5

Malliavin Calculus

In 1976, Malliavin ([74, 75]) proposed a new calculus on Wiener spaces and achieved purely probabilistic proofs of results related to diffusion processes, which, before him, were shown based on outcomes in other mathematical fields like partial differential equations. For example, he proved the existence and smoothness of the transition densities of diffusion processes in a purely probabilistic manner. This method has been developed into a theoretical system, which is nowadays called the **Malliavin calculus** [43, 73, 104, 122]. It plays an important role in stochastic analysis together with the Itô calculus, consisting of stochastic integrals, stochastic differential equations, and so on. The aim of this chapter is to introduce the Malliavin calculus. We will use the fundamental terminologies and notions in functional analysis without detailed explanation. For these, consult [1, 19, 58].

5.1 Sobolev Spaces and Differential Operators

Throughout this chapter, let $T > 0$, $d \in \mathbb{N}$ and $(W_T, \mathscr{B}(W_T), \mu_T)$ be the d-dimensional Wiener space on $[0,T]$ (Definition 1.2.2). For $t \in [0,T]$, define $\theta(t) : W_T \to \mathbb{R}^d$ by $\theta(t)(w) = w(t)$ $(w \in W_T)$. Moreover, let H_T be the Cameron–Martin subspace of W_T. Then, identifying H_T with its dual space H_T^* in a natural way, we obtain the relation $W_T^* \subset H_T^* = H_T \subset W_T$ (see Section 1.2).

Let E be a real separable Hilbert space and $L^p(\mu_T; E)$ be the space of E-valued p-th integrable functions with respect to μ_T on W_T. $L^p(\mu_T; \mathbb{R})$ is simply written as $L^p(\mu_T)$. Denote the norm in $L^p(\mu_T; E)$ by $\|\cdot\|_p$ or $\|\cdot\|_{p,E}$ when emphasizing E is necessary.

Let $\mathscr{P}$ be the set of functions $\phi : W_T \to \mathbb{R}$ of the form $\phi = f(\ell_1, \ldots, \ell_n)$ for $\ell_1, \ldots, \ell_n \in W_T^*$ and a polynomial $f : \mathbb{R}^n \to \mathbb{R}$, that is,

$$\phi(w) = f(\ell_1(w), \ldots, \ell_n(w)) \qquad (w \in W_T).$$

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Set

$$\mathscr{P}(E) = \Big\{\sum_{j=1}^{m} \phi_j e_j\,;\ \phi_j \in \mathscr{P},\ e_j \in E,\ j = 1,\dots,m,\ m \in \mathbb{N}\Big\}.$$

For $\phi = f(\ell_1,\dots,\ell_n) \in \mathscr{P}$, define $\nabla\phi \in \mathscr{P}(H_T)$ by

$$\nabla\phi = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(\ell_1,\dots,\ell_n)\ell_i.$$

Moreover, for $\phi = \sum_{j=1}^{m} \phi_j e_j \in \mathscr{P}(E)$, define $\nabla\phi \in \mathscr{P}(H_T \otimes E)$ by

$$\nabla\phi = \sum_{j=1}^{m} \nabla\phi_j \otimes e_j,$$

where, for real separable Hilbert spaces E_1 and E_2, $E_1 \otimes E_2$ denotes the Hilbert space of Hilbert–Schmidt operators $A : E_1 \to E_2$ and, for $e^{(1)} \in E_1$ and $e^{(2)} \in E_2$, $e^{(1)} \otimes e^{(2)}$ denotes the Hilbert–Schmidt operator such that $E_1 \ni e \mapsto \langle e^{(1)}, e\rangle_{E_1} e^{(2)} \in E_2$. The Hilbert space $E_1 \otimes E_2$ has an inner product given by

$$\langle A, B\rangle_{E_1\otimes E_2} = \sum_{n=1}^{\infty} \langle Ae_n^{(1)}, Be_n^{(1)}\rangle_{E_2} \qquad (A, B \in E_1 \otimes E_2),$$

where $\{e_n^{(1)}\}_{n=1}^{\infty}$ is an orthonormal basis of E_1. It should be noted that the above definition of $\nabla\phi$ does not depend on the expression of $\phi \in \mathscr{P}$, because

$$\langle \nabla\phi(w), h\rangle_{H_T} = \frac{\mathrm{d}}{\mathrm{d}\xi}\Big|_{\xi=0} \phi(w + \xi h) \qquad (w \in W_T,\ h \in H_T). \tag{5.1.1}$$

Example 5.1.1 Let $d = 1$. For $t \in [0, T]$, the coordinate function $\theta(t) : W_T \to \mathbb{R}$ satisfies

$$\langle \nabla\theta(t), h\rangle_{H_T} = h(t) = \int_0^T \mathbf{1}_{[0,t]}(s)\dot{h}(s)\,\mathrm{d}s \qquad (h \in H_T).$$

Hence, defining $\ell_{[0,t]} \in H_T$ by $\dot{\ell}_{[0,t]}(s) = \mathbf{1}_{[0,t]}(s)$ $(s \in [0, t])$, we have $\nabla\theta(t) = \ell_{[0,t]}$.

Lemma 5.1.2 *Let $p > 1$. For $F \in L^p(\mu_T; E)$, $\ell \in W_T^*$, $\phi \in \mathscr{P}$ and $e \in E$, the mapping $\mathbb{R} \ni \xi \mapsto \int_{W_T} \langle F(\cdot + \xi\ell), \phi e\rangle_E\, \mathrm{d}\mu_T$ is differentiable and*

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\Big|_{\xi=0} \int_{W_T} \langle F(\cdot + \xi\ell), \phi e\rangle_E\, \mathrm{d}\mu_T = \int_{W_T} \langle F, e\rangle_E \partial_\ell\phi\, \mathrm{d}\mu_T, \tag{5.1.2}$$

where

$$\partial_\ell\phi(w) = \ell(w)\phi(w) - \langle \nabla\phi(w), \ell\rangle_{H_T}.$$

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In particular, the mapping $\nabla : L^p(\mu_T; E) \supset \mathscr{P}(E) \ni \phi \mapsto \nabla\phi \in L^p(\mu_T; H_T \otimes E)$ *is closable, that is,* ∇ *is extended to a unique closed operator whose domain is a dense subset of* $L^p(\mu_T; E)$*.*

Proof By the Cameron–Martin theorem (Theorem 1.7.2),

$$\int_{W_T} \langle F(w+\xi\ell), \phi(w)e\rangle_E \mu_T(\mathrm{d}w)$$
$$= \int_{W_T} \langle F(w), e\rangle_E \phi(w-\xi\ell) \exp\Bigl(\xi\ell(w) - \frac{\xi^2}{2}\|\ell\|_{H_T}^2\Bigr)\mu_T(\mathrm{d}w).$$

It is easy to see that the right hand side is differentiable in ξ, and, hence, so is the left hand side. Differentiating both sides in $\xi = 0$, we obtain (5.1.2) by (5.1.1).

By (5.1.2), we have, for any $\psi \in \mathscr{P}(E)$, $e \in E$, $\ell \in W_T^*$ and $\phi \in \mathscr{P}$,

$$\int_{W_T} \langle \nabla\psi, \ell \otimes e\rangle_{H_T\otimes E} \phi \,\mathrm{d}\mu_T = \int_{W_T} \langle \psi, e\rangle_E \partial_\ell\phi \,\mathrm{d}\mu_T.$$

Hence, if $\{\psi_n\}_{n=1}^\infty \subset \mathscr{P}(E)$ and $G \in L^p(\mu_T; H_T \otimes E)$ satisfy $\|\psi_n\|_p \to 0$ and $\|\nabla\psi_n - G\|_p \to 0$ $(n \to \infty)$, then

$$\int_{W_T} \langle G, \ell \otimes e\rangle_{H_T\otimes E} \phi \,\mathrm{d}\mu_T = 0.$$

Since this holds for any e, ℓ, and ϕ, we obtain that $G = 0$, μ_T-a.s. and ∇ is closable. □

Remark 5.1.3 (1) By (5.1.1) and (5.1.2),

$$\int_{W_T} \langle \nabla F, \phi\ell \otimes e\rangle_{H_T\otimes E} \,\mathrm{d}\mu_T = \int_{W_T} \langle F, (\partial_\ell\phi)\, e\rangle_E \,\mathrm{d}\mu_T \tag{5.1.3}$$

for any $F \in \mathscr{P}(E)$. Denote the dual operator of ∇ by ∇^*. Then, the left hand side is equal to $\int_{W_T} \langle F, \nabla^*(\phi\ell \otimes e)\rangle_E \,\mathrm{d}\mu_T$. Thus

$$\nabla^*(\phi\ell \otimes e) = (\partial_\ell\phi)\, e \tag{5.1.4}$$

since F is arbitrary. Identity (5.1.3) is a prototype of the integration by parts formula on the Wiener space presented in Section 5.4.

(2) Set $E = \mathbb{R}$ and $\phi = 1$ in (5.1.4). Then

$$(\nabla^*\ell)(w) = \ell(w), \quad \mu_T\text{-a.s.}, \tag{5.1.5}$$

where $\ell \in W_T^*$ is regarded as an H_T-valued constant function on the left hand side and as a random variable $\ell : W_T \to \mathbb{R}$ on the right hand side.

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Moreover, if $\|\ell\|_{H_T} = 1$ and $F = \varphi(\ell)$, then (5.1.3) corresponds to the following elementary identity for a standard normal random variable X,

$$\mathbf{E}[\varphi'(X)] = \int_{\mathbb{R}} \varphi'(x) \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x = \int_{\mathbb{R}} \varphi(x) x \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x = \mathbf{E}[\varphi(X)X].$$

On account of the closability of ∇, we introduce **Sobolev spaces** over the Wiener space.

Definition 5.1.4 Let $p \geqq 1$ and $k \in \mathbb{N}$. For $\phi \in \mathscr{P}(E)$, set

$$\|\phi\|_{(k,p)} = \sum_{j=0}^{k} \|\nabla^j \phi\|_p$$

and denote the completion of $\mathscr{P}(E)$ with respect to $\|\cdot\|_{(k,p)}$ by $\mathbb{D}^{k,p}(E)$. Simply write $\mathbb{D}^{k,p}$ for $\mathbb{D}^{k,p}(\mathbb{R})$. Denote by the same ∇ the extension of $\nabla : \mathscr{P}(E) \to \mathscr{P}(H_T \otimes E)$ to $\mathbb{D}^{k,p}(E)$ and by ∇^* the adjoint operator of the closed operator $\nabla : L^p(\mu_T; E) \to L^p(\mu_T; H_T \otimes E)$.

If $k \leqq k'$ and $p \leqq p'$, then $\mathbb{D}^{k',p'}(E) \subset \mathbb{D}^{k,p}(E)$. By definition, ∇ is defined consistently on each $\mathbb{D}^{k,p}(E)$. Moreover, by (5.1.4), for $F \in \mathscr{P}(H_T \otimes E)$ of the form

$$F = \sum_{j=1}^{m} \phi_j \ell_j \otimes e_j$$

with $\phi_j \in \mathscr{P}$, $\ell_j \in W_T^*$ and $e_j \in E$ $(j = 1, \ldots, m)$, we have

$$\nabla^* F = \sum_{j=1}^{m} (\partial_{\ell_j} \phi_j) e_j.$$

Hence, ∇^* is also defined consistently on each $L^p(\mu_T; E)$. Because of this consistency, we may use the simple notations ∇ and ∇^* without referring to the dependency on k and p.

Example 5.1.5 Let $\ell \in W_T^*$. Set $f(x) = x$ $(x \in \mathbb{R})$ and write $\ell(w) = f(\ell(w))$ $(w \in W_T)$. Then, by definition, the derivative of $\ell : W_T \ni w \to \ell(w) \in \mathbb{R}$ is given by

$$(\nabla \ell)(w) = \ell, \quad \mu_T\text{-a.s. } w \in W_T.$$

Combining this identity with (5.1.5), we obtain

$$\nabla(\nabla^* \ell)(w) = \ell, \quad \mu_T\text{-a.s. } w \in W_T. \tag{5.1.6}$$

We now show that this identity is extended to H_T.

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Let $h \in H_T$ and take $\ell_n \in W_T^*$ $(n = 1, 2, \dots)$ so that $\|\ell_n - h\|_{H_T} \to 0$. Setting

$$A_p = \Big(\int_{\mathbb{R}} |x|^p \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x\Big)^{\frac{1}{p}},$$

we have

$$\int_{W_T} \|\ell_n(w) - \ell_m(w)\|^p \mu_T(\mathrm{d}w) = A_p^p \big\|\ell_n - \ell_m\big\|_{H_T}^p.$$

By (5.1.5), this implies

$$\lim_{n,m\to\infty} \|\nabla^*\ell_n - \nabla^*\ell_m\|_p = 0. \tag{5.1.7}$$

Since ∇^* is a closed operator, h belongs to the domain of ∇^* as a constant H_T-valued function and

$$\lim_{n\to\infty} \|\nabla^*\ell_n - \nabla^* h\|_p = 0.$$

Combining this with (5.1.5), we obtain

$$\nabla^* h = \mathscr{I}(h), \tag{5.1.8}$$

where $\mathscr{I}(h)$ is the Wiener integral of $\dot{h}$ (see Section 1.7).

On the other hand, (5.1.6) implies

$$\lim_{n,m\to\infty} \|\nabla(\nabla^*\ell_n) - \nabla(\nabla^*\ell_m)\|_p = \lim_{n,m\to\infty} \|\ell_n - \ell_m\|_{H_T} = 0.$$

By (5.1.7) and the closedness of ∇, $\nabla^* h$ belongs to the domain of ∇ and

$$\nabla(\nabla^* h) = h. \tag{5.1.9}$$

In order to develop the theory of distributions on Wiener spaces, we need to consider the Sobolev spaces $\mathbb{D}^{k,p}(E)$ for $k \in \mathbb{R}$. For this extension, we introduce the Wiener chaos decomposition of $L^2(\mu_T)$, which plays an important role in several areas of stochastic analysis.

Define the **Hermite polynomials** $\{H_n\}_{n=0}^\infty$ by

$$H_n(x) = \frac{(-1)^n}{n!} \mathrm{e}^{\frac{1}{2}x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-\frac{1}{2}x^2}) \qquad (x \in \mathbb{R}).$$

We have

$$\mathrm{e}^{-\frac{1}{2}(x-y)^2} = \sum_{n=0}^\infty \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-\frac{1}{2}x^2})(-y)^n = \mathrm{e}^{-\frac{1}{2}x^2} \sum_{n=0}^\infty H_n(x) y^n$$

and the generating function for the Hermite polynomials is

$$\sum_{n=0}^\infty H_n(x) y^n = \mathrm{e}^{xy - \frac{1}{2}y^2}. \tag{5.1.10}$$

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$\{H_n\}_{n=0}^\infty$ forms an orthogonal basis of the L^2-space on $\mathbb{R}$ with respect to the standard normal distribution and

$$\int_{\mathbb{R}} H_i(x)H_j(x)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx = \frac{1}{j!}\delta_{ij}.$$

This identity is shown by inserting (5.1.10) into the left hand side of

$$\int_{\mathbb{R}} e^{sx-\frac{1}{2}s^2}e^{tx-\frac{1}{2}t^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx = e^{st}$$

and comparing the coefficients of s^it^j.

By using the Hermite polynomials, we construct an orthonormal basis of $L^2(\mu_T)$ in the following way. Let $\mathscr{A}$ be the set of sequences of non-negative integers with a finite number of non-zero elements:

$$\mathscr{A} = \Big\{\boldsymbol{\alpha} = \{\alpha_j\}_{j=1}^\infty\,;\,\alpha_j \in \mathbb{Z}_+, \sum_{j=1}^\infty \alpha_j < \infty\Big\}.$$

For $\boldsymbol{\alpha} \in \mathscr{A}$, define $|\boldsymbol{\alpha}|$ and $\boldsymbol{\alpha}!$ by

$$|\boldsymbol{\alpha}| = \sum_{j=1}^\infty \alpha_j \quad \text{and} \quad \boldsymbol{\alpha}! = \prod_{j:\alpha_j\neq 0} \alpha_j!.$$

Fix an orthonormal basis $\{h_n\}_{n=1}^\infty$ of the Cameron–Martin subspace H_T and define a family $H_{\boldsymbol{\alpha}}$ ($\alpha \in \mathscr{A}$) of functions on H by

$$H_{\boldsymbol{\alpha}}(w) = \prod_{j=1}^\infty H_{\alpha_j}(\mathscr{I}(h_j)(w)),$$

where $\mathscr{I}(h)$ is the Wiener integral of $h \in H_T$.

Theorem 5.1.6 $\{\sqrt{\alpha!}H_\alpha, \alpha \in \mathscr{A}\}$ *forms an orthonormal basis of* $L^2(\mu_T)$. *Moreover,* $L^2(\mu_T)$ *admits the orthogonal decomposition*

$$L^2(\mu_T) = \bigoplus_{n=0}^\infty \mathscr{H}_n, \tag{5.1.11}$$

where $\mathscr{H}_n$ ($n = 0, 1, 2, \ldots$) *is the closed subspace of* $L^2(\mu_T)$ *spanned by* $\{H_\alpha;$ $|\boldsymbol{\alpha}| = n\}$. $\mathscr{H}_n$ *does not depend on the choice of the orthonormal basis of* H_T.

Proof $\mathscr{I}(h_j)$ is a standard normal random variable. Hence, the orthonormality of $\{\sqrt{n!}H_n\}$ with respect to the standard Gaussian measure implies that of $\{\sqrt{\alpha!}H_\alpha\}$.

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We next show that $\bigoplus_{n=0}^{\infty}\mathscr{H}_n$ is dense. Let $X \in L^2(\mu_T)$ and suppose that $\int_{W_T} XY\,\mathrm{d}\mu_T = 0$ for any $Y \in \bigoplus_{n=0}^{\infty}\mathscr{H}_n$. This implies that X is orthogonal to all polynomials of $\mathscr{I}(h_j)$ and that

$$\int_{W_T} X \exp\Bigl(\mathrm{i}\sum_{j=1}^{n} a_j \mathscr{I}(h_j)\Bigr)\,\mathrm{d}\mu_T = 0$$

for all $n \in \mathbb{N}$ and $a_j \in \mathbb{R}$. Hence,

$$\int_{W_T} X f(\mathscr{I}(h_1), \mathscr{I}(h_2), \ldots, \mathscr{I}(h_n))\,\mathrm{d}\mu_T = 0$$

for any $f \in C_0^\infty(\mathbb{R}^n)$ $(n \in \mathbb{N})$, which means $X = 0$ because, by the Itô–Nisio theorem (Theorem 1.2.5), $\sum_j \mathscr{I}(h_j)h_j$ converges almost surely and the distribution of the limit is μ_T.

If $\|h_n - h\|_{H_T} \to 0$, then $\mathscr{I}(h_n) \to \mathscr{I}(h)$ in $L^2(\mu_T)$. Hence, $\mathscr{H}_n$ does not depend on the choice of the orthonormal basis of H_T. □

Definition 5.1.7 The orthogonal decomposition (5.1.11) of $L^2(\mu_T)$ is called the **Wiener chaos decomposition** and an element in $\mathscr{H}_n$ is called an n-th **Wiener chaos**.

Let $J_n : L^2(\mu_T) \to L^2(\mu_T)$ be the orthogonal projection onto $\mathscr{H}_n$. Extend J_n to $\mathscr{P}(E)$ so that

$$J_n F = \sum_{j=1}^{m} (J_n F_j) e_j$$

for $F = \sum_{j=1}^m F_j e_j$ $(F_j \in \mathscr{P},\ e_j \in E\ (j = 1, \ldots, m))$. If $G \in \mathscr{P}$, then there exist an $N \in \mathbb{N}$ and $c_\alpha \in \mathbb{R}$ $(|\alpha| \leqq N)$ such that

$$G = \sum_{|\alpha| \leqq N} c_\alpha H_\alpha,$$

where H_α is given by $H_\alpha = \prod_{j=1}^{\infty} H_{\alpha_j}(\ell_j)$ with an orthonormal basis $\{\ell_j\}_{j=1}^{\infty}$ such that G is expressed as $G = g(\ell_1, \ldots, \ell_M)$ for some $M \in \mathbb{N}$ and a polynomial $g : \mathbb{R}^M \to \mathbb{R}$. Hence $J_n G \in \mathscr{P}$ and $J_n G = 0$ if $n > N$.

Definition 5.1.8 Let $r \in \mathbb{R}$ and $p > 1$. Define $(I - L)^r : \mathscr{P}(E) \to \mathscr{P}(E)$ by

$$(I - L)^r = \sum_{n=0}^{\infty} (1 + n)^r J_n \tag{5.1.12}$$

and set

$$\|F\|_{r,p} = \|(I - L)^{\frac{r}{2}} F\|_p.$$

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Denote the completion of $\mathscr{P}(E)$ with respect to $\|\cdot\|_{r,p}$ by $\mathbb{D}^{r,p}(E)$ and write $\mathbb{D}^{r,p}$ for $\mathbb{D}^{r,p}(\mathbb{R})$.

The infinite sum on the right hand side of (5.1.12) is a finite one for $G \in \mathscr{P}(E)$. $\mathbb{D}^{0,p}(E)$ is equal to $L^p(\mu_T; E)$. It is known as **Meyer's equivalence** that, for $k \in \mathbb{Z}_+$, Definitions 5.1.4 and 5.1.8 are consistent, that is, they define the same space $\mathbb{D}^{k,p}(E)$.

Theorem 5.1.9 ([104, Theorem 4.4]) *For any $k \in \mathbb{Z}_+$ and $p > 1$, there exist $a_{k,p}$ and $A_{k,p} > 0$ such that*

$$a_{k,p}\|\nabla^k F\|_p \leqq \|F\|_{k,p} \leqq A_{k,p} \sum_{j=0}^{k} \|\nabla^j F\|_p \quad (F \in \mathscr{P}(E)).$$

The family of Sobolev spaces $\mathbb{D}^{r,p}(E)$ $(r \in \mathbb{R}, p > 1)$ has the following consistency.

Theorem 5.1.10 (1) *For $r, r' \in \mathbb{R}$ and $p, p' > 1$ with $r \leqq r'$, $p \leqq p'$, the inclusion mapping $\mathbb{D}^{r',p'}(E) \subset \mathbb{D}^{r,p}(E)$ is a continuous embedding.*
(2) *Let $(\mathbb{D}^{r,p}(E))^*$ be the dual space of $\mathbb{D}^{r,p}(E)$. Under the identification of $(L^p(\mu_T; E))^*$ and $L^q(\mu_T; E)$, where $\frac{1}{p} + \frac{1}{q} = 1$,*

$$\mathbb{D}^{-r,q}(E) = (\mathbb{D}^{r,p}(E))^*.$$

For the proof, we prepare some lemmas. Define L and $T_t : \mathscr{P}(E) \to \mathscr{P}(E)$ $(t > 0)$ by

$$LG = \sum_{n=0}^{\infty}(-n)J_n G \quad \text{and} \quad T_t G = \sum_{n=0}^{\infty} e^{-nt} J_n G \quad (G \in \mathscr{P}(E)), \tag{5.1.13}$$

respectively. Since J_n is the orthogonal projection onto $\mathscr{H}_n$, we have

$$\left\|T_t F\right\|_2^2 = \sum_{n=0}^{\infty} \mathrm{e}^{-2nt}\left\|J_n F\right\|_2^2 \leqq \left\|F\right\|_2^2 \qquad (F \in \mathscr{P}(E)).$$

Since $\mathscr{P}(E)$ is dense in $L^2(\mu_T; E)$, T_t is extended to a contraction operator on $L^2(\mu_T; E)$, which is also denoted by T_t. Moreover,

$$T_t(T_s F) = T_{t+s} F \quad (F \in L^2(\mu_T; E))$$

by definition and $\{T_t\}_{t\geqq 0}$ defines a contraction semigroup on $L^2(\mu_T; E)$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t} T_t F = L T_t F \qquad (F \in \mathscr{P}(E)). \tag{5.1.14}$$

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$\{T_t\}_{t\geqq 0}$ and L are called the **Ornstein–Uhlenbeck semigroup** and the **Ornstein–Uhlenbeck operator**, respectively. Moreover, by definition, for $H_\alpha \in \mathscr{H}_{|\alpha|}$ ($\alpha \in \mathscr{A}$), we have

$$(I-L)^r H_\alpha = (1+|\alpha|)^r H_\alpha, \quad LH_\alpha = -|\alpha|\, H_\alpha, \quad T_t H_\alpha = \mathrm{e}^{-|\alpha|t} H_\alpha. \tag{5.1.15}$$

Lemma 5.1.11 *Let $p > 1$, $F \in \mathscr{P}(E)$ and $G \in L^p(\mu_T; E)$.*
(1) *For any $t \geqq 0$ and $w \in W_T$,*

$$T_t F(w) = \int_{W_T} F(\mathrm{e}^{-t} w + \sqrt{1-\mathrm{e}^{-2t}}\, w')\mu_T(\mathrm{d}w'). \tag{5.1.16}$$

(2) $\|T_t F\|_p \leqq \|F\|_p$ *holds. In particular, $T_t : L^p(\mu_T; E) \supset \mathscr{P}(E) \ni F \mapsto T_t F \in \mathscr{P}(E) \subset L^p(\mu_T; E)$ is extended to a bounded linear operator.*
(3) $\lim_{t\to 0} \|T_t G - G\|_p = 0$ *holds.*

Proof (1) It suffices to show the case where $E = \mathbb{R}$. Let $F \in \mathscr{P}$. Then, there exist an $N \in \mathbb{N}$, a polynomial $f : \mathbb{R}^N \to \mathbb{R}$, and an orthonormal system $\ell_1, \dots, \ell_N \in W_T^*$ of H_T such that $F = f(\ell_1, \dots, \ell_N)$. Set $\boldsymbol{\ell}(w) = (\ell_1(w), \dots, \ell_N(w))$ for $w \in W_T$. Then, denoting the right hand side of (5.1.16) by $S_t F(w)$, we have

$$S_t F(w) = \int_{\mathbb{R}^N} f(\mathrm{e}^{-t}\boldsymbol{\ell}(w) + y) g_N(1-\mathrm{e}^{-2t}, y)\,\mathrm{d}y,$$

where $g_N(s,y) = (2\pi s)^{-\frac{N}{2}} \exp(-\frac{|y|^2}{2s})$. Since

$$\int_{\mathbb{R}^N} g_N(s, z-y) g_N(t,y)\,\mathrm{d}y = g_N(s+t, z) \quad \text{and} \quad \frac{\partial g_N}{\partial s} = \frac{1}{2}\Delta g_N,$$

setting

$$\widetilde{f}(x) = \int_{\mathbb{R}^n} f(\mathrm{e}^{-t}x + y) g_N(1-\mathrm{e}^{-2t}, y)\,\mathrm{d}y, \qquad \widetilde{F} = \widetilde{f}(\ell_1, \dots, \ell_N),$$

we have $S_t F = \widetilde{F}$,

$$S_s(S_t F)(w) = S_{s+t} F(w), \tag{5.1.17}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} S_t F(w) = \Delta f(\boldsymbol{\ell}(w)) - \sum_{j=1}^N \ell_j(w) \frac{\partial f}{\partial x^j}(\boldsymbol{\ell}(w)). \tag{5.1.18}$$

Extend $\{\ell_j\}_{j=1}^N$ to an orthonormal basis $\{\ell_j\}_{j=1}^\infty$ of H_T and set

$$H_\alpha = \prod_{j=1}^\infty H_{\alpha_j}(\ell_j) \quad (\alpha \in \mathscr{A}).$$

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Since $H_n''(x) - xH_n'(x) = -nH_n(x)$, (5.1.18) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} S_t H_\alpha(w) = -|\alpha| H_\alpha(w).$$

Combining this with (5.1.17), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} S_t H_\alpha(w) = -|\alpha| S_t H_\alpha(w).$$

Since $S_0 H_\alpha(w) = H_\alpha(w)$, by this ordinary differential equation and (5.1.15), we obtain

$$S_t H_\alpha(w) = \mathrm{e}^{-|\alpha| t} H_\alpha(w) = T_t H_\alpha(w).$$

Thus (5.1.16) holds for H_α. Since $F \in \mathscr{P}$ is written as a linear combination of H_αs, (5.1.16) is satisfied.

(2) For the same $F = f(\ell_1, \dots, \ell_N)$ as above, we have by Hölder's inequality

$$\begin{aligned}\left\|T_t F\right\|_p^p &\leqq \int_{W_T}\int_{W_T} |F(\mathrm{e}^{-t} w + \sqrt{1-\mathrm{e}^{-2t}}\, w')|^p \mu_T(\mathrm{d}w)\mu_T(\mathrm{d}w') \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |f(x+y)|^p g_N(\mathrm{e}^{-2t}, x) g_N(1-\mathrm{e}^{-2t}, y)\,\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^N} |f(z)|^p g_N(1, z)\,\mathrm{d}z = \left\|F\right\|_p^p.\end{aligned}$$

Hence, $\|T_t F\|_p \leqq \|F\|_p$.

(3) Let $K \in \mathscr{P}(E)$. By (5.1.16), $\lim_{t\to 0} \|T_t K(w) - K(w)\|_E = 0$ $(w \in W_T)$. Since $\|T_t K\|_{2p} \leqq \|K\|_{2p}$ by (2), $\{T_t K\}_{t \in [0,T]}$ is uniformly integrable (Theorem A.3.4). Hence $\lim_{t\to 0} \|T_t K - K\|_p = 0$. Using (2) again, we obtain

$$\|T_t G - G\|_p \leqq \|T_t K - K\|_p + 2\|G - K\|_p.$$

Since $\mathscr{P}(E)$ is dense in $L^p(\mu_T; E)$, this inequality implies the conclusion. □

Lemma 5.1.12 *Let $r, r' \in \mathbb{R}$ and $p, p' > 1$.*
(1) *If $r \leqq r'$ and $p \leqq p'$, then* $\|F\|_{r,p} \leqq \|F\|_{r',p'}$ $(F \in \mathscr{P}(E))$.
(2) *If $F_n \in \mathscr{P}(E)$ satisfies*

$$\lim_{n\to\infty} \|F_n\|_{r,p} = 0, \qquad \lim_{n,m\to\infty} \|F_n - F_m\|_{r',p'} = 0,$$

then $\lim_{n\to\infty} \|F_n\|_{r',p'} = 0$.

Proof (1) Let $s \geqq 0$. Then, by Definition 5.1.8 and (5.1.13),

$$(I - L)^{-s} F = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{e}^{-t} T_t F\,\mathrm{d}t$$

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for $F \in \mathscr{P}(E)$. Since $\|(I-L)^{-s}F\|_p \leqq \|F\|_p$ by Lemma 5.1.11, we obtain

$$\begin{aligned}\|F\|_{r,p} &= \|(I-L)^{-\frac{r'-r}{2}}(I-L)^{\frac{r'}{2}}F\|_p \leqq \|(I-L)^{\frac{r'}{2}}F\|_p \\ &\leqq \|(I-L)^{\frac{r'}{2}}F\|_{p'} = \|F\|_{r',p'}.\end{aligned}$$

(2) Set $G_n = (I-L)^{\frac{r'}{2}}F_n$. Since $\|F_n - F_m\|_{r',p'} \to 0$, $\{G_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^{p'}(\mu_T; E)$. Hence, $\lim_{n\to\infty}\|G_n - G\|_{p'} = 0$ holds for some $G \in L^{p'}(\mu_T; E)$. Since $\|F_n\|_{r,p} \to 0$,

$$\lim_{n\to\infty}\|(I-L)^{\frac{r-r'}{2}}G_n\|_p = 0.$$

Hence, we have, for any $K \in \mathscr{P}(E)$,

$$\begin{aligned}\int_{W_T}\langle G, K\rangle_E \,\mathrm{d}\mu_T &= \lim_{n\to\infty}\int_{W_T}\langle G_n, K\rangle_E \,\mathrm{d}\mu_T \\ &= \lim_{n\to\infty}\int_{W_T}\langle (I-L)^{\frac{r-r'}{2}}G_n, (I-L)^{\frac{r'-r}{2}}K\rangle_E \,\mathrm{d}\mu_T = 0\end{aligned}$$

and $G = 0$. Therefore, $\|F_n\|_{r',p'} = \|G_n\|_{p'} \to 0$. □

Proof of Theorem 5.1.10 (1) The assertion follows from Lemma 5.1.12.
(2) For $p > 1$, $q = \frac{p}{p-1}$ and $G \in \mathscr{P}(E)$, we have

$$\begin{aligned}\|G\|_{-r,q} &= \|(I-L)^{-\frac{r}{2}}G\|_q \\ &= \sup\Big\{\int_{W_T}\langle (I-L)^{-\frac{r}{2}}G, F\rangle_E \,\mathrm{d}\mu_T\,;\, F \in \mathscr{P}(E),\ \|F\|_p \leqq 1\Big\} \\ &= \sup\Big\{\int_{W_T}\langle G, (I-L)^{-\frac{r}{2}}F\rangle_E \,\mathrm{d}\mu_T\,;\, F \in \mathscr{P}(E),\ \|F\|_p \leqq 1\Big\} \\ &= \sup\Big\{\int_{W_T}\langle G, K\rangle_E \,\mathrm{d}\mu_T\,;\, K \in \mathscr{P}(E),\ \|K\|_{r,p} \leqq 1\Big\}.\end{aligned}$$

This implies the assertion (2). □

Definition 5.1.13 (1) Define

$$\begin{aligned}&\mathbb{D}^{r,\infty-}(E) = \bigcap_{p\in(1,\infty)}\mathbb{D}^{r,p}(E), &&\mathbb{D}^{\infty,p}(E) = \bigcap_{r\in\mathbb{R}}\mathbb{D}^{r,p}(E),\\ &\mathbb{D}^{r,1+}(E) = \bigcup_{p\in(1,\infty)}\mathbb{D}^{r,p}(E), &&\mathbb{D}^{-\infty,p}(E) = \bigcup_{r\in\mathbb{R}}\mathbb{D}^{r,p}(E),\\ &\mathbb{D}^{\infty,\infty-}(E) = \bigcap_{r\in\mathbb{R},\ p\in(1,\infty)}\mathbb{D}^{r,p}(E), &&\mathbb{D}^{-\infty,1+}(E) = \bigcup_{r\in\mathbb{R},\ p\in(1,\infty)}\mathbb{D}^{r,p}(E).\end{aligned}$$

(2) An element $\Phi \in \mathbb{D}^{-\infty,1+}(E)$ is called a **generalized Wiener functional**.

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$\mathbb{D}^{\infty,\infty-}(E)$ is a Fréchet space and $\mathbb{D}^{-\infty,1+}(E)$ is its dual space. The value $\Phi(F)$ of $\Phi \in \mathbb{D}^{-\infty,1+}(E) = (\mathbb{D}^{\infty,\infty-}(E))^*$ at $F \in \mathbb{D}^{\infty,\infty-}(E)$ is denoted by $\int_{W_T} \langle F, \Phi\rangle_E \, \mathrm{d}\mu_T$ or $\mathbf{E}[\langle F, \Phi\rangle_E]$:

$$\Phi(F) = \int_{W_T} \langle F, \Phi\rangle_E \, \mathrm{d}\mu_T = \mathbf{E}[\langle \Phi, F\rangle_E].$$

When $E = \mathbb{R}$, we simply write the above as $\int_{W_T} F\Phi \, \mathrm{d}\mu_T$ or $\mathbf{E}[F\Phi]$. Moreover, if $F = 1$, it is also written as $\int_{W_T} \Phi \, \mathrm{d}\mu_T$ or $\mathbf{E}[\Phi]$. These notations come from the fact that, if $F \in L^q(\mu_T; E)$ and $\Phi \in L^p(\mu_T; E)$, then $\langle F, \Phi\rangle_E \in L^1(\mu_T)$ and $\int_{W_T} \langle F, \Phi\rangle_E \mathrm{d}\mu_T$ is a usual integral.

5.2 Continuity of Operators

The aim of this section is to prove the continuity of the operators ∇, ∇^*, and T_t and to present their applications.

Theorem 5.2.1 (1) *For any $r \in \mathbb{R}$ and $p > 1$, $\nabla : \mathscr{P}(E) \to \mathscr{P}(H_T \otimes E)$ is extended to a unique linear operator $\overline{\nabla} : \mathbb{D}^{-\infty,1+}(E) \to \mathbb{D}^{-\infty,1+}(H_T \otimes E)$ whose restriction $\overline{\nabla} : \mathbb{D}^{r+1,p}(E) \to \mathbb{D}^{r,p}(H_T \otimes E)$ is continuous.*
(2) *For any $r \in \mathbb{R}$ and $p > 1$, the adjoint operator ∇^* of ∇ is extended to a unique linear operator $\overline{\nabla}^* : \mathbb{D}^{-\infty,1+}(H_T \otimes E) \to \mathbb{D}^{-\infty,1+}(E)$ whose restriction $\overline{\nabla}^* : \mathbb{D}^{r+1,p}(H_T \otimes E) \to \mathbb{D}^{r,p}(E)$ is continuous.*
(3) *For any $t > 0$ and $p > 1$, $T_t(L^p(\mu_T; E)) \subset \mathbb{D}^{\infty,p}(E)$. In particular, if a measurable function $F : W_T \to E$ is bounded, then $T_tF \in \mathbb{D}^{\infty,\infty-}(E)$.*

In the following, the extensions $\overline{\nabla}$ and $\overline{\nabla}^*$ of ∇ and ∇^* will also be denoted by ∇ and ∇^*.

