

UNBELIEVABLE $O(L^{1.5})$ WORST CASE COMPUTATIONAL COMPLEXITY ACHIEVED BY *spdspd*s ALGORITHM FOR LINEAR PROGRAMMING PROBLEM

©Dr.(Prof.) Keshava Prasad Halemane,
Professor - retired from
Department of Mathematical And Computational Sciences
National Institute of Technology Karnataka, Surathkal
Srinivasnagar, Mangaluru - 575025, India.
Residence: 8-129/12 SASHESHA, Sowjanya Road, Naigara Hills,
Bikarnakatte, Kulshakar Post, Mangaluru - 575005. Karnataka State, India.
k.prasad.h@gmail.com [+919481022946]
<https://arxiv.org/abs/1405.6902>
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<https://hal.archives-ouvertes.fr/hal-03087745>
<https://www.linkedin.com/in/keshavaprasadahalemane/>



ABSTRACT

The Symmetric Primal-Dual *Symplex* Pivot Decision Strategy (*spdspd*s) is a novel iterative algorithm to solve linear programming problems. Here, a simplex pivoting operation is considered simply as an *exchange* between a basic (dependent) variable and a non-basic (independent) variable, in the Tucker's *Compact Symmetric Tableau* (CST) which is a unique symmetric representation common to both the primal as well as the dual of a linear programming problem in its standard canonical form. From this viewpoint, the *classical simplex pivoting* operation of Dantzig may be considered as a restricted special case.

The *infeasibility index* associated with a simplex tableau is defined as the sum of the number of primal variables and the number of dual variables that are infeasible. A *measure of goodness* as a *global effectiveness measure* of a pivot selection is defined/determined as/by the *decrease in the infeasibility index* associated with such a pivot selection. At each iteration the selection of the simplex pivot element is governed by the anticipated *decrease in the infeasibility index* - seeking the best possible *decrease in the infeasibility index* - from among a wide range of candidate choices with non-zero values - limited only by considerations of *potential numerical instability*. The algorithm terminates when further reduction in the infeasibility index is not possible; then the tableau is checked for the *terminal tableau type* to facilitate the *problem classification* - a termination with an *infeasibility index* of zero indicates optimum solution. Even in the absence of an optimum solution, the versatility of the *spdspd*s algorithm allows one to explore/determine the most suitable alternative solutions in a given context, including a comprehensive parametric analysis, etc. The worst case computational complexity of *spdspd*s is shown to be $O(L^{1.5})$.

Keywords: optimization, linear programming, algorithm, simplex, symplex,
symmetric primal dual *symplex*, *spdspd*s, computational complexity

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1. INTRODUCTION

Linear Programming (LP) problem represents one of the most widely used class of computational models, for which any possible improved solution technique would certainly be highly desirable. Of course, there has been several alternative solution strategies suggested including the *classical simplex method* of Dantzig [1] and several variations thereof, followed by recent polynomial time algorithms, namely the Ellipsoid Method of Khachiyan [2] [3] and the Karmarkar Algorithm [4] - both classified now as belonging to Interior Point Algorithms. Terlaky [5] Todd [6] and Adler et al [7] present a summary view of the various developments as well as a projected futuristic view of the overall situation. Let us not get diverted much into the historical developments etc.

The simplex pivoting operation of Dantzig represents a move from one basis/tableau to another basis/tableau, by/through a *single exchange* between an entering *infeasible* non-basic variable and a leaving *feasible* basic variable. For a chosen entering *infeasible* non-basic variable, the leaving *feasible* basic variable is to be selected so as to meet certain restrictive criteria in terms of the corresponding limitation of moving only between two *neighboring extreme/vertex points* of the polytope defined by the set of linear system of inequalities - so as to maintain feasibility while improving the objective function value by moving further towards the optimum.

The proposed *spdspds* approach can be considered as a novel generalization of the simplex method of Dantzig, in terms of lifting all of such restrictions and providing a wider scope for the selection of the pivots - any nonzero element of the coefficient matrix in the tableau can be a potential candidate pivot element. It is indeed true that the very term *simplex pivot* has been redefined here - as a simple/single *exchange* between an entering non-basic variable and a leaving basic variable - maintaining only that combinatorial/structural property of being a simple/single *exchange* between a selected pair - that being the justification for renaming it as *symplex pivot* - emphasizing the primal-dual symmetry therein. The symplex pivot of *spdspds* need not necessarily correspond to a pair consisting of an entering *infeasible* non-basic variable and a leaving *feasible* basic variable; also it does not require to be limited to a move between neighboring extreme/vertex points of the associated polytope - although the move does indeed correspond to one between a pair of *intersection points* defined by the set of linear system of inequalities.

The actual selection of a *spdspds* pivot element is governed by an analysis of the associated measure of goodness of such a pivot choice. A global measure of goodness or a *global effectiveness measure (gem)* for pivot selection is defined, utilizing the novel concept of *infeasibility index* associated with a symplex tableau - defined as the sum of the number of primal variables and the number of dual variables that are infeasible. The change in the infeasibility index associated with a symplex pivot element can be determined by a thorough analysis of the tableau data. To guarantee the best computational performance, it is proposed to select a pivot element corresponding to the best possible decrease in the infeasibility index.

When further decrease in the infeasibility index is not possible, then the pivoting process is terminated. An analysis of the data pattern in the *terminal tableau* can be used to classify the problem into one of the possible *six categories* as explained herein later.

2. TUCKER'S COMPACT SYMMETRIC TABLEAU

We will go through some well known preliminaries for the sake of establishing the notational conventions used in this paper, as used in our earlier reports [8] and [9].

The Symmetric Primal-Dual Pair of LP in the Standard Canonical Form is as follows:

Primal Problem:

$$\begin{array}{llll} \text{maximize} & c.x & = & f \\ \text{s.t.} & A.x & \leq & b \\ & x & \geq & 0 \end{array} \quad (1)$$

Dual Problem:

$$\begin{array}{llll} \text{minimize} & v.b & = & g \\ \text{s.t.} & v.A & \geq & c \\ & v & \geq & 0 \end{array} \quad (2)$$

The descriptions for each of the problem parameters in (1) & (2) above are as follows:

x	Primal decision variables	n x 1 vector
c	Primal objective function coefficients	1 x n vector
f	Primal objective function value	1 x 1 scalar
A	Primal constraint coefficient matrix	m x n matrix
b	Primal constraint upper bound	m x 1 vector
v	Dual decision variables	1 x m vector
g	Dual objective function value	1 x 1 scalar

We introduce the m x 1 vector y of slack variables to (1) and the 1 x n vector u of surplus variables to (2) to write the symmetric primal-dual pair in canonical form as follows:

Primal Problem:

$$\begin{array}{llll} \text{maximize} & c.x + 0.y & = & f \\ \text{s.t.} & A.x + I_m.y & = & b \\ & x, y & \geq & 0 \end{array} \quad (3)$$

Dual Problem:

$$\begin{array}{llll} \text{minimize} & v.b + u.0 & = & g \\ \text{s.t.} & v.A - u.I_n & = & c \\ & v, u & \geq & 0 \end{array} \quad (4)$$

This Symmetric Primal-Dual pair is represented in the Tucker's Compact Symmetric Tableau (CST) as shown in Figure-1.

	x_j	-1	
v_i	a_{ij}	b_i	$= -y_i$
-1	c_j	0	$= f$
	$ $ u_j	$ $ g	

Figure-1: Initial Compact Symmetric Tableau (T₀)

The relations in (3) & (4) corresponding to the Primal and Dual can be combined to give -

$$c.x + u.x = v.A.x = v.b - v.y \quad (5)$$

$$f + u.x = v.A.x = g - v.y \quad (6)$$

For any feasible (basic or non-basic) solution to the P-D pair,

$$u, x, v, y \geq 0 \quad (7)$$

and therefore,

$$(g - f) = (u.x + v.y) \geq 0 \quad (8)$$

For any basic solution (feasible or infeasible) to the P-D pair, the non-basic variables are set to zero; that is,

$$u, y = 0 \quad (9)$$

and therefore,

$$(g - f) = (u.x + v.y) = 0 \quad (10)$$

The entries in the Tucker's Compact Symmetric Tableau (CST) directly correspond to the associated basic solution to the P-D pair, thus establishing a one-to-one correspondence between a solution basis and the associated Tucker's Compact Symmetric Tableau (CST). A pivoting operation on the CST tableau corresponds to the associated move from one basis to another.

