AN IMPROVED RECURSIVE FORMULA

FOR CALCULATING SHOCK RESPONSE SPECTRA

David O. Smallwood
Sandia National Laboratories
Albuquerque, New Mexico 87185

Currently used recursive formulas for calculating the shock response spectra are based on an impulse invariant digital simulation of a single degree of freedom system. This simulation can result in significant errors when the natural frequencies are greater than 1/6 the sample rate. It is shown that a ramp invariant simulation results in a recursive filter with one additional filter weight that can be used with good results over a broad frequency range including natural frequencies which exceed the sample rate.

NOMENCLATURE

$x(t)$ = base input displacement of a single-degree-of-freedom system

$y(t)$ = response displacement of a single-degree-of-freedom system

$\ddot{x}(t)$ = base input acceleration

$\ddot{y}(t)$ = response acceleration

$z(t)$ = relative displacement $y(t) - x(t)$

ζ = fraction of critical damping

ω_n = natural frequency of a single- degree-of-freedom system, rad/sec

s = complex variable

H = transfer function

$L[\ ]$ = Laplace transform

$L^{-1}[\ ]$ = Inverse Laplace transform

Z = z transform

Z^{-1} = Inverse z transform

ω_d = damped natural frequency,

$$\omega_n \sqrt{1 - \zeta^2}$$

T = sample interval

$\delta(t)$ = delta function; $\delta(t) = 1$ for $t = 0$, $\delta(t) = 0$ elsewhere

d_m = digital delta function; $d_m = 1$ for $m = 0$, $d_m = 0$ all other m

SDOF = single degree of freedom

$u(t)$ = units step function; $u(t) = 1$ for $t \geq 0$, $u(t) = 0$ for $t < 0$

t = time

INTRODUCTION

There are many ways to calculate the shock response spectra. A popular technique is to use a digital recursive filter to simulate the single-degree-of-freedom (SDOF) system. The output of the filter using a sampled input is assumed to be a measure of the response of the SDOF system.

The response is then searched for the maximum value. This process is then repeated for each natural frequency of interest. Currently used filters exhibit large errors when the natural frequency exceeds 1/6 the sample rate. This paper will discuss the design of an improved filter which gives much better results at the higher natural frequencies. The companion problem of peak detection of a sampled system will not be discussed in this paper.

MODELS

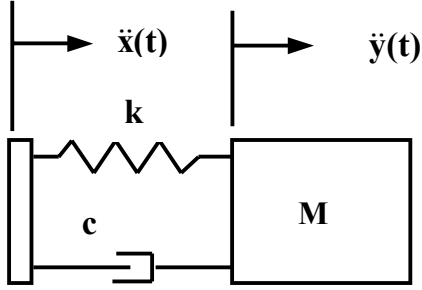Absolute acceleration model – the absolute acceleration model is shown in Fig. 1.

Fig. 1. Absolute acceleration model

The input to the SDOF system is the base acceleration. The response of the system is the absolute acceleration of the mass. The transfer function of this system in the complex Laplace domain is given by

$$H(s) = \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad (1)$$

This is the model most frequently used in shock response spectra calculations.

Relative displacement model – The relative displacement model is shown in Fig. 2.
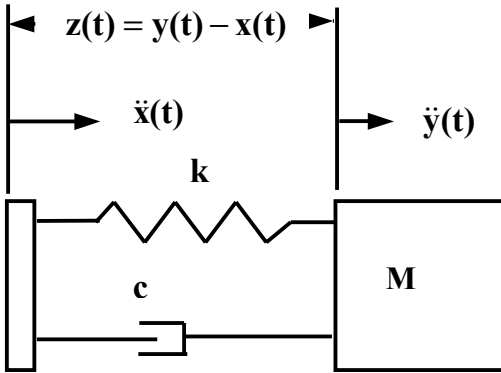

Fig. 2 Relative displacement model

The input to the system is the absolute acceleration of the base. The response of the system is the relative displacement between the base and the mass. The transfer function of this system is given by

$$H(s) = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad (2)$$

If the relative displacement is expressed in terms of equivalent static acceleration,

$$\ddot{y}_{eq} = z\,\omega_n^2 \qquad (3)$$

the transfer function becomes,

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad (4)$$

The relative displacement model is used when the damage potential (perhaps the stress is the support bracket) can be related to the relative displacement. The equivalent static acceleration is used to keep the input and response in the same physical units. Both models give $H(0) = 1$ and have the same denominator. Equations (1) and (4) will be referred to as the absolute acceleration and the relative displacement models respectively.

