COLLATZ-HASSE-SYRACUSE-ULAM-KAKUTANI SEQUENCE
:
CONVERGENCE TO THE TRIVIAL CYCLE PROVED

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ABSTRACT

The convergence of the *Collatz-Hasse-Syracuse-Ulam-Kakutani Sequence* is proved, thus proving the *Collatz Conjecture*, which has been an *unsolved problem*. The proof is based on the *isomorphism* established between the set of positive integers and a carefully designed *system* with a hierarchy (arborescence) of *binary exponential ladder*s defined on the set of positive odd numbers.

Keywords: Collatz-Hasse-Syracuse-Ulam-Kakutani (CHSUK) Sequence; Collatz-Hasse-Syracuse-Ulam-Kakutani (CHSUK) Conjecture; Convergence, Isomorphism, Dedekind-Peano Axioms.

AMS MSC Mathematics Subject Classification: 11B50.

1. INTRODUCTION

The *Collatz-Hasse-Syracuse-Ulam-Kakutani Conjecture* (simply referred to as the *Collatz Conjecture*) states that the Collatz-Hasse-Syracuse-Ulam-Kakutani (CHSUK) Sequence (also referred to as the *Collatz Sequence*) converges to the trivial cycle {(4->2->1)}, starting from any positive integer.

This research report presents a proof for the same, reasoning with the most fundamental *Dedekind-Peano Axioms* and *Modulus Arithmetic* applied to a meticulously designed *Structured System Framework* of *Binary Exponential*

*Ladder*s defined on the set of positive odd numbers, and establishes an *isomorphism* between the *structured system framework* of *Binary Exponential Ladder*s and the set of natural numbers.

2. PROBLEM DESCRIPTION

We define the *Collatz Function* C(n) with a positive integer n as its input argument, in terms of a 'pull-Down' operator D(n) and a 'push-Up' operator U(n) as follows:

if (n is even) C(n) := D(n) = (n / 2); else C(n) := U(n) = (3*n + 1); [Eqn.1]

where the '*pull-Down*' D(n) operator takes only an even number as its input argument whereas the '*push-Up*' operator U(n) takes only an odd number as its input argument and gives an output that is an even number.

For convenience in our study of the Collatz Sequence, we define the *Compact Collatz Function* T(m) by the repeated application of the 'pull-down' operator D(m) wherever applicable, say, ($p \geq 1$) times, that is, $D^*(m) := D^p(m)$ so as to get an output $D^\#(m)$ that is an odd number:

if (m is even) $T(m) := D^*(m) := D^p(m) = (m / 2^p) := D^\#(m)$;
else (m is odd) $T(m) := U(m) = (3 * m + 1) := U^\#(m)$; [Eqn.2]

where $D^\#(m)$ is called the "D-floor number" associated with the input argument m; and $U^\#(m)$ is called the "U-ceiling number" associated with the input argument m.

The *Compact Collatz Function* T(m) may as well be considered to have been redefined with the newly introduced two operators, the "D-floor operator" $D^\#(m)$ and the "U-ceiling operator" $U^\#(m)$ as given in [Eqn.2] above.

This new definition for the *Compact Collatz Function* T(m) facilitates our study of the corresponding *Compact Collatz Sequence*; which is no different from its equivalent Collatz Sequence, once we understand that the repeated application, say, ($p \geq 1$) times, of the 'pull-Down' operator D(m) has now been collapsed into an equivalent single "D-floor operator" $D^\#(m)$ giving the D-floor number $D^\#(m)$ as its output. The push-Up operator U has been simply redefined as the "U-ceiling operator" $U^\#$ for uniformity and elegant completeness.

The *Compact Collatz Sequence* is obtained by the repeated sequential application of the *Compact Collatz Function* T(m) starting with the given initial input number m - represented by an alternating series of $D^\#$ number and $U^\#$ number - except possibly the starting initial 'seed' number m and the final terminating number, which as per the Collatz Conjecture, is anyway a $D^\#$ number that is unity.

