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- **4——**
- 3.

  Denote $P = \{0\} \cup \{p \in \mathcal{P}(\mathrm{F}) : \deg p \; is \; even\}$

  Let $p_1 = x^2 + x, p_2 = -x^2$, thus $p_1, p_2 \in P$

  However $p_1 + p_2 = x$ which is not in P

  Hence P is not closed under addition, which implies it is not a subspace of $\mathcal{P}(\mathrm{F})$

- 5.

  Define $T \in \mathcal{L}(\mathcal{P}_m(\mathrm{F}), \mathrm{F}^{m+1}), Tp = (p(z_1), \dots, p(z_{m+1}))$ for $\forall p \in \mathcal{P}_m(\mathrm{F})$

  Suppose $Tp = 0, p(z_1) = \cdots = p(z_{m+1}) = 0$

  Since $z_1, \dots, z_{m+1}$ are distinct, p have $m+1$ distinct zero, which contradict to the $p \in \mathcal{P}_m(\mathrm{F})$

  Thus $p = 0$, which implies T is injective

  Since $\dim \mathcal{P}_m(\mathrm{F}) = m + 1 = \dim \mathrm{F}^{m+1}$

  T is surjective, which implies that for $\forall (w_1, \dots, w_{m+1}) \in \mathrm{F}^{m+1}$, $\exists p \in \mathcal{P}_m(\mathrm{F})$ s.t.
  $Tp = (p(z_1), \dots, p(z_{m+1})) = (w_1, \dots, w_{m+1})$

  which is also $p(z_j) = w_j$ for $j = 1, \dots, m+1$

- 6.

  Suppose $z_1, \dots, z_m$ are distinct zeros of p, thus $p(z_j) = 0$ for $j = 1, \dots, m$

  For each $j = 1, \dots, m$, since $p(z_j) = 0, p = (z - z_j)q$ where $q \in \mathcal{P}_{m-1}(\mathrm{F})$

  Hence $p' = q + (z - z_j)q'$

  Assume $p'(z_j) = 0, q(z_j) = 0$

  It follows that $q = (z - z_j)s$ where $s \in \mathcal{P}_{m-2}(\mathrm{F})$

  Hence $p = (z - z_j)^2 s$ which implies p have two same zeros z_j

  That contradicts to the distinction of zeros

  Hence $p(z_j)' \neq 0$ for each $j = 1, \dots, m$

  Suppose $z_1, \dots, z_m$ are zeros of p, thus $p(z_j) = 0$ and $p'(z_j) \neq 0$ for $j = 1, \dots, m$

  Assume they are not distinct, we can find $z_s = z_t, z, t \in \{1, \dots, m\}$ and $s \neq t$

  Thus $p = (z - z_s)^2 q$ where $q \in \mathcal{P}_{m-2}$

  $p' = 2(z - z_s)q + (z - z_s)^2 q'$ which implies $p'(z_j) = 0$

  That contradicts to the condition

  Hence $z_1, \dots, z_m$ are distinct

- 7.

Suppose $p \in \mathcal{P}_m(\mathbb{R})$ where m is odd

Thus it can be factorized into $p = c\prod_{i=1}^{n}(x^2 - b_i x + c_i)\prod_{j=1}^{m-2n}(x - \lambda_j)$

Since m is odd, $m - 2n \neq 0$ since $n \in \mathbb{N}$

Thus p have at least one factor as $(x - \lambda_j)$ which implies p has at least one real zero

- 9.

$\overline{p(\overline{z})} = \sum_{k=0}^{m} \overline{a_k} \cdot \overline{(\overline{z})^k} = \sum_{k=0}^{m} \overline{a_k} z^k$

Thus $\overline{p(\overline{z})}$ is a polynomial and so does $q(z) = p(z)\overline{p(\overline{z})}$

Hence $q(z) = \sum_{k=0}^{2m} b_k z^k = p(z)\overline{p(\overline{z})} = \overline{p(\overline{z})\overline{p(z)}} = \overline{q(\overline{z})} = \sum_{k=0}^{2m} \overline{b_k} z^k$

which implies that $b_k = \overline{b_k}$ for each $k = 0, \ldots, 2m$

Hence $q(z)$ has all coefficients real

- **5.A——**
- 1.
  - a.

    If $U \subset \operatorname{null} T$, $\forall u \in U$, $Tu = 0 \in U$

    Thus U is invariant under T
  - b.

    If $U \supset \operatorname{range} T$, since U is a subspace of V, $\forall u \in U, Tu \in \operatorname{range} T \subset U$

    Thus U is invariant under T

- 3.

$\forall u \in \operatorname{range} S, \exists v \in V$ s.t. $Sv = u$

Thus $Tu = TSv = STv \in \operatorname{range} S$

Hence $\operatorname{range} S$ is invariant under T

- 6.

Suppose U is a subspace of V, s.t. $U \neq \{0\}$ and $U \neq V$

Thus $\exists u \in U$ s.t. $u \neq 0$ and $\exists v \in V$ s.t. $v \in V$ but $v \notin U$

Extend u to a basis of V as $u, v_2, \ldots, v_n$

Define $T(u) = w, T(v_j) = 0$ for $j = 2, \ldots, n$

Hence U is not invariant under T

Thus U is invariant under all T iff $U = \{0\}$ or $U = V$

- 7.

