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- **7.C——**
- 1.

  Suppose $T \in \mathcal{L}(\mathbb{R}^2)$, e_1, e_2 is a standard basis

  Let $Te_1 = e_1, Te_2 = -e_2$, define $e_3 = ae_1 + be_2, e_4 = ce_1 + de_2$

  where $a, b, c, d \in \mathbb{R}$

  Since e_3, e_4 is also an orthogonal basis of $\mathbb{R}^2$

  We get $a^2 + b^2 = 1, c^2 + d^2 = 1, ac + bd = 0$

  To let $\langle Te_3, e_3 \rangle \geq 0, \langle Te_4, e_4 \rangle \geq 0$

  We get $a^2 - b^2 \geq 0, c^2 - d^2 \geq 0$

  Thus solving all the equation above $a \geq \frac{1}{2}, b = \sqrt{1 - a^2}, c = a, d = -b$

  Which obviously give at least one solution

  Hence we give a counterexample

- 4.

  $(T^*T)^* = T^*(T^*)^* = T^*T, (TT^*)^* = (T^*)^*T^* = TT^*$

  Hence T^*T，TT^* are self-adjoint

  $\langle T^*Tv, v \rangle = \langle Tv, Tv \rangle \geq 0$

  $\langle TT^*w, w \rangle = \langle T^*w, T^*w \rangle \geq 0$

  Hence T^*T, TT^* are positive operator

- 7.

  If $\langle Tv, v \rangle > 0$ for all $v \in V$, suppose T is not invertible

  $\exists v_0 \in V$ s.t. $Tv_0 = 0$ thus we get $\langle Tv, v \rangle = 0$ which contradicts to the assumption

  If T is invertible, since T is positive, $\exists S$ which is selfs.t. $S^2 = T$

  S is an operator and if $v \neq 0$ s.t. $Sv = 0$ $S^2v = Tv = 0$ which contradicts to the assumption

  Hence S is invertible $Sv \neq 0$ for all $v \in V$

  Hence $\langle Tv, v \rangle = \langle S^2v, v \rangle = \langle Sv, Sv \rangle > 0$

- 8.

  If $\langle \cdot, \cdot \rangle_T$ is an inner product on V, $\langle v, v \rangle_T = \langle Tv, v \rangle > 0$ for all $v \in V, v \neq 0$

  Suppose T is not invertible, $\exists v_0 \in V$ s.t. $Tv_0 = 0, \langle Tv_0, v_0 \rangle = 0$

  which contradicts to the assumption

  $\langle v, w \rangle_T = \langle Tv, w \rangle = \overline{\langle w, v \rangle_T} = \overline{\langle Tw, v \rangle} = \langle v, Tw \rangle$

  Thus T is self-adjoint

Hence T is invertible and positive

If T is invertible positive operator on V

Since T is positive, $\langle v, v\rangle_T = \langle Tv, v\rangle \geq 0$ and $\exists self - adjoint, invertible\ S$ s.t. $S^2 = T$

$\langle v, v\rangle_T = 0$ iff $\langle Tv, v\rangle = 0$ iff $\langle S^2 v, v\rangle = 0$ iff $\langle Sv, Sv\rangle = 0$ iff $Sv = 0$ iff $v = 0$

$\langle v_1 + v_2, w\rangle_T = \langle T(v_1 + v_2), w\rangle = \langle Tv_1, w\rangle + \langle Tv_2, w\rangle = \langle v_1 . w\rangle_T + \langle v_2, w\rangle_T$

$\langle \lambda v, w\rangle_T = \langle \lambda Tv, w\rangle = \lambda\langle Tv, w\rangle = \lambda\langle v, w\rangle_T$

$\langle v, w\rangle_T = \langle Tv, w\rangle = \langle v, Tw\rangle = \overline{\langle Tw, v\rangle} = \overline{\langle w, v\rangle_T}$

Hence it is an inner product

- 10.

  If S is isometry, S^* is isometry too

  Thus (b),(c),(d) holds

  If (d) holds (a) holds by 7.42

- 12.

  Suppose $e_1, \ldots, e_4$ is an orthonormal basis of $\mathbf{F}^4$

  With respect to this basis, define the matrix of T_1 is a diagonal matrix with $2, 2, 5, 7$ being the diagonal extries and T_2 is a giagonal matrix with $2, 5, 7, 7,$ beging the diagonal entries

  Thus since the two matrix equals their transpose, T_1, T_2 are self-adjoint

  Suppose S is isometry s.t. $T_1 = S^* T_2 S$

  Since $SS^* = I = S^* S$, S is invertible, $\exists v \in V$ s.t. Missing superscript or subscript argument 3

  $T_1 v = S^* T_2 S v = S^* T_2 e_3 = 7S^* e_3 = 7S^{-1} e_3 = 7v$

  Hence v is eigenvector corresponding eigenvalues 7

  Similarly, let $w \in V$ s.t. $Sw = e_4$

  Since $\langle Sv, Sw\rangle = \langle v, w\rangle = \langle e_3, e_4\rangle = 0$, v, w are linearly independent

  And with the same step, we get w is eigenvector corresponding to eigenvalues 7

  However $\dim E(T_1, 7) = 1$, so the maximun length of independent list is 1

  which contradicts to the independence of v, w

  Hence such S doesn't exist

- 14.

