第十一届全国大学生数学竞赛决赛
（非数学类）试题与参考答案

一、填空题（本题满分 30 分，每小题 6 分）

1、极限 $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{(1-\sqrt{\sin x})(1-\sqrt[3]{\sin x})\cdots(1-\sqrt[n]{\sin x})}{(1-\sin x)^{n-1}}=$______.

【参考答案】：由等价无穷小，

$$\lim_{x\to\frac{\pi}{2}}\frac{1-\sqrt[k]{\sin x}}{1-\sin x}=\lim_{x\to\frac{\pi}{2}}\frac{1-\sqrt[k]{1+\left(\sin x-1\right)}}{1-\sin x}$$
$$=-\lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{k}\left(\sin x-1\right)}{1-\sin x}=\frac{1}{k}$$

故由极限的乘法法则，得

$$\text{原式}=\lim_{x\to\frac{\pi}{2}}\frac{1-\sqrt{\sin x}}{1-\sin x}\frac{1-\sqrt[3]{\sin x}}{1-\sin x}\cdot\cdots\cdot\frac{1-\sqrt[n]{\sin x}}{1-\sin x}$$
$$=\frac{1}{2}\cdot\frac{1}{3}\cdot\cdots\cdot\frac{1}{n}=\frac{1}{n!}$$

2、设函数$y=f\left(x\right)$由方程$3x-y=2\arctan(y-2x)$所确定，则曲线$y=f\left(x\right)$在点$P\left(1+\dfrac{\pi}{2},3+\pi\right)$处的切线方程为______________.

【参考答案】：对方程$3x-y=2\arctan(y-2x)$两边求导，得

$$3-y'=2\frac{y'-2}{1+(y-2x)^2}$$

将点P的坐标代入，得曲线$y=f(x)$在P点的切线斜率为$y'=\dfrac{5}{2}$. 因此，切线方程为$y-(3+\pi)=\dfrac{5}{2}\left(x-1-\dfrac{\pi}{2}\right)$，即$y=\dfrac{5}{2}x+\dfrac{1}{2}-\dfrac{\pi}{4}$.

3、设平面曲线L的方程为$Ax^2+By^2+Cxy+Dx+Ey+F=0$，且通过五个点$P_1(-1,0),P_2(0,-1),P_3(0,1),P_4(2,-1)$和$P_5(2,1)$， 则$L$上任意两点之间的直线距离最大值为________.

【参考答案】：将所给点的坐标代入方程得

$$\begin{cases} A-D+F=0 \\ B-E+F=0 \\ B+E+F=0 \\ 4A+B-2C+2D-E+F=0 \\ 4A+B+2C+2D+E+F=0 \end{cases}$$

解得曲线L的方程为$x^2+3y^2-2x-3=0$，其标准型为$\dfrac{(x-1)^2}{4}+\dfrac{y^2}{4/3}=1$. 因此曲线$L$上两点间的最长直线距离为 4.

4、设$f(x)=\left(x^2+2x-3\right)^n\arctan^2\dfrac{x}{3}$， 其中$n$为正整数, 则$f^{(n)}(-3)=$________.

【参考答案】: 记$g(x)=(x-1)^n\arctan^2\dfrac{x}{3}$, 则$f(x)=(x+3)^n g(x)$. 利用莱布尼兹法则，可得

$$f^{(n)}(x)=n!g(x)+\sum_{k=0}^{n-1}C_n^k\left[(x+3)^n\right]^{(k)}g^{(n-k)}(x)$$

所以 $f^{(n)}(-3)=n!g(-3)=(-1)^n4^{n-2}n!\pi^2$.

5、设函数$f(x)$的导数$f'(x)$在$[0,1]$上连续，$f(0)=f(1)=0$， 且满足

$$\int_0^1\left[f'(x)\right]^2\mathrm{d}x-8\int_0^1 f(x)\mathrm{d}x+\frac{4}{3}=0$$

则$f(x)=$____________.

