Binomial Generalizado

$$\binom{r}{m} = \frac{\overbrace{r(r-1)\dots(r-(m-1))}^{m \text{ fatores}}}{m!} \rightarrow \text{decresce sucessivamente}$$

$$\binom{r}{0} = 1$$

Folha 4

24
$$\binom{1/2}{3} = \frac{\overbrace{\frac{1}{2}\times\left(\frac{1}{2}-1\right)\times\left(\frac{1}{2}-2\right)}^{3 \text{ fatores}}}{3!} = \frac{\frac{1}{2}\times\left(-\frac{1}{2}\right)\times\left(-\frac{3}{2}\right)}{3\times 2} = \frac{1}{2^4}$$

$$\binom{-2}{3} = \frac{-2\times(-2-1)\times(-2-2)}{3!} = \frac{-2\times(-3)\times(-4)}{3\times 2} = -4$$

$$(1+x)^r = \sum_{m=0}^{\infty}\binom{r}{m}x^m \qquad |x|<1$$

$$(1+x)^r(1+x)^s = (1+x)^{r+s}$$

$$\sum_{n=0}^{\infty}\binom{r}{n}x^n \times \sum_{m=0}^{\infty}\binom{s}{m}x^m = \sum_{n=0}^{\infty}\boxed{\binom{r+s}{m}}x^m$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\binom{r}{k}\binom{s}{m-k}x^m$$

8
$$p(x) = 2x^2 + x$$

$$S_m = \sum_{i=0}^{m} p(i)$$

$$S_m = \sum_{i=0}^{m-1} p(i) + p(m), \quad m \geq 1$$

$$S_m = S_{m-1} + 2m^2 + m$$

$$S_0 = \sum_{i=0}^{0} p(i) = p(0) = 0$$

$$(*) \quad S_m - S_{m-1} = 2m^2 + m \rightarrow \text{fórmula recorrência (??)}$$

Eq. característica

multiplicidade das raízes

$$q - 1 = 0 \Leftrightarrow q = 1$$

Solução da eq. Homogénea

$$S_m - S_{m-1} = 0 \Leftrightarrow S_m^H = K\cdot 1^m = K$$

Solução particular

$$S_m^p = (Am^2 + Bm + C)m^\alpha$$

mesmo grau que a fórmula de recorrência (??)

$\alpha \rightarrow$ multiplicidade de $\boxed{1}$ na eq. característica

$\alpha = 1$

$$S_m^p = Am^3 + Bm^2 + Cm$$

Substituindo em (*) temos

$$\underbrace{Am^3 + Bm^2 + Cm}_{S_m} - \underbrace{\left(A(m-1)^3 + B(m-1)^2 + C(m-1)\right)}_{S_{m-1}} = 2m^2 + m$$

$$\Leftrightarrow Am^3 + Bm^2 + Cm - A(m-1)^3 - B(m-1)^2 - C(m-1) = 2m^2 + m$$

$$\Leftrightarrow \cancel{Am^3} + \cancel{Bm^2} + \cancel{Cm} - A(\cancel{m^3} - 3m^2 + 3m + 1) - B(\cancel{m^2} - 2m + 1) - C(\cancel{m} - 1) = 2m^2 + m$$

$$\Leftrightarrow 3Am^2 - 3Am - A + 2Bm - B + C = 2m^2 + m$$

$$\begin{cases} 3A = 2 \\ -3A + 2B = 1 \\ -A - B + C = 0 \end{cases} \Leftrightarrow \begin{cases} A = \frac{2}{3} \\ -3\cdot\frac{2}{3} + 2B = 1 \\ -\frac{2}{3} - B + C = 0 \end{cases} \Leftrightarrow \begin{cases} \_\_\_ \\ 2B = 3 \\ \_\_\_ \end{cases} \Leftrightarrow \begin{cases} \_\_\_ \\ B = +\frac{3}{2} \\ -\frac{2}{3} - \frac{3}{2} + C = 0 \end{cases} \Leftrightarrow \begin{cases} \_\_\_ \\ \_\_\_ \\ \frac{-4-9}{6} + C = 0 \end{cases} \Leftrightarrow \begin{cases} \_\_\_ \\ \_\_\_ \\ -\frac{13}{6} + C = 0 \end{cases} \Leftrightarrow \begin{cases} \_\_\_ \\ \_\_\_ \\ C = \frac{13}{6} \end{cases}$$

9
$$a_m = \underbrace{(C_1 + C_2 m)2^m + C_3 \cancel{\times 1^m}}_{\text{homogénea}} + \underbrace{4m}_{\text{particular}}$$

