Advancing Theoretical Astrophysics: Astrophysical Structures

The Existence of Giant Planets

The atomic binding energy (per particle) of a system is approximately

$$\varepsilon_{\rm a} \approx \alpha^2 m_e c^2 \approx \frac{q^2}{a_o} \approx q^2 n^{1/3} \tag{1}$$

if the atoms are closely packed (with $na_0^3 \approx 1$), and the gravitational energy per particle is

$$\varepsilon_{\rm g} = \left(\frac{4\pi}{3}\right)^{1/3} G m_p^2 N^{2/3} n^{1/3}. \tag{2}$$

Their ratio is given by

$$\frac{\varepsilon_{\rm a}}{\varepsilon_{\rm g}} \approx \left(\frac{\alpha}{\alpha_G}\right)\left(\frac{1}{N^{2/3}}\right) \equiv \left(\frac{N_G}{N}\right)^{2/3} \approx \left(\frac{10^{54}}{N}\right)^{2/3}. \tag{3}$$

Clearly, the number $N_G \equiv \alpha^{3/2}\alpha_G^{-3/2} \approx 10^{54}$, arising out of fundamental constants, sets the smallest scale in astrophysics, in which the gravitational binding energy becomes as important as the electromagnetic binding energy of matter. The corresponding mass and length scales are

$$M_{\rm planet} = N_G m_p \approx 10^{30}\ {\rm g} \tag{4}$$

and

$$R_{\rm planet} = N_G^{1/3} a_0 \approx 10^{10}\ {\rm cm} \tag{5}$$

and correspond to those of a large planet; for smaller masses gravity is ignorable and matter is homogeneous with constant density so that

$$M \propto R^3. \tag{6}$$

Most of the astrophysically interesting systems have larger mass and require the gravitational force to be balanced by forces other than normal solid-state forces. In general, such systems can be classified in two categories. The first set has the gravitational force balanced by the kinetic energy of classical motion, whereas the second one has the gravitational force balanced by degeneracy pressure. For a system with $na_0^3 \approx 1$, the non-relativistic Fermi energy of electrons is comparable with the atomic binding energy and we can compare $\varepsilon_{\rm a}$ or $\varepsilon_{\rm F}$ with $\varepsilon_{\rm g}$. This is meaningful as long as the temperature of the system is low.

The Existence of Stars

To take into account both thermal and quantum degenerate contributions, we take the matter pressure to be $P \approx nk_BT + n\varepsilon_F$, which is a simple interpolation between the two limits. This pressure can balance the gravitational pressure if $(k_BT + \varepsilon_F) \approx Gm_p^2N^{2/3}n^{1/3}$. Using expression for ε_F for non-relativistic electrons, we get

$$k_BT \approx Gm_p^2N^{2/3}n^{1/3} - \frac{(3\pi^2)^{2/3}}{2}\frac{\hbar^2}{m_e}n^{2/3}. \tag{7}$$

For a classical system, the first term on the right hand side dominates, and we see that the gravitational potential energy and kinetic energy corresponding to the temperature T are comparable; this is merely a restatement of the virial theorem. As the radius of the system R is reduced, the second term on the right-hand side ($\propto n^{2/3}$) grows faster than the first ($\propto n^{1/3}$) and the temperature of the system will increase, reach maximum and decrease again; equilibrium is possible for any of these values with gravity balanced by thermal and degeneracy pressure. The maximum temperature $T_{\rm max}$ is reached when $n = n_c$, with

$$n_c \cong \frac{\alpha_G}{(3\pi^2)^{2/3}}\left(\frac{N^{2/3}}{\lambda\!\!\!^{-}_e}\right); \tag{8}$$

$$k_BT_{\rm max} \approx \frac{\alpha_G^2}{2(3\pi^2)^{2/3}}N^{4/3}m_ec^2 \tag{9}$$

where $\lambda\!\!\!^{-}_e \equiv \hbar/(m_ec)$ and $\alpha_G \equiv Gm_p^2/(\hbar c)$.