【参考答案】: 因为 $\int_0^1 f(x)\mathrm{d}x=-\int_0^1 xf'(x)\mathrm{d}x$，$\int_0^1 f'(x)\mathrm{d}x=0$ 且

$$\int_0^1\left(4x^2-4x+1\right)\mathrm{d}x=\frac{1}{3}$$

所以

$$\begin{aligned} &\int_0^1 f'^2(x)\mathrm{d}x-8\int_0^1 f(x)\mathrm{d}x+\frac{4}{3} \\ =&\int_0^1\left[f'^2(x)+8xf'(x)-4f'(x)+\left(16x^2-16x+4\right)\right]\mathrm{d}x \\ =&\int_0^1\left[f'(x)+4x-2\right]^2\mathrm{d}x=0 \end{aligned}$$

因此 $f'(x)=2-4x$， $f(x)=2x-2x^2+C$. 由 $f(0)=0$ 得 $C=0$. 因此 $f(x)=2x-2x^2$.

二、(12 分) 求极限 $\lim\limits_{n\to\infty}\sqrt{n}\left(1-\sum\limits_{k=1}^{n}\dfrac{1}{n+\sqrt{k}}\right)$.

【参考解答】: 记$a_n=\sqrt{n}\left(1-\sum\limits_{k=1}^{n}\dfrac{1}{n+\sqrt{k}}\right)$, 则

$$a_n=\sqrt{n}\sum_{k=1}^{n}\left(\frac{1}{n}-\frac{1}{n+\sqrt{k}}\right)=\sum_{k=1}^{n}\frac{\sqrt{k}}{\sqrt{n}(n+\sqrt{k})}\le\frac{1}{n\sqrt{n}}\sum_{k=1}^{n}\sqrt{k}$$

因为$\sum_{k=1}^{n}\sqrt{k}\le\sum_{k=1}^{n}\int_{k}^{k+1}\sqrt{x}\mathrm{d}x=\int_{1}^{n+1}\sqrt{x}\mathrm{d}x=\frac{2}{3}((n+1)\sqrt{n+1}-1)$，所以

$$a_n<\frac{2}{3}\cdot\frac{(n+1)\sqrt{n+1}}{n\sqrt{n}}=\frac{2}{3}\left(1+\frac{1}{n}\right)\sqrt{1+\frac{1}{n}}$$

又$\sum_{k=1}^{n}\sqrt{k}\ge\sum_{k=1}^{n}\int_{k-1}^{k}\sqrt{x}\mathrm{d}x=\int_{0}^{n}\sqrt{x}\mathrm{d}x=\frac{2}{3}n\sqrt{n}$，得

$$a_n\ge\frac{1}{\sqrt{n}(n+\sqrt{n})}\sum_{k=1}^{n}\sqrt{k}\ge\frac{2}{3}\cdot\frac{n}{n+\sqrt{n}}$$

于是可得

$$\frac{2}{3}\cdot\frac{n}{n+\sqrt{n}}\le a_n<\frac{2}{3}\left(1+\frac{1}{n}\right)\sqrt{1+\frac{1}{n}}$$

故由夹逼准则，得

$$\lim_{n\to\infty}\sqrt{n}\left(1-\sum_{k=1}^{n}\frac{1}{n+\sqrt{k}}\right)=\lim_{n\to\infty}a_n=\frac{2}{3}$$

三、(12分)设$F(x_1,x_2,x_3)=\int_0^{2\pi}f(x_1+x_3\cos\varphi,x_2+x_3\sin\varphi)\mathrm{d}\varphi$，其中$f(u,v)$具有二阶连续偏导数. 已知$\frac{\partial F}{\partial x_i}=\int_0^{2\pi}\frac{\partial}{\partial x_i}[f(x_1+x_3\cos\varphi,x_2+x_3\sin\varphi)]\mathrm{d}\varphi$，

$$\frac{\partial^2F}{\partial x_i^2}=\int_0^{2\pi}\frac{\partial^2}{\partial x_i^2}[f(x_1+x_3\cos\varphi,x_2+x_3\sin\varphi)]\mathrm{d}\varphi,\quad i=1,2,3$$

试求$x_3\left(\frac{\partial^2F}{\partial x_1^2}+\frac{\partial^2F}{\partial x_2^2}-\frac{\partial^2F}{\partial x_3^2}\right)-\frac{\partial F}{\partial x_3}$并要求化简.