SIMULATION OF CONTINUOUS SYSTEMS

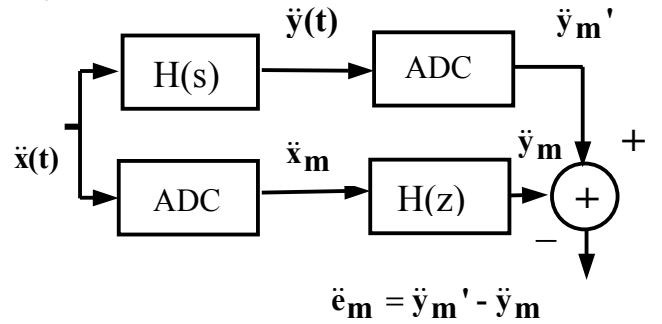Following Stearns [1] a digital simulation will be compared with the continuous system as outlined in Fig. 3.

Fig. 3 Simulation of a Continuous System

The output, $\ddot{y}_m$, of the digital simulation, H(z), will be compared to the sampled output of the continuous system, $y_m{}'$. If the sample set $\ddot{e}_m = 0$ for all m's the simulation is said to be exact.

Impulse invariant simulations --
Stearns shows that if the input is an impulse, i.e.,

$$\ddot{x}(t) = \delta(t)$$

and

$$\ddot{x}_m = d_m$$

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The simulation

$$\widetilde{H}_0(z) = T\,Z\left[L^{-1}\left[H(s)\right]\right] \qquad (5)$$

is exact.

The digital recursive filters given in [2] for the absolute acceleration and relative displacement models can be derived from this formulation. The formulas given in [2] are therefore said to be impulse invariant. It can be shown (using superposition) that if the input is a series of scaled impulses the error of simulation, $\ddot{e}_m$, will be zero. This is true even though the impulses are not band limited. Also H(s) need not be band limited for the simulation to be exact. In the case of a SDOF simulation, the natural frequency can be equal to or above the sampling frequency.

The shock response spectra of a 1 g, 0.64 ms haversine was calculated (Fig. 4), using an impulse invariant simulation of the absolute acceleration model (eq. 6.96 in [2]). The correct shock spectra should be almost a constant 1.0 above a few hundred Hz. Note the gradual decline in the computed shock spectra to a minimum at 1000 Hz (one half the sample rate) and then an increase as 2000 Hz (the sample rate) is approached. As a second example, the shock spectra of an exponentially decaying sinusoid was calculated. The decaying sinusoid was modified to reduce the velocity and displacement change [3]. The input acceleration time history sampled at 2000 sample/s is given by

$$\ddot{x}(t) = u(t)e^{-\eta\omega t}\sin\omega t + u(t+\tau)Ae^{-\nu t}\sin\nu(t+\tau)$$

where

$$A = -0.1995$$
$$\eta = 0.05$$
$$\omega = 2\pi(100)$$
$$\nu = 2\pi(10)$$
$$\tau = -0.015757$$

The shock spectra (Fig. 5) again shows the notch/peak at one half the sample rate and at the sample rate.

The errors cannot be blamed on the sampling theorem as the input is reasonably band limited. If the input is properly band limited, the response will be band limited even if H(s) is not band limited. The errors have long been recognized and the recursive

formulas have not been recommended whenever the sample rate was less than 5 or 6 times the highest natural frequency. However, the author does not believe that the mechanism of the errors has been well understood.

