NCERT -10.3.4.2.2

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I. DIFFERENTIAL EQUATIONS

Question: Five years ago, Nuri was thrice as old as Sonu. Ten years later, Nuri will be twice as old as Sonu. How old are Nuri and Sonu?

Solution: Let Nuri's current age is x and Sonu's current age is y:

- Five years ago, Nuri was thrice as old as Sonu.

$$x - 5 = 3\,(y - 5) \implies x - 3y = -10 \tag{1}$$

- Ten years later, Nuri will be twice as old as Sonu.

$$x + 10 = 2\,(y + 10) \implies x - 2y = 10 \tag{2}$$

Given Equations:

$$x - 3y = -10, \tag{3}$$
$$x - 2y = 10. \tag{4}$$

Eliminate x:

$$(x - 2y) - (x - 3y) = 10 - (-10), \tag{5}$$
$$y = 20. \tag{6}$$

Substitute $y = 20$ into the first equation:

$$x - 3(20) = -10, \tag{7}$$
$$x - 60 = -10, \tag{8}$$
$$x = 50. \tag{9}$$

Final Answer:

$$x = 50, y = 20. \tag{10}$$

Solution using LU Decomposition

The system of equations can be written in matrix form as:

$$\begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -10 \\ 10 \end{pmatrix}. \tag{11}$$

Let:

$$A = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -10 \\ 10 \end{pmatrix}. \tag{12}$$

Doolittle's Algorithm:
The LU decomposition splits A into a lower triangular matrix L and an upper triangular matrix U such that:

$$A = LU \tag{13}$$

Where:

$$L = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ L_{21} & 1 & 0 & \cdots & 0 \\ L_{31} & L_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ L_{n1} & L_{n2} & L_{n3} & \cdots & 1 \end{pmatrix}, \quad U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & \cdots & U_{1n} \\ 0 & U_{22} & U_{23} & \cdots & U_{2n} \\ 0 & 0 & U_{33} & \cdots & U_{3n} \\ \vdots & \vdots & \vdots & \ddots & U_{n-1,n} \\ 0 & 0 & 0 & \cdots & U_{nn} \end{pmatrix}. \tag{14}$$

The Doolittle algorithm is computed as follows:
Elements of the U Matrix: For each column j:

$$U_{ij} = A_{ij} \quad \text{if } i = 0, \tag{15}$$

$$U_{ij} = A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} \quad \text{if } i > 0. \tag{16}$$

Elements of the L Matrix: For each row i:

$$L_{ij} = \frac{A_{ij}}{U_{jj}} \quad \text{if } j = 0, \tag{17}$$

$$L_{ij} = \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}} \quad \text{if } j > 0. \tag{18}$$

We decompose the matrix A into the product of a lower triangular matrix L and an upper triangular matrix U, i.e., $A = LU$.
Let:

$$L = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}. \tag{19}$$

We compute the elements of L and U:

$$u_{11} = a_{11} = 1, \tag{20}$$

$$u_{12} = a_{12} = -3, \tag{21}$$

$$l_{21} = \frac{a_{21}}{u_{11}} = \frac{1}{1} = 1, \tag{22}$$

$$u_{22} = a_{22} - l_{21} u_{12} = -2 - 1 \cdot (-3) = 1. \tag{23}$$

Thus, the LU decomposition is:

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}. \tag{24}$$

First, solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -10 \\ 10 \end{pmatrix}. \tag{25}$$

This gives:

$$y_1 = -10, \tag{26}$$

$$y_1 + y_2 = 10 \implies -10 + y_2 = 10 \implies y_2 = 20. \tag{27}$$

Thus:

$$\mathbf{y} = \begin{pmatrix} -10 \\ 20 \end{pmatrix}. \tag{28}$$

Next, solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$:

$$\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -10 \\ 20 \end{pmatrix}. \tag{29}$$

This gives:

$$y = 20, \tag{30}$$

$$x - 3y = -10 \implies x - 3(20) = -10 \implies x = 50. \tag{31}$$

Final Answer:

$$x = 50, \quad y = 20. \tag{32}$$

So, Nuri is 50 years old and Sonu is 20 years old.