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- **6.C——**
- 3.

  $span(e_1,\ldots,e_m) = span(u_1,\ldots,u_m)$ thus $e_1,\ldots,e_m$ is a spanning list with right length

  Hence it is a basis of U, $\dim U = m$

  Since $e_1,\ldots,e_m,f_1,\ldots,f_n$ is orthonormal basis of V, $\dim V = m+n$ and $f_i \perp e_j$

  Hence $span(f_1,\ldots,f_n) \subset U^\perp$

  Since $V = U \oplus U^\perp$, we get $\dim U^\perp = \dim V - \dim U = n$

  Thus $f_1,\ldots,f_n$ is a linearly independent list with right length

  Hence $f_1,\ldots,f_n$ is a orthonormal basis of $U^\perp$

- 6.

  If $P_U P_W = 0$, $\forall w \in W$, $P_U P_W w = P_U w = 0$ hence $w \in U^\perp$

  Since w is chosen arbitrary, $W \subset U^\perp$

  which implies that $\forall w \in W,\ u \in U$, $\langle u,w\rangle = 0$

  If $\forall w \in W,\ u \in U$, $\langle u,w\rangle = 0$

  $W \in U^\perp,\ U \in W^\perp, \forall v \in V,\ P_W v \in W,\ P_U P_W v = 0$

  which implies that $P_U P_W = 0$

- 7.

  Since $null\ P \subset V,\ range\ P \subset V,\ \dim null\ P + \dim range\ P = \dim V,\ null\ P \perp range\ P$,

  $null\ P \oplus range\ P = V$

  Define $U = range\ P$, $\forall v \in V, \exists w \in null\ P,\ u \in range\ P$, s.t. $v = w + u$

  Since $null\ P \perp range\ P$, $P_U v = u$

  Since $u \in U = range\ P$, $\exists x \in V$, s.t. $Px = u$

  $Pu = P(Px) = Px = u$, hence $Pv = P(w+u) = Pu = u = P_U v$

- 8.

  Suppose $u \in range\ P,\ u \neq 0$, $\exists v_w \in V$, s.t. $u = Pv_w$

  $u = Pv_w = P^2 v_w = Pu$, hence $u \notin null\ P$, since u is chosen arbitrary

  We get $null\ P \cap range\ P = \{0\}$

  Since $\dim null\ P + \dim range\ P = \dim V,\ null\ P, range\ P \subset V, null\ P \oplus range\ P = V$

  $\exists u \in range\ P, w \in null\ P$ s.t. $v = w + u$

  $Pv = P(w+u) = Pu = u = P(u+aw)$ thus $\|u\| = \|P(u+aw)\| \leq \|u+aw\|$ for all $a \in \mathbf{F}$

  Hence $\|u+aw\|^2 - \|u\|^2 = |a|^2\|w\| + a\langle w,u\rangle + \overline{a}\langle u,w\rangle \geq 0$

Let $a = -\langle u, w\rangle / \|w\|$, we get $-|\langle u, w\rangle|^2 \|w\| \geq 0$ thus $\langle u, w\rangle = 0$

Since u, v is chosen arbitrary, $null\ P \perp range\ P$

By Problem 7, we get $P = P_U$

- 10.

  If U and $U^\perp$ are both invariant under T

  $\forall u \in U, Tu \in U$, hence $P_U(Tu) = Tu$, $TP_U u = Tu$

  $\forall w \in U^\perp, Tw \in U^\perp$, hence $P_U(Tw) = 0,\ TP_U w = T0 = 0$

  Since $\forall v \in V, \exists u_0 \in U, w_0 \in U^\perp$ s.t. $v = u_0 + w_0$

  Hence $P_U Tv = P_U T(u_0 + w_0) = TP_U(u_0 + w_0) = TP_U v$

  If $P_U T = TP_U$

  $\forall u \in U, Tu = TP_U u = P_U Tu = u_0$ where $u_0 \in U, w_0 \in U^\perp, u_0 + w_0 = Tu$

  Hence $Tu \in U$

  $\forall w \in U^\perp, P_U Tw = TP_U w = T0 = 0$ which implies $Tw \in U^\perp$

  Hence U and $U^\perp$ is invariant under T

- 11.

