal{M}(\boldsymbol{x}_{j-1}^{f(n)}), \quad n = 1, \ldots, m.$$

(ii) $X_j^f, y_j \to X_j^a$: Incorporate $y_j$ based on least square. 2 methods.

- PO [2, 3]: Stochastic
- ETKF [4, 6, 7]: Deterministic

# Perturbed Observation (PO) methods

(ii) $X_j^f, y_j \to X_j^a$: Replicating observations by adding random perturbations.

$$y_j^{(n)} = y_j + \xi_j^{(n)}, \quad \xi_j^{(n)} \sim N(0, R), \quad n = 1, \ldots, m.$$

Kalman update for each member:

$$\boldsymbol{x}_j^{a(n)} = (I_{N_x} - K_j H)\boldsymbol{x}_j^{f(n)} + K_j y_j^{(n)}, \quad n = 1, \ldots, m.$$

(Kalman gain) $K_j := P_j^f H^\top (H P_j^f H^\top + R)^{-1}$, $P_j^f := \mathrm{Cov}(X_j^f)$.

$\rightarrow$ Simple, but stochastic

Note: The PO method requires many ensemble to reduce errors induced by random perturbations.

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[3] G. Burgers, P. Jan van Leeuwen, and G. Evensen, Analysis Scheme in the Ensemble Kalman Filter, Mon. Wea. Rev., 126, 1719-1724, 1998.

# Ensemble Transform Kalman filter（ETKF）

(ii) $X_j^f, y_j \to X_j^a$: $X_j^f = \overline{\boldsymbol{x}}_j^f + dX_j^f$.

$$\begin{aligned} \text{mean} \quad \overline{\boldsymbol{x}}_j^a &= (I_{N_x} - K_j H)\overline{\boldsymbol{x}}_j^f + K_j y_j, \\ \text{dev.} \quad dX_j^a &= dX_j^f T_j, \end{aligned} \tag{4}$$

where the transform matrix $T_j \in \mathbb{R}^{N \times N}$ holds

$$\mathrm{Cov}(dX_j^f T_j) = (I_{N_x} - K_j H) P_j^f. \tag{5}$$

Finally, put $X^a = \overline{\boldsymbol{x}}^a + dX^a$.

$\to$ Deterministic

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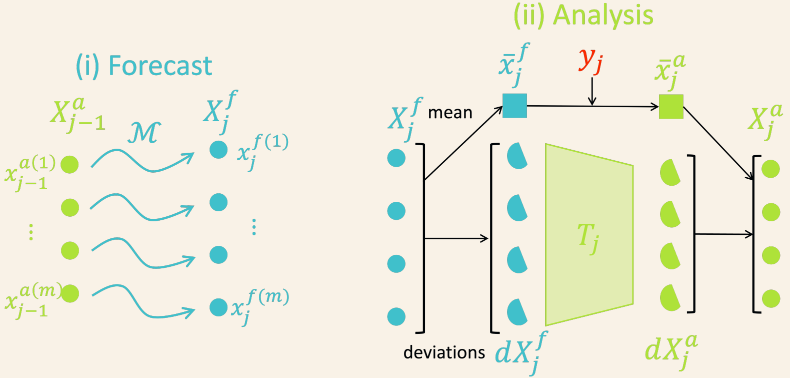
[4] . H. Bishop, B. J. Etherton, and S. J. Majumdar, Adaptive Sampling with the Ensemble Transform Kalman Filter. Part I: Theoretical Aspects, Mon. Wea. Rev., 129, pp. 420–436, 2001.

# ETKF: Illustration



# ETKF: Motivation

In (5), we arrange $T_j$ so that the analysis covariance $P_j^a = \mathrm{Cov}(dX_j^a)$ corresponds to the desired one derived from the KF.

$$P_j^a = \underbrace{\mathrm{Cov}(dX_j^f T_j)}_{\text{analysis cov.}} = \underbrace{(I_{N_x} - K_j H)P_j^f}_{\text{desired cov.}} .$$

→ The ETKF is 'consistent' with the KF when applied to a linear and Gaussian system.

# ETKF: How to get $T_j$

**Remark 1**

$T_j$ is given by

$$T_j = \left( I_m + \frac{1}{m-1} dX_j^{f\top} H^\top R^{-1} H dX_j^f \right)^{-\frac{1}{2}} \in \mathbb{R}^{m \times m}. \quad (6)$$

See the detail derivation in the paper [1].

