

VOL. 1 – CLASSICAL PHYSICS

# The Undergraduate Companion to Theoretical Physics

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*The enchanting charms of this sublime science  
reveal only to those who have the courage to go  
deeply into it.*

— Carl Friedrich Gauss



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# References

Several textbooks, online courses/resources were referenced heavily (to the extend of making this text completely unoriginal, yet hopefully helpful for revision) throughout the writing of these lecture notes. Using a typical bibliography (research paper style) would be a formidable task. Pinpointing exactly where each reference has been used is quite difficult for such a large and well-referenced subject, and would probably change the writing style to a far too formal one for lecture notes. Therefore we instead list the most relevant below giving a brief comment on which topics they were mostly used for:

- A. Steane *Relativity made Relatively Easy*

Fantastic for a first introduction to special and general relativity with mathematical rigor (4-vector approach). Virtually the only reference needed.

- N. Woodhouse *Special Relativity*

Another great, succinct introduction to Special relativity.

# **Part I**

# **Analytical mechanics**

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# A review of Newtonian mechanics

1

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# Lagrangian mechanics

2

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# **Hamiltonan mechanics**

3

## **Part II**

# **Fluid dynamics**

# Fluid properties

## 4.1 Simple fluids

As Batchelor puts it: "A" simple fluid is a material such that the relative positions of elements of the material change by an amount which is not small when suitable chosen forces, however small in magnitude, are applied to the material. That is, a **simple fluid** is a substance that flows (i.e. deforms continuously) under an applied force, irrespective of how small this force is. Such forces are known as shearing forces, they change the shape of the fluid without changing its volume. A fluid may resist this deformation, but this resistance cannot overcome the shearing force, so the "resisting force vanishes with the rate of deformation". Luckily, many fluids that we encounter daily such as water and air may be regarded as simple fluids quite accurately.

Simple fluids may also be understood on a microscopic scale. It is well known that two molecules in a substance interact following a van-der-Waals-like potential shown below:

Fluids and solid are known to have an average separation of order  $r_0$ , and thus strongly interact repulsively, while gas molecules have a separation of order  $10r_0$ , and thus very weakly interact attractively. So, while in statistical mechanics one can often model gases as perfectly non-interacting, in fluid mechanics we are not afforded the same privilege, interactions are key.

At this molecular scale the mass distribution is very uneven, with spikes centered near nuclei, and large regions of emptiness between molecules where the electronic wave-functions have decayed. Working with such non-uniform distributions is a hopeless task, so it is often helpful to regard the properties of a fluid as independent of the fact that the molecular distributions are quasi-discrete or continuous. This is known as the **continuum approximation**, and in the classical regime where low density quantum effects are unimportant it is very successful.

We are therefore allowed to consider a fluid as a mass distribution that is generally continuous in space and time.

## 4.2 Forces acting on fluids

We distinguish between two types of forces acting on matter. One type, known as **long-ranged**, slowly decrease as the separation of the interacting bodies is increased. As a result, long-ranged forces (or **volume forces**) are roughly constant over infinitesimal volume el-

ements  $\delta^3 \mathbf{r}$  and proportional to the size of this element  $dV$ . An example of a volume force is the gravitational force.

The second type, known as **short-ranged**, decrease rapidly as the separation of the interacting bodies is increased. They are only relevant when the separation is on the order of the molecular separation, and thus are appreciable when there is physical contact between the interacting elements. Therefore, short-ranged forces are exerted by a fluid element on another fluid element across their shared boundary surface. As a result, short-ranged forces (or **surface forces**) are roughly constant over an infinitesimal plane surface element  $d^2 \mathbf{r}$ <sup>1</sup> of the shared boundary, and proportional to its area  $dA$ . An example of a surface force is surface tension.

The total volume force acting on a fluid's volume element  $dV$  centered at  $\mathbf{r}$  with density  $\rho$  at time  $t$  is:

$$d\mathbf{F}_{vol} = \mathbf{F}(\mathbf{r}, t) \rho dV \quad (4.2.1)$$

while the total surface force acting on a fluid's surface element  $dA$  centered at  $\mathbf{r}$  with normal  $\mathbf{n}$  at time  $t$  is:

$$d\mathbf{F}_{surf} = \Sigma(\mathbf{r}, \mathbf{n}, t) dA \quad (4.2.2)$$

By convention,  $\Sigma$  is known as the **stress** exerted by the fluid lying on the surface element that  $\mathbf{n}$  points to, on the fluid lying on the side that  $\mathbf{n}$  points away from. By Newton's third law  $\Sigma$  is an odd function in  $\mathbf{n}$ , so  $\Sigma(-\mathbf{n}) = -\Sigma(\mathbf{n})$ .

Consider a tetrahedral volume element with orthogonal surfaces  $\delta A_1, \delta A_2, \delta A_3$  oriented with unit normals  $-\mathbf{x}, -\mathbf{y}, -\mathbf{z}$  respectively, and an inclined surface  $\delta A$  oriented with unit normal  $\mathbf{n}$ . Surface forces from the rest of the fluid will act on this tetrahedron across each of these surfaces:

$$\delta\mathbf{F}_{surf} = \Sigma(\mathbf{n})\delta A + \Sigma(-\mathbf{x})\delta A_1 + \Sigma(-\mathbf{y})\delta A_2 + \Sigma(-\mathbf{z})\delta A_3 \quad (4.2.3)$$

Using the relations  $\delta A_1 = \mathbf{x} \cdot \mathbf{n} \delta A, \delta A_2 = \mathbf{y} \cdot \mathbf{n} \delta A, \delta A_3 = \mathbf{z} \cdot \mathbf{n} \delta A$  we find that:

$$(\delta\mathbf{F}_{surf})_i = [\Sigma(\mathbf{n}) - (x_j \Sigma_i(\mathbf{x}) + y_j \Sigma_i(\mathbf{y}) + z_j \Sigma_i(\mathbf{z}))n_j] \delta A \quad (4.2.4)$$

Now the total forces acting on the tetrahedral volume element are a combination of body forces and surface forces:

$$(\delta m)\mathbf{a} = \delta\mathbf{F}_{vol} + \delta\mathbf{F}_{surf} \quad (4.2.5)$$

As we make  $\delta V \rightarrow 0$ , the LHS of (4.2.5) approaches zero, and so does the body force. Therefore we need  $d\mathbf{F}_{surf} \rightarrow 0$  as  $dV \rightarrow 0$ :

$$\boxed{\Sigma_i(\mathbf{n}) = (x_j \Sigma_i(\mathbf{x}) + y_j \Sigma_i(\mathbf{y}) + z_j \Sigma_i(\mathbf{z}))n_j} \quad (4.2.6)$$

This result tells us that the stress along a given normal can be decomposed into the stresses along three orthogonal directions. We define the **stress tensor**:

$$\Sigma_{ij} = x_j \Sigma_i(\mathbf{x}) + y_j \Sigma_i(\mathbf{y}) + z_j \Sigma_i(\mathbf{z}) \quad (4.2.7)$$

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<sup>1</sup>technically this should be a volume element but the penetration depth of most surface forces is small

yielding:

$$\Sigma_i = \sigma_{ij} n_j \quad (4.2.8)$$

The stress tensor  $\sigma_{ij}$  gives the  $i$ th component of the stress exerted across a surface element normal to the direction  $j$ . The diagonal elements  $\sigma_{ii}$  give the diagonal stresses (compression or expansion), while the off-diagonal elements  $\sigma_{ij}$  give the shearing stresses (deformation while maintaining volume).

We now take the stress torque on a volume element's boundary surface about a point  $\mathcal{O}$  within this element:

$$\int (\mathbf{r} \times \boldsymbol{\sigma})_i \cdot d\mathbf{A} = \int \epsilon_{ijk} r_j \sigma_{kl} n_l dA \quad (4.2.9)$$

where  $\mathbf{r}$  gives the displacement of the surface element  $dA$  relative to our reference point  $\mathcal{O}$ . We can use the divergence theorem to simplify this integral:

$$\int \epsilon_{ijk} \frac{\partial r_j \sigma_{kl}}{\partial r_l} dV = \int \epsilon_{ijk} \left( \sigma_{kj} + r_j \frac{\partial \sigma_{kl}}{\partial r_l} \right) dV \quad (4.2.10)$$

We let the volume collapse onto  $\mathcal{O}$  by keeping the boundary surface fixed, so that  $V \rightarrow 0$ . The total moment and the rate of change of angular momentum scale as  $V^{4/3}$  while the first term scales as  $V$ . Therefore, we need this term to vanish:

$$\boxed{\epsilon_{ijk} \sigma_{kj} = 0} \quad (4.2.11)$$

This relation establishes the symmetry of the stress tensor:  $\sigma_{ij} = \sigma_{ji}$ . It is well known that for any rank two symmetric cartesian tensor, three principal axes may be rotated so that this tensor becomes diagonal in this basis. In other words, we can always orient our axes so that the only stresses acting on the fluid are normal (such stress forces are known as **principal**), a fluid can always be regarded as in a state of tension/compression in three orthogonal directions.

Now consider a fluid of very small volume at rest. In the principal frame, the stress tensor is diagonal, and we write it in the form:

$$\boldsymbol{\sigma} = \begin{pmatrix} \frac{1}{3}\sigma_{ii} & 0 & 0 \\ 0 & \frac{1}{3}\sigma_{ii} & 0 \\ 0 & 0 & \frac{1}{3}\sigma_{ii} \end{pmatrix} + \begin{pmatrix} \sigma_{11} - \frac{1}{3}\sigma_{ii} & 0 & 0 \\ 0 & \sigma_{22} - \frac{1}{3}\sigma_{ii} & 0 \\ 0 & 0 & \sigma_{33} - \frac{1}{3}\sigma_{ii} \end{pmatrix} \quad (4.2.12)$$

The first term is a uniform compressive (normal) stress. Thus it tends to change the volume of the fluid, and can be resisted. The second term however is a shearing stress, and thus cannot be resisted by a fluid since it maintains its volume, it only changes its shape. Consequently, for the fluid to be at rest we need the second term to vanish, making the principal stresses equal to each other:

$$\boldsymbol{\sigma} = \begin{pmatrix} \frac{1}{3}\sigma_{ii} & 0 & 0 \\ 0 & \frac{1}{3}\sigma_{ii} & 0 \\ 0 & 0 & \frac{1}{3}\sigma_{ii} \end{pmatrix} \quad (4.2.13)$$

Note that rotating away from the principal axes will give the same matrix due to the isotropicity of the principal stresses. As a result, , so only normal stresses are active! It

is thus useful to define the **static fluid pressure** as the normal stress  $p = -\frac{1}{3}\text{tr}\sigma = -\frac{1}{3}\sigma_{ii}$  where the minus sign by convention means that positive pressure is compressive. Therefore

$$\boxed{\sigma_{ij} = -p\delta_{ij}} \quad (4.2.14)$$

The surface force across a plane element with normal  $\mathbf{n}$  is given by  $-p\mathbf{n}$ .

