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- **2.A——**
- 1.

  For $\forall v \in V$, since v_1, v_2, v_3, v_4 spans V, $\exists a_1, a_2, a_3, a_4 \in \mathbf{F}$ s.t. $v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4$

  $\Rightarrow v = a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + a_2 + a_3 + a_4)v4$

  Thus $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ also spans V

- 2.
  - a.

    Proof "$\Rightarrow$":

    Since $v \in V$ is linearly independent, if $v = 0, \forall a \neq 0, av = 0$ which contradicts to the independence

    Hence, $v \neq 0$

    Proof "$\Leftarrow$":

    Since $v \neq 0$, $av = 0$ iff $a = 0$

    Hence v is linearly independent
  - b.

    Let $v, w \in V$

    Proof "$\Rightarrow$":

    Since v, w are linearly independent

    Assume $v = kw$

    we can get $v + -kw = 0$ which contradicts to the independence

    Hence, neither vector is a scalar multiple of the other

    Proof "$\Leftarrow$":

    Since neither vector is a scalar multiple of the other

    Assume $\exists a_1, a_2$ not all 0 s.t. $a_1 v + a_2 w = 0$

    Let $a_1 \neq 0, v = -\frac{a_2}{a_1} w$ which contradicts to the assumption

    Hence only to make $a_1 v + a_2 w = 0$ is to make $a_1 = a_2 = 0$, which implies v, w are linearly independent
  - c.

    $a(1,0,0,0) + b(0,1,0,0) + c(0,0,1,0) = 0 \Rightarrow (a,b,c,0) = (0,0,0,0)$

    Hence they are linearly independent
  - d.

    $a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m \equiv 0 \Rightarrow a_0 = a_1 = \cdots = a_m$

    Hence they are linearly independent

- 4.

If $c = 8$, $2(2,3,1)+3(1,-1,2)+(-1)(7,3,8)=(0,0,0)$, which implies they are dependent

If they are independent, $a(2,3,1)+b(1,-1,2)+d(7,3,c)=0$

$\Rightarrow 2a+b+7d=0, 3a-b+3d=0, a+2b+cd=0$

The equation has a solution iff $c=8$

Hence we complete the proof

- 5.
  - a.

    Suppose $a, b \in \mathbb{R}, a(1+i)+b(1-i)=0$

    $\Rightarrow a+b=0, a-b=0$

    Hence $a=b=0$ which implies list $(1+i, 1-i)$ is linearly independent
  - b.

    $i(1+i)+1(1-i)=0$

    Hence list $(1+i, 1-i)$ is linearly dependent

- 6.

  If $\exists a_1, a_2, a_3, a_4$ not all 0, s.t. $a_1(v_1-v_2)+a_2(v_2-v_3)+a_3(v_3-v_4)+a_4v_4=0$

  $a_1+(a_2-a_1)v_2+(a_3-a_2)v_3+(a_4-a_3)v_4=0$

  which contradicts to the independence.

  Hence to make $a_1(v_1-v_2)+a_2(v_2-v_3)+a_3(v_3-v_4)+a_4v_4=0$, only to make $a_1=a_2=a_3=a_4=0$

  which implies they are linearly independent

- 9.

  $(1,0),(0,1)$ and $(-1,0),(0,-1)$ are linearly independent lists

  However $(1,0)+(-1,0),(0,1)+(0,-1)$ is the list $(0,0),(0,0)$ which is not linearly independent

- 10.

  Since $v_1+w, \ldots, v_m+w$ is linearly dependent,

  $\exists a_1, a_2, \ldots, a_m$ not all 0, s.t. $a_1(v_1+w)+\cdots+a_m(v_m+w)=0$

  $\Rightarrow a_1v_1+\cdots+a_mv_m=(a_1+\cdots+a_m)w$

  Since $v_1, \ldots, v_m$ is linearly independent

  $a_1v_1+\cdots+a_mv_m \neq 0$ That is $(a_1+\cdots+a_m) \neq 0$

  $w=\frac{a_1}{a}v_1+\cdots+\frac{a_m}{a}$ where $a=a_1+\cdots+a_m$

  Hence $w=\operatorname{span}(v_1, \ldots, v_m)$

- 11.

