Linear Control of a Quadrotor Using LQR approach

Mohammad Ismail, Ahmad Kourani, Rami Hamdan

***Abstract* – This report gives details about the stabilizing and tracking of quadrotor using LQR and LQI approaches. This investigation was done on the full nonlinear Simulink model of Bouabdallah [2]. These two methods depend on two important ways: modeling the rotor dynamics and designing the control law. The first method uses a LQR controller and feeds back the following variables:**

$$\boldsymbol{\phi},\ \dot{\boldsymbol{\phi}},\ \boldsymbol{\theta},\ \dot{\boldsymbol{\theta}},\ \boldsymbol{\psi},\ \dot{\boldsymbol{\psi}},\ \boldsymbol{z},\ \dot{\boldsymbol{z}},\ \boldsymbol{x},\ \dot{\boldsymbol{x}},\ \boldsymbol{y},\ \dot{\boldsymbol{y}}.$$

The second method uses LQI controller and feeds back the following variables:

$$\boldsymbol{\phi},\ \dot{\boldsymbol{\phi}},\ \boldsymbol{\theta},\ \dot{\boldsymbol{\theta}},\ \boldsymbol{\psi},\ \dot{\boldsymbol{\psi}},\ \boldsymbol{z},\ \dot{\boldsymbol{z}},\ \boldsymbol{x},\ \dot{\boldsymbol{x}},\ \boldsymbol{y},\ \dot{\boldsymbol{y}},\ \dot{\boldsymbol{e}}_z,\ \dot{\boldsymbol{e}}_x,\ \dot{\boldsymbol{e}}_y$$

The achieved performances were acceptable and gave rise to satisfactory results. The success of the project was measured against the quadrotor's ability to track a given input trajectory. Finally, the report concludes a suggestions for future work that enhance trajectory tracking.

Key–*Quadrotor, LQR, Linear model*

I. INTRODUCTION

A. Motivation

Linear flight controllers are the most common flight algorithms, that can perform well during flight and hovering, especially in the case of tracking and stabilising the Unmanned Aerial Vehicle (UAV). The algorithm used in this work is concentrated on Linear Quadratic Regulator (LQR) algorithm considering its easy implementation, optimality and robustness in the case of Multi Input Multi Output (MIMO) systems. LQR will be used for stabilizing the system while Linear Quadratic Integrator (LQI) will be implemented for the purpose of tracking the system.

B. Unmanned Aerial Vehicles

UAV's are recently trending upwards around the globe as they became more recognised for their significant usage in many fields other than the military application which was the only practical thought of UAV's. Advantages such as manoeuvrability and reduced cost are faced with drawbacks especially regarding power consumption.

C. Present and Future Applications

Recently quadrotors are being manufactured for civil use and community services, a large number of them is probably produced as toys or games but on the other hand they have proved to actually serve a great roll helping humans in difficult tasks, dirty tasks, and dangerous ones. This Aerial Vehicle design has many advantages that lets it serve such dangerous and difficult tasks, these pluses are for instance requiring small space for launching and retrials, vertical take-off, vertical landing, sideslip, low speed cruise, etc..

D. Control Techniques

A lot of control techniques were used to control the quadrotor, for linear and nonlinear approaches: full state feedback [1], LQR, PID [2] and H infinity [3] controllers. Comparison between these approaches were also made [4], and most had accepted results.

For the scope of this project, the linear model is considered, and a convenient LQR method is to be used for control. C Balas in [5] introduced a linear model with a PID controller and two different feedback approaches.

II. QUADROTOR MODEL

The dynamic model of the quadrotor is well understood and available online, so the purpose of this chapter is not to derive a model from basic equations, but to show the used one. The main model was taken from [6] under the following assumptions:

- The structure of the quadrotor is symmetric and rigid.
- The center of gravity of the quadrotor coincides with the body fixed frame origin.
- The propellers are rigid, and function linearly, no motor induction.
- Thrust and drag are proportional to the square of the propeller's speed.
- Aerodynamic effects acting on the quadrotor body were neglected.
- Gyroscopic effects are neglected.

E. *Quadrotor Characteristics*

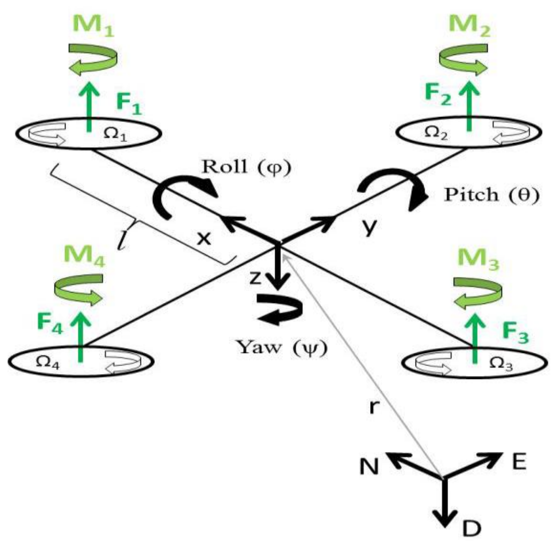


Fig. 1 Quadrotor reference frame with applied forces and moments.

