# Aula 14 (09/04)

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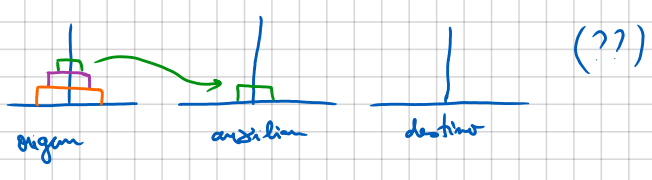
$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

## Torre de Hanói




(??)

## Problema de valor inicial

$$\begin{cases} a_n = 2a_{n-1} + 1, & n \geq 2 \\ a_1 = 1 \end{cases}$$

**Solução:** $a_n = 2^n - 1$

(como verificar?)

$n = 1$

$a_1 = 2^1 - 1 = 2 - 1 = 1$

$n = 2$

$a_2 = 2a_1 + 1 = 2 \times 1 + 1 = 2 + 1 = 3$

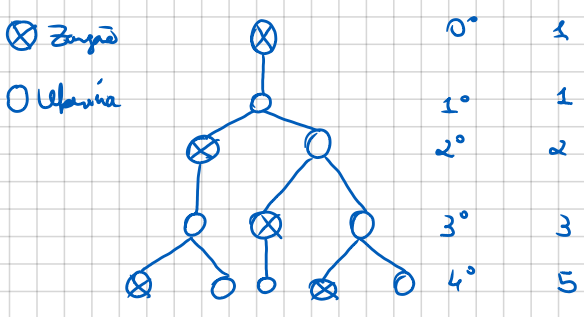
$n = 3$

$a_3 = 2a_2 + 1 = 2 \times 3 + 1 = 6 + 1 = 7$

## Sucessão de Fibonacci

Problema valor inicial

$$\begin{cases} F_{n+2} = F_{n+1} + F_n \\ F_0 = F_1 = 1 \end{cases}$$




## Identidade de recorrência

$$\sum_{k=0}^{n-1} F_k = F_{n+1} - 1$$

**Demonstração**

$$\sum_{k=0}^{n-1} F_k = F_0 + \sum_{k=1}^{n+1} (F_{k+1} - F_{k-1})$$

$F_{k+1} + F_k = F_{k-1}$

Série telescópica

**Obs:**

$$\sum_{k=1}^{n+1} (F_{k+1} - F_{k-1}) = (F_2 - F_1) + (F_3 - F_2) + (F_4 - F_3) + \dots + (F_n - F_{n-2}) = (F_n - F_1)$$

$\sum_{k=0}^{n-1} F_{2k}$ ($F_{par}$ (índice)) → Aula ✳

$\sum_{k=0}^{n-1} F_{2k+1}$ ($F_{ímpar}$ (índice)) → TPC

✳ 1 1 2 3 5 8

**Resolução:**

$$\sum_{k=0}^{n-1} F_{2k} = F_0 + \sum_{k=1}^{n-1} F_{2k} = 1 + \sum_{k=1}^{n-2} (F_{2k-1} + F_{2k-2})$$

$(\dots)$

$n(n-1)! = n!$

$X_n = n!$ satisfaz a eq. $X_n = (\dots)$

$X_n = c_1 X_{n-1} + c_2 X_{n-2} + \dots + c_k X_{n-k}$ $(n \geq k)$ (ordem da eq.)

Vamos procurar soluções da forma

$X_n = q^n$, $q \neq 0$

$q^n = c_1 q^{n-1} + c_2 q^{n-2} + c_3 q^{n-3} + \dots + c_k q^{n-k}$

$\Leftrightarrow q^n - c_1 q^{n-1} - c_2 q^{n-2} - c_3 q^{n-3} - \dots - c_k q^{n-k} = 0$

$\Leftrightarrow \underbrace{q^{n-k}}_{\neq 0} \underbrace{(q^k - c_1 q^{k-1} - c_2 q^{k-2} - \dots)}_{P(q)} = 0$

$P(q)$ – Polinómio Característico

## Exemplo

$X_n = X_{n-1} + 2X_{n-2}$, $n \geq 2$

$X_n = q^n$

$q^n = q^{n-1} + 2q^{n-2}$

$\Leftrightarrow q^n - q^{n-1} - 2q^{n-2} = 0$

$\Leftrightarrow q^{n-2}(q^2 - q - 2) = 0$

$P(q)$ – Polinómio característico

Raízes de $q^2 - q - 2$

$$q = \frac{+1 \pm \sqrt{(-1)^2 - 4 \times 1 \times (-2)}}{2 \times 1}$$

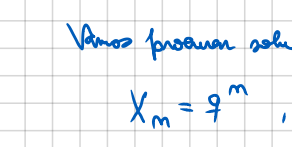
$$q = \frac{+1 \pm \sqrt{9}}{2}$$

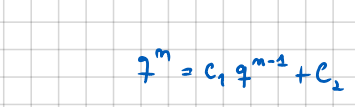
$q = 2 \vee q = -1$

**Solução geral:**

$X_n = \alpha (-1)^n + \beta (2)^n$, $\alpha, \beta \in \mathbb{R}$

$(n \in \mathbb{N})$




$q^2 = q + 2 \times 1$ $\boxed{n=2}$

$\Leftrightarrow q^2 - q - 2 = 0$

**Ex 2**

$X_n = 3X_{n-2} + 2X_{n-3}$ $^{(+0X_{n-2})}$, $n \geq 3$

$\boxed{X_n = q^n}$

$q^n = 3q^{n-2} + 2q^{n-3}$

$\Leftrightarrow q^n - 3q^{n-2} - 2q^{n-3} = 0$

$\boxed{n=3}$

$\Leftrightarrow q^3 - 3q - 2 = 0$

Raízes de $q^3 - 3q - 2$

$q = 1$ raiz da eq.

**Fatorização:** $(q-1)(q^2 + q - 2)$

$\Leftrightarrow (q-1)(q-1)(q+2)$

$\Leftrightarrow (q-1)^2 (q+2)$

multiplicidade 2

$\boxed{X_n = q^n}$

$X_n = \underbrace{\alpha_0 1^n + \alpha_1 \times n \times 1^n}_{\text{Combinação linear associada à raiz } q=1 \text{ de multiplicidade 2}} + \beta (-2)^n$

**Folha 4**

⑥

a) $P(q) = \underbrace{(q-1)^1 (q-3)^1 (q-2)^2}_{\text{Eq. Polinómio característico (fatorizado)}}$

$P(q) = (q-1)(q-3)(q^2 - 4q + 4)$

$= \dots$

$= q^4 - 8q^3 + 23q^2 - 28q + 12$

$P(q) = 0 \Longrightarrow \underbrace{q^n}_{\neq 0} P(q) = q^{n+4} - 8q^{n+3} + 23q^{n+2} - 28q^{n+1} + 12q^n = 0$

$\boxed{X_n = q^n} \Longrightarrow X_{n+4} - 8X_{n+3} + 23X_{n+2} - 28X_{n+1} + 12X_n = 0$

b) $X_n = \alpha 1^n + \beta 3^n + \gamma 2^n + \delta n 2^n$, $\alpha, \beta, \gamma, \delta \in \mathbb{N}$