Exercise 9, Chapter 3

Thomas B. Jørgensen

25 april 2016

Consider the following double integral

$$\int_0^\infty \int_0^x e^{-(x+y)} dy\, dx \tag{1}$$

with the following analytic solution

$$\begin{aligned}
\int_0^\infty \int_0^x e^{-(x+y)} dy\, dx &= \int_0^\infty e^{-x} - e^{-2x} dx & (2)\\
&= \left[-e^{-x} + \frac{e^{-2x}}{2} \right]_0^\infty & (3)\\
&= \frac{1}{2} & (4)
\end{aligned}$$

For y observe that its upper limit $f(x) = x$ exactly divides in two the domain $\Omega = [0, \infty] \times [0, \infty]$, thus we can rewrite (1) so as to obtain the following integral

$$\int_0^\infty \int_0^x e^{-(x+y)} dy\, dx = \frac{1}{2} \cdot \int_0^\infty \int_0^\infty e^{-(x+y)} dy\, dx \tag{5}$$

Now, the rhs of (5) is separable i.e. split the integral into two components

$$\int_0^\infty e^{-x} dx \cdot \int_0^\infty e^{-y} dy \tag{6}$$

Define the function $f(z) = e^{-z}$ with $z \in [0, \infty]$ then it's straightforward to generate values of $f(z)$ using random uniform numbers $U = [0, 1]$ be means of the following composite function

$$g(U) = \frac{f(\frac{1}{U} - 1)}{U^2} = \frac{e^{1-\frac{1}{U}}}{U^2} \tag{7}$$

where we used the substitution $U = \frac{1}{z+1}$. Obviously we must generate two random uniform numbers $U1, U2$ and then taking the product of applying (7) for $U1$ and $U2$. The following program implements the idea

```
n <- 500000
total <- 0
for (i in 1:n) {
    U1 <- runif(1)
    U2 <- runif(2)
    total <- total+exp(1-1/U1)/((U1)^2)*exp(1-1/U2)/((U2)^2)
}
(1/2)*total/n
```

```
## [1] 0.4997001 0.5002075
```

with values very close to (4).