NCERT - 6.5.14

EE224BTECH11044 - Muthyala koushik

I. APPLICATION OF DERIVATIVES

Question: Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximized.

Solution:

Let $z = xy^3$. From the equation $x + y = 60$, we can express x as:

$$x = 60 - y. \tag{1}$$

Substitute this into z:

$$z = xy^3 = (60 - y) \cdot y^3, \tag{2}$$

$$z = 60y^3 - y^4. \tag{3}$$

To maximize z, we take its derivative with respect to y and set it equal to zero:

$$\frac{dz}{dy} = \frac{d}{dy}(60y^3 - y^4), \tag{4}$$

$$\frac{dz}{dy} = 180y^2 - 4y^3, \tag{5}$$

$$\frac{dz}{dy} = y^2(180 - 4y). \tag{6}$$

Setting $\frac{dz}{dy} = 0$:

$$y^2(180 - 4y) = 0. \tag{7}$$

This gives two possibilities:

- $y^2 = 0 \implies y = 0$ (not valid as $y > 0$), and
- $180 - 4y = 0 \implies y = 45$.

Substituting $y = 45$ into $x + y = 60$:

$$x = 60 - 45 \implies x = 15. \tag{8}$$

To confirm this is a maximum, calculate the second derivative of z:

$$\frac{d^2z}{dy^2} = \frac{d}{dy}(180y^2 - 4y^3), \tag{9}$$

$$\frac{d^2z}{dy^2} = 360y - 12y^2. \tag{10}$$

At $y = 45$:

$$\frac{d^2z}{dy^2} = 360(45) - 12(45^2), \tag{11}$$

$$\frac{d^2z}{dy^2} = 16200 - 24300, \tag{12}$$

$$\frac{d^2z}{dy^2} = -8100. \tag{13}$$

Since $\frac{d^2z}{dy^2} < 0$, the value of z is maximized at $y = 45$.

Final Answer: The two numbers are $x = 15$ and $y = 45$.

Solution by the method of Gradient Descent: Using the the method of gradient accent;

$$y_{n+1} = y_n + h * F^{'}(y_n) \tag{14}$$

from the equation above:

$$y_{n+1} = y_n + h * \left(y_n{}^2\,(180 - 4y_n)\right) \tag{15}$$

Choosing $y_0 = 20$, $\alpha = 0.0001$, we get thave value of y as,
x value at maxima = 15.00000114024828
y value at maxima = 44.99999885975172

Solution using geometric programming:
Geometric programming deals with problems where the objective and constraints are expressed as posynomials or monomials.

A monomial is of the form:

$$x_1{}^{a_1} x_2{}^{a_2} \cdots x_n{}^{a_n}, \tag{16}$$

where $x_1, x_2, \ldots, x_n$ are variables, and $a_1, a_2, \ldots, a_n$ are real constants.

A posynomial is a sum of monomials with non-negative coefficients:

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{N} c_i \prod_{j=1}^{n} x_j^{a_{ij}}, \tag{17}$$

where $c_i \geq 0$.

The optimization problem is:

$$\text{Minimize: } f(x_1, x_2, \ldots, x_n), \tag{18}$$

$$\text{Subject to: } g_i(x_1, x_2, \ldots, x_n) \leq 1, \quad i = 1, 2, \ldots, m, \tag{19}$$

where $g_i(x_1, x_2, \ldots, x_n)$ are posynomials.

For $f(x, y) = xy^3$, it is a monomial:

$$f(x, y) = x^1 y^3. \tag{20}$$

The Lagrangian for an objective function $f(x, y)$ and a constraint $g(x, y)$ is given by:

$$L(x, y, \lambda) = f(x, y) + \lambda \cdot g(x, y). \tag{21}$$

We take our objective function to be $f(x, y) = xy^3$, since maximizing xy^3 is equivalent to maximizing its logarithm $\ln(xy^3)$, as ln is an increasing function.