An interesting phenomenon arises if the maximum temperature $T_{\rm max}$ is sufficiently high to trigger nuclear fusion in the system; then we obtain a gravitationally bound, self-sustained nuclear reactor. The condition for triggering nuclear reaction occurs at energy scales higher than $\varepsilon_{\rm nucl} \approx \eta\alpha^2 m_p c^2$, with $\eta \approx 0.1$. The energy corresponding to $k_B T_{\rm max}$ will be larger than $\varepsilon_{\rm nucl}$ when

$$N > (2\eta)^{3/4}(3\pi^2)^{1/2}\left(\frac{m_p}{m_e}\right)^{3/4}\left(\frac{\alpha}{\alpha_G}\right)^{3/2} \approx 4\times10^{56} \tag{10}$$

for $\eta = 0.1$. The corresponding condition on mass is $M > M_*$, where

$$M_* > (2\eta)^{3/4}(3\pi^2)^{1/2}\left(\frac{m_p}{m_e}\right)^{3/4}\left(\frac{\alpha}{\alpha_G}\right)^{3/2} m_p \approx 4\times10^{32}\ \mathrm{g}, \tag{11}$$

which is comparable to the mass of the smallest stars observed in our Universe.

The Existence of Compact Remnants: White Dwarfs, Neutron Stars and Black Holes

When the nuclear fuel in the star is exhausted, the gravitational force will start contracting the matter again and the density will increase. Eventually, the density will be high enough that the quantum degeneracy pressure will dominate over the thermal pressure. The equilibrium condition for such a system will require the degeneracy pressure of matter to be large enough to balance gravitational pressure. Equivalently, the Fermi energy $\varepsilon_F(n)$ must be larger than the gravitational potential energy $\varepsilon_g \approx G m_p^2 N^{2/3} n^{1/3}$. When the particles are non-relativistic they obey the equation: $\varepsilon_F(n) = (\hbar^2/2m_e)(3\pi^2)^{2/3} n^{2/3}$ and the condition $\varepsilon_F \geq \varepsilon_g$ can be satisfied at equality if

$$n^{1/3} = \frac{2}{(3\pi^2)^{2/3}}\left(\frac{G m_p^2 m_e}{\hbar^2}\right) N^{2/3}. \tag{12}$$

With $n = 3N/(4\pi R^3)$ and $N = (M/m_p)$, this reduces to the following mass-radius relation:

$$R M^{1/3} \approx \alpha_G^{-1} \lambda\!\!\!^{-}_e m_p^{1/3} \approx 8.7\times10^{-3} R_\odot M_\odot^{1/3}. \tag{13}$$

When the density is still higher, the Fermi energy has to be supplied by relativistic particles, and ε_F now becomes $\varepsilon_F \approx \hbar c n^{1/3}$, which sclaes as $n^{1/3}$ just like ε_g. Therefore, the

