NCERT - 10.4.ex.4

EE224BTECH11044 - Muthyala koushik

I. QUADRATIC EQUATIONS

Question: Find the roots of the quadratic equation $3x^2 - 2\sqrt{6}x + 2 = 0$
Solution: The given equation:

$$3x^2 - 2\sqrt{6}x + 2 = 3x^2 - \sqrt{6}x - \sqrt{6}x + 2 \tag{1}$$
$$= \sqrt{3}x\left(\sqrt{3}x - \sqrt{2}\right) - \sqrt{2}\left(\sqrt{3}x - \sqrt{2}\right) \tag{2}$$
$$= \left(\sqrt{3}x - \sqrt{2}\right)\left(\sqrt{3}x - \sqrt{2}\right) \tag{3}$$

So, the roots of the equation are the values of x for which

$$\left(\sqrt{3}x - \sqrt{2}\right)\left(\sqrt{3}x - \sqrt{2}\right) = 0 \tag{4}$$
$$\sqrt{3}x - \sqrt{2} = 0 \tag{5}$$
$$x = \sqrt{\frac{2}{3}} \tag{6}$$

Therefore, the roots of $3x^2 - 2\sqrt{6}x + 2 = 0$ are $\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}$

Solution by the method of Fixed point Iteration
Rearrange the equation to $x = g(x)$

$$x = \frac{1}{3}\left(2\sqrt{6} - \frac{2}{x}\right) \tag{7}$$

This gives the iteration function:

$$g(x) = \frac{1}{3}\left(2\sqrt{6} - \frac{2}{x}\right) \tag{8}$$

Iteration: Use the formula repeatedly:

$$x_{n+1} = \frac{1}{3}\left(2\sqrt{6} - \frac{2}{x_n}\right) \tag{9}$$

Stop when $|x_{n+1} - x_n| < \epsilon$
Root: 0.8173993362392574, Iterations: 900
Actual Root:0.81649658092

Solution by the Newton-Raphson method: we have;

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{10}$$
$$x_{n+1} = x_n - \frac{3x^2 - 2\sqrt{6}x + 2}{6x - 2\sqrt{6}x} \tag{11}$$

Iterating and updating the value of x_n, we can obtain the roots of the quadratic equation.
Newton-Raphson Root: 0.8164972809158475, Iterations: 18
Actual Root:0.81649658092

Solution by Matrix method:
The general quadratic equation is expressed as:

$$a\lambda^2 + b\lambda + c = 0 \tag{12}$$

Dividing through by a, we get:

$$\lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0 \tag{13}$$

This equation can be rewritten as:

$$\lambda\left(\lambda + \frac{b}{a}\right) + \frac{c}{a} = 0 \tag{14}$$

$$-\lambda\left(-\lambda - \frac{b}{a}\right) - (-1)\frac{c}{a} = 0 \tag{15}$$

This form is equivalent to the determinant of the following matrix:

$$\det\begin{pmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{pmatrix} = 0 \tag{16}$$

Clearly, the eigenvalues of the matrix:

$$\mathbf{C} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} \tag{17}$$

are the roots of the required quadratic equation. This matrix $\mathbf{C}$ is called the **Companion matrix**.
The companion matrix $\mathbf{C}$ is:

$$\mathbf{C} = \begin{pmatrix} 0 & 1 \\ \frac{-c}{a} & \frac{-b}{a} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{-2}{3} & \frac{2\sqrt{6}}{3} \end{pmatrix} \tag{18}$$

QR Decomposition for Eigenvalues

The eigenvalues of the companion matrix can also be found using the **QR decomposition** method. The steps are:

1. Start with the companion matrix $\mathbf{C}$. 2. Decompose $\mathbf{C}$ into the product of an orthogonal matrix $\mathbf{Q}$ and an upper triangular matrix $\mathbf{R}$, such that:

$$\mathbf{C} = \mathbf{QR}$$

3. Compute a new matrix $\mathbf{C}_1$ as:

$$\mathbf{C}_1 = \mathbf{RQ}$$

This process shifts the eigenvalues of $\mathbf{C}$ closer to the diagonal.

4. Repeat the above steps iteratively until $\mathbf{C}_k$ converges to a diagonal matrix. The diagonal entries of the final matrix are the eigenvalues of $\mathbf{C}$, which correspond to the roots of the quadratic equation.

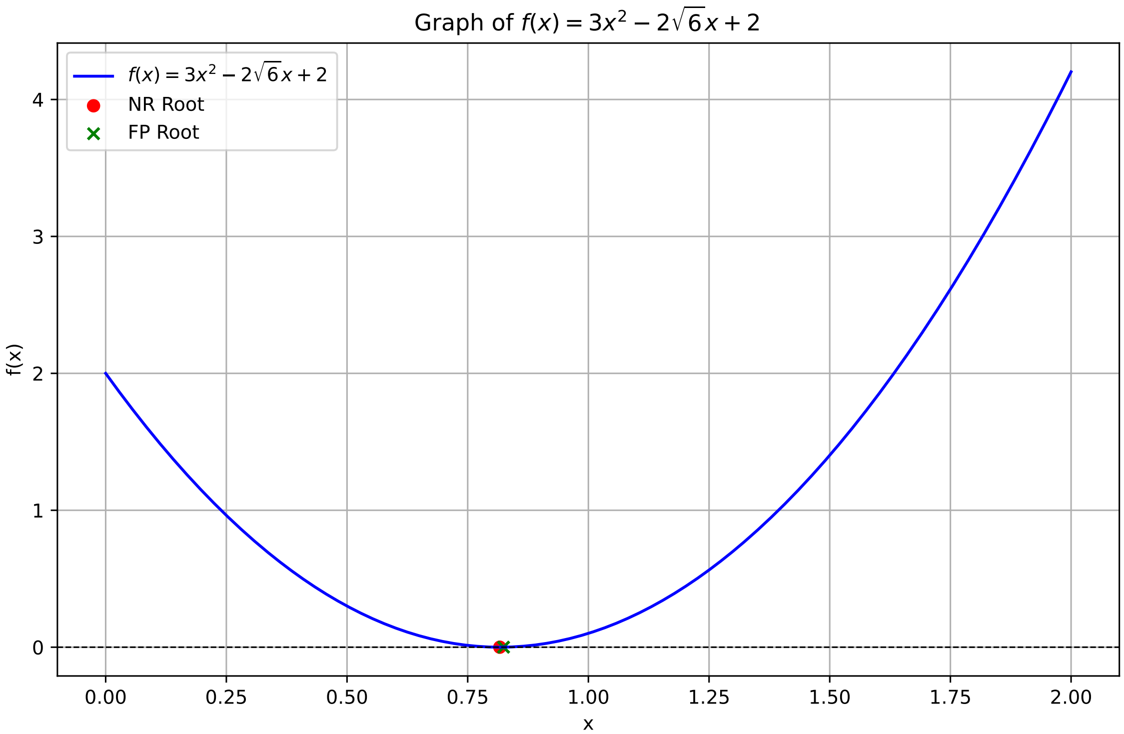

Fig. 0. Solution of given DE