For a proof of the theorem, we prepare a lemma. For $\phi = \{\phi_n\}_{n=0}^\infty \subset \mathbb{R}$, define the mapping $M_\phi : \mathscr{P}(E) \to \mathscr{P}(E)$ by

$$M_\phi F = \sum_{n=0}^\infty \phi_n J_n F \qquad (F \in \mathscr{P}(E)). \tag{5.2.1}$$

Lemma 5.2.2 *For $\phi = \{\phi_n\}_{n=0}^\infty \subset \mathbb{R}$, set $\phi^+ = \{\phi_{n+1}\}_{n=0}^\infty$. Then, for any $F \in \mathscr{P}(E)$,*

$$\nabla M_\phi F = M_{\phi^+} \nabla F.$$

In particular, $\nabla(J_nF) = J_{n-1}(\nabla F)$, $n = 1, 2, \ldots$

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Proof We may assume that $E = \mathbb{R}$ and F is a function of the form $F = H_\alpha = \prod_{j=1}^\infty H_{\alpha_j}(\ell_j)$, $\{\ell_j\}_{j=1}^\infty$ being an orthonormal basis of H_T. Then, since

$$\nabla M_\phi H_\alpha = \phi_{|\alpha|} \nabla H_\alpha$$

and $H_n' = H_{n-1}$, we have

$$\nabla H_\alpha = \sum_{j:\alpha_j>0} H_{\alpha_j-1}(\ell_j)\Big(\prod_{i\neq j} H_{\alpha_i}(\ell_i)\Big)\ell_j.$$

Since $H_{\alpha_j-1}(\ell_j)\Big(\prod_{i\neq j} H_{\alpha_i}(\ell_i)\Big) \in \mathscr{H}_{|\alpha|-1}$, we obtain

$$M_{\phi^+}\nabla H_\alpha = \phi_{|\alpha|}\nabla H_\alpha = \nabla M_\phi H_\alpha. \qquad \square$$

Lemma 5.2.3 (Hypercontractivity of $\{T_t\}$) *Let $p > 1$ and $t \geqq 0$, and set $q(t) = \mathrm{e}^{2t}(p-1)+1$. Then, for any $F \in L^p(\mu_T)$,*

$$\|T_t F\|_{q(t)} \leqq \|F\|_p. \tag{5.2.2}$$

The proof is omitted. See [104, Theorem 2.11].

Lemma 5.2.4 *For any $p > 1$ and $n \in \mathbb{Z}_+$, there exists a constant $b_{p,n} > 0$ such that*

$$\|J_n F\|_p \leqq b_{p,n}\|F\|_p \tag{5.2.3}$$

for any $F \in \mathscr{P}$. In particular, J_n defines a bounded operator on $L^p(\mu_T)$.

Proof For $p > 1$, define $c(p)$ by

$$c(p) = \begin{cases} (p-1)^{\frac{1}{2}} & (p \geqq 2) \\ (p-1)^{-\frac{1}{2}} & (1 < p < 2). \end{cases}$$

First we assume $p \geqq 2$. Let $t \geqq 0$ so that $\mathrm{e}^{2t} = p-1$. Then, since $\|T_t F\|_{1+\mathrm{e}^{2t}} \leqq \|F\|_2$ by Lemma 5.2.3, we have $\|T_t F\|_p \leqq \|F\|_2$. Moreover, since J_n is an orthogonal projection on $L^2(\mu_T)$,

$$\|T_t J_n F\|_p \leqq \|J_n F\|_2 \leqq \|F\|_2 \leqq \|F\|_p.$$

Hence, from the identity $T_t J_n F = \mathrm{e}^{-nt} J_n F = c(p)^{-n} J_n F$, taking $b_{p,n} = c(p)^n$, we obtain (5.2.3).

Second, we assume $1 < p < 2$. Then, since $\frac{p}{p-1} > 2$ and $c(\frac{p}{p-1}) = c(p)$, we obtain from the above consideration

$$\|J_n F\|_{\frac{p}{p-1}} \leqq c(p)^n \|F\|_{\frac{p}{p-1}}.$$

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Due to the duality,

$$\|J_n^* F\|_p \leqq c(p)^n \|F\|_p.$$

Since J_n is an orthogonal projection on $L^2(\mu_T)$, we have $J_n^* F = J_n F$ and obtain (5.2.3), by setting $b_{p,n} = c(p)^n$ again. □

Lemma 5.2.5 *For any $p > 1$ and $n \in \mathbb{Z}_+$, there exists a constant $C_{n,p} > 0$ such that*

$$\|T_t(I - J_0 - \cdots - J_{n-1})F\|_p \leqq C_{n,p} \mathrm{e}^{-nt} \|F\|_p \tag{5.2.4}$$

for any $t > 0$ and $F \in L^p(\mu_T)$.

Proof If $p = 2$, then, by the definition of T_t,

$$\left\|T_t(I - J_0 - \cdots - J_{n-1})F\right\|_2^2 = \sum_{k=n}^{\infty} \mathrm{e}^{-2kt} \left\|J_k F\right\|_2^2 \leqq \mathrm{e}^{-2nt} \left\|F\right\|_2^2$$

and (5.2.4) holds.

Assume that $p > 2$. Set $p = \mathrm{e}^{2t_0} + 1$ for $t_0 > 0$. For $t > t_0$, by Lemma 5.2.3 and the above observation,

$$\begin{aligned}\|T_t(I - J_0 - \cdots - J_{n-1})F\|_p &\leqq \|T_{t-t_0}(I - J_0 - \cdots - J_{n-1})F\|_2 \\ &\leqq \mathrm{e}^{-n(t-t_0)} \|F\|_2 \leqq \mathrm{e}^{nt_0} \mathrm{e}^{-nt} \|F\|_p.\end{aligned}$$

For $t \leqq t_0$, by Lemmas 5.1.11 and 5.2.4,

$$\begin{aligned}\|T_t(I - J_0 - \cdots - J_{n-1})F\|_p &\leqq \|(I - J_0 - \cdots - J_{n-1})F\|_p \\ &\leqq \Big(\sum_{k=0}^{n-1} b_{p,k}\Big) \|F\|_p \leqq \mathrm{e}^{nt_0} \Big(\sum_{k=0}^{n-1} b_{p,k}\Big) \mathrm{e}^{-nt} \|F\|_p.\end{aligned}$$

Hence, we have (5.2.4) also for $p > 2$.

If $p \in (1, 2)$, we can prove the conclusion by the duality between $L^p(\mu_T)$ and $L^{\frac{p}{p-1}}(\mu_T)$ in the same way as in the proof of Lemma 5.2.4. □

Lemma 5.2.6 *Let $\delta > 0$ and $\psi : (-\delta, \delta) \to \mathbb{R}$ be real analytic. Suppose that, for $\alpha \in (0, 1]$, $\phi = \{\phi_n\}_{n=0}^{\infty}$ satisfies $\phi_n = \psi(n^{-\alpha})$ for $n \geqq \delta^{-\frac{1}{\alpha}}$. Then, for each $p > 1$, there exists a constant C_p such that*

$$\|M_\phi F\|_p \leqq C_p \|F\|_p \qquad (F \in \mathscr{P}). \tag{5.2.5}$$

Proof Fix $n \in \mathbb{N}$ so that $\frac{1}{n^\alpha} < \delta$ and set

$$M_\phi^{(1)} = \sum_{k=0}^{n-1} \phi_k J_k \quad \text{and} \quad M_\phi^{(2)} = \sum_{k=n}^{\infty} \phi_k J_k.$$

Since $M_\phi^{(1)}$ is a bounded operator on $L^p(\mu_T)$ by Lemma 5.2.4, it suffices to prove the following inequality:

$$\sup\{\|M_\phi^{(2)}F\|_p\,;\ F \in \mathscr{P},\ \|F\|_p \leqq 1\} < \infty. \tag{5.2.6}$$

First we show (5.2.6) when $\alpha = 1$. Define the operator R by

$$R = \int_0^\infty T_t(I - J_0 - \cdots - J_{n-1})\,\mathrm{d}t.$$

Then, we have

$$R^jF = \int_0^\infty \cdots \int_0^\infty T_{t_1+t_2+\cdots+t_j}(I - J_0 - \cdots - J_{n-1})F\,\mathrm{d}t_1\cdots\mathrm{d}t_j.$$

By Lemma 5.2.5,

$$\|R^jF\|_p \leqq C_{n,p}\frac{1}{n^j}\|F\|_p. \tag{5.2.7}$$

Moreover, by the definition of R,

$$RJ_kF = \frac{1}{k}J_kF, \quad R^jJ_kF = \frac{1}{k^j}J_kF \qquad (k \geqq n).$$

Combining this with the series expansion of ψ,

$$\psi(x) = \sum_{j=0}^\infty a_jx^j \qquad (x \in (-\delta, \delta)),$$

we obtain

$$\phi_kJ_kF = \psi(k^{-1})J_kF = \sum_{j=0}^\infty a_j\frac{1}{k^j}J_kF = \sum_{j=0}^\infty a_jR^jJ_kF.$$

Hence

$$M_\phi^{(2)}F = \sum_{j=0}^\infty a_jR^jF.$$

From this identity and (5.2.7), we obtain

$$\|M_\phi^{(2)}F\|_p \leqq C_{n,p}\sum_{j=0}^\infty |a_j|\Big(\frac{1}{n}\Big)^j\|F\|_p$$

and (5.2.6).

Second, we show (5.2.6) when $\alpha < 1$. For $t \geqq 0$, let ν_t be a probability measure on $[0,\infty)$ such that

$$\int_0^\infty \mathrm{e}^{-\lambda s}\nu_t(\mathrm{d}s) = \mathrm{e}^{-\lambda^\alpha t} \qquad (\lambda > 0).$$

Set

$$Q_t = \int_0^\infty T_s \nu_t(\mathrm{d}s)$$

and define

$$Q = \int_0^\infty Q_t(I - J_0 - \cdots - J_{n-1})\,\mathrm{d}t.$$

By Lemma 5.2.5,

$$\|Q^j F\|_p \leqq C_{n,p}\Big(\frac{1}{n^\alpha}\Big)^j \|F\|_p.$$

Moreover, by definition,

$$Q^j J_k F = \Big(\frac{1}{k^\alpha}\Big)^j J_k F$$

for $k \geqq n$. From these observations we obtain (5.2.6) by a similar argument to the case where $\alpha = 1$. □

By the following lemma, the assertions of Lemmas 5.2.4, 5.2.5, and 5.2.6 also hold if we replace $\mathscr{P}$ and $L^p(\mu_T)$ by $\mathscr{P}(E)$ and $L^p(\mu_T; E)$.

Lemma 5.2.7 *Let K be a real separable Hilbert space and $1 < p \leqq q < \infty$. Suppose that a linear operator $A : \mathscr{P} \to \mathscr{P}(K)$ is extended to a continuous operator $L^p(\mu_T) \to L^q(\mu_T; K)$. Define $A(G\,e) = (AG) \otimes e$ ($G \in \mathscr{P}$, $e \in E$) and extend A to $\mathscr{P}(E)$. Then, $A : \mathscr{P}(E) \to \mathscr{P}(K \otimes E)$ is extended to a continuous linear operator $L^p(\mu_T; E) \to L^q(\mu_T; K \otimes E)$.*

Proof We use the following Khinchin's inequality (see [112]): Let $\{r_n\}_{n=1}^\infty$ be a Bernoulli sequence on a probability space $(\Omega, \mathscr{F}, P)$, that is, $r_1, r_2, \ldots$ are independent and satisfy $P(r_i = 1) = P(r_i = -1) = \frac{1}{2}$ $(i = 1, 2, \ldots)$. Then, for any $p > 1$, $N \in \mathbb{N}$, $e_1, \ldots, e_N \in E$,

$$\frac{1}{B_p}\Big(\mathbf{E}^P\Big[\Big\|\sum_{m=1}^N r_m e_m\Big\|_E^p\Big]\Big)^{\frac{1}{p}} \leqq \Big(\sum_{m=1}^N \|e_m\|_E^2\Big)^{\frac{1}{2}} \leqq B_p\Big(\mathbf{E}^P\Big[\Big\|\sum_{m=1}^N r_m e_m\Big\|_E^p\Big]\Big)^{\frac{1}{p}},$$

where $B_p = (p-1) \vee (\frac{1}{p-1})$.

For $F \in \mathscr{P}(E)$, take an orthonormal basis $\{e_n\}_{n=1}^\infty$ of E so that $F = \sum_{n=1}^N F_n e_n$ for some $N \in \mathbb{N}$ and $F_n \in \mathscr{P}$ $(n = 1, \ldots, N)$. Denoting by L_A the operator norm of $A : L^p(\mu_T) \to L^q(\mu_T; K)$, by Khinchin's inequality, we obtain

$$\big\|AF\big\|_{q,K\otimes E}^q = \int_{W_T}\Big(\sum_{n=1}^N \big\|AF_n(w)\big\|_K^2\Big)^{\frac{q}{2}} \mu_T(\mathrm{d}w)$$

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$$
\begin{aligned}
&\leqq B_q^q \int_{W_T} \mathbf{E}^P\Big[\Big\|\sum_{n=1}^N r_n A F_n(w)\Big\|_K^q\Big]\mu_T(\mathrm{d}w) \\
&\leqq B_q^q L_A^q\Big(\mathbf{E}^P\Big[\int_{W_T}\Big|\sum_{n=1}^N r_n F_n(w)\Big|^p \mu_T(\mathrm{d}w)\Big]\Big)^{\frac{q}{p}} \\
&\leqq B_q^q L_A^q B_p^q\Big(\int_{W_T}\Big(\sum_{n=1}^N F_n(w)^2\Big)^{\frac{p}{2}}\mu_T(\mathrm{d}w)\Big)^{\frac{q}{p}} \\
&= B_q^q L_A^p B_p^q \|F\|_{p,E}^q.
\end{aligned}
$$

Hence, $A : \mathscr{P}(E) \to \mathscr{P}(K \otimes E)$ is extended continuously. $\square$

Proof of Theorem 5.2.1. (1) Let $r \in \mathbb{R}$ and $p > 1$. Define $\phi = \{\phi_n\}_{n=0}^\infty$ by

$$
\phi_0 = 0, \quad \phi_n = \Big(\frac{n}{1+n}\Big)^{\frac{r}{2}} = \Big(\frac{1}{1+\frac{1}{n}}\Big)^{\frac{r}{2}} \qquad (n \geqq 1).
$$

By Lemma 5.2.6, there exists a constant C_p such that

$$
\|M_\phi F\|_p \leqq C_p \|F\|_p
$$

for any $F \in \mathscr{P}(E)$. By Lemma 5.2.2, we have

$$
\nabla M_\phi (I-L)^{\frac{r}{2}} F = (I-L)^{\frac{r}{2}} \nabla F.
$$

Moreover, by Theorem 5.1.9, there exists a constant C_p' such that

$$
\|\nabla F\|_p \leqq C_p' \|(I-L)^{\frac{1}{2}} F\|_p.
$$

Summing up the above observations, we obtain

$$
\begin{aligned}
\|(I-L)^{\frac{r}{2}}\nabla F\|_p &= \|\nabla M_\phi (I-L)^{\frac{r}{2}} F\|_p \\
&\leqq C_p' \|(I-L)^{\frac{1}{2}} M_\phi (I-L)^{\frac{r}{2}} F\|_p = C_p' \|M_\phi (I-L)^{\frac{r+1}{2}} F\|_p \\
&\leqq C_p' C_p \|(I-L)^{\frac{r+1}{2}} F\|_p = C_p' C_p \|F\|_{r+1,p}.
\end{aligned}
$$

Hence, $\nabla : \mathbb{D}^{r+1,p}(E) \to \mathbb{D}^{r,p}(H_T \otimes E)$ is continuous.

(2) The assertion follows from (1) and the duality.

(3) Let $\{\ell_n\}_{n=1}^\infty \subset W_T^*$ be an orthonormal basis of H_T. By (5.1.5), $\nabla^*\ell_n(w) = \ell_n(w)$, μ_T-a.s. $w \in W_T$. Hence, by Lemma 5.1.11,

$$
\begin{aligned}
\langle \nabla T_t F(w), \ell_n\rangle_{H_T} &= \mathrm{e}^{-t}\int_{W_T} \langle (\nabla F)(\mathrm{e}^{-t}w + \sqrt{1-\mathrm{e}^{-2t}}w'), \ell_n\rangle_{H_T}\mu_T(\mathrm{d}w') \\
&= \frac{\mathrm{e}^{-t}}{\sqrt{1-\mathrm{e}^{-2t}}}\int_{W_T} \langle \nabla[F(\mathrm{e}^{-t}w + \sqrt{1-\mathrm{e}^{-2t}}\,\cdot\,)](w'), \ell_n\rangle_{H_T}\mu_T(\mathrm{d}w') \\
&= \frac{\mathrm{e}^{-t}}{\sqrt{1-\mathrm{e}^{-2t}}}\int_{W_T} F(\mathrm{e}^{-t}w + \sqrt{1-\mathrm{e}^{-2t}}w')\ell_n(w')\mu_T(\mathrm{d}w').
\end{aligned}
$$

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Since $\{\ell_n\}_{n=1}^\infty \subset \mathscr{H}_1$ is an orthonormal basis of $\mathscr{H}_1$,

$$\begin{aligned}\big\|\nabla T_t F(w)\big\|_{H_T}^2 &= \sum_{n=1}^\infty \frac{\mathrm{e}^{-2t}}{1-\mathrm{e}^{-2t}}\Big(\int_{W_T} F(\mathrm{e}^{-t}w + \sqrt{1-\mathrm{e}^{-2t}}w')\ell_n(w')\mu_T(\mathrm{d}w')\Big)^2 \\ &= \frac{\mathrm{e}^{-2t}}{1-\mathrm{e}^{-2t}}\big\|J_1[F(\mathrm{e}^{-t}w + \sqrt{1-\mathrm{e}^{-2t}}\,\cdot)]\big\|_2^2.\end{aligned}$$

Set

$$A_p = \Big(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}|x|^p \mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x\Big)^{\frac{1}{p}}.$$

Then, we have

$$\Big(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}}|y|^p \mathrm{e}^{-\frac{y^2}{2t}}\mathrm{d}y\Big)^{\frac{1}{p}} = A_p\sqrt{t}.$$

In particular, since $G \in \mathscr{H}_1$ is a Gaussian random variable with mean 0 and variance $\|G\|_2^2$, $\|G\|_p = A_p\|G\|_2$. Combining this with Lemma 5.2.4, we obtain

$$\begin{aligned}&\int_{W_T} \big\|\nabla T_t F\big\|_{H_T}^p \mathrm{d}\mu_T \\ &= \Big(\frac{\mathrm{e}^{-t}}{\sqrt{1-\mathrm{e}^{-2t}}}\Big)^p \int_{W_T} \big\|J_1[F(\mathrm{e}^{-t}w + \sqrt{1-\mathrm{e}^{-2t}}\,\cdot)]\big\|_2^p \mu_T(\mathrm{d}w) \\ &= A_p^{-p}\Big(\frac{\mathrm{e}^{-t}}{\sqrt{1-\mathrm{e}^{-2t}}}\Big)^p \int_{W_T} \big\|J_1[F(\mathrm{e}^{-t}w + \sqrt{1-\mathrm{e}^{-2t}}\,\cdot)]\big\|_p^p \mu_T(\mathrm{d}w) \\ &\leqq A_p^{-p}b_{p,1}^p\Big(\frac{\mathrm{e}^{-t}}{\sqrt{1-\mathrm{e}^{-2t}}}\Big)^p \int_{W_T} \big\|F(\mathrm{e}^{-t}w + \sqrt{1-\mathrm{e}^{-2t}}\,\cdot)\big\|_p^p \mu_T(\mathrm{d}w) \\ &= A_p^{-p}b_{p,1}^p\Big(\frac{\mathrm{e}^{-t}}{\sqrt{1-\mathrm{e}^{-2t}}}\Big)^p \int_{W_T} T_t|F|^p \mathrm{d}\mu_T = A_p^{-p}b_{p,1}^p\Big(\frac{\mathrm{e}^{-t}}{\sqrt{1-\mathrm{e}^{-2t}}}\Big)^p \big\|F\big\|_p^p,\end{aligned}$$

where the last identity follows from

$$\int_{W_T} G(T_tK)\,\mathrm{d}\mu_T = \int_{W_T} (T_tG)K\,\mathrm{d}\mu_T \quad (G, K \in \mathscr{P}) \quad \text{and} \quad T_t1 = 1.$$

Hence, by Lemma 5.2.7, $\nabla T_t : \mathscr{P}(E) \to \mathscr{P}(H_T \otimes E)$ is extended to a continuous linear operator from $L^p(\mu_T; E)$ into $L^p(\mu_T; H_T \otimes E)$. Therefore,

$$T_t(L^p(\mu_T; E)) \subset \mathbb{D}^{1,p}(E).$$

Repeating the above arguments inductively, we obtain the assertion. □

We end this section by showing the fundamental properties of ∇ and ∇^*.

Theorem 5.2.8 *Let $p, q, r > 1$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and E, E_1, E_2 be real separable Hilbert spaces.*

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(1) *Let* $F \in \mathbb{D}^{1,p}(E_1)$, $G_1 \in \mathbb{D}^{1,q}(E_2)$, $G_2 \in \mathbb{D}^{1,q}(H_T \otimes E_2)$ *and* $K \in \mathbb{D}^{1,p}$. *Then,* $F \otimes G_1 \in \mathbb{D}^{1,r}(E_1 \otimes E_2)$, $KG_2 \in \mathbb{D}^{1,r}(H_T \otimes E_2)$ *and*

$$\nabla(F \otimes G_1) = F \otimes \nabla G_1 + \nabla F \otimes G_1, \tag{5.2.8}$$

$$\nabla^*(KG_2) = K\nabla^* G_2 - \langle \nabla K, G_2 \rangle_{H_T}, \tag{5.2.9}$$

where $E_1 \otimes H_T \otimes E_2$ *is identified with* $H_T \otimes E_1 \otimes E_2$.
(2) *Let* $k \in \mathbb{Z}_+$. *Both of the following mappings are bounded and bilinear:*

$$\mathbb{D}^{k,p}(E_1) \times \mathbb{D}^{k,q}(E_2) \ni (F, G) \mapsto F \otimes G \in \mathbb{D}^{k,r}(E_1 \otimes E_2),$$
$$\mathbb{D}^{k,p}(E) \times \mathbb{D}^{k,q}(E) \ni (F, G) \mapsto \langle F, G \rangle_E \in \mathbb{D}^{k,r}.$$

In particular, if $F, G \in \mathbb{D}^{\infty,\infty-}$, *then* $FG \in \mathbb{D}^{\infty,\infty-}$.

Proof (1) Let $F \in \mathscr{P}(E_1)$ and $G_1 \in \mathscr{P}(E_2)$. By (5.1.1), the $E_1 \otimes E_2$-valued random variable $\langle \nabla(F \otimes G_1), h \rangle_{H_T}$ is obtained by

$$\langle \nabla(F \otimes G_1)(w), h \rangle_{H_T} = \frac{\mathrm{d}}{\mathrm{d}\xi}\Big|_{\xi=0} (F \otimes G_1)(w + \xi h) \qquad (w \in W_T,\ h \in H_T).$$

Hence, $\nabla(F \otimes G_1) = F \otimes \nabla G_1 + \nabla F \otimes G_1$. By the continuity of ∇, (5.2.8) holds for any $F \in \mathbb{D}^{1,p}(E_1)$ and $G_1 \in \mathbb{D}^{1,q}(E_2)$.

Next, let $G_2 \in \mathscr{P}(H_T \otimes E_2)$, $K \in \mathscr{P}$ and $\psi \in \mathscr{P}(E_2)$. By (5.2.8), we have

$$\begin{aligned}\int_{W_T} \langle KG_2, \nabla\psi \rangle_{H_T \otimes E_2} \mathrm{d}\mu_T &= \int_{W_T} \langle G_2, \nabla(K\psi) - \nabla K \otimes \psi \rangle_{H_T \otimes E_2} \mathrm{d}\mu_T \\ &= \int_{W_T} \langle K\nabla^* G_2 - \langle \nabla K, G_2 \rangle_{H_T}, \psi \rangle_{E_2} \mathrm{d}\mu_T.\end{aligned}$$

By the continuity of ∇ and ∇^*, (5.2.9) holds for any $G_2 \in \mathbb{D}^{1,q}(H_T \otimes E_2)$ and $K \in \mathbb{D}^{1,p}$.
(2) The assertion is trivial by (1) and the definition of the inner product. □

Proposition 5.2.9 *For* $G \in \mathbb{D}^{1,2}(E)$, *if* $\nabla G = 0$, *then there exists an* $e \in E$ *such that* $G = e$, μ_T*-a.s.*

Proof We may assume $E = \mathbb{R}$. Let $\{\ell_j\}_{j=1}^\infty \subset W_T^*$ be an orthonormal basis of H_T. As in Theorem 5.1.6, we set $H_{\boldsymbol{\alpha}} = \prod_{j=0}^\infty H_{\alpha_j}(\ell_j)$ for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots) \in \mathscr{A}$. Since $H_n'(x) = xH_n(x) - (n+1)H_{n+1}(x)$ and $\nabla^*\ell_j = \ell_j$, by Theorem 5.2.8,

$$\nabla^*(H_{\boldsymbol{\alpha}}\ell_j) = (\alpha_j + 1)H_{\boldsymbol{\alpha}+\boldsymbol{\delta}_j}, \tag{5.2.10}$$

where $\boldsymbol{\delta}_j = (\delta_{ji})_{i\in\mathbb{N}}$.

For $\alpha \in \mathscr{A}$ with $|\alpha| \neq 0$, fix $j \in \mathbb{N}$ satisfying $\alpha_j \neq 0$. Since $H_\alpha = \nabla^*(\alpha_j^{-1} H_{\alpha-\delta_j}\ell_j)$ by (5.2.10), we have

$$\int_{W_T} G H_\alpha \mathrm{d}\mu_T = \int_{W_T} \langle \nabla G, \alpha_j^{-1} H_{\alpha-\delta_j}\ell_j\rangle_{H_T} \mathrm{d}\mu_T = 0.$$

Hence, by Theorem 5.1.6, G is a constant. □

5.3 Characterization of Sobolev Spaces

The aim of this section is to present explicit criteria for generalized Wiener functionals to belong to $\mathbb{D}^{r,p}(E)$, by using the continuity of ∇ and ∇^*.

The following characterization of $\mathbb{D}^{r,p}(E)$ holds as in the theory of distributions on finite dimensional spaces.

Theorem 5.3.1 *Let $r \in \mathbb{R}$, $k \in \mathbb{Z}_+$ and $p > 1$.*
(1) $\Phi \in \mathbb{D}^{-\infty,1+}(E)$ *belongs to* $\mathbb{D}^{r,p}(E)$ *if and only if*

$$\sup\Big\{\int_{W_T} \langle \Phi, F\rangle_E \,\mathrm{d}\mu_T\,;\, F \in \mathscr{P}(E),\, \|F\|_{-r,q} \leqq 1\Big\} < \infty,$$

where $q = \frac{p}{p-1}$.
(2) $F \in L^p(\mu_T; E)$ *belongs to* $\mathbb{D}^{k,p}(E)$ *if and only if there exists an* $F_k \in L^p(\mu_T; H_T^{\otimes k} \otimes E)$ *such that*

$$\int_{W_T} \langle F, (\nabla^*)^k G\rangle_E \,\mathrm{d}\mu_T = \int_{W_T} \langle F_k, G\rangle_{H_T^{\otimes k}\otimes E} \,\mathrm{d}\mu_T$$

for any $G \in \mathscr{P}(H_T^{\otimes k} \otimes E)$, *where* $H_T^{\otimes k} = \underbrace{H_T \otimes \cdots \otimes H_T}_{k \text{ times}}$. *Moreover, in this case,* $F_k = \nabla^k F$.

Proof (1) The necessity is trivial by the definition. We show the sufficiency. If $\sup\{\cdots\} < \infty$, then the mapping $\mathbb{D}^{-r,q}(E) \ni F \mapsto \int_{W_T} \langle \Phi, F\rangle_E \mathrm{d}\mu_T$ is extended to a bounded linear operator on $\mathbb{D}^{-r,q}(E)$. Since the dual space of $\mathbb{D}^{-r,q}(E)$ is $\mathbb{D}^{r,p}(E)$, there exists a $G \in \mathbb{D}^{r,p}(E)$ such that

$$\int_{W_T} \langle \Phi, F\rangle_E \,\mathrm{d}\mu_T = \int_{W_T} \langle G, F\rangle_E \,\mathrm{d}\mu_T$$

for all $F \in \mathscr{P}(E)$. Hence, $\Phi = G \in \mathbb{D}^{r,p}(E)$.
(2) The necessity and the identity $F_k = \nabla^k F$ are trivial by definition. We only show the sufficiency. Set

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$$R_0 = \sum_{n=1}^{\infty} \frac{1}{n} J_n = \int_0^{\infty} T_t(I - J_0)\,\mathrm{d}t.$$

By Lemma 5.2.5, R_0 is a bounded operator on $L^q(\mu_T; E)$. By Lemma 5.2.4, the operator $R_0(I - L)$,

$$R_0(I - L) = \sum_{n=1}^{\infty} \frac{1+n}{n} J_n = R_0 + (I - J_0),$$

is also a bounded operator on $L^q(\mu_T; E)$. Hence, we have

$$\begin{aligned} \|R_0 F\|_{s+2,q} &= \|(I - L)^{\frac{s+2}{2}} R_0 F\|_q = \|R_0(I - L)(I - L)^{\frac{s}{2}} F\|_q \\ &\leqq \|R_0(I - L)\|_{q\to q} \|F\|_{s,q} \qquad (F \in \mathscr{P}(E)), \end{aligned}$$

where $\|R_0(I-L)\|_{q\to q}$ is the operator norm of $R_0(I-L) : L^q(\mu_T; E) \to L^q(\mu_T; E)$. Hence, $R_0 : \mathbb{D}^{s,q}(E) \to \mathbb{D}^{s+2,q}(E)$ is a bounded operator. In particular, the power

$$(\nabla R_0)^n : \mathbb{D}^{-n,q}(E) \to L^q(\mu_T; H_T^{\otimes n} \otimes E) \qquad (n \in \mathbb{Z}_+)$$

is a bounded operator. Denote the operator norm of $(\nabla R_0)^n$ by $B_{n,q}$.

Let $\{H_\alpha\}_{\alpha \in \mathscr{A}}$ be the orthonormal basis of $L^2(\mu_T)$ as in the proof of Lemma 5.2.2. By (5.2.9), $\nabla^* \nabla H_\alpha = |\alpha| H_\alpha$. Hence,

$$\nabla^* \nabla = \sum_{n=0}^{\infty} n J_n$$

and

$$\nabla^* \nabla R_0 = I - J_0.$$

Suppose that $(\nabla^*)^n (\nabla R_0)^n = I - J_0 - \cdots - J_{n-1} = \sum_{k=n}^{\infty} J_k$. By the commutativity of R_0 and J_k and Lemma 5.2.2, we have

$$\begin{aligned} (\nabla^*)^{n+1} (\nabla R_0)^{n+1} &= \nabla^* \sum_{k=n}^{\infty} J_k \nabla R_0 = \nabla^* \nabla R_0 \sum_{k=n}^{\infty} J_{k+1} \\ &= (I - J_0)(I - J_0 - \cdots - J_n) = I - J_0 - \cdots - J_n. \end{aligned}$$

Hence, by induction,

$$(\nabla^*)^n (\nabla R_0)^n = I - J_0 - \cdots - J_{n-1} \qquad (n \in \mathbb{Z}_+).$$

Thus, for $G \in \mathscr{P}(E)$, $(I - J_0 - \cdots - J_{k-1})F \in L^p(\mu_T; E) \subset \mathbb{D}^{-\infty,1+}(E)$ satisfies

$$\begin{aligned} &\Big| \int_{W_T} \langle (I - J_0 - \cdots - J_{k-1})F, G \rangle_E \,\mathrm{d}\mu_T \Big| \\ &\quad = \Big| \int_{W_T} \langle F, (\nabla^*)^k (\nabla R_0)^k G \rangle_E \,\mathrm{d}\mu_T \Big| = \Big| \int_{W_T} \langle F_k, (\nabla R_0)^k G \rangle_{H_T^{\otimes k} \otimes E} \,\mathrm{d}\mu_T \Big| \\ &\quad \leqq \|(\nabla R_0)^k G\|_q \|F_k\|_p \leqq B_{k,q} \|F_k\|_p \|G\|_{-k,q}. \end{aligned}$$

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By (1), $(I - J_0 - \cdots - J_{k-1})F \in \mathbb{D}^{k,p}(E)$. Since, by Lemma 5.2.4,

$$\|J_n G\|_{k,p} = (1+n)^{\frac{k}{2}}\|J_n G\|_p \leqq (1+n)^{\frac{k}{2}} b_{p,n}\|G\|_p,$$

$J_n(L^p(\mu_T;E)) \subset \mathbb{D}^{k,p}(E)$ holds. Hence, $J_nF \in \mathbb{D}^{k,p}(E)$ $(n = 0,1,\ldots,k-1)$ and $F \in \mathbb{D}^{k,p}(E)$. □

By Theorem 5.3.1, we can prove the chain rule for the composition of differentiable functions and smooth functionals.

Let $C^k_{\exp}(\mathbb{R}^n)$ be the space of C^k-functions $f : \mathbb{R}^n \to \mathbb{R}$ which and whose derivatives of all orders are of at most exponential growth, that is, for each $i_1,\ldots,i_m \in \{1,\ldots,n\}$, $m \leqq k$, there exist positive constants C_1 and C_2 such that $|\frac{\partial^m f}{\partial x^{i_1}\cdots\partial x^{i_m}}(x)| \leqq C_1 e^{C_2|x|}$ $(x \in \mathbb{R}^n)$. Moreover, let $C^k_{\nearrow}(\mathbb{R}^n)$ be the space of C^k-functions $f : \mathbb{R}^n \to \mathbb{R}$ which and whose derivatives of all orders are of at most polynomial growth, that is, for each $i_1,\ldots,i_m \in \{1,\ldots,n\}$, $m \leqq k$, there exist $C \geqq 0$ and $r \in \mathbb{N}$ such that $|\frac{\partial^m f}{\partial x^{i_1}\cdots\partial x^{i_m}}(x)| \leqq C(1+|x|)^r$ $(x \in \mathbb{R}^n)$.

