COLLATZ-HASSE-SYRACUSE-ULAM-KAKUTANI SEQUENCE
:
CONVERGENCE TO THE TRIVIAL CYCLE PROVED

©: Dr.(Prof.) Keshava Prasad Halemane,
Professor - retired from
Department of Mathematical And Computational Sciences,
National Institute of Technology Karnataka Surathkal,
Srinivasnagar, Mangaluru - 575025, India.
https://orcid.org/0000-0003-3483-3521
SASHESHA, 8-129/12 Sowjanya Road, Naigara Hills,
Bikarnakatte, Kulshekar Post, Mangaluru-575005. Karnataka State, India
https://www.linkedin.com/in/keshavaprasadahalemane/

ABSTRACT

This research report presents the *Collatz-Hasse-Syracuse-Ulam-Kakutani* (CHSUK) Theorem, which asserts the convergence of the *Collatz Sequence* to the trivial cycle, thus proving the *Collatz Conjecture*, which has been a long-standing *unsolved problem*. The proof is based on the *bijective isomorphism* established between the set of positive integers and a carefully designed *system* with a hierarchy (arborescence) of *binary-exponential-ladder*s defined on the set of positive odd numbers.

Keywords: Collatz-Hasse-Syracuse-Ulam-Kakutani (CHSUK) Sequence;
Collatz-Hasse-Syracuse-Ulam-Kakutani (CHSUK) Conjecture;
Convergence; Bijection; Isomorphism; Dedekind-Peano Axioms;
CHSUK-Theorem; CHSUK-Generative-Parameters;
Binary-Exponential-Ladder.

AMS MSC Mathematics Subject Classification: 11B50.

1. INTRODUCTION

The *Collatz-Hasse-Syracuse-Ulam-Kakutani Conjecture* simply referred to as the *Collatz Conjecture* [1] states that the Collatz-Hasse-Syracuse-Ulam-Kakutani (CHSUK) Sequence (also referred to as the *Collatz Sequence*) converges to the trivial cycle {(4->2->1)}, starting from any positive integer.

Lagarias [3]&[4] gives an exhaustive annotated bibliography, whereas Lagarias [5]&[6] gives an elaborate overview of the Collatz Problem, also referred to as the 3x+1 problem, referring to it as "The Ultimate Challenge". Guy [2] has been

somewhat pessimistic in advising researchers "Don't Try to Solve These Problems". Lagarius [5]&[6] refers to the statement by the most revered Mathematician and Number Theory Expert Paul Erdos, who said that - "Mathematics is not yet ready for such problems" - about the Collatz Conjecture.

We will not get into any more of the reported literature except the problem definition; but will simply focus on a rather shockingly simple approach to solve this amazing long standing unsolved problem. This research report presents the *Collatz-Hasse-Syracuse-Ulam-Kakutani* (CHSUK) Theorem, asserting the convergence of the Collatz Sequence to the trivial cycle. The proof is by an application of the most fundamental *Dedekind-Peano Axioms* and *Modulus Arithmetic* to a meticulously designed *Structured System Framework* of *Binary-Exponential-Ladder*s defined on the set of positive odd numbers, and establishing a *bijective isomorphism* between the set of natural numbers and the *structured system framework* of *Binary Exponential Ladder*s.

2. PROBLEM DESCRIPTION

We define the *Collatz Function* C(n) with a positive integer n as its input argument, in terms of a 'pull-Down' operator D(n) and a 'push-Up' operator U(n) as follows:

> if (n is even) $C(n) := D(n) = (n / 2)$; else $C(n) := U(n) = (3*n + 1)$;   [Eqn.1]

where the '*pull-Down*' D(n) operator takes only an even number as its input argument whereas the '*push-Up*' operator U(n) takes only an odd number as its input argument and gives an output that is an even number.

For convenience in our study of the Collatz Sequence, we define the *Compact Collatz Function* T(m) by the repeated application of the 'pull-down' operator D(m) wherever applicable, say, ($p \geq 1$) times, that is, $D^*(m) := D^p(m)$ so as to get an output $D^\#(m)$ that is an odd number:

> if (m is even) $T(m) := D^*(m) := D^p(m) = (m / 2^p) := D^\#(m)$;
> else (m is odd) $T(m) := U(m) = (3 * m + 1)   := U^\#(m)$;   [Eqn.2]

where $D^\#(m)$ is called the "D-floor number" associated with the input argument m; and $U^\#(m)$ is called the "U-ceiling number" associated with the input argument m.

The *Compact Collatz Function* T(m) may as well be considered to have been redefined with these newly introduced two operators, the "D-floor operator" $D^\#(m)$ and the "U-ceiling operator" $U^\#(m)$ as given in [Eqn.2] above.

This new definition for the *Compact Collatz Function* T(m) facilitates our study of the corresponding *Compact Collatz Sequence*; which is no different from its equivalent standard Collatz Sequence, once we understand that the repeated

application, say, (p≥1) times, of the 'pull-Down' operator D(m) has now been collapsed into an equivalent single "D-floor operator" $D^{\#}(m)$ giving the D-floor number $D^{\#}(m)$ as its output. The push-Up operator U has been simply redefined as the "U-ceiling operator" $U^{\#}$ for uniformity and elegant completeness.

The *Compact Collatz Sequence* is obtained by the repeated sequential application of the *Compact Collatz Function* T(m) starting with the given initial input number m - represented by an alternating series of $D^{\#}$ number and $U^{\#}$ number - except possibly the starting initial 'seed' number m and the final terminating number, which as per the Collatz Conjecture, is anyway a $D^{\#}$ number that is unity.

