Advancing Theoretical Astrophysics: Energy Scales

Physical Conditions of Astrophysical Matter

The astrophysical systems of interest span a wide range of parameter space and require inputs from several branches of physics (1keV$\equiv 1.602 \times 10^{-9}$ erg $\equiv 2.417 \times 10^{17}$ Hz $\equiv 1.16 \times 10^{7}$K). Typically, the densities can vary from 10^{-25} g cm^{-3} (Interstellar Medium 'ISM') to 10^{15} g cm^{-3} (neutron stars); temperatures from 2.7 K (Cosmic Microwave Background 'CMB') to 10^{9} K (accreting X-ray sources) or even 10^{15} K (early universe); radiation from wavelengths of meters (radio) to fractions of angstroms (gamma-rays); typical speeds of particles can go to $0.999c$ (relativistic jets). Clearly we require inputs from quantum mechanical and relativistic regimes as well as from more familiar classical physics.

Energy Scales of Astrophysical Phenomena

Let us consider a system of N particles ($N \gg 1$), each of mass m. In dealing with the dynamics of such a large collection of particles, it is useful to introduce the concept of pressure. A system is called *ideal* if the kinetic energy dominates over the interaction energy of the particles. In that case ε is essentially the kinetic energy of the particle. With the relations

$$p = \gamma m v, \qquad \varepsilon = (\gamma - 1) m c^2, \qquad \gamma \equiv \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \tag{1}$$

the pressure can be expressed in the form

$$\wp = \frac{1}{3}\int_0^\infty n_\varepsilon\, \varepsilon \left(1 + \frac{2mc^2}{\varepsilon}\right)\left(1 + \frac{mc^2}{\varepsilon}\right)^{-1} d\varepsilon = \begin{cases} \frac{2}{3} < n\epsilon > & \text{if } mc^2 \gg \epsilon \\ \frac{1}{3} < n\epsilon > & \text{if } mc^2 \ll \epsilon. \end{cases} \tag{2}$$

In the non-relativistic limit, this gives $\wp_{\rm NR} \approx (2/3) < n\varepsilon >= (2/3) U_{\rm NR}$, where $U_{\rm NR}$ is the non-relativistic energy density (i.e. energy per unit volume) of the particles. In the relativistic regime, the corresponding expression is $\wp_{\rm ER} \approx (1/3) < n\varepsilon >= (1/3)U_{\rm NR}$. Hence, in general $\wp \approx U$ up to a factor of unity, where U is the energy density.

This result can be converted into a more useful form of equation of state (which relates the density to the pressure) whenever the mean free path of the particles in the system is small compared with the lengths over which the physical parameters of the system change significantly. Then the pressure can be expressed in terms of density and temperature if the energy density can be expressed in terms of these variables. This is possible in several contexts leading to different equations of state. To understand each of these cases it is useful to start by identifying the characteristic energy scales of bulk matter.

Rest-Mass Energy

We can associate the rest mass energy mc^2 with each particle of mass m. In normal matter, made of nucleons and electrons, the rest-mass energy is

$$mc^2 = \begin{cases} m_e c^2 \approx 0.5\text{MeV} \\ m_p c^2 \approx 1\text{GeV}. \end{cases} \tag{3}$$

Because the total mass of the system is mostly due to nucleons, the total rest-mass energy will be $E_{\rm mass} \cong NAm_pc^2 \cong Mc^2$, where A is the atomic mass number, $Am_p \cong m$ is the mass of each nucleus, and $Nm = M$ is the total mass of the system.

If the particles of the system have internal structure (molecular, atomic, nuclear, etc) then we get further energy scales that are characteristic of the interactions. The simplest is the atomic binding energy of atoms and molecules, which arises from the electromagnetic coupling between particles.

Atomic Binding Energy

The characteristic size scale and energy of the ground state of a hydrogen atom with $Z = 1$ can be found using the Hamiltonian describing an electron, moving in the Coulomb field of a nucleus of charge Zq, which is given by $H_0 = (p^2/2m_e) - (Zq^2/r)$:

$$a_0 = \frac{\hbar^2}{m_e q^2} \approx 5.2 \times 10^{-9} \text{ cm} \tag{4}$$

$$\varepsilon_a = \frac{m_e q^4}{2\hbar^2} = \frac{1}{2}\alpha^2 m_e c^2 \approx 13.6 \text{ eV}. \tag{5}$$

Here $\alpha = q^2/(\hbar c) \approx 7.3 \times 10^{-3}$ is the fine structure constant, and a_0 is the Bohr Radius. The wavelength corresponding to ε_a is

$$\lambda = \frac{hc}{\varepsilon_a} = \frac{2\hbar}{\alpha^2 m_e c} \approx 10^3 \text{Å}$$

and lies in the UV. When atoms of size a_0 are closely packed:

$$n_{\text{solid}} \approx (2a_0)^{-3} \approx 10^{24} \text{ cm}^{-3}.$$

The binding energy of such a solid arises essentially because of the residual electromagnetic force between the atoms, and the typical binding energy per particle is $f\varepsilon_a$ with $f \approx 0.1 - 1$.

