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- **3.E——**
- 17.

  Since $V/U$ is finite dimensional, denote $n = \dim V/U$

  Thus suppose $v_1 + U, \ldots, v_n + U$ is basis of $V/U$ where $v_1, .., v_n \in V$

  We can get $v_1, \ldots, v_n$ is linearly independent and $v_i \notin U$

  Let $W = \operatorname{span}(v_1, \ldots, v_n)$

  Thus $\dim W = n = \dim V/U$

  $\forall v \in V, \exists a_1, \ldots a_n \in \mathrm{F}$, s.t. $v + U = a_1(v_1 + U) + \cdots + a_n(v_n + U) = a_1v_1 + \cdots + a_nv_n + U$

  Hence $v - (a_1v_1 + \cdots + a_nv_n) \in U$

  Since $a_1v_1 + \cdots + a_nv_n \in W$, $v = [v - (a_1v_1 + \cdots + a_nv_n)] + [a_1v_1 + \cdots + a_nv_n]$ and $U, W \subset V$

  $V = W + U$

  Suppose $w \in W \cap U$, $w = b_1v_1 + \cdots + b_nv_n$

  Since $w \in U$, $w + U = b_1v_1 + \cdots + b_nv_n + U = 0 + U$

  Thus $b_1 = \cdots = b_n = 0$

  $W \cap U = \{0\}$ Hence $V = W \oplus U$

- 18.

  Suppose such $S$ exist

  $\forall u \in U$, $Tu = S \circ \pi(u) = S(0 + U) = 0$, it follows that $U \subset \operatorname{null} T$

  Suppose $U \subset \operatorname{null} T$

  Define $S \in \mathcal{L}(V/U, W) : S(v + U) = Tv$

  If $v_1 + U = v_2 + U, v_1 - v_2 \in U$, thus $T(v_1 - v_2) = 0 \Rightarrow Tv_1 = Tv_2$

  Thus it is well defined and obvious it is linear

  $\forall v \in V$, $S \circ \pi(v) = S(v + U) = Tv$

  Hence such $S$ satisfies the condition

- 20.
  - a.

    $\forall S_1, S_2 \in \mathcal{L}(V/U, W)$ and $\forall \lambda_1, \lambda_2 \in \mathrm{F}$

    $\Gamma(\lambda_1 S_1 + \lambda_2 S_2) = (\lambda_1 S_1 + \lambda_2 S_2) \circ \pi = \lambda_1 S_1 \circ \pi + \lambda_2 S_2 \circ \pi = \lambda_1 \Gamma(S_1) + \lambda_2 \Gamma(S_2)$

    Thus it is linear
  - b.

    $\Gamma(S) = 0$ iff $S \circ \pi(v) = 0$ for $\forall v \in V$ iff $S(v + U) = 0$ for $\forall v \in V$ iff $S = 0$

    Thus it is injective

- c.

  By problem 18.

  Denote $M=\{T\in\mathcal{L}(V,W):Tu=0\text{ for every }u\in U\}$

  Since $\forall T\in\mathcal{L}(V,W)$ s.t. $U\subset\operatorname{null}T$ are euivalent to exist $S\in\mathcal{L}(V/U,W)$ s.t. $T=S\circ\pi$,

  we get $M=\operatorname{range}\Gamma$

- **3.F——**
- 1.

  Since choose arbitrary $v\in V$, suppose $f\in\mathcal{L}(V,\mathrm{F}),f(v)=a\in\mathrm{F}$

  If $a=0$ for all $v$, $f$ is a zero map

  If not, we can find some $v$ s.t. $f(v)=a\neq 0$

  Thus $\forall\lambda\in\mathrm{F}$, $\frac{\lambda}{a}f(v)=\lambda$ it follows that $f$ is surjective

- 3.

  Extend $v$ to a basis of $V$ as $v,v_2,\ldots,v_n$

  Thus the dual basis of $v,v_2,\ldots,v_n$ is the list $\varphi_1,\ldots,\varphi_n$ s.t. $\varphi_j(v_k)=1$ for $j\neq k$ and $\varphi_j(v_j)=1$

  Hence we can find such $\varphi$

- 7.

  Since $\varphi_j(x^k)=\frac{(x^k)^{(j)}}{j!}(0)=\frac{k(k-1)\cdots(k-j+1)x^{k-j}}{1\cdots j}(0)$

  If $j=k,\varphi_j(x^j)=1$, if not $\varphi_j(x^k)=0$

  Hence it is the basis of the basis $1,x,\ldots,x^m$ of $\mathcal{P}_m(\mathbb{R})$

- 8.
  - a.