For the LP problem pair (1) & (2) or equivalently (3) & (4) the above tableau in Figure-1 represents the initial tableau indicating the *initial basic solution* (IBS) wherein y_i are the primal basic variables associated (one to one permanent association) with v_i the dual non-basic variables, and x_j are the primal non-basic variables associated (one to one permanent association) with u_j the dual basic variables. Note that x_j (and the -1) are column-labels and v_i (and the -1) are row labels in the tableau, and the way to interpret (read) the tableau is as follows:

Primal Problem:

$$\begin{aligned} \sum_{j \in C} a_{ij} \cdot x_j - b_i &= -y_i, \quad i \in R \text{ (row index)} \\ \sum_{j \in C} c_j \cdot x_j - 0 &= f \quad \text{(function to be maximized)} \end{aligned} \quad (11)$$

Dual Problem:

$$\begin{aligned} \sum_{i \in R} v_i \cdot a_{ij} - c_j &= u_j, \quad j \in C \text{ (Column index)} \\ \sum_{i \in R} v_i \cdot b_i - 0 &= g \quad \text{(function to be minimized)} \end{aligned} \quad (12)$$

wherein the variables x_j, y_i, v_i, u_j are all considered to be non-negative.

3. ALGEBRA (ARITHMETIC) OF SYMPLEX PIVOTING PROCESS

With the Tucker's Compact Symmetric Tableau (CST) representation for linear programming, in its standard/canonical form, one can observe that once a pivot element is selected, the actual pivoting process (the *algebra* and hence the *arithmetic* operations) is the same irrespective of the pivot selection; for example whether it is a primal pivot or a dual pivot. Hence it suffices to present here a single (common) set of operations representing that pivoting process. This *expressional elegance* and *computational efficiency* along with *convenience* and *versatility* (as will be evident later) are the reasons why the above representation has been selected for the purpose of our study; motivated by [10].

For the sake of generality, let us imagine that we are somewhere in the middle of solving a LP problem (say after the k^{th} iteration), and have the system model represented by a tableau (T_k) as shown in Figure-2.

By the nature of the sequence of elementary row (column) operations being performed during any pivoting process, the system model represented in Figure-2 is equivalent to that represented by the initial tableau which corresponds to the P-D pair (11) & (12). The transformed version of the primal-dual pair directly expressed by the above tableau is as follows:

	z_j^N	-1	
w_i^N	α_{ij}	β_i	$= -z_i^B$
-1	γ_j	δ	$= f$
	\parallel w_j^B	\parallel g	

Figure-2: Compact Symmetric Tableau (T_k) after k iterations

Primal Problem:

$$z_i^B = \beta_i - \sum_{j \in C} \alpha_{ij} \cdot z_j^N, \quad i \in R \text{ (row index)} \quad (13)$$

$$f = -\delta + \sum_{j \in C} \gamma_j \cdot z_j^N, \quad \text{(function to be maximized)}$$

Dual Problem:

$$w_j^B = -\gamma_j + \sum_{i \in R} w_i^N \cdot \alpha_{ij}, \quad j \in C \text{ (column index)} \quad (14)$$

$$g = -\delta + \sum_{i \in R} w_i^N \cdot \beta_i \quad \text{(function to be minimized)}$$

The effect of a pivoting operation on (13) & (14) performed with a chosen pivot element α_{IJ} is exactly to affect an exchange between the P-D variable pairs indicated by I and J in (13) and (14). That is, z_j^N is entered into primal basis in exchange for z_i^B in (13), and w_i^N is entered into dual basis in exchange for w_j^B in (14). Suppose we have chosen the pivot element α_{IJ} using some appropriate pivot selection scheme, and we would like to derive the resulting tableau (T_{k+1}). Let the resulting tableau (T_{k+1}) be indicated in Figure-3.

$$\begin{array}{ccc}
 & (z_j^N)' & -1 \\
 (w_i^N)' & \boxed{\begin{array}{c|c} (\alpha_{ij})' & (\beta_i)' \end{array}} & = -(z_i^B)' \\
 -1 & \boxed{\begin{array}{c|c} (\gamma_j)' & (\delta)' \end{array}} & = f \\
 & \parallel & \parallel \\
 & (w_j^B)' & g
 \end{array}$$

Figure-3: Compact Symmetric Tableau (T_{k+1}) after $(k+1)$ iterations

The algebra (*arithmetic*) of deriving the above tableau (T_{k+1}) of Figure-3 from the previous tableau (T_k) of Figure-2 is detailed below (directly adaptable for “*omniplex*” pivoting - refer Section-15):

$$\begin{aligned}
 (\alpha_{IJ})' &\leftarrow (\alpha_{IJ})^{-1}; & (\alpha_{Ij})' &\leftarrow (\alpha_{IJ})^{-1}\alpha_{Ij}; & (\beta_I)' &\leftarrow (\alpha_{IJ})^{-1}\beta_I; \\
 (\alpha_{iJ})' &\leftarrow -\alpha_{iJ}(\alpha_{IJ})^{-1}; & (\alpha_{ij})' &\leftarrow \alpha_{ij} - \alpha_{iJ}(\alpha_{IJ})^{-1}\alpha_{Ij}; & (\beta_i)' &\leftarrow \beta_i - \alpha_{iJ}(\alpha_{IJ})^{-1}\beta_I; \\
 (\gamma_J)' &\leftarrow -\gamma_J(\alpha_{IJ})^{-1}; & (\gamma_j)' &\leftarrow \gamma_j - \gamma_J(\alpha_{IJ})^{-1}\alpha_{Ij}; & (\delta)' &\leftarrow \delta - \gamma_J(\alpha_{IJ})^{-1}\beta_I;
 \end{aligned}$$

along with an exchange of labels associated with row I and column J; that is effectively:

$$(z_I^N)' \leftarrow z_I^B; \quad (z_I^B)' \leftarrow z_I^N; \quad (w_J^B)' \leftarrow w_I^N; \quad (w_I^N)' \leftarrow w_J^B;$$

while retaining the very same labels for all the other rows and columns; that is:

$$(z_j^N)' \leftarrow z_j^N; \quad (z_i^B)' \leftarrow z_i^B; \quad (w_j^B)' \leftarrow w_j^B; \quad (w_i^N)' \leftarrow w_i^N;$$

for $i \in R \setminus \{I\}$ and $j \in C \setminus \{J\}$.

It is to be noted here that the Tucker's Compact Symmetric Tableau (CST) is a *unique symmetric* representation *common* to both the primal as well as the dual of a linear programming problem in its standard canonical form. Also, the tableau evolves from T_0 as the initial tableau representing the problem as posed (or close to being so, after some due preprocessing) along the sequence of simplex iterations all the way to the final terminal tableau T^* - while always being a tableau representation of an *equivalent system* of linear inequalities as well as the objective function.

From (13) & (14) above, we can get:

$$(f+\delta) + \sum w_j^B \cdot z_j^N = \sum \sum w_i^N \cdot \alpha_{ij} \cdot z_j^N = (g+\delta) - \sum w_i^N \cdot z_i^B \quad (15)$$

and therefore,

$$(g-f) = \sum w_i^N \cdot z_i^B + \sum w_j^B \cdot z_j^N \quad (16)$$

The values of the primal and the dual basic variables as well as (the primal & the dual) objective function value corresponding to a basis/tableau can be directly read from the entries of the tableau - the primal objective function value being always the same as the dual objective function value for every basis/tableau all along the sequence of simplex iterations - each of the summation terms in both (15) and (16) above being zero, and the *complementary slackness condition being automatically satisfied for every basis/tableau*. The relationships (15) & (16) above indicate the effect of moving away from a basis, as the non-basic variables are moved away from zero; in particular, one can note from (16) above that *for all feasible solutions, the dual objective function value is an upper bound for the primal objective function value and the primal objective function value is a lower bound for the dual objective function value*, which is a well known relationship. A basic solution for the P-D pair is optimal if & only if it is basic feasible for the P-D pair.

Now, one may skip sections 4 to 8 and go directly to section 9 in order to avoid a pedagogical detour, and possibly come back to these sections if and when necessary to fill the gaps in one's detailed understanding.