Corollary 5.3.2 (1) *Let* $f \in C^k_{\exp}(\mathbb{R}^n)$ *and* $\ell_1,\ldots,\ell_n \in W_T^*$. *Then,* $F = f(\ell_1,\ldots,\ell_n) \in \mathbb{D}^{k,\infty-}$ *and*

$$\nabla^j F = \sum_{i_1,\ldots,i_j=1}^{n} \frac{\partial^j f}{\partial x^{i_1}\cdots\partial x^{i_j}}(\ell_1,\ldots,\ell_n)\ell_{i_1}\otimes\cdots\otimes\ell_{i_j}.$$

(2) *Let* $f \in C^k_{\nearrow}(\mathbb{R}^n)$ *and* $F_1,\ldots,F_n \in \mathbb{D}^{\infty,\infty-}$. *Then,* $f(F_1,\ldots,F_n) \in \mathbb{D}^{k,\infty-}$ *and*

$$\nabla(f(F_1,\ldots,F_n)) = \sum_{i=1}^{n}\frac{\partial f}{\partial x^i}(F_1,\ldots,F_n)\nabla F_i. \tag{5.3.1}$$

Proof (1) We may assume that $\ell_1,\ldots,\ell_n$ are orthonormal. Extend this system to an orthonormal basis $\{\ell_k\}_{k=1}^\infty$ of H_T. Since

$$\int_{W_T} g(\ell_1,\ldots,\ell_m)\,d\mu_T = \int_{\mathbb{R}^m} g(x)\frac{1}{\sqrt{2\pi}^m}e^{-\frac{|x|^2}{2}}dx \qquad (m \in \mathbb{N}),$$

we have

$$\int_{W_T} F\nabla^* G\,d\mu_T = \int_{W_T}\Big\langle\sum_{j=1}^{n}\frac{\partial f}{\partial x^i}(\ell_1,\ldots,\ell_n)\ell_i, G\Big\rangle_{H_T} d\mu_T$$

for any $G \in \mathscr{P}(H_T)$. By Theorem 5.3.1, $F \in \mathbb{D}^{1,\infty-}$ and

$$\nabla F = \sum_{j=1}^{n}\frac{\partial f}{\partial x^i}(\ell_1,\ldots,\ell_n)\ell_i.$$

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Applying Theorem 5.2.8, we obtain the conclusion.
(2) If $F_i \in \mathscr{P}$, the conclusion holds by (1). Take $r > 0$ so that

$$\sup_{x\in\mathbb{R}} \frac{|f(x)| + \sum_{i=1}^n \left|\frac{\partial f}{\partial x^i}(x)\right|}{(1+|x|)^r} < \infty$$

and let $p > 2r$. Choose $F_i^m \in \mathscr{P}$ so that $\|F_i^m - F_i\|_{1,p} = 0$ ($m \to \infty$). Then, for $G \in \mathscr{P}(H_T)$,

$$\begin{aligned}
\int_{W_T} f(F_1,\dots,F_n)\nabla^* G\,\mathrm{d}\mu_T &= \lim_{m\to\infty}\int_{W_T} f(F_1^m,\dots,F_n^m)\nabla^* G\,\mathrm{d}\mu_T\\
&= \lim_{m\to\infty}\int_{W_T}\Big\langle\sum_{i=1}^n \frac{\partial f}{\partial x^i}(F_1^m,\dots,F_n^m)\nabla F_i^m, G\Big\rangle_{H_T}\mathrm{d}\mu_T\\
&= \int_{W_T}\Big\langle\sum_{i=1}^n \frac{\partial f}{\partial x^i}(F_1,\dots,F_n)\nabla F_i, G\Big\rangle_{H_T}\mathrm{d}\mu_T.
\end{aligned}$$

Since p is arbitrary, Theorem 5.3.1 implies $f(F_1,\dots,F_n) \in \mathbb{D}^{1,\infty-}$ and (5.3.1). Repeating this argument, we obtain $f(F_1,\dots,F_n) \in \mathbb{D}^{\infty,\infty-}$. □

The operator ∇^* is a generalization of stochastic integrals.[1]

Theorem 5.3.3 (1) *Let $\mathscr{N} \subset \mathscr{B}(W_T)$ be the totality of sets of zero μ_T-outer measure and $\mathscr{F}_t = \sigma(\mathscr{N} \cup \sigma(\{\theta(u), u \leqq t\}))$. Let $\{u(t) = (u^1(t),\dots,u^d(t))\}_{t\in[0,T]}$ be an $\mathbb{R}^d$-valued $\{\mathscr{F}_t\}$-predictable stochastic process such that*

$$\int_{W_T}\Big(\int_0^T |u(t)|^2\,\mathrm{d}t\Big)\mathrm{d}\mu_T < \infty.$$

Define $\Phi_u : W_T \to H_T$ by

$$\Phi_u(w)(t) = \int_0^t u(s)(w)\,\mathrm{d}s \qquad (t \in [0,T]).$$

Then, $\Phi_u \in L^2(\mu_T; H_T)$ and

$$\nabla^*\Phi_u = \sum_{\alpha=1}^d \int_0^T u^\alpha(t)\,\mathrm{d}\theta^\alpha(t). \tag{5.3.2}$$

(2) *Let $f \in C^\infty_{\exp}(\mathbb{R}^d)$. Then, for any $t \in [0,T]$ and $\alpha = 1,\dots,d$,*

$$\int_0^t f(\theta(s))\,\mathrm{d}\theta^\alpha(s) \in \mathbb{D}^{\infty,\infty-}.$$

[1] ∇^* coincides with the Skorohod integral, which is a generalization of stochastic integrals ([28, 92]).

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Remark 5.3.4 In the above assumption on u, μ_T is extended to $\mathscr{F}_T$ naturally, and so is the measurability. Even so, we may think of Φ_u and $\int_0^T u^\alpha(t)\mathrm{d}\theta^\alpha(t)$ as $\mathscr{B}(W_T)$-measurable functions. To see this, let $\mathscr{F}_t^0 = \sigma(\{\theta(s); s \leqq t\})$. Notice that every $\mathscr{F}_t$-measurable F possesses an $\mathscr{F}_t^0$-measurable modification $\widetilde{F}$. Hence every $\nu = \{\nu(t)\}_{t\in[0,T]} \in \mathscr{L}^0(\{\mathscr{F}_t\})$, the $\mathscr{L}^0$-space with respect to $\{\mathscr{F}_t\}$ (Definition 2.2.4), admits $\widetilde{\nu} = \{\widetilde{\nu(t)}\}_{t\in[0,T]} \in \mathscr{L}^0(\{\mathscr{F}_t^0\})$ such that $\mu_T(\nu(t) = \widetilde{\nu}(t)\ (0 \leqq t \leqq T)) = 1$. Therefore, by Proposition 2.2.8, there exists $u_n = \{u_n(t) = (u_n^1(t), \dots, u_n^d(t))\}_{t\in[0,T]} \in \mathscr{L}^0(\{\mathscr{F}_t^0\})$ such that

$$\int_{W_T}\Big(\int_0^T |u_n(t) - u(t)|^2\mathrm{d}t\Big)\mathrm{d}\mu_T \to 0 \quad (n \to \infty).$$

Then, defining

$$\Phi_u(w)^\alpha(t) = \int_0^t \limsup_{n\to\infty} u_n^\alpha(s)\,\mathrm{d}s$$

and

$$\int_0^T u^\alpha(t)\,\mathrm{d}\theta^\alpha(t) = \limsup_{n\to\infty}\int_0^T u_n^\alpha(t)\,\mathrm{d}\theta^\alpha(t) \quad (\alpha = 1, \dots, d),$$

we obtain the desired $\mathscr{B}(W_T)$-modifications.

Proof (1) Take a sequence $\{\{u_n(t) = (u_n^1(t), \dots, u_n^d(t))\}_{t\in[0,T]}\}_{n=1}^\infty$ of $\mathbb{R}^d$-valued stochastic processes with $\{u_n^\alpha(t)\}_{t\in[0,T]} \in \mathscr{L}^0(\{\mathscr{F}_t^0\})$ $(\alpha = 1, \dots, d)$ (see Definition 2.2.4) such that

$$\lim_{n\to\infty}\int_{W_T}\Big(\int_0^T |u_n(t) - u(t)|^2\mathrm{d}t\Big)\mathrm{d}\mu_T = 0.$$

By the definition of $\mathscr{L}^0(\{\mathscr{F}_t^0\})$, there exist an increasing sequence $0 = t_0^n < t_1^n < \cdots < t_k^n < \cdots < t_{m_n}^n = T$ and bounded, $\mathscr{F}_{t_k^n}$-measurable $\mathbb{R}^d$-valued random variables $\xi_{n,k} = (\xi_{n,k}^1, \dots, \xi_{n,k}^d)$ such that

$$u_n^\alpha(t) = \xi_{n,k}^\alpha \qquad (t_k^n < t \leqq t_{k+1}^n,\ k = 0, \dots, m_n - 1,\ \alpha = 1, \dots, d).$$

Since $\mathscr{F}_t^0$ is generated by $\theta(s)$ $(s \leqq t)$, we may assume that, taking a subsequence if necessary, there exist $0 < s_1^{k,n} < \cdots < s_{j_{k,n}}^{k,n} \leqq t_k^n$ and $\phi_{n,k}^\alpha \in C_b^\infty(\mathbb{R}^{dj_{k,n}})$ such that

$$\xi_{n,k}^\alpha = \phi_{n,k}^\alpha(\theta(s_1^{k,n}), \dots, \theta(s_{j_{k,n}}^{k,n})). \tag{5.3.3}$$

For $\alpha = 1, \dots, d$, let $e_\alpha = (\overbrace{0, \dots, 0}^{\alpha-1}, 1, 0, \dots, 0) \in \mathbb{R}^d$. For $0 \leqq s < t \leqq T$, define $\ell_{(s,t]}^\alpha \in H_T$ by

$$\dot{\ell}_{(s,t]}^\alpha(v) = \mathbf{1}_{(s,t]}(v)\mathrm{e}^\alpha \qquad (v \in [0, T]),$$

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that is, $\langle \ell^\alpha_{(s,t]}, h\rangle_{H_T} = h^\alpha(t) - h^\alpha(s)$ $(h \in H_T)$. Then,

$$\Phi_{u_n} = \sum_{k=0}^{m_n-1} \sum_{\alpha=1}^{d} \xi^\alpha_{n,k} \ell^\alpha_{(t^n_k, t^n_{k+1}]}.$$

Hence, by (5.2.9),

$$\nabla^* \Phi_{u_n} = \sum_{k=0}^{m_n-1} \sum_{\alpha=1}^{d} \{\xi^\alpha_{n,k} \nabla^* \ell^\alpha_{(t^n_k, t^n_{k+1}]} - \langle \nabla \xi^\alpha_{n,k}, \ell^\alpha_{(t^n_k, t^n_{k+1}]}\rangle_{H_T}\}.$$

By the expression (5.3.3) and Corollary 5.3.2,

$$\langle \nabla \xi^\alpha_{n,k}, \ell^\alpha_{(t^n_k, t^n_{k+1}]}\rangle_{H_T} = 0 \qquad (k = 0, \ldots, m_n - 1).$$

By (5.1.5), $\{u_n(t)\}_{t\in[0,T]}$ satisfies (5.3.2). In particular, for $F \in \mathscr{P}$, we have

$$\int_{W_T} \langle \Phi_{u_n}, \nabla F\rangle_{H_T} \, d\mu_T = \int_{W_T} \Big(\sum_{\alpha=0}^{d} \int_0^T u^\alpha_n(t) \, d\theta^\alpha(t)\Big) F \, d\mu_T.$$

Letting $n \to \infty$, we arrive at

$$\int_{W_T} \langle \Phi_u, \nabla F\rangle_{H_T} \, d\mu_T = \int_{W_T} \Big(\sum_{\alpha=0}^{d} \int_0^T u^\alpha(t) \, d\theta^\alpha(t)\Big) F \, d\mu_T,$$

which gives (5.3.2).

(2) Define $\{u(s)\}_{s\in[0,T]}$ by $u^\alpha(s) = f(\theta(s))\mathbf{1}_{[0,t]}(s)$ and $u^\beta(s) = 0$ $(\beta \neq \alpha)$. Since $\exp\big(\max_{0\leqq s\leqq T} |\theta(s)|\big) \in \bigcap_{p\in(1,\infty)} L^p(\mu_T)$, Corollary 5.3.2 implies $\Phi_u \in \mathbb{D}^{\infty,\infty-}(H_T)$. (1) and the continuity of ∇^* yields the conclusion. □

By Theorem 5.3.3, we obtain an explicit formula for the integrand in Itô's representation theorem (Theorem 2.6.2) for martingales, as will be seen below. The result is called the **Clark–Ocone formula**, which, for example, plays an important role in the theory of mathematical finance to obtain the hedging strategy for derivatives.

Theorem 5.3.5 *Let $\mathscr{F}_t$ be as in Theorem 5.3.3. For $F \in \mathbb{D}^{1,2}$, set $f^\alpha(t,w) = \mathbf{E}[(\dot{\overbrace{(\nabla F)}}(w))^\alpha(t)|\mathscr{F}_t]$, where $(\dot{\overbrace{(\nabla F)}}(w))^\alpha(t)$ is the α-th component of the value at time t of the derivative of $(\nabla F)(w) \in H_T$ and $\mathbf{E}[\,\cdot\,|\mathscr{F}_t]$ is the conditional expectation with respect to the natural extension of μ_T to $\mathscr{F}_t$. Then,*

$$F = \mathbf{E}[F] + \sum_{\alpha=1}^{d} \int_0^T f^\alpha(t) \, d\theta^\alpha(t). \tag{5.3.4}$$

Proof By Itô's representation theorem (Theorem 2.6.2), there exists some $\{g^\alpha(t)\}_{t\in[0,T]} \in \mathscr{L}^2$ ($\alpha = 1,\dots,d$) such that

$$F = \mathbf{E}[F] + \sum_{\alpha=1}^{d} \int_0^T g^\alpha(t)\,\mathrm{d}\theta^\alpha(t).$$

What is to be shown is $g^\alpha(t) = f^\alpha(t)$ ($\alpha = 1,\dots,d$).

Let $\{u^\alpha(t)\}_{t\in[0,T]}$ be as in Theorem 5.3.3. Since stochastic integrals are isometries (Proposition 2.2.10), Theorem 5.3.3 implies

$$\begin{aligned}
&\int_{W_T} \sum_{\alpha=1}^{d}\Big(\int_0^T u^\alpha(t) g^\alpha(t)\,\mathrm{d}t\Big)\mathrm{d}\mu_T \\
&= \int_{W_T} \Big(\sum_{\alpha=1}^{d}\int_0^T u^\alpha(t)\,\mathrm{d}\theta^\alpha(t)\Big)\Big(\sum_{\alpha=1}^{d}\int_0^T g^\alpha(t)\,\mathrm{d}\theta^\alpha(t)\Big)\mathrm{d}\mu_T \\
&= \int_{W_T} (\nabla^*\Phi_u)(F - \mathbf{E}[F])\,\mathrm{d}\mu_T. \qquad (5.3.5)
\end{aligned}$$

By the definitions of dual operators and the inner product, the last term is rewritten as

$$\int_{W_T} \langle \Phi_u, \nabla F\rangle_{H_T}\,\mathrm{d}\mu_T = \int_{W_T} \sum_{\alpha=1}^{d}\Big(\int_0^T u^\alpha(t)(\overbrace{(\nabla F)}^{\cdot})^\alpha(t)\,\mathrm{d}t\Big)\mathrm{d}\mu_T.$$

Moreover, since $\{u^\alpha(t)\}$ is $\{\mathscr{F}_t\}$-adapted, it is equal to

$$\int_{W_T} \sum_{\alpha=1}^{d}\Big(\int_0^T u^\alpha(t) f^\alpha(t)\,\mathrm{d}t\Big)\mathrm{d}\mu_T$$

by Fubini's theorem. Comparing this with (5.3.5), we obtain $g^\alpha(t) = f^\alpha(t)$ ($\alpha = 1,\dots,d$) since $\{u(t)\}_{t\in[0,T]}$ is arbitrary. □

Next, we show that the Lipschitz continuity of a Wiener functional implies its differentiability.

Theorem 5.3.6 *Suppose that, for $F \in \bigcap_{p\in(1,\infty)} L^p(\mu_T)$, there exists $\widetilde{F}$ with $\widetilde{F} = F$, μ_T-a.s., and a constant C such that*

$$|\widetilde{F}(w+h) - \widetilde{F}(w)| \leqq C\|h\|_{H_T} \qquad (5.3.6)$$

for any $w \in W_T$ and $h \in H_T$. Then, $F \in \mathbb{D}^{1,\infty-}$ and $\|\nabla F\|_{H_T} \leqq C$, μ_T-a.s.

Proof Let $\ell \in W_T^*$ ($\ell \neq 0$). Define $\pi_\ell : W_T \to W_T$ by $\pi_\ell(w) = w - \|\ell\|_{H_T}^{-2}\ell(w)\ell$ and decompose W_T into an orthogonal sum

$$W_T = \pi_\ell(W_T) \oplus \mathbb{R}\ell = \{w' + \xi\ell\,;\, w' \in \pi_\ell(W_T),\ \xi \in \mathbb{R}\}.$$

Then, by the Itô–Nisio theorem (Theorem 1.2.5), we have

$$\mu_T = (\mu_T \circ \pi_\ell^{-1}) \otimes \frac{1}{\sqrt{2\pi\|\ell\|_{H_T}^2}} e^{-\frac{\xi^2}{2\|\ell\|_{H_T}^2}} \mathrm{d}\xi.$$

Let $w' \in \pi_\ell(W_T)$. Since $\mathbb{R} \ni \xi \mapsto \widetilde{F}(w' + \xi\ell)$ is absolutely continuous by the assumption, the set

$$\Big\{\xi \in \mathbb{R}\,;\ \frac{\widetilde{F}(w' + (\xi+\varepsilon)\ell) - \widetilde{F}(w'+\xi\ell)}{\varepsilon} \text{ does not converge as } \varepsilon \to 0\Big\}$$

has Lebesgue measure 0. Hence, setting

$$A(\ell) = \Big\{w \in W_T\,;\ \lim_{\varepsilon\to 0} \frac{\widetilde{F}(w+\varepsilon\ell) - \widetilde{F}(w)}{\varepsilon} \text{ exists}\Big\},$$

we have $\mu_T(A(\ell)) = 1$ by Fubini's theorem. Set

$$G(w,\ell) = \mathbf{1}_{A(\ell)}(w) \lim_{\varepsilon\to 0} \frac{\widetilde{F}(w+\varepsilon\ell) - \widetilde{F}(w)}{\varepsilon} \qquad (w \in W_T).$$

Then, by the assumption,

$$|G(w,\ell)| \leqq C\|\ell\|_{H_T}.$$

for any $w \in W_T$ and $\ell \in W_T^*$.

Let $\{\ell_k\}_{k=1}^\infty \subset W_T^*$ be an orthonormal basis of H_T and set

$$\mathscr{K} = \Big\{\sum_{j=1}^n q_j\ell_j\,;\ q_j \in \mathbb{Q}\ (j = 1,\dots,n),\ n \in \mathbb{N}\Big\}.$$

For $\ell = \sum_{j=1}^n q_j\ell_j \in \mathscr{K}$ and $\phi \in \mathscr{P}$, we have by Lemma 5.1.2

$$\begin{aligned}\int_{W_T} G(\cdot,\ell)\phi\,\mathrm{d}\mu_T &= \lim_{\varepsilon\to 0}\int_{W_T} \frac{\widetilde{F}(\cdot+\varepsilon\ell) - \widetilde{F}(\cdot)}{\varepsilon}\phi\,\mathrm{d}\mu_T = \int_{W_T} \widetilde{F}\partial_\ell\phi\,\mathrm{d}\mu_T \\ &= \sum_{j=1}^n q_j \int_{W_T} \widetilde{F}\partial_{\ell_j}\phi\,\mathrm{d}\mu_T = \int_{W_T} \sum_{j=1}^n q_j G(\cdot,\ell_j)\phi\,\mathrm{d}\mu_T \qquad (5.3.7)\end{aligned}$$

and, for any $\ell \in \mathscr{K}$,

$$G(\cdot,\ell) = \sum_{j=1}^\infty \langle \ell,\ell_j\rangle_{H_T} G(\cdot,\ell_j), \quad \mu_T\text{-a.s.}$$

Hence, setting

$$B = \Big\{w \in \bigcap_{j=1}^\infty A(\ell_j)\,;\ G(w,\ell) = \sum_{j=1}^\infty \langle \ell,\ell_j\rangle_{H_T} G(w,\ell_j),\ \ell \in \mathscr{K}\Big\},$$

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we have $\mu_T(B) = 1$. If $w \in B$, then

$$\Big|\sum_{j=1}^{\infty}\langle \ell, \ell_j\rangle_{H_T} G(w, \ell_j)\Big| = |G(w, \ell)| \leqq C\|\ell\|_{H_T} \qquad (\ell \in \mathscr{K}).$$

Hence, letting $N \in \mathbb{N}$ and taking $k_n \in \mathscr{K}$ with

$$\lim_{n\to\infty}\Big\|k_n - \sum_{j=1}^{N} G(w, \ell_j)\ell_j\Big\|_{H_T} = 0 \quad \text{and} \quad \langle k_n, \ell_j\rangle_{H_T} = 0 \ (j \geqq N+1),$$

we obtain

$$\begin{aligned}\sum_{j=1}^{N} G(w, \ell_j)^2 &= \lim_{n\to\infty}\sum_{j=1}^{N}\langle k_n, \ell_j\rangle_{H_T} G(w, \ell_j) \\ &\leqq \limsup_{n\to\infty} C\|k_n\|_{H_T} = C\Big\{\sum_{j=1}^{N} G(w, \ell_j)^2\Big\}^{\frac{1}{2}}.\end{aligned}$$

Letting $N \to \infty$, we obtain

$$\sum_{j=1}^{\infty} G(w, \ell_j)^2 \leqq C^2 < \infty. \tag{5.3.8}$$

If we set

$$G(w) = \mathbf{1}_B(w)\sum_{j=1}^{\infty} G(w, \ell_j)\ell_j,$$

then $\|G(w)\|_{H_T} \leqq C$ $(w \in W_T)$ by (5.3.8). Moreover, by (5.3.7),

$$\int_{W_T} F\nabla^*(\phi\ell)\,\mathrm{d}\mu_T = \int_{W_T}\langle G, \phi\ell\rangle_{H_T}\,\mathrm{d}\mu_T$$

for $\ell \in \mathscr{K}$ and $\phi \in \mathscr{P}$. Since $\mathscr{K}$ is dense in H_T,

$$\int_{W_T} F\nabla^* K\,\mathrm{d}\mu_T = \int_{W_T}\langle G, K\rangle_{H_T}\,\mathrm{d}\mu_T \quad (K \in \mathscr{P}(H_T)).$$

Therefore, by Theorem 5.3.1, $F \in \mathbb{D}^{1,\infty-}$ and $\nabla F = G$. □

Corollary 5.3.7 *The norm* $\|\theta\| = \max_{0\leqq t\leqq T}|\theta(t)|$ *belongs to* $\mathbb{D}^{1,\infty-}$.

Proof Since $\big|\|w+h\| - \|w\|\big| \leqq \sqrt{T}\,\|h\|_{H_T}$ $(h \in H_T)$, Theorem 5.3.6 implies the assertion. □

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Using the following proposition, we can prove that, when $d = 1$, the derivative of the norm $\|\theta\|$ is given by

$$\overbrace{(\nabla\|\theta\|)}^{\cdot} = \operatorname{sgn}(\theta(\tau))\mathbf{1}_{[0,\tau]}, \quad \mu_T\text{-a.s.}, \tag{5.3.9}$$

where $\tau(w) = \inf\{t \in [0,T]; |w(t)| = \|w\|\}$.

Proposition 5.3.8 (1) *Let* $F \in \mathbb{D}^{1,p}$*. Then,* $F^+ = \max\{F, 0\} \in \mathbb{D}^{1,p}$ *and*

$$\nabla F^+ = \boldsymbol{1}_{(0,\infty)}(F)\nabla F, \quad \mu_T\text{-a.s.}$$

(2) *Let* $F_1, \ldots, F_n \in \mathbb{D}^{1,p}$*. Then,* $\max_{1\leqq i\leqq n} F_i \in \mathbb{D}^{1,p}$ *and*

$$\nabla \max_{1\leqq i\leqq n} F_i = \sum_{i=1}^n \boldsymbol{1}_{A_i}\nabla F_i, \quad \mu_T\text{-a.s.},$$

where $A_i = \{w;\ F_j(w) \leqq F_i(w)\,(j < i),\ F_j(w) < F_i(w)\,(j > i)\}$*.*
(3) *Let* $d = 1$*. Then,* $\max_{0\leqq s\leqq T}\theta(s) \in \mathbb{D}^{1,\infty-}$ *and*

$$\overbrace{(\nabla \max_{0\leqq s\leqq T}\theta(s))}^{\cdot} = \boldsymbol{1}_{[0,\sigma]}, \quad \mu_T\text{-a.s.}, \tag{5.3.10}$$

where $\sigma(w) = \inf\{t \in [0,T];\ w(t) = \max_{0\leqq s\leqq T} w(s)\}$*.*
(4) (5.3.9) *holds.*

Proof (1) Take $\varphi(x) \in C^\infty(\mathbb{R})$ so that $\varphi(x) = 1$ $(x \geqq 1)$ and $\varphi(x) = 0$ $(x \leqq 0)$. Set $\varphi_n(x) = \varphi(nx)$ and define $\psi_n(x) = \int_0^x \varphi_n(y)\,dy$. By the same arguments as in the proof of Corollary 5.3.2, we can show $\psi_n(F) \in \mathbb{D}^{1,p}$. Letting $n \to \infty$, we obtain the conclusion.
(2) If $n = 2$, then the assertion follows from (1) because $\max\{F_1, F_2\} = (F_1 - F_2)^+ + F_2$. By induction we obtain the assertion for general n.
(3) From (2) we have $\max_{0\leqq k\leqq 2^n}\theta(\frac{k}{2^n}) \in \mathbb{D}^{1,\infty-}$ and

$$\overbrace{\Big(\nabla \max_{0\leqq k\leqq 2^n}\theta\Big(\frac{k}{2^n}\Big)\Big)}^{\cdot} = \sum_{k=0}^{2^n}\mathbf{1}_{A_k^n}\mathbf{1}_{[0,\frac{k}{2^n}]}, \tag{5.3.11}$$

where $A_k^n = \{\theta(\frac{j}{2^n}) \leqq \theta(\frac{k}{2^n})\,(j < k),\ \theta(\frac{j}{2^n}) < \theta(\frac{k}{2^n})\,(j > k)\}$. Since

$$\mu_T(\theta(\sigma) > \theta(t),\ t \neq \sigma) = 1$$

(see [56, p.102]), letting $n \to \infty$ in (5.3.11) yields (5.3.10).
(4) Since $\|\theta\| = \max\Big\{\max_{0\leqq s\leqq T}\theta(s), \max_{0\leqq s\leqq T}(-\theta(s))\Big\}$, (1) and (3) yield the conclusion. □

5.4 Integration by Parts Formula

In this section we show an integration by parts formula and, by applying it, we introduce the composition of distributions on $\mathbb{R}^N$ and Wiener functionals.

Definition 5.4.1 $F = (F^1, \ldots, F^N) \in \mathbb{D}^{\infty,\infty-}(\mathbb{R}^N)$ is called **non-degenerate** if

$$\Bigl(\det\Bigl[\bigl(\langle \nabla F^i, \nabla F^j\rangle_{H_T}\bigr)_{i,j=1,\ldots,N}\Bigr]\Bigr)^{-1} \in L^{\infty-}(\mu_T) = \bigcap_{p\in(1,\infty)} L^p(\mu_T). \tag{5.4.1}$$

Example 5.4.2 For $\ell_1, \ldots, \ell_N \in W_T^*$, suppose that

$$\det\Bigl[(\langle \ell_i, \ell_j\rangle_{H_T})_{i,j=1,\ldots,N}\Bigr] \neq 0$$

and set $F = (\ell_1, \ldots, \ell_N)$. Then, $F \in \mathscr{P}(\mathbb{R}^N)$ and $\nabla F^i(w) = \ell_i$. Hence, F is non-degenerate.

In particular, for $t > 0$, $N = d$ and $\ell_i(w) = w^i(t)$ $(i = 1, \ldots, d,\ w \in W_T)$, we have

$$\det\Bigl[(\langle \ell_i, \ell_j\rangle_{H_T})_{i,j=1,\ldots,N}\Bigr] = t^d > 0.$$

Hence, $F = \theta(t)$ is non-degenerate.

Example 5.4.3 Let $\{h_n\}_{n=1}^\infty$ be an orthonormal basis of H_T and $\{a_j\}_{j=1}^\infty \subset \mathbb{R}$ satisfy $\sum_{j=1}^\infty a_j^2 < \infty$. Set

$$F_n = \sum_{j=1}^n a_j\{(\nabla^* h_j)^2 - 1\}.$$

Since $\{\nabla^* h_j\}$ is a sequence of independent identically distributed normal Gaussian random variables, $\|F_n - F_m\|_2^2 = 2\sum_{j=m+1}^n a_j^2$ for $n > m$. Hence, as the limit of F_n in $L^2(\mu_T)$, a random variable

$$F = \sum_{j=1}^\infty a_j\{(\nabla^* h_j)^2 - 1\}$$

is defined.

We have

$$\int_{W_T} \mathrm{e}^{\lambda F_n} \mathrm{d}\mu_T = \prod_{j=1}^n \int_{\mathbb{R}} \mathrm{e}^{\lambda a_j(x^2-1)} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x = \Bigl\{\prod_{j=1}^n (1 - 2\lambda a_j)\mathrm{e}^{2\lambda a_j}\Bigr\}^{-\frac12}$$

for any $\lambda \in \mathbb{R}$ with $a|\lambda| < \frac12$, where $a = \sup_{j\in\mathbb{N}} |a_j|$. Since $\log(1 - x) + x = -\frac{x^2}{2} + o(x^2)$ as $x \to 0$, the infinite product $\prod_{j=1}^\infty (1 - 2\lambda a_j)\mathrm{e}^{2\lambda a_j}$ converges and is not 0. Since $\mathrm{e}^{|y|} \leqq \mathrm{e}^y + \mathrm{e}^{-y}$, applying Fatou's lemma to the sequences

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$\{\int_{W_T} \mathrm{e}^{\pm\lambda F_n} \mathrm{d}\mu_T\}_{n=1}^{\infty}$ (subsequences if necessary), we obtain $\int_{W_T} \mathrm{e}^{|\lambda||F|}\mathrm{d}\mu_T < \infty$. In particular, $F \in L^{\infty-}(\mu_T)$.

Set

$$F' = 2\sum_{j=1}^{\infty} a_j(\nabla^* h_j)h_j.$$

Then, since $\nabla F_n = 2\sum_{j=1}^{n} a_j(\nabla^* h_j)h_j$, we obtain by (5.1.9) and Corollary 5.3.2,

$$\left\|\nabla F_n - F'\right\|_2^2 = \left\|4\sum_{j=n+1}^{\infty} a_j^2(\nabla^* h_j)^2\right\|_1 = 4\sum_{j=n+1}^{\infty} a_j^2 \to 0 \quad (n \to \infty).$$

Hence, for any $G \in \mathscr{P}(H_T)$,

$$\begin{aligned}\int_{W_T} F\nabla^* G \,\mathrm{d}\mu_T &= \lim_{n\to\infty} \int_{W_T} F_n \nabla^* G \,\mathrm{d}\mu_T \\ &= \lim_{n\to\infty} \int_{W_T} \langle \nabla F_n, G\rangle_{H_T} \mathrm{d}\mu_T = \int_{W_T} \langle F', G\rangle_{H_T} \mathrm{d}\mu_T.\end{aligned}$$

Moreover, for $\lambda \in \mathbb{R}$ with $a^2|\lambda| < \frac{1}{2}$, the integrability of $\exp(|\lambda| \left\|F'\right\|_{H_T}^2)$ is shown by a similar argument to that in the preceding paragraph, and $F' \in L^{\infty-}(\mu_T; H_T)$. Thus $F \in \mathbb{D}^{1,\infty-}$ and $\nabla F = F'$.

Furthermore, since $\nabla^2 F_n = 2\sum_{j=1}^{n} a_j h_j \otimes h_j$ and $\nabla^3 F_n = 0$, Theorem 5.2.1 implies

$$F \in \mathbb{D}^{\infty,\infty-}, \qquad \nabla F = 2\sum_{j=1}^{\infty} a_j(\nabla^* h_j)h_j,$$

$$\nabla^2 F = 2\sum_{j=1}^{\infty} a_j h_j \otimes h_j, \qquad \nabla^k F = 0 \quad (k \geqq 3).$$

Finally, we present a sufficient condition for F to be non-degenerate. Suppose that $a_j \neq 0$ for infinitely many js and set $\{j\,;\, a_j \neq 0\} = \{j(1) < j(2) < \cdots\}$. Putting $m_n = \min\{a_{j(k)}^2\,;\, k = 1, \ldots, n\}$, we have $m_n > 0$ and

$$\left\|\nabla F\right\|_{H_T}^2 = \sum_{k=1}^{\infty} a_{j(k)}^2 (\nabla^* h_{j(k)})^2 \geqq m_n \sum_{k=1}^{n} (\nabla^* h_{j(k)})^2 \qquad (n \in \mathbb{N}).$$

Since $\nabla^* h_{j(k)}$ ($k \in \mathbb{N}$) form a sequence of independent identically distributed normal Gaussian random variables, $\left(\sum_{k=1}^{n} (\nabla^* h_{j(k)})^2\right)^{-\frac{1}{2}} \in L^p(\mu_T)$ for $n > p$. Therefore, $\left\|\nabla F\right\|_{H_T}^{-1} \in \bigcap_{p\in(1,\infty)} L^p(\mu_T)$ and F is non-degenerate.

Next we introduce an **integration by parts formula** associated with non-degenerate Wiener functionals. For this purpose we note the following.

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Lemma 5.4.4 *For $G \in \mathbb{D}^{\infty,\infty-}$, assume that $G \geqq 0$, μ_T-a.s., and $\frac{1}{G} \in L^{\infty-}(\mu_T)$. Then, $\frac{1}{G} \in \mathbb{D}^{\infty,\infty-}$. In particular, if $F \in \mathbb{D}^{\infty,\infty-}(\mathbb{R}^N)$ is non-degenerate, then*

$$\Big(\langle \nabla F^i, \nabla F^j \rangle_{H_T}\Big)^{-1}_{i,j=1,\dots,N} \in \mathbb{D}^{\infty,\infty-}(\mathbb{R}^N \otimes \mathbb{R}^N).$$

Proof By Corollary 5.3.2, $(G+\varepsilon)^{-1} \in \mathbb{D}^{\infty,\infty-}$ for any $\varepsilon > 0$. Moreover, we have

$$\nabla^n\Big(\frac{1}{G+\varepsilon}\Big) = \sum_{k=0}^{n} \frac{\phi_k(G)}{(G+\varepsilon)^{k+1}},$$

where $\phi_k(G)$ is a polynomial determined by the tensor products of $\nabla G, \dots, \nabla^n G$. Let $\varepsilon \to 0$ to see $\frac{1}{G} \in \mathbb{D}^{\infty,\infty-}$. □

Theorem 5.4.5 *Suppose that $F \in \mathbb{D}^{\infty,\infty-}(\mathbb{R}^N)$ is non-degenerate and set*

$$\gamma = (\gamma_{ij})_{i,j=1,\dots,N} = \Big(\langle \nabla F^i, \nabla F^j \rangle_{H_T}\Big)^{-1}_{i,j=1,\dots,N}.$$

Define the linear mapping $\xi_{i_1\dots i_n} : \mathbb{D}^{\infty,\infty-} \to \mathbb{D}^{\infty,\infty-}$ $(i_1, \dots, i_n \in \{1, \dots, N\})$ by

$$\xi_i[G] = \sum_{j=1}^{N} \nabla^*(\gamma_{ij} G \nabla F^j), \qquad \xi_{i_1\dots i_n}[G] = \xi_{i_n}[\xi_{i_1\dots i_{n-1}}[G]].$$

Then, for any $p > 1$,

$$\sup\Big\{\int_{W_T} |\xi_{i_1\dots i_n}[G]|\,\mathrm{d}\mu_T;\ G \in \mathbb{D}^{\infty,\infty-},\ \|G\|_{n,p} \leqq 1\Big\} < \infty. \tag{5.4.2}$$

Moreover, for $f \in C^n_{\nearrow}(\mathbb{R}^N)$ and $G \in \mathbb{D}^{\infty,\infty-}$,

$$\int_{W_T} \frac{\partial^n f}{\partial x^{i_1} \cdots \partial x^{i_n}}(F)\, G\, \mathrm{d}\mu_T = \int_{W_T} f(F)\, \xi_{i_1\dots i_n}[G]\, \mathrm{d}\mu_T. \tag{5.4.3}$$

Proof (5.4.2) follows from Theorem 5.2.8 and Corollary 5.4.4. We only prove (5.4.3). Let $f \in C^1_{\nearrow}(\mathbb{R}^N)$. Since

$$\nabla(f(F)) = \sum_{i=1}^{N} \frac{\partial f}{\partial x^i}(F) \nabla F^i,$$

we have

$$\frac{\partial f}{\partial x^i}(F) = \Big\langle \nabla(f(F)), \sum_{j=1}^{N} \gamma_{ij} \nabla F^j \Big\rangle_{H_T}.$$

This implies

$$\int_{W_T} \frac{\partial f}{\partial x^i}(F)\, G\, \mathrm{d}\mu_T = \int_{W_T} f(F)\, \xi_i[G]\, \mathrm{d}\mu_T.$$

By induction we obtain (5.4.3). □

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As an application of the integration by parts formula, we show that a composition of a distribution on $\mathbb{R}^N$ and a non-degenerate Wiener functional is realized as a generalized Wiener functional. By using this result, we present representations as expectations for probability densities and conditional expectations.

Let $\mathscr{S}(\mathbb{R}^N)$ be the space of rapidly decreasing functions on $\mathbb{R}^N$ and $\mathscr{S}'(\mathbb{R}^N)$ be the space of tempered distributions on $\mathbb{R}^N$. For $k \in \mathbb{Z}$, denote by $\mathscr{S}_{2k}(\mathbb{R}^N)$ the completion of $\mathscr{S}(\mathbb{R}^N)$ by the norm

$$\|f\|_{2k} = \sup_{x\in\mathbb{R}^N}\Big|\{I + |x|^2 - \tfrac{1}{2}\Delta\}^k f(x)\Big|,$$

where $\Delta = \sum_{i=1}^N (\frac{\partial}{\partial x^i})^2$. Then, $\mathscr{S}_{2k}(\mathbb{R}^N) \supset \mathscr{S}_{2k+2}(\mathbb{R}^N)$ and $\mathscr{S}_0(\mathbb{R}^N)$ is the space of continuous functions on $\mathbb{R}^N$ satisfying $\lim_{|x|\to\infty} |f(x)| = 0$. Moreover,

$$\mathscr{S}(\mathbb{R}^N) = \bigcap_{k=1}^{\infty} \mathscr{S}_{2k}(\mathbb{R}^N) \quad \text{and} \quad \mathscr{S}'(\mathbb{R}^N) = \bigcup_{k=1}^{\infty} \mathscr{S}_{-2k}(\mathbb{R}^N).$$

Theorem 5.4.6 *Let $p > 1$ and $k \in \mathbb{Z}_+$, and suppose that $F \in \mathbb{D}^{\infty,\infty-}(\mathbb{R}^N)$ is non-degenerate. Then, there exists a constant C such that*

$$\|f(F)\|_{-2k,p} \leqq C\|f\|_{-2k}$$

for any $f \in \mathscr{S}(\mathbb{R}^N)$.