3. OBSERVATIONS ON THE PULL-DOWN OPERATOR

The pull-Down operator D *always* takes only an even number n as its input argument. Every application of this pull-down operator results in an alternating (toggling) effect on the nMOD3 property of the input argument number; that is, a 1MOD3 input gives a 2MOD3 output and a 2MOD3 input gives a 1MOD3 output; whereas a 0MOD3 input gives a 0MOD3 output. Repeated application of D, in case applicable, results in a final output that is an odd number and therefore becomes an input for the push-Up operator. In such a case, we call it a "D-floor operator" $D^{\#}$ as defined in [Eqn.2] above, and its output a "D-floor number" $D^{\#}(n)$ characterized by being a odd number; $D^{\#}(n)$ may be in any one of the three possible types: (1) a 1MOD6 odd number, being a 1MOD3 odd number that is of the type (6m-5); (2) a 5MOD6 odd number, being a 2MOD3 odd number that is of the type (6m-1); (3) a 3MOD6 odd number, being a 0MOD3 odd number that is of the type (6m-3).

4. OBSERVATIONS ON THE PUSH-UP OPERATOR

The push-Up operator U *always* takes only an odd number m as its input argument, and *always* gives an output that is a 4MOD6 number, being a 1MOD3 even number that is of the type (6m-2); irrespective of whether the input is a 1MOD6 odd number or a 3MOD6 odd number or a 5MOD6 odd number. Note that one single application of the 'push-Up' operator U transforms any input odd number m into a 4MOD6 even number that becomes an input to the "D-floor operator $D^{\#}$". That is why we may as well consider the push-Up operator U as the "U-ceiling operator" $U^{\#}$ as defined in [Eqn.2] above.

5. OBSERVATIONS ON THE COMPACT COLLATZ FUNCTION

Start with any positive integer. (1) If the starting initial number n is even, then we apply the D-floor operator $D^{\#}$ operator giving an output that is the D-floor number $D^{\#}(n)$ which is given as input to the U-ceiling operator. Of course, if the starting number is a power of 2 we terminate at unity. Else, we have a D-floor number

$D^{\#}(n)$ that is an odd number greater than unity, in any non-trivial case; as the initial $D^{\#}$ node in the Compact Collatz Sequence. (2) If on the other hand the starting initial seed number n is an odd number, we treat that itself as the initial $D^{\#}$ node in the Compact Collatz Sequence.

Having thus obtained the initial $D^{\#}$ node in the Compact Collatz Sequence, we apply the U-ceiling operator $U^{\#}$ to get the U-ceiling number $U^{\#}$ that is a 4MOD6 even number. That in turn is given as input to the D-floor operator $D^{\#}$. Now the process continues.

Note that the *Compact Collatz Sequence* can therefore be defined by a *trajectory* generated by an *alternating sequence* of a "D-floor number" $D^{\#}$ and a "U-ceiling number" $U^{\#}$, *with its starting initial node being a $D^{\#}$ number*. The Compact Collatz Function as presented in [Eqn.2] defines the unique link (directed arc) from any given D-floor number $D^{\#}$ as the predecessor node to its corresponding unique U-ceiling number $U^{\#}$ as the successor node and also the unique link (directed arc) from any given U-ceiling number $U^{\#}$ as the predecessor node to its corresponding unique D-floor number $D^{\#}$ as the successor node. The unique link (directed arc) from a starting initial even "seed" number leading to the first node (D-floor number $D^{\#}$) in the *trajectory* is similarly defined.

As mentioned earlier, the application of the D-floor operator $D^{\#}$ on a U-ceiling number $U^{\#}$ that is a 4MOD6 even number of the form (6m-2) can lead to a D-floor number $D^{\#}$ that is an odd number that can be either: (1) a 1MOD6 odd number, being a 1MOD3 odd number that is of the type (6m-5); (2) a 5MOD6 odd number, being a 2MOD3 odd number that is of the type (6m-1); but can never be (3) a 3MOD6 odd number, that is a 0MOD3 odd number of the type (6m-3). The only exception, when the D-floor operator $D^{\#}$ gives an output D-floor number $D^{\#}$ that is a 3MOD6 odd number of the type (6m-3) is the situation when its input is a 0MOD6 even number, which is impossible for any U-ceiling number $U^{\#}$; although such an input may come in those special cases wherein the starting initial 'seed' number itself is a 0MOD6 even number that is of the form $(6m-3).2^{p}$ leading to an output $D^{\#}$ that is a 3MOD6 odd number of the form (6m-3).

6. ANALYSIS OF THE COMPACT COLLATZ SEQUENCE

From the above observations, it is clear that corresponding to every positive integer n as the starting initial 'seed' number, there is a starting initial node in the trajectory representing the *Compact Collatz Sequence*, that is a $D^{\#}$ number in exactly one of the three possible forms as mentioned above - that can be an input argument to the U-ceiling operator $U^{\#}$ giving exactly one unique output $U^{\#}$ which itself can be an input to the D-floor operator $D^{\#}$ so that the process continues. Successive application of each of these two operators ($U^{\#}$ and $D^{\#}$) wherever applicable, traces a unique *trajectory*, wherein each node is represented by the

unique output number of the appropriate operation applied to the input number represented by the preceding node in the trajectory.

The anticipated terminating trivial cycle {(4->2->1)} can be obtained only through a final application of the D-floor operator $D^{\#}$ on a 4MOD6 even number that is of the form (6m-2).

7. BINARY-EXPONENTIAL-LADDER WITH ITS DEFINING-BASE-RUNG $D^{\#}$

Here, we present a meticulously designed *Structured System Framework* that *partition*s the *set of positive integers* to facilitate a *general systems analysis* of the *Compact Collatz Sequence*.

Let every positive odd number be associated with a *Binary-Exponential-Ladder*, denoted by BEL(2m-1) and defined as a sequence $\{(2m-1).2^{u} \mid (u \geq 0)\}$; with its *defining-base-rung* (u=0) given by the odd number (2m-1).

$$BEL(2m-1) := \{(2m-1).2^{u} \mid (u \geq 0)\}; \qquad [Eqn.3]$$

Thus, we establish an exact one-to-one correspondence between the *set of positive odd numbers* that form the $D^{\#}$ value for the *defining-base-rung* and the corresponding *Binary Exponential Ladder* BEL($D^{\#}$).