## Covariance inflation numerical technique for EnKF

Underestimation of $P_j^f$ through DA cycle causes a poor estimation.
$\rightarrow$ Stabilize estimation by inflating $P_j^f$ before the analysis step.

### Two basic methods

- additive ($\alpha > 0$): $P_j^f \rightarrow P_j^f + \alpha^2 I_{N_x}$ (EnKF)
- multiplicative ($\alpha > 1$): $P_j^f \rightarrow \alpha^2 P_j^f$ (EnKF) or $dX_j^f \rightarrow \alpha dX_j^f$ (ETKF)

### Remark

Additive inflation technique is not applicable to the ETKF because $P_j^f$ is not explicitly used in the computation of $T_j$:
$T_j = (I_m + \frac{1}{m-1}(dX_j^f)^\top H^\top R^{-1} H dX_j^f)^{-\frac{1}{2}}$

# Numerical Example

OSSE: ETKF($m = 25$) + Lorenz 96($N = 40$)
$H = I$, $R = I$, dt $= 0.1$, obs. per 5 steps.

multiplicative inflation $\alpha = 1.2$

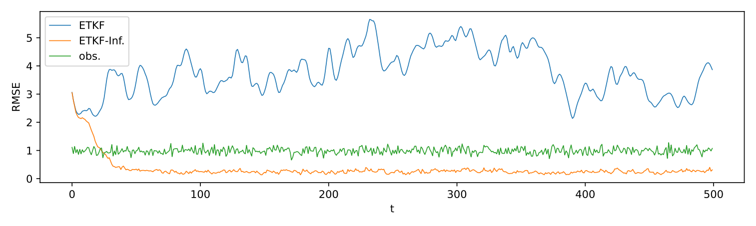



Figure: RMSEs of ETKF, ETKF + inflation, &observation.

# ※ Basic information in DA

Lorenz 96 chaotic advection model on a periodic domain.
$\boldsymbol{x} \in \mathbb{R}^N$, $F$: external force.

$$\frac{dx^i}{dt} = (x^{i+1} - x^{i-2})x^{i-1} - x^i + F, \quad i = 1, \ldots, N,$$
$$x^{-1} = x^{N-1}, x^0 = x^N, x^{N+1} = x^1.$$

OSSE (Observing System Simulation Experiment)

1. Compute a true trajectory $(\boldsymbol{x}_j^\dagger)_{j=1}^J$ for $J \in \mathbb{N}$.
2. Generate observations $(y_j)_{j=1}^J$ from $(\boldsymbol{x}_j^\dagger)_{j=1}^J$.
3. Assimilate with $(y_j)_{j=1}^J$ and obtaining analysis values $(\overline{\boldsymbol{x}}_j^a)_{j=1}^J$.
4. Compute RMSE between $\boldsymbol{x}_j^\dagger$ and $\overline{\boldsymbol{x}}_j^a$:

$$\text{RMSE at time } j = \frac{1}{\sqrt{N_x}} \|\boldsymbol{x}_j^\dagger - \overline{\boldsymbol{x}}_j^a\|.$$

# Target dynamics

Dissipative & Chaotic dynamics such as
Lorenz63, 96 equations, and incompressible 2D Navier-Stokes equations.

Many dynamical models related to geoscience also exhibit this behaviors.

Example: Lorenz 96, $\boldsymbol{x} \in \mathbb{R}^{40}$.

$$\frac{dx_i}{dt} = \underbrace{(x_{i+1} - x_{i-2})x_{i-1}}_{\text{nonlinear conserving}} \quad \underbrace{-x_i}_{\text{linear dissipating}} \quad \underbrace{+F}_{\text{forcing}} \quad .$$

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We can generalize the dynamics by PDE like formulation:

$$\frac{du}{dt} + \mathcal{A}u + \mathcal{B}(u, u) = f, \quad u(0) = u_0.$$

$\mathcal{A}$: linear dissipating, $\mathcal{B}$: nonlinear conserving, $f$: external force.