## 4.3 Static equilibrium in fluids

The total force on a static fluid is a sum of the body and surface forces. Newton's first law then yields:

$$\int \rho \mathbf{F} dV - \int p \mathbf{n} dA = 0 \implies \int (\rho \mathbf{F} - \nabla p) dV = 0 \quad (4.3.1)$$

for any volume  $V$  contained within the fluid. This is only possible if the integrand vanishes inside the fluid, so the equilibrium condition in a fluid is:

$$\boxed{\nabla p = \rho \mathbf{F}} \quad (4.3.2)$$

For a fluid the shear stresses are not necessarily zero, so the equilibrium condition reads

$$\nabla p + \frac{\partial \sigma}{\partial \mathbf{r}} = 0 \quad (4.3.3)$$

Suppose that the body force per unit mass  $\mathbf{F}$  is conservative, with a potential  $\Phi$  so that  $\mathbf{F} = -\nabla\Phi$ . Then:

$$-\rho\nabla\Phi = \nabla p \implies (\nabla p) \times (\nabla\Phi) = 0 \quad (4.3.4)$$

so the unit normal vectors to the level surfaces of  $\rho$  and  $\Gamma$  and  $p$  must all be equal. Consider for example the case of a gravitational force where  $\mathbf{F} = \mathbf{g} \implies \Phi = -\mathbf{g} \cdot \mathbf{r} = gz$ . Then:

$$\nabla p = \rho \mathbf{g} \implies p_2 - p_1 = \rho \mathbf{g} \cdot (\mathbf{r}_2 - \mathbf{r}_1) \implies p_2 + \rho g z_2 = p_1 + \rho g z_1 \quad (4.3.5)$$

This is just an energy conservation law stating that a change in potential energy  $\rho g z_2$  must come at the expense of a pressure difference (we can't have kinetic energies since we are at rest). This result also implies that there are no sudden jumps in pressure within a fluid or within the interface between two fluids. It follows that the surface of a fluid must be horizontal, since any variation in height would give a pressure difference.

Note that due to the arbitrariness of the reference point  $\mathcal{O}$ , we can in general write:

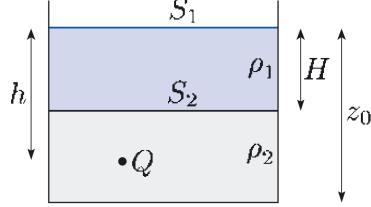
$$p(z) = c - \rho g z \quad (4.3.6)$$

where  $z$  is measured vertically upwards from the reference point, and  $c = p(0)$  is a constant given by the pressure at  $z = 0$ . Suppose for example that the reference point is chosen to be on the interface between an incompressible fluid and the atmosphere. Then the pressure at a depth  $h$  in the fluid is given by:

$$p(h) = p_0 + \rho g h \quad (4.3.7)$$

where  $p_0$  is the atmospheric pressure.

**Example.** An open vessel standing on a horizontal table contains a layer of an incompressible fluid of density  $\rho_1$ , floating on top of a layer of incompressible fluid of density  $\rho_2$ . Let  $S_1$  be the surface separating fluid 1 and the atmosphere, and let  $S_2$  separate fluid 2 from fluid 1. Suppose  $Q$  is a point at a depth  $h$  below the surface  $S_1$ . Let the reference point be



on  $S_2$ . Then for any  $h$  within fluid 2:

$$p_2 = p_1 + \rho_2 g(h - H) \quad (4.3.8)$$

but  $p_1 = p_0 + \rho_1 gH$  so we have that:

$$p_2 = p_0 + \rho_1 gH + \rho_2 g(h - H), \quad H \leq h \leq z_0 \quad (4.3.9)$$

We often refer to the surface between two fluids as a **free surface**.

### Submerged solids

Suppose a flat plate with sides of length  $a, b$  is submerged in a fluid of density  $\rho$  so that its top edge is at a depth  $h_0$ . We let the top-left corner  $A$  of the plate be the origin of our coordinate frame. Consider an infinitesimal surface element  $d\mathbf{A}$  with center  $Q = (x, y)$ . Since pressure acts uniformly on this static system, the force on this element will be  $d\mathbf{f}(x, y) = -p(x, y)d\mathbf{A} = -(p_0 + \rho g(h_0 + x))d\mathbf{A}$ . Integrating over the plate we have that:

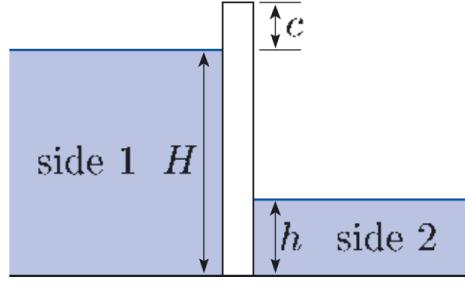
$$\mathbf{f} = -\mathbf{n} \int_0^b \int_0^a (p_0 + \rho g(h_0 + x)) dx dy = -b(p_0 a + \frac{\rho g}{2}(2h_0 a + a^2)) \quad (4.3.10)$$

$$= -ab \left[ p_0 + \rho g \left( h_0 + \frac{a^2}{2} \right) \right] \mathbf{n} \quad (4.3.11)$$

Note that this is just equal to the pressure  $p_M$  at the center of the plate times the area of the plate:

$$\mathbf{f} = -p_M A \mathbf{n} \quad (4.3.12)$$

An interesting application of this result are canal lock gates. One can use the difference in pressure between two bodies of water to block gates from opening in one direction. Suppose we have a fluid of total depth  $H$  on side 1 of a gate of height  $H + c$ , and another fluid of total depth  $h$  on side 2 of the gate. The gate has breadth  $b$ . Intuitively, we expect the gate to be locked in the direction side 2  $\rightarrow$  side 1. Indeed the total force on side 1 is



given by:

$$F_1 = \int_0^b \int_0^H [p_0 + \rho g(H - z)] dz dx + \int_0^b \int_H^{H+c} p_0$$

$$= b(H + c)p_0 + \frac{1}{2}\rho g b H^2$$

where we note that the force due to the fluid pressure is again equal to the  $bH$  times the pressure at the midpoint of the gate in contact with the fluid  $\rho g \frac{H}{2}$ . By the exact same calculations:

$$F_2 = \frac{1}{2}\rho g b h^2 + b p_0 (H + c)$$

giving a net force of:

$$F = \frac{1}{2}\rho g b (H^2 - h^2)$$

directed from side 1 to side 2. For  $b = 3\text{m}$ ,  $h = 2\text{m}$ ,  $H = 5\text{m}$ , and if the fluid is water, then  $F = 3 \times 10^5 \text{N}$ ! If you ever find yourself in a sinking car, don't try to open the doors right away, wait for the water level to equalize and use the water pressure to your advantage.

Now suppose we fully submerge a cube of side length  $a$  so that its center of mass is at a depth  $h_0$ . The pressure force on the lateral sides of the cube cancel out. However, there will be a net force due to the height difference between the top and the base of the cube:

$$\mathbf{f} = \rho g a^2 ((h_0 + 2a) - h_0) \mathbf{z} = \rho g a^3 \mathbf{z}$$

where  $\mathbf{z}$  points upwards.

This force is known as the buoyancy force, and is a general case of **Archimede's principle**: a body submerged in a liquid at rest will experience a resultant force directed upwards equal to the weight of the liquid it has displaced.

Indeed, suppose a body  $\mathcal{V}$  with boundary surface  $\mathcal{S}$  is submerged. Then:

$$\mathbf{f} = - \oint_{\mathcal{S}} p d\mathbf{S}$$

We also have that

$$M\mathbf{g} = \int_{\mathcal{V}} \rho \mathbf{g} dV = \int_{\mathcal{V}} \nabla p dV = \oint_{\mathcal{S}} p d\mathbf{S} = -\mathbf{f}$$

so we do indeed find that  $\mathbf{f} = -M\mathbf{g}$ .

### Atmospheric pressure

Throughout our calculations we have dealt rather mysteriously with the atmospheric pressure  $p_0$ .

We cannot simply model the atmosphere as a fluid of constant density, since this would imply after a short calculation that the atmospheric pressure vanishes at a height of just 8.4km!

Let us model the atmosphere more realistically as an ideal gas with equation of state:

$$p = \rho RT \quad (4.3.20)$$

For an isothermal atmosphere, the temperature is constant implying  $\frac{p}{\rho} = \frac{p_0}{\rho_0}$  where  $p_0$  and  $\rho_0$  are the atmospheric pressure and density at some reference point (e.g. sea level). Consequently:

$$\frac{dp}{dz} = -\rho g = -\frac{p}{p_0} \rho_0 g \quad (4.3.21)$$

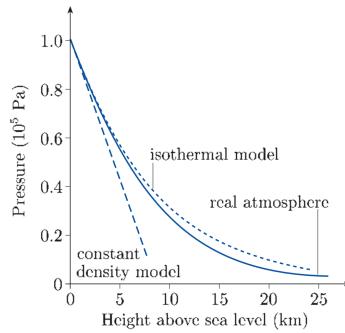
which can be integrated to give:

$$\ln p = -\frac{\rho g}{p_0} z + c \implies p(z) = A e^{-\rho g z / p_0} \quad (4.3.22)$$

and using the condition  $p(0) = p_0$  we find that:

$$p(z) = p_0 e^{-\rho g z / p_0} \quad (4.3.23)$$

The isothermal model is much more accurate than the constant density model, but still differs from experimental data at large heights. Suppose we now model the atmosphere



as a perfect gas satisfying the power law  $p = k\rho^\gamma$  for some constants  $k, \gamma$ . Then:

$$\frac{p}{\rho^\gamma} = \frac{p_0}{\rho_0^\gamma} \implies \rho = \rho_0 \left( \frac{p}{p_0} \right)^{1/\gamma} \quad (4.3.24)$$

Then:

$$\frac{dp}{dz} = k\gamma\rho^{\gamma-1}\frac{d\rho}{dz} = -\rho g \quad (4.3.25)$$

$$\implies \int \rho^{\gamma-2} d\rho = -\frac{g z}{k\gamma} \quad (4.3.26)$$

$$\implies \rho^{\gamma-1} = c - \frac{g z}{k\gamma}(\gamma - 1) \quad (4.3.27)$$

Using the ideal gas law:

$$p = \rho RT \implies T(z) = \frac{k\rho^{\gamma-1}}{R} = A - \frac{(\gamma - 1)g}{\gamma R}z \quad (4.3.28)$$

and using the condition  $T(0) = T_0$  we find that:

$$T(z) = T_0 - \frac{(\gamma - 1)g}{\gamma R}z \quad (4.3.29)$$

## 4.4 Liquid interfaces

We define an interface as the boundary between two media with distinct molecular structures. Consider an interface between a liquid and a gas. Recall from thermodynamics that a reversible isothermal process which increases the interface surface area by  $dA$  will require  $\gamma dA$  work, where  $\gamma$  is known as the **surface tension**. The tensile force is then  $\gamma dl$  along the perpendicular to  $dl$ .

