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半角の公式 (暗記 + 導出)

$$\begin{aligned}\sin^2 \frac{\alpha}{2} &= \frac{1 - \cos \alpha}{2} \\ \cos^2 \frac{\alpha}{2} &= \frac{1 + \cos \alpha}{2} \\ \tan^2 \frac{\alpha}{2} &= \frac{1 - \cos \alpha}{1 + \cos \alpha}\end{aligned}$$

(2つ) 証明)

$$\begin{aligned}\cos 2\alpha &= 1 - 2\sin^2 \alpha = 2\cos^2 \alpha - 1 \\ \downarrow & \qquad \qquad \downarrow \\ \sin^2 \alpha &= \frac{1 - \cos 2\alpha}{2} \quad \cos^2 \alpha = \frac{1 + \cos 2\alpha}{2} \\ \downarrow \alpha \in \frac{\alpha}{2} \text{ であるから} & \\ \sin^2 \frac{\alpha}{2} &= \frac{1 - \cos \alpha}{2} \quad \cos^2 \frac{\alpha}{2} = \frac{1 + \cos \alpha}{2} \\ \tan^2 \frac{\alpha}{2} &= \frac{\sin^2 \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2}} = \frac{\frac{1 - \cos \alpha}{2}}{\frac{1 + \cos \alpha}{2}} = \frac{1 - \cos \alpha}{1 + \cos \alpha}\end{aligned}$$

(例1) $\sin \frac{\pi}{8}$, $\cos \frac{\pi}{8}$, $\tan \frac{\pi}{8}$ の値を求めよ。 $0 < \frac{\pi}{8} < \frac{\pi}{2}$ であるから

$$\sin \frac{\pi}{8} > 0, \quad \cos \frac{\pi}{8} > 0$$

よって

$$\begin{aligned}\sin \frac{\pi}{8} &= \sqrt{\frac{1 - \cos \frac{\pi}{4}}{2}} & \cos \frac{\pi}{8} &= \sqrt{\frac{1 + \cos \frac{\pi}{4}}{2}} \\ &= \sqrt{\frac{1 - \frac{1}{\sqrt{2}}}{2}} & &= \sqrt{\frac{1 + \frac{1}{\sqrt{2}}}{2}} \\ &= \sqrt{\frac{2 - \sqrt{2}}{4}} & &= \sqrt{\frac{2 + \sqrt{2}}{4}} \\ &= \frac{\sqrt{2 - \sqrt{2}}}{2} & &= \frac{\sqrt{2 + \sqrt{2}}}{2}\end{aligned}$$

$$\begin{aligned}\tan \frac{\pi}{8} &= \frac{\sin \frac{\pi}{8}}{\cos \frac{\pi}{8}} \\ &= \frac{\frac{\sqrt{2 - \sqrt{2}}}{2}}{\frac{\sqrt{2 + \sqrt{2}}}{2}} \\ &= \sqrt{\frac{2 - \sqrt{2}}{2 + \sqrt{2}}} \\ &= \sqrt{3 - 2\sqrt{2}} \\ &= \sqrt{2} - 1\end{aligned}$$

(例2) $\frac{\pi}{2} < \alpha < \pi$, $\sin \alpha = \frac{3}{5}$ かつ $\sin \frac{\alpha}{2}$, $\cos \frac{\alpha}{2}$, $\tan \frac{\alpha}{2}$ の値を求めよ。 $\frac{\pi}{2} < \alpha < \pi$ より

$$\cos \alpha < 0$$

であるから

$$\begin{aligned}\cos \alpha &= -\sqrt{1 - \sin^2 \alpha} \\ &= -\sqrt{1 - \left(\frac{3}{5}\right)^2} \\ &= -\frac{4}{5}\end{aligned}$$

また、 $\frac{\pi}{2} < \alpha < \pi$ より $\frac{\pi}{4} < \frac{\alpha}{2} < \frac{\pi}{2}$ であるから

$$\sin \frac{\alpha}{2} > 0, \quad \cos \frac{\alpha}{2} > 0$$

よって

$$\begin{aligned}\sin \frac{\alpha}{2} &= \sqrt{\frac{1 - \cos \alpha}{2}} = \sqrt{\frac{1 - (-\frac{4}{5})}{2}} = \frac{3}{\sqrt{10}} \\ \cos \frac{\alpha}{2} &= \sqrt{\frac{1 + \cos \alpha}{2}} = \sqrt{\frac{1 + (-\frac{4}{5})}{2}} = \frac{1}{\sqrt{10}} \\ \tan \frac{\alpha}{2} &= \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} = \frac{\frac{3}{\sqrt{10}}}{\frac{1}{\sqrt{10}}} = 3\end{aligned}$$