Why Regression?

EC 425/525, Set 3

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Prologue

Schedule

Last time

- The Experimental Ideal
- Fundamentals of `R` (wrap up Lab 1).

Today

What's so great about linear regression and OLS?

Read *MHE* 3.1

Upcoming

Assignment First step of project proposal due April 15th.

Follow up

`return()`

1. `function()` automatically returns the last evaluated value—regardless of `return()`.

2. Hadley Wickham[†] suggests reserving `return` for "early" returns.

† An R god

Regression

Regression

Why?

In our previous discussion, we began moving from simple differences to a regression framework.

Q Why do we[†] care so much about linear regression and OLS?

A As we discussed, regression allows us to control for covariates that *can* assist with (1) causal identification and (2) inference.

There's a deeper reason that we care about *linear* regression and ordinary least squares (OLS): ***the conditional expectation function (CEF).***

† *we* = empirically inclined applied economists

Regression

Why?

Even ignoring causality, we can show important relationships between

1. **the CEF** (the conditional expectation function),
2. the **population regression function**,
3. and the **sampling distribution of regression estimates**.

The *CEF*

Definition The **conditional expectation function** for a dependent variable $\mathbf{Y}_i$, given a $\mathbf{K} \times \mathbf{1}$ vector of covariates $\mathbf{X}_i$, tells us the expected value (population average) of $\mathbf{Y}_i$ with $\mathbf{X}_i$ held constant.

Written as $E[\mathbf{Y}_i \mid \mathbf{X}_i]$, the CEF is a function of $\mathbf{X}_i$.[†]

Examples

- $E[\text{Income}_i \mid \text{Education}_i]$
- $E[\text{Wage}_i \mid \text{Gender}_i]$
- $E[\text{Birth weight}_i \mid \text{Air quality}_i]$

† We'll generally assume $\mathbf{X}_i$ is a random variable, which implies that $E[\mathbf{Y}_i \mid \mathbf{X}_i]$ is also a random variable.

The *CEF*

Formally, for continuous $\mathbf{Y}_i$ with conditional density $f_y(t|\mathbf{X}_i = x)$,

$$E[\mathbf{Y}_i \mid \mathbf{X}_i = x] = \int t f_y(t|\mathbf{X}_i = x)dt$$

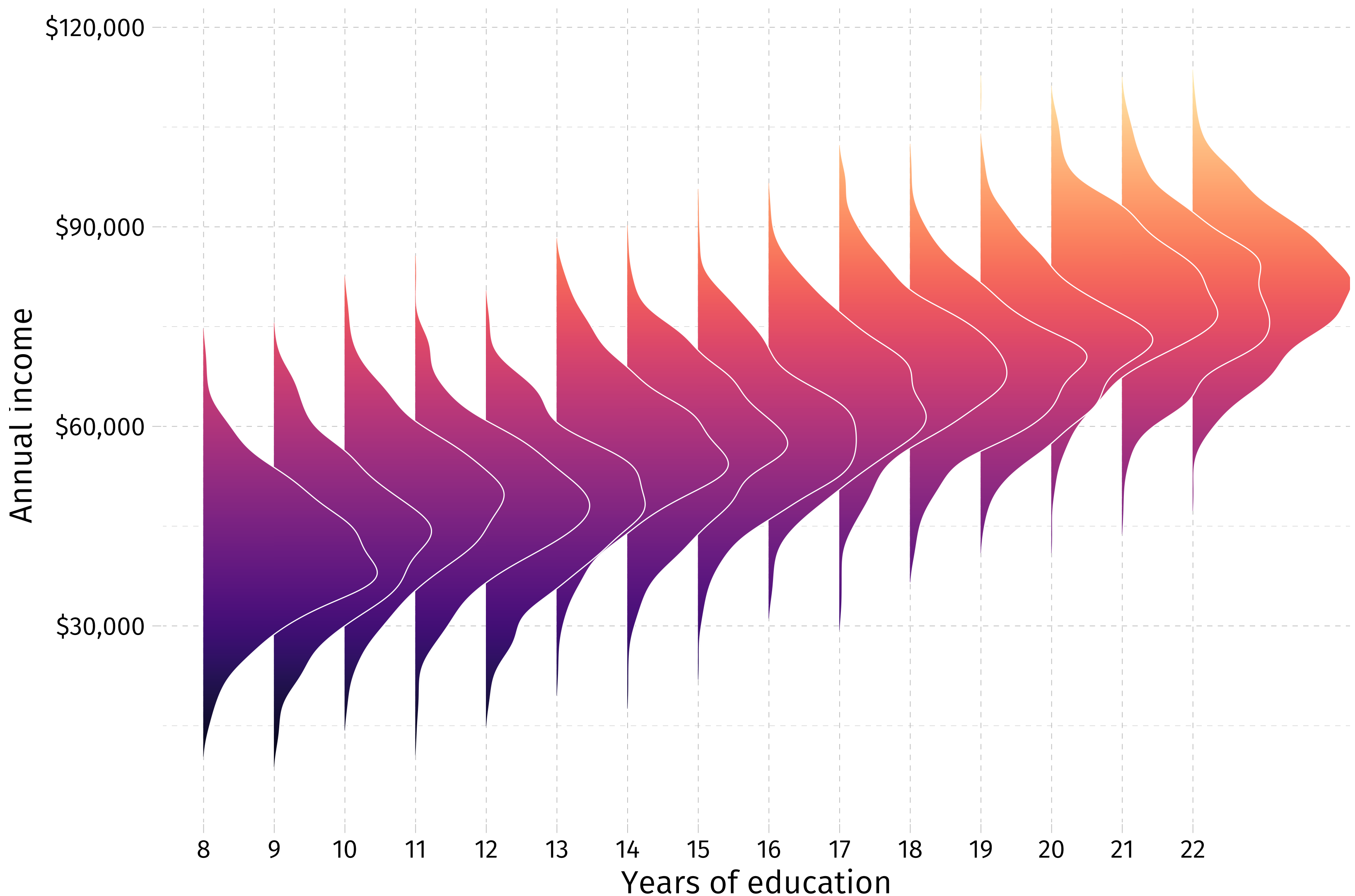
and for discrete $\mathbf{Y}_i$ with conditional p.m.f. $\mathbf{Pr}(\mathbf{Y}_i = t|\mathbf{X}_i = x)$,

$$E[\mathbf{Y}_i \mid \mathbf{X}_i = x] = \sum_t t \, \mathbf{Pr}(\mathbf{Y}_i = t|\mathbf{X}_i = x)$$

