Funções geradoras (Séries formais de potências)

CONCRETA = CONtínua + disCRETA

$|C| = n$ $\qquad C \xrightarrow{f} \{0, 1\}$

$a = a + \underbrace{0x + 0x^2 + \dots}_{\text{adicionar zeros redundantemente}}$

$a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k \underbrace{+ 0x^{k+1} + 0x^{k+2} + \dots}_{\text{adicionar zeros redundantemente}}$

$a_j = \binom{k}{j} \longrightarrow$ Binómio de Newton

Progressão geométrica (Ex1)

$(1-x)(1 + x + x^2 + \dots + x^{k-1})$

$= (1 + \cancel{x} + \cancel{x^2} + \dots + \cancel{x^{k-1}}) - (\cancel{x} + \cancel{x^2} + \dots + \cancel{x^{k-1}} + x^k)$

$= 1 - x^k$

$\therefore (1-x)(x + x^2 + \dots + x^{k-1})$

$= 1 - x^k$

$\Leftrightarrow \underbrace{1 + x + x^2 + \dots + x^{k-1}}_{\text{polinómio}} = \underbrace{\frac{1-x^k}{1-x}}_{\text{função racional}}$

$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ (Série geométrica/uniforme)

no sentido formal

$(x^{\infty} := 0)$

$\frac{d}{dx}(x^k) = k x^{k-1}$ $\quad \left(k = \frac{k!}{(k-1)!}\right)$

$\frac{d^2}{dx^2}(x^k) = \frac{d}{dx}(k x^{k-1}) = k(k-1) x^{k-2}$

Ex 2

Séries Geométricas / Binomial Negativa VS Combinações c/ repetição

$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \Longrightarrow \left(\frac{d}{dx}\right)^{n-1}\left[\frac{1}{1-x}\right] = \sum_{k=0}^{\infty}\left(\frac{d}{dx}\right)^{n-1}[x^k] = \sum_{k=0}^{\infty} \frac{k!}{(k-n+1)!} x^{k-n+1}$

$[n \geq 1]$

$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[(1-x)^{-1}\right] = \frac{1}{(1-x)^2}$ $\quad 1!$

$\frac{d^2}{dx^2}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left(\underbrace{\frac{1}{(1-x)^2}}_{(1-x)^{-2}}\right) = \frac{2}{(1-x)^3}$ $\quad 2!$

$\frac{d^3}{dx^3}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left(\frac{2}{(1-x)^3}\right) = \frac{2 \times 3}{(1-x)^4}$ $\quad 6 = 3!$

$\therefore$

$\frac{1}{(n-1)!}\left(\frac{d}{dx}\right)^{n-1}\left[\frac{1}{1-x}\right] = \sum_{k=0}^{\infty} \underbrace{\frac{k!}{(n-1)!(k-n+1)!}}_{= \binom{k}{n-1}} x^{\boxed{k-n+1} = m}$

C. aux
$k - n + 1 = m$
$k = m + n - 1$

$= \sum_{m=0}^{\infty} \binom{m+n-1}{n-1} x^m = \sum_{m=0}^{\infty} \left(\!\binom{n}{m}\!\right) x^m$

Exemplo 3

$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n, \ n \geq 0$

$\sum_{n=0}^{\infty} L_n x^n = \sum_{n=0}^{\infty}\left(\frac{1+\sqrt{5}}{2}x\right)^n + \sum_{n=0}^{\infty}\left(\frac{1-\sqrt{5}}{2}x\right)^n = \frac{1}{1 - \frac{1+\sqrt{5}}{2}x} + \frac{1}{1 - \frac{1-\sqrt{5}}{2}x}$

$\left|\frac{1-\sqrt{5}}{2}x\right| < \left|\underbrace{\frac{1+\sqrt{5}}{2}}_{\phi}x\right| < 1$

$\Rightarrow \lim \frac{L_n}{L_{n+1}} = \frac{1}{\phi}$

Combinações sem repetição $\longrightarrow \binom{n}{k} \longrightarrow$ Termo do Binómio de Newton $(1+x)^n$

Combinações com repetição $\left(\!\binom{n}{m}\!\right) \longrightarrow$ Binomial Negativo $(1-x)^{-n}$

Arranjos com repetição n^k $\qquad x\frac{d}{dx}$ VS. Série geométrica

$\underbrace{x\frac{d}{dx}}_{N=}(x^n) = x(n x^{n-1})$

$= n x^n$

$\left(x\frac{d}{dx}\right)^k = n^k x^n$

$\sum_{n=0}^{\infty} n^k x^n = \sum_{n=0}^{\infty}\left(x\frac{d}{dx}\right)^k x^n = \left(x\frac{d}{dx}\right)^k \underbrace{\sum_{n=0}^{\infty} x^n}_{\frac{1}{1-x}} = \left(x\frac{d}{dx}\right)^k\left[\frac{1}{1-x}\right]$

