# UNBELIEVABLE $O(L^{1.5})$ WORST-CASE COMPUTATIONAL COMPLEXITY ACHIEVED BY *spdspds* ALGORITHM FOR LINEAR PROGRAMMING PROBLEM

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## ABSTRACT

The Symmetric Primal-Dual Symplex Pivot Decision Strategy (*spdspds*) is a novel iterative algorithm to solve linear programming problems.  A symplex pivoting operation is considered simply as an *exchange* between a basic (dependent) variable and a non-basic (independent) variable, in the *Goldman-Tucker Compact-Symmetric-Tableau* (CST) which is a unique symmetric representation common to both the primal as well as the dual of a linear programming problem in its standard canonical form.  From this viewpoint, the *classical simplex pivoting* operation of Dantzig may be considered as a restricted special case.

The *infeasibility index* associated with a symplex tableau is defined as the sum of the number of primal variables and the number of dual variables that are infeasible.  A *measure of goodness* as a *global effectiveness measure* of a pivot selection is defined/determined as/by the *decrease in the infeasibility index* associated with such a pivot selection.  The selection of the symplex pivot element is made by seeking the best possible anticipated *decrease in the infeasibility index* from among a wide range of candidate choices with non-zero values - limited only by considerations of *potential numerical instability*.  After passing through a non-repeating sequence of CST tableaus, the algorithm terminates when further reduction in the infeasibility index is not possible; then the tableau is checked for the *terminal tableau type* to facilitate the *problem classification* - a termination with an *infeasibility index* of *zero* indicates optimum solution.  Even in the absence of an optimum solution, the versatility of the *spdspds* algorithm allows one to explore/determine the most suitable alternative solutions, including possibly a comprehensive parametric analysis, etc. The worst-case computational complexity of the *spdspds* algorithm is shown to be $O(L^{1.5})$ where L refers to the problem-size expressed in terms of the size(length) of the input data.

Keywords:     optimization, linear programming, algorithm, simplex, symplex,
              symmetric primal dual *sym*plex, *spdspds*, computational complexity

## 1. INTRODUCTION

Linear Programming (LP) problem represents one of the most widely used class of computational models, for which any possible improved solution technique would certainly be highly desirable. Of course, there has been several alternative solution strategies suggested including the *classical simplex method* of Dantzig [1] and several variations thereof, followed by recent polynomial time algorithms, namely the Ellipsoid Method of Khachiyan [2] [3] and the Karmarkar Algorithm [4] - both classified now as belonging to Interior Point Algorithms.  Terlaky [5] Todd [6] and Adler et al [7] present a summary view of the various developments as well as a projected futuristic view of the overall situation. Let us not get diverted much into the historical developments etc.

The simplex pivoting operation of Dantzig represents a move from one basis/tableau to another basis/tableau, by/through a *single exchange* between an entering *infeasible* non-basic variable and a leaving *feasible* basic variable.  For a chosen entering *infeasible* non-basic variable, the leaving *feasible* basic variable is to be selected so as to meet certain restrictive criteria in terms of the corresponding limitation of moving only between two *neighboring extreme/vertex points* of the polytope defined by the set of linear system of inequalities - so as to maintain feasibility while improving the objective function value by moving further towards the optimum.

The proposed *spdspds* approach can be considered as a novel generalization of the simplex method of Dantzig, in terms of lifting all of such restrictions and providing a wider scope for the selection of the pivots - any nonzero element of the coefficient matrix in the tableau can be a potential candidate pivot element.  It is indeed true that the very term *simplex pivot* has been redefined here - as a simple/single *exchange* between an entering non-basic variable and a leaving basic variable - maintaining only the combinatorial/structural property of being a simple/single *exchange* between a selected pair - that being the justification for renaming it as *simplex pivot* - emphasizing the primal-dual symmetry therein.  The symplex pivot of *spdspds* need not necessarily correspond to a pair consisting of an entering *infeasible* non-basic variable and a leaving *feasible* basic variable; also it does not require to be limited to a move between neighboring extreme/vertex points of the associated polytope - although the move does indeed correspond to one between a pair of *intersection points* defined by the set of linear system of inequalities.

The actual selection of a *spdspds* pivot element is governed by an analysis of the associated measure of goodness of such a pivot choice.  A global measure of goodness or a *global effectiveness measure* (*gem*) for pivot selection is defined, utilizing the novel concept of *infeasibility index* associated with a symplex tableau - defined as the sum of the number of primal variables and the number of dual variables that are infeasible.  The change in the infeasibility index associated with a symplex pivot element can be determined by a thorough analysis of the tableau data.  To guarantee the best computational performance, it is proposed to select a pivot element corresponding to the best possible decrease in the infeasibility index.

It will be shown that the spdspds algorithm passes through a non-repeating sequence of CST tableaus and reaches a terminal tableau wherefrom no further reduction in the infeasibility index is possible.  The length of such sequence is limited by the infeasibility index of the initial tableau, except for cases of degeneracies that may possibly cause certain elongated sequence.  An analysis of the data pattern in the *terminal tableau* can be used to classify the problem into one of the possible *six categories* - an *infeasibility index* of *zero* indicates optimum solution.  Even in the absence of an optimum solution, the *spdspds* algorithm allows one to explore/determine the most suitable alternative solutions, including a comprehensive parametric analysis, etc.

## 2. GOLDMAN-TUCKER COMPACT-*SYM*METRIC-TABLEAU

We will go through some well-known preliminaries for the sake of establishing the notational conventions used in this report, as used in our earlier reports [8] and [9].

The Symmetric Primal-Dual Pair of LP in the Standard Canonical Form [10] is as follows:

Primal Problem:

$$
\begin{array}{lllll}
\text{maximize} & c.x & = & f & \\
\text{s.t.} & A.x & \leq & b & \quad\quad (1) \\
& x & \geq & 0 &
\end{array}
$$

Dual Problem:

$$
\begin{array}{lllll}
\text{minimize} & v.b & = & g & \\
\text{s.t.} & v.A & \geq & c & \quad\quad (2) \\
& v & \geq & 0 &
\end{array}
$$

The descriptions for each of the problem parameters in (1) & (2) above are as follows:

| | | |
|---|---|---|
| x | Primal decision variables | n x 1 vector |
| c | Primal objective function coefficients | 1 x n vector |
| f | Primal objective function value | 1 x 1 scalar |
| A | Primal constraint coefficient matrix | m x n matrix |
| b | Primal constraint upper bound | m x 1 vector |
| v | Dual decision variables | 1 x m vector |
| g | Dual objective function value | 1 x 1 scalar |

We introduce the m x l vector y of slack variables to (1) and the 1 x n vector u of surplus variables to (2) to write the symmetric primal-dual pair in canonical form as follows:

Primal Problem:

$$
\begin{array}{llll}
\text{maximize} & c.x + 0.y & = & f \\
\text{s.t.} & A.x + I_m.y & = & b \quad\quad (3) \\
& x, y & \geq & 0
\end{array}
$$

Dual Problem:

$$
\begin{array}{llll}
\text{minimize} & v.b + u.0 & = & g \\
\text{s.t.} & v.A - u.I_n & = & c \quad\quad (4) \\
& v, u & \geq & 0
\end{array}
$$

This Symmetric Primal-Dual pair is represented in the Goldman-Tucker Compact *Sym*metric Tableau (CST) as shown in Figure-1.

$$
\begin{array}{c|c|c|}
 & x_j & -1 \\
\hline
v_i & a_{ij} & b_i \\
\hline
-1 & c_j & 0 \\
\hline
\end{array}
\quad
\begin{array}{l}
= -y_i \\[2em]
= f
\end{array}
$$

$$
\begin{array}{cc}
|| & || \\
u_j & g
\end{array}
$$

**Figure-1: Initial Compact *Sym*metric Tableau (T₀)**

The relations in (3) & (4) corresponding to the Primal and Dual can be combined to give -

$$c.x + u.x \;=\; v.A.x \;=\; v.b - v.y \tag{5}$$

$$f + u.x \;=\; v.A.x \;=\; g - v.y \tag{6}$$

For any feasible (basic or non-basic) solution to the P-D pair,

$$u, x, v, y \quad \geq \quad 0 \tag{7}$$

and therefore,

$$(g - f) \;=\; (u.x + v.y) \;\geq\; 0 \tag{8}$$

For any basic solution (feasible or infeasible) to the P-D pair,
the non-basic variables are set to zero;
that is,

$$u, y \;=\; 0 \tag{9}$$

and therefore,

$$(g - f) \;=\; (u.x + v.y) \;=\; 0 \tag{10}$$

The entries in the Goldman-Tucker Compact-*Sym*metric-Tableau (CST) directly correspond to the associated basic solution to the P-D pair, thus establishing a one-to-one correspondence between a solution basis and the associated Goldman-Tucker Compact-*Sym*metric-Tableau (CST). A pivoting operation on the CST tableau corresponds to the associated move from one basis to another.