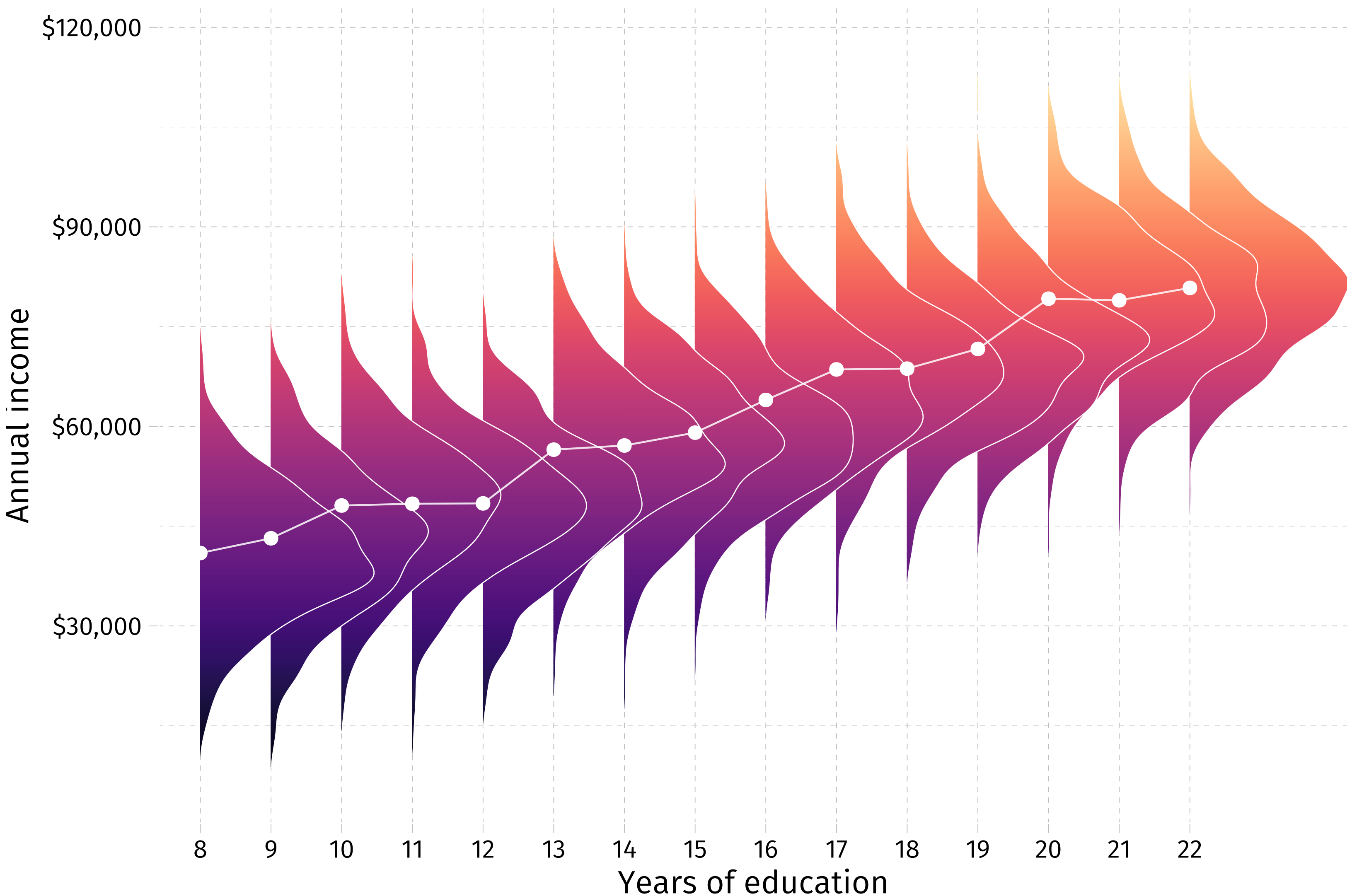
Notice We are focusing on the **population**. We want to build our intuition about the parameters that we will eventually estimate.

Graphically...

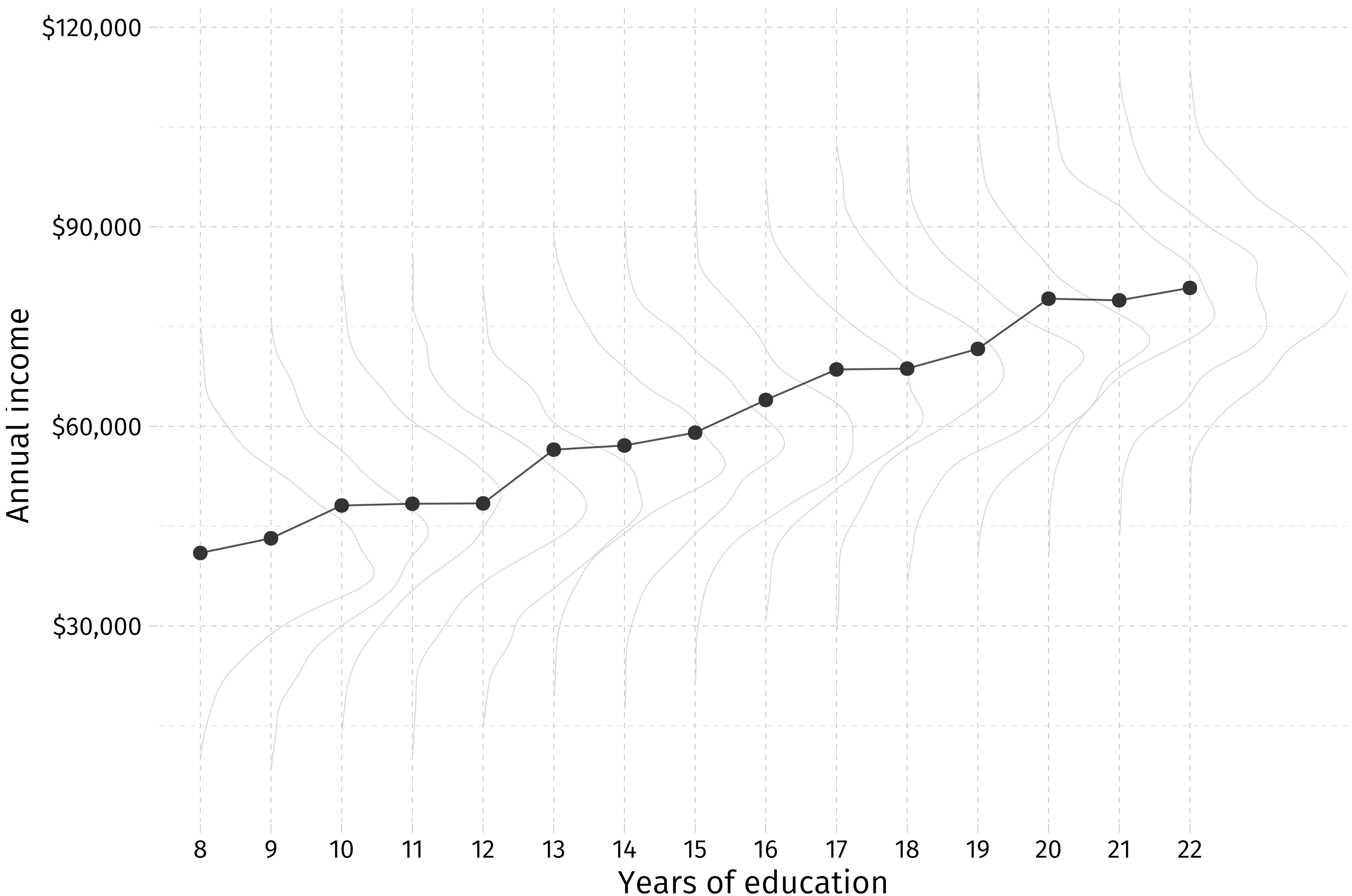
The conditional distributions of $\mathbf{Y}_i$ for $\mathbf{X}_i = x$ in 8, ..., 22.