Every *positive even number* in the form $\{(2m-1).2^{u} \mid (u>0)\}$; for which there exists its corresponding $D^{\#}$ value, $D^{\#}((2m-1).2^{u}) = (2m-1)$; for which there exists exactly one corresponding *Binary-Exponential-Ladder* BEL(2m-1) that contains the given even number $(2m-1).2^{u}$ as B((2m-1),u) as one of the higher rungs in BEL(2m-1).

$$B((2m-1),u) := \{(2m-1).2^{u} \mid (u > 0)\}; \qquad [Eqn.4]$$

Thus, we establish that *the set of all Binary-Exponential-Ladders form a partition of the set of all positive integers*; with an *exact one-to-one correspondence* between each positive odd number $D^{\#}$ and the corresponding *Binary-Exponential-Ladder* for which it is the *defining-base-rung* $D^{\#}$; whereas each of the positive even numbers correspond to exactly one of the higher rungs of a specific Binary-Exponential-Ladder identified by the D-floor number $D^{\#}$ associated with that given positive even number.

$$B((2m-1),u) := \{(2m-1).2^{u} \mid (u \geq 0)\}; \qquad [Eqn.5]$$

This partitioned framework of positive integers goes another step deeper because of the fact that the *defining-base-rung* $D^{\#}$ of a Binary-Exponential-Ladder BEL($D^{\#}$) can itself be in one of the three possible forms 1MOD6 or 5MOD6 or 3MOD6; whereas all the upper rungs of the Binary-Exponential-Ladder are either (1) alternatingly 2MOD6 and 4MOD6 or (2) all being 0MOD6 numbers.

The Collatz Conjecture states that every Collatz Sequence, starting from any positive integer, converges to the trivial cycle {(4->2->1)} which is in BEL(1) that is uniquely identified by its defining-base-rung $D^\#$ value that is unity. Therefore, our focus will be the Binary-Exponential-Ladders BEL(1) and its relationship with every other Binary-Exponential-Ladder BEL($D^\#$).

As seen above, $D^\#$ can be (1) either a 1MOD6 number of the form (6m-5); (2) or a 5MOD6 number of the form (6m-1); (3) or a 3MOD6 number of the form (6m-3). BEL(6m-5) contains the output of $U^\#$ at $(6m-5)2^W$ with w being an even exponent that is of the form (2k) wherein the input of $U^\#$ is given by $[\{(6m-5).2^W -1\}/3]$. BEL(6m-1) contains the output of $U^\#$ at $(6m-1)2^V$ with v being an odd exponent that is of the form (2k-1) wherein the input of $U^\#$ is given by $[\{(6m-1).2^V -1\}/3]$. However, BEL(6m-3) cannot contain any such output of the U-ceiling operator $U^\#$ irrespective of any input argument.

8. IMMEDIATE NEIGHBORHOOD OF A BINARY-EXPONENTIAL-LADDER

The relationship between a pair of Binary-Exponential-Ladders BEL(m) and BEL(n) can be considered to be defined and characterized by the relationship between the corresponding pair of the *defining-base-rung* $D^\#$ values m and n along with the corresponding pair $U^\#(m)$ and $U^\#(n)$.

The immediate-neighborhood of a given Binary-Exponential-Ladder BEL($D^\#$) is defined by the *immediate-predecessors* and *immediate-successors*, considering the U-ceiling operator $U^\#$; since the D-floor operator $D^\#$ is applicable only within a given Binary-exponential-Ladder and not between a pair of them.

8.1 SINGLE UNIQUE IMMEDIATE SUCCESSOR

It turns out that the only *one single unique immediate successor* of BEL(m) is BEL($D^\#(U^\#(m))$) that contains $U^\#(m)$ as one of its higher rungs, with $n := D^\#(U^\#(m))$ as its identifying characteristic D-floor number being its defining-base-rung.

$$S(BEL(m)) = BEL(D^\#(U^\#(m))) := BEL(n); \text{ with } n := D^\#(U^\#(m)); \qquad [Eqn.6]$$

8.2 MULTIPLE IMMEDIATE PREDECESSORS

There exists a *set of immediate-predecessors* for each BEL($D^\#$) of the form BEL(6m-5) and BEL(6m-1) although none for BEL(6m-3). Note that if S(BEL(m)) is BEL(n) than BEL(m) is one of the predecessors of BEL(n).

The *set of immediate-predecessors* for a given BEL(n) is defined by considering the *inverse of the immediate-successor relationship* - as the set of all BELs each of which having its single unique immediate-successor as BEL(n).

$$\{P(BEL(n))\} := \{BEL(m) \mid BEL(n) = S(BEL(m)\}; \qquad [Eqn.7]$$

BEL(1MOD6) or equivalently BEL(6m-5) has, as its set of immediate-predecessors, $\{BEL([(1MOD6).2^W - 1]/3)\}$ or equivalently $\{BEL([(6m-5).2^W - 1]/3)\}$ with w being an positive even exponent of the form (2k), wherein the input of $U^\#$ is given by $[\{(1MOD6).2^W -1\}/3]$ or equivalently $[\{(6m-5).2^W -1\}/3]$ and the output of $U^\#$ being $\{(1MOD6).2^W\}$ or equivalently $\{(6m-5).2^W\}$ that is contained in BEL(1MOD6). Each of the three possible classes of BEL, namely, BEL(1MOD6) and BEL(5MOD6) and BEL(3MOD6) can be the immediate-predecessor of BEL(1MOD6).

$$\{P(BEL(6m-5))\} = \{BEL([(6m-5).2^W - 1]/3)\} \qquad [Eqn.8]$$

BEL(5MOD6) or equivalently BEL(6m-1) has, its set of immediate-predecessors, $\{BEL([(5MOD6).2^V - 1]/3)\}$ or equivalently $\{BEL([(6m-1).2^V - 1]/3)\}$ with v being a positive odd exponent of the form (2k-1), wherein the input of $U^\#$ is given by $[\{(5MOD6).2^V -1\}/3]$ or equivalently $[\{(6m-1).2^V -1\}/3]$ and the output of $U^\#$ being $\{(5MOD6).2^V\}$ or equivalently $\{(6m-1).2^V\}$ that is contained in BEL(5MOD6) or equivalently BEL(6m-1). Each of the three possible classes of BEL, namely, BEL(1MOD6) and BEL(5MOD6) and BEL(3MOD6) can be the immediate-predecessor of BEL(5MOD6).

$$\{P(BEL(6m-1))\} = \{BEL([(6m-1).2^V - 1]/3)\} \qquad [Eqn.9]$$

BEL(3MOD6) or equivalently BEL(6m-3) has no immediate-predecessors.

