NCERT - 6.5.14

EE224BTECH11044 - Muthyala koushik

I. APPLICATION OF DERIVATIVES

Question: Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximized.

Solution:

Let $z = xy^3$. From the equation $x + y = 60$, we can express x as:

$$x = 60 - y. \tag{1}$$

Substitute this into z:

$$z = xy^3 = (60 - y) \cdot y^3, \tag{2}$$
$$z = 60y^3 - y^4. \tag{3}$$

To maximize z, we take its derivative with respect to y and set it equal to zero:

$$\frac{dz}{dy} = \frac{d}{dy}(60y^3 - y^4), \tag{4}$$
$$\frac{dz}{dy} = 180y^2 - 4y^3, \tag{5}$$
$$\frac{dz}{dy} = y^2(180 - 4y). \tag{6}$$

Setting $\frac{dz}{dy} = 0$:

$$y^2(180 - 4y) = 0. \tag{7}$$

This gives two possibilities:
- $y^2 = 0 \implies y = 0$ (not valid as $y > 0$), and
- $180 - 4y = 0 \implies y = 45$.

Substituting $y = 45$ into $x + y = 60$:

$$x = 60 - 45 \implies x = 15. \tag{8}$$

To confirm this is a maximum, calculate the second derivative of z:

$$\frac{d^2z}{dy^2} = \frac{d}{dy}(180y^2 - 4y^3), \tag{9}$$
$$\frac{d^2z}{dy^2} = 360y - 12y^2. \tag{10}$$

At $y = 45$:

$$\frac{d^2z}{dy^2} = 360(45) - 12(45^2), \tag{11}$$

$$\frac{d^2z}{dy^2} = 16200 - 24300, \tag{12}$$

$$\frac{d^2z}{dy^2} = -8100. \tag{13}$$

Since $\frac{d^2z}{dy^2} < 0$, the value of z is maximized at $y = 45$.

Final Answer: The two numbers are $x = 15$ and $y = 45$.

Solution by the method of Gradient Descent: Using the the method of gradient accent;

$$y_{n+1} = y_n + h * F^{'}(y_n) \tag{14}$$

from the equation above:

$$y_{n+1} = y_n + h * \left(y_n{}^2\,(180 - 4y_n)\right) \tag{15}$$

Choosing $y_0 = 20$, $\alpha = 0.0001$, we get thave value of y as,

$$y = 45 \tag{16}$$