3. OBSERVATIONS ON THE PULL-DOWN OPERATOR

The pull-Down operator D *always* takes only an even number n as its input argument. Every application of this pull-down operator results in an alternating (toggling) effect on the nMOD3 property of the input argument number; that is, a 1MOD3 input gives a 2MOD3 output and a 2MOD3 input gives a 1MOD3 output; whereas a 0MOD3 input gives a 0MOD3 output. Repeated application of D, in case applicable, results in a final output that is an odd number and therefore becomes an input for the push-Up operator. In such a case, we call it a "D-floor operator" $D^{\#}$ as defined in [Eqn.2] above, and its output a "D-floor number" $D^{\#}(n)$ characterized by being a odd number; $D^{\#}(n)$ may be in any one of the three possible types: (1) a 1MOD6 odd number, being a 1MOD3 odd number that is of the type (6m-5); (2) a 5MOD6 odd number, being a 2MOD3 odd number that is of the type (6m-1); (3) a 3MOD6 odd number, being a 0MOD3 odd number that is of the type (6m-3).

4. OBSERVATIONS ON THE PUSH-UP OPERATOR

The push-Up operator U *always* takes only an odd number m as its input argument, and *always* gives an output that is a 4MOD6 even number, being a 1MOD3 even number that is of the type (6m-2) - irrespective of whether the input is a 1MOD6 odd number or a 3MOD6 odd number or a 5MOD6 odd number. Note that one single application of the 'push-Up' operator U transforms any input odd number m into a 4MOD6 even number that becomes an input to the "D-floor operator $D^{\#}$". That is why we may as well call the push-Up operator U as the "U-ceiling operator U#" as defined in [Eqn.2] above.

5. OBSERVATIONS ON THE COMPACT COLLATZ FUNCTION

Start with any positive integer. (1) If the starting initial number n is even, then we apply the D-floor operator $D^{\#}$ operator giving an output that is the D-floor number $D^{\#}(n)$ which is given as input to the U-ceiling operator. Of course, if the starting number is a power of 2 we terminate at unity. So, now we have a D-floor number $D^{\#}(n)$ that is an odd number greater than unity, in any non-trivial case, as the initial $D^{\#}$ node in the Compact Collatz Sequence. (2) If on the other hand the starting initial number n is an odd number, we treat that itself as the initial $D^{\#}$ node in the Compact Collatz Sequence.

Having thus obtained the initial $D^{\#}$ node in the Compact Collatz Sequence, we apply the U-ceiling operator $U^{\#}$ to get the U-ceiling number $U^{\#}$ that is a 4MOD6 even number. That in turn is given as input to the D-floor operator $D^{\#}$. Now the process continues.

Note that the Compact Collatz Sequence can therefore be defined by a *trajectory* generated by an alternating sequence of a “D-floor number” $D^{\#}$ and a “U-ceiling number” $U^{\#}$, with its starting initial node being a $D^{\#}$ number. The Compact Collatz Function as presented in [Eqn.2] defines the unique link (directed arc) from any given D-floor number $D^{\#}$ as the predecessor node to its corresponding unique U-ceiling number $U^{\#}$ as the successor node and also the unique link (directed arc) from any given U-ceiling number $U^{\#}$ as the predecessor node to its corresponding unique D-floor number $D^{\#}$ as the successor node. The unique link (directed arc) from a starting initial even “seed” number leading to the first node (D-floor number $D^{\#}$) in the *trajectory* is similarly defined.

As mentioned earlier, the application of the D-floor operator $D^{\#}$ on a U-ceiling number $U^{\#}$ that is a 4MOD6 even number of the form (6m-2) can lead to a D-floor number $D^{\#}$ that is an odd number that can be either: (1) a 1MOD6 odd number, being a 1MOD3 odd number that is of the type (6m-5); (2) a 5MOD6 odd number, being a 2MOD3 odd number that is of the type (6m-1); but can never be (3) a 3MOD6 odd number, that is a 0MOD3 odd number of the type (6m-3). Note that the only situation when the D-floor operator $D^{\#}$ gives an output D-floor number $D^{\#}$ that is a 3MOD6 odd number of the type (6m-3) is when its input is a 0MOD6 even number, which is impossible for any U-ceiling number $U^{\#}$, although such an input may come in those special cases wherein the starting initial ‘seed’ number itself is a 0MOD6 even number that is of the form $(6m-3).2^{p}$ leading to an output $D^{\#}$ that is again a 3MOD6 odd number of the form (6m-3).

