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- **6.A——**
- 1.

  $\langle(1+(-2),1),(2,-1)\rangle = |-2|+|-1| = 3 \neq 7$

  which is $|2|+|-1|+|-4| = \langle(1,1),(2,-1)\rangle + \langle(-2,0),(2,-1)\rangle$

  Addictivity in first slot doesn't hold

  Hence it is not an inner product

- 2.

  $\langle(0,1,0),(0,1,0)\rangle = 0$ however $(0,1,0) \neq 0$

  Hence it is not aninner product

- 4.
  - a.

    $\langle u+v,u-v\rangle = \|u\|^2 + \langle v,u\rangle - \langle u,v\rangle - \|v\|^2$

    Since V is a real space, $\langle u,v\rangle = \langle v,u\rangle$

    Hence $\langle u+v,u-v\rangle = \|u\|^2 - \|v\|^2$
  - b.

    If $\|u\| = \|v\|$, $\langle u+v,u-v\rangle = 0$

    Hence $u+v$ is orthogonal to $u-v$
  - c.

    Suppose quadrangle ABCD is a rhombus

    $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC},\ \overrightarrow{AB}-\overrightarrow{BC}=\overrightarrow{AB}+\overrightarrow{DA}=\overrightarrow{DB}$

    Hence $\langle\overrightarrow{AC},\overrightarrow{DB}\rangle = 0$, which implies that the diagonals are perpendicular to each other

- 6.

  If $\langle u,v\rangle = 0$, then $\|u+av\|^2 = \|u\|^2 + \|av\|^2 \geq \|u\|^2$

  If $\|u+av\| \geq \|u\|$ for all $a \in \mathbf{F}$

  $\|u+av\|^2 - \|u\|^2 = |a|^2\|v\| + a\langle v,u\rangle + \overline{a}\langle u,v\rangle \geq 0$

  Let $a = -\langle u,v\rangle$, we get $-|\langle u,v\rangle|^2 \geq 0$

  Thus $\langle u,v\rangle = 0$

- 7.

  If $\|av+bu\| = \|au+bv\|$ for all $a,b \in \mathbb{R}$

Let $a = 1,\ b = 0$, we get $\|u\| = \|v\|$

If $\|u\| = \|v\|$

$\|au + bv\|^2 = a^2\|u\|^2 + ab(\langle u, v\rangle + \langle v, u\rangle) + b^2\|v\|^2$

$= a^2\|v\|^2 + ab(\langle u, v\rangle + \langle v, u\rangle) + b^2\|u\|^2 = \|av + bu\|^2$

- 10.

  Suppose $v = (x, y),\ u = k(1, 3)$

  Since $u \perp v$, we get $(x, y) \cdot (1, 3) = 0 = x + 3y$

  Also $v + u = (x + k, y + 3k) = (1, 2)$

  Hence $x = \frac{3}{10},\ y = -\frac{1}{10},\ k = \frac{7}{10}$

  We get $v = (\frac{3}{10}, -\frac{1}{10}),\ u = (\frac{7}{10}, \frac{21}{10})$

- 11.

  $(\sqrt{a} \cdot \frac{1}{\sqrt{a}} + \cdots + \sqrt{d} \cdot \frac{1}{\sqrt{d}})^2 = 16 \leq (\sqrt{a}^2 + \cdots + \sqrt{d}^2)(\frac{1}{\sqrt{a}^2} + \cdots + \frac{1}{\sqrt{d}^2}) = (a + \cdots + d)(\frac{1}{a} + \cdots + \frac{1}{d})$

- 15.

  Let $x = (\sqrt{1}a_1, \ldots, \sqrt{n}a_n),\ y = (b_1/\sqrt{1}, .., b_n/\sqrt{n})$

  Since $x \cdot y \leq \|x\| \cdot \|y\|$, substituting the above euqation, we can complete the proof

- 16.

  $\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2 = 18 + 2\|v\|^2 = 52$

  Hence $\|v\| = \sqrt{17}$

- 19.

  $\|u + v\|^2 - \|u - v\|^2 = 2(\langle u, v\rangle + \langle v, u\rangle)$

  Since V is real space, $\langle u, v\rangle = \overline{\langle v, u\rangle} = \langle v, u\rangle$

  $\frac{\|u+v\|^2 - \|u-v\|^2}{4} = \langle u, v\rangle$

- 20.

  $\|u + v\|^2 - \|u - v\|^2 + \|u + iv\|^2 i - \|u - iv\|^2 i = 2(\langle u, v\rangle + \langle v, u\rangle) + 2i(\langle u, iv\rangle + \langle iv, u\rangle)$

  $= 2(\langle u, v\rangle + \langle v, u\rangle + \langle u, v\rangle - \langle v, u\rangle) = 4\langle u, v\rangle$

- 24.

  Positivity

  $\langle u, u\rangle_1 = \langle Su, Su\rangle \geq 0$

  Definiteness

$\langle v,v\rangle_1=\langle Sv,Sv\rangle=0$ iff $Sv=0$ since S is injective $Sv=0$ iff $v=0$

Additivity in first slot

$\langle u+w,v\rangle_1=\langle S(u+w),Sv\rangle=\langle Su,Sv\rangle+\langle Sw,Sv\rangle=\langle u,v\rangle+\langle w,v\rangle$

Homogeneity in first slot

$\langle \lambda u,v\rangle_1=\langle S(\lambda u),Sv\rangle=\langle \lambda Su,Sv\rangle=\lambda\langle Su,Sv\rangle=\lambda\langle u,v\rangle_1$

Conjugate symmetry

$\langle u,v\rangle_1=\langle Su,Sv\rangle=\overline{\langle Sv,Su\rangle}=\overline{\langle v,u\rangle}_1$

- 25.

