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・対称式

> xとyを入れ替えても同じになる式を対称式という。
>
> x, yの対称式は、基本対称式 $x+y$, xy で表せる。

(例) $x=\dfrac{1}{\sqrt{5}+\sqrt{3}}$、$y=\dfrac{1}{\sqrt{5}-\sqrt{3}}$ のとき、次の式の値を求めよ。

(1) $x+y$　(2) xy　(3) x^2+y^2

(4) x^3+y^3　(5) x^4+y^4　(6) x^5+y^5

(1)
$$
\begin{aligned}
x+y&=\frac{1}{\sqrt{5}+\sqrt{3}}+\frac{1}{\sqrt{5}-\sqrt{3}}\\
&=\frac{(\sqrt{5}+\sqrt{3})+(\sqrt{5}-\sqrt{3})}{(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})} \quad \leftarrow \frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}\\
&=\frac{2\sqrt{5}}{2}\\
&=\sqrt{5}_{\,/\!/}
\end{aligned}
$$

(2)
$$
\begin{aligned}
xy&=\frac{1}{\sqrt{5}+\sqrt{3}}\cdot\frac{1}{\sqrt{5}-\sqrt{3}}\\
&=\frac{1}{(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})}\\
&=\frac{1}{2}_{\,/\!/}
\end{aligned}
$$

(3)
$$
\begin{aligned}
x^2+y^2&=(x+y)^2-2xy \quad \leftarrow (x+y)^2=x^2\underbrace{+2xy}_{\text{余分}}+y^2\\
&=(\sqrt{5})^2-2\cdot\frac{1}{2}\\
&=4_{\,/\!/}
\end{aligned}
$$

(4)
$$
\begin{aligned}
x^3+y^3&=(x+y)^3-3xy(x+y) \quad \leftarrow (x+y)^3=x^3\underbrace{+3x^2y+3xy^2}_{\text{余分}}+y^3\\
&=(\sqrt{5})^3-3\cdot\frac{1}{2}\cdot\sqrt{5}\\
&=\frac{7\sqrt{5}}{2}_{\,/\!/}
\end{aligned}
$$

(別解)
$$
\begin{aligned}
x^3+y^3&=(x+y)(x^2-xy+y^2)\\
&=\sqrt{5}\left(4-\frac{1}{2}\right)\\
&=\frac{7\sqrt{5}}{2}_{\,/\!/}
\end{aligned}
$$

(5)
$$
\begin{aligned}
x^4+y^4&=(x^2+y^2)^2-2(xy)^2 \quad \leftarrow (x^2+y^2)^2=x^4\underbrace{+2x^2y^2}_{\text{余分}}+y^4\\
&=4^2-2\cdot\left(\frac{1}{2}\right)^2\\
&=\frac{31}{2}_{\,/\!/}
\end{aligned}
$$

(別解)
$$
\begin{aligned}
x^4+y^4&=(x+y)(x^3+y^3)-xy(x^2+y^2) \quad \leftarrow (x+y)(x^3+y^3)=x^4\underbrace{+xy^3+x^3y}_{\text{余分}}+y^4\\
&=\sqrt{5}\cdot\frac{7\sqrt{5}}{2}-\frac{1}{2}\cdot 4\\
&=\frac{31}{2}_{\,/\!/}
\end{aligned}
$$

(6)
$$
\begin{aligned}
x^5+y^5&=(x^2+y^2)(x^3+y^3)-(xy)^2(x+y) \quad \leftarrow (x^2+y^2)(x^3+y^3)=x^5\underbrace{+x^2y^3+x^3y^2}_{\text{余分}}+y^5\\
&=4\cdot\frac{7\sqrt{5}}{2}-\left(\frac{1}{2}\right)^2\cdot\sqrt{5}\\
&=\frac{55\sqrt{5}}{4}_{\,/\!/}
\end{aligned}
$$

(別解)
$$
\begin{aligned}
x^5+y^5&=(x+y)(x^4+y^4)-xy(x^3+y^3) \quad \leftarrow (x+y)(x^4+y^4)=x^5\underbrace{+xy^4+x^4y}_{\text{余分}}+y^5\\
&=\sqrt{5}\cdot\frac{31}{2}-\frac{1}{2}\cdot\frac{7\sqrt{5}}{2}\\
&=\frac{55\sqrt{5}}{4}_{\,/\!/}
\end{aligned}
$$