8.3 QUARTERNARY-EXPONENTIAL-LADDER

The above property, that *only* the alternating rungs, defined by $(1MOD6).4^u$ of BEL(1MOD6) or $(5MOD6).2.4^u$ of BEL(5MOD6) are the 'active' nodes in the CHSUK-Sequence; makes it convenient to define a system of *Quarternary-Exponential-Ladder*s QEL wherein every rung of QEL becomes an 'active' node in the CHSUK-Sequence. This concept is not directly needed for proving the convergence of the Collatz Sequence, and therefore we will not take up this line of study in this research report.

8.4 BEL(1) AS THE CENTRAL FOCUS

Considering BEL(1) as our central focus of interest, which itself belongs to the type BEL(1MOD6) or equivalently BEL(6m-5); it is interesting to note that it has its *single unique immediate-successor* - as $S(BEL(1)) = BEL(D^\#(U^\#(1))) = S(BEL(1))$ - that is, BEL(1) itself is its single unique immediate-successor, and that it has no other immediate-successor distinct from itself; because of the fact that the trivial cycle $\{(4->2->1)\}$ contained within BEL(1).

8.5. BELnet : NETWORK OF BINARY-EXPONENTIAL-LADDERS

The above discussion about the successor predecessor relationship among the binary-exponential-ladders and its neighborhood leads to the observation that the

network of binary-exponential-ladders, **BELnet**, has countably infinite number of each of the three classes/types of nodes: (1) BEL(1MOD6) or equivalently BEL(6m-5); (2) BEL(5MOD6) or equivalently BEL(6m-1); and (3) BEL(3MOD6) or equivalently BEL(6m-3). Each BEL being a node of the BELnet has a single unique outward directed arc that points towards its single unique immediate-successor, specifically linking onto some higher rung. Multiple (countably infinite number of) inward directed arcs, each linked onto some specific higher rung of a given BEL, emanate from its immediate-predecessor. BEL(1) is an invariant base element or equivalently a sink node in BELnet, the network of binary exponential ladders.

The *connectedness* of the network of binary-exponential-ladders **BELnet** will be analyzed from the *design of a structured system framework* consisting of the entire set of binary-exponential-ladders, that is designed merely as a re-organized *condensation* of the very same BELnet, as presented below.

9. STRUCTURED SYSTEM FRAMEWORK H

From the above discussion we find that it is convenient for our study to consider a *Structured System Framework* H as an infinite (well-ordered) sequence of terms each of which being a set of BELs; that is, $H := \{. . . H_{k-1} < H_k < H_{k+1}. . .\}$ wherein the *well-ordering relationship* between the adjacent terms of the sequence is derived from the successor predecessor relationships among the BELs that form the member elements of these adjacent terms in the sequence.

Specifically, H_k is defined as the set formed by the unique immediate-successor of each BEL belonging to H_{k+1} and also the set of immediate-predecessors of each BEL belonging to H_{k-1}; that is,

$$H := \{. . . H_{k-1} < H_k < H_{k+1} . . .\}; \qquad \text{[Eqn.10]}$$

and

$$H_k := \{S(BEL(m)) \mid BEL(m) \in H_{k+1}\} \cup \{BEL(m) \mid S(BEL(m)) \in H_{k-1}\} \quad \text{[Eqn.11]}$$

Note that the second part of [Eqn.11] here is required to ensure that BELs of the class/type BEL(3MOD6) can be included in each term H_k since each of them have immediate-successor in H_{k-1} although none of them have any predecessors in H_{k+1}.

Now, we may as well define the predecessor relationship as the inverse of the above defined successor relationship, as –

$$H_{k-1} := S(H_k) \qquad \text{and} \qquad H_k := S(H_{k+1}) \qquad \text{[Eqn.12]}$$

and

$$P(H_{k-1}) := H_k \qquad \text{and} \qquad P(H_k) := H_{k+1} \qquad \text{[Eqn.13]}$$

The multiplicity of the *immediate-predecessor* relationship among the BELs requires that the set of all immediate-predecessors of every element of H_{k-1} form

the elements of the set H_k so as to guarantee the strict and complete ordering relation $H_{k-1} < H_k < H_{k+1}$ among these sets, in spite of only a partial ordering relationship among the BELs; and also to guarantee that the entire set of all the BELs are present in H thus making it as a *re-organized structure* for BELnet.

9.1 CLOSED CHAINS AND UNBOUNDED CHAINS AND A SINK NODE IN H

The design of the structured system framework H can *in general* allow for the existence of sink nodes (invariant base elements) and/or unbounded open chains and/or closed chains (loops). That is, the structured system framework H can in general be partitioned into three mutually disjoint and independent components,

$$H := H^s \cup H^\& \cup H^\infty \qquad\qquad [\text{Eqn.}14]$$

where (1) H^s corresponds to the set of all possible terms in H connected with sink nodes; (2) $H^\&$ corresponds to the set of all possible terms in H connected with unbounded open chains; and (3) H^∞ corresponds to the set of all possible terms in H connected with closed chains (loops). In such a situation, each of these components, H^s and $H^\&$ and H^∞ needs to satisfy the well-ordering conditions expressed above in [Eqn.10], [Eqn.11], [Eqn.12] & [Eqn.13].