4. A TYPICAL SYMPLEX PIVOT SELECTION SCHEME

A typical simplex pivot selection scheme can include *four (two pairs) fundamental types of simplex pivot selections* namely Primal Standard Pivot (PSP), Dual Standard Pivot (DSP), Primal Tricky Pivot (PTP) and Dual Tricky Pivot (DTP) that are available for simplex pivoting process in solving linear programming problems. The algebra of these pivot selections are given in Figure-4, along with a schematic representation of the Tableau Data Pattern that leads to such pivot selection.

5. EFFECT OF A TYPICAL PIVOTING OPERATION

It is useful at this point to make a few observations regarding the effect of pivoting operation, in each of the above pivot selections.

DSP {*poxidixi*} brings about an immediate improvement in the primal feasibility w.r.t. the pivot row, without deterioration of dual feasibility. The extent of this improvement in primal feasibility can be measured by the corresponding improvement (decrease) in the value of the dual objective function, given by $|\beta_I \gamma_J / \alpha_{IJ}|$.

PSP {*doxipixi*} brings about an immediate improvement in the dual feasibility w.r.t. the pivot column, without deterioration of primal feasibility. The extent of this improvement in dual feasibility can be measured by the corresponding improvement (increase) in the value of the primal objective function, given by $|\beta_I \gamma_J / \alpha_{IJ}|$.

PTP {*doxopoxi*} brings about an immediate improvement in the primal feasibility w.r.t. the pivot row, without any concern to the dual feasibility. The extent of this improvement in primal feasibility can be measured by the corresponding improvement (increase) in the value of the primal objective function, given by $|\beta_I \gamma_J / \alpha_{IJ}|$.

DTP {*poxodoxi*} brings about an immediate improvement in the dual feasibility (w.r.t. the pivot column, at least) without any concern to the primal feasibility. The extent of this improvement in dual feasibility can be measured by the corresponding improvement (decrease) in the value of the dual objective function, given by $|\beta_I \gamma_J / \alpha_{IJ}|$.

(a) Dual Standard Pivot, DSP: {*poxi**dixi*}

•	—	—
•	•	⊕
+	Θ	•

↓

$$I \in \{i \in R \mid \beta_i < 0\};$$

$$J_{\text{DSP}}(I) \leftarrow \arg\min_{j \in C} \{(\gamma_j / \alpha_{ij}) \mid \gamma_j \leq 0; \alpha_{ij} < 0\};$$

(b) Primal Standard Pivot, PSP: {*doxi**pixi*}

•	•	—
+	•	⊕
+	Θ	•

↑

$$J \in \{j \in C \mid \gamma_j > 0\};$$

$$I_{\text{PSP}}(J) \leftarrow \arg\min_{i \in R} \{(\beta_i / \alpha_{ij}) \mid \beta_i \geq 0; \alpha_{ij} > 0\};$$

(c) Primal Tricky Pivot, PTP: {*doxo**poxi*}

—	•	—
•	•	⊕
*	Θ	•

↑

$$J \in \{j \in C \mid \gamma_j \leq 0\};$$

$$I_{\text{PTP}}(J) \leftarrow \arg\max_{i \in R} \{(\beta_i / \alpha_{ij}) \mid \beta_i < 0; \alpha_{ij} < 0\};$$

(d) Dual Tricky Pivot, DTP: {*poxo**doxi*}

+	•	*
•	•	⊕
+	Θ	•

↓

$$I \in \{i \in R \mid \beta_i \leq 0\};$$

$$J_{\text{DTP}}(I) \leftarrow \arg\max_{j \in C} \{(\gamma_j / \alpha_{ij}) \mid \gamma_j > 0; \alpha_{ij} > 0\};$$

Figure-4:. Four Types of Typical Symplex Pivot Selections

— negative; 0 zero; + positive; Θ non-positive; * any value; ⊕ non-negative; • un-analyzed

6. CELL TYPE

Each potential pivot element/cell in the Compact Symmetric Tableau (CST) is characterized by a “cell-type”. The cell type of a cell in I^{th} row and J^{th} column consists of three components. They are the sign of α_{IJ} (either ‘0’, ‘+’ or ‘—’), the sign of β_i (either ‘Z’, ‘P’ or ‘N’) and the sign of γ_j (either ‘z’, ‘p’ or ‘n’). Hence there will be a total of 27 different cell types. However, if α is zero or numerically near-zero, it will not be a potential pivoting cell, since pivoting will not be performed on such cells. Hence the nine cell types with $\alpha \approx 0$ are combined together and the new cell type given is 0**. Thus there are 19 cell types for our consideration.

7. TYPICAL SYMPLEX PIVOT DECISION STRATEGY

The four distinct types of *typical* pivot selections mentioned above may be considered for designing a pivoting strategy. It is to be noted here that the pivot selection as well as the pivoting operation is symmetrical w.r.t. the primal problem and the dual problem, both represented in the very same Tucker's Compact Symmetric Tableau representation of a linear programming problem.

The four distinct types of typical pivot selections may be considered in the following default preference order: $\{\{DSP,PTP\},\{PSP,DTP\}\}$ to drive towards primal feasibility first; or alternatively $\{\{PSP,DTP\},\{DSP,PTP\}\}$ to drive towards dual feasibility first.

At every iteration, an attempt is made to select a pivot element/cell, by checking the possible pivot selections belonging to one of the above four types of pivot selections in the default preference order as specified above. It is to be noted that depending upon the actual data in the Tableau, a pivot selection of specific type which was not possible in an earlier iteration, can become possible in a later iteration, sometimes even in the very next following iteration. That is why it is a crucial part of an algorithm to check in each (and every) iteration, for each of the four types of possible pivot selections preferably (although not necessarily) in a pre-specified preference order. At each step mentioned above, if there is more than one pivot element of that particular cell type, then the choice can be narrowed by utilizing some measure of goodness for the pivot selection, appropriately defined.

8. MEASURE OF GOODNESS FOR PIVOT SELECTION

From the earlier discussion above, one can observe that the absolute value of the change in the objective function, namely, $abs(\beta_I \gamma_J / \alpha_{IJ})$ can possibly be utilized as a *local effectiveness measure (lem)* of the pivoting operation - applicable for any and every iteration, for both primal and dual - given by $lem(I, J) = abs(\beta_I \gamma_J / \alpha_{IJ})$.

Although it is not specifically suggested here, one can opt to choose a pivot, possibly to maximize this *local effectiveness measure (lem)* in every iteration, either among the possible pivots of a particular type or subtype, or even among of all the possible pivots of all the four possible types. Even, if done so, it cannot be guaranteed (needs further research?) to minimize the overall number of simplex iterations required for reaching an optimum solution.

It requires further research work to thoroughly understand, analyze and incorporate the concept of any "*local effectiveness measure*" (*lem*) for a single simplex pivoting operation to the fullest extent, that would in effect achieve a guaranteed improvement in some corresponding "*global effectiveness measure*" (*gem*) defined appropriately for the given LP problem; in developing a *efficient & robust* solution strategy. For now, let us come to the main algorithm itself.

In order to *achieve guaranteed performance efficiency*, we propose to utilize a *global effectiveness measure (gem)* that characterizes a CST tableau with a number indicating the *largest possible decrease in the infeasibility index* as presented in this paper.

9. INFEASIBILITY INDEX : A GLOBAL EFFECTIVENESS MEASURE

As an *inverse measure of goodness*, the infeasibility index λ of a given CST tableau is defined as the sum of the *primal infeasibility index* μ and the *dual infeasibility index* ν . It corresponds to the number of basic variables in primal & dual which are infeasible in the given tableau. That is, we define $\lambda = \mu + \nu$; and therefore –

$$\lambda = (\mu, \text{ number of rows with } \beta < 0) + (\nu, \text{ number of columns with } \gamma > 0);$$

or

$$\lambda = (\mu, \text{ number of rows with } z_i^B < 0) + (\nu, \text{ number of columns with } w_j^B < 0).$$

If the infeasibility index λ of the given tableau equals to zero then it indicates that the tableau is the terminal tableau which is feasible and optimal. By the definition of the infeasibility index, it can never be negative, nor can it be more than the sum of the number of columns and the number of rows in the Compact Symmetric Tableau. That is,

$$0 \leq \lambda = \{(0 \leq \mu \leq m) + (0 \leq \nu \leq n)\} \leq (m + n).$$

Given a tableau, the *change in the infeasibility index* ($\tau = \Delta\lambda$) can be associated with each cell that can be a potential candidate pivot element (i.e. $\alpha \neq 0$). This change in the infeasibility index consists of two components, one is the *change in the primal infeasibility index* ($\sigma = \Delta\mu$) and the other is the *change in the dual infeasibility index* ($\rho = \Delta\nu$). That is,

$$\text{Change in the infeasibility index } \tau = \Delta\lambda = (\Delta\mu + \Delta\nu) = \sigma + \rho.$$

$$\text{Global Effectiveness Measure } gem = -\tau.$$

9.1 Calculation of the change in the primal infeasibility index ($\sigma = \Delta\mu$)

For each column j , the ratio $R_{ij} = \beta_i/\alpha_{ij}$ is calculated for all rows $i = 1, 2, \dots, m$.