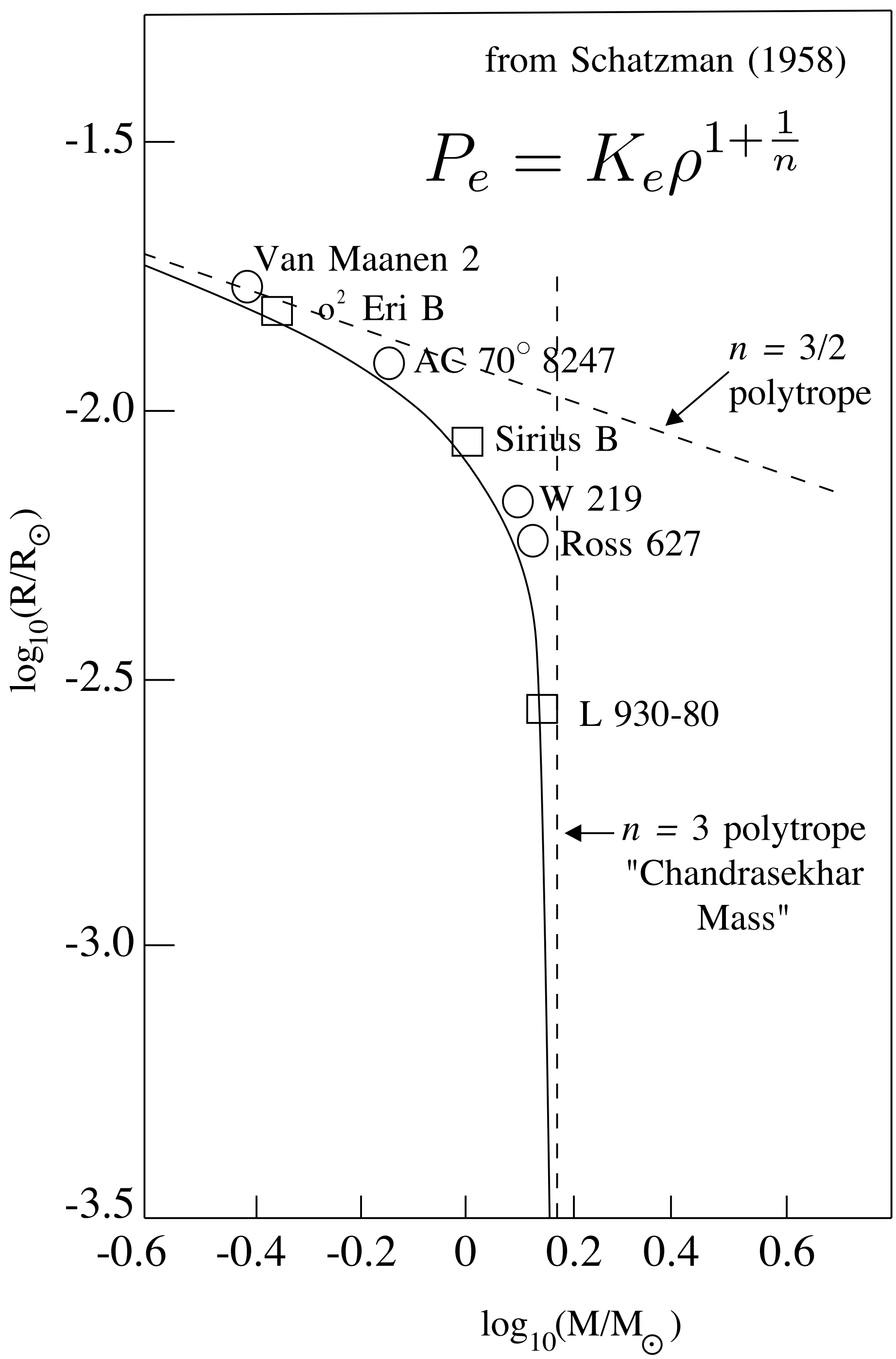

Fig. 1.— White dwarfs in the log R - log M plane.

equilibrium condition is invariant to the number-density; i.e. $\varepsilon_F > \varepsilon_g$ can be satisfied only if $\hbar c \geq G m_p^2 N^{2/3}$ or $N \leq \alpha_G^{-3/2}$. The corresponding mass bound (called the Chandrasekhar limit) is $M_{\rm ch} \leq m_p \alpha_G^{-3/2} \approx 1 M_\odot$. Such structures are called white dwarfs. A white dwarf with $M \approx M_\odot$ will have $R \approx 10^{-2} R_\odot$ and $\rho \approx 10^6 \rho_\odot$ (Figure 1).

As the density increases, electrons combine with protons through inverse beta decay (also called "electron-capture") to form neutrons, which can also provide degeneracy pressure. Equation [12] is still applicable with m_e replaced with m_n; correspondingly the right-hand side of relation [13] is reduced by $(\lambda\!\!\!^{-}_n/\lambda\!\!\!^{-}_e) = (m_e/m_n) \approx 10^{-3}$. Such objects - called neutron stars – will have a radius $R \approx 10^{-5} R_\odot$ and $\rho \approx 10^{15} \rho_\odot$ if $M = M_\odot$. If the mass of the stellar remnant is higher than $\alpha_G^{-3/2} m_p$, no physical process can provide support against the gravitational collapse. In such a case, the star will form a black hole.