6. ANALYSIS OF THE COMPACT COLLATZ SEQUENCE

From the above observations, it is clear that corresponding to every positive integer n as the starting initial ‘seed’ number, there is a starting initial node in the trajectory representing the *Compact Collatz Sequence*, that is a $D^{\#}$ number in exactly one of the three possible forms as mentioned above - that can be an input argument to the U-ceiling operator $U^{\#}$ giving exactly one unique output $U^{\#}$ which itself can be an input to the D-floor operator $D^{\#}$ so that the process continues. Successive application of each of these two operators ($U^{\#}$ and $D^{\#}$) wherever applicable, traces a unique *trajectory*, wherein each node represents a number that is the unique output number of the appropriate operation applied to the input number represented by the preceding node in the trajectory.

The anticipated terminating trivial cycle {(4->2->1)} can be obtained only through a final application of the D-floor operator $D^{\#}$ on a 4MOD6 even number that is of the form (6m-2).

7. BINARY-EXPONENTIAL-LADDER WITH ITS DEFINING-BASE-RUNG $D^{\#}$

Here, we present a meticulously designed *Structured System Framework* that *partition*s the *set of positive integers* to facilitate a *general systems analysis* of the *Compact Collatz Sequence*.

Let every positive odd number be associated with a *Binary-Exponential-Ladder*, denoted by BEL(2m-1) and defined as a sequence $\{(2m-1).2^u\}$ with $(u \geq 0)$; its *defining-base-rung* given by the odd number (2m-1). Thus, we establish an exact one-to-one mapping between the *set of positive odd numbers* that form the $D^{\#}$ value for the *defining-base-rung* and the corresponding *Binary-Exponential-Ladder* BEL($D^{\#}$).

Every positive even number in the form $(2m-1).2^u$ with $(u>0)$; for which there exists its corresponding $D^{\#}$ value, $D^{\#}((2m-1).2^u) = (2m-1)$; for which there exists exactly one corresponding *Binary-Exponential-Ladder* BEL(2m-1) that contains the given even number $(2m-1).2^u$ as one of its higher rungs in that BEL(2m-1) ladder.

Thus, we establish that *the set of all Binary-Exponential-Ladders form a partition of the set of all positive integers*; with an exact one-to-one correspondence between each positive odd number $D^{\#}$ and the corresponding *Binary-Exponential-Ladder* for which it is the *defining-base-rung* $D^{\#}$; whereas each of the positive even numbers correspond to exactly one of the higher rungs of a specific Binary-Exponential-Ladder identified by the D-floor number $D^{\#}$ associated with that given positive even number.

This partitioned framework of positive integers goes another step deeper because of the fact that the *defining-base-rung* $D^{\#}$ of a Binary-Exponential-Ladder BEL($D^{\#}$) can itself be in one of the three possible forms 1MOD6 or 5MOD6 or 3MOD6 whereas all the upper rungs of the Binary-Exponential-Ladder are either (1) alternatingly 2MOD6 and 4MOD6 or (2) all being 0MOD6 numbers.

The Collatz Conjecture states that every Collatz Sequence, starting from any positive integer, converges to the trivial cycle {(4->2->1)} which is in BEL(1) that is uniquely identified by its defining-base-rung $D^{\#}$ value that is unity. Therefore, our focus will be the Binary-Exponential-Ladders BEL(1) and its relationship with every other Binary-Exponential-Ladder BEL($D^{\#}$).

As seen above, $D^{\#}$ can be (1) either a 1MOD6 number of the form (6m-5); (2) or a 5MOD6 number of the form (6m-1); (3) or a 3MOD6 number of the form (6m-3). BEL(6m-5) contains the output of $U^{\#}$ at $(6m-5)2^W$ with w being an even exponent that is of the form (2k) wherein the input of $U^{\#}$ is given by $[\{(6m-5).2^W -1\}/3]$. BEL(6m-1) contains the output of $U^{\#}$ at $(6m-1)2^V$ with v being an odd exponent that is of the form (2k-1) wherein the input of $U^{\#}$ is given by $[\{(6m-1).2^V -1\}/3]$. However, BEL(6m-3) cannot contain any such output of the U-ceiling operator $U^{\#}$ irrespective of any input argument.

8. IMMEDIATE NEIGHBORHOOD OF A BINARY-EXPONENTIAL-LADDER

The relationship between a pair of Binary-Exponential-Ladders BEL(m) and BEL(n) can be considered to be defined and characterized by the relationship between the corresponding pair of the *defining-base-rung* $D^{\#}$ values m and n along with the corresponding pair $U^{\#}$(m) and $U^{\#}$(n).