  By 7.A-21, T is self-adjoint, thus $-T$ is also self-adjoint

  And since $1, \cos x, \ldots, \cos nx, \sin x, \ldots, \sin nx$ is a basis of V

  Also, $-T \cos nx = n^2 \sin nx, -T \sin nx = n^2 \sin nx, -T1 = 0 \times 1$

  Thus this basis consisting of eigenvectors of $-T$ and is corresponding to nonnegative eigenvalues

  Hence $-T$ is positive

- **7.D——**
- 1.
  $\langle Tv, w\rangle = \langle\langle v, u\rangle x, w\rangle = \langle v, u\rangle\langle x, w\rangle = \langle v, \overline{\langle x, w\rangle} u\rangle = \langle v, T^* w\rangle$ thus $T^* v = \langle v, x\rangle u$
  $T^* T v = \|x\|^2 \langle v, u\rangle u$ let $Sv = \frac{\|x\|}{\|u\|}\langle v, u\rangle u$
  $S^2 v = \frac{\|x\|}{\|u\|}\langle v, u\rangle Su = \|x\|^2 \langle v, u\rangle u = T^* T v$ thus $S^2 = T^* T$
  $\langle Sv, w\rangle = \frac{\|x\|}{\|u\|}\langle v, u\rangle\langle u, w\rangle$, $\langle v, Sw\rangle = \frac{\|x\|}{\|u\|}\langle v, u\rangle\langle u, w\rangle$, thus S is self-adjoint
  $\langle Sv, v\rangle = \frac{\|x\|}{\|u\|}\langle v, u\rangle\langle u, v\rangle = \frac{\|x\|}{\|u\|}\|\langle v, u\rangle\|^2 \geq 0$
  Hence S is positive and is the unique positive square root of $T^* T$ denoted by $\sqrt{T^* T}$

- 3.
  For T^*, $\exists S \in \mathcal{L}(V)$ s.t. $T^* = S\sqrt{(T^*)^* T^*}$ which is $T^* = S\sqrt{TT^*}$ and S is self-adjoint
  Since $\sqrt{TT^*}$ is self-adjoint, $T = (T^*)^* = (S\sqrt{TT^*})^* = \sqrt{TT^*} S$

- 5.
  With respect to standard basis of $\mathbb{C}^2$
  $\mathcal{M}(T^* T) = \mathcal{M}(T^*)\mathcal{M}(T) = |1 \quad 0|$
  $|0\ 16|$
  Hence the singular values of T are $1, 4$

- 7.
  With respect to standard basis of F^3
  $\mathcal{M}(T) = |0\ 2\ 0|$ thus the diagonal entries of TT^* are $4, 9, 1$
  $|0\ 0\ 3|$
  $|1\ 0\ 0|$
  Define $L \in \mathcal{L}(\mathrm{F}^3)$ by $L(z_1, \ldots, z_3) = (2z_1, 3z_2, z_3)$
  $L^2 = TT^*$ and L is positive
  Suppose $S \in \mathcal{L}(\mathrm{F}^3)$ s.t. $T = SL$
  Define $S(z_1, z_2, z_3) = (z_3, z_1, z_2)$ we find such S

- 10.
  Since T is self-adjoint, T can be diagonalizable with respect to some basis
  Thus the eigenvalues of T, $\lambda_1, \ldots, \lambda_n$ are diagonal entries of such matrix
  Thus the diagonal entris of matrix $T^* T$ are $\lambda_1\overline{\lambda}_1, \ldots, \lambda_n\overline{\lambda}_n$
  which is also $\|\lambda_1\|^2, \ldots, \|\lambda_n\|^2$
  Hence the singular values of T are $\|\lambda_1\|, \ldots, \|\lambda_n\|$

- 12.

  Let $\mathcal{M}(T) = |0\ 2|$, $\mathcal{M}(T^*) = |0\ 1|$, $\mathcal{M}((T^*)^2T^2) = |4\ 0|$

  $|1\ 0|$ $|2\ 0|$ $|0\ 4|$

  Hence the singular values of T^2 are $2, 2$ however the singular values of T are $2, 1$

- 15.

  If S is isometry, $\sqrt{S^*S} = I$ which eigenvalues are all 1

  If all singular values of S are 1,

  this implies that all eigenvalues of $\sqrt{S^*S}$ are 1

  Since $\sqrt{S^*S}$ is self-adjoint, it can be diagonalizable with respect to some basis

  Hence $\sqrt{S^*S} = I$

  This implies that S is isometry

- 17.
  - a.

    $Te_i = s_i f_i$ for $i = 1, \ldots, n$

    Thus with respect to $e_1, \ldots, e_n; f_1, \ldots, f_n$ T can be diagonalizable

    Hence the matrix of T^* with respect to $f_1, \ldots, f_n; e_1, \ldots, e_n$ is conjugate transpose of the above matrix

    Since the singular value is absolute value

    which can be written as $T^* f_i = s_i e_i$ for $i = 1, \ldots, n$

    Hence $T^* v = \sum s_i \langle v, f_i \rangle e_i$
  - b.

    $T^*Te_i = s_i^2 e_i$

    Hence $T^*Tv = \sum s_i^2 \langle v, e_i \rangle e_i$
  - c.

    Since $e_1, \ldots, e_n$ are eigenvectors of T corresponding to eigenvalues $s_1^2, \ldots, s_n^2$

    $e_1, \ldots, e_n$ are also eigenvectors of $\sqrt{T^*T}$ corresponsing to eigenvalues $s_1, \ldots, s_n$

    Hence denote $S = \sqrt{T^*T}$, $Se_i = s_i e_i$ we get $Sv = \sum s_i \langle v, e_i \rangle e_i$
  - d.

    Since $TT^{-1} = I$

    $e_i = TT^{-1}e_i = T^{-1}Tf_i = s_i T^{-1} e_i$ thus $T^{-1} f_i = \frac{e_i}{s_i}$

    Hence $T^{-1} v = \sum \frac{\langle v, f_i \rangle e_i}{s_i}$