11612733 杜子豪

- **3.C——**
- 1.

  Suppose the matrix of T has at most $\dim \operatorname{range} T - 1$ nonzero entries

  Thus $Tv_1, \ldots, Tv_n$ has at most $\dim \operatorname{range} T - 1$ nonzero vectors

  Since $\operatorname{range} T = \operatorname{span}(Tv_1, \ldots, Tv_n)$

  $\dim \operatorname{range} T \leq N(The\ number\ of\ nonzero\ vectors) = \dim \operatorname{range} T - 1$

  which shows contradiction.

  Hence the matrix of T has at least $\dim \operatorname{range} T$ nonzero entries

- 3.

  Suppose $v_1, \ldots, v_m$ is a basis of $\operatorname{null} T$, we can extend it to a basis of V as $v_1, \ldots, v_m, u_1, \ldots, u_n$

  Thus $\operatorname{range} T = \operatorname{span}(Tu_1, \ldots, Tu_n)$

  Since $\forall v \in V,\ v \notin \operatorname{null} T$, $v = a_1u_1 + \cdots + a_nu_n$, $Tv = a_1Tu_1 + \cdots + a_nTu_n \neq 0$ for $a_1, \ldots, a_n$ not all 0

  Thus the only way is to let $a_1 = \cdots = a_n = 0$ to make $a_1Tu_1 + \cdots + a_nTu_n = 0$

  Hence $Tu_1, \ldots, Tu_n$ is a basis of $\operatorname{range} T$, we can extend it to a basis of W as $Tu_1, \ldots, Tu_n, w_1, \ldots, w_k$

  Thus we can choose $v_1, \ldots, v_m, u_1, \ldots, u_n$ as a basi of V and $Tu_1, \ldots, Tu_n, w_1, \ldots, w_k$ as basis of W

  This way, the matrix of T satisfies the rule in the problem

- 4.

  If $Tv_1 \neq 0$, since $Tv_1 \in W$, we can extend it to a basis of W as $Tv_1, w_2, \ldots, w_m$

  If $Tv_1 = 0$, choose any vector in W as w_1 and extend it to a basis of W

  By both ways of constructions, we can make the matrix of T satisfy the rule in the problem

- 5.

  If $w_1 \in \operatorname{range} T,\ \exists v_1 \in V$, s.t. $Tv_1 = w_1$

  If not choose arbitrary $v_1 \in V$

  Then extend v_1 to a basis of V as $v_1, \ldots, v_n$

  By this way of construction, we can make the matrix of T satisfy the rule in the problem

- 6.

  If exist such T, suppose the respect basis of V is $v_1, \ldots, v_n$, the basis of W is $w_1, \ldots, w_m$

  $Tv_i = w_1 + \cdots + w_m$ for $i = 1, \ldots, n$; and $Tv_1 = \cdots = Tv_n$

  Since $\operatorname{range} T = \operatorname{span}(Tv_1, \ldots, Tv_n)$, $\dim \operatorname{range} T = 1$

  On the contrary, choose $w_1, \ldots, w_m$ as a basis of W, s.t. $w_1 + \cdots + w_m \in \operatorname{range} T$

Thus $\exists v_1 \in V$ s.t. $Tv_1 = w_1 + \cdots + w_m$, extend v_1 to a basis of V as $v_1, \ldots, v_n$

Since $\operatorname{range} T = \operatorname{span}(Tv_1, \ldots, Tv_n)$ and $\dim \operatorname{range} T = 1$

$Tv_1 = \cdots = Tv_n = w_1 + \cdots + w_m$ which makes the matrix of T whith respect to such basis has 1 as all entries

- 14.

  Suppose A is $a \times b$, B is $c \times d$, C is $e \times f$

  Since $(AB)C$ makes sense, then $b = c,\ d = e$, which also ensure A(BC) making sense

  $[(AB)C]_{i,j} = (AB)_{i,\cdot} \times C_{\cdot,j} = (A_{i,\cdot} \times B) \times C_{\cdot,j} = \sum_{1 \le s \le n, 1 \le t \le m} a_{i,s} b_{s,t} c_{t,j}$

  $[A(BC)]_{i,j} = A_{i,\cdot} \times (BC)_{\cdot,j} = A_{i,\cdot} \times (B \times C_{\cdot,j}) = \sum_{1 \le s \le n, 1 \le t \le m} a_{i,s} b_{s,t} c_{t,j}$

  as desire

- **3.D——**
- 1.

