University of Aveiro - Department of Mathematics

Discrete Mathematics - 2024/2025

Worksheet 1 - First-order logic and automated reasoning

1. Identify the free and bounded occurrences of each variable of the following formulas (variables: x, y, z, a, b):
   (a) $\exists y \, P(x, y)$
   (b) $(\forall x(P(x) \to Q(x))) \to (\neg P(x) \vee Q(y))$
   (c) $\exists x(P(y, z) \wedge \forall y(\neg Q(x, y) \vee P(y, z)))$
   (d) $P(a, f(a, b))$
   (e) $\exists x(P(x) \to \neg Q(x))$
   (f) $\forall x((P(x) \wedge C(x)) \to \exists y L(x, y))$

2. Translate the following sentences to first-order logic formulas:
   (a) All birds have feathers.
   (b) All children are younger than their parents.
   (c) All insects are lighter than some mammal.
   (d) No number is less than zero.
   (e) zero is less than any number.
   (f) Some prime numbers are not even.
   (g) Every even number is a prime number.

3. Consider the universe of all texts written in english and the predicates $c(x)$: «x is a clear explanation», $s(x)$: <"x is satisfactory explanation» and $d(x)$: «x is an excuse». Translate the following formulas to English sentences.
   (a) $\forall x \, (c(x) \to s(x))$;
   (b) $\exists x \, (d(x) \wedge \neg s(x))$;
   (c) $\exists x \, (d(x) \wedge \neg c(x))$.

4. Let Π be the set of all subsets of a plane and consider Π the universe for the following predicates:

   $r(x)$: «x is a line»,
   $c(x)$: «x is a circumference»,
   $i(x, y)$: "the intersection of x and y is not empty",

   Using the three predicates, translate the following sentences into first-order logic:
   (a) Every line intersects some circumference.
   (b) Some line doesn't intersect some circumference.

(c) No line intersects all circumferences.

5. Translate the following sentences into first-order logic, using the predicates $\mathrm{House}(x)$: «x is a house»; $\mathrm{Big}(x)$: «x is big»; $\mathrm{Expensive}(x)$: «x is expensive»; $\mathrm{Apartment}(x)$: «x is an apartment»; $\mathrm{Price}(x, y)$: «the price of x is lower than the price of y».

   (a) All big houses are expensive.

   (b) Any apartment costs less than at least one big house.

6. Using the predicate $Like(x, y)$: «x likes y», translate the following sentences into first-order logic formulas:

   (a) Everyone has someone who likes them.

   (b) People that everyone likes also like themselves.

   (c) Formulate the negation of the proposition indicated in (a) and write a sentence in current language (in english) that translates this negation.

7. Obtain, in the simplest form possible, the negation of the following formula

$$\forall y \, \exists x \ (\, (\, q(x) \to p(y) \,) \ \vee \ (\, p(y) \wedge q(x) \,) \,) \ .$$

8. Consider the formula

$$Q : \quad \forall x \, \exists y \, ((t(x) \wedge v(y, x)) \to \neg p(x, y))$$

   for an interpretation with domain $\mathbb{N}$, where $t(x)$: «$x > 1$», $v(y, x)$: «$y = x + 1$» and $p(x, y)$: «x divides y».

   (a) Find the logic value (the truth value) of Q.

   (b) Find the logic value of $(t(x) \wedge v(y, x)) \to \neg p(x, y)$ for the valuation V with $V(x) = 1$ and $V(y) = 2$.

9. Consider an universe X with objects A, B and C (that is, X=$\{A, B, C\}$) and a language where α, β and γ are symbols of contants, f is a symbol of a function with one argument and R is a symbol of predicate with two arguments. Consider the interpretation:

   symbols of constants: $\alpha \mapsto A$, $\beta \mapsto A$ and $\gamma \mapsto B$;

   symbol of function f: $f(A) = B$, $f(B) = C$, $f(C) = C$;

   symbol of predicate R: $\{(B, A), (C, B), (C, C)\}$.

   With this interpretation, evaluate the following formulas:

   (a) $R(\alpha, \beta)$;

   (b) $\exists x \ f(x) = \beta$;

   (c) $\forall w \ R(f(w), w)$.

10. For each formula find, if possible, a model and an interpretation for which it is not valid.

    (a) $\forall x \, (P(x, a) \to \neg Q(x, a))$, where a is a symbol of constant;

(b) $\exists x \, \exists y \, ((P(x,y) \wedge \forall z \, (\neg Q(x,y) \vee P(y,z)))$.

