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- **3.A——**
- 2.

  Suppose $p_1, p_2 \in \mathcal{P}(\mathbb{R})$

  Additivity:

  Let $p = p_1 + p_2$

  $T(p_1+p_2) = (3p(4)+5p'(6)+bp(1)p(2), \int_{-1}^{2} x^3 p(x)dx + c\sin p(0))$

  $p(4) = (p_1+p_2)(4) = p_1(4)+p_2(4)$

  $p'(6) = (p_1+p_2)'(6) = p_1'(6)+p_2'(6)$

  $bp(1)p(2) = b(p_1+p_2)(1)(p_1+p_2)(2) = b[p_1(1)p_1(2)+p_1(1)p_2(2)+p_2(1)p_1(2)p_2(1)p_2(2)]$

  To make $b[p_1(1)p_1(2)+p_1(1)p_2(2)+p_2(1)p_1(2)p_2(1)p_2(2)] = b[p_1(1)p_1(2)+p_2(1)p(2)(2)]$

  Only to make $b = 0$

  $\int_{-1}^{2} x^3 (p_1+p_2)(x)dx = \int_{-1}^{2} x^3 p_1(x)dx + \int_{-1}^{2} x^3 p_2(x)dx$

  $c\sin(p_1+p_2)(0) = c[\sin p_1(0)\cos p_2(0) + \cos p_1(0)\sin p_2(0)]$

  To make $c[\sin p_1(0)\cos p_2(0) + \cos p_1(0)\sin p_2(0)] = c[\sin p_1(0) + \sin p_2(0)]$

  Only to make $c = 0$

  Homogeneity:

  $b(kp)(1)(kp)(2) = bk^2 p(1)p(2)$

  To make $bk^2p(1)p(2) = bkp(1)p(2)$ for $\forall k \in \mathbb{R}$, only to make $b = 0$

  $c\sin(kp)(0) \equiv kc\sin p(0)$ iff $c = 0$

  Which implies that T is linear iff $b = c = 0$

- 3.

  Let $v_k = (A_{1,k}, \dots, A_{m,k})$ $(k = 1, 2, \dots, n)$

  Thus $T(1, 0, \dots, 0) = v_1$

  $T(0, 1, \dots, 0) = v_2$

  …

  $T(0, 0, \dots, 1) = v_n$

  Since $(1, 0, \dots, 0), \dots, (0, 0, \dots, 1)$ is a basis of F^n, $v_1, \dots, v_m$ is a basis of F^m

  Which implies that such distinct T exist

- 4.

  Suppose exists $a_1, \dots, a_m \in \mathrm{F}$ and not all 0 s.t. $a_1v_1 + \cdots + a_nv_n = 0$

  $T(a_1v_1 + \cdots + a_nv_n) = T(0) = 0 = a_1Tv_1 + \cdots + a_nTv_n$

  Thus $Tv_1, \dots, Tv_n$ is linearly dependent, which contradicts to the independence

Hence $v_1, \dots, v_n$ is linearly independent

- 7.

  Since $T \in \mathcal{L}(V, V)$ and $\dim V = 1$, $\exists v, u \in V$ s.t. u and v are two basis of V and $Tv = u$

  Since $\dim V = 1$, $u = av$, thus $Tv = av$

  $\forall w \in V, w = bv$

  $Tw = T(bv) = bTv = bav = a(bv) = aw$

- 8.

  Let $\varphi(x, y) = x^2/y$ if $y \neq 0$ and $\varphi(x, y) = 0$ if $y = 0$

  Thus $\varphi[a(x, y)] = (ax)^2/(ay) = ax^2/y = a\varphi(x, y)$

  However, $\varphi[(1, 2) + (2, 1)] = \varphi(3, 3) = 3 \neq 1/2 + 4$

- 10.

  Suppose $u \in U$,s.t. $Su \neq 0$

  Since $U \subset V$ and $U \neq V$

  We can find $v \in V$ s.t. $v \notin U$

  Thus $v + u \notin U$, $T(v + u) = 0$

  However, $Tv + Tu = Sv + 0 = Sv \neq 0$

- 11.

  Suppose $u_1, \dots, u_m$ is a basis of U

  Since U is a subspace of V, we can extend the basis to $u_1, \dots, u_m, v_{m+1}, \dots, v_n$

  Now, define $Tu_i = Su_i$ for $i = 1, 2, \dots, m$ and $Tv_j = 0$ for $j = m + 1, \dots, n$

  Hence T is a linear map from U to V

  And $\forall u \in U$, we can write u as $u = a_1u_1 + \cdots + a_mu_m$ where $a_i \in \mathrm{F}, i = 1, \dots, m$

  $Tu = T(a_1u_1 + \cdots + a_mu_m) = a_1Tu_1 + \cdots + a_mTu_m = Su$

- 14

  Suppose $v_1, \dots, v_n$ is a basis of V

  Then we can define $Sv_i = v_{i+1}$ for $i = 1, \dots, n - 1$ and $Sv_n = v_1$

  And $Tv_j = v_j$ for $j = 2, \dots, n$ and $Tv_1 = 0$

  $STv_1 = 0$, however, $TSv_1 = v_2 \neq 0$

- **3.B——**
- 3.
  - a.