【参考解答】：令$u=x_1+x_3\cos\varphi,v=x_2+x_3\sin\varphi$，利用复合函数求偏导法则易知

$$\frac{\partial f}{\partial x_1}=\frac{\partial f}{\partial u},\quad\frac{\partial f}{\partial x_2}=\frac{\partial f}{\partial v},\quad\frac{\partial f}{\partial x_3}=\cos\varphi\frac{\partial f}{\partial u}+\sin\varphi\frac{\partial f}{\partial v},$$

$$\frac{\partial^2 f}{\partial x_1^2}=\frac{\partial^2 f}{\partial u^2},\quad\frac{\partial^2 f}{\partial x_2^2}=\frac{\partial^2 f}{\partial v^2},$$

$$\frac{\partial^2 f}{\partial x_3^2}=\frac{\partial^2 f}{\partial u^2}\cos^2\varphi+\frac{\partial^2 f}{\partial u\partial v}\sin 2\varphi+\frac{\partial^2 f}{\partial v^2}\sin^2\varphi$$

所以

$$
\begin{aligned}
& x_3\left(\frac{\partial^2 F}{\partial x_1^2}+\frac{\partial^2 F}{\partial x_2^2}-\frac{\partial^2 F}{\partial x_3^2}\right) \\
& =x_3\left[\begin{array}{l}
\int_0^{2\pi} \frac{\partial^2 f}{\partial u^2} \mathrm{d}\varphi+\int_0^{2\pi} \frac{\partial^2 f}{\partial v^2} \mathrm{d}\varphi \\
\quad-\int_0^{2\pi}\left(\frac{\partial^2 f}{\partial u^2} \cos^2\varphi+\frac{\partial^2 f}{\partial u \partial v} \sin 2\varphi+\frac{\partial^2 f}{\partial v^2} \sin^2\varphi\right) \mathrm{d}\varphi
\end{array}\right] \\
& =x_3 \int_0^{2\pi}\left(\frac{\partial^2 f}{\partial u^2} \sin^2\varphi-\frac{\partial^2 f}{\partial u \partial u} \sin 2\varphi+\frac{\partial^2 f}{\partial v^2} \cos^2\varphi\right) \mathrm{d}\varphi
\end{aligned}
$$

又由于$\frac{\partial F}{\partial x_3}=\int_0^{2\pi}\left(\cos\varphi \frac{\partial f}{\partial u}+\sin\varphi \frac{\partial f}{\partial v}\right) \mathrm{d}\varphi$，利用分部积分，得

$$
\begin{aligned}
\frac{\partial F}{\partial x_3}= & -\int_0^{2x} \sin\varphi\left(\frac{\partial^2 f}{\partial u^2} \frac{\partial u}{\partial \varphi}+\frac{\partial^2 f}{\partial u \partial v} \frac{\partial v}{\partial \varphi}\right) \mathrm{d}\varphi \\
& +\int_0^{2\pi} \cos\varphi\left(\frac{\partial^2 f}{\partial u \partial v} \frac{\partial u}{\partial \varphi}+\frac{\partial^2 f}{\partial v^2} \frac{\partial v}{\partial \varphi}\right) \mathrm{d}\varphi
\end{aligned}
$$