The quadrotor has six degrees of freedom as shown in Fig. 1, three for translation: x, y and z, expressed in the world frame and describing the position of the center of mass of the rotor, and three other orientation Euler angles: the yaw (ψ), the pitch (θ) and the roll(ϕ).

A mini quadrotor model is used with the following characteristics:

$$\begin{aligned} &m = 0.650\ Kg \\ &I_x = I_y = 7.5 * 10^{-3} kg.\, l^2 \\ &I_z = 2.3 * 10^{-2} \\ &l = 0.23\ m \end{aligned} \tag{1}$$

where m is the mass of the quadrotor, I is the inertia, and l is the axel length of the quadrotor.

F. *Rotor dynamics:*

The voltage of each motor (v) can be written as a function of the rotor's velocity, (Ω):

$$v = \frac{R_{mot}}{K_{mot}} J_r \dot{\Omega}_i + K_{mot}\Omega_i + K_M R_{mot} \Omega_i^2 \tag{2}$$

where the equations constants are defined as:

- K_{mot}: motor torque constant
- J_r: inertia of the rotor
- R_{mot}: i^{th} motor resistance
- K_M: aerodynamic moment constant

The rotor dynamics can be approximated to a first order lag transfer function. The desired propeller's speed and the actual speed are mapped by a transfer function [7]:

$$G(s) = \frac{Actual\ rotor\ speed}{Command\ rotor\ speed} = \frac{0.936}{0.178s + 1} \tag{3}$$

From the angular velocity of the propellers, the aerodynamic force and moment of the rotors can be calculated:

$$\begin{aligned} F_i &= K_f \Omega_i^2 \\ M_i &= K_M \Omega_i^2 \end{aligned} \tag{4}$$

Where K_f and K_M are the aerodynamic force and moment constants [6] respectively:

$$\begin{aligned} K_f &= 3.13e^{-5} \\ K_M &= 7.5\ e^{-7} \end{aligned} \tag{5}$$

G. Control Input Vector U:

Control inputs are a combination of voltages applied to each motor, however, from the equations available these voltage can be seen as forces and moments applied directly on the quadrotor. The control inputs are summarized in a vector U:

$$U = [U_1 \quad U_2 \quad U_3 \quad U_4] \tag{6}$$

where

$$\begin{aligned} U_1 &= K_f(\Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2) \\ U_2 &= K_f(-\Omega_2^2 + \Omega_4^2) \\ U_3 &= K_f(\Omega_1^2 - \Omega_3^2) \\ U_4 &= K_M(\Omega_1^2 - \Omega_2^2 + \Omega_3^2 - \Omega_4^2) \end{aligned} \tag{7}$$

These input can be described by the following:

- U_1: resulting upwards force
- U_2: roll force
- U_3: pitch force
- U_4: yaw moment

These combinations will be used in the equations of motion.

The rotor velocities can also be calculated from the control inputs:

$$\begin{aligned} \Omega_1 &= \sqrt{\frac{1}{4K_f}U_1 + \frac{1}{2K_f}U_3 + \frac{1}{4K_M}U_4} \\ \Omega_2 &= \sqrt{\frac{1}{4K_f}U_1 - \frac{1}{2K_f}U_2 - \frac{1}{4K_M}U_4} \\ \Omega_3 &= \sqrt{\frac{1}{4K_f}U_1 - \frac{1}{2K_f}U_3 + \frac{1}{4K_M}U_4} \\ \Omega_4 &= \sqrt{\frac{1}{4K_f}U_1 + \frac{1}{2K_f}U_2 - \frac{1}{4K_M}U_4} \end{aligned} \tag{8}$$

H. Rotational Equations of Motion:

The Inertia matrix is denoted by J:

$$J = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \tag{9}$$

The angular accelerations are given by the following equations:

$$\ddot{\phi} = \frac{l}{I_{xx}}U_2 - \frac{J_r}{I_{xx}}\dot{\theta}\Omega_r + \frac{I_{yy}}{I_{xx}}\dot{\psi}\dot{\theta} - \frac{I_{zz}}{I_{xx}}\dot{\theta}\dot{\psi} \tag{10}$$

$$\ddot{\theta} = \frac{l}{I_{yy}}U_3 - \frac{J_r}{I_{yy}}\dot{\phi}\Omega_r + \frac{I_{zz}}{I_{yy}}\dot{\phi}\dot{\psi} - \frac{I_{xx}}{I_{yy}}\dot{\psi}\dot{\phi} \tag{11}$$

$$\ddot{\psi} = \frac{1}{I_{zz}}U_4 + \frac{I_{xx}}{I_{zz}}\dot{\theta}\dot{\phi} - \frac{I_{yy}}{I_{zz}}\dot{\phi}\dot{\theta} \tag{12}$$