The immediate-neighborhood of a given Binary-Exponential-Ladder BEL($D^{\#}$) is defined by the *immediate-predecessors* and *immediate-successors*, considering the U-ceiling operator $U^{\#}$; since the D-floor operator $D^{\#}$ is applicable only within a given Binary-exponential-Ladder and not between a pair of them.

8.1 SINGLE UNIQUE IMMEDIATE SUCCESSOR

It turns out that the only *one single unique immediate successor* of BEL(m) is BEL($D^{\#}$($U^{\#}$(m))) that contains $U^{\#}$(m) as one of its higher rungs, with $n := D^{\#}(U^{\#}(m))$ as its identifying characteristic D-floor number being its defining-base-rung.

$$S(BEL(m)) = BEL(D^{\#}(U^{\#}(m))) := BEL(n); \qquad \text{[Eqn.3]}$$

However, there exists a *set of immediate-predecessors* for each BEL($D^{\#}$) of the form BEL(6m-5) and BEL(6m-1) although none for BEL(6m-3). Note that if S(BEL(m)) is BEL(n) than BEL(m) is one of the predecessors of BEL(n).

8.2 MULTIPLE IMMEDIATE PREDECESSORS

The set of immediate-predecessors for a given BEL(n) is defined by considering the inverse of the immediate-successor relationship as the set of all BELs each of which having its single unique immediate-successor as BEL(n).

$$\{P(BEL(n))\} := \{BEL(m) \mid BEL(n) = S(BEL(m)\}; \qquad \text{[Eqn.4]}$$

BEL(1MOD6) or equivalently BEL(6m-5) has, as its set of immediate-predecessors, {BEL([(1MOD6).2^W - 1]/3)} or equivalently {BEL([(6m-5).2^W - 1]/3)} with w being an positive even exponent of the form (2k), wherein the input of $U^{\#}$ is given by [{(1MOD6).2^W -1}/3] or equivalently [{(6m-5).2^W -1}/3] and the output of $U^{\#}$ being {(1MOD6).2^W} or equivalently {(6m-5).2^W} that is contained in BEL(1MOD6). Each of the three possible classes of BEL, namely, BEL(1MOD6) and BEL(5MOD6) and BEL(3MOD6) can be the immediate-predecessor of BEL(1MOD6).

BEL(5MOD6) or equivalently BEL(6m-1) has, its set of immediate-predecessors, {BEL([(5MOD6).2^V - 1]/3)} or equivalently {BEL([(6m-1).2^V - 1]/3)} with v being a positive odd exponent of the form (2k-1), wherein the input of $U^{\#}$ is given by [{(5MOD6).2^V -1}/3] or equivalently [{(6m-1).2^V -1}/3] and the output of $U^{\#}$ being {(5MOD6).2^V} or equivalently {(6m-1).2^V} that is contained in BEL(5MOD6) or equivalently BEL(6m-1). Each of the three possible classes of BEL, namely, BEL(1MOD6) and BEL(5MOD6) and BEL(3MOD6) can be the immediate-predecessor of BEL(5MOD6).

BEL(3MOD6) or equivalently BEL(6m-3) has no predecessors.

8.3 QUARTERNARY-EXPONENTIAL-LADDER

The above observed property, that *only* the alternating rungs, defined by (1MOD6).4^u of BEL(1MOD6) or (5MOD6).2.4^u of BEL(5MOD6) being the 'active' nodes in the CHSUK-Sequence; naturally makes it convenient to define a system of *Quarternary-Exponential-Ladder*s QEL wherein every rung of QEL becomes an 'active' node in the CHSUK-Sequence. This concept is not directly needed for proving the convergence of the Collatz Sequence, and therefore we will not take up this line of study in this research report.

8.4 BEL(1) AS THE CENTRAL FOCUS

Considering BEL(1) as our central focus of interest, which itself belongs to the type BEL(1MOD6) or equivalently BEL(6m-5); it is interesting to note that it has its *single unique immediate-successor* as S(BEL(1)) = BEL($D^{\#}$($U^{\#}$(1))) = S(BEL(1)) that is BEL(1) itself because of the trivial cycle {(4->2->1)} contained within BEL(1).