  If S is not injective, $\exists v\in V$ s.t. $v\neq 0,\ Sv=0$

  Hence $\langle v,v\rangle_1=\langle Sv,Sv\rangle=0$

  Thus the difiniteness doesn't hold

- **6.B——**
- 4.

  $\langle\frac{1}{\sqrt{2\pi}},\frac{1}{\sqrt{2\pi}}\rangle=1$, $\langle\frac{\cos kx}{\sqrt{\pi}},\frac{\cos kx}{\sqrt{\pi}}\rangle=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2 kxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos 2kx+1}{2}dx=1$

  Similarly $\langle\frac{\sin kx}{\sqrt{\pi}},\frac{\sin kx}{\sqrt{\pi}}\rangle=1$

  Thus each vector in the list has norm 1

  $\langle\frac{1}{\sqrt{2\pi}},\frac{\cos kx}{\sqrt{\pi}}\rangle=\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\cos kxdx=0$

  $\langle\frac{1}{\sqrt{2\pi}},\frac{\sin kx}{\sqrt{\pi}}\rangle=\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\sin kxdx=0$

  $\langle\frac{\sin kx}{\sqrt{\pi}},\frac{\cos kx}{\sqrt{\pi}}\rangle=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos kx\sin kxdx=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin 2kxdx=0$

  Hence any two vector of the list are orthogonal

  Hence it is an orthonormal list

- 5.

  $\langle 1,1\rangle=1$, hence $e_1=1$

  $x-\langle 1,x\rangle 1=x-\frac{1}{2},\ \|x-\frac{1}{2}\|^2=\frac{1}{12}$, hence $e_2=\sqrt{3}(2x-1)$

  $x^2-\langle 1,x^2\rangle 1-\langle\sqrt{3}(2x-1),x^2\rangle\sqrt{3}(2x-1)=x^2-\frac{1}{3}-(x-\frac{1}{2})=x^2-x+\frac{1}{6}$

  $\langle x^2-x+\frac{1}{6},x^2-x+\frac{1}{6}\rangle=\frac{1}{180}$, hence $e_3=\sqrt{5}(6x^2-6x+1)$

- 6.

  Suppose D is a differentiation operator.

  Since with respect to the basis $1,x,x^2$, $\mathcal{M}(D)$ is an upper-triangular matrix.

  $D(1)\in span(1),D(x)\in span(1,x),D(x^2)\in span(1,x,x^2)$

  Let $e_1=1,e_2=\sqrt{3}(2x-1),e_3=\sqrt{5}(6x^2-6x+1)$

Since $e_1 \in span(1), e_2 \in span(1, x), e_3 \in span(1, x, x^2)$

$D(e_1) = \lambda D(1) \in span(e_1)$

$D(e_2) = \lambda D(1) + \mu D(x) \in span(1, x) = span(e_1, e_2)$

$D(e_3) = \lambda D(1) + \mu D(x) + \varphi D(x^2) \in span(1, x, x^2) = span(e_1, e_2, e_3)$

Hence with respect to e_1, e_2, e_3, $\mathcal{M}(D)$ is also an upper-triangular matrix

- 8.

  Let $\varphi(p) = \int_0^1 p(x)(\cos \pi x)dx$

  Suppose $1, \sqrt{3}(2x - 1), \sqrt{5}(6x^2 - 6x + 1)$ is an orthonormal basis and denote it as e_1, e_2, e_3

  Define $q = \overline{\varphi(e_1)}e_1 + \cdots + \overline{\varphi(e_3)}e_3$

  Hence $\varphi(p) = \varphi(\langle p, e_1\rangle e_1 + \cdots + \langle p, e_3\rangle e_3) = \langle p, q\rangle$

  We get $q = \frac{1}{\pi^2}(12 - 24x)$

- 9.

  Suppose $v_1, \ldots, v_n$ is linearly dependent list

  Suppose $v_1, \ldots, v_{k-1}$ is linearly independent and $v_k = span(v_1, \ldots, v_{k-1})$

  Since $v_1, \ldots, v_{k-1}$ is linearly independent, they can be transformed into $e_1, \ldots, e_{k-1}$

  $v_k = \langle v_k, e_1\rangle e_1 + \cdots + \langle v_k, e_{k-1}\rangle v_{k-1}$

  Hence $v_k - \langle v_k, e_1\rangle e_1 - \cdots - \langle v_k, e_{k-1}\rangle e_{k-1} = 0$

  which makes the construction by Gram-Schmidt fail

- 14.

  $\|e_j - v_j\|^2 = 1 + \|v_j\|^2 - \langle e_j, v_j\rangle - \langle v_j, e_j\rangle < \frac{1}{n}$

  Suppose $v_1, \ldots, v_n$ is linearly dependent

  Thus $\exists a_1, \ldots, a_n$ not all 0 s.t. $a_1v_1 + \cdots + a_nv_n = 0$

  Hence $a_1(e_1 - v_1) + \cdots + a_n(e_n - v_n) = a_1e_1 + \cdots + a_ne_n$

  $\sqrt{a_1^2 + \cdots + a_n^2} = \|a_1e_1 + \cdots + a_ne_n\|$

  $= \|a_1(e_1 - v_1) + \cdots + a_n(e_n - v_n)\| \leq |a_1|\|e_1 - v_1\| + \cdots + |a_n|\|e_n - v_n\|$

  $< \frac{|a_1| + \cdots + |a_n|}{\sqrt{n}}$

  However $\sqrt{\frac{\sum |a_i|^2}{n}} \geq \frac{\sum |a_i|}{n}$

  Hence $a_1 = \cdots = a_n = 0$ which implies $v_1, \ldots, v_n$ is a linearly independent list with right length

  Thus it is a basis