  Applying Gram-Schmidt Procedure to $(1, 1, 0, 0), (1, 1, 1, 2)$,

  We get $e_1 = \frac{1}{\sqrt{2}}(1, 1, 0, 0), e_2 = \frac{1}{\sqrt{5}}(0, 0, 1, 2)$

  Thus e_1, e_2 is an orthonormal basis of U, thus it can be extend to an orthonormal basis of V

  as e_1, e_2, e_3, e_4, denote $\lambda_i = \langle (1, 2, 3, 4), e_i\rangle$ for $i = 1, \ldots, 4$

  Thus $\|u - (1, 2, 3, 4)\| = \|xe_1 + ye_2 - (\lambda_1 e_1 + \cdots + \lambda_4 e_4)\| = |x - \lambda_1| + |y - \lambda_2| + |\lambda_3| + |\lambda_4|$

  Hence $u = \lambda_1 e_1 + \lambda_2 e_2 = \frac{1}{10}(15, 15, 22, 44)$

- 12.

  Define $V = \{p \in \mathcal{P}_3(\mathbb{R}) : p(0) = 0, p'(0) = 0\}$ and $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$, $q(x) = 2 + 3x$

  The basis of V is x^3, x^2, and by Gram-Schmidt Procedure

  It ban be transformed into $e_1 = \sqrt{5}x^2, e_2 = \sqrt{7}(6x^3 - 5x^2)$

  By the proof of Problem 11. $P(x) = \langle q, e_1\rangle e_1 + \langle q, e_2\rangle e_2 = 24x^2 - 10.3x^3$

- **7.A——**
- 1.

  $\langle T(x_1, \ldots, x_n), (y_1, \ldots, y_n)\rangle = \langle (0, x_1, \ldots, x_{n-1}), (y_1, \ldots, y_n)\rangle = x_1 y_2 + \cdots + x_{n-1} y_n$

  $= \langle (x_1, \ldots, x_n), (y_2, \ldots, y_n, 0)\rangle = \langle (x_1, \ldots, x_n), T^*(y_1, \ldots, y_n)\rangle$

  Hence $T^*(z_1, \ldots, z_n) = (z_2, \ldots, z_n, 0)$

- 2.

  λ is an eigenvalue of $T \Leftrightarrow \langle (T-\lambda I)v, w\rangle = 0 \Leftrightarrow \langle v, (T-\lambda I)^* w\rangle = 0$

  $\Leftrightarrow \langle v, (T^* - \overline{\lambda} I)w\rangle = 0$

  Suppose $\overline{\lambda}$ is not an eigenvalue of T^*, $T^* - \overline{\lambda} I$ is suejective

  Thus $\exists u \in V$ s.t. $(T^* - \overline{\lambda} I)u = v$

  We get $\langle v, v\rangle = 0$, However, v is eigenvector of T, $v \neq 0$

  Hence $\overline{\lambda}$ is an eigenvalue of T^*

- 4.
  - a.

    T is injective iff $null\ T = \{0\}$ iff $(range\ T^*)^\perp = \{0\}$ iff $range\ T^* = V$ iff T^* is surjective
  - b.

    T is surjective iff $range\ T = W$ iff $(range\ T)^\perp = \{0\}$ iff $null\ T^* = \{0\}$ iff T^* is injective

- 6.

  Let $p(x) = a_0 + a_1 x + a_2 x^2, q(x) = b_0 + b_1 x + b_2 x^2$
  - a.