8.5. BELnet : NETWORK OF BINARY-EXPONENTIAL-LADDERS

The above discussion about the successor predecessor relationship among the binary-exponential-ladders and its neighborhood leads to the observation that the network of binary-exponential-ladders, **BELnet**, has countably infinite number of each of the three classes/types of nodes: (1) BEL(1MOD6) or equivalently BEL(6m-5); (2) BEL(5MOD6) or equivalently BEL(6m-1); and (3) BEL(3MOD6) or equivalently BEL(6m-3). Each BEL being a node of the BELnet has a single unique outward directed arc that points towards its single unique immediate-successor, specifically linking onto some higher rung. Multiple (countably infinite number of) inward directed arcs, each linked onto some specific higher rung of a given BEL, emanate from its immediate-predecessor BEL.

The *connectedness* of the network of binary-exponential-ladders **BELnet** will be analyzed, from the design of a structured system framework consisting of the entire set of binary-exponential-ladders, that is designed simply as a well-organized condensation of the very same BELnet, as presented below.

9. STRUCTURED SYSTEM FRAMEWORK OF BELs

From the above discussion we find that it is convenient for our study to consider a *Structured System Framework* as an infinite sequence of terms each of which being a set of BELs; that is, $H := \{. . . H_{k-1} < H_k < H_{k+1} . . .\}$ wherein the *unique ordering relationship* between the adjacent terms of the sequence is derived from the uniqueness characteristic of the *immediate-successor* relationship among the BELs that form the member elements of these adjacent terms in the sequence. Specifically, H_{k-1} is defined as the set formed by the unique immediate-successor of each BEL belonging to H_k and H_k is defined as the set formed by the unique immediate-successor for each BEL belonging to H_{k+1} etc.

$$H := \{. . . H_{k-1} < H_k < H_{k+1} . . .\}; \qquad \text{[Eqn.5]}$$

that is,

$$H_{k-1} := \{S(BEL(m)) \mid BEL(m) \in H_k\} \text{ and } H_k := \{BEL(m) \mid S(BEL(m)) \in H_{k-1}\} \qquad \text{[Eqn.6]}$$

Now, we may as well define the predecessor relationship as the inverse of the above defined successor relationship, as –

$$H_{k-1} := S(H_k) \quad \text{and} \quad H_k := S(H_{k+1}) \qquad \text{[Eqn.7]}$$

and

$$P(H_{k-1}) := H_k \quad \text{and} \quad P(H_k) := H_{k+1} \qquad \text{[Eqn.8]}$$

However, the multiplicity of the *immediate-predecessor* relationship among the BELs requires that the set of all immediate-predecessors of every element of H_{k-1} form the elements of the set H_k so as to guarantee the strict and complete ordering relation $H_{k-1} < H_k < H_{k+1}$ among these sets, in spite of only a partial ordering relationship among the BELs.

10. cBELnet : CONDENSED NETWORK OF BELs

The *structured system framework* $H = \{. . . H_{k-1} < H_k < H_{k+1} . . .\}$ defined above is a connected *digraph* that is like a linear directed path (chain) with no forking but merging only at the deeper BEL level, because of the above-mentioned successor predecessor relationships between the BEL elements that are members of the adjacent terms in the sequence, and that H is a system structure framework meticulously designed merely as a well-organized condensation of the entire set of all the BELs.

Now, a question may arise as to the connectedness of the BELnet as mentioned in the earlier section. The structured system framework H can in general allow for the existence of trivial cycles at a deeper BEL level within a single term H_k or even non-trivial cycle (closed chain) consisting of several terms in the sequence.

With the above specific design of the structured system framework H does not allow the existence of isolated islands of such non-trivial cycles, because H itself must be a connected digraph with one single linear directed path (chain). The possibility of the presence of even a single isolated island of non-trivial cycle is excluded by this fundamental property of the structured system framework H.

Note that the presence of trivial cycle among the BELs that form the member elements of any given single term H_k is also excluded, because these member elements of any given single term H_k are not related through any successor predecessor relationships except that each has its immediate-successor in H_{k-1}.

