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- **3.1——**
- 2.

  $r^2+3r+2=0$, hence $r_1=-1, r_2=-2$

  It follows that $y_1=e^{-t}, y_2=e^{-2t}$

  Hence the general solution is $y=c_1e^{-t}+c_2e^{-2t}$ where c_1, c_2 are arbitrary constants

- 4.

  $r^2+5r=0$, hence $r_1=0, r_2=-5$

  It follows that $y_1=1, y_2=e^{-5t}$

  Hence the general solution is $y=c_1+c_2e^{-5t}$ where c_1, c_2 are arbitrary constants

- 15.

  $2r^2-3r+1=0$, hence $r_1=\frac{1}{2}, r_2=1$

  It follows that $y_1=e^{\frac{1}{2}t}, y_2=e^t$

  Hence the general solution is $y=c_1e^{\frac{1}{2}t}+c_2e^t$ where c_1, c_2 are arbitrary constants

  Substituting $y(0)=2, y'(0)=\frac{1}{2}$, we get $c_1+c_2=2, \frac{1}{2}c_1+c_2=\frac{1}{2}$

  Hence $c_1=3, c_2=-1, y=3e^{\frac{1}{2}t}-e^t$

  $y'=\frac{3}{2}e^{\frac{1}{2}t}-e^t$, when $y'=0, e^{\frac{1}{2}t}=\frac{3}{2}$, hence $t=\ln\frac{9}{4}, y(t)=\frac{9}{4}$

  When $y=0, e^{\frac{1}{2}t}=3$, hence $t=\ln 9$

- 16.

  $r^2-r-2=0$, hence $r_1=2, r_2=-1$

  It follows that $y_1=e^{2t}, y_2=e^{-t}$

  Hence the general solution is $y=c_1e^{2t}+c_2e^{-t}$ where c_1, c_2 are arbitrary constants

  Substituting $y(0)=\alpha, y'(0)=2$, we get $c_1+c_2=\alpha, 2c_2-c_2=2$

  Hence $c_1=\frac{2+\alpha}{3}, c_2=\frac{2\alpha-2}{3}, y=\frac{2+\alpha}{3}e^{2t}+\frac{2\alpha-2}{3}e^{-t}$

  Since $t\to\infty, y\to 0$, we get $\frac{2+\alpha}{3}=0, \frac{2\alpha-2}{3}\neq 0$

  Hence $\alpha=-2$

- 18.

  $r^2+(3-\alpha)r-2(\alpha-1)=0$, hence $r_1=\alpha-1, r_2=-2$

  It follows that $y_1=e^{-2t}, y_2=e^{(\alpha-1)t}$

  Hence the general solution is $y=c_1e^{-2t}+c_2e^{(\alpha-1)t}$ where c_1, c_2 are arbitrary constants

If $\alpha - 1 > 0$ which is $\alpha > 1$,

If $c_2 \neq 0, y \to \infty$ as $t \to \infty$

However if $c_2 = 0, y \to 0$ as $t \to \infty$

Hence, such α that makes all solution unbounded when $t \to \infty$ does not exist

If $\alpha - 1 < 0$ which is $\alpha < 1$

$y \to 0$ as $t \to \infty$

- **3.2——**
- 7.

  $t(t-4)y'' + 3ty' + 4y = 2 \Rightarrow y'' + \frac{3t}{t(t-4)}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$

  Let $p(t) = \frac{3t}{t(t-4)}, q(t) = \frac{4}{t(t-4)}, g(t) = \frac{2}{t(t-4)}$

  Thus p, q, g is continuous on $(-\infty, 0), (0, 4), (4, +\infty)$

  Since $y(3) = 0, y'(3) = -1$, the maximum existence interval is $(0, 4)$

- 9.

  $(x-2)y'' + y' + (x-2)(\tan x)y = 0 \Rightarrow y'' + \frac{1}{x-2}y' + (\tan x)y = 0$

  Let $p(t) = \frac{1}{x-2}, q(t) = \tan x$

  Thus p, q is continuous on $(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)(k \in \mathbb{Z}, k \neq 1), (\frac{\pi}{2}, 2), (2, \frac{3\pi}{2})$

  Since $y(3) = 1, y'(3) = 2$, the maximum existence interval is $(2, \frac{3}{2}\pi)$

- 14.

  $f' = 2e^{2t}$ thus suppose g and g'

  $W[f, g] = 3e^{4t} = fg' - f'g = e^{2t}g' - 2e^{2t}g$

  We get $g' - 2g = 3e^{2t} \Rightarrow e^{-2t}g' - 2e^{-2}g = 3 \Rightarrow (e^{-2t}g)' = 3 \Rightarrow g = 3te^{2t} + ce^{2t}$

  where c is an arbitrary constant

- 16.

