Tidal Disruption and the Roche Limit

This problem examines why ring systems about all the giant planets occupy planetocentric distances that are less than ∼2 planetary radii.

(a) Consider a perfectly rigid, spherical satellite of radius R_s, mass m_s, and density ρ_s orbiting a planet of radius R_p, mass m_p, and density ρ_p. Assume the satellite to be in synchronous lock, so that its spin period matches its orbital period. Take the satellite's orbital semi-major axis to be a_s and its orbital eccentricity to be zero.

A marble rests on the surface of this spinning satellite. The spin of the satellite tries to spin it off. The tidal field of the planet also tries to pull it off. The only force trying to keep it glued to the satellite is the satellite's own gravity. For small enough a_s, the marble will fly off. What is this minimum semi-major axis, $a_{s,1}$? Express in terms of ρ_s, ρ_p, and R_p.

Answer: The marble is at its most unstable when it lies right in between the planet and the host satellite. Then the tidal acceleration from the planet acting to pull the marble off is $|(d/da)(Gm_p/a^2)R_s| = 2(Gm_p/a^3)R_s$. The centrifugal acceleration of the satellite acting to spin the marble off is $(Gm_p/a^3)R_s$. At $a = a_{s,1}$, these accelerations add to barely balance the satellite's gravitational pull, Gm_s/R_s^2. Then

$$3\frac{Gm_p}{a_{s,1}^3}R_s = \frac{Gm_s}{R_s^2} \tag{33}$$

Solve for $a_{s,1}$ we find:

$$a_{s,1} = \left(\frac{3\rho_p}{\rho_s}\right)^{1/3} R_p = 1.44\left(\frac{\rho_p}{\rho_s}\right)^{1/3} R_p \tag{34}$$

(b) How does your answer in (a) relate to the radius of the Hill sphere of the satellite, $r_H = (m_s/3m_p)^{1/3}a_s$?

Answer:

When $a = a_{s,1}$, then

$$r_H = (\rho_s/3\rho_p)^{1/3}(R_s/R_p)(3\rho_p/\rho_s)^{1/3}R_p = R_s$$

the satellite just fills its Hill sphere (Roche lobe). When $a < a_{s,1}$, we say the satellite overfills its Roche lobe. When $a > a_{s,1}$, we say the satellite underfills its Roche lobe.

(c) Now consider a marble floating on a perfectly strengthless, fluid, synchronously rotating satellite. The satellite's shape is now free to distort because it is sitting in the tidal field of the planet and because it is spinning.

ESTIMATE (no heroics necessary) the semi-major axis of the satellite, $a_{s,2}$, inside of which the marble flies off the watery satellite. This is a repeat of (a) except that you will need to account for the distorted shape of the satellite; the satellite has an enhanced size due not only to the tide raised on it by the planet, but also due to its spin. You should at least decide whether $a_{s,2}$ should be larger or smaller than $a_{s,1}$.

Answer:

The marble now floats an extra distance away from the center of the satellite. Let's estimate the new distance as $R' = R_s[1+(m_p/m_s)(R_s/a)^3+\omega^2/G\rho_s]$, where the first enhancement factor comes from the tide raised on the watery satellite by the planet, and the second enhancement factor comes from the spin-induced bulge. For a synchronous satellite, the spin $\omega^2 = Gm_p/a^3$. Then $R' = R_s\{1 + [(4\pi+3)/3](m_p/m_s)(R_s/a)^3\}$. Replace R_s in (33) by R' to find

$$3x\left(1+\frac{4\pi+3}{3}x\right)^3 = 1 \tag{35}$$

where $x \equiv (m_p/m_s)(R_s/a)^3$. My pocket calculator gives $x \approx 0.0995$. Then

$$a_{s,2} = x^{-1/3}\left(\frac{m_p}{m_s}\right)^{1/3} R_s \tag{36}$$

$$a_{s,2} = 2.2\left(\frac{\rho_p}{\rho_s}\right)^{1/3} R_p \tag{37}$$

Chandrasekhar did a more proper job and got the coefficient to be 2.46 instead of our 2.2.

(d) At orbital semi-major axes of less than ~2 planetary radii, there are no satellites whose sizes exceed 100 km, but there are rings composed of meter-sized boulders and smaller debris, and 10 km-sized satellites. Given your answers in (a) and (c), explain why these observations make sense.

Answer: Since $\rho_p/\rho_s \approx 1$, we should expect bodies which are held together mostly by self-gravity to be ripped apart by *both* tidal and centrifugal forces if they are found at planetocentric distances less than $[1.4\text{–}2.2]R_p$. Small bodies which are held together mostly by intermolecular cohesive forces rather than self-gravity are immune to this disruption. Thus, plenty of small bodies—ring particles, small satellites—can exist within $\sim 2R_p$, but large bodies are torn apart. The tidal/centrifugal disruption zone is referred to as the "Roche zone."