However, the presence of a trivial cycle within a given BEL is not ruled out. In fact, as observed earlier in the previous section, BEL(1) itself is its single unique immediate-successor and does not have any immediate-successor distinct from itself, although it has multiple immediate-predecessors. Therefore, the above sequence H must necessarily have a term H_0 with BEL(1) being its singleton element, thus forcing the sequence H to be truncated from below at H_0 - the modified and updated sequence becomes the condensed network of binary exponential ladders,

$$\text{cBELnet} := H := \{H_0, H_1, H_2, \ldots\}; \qquad \text{[Eqn.9]}$$

as the redefined structure for the *Structured System Framework*.

From the above discussion we observe that cBELnet, that is the *Structured System Framework* H = $\{H_0, H_1, H_2, \ldots\}$ is an infinite sequence of terms, each term being a countably infinite set of BELs except its 'root' $H_0 := \{\text{BEL}(1)\}$ being a singleton set. The set of k^{th} immediate predecessors of BEL(1) form the set H_k at tier-k level in the hierarchy, if one wishes to consider it as a hierarchy.

Now another question arises, as to why the tier-0 level 'root' $H_0 := \{\text{BEL}(1)\}$ must be a singleton set. The reason for this is based on the characteristic property that every BEL has a single unique immediate-successor and that the cBELnet H is designed such that each of the member elements of H_k has its immediate-successor in H_{k-1} as mentioned above. Since BEL(1) is its own immediate-successor, combining it with any other BELs, each having its single unique immediate-successor, as a member element in the tier-0 level 'root' H_0 would lead to an unacceptable situation wherein BEL(1) must necessarily be present in H_{-1} and again in H_{-2} etc. - a useless and inconvenient situation that can be avoided.

This Structured System Framework H that is cBELnet of BELs has a direct one-to-one correspondence (mapping) with the set of positive integers, considering the distinctly specific rungs of each of the Binary Exponential Ladders; the lowest rung in each BEL($D^{\#}$) being the *defining-base-rung* $D^{\#}$ that is mapped to the corresponding odd number and each of the higher rungs being mapped to the corresponding even number.

Note that there is a strict and complete ordering relation $H_{k-1} < H_k < H_{k+1}$ between the *adjacent terms in the sequence* or the *adjacent levels of the hierarchy* or the *adjacent tier levels*, because of the above-mentioned successor predecessor relationships between them, and a clear *idempotent element* H_0 which is its own successor; completing the preparations for the application of the Dedekind-Peano Axioms in the CHSUK Theorem presented below.

The (1) strict and complete ordering in cBELnet H with a single unique root node H_0; with the above mentioned (2) successor predecessor relationships between the BELs that form the elements of H_k at tier level k and those of H_{k-1} at tier level k-1 in the hierarchy; and the fact that (3) there is an exact one-to-one correspondence between the set of positive integers and the set of all rungs in all the BELs; along with the fact that (4) the cBELnet H is merely a well-organized network of the entire set of all the BELs; imply that - that cBELnet H represents an ***arborescence*** of BELs, wherein there is *exactly one single unique directed path (chain)* with no forking but merging only at the deeper BEL level, from *every positive integer to the trivial cycle* that is *at the defining base rung of* BEL(1).

11. COLLATZ-HASSE-SYRACUSE-ULAM-KAKUTANI (CHSUK) THEOREM

STATEMENT OF THE CHSUK THEOREM

The CHSUK Sequence converges to the trivial cycle {(4->2->1)}.

PROOF

We show that cBELnet, the *Structured System Framework* H by its very design, satisfies the Dedekind-Peano's axioms (replacing the 'successor' by the 'predecessor') and therefore H is isomorphic with the set of natural numbers; and satisfies the above stated convergence statement.

DEDEKIND-PEANO AXIOM : Existence of 0.

$H_0 \in H$.

The trivial cycle {(4->2->1)} is contained in $H_0 \in H$.

DEDEKIND-PEANO AXIOM : Existence of a *successor function*.

By the very design of F, for every positive integer k,

$H_k \in H$ is the *predecessor* of $H_{k-1} \in H$.
Application of the Collatz Function with the input from numbers in H_k yield the output contained in H_{k-1}.

DEDEKIND-PEANO AXIOM : 0 is not a successor.
H_0 is its own predecessor. However, there *does not exist any* $H_k \in H$, $k \neq 0$; that is distinct from H_0; with $H_k \neq H_0$; such that H_0 is the predecessor of H_k.
Once the Collatz Sequence reaches the trivial cycle (sink) there is no exit from it.