Where Ω_r is defined as

$$\Omega_r = -\Omega_1 + \Omega_2 - \Omega_3 + \Omega_4 \tag{13}$$

I. Translational Equations of Motion:

The linear accelerations are given by the following equations:

$$\ddot{x} = \frac{-U_1}{m}(\sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta) \tag{14}$$

$$\ddot{y} = \frac{-U_1}{m}(\cos\phi\sin\psi\sin\theta - \cos\psi\sin\phi) \tag{15}$$

$$\ddot{z} = g - \frac{U_1}{m}(\cos\phi\cos\theta) \tag{16}$$

J. Translational Equations of Motion:

The linear accelerations are given by the following equations:

$$\ddot{x} = \frac{-U_1}{m}(\sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta) \quad (17)$$

$$\ddot{y} = \frac{-U_1}{m}(\cos\phi\sin\psi\sin\theta - \cos\psi\sin\phi) \quad (18)$$

$$\ddot{z} = g - \frac{U_1}{m}(\cos\phi\cos\theta) \quad (19)$$

K. State Model:

The state vector of the quadrotor is defined as

$$X = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8 \quad x_9 \quad x_{10} \quad x_{11} \quad x_{12}]^T \quad (20)$$

Which is mapped to the degrees of freedom of the quadrotor as:

$$X = [\phi \quad \dot{\phi} \quad \theta \quad \dot{\theta} \quad \psi \quad \dot{\psi} \quad z \quad \dot{z} \quad x \quad \dot{x} \quad y \quad \dot{y}\,]^T \quad (21)$$

The following constants are defined that the state space model could be written in a more compact form:

$$a_1 = \frac{I_{yy} - I_{zz}}{I_{xx}}$$

$$a_2 = \frac{J_r}{I_{xx}} \qquad b_1 = \frac{l}{I_{xx}}$$

$$a_3 = \frac{I_{zz} - I_{xx}}{I_{yy}} \qquad b_2 = \frac{l}{I_{yy}} \quad (22)$$

$$a_4 = \frac{J_r}{I_{yy}} \qquad b_3 = \frac{1}{I_{zz}}$$

$$a_5 = \frac{I_{xx} - I_{yy}}{I_{zz}}$$

Where J_r is the rotors inertia:

$$J_r = 6 * 10^{-5} \quad \text{(II.1)}$$

Now the state space representation of the system can be written as follows:

$$f(X,U) = \begin{bmatrix} x_2 \\ x_4 x_6 a_1 - x_4 \Omega_r a_2 + b_1 U_2 \\ x_4 \\ x_2 x_6 a_3 + x_2 \Omega_r a_4 + b_2 U_3 \\ x_6 \\ x_2 x_4 a_5 + b_3 U_4 \\ x_8 \\ g - \frac{U_1}{m}(\cos x_1 \cos x_3\,) \\ x_{10} \\ \frac{-U_1}{m}(\sin x_1 \sin x_5 + \cos x_1 \sin x_3 \cos x_5) \\ x_{12} \\ \frac{U_1}{m}(\sin x_1 \cos x_5 - \cos x_1 \sin x_3 \sin x_5) \end{bmatrix} \quad (23)$$

III. LQR DESIGN FOR ATTITUDE STABILIZATION

In order to stabilize the quad rotor LQR approach was used. The system is nonlinear, so it was linearized around the hovering position. Then state space matrices A & B were generated using the Jacobian with respect to the states and inputs. Then, Controllability was checked by calculating the controllability matrix and it is found that it is fully controllable over the 12 states. Also, it is assumed that all states are available and no observability test will be made.

A. Controller Design

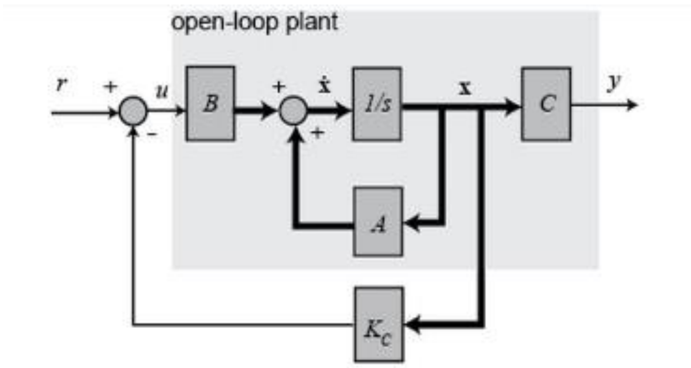


Fig. 2 LQR block diagram [9].

LQR is modern control technique and it's preferred because of its optimality and easy implementation.

