MONTY-HALL THEOREM

BAYES-PRICE RULE (BAYES THEOREM) FOR A THREE PARAMETER EVENT SPACE

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ABSTRACT

This research report presents the statement of the Monty-Hall Theorem and provides a constructive proof by solving the classical Monty-Hall Problem. It establishes the fact that the probability of winning the prize is indeed unaffected by a switched choice – very much unlike the most prevalent and widely accepted position held by the Leading Subject-Matter-Experts.

Keywords: Monty-Hall Theorem; Bayes-Price Rule; Bayes Theorem

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1. INTRODUCTION

The classical "Monty-Hall Problem", also referred to as the "Three-Door Problem" is based on a game show "Let's Make a Deal" wherein the host reveals a losing choice to the guest, who had earlier made an initial choice, and in turn offers the guest an enticing option to switch from the initial choice to a second available choice with an aim to enhance the chances of winning the prize. The most prevalent & widely accepted position, as reported in literature, among the leading Subject-Matter-Experts, mathematicians, statisticians, logicians, and rational intellectuals, is that an appropriate detailed study & analysis of the scenario using the well accepted standard approach of Probability & Statistics, would lead to a recommendation to the guest to switch to the second available choice based on the knowledge obtained from the host revealing a losing choice.

We present the statement of our newly formulated Monty-Hall Theorem, and provide a constructive proof by actually solving the classical Monty-Hall Problem. It is to establish that the probability of the guest winning the prize is indeed 1/2 irrespective of whether the guest stays with the initial choice or opts to go for a switched choice after gathering the information from the host who reveals a losing choice.

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2. PROBLEM DESCRIPTION - INPUT DATA

Let us consider the so-called classical Monty-Hall Problem as reported widely in the literature - with a prize hidden behind one of the three doors; a guest making an initial choice of a door to claim the prize; the host who knows the location of the prize as well as the initial choice made by the guest, now *reveals a distinctly different and yet a losing choice*, by opening a third door. Then the host also offers the guest, an option to switch from the initial choice to the now available second choice, anticipating an enhanced chance of winning the prize, based on the knowledge obtained about a losing choice.

Let us represent the events/actions associated with the three doors:
(1) let xr $\in$ {1,2,3} be the door r behind which the prize x is hidden;
(2) let yp $\in$ {1,2,3} be the initial choice of the door p chosen by the guest y;
(3) let zq $\in$ {1,2,3} be the door q opened by the host z to reveal a losing choice.

Let the symbol ‘ai’ denote the event/action [E{(a=i)}] for any ‘agent’ a $\in$ {x,y,z} and ‘door’ i$\in${r,p,q}={1,2,3}.

It is essential to note here that xr and yp are mutually independent of each other as well as independent of zq; whereas zq itself is dependent on both xr and yp, as per the rules of the game. Also, note that the focus is on the decision-making process & the action to be taken by the guest. So, the *problem formulation* (*modelling*) must necessarily be from the view-point of the guest.

3. ASSUMPTIONS

It is assumed that the prize is hidden randomly behind one of the three doors, each of the events [xr$\in${1,2,3}] being considered equiprobable. Also, the initial choice of the door [yp$\in${1,2,3}] chosen by the guest is also a random (blind) choice.

The host knows the door behind which the prize is hidden and also the door that is the initial choice of the guest. Therefore, the event/action of the host z opening door q, zq$\in${1,2,3} to show a losing choice, is dependent on both yp and xr, as per the rules of the game show. That is, (zq ≠ yp) & (zq ≠ xr). This dependency of zq on yp and xr does indeed limit the available options. It turns out that when yp ≠ xr the host doesn’t have any option except to turn to the one and only one remaining door zq ≠ (yp ≠ xr); whereas when yp = xr the host has the option of choosing between the two doors, that is, zq ≠ (yp = xr). Because the host has this option, at least in a restricted sense, of choosing which of the two doors to open, it introduces an uncertainty for the guest to predict/expect/anticipate the host’s decision/action in this regard. Here, it is *assumed* that whenever zq ≠ (yp = xr) the host’s choice between the two available options is indeed *equiprobable*.

Table-1 lists the 12 mutually-exclusive together-exhaustive possible alternatives for the combined-triple-event space along with the relevant a-priori probabilities.