The ratio can either be positive, zero or negative. It can be represented as in Figure-5.

Let α_{IJ} be the chosen pivot element, then after pivoting, the values of β are given by

$$R_{IJ} = \beta_I/\alpha_{IJ} \quad \text{and}$$

$$(\beta_I)' \leftarrow (\beta_I/\alpha_{IJ})$$

and

$$(\beta_i)' \leftarrow \beta_i - (\beta_I/\alpha_{IJ})\alpha_{iJ}$$

The calculation of σ can be divided into three different cases depending upon the value of R_{IJ} . $\beta = 0$ is considered as feasible.

$$(i) \quad R_{IJ} = 0 \quad (\text{i.e. } \beta_I = 0, \alpha_{IJ} \neq 0)$$

Here, there will not be any change in the β values, and therefore there will not be any change in the infeasibility index. Hence, $\sigma_{IJ} = 0$.

$$(ii) \quad R_{IJ} > 0 \quad (\beta_I > 0, \alpha_{IJ} > 0 : \text{pixi} \quad \text{-or-} \quad \beta_I < 0, \alpha_{IJ} < 0 : \text{poxi}) \quad : \text{b-q}$$

$R_{IJ} < 0$ will not affect the change in the infeasibility index. $R_{IJ} < 0$ can occur in two situations.

One of them is when $\beta_i < 0$ and $\alpha_{ij} > 0$. In this case, it can be seen from the expression for $(\beta_i)'$ given above, that $(\beta_i)'$ will continue to be negative; hence the change in the infeasibility index is not affected.

The other is when $\beta_i > 0$ and $\alpha_{ij} < 0$. In this case, it can be seen from the expression for $(\beta_i)'$ given above, that $(\beta_i)'$ will continue to be positive; hence the change in the infeasibility index is not affected.

$R_{ij} > 0$ will affect the change in the infeasibility index. $R_{ij} > 0$ can occur in two situations.

One is when $\beta_i \geq 0$ and $\alpha_{ij} > 0$. In this case, it can be seen from the expression for $(\beta_i)'$ given above, that $(\beta_i)'$ will continue to be positive for the ratios $R_{ij} > R_{IJ}$, $(\beta_i)'$ will be zero for the ratios $R_{ij} = R_{IJ}$, $(\beta_i)'$ will continue to be positive, and $(\beta_i)'$ will become negative for the ratios $R_{ij} < R_{IJ}$. Hence the change in the infeasibility index is *increased* by the number of ratios R_{ij} which are less than R_{IJ} and greater than or equal to zero.

The other is when $\beta_i < 0$ and $\alpha_{ij} < 0$. In this case, it can be seen from the expression for $(\beta_i)'$ given above, that $(\beta_i)'$ will continue to be negative for the ratios $R_{ij} > R_{IJ}$, $(\beta_i)'$ will become zero for the ratios $R_{ij} = R_{IJ}$, $(\beta_i)'$ will become positive, and $(\beta_i)'$ will become positive for the ratios $R_{ij} < R_{IJ}$. Hence the change in the infeasibility index is *decreased* by the number of ratios R_{ij} which are less than or equal to R_{IJ} and strictly greater than zero.

Therefore,

$$b = \Sigma \text{ No. of rows } i \text{ with } \beta_i \geq 0, \alpha_{ij} > 0 \text{ and } 0 \leq R_{ij} < R_{IJ} \quad : \{pixi\}$$

$$q = \Sigma \text{ No. of rows } i \text{ with } \beta_i < 0, \alpha_{ij} < 0 \text{ and } 0 < R_{ij} \leq R_{IJ} \quad : \{poxi\}$$

and

$$\sigma_{IJ} = b - q \quad : \{pixi - poxi\}$$

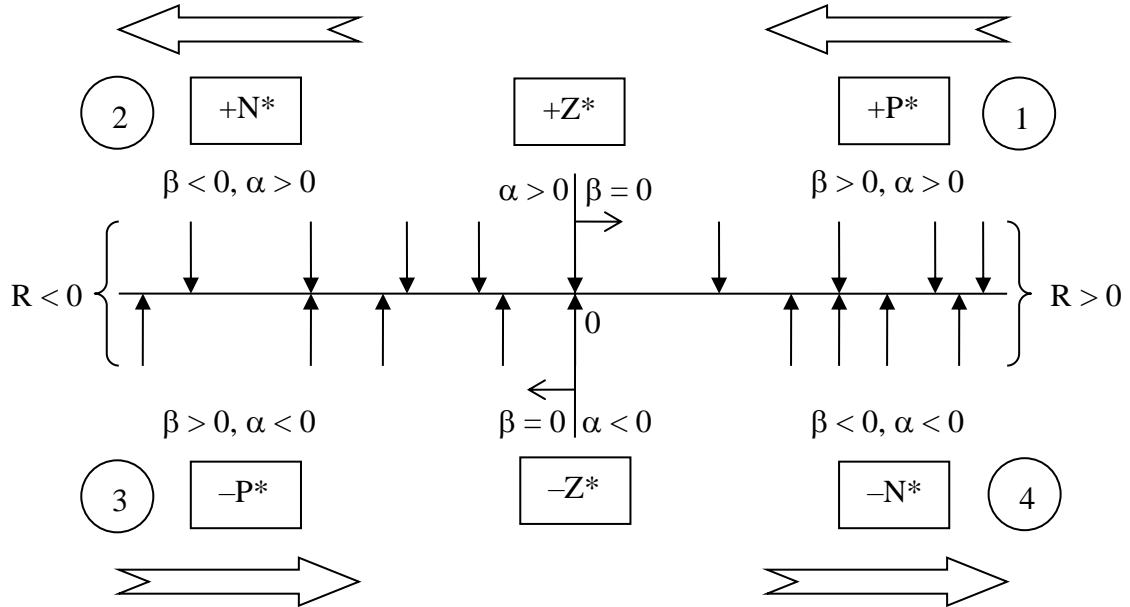


Figure-5: Ordered Pattern of (β_i / α_{ij}) Values for $i = 1, 2, \dots, m$
 $* \Rightarrow ('P', 'N' \text{ or } 'Z')$

(iii) $R_{IJ} < 0$ ($\beta_I \geq 0, \alpha_{IJ} < 0$: *pixo* -or- $\beta_I < 0, \alpha_{IJ} > 0$: *poxo*) : $1+q-b$

$R_{ij} > 0$ will not affect the change in the infeasibility index. $R_{ij} > 0$ can occur in two situations.

One of them is when $\beta_i \geq 0$ and $\alpha_{ij} > 0$. In this case, it can be seen from the expression for $(\beta_i)'$ given above, that $(\beta_i)'$ will continue to be positive; hence the change in the infeasibility index is not affected.

The other is when $\beta_i < 0$ and $\alpha_{ij} < 0$. In this case, it can be seen from the expression for $(\beta_i)'$ given above, that $(\beta_i)'$ will continue to be negative; hence the change in the infeasibility index is not affected.

$R_{ij} < 0$ will affect the change in the infeasibility index. $R_{ij} < 0$ can occur in two situations.

One is when $\beta_i \geq 0$ and $\alpha_{ij} < 0$. In this case, it can be seen from the expression for $(\beta_i)'$ given above, that $(\beta_i)'$ will continue to be positive for the ratios $R_{ij} < R_{IJ}$, $(\beta_i)'$ will be zero for the ratios $R_{ij} = R_{IJ}$, $(\beta_i)'$ will become negative, and $(\beta_i)'$ will become negative for the ratios $R_{ij} > R_{IJ}$. Hence the change in the infeasibility index is increased by the number of ratios R_{ij} which are greater than R_{IJ} and less than or equal to zero (one more, if β_I is positive, since it will become negative).

