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- **5.B——**
- 1.
  - a.
    $(I+T+T^2+\cdots+T^{n-1})(I-T)=(I+T+T^2+\cdots+T^{n-1})-(T+T^2+\cdots+T^{n-1}+T^n)=I$
    Thus $I-T$ is invertible
  - b.
    $(1-x)^{-1}=\sum_{n=0}^{\infty}x^n$
    Substitute x with T we can get the formula

- 4.
  $\forall v\in V, v=Pv+(v-Pv)$
  Since $Pv\in \operatorname{range}P$ and $P(v-Pv)=0$ which implies $v-Pv\in \operatorname{null}P$
  $V=\operatorname{null}P+\operatorname{range}P$
  Suppose $v\in \operatorname{range}P\cap\operatorname{null}P$
  $\exists u\in V$ s.t. $Pu=v$
  $0=Pv=PPu=Pu=v$
  Thus $\operatorname{range}P\cap\operatorname{null}P=\{0\}$

- 6.
  Since U is invariant under T, $u\in U$ implies $Tu\in U$
  Thus $u\in U$ implies $TTu\in U$
  Thus similarly U is invariant under T^n for any $n\in\mathbb{N}^*$
  Since $p(T)=\sum_{n=0}^{\infty}a_nT^n$
  Thus U is invariant under $p(T)$

- 11.
  If $\alpha=p(\lambda)$ where λ is an eigenvalue of T, $p(T)v=p(\lambda)v=av$
  Thus α is an eigenvalue of T
  If α is an eigenvalue of $p(T)$, $p(T)-\alpha I$ is not invertible
  Suppose $p-\alpha=c(x-\lambda_1)\cdots(x-\lambda_n)$ where $c,\lambda_1,\ldots,\lambda_n\in\mathbb{C}$
  $p(T)-\alpha I=c(T-\lambda_1 I)\cdots(T-\lambda_n I)$
  Thus $\exists\lambda_k$ s.t. $T-\lambda_k I$ is not invertible
  $p(T)v=av=p(\lambda_k)v$
- 12.

Suppose $T(x, y) = (-x, y)$

Thus T has no eigenvalue

However $T^2 = I$ which has eigenvalue 1

- 15.

  Can't find such operator....

- 16.

  Define $\phi(p) = p(T)v$ where $\phi \in \mathcal{L}(\mathcal{P}_n(\mathbb{C}), V)$

  $\phi(ap_1 + bp_2) = (ap_1 + bp_2)(T)v = ap_1(T)v + bp_2(T)v = a\phi(p_1) + b\phi(p_2)$

  Thus ϕ is linear.

  Since $\dim \mathcal{P}_n(\mathbb{C}) = n + 1$

  ϕ is not injective

  Hence $\exists p \in \mathcal{P}_n(\mathbb{C})$ s.t. $\phi(p) = 0 = p(T)v$

  Thus we can write $c(T - \lambda_1 I) \cdots (T - \lambda_m I)v = 0$

  Thus $\exists \lambda_j$ s.t. $Tv = \lambda v$

- **5.C——**
- 1.

  Since T is diagonalizable, exist 1-dimensional subspace $U_1, \ldots, U_n$ of V

  For each U_k if $x \in U_k$, $Tx \in U_k$

  Thus if $Tx = 0$, $\operatorname{null} T|_{U_k} = U_k$ otherwise $\operatorname{range} T|_{U_k} = U_k$

  Thus $V = \operatorname{null} T \oplus \operatorname{range} T$

- 3.
  - a to b is obvious
  - b to c

    $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = \dim(\operatorname{null} T + \operatorname{range} T) = \dim V + \dim(\operatorname{range} T \cap \operatorname{null} T)$

    Thus $\dim(\operatorname{range} T \cap \operatorname{null} T) = 0$ which implie $\operatorname{range} T \cap \operatorname{null} T = \{0\}$
  - c to a

    Since $\operatorname{range} T \cap \operatorname{null} T = \{0\}$ and $\dim \operatorname{range} T + \dim \operatorname{null} T = \dim V$

    Hence $\operatorname{range} T + \operatorname{null} T = V$

    which proves (a)

- 6.

  Since T has $\dim V$ distinct eigenvalues, T is diagonalizable

S has the same eigen vectors as T,

Thus with respect to eigenvectors of T, $\mathcal{M}(T)$ has only eigenvalues of T on the diagonal

Thus with respect to eigenvectors of T, $\mathcal{M}(S)$ has also only eigenvalues of S on the diagonal

Thus with respect to this basis $\mathcal{M}(ST) = \mathcal{M}(TS)$ which implies that $ST = TS$

- 9.

  $Tv = \lambda v \Leftrightarrow T^{-1}Tv = \lambda T^{-1}v \Leftrightarrow \frac{1}{\lambda}v = T^{-1}v$

  Hence $E(\lambda, T) = E(\frac{1}{\lambda}, T^{-1})$

- 13.

  Suppose $v_1, \ldots, v_4$ is a basis of $\mathbf{F}^4$

  Define $\mathcal{M}(R)$ with respect to this basis is $|2\ 0\ 0\ 0|$

  $|0\ 6\ 0\ 0|$

  $|0\ 0\ 6\ 0|$

  $|0\ 0\ 0\ 7|$

  And define $\mathcal{M}(T)$ with respect to this basis is $|2\ 0\ 0\ 0|$

  $|1\ 6\ 0\ 0|$

  $|0\ 0\ 6\ 0|$

  $|0\ 0\ 0\ 7|$

  If exist S s.t. $R = S^{-1}TS, T = SRS^{-1}$

  Since S is invertible and R can be diagonalizable

  With respect to $Sv_1, \ldots, Sv_4$, T can be diagonalizable

  However, $\dim E(2, T) = 1, \dim E(6.T) = 1, \dim E(7, T) = 1$ which implies T is not diagonalizable

  Hence such S doesn't exist

- 15.

  To get existance of $(x, y, z) \in \mathbf{F}^3$ s.t. $T(x, y, z) = (17 + 8x, \sqrt{5} + 8y, 2\pi + 8z)$

  We must show that $(T - 8I)(x, y, z) = (17, \sqrt{5}, 2\pi)$ and such (x, y, z) exist

  Since $6, 7$ are eigenvalues of T and T is not diagonalizable

  T has no other rigenvalues

  Thus $\text{null}(T - 8I) = \{0\}$

  Since $T - 8I$ is an operator, it is surjective

  which implies such (x, y, z) must exist