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- **9.B——**
- 1.

  Since S is over real vector space and is an isometry

  With respect to some basis, the matrix can be

  $$\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} or \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

  Thus the matrix of S^2 have a 1 on the diagonal, which implies that 1 is the eigenvalue

  Hence $\exists x \in \mathbb{R}^3$ s.t. $S^2 x = x$

- 2.

  Since every isometry (denoted S) on a odd-dimensional real inner product space have the matrix of the form

  $$\begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_m \end{bmatrix}$$

  with respect to some basis and where each A_i is 2×2 matrix with form

  $$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

  or is 1×1 matrix with 1 or -1

  Thus, an odd-dimensional must have some blocks A_j s.t. A_j is 1×1 matrix form

  Which implies that for the $j-th$ vector of the above basis, denoted e_j and $Se_j = \pm e_j$

- 6.

  Suppose that with respect to the standard basis, the matrix of T is

  $$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

  Thus $T(x, y) = (x, x + y)$

  $\forall x \in span((0,1)), Tx \in span((0,1))$ however $T(1,0) = (1,1) \notin span((1,0))$

- 7.

  $\mathcal{M}(T_1 \cdots T_m) = \mathcal{M}(T)$

  Hence $T = T_1 \cdots T_m$

- **10.A——**
- 1.

  T is invertible $\Leftrightarrow \exists S \in \mathcal{L}(V)$ s.t. $ST = TS = I$

  $\Leftrightarrow \mathcal{M}(S,(v_1,\ldots,v_n))\mathcal{M}(T,(v_1,\ldots,v_n)) = \mathcal{M}(ST,(v_1,\ldots,v_n)) = \mathcal{M}(I,(v_1,\ldots,v_n))$

  $= \mathcal{M}(T,(v_1,\ldots,v_n))\mathcal{M}(S,(v_1,\ldots,v_n))$

  $\Leftrightarrow \mathcal{M}(T,(v_1,\ldots,v_n))$ is invertible

- 4.

  $\mathcal{M}(T,(v_1,\ldots,v_n)) = \mathcal{M}(I,(u_1,\ldots,u_n),(v_1,\ldots,v_n))\mathcal{M}(T,(v_1,\ldots,v_n),(u_1,\ldots,u_n))$

  $= \mathcal{M}(I,(u_1,\ldots u_n),(v_1,\ldots,v_n))I = \mathcal{M}(I,(u_1,\ldots u_n),(v_1,\ldots,v_n))$

- 6.

  With respect to the standard basis of $\mathbb{R}^2$

  Let the matrix of T be

  $$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

  Thus T^2 is

  $$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

  Hence $trace\, T^2 < 0$

- 8.

  Extend $\frac{v}{\|v\|}$ to a orthonoramal basis of V as $e_1 \doteq \frac{v}{\|v\|}, e_2, \ldots, e_n$

  Thus $Te_1 = T_{1,1}e_1 + \cdots + T_{n,1}e_n$

  It follows that $T_{1,1} = \langle Te_1, e_1\rangle$

  Thus $trace\, T = T_{1,1} + \cdots + T_{n,n} = \langle Te_1, e_1\rangle + \cdots + \langle Te_n, e_n\rangle$

  which is $\langle w, v\rangle$

- 9.

  Suppose $v_1,\ldots,v_n$ is a basis of $range\, T$, $u_1,\ldots,u_m$ is a basis of $null\, T$

  Thus $v_1,\ldots,v_n,u_1,\ldots,u_m$ is a basis of V

  $Pu_i = 0$ for all $i = 1,\ldots,m$

  Suppose $Pv = v_i$, we get $Pv = P^2v = P(Pv) = Pv_i = v_i$

  Thus $Pv_i = v_i$ for all $i = 1,\ldots,n$

  Hence $trace\, T = n = \dim range\, T$

- 11.

  Since T is positive, it is self-adjoint and its eigenvalues are nonnegative

  Thus with respect to some orthonormal basis, the matrix of T is diagonal matrix

  We denote such matrix A

  Since $trace\, T = 0$, $trace\, A = 0$ and A have only nonnegative entries on the diagonal

  we get $A = 0$, which is $T = 0$

- 17.

  Suppose $T \neq 0$, that is $\exists v \in V$ s.t. $Tv \neq 0$

  Thus extend it to a basis of V as $v_1 \doteq Tv, v_2, \ldots, v_n$

  Define $Sv_1 = v, Sv_j = 0$ for all $j = 2, \ldots, n$

  Thus $TSv_1 = Tv = v_1, TSv_j = T0 = 0$ for all $j = 2, \ldots, n$

  It follows that $trace\, TS = 1 = trace\, ST$

  Hence $T = 0$

- 18.

  Let $S = T^*T$, and suppose $e_1, \ldots, e_n$ is an orthonormal basis of V

  $Se_1 = S_{1,1}e_1 + \cdots + S_{n,1}e_n$ thus $S_{1,1} = \langle Se_1, e_1\rangle$

  Similarly $S_{i,i} = \langle Se_i, e_i\rangle$ for all $i = 1, \ldots, n$

  Thus $trace\, T^*T = trace\, S = \langle T^*Te_1, e_1\rangle + \cdots + \langle T^*Te_n, e_n\rangle = \langle Te_1, Te_1\rangle + \cdots + \langle Te_n, e_n\rangle$

  which is $\|Te_1\|^2 + \cdots + \|Te_n\|^2$

- 20.

  Suppose such orthonormal basis is $e_1, \ldots, e_n$

  $\sum\sum |A_{j,k}|^2 = trace\, \mathcal{M}(T, (e_1, \ldots e_n))\mathcal{M}(T^*, (e_1, \ldots e_n)) = trace\, T^*T$

  Since T has an block diagonal matrix, with each block being upper-triangular matrix with the size of eigenvalue's multiplicity square and eigenvalue on the diagonal with reapect to some basis consisting of generalized eigenvalues

  Thus T^* is the hermite matrix with such basis

  It follows that $trace\, T^*T \geq |\lambda_1|^2 + \cdots + |\lambda_n|^2$