  Suppose $u \in U$, since S, T are invertible, $STu = 0$ iff $Tu = 0$ iff $u = 0$

  Thus $\operatorname{null} ST = 0$, ST is injective

  Since S, T are invertible U, V, W are isomorphic, which implies $\dim W = \dim U = 0 + \operatorname{range} ST$

  It follows that ST is surjective

  Hence ST is invertible and $(ST)(ST)^{-1} = I, STT^{-1}S^{-1} = SIS^{-1} = I$

  Since the invert is unique, $(ST)^{-1} = T^{-1}S^{-1}$

  as desire

- 3.

  If such T exist, $Su = 0$ iff $Tu = 0$ iff $u = 0$, which implies that S is injective

  If S is injective:

  Suppose $u_1, \ldots, u_n$ is a basis of U, we can extend it to a basis of V as $u_1, \ldots, u_n, w_1, \ldots w_m$

  $\operatorname{range} S = \operatorname{span}(Su_1, \ldots, Su_n)$

  S is injective, thus $\dim \operatorname{range} S = \dim U = n$

  Thus $Su_1, \ldots, Su_n$ is a span list of right length, which implies it is a basis of $\operatorname{range} S$

  Since $\operatorname{range} S$ is a subspace of V

  We can extend the basis above to a basis of V as $Su_1, \ldots, Su_n, v_1, \ldots, v_m$

  Let $Tu_i = Su_i,\ Tw_j = v_j$

  This way T is injective and $Tu = Su$ for all $u \in U$

- 5.

  If $\operatorname{range} T_1 = \operatorname{range} T_2$, $\dim \operatorname{null} T_1 = \dim \operatorname{null} T_2$

  Suppose $v_1, \ldots, v_n$ is a basis of $\operatorname{null} T_1$, we can extend it to a basis of V as $v_1, \ldots, v_n, u_1, \ldots u_m$

Thus $\operatorname{range} T_1 = \operatorname{span}(T_1 u_1, \ldots, T_1 u_m)$

This list have the right length, which implies that it is linearly independent

Then we can find $l_1, \ldots l_m$ s.t. $T_1 u_i = T_2 l_i$

We can extend it to a basis of V as $w_1, \ldots, w_n, l_1, \ldots l_m$

Thus $w_1, \ldots w_n$ is a basis of $\operatorname{null} T_2$

Now let $Sv_i = w_i, Su_j = l_j$, $T_1 = T_2 S$ and S is an operator with injectivity

Hence such S is invertible

On the contrary, if such S exists

Since $T_1 v = T_2 S v$ for $\forall v \in V$, $\operatorname{range} T_1 \subset \operatorname{range} T_2$

Suppose the invert of S is S^{-1}

Since $S^{-1} T_1 v = T_2 v$ for $\forall v \in V$, $\operatorname{range} T_1 \supset \operatorname{range} T_2$

Hence $\operatorname{range} T_1 = \operatorname{range} T_2$

- 7.
  - a.

    Suppose $T_1, T_2 \in E$

    Since $T_1, T_2 \in \mathcal{L}(V, W)$, $(T_1 + T_2)v = T_1 v + T_2 v = 0$

    $aT_1 v = 0$

    Hence it is a subspace of $\mathcal{L}(V, W)$
  - b.

    Extend such v to a basis of V as $v, v_2, \ldots, v_n$

    Now for $Tv_2, .., Tv_n$ extend it to a basis of V as $Tv_2, .., Tv_n, w_1, \ldots w_k$

    Since $Tv = A_{1,1} Tv_2 + \cdots + Am, 1w_k = 0$

    $\mathcal{M}(T)$ have all 0's on the first colum

    Thus $\dim E = \dim W(\dim V - 1)$

- 8.

  Suppose $v_1, \ldots v_m$ is a basis of $\operatorname{null} T$ we can extend it to a basis of V as $v_1, \ldots, v_m, u_1, \ldots u_n$

  Let $U = \operatorname{span}(u_1, \ldots, u_m)$

  Since T is surjective, $\dim U = n = \dim \operatorname{range} T = \dim W$

  Thus U and W are isomorphic

  What's more, $T|_U(u_i) = Tu_i$ and $\operatorname{range} T = \operatorname{range} T|_U$

  Thus $T|_U$ is surjective and since $\dim U = \dim W$ it is also invertible as desire

- 9.

  If S, T are both invertible

$SS^{-1} = I,\ TT^{-1} = I$ and $STT^{-1}S^{-1} = I$

If ST is invertible, exist P s.t. $P(ST) = (ST)P = I$

If $v \in \operatorname{null} T$, $v = Iv = P(ST)v = PS(Tv) = 0$

Thus T is injective. Similarly, S is injective

Since S, T are operator, they are invertible

- 11.