DEDEKIND-PEANO AXIOM : Successor function is a unique one-to-one mapping.
If H_u is the predecessor of H_v and also H_u is the predecessor of H_w;
then it necessarily implies $H_v = H_w$ by the very design of H;
and,
If H_v is the predecessor of H_u and also H_w is the predecessor of H_u;
then it necessarily implies $H_v = H_w$ by the very design of H;
This is because the predecessor relation in H is a unique one-to-one mapping (bijection).
Also, note that for each positive integer k there corresponds a unique set $H_k \in H$, and for each $H_k \in H$ there corresponds a unique positive integer k; thus, establishing a one-to-one mapping (bijection) between the set of positive integers and cBELnet, the *Structured System Framework* H.

This guarantees the Compact Collatz Sequence to be a linear directed path (chain) with no forking or merging in H (although merging is observed deeper at the level of the BELs) and the path traces through . . . H_{k+1} onto H_k onto H_{k-1} . . . . etc in that order, each of these terms in H forming a hypernode in the CHSUK-Sequence.

DEDEKIND-PEANO AXIOM : Principle of induction.
Collatz Sequence starting with numbers from H_0 converge in the trivial cycle that is contained in H_0.
Collatz Sequence starting with any positive number that passes through H_k must necessarily pass through H_{k-1} because by design $H_{k-1} := S(H_k)$.

Therefore, the Collatz Sequence starting with any positive integer being contained in some $H_k \in H$, $k \geq 0$; must necessarily reach H_0 and therefore converge in the trivial cycle.

Thus, we establish a direct isomorphism between the *Structured System Framework* H and the set of Natural Numbers N; and the proof of convergence of the Collatz Sequence is an immediate consequence of this isomorphism according to the property of induction as mentioned above.

END OF PROOF

12. SOME EXPLICIT FORMS FOR THE BEL-NEIGHBORHOOD

We can perform some simple algebraic manipulation to get the parametric relation [Eqn.10] that gives a generic form for the set H_k that is the set of k^{th} predecessors of H_0 = {BEL(1)}; that is, the set H_k corresponds to the set of tier-k level of the hierarchy with the set of Binary-Exponential-Ladders {BEL(m)} each with its defining-base-rung m being a positive odd number m>1.

$$m = [2^z - \{3^0.2^{z0} + 3^1.2^{z1} + 3^2.2^{z2} + \ldots + 3^{k-1}.2^{z_{k-1}}\}] / 3^k \qquad \text{[Eqn.10]}$$

wherein k>0 is the tier-level whereas z>0 and the k-tuple (z_0, z_1, z_2, . . . z_{k-1}) form the set of non-negative integer exponents in [Eqn.10] each of which takes a unique value corresponding to each positive odd number m>1. That is, each positive odd number m>1 can be considered to be defined by the corresponding unique set of these parameters. Here the set of values for the k-tuple (z_0, z_1, z_2, . . . z_{k-1}) are of decreasing values all less than z; (z_k := 0; z_{k-1} = 0 for positive odd number m>1).

Now, define p_0 := (z - z_0); p_j := (z_{j-1} - z_j); where p_j corresponds to the number of rungs in BEL{H_j} above the *defining-base-rung* of BEL{H_j} for the node located in BEL{H_j} that the Collatz sequence/trajectory passes through; BEL{H_j} being the *Binary-Exponential-Ladder* at tier-j with j=0,1,2, . . . k. Thus, we may as well redefine the set of (k+1) parameters as a tuple (P_k) := (p_0, p_1, p_2, . . . p_k) the set of (k+1) CHSUK generative parameters that generate each positive integer n as per the parametric relation [Eqn.10] given above (p_k = 0 for positive odd number m).

For any k>0, the above set of exponents z, z_0, z_1, z_2, z_3, . . . z_k, can be redefined in terms of the newly defined CHSUK generative parameters, by rewriting the above definition as z := (z_0 + p_0); z_{j-1} := (z_j + p_j); z_k := 0; p_k = 0 for positive odd number m.