    $\langle Tp, q\rangle = \int_0^1 a_1 x(b_0 + b_1 x + b_2 x^2)dx = a_1(\frac{1}{2}b_0 + \frac{1}{3}b_1 + \frac{1}{4}b_2)$

    $\langle p, Tq\rangle = \int_0^1 b_1 x(a_0 + a_1 x + a_2 x^2)dx = b_1(\frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2)$

    Hence $\langle Tp, q\rangle \neq \langle p, Tq\rangle$ which implies that T is not self-adjoint
  - b.

    Since the basis $1, x, x^2$ is not an orthonormal basis

- 11.

  If $P = P_U$ for some U as a subspace of V

  $\forall v_1, v_2 \in V$, $\exists u_1, u_2 \in U, w_1, w_2 \in U^\perp$ s.t. $v_1 = u_1 + w_1, v_2 = u_2 + w_2$

  $\langle Pv_1, v_2\rangle = \langle u_1, u_2 + w_2\rangle = \langle u_1, u_2\rangle = \langle u_1, u_2 + w_2\rangle = \langle v_1, Pv_2\rangle$

  Hence P is self-adjoint

  If P is self-adjoint

  $nullP\ = (range\ P^*)^\perp = (range\ P)^\perp$

  Hence By Problem 6.C 7, we get $P = P_{range\ P}$
- 13.

  With respect to standard basis $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$

  Let $\mathcal{M}(T) = |1\ 0\ 0\ 0|$ The $\mathcal{M}(T^*)$ is the conjugate transpose of $\mathcal{M}(T)$ as $|1\ 0\ 0\ 0|$

  $|0\ 0\ 0\ 1|$ $|0\ 0\ 1\ 0|$

  $|0\ 1\ 0\ 0|$ $|0\ 0\ 0\ 1|$

$|0\ 0\ 1\ 0|\ |0\ 1\ 0\ 0|$

Hence $\mathcal{M}(TT^*) = I = \mathcal{M}(T^*T)$ however $T \neq T^*$

- 15.
    - a.

      T is self-adjoint iff $\langle T^*v, w\rangle = \langle v, Tw\rangle = \langle v, x\rangle\langle u, w\rangle = \langle\langle v, x\rangle u, w\rangle$

      iff $T^*v = \langle v, x\rangle u$

      If $u = kx$ for some $k \in \mathbb{R}$, $Tv = \langle v, kx\rangle x = \langle v, x\rangle kx = T^*v$

      If $T^*v = \langle v, x\rangle u$, $T^*v - Tv = 0 = \langle v, x\rangle u - \langle v, u\rangle x$

      Let $v = u$ and $u \neq 0$, $\langle u, x\rangle u - \langle u, u\rangle x = 0$

      Since $\langle u, u\rangle \neq 0$, u, x are linearly dependent

      Hence T is self-adjoint iff u, x are linearly dependent
    - b.

- 18.

  Let $V = \mathbb{R}^2$, and with respect to standard orthonormal basis $e_1 = (1, 0), e_2 = (0, 1)$

  Define $Te_1 = e_1 + e_2, Te_2 = e_1 - e_2$

  Thue $T^*e_1 = e_1 - e_2, T^*e_2 = e_2 - e_1$

  $\|Te_1\| = \|T^*e_1\|, \|Te_2\| = \|T^*e_2\|$ however $\mathcal{M}(T) \neq \mathcal{M}(T^*)$

- 19.

  Since T is normal $\forall v \in null\ T$, $\|T^*v\| = \|Tv\| = 0$ thus $v \in null\ T^*$ which is $null\ T \subseteq null\ T^*$

  Similarly $null\ T^* \subseteq null\ T$ hence $null\ T = null\ T^*$

  $\langle T^*(z_1, z_2, z_3), (1, 1, 1)\rangle = 0 = \langle (z_1, z_2, z_3), T(1, 1, 1)\rangle = \langle (z_1, z_2, z_3), (2, 2, 2)\rangle$

  Thus $2z_1 + 2z_2 + 2z_3 = 0$ which is $z_1 + z_2 + z_3 = 0$