The tensile force acting on a surface  $\mathcal{S}$  along a curve  $\Gamma$  is given by:

$$\mathbf{F}_\gamma = -\gamma \oint_{\Gamma} \mathbf{n} \times d\mathbf{l} \quad (4.4.1)$$

To see where this comes from, note that:

$$dW = d\mathbf{F} \cdot d\mathbf{x} = -\gamma(\mathbf{n} \times d\mathbf{l}) \cdot d\mathbf{x} = -\gamma(d\mathbf{l} \times d\mathbf{x}) \cdot \mathbf{n} = \gamma dA \quad (4.4.2)$$

as desired. Suppose the surface is defined by  $\mathcal{S} : z - f(x, y) = 0$  giving a normal  $\mathbf{n} = (-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1)$ .

$$\mathbf{F}_\gamma = -\gamma \oint_{\Gamma} \left( -\frac{\partial f}{\partial x} dy + \frac{\partial f}{\partial y} dx \right) \approx \gamma \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) dA \quad (4.4.3)$$

giving a pressure

$$p = \gamma \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) = \gamma \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \quad (4.4.4)$$

Equilibrium is only achieved when this outwards pressure is countered by an equal inwards pressure, known as a Laplace pressure:

$$p = \gamma \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \quad (4.4.5)$$

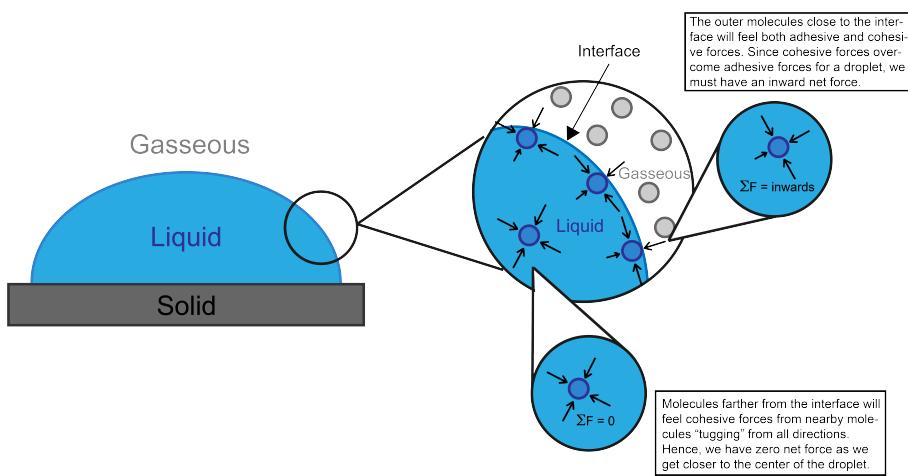
We will discuss these effects due to surface tension more precisely in the next chapter.

# Surface tension

## 5.1 Defining Surface Tension

It would be impossible for a person not to encounter the enthralling phenomenon that is *surface tension*. From taking a shower to washing hands with soap, and even the movement of foam when making tea, surface tension plays a central role, governing the dynamics and statics of interactions between fluids.

To understand how surface tension occurs, imagine looking at a water droplet under an immensely strong electron microscope, capable of clearly displaying individual water molecules. Firstly, consider the water molecules at the center of the droplet. These will feel cohesive forces from neighbouring water molecules. Hence, they will be "pulled" in all directions and will experience no net force. Let us now consider water molecules at the edge of the droplet, adjacent to the so-called "interface" surface (an imaginary surface delimiting two phases, such as water and air). Clearly, these molecules will experience both a cohesive force from neighbouring water molecules, but also adhesive forces from the nearby air molecules. Due to the imbalance between cohesive and adhesive forces, a net inward force will act on the outer layer of the droplet, giving it a spherical shape (see Figure 2.1)



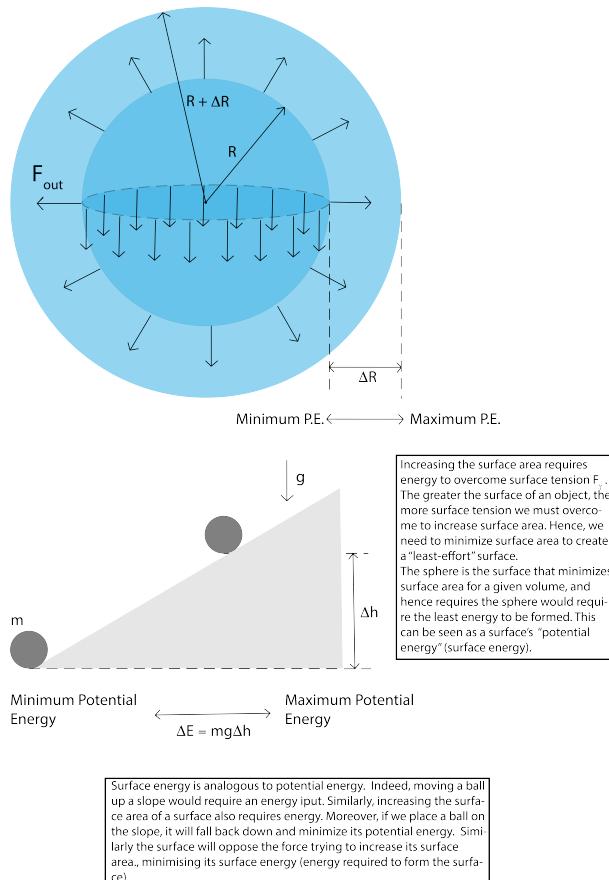
**Figure 5.1.** Diagram showing dynamics behind surface tension in a droplet

One might wonder why a droplet doesn't take a rectangular shape, or a pyramidal shape. The answer lies in surface optimization and the tendency of nature to minimize potential

## 5.1. DEFINING SURFACE TENSION

energy. Indeed, surface tension  $\gamma$  has units  $\frac{J}{m^2}$ , energy per unit area. It is the energy needed to increase the droplet's surface area. If insufficient force acts on the droplet trying to increase its surface, then it will try to minimize it, "pushing" or "resisting" against any such force.

To better illustrate this idea, consider an arbitrary volume of water. We are asked to create a surface from that volume of water requiring the least effort. To do so, it is essential to note that the molecules forming this surface will oppose resistance to any increase in surface area due to cohesive forces, especially near the interface. In other words, the greater the surface area of this surface, the more work will be done to construct it. To use the least energy, the droplet will therefore have largest possible Volume-Surface ratio, trying to fit in the volume of water in the smallest possible surface area (the problem of defining a shape with the largest volume-surface ratio is known as the *Isoperimetric Inequality*). It can be proven that the sphere has the largest  $V - S$  ratio. Any liquid will naturally rearrange itself into a sphere, as it requires the least energy (more formally, it has the least *surface energy*).



**Figure 5.2.** Analogy between surface energy and potential energy minimization.

By taking a spherical shape, the least work is required to increase surface area, the droplet essentially minimizes its surface energy (the energy needed to create the surface). This is

quite similar to a ball rolling up a hill. When we increase the surface area of a droplet, this requires energy input, just like trying to roll a ball up a slope. If we release the ball on this slope, it will try to minimize its potential energy, and fall back down. Analogously, the droplet will try minimizing its surface energy, resisting against any attempt at increasing surface area.

We can therefore define surface tension as the energy needed to increase the surface area of a liquid by unit area. It may also be considered as the force acting against deforming the surface per unit length. The higher the surface tension of a surface is, the less it is prone to increasing its surface area.

As hinted earlier, surface tension governs several phenomena we observe daily. As soon as we wake up, we usually go to the bathroom, and wash our face/hands. The reason we use soap is not only because of its antibacterial qualities, but also because it reduces surface tension of water. Thus, when coming into contact with our hands, water will be more easily deformed, entering into the crevices and wrinkles on our palms, and removing dirt. Another common way to observe surface tension is when making tea or coffee. Indeed, the reader has probably noticed the formation of foam and bubbles on the surface of the liquid. Usually, this foam collects either at the center of the surface, or at its circumference, for reasons we will allude to later on. The goal of this chapter will be to develop the physical laws describing surface tension, and related phenomena.

## 5.2 Deriving the Young-Laplace Equation

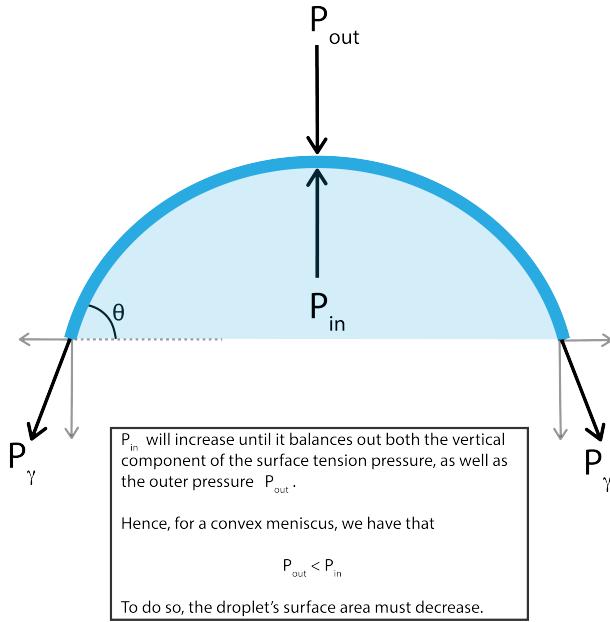
We shall now derive the Young-Laplace equation, without which most of the results in the rest of the chapter wouldn't be known. We will closely follow the derivation given in *Siqvel and Skjæveland*, (2015). Surface tension can be defined as the force per unit length exerted on any fluid against increasing its surface area. Consider a curved surface, as shown below, representing the boundary between a liquid region and a gaseous region (known as the interface). Recall that when crossing this surface, there will be a change in pressure, called the Laplace Pressure (see Fig 1.3). As we can see, the internal pressure must counteract the outer atmospheric pressure, as well as the surface tension acting tangentially on its surface. This means that the pressure inside the droplet will obviously be larger than the outer pressure. Furthermore, this discontinuity in pressure when crossing the interface may also be seen as an explanation of why droplets try minimizing surface area. The internal pressure will keep increasing until it counteracts both the outward pressure and surface tension, which can be done by reducing the surface area of the droplet.

