PORTAL DA MATEMÁTICA
OBMEP

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Lista 4 - EDOs

1) Resolva as EDOs dada:

a) $xy'+2y = x^2 - x + 1$ para $y(1) = \frac{1}{2}$ e $x > 0$

$$y' + \underbrace{\frac{2}{x}}_{P(x)} y = \underbrace{x - 1 + \frac{1}{x}}_{Q(x)} \qquad x > 0$$

$$\to I = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$$

$$\Rightarrow x^2 y' + 2xy = x^3 - x^2 + x$$

$$\frac{d(x^2 y)}{dx} = x^3 - x^2 + x$$

$$\Rightarrow d(x^2 y) = (x^3 - x^2 + x)dx \Rightarrow x^2 y = \int (x^3 - x^2 + x)dx = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} + C$$

$$\Rightarrow y = \frac{x^2}{4} - \frac{x}{3} + \frac{1}{2} + \frac{C}{x^2} \Rightarrow \frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C \Rightarrow C = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\therefore \boxed{y = \frac{x^2}{4} - \frac{x}{3} + \frac{1}{2} + \frac{1}{12x^2}}$$

b) $xy'+2y = \operatorname{sen} x$ para $y(\frac{\pi}{2}) = 1$

$$y' + \frac{2}{x} y = \frac{\operatorname{sen} x}{x} \to I = x^2 \qquad x \neq 0$$

$$x^2 y' + 2xy = x \operatorname{sen} x$$

$$\frac{d(x^2 y)}{dx} = x \operatorname{sen} x \Rightarrow x^2 y = \int x \operatorname{sen} x \, dx$$

$$\int x \operatorname{sen} x \, dx = \int u \, dv \, dx = -x\cos x - \int -\cos x \, dx = -x\cos x + \operatorname{sen} x + C$$

$u = x$

$dv = \operatorname{sen} x \Rightarrow v = -\cos x$

$$\Rightarrow y = \frac{\operatorname{sen} x}{x^2} - \frac{\cos x}{x} + \frac{C}{x^2}$$

$$1 = \frac{1}{(\frac{\pi}{2})^2} - 0 + \frac{C}{(\pi/2)^2} \Rightarrow C + 1 = \frac{\pi^2}{4} \Rightarrow C = \frac{\pi^2}{4} - 1$$

$$\therefore \boxed{y = \frac{\operatorname{sen} x}{x^2} - \frac{\cos x}{x} + \frac{\frac{\pi^2}{4} - 1}{x^2}}$$

c) $\frac{dy}{dx} = \frac{2x}{1+2y}$ para $y(2) = 0$

$(1+2y)\,dy = 2x\,dx$

$\int (1+2y)\,dy = \int 2x\,dx \Rightarrow y + y^2 = x^2 + C$

$0 + 0 = 2^2 + C \Rightarrow C = -4$

$\therefore \boxed{y^2 + y = x^2 - 4}$

d) $\frac{dy}{dx} = \frac{1+3x^2}{3y^2 - 6y}$ para $y(0) = 1$

$(3y^2 - 6y)\,dy = (1+3x^2)\,dx$

$\int (3y^2 - 6y)\,dy = \int (1+3x^2)\,dx$

$$\left.\begin{array}{l} y^3 - 3y^2 = x + x^3 + C \\ 1 - 3 = 0 + C \Rightarrow C = -2 \end{array}\right\} \boxed{y^3 - 3y^2 = x^3 + x - 2}$$

e) $y^2 \frac{dy}{dx} = \frac{arc\ sen\ x}{(1-x^2)^{1/2}}$ para $y(\frac{\sqrt{2}}{2}) = 1$

$y^2\,dy = \frac{\arcsen x}{\sqrt{(1-x^2)}}\,dx \Rightarrow y^2 = \int \frac{\arcsen x}{\sqrt{1-x^2}}\,dx = \int u\,du = \frac{u^2}{2} + C$