Note that in this research report, our focus is only on H^s; whereas $H^\&$ and H^∞ are left out for further study/research – including the question of their possible existence itself – although those questions and the related details are indeed *inconsequential* in the study of convergence of the CHSUK Sequence, as will be evident later in this research report.

9.2 A SINK NODE H_0 IN H^s

We have observed earlier that BEL(1) itself is its single unique immediate-successor and does not have any immediate-successor distinct from itself, although it has multiple immediate-predecessors. That is, BEL(1) is an invariant-base-element or equivalently a sink node in BELnet. Therefore, the component H^s must necessarily have a term H_0 as its *invariant-base-element* or equivalently a *sink node*, that is, $H_0 := \{BEL(1)\}$; with BEL(1) being its singleton member element. That is,

$$H^s := \{H_0, H_1, H_2, . . .\}; \qquad\qquad [\text{Eqn.}15]$$

From the above discussion we observe that $H^s := \{H_0, H_1, H_2, . . .\}$ is, by its very design, an infinite (well-ordered) sequence of terms, each term being a countably infinite set of BELs with an exception that the 'root' $H_0 := \{BEL(1)\}$ is a singleton set. The set of kth immediate predecessors of BEL(1) form the set H_k at tier-k level in the hierarchy, if one wishes to consider it as a hierarchy.

The Collatz-Hasse-Syracuse-Ulam-Kakutani (CHSUK) Theorem is presented and proved below, which establishes a <mark>*bijection isomorphism*</mark> between H^s and the set of positive integers, thus proving the convergence of the CHSUK Sequence starting from any given positive integer contained in any BEL that is in H^s to the trivial cycle $\{(4->2->1)\}$ at the base of BEL(1) which itself is in H^s.

10. COLLATZ-HASSE-SYRACUSE-ULAM-KAKUTANI (CHSUK) THEOREM

STATEMENT OF THE CHSUK THEOREM

The CHSUK Sequence converges to the trivial cycle $\{(4->2->1)\}$.

PROOF
We show that H^s satisfies the Dedekind-Peano's axioms (replacing the 'successor' by the 'predecessor') and therefore H^s is isomorphic with the set of natural numbers; and satisfies the above stated convergence statement.

DEDEKIND-PEANO AXIOM : Existence of **1** as the invariant base element.
$H_0 \in H^s$. H_0 is the invariant-base-element of H^s.
The trivial cycle $\{(4->2->1)\} \in$ BEL(1) is contained in $H_0 \in H^s$.

DEDEKIND-PEANO AXIOM : Existence of a *successor function*.
By the very design of $H^s := \{H_0, H_1, H_2, \ldots\}$, for every positive integer k,
$H_k \in H^s$ is the *predecessor* of $H_{k-1} \in H^s$.
Application of the Compact Collatz Function with the input from numbers contained in some BEL that is a member of H_k yields the single unique output number contained in some immediate-successor BEL that is a member of H_{k-1}; because of the definition of the successor predecessor relationship between H_k and H_{k-1}.

DEDEKIND-PEANO AXIOM : **1** is not a successor; **1** has no predecessor; **1** is a source node in the sequence of natural numbers.
H_0 is not a predecessor to any other H_k. There *does not exist any* $H_k \in H^s$, k ≠ 0; that is distinct from H_0; with $H_k \neq H_0$; such that H_0 is the predecessor of H_k.
H_0 does not have any successor distinct from itself.
H_0 is a sink node in the sequence $H^s := \{H_0, H_1, H_2, \ldots\}$.

Once the Collatz Sequence reaches the trivial cycle (sink) there is no exit from it.

DEDEKIND-PEANO AXIOM : Successor function is a unique one-to-one mapping.
If H_u is the predecessor of H_v and also H_u is the predecessor of H_w;
then it necessarily implies $H_v = H_w$ by the very design of H^s;
and,
If H_v is the predecessor of H_u and also H_w is the predecessor of H_u;
then it necessarily implies $H_v = H_w$ by the very design of H^s;
This is because the predecessor relation in H^s is a unique one-to-one mapping.

Also, note that for each positive integer k there corresponds a unique set $H_k \in H^s$, and for each $H_k \in H^s$ there corresponds a unique positive integer k; thus, establishing a one-to-one mapping (***bijection***) between H^s and the set of positive integers.

This guarantees the Compact Collatz Sequence to be a linear directed path (chain) with no forking or merging in H^s (although merging is observed deeper at the level of the BELs) and the path traces through . . . H_{k+1} onto H_k onto H_{k-1} . . . etc in that order, wherein each of these terms in H^s correspond to a node (either a $D^\#$ node in H_{k+1} followed by a $U^\#$ node in H_k followed by a $D^\#$ node in H_k followed by a $U^\#$ node in H_{k-1} and so on) in the Compact CHSUK Sequence which is itself an alternating sequence of these $D^\#$ nodes and $U^\#$ nodes as observed earlier.

DEDEKIND-PEANO AXIOM : Principle of induction.
Collatz Sequence starting with any number from $BEL(1) \in H_0$ converges in the trivial cycle $\{(4\text{->}2\text{->}1)\} \in BEL(1) \in H_0$.
Collatz Sequence starting with any positive number that passes through H_k must necessarily pass through H_{k-1} because by design $H_{k-1} := S(H_k)$.

Therefore, the Collatz Sequence starting with any positive integer being contained in some $H_k \in H^s$, k ≥ 0; must necessarily reach H_0 and therefore converge in the trivial cycle.

Thus, we establish a direct ==bijection isomorphism== between H^s and the set of Natural Numbers N; and the proof of convergence of the Collatz Sequence is an immediate consequence of this ==bijection isomorphism==.
END OF PROOF

11. CONCERNS REGARDING EXTRANEOUS OBJECTS H[&] AND/OR H[∞] IN H

The CHSUK Theorem safeguards the CHSUK Conjecture from being affected by any concerns regarding the presence of and/or the nature of any possible *rogue elements* and/or *extraneous objects or sub-systems* that may in general be considered undesirable. However, this author claims that there cannot exist any such *rogue elements or extraneous objects* of the types $H^\&$ and/or H^∞ in the structured system framework H; esplly after having established the ***bijection between H^s and the set of natural numbers*** along with the ***isomorphism*** between them, as proved in the CHSUK Theorem presented above.

