NCERT - 8.3.17

EE24BTECH11044 - MUTHYALA KOUSHIK*

Question

The area bounded by the curve $y = x\left|x\right|$, x-axis and the ordinates $x = -1$ and $x = 1$ is given by

Solution

- The function $y = x|x|$ is piecewise defined:

$$y = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$

- The area is computed as:

$$A = \int_{-1}^{1} |y| \, dx.$$

Splitting the Integral

- The function changes at $x = 0$, so split the integral:

$$A = \int_{-1}^{0} \left|-x^2\right| \, dx + \int_{0}^{1} x^2 \, dx.$$

- We evaluate each part separately.

Evaluating the Integrals

- For the first integral:

$$A_1 = \int_{-1}^{0} \left|-x^2\right| \, dx = \left[\frac{x^3}{3}\right]_{-1}^{0} = \frac{1}{3}.$$

- For the second integral:

$$A_2 = \int_{0}^{1} x^2 \, dx = \left[\frac{x^3}{3}\right]_{0}^{1} = \frac{1}{3}.$$

- The total area is:

$$A = A_1 + A_2 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

Therefore, the area is:

$$A = 0.66666666667.$$

Numerical Solution Using Trapezoidal Rule

Numerical Solution: We can approximate the integral using the trapezoidal rule:

$$J = \int_a^b f(x)\,dx \approx h\left(\frac{1}{2}f(a) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2}f(b)\right),$$

where $h = \frac{b-a}{n}$ is the step size.
For our problem:

$$A = A_n, \text{ where } A_{i+1} = A_i + h \cdot \frac{f(x_{n+1}) + f(x_n)}{2}.$$

$$A_{i+1} = A_i + \frac{h}{2}\left(f(x_{n+1}) + f(x_n)\right).$$

Numerical Solution

Iteration Formula:

- $$A_{i+1} = A_i + \frac{h}{2}\left({x_{n+1}}^2 + {x_n}^2\right), \quad x_{n+1} = x_n + h.$$

The step size $h = \frac{2}{n}$, where n is the number of subintervals. For this example, assume $n = 1000$. The initial conditions are:

- $a = -1$,
- $b = 1$,
- $A_0 = 0$,
- $h = \frac{2}{n}$.

We compute A iteratively until we reach the final area.

Theoretical vs Computational Results

The theoretical value of the area is:

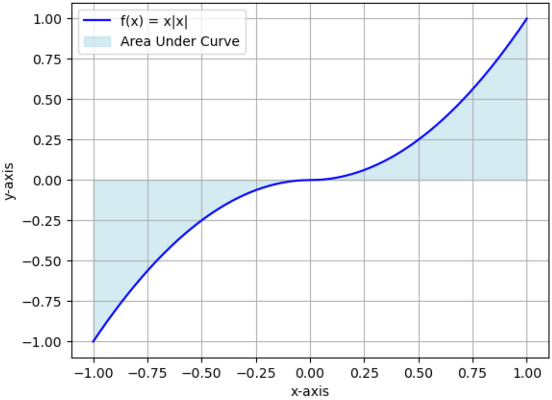
$$A = \frac{2}{3} \approx 0.66666666667.$$

Using the trapezoidal rule, the computed value of the area is:

$$A \approx 0.6666679999999998.$$

Thus, the computed value is very close to the theoretical value, showing that the trapezoidal rule provides a good approximation.

Plot



f(x) = x|x|
Area Under Curve
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
y-axis
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
x-axis