$u = \arcsen x$

$du = \frac{1}{\sqrt{1-x^2}}\,dx$

$$\left.\begin{array}{l} \Rightarrow y^2 = \frac{(\arcsen x)^2}{2} + C \\ 1 = \frac{\left(\frac{\pi}{4}\right)^2}{2} + C \Rightarrow C = \frac{2 - \frac{\pi^2}{16}}{2} \Rightarrow C = \frac{32 - \pi^2}{32} \end{array}\right\} \boxed{y^2 = \frac{(\arcsen x)^2}{2} + \frac{32-\pi^2}{32}}$$

f) $t\,y' + 2y = 4t^2$ com $y(1) = 2$

$$y' + \frac{2}{t}y = 4t \quad t \neq 0$$

$$t^2y' + 2ty = 4t^3$$

$$d(t^2y) = 4t^3\,dt \Rightarrow t^2y = t^4 + C \Rightarrow 1^2 \cdot 2 = 1 + C \Rightarrow C = 1$$

$$\therefore \boxed{y = t^2 + \frac{1}{t^2}}$$

g) $y' + (\cot x)y = 2\csc x$ para $y\left(\frac{\pi}{2}\right) = 1$

$$I = e^{\int \cot x\,dx} = e^{\ln(\operatorname{sen} x)} = \operatorname{sen} x$$

$$\operatorname{sen} x\, y' + \cos x\, y = 2$$

$$d(\operatorname{sen}(x) \cdot y) = 2\,dx \Rightarrow \operatorname{sen}(x) \cdot y = 2x + C \Rightarrow 1 = \pi + C \Rightarrow C = 1 - \pi$$

$$\therefore \boxed{y = \frac{2x + 1 - \pi}{\operatorname{sen} x \neq 0}}$$

h) $y' - \frac{1}{x} y = x^{1/2}$ para $y(2) = 1$

$I = e^{\int -\frac{1}{x} dx} = e^{-\ln(x)} = x^{-1}$

$x^{-1} y' - x^{-2} y = x^{-1/2}$

$d(x^{-1} y) = x^{-1/2} dx \Rightarrow x^{-1} y = \int x^{-1/2} dx = 2x^{1/2} + C \Rightarrow \frac{1}{2} = 2\sqrt{2} + C \Rightarrow C = \frac{1 - 4\sqrt{2}}{2}$

$\therefore \boxed{y = 2x^{3/2} + \frac{(1 - 4\sqrt{2})x}{2}}$

2) Determine as soluções das equações diferenciais de variáveis separáveis abaixo:

(a) $y' = y^2$

$\frac{dy}{dx} = y^2$

$\left(\frac{1}{y^2}\right) dy = dx$

$\int \frac{1}{y^2} dy = \int dx$

$-y^{-1} = x + C$

$\frac{1}{y} = -x + C$

$\boxed{y = \frac{-1}{x + C}}$

(b) $xy' = y$

$x \frac{dy}{dx} = y$

$\frac{1}{y} dy = \frac{1}{x} dx$

$\ln y = \ln x + C$

$y = e^{\ln x + c}$

$\boxed{y = Cx}$

(c) $yy' = x$

$y \frac{dy}{dx} = x$

$y\,dy = x\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + C$

$\boxed{y^2 = x^2 + C}$

(d) $y' = (1 - y)(2 - y)$

$\frac{1}{(1-y)(2-y)} dy = dx$

$\left(\frac{A}{1-y} + \frac{B}{2-y}\right) dy = dx$

$2A - Ay + B - By = 1$

$\Rightarrow A + B = 0 \Rightarrow -A = B$

$\Rightarrow 2A + B = 1 \Rightarrow A = 1 \text{ e } B = -1$

$\int \left(\frac{1}{1-y} - \frac{1}{2-y}\right) dy = x + C$

$-\ln(1-y) + \ln(2-y) = x + C$

$\ln\left(\frac{2-y}{1-y}\right) = x + C \Rightarrow \frac{2-y}{1-y} = Ce^x \Rightarrow 2 - y = ce^x - ce^x y \Rightarrow y(Ce^x - 1) = Ce^x - 2$

$\therefore \boxed{y = \frac{ce^x - 2}{ce^x - 1}}$

(e) $y' = e^{x-2y}$

$$\frac{dy}{dx} = e^{x} \cdot e^{-2y} \Rightarrow e^{2y}\,dy = e^{x}\,dx \Rightarrow \int e^{2y}\,dy = \int e^{x}\,dx \Rightarrow \frac{1}{2} e^{2y} = e^{x} + C$$

$$\Rightarrow e^{2y} = 2e^{x} + C \Rightarrow 2y = \ln(2e^{x} + C) \Rightarrow \boxed{y = \frac{\ln(2e^{x} + C)}{2}}$$