The CEF, $E[\mathbf{Y}_i \mid \mathbf{X}_i]$, connects these conditional distributions' means.



Focusing in on the CEF, $E[\mathbf{Y}_i \mid \mathbf{X}_i]$...



Q How does the CEF relate to/inform regression?

The *CEF*

As we derive the properties and relationships associated with the CEF, regression, and a host of other estimators, we will frequently rely upon ***the Law of Iterated Expectations*** (LIE).

$$E[\mathrm{Y}_i] = E\bigg(E[\mathrm{Y}_i \mid \mathrm{X}_i] \bigg)$$

which says that the **unconditional expectation** is equal to the **unconditional average** of the **conditional expectation function**.

Regression

A proof of the LIE

First, we need notation...

Let $f_{x,y}(u, t)$ denote the joint density for continuous RVs $(\mathbf{X}_i, \mathbf{Y}_i)$.

Let $f_{y|x}(t \mid \mathbf{X}_i = u)$ denote the conditional distribution of $\mathbf{Y}_i$ given $\mathbf{X}_i = u$.

And let $g_y(t)$ and $g_x(u)$ denote the marginal densities of $\mathbf{Y}_i$ and $\mathbf{X}_i$.

Regression

A proof of the LIE

$$
\begin{aligned}
E\left(E[\mathrm{Y}_i \mid \mathrm{X}_i] \right)
&= \int E[\mathrm{Y}_i \mid \mathrm{X}_i = u]\, g_x(u)du \\
&= \int \left[\int t\, f_{y|x}(t \mid \mathrm{X}_i = u)dt \right] g_x(u)du \\
&= \int \int t\, f_{y|x}(t \mid \mathrm{X}_i = u)\, g_x(u)du\, dt \\
&= \int t \left[\int f_{y|x}(t \mid \mathrm{X}_i = u)\, g_x(u)du \right] dt \\
&= \int t \left[\int f_{x,y}(u, t)du \right] dt \\
&= \int t\, g_y(t)\, dt \\
&= E[\mathrm{Y}_i]
\end{aligned}
$$

🥳

Great. What's the point?

The *LIE* and the *CEF*

Theorem The CEF decomposition property (3.1.1)

The LIE allows us to **decompose random variables** into two pieces

$$\mathrm{Y}_i = E[\mathrm{Y}_i \mid \mathrm{X}_i] + \varepsilon_i$$

1. **the conditional expectation function**
2. **a residual** with special powers†
   i. ε_i is mean independent of $\mathbf{X}_i$, *i.e.*, $E[\varepsilon_i \mid \mathbf{X}_i] = \mathbf{0}$.
   ii. ε_i is uncorrelated with any function of $\mathbf{X}_i$.

Important It might not seem like much, but these results are **huge** for building intuition, theory, *and* application. Put a ⭐ here!

† Angrist and Pischke go with *special properties*.

The *LIE* and the *CEF*

Proof The CEF decomposition property (properties i. and ii. of ε_i)

Mean independence, $E[\varepsilon_i \mid \mathrm{X}_i] = 0$

$$
\begin{aligned}
&E[\varepsilon_i \mid \mathrm{X}_i] \\
&= E\left(\mathrm{Y}_i - E[\mathrm{Y}_i \mid \mathrm{X}_i] \middle| \mathrm{X}_i\right) \\
&= E[\mathrm{Y}_i \mid \mathrm{X}_i] - E\left(E[\mathrm{Y}_i \mid \mathrm{X}_i] \middle| \mathrm{X}_i\right) \\
&= E[\mathrm{Y}_i \mid \mathrm{X}_i] - E[\mathrm{Y}_i \mid \mathrm{X}_i] \\
&= 0
\end{aligned}
$$

Zero correlation btn. ε_i and $h(\mathrm{X}_i)$

$$
\begin{aligned}
&E[h(\mathrm{X}_i)\varepsilon_i] \\
&= E\left(E[h(\mathrm{X}_i)\varepsilon_i \mid \mathrm{X}_i] \right) \\
&= E\left(h(\mathrm{X}_i)\, E[\varepsilon_i \mid \mathrm{X}_i] \right) \\
&= E[h(\mathrm{X}_i) \times 0] \\
&= 0
\end{aligned}
$$

The *LIE* and the *CEF*

The CEF decomposition property

says that we can decompose any random variable (*e.g.*, $\mathbf{Y}_i$) into

1. a part that is explained by $\mathbf{X}_i$ (*i.e.*, the CEF $E[\mathbf{Y}_i \mid \mathbf{X}_i]$),
2. a part that is *orthogonal to*† any function of $\mathbf{X}_i$ (*i.e.*, ε_i).

Why the CEF?

The CEF also presents an intuitive summary of the relationship between $\mathbf{Y}_i$ and $\mathbf{X}_i$, since we are often use means to characterize random variables.

But (of course) there are more reasons to use the CEF...

† "orthogonal to" = "uncorrelated with"

The *LIE* and the *CEF*

Theorem The CEF prediction property (3.1.2)

Let $m(\mathbf{X}_i)$ be *any* function of $\mathbf{X}_i$. The CEF solves

$$E[\mathbf{Y}_i \mid \mathbf{X}_i] = \underset{m(\mathbf{X}_i)}{\mathbf{arg\,max}} \;\; E\Big[\left(\mathbf{Y}_i - m(\mathbf{X}_i)\right)^2\Big]$$

In other words, the **CEF** is the minimum mean-squared error (MMSE) predictor of $\mathbf{Y}_i$ given $\mathbf{X}_i$.

Notice

1. We haven't restricted m to any class of functions—it can be nonlinear.
2. We're talking about *prediction* (specifically predicting $\mathbf{Y}_i$).

Proof The CEF prediction property

$$\Big(\mathrm{Y}_i - m(\mathrm{X}_i)\Big)^2 \quad (1)$$

$$= \Big(\{\mathrm{Y}_i - E[\mathrm{Y}_i \mid \mathrm{X}_i]\} + \{E[\mathrm{Y}_i \mid \mathrm{X}_i] - m(\mathrm{X}_i)\}\Big)^2$$

$$= \Big(\mathrm{Y}_i - E[\mathrm{Y}_i \mid \mathrm{X}_i]\Big)^2 \quad (a)$$

$$+2\Big(E[\mathrm{Y}_i \mid \mathrm{X}_i] - m(\mathrm{X}_i)\Big) \times \Big(\mathrm{Y}_i - E[\mathrm{Y}_i \mid \mathrm{X}_i]\Big) \quad (b)$$

$$+\Big(E[\mathrm{Y}_i \mid \mathrm{X}_i] - m(\mathrm{X}_i)\Big)^2 \quad (c)$$

Recall: We want to choose the $m(\mathrm{X}_i)$ that minimizes (**1**) in expectation.