The other is when $\beta_i < 0$ and $\alpha_{ij} > 0$. In this case, it can be seen from the expression for $(\beta_i)'$ given above, that $(\beta_i)'$ will continue to be negative for the ratios $R_{ij} < R_{IJ}$, $(\beta_i)'$ will become zero for the ratios $R_{ij} = R_{IJ}$, $(\beta_i)'$ will become negative, and $(\beta_i)'$ will become positive for the ratios $R_{ij} > R_{IJ}$. Hence the change in the infeasibility index is decreased by the number of ratios R_{ij} which are greater than or equal to R_{IJ} and strictly less than zero (one less, if β_I is negative, since it will continue to be negative).

Therefore,

$$\begin{aligned} q &= \Sigma \text{ No. of rows } i \text{ with } \beta_i \geq 0, \alpha_{ij} < 0 \text{ and } R_{IJ} < R_{ij} \leq 0 & : \{ \text{pixo} \} \\ b &= \Sigma \text{ No. of rows } i \text{ with } \beta_i < 0, \alpha_{ij} > 0 \text{ and } R_{IJ} \leq R_{ij} < 0 & : \{ \text{poxo} \} \end{aligned}$$

and

$$\sigma_{IJ} = (q + 1) - b \quad \text{if } (\beta_I > 0)$$

$$\text{or } \sigma_{IJ} = q - (b - 1) \quad \text{if } (\beta_I < 0)$$

Hence we get,

$$\sigma_{IJ} = 1 + q - b \quad : \{ I \text{pixo-poxo} \}$$

9.2 Calculation of the change in the dual infeasibility index ($p = \Delta v$)

For each row i , the ratio $R_{ij} = \gamma_j / \alpha_{ij}$ is calculated for all columns $j = 1, 2, \dots, n$.

The ratio can either be positive or negative. It can be represented as in Figure-6.

Let α_{IJ} be the chosen pivot element, then after pivoting, the values of γ are given by

$$R_{IJ} = \gamma_I / \alpha_{IJ} \quad \text{and}$$

$$(\gamma_I)' \leftarrow -(\gamma_I / \alpha_{IJ})$$

and

$$(\gamma_j)' \leftarrow \gamma_j - (\gamma_I / \alpha_{IJ}) \alpha_{Ij}$$

The calculation of ρ can be divided into three different cases depending upon the value of R_{IJ} . $\gamma = 0$ is considered as feasible.

(i) $R_{IJ} = 0$ (i.e. $\gamma_I = 0$, $\alpha_{IJ} \neq 0$)

Here there will not be any change in the γ values, and therefore there will not be any change in the infeasibility index. Hence, $\rho_{IJ} = 0$.

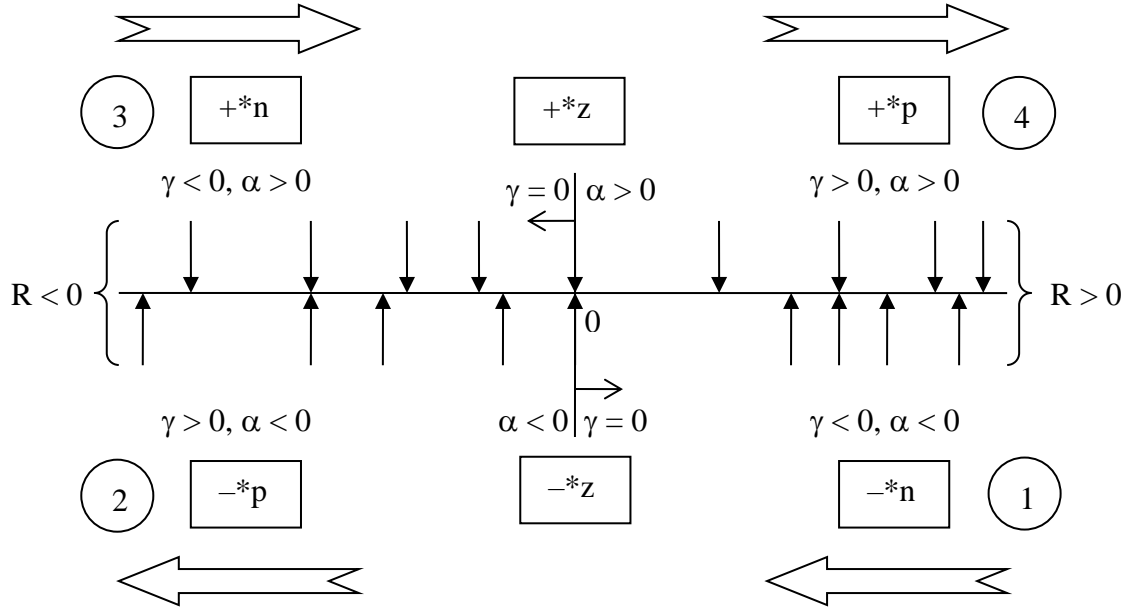


Figure-6: Ordered Pattern of (γ_j / α_{Ij}) Values for $j = 1, 2, \dots, n$

* \Rightarrow ('p', 'n' or 'z')

(ii) $R_{IJ} > 0$ ($\gamma_I < 0$, $\alpha_{IJ} < 0$: *dixi* -or- $\gamma_I > 0$, $\alpha_{IJ} > 0$: *doxi*) : p-d

$R_{IJ} < 0$ will not affect the change in the infeasibility index. $R_{IJ} < 0$ can occur in two situations.

One of them is when $\gamma_j < 0$ and $\alpha_{Ij} > 0$. In this case, it can be seen from the expression for $(\gamma_j)'$ given above, that $(\gamma_j)'$ will continue to be negative; hence the change in the infeasibility index is not affected.

The other is when $\gamma_j > 0$ and $\alpha_{Ij} < 0$. In this case, it can be seen from the expression for $(\gamma_j)'$ given above, that $(\gamma_j)'$ will continue to be positive; hence the change in the infeasibility index is not affected.

$R_{IJ} > 0$ will affect the change in the infeasibility index. $R_{IJ} > 0$ can occur in two situations.

One is when $\gamma_j \leq 0$ and $\alpha_{Ij} < 0$. In this case, it can be seen from the expression for $(\gamma_j)'$ given above, that $(\gamma_j)'$ will continue to be negative for the ratios $R_{IJ} > R_{IJ}$, $(\gamma_j)'$ will be zero for the ratios $R_{IJ} = R_{IJ}$, $(\gamma_j)'$ will continue to be negative, and $(\gamma_j)'$ will become positive for the ratios $R_{IJ} < R_{IJ}$. Hence the change in the infeasibility index is *increased* by the number of ratios R_{IJ} which are less than R_{IJ} and greater than or equal to zero.

The other is when $\gamma_j > 0$ and $\alpha_{lj} > 0$. In this case, it can be seen from the expression for $(\gamma_j)'$ given above, that $(\gamma_j)'$ will continue to be positive for the ratios $R_{lj} > R_{lj}$, $(\gamma_j)'$ will become zero for the ratios $R_{lj} = R_{lj}$, $(\gamma_j)'$ will become negative, and $(\gamma_j)'$ will become negative for the ratios $R_{lj} < R_{lj}$. Hence the change in the infeasibility index is *decreased* by the number of ratios R_{lj} which are less than or equal to R_{lj} and strictly greater than zero.

Therefore,

$$p = \Sigma \text{ No. of columns } j \text{ with } \gamma_j \leq 0, \alpha_{lj} < 0 \text{ and } 0 \leq R_{lj} < R_{lj} : \{dixi\}$$

$$d = \Sigma \text{ No. of columns } j \text{ with } \gamma_j > 0, \alpha_{lj} > 0 \text{ and } 0 < R_{lj} \leq R_{lj} : \{doxi\}$$

and

$$\rho_{lj} = p - d : \{dixi-doxi\}$$

$$(iii) \quad R_{lj} < 0 \quad (\gamma_j \leq 0, \alpha_{lj} > 0 : dixo \text{ -or- } \gamma_j > 0, \alpha_{lj} < 0 : doxo) : 1+d-p$$

$R_{lj} > 0$ will not affect the change in the infeasibility index. $R_{lj} > 0$ can occur in two situations.