For an infinitesimal patch of this surface, the net force  $F_P$  caused by the pressure difference  $\Delta P$  will be:

$$dF_P = \Delta P dS = \Delta P dL_1 dL_2 \quad (5.2.1)$$

We can then use the fact that  $dL_1 = 2R_1 d\theta_1$  and  $dL_2 = 2R_2 d\theta_2$ :

$$dF_P = \Delta P dS = \Delta P (2R_1 d\theta_1)(2R_2 d\theta_2) \quad (5.2.2)$$



**Figure 5.3.** Pressure inside and outside of a spherical droplet or convex meniscus

Consider the surface tension force acting on the patch. Since we defined surface tension  $\gamma$  to be the force per unit length acting on the patch, then the total surface tensile force acting on it will be  $\gamma \times$  infinitesimal length:

$$dF_\gamma^{(1)} = \gamma dL_2, \quad dF_\gamma^{(2)} = \gamma dL_1 \quad (5.2.3)$$

We can now take the total components acting vertically against  $F_P$ . Since there will be two forces of magnitude  $dF_\gamma^{(1)}$  and two forces of magnitude  $dF_\gamma^{(2)}$ , we get that this component is:

$$2dF_\gamma^{(1)} \sin \theta_2 + 2dF_\gamma^{(2)} \sin \theta_1 \quad (5.2.4)$$

Recall that surface tension acts tangentially to the surface. Hence,  $dF_\gamma^{(1)}$  acts on the principal line 2, and  $dF_\gamma^{(2)}$  acts on the principal line 1.

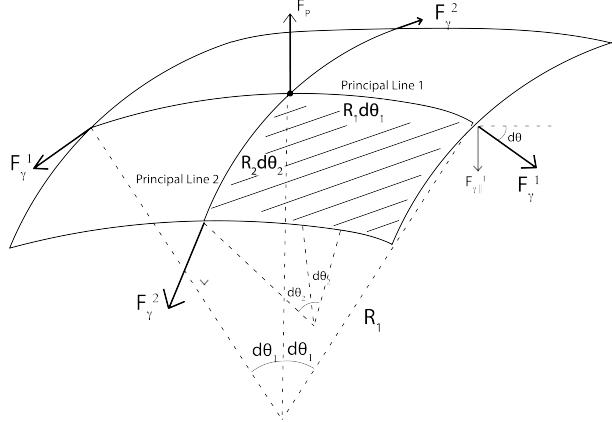
We then use the small angle approximation  $\sin x \approx x$  for simplification, and substituting our expressions for  $dL_1, 2$ :

$$dF_\gamma = \gamma(2(2R_1 d\theta_1) d\theta_2 + 2(2R_2 d\theta_2) d\theta_1). \quad (5.2.5)$$

For a static droplet, we must have that:  $dF_P = dF_\lambda$ . Hence:

$$\overbrace{\Delta P}^{Nm^{-2}} = \underbrace{\gamma}_{Fm^{-1}} \underbrace{\frac{R_1 + R_2}{R_1 R_2}}_{m^{-1}} \quad (5.2.6)$$

and finally we reach the Young-Laplace equation:



**Figure 5.4.** Infinitesimal patch over which surface tension acts along the edges

### Young-Laplace Equation

$$\Delta P = \gamma \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \quad (5.2.7)$$

Let's stop a moment to analyze what this equation represents. The term on the right,  $\Delta P$ , is the change in pressure when moving through the interface. On the right hand side, we have an expression with units  $N \cdot m^{-2}$ . This tells us that the more curved a surface is (the smaller  $R_{1,2}$  are), the greater the Laplace pressure. This is expected, as we have a greater component due to surface tension against which internal pressure must act against. This widens the gap between pressure inside and outside.

Furthermore, as  $R_{1,2} \rightarrow \infty$  (we get a flat surface, a plane), we have that the Laplace pressure decreases very quickly  $\Delta P \rightarrow 0$ . This makes sense, and agrees with the well known result that the pressure difference acting on a flat surface must be zero for equilibrium to be satisfied, a property that isn't necessarily true for curved surfaces as demonstrated.

We extend our formula further, using some notions of differential geometry. Note that the mean curvature of a surface at a point,  $H$  can be defined as the arithmetic mean of the minimum and maximum curvature (principal curvatures):

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) \quad (5.2.8)$$

where  $\kappa_1, \kappa_2$ , the principal curvatures of the surface. This then clearly yields for our patch:

$$H = \frac{1}{2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \quad (5.2.9)$$

where, as discussed earlier,  $R_1$  and  $R_2$  are the radii of principal curvatures at point P.

$$\Delta P = \gamma(2H) \quad (5.2.10)$$

Using Frenet-Serret equations, we know that the mean curvature is:

$$H = -\frac{1}{2}\nabla \cdot \hat{\mathbf{n}} = -\frac{1}{2}|\nabla \cdot \left(\frac{\nabla f}{|\nabla f|}\right)| \quad (5.2.11)$$

which allows us to write more generally:

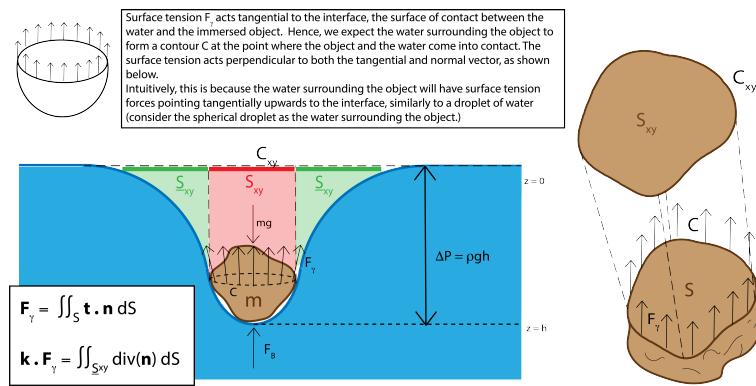
$$\Delta P = -\gamma(\nabla \cdot \hat{\mathbf{n}}) \quad (5.2.12)$$

This is a non linear partial differential equation, which relates the pressure difference through an interface and the shape of the interface surface. For positive curvature (e.g. concave meniscus), the Laplace pressure will be negative, whereas for negative curvature (e.g. convex meniscus), the Laplace pressure will be positive. Oddly, for a convex meniscus, the pressure outside is actually greater than the pressure just under the meniscus. As we will see later, the Young-Laplace equation is extremely powerful when solving problems for curved liquid surfaces. Three such problems are the floating body, the shape of the meniscus and the profile of a water droplet. These can all be solved by utilizing the Young-Laplace PDE.

### Floating Bodies

We shall now address the first problem introduced earlier, the statics of floating bodies, using the approach in *D. J. Vella*, (2007).

Consider an object (such as a metal pin, which is denser than water) of sufficiently small mass  $m$  "suspended" on a liquid. This object is not submerged, assuming the surface tension forces  $F_\gamma$  between the molecules of the liquid are strong enough not to let the interface surface rip. However, the object displaces water to its sides, and will therefore feel a buoyant force  $F_B$ .



**Figure 5.5.** Body placed on a fluid depresses its surface, but doesn't necessarily sink due to surface tension and buoyancy.

Hence, for equilibrium we must have that  $F_B + F_{\gamma\parallel} = mg$ , where  $F_{\gamma\parallel}$  is the component of  $F_\gamma$  acting vertically. Moreover, since pressure is defined as the force applied per unit

surface area,  $F = -\iint_S P dA$ . Using the Laplace-Young equation:

$$\mathbf{k} \cdot \mathbf{F}_\gamma = F_{\gamma\parallel} = -\gamma \int_{S_{xy}} \nabla \cdot \hat{\mathbf{n}} \, dA \quad (5.2.13)$$

where  $S$  is the surface of contact between the floating body and the liquid, the interface surface. Furthermore,  $S_{xy}$  is the projection of  $S$  on the  $x-y$  plane, and  $\bar{S}_{xy}$  is  $\mathbb{R}^2 \setminus S_{xy}$ . We could also derive this result using the definition of surface tension as the force per length applied on the interface surface. Indeed, defining  $C$  as the contact line between the object and water expressed as an arc parametrized vector function  $\mathbf{r} = \mathbf{r}(l)$ , we get that:

$$\mathbf{F}_\gamma = \gamma \int_C \dot{\mathbf{r}} \times \hat{\mathbf{n}} \, dl \quad (5.2.14)$$

We took the cross product  $\dot{\mathbf{r}} \times \hat{\mathbf{n}}$  because the surface tension force is orthogonal to both the vector tangent to  $\mathbf{r}$  and the normal to the liquid surface  $\mathbf{n}$ . Integrating over  $C$  then gives the length and direction over which surface tension acts. We can now evaluate the component of the tensile force acting vertically:

$$F_{\gamma\parallel} = \gamma \int_C \mathbf{k} \cdot (\dot{\mathbf{r}} \times \hat{\mathbf{n}}) \, dl \quad (5.2.15)$$

This is equivalent to projecting the contact line on the  $x-y$  plane, forming  $C_{xy}$ , and then evaluating the component of the surface tension force acting vertically along  $\hat{\mathbf{n}}$ . This new path will have arc length  $l'$  and normal vector  $\mathbf{n}'$  (this can be seen as transforming the integral using  $l \rightarrow l'$  and  $\dot{\mathbf{r}} \times \mathbf{n} \rightarrow \mathbf{n}'$ ) so that we get:

$$F_{\gamma\parallel} = \gamma \int_{C_{xy}} \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}' \, dl' = \gamma \int_{C_{xy}} \hat{\mathbf{n}} \cdot d\hat{\mathbf{l}}' \quad (5.2.16)$$

We can now use the two dimensional Divergence Theorem to simplify this integral.