For the LP problem P-D pair (1) & (2) or equivalently (3) & (4) the above tableau in Figure-1 represents the initial tableau indicating the *initial basic solution* (IBS) wherein $y_i$ are the primal basic variables associated (one-to-one permanent association) with $v_i$ the dual non-basic variables, whereas $x_j$ are the primal non-basic variables associated (one-to-one permanent association) with $u_j$ the dual basic variables. Note that the column-labels $x_j$ (along with the label –1 for the RHS column) and the row-labels $v_i$ (along with the label –1 for the objective function row) play a significant role in the CST tableau, and the way to interpret (read) the CST tableau is as follows:

Primal Problem:

$$\sum_{j \in C} a_{ij} . x_j - b_i = -y_i , \quad i \in R \text{ (row index)}$$

(11)

$$\sum_{j \in C} c_j . x_j - 0 = f \qquad \text{(function to be maximized)}$$

Dual Problem:

$$\sum_{i \in R} v_i . a_{ij} - c_j = u_j , \quad j \in C \text{ (Column index)}$$

(12)

$$\sum_{i \in R} v_i . b_i - 0 = g \qquad \text{(function to be minimized)}$$

wherein the variables $x_j$, $y_i$, $v_i$, $u_j$ are all considered to be non-negative.

## 3. ALGEBRA *(ARITHMETIC)* OF *SYM*PLEX PIVOTING PROCESS

With the Goldman-Tucker Compact-*Sym*metric-Tableau (CST) representation for linear programming, in its standard/canonical form, one can observe that once a pivot element is selected, the actual pivoting process (the *algebra* and hence the *arithmetic* operations) is the same irrespective of the pivot selection; for example whether it is a primal pivot or a dual pivot. Hence it suffices to present here a single (common) set of operations representing the actual pivoting process - be it primal or dual. This *expressional elegance* and *computational efficiency* along with the *convenience* and the *versatility* (as will be evident later) are the reasons why the above representation has been selected for the purpose of our study.

For the sake of generality, let us imagine that we are somewhere in the middle of solving a LP problem (say after the $k^{th}$ iteration), and have the system model represented by a tableau ($T_k$) as shown in Figure-2.

By the nature of the sequence of elementary row (column) operations being performed during any pivoting process, the system model represented in Figure-2 is equivalent to that represented by the initial tableau which corresponds to the P-D pair (11) & (12). The transformed version of the primal-dual pair directly expressed by the above tableau is as follows:

$$
\begin{array}{c|c|c|}
 & z_j^N & -1 \\
\hline
w_i^N & \alpha_{ij} & \beta_i \\
\hline
-1 & \gamma_j & \delta \\
\hline
\end{array}
\begin{array}{l}
\\
= -z_i^B \\
= f
\end{array}
$$

$$
\begin{array}{cc}
\| & \| \\
w_j^B & g
\end{array}
$$

**Figure-2: Compact *Sym*metric Tableau (T$_k$) after k iterations**

Primal Problem:

$$z_i^B = \beta_i - \sum_{j \in C} \alpha_{ij} . z_j^N, \quad i \in R \text{ (row index)}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (13)$$

$$f = -\delta + \sum_{j \in C} \gamma_j . z_j^N, \quad \text{(function to be maximized)}$$

Dual Problem:

$$w_j^B = -\gamma_j + \sum_{i \in R} w_i^N . \alpha_{ij}, \quad j \in C \text{ (column index)}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (14)$$

$$g = -\delta + \sum_{i \in R} w_i^N . \beta_i \quad \text{(function to be minimized)}$$

The effect of a pivoting operation on (13) & (14) performed with a chosen pivot element $\alpha_{IJ}$ is exactly to affect an exchange between the P-D variable pairs indicated by I and J in (13) and (14). That is, $z_J^N$ is entered into primal basis in exchange for $z_I^B$ in (13), and $w_I^N$ is entered into dual basis in exchange for $w_J^B$ in (14). Suppose we have chosen the pivot element $\alpha_{IJ}$ using some appropriate pivot selection scheme, and we would like to derive the resulting tableau (T$_{k+1}$). Let the resulting tableau (T$_{k+1}$) be indicated in Figure-3.

$$
\begin{array}{c|c|c|}
 & (z_j^N)' & -1 \\
\hline
(w_i^N)' & (\alpha_{ij})' & (\beta_i)' \quad = -(z_i^B)' \\
\hline
-1 & (\gamma_j)' & (\delta)' \quad = f \\
\end{array}
$$

$$
\begin{array}{cc}
\| & \| \\
(w_j^B)' & g
\end{array}
$$

**Figure-3: Compact *Sym*metric Tableau ($T_{k+1}$) after (k+1) iterations**

The algebra (*arithmetic*) of deriving the above tableau ($T_{k+1}$) of Figure-3 from the previous tableau ($T_k$) of Figure-2 is detailed below (spdspds "*omni*plex" pivoting operation - refer Section-16):

| | | |
|---|---|---|
| $(\alpha_{IJ})' \leftarrow (\alpha_{IJ})^{-1}$ | $(\alpha_{Ij})' \leftarrow (\alpha_{IJ})^{-1}\alpha_{Ij}$ | $(\beta_I)' \leftarrow (\alpha_{IJ})^{-1}\beta_I$ |
| $(\alpha_{iJ})' \leftarrow -\alpha_{iJ}(\alpha_{IJ})^{-1}$ | $(\alpha_{ij})' \leftarrow \alpha_{ij} - \alpha_{iJ}(\alpha_{IJ})^{-1}\alpha_{Ij}$ | $(\beta_i)' \leftarrow \beta_i - \alpha_{iJ}(\alpha_{IJ})^{-1}\beta_I$ |
| $(\gamma_J)' \leftarrow -\gamma_J(\alpha_{IJ})^{-1}$ | $(\gamma_j)' \leftarrow \gamma_j - \gamma_J(\alpha_{IJ})^{-1}\alpha_{Ij}$ | $(\delta)' \leftarrow \delta - \gamma_J(\alpha_{IJ})^{-1}\beta_I$ |

*along with an exchange of label*s associated with row I and column J; that is effectively:

$$(z_J^N)' \leftarrow z_I^B; \quad (z_I^B)' \leftarrow z_J^N; \quad (w_J^B)' \leftarrow w_I^N; \quad (w_I^N)' \leftarrow w_J^B;$$

while retaining the very same labels for all the other rows and columns; that is:

$$(z_j^N)' \leftarrow z_j^N; \quad (z_i^B)' \leftarrow z_i^B; \quad (w_j^B)' \leftarrow w_j^B; \quad (w_i^N)' \leftarrow w_i^N;$$

for i ε R\{I} and j ε C\{J}.

It is to be noted here that the Goldman-Tucker Compact-*Sym*metric-Tableau (CST) is a *unique symmetric* representation *common* to both the primal as well as the dual of a linear programming problem in its standard canonical form. Also, the tableau evolves from $T_0$ as the initial tableau representing the problem in its standard canonical form, while following the sequence of symplex iterations all the way to the final terminal tableau $T_*$, while always being a tableau representation of an *equivalent system* of linear inequalities along with the corresponding objective function.

From (13) & (14) above, we can get:

$$(f+\delta) + \sum w_j^B \cdot z_j^N = \sum \sum w_i^N \cdot \alpha_{ij} \cdot z_j^N = (g+\delta) - \sum w_i^N \cdot z_i^B \quad (15)$$

and therefore,

$$(g - f) = \sum w_i^N \cdot z_i^B + \sum w_j^B \cdot z_j^N \quad (16)$$

The values of the primal and the dual basic variables as well as (the primal & the dual) objective function value corresponding to a basis/tableau can be directly read from the entries of the tableau - the primal objective function value being always the same as the dual objective function value for every basis/tableau all along the sequence of symplex iterations - each of the summation terms in both (15) and (16) above being zero, and the *complementary slackness condition being automatically satisfied for every basis/tableau*. The relationships (15) & (16) above indicate the effect of moving away from a basis, as the non-basic variables are moved away from zero; in particular, one can note from (16) above that *for all feasible solutions, the dual objective function value is an upper bound for the primal objective function value* and *the primal objective function value is a lower bound for the dual objective function value*, which is a well known relationship. A basic solution  for the P-D pair  is  optimal  if & only if  it is  basic feasible  for the P-D pair.

Now, one may skip Sections 4 to 8 and go directly to Section 9 in order to avoid a pedagogical detour, and possibly come back to these sections if and when necessary to fill the gaps in one's detailed understanding.


## 4. A *TYPICAL SYM*PLEX PIVOT SELECTION

A typical symplex pivot selection scheme can include *four* (*two pairs*) *fundamental types of symplex pivot selections* - the Primal Standard Pivot (PSP), Dual Standard Pivot (DSP) are the classical pair defined by Dantzig [1]. Here, we define a new pair of pivots - Primal Tricky Pivot (PTP) and Dual Tricky Pivot (DTP). The algebra of these pivot selections is given in Figure-4, along with a schematic representation of the Tableau Data Pattern that leads to such pivot selection.


## 5. EFFECT OF A *TYPICAL* PIVOTING OPERATION

It is useful at this point to make a few observations regarding the effect of pivoting operation, in each of the above pivot selections.

*DSP* {*poxidixi*} brings about an immediate improvement in the primal feasibility w.r.t. the pivot row, without deterioration of dual feasibility. The extent of this improvement in primal feasibility can be measured by the corresponding improvement (decrease) in the value of the dual objective function, given by a local effectiveness measure $lem(I, J) = abs\{\gamma_J(\alpha_{IJ})^{-1}\beta_I\}$ [refer Section-8].