Table-1 : Some typical CHSUK generative parameter tuples

p0	p1	p2	p3	p4	p5	p6		n		z	z0	z1	z2	z3	z4	z5	z6
0								1		2	0						
1								2									
2								4									
3								8									
4								16									
4	0							5		4	0	0					
4	1	0						3		5	1	0	0				
4	3	0						13		7	3	0	0				
4	5	0						85		9	5	0	0				
4	3	2	0					17		9	5	2	0	0			
4	3	2	1	0				11		10	6	3	1	0	0		
4	3	2	1	1	0			7		11	7	4	2	1	0	0	
4	3	2	1	1	2	0		9		13	9	6	4	3	2	0	0

Table-1 : Some typical CHSUK generative parameter tuples

Table-1 gives some of the possible set of valid CHSUK generative parameters and therefore the corresponding valid values of the exponents in [Eqn.10] above along with the resultant $n(P_k) := n(p_0, p_1, p_2, \ldots p_k)$ values. Note that the set of valid values for the CHSUK generative parameters and therefore for the exponents in [Eqn.10] above are governed by certain rules as can be seen from the earlier observations above, regarding the matching relationship between the $((D^{\#})MOD3)$ of the predecessor and the $((U^{\#})MOD3)$ of the successor in the CHSUK Sequence.

13. A CHALLENGE TO MY COOL-HEADED BRAVE-HEARTS

If you can prove that corresponding to every positive odd number m>1 there exists a unique valid set of CHSUK generative parameters $\{p_0, p_1, p_2, \ldots p_k\}$ and therefore the corresponding valid set of exponents $\{z, z_0, z_1, z_2, z_3, \ldots z_{k-1}, z_k\}$ in the parametric equation [Eqn.10] given above that generates every positive odd number m>1, then you can directly prove the CHSUK Conjecture establishing the convergence of the CHSUK Sequence to the trivial cycle {(4->2->1)}.

14. CONCLUSION

We have presented a meticulously designed *structured system framework* of *Binary-Exponential-Ladders* and established its isomorphism with the set of positive integers, that directly leads to a simple and elegant proof of the convergence of the CHSUK Sequence.

15. RECOMMENDED READING

[1]. Wikipedia Page – https://en.wikipedia.org/wiki/Collatz_conjecture

[2]. Jeffrey C Lagarias;
"The 3x+1 problem: An annotated bibliography (1963--1999) (sorted by author)";
https://arxiv.org/abs/math/0309224

[3]. Jeffrey C Lagarias;
"The 3x+1 Problem: An Annotated Bibliography, II (2000-2009)";
https://arxiv.org/abs/math/0608208

[4]. Jeffrey C Lagarias;
"The 3x + 1 Problem : An Overview"
https://arxiv.org/abs/2111.02635

[5]. Halemane, K. P. (2014);
"Unbelievable O($L^{1.5}$) worst case computational complexity achieved by *spdspds* algorithm for linear programming problem";
https://arxiv.org/abs/1405.6902 (2025).

[6]. Halemane, K. P. (2025);
"Monty-Hall Theorem";
https://engrxiv.org/preprint/view/5594

15. ACKNOWLEDGEMENT

I acknowledge the fact that the most revered Number-Theory Expert Paul Erdos once said about the Collatz Conjecture - "Mathematics is not yet ready for such problems" as quoted by Jeffrey Lagarias [4].

I must necessarily confess here that the *core idea behind this analysis is so stunningly & elusively simple*, that one may simply be taken aback in a profound wonder-struck jaw-drop-silence, maybe with an after-thought: "*oh my goodness, how could it be that it never flashed on me any time earlier*"! as was also the case in earlier research reports [5]&[6].

16. DEDICATION

To my ಅಜ್ಜ(ajja) Karinja Halemane Keshava Bhat & ಅಜ್ಜಿ(ajji) Thirumaleshwari, ಅಪ್ಪ(appa) Shama Bhat & ಅಮ್ಮ(amma) Thirumaleshwari, for their *teachings through love*, that *quality matters more than quantity*; to my wife Vijayalakshmi for her *ever consistent love & support*; to my daughter Sriwidya.Bharati and my twin sons Sriwidya.Ramana & Sriwidya.Prawina for their *love & affection*.

To all the *cool-headed brave-hearts*, eagerly awaited but probably yet to be visible among the world professionals, especially the *Subject-Matter-Experts*, who would be attracted to and certainly capable of effectively understanding without any prejudice and appreciating the deeper insights enshrined in this research report.

ॐतत्सत्