The linear time invariant system in state space is:

$$\dot{X} = AX + BU$$
$$Y = CX + DU \tag{24}$$

Where A is the system matrix, B is the input matrix, C is the output matrix, D is the direct transition matrix, U is the input, and X is the states of the system.

$$A = \frac{\partial f}{\partial X} \;\&\; B = \frac{\partial f}{\partial U} \tag{25}$$

LQR aims to minimize the cost function:

$$J = \int_0^{\infty} \left(X^T(t)QX(t) + U^T(t)RU(t)\right)dt \tag{26}$$

where $U(t) = -KX$, K is the feedback of the controller, Q and R are the weighting matrices which represent the weight on states and inputs respectively. After building the model it was simulated for many values of Q. For small values of Q on z very small response was obtained and for high values instability occurs. So, after many iterations a value of three was set for weight on z. Higher weight was applied on roll, and pitch angles since any change in them would cause instability. After many iterations the matrix Q that gave the gain K was obtained which was able to stabilize the Quadrotor. The state cost matrix is finally set to be:

$$diag(Q) = [10 \quad 10 \quad 10 \quad 10 \quad 10 \quad 1 \quad 3 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1] \tag{27}$$

The input cost matrix is taken from [5]:

$$diag(R) = [0.01 \quad 0.1 \quad 0.1 \quad 0.1] \tag{28}$$

Then the feedback gain is calculated using the MATLAB command "LQR (A,B,Q,R)":

$$K_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -18 & -11 & 0 & 0 & 0 & 0 \\ 49 & 10 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 13 & 11 \\ 0 & 0 & 49 & 10 & 0 & 0 & 0 & 1 & -13 & -11 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{29}$$

B. Test and results

In order to test the stability of the system, it was subjected to disturbance on the inputs. The disturbance was in the form of Gaussian noise block with the following standard deviations: $\sigma_1 = \sqrt{0.3}$, $\sigma_2 = \sqrt{0.1}$, $\sigma_3 = \sqrt{0.1}$, and $\sigma_4 = \sqrt{0.001}$.

These noises are applied on inputs u_1, u_2, u_3, and u_4 respectively. As seen in Fig. 3 and Fig. 4 roll and yaw angles variation was between -0.5 and 0.5 degrees. Whereas, X and Z fluctuated for few centimeters around the hovering position. Since there was a small variation in states around hovering position due to disturbance applied on inputs then stability is achieved.

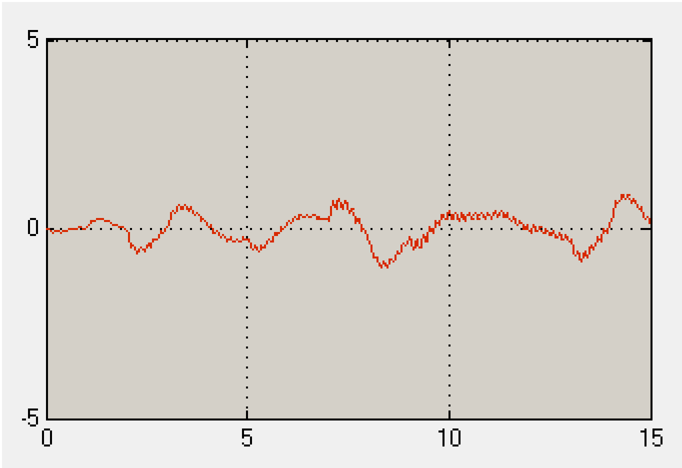


Fig. 3 Theta variation due to disturbance.

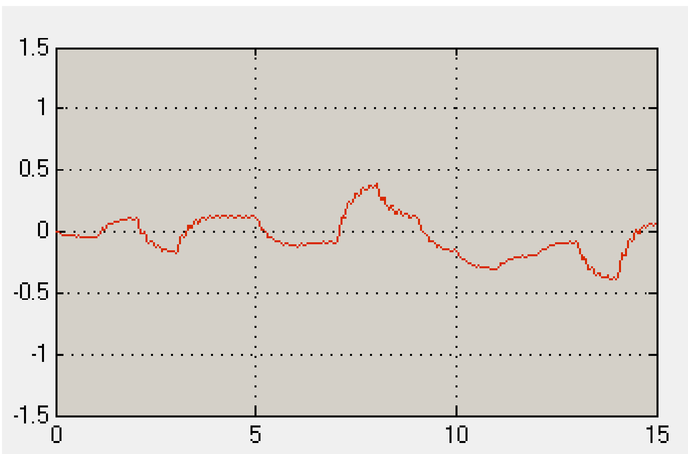


Fig. 4 PSI variation due to disturbance.

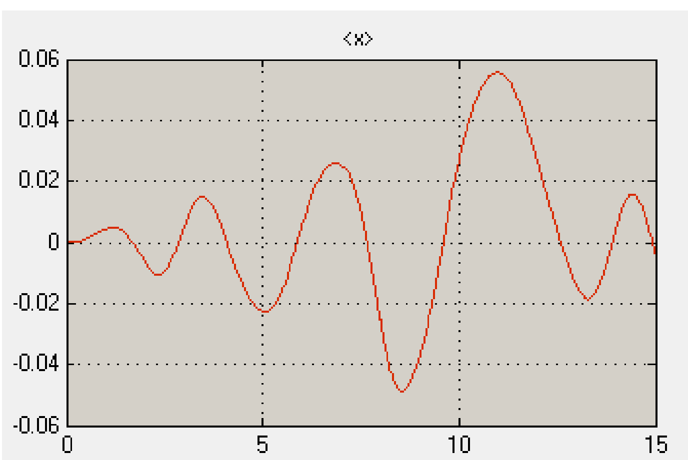


Fig. 5 X variation due to disturbance.