The constraint is $g(x, y) = x + y - 60$. Applying the Lagrangian, we get:

$$L(x, y, \lambda) = xy^3 + \lambda(x + y - 60). \tag{22}$$

To find the optimal solution, we take the partial derivatives of L with respect to x, y, and λ, and set them to zero:

1. Partial derivative with respect to x:

$$\frac{\partial L}{\partial x} = y^3 + \lambda = 0 \implies \lambda = -y^3. \tag{23}$$

2. Partial derivative with respect to y:

$$\frac{\partial L}{\partial y} = 3xy^2 + \lambda = 0 \implies \lambda = -3xy^2. \tag{24}$$

3. Partial derivative with respect to λ:

$$\frac{\partial L}{\partial \lambda} = x + y - 60 = 0 \implies x + y = 60. \tag{25}$$

From the first two equations, we equate λ:

$$-y^3 = -3xy^2. \tag{26}$$

Solving for x in terms of y:

$$x = \frac{y}{3}. \tag{27}$$

Substitute this into the constraint $x + y = 60$:

$$\frac{y}{3} + y = 60 \implies \frac{4y}{3} = 60 \implies y = 45. \tag{28}$$

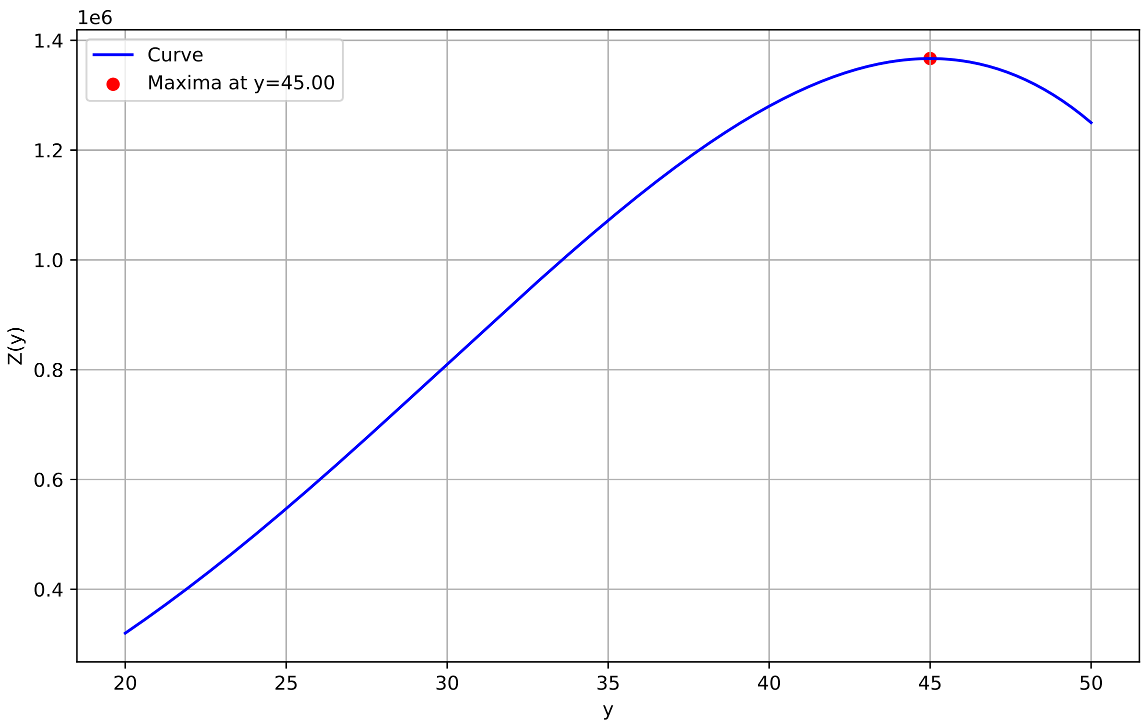
Using $x = \frac{y}{3}$:

$$x = \frac{45}{3} = 15. \tag{29}$$

Thus, the optimal solution is:

$$x = 15 \quad \text{and} \quad y = 45. \tag{30}$$

x using geometric programming 14.99878409289183
y using geometric programming 45.00121536318684


1e6
1.4
1.2
1.0
0.8
0.6
0.4
Z(y)
Curve
Maxima at y=45.00
20
25
30
35
40
45
50
y