12. SOME EXPLICIT FORMS FOR THE BEL-NEIGHBORHOOD

We can perform some simple algebraic manipulation to get the parametric relation [Eqn.16] that gives a generic form for the set H_k that is the set of kth predecessors of $H_0 = \{BEL(1)\}$; that is, the

set H_k corresponds to the set of tier-k level of the hierarchy with the set of Binary-Exponential-Ladders {BEL(m)} each with its defining-base-rung m being a positive odd number m>1.

$$m = [2^z - \{3^0.2^{z0} + 3^1.2^{z1} + 3^2.2^{z2} + \ldots + 3^{k-1}.2^{zk-1}\}] / 3^k \qquad \text{[Eqn.16]}$$

wherein k>0 is the tier-level whereas z>0 and the k-tuple $(z_0, z_1, z_2, \ldots z_{k-1})$ form the set of non-negative integer exponents in [Eqn.16] each of which takes a unique value corresponding to each positive odd number m>1. That is, each positive odd number m>1 can be considered to be defined by the corresponding unique set of these parameters. Here the set of values for the k-tuple $(z_0, z_1, z_2, \ldots z_{k-1})$ are of decreasing values all less than z; ($z_k := 0$; $z_{k-1} = 0$ for positive odd number m>1).

Now, define $p_0 := (z - z_0)$; $p_j := (z_{j-1} - z_j)$; where p_j corresponds to the number of rungs in BEL{H_j} above the *defining-base-rung* of BEL{H_j} for the node located in BEL{H_j} that the Collatz sequence/trajectory passes through; BEL{H_j} being the *Binary-Exponential-Ladder* at tier-j with j=0,1,2, . . . k. Thus, we may as well redefine the set of (k+1) parameters as a tuple (GPT) := $(p_0, p_1, p_2, \ldots p_k)$ the set of (k+1) CHSUK generative parameters that generate each positive integer n as per the parametric relation [Eqn.16] given above ($p_k = 0$ for positive odd number m).

For any k>0, the above set of exponents z, $z_0, z_1, z_2, z_3, \ldots z_k$, can be redefined in terms of the newly defined CHSUK generative parameters, by rewriting the above definition as $z := (z_0 + p_0)$; $z_{j-1} := (z_j + p_j)$; $z_k := 0$; $p_k = 0$ for positive odd number m.

Table-1 : Some typical CHSUK generative parameter tuples GPT(n)																
p0	p1	p2	p3	p4	p5	p6		n	z	z0	z1	z2	z3	z4	z5	z6
0								1	2	0						
1								2								
2								4								
3								8								
4								16								
4	0							5	4	0	0					
4	1	0						3	5	1	0	0				
4	3	0						13	7	3	0	0				
4	5	0						85	9	5	0	0				
4	3	2	0					17	9	5	2	0	0			
4	3	2	1	0				11	10	6	3	1	0	0		
4	3	2	1	1	0			7	11	7	4	2	1	0	0	
4	3	2	1	1	2	0		9	13	9	6	4	3	2	0	0
Table-1 : Some typical CHSUK generative parameter tuples GPT(n)																

Table-1 gives some of the possible set of valid values for the CHSUK generative parameters and therefore the corresponding valid values of the exponents in [Eqn.16] above along with their resultant n(GPT) := $n(p_0, p_1, p_2, \ldots p_k)$ values.

13. A CHALLENGE TO MY COOL-HEADED BRAVE-HEARTS

If you can prove that corresponding to every positive odd number m>1 there exists a unique valid set of CHSUK generative parameters GPT(m) = {$p_0, p_1, p_2, \ldots p_k$} and therefore the corresponding valid set of exponents {z, $z_0, z_1, z_2, z_3, \ldots z_{k-1}, z_k$} in the parametric equation [Eqn.16] given above that generates every positive odd number m>1, then you can directly prove the CHSUK Conjecture establishing the convergence of the CHSUK Sequence to the trivial cycle {(4->2->1)}.

13.1 RESTRICTIONS ON THE CHSUK GENERATIVE PARAMETERS

Note that the set of valid values for the CHSUK generative parameters and therefore for the exponents in [Eqn.16] above, are governed by certain rules as can be seen from the earlier observations, regarding the matching relationship between the $((D^\#)MOD3)$ of the predecessor and the $((U^\#)MOD3)$ of the successor in the CHSUK Sequence.

Specifically, [Eqn.17] states the relationship satisfied among (i) the $[x]MOD3$ value of the exponent x for $U^\# = \{(6m-5).4^x\}$ at some higher rung in QEL(6m-5); (ii) with its defining-base-rung at (6m-5); and (iii) its predecessor $D^\# = [\{(6m-5).4^x - 1\}/3]$. Similarly, [Eqn.18] states the relationship satisfied among (i) the $[y]MOD3$ value of the exponent x for $U^\# = \{(6m-1).2.4^y\}$ at some higher rung in QEL(6m-1); (ii) with its defining-base-rung at (6m-1); and (iii) its predecessor $D^\# = [\{(6m-1).2.4^y - 1\}/3]$.

$$[\{(6m-5).4^x - 1\}/3]MOD3 = [[x]MOD3 - [m]MOD3 +1]MOD3; \qquad [Eqn.17]$$
and
$$[\{(6m-1).2.4^y - 1\}/3]MOD3 = [[y]MOD3 + [m]MOD3 -1]MOD3; \qquad [Eqn.18]$$

Rewriting [Eqn.17]&[Eqn.18] for the Binary-Exponential-Ladders, we get the equivalent set of equations as:

$$[\{(6m-5).2^w - 1\}/3]MOD3 = [[w/2]MOD3 - [m]MOD3 + 1]MOD3; \qquad [Eqn.19]$$
and
$$[\{(6m-1).2^v - 1\}/3]MOD3 = [[(v-1)/2]MOD3 + [m]MOD3 -1]MOD3; \qquad [Eqn.20]$$