11. Transform the following formulas into prenex disjunctive normal form and prenex conjunctive normal form::

    (a) $(\forall x \, S(x)) \rightarrow (\exists z \, P(z))$;
    (b) $\neg(\forall x \, (S(x) \rightarrow P(x)))$;
    (c) $\forall x \, (P(x) \rightarrow (\exists y \, Q(x,y)))$;
    (d) $\exists x \, (\neg(\exists y \, P(x,y)) \rightarrow (\exists z \, (Q(z) \rightarrow R(x))))$;
    (e) $\forall x \, \exists y \, \exists z \, ((\neg P(x,y) \wedge Q(x,z)) \vee R(x,y,z))$.

12. Find the Skolem standard form for each formula.

    (a) $\neg((\forall x \, P(x)) \rightarrow (\exists y \, P(y)))$
    (b) $\neg((\forall x \, P(x)) \rightarrow (\exists y \, \forall z \, Q(y,z)))$
    (c) $\forall x \, \exists y \, \exists z \, ((\neg P(x,y) \wedge Q(x,z)) \vee R(x,y,z))$

13. Show that the set

$$S = \{P \vee R, \neg Q \vee R, \neg S \vee Q, \neg P \vee S, \neg Q, \neg R\}$$

    is inconsistent.

14. In each case, compute $E\Theta$:

    (a) $\Theta = \{a/x, f(z)/y, g(x)/z\}$, $E = P(h(x), z, f(z))$;
    (b) $\Theta = \{f(y)/x, a/y\}$, $E = F(a, h(a), x, h(y))$.

15. For each of the following sets of formulas, verify if it is a unified set and, in affirmative case, find a most general unifier. Note that «a» and «b» denote symbols of constants.

    (a) $\{P(f(x), z), P(y, a)\}$;
    (b) $\{P(f(x), x), P(z, a)\}$;
    (c) $\{P(a, x, f(g(y))), P(b, h(z, w), f(w))\}$;
    (d) $\{S(x, y, z), S(u, g(v, v), v)\}$;
    (e) $\{P(x, x), P(y, f(y))\}$;
    (f) $\{Q(f(a), g(x)), Q(y, y)\}$;
    (g) $\{Q(f(x), y), Q(z, g(w))\}$.

16. Consider the set of formulas

$$E = \{C(x, \mathrm{LordRings}, y), C(\mathrm{Mary}, z, f(t)), C(w, \mathrm{LordRings}, f(\mathrm{BlueTable}))\}.$$

    Verify if E is a unified set and, in affirmative case, find a most general unifier.

17. Verify if the following clauses admit a factor. If so, determine it.

    (a) $P(x) \vee P(a) \vee Q(f(x)) \vee Q(f(a))$;

(b) $P(x) \vee P(f(y)) \vee Q(x, y)$.

18. Find resolvents (if any) for the following pairs of clauses:

    (a) $C_1 : \neg P(x) \vee Q(x, b)$ e $C_2 : P(a) \vee Q(a, b)$;
    (b) $C_1 : \neg P(x) \vee Q(x, x)$ e $C_2 : \neg Q(a, f(a))$.

19. Consider the following first-order logic formulas:

    F1: $\forall x\, (G(x) \rightarrow \forall y (P(y) \rightarrow L(x, y)))$
    F2: $\exists x\, G(x)$
    F3: $\exists x\, \forall y\, (P(y) \rightarrow L(x, y))$

    Using the principle of resolution show that F3 is a consequence of F1 and F2.

20. Consider the following statements:

    - Every student at the University of Aveiro who studies hard passes Discrete Mathematics.
    - João is a student at the University of Aveiro.
    - João studies hard.

    (a) Express the above statements as well-formed formulas of the predicate calculus.
    (b) Prove, using the resolution principle, that João passes Discrete Mathematics.

21. Consider the following statements, in the universe of animals:

    - Animals with fur are mammals.
    - Bears are furry animals.
    - Rabbits are mammals.
    - Winnie is a bear.
    - Bugsbunny is a rabbit.
    - Sylvester is a furry animal.

    (a) Represent them in first-order logic.
    (b) Using the Resolution Principle, answer the following questions:
        (i) Winnie is a mammal?
        (ii) Which animals are mammals?
        (iii) Which animals have fur?

22. Consider each of the predicate symbols $SH(x)$, $IH(x)$ and $HSP(x)$ whose interpretation is as follows:

    - $SH(x)$ represents «x is a superhero»;
    - $IH(x)$ represents «x is an infra-hero»;
    - $SP(x)$ represents «x has superpowers».

Let's assume that the following facts are known:

(i) Superheroes have superpowers;

(ii) There is someone who doesn't have superpowers;

(iii) There are only superheroes or infra-heroes.