PSP {*doxipixi*} brings about an immediate improvement in the dual feasibility w.r.t. the pivot column, without deterioration of primal feasibility. The extent of this improvement in dual feasibility can be measured by the corresponding improvement (increase) in the value of the primal objective function, given by the same local effectiveness measure as mentioned here above.

PTP {*poxidoxo*} brings about an immediate improvement in the primal feasibility w.r.t. the pivot row, without any concern to the dual feasibility. The extent of this improvement in primal feasibility can be measured by the corresponding improvement (increase) in the value of the primal objective function, given by the same local effectiveness measure as mentioned here above.

DTP {*doxipoxo*} brings about an immediate improvement in the dual feasibility w.r.t. the pivot column, without any concern to the primal feasibility. The extent of this improvement in dual feasibility can be measured by the corresponding improvement (decrease) in the value of the dual objective function, given by the same local effectiveness measure as mentioned here above.

**(a)** <mark>Dual Standard Pivot, DSP</mark>: {*poxidixi*}

| | | |
|---|---|---|
| ● | − | − | ←
| ● | ● | ⊕ |
| + | Θ | ● |

↓

$I \in \{ i \in R \mid \beta_i < 0 \};$
$J_{DSP}(I) \leftarrow \text{arg-min} \{ (\gamma_j / \alpha_{Ij}) \mid \gamma_j \leq 0; \alpha_{Ij} < 0 \};$
$\qquad\qquad j \in C$

**(b)** <mark>Primal Standard Pivot, PSP</mark>: {*doxipixi*}

| | | |
|---|---|---|
| ● | ● | − |
| + | ● | ⊕ | →
| + | Θ | ● |

↑

$J \in \{ j \in C \mid \gamma_j > 0 \};$
$I_{PSP}(J) \leftarrow \text{arg-min} \{ (\beta_i / \alpha_{iJ}) \mid \beta_i \geq 0; \alpha_{iJ} > 0 \};$
$\qquad\qquad i \in R$

**(c)** <mark>Primal Tricky Pivot, PTP</mark>: {*poxidoxo*}

| | | |
|---|---|---|
| − | ● | − | ←
| ● | ● | ⊕ |
| + | Θ | ● |

↓

$I \in \{ i \in R \mid \beta_i < 0 \};$
$J_{PTP}(I) \leftarrow \text{arg-max} \{ \text{abs}( \gamma_j / \alpha_{Ij}) \mid \alpha_{Ij} < 0; \gamma_j > 0 \};$
$\qquad\qquad j \in C$

**(d)** <mark>Dual Tricky Pivot, DTP</mark>: {*doxipoxo*}

| | | |
|---|---|---|
| + | ● | − | →
| ● | ● | ⊕ |
| + | Θ | ● |

↑

$J \in \{ j \in C \mid \gamma_j > 0 \};$
$I_{DTP}(J) \leftarrow \text{arg-max} \{ \text{abs}( \beta_i / \alpha_{iJ}) \mid \alpha_{iJ} > 0; \beta_i < 0 \};$
$\qquad\qquad i \in R$

**Figure-4:. Four Types of *Typical Sym*plex Pivot Selections**

— negative; 0 zero; + positive; Θ non-positive; * any value; ⊕ non-negative; ● un-analyzed

## 6. CHARACTERIZATION OF A PIVOT ELEMENT/CELL

Each potential pivot element/cell in the Compact *Sym*metric Tableau (CST) is characterized by a "cell-type". The cell type of a cell in I[th] row and J[th] column consists of three components. They are the sign of $\alpha_{IJ}$ (either '0', '+' or '−'), the sign of $\beta_I$ (either 'Z', 'P' or 'N') and the sign of $\gamma_J$ (either 'z', 'p' or 'n'). Hence there will be a total of 27 different cell types. However, if α is zero or numerically near-zero, it will not be a potential pivoting cell, since pivoting will not be performed on such cells. Hence the nine cell types with $\alpha \approx 0$ are combined together and the new cell type given is 0**. Thus there are 19 cell types for our consideration.

<mark>It will be clear from the foregoing discussion (Section-9) that any specific pivoting operation will affect/change the feasibility of a primal-basic-variable [dual-basic-variable] if the ratio $\beta_i/\alpha_{iJ}$ is closer to zero than the *pivoting-ratio* $\beta_I/\alpha_{IJ}$ [if the ratio $\gamma_j/\alpha_{Ij}$ is closer to zero than the *pivoting-ratio* $\gamma_J/\alpha_{IJ}$]: the feasibility/infeasibility of those rows [columns] with the associated ratios farther away from zero beyond (relative to) the corresponding *pivoting-ratio* will not get affected by such pivoting operation.</mark>

## 7. *TYPICAL SYMPLEX PIVOT DECISION STRATEGY*

The four distinct types of *typical* pivot selections mentioned above may be considered for designing a pivoting strategy. It is to be noted here that the pivot selection as well as the pivoting operation is symmetrical w.r.t. the primal problem and the dual problem, both represented in the very same CST representation of a linear programming problem.

The four distinct types of typical pivot selections may be considered in the following default preference order: {{DTP,PSP},{PTP,DSP}} to drive towards dual feasibility first; or alternatively {{PTP,DSP},{DTP,PSP}} to drive towards primal feasibility first.

At every iteration, an attempt is made to select a pivot element/cell, by checking the possible pivot selections belonging to one of the above four types of pivot selections in the default preference order as specified above. It is to be noted that depending upon the actual data in the Tableau, a pivot selection of specific type which was not possible in an earlier iteration, can become possible in a later iteration, sometimes even in the very next following iteration. That is why it is a crucial part of an algorithm to check in each (and every) iteration, for each of the four types of possible pivot selections preferably (although not necessarily) in a pre-specified preference order. At each step mentioned above, if there is more than one pivot element of that particular cell type, then the choice can be narrowed by utilizing some measure of goodness for the pivot selection, appropriately defined.

## 8. MEASURE OF GOODNESS FOR PIVOT SELECTION

From the earlier discussion above, one can observe that the absolute value of the change in the objective function, can possibly be utilized as a *local effectiveness measure* (*lem*) of the pivoting operation - applicable for any and every iteration, for both primal as well as dual – that is, given by $lem(I, J) = abs\{\gamma_J(\alpha_{IJ})^{-1}\beta_I\}$.

Although it is not specifically suggested here, one can opt to choose a pivot, possibly to maximize this *local effectiveness measure* (*lem*) in every iteration, either among the possible pivots of a particular type or subtype, or even among of all the possible pivots of all the four possible types. Even, if done so, it cannot be guaranteed (needs further research?) to minimize the overall number of symplex iterations required for reaching an optimum solution.

It may require further research work to thoroughly understand, analyze and incorporate the concept of any "*local effectiveness measure*" (*lem*) for a single symplex pivoting operation to the fullest extent, that would in effect achieve a guaranteed improvement in some corresponding "*global effectiveness measure*" (*gem*) defined appropriately for the given LP problem; in developing an *efficient* & *robust* solution strategy. For now, let us come to the main algorithm itself.

In order to *achieve a guaranteed performance efficiency,* we propose to utilize a *global effectiveness measure* (*gem*) that characterizes a CST tableau with a number indicating the *largest possible decrease in the infeasibility index*.

## 9. INFEASIBILITY INDEX : A GLOBAL EFFECTIVENESS MEASURE

As an *inverse measure of goodness*, the infeasibility index $\lambda$ of a given CST tableau is defined as the sum of the *primal infeasibility index* $\mu$ and the *dual infeasibility index* $\nu$. It corresponds to the number of basic variables in primal & dual which are infeasible in the given tableau. That is, we define $\lambda = \mu + \nu$; and therefore -

$\lambda = (\mu,$ number of rows with $\beta < 0)$ + $(\nu,$ number of columns with $\gamma > 0)$;

or

$\lambda = (\mu,$ number of rows with $z_i^B < 0)$ + $(\nu,$ number of columns with $w_j^B < 0)$.

*If the infeasibility index $\lambda$ of the given tableau equals to zero then it indicates that the tableau is the terminal tableau which is feasible and optimal.* By the definition of the infeasibility index, it can never be negative, *nor can it be more than the sum of the number of columns and the number of rows in the Compact Symmetric Tableau.* That is,

$0 \leq \lambda = \{(0 \leq \mu \leq m) + (0 \leq \nu \leq n)\} \leq (m + n).$

Given a tableau, the *change in the infeasibility index $(\tau = \Delta\lambda)$* can be associated with each cell that can be a potential candidate pivot element (i.e. $\alpha \neq 0$). This change in the infeasibility index consists of two components, one is the *change in the primal infeasibility index $(\sigma = \Delta\mu)$* and the other is the *change in the dual infeasibility index $(\rho = \Delta\nu)$*. That is,

Change in the infeasibility index, $\tau = \Delta\lambda = (\Delta\mu + \Delta\nu) = \sigma + \rho.$
Global Effectiveness Measure, *gem* $= -\tau,$ that is, the decrease in the infeasibility index.