    Suppose $a_m(x-5)^m+\cdots+a_1(x-5)+a_0=0$

    Since only the first part contains $x^m$ thus $a_m$ must equals 0

    Thus we reduce the formula to $m-1$ degree $a_{m-1}(x-5)^{m-1}+\cdots+a_1(x-5)+a_0$

    Repeat this procedure, we get $a_m=\cdots=a_0$

    Thus it is a linearly independent list with right length

    It is a basis
  - b.

    Define $\varphi_j(p)=\frac{p^{(j)}(5)}{j!}$

    Thus by problem 7, it is a basis of the dual space of $\mathcal{P}_m(\mathbb{R})$

- 11.

  Suppose such $(c_1,\ldots,c_m)$ and $(d_1,\ldots,d_n)$ exist

Hence $A = [c_1, \ldots, c_m]^T [d_1, \ldots, d_n] = [d_1 c, d_2 c, \ldots, d_n c]$

This implies that each column is a scalar multiple of $c$

Hence $\operatorname{span}(d_1 c, \ldots, d_n c) = \operatorname{span}(c)$, $\operatorname{rank} A = 1$

On the contrary, if $\operatorname{rank} A = 1$, all the column is a scalar multiple of a vector

Thus $A = [d_1 c, .., d_n c]$ and can be written as $A = [c_1, \ldots, c_m]^T [d_1, \ldots, d_n]$

- 14.
  - a.

    $T'(\varphi) = \varphi \circ T(p) = \varphi(x^2 p + p'') = (2xp + x^2 p' + p''')|_{x=4} = 8p(4) + 16p'(4) + p'''(4)$
  - b.

    $T'(\varphi)(x^3) = \varphi \circ T(x^3) = \varphi(x^5 + 6x) = \int_0^1 (x^5 + 6x) dx = \frac{1}{6} + 3 = \frac{19}{6}$

- 16.

  Define $S(T) = T'$

  Thus $S(T) = 0$ iff $T' = 0$ iff $\varphi \circ T = 0$ iff $T = 0$ Hence $S$ is injective

  Since $\dim \mathcal{L}(V, W) = \dim \mathcal{L}(W', V')$ and $S \in \mathcal{L}(\mathcal{L}(V, W), \mathcal{L}(W', V'))$

  $S$ is surjective

  Hence $S$ is isomorphic

- 20.

  $\forall \varphi \in W^0$, $\forall w \in W, \varphi(w) = 0$

  Since $U \subset W$, $\forall u \in U$, we have $\varphi(u) = 0$

  Hence $W^0 \subset U^0$

- 23.

  $\forall \varphi \in (U \cap W)^0, \forall w \in W, u \in U, \varphi(w) = 0, \varphi(u) = 0$

  Hence $(U \cap W)^0 \subset U^0 + W^0$

  From problem 20, we have $(U \cap W)^0 \supset U^0 + W^0$

  Hence they are equal

- 26.

  Denote $W = \{v \in V : \varphi(v) = 0 \ for\ every\ \varphi \in \Gamma\}$

  Thus $\forall \varphi \in W^0, \forall v \in W, \varphi(v) = 0$

  This implies that $W^0 = \Gamma$

- 28.

Since $\operatorname{null} T' = \operatorname{span}(\varphi)$, $\varphi \circ Tv = 0$ for all $v \in V$

Hence $\operatorname{range} T \subset \operatorname{null} \varphi$

Since $\operatorname{null} T = (\operatorname{range} T)^0 = \operatorname{span}(\varphi)$, $\dim(\operatorname{range} T)^0 = 1$

$\dim \operatorname{range} T + \dim(\operatorname{range} T)^0 = \dim W \Rightarrow \dim \operatorname{range} T = \dim W - 1$

- 32.
  - a to b

    Since $T$ is invertible, $\dim \operatorname{range} T = \dim V = n$, which is also colomn rank of $\mathcal{M}(T)$

    Thus the colomn of $\mathcal{M}(T)$ is linearly independent
  - b to c

    It is a linearly independent list with right length

    Hence it spans $\mathbf{F}^{n,1}$
  - c to d

    Since the rank of row of $\mathcal{M}(T)$ equals the rank of colomn of $\mathcal{M}(T)$, which is $n$

    Hence the row is linearly independent
  - d to e

    It is a linearly independent list with right length

    Hence it spans $\mathbf{F}^{1,n}$
  - e to a

    Since it spans $\mathbf{F}^{1,n}$

    the rank of row of $\mathcal{M}(T)$ equals the rank of colomn of $\mathcal{M}(T)$, which is $n$

    Thus $\dim \operatorname{range} T = n = \dim V$

    Which implise $T$ is invertible