One of them is when $\gamma_j < 0$ and $\alpha_{lj} < 0$. In this case, it can be seen from the expression for $(\gamma_j)'$ given above, that $(\gamma_j)'$ will continue to be negative; hence the change in the infeasibility index is not affected.

The other is when $\gamma_j > 0$ and $\alpha_{lj} > 0$. In this case, it can be seen from the expression for $(\gamma_j)'$ given above, that $(\gamma_j)'$ will continue to be positive; hence the change in the infeasibility index is not affected.

$R_{lj} < 0$ will affect the change in the infeasibility index. $R_{lj} < 0$ can occur in two situations.

One is when $\gamma_j \leq 0$ and $\alpha_{lj} > 0$. In this case, it can be seen from the expression for $(\gamma_j)'$ given above, that $(\gamma_j)'$ will continue to be negative for the ratios $R_{lj} < R_{lj}$, $(\gamma_j)'$ will be zero for the ratios $R_{lj} = R_{lj}$, $(\gamma_j)'$ will become positive, and $(\gamma_j)'$ will become positive for the ratios $R_{lj} > R_{lj}$. Hence the change in the infeasibility index is increased by the number of ratios R_{lj} which are greater than R_{lj} and less than or equal to zero (one more, if γ_j is negative, since it will become positive).

The other is when $\gamma_j > 0$ and $\alpha_{lj} < 0$. In this case, it can be seen from the expression for $(\gamma_j)'$ given above, that $(\gamma_j)'$ will continue to be positive for the ratios $R_{lj} < R_{lj}$, $(\gamma_j)'$ will become zero for the ratios $R_{lj} = R_{lj}$, $(\gamma_j)'$ will become positive, and $(\gamma_j)'$ will become negative for the ratios $R_{lj} > R_{lj}$. Hence the change in the infeasibility index is decreased by the number of ratios R_{lj} which are greater than or equal to R_{lj} and strictly less than zero (one less, if γ_j is positive, since it will continue to be positive).

Therefore,

$$d = \Sigma \text{ No. of columns } j \text{ with } \gamma_j \leq 0, \alpha_{lj} > 0 \text{ and } R_{lj} < R_{lj} \leq 0 : \{dixo\}$$

$$p = \Sigma \text{ No. of columns } j \text{ with } \gamma_j > 0, \alpha_{lj} < 0 \text{ and } R_{lj} \leq R_{lj} < 0 : \{doxo\}$$

and

$$\rho_{lj} = (d + 1) - p_2 \quad \text{if } (\gamma_j < 0)$$

$$\text{or } \rho_{lj} = d - (p_2 - 1) \quad \text{if } (\gamma_j > 0)$$

Hence we get,

$$\rho_{lj} = 1 + d - p : \{1dixo-doxo\}$$

10. CST-SIGNATURE OF A COMPACT SYMMETRIC TABLEAU

The CST-*signature* of a Compact Symmetric Tableau is a string of length $n+m$ (number of columns + number of rows). The first n entries are chosen from the characters n, p or z depending up on whether the γ value is negative, positive or zero respectively; the next m entries are chosen from the characters N, P or Z depending up on whether the β value is negative, positive or zero respectively. The positions of these entries are fixed with respect to the initial tableau - considering the *lexicographic ordering* of these $n+m$ parameters. During the pivoting operation, for every subsequent tableau, the CST-*signature* string is generated, and it is compared with the signatures of all the previous tableaus stored in a dictionary, to facilitate the detection of any possible imminent cycle; since each CST-*tableau* has a unique CST-*signature* and each CST-*signature* uniquely identifies a CST-*tableau*. The CST-*signature* of an anticipated resultant CST-*tableau* corresponding to a specific possible choice of pivot element can indeed be pre-determined, in advance, even before performing a pivot operation.

11. ALGORITHM TERMINATION - TERMINAL TABLEAU TYPES

When further pivoting is not possible, the tableau is checked & confirmed to be the *terminal tableau*. The possible terminal tableau types as shown in Figure-7 belong to one of the six possible categories listed here. Label F is for *feasible-basic-finite*, ∞ for *feasible basic-finite/non-basic-infinite* and Φ for *infeasible* - corresponding to both primal and dual *variables*.

(1) $\{P=F, D=F\}$; (2) $\{P=F, D=\infty\}$; (3) $\{P=\infty, D=F\}$; (4) $\{P=\infty, D=\Phi\}$; (5) $\{P=\Phi, D=\infty\}$; (6) $\{P=\Phi, D=\Phi\}$

This classification into *six categories* as presented here is a refinement over the well-known classical approach; one that enables to distinctly identify the case wherein primal (dual) has a *feasible basic-finite/non-basic-infinite* optimum with finite value for the objective function, while the dual (primal) has a *feasible-basic-finite* optimum. This distinction from the classical approach arises because we give primary emphasis on the classification based on the nature of the decision variables at termination, and give secondary emphasis on the finiteness (or otherwise) of the objective function value. The set of criteria for this classification scheme, to be used after ensuring that the tableau is indeed the *terminal tableau*, is given below:

- $P=F : \mu = 0; \quad [\forall (I \in R, J \in C): \{0 \leq \sigma_{IJ}\}] \ \& \ [\nexists (i \in R, j \in C): \{(\alpha_{ij} < 0)(\beta_i \geq 0)(\gamma_j \geq 0)\}];$
- $P=\infty : \mu = 0; \quad [\forall (I \in R, J \in C): \{0 \leq \sigma_{IJ}\}] \ \& \ [\exists (i \in R, j \in C): \{(\alpha_{ij} < 0)(\beta_i \geq 0)(\gamma_j \geq 0)\}];$
- $P=\Phi : \mu > 0; \quad [\forall (I \in R, J \in C): \{0 \leq \sigma_{IJ}\}];$
- $D=F : v = 0; \quad [\forall (J \in C, I \in R): \{0 \leq \rho_{IJ}\}] \ \& \ [\nexists (j \in C, i \in R): \{(\alpha_{ij} > 0)(\beta_i \leq 0)(\gamma_j \leq 0)\}];$
- $D=\infty : v = 0; \quad [\forall (J \in C, I \in R): \{0 \leq \rho_{IJ}\}] \ \& \ [\exists (j \in C, i \in R): \{(\alpha_{ij} > 0)(\beta_i \leq 0)(\gamma_j \leq 0)\}];$
- $D=\Phi : v > 0; \quad [\forall (J \in C, I \in R): \{0 \leq \rho_{IJ}\}];$

Note that the situations corresponding to primal/dual *degeneracy* with the resultant dual/primal *multiplicity* and also that of *infinite rays* are all easily discernible in this classification scheme.

In case of terminal infeasibility, a judicious use of either the *change in primal infeasibility index* ($\sigma=\Delta\mu$) or *change in dual infeasibility index* ($\rho=\Delta\nu$) instead of the *change in overall infeasibility index* ($\tau=\Delta\lambda$) can be utilized to arrive at an *almost primal feasible* or an *almost dual feasible* tableau, if one requires such an output for further problem analysis etc. This maneuverability can be adapted to the needs of the problem and that shows the *versatility* of the *spdspd*s approach.

	D = F	D = ∞	D = Φ															
P = F	<table><tr><td>•</td><td>Θ</td><td>0</td></tr><tr><td>⊕</td><td>*</td><td>+</td></tr><tr><td>0</td><td>−</td><td>•</td></tr></table>	•	Θ	0	⊕	*	+	0	−	•	<table><tr><td>+</td><td>0</td></tr><tr><td>*</td><td>+</td></tr><tr><td>Θ</td><td>•</td></tr></table>	+	0	*	+	Θ	•	
•	Θ	0																
⊕	*	+																
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+	0																	
*	+																	
Θ	•																	
P = ∞	<table><tr><td>−</td><td>*</td><td>⊕</td></tr><tr><td>0</td><td>−</td><td>•</td></tr></table>	−	*	⊕	0	−	•		<table><tr><td>•</td><td>Θ</td><td>⊕</td></tr><tr><td>Θ</td><td>+</td><td>•</td></tr></table>	•	Θ	⊕	Θ	+	•			
−	*	⊕																
0	−	•																
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Θ	+	•																
P = Φ		<table><tr><td>•</td><td>⊕</td></tr><tr><td>⊕</td><td>−</td></tr><tr><td>Θ</td><td>•</td></tr></table>	•	⊕	⊕	−	Θ	•	<table><tr><td>•</td><td>Θ</td><td>⊕</td></tr><tr><td>⊕</td><td>0</td><td>−</td></tr><tr><td>Θ</td><td>+</td><td>•</td></tr></table>	•	Θ	⊕	⊕	0	−	Θ	+	•
•	⊕																	
⊕	−																	
Θ	•																	
•	Θ	⊕																
⊕	0	−																
Θ	+	•																

Figure-7. Six Categories for Terminal Tableau Data Pattern.