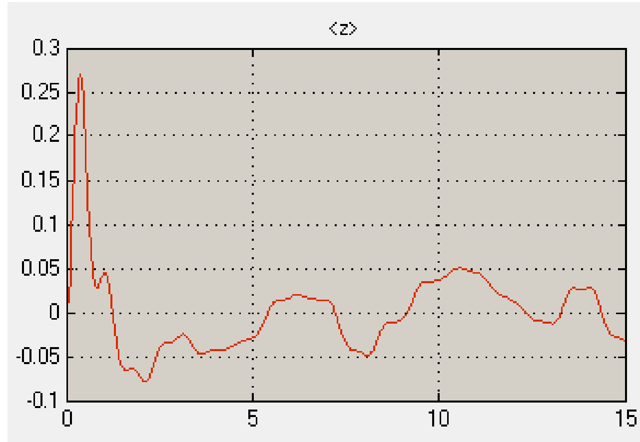


Fig. 6 Variation of Z due to disturbance.

IV. POSITION TRACKING USING LQI

After the design of the LQR that allows the quadrotor to hover and maintain its initial position, a further step can be done to make the quadrotor follow a desired position. The LQI method was chosen to achieve this.

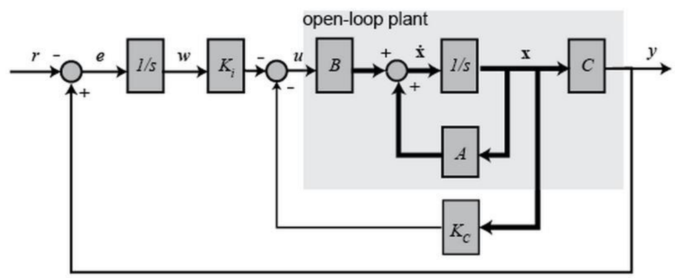


Fig. 7 LQI block diagram [9].

Adding an integrator in series with the plant will remove the steady state error between the output and the step reference, the block diagram of the LQI method is shown in Fig. 7.To implement this method, the state matrix A need to be augmented with the correct states, and since the objective is to track x, y and z position, the following equations will be introduced:

$$\begin{aligned} \dot{e}_z &= z - z_{des} \\ \dot{e}_x &= x - x_{des} \end{aligned} \tag{30}$$

$$\dot{e}_y = y - y_{des}$$

A test was done and proved that the system is still state controllable. The augmented state vector is

$$X = [\phi \quad \dot{\phi} \quad \theta \quad \dot{\theta} \quad \psi \quad \dot{\psi} \quad z \quad \dot{z} \quad x \quad \dot{x} \quad y \quad \dot{y} \quad \dot{e}_z \quad \dot{e}_x \quad \dot{e}_y]^T \tag{31}$$

The same procedure used for LQR is to be followed to find the augmented K matrix:

$$K_a = [K_c \quad K_i] \tag{32}$$

The new state costs are set to be:

$$diag(Q_a) = [15 \quad 10 \quad 15 \quad 10 \quad 10 \quad 1 \quad 1 \quad 1] \quad 2 \quad 1 \quad 2 \quad 1 \quad 1 \quad 5 \quad 5] \tag{33}$$

The input costs [5] are the same as for the LQR:

$$diag(Ra) = [0.01 \quad 0.1 \quad 0.1 \quad 0.1] \tag{34}$$

It can be noticed that the weight on z was decreased, and that is because the z error term was introduced which also affect the motors response speed in the same manner as z component. The rise in x and y errors cost is to increase speed and reduce and reduce the overshoot, but to achieve this efficiently the costs on ϕ and θ need to be increased to maintain stability. The reason beyond this is that if x and y position are as (or closely enough) important as the ϕ and θ values, the quadrotor will tend to move faster when x and y are stimulated, and this happens only by higher values of ϕ and θ, which means less stability.

The resulting gain matrix is as divided:

$$K_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -18 & -11 & 0 & 0 & 0 & 0 \\ 49 & 10 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 13 & 11 \\ 0 & 0 & 49 & 10 & 0 & 0 & 0 & 1 & -13 & -11 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{35}$$

$$K_i = \begin{bmatrix} -10 & 0 & 0 \\ 0 & 0 & 7 \\ 0 & -7 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{36}$$

C. *Test and results*

These gain values were tested for several x, y and z positions, it will maintain stability for some step combination ranging from at least $(1,1,1)$ to $(-1,-1,-1)$. The settling time for all states is about six seconds.

Higher steps may result in simulation error. Mainly, this error is not due to excessive state variation, but to a negative input signal to the rotors, having their calculation involving a square root. However, the controller need to be upgraded to not allow the rotors speeds to fall under a minimum threshold, and still follow the desired position.

