Assignment 2- Probability and Random Variables

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1 GATE PROBLEM : 76

Let X and Y be two continuous random variables with the joint probability density function

$$f(x, y) = \begin{cases} 2 & 0 < x + y < 1, x > 0, y > 0 \\ 0 & \text{elsewhere.} \end{cases} \quad (1.0.1)$$

Then $E\left(X|Y = \frac{1}{2}\right)$ is

2 SOLUTION

Given X and Y are two continuous random variables with joint probability density function,

$$f(x, y) = \begin{cases} 2 & 0 < x + y < 1, x > 0, y > 0 \\ 0 & \text{elsewhere.} \end{cases} \quad (2.0.1)$$

We know that,
$0 < x + y < 1 \implies 0 < y < 1 - x$ for $0 < x < 1$.
Then,

$$f_X(x) = \int f_{XY}(x, y)\, dy \quad (2.0.2)$$

$$= \int_0^{1-x} (2)dy \quad (2.0.3)$$

$$= 2(1 - x) \quad (2.0.4)$$

$$\implies f_X(x) = \begin{cases} 2(1 - x) & 0 \le x < 1 \\ 0 & \text{otherwise.} \end{cases} \quad (2.0.5)$$

Similarly,
$0 < x + y < 1 \implies 0 < x < 1 - y$ for $0 < y < 1$
Then,

$$f_y(y) = \int f_{XY}(x, y)\, dx \quad (2.0.6)$$

$$= \int_0^{1-y} (2)dx \quad (2.0.7)$$

$$= 2(1 - y) \quad (2.0.8)$$

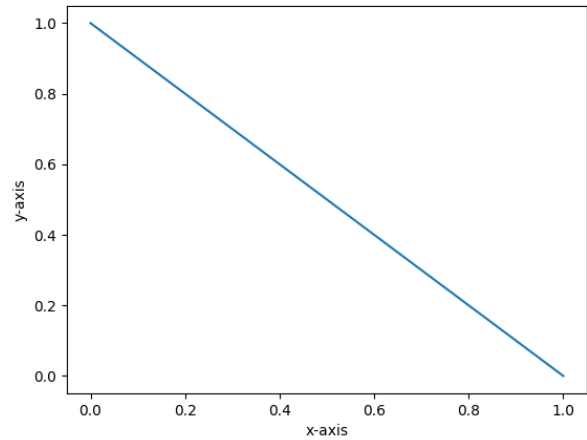$$\implies f_Y(y) = \begin{cases} 2(1 - y) & 0 \le y < 1 \\ 0 & \text{otherwise.} \end{cases} \quad (2.0.9)$$

Fig. 0: Graph of x+y=1

Therefore ,

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)} \quad (2.0.10)$$

$$= \begin{cases} \frac{2}{2(1-y)} & \text{if } 0 \le x + y < 1 \\ 0 & \text{otherwise} \end{cases} \quad (2.0.11)$$

Then,

$$E\left(X|Y = \frac{1}{2}\right) = \int_{-\infty}^{\infty}(x)\left(\frac{1}{1-y}\right)dx \qquad (2.0.12)$$

$$= \frac{1}{1-y}\int_{0}^{1-y}(x)dx \qquad (2.0.13)$$

$$= \frac{1}{1-y}\left[\frac{x^2}{2}\right]_{0}^{1-y} \qquad (2.0.14)$$

$$= \frac{1-y}{2} \qquad (2.0.15)$$

$$= \frac{1-\frac{1}{2}}{2} \qquad (2.0.16)$$

$$= \frac{1}{4} \qquad (2.0.17)$$

$$\therefore E\left(X|Y = \frac{1}{2}\right) = \frac{1}{4} \qquad (2.0.18)$$