### 9.1 Calculation of the change in the primal infeasibility index ($\sigma = \Delta\mu$)

For each column j, the ratio $r_{ij} = \beta_i/\alpha_{ij}$ is calculated for all rows i = 1, 2, …, m.
The ratio can either be positive, zero or negative. It can be represented as in Figure-5.

Let $\alpha_{IJ}$ be the chosen pivot element, then after pivoting, the values of $\beta$ are given by

$$\mathbf{r_{IJ}} = \beta_I/\alpha_{IJ} \qquad \text{and}$$
$$(\beta_I)' \leftarrow (\beta_I/\alpha_{IJ})$$
and $\qquad (\beta_i)' \leftarrow \beta_i - (\beta_I/\alpha_{IJ})\alpha_{iJ}$

The calculation of $\sigma$ can be divided into three different cases depending upon the value of $\mathbf{r_{IJ}}$. $\beta = 0$ is considered as feasible.

**(i)  $\mathbf{r_{IJ} = 0}$  ($\beta_I = 0, \alpha_{IJ} \neq 0$)**

Here, there will not be any change in the $\beta$ values, and therefore there will not be any change in the infeasibility index. Hence, $\qquad \sigma_{IJ} = 0.$

**(ii)  $\mathbf{0 < r_{IJ}}$  ($0 \leq \beta_I , 0 < \alpha_{IJ} :$ *pixi* -or- $\beta_I < 0, \alpha_{IJ} < 0 :$ *poxi*)          : b–p**

$\mathbf{r_{iJ}} < 0$ will not affect the change in the infeasibility index. $\mathbf{r_{iJ}} < 0$ can occur in two situations.

One of them is when $\beta_i < 0$ and $\alpha_{iJ} > 0$.  In this case, it can be seen from the expression for $(\beta_i)'$ given above, that $(\beta_i)'$ will continue to be negative; hence the change in the infeasibility index is not affected.

The other is when $\beta_i > 0$ and $\alpha_{iJ} < 0$.  In this case, it can be seen from the expression for $(\beta_i)'$ given above, that $(\beta_i)'$ will continue to be positive; hence the change in the infeasibility index is not affected.

$\mathbf{r_{iJ}} > 0$ will affect the change in the infeasibility index. $\mathbf{r_{iJ}} > 0$ can occur in two situations.

One is when $\beta_i \geq 0$ and $\alpha_{iJ} > 0$. In this case, it can be seen from the expression for $(\beta_i)'$ given above, that $(\beta_i)'$ will continue to be positive for the ratios $\mathbf{r_{iJ}} > \mathbf{r_{IJ}}$, $(\beta_i)'$ will be zero for the ratios $\mathbf{r_{iJ}} = \mathbf{r_{IJ}}$, $(\beta_I)'$ will continue to be positive, and $(\beta_i)'$ will become negative for the ratios $\mathbf{r_{iJ}} < \mathbf{r_{IJ}}$. Hence the change in the infeasibility index is *increased* by the number of ratios $\mathbf{r_{iJ}}$ which are less than $\mathbf{r_{IJ}}$ and greater than or equal to zero.

The other is when $\beta_i < 0$ and $\alpha_{iJ} < 0$. In this case, it can be seen from the expression for $(\beta_i)'$ given above, that $(\beta_i)'$ will continue to be negative for the ratios $\mathbf{r_{iJ}} > \mathbf{r_{IJ}}$, $(\beta_i)'$ will become zero for the ratios $\mathbf{r_{iJ}} = \mathbf{r_{IJ}}$, $(\beta_I)'$ will become positive, and $(\beta_i)'$ will become positive for the ratios $\mathbf{r_{iJ}} < \mathbf{r_{IJ}}$. Hence the change in the infeasibility index is *decreased* by the number of ratios $\mathbf{r_{iJ}}$ which are less than or equal to $\mathbf{r_{IJ}}$ and strictly greater than zero.

Therefore,

$b = \Sigma$ No. of rows i with $0 \leq \beta_i$ , $0 < \alpha_{iJ}$ and $0 \leq \mathbf{r_{iJ}} < \mathbf{r_{IJ}}$          : {*pixi*}

$p = \Sigma$ No. of rows i with $\beta_i < 0$ , $\alpha_{iJ} < 0$   and $0 < \mathbf{r_{iJ}} \leq \mathbf{r_{IJ}}$          : {*poxi*}

and

$\sigma_{IJ} = b - p$          : {*pixi-poxi*}
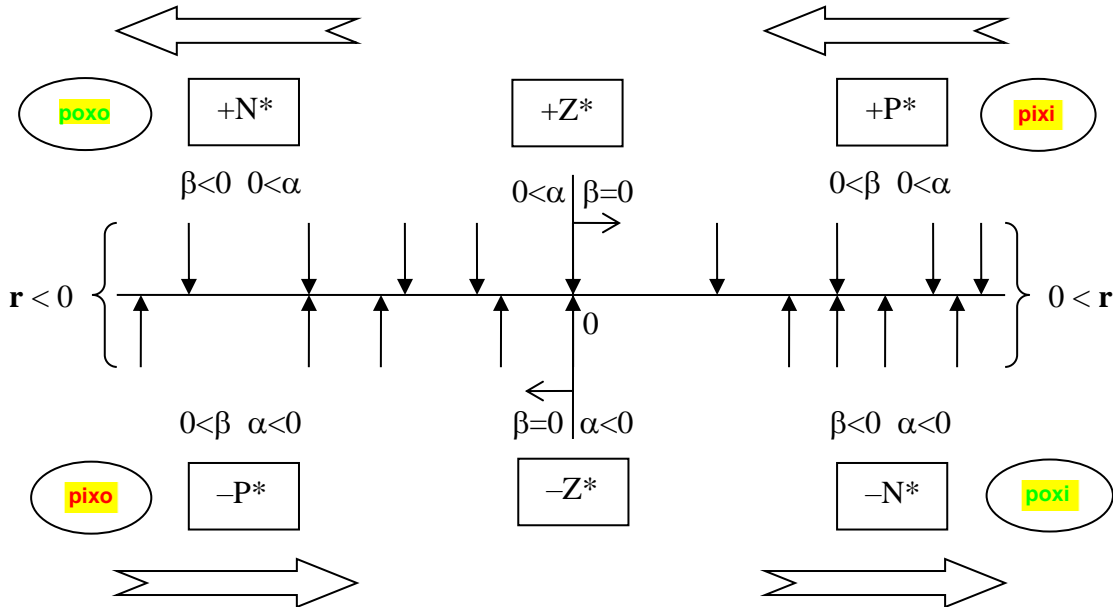


**Figure-5: Ordered Pattern of $(\beta_i / \alpha_{iJ})$ Values for i = 1, 2, …, m**
* => ('P', 'N' or 'Z')

**(iii)**    $\mathbf{r_{IJ} < 0}$    $(\mathbf{0 \leq \beta_I}$ , $\mathbf{\alpha_{IJ} < 0}$ : *pixo* **-or-** $\mathbf{\beta_I < 0, 0 < \alpha_{IJ}}$ : *poxo*)        : 1+q–d

$r_{iJ} > 0$ will not affect the change in the infeasibility index. $r_{iJ} > 0$ can occur in two situations.

One of them is when $\beta_i \geq 0$ and $\alpha_{iJ} > 0$. In this case, it can be seen from the expression for $(\beta_i)'$ given above, that $(\beta_i)'$ will continue to be positive; hence the change in the infeasibility index is not affected.

The other is when $\beta_i < 0$ and $\alpha_{iJ} < 0$. In this case, it can be seen from the expression for $(\beta_i)'$ given above, that $(\beta_i)'$ will continue to be negative; hence the change in the infeasibility index is not affected.

$r_{iJ} < 0$ will affect the change in the infeasibility index. $r_{iJ} < 0$ can occur in two situations.

One is when $\beta_i \geq 0$ and $\alpha_{iJ} < 0$. In this case, it can be seen from the expression for $(\beta_i)'$ given above, that $(\beta_i)'$ will continue to be positive for the ratios $r_{iJ} < r_{IJ}$, $(\beta_i)'$ will be zero for the ratios $r_{iJ} = r_{IJ}$, $(\beta_I)'$ will become negative, and $(\beta_i)'$ will become negative for the ratios $r_{iJ} > r_{IJ}$. Hence the change in the infeasibility index is increased by the number of ratios $r_{iJ}$ which are greater than $r_{IJ}$ and less than or equal to zero (one more, if $\beta_I$ is positive, since it will become negative).

The other is when $\beta_i < 0$ and $\alpha_{iJ} > 0$. In this case, it can be seen from the expression for $(\beta_i)'$ given above, that $(\beta_i)'$ will continue to be negative for the ratios $r_{iJ} < r_{IJ}$, $(\beta_i)'$ will become zero for the ratios $r_{iJ} = r_{IJ}$, $(\beta_I)'$ will become negative, and $(\beta_i)'$ will become positive for the ratios $r_{iJ} > r_{IJ}$. Hence the change in the infeasibility index is decreased by the number of ratios $r_{iJ}$ which are greater than or equal to $r_{IJ}$ and strictly less than zero (one less, if $\beta_I$ is negative, since it will continue to be negative).