**Theorem 5.1** (2D Divergence Theorem). *Let  $S$  be a region enclosed by a smooth curve  $\partial S$ , with normal vector  $\hat{\mathbf{n}}$ ). Then, the following holds for any vector field  $\mathbf{F}$  such that  $\nabla \cdot \mathbf{F} \neq 0$ :*

$$\iint_S \nabla \cdot \mathbf{F} \, dA = \int_{\partial S} \mathbf{F} \cdot d\hat{\mathbf{n}} \quad (5.2.17)$$

Since  $C_{xy}$  is simply the boundary of  $S_{xy}$ , we can define  $\bar{S}_{xy}$  as  $\mathbb{R}^2 \setminus S_{xy}$ , with boundary  $\bar{C}_{xy}$ . We then have that:

$$\int_{C_{xy}} \hat{\mathbf{n}} \cdot d\hat{\mathbf{l}}' = - \int_{C_{xy}} -\hat{\mathbf{n}} \cdot d\hat{\mathbf{l}}' = - \int_{\bar{S}_{xy}} \nabla \cdot \hat{\mathbf{n}} \, dA \quad (5.2.18)$$

so that we finally reach:

$$F_{\gamma\parallel} = -\gamma \iint_{\bar{S}_{xy}} \nabla \cdot \hat{\mathbf{n}} \, dA \quad (5.2.19)$$

as required. By taking into account buoyant forces, we can then write by balancing forces:

$$-\iint_{\bar{S}_{xy}} \nabla \cdot \hat{\mathbf{n}} \, dA + \frac{F_B}{\gamma} = \frac{mg}{\gamma} \quad (5.2.20)$$

Consider now the two fluids separated by an interface. Denoting the liquid density  $\rho$ , the variation in vertical pressure between the two phases (liquid and gas) will be  $\Delta P = \rho gh$ , where  $h$  is the depression in the liquid. We can then write:

$$\iint_{\bar{S}_{xy}} \rho gh \, dA = mg - F_B \quad (5.2.21)$$

where we have used the Young-Laplace equation. Let us now evaluate  $F_B$ , the buoyant force acting on this object. We get using the Archimedean principle:

$$F_B = \rho gh S_{x,y} \quad (5.2.22)$$

Taking the ratio between the two yields:

$$\frac{F_B}{F_\gamma} = \frac{S_{xy}}{\bar{S}_{xy}} \quad (5.2.23)$$

This demonstrates that as the size of the object decreases,  $S_{xy}$  will also decrease, and hence this ratio will decrease. For smaller objects, such as needle pins, contrary to common belief, buoyancy is not the main reason they float, it is surface tension. Thus, we can conclude that objects with greater density than water can still float as long as they are small enough.

### 5.3 Capillary Action

Capillary action is another mechanism caused by surface tension, and leads to various interesting phenomena, such as the formation of menisci and the shape puddles and droplets.

When a liquid is placed in a narrow tube or cylinder of sufficiently small radius, this liquid may "rise" upwards. The adhesive forces overcome the cohesive forces, then the liquid molecules will be pulled by the walls of the container, rising.

#### Jurin's Law

Before we address the meniscus problem introduced earlier, it is important to allude to the phenomenon of capillary action to truly understand the mechanism behind which menisci actually form, using two standard arguments to derive Jurin's Law.

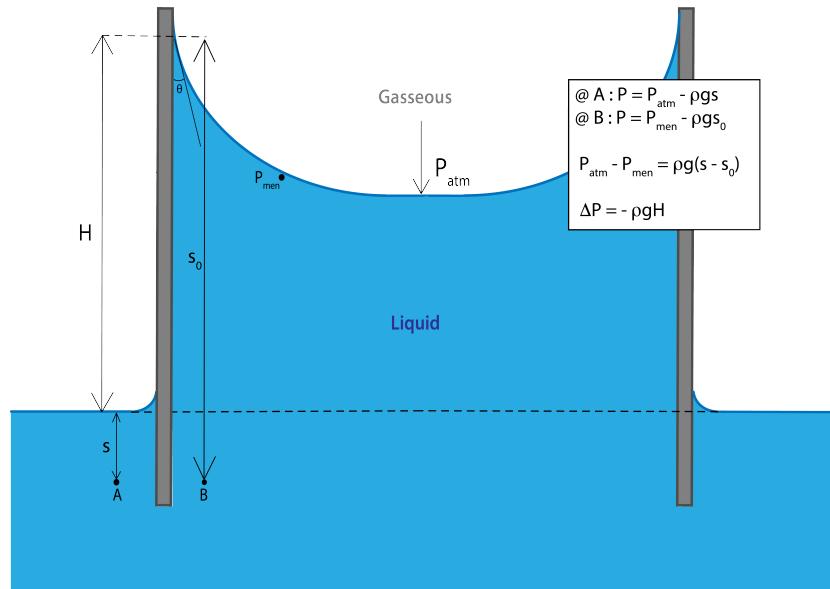
Let us consider a cylinder filled with water, of radius  $R$ . Assuming that the meniscus has a spherical shape (we'll see later on how to more accurately define the profile of a meniscus) with contact angle  $\theta$  with the cylinder's walls. It can be shown that the radius of curvature of the interface is  $R \sec \theta$ , so that the Laplace pressure is:

$$\Delta P = -\frac{2\gamma}{R \sec \theta} \quad (5.3.1)$$

Furthermore, the Laplace pressure can also be defined as:

$$\Delta P = P_{atm} - P_{men} = -\frac{2\gamma}{R \sec \theta} \quad (5.3.2)$$

Let us now consider two communicating vases as shown below.



**Figure 5.6.** Capillary rise for a concave meniscus

Since pressure at equal heights in communicating vases must be equal, it follows that the pressure at an arbitrary height  $s$  under the water level is:

$$P_s = P_{atm} - \rho g s \quad (5.3.3)$$

for the "outer" vase. For the inner vase, the tube, we have that:

$$P_s = P_{men} - \rho g s_0 \quad (5.3.4)$$

where  $s_0$  is defined as shown in Figure,  $P_{men}$  is the pressure at the meniscus. Equating these two expressions finally gives:

$$P_{atm} - P_{men} = \rho g (s - s_0) = -\rho g H \quad (5.3.5)$$

Using the Young-Laplace equation, we arrive at:

$$\rho g H = \frac{2\gamma}{R \sec \theta} \quad (5.3.6)$$

Rearranging we get Jurin's famous law for capillary rise

### Jurin's Law

$$H = \frac{2\gamma \cos \theta}{R\rho g} \quad (5.3.7)$$

We could have also derived this result by equating forces as shown in *J. Pellicer et al.* (1995). The weight due to the column of water must be equal to the surface tension forces acting along the meniscus perimeter, causing the liquid to rise:

$$F_\gamma = 2\pi R\gamma \cos \theta = F_g = \pi\rho R^2 g H \quad (5.3.8)$$

Again, we should check boundary conditions to see if our answer makes physically sense. Jurin's law tells us that  $H \propto \frac{1}{R\rho}$ . The denser the liquid, the higher it will rise (there are some exceptions such as mercury). Moreover, the narrower the tube, the greater the rise. Both sound physically intuitive and are correct.

### The Concave Meniscus ( $\theta < \frac{\pi}{2}$ )

Notice that the size of the meniscus plays a huge role in the derivation of Jurin's law. As seen earlier, the balance between cohesive and adhesive forces determines the shape of a meniscus (more specifically the contact angle formed with a wall, which defines the shape of a meniscus).

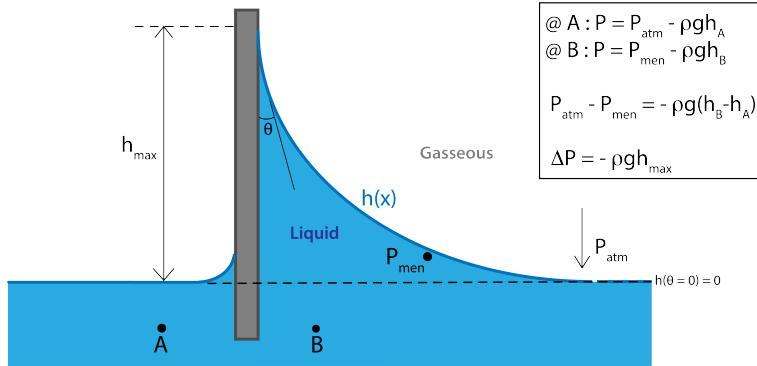
Consider once again a liquid placed in a tube. If liquid molecules are more attracted to the walls than to other liquid molecules (when adhesive forces overcome cohesive forces), one intuitively expects the meniscus to be concave. The molecules at the edges will be "dragged" upwards by adhesive forces, similar to a water column in a capillary tube would rise. Nearby molecules will move alongside as a result of cohesive forces. This may be easier to imagine if we interpret the cohesive forces as "chains": if the molecules at the edges move upwards, nearby molecules will move too. As we get farther from the tubes, these effects become more and more negligible, until they are null at the center of the container.

Instead, if the cohesive forces overcome the adhesive forces, then the meniscus will have a convex profile (similar to a droplet or puddle), since liquid molecules will try to "clump" all together, amassing near the center of the meniscus. Another major consequence is that we will have the "opposite" of capillary action, capillary fall. Instead of rising, the liquid molecules will try to "stick together", and actually fall (often observed in mercury). It follows that the equations governing puddles, droplets and menisci will be the same, as the conditions leading to their formation are identical.

Observe the right side of a meniscus in a cylinder containing a liquid columns. We will set  $z = 0$  as the height at which the meniscus "converges" towards, and denote the contact angle with the wall of the cylinder as  $\theta$ , as shown below.

Using (5.3.5), and following *Berg, 2009*, we can write that:

$$\gamma \nabla \cdot \hat{\mathbf{n}} = -\rho g z \quad (5.3.9)$$



**Figure 5.7.** Profile of a concave meniscus

which can be used to define the shape of the meniscus. Indeed, if we define the surface of the meniscus as  $f(x, z) = z - h(x)$ , then:

$$\hat{\mathbf{n}} = \frac{\nabla f}{|\nabla f|} = \frac{\hat{\mathbf{z}} - h_x(x)\hat{\mathbf{x}}}{\sqrt{1 + h_x(x)^2}} \quad (5.3.10)$$

and we finally reach:

$$\nabla \cdot \hat{\mathbf{n}} = \frac{-h_{xx}(x)}{(1 + h_x(x)^2)^{\frac{3}{2}}} \approx -h_{xx}(x) \quad (5.3.11)$$

for  $h_x(x) < 1$ . Substituting into (5.3.9) we get the second order partial differential equation:

$$-\gamma \frac{\partial^2 h}{\partial x^2} = \rho g h, \quad h_x(0) = -\cot \theta, \quad (5.3.12)$$

whose solution is:

$$h(x) = \lambda \cot \theta e^{-\frac{x}{\lambda}} \quad (5.3.13)$$

where  $\lambda = \sqrt{\frac{\gamma}{\rho g}}$  is the so called "Capillary length". This is another fundamental variable in the study of capillarity.<sup>1</sup> This expression gives the profile of a meniscus at a distance  $x$  from the wall of the container. As the capillary length increases, we expect that meniscus to be more curved (see next section), which agrees with our expression. A similar argument holds for the contact angle  $\theta$ .