Assume $w \notin \mathrm{span}(v_1, \ldots, v_m)$

If $v_1, \ldots, v_m, w$ is linearly dependent, $w \in \mathrm{span}(v_1, \ldots, v_m)$ which is contradictive

Hence, $v_1, \ldots, v_m, w$ is linearly independent

Assume $v_1, \ldots, v_m, w$ is linearly independent

If $w \in \mathrm{span}(v_1, \ldots, v_m)$, $\exists a_1, \ldots, a_m$ not all 0, s.t. $w = a_1 v_1 + \cdots + a_m v_m$

$a_1 v_1 + \cdots + a_m v_m - w = 0$

which contradicts to the independence

Hence, $w \notin \mathrm{span}(v_1, \ldots, v_m)$

Here, we complete the proof.

- 12.

  The list $(1, z, z^2, z^3, z^4)$ can span $\mathcal{P}_4(\mathrm{F})$

  By lemma, the length basis must less or equal the length of spanning list

  Thus the length of basis cannot be more than 5

- 13.

  The list $(1, z, z^2, z^3, z^4)$ can span $\mathcal{P}_4(\mathrm{F})$ and is linearly independent

  Hence it is a basis

  By lemma, the length basis must less or equal the length of spanning list

  Thus the length of basis cannot be less than 5

- 14.

  Proof "$\Rightarrow$":

  For $m = 1$, a vector v_1 is linearly independent is easy to find

  For $m = k$, assume there is a list $(v_1, \ldots, v_m)$ that is linearly independent

  For $m = k + 1$, since V is infinite-dimensional, $v_1, \ldots, v_m$ cannot span V

  Thus we can find $v_{m+1} \notin \mathrm{span}(v_1, \ldots, v_m)$

  Hence $v_1, \ldots, v_m, v_{m+1}$ is linearly independent

  Proof "$\Leftarrow$":

  By definition, it is obvious

- **2.B——**
- 1.

  (The answer is $\{0\}$, but if it is, the basis can only be $v = 0$. However if $v = 0$, v is not linearly independent)

- 2.

- a.

  obvious
- b.

  $(1,2),(3,5)$ is linearly independent

  $(x,y)=-(5x-3y)(1,2)+(2x-y)(3,5)$
- c.

  $(8.-3,0)$ cannot be expressed as $a(1,2,-4)+b(7,-5,6)$
- d.

  $(4,13)=19(1,2)-5(3,5)$
- e.

  $(1,1,0),(0,0,1)$ is obviously linearly independent

  $(x,x,y)=x(1,1,0)+y(0,0,1)$
- f.

  $(1,-1,0),(1,0,-1)$ is obviously linearly independent

  $(-y-z,y,z)=-y(1,-1,0)-z(1,0,-1)$
- g.

  obvious

* 4
  - a.

    $(1,6,0,0,0),(0,0,2,-1,0),(0,0,3,0,-1)$
  - b.

    $(1,6,0,0,0),(0,0,2,-1,0),(0,0,3,0,-1),(0,1,0,0,0),(0,0,1,0,0)$
  - c.

    $W=\mathrm{span}[(0,1,0,0,0),(0,0,1,0,0)]$

* 7.

  Counterexample

  Basis is $(1,0,0,0),(0,1,0,0),(0,0,1,0),(1,1,1,1)$

  $U=\{(x,y,0,z)\}$

* 8.

  $\forall v\in V,\exists u\in U,w\in W$ s.t. $v=u+w$

  Since $u_1,\dots,u_m$ is a basis of U and $w_1,\dots,w_n$ is a basis of W

  $v=a_1v_1+\cdots+a_mv_m+b_1w_1+\cdots+b_nw_n$

  Thus $u_1,\dots,u_m,w_1,\dots,w_n$ spans V

$0 = u + w$ implies $u = w = 0$ which is $a_1v_1 + \cdots + a_mv_m + b_1w_1 + \cdots + b_nw_n = 0$ only if $a's$ and $b's$ are all 0

Thus $u_1, \ldots, u_m, w_1, \ldots w_n$ is linearly independent

Hence it is a basis

- **2.C——**
- 1.