(**a**) is irrelevant, *i.e.*, it does not depend upon $m(\mathrm{X}_i)$.

(**b**) equals zero in expectation: $E[h(\mathrm{X}_i) \times \varepsilon_i] = 0$.

(**c**) is minimized by $m(\mathrm{X}_i) = E[\mathrm{Y}_i \mid \mathrm{X}_i]$, *i.e.*, when $m(\mathrm{X}_i)$ is the CEF.

The *LIE* and the *CEF*

∴ the CEF is the function that minimizes the mean-squared error (MSE)

$$E[\mathrm{Y}_i \mid \mathrm{X}_i] = \underset{m(\mathrm{X}_i)}{\arg\max}\; E\Big[\big(\mathrm{Y}_i - m(\mathrm{X}_i)\big)^2\Big]$$

The *LIE* and the *CEF*

One final property of the CEF (very similar to the decomposition property)

Theorem The ANOVA theorem (3.1.3)

$$\mathbf{Var}(\mathbf{Y}_i) = \mathbf{Var}(E[\mathbf{Y}_i \mid \mathbf{X}_i]) + E[\mathbf{Var}(\mathbf{Y}_i \mid \mathbf{X}_i)]$$

which says that we can decompose the variance in $\mathbf{Y}_i$ into

1. the variance in the CEF
2. the variance of the residual

Example Decomposing wage variation into (**1**) variation explained by workers' characteristics and (**2**) unexplained (residual) variation

The proof centers on the fact that the decomposition property of the CEF.

We now understand the CEF a bit better.

But how does the CEF actually relate to regression?

The *CEF* and regression

We've discussed how the CEF summarizes empirical relationships.

Previously we discussed how regression provides simple empirical insights.

Let's link these two concepts.

The *CEF* and regression

Population least-squares regression

We will focus on β, the vector (a $K \times 1$ matrix) of population, least-squares regression coefficients, *i.e.*,

$$\beta = \underset{b}{\arg\min}\ E\Big[\big(\mathrm{Y}_i - \mathrm{X}_i' b\big)^2\Big]$$

where b and X_i are also $K \times 1$, and Y_i is a scalar.

Taking the first-order condition gives

$$E\big[\mathrm{X}_i \left(\mathrm{Y}_i - \mathrm{X}_i' b\right)\big] = 0$$

The *CEF* and regression

From the first-order condition

$$E\left[\mathrm{X}_i\left(\mathrm{Y}_i - \mathrm{X}_i' b\right)\right] = 0$$

we can solve for b. We've defined the optimum as β. Thus,

$$\beta = E\left[\mathrm{X}_i \mathrm{X}_i'\right]^{-1} E[\mathrm{X}_i \mathrm{Y}_i]$$

Note The first-order conditions tell us that our least-squares population regression residuals $\left(e_i = \mathrm{Y}_i - \mathrm{X}_i' \beta\right)$ are uncorrelated with X_i.

Anatomy

Our "new" result: $\beta = E\left[\mathrm{X}_i \mathrm{X}_i'\right]^{-1} E[\mathrm{X}_i \mathrm{Y}_i]$

In **simple linear regression** (an intercept and one regressor x_i),

$$\beta_1 = \frac{\mathrm{Cov}(\mathrm{Y}_i,\, x_i)}{\mathrm{Var}(x_i)} \qquad \beta_0 = E[\mathrm{Y}_i] - \beta_1\, E[x_i]$$

For **multivariate regression**, the coefficient on the k$^{\text{th}}$ regressor x_{ki} is

$$\beta_k = \frac{\mathrm{Cov}(\mathrm{Y}_i,\, \tilde{x}_{ki})}{\mathrm{Var}(\tilde{x}_{ki})}$$

where $\tilde{x}_{ki}$ is the residual from a regression of x_{ki} on all other covariates.

Anatomy

This alternative formulation of least-squares coefficients is quite powerful.

$$\beta_k = \frac{\text{Cov}(\text{Y}_i,\ \tilde{x}_{ki})}{\text{Var}(\tilde{x}_{ki})}$$

Why? This expression illustrates how each coefficient in a least-squares regression represents the bivariate slope coefficient after controlling for the other covariates.

Anatomy

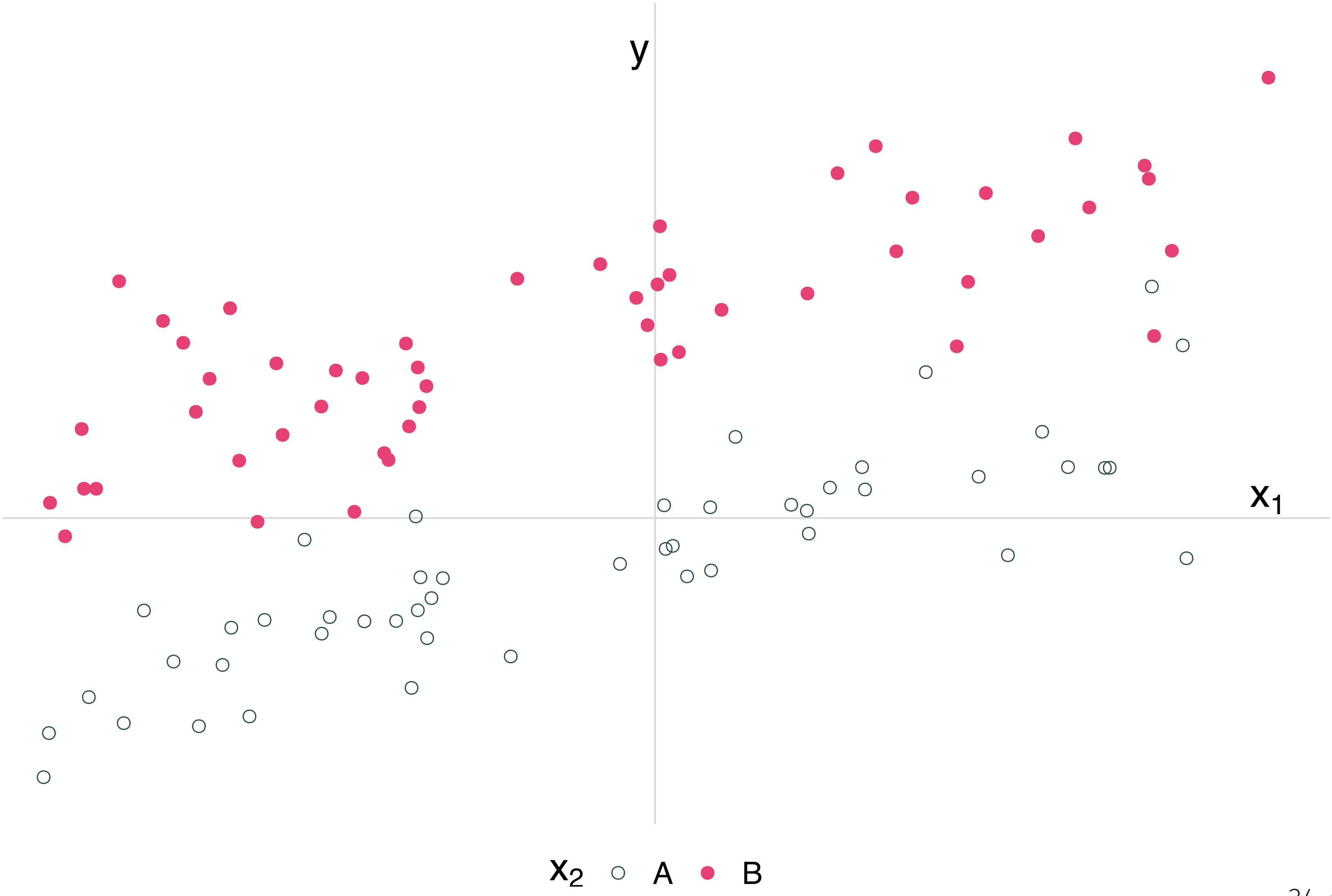
In fact, we can re-write our coefficients to further emphasize this point