Folha 4

18 a)

$\sum_{n=0}^{\infty} n k^n x^n$ $\qquad \left[n\frac{d}{dx}(x^n) = n x^n\right]$

constante

$= \sum_{n=0}^{\infty}\left(x\frac{d}{dx}\right)\underbrace{(k^n x^n)}_{=(kx)^n}$

$= x\frac{d}{dx}\left(\sum_{n=0}^{\infty}(kx)^n\right) \Longrightarrow$ Progressão geométrica de razão kx

$= x\underbrace{\frac{d}{dx}\left(\frac{1}{1-kx}\right)}_{} = \frac{kx}{(1-kx)^2}$

$= \frac{d}{dx}\left((1-kx)^{-1}\right)$

$= k(1-kx)^{-2}$

b)

$\sum_{n=0}^{\infty} L_n x^n = \frac{2-x}{1-x-x^2}$

Nºs de Lucas

$\begin{cases} L_n = L_{n-1} + L_{n-2}, \ n \geq 2 \\ L_1 = 2 \\ L_0 = 1 \end{cases}$

$\mathcal{L} = \sum_{n=0}^{\infty} L_n x^n$ $\longrightarrow$ Função geradora dos nºs de Lucas

$\mathcal{L} = L_0 + L_1 x + \sum_{n=2}^{\infty} L_n x^n$

$= 2 + x + \sum_{n=2}^{\infty}(L_{n-1} + L_{n-2}) x^n$

$= 2 + x + x\underbrace{\sum_{\boxed{n=2}}^{\infty} L_{n-1} x^{n-1}}_{= \mathcal{L} - L_0} + x^2\underbrace{\sum_{n=2}^{\infty} L_{n-2} x^{n-2}}_{= \mathcal{L}}$

$= 2 + x + x|\mathcal{L} - 2| + x^2\mathcal{L}$

$= 2 - x + (x + x^2)\mathcal{L}$

Portanto $1\mathcal{L} = 2 - x + (x + x^2)\mathcal{L}$

$\Leftrightarrow 1 - (x + x^2)\mathcal{L} = 2 - x$

$\Leftrightarrow (1 - x - x^2)\mathcal{L} = 2 - x$

$\Leftrightarrow \mathcal{L} = \frac{2-x}{1-x-x^2}$

18 a) [Matemática Contínua]

$\sum_{n=0}^{\infty} n k^n x^n = \frac{kx}{(1-kx)^2}$

b) $\sum_{n=0}^{\infty} c_n x^n = ??$

$c_n = k + 2k^2 + \dots + n k^n$

$(c_n)_{n \in \mathbb{N}}$ Como relação de recorrência

(R) $\begin{cases} c_n = c_{n-1} + n k^n \\ c_0 = 0 \end{cases}$

$C = \sum_{n=0}^{\infty} c_n x^n = \underbrace{c_0}_{0} + \sum_{n=1}^{\infty}(c_{n-1} + n k^n) x^n$

$= x\underbrace{\sum_{n=1}^{\infty} c_n x^{n-1}}_{C} + \underbrace{\sum_{n=1}^{\infty} n k^n x^n}_{= \frac{kx}{(1-kx)^2}}$

$\therefore C = xC + \underbrace{\frac{kx}{(1-kx)^2}}_{\text{Solução do 18 a)}}$

$\Leftrightarrow (1-x)C = \frac{kx}{(1-kx)^2}$

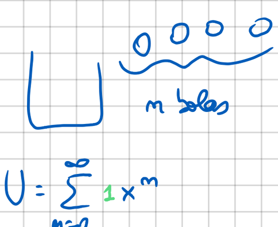
$\Leftrightarrow C = \frac{kx}{(1-x)(1-kx)^2}$

$A = \sum_{n=0}^{\infty} a_n x^n$

$B = \sum_{n=0}^{\infty} b_n x^n$

$AB = C$

$C = \sum_{n=0}^{\infty} c_n x^n$ $\qquad c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$



n bolas

$U = \sum_{n=0}^{\infty} 1 x^n$

$U^2 = U \cdot U = \frac{1}{(1-x)^2}$

$= \sum_{n=0}^{\infty}\left(\!\binom{2}{n}\!\right) x^n$ $\quad$ nº de bolas

Solução $\left(\!\binom{2}{n}\!\right)$