  $y_3' = a_1y_1' + a_2y_2', y_4' = b_1y_1' + b_2y_2'$

  $W[y_3, y_4] = y_3y_4' - y_4y_3' = (a_1y_1 + a_2y_2)(b_1y_1' + b_2y_2') - (b_1y_1 + b_2y_2)(a_1y_1' + a_2y_2')$

  which equals $(a_1b_2 - b_1a_2)y_1y_2' + (a_2b_1 - a_1b_2)y_2y_1' = (a_1b_2 - a_2b_1)W[y_1, y_2]$

  If $a_1b_2 \neq a_2b_1,\ y_3, y_4$ is also a fundamental set of soluiton

  If not, then y_3, y_4 is not such set

- 25.

  $p(t) = \frac{2x}{x^2-1}$

  $W[y_1, y_2](t) = c\exp(-\int p(t)dt) = c\exp(-\ln(x^2-1)) = \frac{c}{x^2-1}$ where c is an arbitrary constant

- **3.3——**
- 8.

  $r^2+6r+13=0$, hence $r_1=-3-2i, r_2=-3+2i$

  It follows that $y_1=e^{-3t}\cos(2t), y_2=e^{-3t}\sin(2t)$

  Hence, the general solution is $y=c_1e^{-3t}\cos(2t)+c_2e^{-3t}\sin(2t)$ where c_1, c_2 are arbitrary constants

- 21.

  $e^{it}=\sum_{n=0}^{\infty}\frac{(it)^n}{n!}=\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k}}{(2k)!}+i\sum_{k=0}^{\infty}\frac{(-1)^k t(2k+1)}{(2k+1)!}=\cos t+i\sin t$

  $e^{-it}=\sum_{n=0}^{\infty}\frac{(it)^n}{n!}=\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k}}{(2k)!}+i\sum_{k=0}^{\infty}\frac{(-1)^{(k+1)} t(2k+1)}{(2k+1)!}=\cos t-i\sin t$

  Hence $\frac{e^{-it}+e^{-it}}{2}=\cos t, \frac{e^{it}-e^{-it}}{2}=\sin t$

- 23.

  $u'(t)=\lambda e^{\lambda t}\cos(\mu t)-e^{\lambda t}\mu\sin(\mu t)=\lambda u(t)-\mu v(t)$

  $v'(t)=\lambda e^{\lambda t}\sin(\mu t)+e^{\lambda t}\mu\cos(\mu t)=\lambda v(t)+\mu u(t)$

  $u''(t)=\lambda(\lambda u(t)-\mu v(t))-\mu(\lambda v(t)+\mu u(t))=(\lambda^2-\mu^2)u(t)-2\mu\lambda v(t)$

  $v''(t)=\lambda(\lambda v(t)+\mu u(t))+\mu(\lambda u(t)-\mu v(t))=(\lambda^2-\mu^2)v(t)+2\mu\lambda u(t)$

  Since $\lambda=-\frac{b}{2a}, \mu=\pm\frac{\sqrt{4ac-b^2}}{2a}$

  $au''+bu'+cu=[a(\lambda^2-\mu^2)+b\lambda+c]u-[a2\mu\lambda+b\mu]v$

  $=[\frac{b^2}{4a}-\frac{4ac-b^2}{4a}-\frac{b^2}{2a}+c]u-[\frac{-b\sqrt{4ac-b^2}}{2a}+\frac{b\sqrt{4ac-b^2}}{2a}]v=0$

  $av''+bv'+cv=[a(\lambda^2-\mu^2)+b\lambda+c]v+[a2\mu\lambda+b\mu]u$

  $=[\frac{b^2}{4a}-\frac{4ac-b^2}{4a}-\frac{b^2}{2a}+c]v+[\frac{-b\sqrt{4ac-b^2}}{2a}+\frac{b\sqrt{4ac-b^2}}{2a}]u=0$

  which implies $u(t), v(t)$ are solutions

- 25.
  - a.

    $dy/dt=(dy/dx)(dx/dt)=t^{-1}dy/dx,\ d^2y/dt^2=[d(t^{-1}dy/dx)/dx](dx/dt)$

    which is $t^{-1}(-t^{-1}dy/dx+t^{-1}d^2y/dx^2)=t^{-2}(d^2y/dx^2-dy/dx)$
  - b.

    The equation can transformed into

    $\frac{d^2y}{dx^2}-\frac{dy}{dx}+\alpha\frac{dy}{dx}+\beta y=0$

    which is equation (34)

- 27.