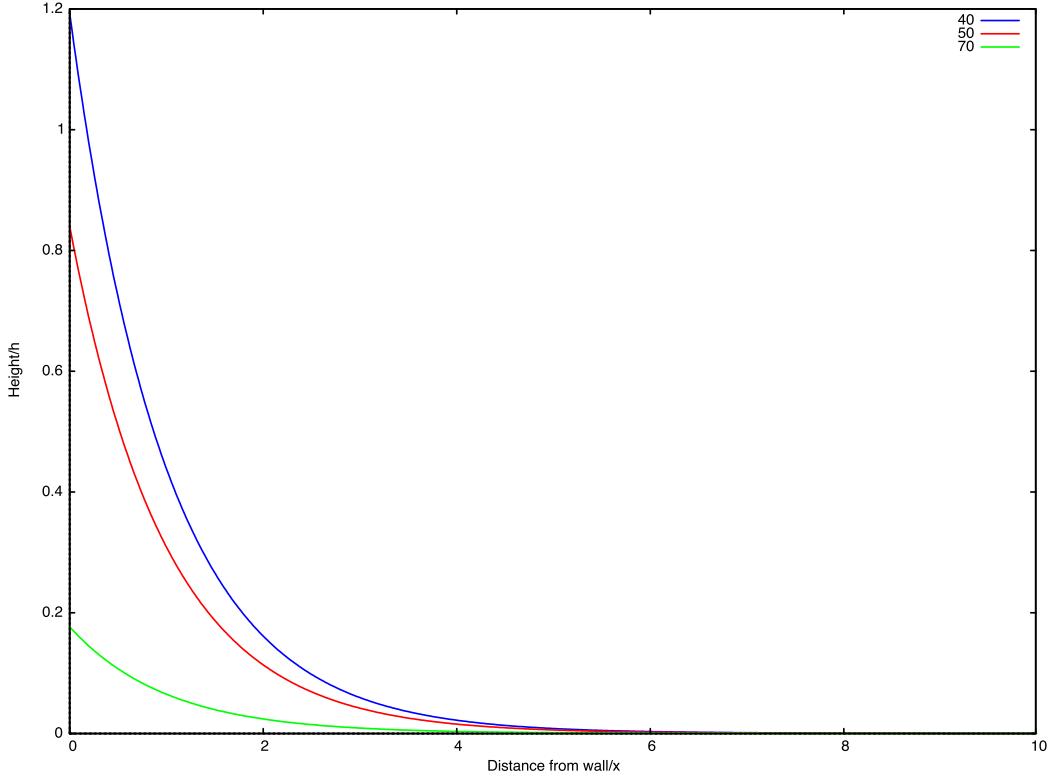
Some profiles for concave menisci at different contact angles are given.

### Capillary Length $\lambda$ and the shape of puddles

Consider a water droplet or puddle on a solid surface. We then know that the Laplace pressure at two points A, B inside this droplet, of radius of curvature  $R_A, R_B$  respectively

<sup>1</sup>We could have derived a similar result for a convex meniscus ( $\theta > \frac{\pi}{2}$ ), by applying the initial condition  $h_x(0) = \tan \theta$ , giving as a solution:

$$h(x) = -\lambda \tan \theta e^{-\frac{x}{\lambda}} \quad (5.3.14)$$



**Figure 5.8.** Menisci for contact angles  $40^\circ, 50^\circ, 70^\circ$ , setting  $\lambda = 1$

is then:

$$\Delta P_A = \frac{2\gamma}{R_A}, \quad \Delta P_B = \frac{2\gamma}{R_B} \quad (5.3.15)$$

We then have that the pressure difference between these two points is:

$$\Delta P_A - \Delta P_B = 2\gamma \left( \frac{1}{R_A} - \frac{1}{R_B} \right) \quad (5.3.16)$$

This is equal to the vertical hydrostatic pressure difference  $\rho gh$ , where  $h$  is the height difference between the A and B. Equating the two yields:

$$\left( \frac{1}{R_A} - \frac{1}{R_B} \right) = \frac{h}{2\frac{\gamma}{\rho g}} \quad (5.3.17)$$

Using dimensional analysis, we can conclude that the term  $\frac{\gamma}{\rho g}$  must have units of  $[L]^2$ , so that we may define the capillary length as:

$$\lambda = \sqrt{\frac{\gamma}{\rho g}} \quad (5.3.18)$$

The physical interpretation for this value is the distance over which a liquid-gas interface is curved. Thus, it follows that capillary length plays a vital role in determining the shape of a droplet or puddle. This is evident when analyzing cases where  $h \gtrless \lambda$ .

**Case 1:  $h > \lambda$** 

For droplets where  $h > \lambda$ , we then have using (5.3.17) that

$$\frac{1}{R_A} - \frac{1}{R_B} \gg 0 \quad (5.3.19)$$

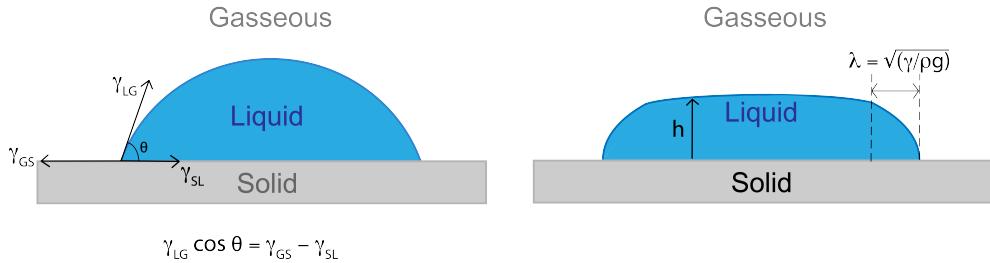
It follows that the radius of curvature at point B will be greater than at point A. This means that the curvature decreases as we move upwards, and hence we expect the top part of the droplet to be flat, and become more curved as we move downwards.

**Case 2:  $h < \lambda$** 

Using our physical intuition, for  $h < \lambda$  we expect the droplet to have a spherical shape. Indeed, using the same procedure, we get that:

$$\frac{1}{R_A} - \frac{1}{R_B} \approx 0 \quad (5.3.20)$$

This means that the radius of curvature between any two arbitrary points A, B inside the droplet is the same. Hence, we must have a spherical droplet.



**Figure 5.9.** Comparison between puddles with  $h > \lambda$  and  $h < \lambda$ .

One might now wonder how to calculate the maximum height of a droplet on an ideal smooth hydrophobic surface. We have that the net energy density or net force per unit length acting on the contour of the three phases (solid, liquid, gas), the interface, must be null. If we denote the surface tension of the solid-liquid, liquid-gas and gas-solid interface as  $\gamma_{SL}$ ,  $\gamma_{LG}$ ,  $\gamma_{GS}$ , and the contact angle as  $\theta$ , then balancing force per unit length between these three phases:

$$\gamma_{SL} + \gamma_{LG} \cos \theta = \gamma_{GS} \quad (5.3.21)$$

which can be rearranged into the Young equation (not to be mistaken with the Young-Laplace equation):

**Young Equation**

$$\cos \theta = \frac{\gamma_{GS} - \gamma_{SL}}{\gamma_{LG}} \quad (5.3.22)$$

As the surface tension between the liquid and gas phases increases, the angle of contact must decrease. This agrees with Young's equation. Since surface tension is essentially how

much a surface pushes against increasing its surface area, if surface tension is greater, we expect its surface area to get smaller and smaller, causing a decrease in contact angle.

Going back to the derivation in subsection 2.3.2, we can write:

$$-\rho gx = \frac{\gamma g_{xx}(x)}{(1 + g_x(x)^2)^{\frac{3}{2}}} \quad (5.3.23)$$

Substituting  $q = g_x(x)$ , we can solve this ODE:

$$-\frac{1}{2}\rho gh^2 = \frac{\gamma q}{\sqrt{1 + q^2}} + C \quad (5.3.24)$$

$$= \gamma \cos \theta + C \quad (5.3.25)$$

We can now set initial conditions  $h(\theta = 0) = 0$ , so that  $C = -\gamma$ . Finally, we have the result:

$$h = \sqrt{\frac{2\gamma}{\rho g}(1 - \cos \theta)} \quad (5.3.26)$$

which can be rewritten using Young's equation:

$$h = \sqrt{\frac{2}{\rho g}(\gamma - \gamma_{GS} + \gamma_{SL})} \quad (5.3.27)$$

The same result can be rewritten as

$$h = 2\lambda \sin\left(\frac{\theta}{2}\right) \quad (5.3.28)$$

Observing figure 2.9, notice that we can define a *spread factor*  $S$  as the difference between the surface energies trying to "spread" the droplet (pointing outwards), and the surface energies pointing inwards:

$$S = \gamma + \gamma_{SL} - \gamma_{GS} \quad (5.3.29)$$

so that we finally reach:

$$h = \sqrt{\frac{2S}{\rho g}} \quad (5.3.30)$$

This makes sense from a physical standpoint. Indeed, one would expect that for a puddle of droplet to have a greater maximum height, it would be "pushing inwards" more than it would be "pushing outwards". In other words, the greater the spreading parameter  $S$  is, the more spherical we'd expect the droplet to be. This agrees with (5.3.30). To conclude, we provide a table to summarize our results on menisci.

Concave Meniscus	Convex Meniscus
$h(x) \approx \lambda \cot \theta e^{-\frac{x}{\lambda}}$	$h(x) \approx -\lambda \tan \theta e^{-\frac{x}{\lambda}}$
$h_{max} = \sqrt{\frac{2\gamma}{\rho g}(1 - \sin \theta)}$	$h_{max} = \sqrt{\frac{2\gamma}{\rho g}(1 - \cos \theta)}$
$h_{max} \approx \lambda \cot \theta$	$h_{max} \approx \lambda \tan \theta$

**Table 5.1.** Table Summarizing Shape of Menisci

2

## 5.4 Minimal Surfaces

Consider a soap film produced when immersing a frame into a water-soap solution. Since we have no change in pressure when moving through the interface layer, it follows from Laplace's equation that the mean curvature of this soap film must be zero. Such types of surfaces that minimize surface area by having zero mean curvature at all points are called minimal surfaces. Indeed, *A. Presley (2012)* gives the definition of a minimal surface as:

*A minimal surface is a surface whose mean curvature is zero everywhere.*

They are given by the solutions to the *minimal surface equation*:

### Minimal Surface Equation

$$\nabla \cdot \left( \frac{\nabla f}{(1 + |\nabla f|^2)^{\frac{1}{2}}} \right) = 0 \quad (5.4.1)$$

Quite obviously, a simple plane would satisfy this equation. Another solution is the Helicoid, the second non-trivial solution to be discovered after the Catenoid (see Fig. 2.10).