$$
\begin{aligned}
& =x_3 \int_0^{2\pi}\left(\frac{\partial^2 f}{\partial u^2} \sin^2\varphi-\frac{1}{2} \sin 2\varphi \frac{\partial^2 f}{\partial u \partial v}\right) \mathrm{d}\varphi-x_3 \int_0^{2n}\left(\frac{1}{2} \sin 2\varphi \frac{\partial^2 f}{\partial u \partial v}-\cos^2\varphi \frac{\partial^2 f}{\partial v^2}\right) \mathrm{d}\varphi \\
& =x_3 \int_0^{2\pi}\left(\frac{\partial^2 f}{\partial u^2} \sin^2\varphi-\frac{\partial^2 f}{\partial u \partial v} \sin 2\varphi+\frac{\partial^2 f}{\partial v^2} \cos^2\varphi\right) \mathrm{d}\varphi
\end{aligned}
$$

所以$x_3\left(\frac{\partial^2 F}{\partial x_1^2}+\frac{\partial^2 F}{\partial x_2^2}-\frac{\partial^2 F}{\partial x_3^2}\right)-\frac{\partial F}{\partial x_3}=0$.

四、(10 分) 函数$f(x)$在$[0,1]$上具有连续导数，且

$$\int_0^1 f(x)\mathrm{d}x=\frac{5}{2},\int_0^1 x f(x)\mathrm{d}x=\frac{3}{2}$$

证明：存在 $\xi \in(0,1)$，使得$f'(\xi)=3$.

【参考解答】：**【思路一】** 考虑积分$\int_0^1 x(1-x)[3-f'(x)]\mathrm{d}x$. 利用分部积分及题设条件，得

$$
\begin{aligned}
& \int_0^1 x(1-x)[3-f'(x)] \mathrm{d}x \\
& =x(1-x)[3x-f(x)]_0^1-\int_0^1(1-2x)[3x-f(x)] \mathrm{d}x \\
& =\int_0^1 3x(2x-1) \mathrm{d}x+\int_0^1(1-2x) f(x) \mathrm{d}x \\
& =\left.\left(2x^3-\frac{3}{2} x^2\right)\right|_0^1+\int_0^1 f(x) \mathrm{d}x-2 \int_0^1 x f(x) \mathrm{d}x \\
& =2-\frac{3}{2}+\frac{5}{2}-3=0
\end{aligned}
$$

根据积分中值定理，存在$\xi \in(0,1)$，使得 $\xi(1-\xi)[3-f'(\xi)]=0$ ，即

$$f'(\xi)=3.$$

【思路二】由定积分的分部积分法，有

$$\int_0^1 f(x)\,\mathrm{d}x = xf(x)\Big|_0^1 - \int_0^1 xf'(x)\,\mathrm{d}x = \frac{5}{2} \quad (*)$$

$$\int_0^1 xf(x)\,\mathrm{d}x = \frac{1}{2}f(x)\cdot x^2\Big|_0^1 - \frac{1}{2}\int_0^1 x^2 f'(x)\,\mathrm{d}x = \frac{3}{2} \quad (**)$$

用$(*)\times\frac{1}{2}-(**)$，得

$$\begin{aligned}&\frac{1}{2}\int_0^1 f(x)\,\mathrm{d}x - \int_0^1 xf(x)\,\mathrm{d}x\\ =& -\frac{1}{2}\int_0^1 xf'(x)\,\mathrm{d}x + \frac{1}{2}\int_0^1 x^2 f'(x)\,\mathrm{d}x = -\frac{1}{4}\end{aligned}$$

整理得

$$\int_0^1 f'(x)x\left(x-1\right)\mathrm{d}x = -\frac{1}{2}$$

由于$f'\left(x\right)$连续，而$x\left(x-1\right)$在$\left[0,1\right]$上不改变符号，故由第一积分中值定理知，存在$\xi\in(0,1)$，使得

$$f'(\xi)\int_0^1\left(x^2-x\right)\mathrm{d}x = -\frac{1}{2}$$

其中$\int_0^1\left(x^2-x\right)\mathrm{d}x = -\frac{1}{6}$，即$f'(\xi)=3$成立.