3 constantes $\longrightarrow$ grau 3

Soluções eq. característica

2 , 1

2 → multiplicidade 2

1 → multiplicidade 1 (raiz simples)

$$(q-2)^2(q-1)$$
$$= (q^2 - 4q + 4)(q-1)$$
$$= q^3 - 4q^2 + 4q - q^2 + 4q - 4$$
$$= q^3 - 5q^2 + 8q - 4$$

$$a_m - 5a_{m-1} + 8a_{m-2} - 4a_{m-3} = 0 \longrightarrow \text{Eq. homogénea}$$

$$\Leftrightarrow a_m - 5a_{m-1} + 8a_{m-2} - 4a_{m-3} = f(m)$$

$$\Leftrightarrow 4m - 5(4(m-1)) + 8(4(m-2)) - 4(4(m-3)) = f(m)$$

$$\Leftrightarrow \cancel{4m} - \cancel{20m} + 20 + \cancel{32m} - 64 - \cancel{16m} + 48 = f(m)$$

$$\Leftrightarrow -44 + 48 = f(m)$$

$$\Leftrightarrow 4 = f(m)$$

$$\boxed{a_m - 5a_{m-1} + 8a_{m-2} - 4a_{m-3} = 4}$$

Teste 2 (2022/2023)

1
a) Tinta azul e verde – 0 / 1 / 2 / 3 quantos

outras – 0 / 1 / 2 / 3 / 4 / ...

$$\underbrace{(1 + x + x^2 + x^3)}_{U - x^4 U}{}^{2 \to \text{azul e verde}} \underbrace{(1 + x + x^2 + x^3 + \dots)}_{U}{}^{3 \to \text{cores restantes}} = (1-x^4)^2 U^2 U^3 = (1-x^4)^2 \cdot U^5 = (1-x^4)^2 \times \frac{1}{(1-x)^5} = \frac{(1-x^4)^2}{(1-x)^5}$$

↳ Série geradora

$$\boxed{1 + x + \dots + x^p = U - x^{p+1}U}$$

b)

desenvolver binómio

$$\frac{(1-x^4)^2}{(1-x)^5} = \sum_{m=0}^{\infty} a_m x^m, \quad a_{20} = ?$$

$$(1 - 2x^4 + x^8)\cdot\frac{1}{(1-x)^5} = (1 - 2x^4 + x^8)\cdot\sum_{m=0}^{\infty}\left(\!\binom{5}{m}\!\right)x^m$$

$$a_{20} = \left(\!\binom{5}{20}\!\right) - 2\left(\!\binom{5}{16}\!\right) + \left(\!\binom{5}{12}\!\right)$$
$$= \binom{24}{20} - 2\binom{20}{16} + \binom{16}{12}$$

Exame recurso (22/23)

5
$$\frac{x}{(1-x)^3} = x\,\frac{1}{(1-x)^3} = x\cdot\sum_{m=0}^{\infty}\left(\!\binom{3}{m}\!\right)x^m$$

$$= \sum_{m=0}^{\infty}\left(\!\binom{3}{m}\!\right)x^{m+1} = \sum_{m=1}^{\infty}\underbrace{\left(\!\binom{3}{m-1}\!\right)}_{a_m}x^m$$

$m = m+1 - 1$

$m - 1$

$m = 1$

↳ avança 1 termo para excluir o 1º termo

$$a_m = \left(\!\binom{3}{m-1}\!\right), \quad m \geq 1$$

$$a_0 = 0$$

6
$$\begin{cases} m a_m - (5m-5)a_{m-1} = 5^m + 4 \\ a_0 = 10 \end{cases}$$

$$m a_m - 5(m-1)a_{m-1} = 5^m + 4, \quad m a_m = b_m$$

$$\Leftrightarrow b_m - 5b_{m-1} = 5^m + 4$$

profundidade = 1 → grau eq. característica

Eq. Característica

$$q - 5 = 0$$
$$\Leftrightarrow q = 5$$

Solução da eq. homogénea

$$b_m^H = A5^m$$

(1)

$$b_m^{p_1} - 5b_{m-1}^{p_1} = 5^m$$

$$b_m^{p_1} = B5^m m^\alpha = B5^m$$

$\alpha \rightarrow$ multiplicidade de $\boxed{5}$ na eq. característica

$\alpha = 1$

$$B\cancel{5^m}m - 5B\cancel{5^{m-1}}(m-1) = \cancel{5^m}$$

$$\cancel{5Bm} - \cancel{5Bm} + 5B = 5$$

$$B = 1$$

$$b_m^{p_1} = 5^m m$$

(2)

$$b_m^{p_2} - 5b_{m-1}^{p_2} = 4$$

$$b_m^{p_2} = Cm^\alpha, \quad \alpha \rightarrow \text{multiplicidade de } \boxed{1} \text{ na eq. característica} = 0$$

$$b_m^{p_2} = C$$

$$C - 5C = 4$$

$$-4C = 4$$

$$C = -1$$

$$b_m^{p_2} = -1$$

$$b_m = A5^m + 5^m m - 1$$

$$0 = A - 1$$

$$\Leftrightarrow A = 1$$

$$b_m = 5^m + 5^m m - 1$$

$a_0 = 10$

$b_m = m a_m$

$b_0 = 0 a_0 = 0 \times 10 = 0$

→ $m \neq 0$

$$a_m = \frac{b_m}{m} = \frac{5^m + 5^m m - 1}{m}$$

$$a_0 = 10$$