(f) $x^2 y^2 y' = 1 + x^2$

$$y^2\,dy = \left(\frac{1}{x^2} + 1\right) dx$$

$$\frac{y^3}{3} = \frac{-1}{x} + x + C \Rightarrow \boxed{y^3 = 3\left(x - \frac{1}{x}\right) + C}$$

3) Resolva as seguintes equações homogêneas de primeira ordem:

(a) $y' = \frac{x+y}{x}$ $\Rightarrow xy' - y = x \Rightarrow y' - \frac{1}{x}y = 1 \Rightarrow \frac{1}{x}y' - \frac{1}{x^2}y = \frac{1}{x} \Rightarrow d\left(\frac{1}{x}y\right) = \frac{1}{x}dx$

$I = e^{\int -\frac{1}{x}dx} = x^{-1}$ $\Big|$ $\Rightarrow \frac{1}{x}y = \ln(x) + C \Rightarrow \boxed{y = x\ln x + C}$

(b) $2y - xy' = 0$

$\frac{dy}{dx}x = 2y \Rightarrow \frac{1}{2y}dy = \frac{1}{x}dx \Rightarrow \frac{\ln y}{2} = \ln(x) + C \Rightarrow y = \left(e^{\ln x + c}\right)^2 = (cx)^2 = cx^2$

$\therefore \boxed{y = Cx^2}$

(c) $y' = \frac{x^2 + xy + y^2}{x^2}$ $\Rightarrow \underbrace{x^2 dy = (x^2 + xy + y^2)dx}_{\text{Homogênea}}$ $\Rightarrow x^2(t\,dx + x\,dt) = (x^2 + x^2t + x^2t^2)dx$

$\Rightarrow x^3 dt = (x^2 + x^2t^2)dx$

$\Rightarrow x\,dt = (1 + t^2)dx$

$t = \frac{y}{x} \Rightarrow y = tx \Rightarrow dy = t\,dx + x\,dt$

$\frac{1}{1+t^2}dt = \frac{1}{x}dx$

$\Rightarrow \arctan(t) = \ln x + C \Rightarrow t = \tan(\ln x + C) \Rightarrow \boxed{y = x\tan(\ln x + C)}$

(d) $y' = \frac{y^2 + 2xy}{x^2}$

$\underbrace{x^2 dy = (y^2 + 2xy)dx}_{\text{Homogênea}}$ $\Rightarrow x^2(t\,dx + x\,dt) = (t^2x^2 + 2x^2t)dx$

$\Rightarrow t\,dx + x\,dt = (t^2 + 2t)dx \Rightarrow x\,dt = (t^2 + t)dx \Rightarrow \frac{1}{t^2+t}dt = \frac{1}{x}dx$

$t = \frac{y}{x} \Rightarrow y = tx \Rightarrow dy = t\,dx + x\,dt$ $\Big|$ $\int \frac{dt}{t(t+1)} = \ln x + C \Rightarrow \int\left(\frac{1}{t} - \frac{1}{t+1}\right)dt = \ln(x) + C$

$\frac{A}{t} + \frac{B}{t+1} \Rightarrow At + A + Bt = 1 \Rightarrow A = 1$ e $B = -1$ $\Big|$ $\Rightarrow \ln t - \ln(t+1) = \ln(x) + c$

$\ln\left(\frac{t}{t+1}\right) = \ln(x) + C$

$\Rightarrow \frac{t}{t+1} = Cx \Rightarrow t - Cxt = Cx \Rightarrow t = \frac{Cx}{1 - Cx} \Rightarrow \boxed{y = \frac{Cx^2}{1 - Cx} = \frac{-x^2}{x + C}}$

(e) $y' = \dfrac{4y-3x}{2x-y}$

$$\underbrace{(2x-y)\,dy = (4y-3x)\,dx}_{\text{homogênea}} \Rightarrow (2x-tx)\,dy = (4tx-3x)\,dx$$

$$\Rightarrow (2-t)(x\,dt + t\,dx) = (4t-3)\,dx$$

$$\Rightarrow (2-t)\,x\,dt + (2-t)t\,dx = (4t-3)\,dx$$

$$t = \frac{y}{x} \Rightarrow y = xt \Rightarrow dy = x\,dt + t\,dx$$

$$\Rightarrow (2-t)x\,dt = (5t-5)\,dx$$

$$\Rightarrow \frac{(2-t)}{5(t-1)}\,dt = \frac{1}{x}\,dx$$

$$\Rightarrow \frac{t-2}{t-1}\,dt = \frac{-5}{x}\,dx \Rightarrow \int\left(1 - \frac{1}{t-1}\right)dt = -5\ln(x) + C$$

$$\Rightarrow t - \ln(t-1) = -5\ln x + C$$

$$\frac{y}{x} - \ln(y-x) + \ln x = -5\ln x + C$$

$$\boxed{y - x\ln(y-x) = -6x\ln x + Cx}$$

4) Resolva os seguintes problemas de valor inicial:

(a) $y' - y = 2xe^{2x}$, $y(0) = 1$ $\Rightarrow e^{-x}y' - e^{-x}y = 2xe^{x}$

$P(x)$

$I = e^{\int -1\,dx} = e^{-x}$

$\Rightarrow d(e^{-x}y) = 2xe^{x}\,dx$

$\Rightarrow e^{-x}y = 2\int xe^{x}dx = 2\left(xe^{x} - \int e^{x}dx\right) = 2\left(e^{x}(x-1) + C\right)$

$\Rightarrow y = 2e^{2x}(x-1) + Ce^{x}$

$1 = 2(-1) + C \cdot 1 \Rightarrow C = 3$

$$\boxed{y = 2e^{2x}(x-1) + 3e^{x}}$$

$u = x \Rightarrow du = 1$

$dv = e^{x} \Rightarrow v = e^{x}$

(b) $y' + 2y = xe^{-2x}$, $y(1) = 0$ $\Rightarrow e^{2x}y' + 2e^{2x}y = x$

$P(x)$

$I = e^{\int 2\,dx} = e^{2x}$

$\Rightarrow d(e^{2x}y) = x\,dx$

$\Rightarrow e^{2x}y = \frac{x^2}{2} + C \Rightarrow 0 = \frac{1}{2} + C \Rightarrow C = \frac{-1}{2}$

$$\therefore \boxed{y = \frac{x^2 - 1}{2e^{2x}}}$$

(c) $x^2y' + 2xy = \cos x$, $y(\pi) = 0$

$d(x^2y) = \cos x\,dx$

$x^2y = \operatorname{sen} x + C \Rightarrow \pi^2 \cdot 0 = 0 + C \Rightarrow C = 0$

$$\therefore \boxed{y = \frac{\operatorname{sen} x}{x^2}}$$

(d) $xy' + 2y = \operatorname{sen} x$, $y(\pi/2) = 1$

$d(x^2y) = x \operatorname{sen} x\,dx$

$x^2y = \int x \operatorname{sen} x\,dx$

$x^2y = \operatorname{sen} x - x\cos x + C$

$\left(\frac{\pi}{2}\right)^2 = 1 - \frac{\pi}{2} \cdot 0 + C \Rightarrow C = \frac{\pi^2}{4} - 1$

$$\therefore \boxed{y = \frac{\operatorname{sen} x - x\cos x + \frac{\pi^2}{4} - 1}{x^2}}$$

(e) $y' = x + y$, $y(0) = 1$

$y' - y = x \Rightarrow d(e^{-x}y) = xe^{-x}dx$

$I = e^{\int -1\,dx} = e^{-x}$

$\Rightarrow e^{-x}y = \int xe^{-x}dx = -xe^{-x} - \int -e^{-x}dx = -xe^{-x} - e^{-x} + C$

$\Rightarrow y = -x - 1 + Ce^{x} \Rightarrow 1 = -1 + C \Rightarrow C = 2$

$u = x \Rightarrow du = 1$

$dv = e^{-x} \Rightarrow v = -e^{-x}$

$$\therefore \boxed{y = 2e^{x} - x - 1}$$

5) Suponha que o valor y foi investido numa conta de poupança na qual os juros são continuamente capitalizados numa taxa constante de 5.5% ao ano. A equação y ′ = ky descreve a taxa de crescimento do montante investido, onde k é a taxa de juros. Se $5000 foram inicialmente investidos, qual é o montante após 3 anos?

$$\frac{dy}{dt} = ky \Rightarrow \frac{1}{y}\,dy = k\,dt \Rightarrow \ln y = kt + C \Rightarrow y = Ce^{kt}$$

$$\text{Em } t=0,\ y = 5000 \Rightarrow C = 5000 \Rightarrow y = 5000\,e^{\frac{105,5}{100}t}$$

$$\Rightarrow \boxed{y(3) = 5000\,e^{\frac{316,5}{100}} \approx \$118.443,72}$$