Therefore,
        $q = \Sigma$ No. of rows i with  $0 \leq \beta_i$ , $\alpha_{iJ} < 0$  and  $r_{IJ} < r_{iJ} \leq 0$            : {*pixo*}
        $d = \Sigma$ No. of rows i with  $\beta_i < 0$ ,  $0 < \alpha_{iJ}$  and  $r_{IJ} \leq r_{iJ} < 0$            : {*poxo*}
and
        $\sigma_{IJ} = (q + 1) - d$        if $(\beta_I > 0)$
or      $\sigma_{IJ} = q - (d - 1)$        if $(\beta_I < 0)$
Hence we get,
        $\sigma_{IJ} = 1 + q - d$                                                    : {*1pixo-poxo*}


## 9.2  Calculation of the change in the dual infeasibility index ($\rho = \Delta v$)

For each row i, the ratio $r_{ij} = \gamma_j / \alpha_{ij}$ is calculated for all columns j = 1, 2, …, n.
The ratio can either be positive or negative. It can be represented as in Figure-6.

Let $\alpha_{IJ}$ be the chosen pivot element, then after pivoting, the values of $\gamma$ are given by

$$\mathbf{r_{IJ}} = \gamma_J / \alpha_{IJ} \qquad \text{and}$$
$$(\gamma_J)' \leftarrow - (\gamma_J / \alpha_{IJ})$$
and      $(\gamma_j)' \leftarrow \gamma_j - (\gamma_J / \alpha_{IJ}) \alpha_{Ij}$

The calculation of $\rho$ can be divided into three different cases depending upon the value of $\mathbf{r_{IJ}}$. $\gamma = 0$ is considered as feasible.

**(i)      $\mathbf{r_{IJ} = 0}$     ($\boldsymbol{\gamma_J = 0,\ \alpha_{IJ} \neq 0}$)**

Here there will not be any change in the $\gamma$ values, and therefore there will not be any change in the infeasibility index. Hence,          $\rho_{IJ} = 0$.
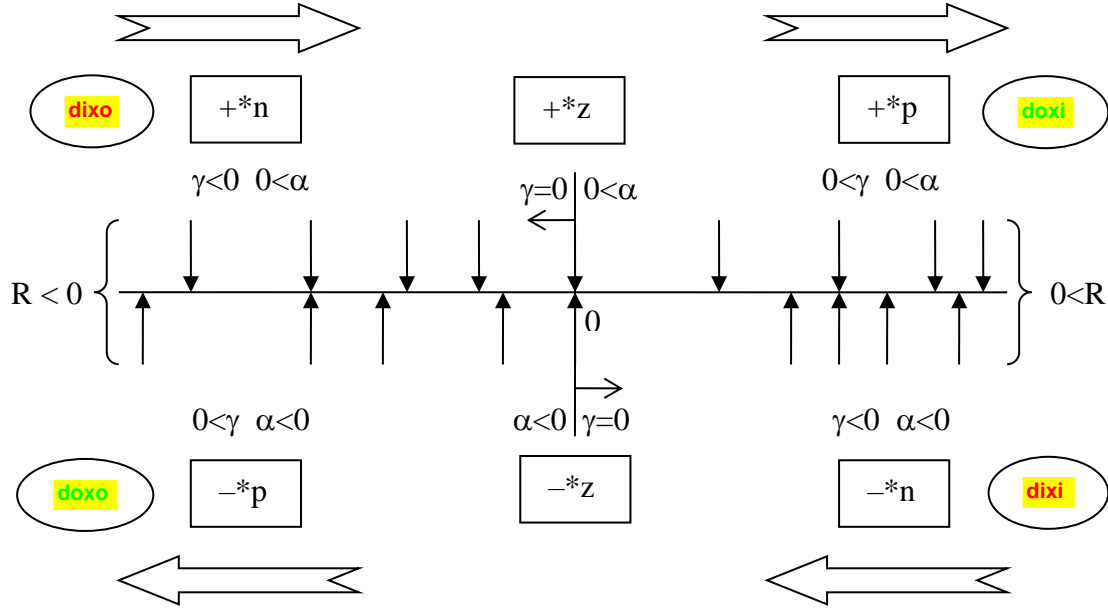


**Figure-6: Ordered Pattern of ($\gamma_j\, /\, \alpha_{Ij}$) Values for j = 1, 2, …, n**
* => ('p', 'n' or 'z')

**(ii)      $\mathbf{0 < r_{IJ}}$     ($\boldsymbol{\gamma_J \leq 0, \alpha_{IJ} < 0}$ : *dixi* -or- $\boldsymbol{\gamma_J > 0, \alpha_{IJ} > 0}$ : *doxi*)                    : p–b**

$\mathbf{r_{Ij}} < 0$ will not affect the change in the infeasibility index. $\mathbf{r_{Ij}} < 0$ can occur in two situations.

One of them is when $\gamma_j < 0$ and $\alpha_{Ij} > 0$. In this case, it can be seen from the expression for $(\gamma_j)'$ given above, that $(\gamma_j)'$ will continue to be negative; hence the change in the infeasibility index is not affected.

The other is when $\gamma_j > 0$ and $\alpha_{Ij} < 0$. In this case, it can be seen from the expression for $(\gamma_j)'$ given above, that $(\gamma_j)'$ will continue to be positive; hence the change in the infeasibility index is not affected.

$\mathbf{r_{Ij}} > 0$ will affect the change in the infeasibility index. $\mathbf{r_{Ij}} > 0$ can occur in two situations.

One is when $\gamma_j \leq 0$ and $\alpha_{Ij} < 0$. In this case, it can be seen from the expression for $(\gamma_j)'$ given above, that $(\gamma_j)'$ will continue to be negative for the ratios $\mathbf{r_{Ij}} > \mathbf{r_{IJ}}$, $(\gamma_j)'$ will be zero for the ratios $\mathbf{r_{Ij}} = \mathbf{r_{IJ}}$, $(\gamma_J)'$ will continue to be negative, and $(\gamma_j)'$ will become positive for the ratios $\mathbf{r_{Ij}} < \mathbf{r_{IJ}}$. Hence the change in the infeasibility index is *increased* by the number of ratios $\mathbf{r_{Ij}}$ which are less than $\mathbf{r_{IJ}}$ and greater than or equal to zero.

The other is when $\gamma_j > 0$ and $\alpha_{Ij} > 0$. In this case, it can be seen from the expression for $(\gamma_j)$' given above, that $(\gamma_j)$' will continue to be positive for the ratios $r_{Ij} > r_{IJ}$, $(\gamma_j)$' will become zero for the ratios $r_{Ij} = r_{IJ}$, $(\gamma_j)$' will become negative, and $(\gamma_j)$' will become negative for the ratios $r_{Ij} < r_{IJ}$. Hence the change in the infeasibility index is *decreased* by the number of ratios $r_{Ij}$ which are less than or equal to $r_{IJ}$ and strictly greater than zero.

Therefore,

$\quad\quad$ p = $\Sigma$ No. of columns j with $\ \gamma_j \leq 0\ ,\ \alpha_{Ij} < 0\ $ and $\ 0 \leq r_{Ij} < r_{IJ}\ $ $\quad$ : {*dixi*}

$\quad\quad$ b = $\Sigma$ No. of columns j with $\ 0 < \gamma_j\ ,\ 0\ < \alpha_{Ij}\ $ and $\ 0 < r_{Ij} \leq r_{IJ}\ $ $\quad$ : {*doxi*}

and

$\quad\quad$ $\rho_{IJ}$ = p − b $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ : {*dixi*-*doxi*}

**(iii)**$\quad$ **$r_{IJ} < 0$**$\quad$ (**$\gamma_J \leq 0, \alpha_{IJ} > 0$ :** *dixo* **-or-**$\ $ **$\gamma_J > 0, \alpha_{IJ} < 0$ :**$\ $ *doxo*)$\quad\quad$ : 1+d−q

$r_{Ij} > 0$ will not affect the change in the infeasibility index. $r_{Ij} > 0$ can occur in two situations.

One of them is when $\gamma_j < 0$ and $\alpha_{Ij} < 0$. In this case, it can be seen from the expression for $(\gamma_j)$' given above, that $(\gamma_j)$' will continue to be negative; hence the change in the infeasibility index is not affected.

The other is when $\gamma_j > 0$ and $\alpha_{Ij} > 0$. In this case, it can be seen from the expression for $(\gamma_j)$' given above, that $(\gamma_j)$' will continue to be positive; hence the change in the infeasibility index is not affected.

$r_{Ij} < 0$ will affect the change in the infeasibility index. $r_{Ij} < 0$ can occur in two situations.

One is when $\gamma_j \leq 0$ and $\alpha_{Ij} > 0$. In this case, it can be seen from the expression for $(\gamma_j)$' given above, that $(\gamma_j)$' will continue to be negative for the ratios $r_{Ij} < r_{IJ}$, $(\gamma_j)$' will be zero for the ratios $r_{Ij} = r_{IJ}$, $(\gamma_J)$' will become positive, and $(\gamma_j)$' will become positive for the ratios $r_{Ij} > r_{IJ}$. Hence the change in the infeasibility index is increased by the number of ratios $r_{Ij}$ which are greater than $r_{IJ}$ and less than or equal to zero (one more, if $\gamma_J$ is negative, since it will become positive).