    If $\operatorname{span}(v_1, \dots, v_m) = V$, $\operatorname{range} T = \operatorname{span}(v_1, \dots, v_m) = V$

Hence T is surjective

  - b.

    If $v_1, \ldots, v_m$ is linearly independent, $\forall u \in \operatorname{span}(v_1, \ldots, v_m)$

    $\exists unique\, a_1, \ldots, a_m \in \mathrm{F}$ s.t. $u = a_1 v_1 + \cdots + a_m v_m$

    This implies that $T(z_1, \ldots, z_m) = T(s_1, \ldots, s_m)$ iff $(z_1, \ldots, z_m) = (s_1, \ldots, s_m)$

    Hence T is injective

- 4.

  Denote $\mathcal{M} = \{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) : \dim \operatorname{null} T > 2\}$

  Suppose $T_1(v, w, x, y, z) = (0, 0, x, y, z), T_2(v, w, x, y, z) = (0, 0, -x, y, z)$

  Hence $\dim \operatorname{null} T_1 = 3 = \dim \operatorname{null} T_2$ which implies $T_1, T_2 \in \mathcal{M}$

  However $(T_1 + T_2)(v, w, x, y, z) = (0, 0, 0, y, z)$, $\dim \operatorname{null}(T_1 + T_2) = 2$

- 5.

  Let $T(w, x, y, z) = (0, 0, y, z)$

- 7.

  Denote $\mathcal{M} = \{T \in \mathcal{L}(V, W) : T\, is\ not\ injective\}$

  Suppose $v_1, \ldots, v_m$ is a basis of V, $w_1, \ldots, w_n$ is a basis of W and $2 \leq m \leq n$

  Let $T_1 v_1 = 0$, $T_1 v_i = w_i$ for $i = 2, \ldots, m$ and $T_2 v_j = w_j$ for $j = 1, \ldots, m-1$, $T_2 v_n = 0$

  $\dim \operatorname{range} T_1 = \dim \operatorname{range} T_2 = m - 1$ which implies that $T_1, T_2 \in \mathcal{M}$

  However $(T_1 + T_2)(v_i) = 2w_i$ for $i = 2, \ldots, m-1$ and $(T_1 + T_2)(v_j) = w_j$ for $j = 1, m$

  $\dim \operatorname{range}(T_1 + T_2) = m$ which implies $(T_1 + T_2) \notin \mathcal{M}$

- 9.

  Assume that $Tv_1, \ldots, Tv_n$ is not linearly independent

  Thus Tv_n can be written as $Tv_n = a_1 Tv_1 + \cdots + a_{n-1} Tv_{n-1}$ where $a_1, \ldots, a_{n-1} \in \mathrm{F}$ and not all 0

  $\Rightarrow Tv_n = T(a_1 v_1 + \cdots + a_{n-1} v_{n-1})$

  Since $v_1, \ldots, v_n$ is linearly independent, $v_n \neq a_1 v_1 + \cdots + a_{n-1} v_{n-1}$

  However T is injective

  Hence $Tv_1, \ldots, Tv_n$ can only be linearly independent

- 11.

  Since S_1 is injective, $S_1(S_2 \cdots S_n v) = 0$ iff $S_2(S_3 \cdots S_n v) = 0 \cdots$ iff $S_n v = 0$ iff $v = 0$

  Hence $S_1 \cdots S_n$ is injective

- 15.

  From the question, we know that $\dim \operatorname{null} T = 2$

  However $\dim \mathrm{F}^5 = 5 = \dim \operatorname{null} T + \dim \operatorname{range} T$, $\dim \operatorname{range} T = 3 > \dim \mathrm{F}^2$

  Hence such T does not exist

- 17.

  If T is injective, $\dim V = 0 + \dim \operatorname{range} T \leq \dim W$

  If $\dim V \leq \dim W$, let $v_1, \ldots, v_n$ be basis of V and $w_1, \ldots, w_m$ be basis of W

  Define $Tv_i = w_i$ for $i = 1, \ldots, n$

  Then $Tv = 0$ iff v=0 which implies T is injective

- 20.

  If T is injective, define $S(Tv) = v$, since Tv is unique determined by v, ST is well defined on $\mathcal{L}(V, V)$

  If exist such S, suppose $u, v \in V$ s.t. $Tu = Tv$

  $v = (ST)v = S(Tv) = S(Tu) = (ST)u = u$, hence T is injective