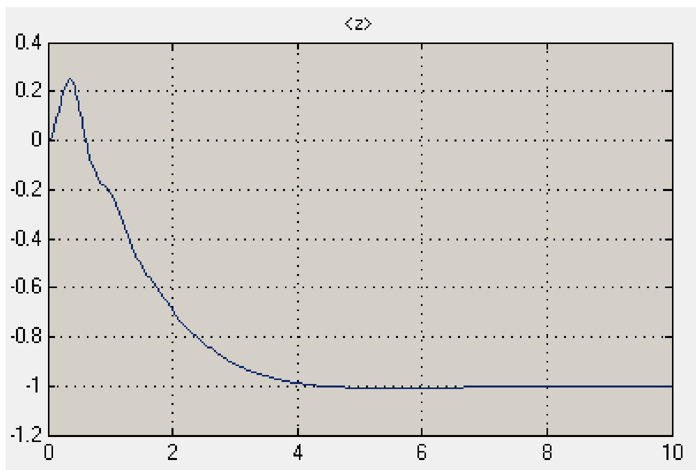


Fig. 8 1 Z variations for (1,1,-1) step.

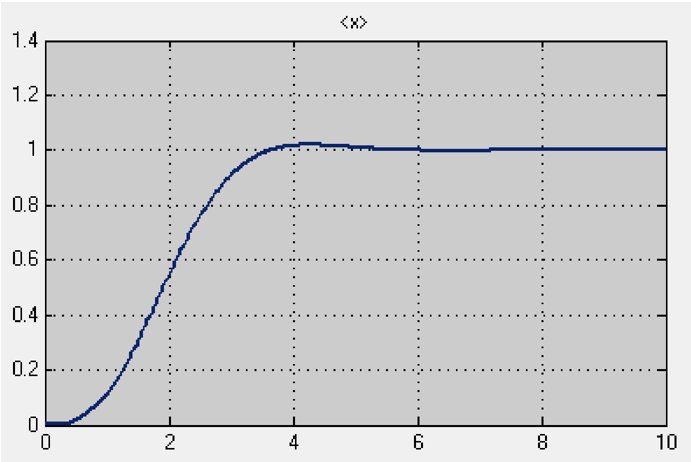


Fig. 9 X variations for (1,1,-1) step.

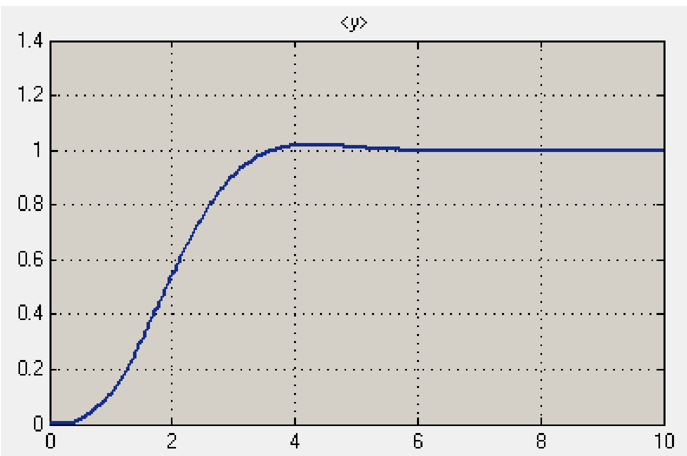


Fig. 10 Y variations for (1,1,-1) step.

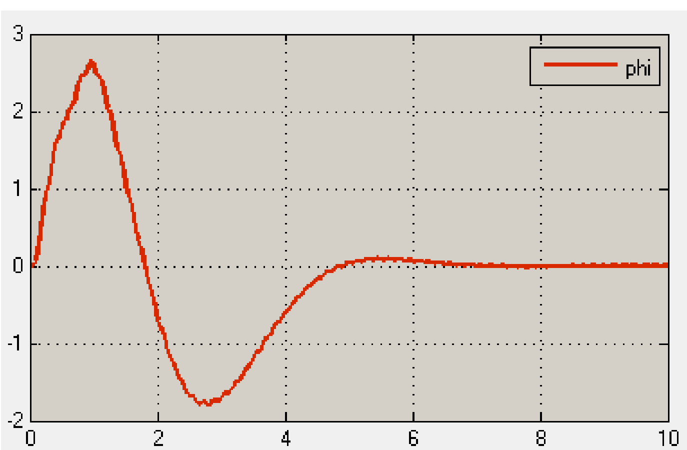


Fig. 11 ϕ variations for (1,1,-1) step.

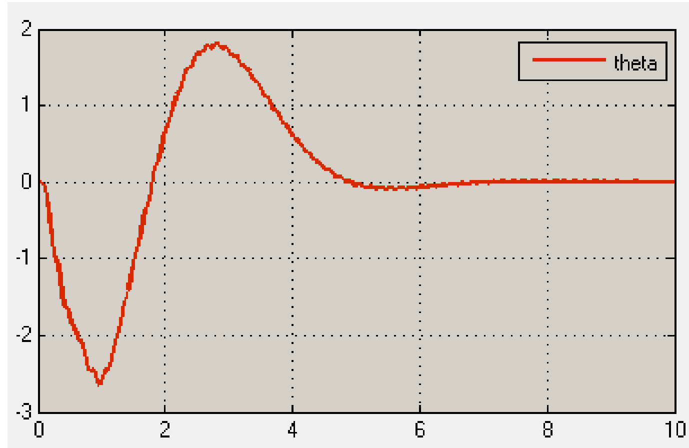


Fig. 12 θ variations for (1,1,-1) step.