Proof Define $\eta : \mathbb{D}^{\infty,\infty-} \to \mathbb{D}^{\infty,\infty-}$ by

$$\eta[G] = G + |F|^2 G - \frac{1}{2}\sum_{i=1}^N \xi_{ii}[G].$$

By Theorem 5.4.5,

$$\int_{W_T} (\{I + |x|^2 - \tfrac{1}{2}\Delta\}^k f)(F) G \,\mathrm{d}\mu_T = \int_{W_T} f(F)\eta^k[G]\,\mathrm{d}\mu_T.$$

This implies

$$\int_{W_T} f(F)G\,\mathrm{d}\mu_T = \int_{W_T} (\{I + |x|^2 - \tfrac{1}{2}\Delta\}^{-k} f)(F)\eta^k[G]\,\mathrm{d}\mu_T$$

for any $G \in \mathbb{D}^{\infty,\infty-}$. As we have shown in the proof of Theorem 5.1.10, we have

$$\|f(F)\|_{-2k,p} = \sup\Big\{\int_{W_T} f(F)G\,\mathrm{d}\mu_T\,;\ G \in \mathbb{D}^{\infty,\infty-},\ \|G\|_{2k,q} \leqq 1\Big\},$$

where $q = \frac{p}{p-1}$. Combining this with (5.4.2), we obtain the conclusion. □

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Corollary 5.4.7 *If $F \in \mathbb{D}^{\infty,\infty-}(\mathbb{R}^N)$ is non-degenerate, then, for any $p > 1$ and $k \in \mathbb{Z}_+$, the mapping $\mathscr{S}(\mathbb{R}^N) \ni f \mapsto f(F) \in \mathbb{D}^{k,p}$ is extended to a continuous linear mapping $\Phi_F : \mathscr{S}_{-2k} \to \mathbb{D}^{-2k,p}$.*

Definition 5.4.8 Suppose that $F \in \mathbb{D}^{\infty,\infty-}(\mathbb{R}^N)$ is non-degenerate. For $u \in \mathscr{S}_{-2k}(\mathbb{R}^N)$, the generalized Wiener functional $\Phi_F(u) \in \mathbb{D}^{-2k,p}$ in Corollary 5.4.7 is denoted by $u(F)$ and called the **pull-back** of u by F.

By Corollary 5.4.7 we obtain the following.

Corollary 5.4.9 *Assume that $F \in \mathbb{D}^{\infty,\infty-}(\mathbb{R}^N)$ is non-degenerate. Let $p > 1$ and $U \subset \mathbb{R}^n$ be an open set.*
(1) *Let $m \in \mathbb{Z}_+$. If the mapping $U \ni z \mapsto u_z \in \mathscr{S}_{-2k}$ is of C^m-class, then so is the mapping $U \ni z \mapsto u_z(F) \in \mathbb{D}^{-2k,p}$.*
(2) *Assume that $U \ni z \mapsto u_z \in \mathscr{S}_{-2k}$ is continuous and admits the Bochner integral $\int_U u_z \mathrm{d}z$. Then, $z \mapsto u_z(F)$ is Bochner integrable as a $\mathbb{D}^{-2k,p}$-valued function and*

$$\Big(\int_U u_z \,\mathrm{d}z\Big)(F) = \int_U u_z(F)\,\mathrm{d}z.$$

Remark 5.4.10 In the above, for a Banach space E, the derivative of an E-valued function $\psi : U \to E$ at $z \in U$ is, by definition, a continuous linear mapping $\psi'(z) : \mathbb{R}^n \to E$ such that $\|\frac{1}{\varepsilon}\{\psi(z+\varepsilon\xi) - \psi(z)\} - [\psi'(z)](\xi)\|_E \to 0$ $(\varepsilon \to 0)$ for any $\xi \in \mathbb{R}^n$. The higher order derivatives are defined inductively. For the Bochner integral, see [133].

Using a composition of a non-degenerate functional and a distribution, we have the following expression of the probability density via a generalized Wiener functional. Let δ_x be the **Dirac measure** on $\mathbb{R}^N$ concentrated at $x \in \mathbb{R}^N$.

Theorem 5.4.11 *Suppose that $F \in \mathbb{D}^{\infty,\infty-}(\mathbb{R}^N)$ is non-degenerate.*
(1) *Let $p_F(x)$ be the value of $\delta_x(F) \in \mathbb{D}^{-\infty,1+}$ at $1 \in \mathbb{D}^{\infty,\infty-}$;*

$$p_F(x) = \mathbf{E}[\delta_x(F)] = [\delta_x(F)](1).$$

Then, p_F is of C^∞-class and the probability density of F:

$$\mu_T(F \in A) = \int_A p_F(x)\,\mathrm{d}x \qquad (A \in \mathscr{B}(\mathbb{R}^N)).$$

(2) *Let $G \in \mathbb{D}^{\infty,\infty-}$ and set $p_{G|F}(x) = \mathbf{E}[\delta_x(F)G]$. Then,*

$$\int_{W_T} f(F)G\,\mathrm{d}\mu_T = \int_{\mathbb{R}^N} f(x)p_{G|F}(x)\,\mathrm{d}x$$

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for any $f \in \mathscr{S}(\mathbb{R}^N)$*. In particular,* $p_{G|F}(x) = p_F(x)\mathbf{E}[G|F = x]$ *holds for almost all* $x \in \mathbb{R}^N$ *with* $p_F(x) > 0$*.*

Proof For $k \in \mathbb{Z}_+$, the mapping $\mathbb{R}^N \ni x \mapsto \delta_x \in \mathscr{S}_{-2([\frac{N}{2}]+1+k)}$ is of C^{2k}-class (see [45, Lemma V-9.1]). Hence, by Corollary 5.4.9, both p_F and $p_{G|F}$ are of C^∞-class.

For $f \in \mathscr{S}(\mathbb{R}^N)$, the integral $\int_{\mathbb{R}^N} f(x)\delta_x \mathrm{d}x$ of the $\mathscr{S}_{-2([\frac{N}{2}]+1+k)}$-valued function $x \mapsto f(x)\delta_x$ coincides with f. By Corollary 5.4.9 again,

$$\int_{\mathbb{R}^N} f(x)\delta_x(F)\,\mathrm{d}x = f(F).$$

Hence, by Corollary 5.4.9 and the commutativity between Bochner integrals and linear continuous operators, we obtain, for $G \in \mathbb{D}^{\infty,\infty-}$,

$$\int_{W_T} f(F)G\,\mathrm{d}\mu_T = \int_{\mathbb{R}^N} f(x)\mathbf{E}[\delta_x(F)G]\,\mathrm{d}x. \tag{5.4.4}$$

Setting $G = 1$ in (5.4.4), we see that p_F is the probability density of F. Moreover, since the left hand side of (5.4.4) is equal to $\int_{\mathbb{R}^N} f(x)\mathbf{E}[G|F = x]p_F(x)\,\mathrm{d}x$, the identity

$$p_{G|F}(x) = p_F(x)\mathbf{E}[G|F = x]$$

holds for almost all $x \in \mathbb{R}^N$ with $p_F(x) > 0$. □

Positive distributions on $\mathbb{R}^n$ are realized by measures ([36]). Similar facts holds for generalized Wiener functionals.

Definition 5.4.12 $\Phi \in \mathbb{D}^{-\infty,1+}$ is said to be positive ($\Phi \geqq 0$ in notation) if

$$\int_{W_T} F\Phi\,\mathrm{d}\mu_T \geqq 0$$

holds for any non-negative $F \in \mathbb{D}^{\infty,\infty-}$.

If $\Phi \in L^p(\mu_T)$, then the condition in the above definition is equivalent to $\Phi \geqq 0$, μ_T-a.s.

Set

$$\mathscr{F}C_{\mathrm{b}}^\infty = \{F\,;\, F = f(\ell_1, \dots, \ell_n),\ f \in C_{\mathrm{b}}^\infty(\mathbb{R}^n),\ \ell_1, \dots, \ell_n \in W_T{}^*,\ n \in \mathbb{N}\}.$$

Since $\mathscr{F}C_{\mathrm{b}}^\infty$ is dense in $\mathbb{D}^{r,p}$, Φ is positive if and only if $\int_{W_T} F\Phi\,\mathrm{d}\mu_T \geqq 0$ for any non-negative $F \in \mathscr{F}C_{\mathrm{b}}^\infty$.

Proposition 5.4.13 *If* $F \in \mathbb{D}^{\infty,\infty-}(\mathbb{R}^N)$ *is non-degenerate,* $\delta_x(F) \in \mathbb{D}^{-\infty,1+}$ *is positive.*

Proof The assertion follows from the identity

$$\int_{W_T} G\delta_x(F)\,\mathrm{d}\mu_T = \mathbf{E}[G|F=x]p_F(x) \quad (G \in \mathbb{D}^{\infty,\infty-}). \qquad \square$$

Lemma 5.4.14 *Let $\Phi \in \mathbb{D}^{-\infty,1+}$ be positive. Then, $\Phi = 0$ if and only if $\int_{W_T} \Phi\,\mathrm{d}\mu_T = 0$.*

Proof Obviously $\int_{W_T} \Phi\,\mathrm{d}\mu_T = 0$ if $\Phi = 0$. We show the converse. Suppose that $F \in \mathscr{F}C_{\mathrm{b}}^\infty$ is non-negative and set $M = \sup_{w\in W_T} F(w)$. Since $M - F \in \mathscr{F}C_{\mathrm{b}}^\infty$ is non-negative, we have

$$0 \leqq \int_{W_T} (M-F)\Phi\,\mathrm{d}\mu_T = -\int_{W_T} F\Phi\,\mathrm{d}\mu_T \leqq 0.$$

Hence $\int_{W_T} F\Phi\,\mathrm{d}\mu_T = 0$. Since F is arbitrary, we obtain $\Phi = 0$. $\square$

Theorem 5.4.15 *For any positive $\Phi \in \mathbb{D}^{-\infty,1+}$, there exists a finite measure ν_Φ on W_T such that*

$$\int_{W_T} F\Phi\,\mathrm{d}\mu_T = \int_{W_T} F\,\mathrm{d}\nu_\Phi \tag{5.4.5}$$

for any $F \in \mathscr{F}C_{\mathrm{b}}^\infty$.

Remark 5.4.16 If $p > 1$ and $\Phi \in L^p(\mu_T)$, then $\mathrm{d}\nu_\Phi = \Phi\,\mathrm{d}\mu_T$.

Proof Let $\Phi \in \mathbb{D}^{-r,p}$, $\Phi \neq 0$ $(r \in \mathbb{R}, p > 1)$. By Lemma 5.4.14, we may assume $\int_{W_T} \Phi\,\mathrm{d}\mu_T = 1$. Set

$$\mathbf{D} = \Big\{\frac{k}{2^n}\,;\ n \in \mathbb{Z}_+,\ k \in \mathbb{Z}_+,\ k \leqq 2^n T\Big\}.$$

For $t_j \in \mathbf{D}$ $(j = 1,\dots,n)$ with $0 \leqq t_1 < \cdots < t_n$, define $u_{t_1\dots t_n} : \mathscr{S}((\mathbb{R}^d)^n) \to \mathbb{R}$ by

$$u_{t_1\dots t_n}(f) = \int_{W_T} f(\theta(t_1),\dots,\theta(t_n))\Phi\,\mathrm{d}\mu_T,$$

where $\{\theta(t)\}_{t\in[0,T]}$ is the coordinate process. Then $u_{t_1\dots t_n}$ is a positive distribution. Hence, there exists a probability measure $\nu_{t_1\dots t_n}$ on $(\mathbb{R}^d)^n$ such that

$$\int_{W_T} f(\theta(t_1),\dots,\theta(t_n))\Phi\,\mathrm{d}\mu_T = \int_{(\mathbb{R}^d)^n} f(x_1,\dots,x_n)\nu_{t_1\dots t_n}(\mathrm{d}x_1\cdots\mathrm{d}x_n)$$

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for any $f \in \mathscr{S}((\mathbb{R}^d)^n)$. Since $\{\nu_{t_1\ldots t_n};\, t_1, \ldots, t_n \in \mathbf{D}, n \in \mathbb{N}\}$ is consistent, by Kolmogorov's extension theorem (see, e.g., [56, 114]), there exists a probability measure ν_Φ on $(\mathbb{R}^d)^{\mathbf{D}}$ such that

$$\nu_\Phi((X(t_1), \ldots, X(t_n)) \in A) = \nu_{t_1\ldots t_n}(A). \tag{5.4.6}$$

for any $A \in \mathscr{B}((\mathbb{R}^d)^n)$ $(t_1 < \cdots < t_n \in \mathbf{D})$, where $X(t) : (\mathbb{R}^d)^{\mathbf{D}} \to \mathbb{R}^d$ is given by $X(t, \phi) = \phi(t)$ $(\phi \in (\mathbb{R}^d)^{\mathbf{D}})$.

By Lemma 5.2.6, there exists a constant C such that

$$\|G\|_{r,q} \leqq C\|G\|_q$$

for any $G \in \bigoplus_{n=0}^4 \mathscr{H}_n$, where $q = \frac{p}{p-1}$. Since $|\theta(t) - \theta(s)|^4 \in \bigoplus_{n=0}^4 \mathscr{H}_n$ for any $t, s \in \mathbf{D}$, we have

$$\begin{aligned}\int_{(\mathbb{R}^d)^{\mathbf{D}}} |X(t) - X(s)|^4 \mathrm{d}\nu_\Phi &= \int_{W_T} |\theta(t) - \theta(s)|^4 \Phi \,\mathrm{d}\mu_T \\ &\leqq C\|\Phi\|_{-r,p} \Big(\int_{\mathbb{R}^d} |x|^{4q} \frac{1}{(2\pi)^{\frac{d}{2}}} \mathrm{e}^{-\frac{|x|^2}{2}} \mathrm{d}x\Big)^{\frac{1}{q}} |t - s|^2.\end{aligned}$$

Hence, by Kolmogorov's continuity theorem (Theorem A.5.1), $\{X(t)\}_{t\in\mathbf{D}}$ is extended to a stochastic process $\{X(t)\}_{t\in[0,T]}$, which is continuous almost surely with respect to ν_Φ. Therefore, ν_Φ is regarded as a probability measure on W_T.

Let $f \in C_{\mathrm{b}}^\infty((\mathbb{R}^d)^n)$. By (5.4.6), we have for $t_1 < \cdots < t_n \in \mathbf{D}$

$$\int_{W_T} f(\theta(t_1), \ldots, \theta(t_n))\Phi \,\mathrm{d}\mu_T = \int_{W_T} f(\theta(t_1), \ldots, \theta(t_n)) \,\mathrm{d}\nu_\Phi.$$

Since $\mathbf{D}$ is dense in $[0, T]$, this identity continues to hold for any $t_1 < \cdots < t_n \in [0, T]$. We have now proved the conclusion because the elements of the form $f(\theta(t_1), \ldots, \theta(t_n))$ form a dense subset in $\mathbb{D}^{\infty,\infty-}$. □

Example 5.4.17 Let $\eta_1, \ldots, \eta_n \in W_T^*$ form an orthonormal system in H_T. Then, by Example 5.4.2, $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_n) \in \mathbb{D}^{\infty,\infty-}(\mathbb{R}^n)$ is non-degenerate and $\delta_x(\boldsymbol{\eta})$ is positive by Proposition 5.4.13.

Take $\varphi \in C_0^\infty(\mathbb{R}^n)$ so that $\varphi(y) = 1$ for $|y| \leqq 1$ and set $\varphi_m(y) = m^n\varphi(\frac{y-x}{m})$. By Theorem 5.4.11 we have

$$\begin{aligned}\frac{1}{\sqrt{2\pi}^n}\mathrm{e}^{-\frac{|x|^2}{2}} &= \int_{W_T} \delta_x(\boldsymbol{\eta}) \,\mathrm{d}\mu_T = \int_{W_T} \varphi_m(\boldsymbol{\eta})\delta_x(\boldsymbol{\eta}) \,\mathrm{d}\mu_T \\ &= \int_{W_T} \varphi_m(\boldsymbol{\eta}) \,\mathrm{d}\nu_{\delta_x(\boldsymbol{\eta})} \to \nu_{\delta_x(\boldsymbol{\eta})}(\{\boldsymbol{\eta} = x\}) \quad (m \to \infty).\end{aligned}$$

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Hence

$$\nu_{\delta_x(\boldsymbol{\eta})}(\{\boldsymbol{\eta} = y\}) = \begin{cases} \dfrac{1}{(2\pi)^{\frac{n}{2}}} \mathrm{e}^{-\frac{|x|^2}{2}} & (y = x), \\ 0 & (y \neq x). \end{cases}$$

Thus $\nu_{\delta_x(\boldsymbol{\eta})}$ is a measure concentrated on the "hyperplane" $\{w\,;\,\boldsymbol{\eta}(w) = x\}$ on the Wiener space. Since $\mu_T(\boldsymbol{\eta} = x) = 0$, $\nu_{\delta_x(\boldsymbol{\eta})}$ is singular with respect to μ_T.

5.5 Application to Stochastic Differential Equations

We present applications of the Malliavin calculus to stochastic differential equations. Throughout this section, let $V_0, V_1, \ldots, V_d : \mathbb{R}^N \to \mathbb{R}^N$ be C^∞ functions on $\mathbb{R}^N$ with bounded derivatives of all orders.

In this section, we think of $\{\theta(t)\}_{t\in[0,T]}$ as an $\{\mathscr{F}_t\}$-Brownian motion as described in Theorem 5.3.3. However, as mentioned in the remark after the theorem, all random variables are $\mathscr{B}(W_T)$-measurable.

Denote by $\{X(t,x)\}_{t\in[0,T]}$ the unique strong solution of the stochastic differential equation

$$\mathrm{d}X(t) = \sum_{\alpha=1}^{d} V_\alpha(X(t))\,\mathrm{d}\theta^\alpha(t) + V_0(X(t))\,\mathrm{d}t, \quad X(0) = x \tag{5.5.1}$$

(Theorem 4.4.5). By Theorem 4.10.8, $X(t,\cdot)$ is of C^∞-class and the Jacobian matrix $Y(t,x) = (\frac{\partial X^i(t,x)}{\partial x^j})_{i,j=1,\ldots,N}$ satisfies the stochastic differential equation

$$\begin{aligned} \mathrm{d}Y(t,x) &= \sum_{\alpha=1}^{d} V_\alpha'(X(t,x))Y(t,x)\,\mathrm{d}\theta^\alpha(t) + V_0'(X(t,x))Y(t,x)\,\mathrm{d}t, \\ Y(0,x) &= I, \end{aligned} \tag{5.5.2}$$

where $V_\alpha'(x) = (\frac{\partial V_\alpha^i}{\partial x^j}(x))_{i,j=1,\ldots,N}$ $(\alpha = 0,1,\ldots,d)$. Moreover, $Y(t,x)$ is non-degenerate and the inverse matrix

$$Z(t,x) = Y(t,x)^{-1}$$

satisfies the stochastic differential equation

$$\begin{aligned} \mathrm{d}Z(t,x) = &- \sum_{\alpha=1}^{d} Z(t,x)V_\alpha'(X(t,x))\,\mathrm{d}\theta^\alpha(t) - Z(t,x)V_0'(X(t,x))\,\mathrm{d}t \\ &+ \sum_{\alpha=1}^{d} Z(t,x)(V_\alpha'(X(t,x)))^2\,\mathrm{d}t. \end{aligned} \tag{5.5.3}$$

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From these observations, we have, in particular,

$$\sup_{x\in\mathbb{R}^N}\int_{W_T}\sup_{0\leqq t\leqq T}\{|Y(t,x)|^p+|Z(t,x)|^p\}\,\mathrm{d}\mu_T<\infty \tag{5.5.4}$$

for any $p>1$, where $|A|=\left(\sum_{i,j=1}^N a_{ij}^2\right)^{\frac{1}{2}}$ for a matrix $A=(a_{ij})_{i,j=1,\dots,N}$.

Theorem 5.5.1 *Let $t\in[0,T]$. Then, $X(t,x)\in\mathbb{D}^{\infty,\infty-}(\mathbb{R}^N)$ and*

$$(\nabla X^i(t,x))^\alpha(u)=\sum_{j,k=1}^N Y_j^i(t,x)\int_0^{t\wedge u}Z_k^j(v,x)V_\alpha^k(X(v,x))\,\mathrm{d}v \qquad (\alpha=1,\dots,d), \tag{5.5.5}$$

where, for $h\in H_T$, $h^\alpha(u)$ is the value of the α-th component $h^\alpha:[0,T]\to\mathbb{R}$ of h at time $u\in[0,T]$.

Proof For $n\in\mathbb{N}$, set $[s]_n=\frac{[2^n s]}{2^n}$ and define $\{X_n(s)\}_{s\in[0,T]}$ by

$$\begin{aligned}X_n(0)&=x,\\ X_n(s)&=X_n([s]_n)+\sum_{\alpha=1}^d V_\alpha(X_n([s]_n))\{\theta^\alpha(s)-\theta^\alpha([s]_n)\}\\ &\qquad+V_0(X_n([s]_n))\{s-[s]_n\}.\end{aligned}$$

By definition,

$$X_n(t)\in\mathbb{D}^{\infty,\infty-}(\mathbb{R}^N).$$

Moreover, using the expression

$$\mathrm{d}X_n(s)=\sum_{\alpha=1}^d V_\alpha(X_n([s]_n))\,\mathrm{d}\theta^\alpha(s)+V_0(X_n([s]_n))\,\mathrm{d}s, \tag{5.5.6}$$

we observe

$$\int_{W_T}\sup_{0\leqq s\leqq T}|X_n(s)-X(s,x)|^2\mathrm{d}\mu_T\to 0\quad(n\to\infty). \tag{5.5.7}$$

To see this, set

$$\begin{aligned}R_n(s)&=\sum_{\alpha=1}^d\int_0^s\{V_\alpha(X_n(u))-V_\alpha(X_n([u]_n))\}\,\mathrm{d}\theta^\alpha(u)\\ &\qquad+\int_0^s\{V_0(X_n(u))-V_0(X_n([u]_n))\}\,\mathrm{d}u.\end{aligned}$$

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In the same way as for Theorem 4.3.9, we obtain from (5.5.6)

$$\sup_{n\in\mathbb{N}} \int_{W_T} \sup_{0\leqq s\leqq T} |X_n(s)|^p \mathrm{d}\mu_T < \infty$$

for any $p > 1$. Hence, by the definition of $X_n(s)$, there exists a constant C_1 such that

$$\int_{W_T} |X_n(u) - X_n([u]_n)|^2 \mathrm{d}\mu_T \leqq C_1 2^{-n}.$$

By this estimate, the Lipschitz continuity of V_α and the Burkholder–Davis–Gundy inequality (Theorem 2.4.1), there exists a constant C_2 such that

$$\int_{W_T} \sup_{0\leqq s\leqq T} |R_n(s)|^2 \mathrm{d}\mu_T \leqq C_2 2^{-n} \qquad (n = 1, 2, \dots).$$

Moreover, since

$$\begin{aligned} X(s) - X_n(s) = \sum_{\alpha=1}^{n} \int_0^s \{V_\alpha(X(u)) - V_\alpha(X_n(u))\}\, \mathrm{d}\theta^\alpha(u) \\ + \int_0^s \{V_0(X(u)) - V_0(X_n(u))\}\, \mathrm{d}u + R_n(s), \end{aligned}$$

we see, by using the Burkholder–Davis–Gundy inequality again, that there exist constants C_3 and C_4 such that

$$\begin{aligned} &\int_{W_T} \sup_{0\leqq u\leqq s} |X_n(u) - X(u)|^2 \mathrm{d}\mu_T \\ &\qquad \leqq C_3 2^{-n} + C_4 \int_0^s \Big(\int_{W_T} \sup_{0\leqq u\leqq v} |X_n(u) - X(u)|^2 \mathrm{d}\mu_T \Big) \mathrm{d}v \quad (n = 1, 2, \dots). \end{aligned}$$

Hence, by Gronwall's inequality, we obtain (5.5.7).

Let $h \in H_T$ and set

$$J_{n,h}(s) = \langle \nabla X_n(s), h \rangle_{H_T} \qquad (s \in [0, T]).$$

By the definition of $X_n(s)$,

$$\begin{aligned} \mathrm{d}J_{n,h}(s) = \sum_{\alpha=1}^{d} V'_\alpha(X_n([s]_n)) J_{n,h}([s]_n)\, \mathrm{d}\theta^\alpha(s) + V'_0(X_n([s]_n)) J_{n,h}([s]_n)\, \mathrm{d}s \\ + \sum_{\alpha=1}^{d} V_\alpha(X_n([s]_n)) \dot{h}^\alpha(s)\, \mathrm{d}s. \end{aligned}$$

Let an $\mathbb{R}^N$-valued stochastic process $\{J_h(s)\}_{s\in[0,T]}$ be the solution of

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$$
\begin{aligned}
\mathrm{d}J_h(s) = \sum_{\alpha=1}^{d} V'_\alpha(X(s,x))J_h(s)\,\mathrm{d}\theta^\alpha(s) + V'_0(X(s,x))J_h(s)\,\mathrm{d}s \\
+ \sum_{\alpha=1}^{d} V_\alpha(X(s,x))\dot{h}^\alpha(s)\,\mathrm{d}s
\end{aligned}
\tag{5.5.8}
$$

satisfying $J_h(0) = 0$ and set

$$
\begin{aligned}
R'_{n,h}(s) = \sum_{\alpha=1}^{d} \int_0^s \{V'_\alpha(X_n([u]_n))J_{n,h}([u]_n) - V'_\alpha(X(u,x))J_{n,h}(u)\}\,\mathrm{d}\theta^\alpha(u) \\
+ \int_0^s \{V'_0(X_n([u]_n))J_{n,h}([u]_n) - V'_0(X(u,x))J_{n,h}(u)\}\,\mathrm{d}u \\
+ \sum_{\alpha=1}^{d} \int_0^s \{V_\alpha(X_n([u]_n)) - V_\alpha(X(u,x))\}\dot{h}^\alpha(u)\,\mathrm{d}u.
\end{aligned}
$$

Rewriting as

$$
\begin{aligned}
& V'_\alpha(X_n([u]_n))J_{n,h}([u]_n) - V'_\alpha(X(u,x))J_{n,h}(u) \\
& \quad = \{V'_\alpha(X_n([u]_n)) - V'_\alpha(X(u,x))\}J_{n,h}([u]_n) \\
& \qquad + V'_\alpha(X(u,x))\{J_{n,h}([u]_n) - J_{n,h}(u)\}
\end{aligned}
$$

and using the estimate

$$
\sup_{n\in\mathbb{N}} \int_{W_T} \sup_{0\leqq s\leqq T} |J_{n,h}(s)|^p \mathrm{d}\mu_T < \infty
$$

for any $p > 1$, we obtain

$$
\lim_{n\to\infty} \int_{W_T} \sup_{0\leqq s\leqq T} |R'_{n,h}(s)|^2 \mathrm{d}\mu_T = 0.
$$

Hence, by the expression

$$
\begin{aligned}
J_{n,h}(s) - J_h(s) = \sum_{\alpha=1}^{d} \int_0^s V'_\alpha(X(u,x))\{J_{n,h}(u) - J_h(u)\}\,\mathrm{d}\theta^\alpha(u) \\
+ \int_0^s V'_0(X(u,x))\{J_{n,h}(u) - J_h(u)\}\,\mathrm{d}u + R'_{n,h}(s),
\end{aligned}
$$

a similar argument to that in (5.5.7) yields

$$
\int_{W_T} |J_{n,h}(t) - J_h(t)|^2 \mathrm{d}\mu_T \to 0 \quad (n \to \infty). \tag{5.5.9}
$$

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Define an $H_T \otimes \mathbb{R}^N$-valued random variable $F(t) = (F^1(t), \dots, F^N(t)) \in \mathbb{D}^{0,\infty-}(H_T \otimes \mathbb{R}^N)$ by

$$\langle F^i(t), g\rangle_{H_T} = \sum_{j,k=1}^N \sum_{\alpha=1}^d Y^i_j(t,x) \int_0^t Z^j_k(v,x) V^k_\alpha(X(v,x))\, \dot{g}^\alpha(v)\, \mathrm{d}v \quad (g \in H_T)$$

for $i = 1, \dots, N$. Then, by (5.5.8), we have

$$\langle F(t), h\rangle_{H_T} = J_h(t). \tag{5.5.10}$$

Let $\phi \in \mathscr{P},\ h \in H_T,\ i = 1, \dots, N$. Then, by (5.5.7),

$$\begin{aligned}\int_{W_T} X^i(t,x)\nabla^*(\phi \cdot h)\, \mathrm{d}\mu_T &= \lim_{n\to\infty} \int_{W_T} X^i_n(t)\nabla^*(\phi \cdot h)\, \mathrm{d}\mu_T \\ &= \lim_{n\to\infty} \int_{W_T} \langle \nabla X^i_n(t), \phi \cdot h\rangle_{H_T} \mathrm{d}\mu_T = \lim_{n\to\infty} \int_{W_T} J^i_{n,h}(t)\phi\, \mathrm{d}\mu_T.\end{aligned}$$

By (5.5.9) and (5.5.10), the right hand side coincides with

$$\int_{W_T} J^i_h(t)\phi\, \mathrm{d}\mu_T = \int_{W_T} \langle F^i(t), \phi \cdot h\rangle_{H_T} \mathrm{d}\mu_T,$$

and hence we obtain from Theorem 5.3.1,

$$X(t,x) \in \mathbb{D}^{1,\infty-}(\mathbb{R}^N) \quad \text{and} \quad \nabla X(t,x) = F(t).$$

Using the result $\nabla X(t,x) = F(t)$ and repeating a similar argument to the above, we can show $X(t,x) \in \mathbb{D}^{\infty,\infty-}(\mathbb{R}^N)$. We omit the details and refer to [104]. □

On the non-degeneracy of $X(t,x)$, we have the following.

Theorem 5.5.2 *Set $a^{ij}(y) = \sum_{\alpha=1}^d V^i_\alpha(y)V^j_\alpha(y)$. If $a(x) = (a^{ij}(x))_{i,j=1,\dots,N}$ is positive definite at the starting point x of $\{X(t,x)\}_{t\in[0,T]}$, then $X(t,x)$ is non-degenerate for any $t \in (0,T]$.*

We give a lemma for the proof.

Lemma 5.5.3 *Let $\{u_\alpha(t)\}_{t\in[0,T]}$ $(\alpha = 0, 1, \dots, d)$ be $\{\mathscr{F}_t\}$-predictable and bounded (see Theorem 5.3.3 for $\mathscr{F}_t$) and set*

$$M = \sup\{|u_\alpha(t,w)|;\ t \in [0,T],\ w \in W_T,\ \alpha = 0, 1, \dots, d\}.$$

Define a stochastic process $\{\xi(t)\}_{t\in[0,T]}$ by

$$\xi(t) = x + \sum_{\alpha=1}^d \int_0^t u_\alpha(s)\, \mathrm{d}\theta^\alpha(s) + \int_0^t u_0(s)\, \mathrm{d}s$$

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and, for $\varepsilon > 0$, set

$$\sigma_\varepsilon = \inf\{t \geqq 0; |\xi(t) - x| > \varepsilon\}.$$

Then,

$$\mu_T(\sigma_\varepsilon \leqq t) \leqq \int_{\frac{\varepsilon}{2\sqrt{dM^2 t}}}^{\infty} \sqrt{\frac{2}{\pi}}\, \mathrm{e}^{-\frac{y^2}{2}} \mathrm{d}y$$

holds for $t < \frac{\varepsilon}{2M}$. In particular, $\frac{1}{\sigma_\varepsilon} \in \bigcap_{p\in(1,\infty)} L^p(\mu_T)$.

Proof Let $t < \frac{\varepsilon}{2M}$. Then, since

$$\Big|\sum_{\alpha=1}^{d} \int_0^s u_\alpha(v)\, \mathrm{d}\theta^\alpha(v)\Big| \geqq |\xi(s) - x| - Mt$$

for any $s \in [0, t]$, we have

$$\{\sigma_\varepsilon < t\} \subset \Big\{ \sup_{0\leqq s\leqq t} \Big|\sum_{\alpha=1}^{d} \int_0^s u_\alpha(v)\, \mathrm{d}\theta^\alpha(v)\Big| > \frac{\varepsilon}{2}\Big\}.$$

By Theorem 2.5.5, there exists a Brownian motion $\{\beta(t)\}_{t\geqq 0}$ such that[2]

$$\sum_{\alpha=1}^{d} \int_0^s u_\alpha(v)\, \mathrm{d}\theta^\alpha(v) = \beta(\phi(s)),$$

where

$$\phi(s) = \sum_{\alpha=1}^{d} \int_0^s (u_\alpha(v))^2 \mathrm{d}v.$$

Since $\phi(s) \leqq dM^2 s$,

$$\Big\{ \sup_{0\leqq s\leqq t} \Big|\sum_{\alpha=1}^{d} \int_0^s u_\alpha(v)\, \mathrm{d}\theta^\alpha(v)\Big| > \frac{\varepsilon}{2}\Big\} \subset \Big\{ \max_{0\leqq s\leqq dM^2 t} |\beta(s)| > \frac{\varepsilon}{2}\Big\}.$$

Applying Corollary 3.1.8, we obtain the conclusion. □

Proof of Theorem 5.5.2 Let $t \in (0, T]$ and set

$$A(t, x) = \int_0^t Z(s, x) a(X(s, x)) Z(s, x)^* \mathrm{d}s,$$

where Z^* is the transposed matrix of Z. By Theorem 5.5.1,

$$\Big(\langle \nabla X^i(t, x), \nabla X^j(t, x)\rangle_{H_T}\Big)_{i,j=1,\ldots,N} = Y(t, x) A(t, x) Y(t, x)^*.$$

[2] Strictly speaking, we need to extend the probability space. We suppose here that the probability space is already extended and we do not write it explicitly. For details, see Theorem 2.5.5.

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Since $\frac{1}{\det Y(t,x)} = \det Z(t,x)$, it suffices to show

$$\frac{1}{\det A(t)} \in \bigcap_{p\in(1,\infty)} L^p(\mu_T) \tag{5.5.11}$$

because of (5.5.4).

Fix a sufficiently small $\varepsilon > 0$. By the positivity of $a(x)$, there exists a $\delta > 0$ such that

$$a(y) \geqq \varepsilon I \quad (y \in B(x,\delta) = \{y; |y-x| < \delta\}).$$

For $\eta > 0$, define stopping times τ_η and σ_η by

$$\tau_\eta = \inf\{s > 0; |X(t,x) - x| > \eta\} \text{ and } \sigma_\eta = \inf\{s > 0; |Z(s,x) - I| > \eta\}.$$

By the definitions, we have

$$A(t) \geqq \varepsilon \int_0^{t\wedge\tau_\delta\wedge\sigma_{\frac{1}{4}}} Z(s,x)Z(s,x)^* \mathrm{d}s \geqq \frac{9\varepsilon}{16}(t\wedge\tau_\delta\wedge\sigma_{\frac{1}{4}})I.$$

In particular,

$$\det A(t) \geqq \Big(\frac{9\varepsilon}{16}(t\wedge\tau_\delta\wedge\sigma_{\frac{1}{4}})\Big)^N.$$

Applying Lemma 5.5.3 to $\{|X(s\wedge\tau_1, x) - x|^2\}_{s\in[0,T]}$ and $\{|Z(s\wedge\sigma_1, x)|^2\}_{s\in[0,T]}$, we obtain $(t\wedge\tau_\delta\wedge\sigma_{\frac{1}{4}})^{-1} \in \bigcap_{p\in(1,\infty)} L^p(\mu_T)$ and (5.5.11). □

As will be mentioned below, the non-degeneracy in Theorem 5.5.2 holds under weaker conditions. Denote the space of $\mathbb{R}^N$-valued C^∞ functions on $\mathbb{R}^N$ by $C^\infty(\mathbb{R}^N;\mathbb{R}^N)$ and identify each element U of $C^\infty(\mathbb{R}^N;\mathbb{R}^N)$ with the differential operator $\sum_{i=1}^N U^i(x)\frac{\partial}{\partial x^i}$. For $U, V \in C^\infty(\mathbb{R}^N;\mathbb{R}^N)$, let $[U,V]$ be the Lie bracket of U and $V : [U,V] = U\circ V - V\circ U$. By the identification mentioned above, $[U,V] \in C^\infty(\mathbb{R}^N;\mathbb{R}^N)$.

Theorem 5.5.4 *Let $x \in \mathbb{R}^N$ and $\mathscr{L}(x)$ be the subspace of $\mathbb{R}^N$ spanned by $V_\alpha(x)$, $[V_{k_1},[V_{k_2},\dots,[V_{k_n},V_\alpha]\dots]](x)$ $(\alpha = 1,\dots,d,\ k_j = 0,1,\dots,d,\ j = 1,\dots,n,\ n \geqq 1)$. If $\dim\mathscr{L}(x) = N$, then $X(t,x)$ is non-degenerate for any $t \in [0,T]$.*

As Theorem 5.5.2, this theorem is proven by showing the integrability of $\frac{1}{\det A(t)}$. The condition in the theorem is called **Hörmander's condition**. For details, see [104].