$$\beta_k = \frac{\textbf{Cov}\left(\widetilde{\mathrm{Y}}_i,\, \tilde{x}_{ki}\right)}{\textbf{Var}(\tilde{x}_{ki})}$$

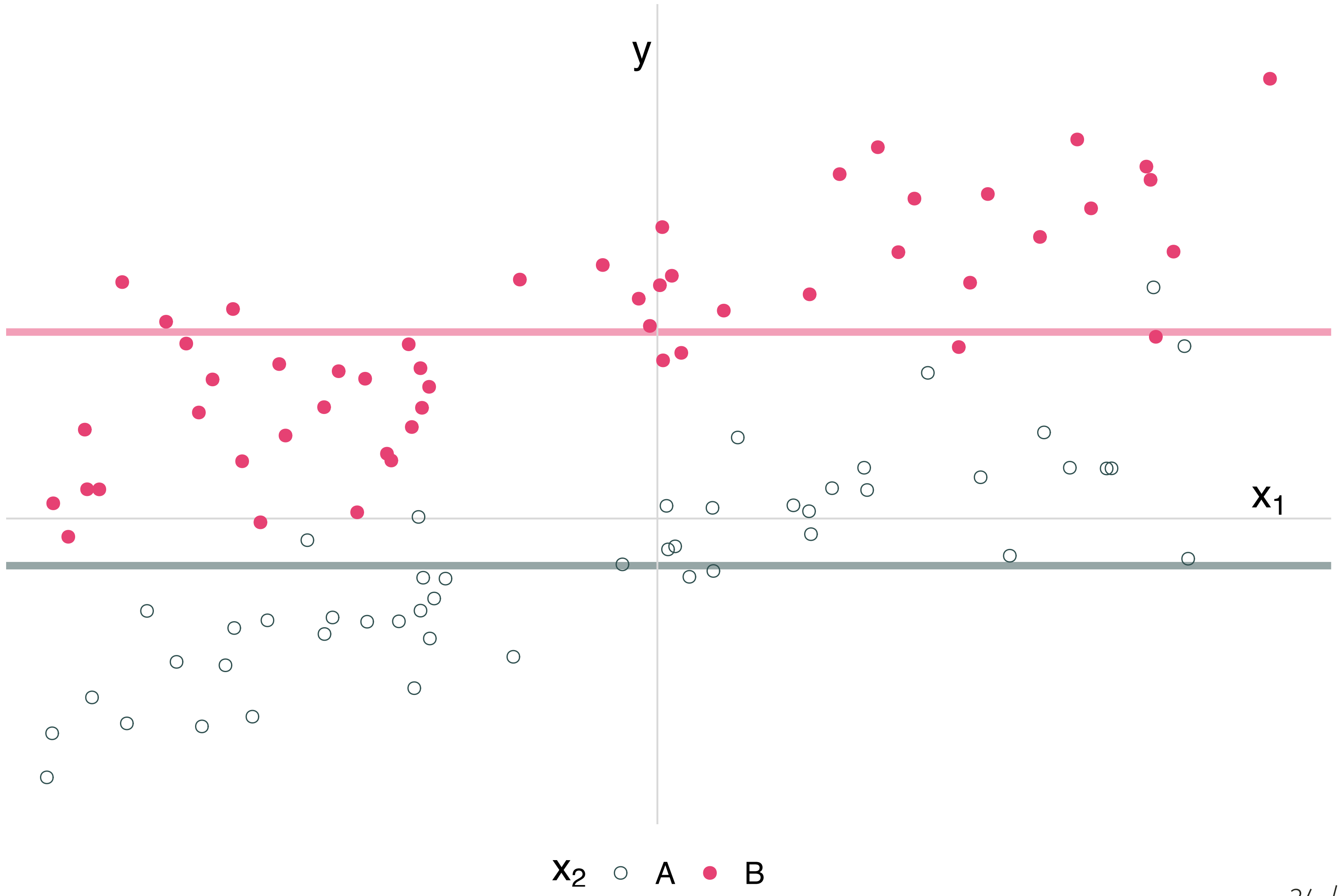
$\widetilde{\mathrm{Y}}_i$ denotes the residual from regressing $\mathbf{Y}_i$ on all regressors except x_{ki}.

