Silver Ratio

In mathematics, two quantities are in the silver ratio (also silver mean or silver constant) if the ratio of the sum of the smaller and twice the larger of those quantities, to the larger quantity, is the same as the ratio of the larger one to the smaller one (see below). This defines the silver ratio as an irrational mathematical constant, whose value of one plus the square root of 2 is approximately 2.4142135623.

The relation described above can be expressed algebraically:

$$\frac{2a+b}{a} = \frac{a}{b} \equiv \delta_S$$

or equivalently,

$$2 + \frac{b}{a} = \frac{a}{b} \equiv \delta_S$$

Its name is an allusion to the golden ratio; analogously to the way the golden ratio is the limiting ratio of consecutive Fibonacci numbers, the silver ratio is the limiting ratio of consecutive Pell numbers. The silver ratio is denoted by δS.

$$2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}} = \delta_S$$

The convergents of this continued fraction (2/1, 5/2, 12/5, 29/12, 70/29, ...) are ratios of consecutive Pell numbers. These fractions provide accurate rational approximations of the silver ratio, analogous to the approximation of the golden ratio by ratios of consecutive Fibonacci numbers.

Properties[edit]

Number-theoretic properties

The silver ratio is a Pisot–Vijayaraghavan number (PV number), as its conjugate $1 - \sqrt{2} = -1/\delta_S \approx -0.41$ has absolute value less than 1. In fact it is the second smallest quadratic PV number after the golden ratio. This means the distance from δ_S^n to the nearest integer is $1/\delta_S^n \approx 0.41^n$. Thus, the sequence of fractional parts of δ_S^n, $n = 1, 2, 3, \ldots$ (taken as elements of the torus) converges. In particular, this sequence is not equidistributed mod 1.

Powers

The lower powers of the silver ratio are

$$\begin{aligned}
\delta_S^{-1} &= 1\delta_S - 2 = [0; 2, 2, 2, 2, 2, \ldots] \approx 0.41421 \\
\delta_S^{0} &= 0\delta_S + 1 = [1] = 1 \\
\delta_S^{1} &= 1\delta_S + 0 = [2; 2, 2, 2, 2, 2, \ldots] \approx 2.41421 \\
\delta_S^{2} &= 2\delta_S + 1 = [5; 1, 4, 1, 4, 1, \ldots] \approx 5.82842 \\
\delta_S^{3} &= 5\delta_S + 2 = [14; 14, 14, 14, \ldots] \approx 14.07107 \\
\delta_S^{4} &= 12\delta_S + 5 = [33; 1, 32, 1, 32, \ldots] \approx 33.97056
\end{aligned}$$

The powers continue in the pattern

$$\delta_S^n = K_n \delta_S + K_{n-1}$$

where

$$K_n = 2K_{n-1} + K_{n-2}$$

For example, using this property:

$$\delta_S^5 = 29\delta_S + 12 = [82; 82, 82, 82, \ldots] \approx 82.01219$$

Using $K_0 = 1$ and $K_1 = 2$ as initial conditions, a Binet-like formula results from solving the recurrence relation

$$K_n = 2K_{n-1} + K_{n-2}$$

which becomes

$$K_n = \frac{1}{2\sqrt{2}} \left(\delta_S^{n+1} - (2 - \delta_S)^{n+1} \right)$$

Trigonometric properties

$$\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2} = \frac{\sqrt{1-1/\delta_s}}{2}$$

$$\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2} = \frac{\sqrt{1+\delta_s}}{2}$$

$$\tan\frac{\pi}{8} = \sqrt{2}-1 = \frac{1}{\delta_s}$$

$$\cot\frac{\pi}{8} = \tan\frac{3\pi}{8} = \sqrt{2}+1 = \delta_s$$