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Example 5.5.5 Let $d = 1$ and $N = 2$. Define the vector fields V_1 and V_2 on $\mathbb{R}^2$ by

$$V_1(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad V_0(x) = \begin{pmatrix} 0 \\ x \end{pmatrix}.$$

A diffusion process on $\mathbb{R}^2$ defined by the solution of the stochastic differential equation

$$\mathrm{d}X^1(t) = \mathrm{d}\theta^1(t), \quad \mathrm{d}X^2(t) = X^1(t)\,\mathrm{d}t, \quad X(0, x) = (x^1, x^2)$$

is called the **Kolmogorov diffusion**. By Theorem 5.5.1, $X(t, x) \in \mathbb{D}^{\infty,\infty-}(\mathbb{R}^2)$. Moreover, since $[V_0, V_1] = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $\dim \mathscr{L}(x) = 2$ and $X(t, x)$ is non-degenerate by Theorem 5.5.4. Hence, $\mathbf{E}[\delta_y(X(t, x))]$ gives the transition density $p(t, x, y)$ of the diffusion process $\{X(t, x)\}_{t \geqq 0}$.

The above results can be seen in a more straightforward manner. In fact, the solution of this stochastic differential equation is explicitly given by

$$X(t, x) = \begin{pmatrix} x^1 + \theta^1(t) \\ x^2 + x^1 t + \int_0^t \theta^1(s)\,\mathrm{d}s \end{pmatrix}.$$

This immediately implies $X(t, x) \in \mathbb{D}^{\infty,\infty-}(\mathbb{R}^2)$. Moreover, since

$$(\langle \nabla X^i(t, x), \nabla X^j(t, x) \rangle_{H_T})_{i,j=1,2} = \begin{pmatrix} t & \frac{t^2}{2} \\ \frac{t^2}{2} & \frac{t^3}{3} \end{pmatrix},$$

the non-degeneracy of $X(t, x)$ follows.

Furthermore, $p(t, x, y)$ admits an explicit expression. In fact, the distribution of $X(t, x)$ is Gaussian with mean $\begin{pmatrix} x^1 \\ x^2 + x^1 t \end{pmatrix}$ and covariance matrix $\begin{pmatrix} t & \frac{t^2}{2} \\ \frac{t^2}{2} & \frac{t^3}{2} \end{pmatrix}$. Hence

$$p(t, x, y) = \frac{\sqrt{3}}{\pi t^2} \exp\{-2t^{-1}(y^1 - x^1)^2 + 6t^{-2}(y^1 - x^1)(y^2 - x^2 - tx^1) \\ - 6t^{-3}(y^2 - x^2 - tx^1)^2\},$$

where $x = (x^1, x^2)$ and $y = (y^1, y^2)$.

The generator $\frac{1}{2} \frac{\partial^2}{\partial (x^1)^2} + x^2 \frac{\partial}{\partial x^2}$ is called the **Kolmogorov operator** and it is referred to as a typical degenerate and hypoelliptic operator in the original paper by Hörmander [37]. See [43] and [113] for recent related studies.

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Example 5.5.6 Let $d = 2$ and $N = 3$. Define the vector fields V_0, V_1, and V_2 on $\mathbb{R}^3$ by $V_0 = 0$,

$$V_1(x) = \begin{pmatrix} 1 \\ 0 \\ -\frac{x^2}{2} \end{pmatrix}, \quad \text{and} \quad V_2(x) = \begin{pmatrix} 0 \\ 1 \\ \frac{x^1}{2} \end{pmatrix} \qquad (x = (x^1, x^2, x^3) \in \mathbb{R}^3).$$

The solution $X(t, x)$ of the corresponding stochastic differential equation

$$\begin{aligned} &\mathrm{d}X^1(t) = \mathrm{d}\theta^1(t), \qquad \mathrm{d}X^2(t) = \mathrm{d}\theta^2(t), \\ &\mathrm{d}X^3(t) = \frac{1}{2}X^1(t)\,\mathrm{d}\theta^2(t) - \frac{1}{2}X^2(t)\,\mathrm{d}\theta^1(t) \end{aligned}$$

belongs to $\mathbb{D}^{\infty,\infty-}(\mathbb{R}^3)$. Moreover, since

$$[V_1, V_2](x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$X(t, x)$ is non-degenerate. $X(t, x)$ is explicitly written as

$$\begin{aligned} X^\alpha(t, x) &= x^\alpha + \theta^\alpha(t) \qquad (\alpha = 1, 2), \\ X^3(t, x) &= x^3 + \frac{1}{2}\{x^1\theta^2(t) - x^2\theta^1(t)\} \\ &\qquad + \frac{1}{2}\Big\{\int_0^t \theta^1(s)\,\mathrm{d}\theta^2(s) - \int_0^t \theta^2(s)\,\mathrm{d}\theta^1(s)\Big\}. \end{aligned}$$

The stochastic process

$$\mathfrak{s}(t) = \frac{1}{2}\Big\{\int_0^t \theta^1(s)\,\mathrm{d}\theta^2(s) - \int_0^t \theta^2(s)\,\mathrm{d}\theta^1(s)\Big\}$$

which appears in the expression for $X^3(t, x)$ is called **Lévy's stochastic area** and plays an important role in various fields related to stochastic analysis. The explicit form of the characteristic function of $\mathfrak{s}(T)$ is well known (Theorem 5.8.4) and is called **Lévy's formula**.

Next we apply the Malliavin calculus to Schrödinger operators on $\mathbb{R}^d$. First we consider Brownian motions, that is the case where $N = d$ and $X(t, x) = x + \theta(t)$ $(x \in \mathbb{R}^d)$. We presented a probabilistic representation for the corresponding heat equations in Chapter 3. We here consider Schrödinger operators with magnetic fields and give representations for the **fundamental solutions** by using the results in the previous section.

Let $V, \Theta_1, \ldots, \Theta_d \in C^\infty_{\exp}(\mathbb{R}^d)$ and assume that

$$\inf_{x\in\mathbb{R}^d} V(x) > -\infty. \tag{5.5.12}$$

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The differential operator H given by

$$H = -\frac{1}{2}\sum_{\alpha=1}^{d}\Big(\frac{\partial}{\partial x^\alpha} + \mathrm{i}\,\Theta_\alpha\Big)^2 + V$$

is called a Schrödinger operator with vector potential $\Theta = (\Theta_1, \dots, \Theta_d)$ and scalar potential V. The fundamental solution for the heat equation

$$\frac{\partial u}{\partial t} = -Hu, \quad u(0,\cdot) = f \in C^\infty_{\exp}(\mathbb{R}^d) \tag{5.5.13}$$

associated with H is a function $p(t,x,y)$ such that

$$u(t,x) = \int_{\mathbb{R}^d} f(y)p(t,x,y)\,\mathrm{d}y$$

is a solution of (5.5.13). We construct the fundamental solution by applying the Malliavin calculus. It is easy to see $\int_0^t V(x+\theta(s))\,\mathrm{d}s \in \mathbb{D}^{\infty,\infty-}$ and, from the assumption (5.5.12), we have

$$\exp\Big(-\int_0^t V(x+\theta(s))\,\mathrm{d}s\Big) \in \mathbb{D}^{\infty,\infty-} \qquad (t\in[0,T],\ x\in\mathbb{R}^d).$$

Set

$$L(t,x;\Theta) = \sum_{\alpha=1}^{d}\int_0^t \Theta_\alpha(x+\theta(s))\circ \mathrm{d}\theta^\alpha(s).$$

By Theorem 5.3.3, $L(t,x;\Theta) \in \mathbb{D}^{\infty,\infty-}$. Hence, by Corollary 5.3.2,

$$e(t,x) = \exp\Big(\mathrm{i}\,L(t,x;\Theta) - \int_0^t V(x+\theta(s))\,\mathrm{d}s\Big) \in \mathbb{D}^{\infty,\infty-}.$$

Theorem 5.5.7 *The function $p(t,x,y)$ ($t>0,\ x,y\in\mathbb{R}^d$) defined by*

$$p(t,x,y) = \mathbf{E}[e(t,x)\delta_y(x+\theta(t))] = \int_{W_T} e(t,x)\delta_y(x+\theta(t))\,\mathrm{d}\mu_T$$

is the fundamental solution for the heat equation (5.5.13) *associated with the Schrödinger operator H.*

Proof Let $f\in C^\infty_{\exp}(\mathbb{R}^d)$. Then $Hf \in C^\infty_{\exp}(\mathbb{R}^d)$. Setting

$$v(t,x;f) = \int_{W_T} f(x+\theta(t))e(t,x)\,\mathrm{d}\mu_T,$$

we can prove, by Lebesgue's convergence theorem, that $v(t,\cdot\,;f) \in C^\infty_{\exp}(\mathbb{R}^d)$. By Itô's formula,

$$v(t,x;f) = f(x) + \int_0^t v(s,x;-Hf)\,\mathrm{d}s.$$

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Hence, we obtain

$$\frac{\partial v(t,x;f)}{\partial t} = v(t,x;-Hf) \tag{5.5.14}$$

and, by the Markov property of Brownian motions,

$$v(t,x;f) = v(s,x;v(t-s,\cdot\,;f)) \qquad (s \leqq t).$$

Differentiate both sides with respect to s. Then, since the mapping $f \mapsto v(s,x;f)$ is linear, by (5.5.14), we obtain

$$0 = v(s,x;-Hv(t-s,\cdot\,;f)) - v(s,x;v(t-s,\cdot\,;-Hf)).$$

Setting $s=0$, we see

$$-Hv(t,x;f) = v(t,x;-Hf).$$

Hence, by (5.5.14),

$$\frac{\partial v(t,x;f)}{\partial t} = -Hv(t,x;f). \tag{5.5.15}$$

By Theorem 5.4.11,

$$v(t,x;f) = \int_{\mathbb{R}^d} f(y)p(t,x,y)\,\mathrm{d}y \tag{5.5.16}$$

for any $f \in \mathscr{S}(\mathbb{R}^d)$. For $\boldsymbol{a} = (a_1,\dots,a_d) \in \mathbb{R}^d$, set $g_{\boldsymbol{a}}(x) = \cosh\Big(\sum_{\alpha=1}^d a_\alpha x^\alpha\Big)$. Moreover, take $\phi_n \in C_0^\infty(\mathbb{R}^d)$ such that $\phi_n(x) = 1$ for $|x| \leqq n$ and $\phi_n(x) = 0$ for $|x| > n+1$, and set $g_{\boldsymbol{a},n} = g_{\boldsymbol{a}}\phi_n$. Since $g_{\boldsymbol{a}} \in C_{\exp}^\infty(\mathbb{R}^d)$, by the monotone convergence theorem and (5.5.16),

$$\begin{aligned}\int_{\mathbb{R}^d} g_{\boldsymbol{a}}(y)p(t,x,y)\,\mathrm{d}y &= \lim_{n\to\infty}\int_{\mathbb{R}^d} g_{\boldsymbol{a},n}(y)p(t,x,y)\,\mathrm{d}y \\ &= \lim_{n\to\infty} v(t,x;g_{\boldsymbol{a},n}) = v(t,x;g_{\boldsymbol{a}}) < \infty.\end{aligned}$$

If $f \in C_{\exp}^\infty(\mathbb{R}^d)$, there exists an $\boldsymbol{a} = (a_1,\dots,a_d) \in \mathbb{R}^d$ such that $|f| \leqq g_{\boldsymbol{a}}$. Hence, by Lebesgue's convergence theorem and (5.5.16), we obtain

$$\begin{aligned}v(t,x;f) &= \lim_{n\to\infty} v(t,x;f\phi_n) \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^d}(f\phi_n)(y)p(t,x,y)\,\mathrm{d}y = \int_{\mathbb{R}^d} f(y)p(t,x,y)\,\mathrm{d}y.\end{aligned}$$

Combining this with (5.5.15), we see that $p(t,x,y)$ is the fundamental solution for the heat equation (5.5.13). $\square$

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Remark 5.5.8 By Corollary 5.4.9, $p \in C^\infty((0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$. Moreover, by Theorem 5.4.11(2), we have

$$p(t,x,y) = \mathbf{E}[e(t,x)|x+\theta(t)=y] \times \frac{1}{(2\pi t)^{\frac{d}{2}}}\mathrm{e}^{-\frac{|x-y|^2}{2t}}.$$

The expression in Theorem 5.5.7 above is essentially a conditional expectation.

The above result is naturally extended to solutions of general stochastic differential equations. Let $\{X(t,x)\}_{t\in[0,\infty)}$ be the solution of the stochastic differential equation (5.5.1). Assume that the functions $V, \Theta_1, \dots, \Theta_d \in C^\infty_{\nearrow}(\mathbb{R}^N)$ satisfy (5.5.12). Define the Schrödinger operator $\widetilde{H}$ by

$$\begin{aligned}\widetilde{H}f = \frac{1}{2}\sum_{i,j=1}^N a^{ij}\frac{\partial^2 f}{\partial x^i \partial x^j} + \sum_{i=1}^N\Big(V_0^i + \mathrm{i}\sum_{j=1}^N a^{ij}\Theta_j\Big)\frac{\partial f}{\partial x^i}\\ + \Big\{\mathrm{i}\Big(\frac{1}{2}\sum_{i,j=1}^N a^{ij}\frac{\partial \Theta_i}{\partial x^j} + \sum_{i=1}^N V_0^i\Theta_i\Big) - V - \frac{1}{2}\sum_{i,j=1}^N a^{ij}\Theta_i\Theta_j\Big\}f,\end{aligned}$$

where $a^{ij} = \sum_{\alpha=1}^N V_\alpha^i V_\alpha^j$. Set

$$\widetilde{e}(t,x) = \exp\Big(\mathrm{i}\sum_{i=1}^N\int_0^t \Theta_i(X(s,x))\circ \mathrm{d}X^i(s,x) - \int_0^t V(X(s,x))\,\mathrm{d}s\Big).$$

Then, $\widetilde{e}(t,x) \in \mathbb{D}^{\infty,\infty-}$ and the following holds as in the case of Brownian motions.

Theorem 5.5.9 *Suppose that Hörmander's condition holds at every $x \in \mathbb{R}^N$. Then the function $q(t,x,y) = \int_{W_T}\widetilde{e}(t,x)\delta_y(X(t,x))\,\mathrm{d}\mu_T$ is the fundamental solution of the heat equation*

$$\frac{\partial u}{\partial t} = \widetilde{H}u, \quad u(0,\cdot) = f \in C^\infty_{\nearrow}(\mathbb{R}^N)$$

associated with the Schrödinger operator $\widetilde{H}$. That is, the function $u(t,x) = \int_{\mathbb{R}^N} f(y)q(t,x,y)\,\mathrm{d}y$ is the solution of this heat equation.

Proof For $\beta = (\beta_1,\dots,\beta_N) \in \mathbb{Z}_+^N$, let ∂^β be the differential operator

$$\partial^\beta = \Big(\frac{\partial}{\partial x^1}\Big)^{\beta_1}\cdots\Big(\frac{\partial}{\partial x^N}\Big)^{\beta_N}.$$

Since the mapping

$$x \mapsto \int_{W_T}|\partial^\beta X(t,x)|^p\,\mathrm{d}\mu_T$$

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is at most of polynomial growth for any $p > 1$, a repetition of the arguments in the proof of Theorem 5.5.7 yields the conclusion. □

Another application of the Malliavin calculus to a study of Greeks in mathematical finance will be discussed in the next chapter.

5.6 Change of Variables Formula

The integration by parts formula and the change of variables formula are fundamental in calculus. We have discussed the integration by parts formula on Wiener spaces and its applications. In this section we investigate a change of variables formula on a Wiener space.

Let E be a real separable Hilbert space. For $A \in E^{\otimes 2}$, we define the regularized determinant $\det_2(I + A)$ of $I + A$ so that

$$\det_2(I + A) = \det(I + A)\mathrm{e}^{-\operatorname{tr} A}$$

if A is of trace class, where I is the identity mapping of E. For details, see [19, XI.9] and [107, Chapter 9].

With the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ of A, repeated according to multiplicity, the regularized determinant is written as

$$\det_2(I + A) = \prod_{j=1}^{\infty}(1 + \lambda_j)\mathrm{e}^{-\lambda_j}. \tag{5.6.1}$$

The following change of variables formula holds on W_T.

Theorem 5.6.1 *Let $F \in \mathbb{D}^{\infty,\infty-}(H_T)$. Suppose that there exists a $q > \frac{1}{2}$ such that*

$$\mathrm{e}^{-\nabla^* F + q\|\nabla F\|^2_{H_T^{\otimes 2}}} \in L^{1+}(\mu_T) = \bigcup_{p\in(1,\infty)} L^p(\mu_T). \tag{5.6.2}$$

Then, for any $f \in C_{\mathrm{b}}(W_T)$,

$$\int_{W_T} f(\iota + F)\det_2(I + \nabla F)\mathrm{e}^{-\nabla^* F - \frac{1}{2}\|F\|^2_{H_T}}\, \mathrm{d}\mu_T = \int_{W_T} f\, \mathrm{d}\mu_T, \tag{5.6.3}$$

where $\iota(w) = w$ ($w \in W_T$).

The left hand side is well defined because

$$\bigl|\det_2(I + A)\bigr| \leqq \exp\Bigl(\frac{1}{2}\bigl\|A\bigr\|^2_{E^{\otimes 2}}\Bigr). \tag{5.6.4}$$

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This estimate is obtained by combining (5.6.1) with the inequality

$$\left|(1+x)e^{-x}\right|^2 \leqq e^{x^2} \qquad (x \in \mathbb{R}).$$

Remark 5.6.2 (1) By (5.6.3), if $\det_2(I + \nabla F) \geqq 0$, μ_T-a.s., the measure on W_T with density

$$\det{}_2(I + \nabla F)e^{-\nabla^* F - \frac{1}{2}\|F\|_{H_T}^2}$$

with respect to μ_T is a probability measure and the distribution of the W_T-valued function $\iota + F$ under this probability measure is the Wiener measure. In this case, (5.6.3) also holds for any bounded measurable $f : W_T \to \mathbb{R}$.
(2) Suppose that $G : W_T \to H_T$ is continuous. If there exists an $F \in \mathbb{D}^{\infty,\infty-}(H_T)$ such that the conditions of Theorem 5.6.1 are fulfilled and $(\iota + G) \circ (\iota + F) = \iota$, then

$$\int_{W_T} f(\iota + G)\, d\mu_T = \int_{W_T} f \det{}_2(I + \nabla F)e^{-\nabla^* F - \frac{1}{2}\|F\|_{H_T}^2}\, d\mu_T$$

for any $f \in C_b(W_T)$, that is, the distribution of $\iota + G$ under μ_T coincides with the probability measure $\widehat{\mu}_T$ given by

$$\widehat{\mu}_T(A) = \int_A \det{}_2(I + \nabla F)e^{-\nabla^* F - \frac{1}{2}\|F\|_{H_T}^2}\, d\mu_T \qquad (A \in \mathscr{B}(W_T)).$$

(3) The Cameron–Martin theorem (Theorem 1.7.2) is a special case of this theorem. In fact, if F is an H_T-valued constant function, say $F = h$ ($h \in H_T$), then $\nabla F = 0$ and $\|F\|_{H_T} = \|h\|_{H_T}$. Moreover, by Example 5.1.5, $\nabla^* F = \mathscr{I}(h)$. Hence, by Theorem 5.6.1, we have

$$\int_{W_T} f(w + h)e^{-\mathscr{I}(h) - \frac{1}{2}\|h\|_{H_T}^2} \mu_T(dw) = \int_{W_T} f\, d\mu_T$$

for any $f \in C_b(W_T)$.[3]
(4) Girsanov's theorem (Theorem 4.6.2) is also derived from Theorem 5.6.1. To show this, let $\{u(t) = (u_1(t), \ldots, u_d(t))\}_{t \in [0,T]}$ be an $\{\mathscr{F}_t\}$-predictable and bounded $\mathbb{R}^d$-valued stochastic process. As in Theorem 5.3.3, we define $\Phi_u : W_T \to H_T$ by $\dot{\Phi}_u(t) = u(t)$ ($t \in [0, T]$). Assume that $\Phi_u \in \mathbb{D}^{\infty,\infty-}(H_T)$. Since, by Theorem 5.3.3,

$$\nabla^* \Phi_u = \sum_{\alpha=1}^{d} \int_0^T u_\alpha(t)\, d\theta^\alpha(t),$$

[3] Research on the change of variables formula on W_T started from a series of studies by Cameron and Martin in the 1940s, including this Cameron–Martin theorem.

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we can rewrite Girsanov's theorem as

$$\int_{W_T} f(\iota+\Phi_u)\mathrm{e}^{-\nabla^*\Phi_u-\frac{1}{2}\|\Phi_u\|_{H_T}^2}\mathrm{d}\mu_T = \int_{W_T} f\,\mathrm{d}\mu_T. \tag{5.6.5}$$

We take $\mathbb{R}^d$-valued stochastic processes $\{u_n(t)=(u_n^1(t),\dots,u_n^d(t))\}_{t\in[0,T]}$ with components $\{u_n^\alpha(t)\}_{t\in[0,T]}\in\mathscr{L}^0$ ($\alpha=1,\dots,d$) (see Definition 2.2.4) such that

$$\lim_{n\to\infty}\int_{W_T}\Big(\int_0^T|u_n(t)-u(t)|^2\mathrm{d}t\Big)\mathrm{d}\mu_T=0 \quad\text{and}\quad M:=\sup_{n\in\mathbb{N},w\in W_T}|u_n(t,w)|<\infty.$$

Moreover, we may assume that each $u_n^\alpha(t)$ is written as

$$u_n^\alpha(t)=\xi_{n,k}^\alpha,\quad t_k^n<t\leqq t_{k+1}^n,\ k=0,1,\dots,m_n-1 \qquad (\alpha=1,\dots,d),$$

where the random variables $\xi_{n,k}^\alpha$ are given by

$$\xi_{n,k}^\alpha=\phi_{n,k}^\alpha(\theta(s_1^{k,n}),\dots,\theta(s_{j_{k,n}}^{k,n}))$$

for a monotone increasing sequence $0=t_0^n<t_1^n<\cdots<t_k^n<\cdots<t_{m_n}^n=T$ and $0<s_1^{k,n}<\cdots<s_{j_{k,n}}^{k,n}\leqq t_k^n$ and $\phi_{n,k}^\alpha\in C_{\mathrm{b}}^\infty((\mathbb{R}^d)^{j_{k,n}})$ (see the proof of Theorem 5.3.3). Then, we have

$$\Phi_{u_n}=\sum_{k=0}^{m_n-1}\sum_{\alpha=1}^d\xi_{n,k}^\alpha\ell_{(t_k^n,t_{k+1}^n]}^\alpha,$$

where $e_\alpha=(\overbrace{0,\dots,0}^{\alpha-1},1,0,\dots,0)\in\mathbb{R}^d$ and $\ell_{(s,t]}^\alpha\in H_T$ is defined by $\dot{\ell}_{(s,t]}^\alpha(v)=\mathbf{1}_{(s,t]}(v)e_\alpha$ ($v\in[0,T]$). By Corollary 5.3.2,

$$\nabla\Phi_{u_n}=\sum_{k=0}^{m_n-1}\sum_{i=1}^{j_{k,n}}\sum_{\alpha,\beta=1}^d\frac{\partial\phi_{n,k}^\alpha}{\partial x_i^\beta}(\theta(s_1^{k,n}),\dots,\theta(s_{j_{k,n}}^{k,n}))\ell_{(0,s_i^{k,n}]}^\beta\otimes\ell_{(t_k^n,t_{k+1}^n]}^\alpha,$$

where the coordinate of $(\mathbb{R}^d)^{j_{k,n}}$ is $(x_1^1,\dots,x_1^d,\dots,x_{j_{k,n}}^1,\dots,x_{j_{k,n}}^d)$. Hence, there exist an $N\in\mathbb{N}$, an orthonormal system $g_1,\dots,g_N$ of H_T, and random variables a_{ij} ($i,j=1,\dots,N$) such that

$$\nabla\Phi_{u_n}=\sum_{1\leqq i<j\leqq N}a_{ij}g_i\otimes g_j.$$

Since all the eigenvalues of upper triangular matrices are zero, $\det_2(I+\nabla\Phi_{u_n})=1$. Hence, by Theorem 5.6.1, (5.6.5) holds for $u=u_n$.

Since $\{\mathrm{e}^{-2\sum_{\alpha=1}^d\int_0^t u_{n,\alpha}(s)\mathrm{d}\theta^\alpha(s)-2\int_0^t|u_n(s)|^2\mathrm{d}s}\}_{t\in[0,T]}$ is a martingale,

$$\int_{W_T}\mathrm{e}^{2\{-\nabla^*\Phi_n-\frac{1}{2}\|\Phi_{u_n}\|_{H_T}^2\}}\mathrm{d}\mu_T\leqq\mathrm{e}^{M^2T}$$

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and $e^{-\nabla^*\Phi_n-\frac{1}{2}\|\Phi_{u_n}\|_{H_T}^2}$ ($n \in \mathbb{N}$) is uniformly integrable. Hence, setting $u = u_n$ in (5.6.5) and letting $n \to \infty$, we obtain (5.6.5) for a bounded stochastic process $\{u(t)\}_{t\in[0,T]}$.

(5) The change of variables formula via the regularized determinant $\det_2$ and the derivative ∇ on Wiener spaces as in the theorem was first studied by Kusuoka [66] and his result was applied to the degree theorem on Wiener spaces in [30]. The proof of the theorem below is based on the arguments in Üstünel and Zakai [120].

For a proof of Theorem 5.6.1, we show a change of variables formula for the integrals with respect to Gaussian measures on $\mathbb{R}^n$. Denote the inner product in $\mathbb{R}^n$ by $\langle\cdot,\cdot\rangle$ and the norm on $\mathbb{R}^n$ by $\|\cdot\|$, where the norm was denoted by $|\cdot|$ in the previous chapters. This change is to make notation analogous to that for H_T. The gradient operator on $\mathbb{R}^n$ is also denoted by ∇, that is, $\nabla f = (\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^n})$. Let ν_n be the probability measure on $\mathbb{R}^n$ defined by

$$\nu_n(\mathrm{d}x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{\|x\|^2}{2}}\,\mathrm{d}x \tag{5.6.6}$$

and ∇^* be the formal adjoint operator of ∇ with respect to ν_n,

$$\nabla^* F(x) = \langle x, F(x)\rangle - \mathrm{tr}(\nabla F(x)) \qquad (F \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)),$$

where, for $F = (F^1, \dots, F^n) \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$,

$$\nabla F = \Big(\frac{\partial F^i}{\partial x^j}\Big)_{i,j=1,\dots,n}.$$

Moreover, the space of $\mathbb{R}^k$-valued C^∞ functions F on $\mathbb{R}^n$ such that $\nabla^j F \in L^{\infty-}(\nu_n;\mathbb{R}^{kn^j})$ for any $j \in \mathbb{Z}_+$ is denoted by $\mathscr{D}^{\infty,\infty-}(\mathbb{R}^n;\mathbb{R}^k)$. When $k = 1$, we simply write $\mathscr{D}^{\infty,\infty-}(\mathbb{R}^n)$.

For $F \in \mathscr{D}^{\infty,\infty-}(\mathbb{R}^n;\mathbb{R}^n)$, set

$$\Lambda_F = \det{}_2(I + \nabla F)e^{-\nabla^* F-\frac{1}{2}\|F\|^2} = \det(I + \nabla F)e^{-\langle F,\cdot\rangle-\frac{1}{2}\|F\|^2}.$$

For a matrix $A \in \mathbb{R}^n \otimes \mathbb{R}^n$, let $A^\sim$ be its cofactor matrix, that is, letting $\widehat{a}_{ij}$ be the (i, j)-cofactor of A, $A^\sim = (\widehat{a}_{ji})_{i,j=1,\dots,n}$. The product of A and $A^\sim$ is

$$A\,(A^\sim) = \det A \times I.$$

By this identity we can define $\Lambda_F(I + \nabla F)^{-1}(x) \in \mathbb{R}^n \otimes \mathbb{R}^n$ by

$$\Lambda_F(I + \nabla F)^{-1}(x) = e^{-\langle x,F(x)\rangle-\frac{1}{2}\|F(x)\|^2}(I + \nabla F(x))^\sim \tag{5.6.7}$$

regardless of the regularity of the matrix $I + \nabla F(x) \in \mathbb{R}^n \otimes \mathbb{R}^n$.

Lemma 5.6.3 *For $x, v \in \mathbb{R}^n$,*

$$\nabla^*(\Lambda_F(I+\nabla F)^{-1}v)(x) = \Lambda_F(x)\langle v, x+F(x)\rangle.$$

Proof Let $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$. By (5.6.7) and the definition of ∇^*,

$$\nabla^*(\Lambda_F(I+\nabla F)^{-1}v) = \Lambda_F\langle v, \cdot + F\rangle - \mathrm{e}^{-\langle F,\cdot\rangle - \frac{1}{2}\|F\|^2}\sum_{i,j=1}^n \frac{\partial}{\partial x^i}(I+\nabla F)^{\sim}_{ij}v_j.$$

Hence it suffices to show

$$\sum_{i,j=1}^n \frac{\partial}{\partial x^i}(I+\nabla F)^{\sim}_{ij}v_j = 0. \tag{5.6.8}$$

For $x \in \mathbb{R}^n$ and $\zeta \in \mathbb{C}$, set

$$f(x,\zeta) = \sum_{i,j=1}^n \frac{\partial}{\partial x^i}(I+\zeta\nabla F)^{\sim}_{ij}v_j.$$

Suppose that $x \in \mathbb{R}^n$ and $\zeta \in \mathbb{C}$ satisfy $\det(I+\zeta\nabla F(x)) \neq 0$. Since $\det(I+\zeta\nabla F(\cdot)) \neq 0$ in a neighborhood of x,

$$(I+\zeta\nabla F)^{\sim}_{ij} = \det(I+\zeta\nabla F)((I+\zeta\nabla F)^{-1})_{ij}.$$

Since $\frac{\partial}{\partial a_{pq}}\det A = \det A\,(A^{-1})_{qp}$ for $A = (a_{ij})_{i,j=1,\ldots,n}$, a straightforward computation yields

$$\sum_{i=1}^n \frac{\partial}{\partial x^i}(I+\zeta\nabla F)^{\sim}_{ij} = 0.$$

Hence, if $\det(I+\zeta\nabla F(x)) \neq 0$, then $f(x,\zeta) = 0$. For each $x \in \mathbb{R}^n$ there are at most n ζs such that $\det(I+\zeta\nabla F(x)) = 0$. Therefore we obtain $f(x,\zeta) \equiv 0$ and (5.6.8). □

Lemma 5.6.4 *Let $F \in \mathscr{D}^{\infty,\infty-}(\mathbb{R}^n;\mathbb{R}^n)$. Suppose that there exist $\gamma > 0$ and $q > \frac{1}{2}$ such that*

$$\mathrm{e}^{-\nabla^* F + q\|\nabla F\|^2} \in L^{1+\gamma}(\nu_n). \tag{5.6.9}$$

Then, for any $v \in \mathbb{R}^n$,

$$\Lambda_F(I+\nabla F)^{-1}v \in L^{1+\gamma}(\nu_n), \quad \Lambda_F\langle \cdot + F, v\rangle \in \bigcap_{p\in(1,1+\gamma)} L^p(\nu_n).$$

Moreover, for any $G \in \mathscr{D}^{\infty,\infty-}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}\langle \nabla G, \Lambda_F(I+\nabla F)^{-1}v\rangle\,\mathrm{d}\nu_n = \int_{\mathbb{R}^n} G\Lambda_F\langle \cdot + F, v\rangle\,\mathrm{d}\nu_n.$$

Proof By Lemma 5.6.3, it suffices to show the integrability of the first two Wiener functionals.

First we show, for $A \in \mathbb{R}^n \otimes \mathbb{R}^n$,

$$\left\|\det_2(I+A)\{(I+A)^{-1}-I\}\right\| \leqq \exp\Bigl(\frac{1}{2}(\|A\|+1)^2\Bigr). \tag{5.6.10}$$

Since $\zeta \mapsto \det_2(I+A+\zeta B)$ $(B \in \mathbb{R}^{n\times n})$ is holomorphic and

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\Big|_{\zeta=0}\det_2(I+A+\zeta B) = \det_2(I+A)\mathrm{tr}[\{(I+A)^{-1}-I\}B],$$

by Cauchy's integral formula and (5.6.4), we obtain

$$\begin{aligned}\left|\det_2(I+A)\mathrm{tr}[\{(I+A)^{-1}-I\}B]\right| &= \left|\frac{1}{2\pi}\int_0^{2\pi}\frac{\det_2(I+A+\mathrm{e}^{\mathrm{i}s}B)}{\mathrm{e}^{\mathrm{i}s}}\,\mathrm{d}s\right| \\ \leqq \frac{1}{2\pi}\int_0^{2\pi}\exp\Bigl(\frac{1}{2}\|A+\mathrm{e}^{\mathrm{i}s}B\|^2\Bigr)\mathrm{d}s &\leqq \exp\Bigl(\frac{1}{2}(\|A\|+\|B\|)^2\Bigr).\end{aligned}$$

Hence, since $\|T\| = \sup\{|\mathrm{tr}(TB)| \mid \|B\| \leqq 1\}$, we have (5.6.10).

Second, by using (5.6.4), (5.6.10), and an elementary inequality $\frac{1}{2}(a+1)^2 \leqq qa^2 + \frac{q}{2q-1}$ $(a>0)$, we obtain

$$\begin{aligned}\|\Lambda_F(I+\nabla F)^{-1}v\| &\leqq 2\mathrm{e}^{-\nabla^*F-\frac{1}{2}\|F\|^2+q\|\nabla F\|^2+\frac{q}{2q-1}}\|v\| \\ &\leqq 2\mathrm{e}^{-\nabla^*F+q\|\nabla F\|^2+\frac{q}{2q-1}}\|v\|.\end{aligned}$$

This implies the first assertion.

Since $\langle\cdot+F,v\rangle \in L^{\infty-}(\nu_n)$, the second assertion follows from (5.6.4) and the assumption. □

Lemma 5.6.5 *If* $F \in \mathscr{D}^{\infty,\infty-}(\mathbb{R}^n;\mathbb{R}^n)$ *satisfies* (5.6.9), *then*

$$\int_{\mathbb{R}^n} f(x+F(x))\Lambda_F(x)\nu_n(\mathrm{d}x) = \int_{\mathbb{R}^n}\Lambda_F\,\mathrm{d}\nu_n\int_{\mathbb{R}^n} f\,\mathrm{d}\nu_n \tag{5.6.11}$$

for any $f \in C_b(\mathbb{R}^n)$.

Proof For $v \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, set $f_\lambda(x) = \exp(\mathrm{i}\,\lambda\langle x,v\rangle)$. Since

$$\langle\nabla(f_\lambda(\cdot+F)),k\rangle = \mathrm{i}\,\lambda\langle v,(I+\nabla F)k\rangle f_\lambda(\cdot+F) \qquad (k \in \mathbb{R}^n),$$

setting $k = \Lambda_F(I+\nabla F)^{-1}v$, we have

$$\langle\nabla(f_\lambda(\cdot+F)),\Lambda_F(I+\nabla F)^{-1}v\rangle = \mathrm{i}\,\lambda\|v\|^2 f_\lambda(\cdot+F)\Lambda_F.$$

Combining this identity with Lemma 5.6.3, we obtain

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$$\begin{aligned}
&\frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\lambda}\int_{\mathbb{R}^n} f_\lambda(x+F(x))\Lambda_F(x)\nu_n(\mathrm{d}x)\\
&\quad=\int_{\mathbb{R}^n} f_\lambda(x+F(x))\Lambda_F(x)\langle x+F(x),v\rangle\nu_n(\mathrm{d}x)\\
&\quad=\mathrm{i}\,\lambda\|v\|^2\int_{\mathbb{R}^n} f_\lambda(x+F(x))\Lambda_F(x)\nu_n(\mathrm{d}x).
\end{aligned}$$

Solving this ordinary differential equation, we arrive at

$$\int_{\mathbb{R}^n} f_\lambda(x+F(x))\Lambda_F(x)\nu_n(\mathrm{d}x)=\mathrm{e}^{-\frac{1}{2}\lambda^2\|v\|^2}\int_{\mathbb{R}^n}\Lambda_F(x)\nu_n(\mathrm{d}x).$$

Since $\mathrm{e}^{-\frac{1}{2}\lambda^2\|v\|^2}=\int_{\mathbb{R}^n} f_\lambda\,\mathrm{d}\nu_n$, we obtain (5.6.11).