Graphical example

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \varepsilon_i$$



β_1 gives the relationship between y and x_1 *after controlling for* x_2



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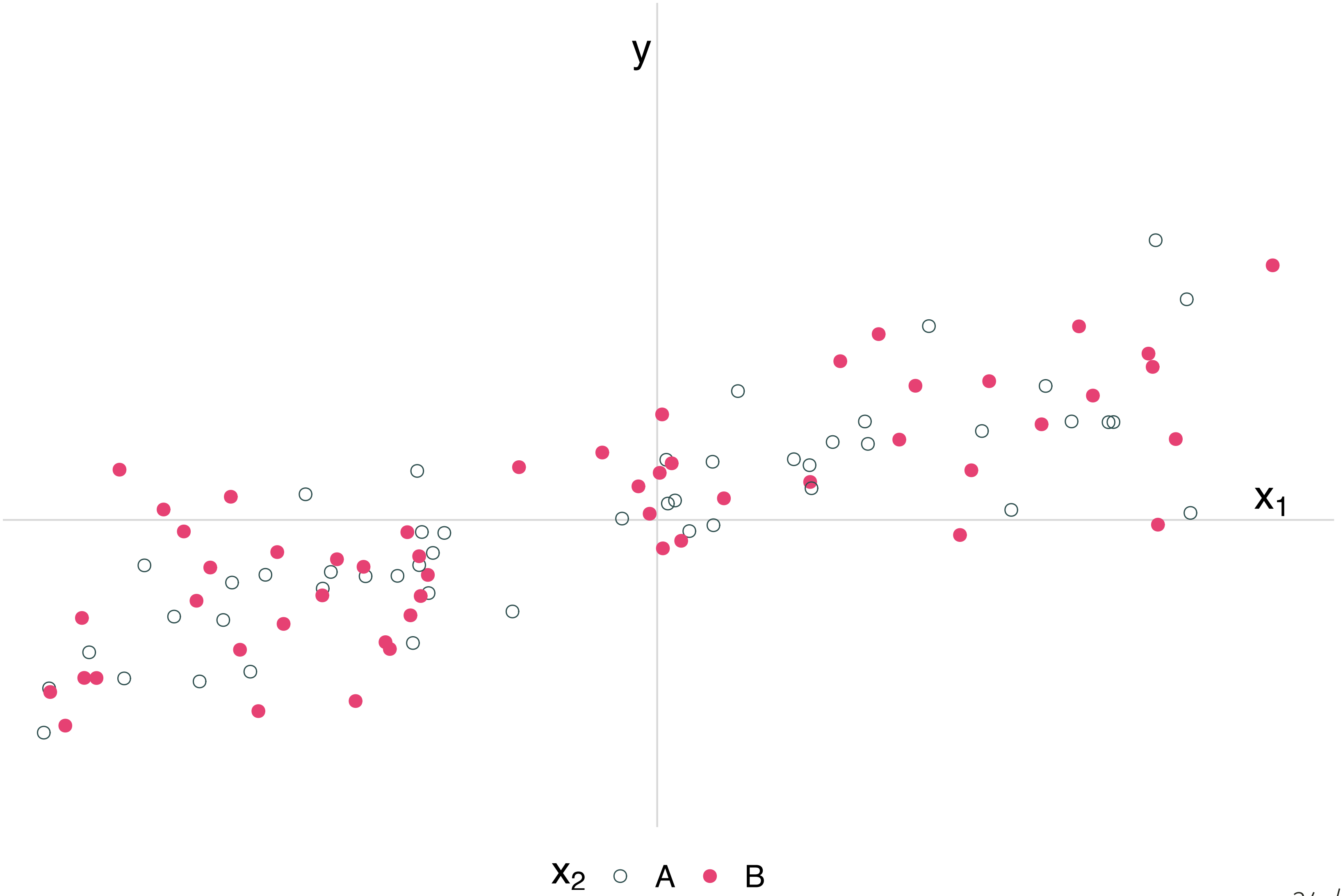
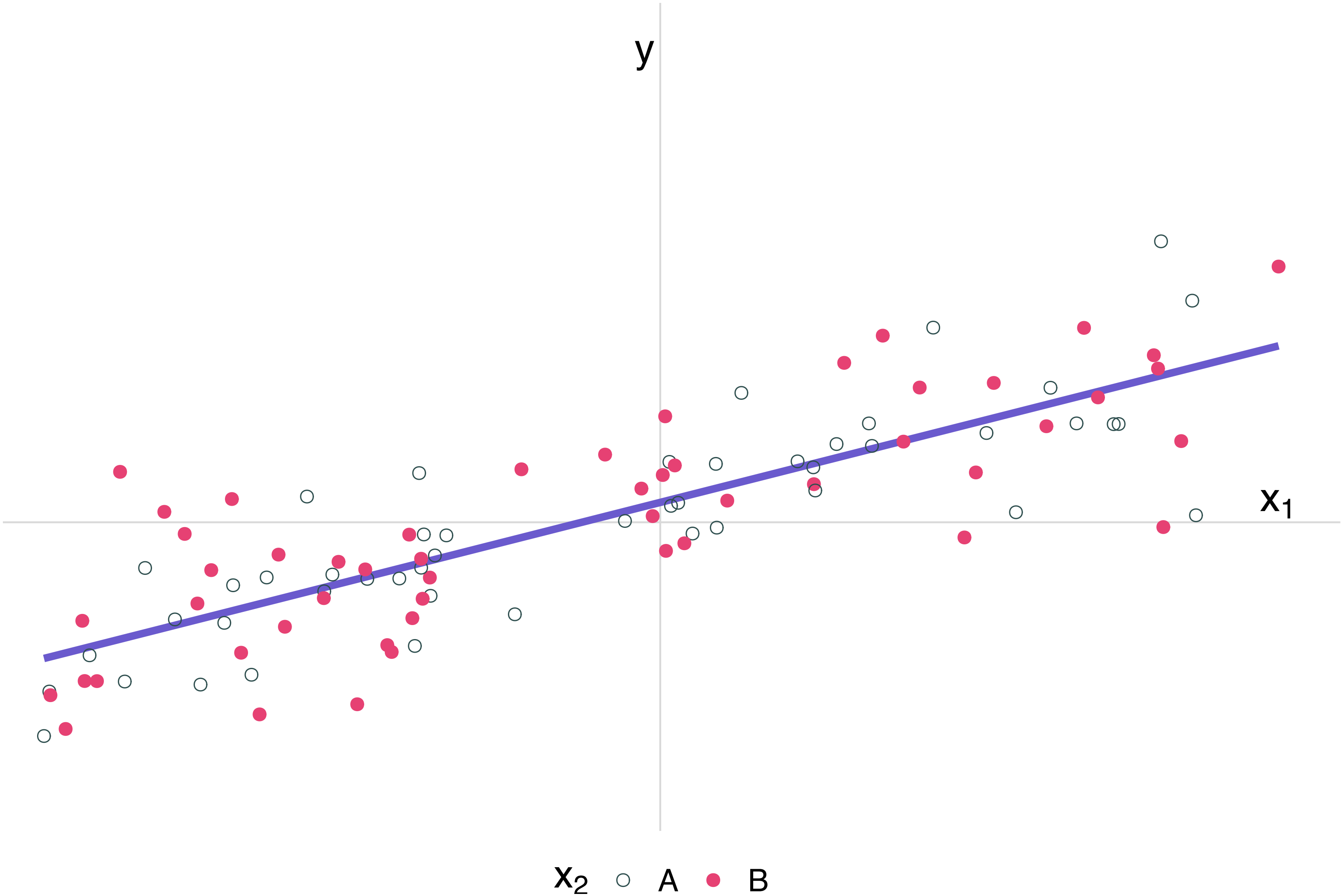


β_1 gives the relationship between y and x_1 *after controlling for* x_2



Now that we've refreshed/deepened our regression knowledge, let's connect regression and the CEF.

Regression and the *CEF*

Angrist and Pischke make the case that

> ... you should be interested in regression parameters if you are interested in the CEF. (*MHE*, p.36)

Q What is the reasoning/connection?

A We'll cover three reasons.

1. *If the CEF is linear*, then the population regression line is the CEF.
2. The function $\mathbf{X}_i'\beta$ is the min. MSE *linear* predictor of $\mathbf{Y}_i$ given $\mathbf{X}_i$.
3. The function $\mathbf{X}_i'\beta$ gives the min. MSE *linear* approximation to the CEF.