Note that, except for the plane, all other solutions of the minimal surface equation will have non-zero curvature at some points. However, they average out at every point to be zero.

<sup>2</sup>We could have also derived the formula for a concave meniscus. Consider:

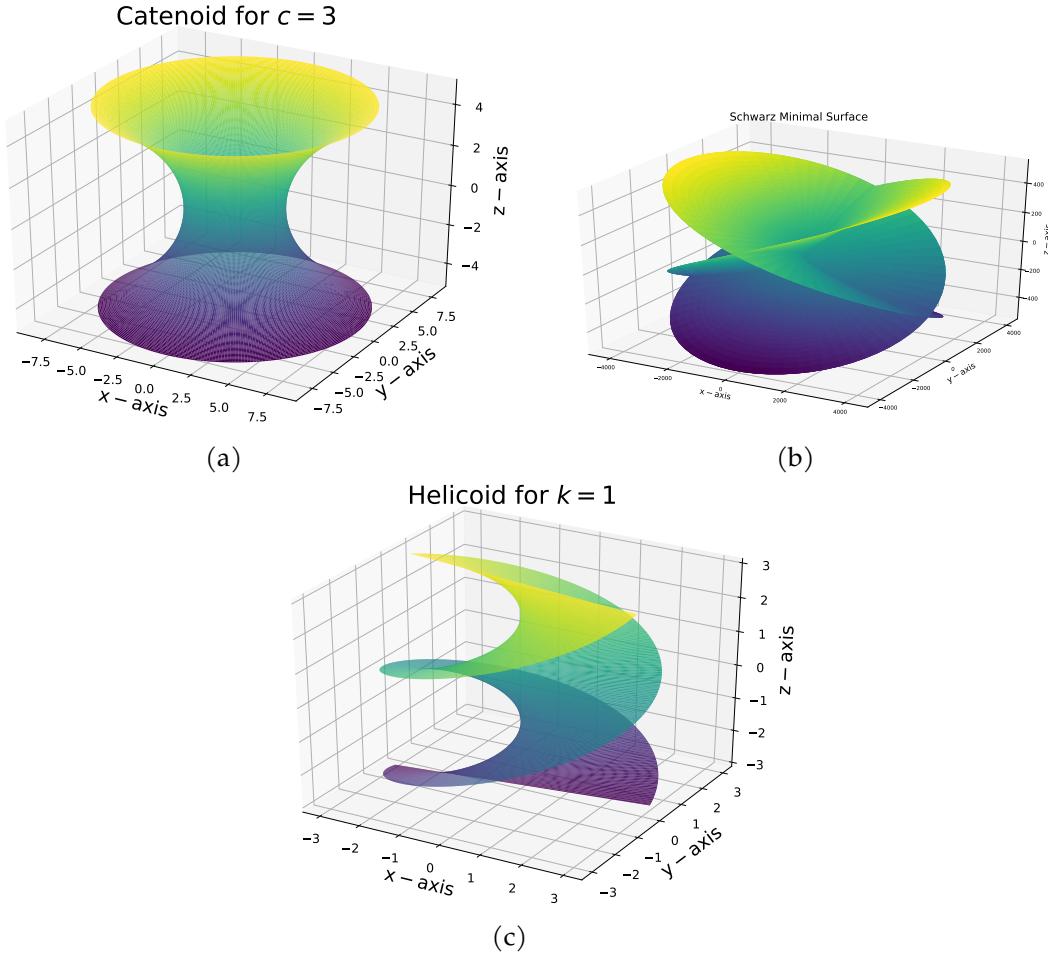
$$-\frac{1}{2}\rho gh^2 = \frac{\gamma q}{\sqrt{1+q^2}} + C = \gamma \frac{\tan \theta}{\sqrt{1+\tan^2 \theta}} + C = \gamma \sin \theta$$

Using the initial condition that  $h(q = \infty) = 0$ :

$$h = \sqrt{\frac{2\gamma}{\rho g}(1 - \sin \theta)}$$

One could have also used trigonometric identities and have shown that:

$$h = \sqrt{\frac{2}{\rho g}(1 - \cos \psi)} = \sqrt{\frac{2}{\rho g}(1 - \cos(90 - \pi))} = \sqrt{\frac{2}{\rho g}(1 - \sin \theta)}$$



**Figure 5.10.** a) Plot of a Catenoid b) Plot of Henneberg Surface c) Plot of Helicoid

Moreover, all these surfaces have a "soap film frame", the frame that contains the set of all points the surface must contain, while still minimizing its surface area. For the Catenoid and the Helicoid, these frames are quite easy to imagine. The former is generated when immersing two elliptical rings parallel to each other in soap, whereas the latter is formed when using a helix.

# Fluid kinematics

We now turn our attention to the non-static case. We will deal with the distribution of the fluid velocity  $\mathbf{v} = \mathbf{v}(x, y, z, t)$ , which together with the fluid pressure  $p(x, y, z, t)$  and density  $\rho(x, y, z, t)$  fully determine all physical properties of the fluid. Indeed the fundamental equations of fluid dynamics which express conservation laws are expressed using these quantities.

## 6.1 Mass conservation

One important conservation law that applies to fluids is the local conservation of mass. Let us consider a region  $\mathcal{V}$  of space, so that the total mass of the fluid contained within this region is  $\int_{\mathcal{V}} \rho d^3\mathbf{r}$ . On the other hand,  $\rho\mathbf{v}$  represents the mass per unit area (cross-section) per unit time passing through a region, so that  $\rho\mathbf{v} \cdot d\mathbf{A}$  gives the mass per unit time flowing through the surface element  $d\mathbf{A}$ . When integrated over  $\partial\mathcal{V}$  this gives the total fluid mass flux through the region. For mass to be conserved in  $V$  we need the rate of change of the mass in  $V$  summed with the mass flux out of  $V$  to be equal to zero. Therefore

$$\frac{\partial}{\partial t} \int_{\mathcal{V}} \rho d^3\mathbf{r} + \oint_{\partial\mathcal{V}} \rho\mathbf{v} \cdot d\mathbf{A} = 0 \quad (6.1.1)$$

We can use the Divergence theorem to write the surface integral as a volume integral

$$\int_{\mathcal{V}} \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) \right) d^3\mathbf{r} = 0 \quad (6.1.2)$$

Since this must hold for any  $\mathcal{V}$ , the integrand has to vanish, implying that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0$$

(6.1.3)

This equation is the continuity equation, it states that any local change in density must be accounted for by a net divergence of mass flux. In other words fluid must be displaced out of a region for the mass within this region to change. Expanding the divergence allows us to write the continuity equation as

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho = 0 \quad (6.1.4)$$

The quantity  $\rho\mathbf{v}$  has played a fundamental role in this discussion and thus deserves its own name. It is known as the mass flux density  $\mathbf{j}$ , it is a vector field whose direction gives the direction of the fluid's motion, and whose magnitude yields the mass flux per unit time to a surface perpendicular to  $\mathbf{v}$ .

## 6.2 Velocity conservation

We again consider some arbitrary region  $\mathcal{V}$  in the fluid. The total surface force acting on this volume is given by

$$\mathbf{F}_{\text{surf}} = - \oint_{\partial\mathcal{V}} p d\mathbf{A} = - \int_{\mathcal{V}} \nabla p \cdot d^3\mathbf{r} \quad (6.2.1)$$

so using Newton's second law one finds that

$$\rho \frac{d\mathbf{v}}{dt} + \nabla p = 0 \quad (6.2.2)$$

If we also had volume forces  $\mathbf{f}_{\text{vol}}$  e.g. gravity then we would find

$$\rho \frac{d\mathbf{v}}{dt} + \nabla p = \mathbf{f}_{\text{vol}} \quad (6.2.3)$$

This is a conservation law! It says that any local change in the velocity of the fluid must be accounted for by a pressure gradient. Unlike previously however, we have a full derivative in time rather than a partial derivative. This poses some problems because in general  $\mathbf{r}$  is time dependent, so  $\mathbf{v} = \mathbf{v}(\mathbf{r}(t), t)$ .

What we are interested in is the change in velocity  $d\mathbf{v}$  in an infinitesimal time interval  $dt$ . There will be two contributions, one due to the change  $d\mathbf{r}$  in position during  $dt$  yielding a change in  $\mathbf{v}$ , and the other due to the change in time (at constant  $\mathbf{r}$ ) directly/explicitly yielding a change in  $\mathbf{v}$ . The second contribution is easiest to work out, it is just  $\frac{\partial \mathbf{v}}{\partial t}|_{\mathbf{r} dt}$ . The first contribution, on the other hand, is given by

$$dx \frac{\partial \mathbf{v}}{\partial x} + dy \frac{\partial \mathbf{v}}{\partial y} + dz \frac{\partial \mathbf{v}}{\partial z} \quad (6.2.4)$$

which is just the inner product of the directional derivative and the displacement  $d\mathbf{r}$ : <sup>1</sup>

$$d\mathbf{r} \cdot \nabla \mathbf{v} \quad (6.2.5)$$

Therefore we find that

$$d\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} dt + (d\mathbf{r} \cdot \nabla) \mathbf{v} \quad (6.2.6)$$

and therefore

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v}$$

(6.2.7)

We define  $\frac{d}{dt}$  in this context to be the convective derivative, it is extremely useful as any local conservation law can be written as  $\frac{dA}{dt} = 0$ .

