Minimize SSE for Least Square Method

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The following proof was introduced by Eric Iksoon (1996).

The standard bivariate linear regression model: $Y_i = \alpha + \beta X + \epsilon$, where $i = 1, \ldots, n$, then we can rewrite it as $e_i = Y_i - a - bX_i$, where a and b can be any estimators of α and β, respectively. Here, $\overline{Y}$ and $\overline{X}$ denote the respective sample means of Y_i and X_i; x_i and y_i denote deviation of X_i and Y_i. We can work out the lower bound for $SSE(a, b)$ that is the error sum of squares as a function of a and b.

$$
\begin{aligned}
SSE(a,b) &= \sum [Y_i - a - bX_i]^2 \\
&= \sum [(\overline{Y} + y_i) - a - b(\overline{X} + x_i)]^2 \\
&= \sum [q - (bx_i - y_i)]^2 \\
&= nq^2 + \sum (bx_i - y_i)^2 \\
&= nq^2 + \sum (b^2 x_i - 2bx_i y_i + y_i^2) \\
&= nq^2 + \left(b^2 - 2b\frac{\sum x_i y_i}{\sum x_i^2} + \frac{\sum y_i^2}{\sum x_i^2}\right) \sum x_i^2 \\
&= nq^2 + \left(\underbrace{b^2 - 2b\frac{\sum x_i y_i}{\sum x_i^2} + \left[\frac{\sum x_i y_i}{\sum x_i^2}\right]^2}\ - \underbrace{\left[\frac{\sum x_i y_i}{\sum x_i^2}\right]^2 + \frac{\sum y_i^2}{\sum x_i^2}}\right) \sum x_i^2 \\
&= \underbrace{nq^2}_{\geq 0} + \underbrace{\left(b - \frac{\sum x_i y_i}{\sum x_i^2}\right)^2 \sum x_i^2}_{\geq 0} + \sum y_i^2 \left(1 - \left(\frac{\sum x_i y_i}{\sqrt{\sum x_i^2 \sum y_i^2}}\right)^2\right) \\
&\geq \sum y_i^2 \left(1 - \left(\frac{\sum x_i y_i}{\sqrt{\sum x_i^2 \sum y_i^2}}\right)^2\right)
\end{aligned}
$$

The conditions for SSE lower bound is that the two nonnegative items are all zeros. Then we have two equations:

$$
\begin{aligned}
q = \overline{Y} - \hat{a} - \hat{b}\overline{X} &= 0 \\
\hat{b} - \frac{\sum x_i y_i}{\sum x_i^2} &= 0
\end{aligned}
$$

then,

$$
\begin{aligned}
\hat{a} &= \overline{Y} - \hat{b}\overline{X} \\
\hat{b} &= \frac{\sum x_i y_i}{\sum x_i^2}
\end{aligned}
$$

where $\hat{a_0}$ and $\hat{a_1}$ shown in the MOOCS slides are $\hat{a}$ and $\hat{b}$

References

Eric Iksoon, I. (1996), A Note On Derivation of the Least Squares Estimator, Working Papers 199611, University of Hawaii at Manoa, Department of Economics.
URL: *https://ideas.repec.org/p/hai/wpaper/199611.html*