  From problem 9, U is invertible, thus $STU = UST = I$

  Thus again T is invertible, thus $TT^{-1} = I = UST$ which implies $T^{-1} = US$

- 15.

  $F^{n,1}$ has length n basis and $F^{m,1}$ has length m basis

  Thus with this basis $\mathcal{M}(T)$ is a $m \times n$ matrix

  $\mathcal{M}(Tx) = \mathcal{M}(T)\mathcal{M}(x)$

  Since x is a matrix already, if we choose the standard basis as 1 in given position and al the others are $0's$

  $\mathcal{M}(x) = x$

  Hence exist such matrix and $A = \mathcal{M}(T)$ with respect to the basis choosen above

- 18.

  If V is finite, $\dim V = \dim F \times \dim V$ which is obvious

  If V is infinite

  Suppose $f_{a,v}(a) = v$ for some $v \in V,\ a \in \mathbf{F}$

  Let $\varphi(v) = f_{1,v}$

  Since f is linear

  $\forall f_{a,v}(ab) = af_{a,v}(b) = bv = bf_{1,v}(1) = f_{1,v}(b)$

  which implies φ is surjective

  $\varphi(v) = 0$ iff $f_{1,v}(\lambda) = \lambda v = 0$ for all λ iff $v = 0$

  Hence φ is invertible

  Thus we construct a linear map which is invertible between V and $\mathcal{L}(\mathbf{F}, V)$

  as desire

- 20.

  Let matrix $(A)_{i,j} = A_{i,j}$ and $X = (x_1, \ldots, x_n)^T$ and $c \in \mathbf{F}^n$

  Hence this question is equivalent to

  $X = 0$ is the only solution of $AX = 0$ (a.) iff $AX = c$ has a solution for all c (b.)

  Let $\mathcal{M}(T) = A$ with respect to standard basis (a.) is equivalent to $T's$ injectivity

(b.) is equivalent to $T's$ surjectivity

Since T is an operator, they are both equivalent

- **3.E——**
- 1.

  Suppose $(v_1, Tv_1), (v_2, Tv_2) \in V \times W$

  If T is linear

  $(v_1, Tv_1) + (v_2, Tv_2) = (v_1 + v_2, T(v_1 + v_2))$ where $T(v_1 + v_2) \in W$

  Hence graph of T is closed under addition

  $a(v_1, Tv_1) = (av_1, T(av_1))$ where $T(av_1) \in W$

  Hence graph of T is closed under multiplication

  Since $(0, 0)$ is in the graph of T, graph of T is a subspace

  If graph of T is a subspace

  $(v_1, Tv_1) + (v_2, Tv_2) = (v_1 + v_2, Tv_1 + Tv_2) \in V \times W$ and $(v_1 + v_2, T(v_1 + v_2)) \in V \times W$

  Hence $T(v_1 + v_2) = Tv_1 + Tv_2$

  $a(v_1, Tv_1) = (av_1, aTv_1) \in V \times W$ and $(av_1, T(av_1)) \in V \times W$

  Hence $aTv_1 = T(av_1)$

  which implies T is linear
- 2.

  $\dim(V_1 \times \cdots \times V_m) = \sum_{k=1}^{m} \dim V_k < +\infty$

  Thus $\dim V_j \leq \sum_{k=1}^{m} \dim V_k$ which implies V_j is finite for $j = 1, \ldots m$

- 5.

  Suppose $f_i \in \mathcal{L}(V, W_i)$ where $i = 1, \ldots, m$ and define $f(v) = (f_1(v), \ldots, f_m(v))$

  Hence $f \in \mathcal{L}(V, W_1 \times \cdots \times W_m)$

  Define $g(f_1, \ldots, f_m) = f$

  Suppose $h_i \in \mathcal{L}(V, W_i)$ for $i = 1, \ldots, m$

  $g(f_1, \ldots, f_m) + g(h_1, \ldots, h_m) = (f_1(v), \ldots, f_m(v)) + (h_1(v), \ldots, h_m(v))$

  $= (f_1(v) + h_1(v), \ldots, f_m(v) + h_m(v)) = g[(f_1, \ldots, f_m) + (h_1, \ldots, h_m)]$

  $ag(f_1, \ldots, f_m) = a(f_1(v), \ldots, f_m(v)) = (af_1(v), \ldots, af_m(v)) = g[a(f_1, \ldots, f_m)]$

  Thus g is linear map from $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m) \to \mathcal{L}(V, W_1 \times \cdots \times W_m)$

  $g(f_1, \ldots, f_m) = 0$ iff $f_1(v) = \cdots = f_m(v) = 0$ for all $v \in V$ which implies g is injective

  Since $\forall F \in \mathcal{L}(V, W_1 \times \cdots \times W_m)$ define $F_i = \{x : x\ is\ the\ ith\ slot\ of\ F(v)\}$

  Obviously, $g(F_1, \ldots, F_m) = F$

  Hence g is surjective and implies that g is invertible

Thus we can get that the two given spaces are isomorphic

- 6.