Sl.No.	[xr]	[yp]	[xr&yp]	[zq]	[xr&yp&zq]		P[xr]	P[yp]	P[zq | (xr & yp)]	P[xr&yp&zq]
01	1	1	11	2	112		1/3	1/3	1/2	1/18
02	1	1	11	3	113		1/3	1/3	1/2	1/18
03	1	2	12	3	123		1/3	1/3	1	1/9
04	1	3	13	2	132		1/3	1/3	1	1/9
05	2	1	21	3	213		1/3	1/3	1	1/9
06	2	2	22	1	221		1/3	1/3	1/2	1/18
07	2	2	22	3	223		1/3	1/3	1/2	1/18
08	2	3	23	1	231		1/3	1/3	1	1/9
09	3	1	31	2	312		1/3	1/3	1	1/9
10	3	2	32	1	321		1/3	1/3	1	1/9
11	3	3	33	1	331		1/3	1/3	1/2	1/18
12	3	3	33	2	332		1/3	1/3	1/2	1/18

Table-1: Twelve *combined-triplet-event* possibilities along with its *joint-probabilities*.

[xr]: prize x behind door r; [yp]: guest y choses door p; [zq]: host z reveals door q

Twelve Mutually-Exclusive Together-Exhaustive Alternative-Possibilities

4. MONTY-HALL THEOREM

Given that the initial choice of the guest is, say door-1 (event [y1]); and that the host opens the door, say door-3 (event [z3]) to reveal a losing choice, that is different from the door behind which the prize is hidden, and also different from the initial choice of the guest; then the probability of the guest winning the prize is given by the *a-posteriori* (conditional to [z3]) *probability* of the prize being hidden behind the door-1 (event [x1]); that is, P[x1 | z3]. In the case of the classical Monty-Hall Problem, this value may be computed by the application of the *Bayes-Price Rule* (*Bayes Theorem*) for the case of *three parameter event(sample)space*; and it is equal to 0.50 - therefore the option of the switched choice doesn't yield any enhancement in the chances of winning the prize.

PROOF

The proof is simply by solving the problem, following the below enumerated steps. For each required value, a general expression is given first; followed by the classical case.

(1) INPUT DATA

P[x1]; P[x2]; P[x3]; P[y1]; P[y2]; P[y3];
P[z3 | x1y1]; P[z3 | x1y2]; P[z3 | x2y1]; P[z3 | x2y2];
P[z2 | x1y1]; P[z2 | x1y3]; P[z2 | x3y1]; P[z2 | x3y3];
P[z1 | x2y2]; P[z1 | x2y3]; P[z1 | x3y2]; P[z1 | x3y3];

(2) JOINT PROBABILITIES FOR INDEPENDENT EVENTS [xr & yp]

P[x1y1] = P[x1]*P[y1]; P[x1y2] = P[x1]*P[y2]; P[x1y3] = P[x1]*P[y3];
P[x2y1] = P[x2]*P[y1]; P[x2y2] = P[x2]*P[y2]; P[x2y3] = P[x2]*P[y3];
P[x3y1] = P[x3]*P[y1]; P[x3y2] = P[x3]*P[y2]; P[x3y3] = P[x3]*P[y3];

(3) VALIDITY CHECK FOR NON-ZERO A-PRIORI JOINT PROBABILITIES

Check and confirm the *validity of input data values* for application of Bayes-Price Rule (Bayes Theorem). The presence of *zero-value* for any of the *a-priori probabilities* leading to the intended conditional used to derive the required *a-posteriori* (*conditional*) *probabilities*, can result in *spurious results*. Appropriate alternative approach may be needed in such cases. For the classical Monty-Hall Problem, the *joint probabilities* listed above leading to the required conditionality of the host opening a door (say z3) will be used in the below calculations.

(4) A-PRIORI PROBABILITY FOR [z3] AS PER THE RULES OF THE GAME

P[z3] = P[z3 | x1y1]*P[x1y1] + P[z3 | x2y1]*P[x2y1] + P[z3 | x1y2]*P[x1y2] + P[z3 | x2y2]*P[x2y2];
= P[x1y1z3] + P[x1y2z3] + P[x2y1z3] + P[x2y2z3];
= 1/18 + 1/9 + 1/9 + 1/18;
= 1/3;

(5) A-PRIORI CONDITIONAL (w.r.t. x1) MARGINAL (w.r.t. yp) PROBABILITY FOR z3

P[z3 | x1] = (P[z3 | x1y1] * P[x1y1] + P[z3 | x1y2] * P[x1y2]) / (P[x1]);
= (P[z3x1y1] + P[z3x1y2]) / (P[x1]);
= (1/18 + 1/9) / (1/3);
= 1/2;

(6) A-POSTERIORI CONDITIONAL (w.r.t. z3) MARGINAL (w.r.t. yp) PROBABILITY FOR x1

P[x1 | z3] = (P[z3 | x1] * P[x1]) / (P[z3]);
= (1/2 * 1/3) / (1/3);
= 1/2;

END OF PROOF

5. DISCUSSION

It is to be noted here that the theorem and the proof uses some specific labels for the doors, just for convenience; namely, door-3 [z3] for the door opened by the host to reveal a losing choice, and door-1 [y1] for the guest's initial choice, thus leading to a decision making problem for the guest that requires the computation of the value P[x1 | z3] and its complementary value P[x2 | z3]. However, the result is neither restricted by nor dependent on these specific labels. This is evident from the symmetry in the data entries in Table-1, which shows that xr and yp are interchangeable for any given zq, and that the entries are identical for the three subsets corresponding to each of the values for zq $\in$ {1,2,3}. Or, if one wishes, one can re-write the theorem in three parts, one corresponding to each of the cases with the host opening a door zq $\in$ {1,2,3}.