Regression and the *CEF*

Theorem The linear CEF theorem (3.1.4)

If the CEF is linear, then the population regression is the CEF.

Proof Let the CEF equal some linear function, *i.e.*, $E[\mathrm{Y}_i \mid \mathrm{X}_i] = \mathrm{X}_i'\beta^\star$.

From the CEF decomposition property, we know $E[\mathrm{X}_i\varepsilon_i] = \mathbf{0}$.

$$\implies E[\mathrm{X}_i\,(\mathrm{Y}_i - E[\mathrm{Y}_i \mid \mathrm{X}_i])] = 0$$

$$\implies E\left[\mathrm{X}_i\left(\mathrm{Y}_i - \mathrm{X}_i'\beta^\star\right)\right] = 0$$

$$\implies E[\mathrm{X}_i\mathrm{Y}_i] - E\left[\mathrm{X}_i\mathrm{X}_i'\beta^\star\right] = 0$$

$\implies \beta^\star = E\left[\mathrm{X}_i\mathrm{X}_i'\right]^{-1} E[\mathrm{X}_i\mathrm{Y}_i] = \beta$, our population regression coefficients.

Regression and the *CEF*

> **Theorem** The linear CEF theorem (3.1.4)
>
> If the CEF is linear, then the population regression is the CEF.

Linearity can be a strong assumption. When might we expect linearity?

1. Situations in which $(\mathbf{Y}_i, \mathbf{X}_i)$ follows a multivariate normal distribution.
   Concern Might be limited—especially when $\mathbf{Y}_i$ or $\mathbf{X}_i$ are not continuous.

1. Saturated regression models
   Example A model with two binary indicators and their interaction.

Regression and the *CEF*

Theorem The best linear predictor theorem (3.1.5)

The function $\mathbf{X}_i'\beta$ the best linear predictor of $\mathbf{Y}_i$ given $\mathbf{X}_i$ (minimizes MSE).

Proof We defined β as the vector that minimizes MSE, *i.e.*,

$$\beta = \mathop{\mathbf{arg\,min}}_{b}\ E\left[\left(\mathbf{Y}_i - \mathbf{X}_i'b\right)^2\right]$$

so $\mathbf{X}_i'\beta$ is literally defined as the minimum MSE linear predictor of $\mathbf{Y}_i$.

- The population-regression function $\left(\mathbf{X}_i'\beta\right)$ is the best (min. MSE) *linear* predictor of $\mathbf{Y}_i$ given $\mathbf{X}_i$.
- The CEF ($E[\mathbf{Y}_i \mid \mathbf{X}_i]$) is the best predictor (min. MSE) of $\mathbf{Y}_i$ given $\mathbf{X}_i$ across *all classes* of functions.

Regression and the *CEF*

Q If $\mathbf{X}_i'\beta$ is **the best linear predictor** of $\mathbf{Y}_i$ given $\mathbf{X}_i$, then why is there so much interest machine learning for prediction (opposed to regression)?

A A few reasons

1. Relax *linearity*
2. Model selection
   - choosing $\mathbf{X}_i$ is not always obvious
   - overfitting is bad
3. It's fancy/shiny and new
4. Some ML methods boil down to regression
5. Others?

Counter Q Why are we (still) using regression?

Regression and the *CEF*

Theorem The regression CEF theorem (3.1.6)

The population regression function $\mathrm{X}_i'\beta$ provides the minimum MSE linear approximation to the CEF $E[\mathrm{Y}_i \mid \mathrm{X}_i]$, *i.e.*,

$$\beta = \underset{b}{\arg\min}\ E\left\{\left(E[\mathrm{Y}_i \mid \mathrm{X}_i] - \mathrm{X}_i'b\right)^2\right\}$$

Put simply Regression gives us the *best* linear approximation to the CEF.

Proof First, recall that, in expectation, β is the b that minimizes $\left(\mathrm{Y}_i - \mathrm{X}_i' b\right)$

$$
\begin{aligned}
&\left(\mathrm{Y}_i - \mathrm{X}_i' b\right)^2 && (1)\\
&\quad = \left(\{\mathrm{Y}_i - E[\mathrm{Y}_i \mid \mathrm{X}_i]\} + \{E[\mathrm{Y}_i \mid \mathrm{X}_i] - \mathrm{X}_i' b\} \right)^2 \\
&\quad = \left(\mathrm{Y}_i - E[\mathrm{Y}_i \mid \mathrm{X}_i]\right)^2 && (a)\\
&\qquad + \left(E[\mathrm{Y}_i \mid \mathrm{X}_i] - \mathrm{X}_i' b \right)^2 && (b)\\
&\qquad + 2\left(\mathrm{Y}_i - E[\mathrm{Y}_i \mid \mathrm{X}_i]\right)\left(E[\mathrm{Y}_i \mid \mathrm{X}_i] - \mathrm{X}_i' b \right) && (c)
\end{aligned}
$$

We want to minimize (**b**), and we know β minimizes (**1**).

(**a**) is irrelevant, *i.e.*, it does not depend upon b.

(**c**) can be written as $2\varepsilon_i\, h(\mathrm{X}_i)$, which equals zero in expectation.

$\therefore$ (In expectation) If $b = \beta$ minimizes (**1**), then $b = \beta$ minimizes (**b**).

Regression and the *CEF*

Let's review our new(-ish) regression results

1. When the CEF is linear, the regression function *is* the CEF.
   Too small Very specific circumstances—or big assumptions.

2. Regression gives us the best *linear* predictor of $\mathbf{Y}_i$ (given $\mathbf{X}_i$)
   Off point We're often interested in β—not $\widehat{\mathbf{Y}}_i$.