  Suppose $u_1, u_2, \cdots, u_n$ is a basis of U and $n = \dim U$

  Then $u_1, u_2, \cdots, u_n$ is linearly independent list

  Since $n = \dim U = \dim V$, $u_1, u_2, \cdots, u_n$ is independent list with right length $\dim V$

  Hence $u_1, u_2, \cdots, u_n$ is a basis of V

  which implies that $U = V$

- 3.

  The dimension of subspace of $\mathbb{R}^3$ can only be $0, 1, 2, 3$.

  Suppose U is a subspace of $\mathbb{R}^3$

  - $\dim U = 0 : U = \{0\}$ obviously
  - $\dim U = 1 : \forall u \in U,\ ku \in U$

    Then U is a line in $\mathbb{R}^3$

    Since $\{0\} \in U$, U is a line through origin
  - $\dim U = 2 : \exists \text{basis}\, u_1, u_2 \in \mathbb{R}^3$ s.t. $U = \{au_1 + bu_2 : a, b \in \mathbb{R}\}$

    Hence U is a plane in $\mathbb{R}^3$

    Since $\{0\} \in U$, U is a plane through origin
  - $\dim U = 3 : U = \mathbb{R}^3$ obviously

- 4.
  - a.

    $(x-6), (x-6)x, (x-6)x^2, (x-6)x^3$
  - b.

    $1, (x-6), (x-6)x, (x-6)x^2, (x-6)x^3$
  - c.

    Let $W = \{c : c \in \mathbb{R}\}$

- 7.
  - a.

    $1, (x-2)(x-5)(x-6), x(x-2)(x-5)(x-6)$
  - b.

$1, x, x^2, (x-2)(x-5)(x-6), x(x-2)(x-5)(x-6)$

- c.

  Let $W = \{a_1x + a_2x^2 : a_1, a_2 \in \mathbb{R}\}$

- 8.
  - a.

    $5x^4 - 1, x^3, 3x^2 - 1, x$

  - b.

    $1, 5x^4 - 1, x^3, 3x^2 - 1, x$

  - c.

    Let $W = \{c : c \in \mathbb{R}\}$

- 9.

  $v_k - v_1 = (v_k + w) - (v_1 + w)$ where $k = 1, 2, \cdots, m$

  Then $v_k - v_1 \in \operatorname{span}(v_1 + w, \cdots, v_m + w)$

  Since $v_2 - v_1, \cdots, v_m - v_1$ is linearly independent

  which implie that $\dim \operatorname{span}(v_1 + w, \cdots, v_m + w) \geq m - 1$

- 13.

  $\dim(U + W) = \dim U + \dim W - \dim(U \cap W) = 8 - \dim(U \cap W)$

  Since $4 \leq \dim(U + W) \leq 6,\ 2 \leq \dim(U \cap W) \leq 4$

  Thus we can find at least length 2 linearly independent list

- 15.

  Since $\dim V = n$, we can find a basis $v_1, v_2, \cdots, v_n$

  Thus to make $a_1v_1 + \cdots + a_nv_n = 0$ only to make $a_1 = \cdots = a_n = 0$

  Now, Let $U_k = \{kv_k : k \in \mathrm{F}\}$

  Thus $U_1 + U_2 + \cdots + U_n = V$

  Hence $V = U_1 \oplus \cdots \oplus U_n$

- 17.

  Let $U_1 = \{(x, 0)\}, U_2 = \{(0, y)\}, U_3 = \{(z, z)\}$

  $\dim(U_1 + U_2 + U_3) = 2$

  However,
  $\dim U_1 + \dim U_2 + \dim U_3 - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3)$

  $= 1 + 1 + 1 - 0 - 0 - 0 + 0 = 3$

  which shows the contradiction