  The equation can be transformed into $y''+3y'+2y=0$

$r^2 + 3r + 2 = 0$, hence $r_1 = -1, r_2 = -2$

It follows that $y_1 = e^{-x},\ y_2 = e^{-2x}$ which is $y_1 = t^{-1},\ y_2 = t^{-2}$

Hence the general solution is $y = c_1 t^{-1} + c_2 t^{-2}$ where c_1, c_2 are arbitrary constants

- **3.4——**
- 6.

  $4r^2 + 17r + 4 = 0$, hence $r_1 = -\frac{1}{4},\ r_2 = -4$

  It follows that $y_1 = e^{-t/4},\ y_2 = e^{-4t}$

  Hence the general solution is $y = c_1 e^{-t/4} + c_2 e^{-4t}$ where c_1, c_2 are arbitrary constants

- 12.

  $r^2 - r + \frac{1}{4} = 0$, hence $r_1 = r_2 = \frac{1}{2}$

  It follows that $y_1 = e^{t/2}$

  Let $y = v(t)e^{t/2}$, $y' = v'e^{t/2} + \frac{1}{2}ve^{t/2} = (v' + \frac{1}{2}v)e^{t/2},\ y'' = (v'' + v' + \frac{1}{4}v)e^{t/2}$

  We can get $v'' + v' + \frac{1}{4} - v' - \frac{1}{2} + \frac{1}{4} = 0$ which is $v'' = 0$

  Hence $v(t) = c_1 + c_2 t,\ y = c_1 e^{t/2} + c_2 te^{t/2}$ where c_1, c_2 are arbitrary constants

  Substituting $y(0) = 2,\ y'(0) = b$, we get $c_1 = 2,\ c_2 = b - 1$

  Hence $y = 2e^{t/2} + (b-1)te^{t/2}$

  Thus the critical value of b is 1

- 15.
  - a.

    $r^2 + 2ar + a^2 = 0$, hence $r_1 = r_2 = -a$
  - b.

    $p(t) = 2a$

    $W(t) = c\exp(-\int p(t)dt) = ce^{-2at}$ where c is an arbitrary constant
  - c.

    $y_1(t)y_2'(t) - y_2(t)y_1'(t) = e^{-at}y_2' + ae^{-at}y_2 = ce^{-2at}$

    Hence $y_2' + ay_2 = ce^{-at} \Rightarrow e^{at}y_2' + ae^{at}y_2 = c \Rightarrow (e^{at}y_2)' = c$

    We get $y_2 = c_1 te^{-at} + c_2 e^{-at}$ where c_1, c_2 are arbitrary constants

    Now let $c_1 = 1,\ c_2 = 0$, $y_2(t) = te^{-at}$

- 20.

  $y'' + \frac{3}{t}y' + \frac{1}{t^2}y = 0$, let $y(t) = v(t)y_1(t) = \frac{v}{t}$

  Substituting $y = \frac{v}{t}$, we get $v''t^{-1} + (-2t^{-2} + 3t^{-2})v' = 0$

  Hence $v'' + t^{-1}v' = 0$ and we get $v' = c_1 t^{-1}$, it follows that $v = c_1 \int t^{-1}dt = c_1 \ln t + c_2$

Here c_1, c_2 are arbitrary constants

$y = c_1 t^{-1} \ln t + c_2 t^{-1}$, the secend solution is $y_2 = t^{-1} \ln t$

- 28.

  Suppose r_1, r_2 are the roots of $ar^2 + br + c = 0$

  If $r_1, r_2 \in \mathbb{R}$, $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

  $r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} < 0,\ r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} < 0$

  Hence $y \to 0$ as $t \to \infty$

  If $r_1, r_2 \notin \mathbb{R}$, $r_1 = \lambda + i\mu,\ r_2 = \lambda - i\mu$ where $\lambda, \mu \in \mathbb{R}$

  $y = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t)$

  Since $\lambda = -\frac{b}{2a} < 0$ and $\cos(\mu t), \sin(\mu t)$ are bounded

  $y \to 0$ as $t \to \infty$

  If $r_1 = r_2$, $y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$

  $r_1 = r_2 = -\frac{b}{2a} < 0$

  Since $\lim_{t\to\infty} \frac{t}{e^{-r_1 t}} = \lim_{t\to\infty} \frac{t'}{(e^{-r_1 t})'} = \lim_{t\to\infty} \frac{1}{-r_1 e^{-r_1 t}} = 0$

  Hence $y \to 0$ as $t \to \infty$