Nuclear-Energy Scales

Atomic nuclei are bound by the strong interaction force that produces a binding energy per particle $\sim$ 8 MeV, which is the characteristic scale for nuclear energy levels. In the astrophysical context, a more relevant energy scale is the one at which nuclear reactions can be triggered. For two protons to fuse together, while undergoing nuclear reaction, it is necessary that they are brought within the range of attractive nuclear force, which is approximately $l = 2\pi\hbar/(m_p c)$. Because this requires overcoming the the Coulomb repulsion, such direct interaction can take place only if the kinetic energy of colliding particles is of the order of the electrostatic potential energy at the separation l. This requires energies of the order of

$$\varepsilon \approx \frac{q^2}{l} = \frac{\alpha}{2\pi} m_p c^2 \approx 1 \text{ MeV}. \tag{6}$$

It is, however, possible for nuclear reactions to occur through quantum-mechanical tunneling when the de Broglie wavelength $\lambda_{\rm deB} = h/(m_p v) = l(c/v)$ of the two protons overlap. This occurs when the energy is approximately

$$\varepsilon_{\rm nucl} \approx \frac{\alpha^2}{2\pi^2}\, m_p c^2 \approx 1 \text{ keV}. \tag{7}$$

It is conventional to write this expression as $\varepsilon_{\rm nucl} \approx \eta\alpha^2 m_p c^2$, with $\eta \approx 0.1$.

Gravitational Binding Energy

In the non-relativistic, Newtonian theory of gravity, the gravitational energy of a system of size R and mass M will be $E_{\rm grav} \approx GM^2/R = (Gm_p^2/R)N^2$. The potential energy per particle varies as

$$\varepsilon_g = \frac{E_{\rm grav}}{N} = \frac{Gm_p^2}{R} N = \frac{4\pi}{3}^{1/3} Gm_p^2 N^{2/3} n^{1/3}, \tag{8}$$

where $n = 3N/(4\pi R^3)$ is the number density of particles. The pressure due to gravitational force near the center of an object will be approximately

$$\wp_g \approx \frac{(GM^2/R^2)}{(4\pi R^2)} \approx \frac{1}{3}\left(\frac{4\pi}{3}\right)^{1/3} Gm_p^2 N^{2/3} n^{4/3} \cong \frac{1}{3}\left(\frac{E_{\rm grav}}{V}\right). \tag{9}$$

General relativistic effects become important when $R_{\rm gm} \equiv E_{\rm grav}/E_{\rm mass} \sim 1$, where $R_{\rm gm} \approx 0.7(M/10^{33}\text{ g})(R/1\text{ km})^{-1}$.

Thermal and Degeneracy Energy

A system in local thermodynamical equilibrium can be characterized by a temperature T and energy

$$\varepsilon \approx k_B T. \tag{10}$$

The probability for occupying a state with energy E will then scale as $\propto \exp[-(E/k_B T)]$. The typical momentum of the particle when the temperature is T is given by

$$p \approx mc\left[\frac{2k_B T}{mc^2} + \left(\frac{k_B T}{mc^2}\right)^2\right]^{1/2} \approx \begin{cases} (2mk_B T)^{1/2} & \text{if } mc^2 \gg k_B T \\ k_B T/c & \text{if } mc^2 \ll k_B T \end{cases} \tag{11}$$

In this case, the momentum and the kinetic energy of the particles vanish when $T \to 0$.

