Assignment 4- Probability and Random Variables

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Download all python codes from

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https://github.com/KoteshSatvik/AI1103-Probability_and_Random_Variables/tree/main/Assignment-4/codes
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and latex-tikz codes from

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https://github.com/KoteshSatvik/AI1103-Probability_and_Random_Variables/blob/main/Assignment-4/Assignment4.tex
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1 Gate 2016 (cs-set 1) Q:29

Consider the experiment with the following steps.

1) Flip a coin twice.
2) If the outcomes are (TAILS, HEADS) then output Y and stop.
3) If the outcomes are either (HEADS, HEADS) or (HEADS, TAILS), then output N and stop.
4) If the outcomes are (TAILS, TAILS), then go to Step 1.

The probability that the output of the experiment is Y is (upto two decimal places)......

2 Solution

Given a fair coin is flipped twice.
Let us define a Markov chain with states {1,2,3,4}, such that

State	Events
1	Event of tossing a fair coin twice
2	Event of obtaining 'N' as the output
3	Event of obtaining 'Y' as the output
4	Event of obtaining (TAIL,TAIL) as the output

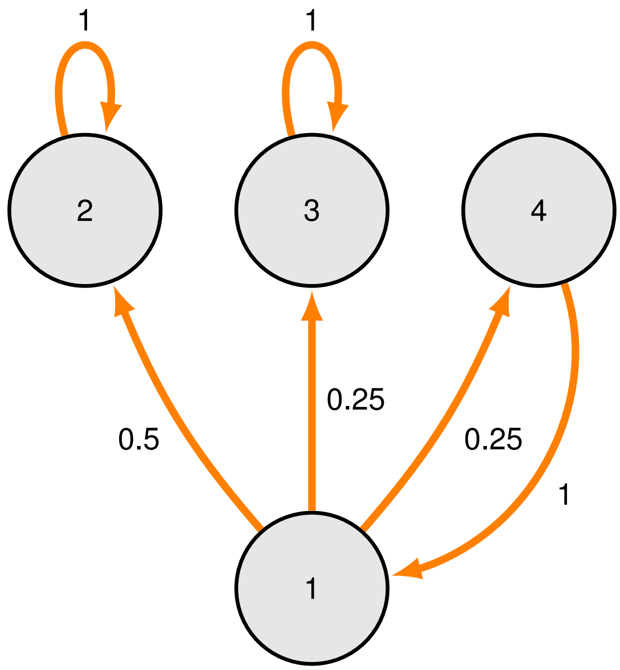
TABLE 4: Representation of different events

We know that when a fair coin is tossed,

$$\Pr(\text{HEAD}) = 1/2 \text{ and} \tag{2.0.1}$$

$$\Pr(\text{TAIL}) = 1/2. \tag{2.0.2}$$

Then,
Markov chain diagram :



The state transition matrix (P) for the Markov chain is

$$P = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1\\2\\3\\4 \end{array} & \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{2.0.3}$$

By the definition of transient and absorbing states, we can say that 1,4 are transient states whereas 2,3 are absorbing.
Then, the canonical form of the transition matrix is,

$$P = \begin{array}{cc} & \begin{array}{cccc} 2 & 3 & 1 & 4 \end{array} \\ \begin{array}{c} 2\\3\\1\\4 \end{array} & \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 \end{array}\right] \end{array} \tag{2.0.4}$$

The canonical form divides the transition matrix into four sub-matrices based on the states as listed below.

$$
\begin{array}{r c}
 & \begin{array}{cc} \text{Absorbing} & \text{Non-Absorbing} \end{array} \\
\begin{array}{r} \text{Absorbing} \\ \text{Non-Absorbing} \end{array} & \left[\begin{array}{c|c} \text{I} & \text{O} \\ \hline \text{A} & \text{B} \end{array}\right]
\end{array}
$$

where,

Variable	Type of Matrix
I	Identity matrix
O	Zero matrix
A, B	Some matrices

TABLE 4: Representation of different matrices

and From (2.0.4),

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & 0 \end{bmatrix} \tag{2.0.5}$$

The fundamental matrix F for the absorbing Markov chain is defined as

$$F = (I - B)^{-1} \tag{2.0.6}$$

Then,

$$F = \begin{bmatrix} 1 & -\frac{1}{4} \\ -1 & 1 \end{bmatrix}^{-1} \tag{2.0.7}$$

$$\implies F = \begin{bmatrix} 1.33 & 0.33 \\ 1.33 & 1.33 \end{bmatrix} \tag{2.0.8}$$

Therefore,

$$FA = \begin{bmatrix} 0.67 & 0.33 \\ 0.67 & 0.33 \end{bmatrix} \tag{2.0.9}$$

Then the limiting matrix for the markov chain is

$$\bar{P} = \begin{bmatrix} I & O \\ FA & O \end{bmatrix} \tag{2.0.10}$$

where the element p_{ij} of $\bar{P}$ represents the probability of absorption in state j, when the initial state is i.

$$\therefore \bar{P} = \begin{array}{c c} & \begin{array}{cccc} 2 & 3 & 1 & 4 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 1 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.67 & 0.33 & 0 & 0 \\ 0.67 & 0.33 & 0 & 0 \end{bmatrix} \end{array} \tag{2.0.11}$$

Therefore,

$$\text{Req. Probability } = p_{13} = 0.33 \tag{2.0.12}$$