– negative; 0 zero; + positive;
 Θ non-positive; * any value; \oplus non-negative; • un-analyzed

12. SYMMETRIC PRIMAL DUAL SYMPLEX PIVOT SELECTION STRATEGY

(1) The proposed *spdspd*s approach can be used to solve any LP problem, by first converting it into the standard/canonical form before proceeding further. In performing such transformation, it is possible to enhance the overall efficiency by the following approach:

- Free variables can be replaced by non-negative variables not double in number, but only just one extra in number; Equations can be replaced by inequalities not double in number, but only just one extra in number;
- No need for use of artificial variables; the *initial basic solution* need not necessarily be feasible; and therefore no need to rely on two-phase method or big-M method, etc.

(2) Any nonzero element of the coefficient matrix in the tableau can be a potential pivot element. The earlier defined typical simplex pivot selection scheme and the associated set of four types of pivot candidates {DSP, PTP, PSP, DTP} is only a *typical* sample, presented for *pedagogical* reasons, to eventually take the reader towards the proposed *spdspds*. For example, a pivot selection with cell type $+P_n$ or $-P_n$ may usually seem to be simply unacceptable or extremely unlikely. However, it may turn out that such a choice can yield a surprisingly large decrease in the infeasibility index, depending on the tableau data, especially as an intermediary pivoting iteration, and therefore worth the consideration.

(3) The proposed concept of *infeasibility index* along with the use of a *measure of goodness* for the pivot selection determined as/by the anticipated *decrease in the infeasibility index* considered as a *global effectiveness measure (gem)* arising due to the specific pivot selection, turns out to be a great grand breakthrough in achieving the ultimate performance challenge in the use of simplex method for solving linear programming problems.

(4) The concept of the CST-signature explained earlier can be utilized to *detect & avoid* cycling as well as to keep a record of the actual computational path taken by the algorithm - as a track record, from the initial tableau all the way up to the terminal/final tableau. It may be noted here that a more *compact* CST-signature can also be defined, that indicates only the *combinatorial information*, that is, only about the *partitioning of the set of variables into the two disjoint subsets*, namely, the *set of basic variables* and the *set of non-basic variables*; without giving any specific detail about the feasibility/infeasibility or the numerical sign of the variable.

(5) Problem of *cycling* (en-route towards the terminal tableau, caused by *intermediate degeneracy*) will get *prevented* because of the very nature of the *spdspds* pivot selection strategy, seeking the best possible decrease of the infeasibility index - a primal/dual degenerate pivot doesn't result in any decrease in the primal/dual infeasibility index. When encountered with a possible degeneracy, the *lexicographic ordering of the variables* is used as a tie-breaking mechanism to guide the choice of pivot element, thus coursing through a part of the cycle, until a point is reached wherefrom *spdspds* finds a step down the infeasibility path again, unless it happens to be the terminal tableau. So, *spdspds* is effectively *immune* to problem of *cycling* caused by *intermediate degeneracy*.

(6) Potential possibility of *numerical instability* can be avoided by careful elimination of poor choices of the pivot element, using *appropriate filters* - even if it requires going for a next-best (sub-optimal) pivot choice in terms of *gem (decrease in the infeasibility index)* - in order to avoid such treacherous pathways leading to numerically disastrous computational behavior - effectively using an appropriate *pre-conditioner* without actually doing so - which is like first exchanging I for K followed by an exchange of K for J in order to achieve an exchange of I for J.

13. COMPUTATIONAL COMPLEXITY

For a linear programming problem represented in its standard canonical form using the Tucker's Compact Symmetric Tableau of size $(m) \times (n)$ the problem size L can be taken as $[(m+1) \times (n+1)]$.

In the worst case, the infeasibility index of the initial tableau will at most be $\{(m)+(n)\}$ and at each *spdspds* simplex pivoting iteration the infeasibility index gets reduced by at least *one* so that it takes at most $\{(m)+(n)\}$ *spdspds* simplex pivoting iterations to reach the optimum solution if one exists; or may even be well before that in order to report the infeasibility status of the given problem. However, the expected number of *spdspds* simplex pivoting iterations is $\leq \min\{(m),(n)\}$ because the basic / non-basic status of each variable is not expected to switch around much except in rare

instances of potential numerical instability etc. Therefore, the number of *spdspd*s simplex pivoting iterations required to solve the linear programming problem is, $O(L^{1/2})$ in the worst case.

Each *spdspd*s simplex pivoting operation requires a complete *analysis of the infeasibility status* before performing the actual pivoting operation, and that requires $\leq [2*(m)*(n)]$ floating point operations, that is $O(L)$. Each *spdspd*s simplex pivoting operation itself requires $\leq [3*(m+1)*(n+1)]$ floating point operations, which is again $O(L)$.

Therefore, the total computational work involved in solving the linear programming problem is $\leq \{(m)+(n)\}*[5*m*n + 3*m + 3*n]$ *floating point arithmetic operations*, which is of course $O(L^{1.5})$ – that is *super-linear sub-quadratic polynomial complexity bound*.

It can be shown easily that the *space complexity* is only $O(L)$ - that is, linear complexity bound.

Note that *spdspd*s has *finite termination* and provides a definite *output*, that is, either the optimum solution if & when one exists, or a point of *minimal infeasibility* beyond which no further *spdspd*s simplex pivoting is possible towards any improvement in the feasibility of the given problem - which itself may provide insight as to the possible refinements in the problem formulation itself.

14. A POSER TO THE COOL HEADED BRAVE HEARTS

Here i present a poser to you, my cool headed brave heart counterpart:

Refer to the notation used in Figure-4 and consider the two scenarios described below:

$$[\forall I \in R]: [\exists J \in C \{ \{I = I_{PSP}(J)\} \vee \{I = I_{PTP}(J)\} \}]; \& [\forall J \in C]: [\exists I \in R \{ \{J = J_{DSP}(I)\} \vee \{J = J_{DTP}(I)\} \}];$$

or the scenarios described below:

$$[\forall I \in R]: [\exists J \in C \{ \forall i \in R \{ (\beta_i \geq 0) \wedge ((0 \leq \beta_i/\alpha_{ij} < \beta_i/\alpha_{ij}) \vee (\beta_i/\alpha_{ij} < \beta_i/\alpha_{ij} \leq 0)) \} \vee \{ (\beta_i < 0) \wedge ((0 \leq \beta_i/\alpha_{ij} < \beta_i/\alpha_{ij}) \vee (\beta_i/\alpha_{ij} < \beta_i/\alpha_{ij} \leq 0)) \} \}];$$

&

$$[\forall J \in C]: [\exists I \in R \{ \forall j \in C \{ (\gamma_j \leq 0) \wedge ((0 \leq \gamma_j/\alpha_{ij} < \gamma_j/\alpha_{ij}) \vee (\gamma_j/\alpha_{ij} < \gamma_j/\alpha_{ij} \leq 0)) \} \vee \{ (\gamma_j > 0) \wedge ((0 \leq \gamma_j/\alpha_{ij} < \gamma_j/\alpha_{ij}) \vee (\gamma_j/\alpha_{ij} < \gamma_j/\alpha_{ij} \leq 0)) \} \}];$$

and then may even possibly consider the scenarios described below:

$$[\forall I \in R]: [\exists J \in C \{ \forall i \in R \{ (\sigma_{ij} < \sigma_{ij} \leq 0) \} \} \& [\forall J \in C]: [\exists I \in R \{ \forall j \in C \{ (\rho_{ij} < \rho_{ij} \leq 0) \} \}];$$

A careful reading of the above will reveal that the underlying reasoning here is very similar to that used in the concept of *infeasibility index* as a *global effectiveness measure* for a simplex pivoting operation that form the conceptual foundation for the design of *spdspd*s expressed by the following:

$$\neg[(\exists I \in R)(\exists J \in C)]: [\{ (\forall i \in R)(\forall j \in C)(i \neq I)(j \neq J) \{ \tau_{ij} < \tau_{ij} \leq 0 \}]$$

15. DIRECTIONS FOR FUTURE RESEARCH

0: The spdsps algorithm provides a scope for future research work with a motivation to further enhance the worst case computational complexity possibly down to $O(L)$ - that is, linear w.r.t. the size of the input data set. The idea is to *exploit the complete information content* that is made available through the *infeasibility analysis* of initial tableau T_0 . Note that the infeasibility analysis corresponding to every potential pivot element (I, J) provides information as to the feasibility status (*positive/zero/negative*) of the primal/dual variables in the associated resultant tableau.