$T(x, y) = \lambda(x, y) = (\lambda x, \lambda y) = (-3y, x)$

Thus $\lambda x = -3y, \lambda y = x$, we get $\lambda^2 y = -3y$

Since $T \in \mathcal{L}(\mathbb{R}^2)$, $\lambda^2 \neq -3$, we get $y = 0$

Thus $x = y = 0$ which contradicts since (x, y) is an eigenvector

Hence T has no eigenvalues

- 9.

  $T(z_1, z_2, z_3) = \lambda(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$

  Thus $\lambda z_1 = 2z_2, \lambda z_2 = 0, \lambda z_3 = 5z_3$

  Hence we get $\lambda = 5$ and corresponding eigenvectors are $(0, 0, z_3)$

  and $\lambda = 0$ and corresponding eigenvectors are $(z_1, 0, 0)$

- 12.

  $Tp(x) = \lambda p(x) = xp'(x)$

  Suppose $p = \sum_{k=0}^{4} a_k x^k$, we get $\sum_{k=0}^{4} \lambda a_k x^k = \sum_{k=0}^{4} k a_k x^k$

  Thus $\lambda = 0, 1, 2, 3, 4$ and corresponding eigenverctors are $p = ax^\lambda$

- 14.

  $P(u + w) = \lambda(u + w) = u$

  Since $U + W$ is a direct sum,

  $\lambda = 1$ and corresponding vectors are $u \in U$ or

  $\lambda = 0$ and corresponding vectors are $w \in W$

- 16.

  Suppose such basis of V is $v_1, \ldots, v_n$

  Thus $T(v_j) = A_{1,j} v_1 + \cdots + A_{n,j} v_n$

  Suppose λ is an eigenvalue and corresponsing vectors are $v = a_1 v_1 + \cdots + a_n v_n$

  Hence $T(v) = \sum_{i=1}^{n} a_i (\sum_{j=1}^{n} A_{j,i} v_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i A_{j,i} v_j = \lambda \sum_{i=1}^{n} a_i v_i$

- 19.

  Suppose $\sum_{i=0}^{n} x_i = x$, thus $T(x_1, \ldots, x_n) = (x, \ldots, x) = \lambda(x_1, \ldots, x_n)$

  Thus $\lambda = 0$ or $x_1 = \cdots = x_n$

  Hence the eigenvalues are $0, n$,

  and corresponding eigenvectors are $(x_1, \ldots, x_n) : x_1 + \cdots + x_n = 0, (x, \ldots, x) \in \mathrm{F}^n$

- 21.

  $T^{-1} x = \frac{1}{\lambda} x \Leftrightarrow \lambda T T^{-1} x = Tx \Leftrightarrow Tx = \lambda x$

- 26.

- 28.

  Suppose $v_1, \dots, v_n$ is a basis of V

  For each $v_j, j = 1, \dots, n$

  $T(v_j) = A_{1,j}v_1 + \cdots + A_{n,j}v_n$

  Since T is invariant in any 2-dimensional subspace

  Suppose $U = (v_j, v_k)$ where $k \neq j$ and $k = 1, \dots, n$

  Thus $T(v_j) \in \operatorname{span}(v_j, v_k)$ which implies that $A_{i,j} = 0$ where $i = 1, \dots, n$ and $i \neq j, i \neq k$

  Since k is choosen arbitrary other than j

  $T(v_j) = \lambda_j v_j$

  Hence λ_j is an eigenvalue of T

  Since the basis can be choose arbitrary, any vector in V canbe $T's$ eigrnvector

  By problem 26, $T = aI$

- 31.

  If $v_1, . . , v_m$ are eigrnvectors of some T corresponding to distinct eigenvalues,

  they are linearlu independent obviously

  If $v_1, \dots, v_m$ are linearly indeendent

  Extend it to a basis of V as $v_1, \dots, v_m, v_{m+1}, \dots, v_n$

  Define $T(v_i) = a_i v_i$ where $i = 1, \dots, n$ and a_i are distict

  Thus a_i are $T's$ distict eigenvalue, which implies such T exist

- 34.

  $T/(\operatorname{null} T)(x + \operatorname{null} T) = Tx + \operatorname{null} T$

  Thus it is a injective map

  iff $T/(\operatorname{null} T)(x + \operatorname{null} T) = 0 + \operatorname{null} T$ implies $x \in \operatorname{null} T$

  iff $Tx \in \operatorname{null} T$ implies $x \in \operatorname{null} T$

  Suppose $\operatorname{null} T \cap \operatorname{range} T \neq \{0\}$

  We can find some $v \in \operatorname{null} T \cap \operatorname{range} T$ s.t. $v \neq 0$

  Thus $\exists u \in V$ s.t. $Tu = v$

  Hence $Tu = v \in \operatorname{null} T$ which implies that $u \in \operatorname{null} T$

  That is $v = 0$

  which proves that $Tx \in \operatorname{null} T$ implies $x \in \operatorname{null} T$ iff $\operatorname{null} T \cap \operatorname{range} T = \{0\}$