3. Regression provides the best *linear* approximation of the CEF.
   Just right? (Depends on your goals)

Regression and the *CEF*

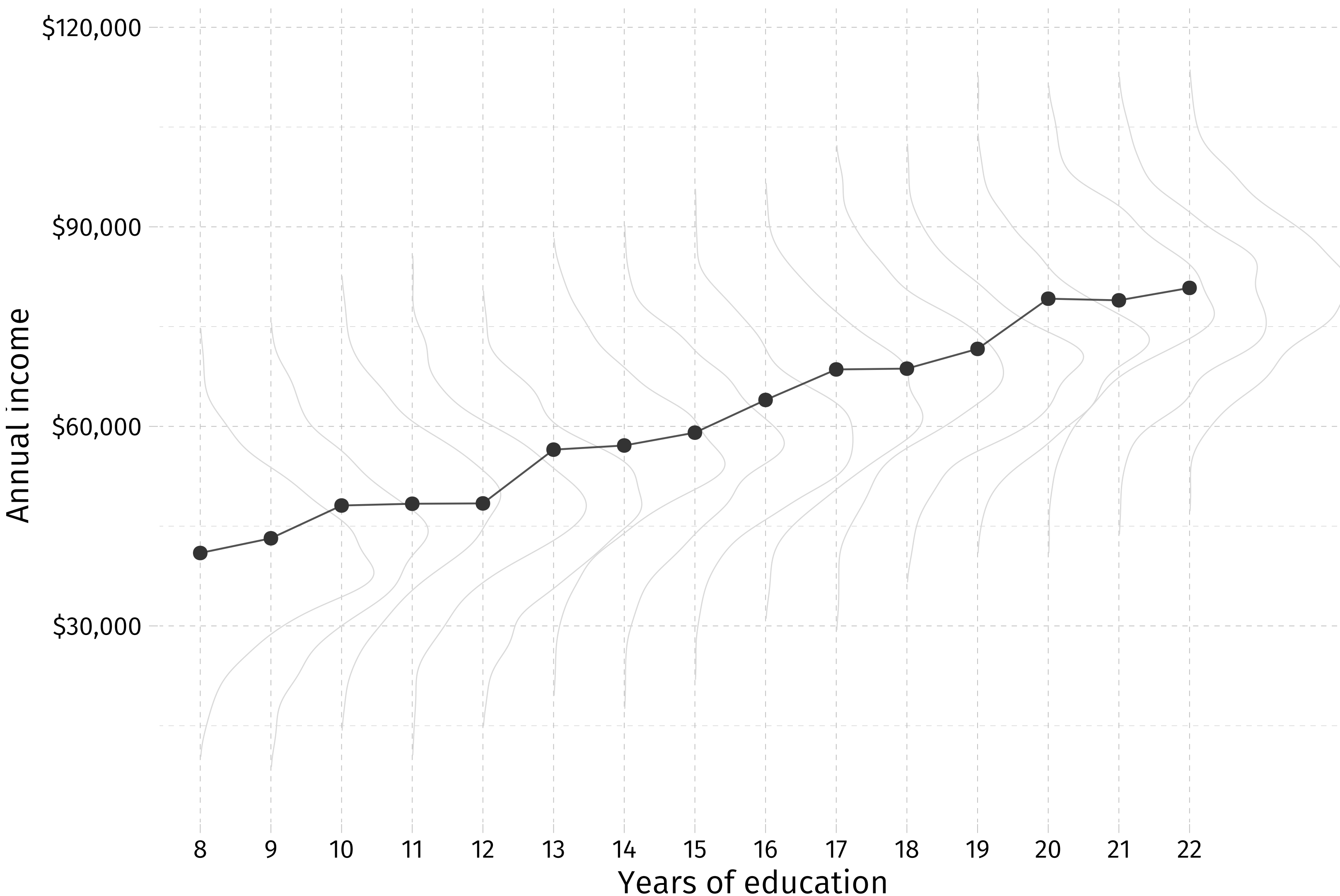
Motivation (**3**) tends to be the most compelling.

Even when the CEF is not linear, regression recovers the best linear approximation to the CEF.

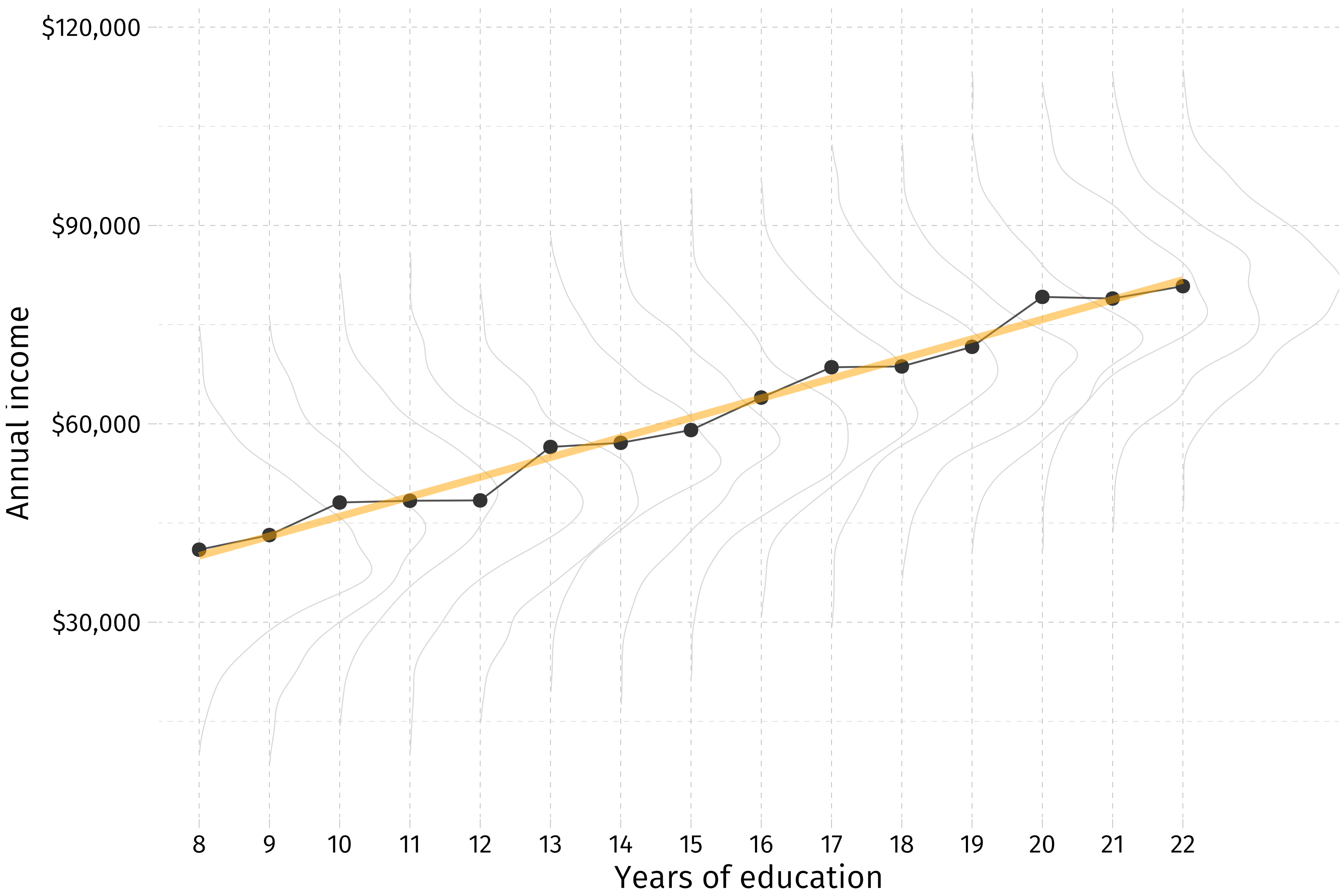
> The statement that regression approximates the CEF lines up with our view of empirical work as an effort to describe the essential features of statistical relationships without necessarily trying to pin them down exactly. (*MHE*, p.39, emphasis added)

Let's dig into this linear-approximate to the CEF a little more...

Returning to our **CEF**



Adding the population **regression function**



Regression and the *CEF*

As the previous figure suggests, one way to think about least-squares regression is **estimating a weighted regression on the CEF** rather than the individual observations.

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