<sup>1</sup>recall that  $\mathbf{n} \cdot \nabla f$  gives the change in  $f$  along  $\mathbf{n}$

Substituting (6.2.7) into (6.2.3) then we get Euler's equation

$$\boxed{\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{1}{\rho} \nabla p = 0} \quad (6.2.8)$$

Note that throughout this derivation we have not taken into account any sources for energy dissipation, such as viscosity. Therefore Euler's equation only holds for ideal fluids. Note that since there is no heat exchange in the fluid, the motion we have described is reversible and adiabatic, or isentropic ( $dS = 0$ ). Since entropy is constant, we can write that the convective derivative is zero

$$\frac{\partial S}{\partial t} + (\mathbf{v} \cdot \nabla) S \quad (6.2.9)$$

which, together with the continuity equation, gives the entropic equation of continuity

$$\frac{\partial(\rho S)}{\partial t} + \nabla \cdot (\rho S \mathbf{v}) = 0 \quad (6.2.10)$$

Often the entropy will also be homogeneous across the fluid, in which case  $\nabla S = 0$  and thus  $S$  will be constant in time as well. For isentropic processes the specific enthalpy  $h$  satisfies  $dh = Tds + \frac{dp}{\rho} = \frac{dp}{\rho}$ , and thus, if the flow is steady (so that we are at equilibrium), then

$$\nabla h = \frac{1}{\rho} \nabla p \quad (6.2.11)$$

This allows us to write Euler's equation as

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla h = 0 \quad (6.2.12)$$

Yet another useful form of Euler's equation comes from applying the vector identity <sup>2</sup>

$$\frac{1}{2} \nabla v^2 = \mathbf{v} \times (\nabla \times \mathbf{v}) + (\mathbf{v} \cdot \nabla) \mathbf{v} \quad (6.2.14)$$

then Euler's equation takes the form (letting  $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ )

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left( h + \frac{1}{2} v^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = 0 \quad (6.2.15)$$

or, taking the curl of both sides

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) \quad (6.2.16)$$

As with all PDEs, a particular solution can only be found if we are also given sufficient boundary and initial value conditions. For ideal fluids in contact with a solid surface with

<sup>2</sup>Here's the proof

$$[\mathbf{v} \times (\nabla \times \mathbf{v})]^i = \epsilon^{ijk} v_j \epsilon_{klm} \partial^l v^m = (\delta_l^i \delta_m^j - \delta_m^i \delta_l^j) v_j \partial^l v^m = \frac{1}{2} \partial^i (v_m v^m) - v_l \partial^l v^i \quad (6.2.13)$$

unit normal  $\mathbf{n}$ , it is implicit that  $\mathbf{n} \cdot \mathbf{v} = w$  where  $w$  is the speed of the surface. If instead we are dealing with two immiscible fluids forming a boundary, then we must require the pressure and  $\mathbf{n} \cdot \mathbf{v}$  to be continuous across the boundary.

### 6.3 Energy conservation

The energy per unit volume of a fluid is given by

$$\frac{1}{2}\rho v^2 + \rho\epsilon \quad (6.3.1)$$

where  $\epsilon$  is the specific internal energy. We are interested in the rate of change of this energy with respect to time

$$\frac{\partial}{\partial t} \left( \frac{1}{2}\rho v^2 + \rho\epsilon \right) = \frac{1}{2}v^2 \frac{\partial\rho}{\partial t} + \rho\mathbf{v} \cdot \frac{\partial\mathbf{v}}{\partial t} + \frac{\partial(\rho\epsilon)}{\partial t} \quad (6.3.2)$$

To compute the last partial derivative we use the First law of thermodynamics

$$d\epsilon = Tds + \frac{p}{\rho^2}d\rho \quad (6.3.3)$$

to yield

$$d(\rho\epsilon) = \rho T ds + \left( \epsilon + \frac{p}{\rho} \right) d\rho = \rho T ds + wd\rho \quad (6.3.4)$$

and thus

$$\frac{\partial(\rho\epsilon)}{\partial t} = w \frac{\partial\rho}{\partial t} + \rho T \frac{\partial s}{\partial t} \quad (6.3.5)$$

We can use the entropic continuity equation to write this as

$$\frac{\partial(\rho\epsilon)}{\partial t} = w \frac{\partial\rho}{\partial t} - \rho T (\mathbf{v} \cdot \nabla)s \quad (6.3.6)$$

Similarly, we can massage the rest of (6.3.2) by using the continuity equation and Euler's equation:

$$\frac{\partial}{\partial t} \left( \frac{1}{2}\rho v^2 \right) = -\frac{1}{2}v^2 \nabla \cdot (\rho\mathbf{v}) - \mathbf{v} \cdot \nabla p - \rho\mathbf{v} \cdot ((\mathbf{v} \cdot \nabla)\mathbf{v}) \quad (6.3.7)$$

$$= -\frac{1}{2}v^2 \nabla \cdot (\rho\mathbf{v}) - \rho\mathbf{v} \cdot \nabla \left( \frac{1}{2}v^2 + h \right) + \rho T \mathbf{v} \cdot (\nabla s) \quad (6.3.8)$$

where we used  $\nabla p = \rho\nabla h - \rho T\nabla s$ . Finally, the condition for energy to be conserved thus becomes

$$\frac{\partial}{\partial t} \left( \frac{1}{2}\rho v^2 + \rho\epsilon \right) = -\left( \frac{1}{2}v^2 + h \right) \nabla \cdot (\rho\mathbf{v}) - \rho\mathbf{v} \cdot \nabla \left( \frac{1}{2}v^2 + h \right) \quad (6.3.9)$$

or more simply

$$\frac{\partial}{\partial t} \left( \frac{1}{2}\rho v^2 + \rho\epsilon \right) + \nabla \cdot \left[ \rho\mathbf{v} \left( \frac{1}{2}v^2 + h \right) \right] = 0 \quad (6.3.10)$$

This is yet another conservation law, it tells us that any change in the energy of the fluid contained within a region must be accounted for by an energy flux divergence.

In the case of steady flow (defined by a time-independent velocity field  $\mathbf{v}(x, y, z)$ ), Euler's equations simplify a great deal into

$$\nabla \left( \frac{1}{2} v^2 + h \right) - \mathbf{v} \times \boldsymbol{\omega} = 0 \quad (6.3.11)$$

Dotting to the left with  $\mathbf{v}$  and using the fact that  $\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial h}{\partial t} = 0$  we then find that

$$\mathbf{v} \cdot \nabla \left( \frac{1}{2} v^2 + h \right) = 0 \implies \frac{d}{dt} \left( \frac{1}{2} v^2 + h \right) = 0 \quad (6.3.12)$$

or in other words

$$\frac{1}{2} v^2 + h = \text{const.} \quad (6.3.13)$$

If we also add an external body force with potential  $\Phi$  then we get

$$\boxed{\frac{1}{2} v^2 + h + \Phi = \text{const}} \quad (6.3.14)$$

which is known as Bernoulli's equation.

## 6.4 Momentum conservation

We can try to combine the mass conservation and velocity conservation to obtain a momentum conservation law. In index notation we find that

$$\frac{\partial}{\partial t} (\rho v_i) = \rho \frac{\partial v_i}{\partial t} + v_i \frac{\partial \rho}{\partial t} \quad (6.4.1)$$

$$= -\rho v_k \frac{\partial v_i}{\partial x_k} - \frac{\partial p}{\partial x_i} - \frac{\partial(\rho v_k)}{\partial x_k} v_i \quad (6.4.2)$$

$$= -\frac{\partial}{\partial x_k} (\rho v_i v_k) - \delta_{ik} \frac{\partial p}{\partial x_k} = -\frac{\partial \sigma_{ik}}{\partial x_k} \quad (6.4.3)$$

where we used the stress tensor

$$\boxed{\sigma_{ik} = \rho v_i v_k + \delta_{ik} p \iff \boldsymbol{\sigma} = \rho \mathbf{v} \otimes \mathbf{v} + p \mathbb{1}} \quad (6.4.4)$$

The physical interpretation of this stress tensor is best understood by integrating (6.4.3):

$$\frac{\partial}{\partial t} \int \rho v_i d^3 \mathbf{r} = - \oint \sigma_{ik} dA_k \quad (6.4.5)$$

so just like the static case the stress tensor component  $\sigma_{ik}$  gives the  $i$ th component of the momentum flux per unit time passing through an area perpendicular to the  $x_k$ -axis. This time however, the presence of dynamics added a velocity term  $\rho \mathbf{v} \otimes \mathbf{v}$ . We therefore see

that

$$\boxed{\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot \boldsymbol{\sigma} = 0} \quad (6.4.6)$$

is a conservation law for momentum!

## 6.5 Circulation conservation

Let  $\Gamma$  be a closed contour within the fluid at some instant in time, and let

$$C = \oint_{\Gamma} \mathbf{v} \cdot d\mathbf{l} \quad (6.5.1)$$

be the velocity circulation around  $\Gamma$ . This curve will be composed of the fluid particles at some instant, so as the particles move the contour will also move with the fluid. Since we are interested in the time derivative of the circulation, it will be helpful to rewrite the line integral as

$$C = \lim_{N \rightarrow \infty} \sum \mathbf{v} \cdot \delta \mathbf{r}_k \quad (6.5.2)$$

It follows that

$$\frac{dC}{dt} = \lim_{N \rightarrow \infty} \sum_{k=0}^N \left[ \frac{d\mathbf{v}_k}{dt} \cdot \delta \mathbf{r}_k + \mathbf{v}_k \cdot \frac{d(\delta \mathbf{r}_k)}{dt} \right] \quad (6.5.3)$$

$$= \lim_{N \rightarrow \infty} \sum_{k=0}^N \left[ \frac{d\mathbf{v}_k}{dt} \cdot \delta \mathbf{r}_k + \frac{1}{2} \delta(v_k^2) \right] \quad (6.5.4)$$

$$= \oint_{\Gamma} \frac{d\mathbf{v}}{dt} \cdot d\mathbf{l} \quad (6.5.5)$$

since the line integral of a full differential vanishes. Using  $\frac{d\mathbf{v}}{dt} = -\nabla h$  we then get that

$$\frac{d}{dt} \left( \oint_{\Gamma} \mathbf{v} \cdot d\mathbf{l} \right) = - \oint_{\Gamma} (\nabla h) \cdot d\mathbf{l} \quad (6.5.6)$$

So we find that the velocity circulation is a conserved quantity

$$\boxed{\frac{d}{dt} \left( \oint_{\Gamma} \mathbf{v} \cdot d\mathbf{l} \right) = \frac{d}{dt} \left( \int_S \boldsymbol{\omega} \cdot d\mathbf{A} \right) = 0} \quad (6.5.7)$$

Note that we used Euler's equation to write  $\frac{d\mathbf{v}}{dt}$  as a gradient, so this result again only holds for ideal fluids.

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# **Vorticity**

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# Viscosity

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# Water and gravity waves

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# Turbulence

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# Magnetohydrodynamics

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# **Plasma physics**

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## **Part III**

# **Continuum mechanics**

## **Part IV**

# **Chaos theory and stochastic dynamics**

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## Acknowledgments

This is the most common positions for acknowledgments. A macro is available to maintain the same layout and spelling of the heading.

**Note added.** This is also a good position for notes added after the paper has been written.

# Bibliography

- [1] Author, *Title*, *J. Abbrev.* **vol** (year) pg.
- [2] Author, *Title*, arxiv:1234.5678.
- [3] Author, *Title*, Publisher (year).