NCERT - 6.5.14

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Question

Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximized.

Solution

- Let $z = xy^3$. From $x + y = 60$, we get:

$$x = 60 - y$$

- Substitute into z:

$$z = (60 - y)y^3 = 60y^3 - y^4$$

- Find critical points:

$$\frac{dz}{dy} = 180y^2 - 4y^3, \quad \text{solve } y^2(180 - 4y) = 0$$

- Solutions:

$$y = 45, \quad x = 15$$

Gradient Accent Method

- Using the the method of gradient accent;

$$y_{n+1} = y_n + h * F^{'}(y_n)$$

- Update rule:

$$y_{n+1} = y_n + h \cdot y_n^2(180 - 4y_n)$$

- Steps:
  - Start with an initial guess, $y_0 = 20$.
  - Choose a small step size, $h = 0.0001$.
  - Iteratively update y using the rule above until convergence.
- Results:
  - $x \approx 15.000001$
  - $y \approx 44.999999$

Using the Logarithmic Lagrangian

- Objective Function:

$$f(x, y) = xy^3$$

- Constraint:

$$x + y = 60$$

- To simplify, take the natural logarithm of the objective:

$$\ln(f(x, y)) = \ln(xy^3) = \ln(x) + 3\ln(y)$$

- Lagrangian Formulation:

$$\mathcal{L}(x, y, \lambda) = \ln(x) + 3\ln(y) + \lambda(60 - x - y)$$

Solution Steps

- Partial Derivatives:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{1}{x} - \lambda = 0 \quad \Rightarrow \quad \lambda = \frac{1}{x}$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{3}{y} - \lambda = 0 \quad \Rightarrow \quad \lambda = \frac{3}{y}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 60 - x - y = 0$$

- Equating λ:

$$\frac{1}{x} = \frac{3}{y} \quad \Rightarrow \quad y = 3x$$

- Substitute into the constraint:

$$x + y = 60 \quad \Rightarrow \quad x + 3x = 60 \quad \Rightarrow \quad x = 15,\ y = 45$$

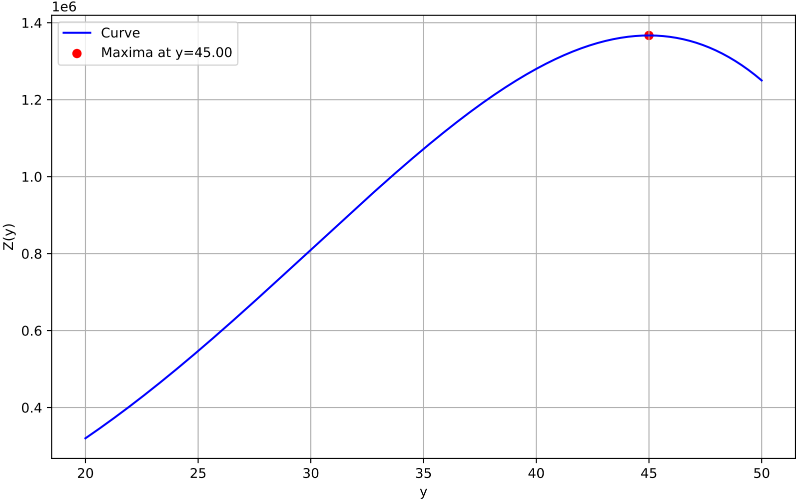
Results and Visualization

- Using Geometric Programming

$$x \approx 14.9988, \quad y \approx 45.0012$$

- Final Answer:
  - $x = 15$, $y = 45$
  - Maximum Value: $f(x, y) = xy^3 = 15 \cdot 45^3 = 136125$

Plot


1e6
1.4
1.2
1.0
0.8
0.6
0.4
Z(y)
20
25
30
35
40
45
50
y
Curve
Maxima at y=45.00

Python Implementation

```python
# Import necessary libraries
import numpy as np
import matplotlib.pyplot as plt
import cvxpy as cp

# Define the decision variables (positive)
x = cp.Variable(pos=True)  # x > 0
y = cp.Variable(pos=True)  # y > 0

# Define the equality constraint
constraints = [x + y == 60]  # Constraint: x + y = 60

# Define the objective function using logarithmic transformation
objective = cp.Maximize(cp.log(x) + 3 * cp.log(y))  # Maximize log(x) + 3*log(y)

# Formulate the optimization problem
problem = cp.Problem(objective, constraints)

# Solve the problem
problem.solve()

# Extract and print the results
optimal_x = x.value
optimal_y = y.value
optimal_f = optimal_x * (optimal_y ** 3)  # f(x) = x * y^3

# Output results
print(f"Optimal x: {optimal_x}")
print(f"Optimal y: {optimal_y}")
print(f"Maximum f(x, y): {optimal_f}")
```