The mean energy of a system of electrons will not vanish at zero temperature because electrons obey the Pauli exclusion principle, which requires that the number of electrons that can occupy any quantum state be two, one spin up and another with spin down. The uncertainty principle requires that $\Delta x \Delta p_x \geq h$, we can associate $(d^3x d^3p)/(2\pi\hbar)^3$ micro states with a phase volume $d^3x d^3p$. Therefore the number of quantum states with momentum less than p is $V(4\pi p^3/3)/(2\pi\hbar)^3$, where V is the spatial volume available for the system. The lowest energy state will be the one in which the N electrons fill all levels up to some momentum p_F, called the Fermi momentum This requires that $n = N/V = 2(4\pi p_F^3/3)/(2\pi\hbar)^3$, giving

$$p_F = \hbar(3\pi^2 n)^{1/3}. \tag{12}$$

The quantity ε_F sets the quantum mechanical scale of the energy

$$\varepsilon_F = \sqrt{p_F^2 c^2 + m^2 c^4} - mc^2 \approx \left\{ \begin{array}{ll} \frac{p_F^2}{2m} = \left(\frac{\hbar^2}{2m}\right)(3\pi^2 n)^{2/3} & \text{NR, if } n \ll (\hbar/mc)^{-3} \\ p_F c = (\hbar c)(3\pi^2 n)^{1/3} & \text{ER, if } n \gg (\hbar/mc)^{-3} \end{array} \right\} \tag{13}$$

where the two limits are valid for $p_F c \ll mc^2$ (NR) and $p_F c \gg mc^2$ (extremely relativistic), respectively. Electrons have $\varepsilon_F \sim m_e c^2$ for

$$n = \left(\frac{\hbar}{m_e c}\right)^{-3} \approx 10^{31}\ \mathrm{cm}^{-3} \tag{14}$$

$$\rho \sim n m_p \approx 10^7\ \mathrm{g\ cm}^{-3}. \tag{15}$$

The quantum effects will dominate thermal effects if $k_B T \ll \varepsilon_F$ (degenerate) and classical theory will be valid for $k_B T \gg \varepsilon_F$.

The energy scales which characterize these interactions determine the properties of a collection of particles. For example, if the interaction energy is larger than the binding energy of the atomic system, the atoms will be ionized and the electrons will be separated from the atoms. The familiar case in which this happens is at high temperatures with

$$k_B T \geq \varepsilon_a \approx 13.6\mathrm{eV} \tag{16}$$

when the system is made of free electrons and positively charged ions, whereas, if $k_B T \ll \varepsilon_a$, the system will be neutral. The transition temperature at which nearly half the number of atoms are ionized occurs around

$$k_B T \approx \left(\frac{\varepsilon_a}{10}\right), \tag{17}$$

which is $\sim 10^4$K for hydrogen. For $T \gg 10^4$K, the kinetic energy of the free electrons in the hydrogen plasma will be $\sim k_B T$.

The electrons can be stripped off the atoms in another context. When the density is high enough (the atoms are close enough), the electrons form a common pool

$$\varepsilon_F \geq \varepsilon_a. \tag{18}$$

In this case, the electrons will be quantum mechanical if

$$n > \frac{1}{3\pi^2}\left(\frac{m_e^2 q^4}{\hbar^4}\right)^{3/2} \approx 10^{23}\ \mathrm{cm}^{-3} \tag{19}$$

and the relevant energy scale will be ε_F. The temperature does not enter into the picture if $k_B T \ll \varepsilon_F$ and we may call this a zero-temperature plasma. Conventionally such systems are called degenerate (for normal metals in the laboratory the Fermi energy is comparable with the binding energy within an order of magnitude). For $\epsilon_F \ll m_e c^2$ (NR), $\varepsilon_F \sim k_B T$ occurs at

$$nT^{-3/2} = \frac{(m_e k_B)^{3/2}}{\hbar^3} = 3.6 \times 10^{16} (\text{cgs}). \tag{20}$$

In general the kinetic energy of a particle will have contributions from the temperature as well as from the Fermi energy. If we are interested in only the asymptotic limits, we can take the total kinetic energy per particle to be $\varepsilon \approx \varepsilon_F(n) + k_B T$. By using $\wp \cong n\varepsilon$ (equation [2]), we can obtain the equation of state. First, for a quantum-mechanical gas of fermionic particles with $k_B T \ll \varepsilon_F$ and $\varepsilon \approx \varepsilon_F$, it follows from equation [13] that $\wp_{\rm NR} \propto n^{5/3}$ and $\wp_{\rm ER} \propto n^{4/3}$. Whether the system is relativistic or not is decided by the ratio $p_F/(mc)$ or – equivalently – the ratio $\varepsilon_F/(mc^2)$. The transition occurs at

$$n = \left(\frac{\hbar}{m_e c}\right)^{-3} \approx 10^{31}\ \text{cm}^{-3} \tag{21}$$

$$\rho \sim n m_p \approx 10^7\ \text{g cm}^{-3}. \tag{22}$$

Second, if the system is classical with $k_B T \gg \varepsilon_F$ so that $\varepsilon \approx k_B T$, then $\wp \approx n k_B T$ in both NR and ER limits.