The other is when $\gamma_j > 0$ and $\alpha_{Ij} < 0$. In this case, it can be seen from the expression for $(\gamma_j)$' given above, that $(\gamma_j)$' will continue to be positive for the ratios $r_{Ij} < r_{IJ}$, $(\gamma_j)$' will become zero for the ratios $r_{Ij} = r_{IJ}$, $(\gamma_J)$' will become positive, and $(\gamma_j)$' will become negative for the ratios $r_{Ij} > r_{IJ}$. Hence the change in the infeasibility index is decreased by the number of ratios $R_{Ij}$ which are greater than or equal to $r_{IJ}$ and strictly less than zero (one less, if $\gamma_J$ is positive, since it will continue to be positive).

Therefore,

$\quad\quad$ d = $\Sigma$ No. of columns j with $\ \gamma_j \leq 0\ ,\ 0\ < \alpha_{Ij}\ $ and $\ r_{IJ} < r_{Ij} \leq 0\ $ $\quad$ : {*dixo*}

$\quad\quad$ q = $\Sigma$ No. of columns j with $\ 0 < \gamma_j\ ,\ \alpha_{Ij} < 0\ $ and $\ r_{IJ} \leq r_{Ij} < 0\ $ $\quad$ : {*doxo*}

and

$\quad\quad$ $\rho_{IJ}$ = (d + 1) − q $\quad\quad$ if $\ (\gamma_J < 0)$

or$\quad$ $\rho_{IJ}$ = d − (q − 1) $\quad\quad$ if $\ (\gamma_J > 0)$

Hence we get,

$\quad\quad$ $\rho_{IJ}$ = 1 + d − q $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ :{*1dixo*-*doxo*}

## 10.  CST-SIGNATURE  OF  A  COMPACT  *SYM*METRIC  TABLEAU

We have designed a unique CST-*signature* that distinctly identifies any specific Goldman-Tucker Compact *Sym*metric Tableau associated with a given Linear Programming Problem.  It is a string of length $n+m$ (number of columns + number of rows).  The first $n$ entries are chosen from the character set $\{n, z, p\}$ depending upon whether the $\gamma$ value is negative, zero or positive respectively; the next $m$ entries are chosen from the character set $\{P, Z, N\}$ depending upon whether the $\beta$ value is positive, zero or negative respectively.  The positions of these entries in the signature string are fixed conveniently with respect to the initial tableau, as in considering the *lexicographic ordering* of these $n+m$ parameters.  As part of the pivot selection process, during the infeasibility analysis, corresponding to a specific possible choice of the pivot element, the CST-*signature* string can be generated for the anticipated resultant tableau. It can be compared with the signatures of all the previous tableaus stored in a dictionary, to facilitate the detection of any possible imminent cycle, since each CST-*tableau* has a unique CST-*signature* and each CST-*signature* uniquely identifies a CST-*tableau*. If&when an imminent cycle is detected, such pivot choice can be avoided and possible alternative pivot choice may be considered for actual pivoting operation.  This methodology will be useful especially in situations of degeneracy.  Note that an alternative binary string CST-signature may be defined by mapping the alphabet symbols $\{n,z,p\}$ to 0 indicating the primal non-basic status, and mapping the alphabet symbols $\{p,Z,N\}$ to 1 indicating primal basic status, of the corresponding parameters, as a framework to represent only the combinatorial information without any reference to the feasibility information.


## 11. DEGENERACY

When no further decrease in the infeasibility index is indicated by every/all possible pivot selections (the best pivot choice corresponding to the largest possible decrease - non-increase - in the infeasibility index is itself a case of no change in the infeasibility index) the situation may correspond to either of the two cases: (1) the tableau is indeed the terminal tableau, or (2) the best possible choice of the pivot element is in some degenerate row/column.  In the first case, the *spdspds* algorithm terminates and the tableau may be analyzed to classify it into one of the six possible categories as described later.  The presence of primal/dual degeneracy (either in the terminal tableau itself, as in the first case, or may even be some temporary/intermediate degeneracy encountered enroute towards the optimum/terminal tableau, as in the second case) may be identified by the presence of *zero* in some row/column.  The situation wherein the best pivot choice is itself in some degenerate row/column, requires further analysis, before declaring that the tableau is indeed the terminal tableau - it may indeed be either a case of temporary/intermediate degeneracy or a case of degeneracy in a non-optimum but terminal tableau.

A pivot selection on any degenerate row/column with primal/dual degeneracy doesn't lead to any change in the primal/dual feasibility (refer to the algebra/arithmetic of the pivoting process presented in Section-3) although the coefficient matrix entries will surely be changed in the resulting tableau.  Therefore, the infeasibility analysis needs to be carried out for the resulting tableau so as to check & confirm whether any further reduction in the infeasibility index is now attainable - if so, the usual iteration process continues; if not, then further pivoting on the degenerate rows/columns needs to be continued (as per a *predetermined lexicographically ordered sequence of the problem parameters*) among those possible pivot options while also keeping track of the CST-signature of the tableaus to detect & avoid possible impending cycling.

## 12. ALGORITHM TERMINATION - TERMINAL TABLEAU TYPES

When further reduction in the infeasibility index is not possible, the tableau is checked & confirmed to be the *terminal tableau*. The possible terminal tableau types as shown in Figure-7 belong to one of the six possible categories listed here. Label F is for *feasible-basic-finite*, ∞ for *feasible basic-finite/non-basic-infinite* and Φ for *infeasible* - corresponding to both primal and dual *variables*.

(1) {P=F,D=F}; (2) {P=F,D=∞}; (3) {P=∞,D=F}; (4) {P=∞,D=Φ}; (5) {P=Φ,D=∞}; (6) {P= Φ,D= Φ}



**Figure-7. Six Categories for Terminal Tableau Data Pattern.**
−  negative;      0 zero;      +  positive;
Θ non-positive; * any value; ⊕ non-negative; • un-analyzed

This classification into *six categories* as presented here is a refinement over the well-known classical approach; one that enables to distinctly identify the case wherein primal (dual) has a *feasible basic-finite/non-basic-infinite* optimum with finite value for the objective function, while the dual (primal) has a *feasible-basic-finite* optimum. This distinction from the classical approach arises because we give primary emphasis on the classification based on the nature of the decision variables at termination, and give secondary emphasis on the finiteness (or otherwise) of the objective function value. The set of criteria for this classification scheme, to be used after ensuring that the tableau is indeed the *terminal tableau*, is given along with Figure-7.

Note that the situations corresponding to primal/dual *degeneracy* with the resultant dual/primal *multiplicity* and also that of *infinite rays* are all easily discernible in this classification scheme.

In case of terminal infeasibility, a judicious use of either the *change in primal infeasibility index* ($\sigma=\Delta\mu$) or *change in dual infeasibility index* ($\rho=\Delta\nu$) instead of the *change in overall infeasibility index* ($\tau=\Delta\lambda$) can be utilized to arrive at an *almost primal feasible* or an *almost dual feasible* tableau, if one requires such an output for further problem analysis etc.  This maneuverability can be adapted to the needs of the problem and that shows the *versatility* of the *spdspds* approach.

- P=F  : $(\mu = 0)$  &  $[\forall(I\in R,J\in C):\{0 \leq \sigma_{IJ}\}]$ & $[\nexists(i\in R,j\in C):\{(\alpha_{ij} <0)(\beta_i \geq 0)(\gamma_j \geq 0)\}]$;

- P=$\infty$  : $(\mu = 0)$  &  $[\forall(I\in R,J\in C):\{0 \leq \sigma_{IJ}\}]$ & $[\exists(i\in R,j\in C):\{(\alpha_{ij} <0)(\beta_i \geq 0)(\gamma_j \geq 0)\}]$;

- P=$\Phi$  : $(\mu > 0)$  &  $[\forall(I\in R,J\in C):\{0 \leq \sigma_{IJ}\}]$;

- D=F  : $(\nu = 0)$  &  $[\forall(J\in C,I\in R):\{0 \leq \rho_{IJ}\}]$ & $[\nexists(j\in C,i\in R):\{(\alpha_{ij} >0)(\beta_i \leq 0)(\gamma_j \leq 0)\}]$;

- D=$\infty$  : $(\nu = 0)$  &  $[\forall(J\in C,I\in R):\{0 \leq \rho_{IJ}\}]$ & $[\exists(j\in C,i\in R):\{(\alpha_{ij} >0)(\beta_i \leq 0)(\gamma_j \leq 0)\}]$;

- D=$\Phi$  : $(\nu > 0)$  &  $[\forall(J\in C,I\in R):\{0 \leq \rho_{IJ}\}]$;


## 13. SYMMETRIC PRIMAL DUAL *SYM*PLEX PIVOT SELECTION STRATEGY

(1) The proposed *spdspds* approach can be used to solve any LP problem, by first converting it into the standard/canonical form before proceeding further.  In performing such transformation, it is possible to enhance the overall efficiency by the following approach:

  (a)  Free/bounded variables can be replaced by non-negative variables not double in number, but *only just one extra in number*.  Equations can be replaced by inequalities not double in number, but *only just one extra in number*.

  (b)  No need for use of artificial variables; the *initial basic solution* need not necessarily be feasible; and therefore no need to rely on two-phase method or big-M method, etc.