13.2 PERFECT SYMMETRY IN THE BELnet ARBORESCENCE

From the above discussion one can notice that BELnet forms an arborescence in H^s with a perfect symmetry. {BEL(1)} stands at the center, with its trivial cycle at its defining-base-rung. At every tier-level k corresponding to the kth term H_k in the sequence H^s, Binary-Exponential-Ladders of all the three classes/types are present, each being equal in number; each having its single unique immediate-successor in H_{k-1}; each {BEL(6m-5)} and each {BEL(6m-1)} has its immediate-predecessors in H_{k+1}; whereas {BEL(6m-3)} remain as leaf-nodes since they can't have any immediate-predecessors. Thus, one-third of the BELs remain as leaf nodes; the other two-thirds become intermediate nodes that propagate the arborescence structure unboundedly to infinity.

13.3 MOST INTRIGUING AND REASSURING OBSERVATION

Suppose we move forward from H_{k-1} to H_k and then onto H_{k+1}; that is, trace backwards along the CHSUK Sequence, in the reverse direction; moving from $BEL(n_{k-1})$ to $BEL(n_k)$ and then onto $BEL(n_{k+1})$; with $BEL(n_{k-1}) = S(BEL(n_k))$ and $BEL(n_k) = S(BEL(n_{k+1}))$; every time selecting the immediate-predecessor BEL of the *lowest denomination* with the *lowest value of the defining-base-rung*, or in other

words, $BEL(n_{k-1}) \in H_{k-1}$; as represented by the lowest value of 2 for w in [Eqn.8] or the lowest value of 1 for v in [Eqn.9] above.

We find that [Eqn.8] with w=2 leads to a situation wherein $n_{k-1} < n_k$; that is,
$$BEL(6m-5) = S(BEL([(6m-5).2^2 - 1]/3)) = S(BEL(8m-7)); \qquad [Eqn.21]$$
whereas [Eqn.9] with v=1 leads to a situation wherein $n_{k-1} > n_k$; that is,
$$BEL(6m-1) = S(BEL([(6m-5).2^1 - 1]/3)) = S(BEL(4m-1)); \qquad [Eqn.22]$$

Thus, we find that moving forward from H_{k-1} to H_k and then onto H_{k+1}; every time selecting the immediate-predecessor BEL of the lowest denomination, we encounter the three distinct situations :– (1) $BEL(6m-3) \in H_{k-1}$ being the leaf node of the BELnet, does not have any predecessors; (2) $BEL(6m-5) \in H_{k-1}$ has its immediate-predecessor BEL of lowest denomination being $BEL(8m-7) \in H_k$; (3) $BEL(6m-1) \in H_{k-1}$ has its immediate-predecessor BEL of lowest denomination being $BEL(4m-1) \in H_k$.

This establishes the fact that apart from the one-third of the BELs that lead to the leaf-nodes, the remaining two-thirds of the BELs are equally distributed between the two cases - BEL(6m-5) leads to the situation wherein the arborescence grows forward with increasing values for the defining-base-rung; whereas BEL(6m-1) leads to the situation wherein the arborescence grows forward with decreasing values for the defining-base-rung, thus *reaching-down* to some positive odd number of the form (4m-1) that was *left out in the earlier stages/tier-levels* of the hierarchy.

Note that there can be only three types of positive odd numbers – odd multiples of three that are of the form (6m-3) or numbers of the form (6m-5) or numbers of the form (6m-1) – of which the odd multiples of three correspond to the defining-base-rung of the leaf nodes in the BELnet arborescence. Of the remaining two cases, it is of utmost significance that every positive odd number of the form (6m-1) forming the defining-base-rung of $BEL(6m-1) \in H_{k-1}$ having its immediate-predecessor BEL of lowest denomination that is of the form $BEL(4m-1) \in H_k$. This is how as we move forward from H_{k-1} to H_k and then onto H_{k+1}; we encounter some positive odd numbers of the form (4m-1) that *could not be reached in the earlier stages/tier-levels*; as indicated by the presence of 1 in the CHSUK generative parameter tuple. It is intriguing to observe that this specific process of reaching-down to such positive odd numbers of the form (4m-1) can indeed be a repeated contiguous operation at times, as can be seen by the presence of repeated 1's in the CHSUK generative parameter tuple.

The CHSUK generative parameter tuple corresponding to every positive odd integer of the form (4m-1) must necessarily end with the sub-sequence (..., 1, 0); as for example, GPT(11) = (4, 3, 2, 1, 0); whereas the presence of a positive odd integer of the form (4m-1) anywhere in the CHSUK Sequence is indicated by the presence of a 1 in the corresponding position in its CHSUK generative parameter tuple; as for example, GPT(9) = (4, 3, 2, 1, 1, 2, 0); with the CHSUK Sequence for

9 is 9->7->11->17->13->5->1; the two repeated occurrence of 1s corresponding to the sub-sequence 7->11->17 in the trajectory. Note that the length of the CHSUK generative parameter tuple required to generate a given positive odd number m is clearly limited to be no more than m; 42 for m=27 and 40 for m=31 being the only two exceptions. Explain why.

14. CONCLUSION

We have presented a meticulously designed *structured system framework* of *Binary-Exponential-Ladders* H merely as a well-organized condensation of the network of Binary-Exponential-Ladders BELnet. We established a *bijective isomorphism* between the set of positive integers and the relevant component of the structured system framework, H^s that is shown to be an infinite (well-ordered) sequence or a hierarchy (arborescence) of the *Binary-Exponential-Ladders* having its root at H_0; that directly leads to a simple and elegant proof of the convergence of the CHSUK Sequence. We have also presented a possible approach to prove the same result, using modulus arithmetic for conditions to be satisfied by the CHSUK generative parameters or equivalently the exponents in a closed form expression, corresponding to every positive odd number.