# Demonstration: Lorenz 63

See the typical behaviors $\downarrow$
Lorenz simulator on a web browser:
`https://kotatakeda.github.io/lorenz-webgl/`

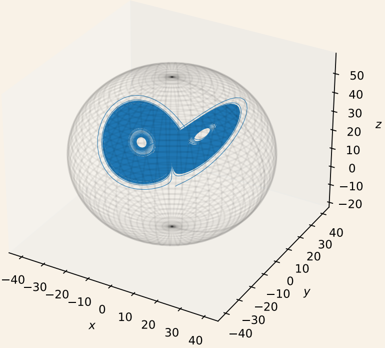
# Mathematical description: dissipative

Energy dissipation $\rightarrow$ the trajectory is bounded on the state space.

**Bounded by $\rho > 0$**

$\exists \rho > 0$, s.t. $\forall x \in B(\rho) \coloneqq \{x \in \mathcal{X} \mid \|x\| \leq \rho\}$, $\mathcal{M}(x) \in B(\rho)$.

The bounded trajectory of Lorenz'63 model $\rightarrow$



# Mathematical description: Chaotic

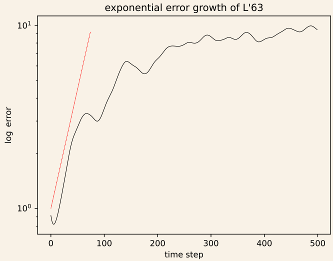
**Error growth rate $\beta > 0$**

$\exists \beta > 0$ s.t. $\forall x \in B(\rho)$, $\forall z \in \mathcal{X}$,

$$|\mathcal{M}(x) - \mathcal{M}(z)| \leq e^{\beta h}|x - z|. \tag{7}$$

$\beta$ provides the maximum of the exponential error growth rate.

Exponential error growth of Lorenz'63 model $\rightarrow$




[2] D. Kelly et al., Well-Posedness and Accuracy of the Ensemble Kalman Filter in Discrete and Continuous Time, Nonlinearity, 27, 2579–2603, 2014.

# How can we calculate the constants?

The bound radius $\rho$ and the exponential error growth rate $\beta$ have been estimated for Lorenz'63, '96 models and 2D Navier-Stokes equations [9, 10, 11].

However, for the complex models used in ocean and weather simulations, it is impossible to compute their constants analytically.
$\rightarrow$ required to estimate them numerically.

[9] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Springer, 1997. [10] K. Hayden, E. Olson, and E. S. Titi, Discrete Data Assimilation in the Lorenz and 2D Navier-Stokes Equations, Physica D: Nonlinear Phenomena, 240, 1416–1425, 2011. [11] K. J. H. Law, et al., Filter accuracy for the lorenz 96 model: Fixed versus adaptive observation operators., PHYSICA D-NONLINEAR PHENOMENA, 325, 1–13, JUN 15 2016

# Assumption to dynamics

Assume these conditions for models dynamics $\mathcal{M}$ in the following analysis.

**Bounded by $\rho > 0$**

$\exists \rho > 0$, s.t. $\forall x \in B(\rho) := \{x \in \mathcal{X} \mid \|x\| \leq \rho\}$, $\mathcal{M}(x) \in B(\rho)$.

**Error growth rate $\beta > 0$**

$\exists \beta > 0$ s.t. $\forall x \in B(\rho)$, $\forall z \in \mathcal{X}$,

$$|\mathcal{M}(x) - \mathcal{M}(z)| \leq e^{\beta h}|x - z|. \tag{8}$$

# Mean Square Error (MSE)

Here, we evaluate the error between a variable $\boldsymbol{z} \in \mathbb{R}^{N_x}$ and the true state $\boldsymbol{x}^\dagger \in \mathbb{R}^{N_x}$ by the Mean Square Error (MSE).

$$\mathrm{MSE} \; = \frac{1}{N_x}\|\boldsymbol{x}^\dagger - \boldsymbol{z}\|^2 = \frac{1}{N_x}\sum_{i=1}^{N_x}|x_i^\dagger - z_i|^2.$$

The analysis MSE $E_j$ is given by

$$E_j = \frac{1}{N_x}\|\boldsymbol{x}_j^\dagger - \overline{\boldsymbol{x}}_j^a\|^2.$$

The analysis MSE of each member $E_j^{(n)}$ is given by

$$E_j^{(n)} = \frac{1}{N_x}\|\boldsymbol{x}_j^\dagger - \boldsymbol{x}_j^{a(n)}\|^2,$$

for $n = 1, \ldots, m$. We consider expectations of these values.