(2) Any nonzero element of the coefficient matrix in the tableau can be a potential pivot element. The earlier defined typical simplex pivot selection scheme and the associated set of four types of pivot candidates {{DTP, PSP},{PTP, DSP}} is a *typical* sample, presented (refer Sections 4-to-8) only for *pedagogical* reasons, to eventually take the reader towards the proposed *spdspds*.  Note that the classical PSP (*doxipixi* pivot) with cell-type +Pp and DSP (*poxidixi* pivot) with cell-type – Nn are the most often encountered ones simply because of their classical appeal.  Although a *doxopixo* pivot with cell type –Pp or a *poxodixo* pivot with cell-type +Nn may usually seem to be simply unacceptable or extremely unlikely, it may indeed turn out that such a pivot choice can yield a surprisingly large decrease in the infeasibility index, depending on the tableau data entries, especially as an intermediary pivoting iteration, and therefore worth the consideration.  The versatility of our spdspds algorithm allows for such pivot selection decisions.

(3) The proposed concept of *infeasibility index* along with the use of a *measure of goodness* for the pivot selection determined as/by the anticipated *decrease in the infeasibility index* considered as a *global effectiveness measure* (*gem* = - $\tau$) arising due to the specific pivot selection, turns out to be a great grand breakthrough in achieving the ultimate performance challenge in the use of symplex method for solving linear programming problems.  One can use either $\sigma$ (if m $\leq$ n) or $\rho$ (if n $\leq$ m) as a tie-breaking measure to choose among alternatives having the same value for $\tau$.  Also note that the *entire infeasibility analysis is independent of and unaffected by any scaling of rows/columns*.

(4) Problem of *cycling* en-route towards the terminal tableau, possibly caused by some *intermediate degeneracy* will get *prevented* because of the very nature of the *spdspds* pivot selection strategy, seeking the best possible decrease of the infeasibility index through a non-degenerate pivot. If and when encountered with a possible degeneracy that seems to block further progress in decreasing the infeasibility index, the *lexicographic ordering of the variables* may be used as a tie-breaking mechanism to guide the choice of pivot element, thus coursing through a part of the cycle, until a point is reached wherefrom *spdspds* finds a step down the infeasibility path again, unless it happens to be the terminal tableau. So, *spdspds is effectively immune to problem of cycling caused by any possible intermediate degeneracy*.

(5) Potential possibility of *numerical instability* can be avoided by careful elimination of poor choices of the pivot element, using *appropriate filters* - even if it requires going for a next-best pivot choice in terms of *gem* (*decrease in the infeasibility index*) - in order to avoid such treacherous pathways leading to numerically disastrous computational behavior - that is effectively like first exchanging I for K followed by an exchange of K for J in order to achieve an exchange of I for J - equivalent to using an appropriate *pre-conditioner* without actually doing so.

## 14. COMPUTATIONAL COMPLEXITY

For a linear programming problem represented in its standard canonical form using the CST of size (m)x(n) the size(length) of the input data (problem size) $L$ can be taken as ~ [(m+1)*(n+1)].

In the worst-case, the infeasibility index of the initial tableau will at most be {(m)+(n)} and at each *spdspds* symplex pivoting iteration the infeasibility index gets reduced by at least *one* so that it takes at most {(m)+(n)} *spdspds* symplex pivoting iterations to reach the optimum solution if one exists; or may even be well before that in order to report the infeasibility status of the given problem. Therefore, the number of *spdspds* symplex pivoting iterations required to solve the linear programming problem is, $O(L^{1/2})$ in the worst-case. However, the expected number of *spdspds* symplex pivoting iterations is ≤min{(m),(n)} because *every symplex pivot choice corresponds to a decrease in the infeasibility index and also that the basic/non-basic status of each variable is not expected to switch around much except in rare instances of potential numerical instability etc*.

Each *spdspds* symplex pivoting operation requires a complete analysis of the infeasibility status before performing the actual pivoting operation, and that itself requires ≤[2*(m)*(n)] floating point operations, that is $O(L)$. Each *spdspds* symplex pivoting operation requires ≤[3*(m+1)*(n+1)] floating point operations, which is again $O(L)$.

Thus, the total computational work involved in solving the linear programming problem is ≤{(m)+(n)}*[5*m*n + 3*m + 3*n] *floating-point arithmetic operations*, which is of course $O(L^{1.5})$ - that is *super-linear sub-quadratic polynomial time complexity bound*. We assert that this is indeed the lower limit in worst-case computational complexity for any symplex/simplex based algorithm for LP problem and that in fact it represents the computational complexity of LP problem itself.

It can be shown easily that the *space complexity* is only $O(L)$ - that is, linear complexity bound.

Note that *spdspds* has *finite termination* and provides a definite *output*, that is, either the optimum solution if & when one exists, or a point of *minimal infeasibility* beyond which no further *spdspds* symplex pivoting is possible towards any improvement in the feasibility of the given problem - which itself may provide insight as to the possible refinements in the problem formulation itself.

## 15. DIRECTIONS FOR FUTURE RESEARCH

0: The spdspds algorithm provides a scope for future research work with a motivation to further enhance the computational efficiency - by *exploiting the complete information content* that is made available through the *infeasibility analysis* of initial tableau.

1: Analyze the entries in the initial tableau $T_0$ and make a statement regarding the basic/non-basic status of each of the variables in the optimum tableau $T_*$. This question is conceptually equivalent to asking, in the context of non-linear programming [11] as to whether a constraint will be *active* (non-basic) or otherwise at the optimum - although of course expecting an answer in that context may certainly not be practical. However, the situation with linear systems can be more promising so as to expect a possible attempt in answering that question - based only on a thorough analysis of either the initial tableau $T_0$, or equivalently, any tableau $T_k$ en-route towards the optimum - using a radically *redefined* concept of binding/nonbinding [8][12][13] constraints - implicitly achieved by the *spdspds* algorithm as explained herein below:

2: For any column $J \in C$ the set of rows $\{ineg\tau(J)\}$ with $\tau_{iJ} < 0$ that is $\{(i \in R):(\tau_{iJ}<0)]\}$ along with the associated *set of ratios* $\{r_{ineg\tau(J)}\}$ that is $\{\beta_{ineg\tau(J)}/\alpha_{ineg\tau(J).J}\}$ may be determined. Suppose there exists a row $I \in R$ having its ratio $\beta_I/\alpha_{IJ}$ for every column $J \in C$ that is farther away from zero beyond (relative to) the set of ratios $\{r_{ineg\tau(J)}\}$.
$(\exists I \in R)(\forall J \in C)[\{\alpha_{IJ}=0\} \lor (\forall i \in R\backslash I)[(0 \leq \{\beta_{ineg\tau(J)}/\alpha_{ineg\tau(J).J}\}<\beta_I/\alpha_{IJ}) \lor (\beta_I/\alpha_{IJ}<\{\beta_{ineg\tau(J)}/\alpha_{ineg\tau(J).J}\} \leq 0)]]$
Note that the feasibility/infeasibility of such a row (*non-binding?row*) will not be affected by any of our symplex pivoting operation.

Exactly on the same lines, for any row $I \in R$ the set of columns $\{jneg\tau(I)\}$ with $\tau_{Ij} < 0$ that is $\{(j \in C):(\tau_{Ij}<0)]\}$ along with the associated *set of ratios* $\{r_{jneg\tau(I)}\}$ that is $\{\gamma_{jneg\tau(I)}/\alpha_{I.jneg\tau(I)}\}$ may be determined. Suppose there exists a column $J \in C$ having its ratio $\gamma_J/\alpha_{IJ}$ for every row $I \in R$ that is farther away from zero beyond (relative to) the set of ratios $\{r_{jneg\tau(I)}\}$.
$(\exists J \in C)(\forall I \in R)[\{\alpha_{IJ}=0\} \lor (\forall j \in C\backslash J)[(0 \leq \{\gamma_{jneg\tau(I)}/\alpha_{I.jneg\tau(I)}\}<\gamma_J/\alpha_{IJ}) \lor (\gamma_J/\alpha_{IJ}<\{\gamma_{jneg\tau(I)}/\alpha_{I.jneg\tau(I)}\} \leq 0)]]$
Note that the feasibility/infeasibility of such a column (*non-binding?column*) will not be affected by any of our symplex pivoting operation.

This explains how *spdspds* algorithm effectively avoids any *unnecessary swapping back and forth of the basic/non-basic status of each variable* (that has been indeed the root cause of the severe computational inefficiency in the classical simplex method and all its variants - that has never been effectively addressed till now) by the very choice of the *spdspds* symplex pivot corresponding to the largest possible decrease in the infeasibility index at each symplex iteration.

Q: is there any further improvement possible beyond this? Suppose there exists a row $i \in R$ having its ratio $\beta_i/\alpha_{ij}$ for every column $j \in C$ that is closer to (no farther from) zero relative to the set of ratios $\{r_{ineg\tau(j)}\}$ - can we say that such a row (*binding?row*) will certainly be pushed to primal non-basic status by some pivoting operation? Suppose there exists a column $j \in C$ having its ratio $\gamma_j/\alpha_{ij}$ for every row $i \in R$ that is closer to (no farther from) zero relative to the set of ratios $\{r_{jneg\tau(i)}\}$ - can we say that such a column (*binding?column*) will certainly be pushed to primal basic status by some pivoting operation?