15. RECOMMENDED READING

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[8].   Halemane, K. P. (2025);
         "Monty-Hall Theorem";
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16. ACKNOWLEDGEMENT

I must necessarily confess here that the *core idea behind this analysis is so stunningly & elusively simple*, that one may simply be taken aback in a profound wonder-struck jaw-drop-silence, maybe with an after-thought: "*oh my goodness, how could it be that it never flashed on me any time earlier*"! as was also the case in earlier research reports [7]&[8].

17. DEDICATION

To my ಅಜ್ಜ(ajja) Karinja Halemane Keshava Bhat & ಅಜ್ಜಿ(ajji) Thirumaleshwari, ಅಪ್ಪ(appa) Shama Bhat & ಅಮ್ಮ(amma) Thirumaleshwari, for their *teachings through love*, that *quality matters more than quantity*; to my wife Vijayalakshmi for her *ever consistent love & support*; to my daughter Sriwidya.Bharati and my twin sons Sriwidya.Ramana & Sriwidya.Prawina for their *love & affection*.

Whereas *this Original Author-Creator* holds the (PIPR:©:) Perpetual Intellectual Property Rights, his legal heirs (three children mentioned above) may avail the same for perpetuity.

To all the *cool-headed brave-hearts*, eagerly awaited but probably yet to be visible among the world professionals, especially the *Subject-Matter-Experts*, who would be attracted to and certainly capable of effectively understanding without any prejudice and appreciating the deeper insights enshrined in this research report. Let me present to you a gift of mine – *a perpetual calendar in one single page* - that I designed during my 1966 summer holidays after completion of my tenth grade high school studies; which effectively exploits the modulus arithmetic relationship among the *days-of-the-week* the *months-of-the-year* and the *leap-year* cycles; it can be extended both ways, forward and backward, and its validity is guaranteed as long as the number of days of each month and the rules for reckoning the leap-year remain the same, and may only need some appropriate tweaking if-&-when those rules are changed.

KpH(@1966)Calendar

$$Y(y)=MOD7[\{y+FLOOR(y/4)-FLOOR(y/100)+FLOOR(y/400)\}]; \quad M=M(m); \quad W=W(w); \quad D(d)=MOD7(d); \quad W=MOD7(Y+M+D);$$

								SUN	MON	TUE	WED	THU	FRI	SAT
OCT6 JAN6	APR5 JUL5 5jan	SEP4 DEC4	JUN3	FEB2 MAR2 NOV2	AUG1 1feb	MAY0	>6>	0 / 0	1 / 1	2 / 2	3 / 3	4 / 4	5 / 5	6 / 6
APR5 JUL5 5jan	SEP4 DEC4	JUN3	FEB2 MAR2 NOV2	AUG1 1feb	MAY0	OCT6 JAN6	>5>	6 SAT 6	0 SUN 0	1 MON 1	2 TUE 2	3 WED 3	4 THU 4	5 FRI 5
SEP4 DEC4	JUN3	FEB2 MAR2 NOV2	AUG1 1feb	MAY0	OCT6 JAN6	APR5 JUL5 5jan	>4>	5 FRI 5	6 SAT 6	0 SUN 0	1 MON 1	2 TUE 2	3 WED 3	4 THU 4
JUN3	FEB2 MAR2 NOV2	AUG1 1feb	MAY0	OCT6 JAN6	APR5 JUL5 5jan	SEP4 DEC4	>3>	4 THU 4	5 FRI 5	6 SAT 6	0 SUN 0	1 MON 1	2 TUE 2	3 WED 3
FEB2 MAR2 NOV2	AUG1 1feb	MAY0	OCT6 JAN6	APR5 JUL5 5jan	SEP4 DEC4	JUN3	>2>	3 WED 3	4 THU 4	5 FRI 5	6 SAT 6	0 SUN 0	1 MON 1	2 TUE 2
AUG1 1feb	MAY0	OCT6 JAN6	APR5 JUL5 5jan	SEP4 DEC4	JUN3	FEB2 MAR2 NOV2	>1>	2 TUE 2	3 WED 3	4 THU 4	5 FRI 5	6 SAT 6	0 SUN 0	1 MON 1
MAY0	OCT6 JAN6	APR5 JUL5 5jan	SEP4 DEC4	JUN3	FEB2 MAR2 NOV2	AUG1 1feb	>0>	1 MON 1	2 TUE 2	3 WED 3	4 THU 4	5 FRI 5	6 SAT 6	0 SUN 0

^0^	^1^	^2^	^3^	^4^	^5^	^6^	KpH(@1966)	!!!	!!!	!!!	!!!	!!!	!!!	!!!
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1939 1967 1995 2023 2051		1940 1968 1996 2024 2052	1941 1969 1997 2025 2053	1942 1970 1998 2026 2054	1943 1971 1999 2027 2055		P R A S A D a	8	9	10	11	12	13	14
1944 1972 2000 2028 2056	1945 1973 2001 2029 2057	1946 1974 2002 2030 2058	1947 1975 2003 2031 2059		1948 1976 2004 2032 2060	1949 1977 2005 2033 2061	H a l e	15	16	17	18	19	20	21
1950 1978 2006 2034 2062	1951 1979 2007 2035 2063		1952 1980 2008 2036 2064	1953 1981 2009 2037 2065	1954 1982 2010 2038 2066	1955 1983 2011 2039 2067	m a n e	22	23	24	25	26	27	28
	1956 1984 2012 2040 2068	1957 1985 2013 2041 2069	1958 1986 2014 2042 2070	1959 1987 2015 2043 2071		1960 1988 2016 2044 2072		29	30	31				

$$Y(y)=MOD7[\{y+FLOOR(y/4)-FLOOR(y/100)+FLOOR(y/400)\}]; \quad M=M(m); \quad W=W(w); \quad D(d)=MOD7(d); \quad W=MOD7(Y+M+D);$$

KpH(@1966)Calendar