五、(12 分) 设 $B_1,B_2,\cdots,B_{2021}$ 为空间$\mathbf{R}^3$中半径不为零的 2021 个球，$A=\left(a_{ij}\right)$为 2021 阶方阵，其(i,j)元a_{ij}为球B_i与B_j相交部分的体积. 证明：行列式$|\,E+A\,|>1$，其中E为单位矩阵.

【参考解答】：记Ω为以原点O为球心且包含$B_1,B_2,\cdots,B_{2021}$在内的球，考察二次型$f=\sum_{i=1}^{2021}\sum_{j=1}^{2011}a_{ij}z_iz_j$.注意到

$$a_{ij}=\iiint_\Omega \chi_i(t,u,v)\chi_j(t,u,v)\mathrm{d}t\mathrm{d}u\mathrm{d}v$$

其中$\chi_i(t,u,v)$的定义为$\chi_i(t,u,v)=\begin{cases}1, & (t,u,v)\in B_i\\ 0, & (t,u,v)\in\Omega\setminus B_i\end{cases}$于是有

$$\begin{aligned}f&=\sum_{i=1}^{2021}\sum_{j=1}^{201}a_{ij}z_iz_j\\ &=\sum_{i=1}^{2011}\sum_{j=1}^{2021}\iiint\left[\chi_i(t,u,v)z_i\right]\left[\chi_j(t,u,v)z_j\right]\mathrm{d}t\mathrm{d}u\mathrm{d}v\\ &=\iiint_\Omega\sum_{i=1}^{2021}\left[\chi_i(t,u,v)z_i\right]^2\mathrm{d}t\mathrm{d}u\mathrm{d}v\ge 0\end{aligned}$$

另一方面，存在正交变换$Z=PY$使得f化为

$$f=\lambda_1y_1^2+\lambda_2y_2^2+\cdots+\lambda_{2021}y_{2021}^2$$

其中$\lambda_1,\lambda_2,\cdots,\lambda_{2021}$为$A$的全部特征值. 因为二次型$f\geq 0$，所以$A$的特征值$\lambda_i\geq 0,(i=1,2,\cdots 2021)$. 于是

$$\begin{aligned}|E+A|&=\left|P^{-1}(E+A)P\right|\\&=\left(1+\lambda_1\right)\left(1+\lambda_2\right)\cdots\left(1+\lambda_{2021}\right)\geq 1.\end{aligned}$$

注意到A不是零矩阵，所以至少有一个特征值$\lambda_i>0$，故$|E+A|>1$.

六、(12 分) 设Ω是由光滑的简单封闭曲面Σ围成的有界闭区域，函数$f(x,y,z)$在Ω上具有连续二阶偏导数，且$f(x,y,z)\big|_{(x,y,z)\in\Sigma}=0$. 记$\nabla f$为$f(x,y,z)$的梯度，并令

$$\Delta f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}.$$

证明：对任意常数$C>0$，恒有

$$C\iiint_\Omega f^2\mathrm{d}x\mathrm{d}y\mathrm{d}z+\frac{1}{C}\iiint_\Omega(\Delta f)^2\mathrm{d}x\mathrm{d}y\mathrm{d}z\geq 2\iiint_\Omega|\nabla f|^2\ \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

【参考解答】：首先利用 Gauss 公式，得

$$\iint_\Sigma f\frac{\partial f}{\partial x}\mathrm{d}y\mathrm{d}z+f\frac{\partial f}{\partial y}\mathrm{d}z\mathrm{d}x+f\frac{\partial f}{\partial z}\mathrm{d}x\mathrm{d}y=\iiint_\Omega\left(f\Delta f+|\nabla f|^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