3: The combined effect of a simultaneous application of more than one symplex pivot (rather than a *sequence of symplex pivoting operations*) is indeed worth further detailed study, and can possibly lead to what may be called as the general "spdspds *omni*plex pivoting" operation.

4: The versatility of the spdspds algorithm allows for extensive combinatorial analysis along with numerical experimentation to explore further possibilities, including possible applications in *polyhedral combinatorics*, *network optimization*, etc.  Also, for example, as in the classical approaches, one can possibly seek to first achieve primal or dual feasibility, more efficiently, by utilizing the best possible decrease in the corresponding component - either $\sigma$ or $\rho$ rather than $\tau$, before going further with later iterations.

## 16.  A POSER TO THE COOL-HEADED BRAVE-HEARTS

Here, we present a poser to you, our cool-headed brave-heart counter-part:

Refer to the notation used in Figure-4 and consider the two scenarios described below:

$[\forall I \in R]:[\nexists J \in C\{\{I=I_{PSP}(J)\} \vee \{I=I_{PTP}(J)\}\}]; \& [\forall J \in C]:[\nexists I \in R\{\{J=J_{DSP}(I)\} \vee \{J=J_{DTP}(I)\}\}];$

or the scenarios described below:

$[\forall I \in R]:[\nexists J \in C\{\forall i \in R\{(\beta_I \geq 0) \wedge ((0 \leq \beta_I/\alpha_{IJ} < \beta_i/\alpha_{iJ}) \vee (\beta_i/\alpha_{Ij} < \beta_I/\alpha_{IJ} \leq 0))\} \vee \{(\beta_I < 0) \wedge ((0 \leq \beta_i/\alpha_{iJ} < \beta_I/\alpha_{IJ}) \vee (\beta_I/\alpha_{IJ} < \beta_i/\alpha_{iJ} \leq 0))\}\}];$

&

$[\forall J \in C]:[\nexists I \in R\{\forall j \in C\{(\gamma_J \leq 0) \wedge ((0 \leq \gamma_J/\alpha_{IJ} < \gamma_j/\alpha_{Ij}) \vee (\gamma_j/\alpha_{Ij} < \gamma_J/\alpha_{IJ} \leq 0))\} \vee \{(\gamma_J > 0) \wedge ((0 \leq \gamma_j/\alpha_{Ij} < \gamma_J/\alpha_{IJ}) \vee (\gamma_J/\alpha_{IJ} < \gamma_j/\alpha_{Ij} \leq 0))\}\}];$

and then may even possibly consider the scenarios described below:

$[\{(\forall I \in R)(\nexists J \in C)\}:\{(\forall i \in R)(\sigma_{IJ} < \sigma_{iJ} \leq 0)\}] \ \& \ [\{(\forall J \in C)(\nexists I \in R)\}:\{(\forall j \in C)(\rho_{IJ} < \rho_{Ij} \leq 0)\}];$

A careful reading of the above will reveal that the underlying reasoning here is what leads to the concept of *infeasibility index* as a *global effectiveness measure* for a symplex pivoting operation that form the conceptual foundation for the design of *spdspds* expressed by the following:

$$\neg[(\exists I \in R)(\exists J \in C)]:[\{(\forall i \in R)(\forall j \in C)(i \neq I)(j \neq J)\{\tau_{IJ} < \tau_{ij} \leq 0\}]$$

A careful reading of this entire report will convince you that the *spdspds* algorithm has been designed with strong/rigorous logical/mathematical justification at every step.

## 17.  CONCLUSION

The proposed concept of *infeasibility index* is an *inverse measure of goodness* associated with a CST tableau.  The *decrease in the infeasibility index* as a *global effectiveness measure* (*gem*) associated with each potential pivot element forms the basic foundation for the proposed *spdspds* algorithm.  Also note that the *entire infeasibility analysis is independent of and unaffected by any scaling of rows/columns*.

The proposed *spdspds* algorithm provides a novel viewpoint to the very same classic framework of Goldman-Tucker Compact-Symmetric-Tableau representation for LP - with a newly defined concept of *symplex* pivot; leading to an *efficient*, *robust* and *versatile* iterative solution strategy

passing through a non-repetitive sequence of symplex tableaus requiring *minimum number of symplex iterations*, with a worst-case computational complexity of $O(L^{1.5})$.

The proposed concept of the *CST-signature* can be utilized to keep track of the computational path from the initial tableau to the terminal tableau, and for advance detection of possible cycling before it actually occurs - even though of course the *spdspds* algorithm is effectively immune to any cycling caused by possible intermediate degeneracy.

For the first time in the history of Linear Programming, we have achieved -
(0.0) effective utilization of the Goldman-Tucker Compact-*Sym*metric-Tableau (CST) which is a *unique symmetric* representation *common* to both the primal as well as the dual of a linear programming problem in its *standard canonical form*, and in which the algebra/arithmetic of the pivoting operation also gets represented by a unique single set of operations irrespective of whether it is a primal pivot or a dual pivot;
(1.0) with '*symplex*' - a new avataar of 'simplex' in the 21st century, the concept of
(2.0) *infeasibility-index* for a P-D pair has been defined and effectively used as a
(3.0) *global-effectiveness-measure*, in the design of 'spdspds' algorithm,
to guide the pivoting process, with *five guarantees* -
(3.1)* improvement of the overall P-D feasibility at every iteration,
(3.2)* immunity against cycling,
(3.3)* finite termination,
(3.4)* minimum number of spdspds symplex pivoting iterations,
(3.5)* worst case computational complexity of $O(L^{1.5})$ -
(4.0) with a *re-defined concept of binding / non-binding constraints implicitly incorporated* -
(5.0) by this Game-Changer 'spdspds' Algorithm,
claiming to be the Great Grand Breakthrough, having successfully resolved-&-reposed the *Linear Programming Performance Challenge of the millennium* -
refer to the 9th among the 18 "Mathematical Problems for the Next Century" [14][15] -
(6.0) with consequent *immediate* as well as *lasting*, *deep* as well as *far-reaching* impact on the study, teaching and application of *Linear Programming* and *Algorithms* in particular as well as *Computational Complexity Analysis* in general.

The Linear-Programming Grand-Master reads '**spdspds**' as '**y**es-s**p**ee**d**y-yes-s**p**ee**d**y-ye**s**' (**y**es! - wh**y** silent?) while rejoicing its 180-degree rotational *sym*metry as a string symbol.


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## 19.  ACKNOWLEDGEMENT

I wish to put on record my gratitude towards Prof. Gerald L Thompson of CMU GSIA who introduced the use of  Goldman-Tucker Compact-*Sym*metric-Tableau for LP  to me in his classes.

An earlier version of this report [*Ms.No.MAPR-D-11-00001*] was *rejected* [*2011JAN18*] for the reason - "*the content is not sufficient for the report to appear in MPA*" - therefore making me acknowledge the view-point : "*rejection can indeed be turned around towards perfection*"!

I must necessarily confess that the core idea behind '*spdspds*' is so stunningly & elusively simple, that one may simply be taken aback in a profound wonder-struck jaw-drop-silence, possibly with an after-thought : "*oh my goodness, how could it be that it never flashed on me any time earlier*"!

I acknowledge that this report was *not accepted* (2020JUN01) for *uploading* onto the Optimization Online E-print Repository; the reason given was: "*The report seems wrong*"!

On the auspicious vidyaa(vijaya)daSami day [2020OCT25] I wish to share my pleasure in giving a new name as '*sym*plex pivot' to the now redefined *sim*plex pivot, emphasizing the primal-dual symmetry therein, along with the expanded scope in terms of the far wider range of choices available in the selection of such pivot element, while maintaining the minimal combinatorial/structural requirement of being simply an exchange between a basic (dependent) variable and a non-basic (independent) variable, in the Goldman-Tucker Compact *Sym*metric Tableau (CST) which is a unique symmetric representation common to both the primal as well as the dual of a linear programming problem in its standard canonical form.  This renaming acknowledges that our  '*symplex*' is indeed a rebirth of the  classical  *simplex*  defined by Dantzig, although after well over seven decades, now as a new avataar in the twenty first century.

## 20.  DEDICATION

To my ಅಜ್ಜ (ajja) & ಅಜ್ಜಿ (ajji) &  Karinja Halemane Keshava Bhat  & Thirumaleshwari  and to my ಅಪ್ಪ (appa) & ಅಮ್ಮ (amma) Shama Bhat  & Thirumaleshwari  for their teachings through love, that quality matters more than quantity; to my wife Vijayalakshmi  for her ever consistent love & support; to my daughter Sriwidya.Bharati and my twin sons Sriwidya.Ramana  & Sriwidya.Prawina  for their love & affection.

The Perpetual Intellectual Property Rights (PIPR:©:) resides with this Original Author-Creator and to be availed by his legal heirs for perpetuity.

To all the *cool-headed brave-hearts*, eagerly awaited but probably yet to be visible among the world optimization professionals, who would be attracted to and certainly capable of effectively understanding without any prejudice and appreciating the deeper details enshrined in '*spdspds*', who may opt for '*symplex*' over '*simplex*' and '*spdspds*' over any other strategy for solving linear programming problems.

<div align="center">ॐतत्सत्</div>