(a) Express facts (i), (ii), and (iii) with well-formed formulas from first-order logic, using the predicate symbols defined above.

(b) Accept (i), (ii) and (iii) as true facts. Applying the principle of resolution, show there is at least one infra-hero.

23. The following facts are known:

- Every horse is faster than every greyhound;
- There is at least one greyhound that is faster than all rabbits;
- For any x, y, and z, if x is faster than y and y is faster than z, then x is faster than z.
- Roger is a rabbit;
- Harry is a horse.

(a) Transcribe the facts into first-order logic, using the predicates
   - $Horse(x)$ represents «x is a horse»;
   - $Greyhound(x)$ represents «x is a greyhound»;
   - $Rabbit(x)$ represents «x is a rabbit»;
   - $Faster(x, y)$ represents «x is faster than y».

(b) Applying resolution, show that Harry is faster than Roger.

Some solutions:

1

a) x is free, y is bounded;

b) x has free and bounded occurrences, y is free;

c) x is bounded, y has free and bounded occurrences, z is free;

d) a and b are free;

e) x is bounded;

f) x and y are bounded.

2

a) $\forall x\,(bird(x) \to Feathers(x))$

b) $\forall x \forall y\,((child(x) \wedge parent(x,y)) \to younger(x,y))$

c) $\forall x\ (insect(x) \to \exists y\ (mammal(y) \wedge lighter(x,y)))$

d) $\forall x\ (number(x) \to x \geq 0)$

e) $\forall x\ (number(x) \to 0 < x)$

f) $\exists x\,(prime(x) \wedge \neg even(x))$

g) $\forall x\,(even(x) \to prime(x))$

3

a) All clear explanations are satisfactory;

b) Some excuses are not satisfactory;

c) Some excuses are not clear explanations.

4

a) $\forall x\ (\ r(x)\ \to\ \exists y\ \ (\ c(y)\ \wedge\ i(x,y)\ )\ )$

b) $\exists x\ \exists y\ (\ r(x)\ \wedge\ c(y)\ \wedge\ \neg\ i(x,y)\ )$

c) $\forall x\ (\ r(x)\ \to \exists y\ (\ c(y)\ \wedge\ \neg i(x,y))\ )$

5

a) $\forall x\,((\text{House}(x) \wedge \text{Big}(x)) \to \text{Expensive}(x))$

b) $\forall x\,(\text{Apartment}(x) \to \exists y\,(\text{House}(y) \wedge \text{Big}(y) \wedge \text{Price}(x,y)))$

6

a) $\forall x\ \exists y\ \text{Like}(y,x)$

b) $\forall x\ ((\forall y\ \text{Like}(y,x))\ \to\ \text{Like}(x,x))$

c) $\exists x\ \forall y\ \neg\text{Like}(y,x)$; There's someone nobody likes.

7 $\exists y \forall x\ \neg\,(q(x) \to p(y))$

8

a) True.
b) True.

9

a) False;
b) False;
c) True.

11

a) $\exists x\, \exists z\, (\neg S(x) \vee P(z))$;
b) $\exists x\, (S(x) \wedge \neg P(x))$;
c) $\forall x\, \exists y\, (\neg P(x) \vee Q(x, y))$;
d) $\exists x\, \exists y\, \exists z\, (P(x, y)\ \vee\ \neg Q(z)\ \vee\ R(x))$.
e) The formula is in the disjunctive normal form.
Conjunctive normal form: $\forall x \exists y \exists z\, ((\neg P(x, y) \vee R(x, y, z)) \wedge (Q(x, z) \vee R(x, y, z)))$.

12

a) $\forall x\, \forall y\, (P(x)\ \wedge\ \neg P(y)) \equiv \forall z\ \perp$
b) $\forall x\, \forall y\, (P(x)\ \wedge\ \neg Q(y, f(x, y)))$
c) $\forall x\, ((\neg P(x, f(x))\ \vee\ R(x, f(x), g(x)))\ \wedge\ (Q(x, g(x))\ \vee\ R(x, f(x), g(x))))$

14

a) $E\Theta = P(h(a), g(x), f(g(x)))$
b) $E\Theta = F(a, h(a), f(y), h(a))$

15

a) $\{f(x)/y, a/z\}$
b) $\{a/x, f(a)/z\}$
c) It isn't a unified set.
d) $\{u/x, g(v, v)/y, v/z\}$
e) It isn't a unified set.
f) It isn't a unified set.
g) $\{f(x)/z, g(w)/y\}$

17

a) $P(a) \vee Q(f(a))$
b) $P(f(y)) \vee Q(f(y), y)$

18

a) $Q(a, b)$
b) Doesn't exist.