其中Σ取外侧. 因为$f(x,y,z)\big|_{(x,y,z)\in\Sigma}=0$，所以上式左端等于零. 利用 Cauchy 不等式，得

$$\begin{aligned}\iiint_\Omega|\nabla f|^2\ \mathrm{d}x\mathrm{d}y\mathrm{d}z&=-\iiint_\Omega(f\Delta f)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\&\leq\left(\iiint_\Omega f^2\mathrm{d}x\mathrm{d}y\mathrm{d}z\right)^{1/2}\left(\iiint_\Omega(\Delta f)^2\mathrm{d}x\mathrm{d}y\mathrm{d}z\right)^{1/2}\end{aligned}$$

故对任意常数$C>0$，恒有(利用均值不等式)

$$\begin{aligned}&C\iiint_\Omega f^2\mathrm{d}x\mathrm{d}y\mathrm{d}z+\frac{1}{C}\iiint_\Omega(\Delta f)^2\mathrm{d}x\mathrm{d}y\mathrm{d}z\\&\geq 2\left(\iiint_\Omega f^2\mathrm{d}x\mathrm{d}y\mathrm{d}z\right)^{1/2}\left(\iiint_\Omega(\Delta f)^2\mathrm{d}x\mathrm{d}y\mathrm{d}z\right)^{1/2}\\&\geq 2\iiint_\Omega|\nabla f|^2\ \mathrm{d}x\mathrm{d}y\mathrm{d}z\end{aligned}$$

七、(12 分) 设$\left\{u_n\right\}$是正数列，满足$\dfrac{u_{n+1}}{u_n}=1-\dfrac{\alpha}{n}+O\left(\dfrac{1}{n^\beta}\right)$，其中常数$\alpha>0,\beta>1$.

(1) 对于$v_n=n^\alpha u_n$，判断级数$\sum\limits_{n=1}^{\infty}\ln\dfrac{v_{n+1}}{v_n}$的敛散性；

(2) 讨论级数$\sum\limits_{n=1}^{\infty}u_n$的敛散性.

[注：设数列$\left\{a_n\right\},\left\{b_n\right\}$满足$\lim\limits_{n\to\infty}a_n=0,\lim\limits_{n\to\infty}b_n=0$，则$a_n=O\left(b_n\right)\Leftrightarrow$存在常数 $M>0$ 及正整数N，使得$\left|a_n\right|\leq M\left|b_n\right|$对任意$n>N$成立.]

【参考解答】：(1) 注意到

$$\begin{aligned}\ln\frac{v_{n+1}}{v_n}&=\alpha\ln\left(1+\frac{1}{n}\right)+\ln\frac{u_{n+1}}{u_n}\\&=\left(\frac{\alpha}{n}+O\left(\frac{1}{n^2}\right)\right)+\left(-\frac{\alpha}{n}+\frac{\alpha^2}{n^2}+O\left(\frac{1}{n^\beta}\right)\right)=O\left(\frac{1}{n^\gamma}\right)\end{aligned}$$

其中$\gamma=\min\{2,\beta\}>1$，故存在常数$C>0$及正整数N，使得

$$\left|\ln\frac{v_{n+1}}{v_n}\right|\le C\left|\frac{1}{n^\gamma}\right|$$

对任意$n>N$成立，所以级数$\sum_{n=1}^{\infty}\ln\frac{v_{n+1}}{v_n}$收敛.

(2) 因为$\sum_{k=1}^{n}\ln\frac{v_{k+1}}{v_k}=\ln v_{n+1}-\ln v_1$，所以由(1)的结论可知，极限$\lim_{n\to\infty}\ln v_n$存在. 令$\lim_{n\to\infty}\ln v_n=a$，则$\lim_{n\to\infty}v_n=e^a>0$，即$\lim_{n\to\infty}\frac{u_n}{1/n^\alpha}=e^a>0$.根据正项级数的比较判别法，级数$\sum_{n=1}^{\infty}u_n$当$\alpha>1$时收敛，当$\alpha\le 1$时发散.