# Strategy of the error analysis

Let $E_j^a$ and $E_j^f$ be the analysis and forecast MSE at time $j$.
Strategy: estimating the change of the MSE in one DA cycle.

$$E_{j-1}^a \underset{\times \theta_j^f}{\overset{(i) forecast}{\rightarrow}} E_j^f \underset{\times \theta_j^a}{\overset{(ii) analysis}{\rightarrow}} E_j^a$$

The upper bound of MSE is

- (i) expanded by $\mathcal{M} \rightarrow$ rate $\theta_j^f$
- (ii) contracted by assimilating the observation $\rightarrow$ rate $\theta_j^a$ (& increased with observation noise).

Key

To obtain the bounded MSE, it is necessary to show

$$\text{total rate } \theta_j = \theta_j^f \times \theta_j^a < 1.$$

# PO + additive inflation (previous result)

Consider dissipative & chaotic dynamics $\mathcal{M}$. Let an inflation parameter $\alpha > 0$. Assume that $H = I_{N_x}$, $R = r^2 I_{N_x}$ for $r > 0$. Let $E_j^{(n)}$ be MSE of each member $n = 1, \ldots, m$.

**Error analysis for the PO + additive inflation [2]**

For any $n = 1, \ldots, m$ and $j \in \mathbb{N}$,

$$E_j^{(n)} \leq \theta^j E_0^{(n)} + \frac{2\min\{m, N_x\}}{N_x} r^2 \frac{1-\theta^j}{1-\theta}. \tag{9}$$

where $\theta$ is

$$\theta = \left(\frac{r^2}{r^2+\alpha^2}\right)^2 \quad \times \quad e^{2\beta h}$$

contraction at (ii) expansion at (i)

Note: $\theta \to 0 \quad (\alpha \to \infty)$.

# Cor: PO + additive inflation (previous result)

**Error bound for the PO + additive inflation [2]**

Take $\alpha \gg 1$, then we have $\theta < 1$ and for any $n = 1, \dots, m$,

$$\limsup_{j} E_j^{(n)} \leq \frac{2m}{N_x(1-\theta)} r^2 = O(r^2). \tag{10}$$

order of observation noise

# Essence of proof

**flow**

Deriving the inequality between before and after a DA cycle.
(i) error is expanded ($\theta_j^f = e^{2\beta h}$) by $\mathcal{M}$.
(ii) error contracts ($\theta_j^a = \left(\frac{r^2}{r^2+\alpha^2}\right)^2$) $\rightarrow$ detail in next slide
& observation noise is added twice [a] ($\frac{2m}{N_x} r^2$).

[a] since an extra noise is added when perturbing observation.

$$
\begin{aligned}
E_j^{(n)} &\leq \theta_j^a \theta_j^f E_{j-1}^{(n)} + \frac{2m}{N_x} r^2 \\
&\leq \theta_j^a \theta_j^f \left( \theta_j^a \theta_j^f E_{j-2}^{(n)} + \frac{2m}{N_x} r^2 \right) + \frac{2m}{N_x} r^2 \\
&\leq \dots \text{ repeat up to } j = 0.
\end{aligned}
$$

# Essence of proof: where $\theta_j^a$ comes from?

Important relation in the EnKF algorithms:

$$\boldsymbol{x}_j^{a(n)} = (I_{N_x} - K_j H)\boldsymbol{x}_j^{f(n)} + K_j y_j \tag{11}$$

Apparently,

$$\boldsymbol{x}_j^{\dagger} = (I_{N_x} - K_j H)\boldsymbol{x}_j^{\dagger} + K_j H \boldsymbol{x}_j^{\dagger}$$

By subtracting both sides, we have

$$e_j^{a(n)} = (I_{N_x} - K_j H) e_j^{f(n)} + \text{ \{obs. noise term\}}.$$

Hence,

$$E_j^{a(n)} \leq \|(I_{N_x} - K_j H)\|_{op}^2 E_j^{f(n)} + \text{ \{obs. noise term\}},$$

where the operator norm $\|A\|_{op} = \sup_{\boldsymbol{x} \neq 0} \frac{\|A\boldsymbol{x}\|}{\|\boldsymbol{x}\|}$.