  From problem 5, we can get $\mathcal{L}(\mathrm{F}^n, V)$ and $\mathcal{L}(F, V)^n$ are isomorphic

  Now define $f_{v_i}(a) = av_i$ for given $v_i \in V$ and define $g(v_1, \ldots, v_n) = (f_{v_1}, \ldots, f_{v_n})$

  Thus $g(v_1, \ldots, v_n) = 0$ iff $f_{v_1} = \cdots = f_{v_n} = 0$ for all $v_i \in V$ iff $v_1 = \cdots = v_n = 0$

  Hence g is injective.

  $\forall f \in \mathcal{L}(\mathrm{F}, V)$ if $f(x) = v$, $f(x) = f_v(1)$, which implise it is a linear combination of f_v

  Hence g is surjective, and it follows that g is invertible

  Hence V^n and $\mathcal{L}(F, V)^n$ are isomorphic, so does V^n and $\mathcal{L}(\mathrm{F}^n, V)$

- 8.

  If A is an affine subset of V, exist some $x \in V$ and some subspace $U \subset V$, s.t. $A = x + U$

  Since $v, w \in A$, exist $u_1, u_2 \in U$ s.t. $v = x + u_1, w = x + u_2$

  $\lambda v + (1-\lambda)w = \lambda x + (1-\lambda)x + [\lambda u_1 + (1-\lambda)u_2] = x + [\lambda u_1 + (1-\lambda)u_2] \in A$

  If $\forall \lambda \in \mathrm{F}, v, w \in A$, $\lambda v + (1-\lambda)w \in A$

  Thus $\forall x \in A$, $\lambda(v-x) + (1-\lambda)(w-x) = \lambda v + (1-\lambda)w - x \in A - x$

  Let $w = x$, $\lambda(v-x) \in A - x$

  Let $\lambda = \frac{1}{2}$, $\frac{v-x+w-x}{2} \in A - x$ hence $v - x + w - x \in A - x$

  Thus $A - x$ is a subspace, $A = x + (A - x)$ is an affine subset

- 9.

  Since A_1, A_2 are affine subsets of V

  According to problem 8

  $\forall v_1, w_1 \in A_1$, $\lambda v_1 + (1-\lambda)w_1 \in A_1$ for all $\lambda \in \mathrm{F}$

  $\forall v_2, w_2 \in A_2$, $\lambda v_2 + (1-\lambda)w_2 \in A_2$ for all $\lambda \in \mathrm{F}$

  Hence $\forall v, w \in A_1 \cap A_2$, $\lambda v + (1-\lambda)w \in A_1 \cap A_2$ for all $\lambda \in \mathrm{F}$

  According to problem 8

  This implies $A_1 \cap A_2$ is an affine subset

- 11.
  - a.

    Suppose $v = \lambda_1 v_1 + \cdots + \lambda_m v_m$, $w = \mu_1 v_1 + \cdots + \mu_m v_m$

    where $\lambda_1 + \cdots + \lambda_m = \mu_1 + \cdots + \mu_m = 1$

    Thus $\lambda v + (1-\lambda)w = \sum_{k=1}^m (\lambda\lambda_k + (1-\lambda)\mu_k)v_k$

    Since $\sum_{k=1}^m (\lambda\lambda_k + (1-\lambda)\mu_k) = \lambda \sum_{k=1}^m \lambda_k + (1-\lambda)\sum_{k=1}^m \mu_k = 1$

$\lambda v + (1-\lambda)w \in A$

According to peoblem 8, A is an affine subset

- b.

- c.

- 13.

  $U \cap V/U = \{v + U : v \in U\} = 0 + U$

  Hence the only way to get $a_1 v_1 + \cdots + a_m v_m + b_1 u_1 + \cdots + b_n u_n = 0$ is to

  make $a_1 = \cdots = a_m = b_1 = \cdots = n_n = 0$

  Hence $v_1, \ldots, v_m, u_1, \ldots, u_n$ is linearly independent

  Since $m + n = \dim U + \dim(V - U) = \dim U + \dim V - \dim U = \dim V$

  This list is a linearly independent list with right length

  Hence it is a basis of V