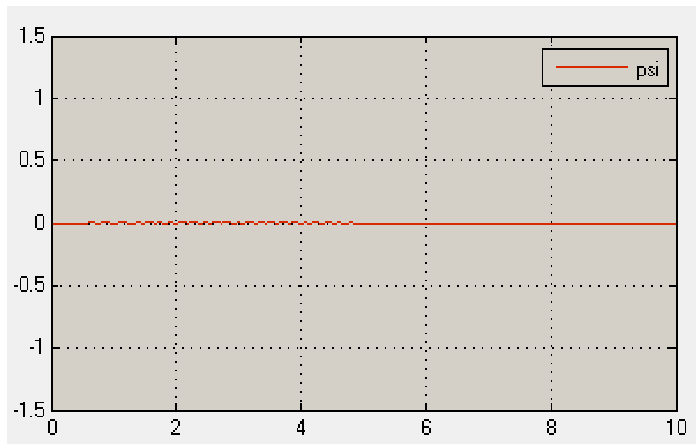


Fig. 13 ψ variations for (1,1,-1) step.

V. CONCLUSION

The report gave satisfactory results in both stability and tracking. But, in order to give better performance the rotors should be replaced in order to avoid steady state errors and un-expectable failures since dynamic performance and robustness of the designed closed loop system depend on the upper limits of the rotor voltages. Also observer design and further implementation should be done in order to test the controllers in real flight.

APPENDIX A

State matrix:

$$A=\begin{bmatrix}0&1&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&-g&0&0&0&0&0&0&0&0&1\\g&0&0&0&0&0&0&0&0&0&0&0\end{bmatrix} \tag{1}$$

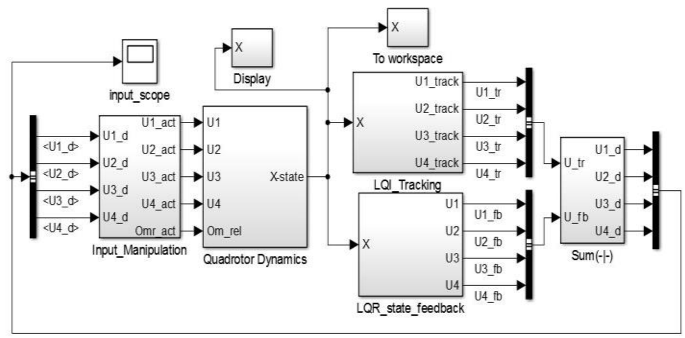
Input matrix:

$$B=\begin{bmatrix}0&0&0&0\\0&b_1&0&0\\0&0&0&0\\0&0&b_2&0\\0&0&0&0\\0&0&0&b_3\\0&0&0&0\\-\frac{1}{m}&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \tag{2}$$

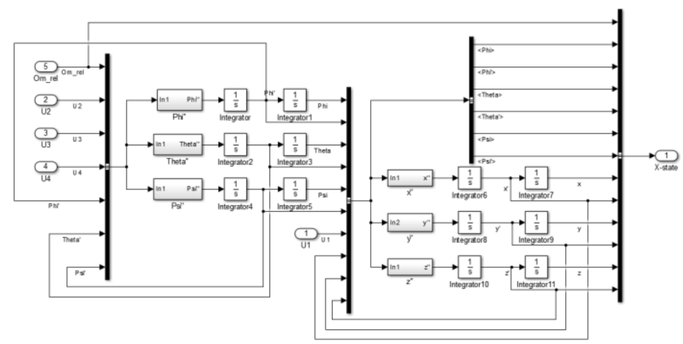
Controllability and observability matrices can be easily calculation throw MATLAB. They have been found of full rank.

APPENDIX B SYMULINK MODELS

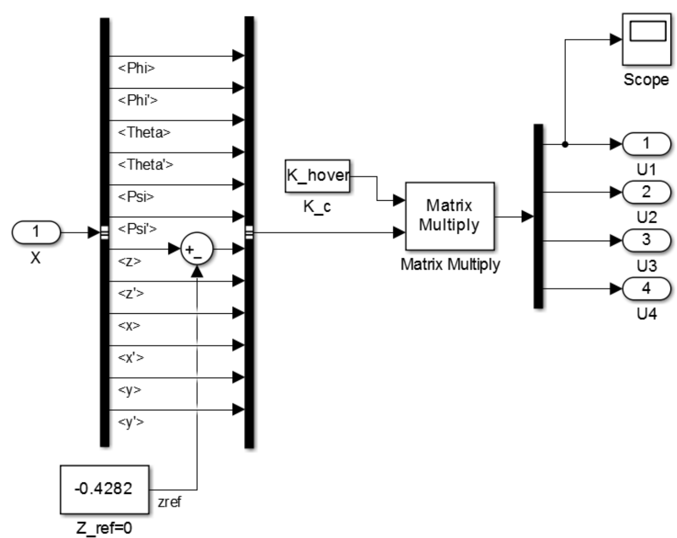
UAV full Model:



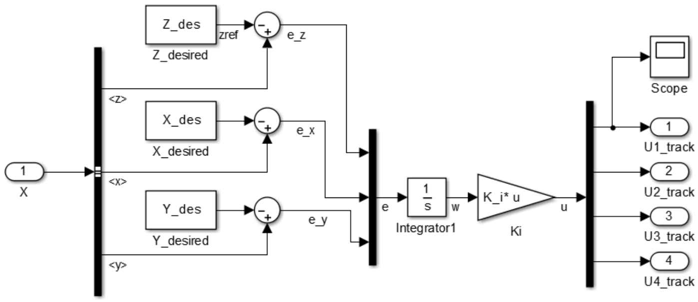
Quadrotor Dynamics:



LQR Block:



LQI Tracking Block:



APPENDIX C GRAPHYCAL USER INTERFACE

A GUI was designed to allow more interaction with the model, Fig. 14 shows the GUI for zero position order and noise described in

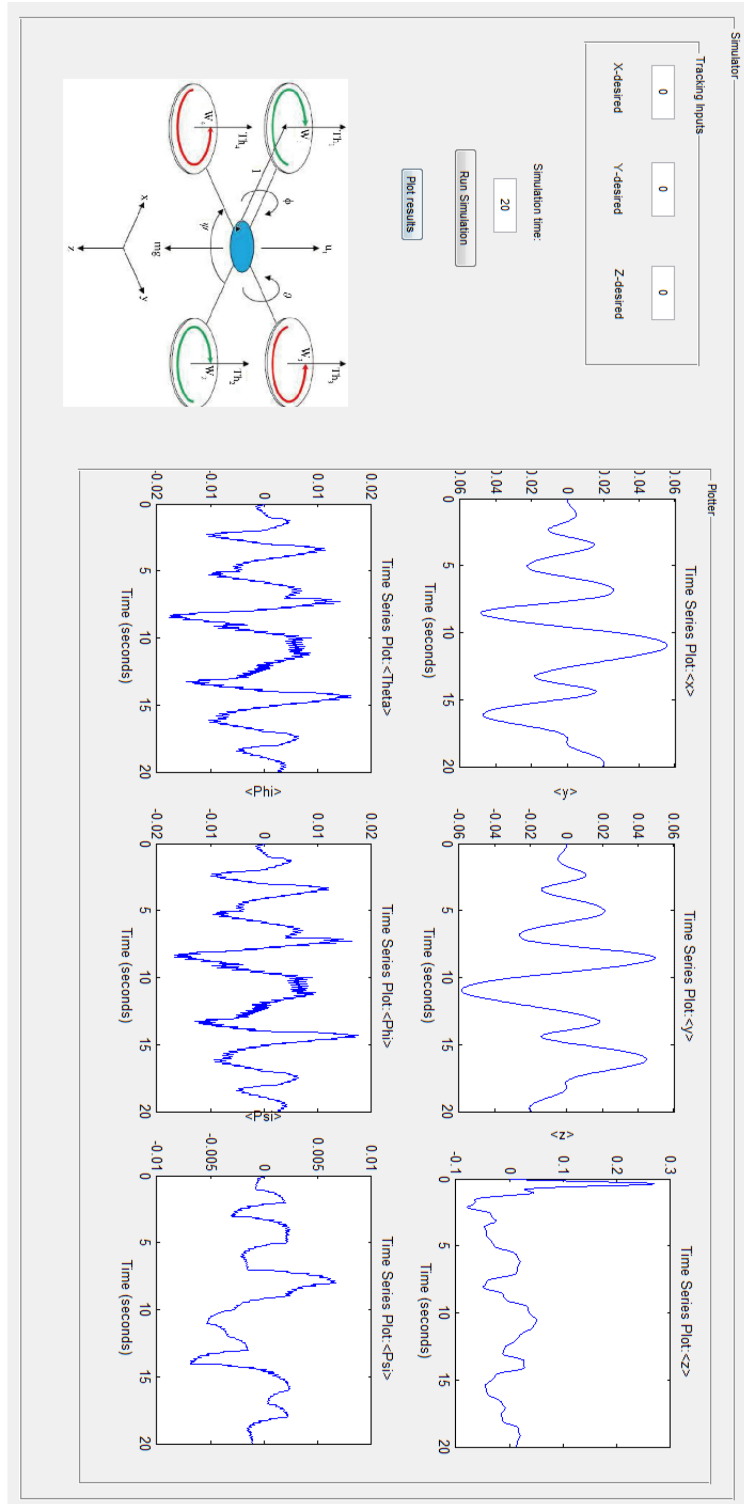


Fig. 14 GUI designed for the quadrotor input control.

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