For general $f\in C_{\mathrm{b}}(\mathbb{R}^n)$, approximating it by elements in $\mathscr{S}(\mathbb{R}^n)$ and expressing elements of $\mathscr{S}(\mathbb{R}^n)$ in terms of Fourier transforms, we obtain the assertion from the identity above. □

Lemma 5.6.6 *Suppose that* $F\in\mathscr{D}^{\infty,\infty-}(\mathbb{R}^n;\mathbb{R}^n)$ *satisfies* (5.6.9). *Then,*

$$\int_{\mathbb{R}^n} f(x+F(x))\det{}_2(I+\nabla F(x))\mathrm{e}^{-\nabla^*F(x)-\frac{1}{2}\|F(x)\|^2}\nu_n(\mathrm{d}x)=\int_{\mathbb{R}^n} f\,\mathrm{d}\nu_n \tag{5.6.12}$$

for any $f\in C_{\mathrm{b}}(\mathbb{R}^n)$.

Proof Let $t\in[0,1]$. Since $a^t\leqq 1+a$ $(a\geqq 0)$,

$$\mathrm{e}^{-\nabla^*(tF)+q\|\nabla(tF)\|^2}\leqq \mathrm{e}^{t(-\nabla^*F+q\|\nabla F\|^2)}\leqq 1+\mathrm{e}^{-\nabla^*F+q\|\nabla F\|^2}.$$

Hence tF also satisfies (5.6.9) and the mapping $t\mapsto\int_{\mathbb{R}^n}\Lambda_{tF}\mathrm{d}\nu_n$ is continuous. If $\int_{\mathbb{R}^n}\Lambda_{tF}\mathrm{d}\nu_n\in\mathbb{Z}$ $(t\in[0,1])$, then $\int_{\mathbb{R}^n}\Lambda_{tF}\mathrm{d}\nu_n=\int_{\mathbb{R}^n}\Lambda_0\mathrm{d}\nu_n=1$ and we obtain (5.6.12) by Lemma 5.6.5. Hence we show $\int_{\mathbb{R}^n}\Lambda_F\mathrm{d}\nu_n\in\mathbb{Z}$.

Let $\{E_k\}_{k=1}^\infty$ be a sequence of disjoint Borel sets such that

$$\bigcup_{k=1}^\infty E_k=\{x\in\mathbb{R}^n\,;\,\det(I+\nabla F(x))\neq 0\}$$

and, in a neighborhood of each E_k, the mapping $x\mapsto T(x)=x+F(x)$ is a diffeomorphism. By the change of variables formula with respect to the Lebesgue measure, we have

$$\begin{aligned}
&\int_{E_k} f(T(x))|\Lambda_F(x)|\nu_n(\mathrm{d}x)\\
&\quad=\int_{E_k} f(T(x))|\det(\nabla T(x))|(2\pi)^{-\frac{n}{2}}\mathrm{e}^{-\frac{1}{2}\|T(x)\|^2}\mathrm{d}x\\
&\quad=\int_{T(E_k)} f(x)\nu_n(\mathrm{d}x)\qquad (k=1,2,\dots,\ f\in C_{\mathrm{b}}(\mathbb{R}^n)).
\end{aligned}\tag{5.6.13}$$

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Since $\Lambda_F(x) = 0$ if $\det(I + \nabla F(x)) = 0$, this implies

$$\int_{\mathbb{R}^n} f(T)|\Lambda_F|\,\mathrm{d}\nu_n = \int_{\{\det(I+\nabla F)\neq 0\}} f(T)|\Lambda_F|\,\mathrm{d}\nu_n = \sum_{k=1}^{\infty}\int_{T(E_k)} f\,\mathrm{d}\nu_n$$

for any $f \in C_{\mathrm{b}}(\mathbb{R}^n)$. Setting $f = 1$, we obtain

$$\int_{\mathbb{R}^n}\Big(\sum_{k=1}^{\infty}\mathbf{1}_{T(E_k)}\Big)\mathrm{d}\nu_n < \infty. \tag{5.6.14}$$

Denote by $s_k \in \{\pm 1\}$ the signature of $\det(I + \nabla F)$ on E_k. By Lemma 5.6.5 and (5.6.13), we have

$$\int_{\mathbb{R}^n}\Lambda_F \mathrm{d}\nu_n \int_{\mathbb{R}^n} f\,\mathrm{d}\nu_n = \int_{\mathbb{R}^n} f(\cdot + F)\Lambda_F \mathrm{d}\nu_n = \sum_{k=1}^{\infty} s_k \int_{T(E_k)} f\,\mathrm{d}\nu_n$$

for any $f \in C_{\mathrm{b}}(\mathbb{R}^n)$. The sum $\sum_{k=1}^{\infty} s_k\mathbf{1}_{T(E_k)}$ is dominated by $\sum_{k=1}^{\infty}\mathbf{1}_{T(E_k)}$ and, by (5.6.14), converges absolutely ν_n-a.e. Hence we have

$$\int_{\mathbb{R}^n}\Lambda_F \mathrm{d}\nu_n \int_{\mathbb{R}^n} f\,\mathrm{d}\nu_n = \int_{\mathbb{R}^n}\Big(\sum_{k=1}^{\infty} s_k\mathbf{1}_{T(E_k)}(x)\Big) f(x)\,\nu_n(\mathrm{d}x)$$

for any $f \in C_{\mathrm{b}}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n}\Lambda_F \mathrm{d}\nu_n = \sum_{k=1}^{\infty} s_k\mathbf{1}_{T(E_k)}, \qquad \nu_n\text{-a.e.}$$

In particular, $\int_{\mathbb{R}^n}\Lambda_F \mathrm{d}\nu_n \in \mathbb{Z}$. □

We extend the identity (5.6.12) on $\mathbb{R}^n$ to that on W_T. For this purpose we prepare some notation. Let $\{\ell_i\}_{i=1}^{\infty} \subset W_T^*$ be an orthonormal basis of H_T. For each $n \in \mathbb{N}$, let $\mathscr{G}_n$ be the σ-field generated by the random variables $\ell_1, \ldots, \ell_n$: $\mathscr{G}_n = \sigma(\ell_1, \ldots, \ell_n)$. Define the projection $\pi_n : W_T \to W_T^* \subset H_T \subset W_T$ by

$$\pi_n w = \sum_{j=1}^{n} \ell_j(w)\ell_j \qquad (w \in W_T).$$

Moreover, for $j \in \mathbb{N}$, define $\pi_n^{\otimes j} : H_T^{\otimes j} \to H_T^{\otimes j}$ by

$$\pi_n^{\otimes j}(h_1 \otimes \cdots \otimes h_j) = \pi_n h_1 \otimes \cdots \otimes \pi_n h_j.$$

Denote by $\mathbf{E}_n$ the conditional expectation given $\mathscr{G}_n$, $\mathbf{E}_n(F) = \mathbf{E}[F|\mathscr{G}_n]$, and extend it to the $H_T^{\otimes j}$-valued random variable $G \in L^2(\mu_T; H_T^{\otimes j})$ by

$$\mathbf{E}_n(G) = \sum_{i=1}^{\infty}\mathbf{E}_n(\langle G, \psi_i\rangle_{H_T^{\otimes j}})\psi_i, \tag{5.6.15}$$

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where $\{\psi_i\}_{i=1}^\infty$ is an orthonormal basis of $H_T^{\otimes j}$. The above infinite sum converges in $H_T^{\otimes j}$ almost surely and in the L^2-sense. Specifically, since $\{\mathbf{E}_n(G)\}_{n=1}^\infty$ is a discrete time martingale, by the monotone convergence theorem, Doob's inequality, and Jensen's inequality, we have

$$\begin{aligned}\int_{W_T} \sup_{n\in\mathbb{N}}(\mathbf{E}_n(\langle G,\psi_i\rangle_{H_T^{\otimes j}}))^2 \mathrm{d}\mu_T &= \lim_{m\to\infty}\int_{W_T}\max_{n\leqq m}(\mathbf{E}_n(\langle G,\psi_i\rangle_{H_T^{\otimes j}}))^2\mathrm{d}\mu_T \\ &\leqq 4\limsup_{m\to\infty}\int_{W_T}(\mathbf{E}_m(\langle G,\psi_i\rangle_{H_T^{\otimes j}}))^2\mathrm{d}\mu_T \\ &\leqq 4\limsup_{m\to\infty}\int_{W_T}\mathbf{E}_m(\langle G,\psi_i\rangle_{H_T^{\otimes j}}^2)\,\mathrm{d}\mu_T \\ &= 4\int_{W_T}\langle G,\psi_i\rangle_{H_T^{\otimes j}}^2\mathrm{d}\mu_T \qquad (i\in\mathbb{N}).\end{aligned}$$

Hence we obtain

$$\int_{W_T}\sum_{i=1}^\infty\Bigl(\sup_{n\in\mathbb{N}}\Bigl(\mathbf{E}_n(\langle G,\psi_i\rangle_{H_T^{\otimes j}})\Bigr)^2\Bigr)\mathrm{d}\mu_T \leqq 4\int_{W_T}\|G\|_{H_T^{\otimes j}}^2\mathrm{d}\mu_T. \tag{5.6.16}$$

The almost sure and L^2-convergence of the right hand side of (5.6.15) follows from this estimate.

Lemma 5.6.7 (1) $\mathbf{E}_n(G)$ *is an* $H_T^{\otimes j}$*-valued random variable, unique up to* μ_T*-null sets, such that*

$$\int_{W_T}\langle G,G'\rangle_{H_T^{\otimes j}}\mathrm{d}\mu_T = \int_{W_T}\langle\mathbf{E}_n(G),G'\rangle_{H_T^{\otimes j}}\mathrm{d}\mu_T$$

for any $\mathscr{G}_n$*-measurable* $G'\in L^2(\mu_T;H_T^{\otimes j})$*. In particular,* $\mathbf{E}_n(G)$ *is independent of the choice of the orthonormal basis* $\{\psi_i\}_{i=1}^\infty$.
(2) *For any* $G,K\in L^2(\mu_T;H_T^{\otimes j})$,

$$\begin{aligned}\int_{W_T}\langle\mathbf{E}_n(G),K\rangle_{H_T^{\otimes j}}\mathrm{d}\mu_T &= \int_{W_T}\langle\mathbf{E}_n(G),\mathbf{E}_n(K)\rangle_{H_T^{\otimes j}}\mathrm{d}\mu_T \\ &= \int_{W_T}\langle G,\mathbf{E}_n(K)\rangle_{H_T^{\otimes j}}\mathrm{d}\mu_T.\end{aligned}\tag{5.6.17}$$

Moreover, for $G\in L^p(\mu_T;H_T^{\otimes j})$, $p\geqq 1$,

$$\|\mathbf{E}_n(G)\|_{H_T^{\otimes j}}^p \leqq \mathbf{E}_n(\|G\|_{H_T^{\otimes j}}^p). \tag{5.6.18}$$

(3) *The following convergence holds:*

$$\lim_{n\to\infty}\int_{W_T}\|\mathbf{E}_n(\pi_nG)-G\|_{H_T^{\otimes j}}^2\mathrm{d}\mu_T = 0.$$

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(4) *Let $G \in \mathscr{P}$. Then, for μ_T-a.s. $w \in W_T$,*

$$\mathbf{E}_n(G)(w) = \int_{W_T} G(\pi_n w + (1 - \pi_n)w')\mu_T(\mathrm{d}w'). \tag{5.6.19}$$

Proof (1) Let $G' \in L^2(\mu_T; H_T^{\otimes j})$ be $\mathscr{G}_n$-measurable. By the expansion with respect to $\{\psi_i\}_{i=1}^\infty$,

$$\langle \mathbf{E}_n(G), G'\rangle_{H_T^{\otimes j}} = \sum_{i=1}^\infty \mathbf{E}_n(\langle G, \psi_i\rangle_{H_T^{\otimes j}})\langle G', \psi_i\rangle_{H_T^{\otimes j}}.$$

By the identities

$$\sum_{i=1}^\infty \langle G', \psi_i\rangle_{H_T^{\otimes j}}^2 = \|G'\|_{H_T^{\otimes j}}^2, \quad \sum_{i=1}^\infty \langle G, \psi_i\rangle_{H_T^{\otimes j}}^2 = \|G\|_{H_T^{\otimes j}}^2$$

and (5.6.16), we can apply Lebesgue's convergence theorem to obtain the desired equality

$$\begin{aligned}
\int_{W_T} \langle \mathbf{E}_n(G), G'\rangle_{H_T^{\otimes j}} \mathrm{d}\mu_T &= \sum_{i=1}^\infty \int_{W_T} \mathbf{E}_n(\langle G, \psi_i\rangle_{H_T^{\otimes j}})\langle G', \psi_i\rangle_{H_T^{\otimes j}} \mathrm{d}\mu_T \\
&= \sum_{i=1}^\infty \int_{W_T} \langle G, \psi_i\rangle_{H_T^{\otimes j}}\langle G', \psi_i\rangle_{H_T^{\otimes j}} \mathrm{d}\mu_T \\
&= \int_{W_T} \langle G, G'\rangle_{H_T^{\otimes j}} \mathrm{d}\mu_T.
\end{aligned}$$

because $\langle G', \psi_i\rangle_{H_T^{\otimes j}}$ is $\mathscr{G}_n$-measurable.

The uniqueness is shown in the same way as the usual conditional expectation.

(2) Set $G = K$ and $G' = \mathbf{E}_n(G)$ in (1) to obtain the first identity of (5.6.17). The second identity is obtained by changing G and K in the first one.

Next we show (5.6.18). It suffices to prove it in the case when $p = 1$ because the general case is obtained by Jensen's inequality $(\mathbf{E}_n(X))^p \leqq \mathbf{E}_n(X^p)$ for conditional expectations.

Let $g \in H_T^{\otimes j}$. For any $\mathscr{G}_n$-measurable $\phi \in L^2(\mu_T)$, set $G' = \phi \cdot g$ in the identity described in (1). Then we have

$$\int_{W_T} \langle \mathbf{E}_n(G), g\rangle_{H_T^{\otimes j}} \phi \,\mathrm{d}\mu_T = \int_{W_T} \langle G, g\rangle_{H_T^{\otimes j}} \phi \,\mathrm{d}\mu_T.$$

Hence

$$\langle \mathbf{E}_n(G), g\rangle_{H_T^{\otimes j}} = \mathbf{E}_n(\langle G, g\rangle_{H_T^{\otimes j}}).$$

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In particular, since $|\langle G, g\rangle_{H_T^{\otimes j}}| \leqq \|G\|_{H_T^{\otimes j}}\|g\|_{H_T^{\otimes j}}$, we obtain

$$|\langle \mathbf{E}_n(G), g\rangle_{H_T^{\otimes j}}| \leqq \mathbf{E}_n(\|G\|_{H_T^{\otimes j}})\|g\|_{H_T^{\otimes j}}. \tag{5.6.20}$$

Since the space $H_T^{\otimes j}$ is separable, there exists a countable sequence $\{g_i\}_{i=1}^{\infty}$ with $\|g_i\|_{H_T^{\otimes j}} \leqq 1$ such that

$$\|\xi\|_{H_T^{\otimes j}} = \sup_{i\in\mathbb{N}} |\langle \xi, g_i\rangle_{H_T^{\otimes j}}| \qquad (\xi \in H_T^{\otimes j}).$$

Combining this with (5.6.20), we obtain (5.6.18) when $p = 1$.
(3) By the linearity of $\mathbf{E}_n$,

$$\int_{W_T} \left\|\mathbf{E}_n(\pi_n G) - G\right\|^2_{H_T^{\otimes j}} \mathrm{d}\mu_T \\ \leqq 2\int_{W_T} \left\|\mathbf{E}_n(\pi_n G - G)\right\|^2_{H_T^{\otimes j}} \mathrm{d}\mu_T + 2\int_{W_T} \left\|\mathbf{E}_n(G) - G\right\|^2_{H_T^{\otimes j}} \mathrm{d}\mu_T.$$

Using (5.6.18) for $p = 2$, we obtain

$$\int_{W_T} \left\|\mathbf{E}_n(\pi_n G - G)\right\|^2_{H_T^{\otimes j}} \mathrm{d}\mu_T \leqq \int_{W_T} \left\|\pi_n G - G\right\|^2_{H_T^{\otimes j}} \mathrm{d}\mu_T \to 0 \quad (n \to \infty).$$

By (5.6.16), the martingale convergence theorem (Theorem 1.4.21) and the dominated convergence theorem, we have

$$\int_{W_T} \left\|\mathbf{E}_n(G) - G\right\|^2_{H_T^{\otimes j}} \mathrm{d}\mu_T = \int_{W_T} \sum_{i=1}^{\infty} \{\mathbf{E}_n(\langle G, \psi_i\rangle_{H_T^{\otimes j}}) - \langle G, \psi_i\rangle_{H_T^{\otimes j}}\}^2 \mathrm{d}\mu_T \\ \longrightarrow 0 \quad (n \to \infty).$$

(4) Let $G \in \mathscr{P}$ be of the form

$$G(w) = f(\eta_1(w), \ldots, \eta_m(w)) \qquad (w \in W_T)$$

with a polynomial $f : \mathbb{R}^m \to \mathbb{R}$ and $\eta_1, \ldots, \eta_m \in W_T^*$. By the embedding $W_T^* \subset H_T^* = H_T \subset W_T$, we have

$$\eta_i(\ell_j) = \langle \eta_i, \ell_j\rangle_{H_T} = \ell_j(\eta_i).$$

Hence, for any $w \in W_T$,

$$\eta_i(\pi_n w) = (\pi_n \eta_i)(w), \quad \eta_i((I - \pi_n)w) = ((I - \pi_n)\eta_i)(w). \tag{5.6.21}$$

Since $\langle \pi_n \eta_i, (I - \pi_n)\eta_j\rangle_{H_T} = 0$, $\{\eta_i \circ \pi_n\}_{i=1}^m$ and $\{\eta_i \circ (I - \pi_n)\}_{i=1}^m$ are independent. Define a polynomial $\widetilde{f}$ by

$$\widetilde{f}(x_1, \ldots, x_m) = \int_{W_T} f(x_1 + \eta_1((I - \pi_n)w'), \ldots, x_m + \eta_m((I - \pi_n)w'))\mu_T(\mathrm{d}w').$$

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Then, since $\{\eta_i \circ \pi_n\}_{i=1}^m$ is $\mathscr{G}_n$-measurable, by Proposition 1.4.2, we obtain

$$\begin{aligned}\mathbf{E}_n(G) &= \mathbf{E}[f(\eta_1 \circ \pi_n + \eta_1 \circ (I - \pi_n), \ldots, \eta_m \circ \pi_n + \eta_m \circ (I - \pi_n)) \big| \mathscr{G}_n] \\ &= \widetilde{f}(\eta_1 \circ \pi_n, \ldots, \eta_m \circ \pi_n) = \int_{W_T} G(\pi_n \cdot + (I - \pi_n) w') \mu_T(\mathrm{d}w'). \end{aligned}$$ □

Lemma 5.6.8 *Let* $F \in \mathbb{D}^{\infty,\infty-}(H_T^{\otimes j})$ *and* $F_1 \in \mathbb{D}^{\infty,\infty-}(H_T)$. *Then* $\mathbf{E}_n(\pi_n^{\otimes j} F) \in \mathbb{D}^{\infty,\infty-}(H_T^{\otimes j})$ *and*

$$\nabla(\mathbf{E}_n(\pi_n^{\otimes j} F)) = \pi_n^{\otimes j+1}(\mathbf{E}_n(\nabla F)) = \mathbf{E}_n(\pi_n^{\otimes j+1}(\nabla F)), \tag{5.6.22}$$

$$\nabla^*(\mathbf{E}_n(\pi_n F_1)) = \mathbf{E}_n(\nabla^* F_1), \tag{5.6.23}$$

$$\big\|\nabla(\mathbf{E}_n(\pi_n F_1))\big\|_{H_T^{\otimes 2}}^2 \leqq \mathbf{E}_n(\big\|\nabla F_1\big\|_{H_T^{\otimes 2}}^2). \tag{5.6.24}$$

Proof First let $F \in \mathscr{P}(H_T^{\otimes j})$. We prove $\mathbf{E}_n(\pi_n^{\otimes j} F)$ belongs to $\mathscr{P}(H_T^{\otimes j})$ and (5.6.22) holds. To do this, it suffices to show it in the case where $F = Ge$ for the same $G \in \mathscr{P}$ as in the proof of Lemma 5.6.7 (4) and $e \in H_T^{\otimes j}$. We use the same notation as in the proof of Lemma 5.6.7 (4).

Then, since $\mathbf{E}_n(G) = \widetilde{f}(\eta_1 \circ \pi_n, \ldots, \eta_m \circ \pi_n) \in \mathscr{P}$, we have $\mathbf{E}_n(\pi_n^{\otimes j} F) = \mathbf{E}_n(G) \pi_n^{\otimes j} e \in \mathscr{P}(H_T^{\otimes j})$. Hence, by Lemma 5.6.7 (4),

$$\begin{aligned}\nabla(\mathbf{E}_n(\pi_n^{\otimes j} F)) &= \nabla(\mathbf{E}_n(G)) \pi_n^{\otimes j} e \\ &= \sum_{i=1}^m \frac{\partial \widetilde{f}}{\partial x^i}(\eta_1 \circ \pi_n, \ldots, \eta_m \circ \pi_n)(\eta_i \circ \pi_n) \otimes \pi_n^{\otimes j} e.\end{aligned}$$

On the other hand, since $\pi_n^{\otimes j+1}(\nabla F) = \sum_{i=1}^m \frac{\partial f}{\partial x_i}(\eta_1, \ldots, \eta_m)(\pi_n \eta_i) \otimes \pi_n^{\otimes j} e$, using again Lemma 5.6.7 (4), we obtain

$$\mathbf{E}_n(\pi_n^{\otimes j+1}(\nabla F)) = \sum_{i=1}^m \frac{\partial \widetilde{f}}{\partial x^i}(\eta_1 \circ \pi_n, \ldots, \eta_m \circ \pi_n)(\pi_n \eta_i) \otimes \pi_n^{\otimes j} e.$$

By (5.6.21), (5.6.22) holds for $F \in \mathscr{P}(H_T^{\otimes j})$.

Second, let $F \in \mathbb{D}^{\infty,\infty-}(H_T^{\otimes j})$. For $k \in \mathbb{N}$, $p > 1$, take $F_m \in \mathscr{P}(H_T^{\otimes j})$ so that $\lim_{m\to\infty} \|F_m - F\|_{(k,p)} = 0$ (see Definition 5.1.4). Then, applying (5.6.22) to F_m and using (5.6.18), we obtain for $\ell \leqq k$

$$\begin{aligned}&\big\|\nabla^\ell(\mathbf{E}_n(\pi_n^{\otimes j} F_m)) - \mathbf{E}_n(\pi_n^{\otimes j+\ell}(\nabla^\ell F))\big\|_p \\ &\quad \leqq \big\|\mathbf{E}_n(\pi_n^{\otimes j+\ell}(\nabla^\ell F_m)) - \mathbf{E}_n(\pi_n^{\otimes j+\ell}(\nabla^\ell F))\big\|_p \\ &\quad \leqq \big\|\nabla^\ell F_m - \nabla^\ell F\big\|_p \longrightarrow 0 \quad (m \to \infty).\end{aligned}$$

Hence, we have $\mathbf{E}_n(\pi_n^{\otimes j} F) \in \mathbb{D}^{k,p}(H_T^{\otimes j})$ and (5.6.22). Since k and p are arbitrary, $F \in \mathbb{D}^{\infty,\infty-}(H_T^{\otimes j})$.

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Third, we show (5.6.23). Let $K \in \mathscr{P}$. By the symmetry of $\mathbf{E}_n$ mentioned in (5.6.17), the symmetry of π_n in H_T, the commutativity of $\mathbf{E}_n$ and π_n, and (5.6.22) for $j = 0$, we have

$$\begin{aligned}\int_{W_T} K\nabla^*(\mathbf{E}_n(\pi_n F_1))\,\mathrm{d}\mu_T &= \int_{W_T} \langle \nabla K, \mathbf{E}_n(\pi_n F_1)\rangle_{H_T}\,\mathrm{d}\mu_T\\ &= \int_{W_T} \langle \pi_n(\mathbf{E}_n(\nabla K)), F_1\rangle_{H_T}\,\mathrm{d}\mu_T = \int_{W_T} \langle \mathbf{E}_n(\pi_n(\nabla K)), F_1\rangle_{H_T}\,\mathrm{d}\mu_T\\ &= \int_{W_T} \langle \nabla(\mathbf{E}_n K), F_1\rangle_{H_T}\,\mathrm{d}\mu_T = \int_{W_T} K\mathbf{E}_n(\nabla^* F_1)\,\mathrm{d}\mu_T.\end{aligned}$$

Thus we obtain (5.6.23).

Finally, we show (5.6.24). By (5.6.22) and (5.6.18), we obtain

$$\begin{aligned}\big\|\nabla(\mathbf{E}_n(\pi_n F_1))\big\|^2_{H_T^{\otimes 2}} &= \big\|\mathbf{E}_n(\pi_n^{\otimes 2}(\nabla F_1))\big\|^2_{H_T^{\otimes 2}}\\ &\leqq \mathbf{E}_n\big(\big\|\pi_n^{\otimes 2}(\nabla F_1)\big\|^2_{H_T^{\otimes 2}}\big) \leqq \mathbf{E}_n\big(\big\|\nabla F_1\big\|^2_{H_T^{\otimes 2}}\big).\end{aligned} \qquad \square$$

Proof of Theorem 5.6.1 By (5.6.2), there exist $\gamma > 0$ and $q > \frac{1}{2}$ such that

$$\mathrm{e}^{-\nabla^* F + q\|\nabla F\|^2_{H_T^{\otimes 2}}} \in L^{1+\gamma}(\mu_T).$$

Let $\mathbf{E}_n$ and π_n be as above and set $F_n = \mathbf{E}_n(\pi_n F)$. By (5.6.23), (5.6.24), and Jensen's inequality for conditional expectations, we have

$$\begin{aligned}\int_{W_T} \mathrm{e}^{(1+\gamma)(-\nabla^* F_n + q\|\nabla F_n\|^2_{H_T^{\otimes 2}})}\mathrm{d}\mu_T &\leqq \int_{W_T} \mathrm{e}^{\mathbf{E}_n((1+\gamma)\{-\nabla^* F + q\|\nabla F\|^2_{H_T^{\otimes 2}}\})}\mathrm{d}\mu_T\\ &\leqq \int_{W_T} \mathbf{E}_n\Big(\mathrm{e}^{(1+\gamma)\{-\nabla^* F + q\|\nabla F\|^2_{H_T^{\otimes 2}}\}}\Big)\mathrm{d}\mu_T\\ &= \int_{W_T} \mathrm{e}^{(1+\gamma)\{-\nabla^* F + q\|\nabla F\|^2_{H_T^{\otimes 2}}\}}\mathrm{d}\mu_T.\end{aligned} \tag{5.6.25}$$

Identify $\pi_n(W_T)$ with $\mathbb{R}^n$ in a natural way. By Lemma 5.6.8, applying the Sobolev embedding theorem ([1]), we may regard $F_n \in \mathscr{D}^{\infty,\infty-}(\mathbb{R}^n;\mathbb{R}^n)$. By (5.6.25), F_n satisfies (5.6.9). Thus, by Lemma 5.6.6, (5.6.3) holds for $F = F_n$.

By the martingale convergence theorem (Theorem 1.4.21), Lemma 5.6.7 (3), (5.6.22), and (5.6.23), we may suppose that F_n, ∇F_n, and $\nabla^* F_n$ converges almost surely to F, ∇F, and $\nabla^* F$, respectively, taking a subsequence if necessary. By (5.6.4), we have

$$|\det{}_2(I + \nabla F_n)|\mathrm{e}^{-\nabla^* F_n - \frac{1}{2}\|F_n\|^2_{H_T}} \leqq \mathrm{e}^{-\nabla^* F_n + \frac{1}{2}\|\nabla F_n\|^2_{H_T^{\otimes 2}}}.$$

By (5.6.25),

$$\Big\{f(\iota + F_n)\det{}_2(I + \nabla F_n)\exp\Big(-\nabla^* F_n - \frac{1}{2}\big\|F_n\big\|^2_{H_T}\Big)\Big\}_{n\in\mathbb{N}}$$

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is uniformly integrable (Theorem A.3.4). Hence, letting $n \to \infty$ in (5.6.3) for $F = F_n$, we obtain the desired identity (5.6.3) for F. □

5.7 Quadratic Forms

As in other fields of analysis, quadratic forms on the Wiener space play fundamental roles in stochastic analysis. In this section we show a general theory on quadratic Wiener functionals and, in the next section, we present concrete examples.

Definition 5.7.1 Regarding a symmetric $A \in H_T^{\otimes 2}$ as an $H_T^{\otimes 2}$-valued constant function on W_T, set

$$Q_A = (\nabla^*)^2 A, \quad L_A = \nabla^* A.$$

Q_A is called a **quadratic form** associated with A.

By Theorem 5.2.1, $Q_A \in \mathbb{D}^{\infty,\infty-}$ and $L_A \in \mathbb{D}^{\infty,\infty-}(H_T)$.

Since the Hilbert–Schmidt operators are compact operators, by the spectral decomposition for compact operators ([58]), A is diagonalized as

$$A = \sum_{n=1}^{\infty} a_n h_n \otimes h_n, \tag{5.7.1}$$

where $\{h_n\}_{n=1}^{\infty}$ is an orthonormal basis of H_T and $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers with $\sum_{n=1}^{\infty} a_n^2 < \infty$. By using this decomposition, Q_A and L_A are represented as infinite sums.

Lemma 5.7.2 (1) *The following convergence in the L^2-sense holds:*

$$Q_A = \sum_{n=1}^{\infty} a_n \{(\nabla^* h_n)^2 - 1\} \quad \textit{and} \quad L_A = \sum_{n=1}^{\infty} a_n (\nabla^* h_n) h_n.$$

(2) *Set $\|A\|_{\mathrm{op}} = \sup_n |a_n|$. For $\lambda \in \mathbb{R}$ with $|\lambda|\,\|A\|_{\mathrm{op}} < \frac{1}{2}$, $\mathrm{e}^{\lambda Q_A} \in L^{1+}(\mu_T)$.*
(3) *For $\lambda \in \mathbb{R}$ with $|\lambda|\,\|A\|_{\mathrm{op}} < \frac{1}{2}$,*

$$\int_{W_T} \mathrm{e}^{\lambda Q_A} \mathrm{d}\mu_T = \{\det{}_2(I - 2\lambda A)\}^{-\frac{1}{2}}.$$

The aim of this section is to extend the assertion (3) to general integrals of the form $\int_{W_T} \mathrm{e}^{\lambda Q_A} f \,\mathrm{d}\mu_T$ ($f \in C_{\mathrm{b}}(W_T)$) (Theorem 5.7.6), applying the change of variables formula on W_T as shown in the previous section.

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Remark 5.7.3 Since $\{\nabla^* h_n\}_{n=1}^{\infty}$ is a sequence of independent standard Gaussian random variables, by the Itô–Nisio theorem (Theorem 1.2.5), we have

$$\theta = \sum_{n=1}^{\infty} (\nabla^* h_n) h_n.$$

Combining this with (5.7.1), we have a formal expression for Lemma 5.7.2(1):

$$Q_A = \langle \theta, A\theta \rangle_{H_T} - \mathrm{tr}(A).$$

While this expression is "$\infty - \infty$" in general because $H_T \subsetneq W_T$ and A is not necessarily of trace class, it suggests the origin of the name of quadratic forms associated with A.

Proof of Lemma 5.7.2 (1) First we show

$$(\nabla^*)^2 (h \otimes g) = (\nabla^* h)(\nabla^* g) - \langle h, g \rangle_{H_T} \qquad (h, g \in H_T). \tag{5.7.2}$$

For this purpose, let E be a separable Hilbert space, $G \in \mathbb{D}^{\infty,\infty-}(H_T \otimes E)$ and $e \in E$. We have $\langle G, e \rangle_E \in \mathbb{D}^{\infty,\infty-}(H_T)$. Since

$$\begin{aligned} \int_{W_T} \langle \nabla^* G, e \rangle_E \phi \, \mathrm{d}\mu_T &= \int_{W_T} \langle G, \nabla(\phi \cdot e) \rangle_{H_T \otimes E} \mathrm{d}\mu_T \\ &= \int_{W_T} \langle G, (\nabla \phi) \otimes e \rangle_{H_T \otimes E} \mathrm{d}\mu_T = \int_{W_T} \langle \langle G, e \rangle_E, \nabla \phi \rangle_{H_T} \mathrm{d}\mu_T \end{aligned}$$

for any $\phi \in \mathscr{P}$,

$$\langle \nabla^* G, e \rangle_E = \nabla^* (\langle G, e \rangle_E).$$

Using this identity with $E = H_T$, we obtain

$$\langle \nabla^* (h \otimes g), h_n \rangle_{H_T} = (\nabla^* h) \langle g, h_n \rangle_{H_T} \qquad (n = 1, 2, \ldots).$$

Hence we have

$$\nabla^* (h \otimes g) = (\nabla^* h) g. \tag{5.7.3}$$

Since $\nabla(\nabla^* h) = h$ (Example 5.1.5), by Theorem 5.2.8, we obtain (5.7.2).

Setting $K_m = \sum_{n=1}^{m} a_n h_n \otimes h_n$, by (5.7.2) and (5.7.3), we have

$$\nabla^* K_m = \sum_{n=1}^{m} a_n (\nabla^* h_n) h_n \quad \text{and} \quad (\nabla^*)^2 K_m = \sum_{n=1}^{m} a_n \{(\nabla^* h_n)^2 - 1\}.$$

By the continuity of ∇^* and a similar argument to that in Example 5.4.3, we obtain the conclusion.

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(2) It suffices to show $e^{\lambda Q_A} \in L^1(\mu_T)$ for $\lambda \in \mathbb{R}$ with $|\lambda|\,\|A\|_{\mathrm{op}} < \frac{1}{2}$. By (1), there exists a subsequence $\{m_n\}_{n=1}^{\infty}$ such that

$$F_n = \sum_{k=1}^{m_n} a_k\{(\nabla^* h_k)^2 - 1\} \to Q_A, \quad \mu_T\text{-a.s.}$$

Since $\{\nabla^* h_n\}$ is a sequence of independent standard Gaussian random variables by (5.1.8), we have

$$\begin{aligned}\int_{W_T} e^{\lambda Q_A} d\mu_T &\leqq \liminf_{n\to\infty} \int_{W_T} e^{\lambda F_n} d\mu_T \\ &= \liminf_{n\to\infty} \prod_{k=1}^{m_n} (1 - 2\lambda a_k)^{-\frac{1}{2}} e^{-\lambda a_k} = \det{}_2(1 - 2\lambda A)^{-\frac{1}{2}} < \infty.\end{aligned}$$

(3) By the proof of (2), $\{e^{\lambda F_n}\}_{n=1}^{\infty}$ is uniformly integrable. Hence, in the same way as in (2), we have

$$\begin{aligned}\int_{W_T} e^{\lambda Q_A} d\mu_T &= \lim_{n\to\infty} \int_{W_T} e^{\lambda F_n} d\mu_T \\ &= \lim_{n\to\infty} \prod_{k=1}^{m_n} (1 - 2\lambda a_k)^{-\frac{1}{2}} e^{-\lambda a_k} = \det{}_2(1 - 2\lambda A)^{-\frac{1}{2}}.\end{aligned}$$

□

By Lemma 5.7.2 (1), $\nabla^3 Q_A = 0$. The converse is also true.

Proposition 5.7.4 *Let $F \in \mathbb{D}^{\infty,\infty-}$. If $\nabla^3 F = 0$, then there exist a symmetric operator $A \in H_T^{\otimes 2}$, $h \in H_T$, and $c \in \mathbb{R}$ such that*

$$F = c + \nabla^* h + \frac{1}{2} Q_A. \tag{5.7.4}$$

Moreover,

$$c = \int_{W_T} F \, d\mu_T \quad \text{and} \quad h = \int_{W_T} \nabla F \, d\mu_T. \tag{5.7.5}$$

Proof By Proposition 5.2.9, there exists an $A \in H_T^{\otimes 2}$ such that $\nabla^2 F = A$. In particular, A is symmetric.

Set $F_1 = F - \frac{1}{2} Q_A$. Then, by Lemma 5.7.2, $\nabla^2 F_1 = 0$. By Proposition 5.2.9 again, there exists an $h \in H_T$ such that $\nabla F_1 = h$.

Next set $F_2 = F_1 - \nabla^* h$. By Example 5.1.5, $\nabla F_2 = 0$. Hence, there exists a $c \in \mathbb{R}$ such that $F_2 = c$. From these observations, we obtain (5.7.4).

By Lemma 5.7.2 and Example 5.1.5, we have

$$\int_{W_T} Q_A \, d\mu_T = 0, \quad \int_{W_T} \nabla Q_A \, d\mu_T = 0, \quad \int_{W_T} \nabla^* h \, d\mu_T = 0.$$

Hence, (5.7.5) follows from (5.7.4). □

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Remark 5.7.5 From the expression (5.7.4), we can prove that the distribution of F is infinitely divisible ([101]) and can compute the corresponding Lévy measure. For details, see [82].