Also, one can always consider a scenario wherein the three doors are exactly identical from the viewpoint of the guest, and that the initial choice of the guest is then labelled as door-1, and that the door that is opened by the host is then labelled as door-3, thus leaving the remaining door to be labelled as door-2. Therefore, it gets established that irrespective of whichever be the door opened by the host, each of the remaining two doors have equal probability of having the prize hidden behind it.

6. EARLIER ERRONEOUS RESULT

The Monty-Hall Theorem reaffirms common-sense based rational & intellectual reasoning, confirmed by the results obtained through the computations shown in the proof. Note that the Monty-Hall Problem is not a problem with possibly multiple correct solutions. Therefore, the above theorem indirectly points out the erroneous result that has been the widely accepted position by the Leading Subject Matter Experts, claiming that a switched choice has a clear advantage, with the chances of winning the prize being 2/3 as against only 1/3 for staying with the initial choice.

There seems to be various approaches adopted by the Leading Subject Matter Experts to derive the very same erroneous result – almost all centered around the a-priori probabilities of events leading to the intended conditional that is used to derive the required *a-posteriori* (*conditional*) *probabilities*. One of the most striking errors is the wrong application of the Bayes-Price Rule (Bayes Theorem) in a situation with zero value associated with a-priori probability - an issue of concern that has been clearly mentioned in step (3) in the above proof while insisting on a check for the validity of input data before further processing. Some tend to impose the conditionality of the initial choice of the guest while computing the a-posteriori probabilities and later lift that condition to recommend a switched choice, which is indeed a very serious case of a Logical Fallacy.

7. CONCLUSION

The Monty-Hall Theorem establishes the correct approach in solving the classical Monty-Hall Problem. It establishes the fact that the probability of winning the prize is indeed unaffected by a switched choice – very much unlike the most prevalent and widely accepted position held by the Leading Subject-Matter-Experts.

8. RECOMMENDED READING

[1]. Wikipedia Page –
https://en.wikipedia.org/wiki/Monty_Hall_problem

[2]. Jason Rosenhouse; “The Monty Hall Problem:
The Remarkable Story of Math’s Most Contentious Brain Teaser”;
Oxford University Press, ISBN 978-0-19-536789-8, 2009.

[3]. Jason Rosenhouse;
“Games-for-Your-Mind_History-&-Future-of-Logic-Puzzles”;
Princeton University Press, 2020.

[4]. Anthony B. Morton;
“Prize insights in probability, and one goat of a recycled error”;
Arxiv:1011.3400v2 2010.

[5]. Matthew A. Carlton;
“Pedigrees, Prizes, and Prisoners: The Misuse of Conditional Probability”;
Journal of Statistics Education Volume 13, Number 2 (2005);
ww2.amstat.org/publications/jse/v13n2/carlton.html

[6]. Richard D. Gill; “The Monty Hall Problem is not a Probability Puzzle :
It's a challenge in mathematical modelling”;
arXiv:1002.0651v4 2023.

[7]. Torsten Enßlin and Margret Westerkamp;
“The rationality of irrationality in the Monty Hall problem”;
arXiv:1804.04948v4 2018.

[8]. A.P. Flitney_, D. Abbott; “Quantum version of the Monty Hall problem”;
arXiv:quant-ph/0109035v3 2024.

[9]. Jeffrey S. Rosenthal; “Monty Hall, Monty Fall, Monty Crawl”;
probability.ca/jeff/writing_montyfall

[10]. Christopher A. Pynes; “IF MONTY HALL FALLS OR CRAWLS”;
EuJAP Vol.9, No.2, pp 33-47; 2013.

[11]. Andrew Vazsonyi, Feature Editor; “Which Door Has the Cadillac?”;
The Real-Life Adventures of a Decision Scientist – featured column
www.decisionsciences.org/DecisionLine/Vol30/30_1/vazs30_1.pdf

[12]. Halemane, K.P. (2025);
“Refutation of the Logical Fallacy Committed by the Subject Matter Experts
on the Monty-Hall Problem”;
https://engrxiv.org/preprint/view/5102