# Essence of proof: where $\theta_j^a$ comes from?

$$E_j^{a(n)} \leq \|(I_{N_x} - K_j H)\|_{op}^2 E_j^{f(n)} + \ \{\text{obs. noise term}\},$$

then,

$$\theta_j^a = \|I_{N_x} - K_j H\|_{op}^2$$

By using Woodbly's matrix Lemma, it is transformed:

$$I_{N_x} - K_j H = (I_{N_x} + P_j^f H^\top R^{-1} H)^{-1}.$$

Therefore,

**Key relation**

For $G_j = P_j^f H^\top R^{-1} H$,

$$\theta_j^a = \|(I_{N_x} + G_j)^{-1}\|_{op}^2.$$

# Essence of proof: where $\theta_j^a$ comes from?

**Key relation**

For $G_j = P_j^f H^\top R^{-1} H$,

$$\theta_j^a = \|(I_{N_x} + G_j)^{-1}\|_{op}^2.$$

In the situation of [2], $G_j = r^{-2}(P_j^f + \alpha^2 I_{N_x})$ and this is symmetric. Hence, by using a formula to calculate the operator norm of a symmetric matrix

$$\|(I_{N_x} + G_j)^{-1}\|_{op}^2 = \left(\frac{1}{1 + \lambda_{min}(G_j)}\right)^2 = \left(\frac{1}{1 + \alpha^2/r^2}\right)^2.$$

Therefore,

$$\theta_j^a = \left(\frac{1}{1 + \alpha^2/r^2}\right)^2 = \left(\frac{r^2}{r^2 + \alpha^2}\right)^2.$$

# Effect of additive inflation

## Shifting eigenvalues

The minimum eigenvalue $\lambda_{min}(P_j^f)$ can be 0 in general.

$\underset{\text{additive inflation}}{\rightarrow}$ $\lambda_{min}(P_j^f + \alpha^2 I_{N_x}) > \alpha^2$ (positive!)

The information of $P_j^f$ is ignored in the analysis above!

## Increasing the estimated uncertainty of forecast

By increasing $\alpha > 0$, we can increase uncertainty of forecast $X_j^f$ compared with observation $y_j$.

$\rightarrow$ weight of $\boldsymbol{x}_j^{f(n)}$ is reduced when averaged into $\boldsymbol{x}_j^{a(n)}$.

$\rightarrow$ contraction of the error $\left(\frac{r^2}{r^2+\alpha^2}\right)^2 < 1$.

# ETKF + multiplicative inflation

Consider dissipative & chaotic dynamics $\mathcal{M}$. Let $\epsilon > 0$. Assume that $H = I_{N_x}$, $R = r^2 I_{N_x}$ for $r > 0$ and $m \gg N_x$ so that $\lambda_{min}(P_0^a) > 0$. Let $E_j$ be the analysis MSE.

**Error bound for the ETKF [1, T.-Sakajo]**

Take $\alpha \gg 1$, then $\lambda_{min}(P_j^f) > \lambda_*$ for any $j \in \mathbb{N}$ and $\theta < 1$,

$$\limsup_j E_j \leq \frac{r^2}{1-\theta} + O(r^4) = O(r^2). \tag{12}$$

order of observation noise

where $\theta$ is

$$\theta = \left(\frac{r^2}{r^2+\alpha^2\lambda_*}\right)^2 \times e^{2(\beta+\epsilon)h}$$

contraction at (ii) expansion at (i)

# Essence of proof

Goal: to apply the same strategy with the previous work, we need to check that $\lambda_{min}(P_j^f)$ is always positive.

**Using assumption $m > N_x$**

However, multiplicative inflation does not ensure this. Instead, assumption $m > N_x$ avoids $\lambda_{min}(P_0^f) = 0$ .

Then, an appropriate choice of $\alpha$ sustains the positivity: $\lambda_{min}(P_j^f) \geq \lambda_*$ for some $\lambda_* > 0$. Proved by considering the time evolution of the minimum eigenvalue.
$\rightarrow$ original point of our analysis (in the next slide)

The rest is similar with the previous work.