Develop a symmetric operator $A \in H_T^{\otimes 2}$ as (5.7.1). For $\lambda \in \mathbb{R}$ with $2|\lambda|\,\|A\|_{\mathrm{op}} < 1$, set

$$s_n^{A,\lambda} = (1-2\lambda a_n)^{-\frac{1}{2}} - 1 \quad \text{and} \quad S^{A,\lambda} = \sum_{n=1}^{\infty} s_n^{A,\lambda} h_n \otimes h_n.$$

Since

$$|s_n^{A,\lambda}| \leqq \frac{|2\lambda a_n|}{\sqrt{1-2|\lambda|\,\|A\|_{\mathrm{op}}}}, \tag{5.7.6}$$

$S^{A,\lambda}$ is a symmetric Hilbert–Schmidt operator. If $|\lambda|$ is sufficiently small, for example, if $|\lambda|\,\|A\|_{\mathrm{op}} < \frac{3}{16}$, then $\|S^{A,\lambda}\|_{\mathrm{op}} < \frac{1}{2}$.

In connection with quadratic forms, the following change of variables formula holds.

Theorem 5.7.6 *For $\lambda \in \mathbb{R}$ with $|\lambda|\,\|A\|_{\mathrm{op}} < \frac{3}{16}$ and $f \in C_{\mathrm{b}}(W_T)$,*

$$\int_{W_T} \mathrm{e}^{\lambda Q_A} f \, \mathrm{d}\mu_T = \{\det_2(I-2\lambda A)\}^{-\frac{1}{2}} \int_{W_T} f(\iota + L_{S^{A,\lambda}})\, \mathrm{d}\mu_T. \tag{5.7.7}$$

For a proof, we give a lemma.

Lemma 5.7.7 (1) *For each $h \in H_T$, there exists an H_T-invariant $X_h \in \mathscr{B}(W_T)$ with $\mu_T(X_h) = 1$ such that*

$$(\nabla^* h)(w+g) = (\nabla^* h)(w) + \langle h, g\rangle_{H_T} \tag{5.7.8}$$

for any $w \in X_h$ and $g \in H_T$, where the H_T-invariance of X_h means that $X_h + g = X_h$ for any $g \in H_T$.
(2) *For any symmetric operator $A \in H_T^{\otimes 2}$, there exists an H_T-invariant $X_A \in \mathscr{B}(W_T)$ with $\mu(X_A) = 1$ such that*

$$Q_A(w+g) = Q_A(w) + 2\langle L_A(w), g\rangle_{H_T} + \langle Ag, g\rangle_{H_T} \tag{5.7.9}$$

for any $w \in X_A$ and $g \in H_T$.

$\nabla^* h, L_A$ and Q_A are defined up to null sets and the assertions of the lemma include the problem of the choice of modifications. We give an answer in the proof below. In order to expand $Q_A(w + F(w))$, which appears in the change of variables formula on W_T, for each w, we need to consider the H_T-invariant sets as in the lemma.

Proof (1) Let $\{\ell_n\}_{n=1}^\infty \subset W_T^*$ be an orthonormal basis of H_T. Set

$$\widetilde{h}_n = \sum_{k=1}^n \langle h, \ell_k\rangle_{H_T} \ell_k.$$

By (5.1.4), if $\ell_n \in W_T^*$, then $\nabla^* \ell_n = \ell_n$, μ_T-a.s. Hence, we can take the modification of $\nabla^* \widetilde{h}_n$ so that

$$\nabla^* \widetilde{h}_n = \widetilde{h}_n.$$

Moreover, we may assume that $\widetilde{h}_n \to \nabla^* h$, μ_T-a.s., choosing a subsequence if necessary. Set

$$X_h = \Big\{ w \in W_T;\ \lim_{n,m\to\infty} |\widetilde{h}_n(w) - \widetilde{h}_m(w)| = 0 \Big\}.$$

X_h is H_T-invariant because $\lim_{n\to\infty} \|\widetilde{h}_n - h\|_{H_T} = 0$. Moreover, since $\widetilde{h}_n \to \nabla^* h$, μ_T-a.s., $\mu_T(X_h) = 1$. Define a modification of $\nabla^* h$ by

$$(\nabla^* h)(w) = \begin{cases} \lim_{n\to\infty} \widetilde{h}_n(w) & (w \in X_h) \\ \quad 0 & (w \notin X_h). \end{cases}$$

Then, since

$$\widetilde{h}_n(w+g) = \widetilde{h}_n(w) + \langle \widetilde{h}_n, g\rangle_{H_T} \qquad (w \in W_T,\ g \in H_T),$$

we obtain (5.7.8) by letting $n \to \infty$.

(2) Develop A as in (5.7.1) with an orthonormal basis $\{h_n\}_{n=1}^\infty$ of H_T. For each h_n, define X_{h_n} and $\nabla^* h_n$ by (1). Let X_A be the set of $w \in \bigcap_{n=1}^\infty X_{h_n}$ such that $\sum_{n=1}^\infty a_n^2 (\nabla^* h_n)^2(w) < \infty$ and $\sum_{n=1}^\infty a_n \{(\nabla^* h_n)^2(w) - 1\}$ converges. By (5.7.8), X_A is H_T-invariant. Since $\sum_{n=1}^\infty a_n\{(\nabla^* h_n)^2 - 1\}$ converges in L^2 and $\sum_{n=1}^\infty a_n^2 (\nabla^* h_n)^2$ is integrable, by the assertion (1), $\mu_T(X_A) = 1$.

Let $w \in X_A$ and $g \in H_T$. Recalling Lemma 5.7.2, set

$$Q_A(w) = \begin{cases} \sum_{n=1}^\infty a_n \{(\nabla^* h_n)^2(w) - 1\} & (w \in X_A), \\ 0 & (w \notin X_A), \end{cases}$$

$$L_A(w) = \begin{cases} \sum_{n=1}^\infty a_n (\nabla^* h_n)(w) h_n & (w \in X_A), \\ 0 & (w \notin X_A). \end{cases}$$

Then, we have (5.7.9). □

Proof of Theorem 5.7.6 For simplicity of notation, write $S = S^{A,\lambda}$ and $s_n = s_n^{A,\lambda}$, and set $F = L_S$. Then, $\nabla F = S$ and $\nabla^* F = Q_S$.

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If $|\lambda|\,\|A\|_{\mathrm{op}} < \frac{3}{16}$, then $\|S\|_{\mathrm{op}} < \frac{1}{2}$ by (5.7.6). Lemma 5.7.2(2) implies that $\mathrm{e}^{-\nabla^* F + \|\nabla F\|^2_{H_T^{\otimes 2}}} \in L^{1+}(\mu_T)$. Hence, the conditions of Theorem 5.6.1 hold with $q = 1$. Moreover, by Remark 5.6.2(1), we have

$$\det{}_2(I+S)\int_{W_T} f(\iota+F)\mathrm{e}^{\lambda Q_A\circ(\iota+F)}\mathrm{e}^{-\nabla^* F-\frac{1}{2}\|F\|^2_{H_T}}\mathrm{d}\mu_T = \int_{W_T} f\mathrm{e}^{\lambda Q_A}\mathrm{d}\mu_T. \tag{5.7.10}$$

Since $(1+s_n)^2(1-2\lambda a_n) = 1$,

$$\begin{aligned} &\lambda a_n + 2\lambda a_n s_n + a_n s_n^2 - s_n - \frac{1}{2}s_n^2 = 0,\\ &\lambda A + 2\lambda AS + AS^2 - S - \frac{1}{2}S^2 = 0. \end{aligned} \tag{5.7.11}$$

From these identities, it follows that

$$\det{}_2(I+S) = \Big\{\prod_{n=1}^{\infty}(1+s_n)^2\mathrm{e}^{-2s_n}\Big\}^{\frac{1}{2}} = \{\det{}_2(I-2\lambda A)\}^{-\frac{1}{2}}\mathrm{e}^{\mathrm{tr}(\lambda A - S)}. \tag{5.7.12}$$

By Lemma 5.7.7,

$$Q_A\circ(\iota+F) = Q_A + 2\langle L_A, F\rangle_{H_T} + \langle AF, F\rangle_{H_T}.$$

By Lemma 5.7.2,

$$\begin{aligned} &\langle L_A, F\rangle_{H_T} = Q_{AS} + \mathrm{tr}\,AS, \qquad \langle AF, F\rangle_{H_T} = Q_{AS^2} + \mathrm{tr}\,AS^2,\\ &\big\|F\big\|^2_{H_T} = Q_{S^2} + \mathrm{tr}\,S^2. \end{aligned}$$

Thus, by the linearity $pQ_B + qQ_C = Q_{pB+qC}$ and (5.7.11),

$$\lambda Q_A\circ(\iota+F) - \nabla^* F - \frac{1}{2}\big\|F\big\|^2_{H_T} = \mathrm{tr}(S-\lambda A).$$

Plugging this and (5.7.12) into (5.7.10), we obtain (5.7.7). □

Corollary 5.7.8 *For $\lambda\in\mathbb{R}$ with $|\lambda|\,\|A\|_{\mathrm{op}} < \frac{1}{2}$ and $g\in H_T$,*

$$\int_{W_T}\mathrm{e}^{\lambda Q_A+\nabla^* g}\mathrm{d}\mu_T = \{\det{}_2(I-2\lambda A)\}^{-\frac{1}{2}}\mathrm{e}^{\frac{1}{2}\langle (I-2\lambda A)^{-1}g, g\rangle_{H_T}}. \tag{5.7.13}$$

Proof It suffices to show (5.7.13) when $|\lambda|\,\|A\|_{\mathrm{op}} < \frac{3}{16}$, because, by analytic continuation and Lemma 5.7.2(2), (5.7.13) holds also when $|\lambda|\,\|A\|_{\mathrm{op}} < \frac{1}{2}$.

Develop A as in (5.7.1) and set $g_n = \sum_{k=1}^n \langle g, h_k\rangle_{H_T} h_k$. By Lemmas 5.7.7 and 5.7.2, we have

$$\begin{aligned} \nabla^* g_n\circ(\iota + L_{S^{A,\lambda}}) &= \nabla^* g_n + \langle g_n, L_{S^{A,\lambda}}\rangle_{H_T}\\ &= \sum_{k=1}^n \langle g, h_k\rangle_{H_T}(1-2\lambda a_k)^{-\frac{1}{2}}\nabla^* h_k. \end{aligned}$$

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Since $\{\nabla^* h_k\}$ is a sequence of independent standard Gaussian random variables,

$$\int_{W_T} e^{\nabla^* g_n \circ (\iota + L_{SA,\lambda})} d\mu_T = e^{\frac{1}{2}\langle (I-2\lambda A)^{-1} g_n, g_n\rangle_{H_T}}.$$

Hence, by Theorem 5.7.6, we obtain

$$\int_{W_T} e^{\lambda Q_A + \nabla^* g_n} d\mu_T = \{\det_2(I-2\lambda A)\}^{-\frac{1}{2}} e^{\frac{1}{2}\langle (I-2\lambda A)^{-1} g_n, g_n\rangle_{H_T}}. \tag{5.7.14}$$

Since the distribution of $\nabla^* g_n$ is Gaussian with mean 0 and variance $\|g_n\|_{H_T}^2$, for any $p > 0$

$$\int_{W_T} e^{p\nabla^* g_n} d\mu_T = e^{\frac{1}{2}p^2 \|g_n\|_{H_T}^2} \leqq e^{\frac{1}{2}p^2 \|g\|_{H_T}^2}.$$

Hence, by Lemma 5.7.2(2), $\{e^{\lambda Q_A + \nabla^* g_n}\}_{n=1}^\infty$ is uniformly integrable. Since $\nabla^* g_n$ converges to $\nabla^* g$ in L^2, we obtain (5.7.13) by letting $n \to \infty$ in (5.7.14). □

Corollary 5.7.9 *Let $\eta_1, \dots, \eta_n \in W_T^*$ be an orthonormal system in H_T. Define $\pi : W_T \to W_T$ by $\pi(w) = \sum_{i=1}^n \eta_i(w)\eta_i$, and A_0 and $A_1 : H_T \to H_T$ by $A_0 = (I-\pi)A(I-\pi)$ and $A_1 = \pi A \pi$, respectively. Write $\delta_0(\boldsymbol{\eta})d\mu_T$ for the measure $\nu_{\delta_0(\boldsymbol{\eta})}$ in Theorem 5.4.15. Then, for $\lambda \in \mathbb{R}$ with $|\lambda|\,\|A\|_{\mathrm{op}} < \frac{1}{2}$ and $g \in H_T$,*

$$\int_{W_T} e^{\lambda Q_A + \nabla^* g} \delta_0(\boldsymbol{\eta}) d\mu_T$$
$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \{\det_2(I-2\lambda A_0)\}^{-\frac{1}{2}} e^{-\lambda tr(A_1)} e^{\frac{1}{2}\langle (I-2\lambda A_0)^{-1}(I-\pi)g, (I-\pi)g\rangle_{H_T}}. \tag{5.7.15}$$

Proof Write $w = (I-\pi)(w) + \sum_{i=1}^n \eta_i(w)\eta_i$. Then, by Example 5.4.17,

$$\eta_i(w) = 0, \quad \delta_0(\boldsymbol{\eta})d\mu_T\text{-a.e. } w \in W_T, \quad \eta_i \circ (I-\pi) = 0 \quad (i = 1, \dots, n). \tag{5.7.16}$$

Hence, for $F = f(\ell_1, \dots, \ell_m)$ with $\ell_1, \dots, \ell_m \in W_T^*$ and $f \in C_{\nearrow}^\infty(\mathbb{R}^m)$, we have

$$F = F \circ (I-\pi), \quad \delta_0(\boldsymbol{\eta})d\mu_T\text{-a.e.} \tag{5.7.17}$$

On the other hand, since $I-\pi$ and $\boldsymbol{\eta}$ are independent under μ_T,

$$\int_{W_T} (F\circ(I-\pi))\varphi(\boldsymbol{\eta})\, d\mu_T = \int_{W_T} (F\circ(I-\pi))\, d\mu_T \times \int_{W_T} \varphi(\boldsymbol{\eta})\, d\mu_T$$
$$= \int_{W_T} (F\circ(I-\pi))\, d\mu_T \times \int_{\mathbb{R}^n} \varphi(x) \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}|x|^2} dx$$

for any $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Hence, by taking $\{\varphi_k\}_{k=1}^\infty \in \mathscr{S}(\mathbb{R}^n)$ converging to δ_0 and letting $k \to \infty$, by Corollary 5.4.7 and (5.7.16), we obtain

$$\int_{W_T} F\delta_0(\boldsymbol{\eta}) d\mu_T = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{W_T} (F\circ(I-\pi))\, d\mu_T. \tag{5.7.18}$$

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Extend $\eta_1,\dots,\eta_n$ to an orthonormal basis $\{\eta_i\}_{i=1}^{\infty}$ of H_T. Set $c_i=\langle g,\eta_i\rangle_{H_T}$, $a_{ij}=\langle \eta_i, A\eta_j\rangle_{H_T}$,

$$g_N=\sum_{i=1}^{N} c_i\eta_i \quad \text{and} \quad A_N=\sum_{i,j=1}^{N} a_{ij}\eta_i\otimes\eta_j \qquad (N\geqq n).$$

Then, we have

$$Q_{A_N}=Q_{(I-\pi)A_N(I-\pi)}-\mathrm{tr}(\pi A_N\pi), \quad \delta_0(\boldsymbol{\eta})\mathrm{d}\mu_T\text{-a.e.,} \tag{5.7.19}$$
$$Q_{(I-\pi)A_N(I-\pi)}\circ(I-\pi)=Q_{(I-\pi)A_N(I-\pi)}, \quad \mu_T\text{-a.s.} \tag{5.7.20}$$

In fact, by (5.7.2),

$$Q_{A_N}=\sum_{i,j=1}^{N} a_{ij}\{\eta_i\eta_j-\delta_{ij}\} \quad \text{and} \quad Q_{(I-\pi)A_N(I-\pi)}=\sum_{n<i,j\leqq N} a_{ij}\{\eta_i\eta_j-\delta_{ij}\}.$$

Then we obtain (5.7.19) from $\mathrm{tr}(\pi A_n\pi)=\sum_{i=1}^{n} a_{ii}$ and (5.7.16). Moreover, the identity

$$Q_{(I-\pi)A_N(I-\pi)}=\sum_{n<i,j\leqq N} a_{ij}\{((I-\pi)\eta_i)((I-\pi)\eta_j)-\langle (I-\pi)\eta_i,\eta_j\rangle_{H_T}\}$$

yields (5.7.20).

Since $\nabla^* g_N=\sum_{i=1}^{N} c_i\eta_i$ and $\nabla^*((I-\pi)g_N)=\sum_{n<i\leqq N} c_i\eta_i$, by (5.7.16) again, we have

$$\nabla^* g_N=\nabla^*((I-\pi)g_N), \quad \delta_0(\boldsymbol{\eta})\mathrm{d}\mu_T\text{-a.e.,}$$
$$\nabla^*((I-\pi)g_N)=\nabla^*((I-\pi)g_N)\circ(I-\pi), \quad \mu_T\text{-a.s.}$$

Recalling the inequality $\|(I-\pi)A_N(I-\pi)\|_{\mathrm{op}}\leqq\|A\|_{\mathrm{op}}$ and then applying (5.7.18), we obtain

$$\begin{aligned}&\int_{W_T} \mathrm{e}^{\lambda Q_{A_N}+\nabla^* g_N}\delta_0(\boldsymbol{\eta})\mathrm{d}\mu_T\\ &\quad=\int_{W_T} \mathrm{e}^{\lambda\{Q_{(I-\pi)A_N(I-\pi)}-\mathrm{tr}(\pi A_N\pi)\}+\nabla^*((I-\pi)g_N)}\delta_0(\boldsymbol{\eta})\mathrm{d}\mu_T\\ &\quad=\frac{\mathrm{e}^{-\lambda\mathrm{tr}(\pi A_N\pi)}}{(2\pi)^{\frac{n}{2}}}\int_{W_T} \mathrm{e}^{\lambda Q_{(I-\pi)A_N(I-\pi)}+\nabla^*((I-\pi)g_N)}\mathrm{d}\mu_T.\end{aligned}$$

The uniform integrability of the integrands can be seen in the same way as in the proof of Lemma 5.7.2(2). Therefore, letting $N\to\infty$, we obtain

$$\int_{W_T} \mathrm{e}^{\lambda Q_A+\nabla^* g}\delta_0(\boldsymbol{\eta})\mathrm{d}\mu_T=\frac{\mathrm{e}^{-\lambda\mathrm{tr}(\pi A\pi)}}{(2\pi)^{\frac{n}{2}}}\int_{W_T} \mathrm{e}^{\lambda Q_{A_0}+\nabla^*((I-\pi)g)}\mathrm{d}\mu_T.$$

In conjunction with Corollary 5.7.8 the conclusion follows. □

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Remark 5.7.10 By analytic continuation, Corollaries 5.7.8 and 5.7.9 hold for $\lambda \in \mathbb{C}$ with $|\mathrm{Re}(\lambda)|\,\|A\|_{\mathrm{op}} < \frac{1}{2}$.

5.8 Examples of Quadratic Forms

In this section, the results in the previous section are applied to concrete examples: harmonic oscillators, Lévy's stochastic area, and sample variance.

5.8.1 Harmonic Oscillators

Let $d = 1$ and W_T be the one-dimensional Wiener space. Set

$$\mathfrak{h}_T(w) = \int_0^T w(t)^2 \mathrm{d}t \qquad (w \in W_T).$$

The functional $\mathfrak{h}_T$ is closely related to the **harmonic oscillator** $-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \lambda x^2$, which is one of the fundamental Schrödinger operators.

First we present the Laplace transforms of the probability law of $\mathfrak{h}_T$.

Theorem 5.8.1 *For* $\lambda > -\frac{\pi^2}{4T^2}$,

$$\int_{W_T} \mathrm{e}^{-\frac{1}{2}\lambda\mathfrak{h}_T} \mathrm{d}\mu_T = \sqrt{\frac{1}{\cosh(\sqrt{\lambda}\,T)}}, \tag{5.8.1}$$

$$\int_{W_T} \mathrm{e}^{-\frac{1}{2}\lambda\mathfrak{h}_T} \delta_0(\theta(T))\mathrm{d}\mu_T = \frac{1}{\sqrt{2\pi T}}\sqrt{\frac{\sqrt{\lambda}\,T}{\sinh(\sqrt{\lambda}\,T)}}. \tag{5.8.2}$$

Proof First we show that $\mathfrak{h}_T \in \mathbb{D}^{\infty,\infty-}$ and

$$\mathfrak{h}_T = Q_A + \frac{T^2}{2}, \tag{5.8.3}$$

where $A : H_T \to H_T$ is given by

$$(\dot{Ah})(t) = \int_t^T h(s)\,\mathrm{d}s \qquad (t \in [0,T],\ h \in H_T). \tag{5.8.4}$$

For $n \in \mathbb{N}$, set

$$\mathfrak{h}_T^{(n)} = \frac{T}{n}\sum_{i=0}^{n-1} \theta\Big(\frac{i}{n}T\Big)^2.$$

$\mathfrak{h}_T^{(n)} \in \mathscr{P}$ and, by (5.1.1),

$$\langle \nabla \mathfrak{h}_T^{(n)}, h\rangle_{H_T} = \frac{2T}{n}\sum_{i=0}^{n-1} \theta\Big(\frac{i}{n}T\Big)h\Big(\frac{i}{n}T\Big) \qquad (h \in H_T).$$

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Hence, defining $\ell_{[0,t]} \in W_T^* \subset H_T$ by $\ell_{[0,t]}(w) = w(t)$ $(w \in W_T)$, we have

$$\nabla \mathfrak{h}_T^{(n)} = \frac{2T}{n} \sum_{i=0}^{n-1} \theta\Big(\frac{i}{n}T\Big)\ell_{[0,\frac{i}{n}T]}.$$

From this expression, using (5.1.1) again, we obtain

$$\nabla^2 \mathfrak{h}_T^{(n)} = \frac{2T}{n} \sum_{i=0}^{n-1} \ell_{[0,\frac{i}{n}T]} \otimes \ell_{[0,\frac{i}{n}T]}, \quad \nabla^3 \mathfrak{h}_T^{(n)} = 0.$$

Let $n \to \infty$. Then, for any $p > 1$, we have the following L^p-convergence:

$$\mathfrak{h}_T^{(n)} \to \mathfrak{h}_T, \quad \nabla \mathfrak{h}_T^{(n)} \to 2\int_0^T \theta(t)\ell_{[0,t]}\,\mathrm{d}t, \quad \nabla^2 \mathfrak{h}_T^{(n)} \to 2\int_0^T \ell_{[0,t]} \otimes \ell_{[0,t]}\,\mathrm{d}t.$$

Hence, $\mathfrak{h}_T \in \mathbb{D}^{\infty,\infty-}$ and

$$\nabla \mathfrak{h}_T = 2\int_0^T \theta(t)\ell_{[0,t]}\,\mathrm{d}t, \quad \nabla^2 \mathfrak{h}_T = 2\int_0^T \ell_{[0,t]} \otimes \ell_{[0,t]}\,\mathrm{d}t, \quad \nabla^3 \mathfrak{h}_T = 0.$$

Since the integration by parts yields

$$\Big\langle\Big(\int_0^T \ell_{[0,t]} \otimes \ell_{[0,t]}\,\mathrm{d}t\Big)[h], g\Big\rangle_{H_T} = \int_0^T h(t)g(t)\,\mathrm{d}t = \int_0^T \Big(\int_t^T h(s)\mathrm{d}s\Big)\dot{g}(t)\,\mathrm{d}t,$$

we have

$$\nabla^2 \mathfrak{h}_T = 2A.$$

Moreover, by the above expression, we have

$$\int_{W_T} \nabla \mathfrak{h}_T\,\mathrm{d}\mu_T = 0 \quad \text{and} \quad \int_{W_T} \mathfrak{h}_T\,\mathrm{d}\mu_T = \frac{T^2}{2}.$$

Thus, by Proposition 5.7.4, (5.8.3) holds.

Second, we compute the eigenvalues and eigenfunctions of the Hilbert–Schmidt operator A. By (5.8.4), the equation $\lambda h = Ah$ is equivalent to

$$\lambda \dot{h}(t) = \int_t^T h(s)\,\mathrm{d}s \quad (t \in [0,T]) \quad \text{and} \quad h(0) = 0.$$

From this, A does not have a zero eigenvalue since $\lambda = 0$ implies $h = 0$. For $\lambda \neq 0$, we rewrite the above equation as

$$\lambda \ddot{h} + h = 0, \quad \dot{h}(T) = 0, \quad h(0) = 0.$$

The solution of this second order ordinary differential equation is given by a linear combination of $\cos(\lambda^{-\frac{1}{2}}t)$ and $\sin(\lambda^{-\frac{1}{2}}t)$. By the initial condition $\dot{h}(T) = 0$, $h(0) = 0$, we obtain

$$\lambda^{-\frac{1}{2}} = \frac{(n+\frac{1}{2})\pi}{T} \quad \text{and} \quad h(t) = c\sin\Big(\frac{(n+\frac{1}{2})\pi t}{T}\Big),$$

where c is a non-zero constant. Set

$$h_n(t) = \frac{\sqrt{2T}}{(n+\frac{1}{2})\pi}\sin\Big(\frac{(n+\frac{1}{2})\pi t}{T}\Big).$$

Then, $\{h_n\}_{n=0}^{\infty}$ is an orthonormal basis of H_T and each h_n is an eigenfunction of A corresponding to the eigenvalue $\frac{T^2}{(n+\frac{1}{2})^2\pi^2}$. Hence, A is diagonalized as

$$A = \sum_{n=0}^{\infty} \frac{T^2}{(n+\frac{1}{2})^2\pi^2} h_n \otimes h_n.$$

We have $\|A\|_{\mathrm{op}} = \frac{4T^2}{\pi^2}$. Moreover, since $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$, $\mathrm{tr}(A) = \frac{T^2}{2}$.

By Corollary 5.7.8, for $\lambda \in \mathbb{R}$ with $|\lambda| < \frac{\pi^2}{4T^2}$,

$$\begin{aligned}\int_{W_T} \mathrm{e}^{-\frac{1}{2}\lambda\mathfrak{h}_T}\mathrm{d}\mu_T &= \int_{W_T} \mathrm{e}^{-\frac{1}{2}\lambda Q_A}\mathrm{d}\mu_T \cdot \mathrm{e}^{-\frac{1}{4}\lambda T^2} = \{\det{}_2(I+\lambda A)\}^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{4}\lambda T^2}\\ &= \Big\{\prod_{n=0}^{\infty}\Big(1+\frac{4\lambda T^2}{(2n+1)^2\pi^2}\Big)\Big\}^{-\frac{1}{2}}.\end{aligned}$$

Combining this with the identity

$$\cosh x = \prod_{n=0}^{\infty}\Big(1+\frac{4x^2}{(2n+1)^2\pi^2}\Big),$$

we see that (5.8.1) holds for $\lambda \in \mathbb{R}$ with $|\lambda| < \frac{\pi^2}{4T^2}$. By analytic continuation, (5.8.1) holds for $\lambda \in \mathbb{R}$ with $\lambda > -\frac{\pi^2}{4T^2}$.

Next we show (5.8.2). As above, it suffices to show (5.8.2) for $\lambda \in \mathbb{R}$ with $|\lambda| < \frac{\pi^2}{4T^2}$. Define $\eta \in H_T$ by $\eta(t) = \frac{t}{\sqrt{T}}$ ($t \in [0,T]$) and π, A_0, A_1 as in Corollary 5.7.9. A_0 is given by

$$(A_0\dot{h})(t) = \int_t^T h(s)\,\mathrm{d}s - \frac{1}{T}\int_0^T\Big(\int_s^T h(u)\,\mathrm{d}u\Big)\mathrm{d}s$$

for $h \in H_T$ with $\pi h = 0$ or $h(T) = 0$. By a similar argument to the above, A_0 is developed as

$$A_0 = \sum_{n=1}^{\infty}\frac{T^2}{n^2\pi^2}k_n\otimes k_n, \qquad k_n(t) = \frac{\sqrt{2T}}{n\pi}\sin\Big(\frac{n\pi t}{T}\Big).$$

Since $\delta_0(\theta(T)) = \frac{1}{\sqrt{T}}\delta_0(\boldsymbol{\eta})$ and $\mathrm{tr}(A) = \mathrm{tr}(A_0) + \mathrm{tr}(A_1)$, by Corollary 5.7.9, we obtain

$$\int_{W_T}\mathrm{e}^{-\frac{1}{2}\lambda\mathfrak{h}_T}\delta_0(\theta(T))\mathrm{d}\mu_T = \int_{W_T}\mathrm{e}^{-\frac{1}{2}\lambda Q_A}\delta_0(\theta(T))\mathrm{d}\mu_T\,\mathrm{e}^{-\frac{1}{2}\lambda\mathrm{tr}(A)}$$

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$$= \frac{1}{\sqrt{2\pi T}}\{\det_2(I + \lambda A_0)\}^{-\frac{1}{2}} e^{-\frac{1}{2}\lambda\{\mathrm{tr}(A)-\mathrm{tr}(A_1)\}}$$
$$= \frac{1}{\sqrt{2\pi T}}\Big\{\prod_{n=1}^{\infty}\Big(1 + \frac{\lambda T^2}{n^2\pi^2}\Big)^2\Big\}^{-\frac{1}{2}}.$$

By the identity

$$\sinh x = x\prod_{n=1}^{\infty}\Big(1 + \frac{x^2}{n^2\pi^2}\Big), \tag{5.8.5}$$

we arrive at (5.8.2). □

By using Theorem 5.8.1, we show the explicit formula for the heat kernel of the Schrödinger operator investigated in Theorem 5.5.7 when $d = 1$, $\Theta = 0$, and $V(x) = x^2$.

Theorem 5.8.2 *Fix $\lambda > 0$. Then, for $x, y \in \mathbb{R}$ and $T > 0$,*

$$\int_{W_T} \exp\Big(-\frac{\lambda^2}{2}\int_0^T (x + \theta(t))^2 \mathrm{d}t\Big)\delta_y(x + \theta(T))\, \mathrm{d}\mu_T \tag{5.8.6}$$
$$= \frac{1}{\sqrt{2\pi T}}\sqrt{\frac{\lambda T}{\sinh(\lambda T)}}\exp\Big(-\frac{\lambda}{2}\coth(\lambda T)\{x^2 - 2xy\,\mathrm{sech}(\lambda T) + y^2\}\Big).$$

Proof Let $\phi : [0, T] \to \mathbb{R}$ be the unique solution for the ordinary differential equation[4]

$$\phi'' - \lambda^2\phi = 0, \quad \phi(0) = x, \ \ \phi(T) = y. \tag{5.8.7}$$

Define $h \in H_T$ by $h(t) = \phi(t) - x$. Since

$$\int_0^T \phi'(t)\, \mathrm{d}\theta(t) = \theta(T)\phi'(T) - \int_0^T \phi''(t)\theta(t)\, \mathrm{d}t$$

and ϕ satisfies (5.8.7), applying the Cameron–Martin theorem (Theorem 1.7.2), we obtain

$$\int_{W_T} \exp\Big(-\frac{\lambda^2}{2}\int_0^T (x + \theta(t))^2 \mathrm{d}t\Big)\delta_y(x + \theta(T))\, \mathrm{d}\mu_T$$
$$= \int_{W_T} \exp\Big(-\frac{\lambda^2}{2}\int_0^T (x + \theta(t) + h(t))^2 \mathrm{d}t\Big)\, e^{-\nabla^* h - \frac{1}{2}\|h\|_{H_T}^2}$$
$$\times\, \delta_y(x + \theta(T) + h(T))\, \mathrm{d}\mu_T$$
$$= \exp\Big(-\frac{1}{2}\int_0^T \{\lambda^2\phi(t)^2 + (\phi'(t))^2\}\, \mathrm{d}t\Big)\int_{W_T} e^{-\frac{1}{2}\lambda^2\mathfrak{h}_T}\delta_0(\theta(T))\, \mathrm{d}\mu_T.$$

[4] This equation is the Lagrange equation corresponding to the action integral $S_T(\phi) = \int_0^T L(\phi(t), \dot\phi(t))\, \mathrm{d}t$ associated with the Lagrangian $L(p, q) = \frac{1}{2}\{p^2 + q^2\}$. See Section 7.1.

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By (5.8.7), we have

$$\int_0^T \{\lambda^2\phi(t)^2 + (\phi'(t))^2\}\,\mathrm{d}t = y\phi'(T) - x\phi'(0).$$

Then, plugging in the explicit form of ϕ,

$$\phi(t) = \frac{y - \mathrm{e}^{-\lambda T}x}{\mathrm{e}^{\lambda T} - \mathrm{e}^{-\lambda T}}\mathrm{e}^{\lambda t} - \frac{y - \mathrm{e}^{\lambda T}x}{\mathrm{e}^{\lambda T} - \mathrm{e}^{-\lambda T}}\mathrm{e}^{-\lambda t} \qquad (t \in [0, T])$$

and using Theorem 5.8.1, we obtain (5.8.6). □

Remark 5.8.3 We have derived (5.8.6) by applying (5.8.2) in Theorem 5.8.1. The identity (5.8.6) can be shown in a direct and functional analytical way associated with the Schrödinger operator $H_\lambda = -\frac{1}{2}(\frac{\mathrm{d}}{\mathrm{d}x})^2 + \frac{\lambda^2}{2}x^2$ ($\lambda > 0$) on $\mathbb{R}$. The method is as follows.

Realize H_λ as a self-adjoint operator on $L^2(\mathbb{R})$, the Hilbert space of square-integrable functions with respect to the Lebesgue measure. The spectrum of H_λ consists only of the eigenvalues $\{\lambda(n + \frac{1}{2})\}_{n=0}^\infty$ with multiplicity one and the corresponding normalized eigenfunction ϕ_n is given by

$$\phi_n(x) = \sqrt{n!}\Big(\frac{\lambda}{\pi}\Big)^{\frac{1}{4}}\mathrm{e}^{-\frac{1}{2}\lambda x^2}H_n(\sqrt{2\lambda}\,x),$$

where $H_n(x)$ is a Hermite polynomial. Since $p(t, x, y)$ admits the eigenfunction expansion

$$p(t, x, y) = \sum_{n=0}^\infty \mathrm{e}^{-\lambda(n+\frac{1}{2})t}\phi_n(x)\phi_n(y),$$

a well-known formula for the Hermite polynomials

$$\sum_{n=0}^\infty n!H_n(x)H_n(y)t^n = \frac{1}{\sqrt{1-t^2}}\exp\Big(-\frac{1}{2}\,\frac{1}{1-t^2}\,(t^2x^2 - 2txy + t^2y^2)\Big)$$

yields (5.8.6). For this identity, see [67].

5.8.2 Lévy's Stochastic Area

Let W_T be the two-dimensional Wiener space and consider Lévy's stochastic area $\mathfrak{s}(T)$ (Example 5.5.6).