1: Analyze the entries in the initial tableau T_0 and make a statement declaring the status of each of the $n+m$ variables in the optimum tableau T^* ; that is, as to whether it will be a basic variable or a non-basic variable in the optimum tableau. This question is conceptually equivalent to asking, in the context of non-linear programming [11] as to whether a constraint will be *active* (non-basic) or otherwise at the optimum - although of course expecting an answer in that context may certainly not be practical. However, the situation with linear systems can be more promising so as to expect a possible attempt in answering that question - based only on a thorough analysis of either the initial tableau T_0 , or equivalently, any tableau T_k en-route towards the optimum - maybe by using a radically *redefined* concept of *binding/nonbinding* [8][12][13] *constraints* - worth investigating.

2. Refer to Figure-2 and relations 13 & 14. A row i having a negative/positive dot-product $(\alpha_i) \cdot (\gamma)$ is painted green/red. Similarly, a column j having a positive/negative the dot-product $(\beta) \cdot (\alpha_j)$ is painted green/red. A detailed study is required to understand the effect of pivoting on the color of a row/column and also the color of the rows/columns in the optimum tableau, etc.

3: For every column j the minimum value of (negative) τ_{ij} (largest decrease in τ_{ij}) associated with some $I_r(j)=[\arg(i)\{\min(\tau_{ij})\}]$ may be determined. Also, for every row i the minimum value of (negative) τ_{ij} (largest decrease in τ_{ij}) associated with some $J_r(i)=[\arg(j)\{\min(\tau_{ij})\}]$ may be determined. Now the question is whether a statement can be made that, a row that is *not* $I_r(j)$ for any of the columns j (a column that is *not* $J_r(i)$ for any of the rows i) is likely to be a “*non-binding protected row*” (“*non-binding protected column*”) untouched by any pivot selection process? Also, what if in this analysis the $I_r(j)=[\arg(i)\{\min(\tau_{ij})\}]$ is replaced by $I_\sigma(j)=[\arg(i)\{\min(\sigma_{ij})\}]$ and $J_r(i)=[\arg(j)\{\min(\tau_{ij})\}]$ is replaced by $J_\rho(i)=[\arg(j)\{\min(\rho_{ij})\}]$? Investigation on these lines may be used in avoiding unnecessary pivoting and instead focus on those required to achieve the right combination of the basic & non-basic variables at the optimum.

4: A binary string *CST-signature* may be defined by mapping the alphabet symbols $\{n,z,P,Z\}$ to 1 indicating feasibility and mapping the alphabet symbols $\{p,N\}$ to 0 indicating infeasibility. Let the binary string *CST-signature* of the initial tableau T_0 be S_0 and that of $T_0(I,J)$ be $S_0(I,J)$. For the initial tableau T_0 create a list L_0 of the binary strings $S_0(I,J)$ corresponding to each of the potential spdsps pivot elements (I,J) associated with a decrease $0 > \tau = \Delta\lambda = (\Delta\mu + \Delta\nu) = \sigma + \rho$ in the overall infeasibility index λ . A detailed analysis of how feasibility status of each of the primal and dual variables get affected by each choice of the potential pivot element, requires the study of the components σ_1 & σ_4 or σ_3 & σ_2 along with ρ_1 & ρ_4 or ρ_3 & ρ_2 as appropriate. Such an effort may require an exploration to further strengthen the correspondence among - linear programming and network optimization along with combinatorial optimization - while also leading towards a design of an efficient solution strategy for the LP - with very similar or even better performance efficiency than that of *spdsps*.

5: The combined effect of a simultaneous application of more than one simplex pivot (rather than a *sequence of simplex pivoting operations*) is indeed worth further detailed study, and can possibly lead to what may be called as an “*omniplex pivoting operation*”.

6: The versatility of the *spdspds* algorithm allows for extensive combinatorial analysis along with numerical experimentation that may need to be conducted to explore further possibilities, including possible applications in *polyhedral combinatorics*, etc. Also, for example, as in the classical approaches, one can possibly seek to first achieve primal or dual feasibility, more efficiently, by utilizing the best possible decrease in the corresponding component σ or ρ rather than τ , before going further with later iterations.

16. CONCLUSION

The proposed concept of *infeasibility index* is an *inverse measure of goodness* associated with a CST tableau. The anticipated *decrease in the infeasibility index* as a *global effectiveness measure* associated with each pivot element forms the basic foundation for the proposed *spdspds* algorithm. The proposed *spdspds* provides a novel viewpoint to the very same classic framework of Tucker’s Compact Symmetric Tableau (CST) representation for LP - with a newly defined concept of *simplex pivot*; while also leading to an *efficient, robust and versatile* iterative solution strategy requiring *minimum number of simplex iterations*, with a worst case computational complexity of $O(L^{1.5})$.

The proposed concept of the *CST-signature* can be utilized to keep track of the computational path from the initial tableau to the terminal tableau, and in turn also to detect cycling. It is expected that an implementation of this proposed *spdspds* approach can *achieve the ultimate performance challenge in solving linear programming problems* - and thus lead towards consequent immediate as well as lasting, deep and far reaching impact on the development & study of optimization algorithms and their computational complexity analyses.

The *spdspds* algorithm claims -

- (1) guaranteed improvement in the feasibility at every step;
- (2) guaranteed finite termination;
- (3) guaranteed immunity against cycling;
- (4) guaranteed minimum number of pivoting iterations;
- (5) guaranteed worst case computational complexity of $O(L^{1.5})$.

It is suggested that ‘*spdspds*’ may be read as ‘*yes-speedy-yes-speedy-yes*’ (yes! - why *silent*?) while rejoicing its 180-degree rotational *symmetry* as a string symbol.

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An earlier version of this paper [*Ms.No.MAPR-D-11-00001*] was *rejected* [2011JAN18] for the reason - “*the content is not sufficient for the paper to appear in MPA*” - therefore making me acknowledge the view-point : “*rejection can indeed be turned around towards perfection*”!

I must necessarily confess that the core idea behind '*spdspds*' is so stunningly simple, that one may simply be taken aback in a profound wonder-struck jaw drop silence, possibly with an after-thought : "*oh my goodness, how could it be that it never flashed on me any time earlier*"!

I acknowledge that this paper was *not accepted* (2020JUN01) for *uploading* onto the Optimization Online E-print Repository; the reason given was: "*The paper seems wrong*"!

On the auspicious vidyaa(vijaya)daSami day [2020OCT25] I wish to share my pleasure in giving a new name as '*symplex pivot*' to the now redefined *simplex* pivot, emphasizing the primal-dual symmetry therein, along with the expanded scope in terms of the far wider range of choices available in the selection of such pivot element, while maintaining the minimal combinatorial/structural requirement of being simply an exchange between a basic (dependent) variable and a non-basic (independent) variable, in the Tucker's Compact Symmetric Tableau (CST) which is a unique symmetric representation common to both the primal as well as the dual of a linear programming problem in its standard canonical form. This renaming acknowledges that '*symplex*' is indeed a rebirth of the classical *simplex* defined by Dantzig, now as a new avataar in the twenty first century.

19. DEDICATION

To my ಅಜ್ಜ (ajja) & ಅಜ್ಜಿ (ajji) *Karinja Halemane Keshava Bhat & Thirumaleshwari* and to my ಅಪ್ಪ (appa) & ಅಮ್ಮ (amma) *Shama Bhat & Thirumaleshwari* for their teachings through love, that *quality* matters more than *quantity*; to my wife *Vijayalakshmi* for her ever consistent love; to my daughter *Sriwidya Bharati* and my twin sons *Sriwidya Ramana & Sriwidya Prawina* for their love and affection.

To the *cool headed brave hearts* among the world optimization professionals who may opt for *symplex* over *simplex* and *spdspds* over any other strategy for solving linear programming problems.