# The time evolution of the minimum eigenvalue

Set $\lambda_j^f = \lambda_{min}(P_j^f)$ and $\lambda_j^a = \lambda_{min}(P_j^a)$. Let us estimate the change

$$\lambda_{j-1}^f \to \lambda_{j-1}^a \to \lambda_j^f.$$

$\lambda_{j-1}^f \to \lambda_{j-1}^a$: inflation and analysis

$$\lambda_{j-1}^a = \frac{\alpha^2 \lambda_{j-1}^f}{1 + \frac{\alpha^2}{\gamma^2} \widehat{\lambda}_{j-1}^f}$$

$\lambda_{j-1}^a \to \lambda_j^f$: forecast

$$\lambda_j^f \geq e^{-ch} \lambda_{j-1}^a.$$

where $c = 8 \frac{m}{m-1} \beta \rho^2$.
Finally, we have

$$\lambda_j^f \geq \frac{e^{-ch} \alpha^2 \lambda_{j-1}^f}{1 + \frac{\alpha^2}{\gamma^2} \lambda_{j-1}^f}.$$

# The time evolution of the minimum eigenvalue

By considering an asymptotic behavior of the right hand side

$$\lambda_j^f \geq \frac{e^{-ch}\alpha^2\lambda_{j-1}^f}{1 + \frac{\alpha^2}{\gamma^2}\lambda_{j-1}^f},$$

we have

$$\liminf_{j\to\infty} \lambda_j^f \geq \frac{\gamma^2}{\alpha^2}(e^{-ch}\alpha^2 - 1) > 0,$$

if and only if $e^{-ch}\alpha^2 > 1$.

The uniform-in-time positivity of $\lambda_{min}(P_j^f)$ is ensured by a sufficiently large inflation parameter $\alpha$.

# Effect of multiplicative inflation

**Not improving eigenvalue**

If $\lambda_{min}(P_j^f) = 0, \underset{\text{multiplicative inflation}}{\rightarrow} \lambda_{min}(\alpha^2 P_j^f) = 0.$

**Sustaining positivity (against contraction of $P^f$)**

If $\lambda_{min}(P_0^f) > 0$, we have $\lambda_{min}(P_j^f) > 0$ with sufficiently large $\alpha > 1$.

**Increasing the estimated uncertainty of forecast**

By increasing $\alpha > 1$,

we obtain contraction of the error $\left(\frac{r^2}{r^2+\alpha^2\lambda_*}\right)^2 < 1$.

# Summary

EnKFs: PO(stochastic) & ETKF(deterministic)
Error Analysis of EnKF:

- Key: $G_j = P_j^f H^\top R^{-1} H, \quad \theta_j^a = \|(I_{N_x} + G_j)^{-1}\|_{op}^2$.
- Desired: $\theta_j^a < 1$.

Results:

- (previous) PO + additive inflation:
  $\alpha \gg 1 \rightarrow$ bounded MSE
- (our) ETKF + multiplicative inflation:
  $\alpha \gg 1$, $m > N_x \rightarrow$ bounded MSE

## Non-degeneracy of $G_j$ is important!

(in the current analysis)

# Discussion

**Next direction: analysis in realistic settings**

- partial observation: $rank(H) < N_x$
- small ensemble size: $m < N_x$

Show MSE $\leq Cr^2$ for some $0 < C < 1$.

In these situations, $G_j$ is degenerated.

**Ideas**:

- Dimension reduction by Lyapunov Analysis
- Localization & others

# Idea1: Dimension reduction by Lyapunov Analysis

dimension reduction by introducing splitting of state space into stable/unstable space with respect to $\mathcal{M}$, i.e. there exists $\lambda_+ > 1 > \lambda_-$ such that for any point $\boldsymbol{x}$ in the attractor,

$$\mathbb{R}^{N_x} = E_x^u \oplus E_x^s,$$
$$\|J_x v\| \geq \lambda_+ \|v\|, \forall v \in E_x^u, \quad \|J_x v\| \leq \lambda_- \|v\|, \forall v \in E_x^s,$$

where $J_x$ is the Jacobian of $\mathcal{M}$.

**reducing to analysis in low dimension**

Only focus on the error contraction in $E_x^u$.
$\to$ Show $\|(P_x^u(I_{N_x} + G_j)P_x^u)^{-1}\|_{op} < 1$ for the projection $P_x^u$ onto $E_x^u$.

# Idea2: Localization & others

- partial observation: $rank(H) < N_x \to$ delay embedding.
- small ensemble size: $m < N_x \to$ covariance localization.
- improve bound: $C < 1 \to$ observation localization.

We need to use information of spatial correlations/interactions of model dynamics.

For example, Lorenz'96 only affected by neighbor grid points.

$$\frac{dx_i}{dt} = (x_{i+1} - x_{i-2})x_{i-1} - x_i + F.$$

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