Theorem 5.8.4 *For $\lambda \in \mathbb{R}$ with $|\lambda| < \frac{\pi}{T}$,*

$$\int_{W_T} \mathrm{e}^{\lambda\mathfrak{s}(T)}\mathrm{d}\mu_T = \frac{1}{\cos(\frac{1}{2}\lambda T)}, \tag{5.8.8}$$

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$$\int_{W_T} \mathrm{e}^{\lambda\mathfrak{s}(T)}\delta_0(\theta(T))\mathrm{d}\mu_T = \frac{1}{2\pi T}\frac{\frac{1}{2}\lambda T}{\sin(\frac{1}{2}\lambda T)}. \tag{5.8.9}$$

Proof First we show $\mathfrak{s}(T) \in \mathbb{D}^{\infty,\infty-}$ and the expression

$$\mathfrak{s}(T) = \frac{1}{2}Q_A, \tag{5.8.10}$$

where $A : H_T \to H_T$ is given by

$$(\dot{Ah})(t) = J\Big(h(t) - \frac{1}{2}h(T)\Big) \qquad (t \in [0,T],\ h \in H_T)$$

and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. For $n \in \mathbb{N}$, define $\mathfrak{s}^{(n)}(T) \in \mathscr{P}$ by

$$\mathfrak{s}^{(n)}(T) = \frac{1}{2}\sum_{i=0}^{n-1}\Big\langle J\theta\Big(\frac{i}{n}T\Big), \theta\Big(\frac{i+1}{n}T\Big) - \theta\Big(\frac{i}{n}T\Big)\Big\rangle_{\mathbb{R}^2}.$$

By (5.1.1), we have for $h \in H_T$

$$\begin{aligned}\langle \nabla\mathfrak{s}^{(n)}(T), h\rangle_{H_T} &= \frac{1}{2}\sum_{i=0}^{n-1}\Big\langle Jh\Big(\frac{i}{n}T\Big), \theta\Big(\frac{i+1}{n}T\Big) - \theta\Big(\frac{i}{n}T\Big)\Big\rangle_{\mathbb{R}^2} \\ &\quad + \frac{1}{2}\sum_{i=0}^{n-1}\Big\langle J\theta\Big(\frac{i}{n}T\Big), h\Big(\frac{i+1}{n}T\Big) - h\Big(\frac{i}{n}T\Big)\Big\rangle_{\mathbb{R}^2}.\end{aligned}$$

Since $\theta(0) = h(0) = 0$, an algebraic manipulation yields

$$\begin{aligned}\langle \nabla\mathfrak{s}^{(n)}(T), h\rangle_{H_T} &= \frac{1}{2}\sum_{i=1}^{n-1}\Big\langle J\theta\Big(\frac{i}{n}T\Big), h\Big(\frac{i+1}{n}T\Big) - h\Big(\frac{i-1}{n}T\Big)\Big\rangle_{\mathbb{R}^2} \\ &\quad - \frac{1}{2}\Big\langle J\theta(T), h\Big(\frac{n-1}{n}T\Big)\Big\rangle_{\mathbb{R}^2}.\end{aligned}$$

Using (5.1.1) again, we obtain for $h, g \in H_T$

$$\begin{aligned}\langle [\nabla^2\mathfrak{s}^{(n)}(T)](g), h\rangle_{H_T} &= \frac{1}{2}\sum_{i=1}^{n-1}\Big\langle Jg\Big(\frac{i}{n}T\Big), h\Big(\frac{i+1}{n}T\Big) - h\Big(\frac{i-1}{n}T\Big)\Big\rangle_{\mathbb{R}^2} \\ &\quad - \frac{1}{2}\Big\langle Jg(T), h\Big(\frac{n-1}{n}T\Big)\Big\rangle_{\mathbb{R}^2}.\end{aligned}$$

Letting $n \to \infty$, we see that $\mathfrak{s}(T) \in \mathbb{D}^{\infty,\infty-}$,

$$\langle \nabla\mathfrak{s}(T), h\rangle_{H_T} = \int_0^T \langle J\theta(t), \dot{h}(t)\rangle_{\mathbb{R}^2}\,\mathrm{d}t - \frac{1}{2}\langle J\theta(T), h(T)\rangle_{\mathbb{R}^2}$$

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and

$$\begin{aligned}\langle [\nabla^2 \mathfrak{s}(T)](g), h\rangle_{H_T} &= \int_0^T \langle Jg(t), \dot{h}(t)\rangle_{\mathbb{R}^2}\,\mathrm{d}t - \frac{1}{2}\langle Jg(T), h(T)\rangle_{\mathbb{R}^2} \\ &= \int_0^T \Big\langle J\Big(g(t) - \frac{1}{2}g(T)\Big), \dot{h}(t)\Big\rangle_{\mathbb{R}^2}\mathrm{d}t \qquad (h, g \in H_T).\end{aligned}$$

From these observations we obtain

$$\nabla^2 \mathfrak{s}(T) = A, \quad \int_{W_T} \nabla \mathfrak{s}(T)\,\mathrm{d}\mu_T = 0, \quad \int_{W_T} \mathfrak{s}(T)\,\mathrm{d}\mu_T = 0.$$

By Proposition 5.7.4, (5.8.10) holds.

Second, we compute the eigenvalues and eigenfunctions of A. The equation $Ah = \lambda h$ is equivalent to

$$\lambda \ddot{h} = J\dot{h}, \quad h(0) = 0, \quad \lambda \dot{h}(0) = -\frac{1}{2}Jh(T).$$

Solving this ordinary differential equation, we see that the following functions h_n and $\widehat{h}_n$ are the eigenfunctions corresponding to the eigenvalue $\lambda_n = \frac{T}{(2n+1)\pi}$:

$$h_n(t) = \frac{\sqrt{T}}{(2n+1)\pi}\begin{pmatrix}\cos(\frac{(2n+1)\pi t}{T}) - 1 \\ \sin(\frac{(2n+1)\pi t}{T})\end{pmatrix} \quad \text{and} \quad \widehat{h}_n = Jh_n \qquad (n \in \mathbb{Z}).$$

Moreover, $\{h_n, \widehat{h}_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis of H_T. Hence, we have the expansion

$$A = \sum_{n\in\mathbb{Z}} \frac{T}{(2n+1)\pi}\{h_n \otimes h_n + \widehat{h}_n \otimes \widehat{h}_n\}.$$

The multiplicity of each eigenvalue $\frac{T}{(2n+1)\pi}$ is two and $\|A\|_{\mathrm{op}} = \frac{T}{\pi}$.

If $|\lambda| < \frac{\pi}{T}$, then, by Corollary 5.7.8,

$$\begin{aligned}\int_{W_T} \mathrm{e}^{\lambda \mathfrak{s}(T)}\mathrm{d}\mu_T &= \{\det{}_2(I - \lambda A)\}^{-\frac{1}{2}} \\ &= \Big\{\prod_{n\in\mathbb{Z}}\Big(1 - \frac{\lambda T}{(2n+1)\pi}\Big)\mathrm{e}^{\frac{\lambda T}{(2n+1)\pi}}\Big\}^{-1} = \Big\{\prod_{n=0}^{\infty}\Big(1 - \frac{\lambda^2 T^2}{(2n+1)^2\pi^2}\Big)\Big\}^{-1}.\end{aligned}$$

By the identity

$$\cos x = \prod_{n=0}^{\infty}\Big(1 - \frac{4x^2}{(2n+1)^2\pi^2}\Big),$$

we obtain (5.8.8).

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Next we show (5.8.9). Let $e_1 = (1,0)$ and $e_2 = (0,1) \in \mathbb{R}^2$. Define $\eta_i \in H_T$ by $\eta_i(t) = \frac{t}{\sqrt{T}} e_i$ $(i = 1,2)$. Moreover, define π, A_0, and A_1 as in Corollary 5.7.9. For $h \in H_T$ with $\pi h = 0$ or $h(T) = 0$, we have

$$(\dot{A_0 h})(t) = J(h(t) - \overline{h}),$$

where $\overline{h} = T^{-1} \int_0^T h(s)\,\mathrm{d}s$. Hence, in the same way as above,

$$A_0 = \sum_{n \in \mathbb{Z} \setminus \{0\}}^{\infty} \frac{T}{2n\pi} \{k_n \otimes k_n + \widehat{k_n} \otimes \widehat{k_n}\},$$

where

$$k_n(t) = \frac{\sqrt{T}}{2n\pi} \begin{pmatrix} \cos(\frac{2n\pi t}{T}) - 1 \\ \sin(\frac{2n\pi t}{T}) \end{pmatrix} \quad \text{and} \quad \widehat{k_n} = Jk_n.$$

Furthermore, since

$$\operatorname{tr}(A_1) = \sum_{i=1}^{2} \langle \eta_i, A\eta_i \rangle_{H_T} = \sum_{i=1}^{2} \int_0^T \frac{t-T}{T} \langle e_i, Je_i \rangle_{\mathbb{R}^2} \mathrm{d}t = 0$$

and $\delta_0(\theta(T)) = \frac{1}{T}\delta_0(\boldsymbol{\eta})$, by (5.7.9), we obtain

$$\begin{aligned} \int_{W_T} \mathrm{e}^{\lambda \mathfrak{s}(T)} \delta_0(\theta(T)) \mathrm{d}\mu_T &= \int_{W_T} \mathrm{e}^{\frac{1}{2}\lambda Q_A} \delta_0(\theta(T)) \mathrm{d}\mu_T = \frac{1}{2\pi T} \{\det{}_2(I - \lambda A_0)\}^{-\frac{1}{2}} \\ &= \frac{1}{2\pi T} \Big\{ \prod_{n \in \mathbb{Z} \setminus \{0\}} \Big(1 - \frac{\lambda T}{2n\pi}\Big) \Big\}^{-1} = \frac{1}{2\pi T} \Big\{ \prod_{n=1}^{\infty} \Big(1 - \frac{\lambda^2 T^2}{4n^2\pi^2}\Big) \Big\}^{-1}. \end{aligned}$$

By the identity

$$\sin x = x \prod_{n=1}^{\infty} \Big(1 - \frac{x^2}{n^2\pi^2}\Big)$$

we obtain (5.8.9). □

As Theorem 5.8.2, Theorem 5.8.4 is applicable to compute the heat kernel.

Theorem 5.8.5 *Let* $\Theta(x) = (-\frac{x^2}{2}, \frac{x^1}{2})$ $(x = (x^1, x^2) \in \mathbb{R}^2)$*. Define* $L(t, x; \Theta)$ *as in Theorem 5.5.7. Then, for* $\lambda \in \mathbb{R}$*,*

$$\begin{aligned} &\int_{W_T} \mathrm{e}^{\mathrm{i}\,\lambda L(T,x;\Theta)} \delta_y(x + \theta(T))\,\mathrm{d}\mu_T \\ &\qquad = \frac{\lambda}{4\pi \sinh(\frac{1}{2}\lambda T)} \exp\Big(\frac{\mathrm{i}\,\lambda}{2} \langle Jx, y \rangle_{\mathbb{R}^2} - \frac{\lambda}{4} \coth\Big(\frac{1}{2}\lambda T\Big) |y - x|^2\Big). \end{aligned}$$

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Proof Let $x, y \in \mathbb{R}^2$. If we show

$$\int_{W_T} e^{\alpha L(T,x;\Theta)} \delta_y(x + \theta(T))\, d\mu_T = \frac{\alpha}{4\pi \sin(\frac{1}{2}\alpha T)} \exp\Big(\frac{\alpha}{2}\langle Jx, y\rangle_{\mathbb{R}^2} - \frac{\alpha}{4}\cot\Big(\frac{1}{2}\alpha T\Big)|y - x|^2\Big) \quad (5.8.11)$$

for sufficiently small $\alpha \in \mathbb{R}$, we obtain the conclusion by analytic continuation.

Let $\phi : [0, T] \to \mathbb{R}^2$ be the solution of the ordinary differential equation

$$\ddot{\phi} - \alpha J\dot{\phi} = 0, \quad \phi(0) = x, \ \ \phi(T) = y \quad (5.8.12)$$

and define $h \in H_T$ by $h(t) = \phi(t) - x$. Since

$$L(t, x; \Theta) = \frac{1}{2}\int_0^T \langle J(x + \theta(t)), d\theta(t)\rangle_{\mathbb{R}^2}$$

and

$$L(t, x; \Theta)(\cdot + h) = \mathfrak{s}(T) + \int_0^T \langle J\theta(t), \phi'(t)\rangle_{\mathbb{R}^2} dt - \frac{1}{2}\langle J\theta(T), \phi(T)\rangle_{\mathbb{R}^2} + \frac{1}{2}\int_0^T \langle J\phi(t), \phi'(t)\rangle_{\mathbb{R}^2} dt,$$

by the Cameron–Martin theorem, we obtain

$$\int_{W_T} e^{\alpha L(T,x;\Theta)} \delta_y(x + \theta(T))\, d\mu_T = \exp\Big(\frac{1}{2}\int_0^T \{\langle \alpha J\phi(t), \phi'(t)\rangle_{\mathbb{R}^2} - |\phi'(t)|^2\}\, dt\Big) \int_{W_T} e^{\alpha \mathfrak{s}(T)} \delta_0(\theta(T))\, d\mu_T. \quad (5.8.13)$$

By integration by parts on $[0, T]$ and (5.8.12),

$$\int_0^T \{\langle \alpha J\phi(t), \phi'(t)\rangle_{\mathbb{R}^2} - |\phi'(t)|^2\}\, dt = \langle \phi'(0), x\rangle_{\mathbb{R}^2} - \langle \phi'(T), y\rangle_{\mathbb{R}^2}.$$

The solution of (5.8.12) is explicitly given by

$$\phi(t) = x + \frac{1}{\alpha} J(I - e^{\alpha t J})\gamma \qquad (t \in [0, T]),$$

where

$$\gamma = \frac{\alpha}{2\sin(\frac{1}{2}\alpha T)} \begin{pmatrix} \cos(\frac{1}{2}\alpha T) & \sin(\frac{1}{2}\alpha T) \\ -\sin(\frac{1}{2}\alpha T) & \cos(\frac{1}{2}\alpha T) \end{pmatrix} (y - x).$$

Hence

$$\phi'(0) = \gamma \quad \text{and} \quad \phi'(T) = e^{\alpha T J}\gamma = \alpha J(y - x) + \gamma.$$

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Moreover, we have

$$\begin{aligned}\langle \phi'(0), x\rangle_{\mathbb{R}^2} - \langle \phi'(T), y\rangle_{\mathbb{R}^2} &= \alpha\langle Jx, y\rangle_{\mathbb{R}^2} + \langle \gamma, x-y\rangle_{\mathbb{R}^2} \\ &= \alpha\langle Jx, y\rangle_{\mathbb{R}^2} - \frac{\alpha\cos(\frac{1}{2}\alpha T)}{2\sin(\frac{1}{2}\alpha T)}|x-y|^2.\end{aligned}$$

Plugging this and (5.8.9) into (5.8.13), we obtain (5.8.11). □

Remark 5.8.6 Lévy [70] showed the results in this section by using the Fourier expansion of Brownian motion. Moreover, several proofs are known (see [4]).

5.8.3 Sample Variance

Let W_T be the one-dimensional Wiener space and set

$$\mathfrak{v}_T(w) = \int_0^T (w(t) - \overline{w})^2 \mathrm{d}t \qquad (w \in W_T),$$

where $\overline{w} = \frac{1}{T}\int_0^T w(t)\,\mathrm{d}t$.

Theorem 5.8.7 *For $\lambda \in \mathbb{R}$ with $\lambda > -\frac{\pi^2}{T^2}$,*

$$\int_{W_T} \mathrm{e}^{-\frac{1}{2}\lambda\mathfrak{v}_T}\mathrm{d}\mu_T = \Big(\frac{\sqrt{\lambda}\,T}{\sinh(\sqrt{\lambda}\,T)}\Big)^{\frac{1}{2}}, \tag{5.8.14}$$

$$\int_{W_T} \mathrm{e}^{-\frac{1}{2}\lambda\mathfrak{v}_T}\delta_0(\theta(T))\,\mathrm{d}\mu_T = \frac{\frac{1}{2}\sqrt{\lambda}\,T}{\sinh(\frac{1}{2}\sqrt{\lambda}\,T)}. \tag{5.8.15}$$

Proof First we show (5.8.14). Define $A : H_T \to H_T$ by

$$(\dot{Ah})(t) = \int_t^T (h(s) - \overline{h})\,\mathrm{d}s \qquad (t \in [0,T],\ h \in H_T).$$

In the same way as in Theorem 5.8.1, we can show for $h, g \in H_T$

$$\langle \nabla\mathfrak{v}_T, h\rangle_{H_T} = 2\int_0^T (\theta(t) - \overline{\theta})(h(t) - \overline{h})\,\mathrm{d}t$$

and

$$\begin{aligned}\langle [\nabla\mathfrak{v}_T](g), h\rangle_{H_T} &= 2\int_0^T (g(t) - \overline{g})(h(t) - \overline{h})\,\mathrm{d}t = 2\int_0^T (g(t) - \overline{g})h(t)\,\mathrm{d}t \\ &= 2\int_0^T \Big(\int_t^T (g(s) - \overline{g})\,ds\Big)\dot{h}(t)\,\mathrm{d}t.\end{aligned}$$

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From these identities, we obtain

$$\nabla^2 \mathfrak{v}_T = 2A, \quad \int_{W_T} \nabla \mathfrak{v}_T \, \mathrm{d}\mu_T = 0, \quad \int_{W_T} \mathfrak{v}_T \, \mathrm{d}\mu_T = \frac{T^2}{6}.$$

Hence, by Proposition 5.7.4, we obtain

$$\mathfrak{v}_T = Q_A + \frac{T^2}{6}.$$

The equation $Ah = \lambda h$ is equivalent to

$$\lambda \dddot{h} = -\dot{h}, \quad h(0) = 0, \quad \dot{h}(0) = \dot{h}(T) = 0.$$

Solving this equation, we obtain the expansion

$$A = \sum_{n=1}^{\infty} \Big(\frac{T}{n\pi}\Big)^2 h_n \otimes h_n$$

where $h_n(t) = \frac{\sqrt{2T}}{n\pi}\Big\{\cos\Big(\frac{n\pi t}{T}\Big) - 1\Big\}$. In particular, we have $\|A\|_{\mathrm{op}} = \frac{T^2}{\pi^2}$. If $|\lambda| < \frac{\pi^2}{T^2}$, then

$$\int_{W_T} \mathrm{e}^{-\frac{1}{2}\lambda \mathfrak{v}_T} \mathrm{d}\mu_T = \{\det_2(I + \lambda A)\}^{-\frac{1}{2}} = \Big\{\prod_{n=1}^{\infty}\Big(1 + \frac{\lambda T^2}{n^2\pi^2}\Big)\Big\}^{-\frac{1}{2}}$$

by Corollary 5.7.8. Combining this with (5.8.5), we obtain (5.8.14) for $\lambda \in \mathbb{R}$ with $|\lambda| < \frac{\pi^2}{T^2}$. By analytic continuation, (5.8.14) holds also for $\lambda \in \mathbb{R}$ with $\lambda > -\frac{\pi^2}{T^2}$.

Second, we show (5.8.15). It suffices to show it when $|\lambda| < \frac{\pi^2}{T^2}$.

Define $\eta \in H_T$ by $\eta(t) = \frac{t}{\sqrt{T}}$ $(t \in [0, T])$. Define π, A_0, and A_1 as in Corollary 5.7.9. For $h \in H_T$ with $\pi h = 0$ or $h(T) = 0$, we have

$$(\dot{A_0 h})(t) = \int_t^T (h(s) - \overline{h})\,\mathrm{d}s - \frac{1}{T}\int_0^T \Big(\int_s^T (h(u) - \overline{h})\,\mathrm{d}u\Big)\mathrm{d}s.$$

From this, by a similar argument to the above, we obtain

$$A_0 = \sum_{n=1}^{\infty}\Big(\frac{T}{2n\pi}\Big)^2\{k_n \otimes k_n + \widehat{k}_n \otimes \widehat{k}_n\},$$

where the eigenfunctions are given by

$$k_n(t) = \frac{\sqrt{2T}}{2n\pi}\sin\Big(\frac{2n\pi t}{T}\Big), \qquad \widehat{k}_n(t) = \frac{\sqrt{2T}}{2n\pi}\Big\{\cos\Big(\frac{2n\pi t}{T}\Big) - 1\Big\}.$$

Hence, since $\delta_0(\theta(T)) = \frac{1}{\sqrt{T}}\delta(\boldsymbol{\eta})$ and $\mathrm{tr}(A) = \mathrm{tr}(A_0) + \mathrm{tr}(A_1)$, by Corollary 5.7.9, we have

$$\int_{W_T} \mathrm{e}^{-\frac{1}{2}\lambda \mathfrak{v}_T}\delta_0(\theta(T))\,\mathrm{d}\mu_T = \Big\{\prod_{n=1}^{\infty}\Big(1 + \frac{\lambda T^2}{(2n\pi)^2}\Big)^2\Big\}^{-\frac{1}{2}}.$$

By (5.8.5), we obtain (5.8.15) for $\lambda \in \mathbb{R}$ with $|\lambda| < \frac{\pi^2}{T^2}$. □

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5.9 Abstract Wiener Spaces and Rough Paths

An **abstract Wiener space** is a triplet $(\mathscr{W}, \mathscr{H}, \nu)$ such that

(i) $\mathscr{W}$ is a real separable Banach space,
(ii) $\mathscr{H}$ is a real separable Hilbert space embedded in $\mathscr{W}$ densely and continuously,
(iii) ν is a probability measure on $(\mathscr{W}, \mathscr{B}(\mathscr{W}))$ under which every $\ell \in \mathscr{W}^*$ is a Gaussian random variable with mean 0 and variance $\|\ell\|_{\mathscr{H}}^2$, where we have used the inclusion $\mathscr{W}^* \subset \mathscr{H}^* = \mathscr{H} \subset \mathscr{W}$.

Example 5.9.1 Let (W_T, H_T, μ_T) be the d-dimensional Wiener space.
(1) By Lemma 1.7.1, (W_T, H_T, μ_T) is an abstract Wiener space.
(2) Let $W_{T,0} = \{w \in W_T; w(T) = 0\}$, $H_{T,0} = \{h \in H_T; h(T) = 0\}$ and $\mu_{T,0} = (2\pi T)^{\frac{d}{2}}\delta_0(\theta(T))$, where we have thought of the positive generalized Wiener functional $\delta_0(\theta(T))$ as a Borel measure on W_T by Theorem 5.4.15. Then $(W_{T,0}, H_{T,0}, \mu_{T,0})$ is an abstract Wiener space. To see this, define $\ell_\alpha \in W_T^*$ $(\alpha = 1, \dots, d)$ by $\ell_\alpha(w) = \frac{1}{\sqrt{T}} w^\alpha(T)$ $(w = (w^1, \dots, w^d) \in W_T)$. Observe that $\|\ell_\alpha\|_{H_T} = 1$. Take $\ell_j \in W_T^*$ $(j \geqq d+1)$ so that $\{\ell_n\}_{n=1}^\infty$ is an orthonormal basis of H_T. Then, thinking of $\{\ell_n\}_{n=1}^\infty$ as a sequence of independent Gaussian random variables with mean 0 and variance 1, μ_T can be decomposed as a product measure of a d-dimensional standard normal distribution and $\mu_{T,0}$. This implies the desired result. We left the details to the reader.
(3) Let γ be the distribution on W_T of a continuous d-dimensional Gaussian process with mean 0. For $Z \in \mathbf{H}$, the $L^2(\gamma)$-closure of the span of $\theta^\alpha(t)$ $(\alpha = 1, \dots, d, 0 \leqq t \leqq T)$, define $h_Z \in W_T$ by $h_Z(t) = \int_{W_T} Z\theta(t)\mathrm{d}\gamma$. Let $\mathscr{H} = \{h_Z; Z \in \mathbf{H}\}$ and $\mathbf{W}$ be the closure of $\mathscr{H}$ in W_T with respect to the uniform norm. Then, $(\mathbf{W}, \mathbf{H}, \gamma)$ becomes an abstract Wiener space. For details, see [7, 68].

Repeating the arguments in the preceding sections with an abstract Wiener space $(\mathscr{W}, \mathscr{H}, \nu)$ instead of (W_T, H_T, μ_T), we can define the Sobolev spaces $\mathbb{D}^{p,k}(E)$, the operator ∇, and other things on $\mathscr{W}$ similarly. All assertions and proofs there, except the proof of Theorem 5.4.15, continue to be true without any changes. For the proof of Theorem 5.4.15, an additional observation is necessary (see [116]).

As an extension of Theorem 5.4.11, we have the following assertion on absolute continuity.

Proposition 5.9.2 *Let $(\mathscr{W}, \mathscr{H}, \nu)$ be an abstract Wiener space and $F : \mathscr{W} \to \mathbb{R}$ be of class $\mathbb{D}^{\infty,\infty-}$. Then the distribution of F under $\|\nabla F\|_{\mathscr{H}}^2 \mathrm{d}\nu$ has a density*

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function $p(x) = \mathbf{E}[\mathbf{1}_{[x,\infty)}(F)\nabla^*\nabla F]$ *with respect to the Lebesgue measure. In particular, if* $\nabla F \neq 0$ ν*-a.s., then the distribution of* F *is absolutely continuous with respect to the Lebesgue measure.*

Proof Let $\hat{\mathscr{W}} = \mathscr{W} \times W_1^1$, $\hat{\mathscr{H}} = \mathscr{H} \times H_1^1$, and $\hat{\nu} = \nu \times \mu_1^1$. Then $(\hat{\mathscr{W}}, \hat{\mathscr{H}}, \hat{\nu})$ is an abstract Wiener space. Denote by $\hat{\nabla}$ and $\hat{\mathbb{D}}^{\infty,\infty-}$ the gradient operator and the $\mathbb{D}^{\infty,\infty-}$-space on $\hat{\mathscr{W}}$, respectively. By a natural inclusion, $\mathbb{D}^{\infty,\infty-} \subset \hat{\mathbb{D}}^{\infty,\infty-}$, $\hat{\nabla}|_{\mathbb{D}^{\infty,\infty-}} = \nabla$, and $\hat{\nabla}^*|_{\mathbb{D}^{\infty,\infty-}(\mathscr{H})} = \nabla^*$.

For $\varepsilon > 0$, define $\hat{F}_\varepsilon : \hat{\mathscr{W}} \to \mathbb{R}$ by $\hat{F}_\varepsilon(w, w') = F(w) + \varepsilon e^{\xi(w')}$ ($(w, w') \in \hat{\mathscr{W}}$), where $\xi(w') = w'(1)$. Then, $\hat{\nabla}\hat{F}_\varepsilon = \nabla F + \varepsilon e^{\xi}\nabla'\xi$, where ∇' stands for the gradient operator on W_1^1. In particular,

$$\|\hat{\nabla}\hat{F}_\varepsilon\|^2_{\hat{\mathscr{H}}} = \|\nabla F\|^2_{\mathscr{H}} + \varepsilon^2 e^{2\xi} \quad \text{and} \quad \hat{\nabla}^*\hat{\nabla}\hat{F}_\varepsilon = \nabla^*\nabla F + \varepsilon(\xi - 1)e^{\xi},$$

where we have used Theorem 5.2.8 to see the second identity. Thus, $\hat{F}_\varepsilon$ is of class $\mathbb{D}^{\infty,\infty-}$ and non-degenerate. By Theorem 5.4.11, the integration by parts formula and Theorem 5.2.1,

$$\begin{aligned} \mathbf{E}[f(\hat{F}_\varepsilon)\|\hat{\nabla}\hat{F}_\varepsilon\|^2_{\hat{\mathscr{H}}}] &= \int_{\mathbb{R}} f(x)\mathbf{E}[\delta_x(\hat{F}_\varepsilon)\|\hat{\nabla}\hat{F}_\varepsilon\|^2_{\hat{\mathscr{H}}}]\mathrm{d}x \\ &= \int_{\mathbb{R}} f(x)\mathbf{E}[\mathbf{1}_{[x,\infty)}(\hat{F}_\varepsilon)\hat{\nabla}^*\hat{\nabla}\hat{F}_\varepsilon]\mathrm{d}x \quad (f \in C_{\mathrm{b}}(\mathscr{W})). \end{aligned}$$

Letting $\varepsilon \to 0$, we arrive at

$$\mathbf{E}[f(F)\|\nabla F\|^2_{\mathscr{H}}] = \int_{\mathbb{R}} f(x)p(x)\mathrm{d}x.$$

This implies the first assertion. The second assertion is an immediate consequence of the first one. □

It was shown by Bouleau and Hirsch [8] that the second assertion continues to hold for $\mathbb{R}^n$-valued Wiener functionals. See also [92].

Proposition 5.9.3 *Let* $F = (F^1, \dots, F^n) : \mathscr{W} \to \mathbb{R}^n$ *be of class* $\mathbb{D}^{1,p}$ *for some* $p > 1$. *Suppose* $\det[(\langle\nabla F^i, \nabla F^j\rangle)_{i,j=1,\dots,n}] \neq 0$ ν*-a.s. Then, the distribution of* F *on* $\mathbb{R}^n$ *is absolutely continuous with respect to the Lebesgue measure.*

An application of the Malliavin calculus on abstract Wiener spaces is the one to stochastic differential equations extended by the theory of rough paths. The theory of rough paths was initiated by Lyons in the 1990s, and developed widely to produce several monographs [26, 27, 71, 72]. In the remainder of this section, we shall give a glance at the theory of rough paths, following [26].

For a while, we work in the deterministic setting. Let V be a Banach space. A rough path $\boldsymbol{X} = (X, \mathbb{X})$ is a pair of continuous functions $X : [0, T] \to V$ and $\mathbb{X} : [0, T]^2 \to V \otimes V$, satisfying

$$\mathbb{X}(s,t) - \mathbb{X}(s,u) - \mathbb{X}(u,t) = X(s,u) \otimes X(u,t), \quad \text{where } X(s,t) = X(t) - X(s).$$

For $\frac{1}{3} < \alpha \leqq \frac{1}{2}$, $\mathcal{C}^\alpha = \mathcal{C}^\alpha([0,T], V)$ denotes the space of rough paths $\boldsymbol{X} = (X, \mathbb{X})$ such that

$$\|X\|_\alpha = \sup_{s \neq t \in [0,T]} \frac{|X(s,t)|}{|t-s|^\alpha} < \infty, \quad \|\mathbb{X}\|_{2\alpha} = \sup_{s \neq t \in [0,T]} \frac{|\mathbb{X}(s,t)|}{|t-s|^{2\alpha}} < \infty.$$

Moreover, $\mathcal{C}^\alpha_{\mathrm{g}}$ stands for the space of rough paths $\boldsymbol{X} = (X, \mathbb{X}) \in \mathcal{C}^\alpha$ such that $\mathrm{Sym}(\mathbb{X}(s,t)) = \frac{1}{2} X(s,t) \otimes \frac{1}{2} X(s,t)$, where Sym is the symmetrizing operator.

For $X \in C^\alpha([0,T], V)$, $Y \in C^\alpha([0,T], \overline{W})$, $\overline{W}$ being a Banach space, is said to be controlled by X if there exists $Y' \in C^\alpha([0,T], \mathcal{L}(V, \overline{W}))$, where $\mathcal{L}(V, \overline{W})$ is the space of continuous linear mappings of V to $\overline{W}$, such that $R^Y(s,t) = Y(s,t) - Y'(s)X(s,t)$ satisfies $\|R^Y\|_{2\alpha} < \infty$. The space of such pairs (Y, Y') is denoted by $\mathcal{D}^{2\alpha}_X([0,T], \overline{W})$.

Let $\alpha > \frac{1}{3}$. If $\boldsymbol{X} = (X, \mathbb{X}) \in \mathcal{C}^\alpha([0,T], V)$ and $(Y, Y') \in \mathcal{D}^{2\alpha}_X([0,T], \mathcal{L}(V, W))$, then, for every $s < t \leqq T$, $\lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} (Y(u)X(u,v) + Y'(u)\mathbb{X}(u,v))$ exists, where $\mathcal{P}$ is a partition of $[s,t]$. The limit is called the integration of Y against the rough path $\boldsymbol{X}$, and denoted by $\int_s^t Y(r) \mathrm{d}\boldsymbol{X}(r)$.

Using integrations against rough paths, a differential equation driven by a rough path, say a rough differential equation, can be defined; the rough differential equation

$$\mathrm{d}Y = f(Y)\mathrm{d}\boldsymbol{X}, \quad Y_0 = \xi$$

means the integral equation

$$Y(t) = \xi + \int_0^t f(Y(s))\mathrm{d}\boldsymbol{X}(s).$$

We now proceed to the stochastic setting. First we shall see that a rough differential equation extends a stochastic differential equation. To do this, let $B = \{B(t)\}_{t \geqq 0}$ be a d-dimensional standard Brownian motion. Set

$$\mathbb{B}^{\text{Itô}} = \int_s^t B(s,r) \otimes \mathrm{d}B(r) \in \mathbb{R}^d \otimes \mathbb{R}^d.$$

Then $\boldsymbol{B}^{\text{Itô}} = (B, \mathbb{B}^{\text{Itô}}) \in \mathcal{C}^\alpha([0,T], \mathbb{R}^d)$ a.s. for any $\alpha \in (\frac{1}{3}, \frac{1}{2})$ and $T > 0$. Similarly, if we set

$$\mathbb{B}^{\text{Strat}} = \int_s^t B(s,r) \otimes \circ \mathrm{d}B(r) \in \mathbb{R}^d \otimes \mathbb{R}^d,$$

then $\boldsymbol{B}^{\mathrm{Strat}} = (B, \mathbb{B}^{\mathrm{Strat}}) \in C_{\mathrm{g}}^{\alpha}([0,T], \mathbb{R}^d)$ a.s. for any $\alpha \in (\frac{1}{3}, \frac{1}{2})$ and $T > 0$. It may be worthwhile to notice that, if $s = 0$, then the anti-symmetric parts of $\mathbb{B}^{\mathrm{Itô}}$ and $\mathbb{B}^{\mathrm{Strat}}$ coincide with Lévy's stochastic area.

If $(Y(\omega), Y'(\omega)) \in \mathcal{D}^{2\alpha}_{B(\omega)}$ for a.a. ω, and Y, Y' are both predictable, then

$$\int_0^T Y \mathrm{d}\boldsymbol{B}^{\mathrm{Itô}} = \int_0^T Y(t)\mathrm{d}B(t) \quad \text{and} \quad \int_0^T Y \mathrm{d}\boldsymbol{B}^{\mathrm{Strat}} = \int_0^T Y(t) \circ \mathrm{d}B(t).$$

Moreover, for $f \in C_{\mathrm{b}}^3(\mathbb{R}^e, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e))$, Lipschitz continuous $f_0 : \mathbb{R}^e \to \mathbb{R}^e$, and $\xi \in \mathbb{R}^e$, (i) for a.a. ω, there is a unique solution $(Y(\omega), f(Y(\omega))) \in \mathcal{D}^{2\alpha}_{B(\omega)}$ to the rough differential equation

$$\mathrm{d}Y(t,\omega) = f_0(Y(t,\omega))\mathrm{d}t + f(Y(t,\omega))\mathrm{d}\boldsymbol{B}^{\mathrm{Itô}}(t,\omega), \quad Y(0,\omega) = \xi,$$

and (ii) $Y = \{Y(t,\omega)\}_{t\geq 0}$ is a strong solution to the Itô stochastic differential equation

$$\mathrm{d}Y(t) = f_0(Y(t))\mathrm{d}t + f(Y(t))\mathrm{d}B(t), \quad Y(0) = \xi.$$

A similar assertion holds with "Strat" instead of "Itô".

We now investigate rough paths arising from Gaussian processes, for which the Malliavin calculus on abstract Wiener spaces works. Let X be a continuous, centered Gaussian process with values in $\mathbb{R}^d$. In what follows, we work on the abstract Wiener space given in Example 5.9.1 (3). The rectangle increment of the covariance is defined by

$$R\begin{pmatrix} s,t \\ s',t' \end{pmatrix} = \mathbb{E}[X(s,t) \otimes X(s',t')].$$

Its ρ-variation on a rectangle $I \times I'$, where I and I' are both rectangles in $\mathbb{R}^d$, is given by

$$\|R\|_{\rho, I\times I'} = \Biggl(\sup_{\substack{\mathcal{P}\subset I,\\ \mathcal{P}'\subset I'}} \sum_{\substack{[s,t]\in\mathcal{P}\\ [s',t']\in\mathcal{P}'}} \left| R\begin{pmatrix} s,t \\ s',t' \end{pmatrix}\right|^{\rho}\Biggr)^{\frac{1}{\rho}},$$

where $\mathcal{P}$ (resp. $\mathcal{P}'$) is a partition of I (resp. I').

Let $\rho \in [1, \frac{3}{2})$, $\alpha \in (\frac{1}{3}, \frac{1}{2\rho})$, and $\{X(t)\}_{0\leqq t\leqq T}$ be a d-dimensional, continuous, centered Gaussian process with independent components such that

$$\sup_{0\leqq s<t\leqq T} \frac{\|R_{X^i}\|_{\rho,[s,t]^2}}{|t-s|^{\frac{1}{\rho}}} < \infty \quad (i = 1, \ldots, d).$$

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Define

$$\mathbb{X}^{i,j}(s,t) = \begin{cases} L^2 - \lim\limits_{\substack{\mathcal{P}\in\Pi_{[s,t]} \\ |\mathcal{P}|\to 0}} \sum\limits_{[u,v]\in\mathcal{P}} X^i(s,u)X^j(u,v), & \text{if } i < j, \\ \frac{1}{2}(X^i_{s,t})^2, & \text{if } i = j, \\ -\mathbb{X}^{j,i} + X^i(s,t)X^j(s,t), & \text{if } i > j, \end{cases}$$

where $\Pi_{[s,t]}$ is the set of partitions of $[s,t]$. Then $\boldsymbol{X} = (X, \mathbb{X}) \in C^\alpha_{\mathrm{g}}$. For this $\boldsymbol{X}$ and $V_1, \dots, V_d \in C^\infty_{\mathrm{b}}(\mathbb{R}^e, \mathbb{R}^e)$, let $\{Y(t)\}_{0\leqq t\leqq T}$ be the solution to the rough differential equation

$$\mathrm{d}Y = V(Y)\mathrm{d}\boldsymbol{X}, \quad Y(0) = y_0 \in \mathbb{R}^e,$$

where $V = (V_1, \dots, V_d)$. As an application of Proposition 5.9.3, we have the following.

Theorem 5.9.4 *Assume that*
(1) *For* $f \in C^\alpha([0,t], \mathbb{R}^d)$, $f = 0$ *if and only if* $\sum_{j=1}^d \int_0^t f_j \mathrm{d}h^j = 0$ *for all* $h \in \mathscr{H}$.
(2) *For a.a.* ω, $X(\omega)$ *is truly rough, at least in a right neighborhood of* 0, *that is, there is a dense subset* A *of a right neighborhood of* 0 *such that for any* $s \in A$,

$$\limsup_{t\downarrow s} \frac{|\langle v, X(s,t)\rangle|}{|t-s|^{2\alpha}} = \infty, \quad \textit{for any } v \in \mathbb{R}^d \setminus \{0\}.$$

Moreover, suppose that $\mathrm{Lie}(V_1, \dots, V_d)\big|_{y_0} = \mathbb{R}^e$. *Then, for any* $t > 0$, *the distribution of* $Y(t)$ *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}^e$.

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