The errors can be explained using the following argument. Consider, the response to a square wave represented by two impulses. That is, the original square wave is sampled. The function is now represented by a series of scaled impulses at each sampling time. Note that the sample rate is such that only two non-zero samples are observed. Set the natural frequency of the SDOF system equal to one half the sample rate (see Fig. 6).
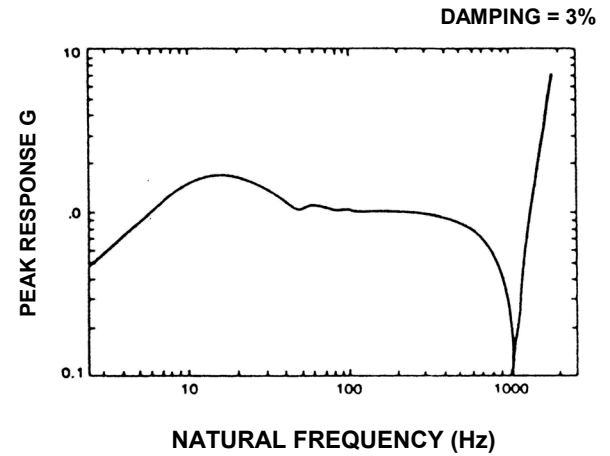
DAMPING = 3%

Fig. 4   Shock Response Spectra of a 64 ms Haversine with Unity Amplitude Sampled at 2000 Samples/sec Using an Impulse Invariant Filter
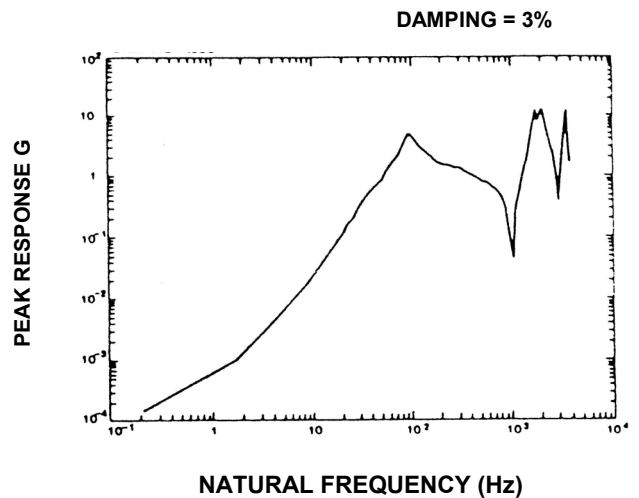
DAMPING = 3%

Fig. 5   Shock Response Spectra of a 100 Hz Decaying Sinusoid Sampled at 2000 Samples/Sec Using an Impulse Invariant Filter
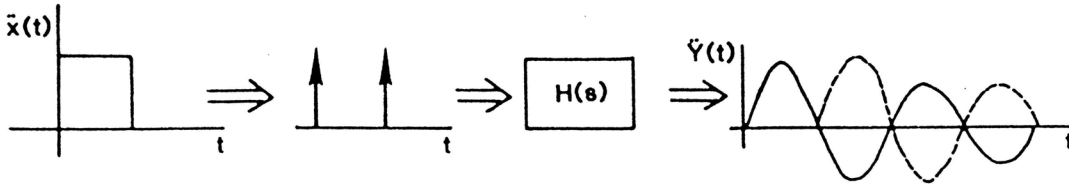
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Fig. 6. Response to Two Impulses by a Single Degree of Freedom System

The solid line represents the response to the first impulse and the dashed line represents the response to the second impulse. The total response will be the sum of the two curves. Clearly the total response will be quite small except for the first half cycle of response due to the destructive interference of the two impulse responses. The actual response of the system to a square wave will be large than the response to two impulses. The simulation of the square wave input by two impulses is not very good in this example. This argument can be extended to more complicated waveforms simulated by a series of impulses. In general, using an impulse invariant simulation of a SDOF system, a minimum in the observed response will be found when the natural frequency is near one half the sample rate due to the destructive interference of successive impulses. A maximum (constructive interference) will be found when the natural frequency is near the sample rate.

Ramp invariant simulations [1] – Let the input to the system be a generalized ramp function;

$$\ddot{x}(t) = A \cdot (t - mT) \cdot u(t - mT)$$

i.e., a ramp with slope A beginning at time t=mT. A ramp invariant simulation can then be found from

$$\tilde{H}(z) = \frac{(z-1)^2}{Tz} Z\left[L^{-1}\left[\frac{H(s)}{s^2}\right]\right] \qquad (6)$$

Using superposition, an input composed of straight lines connecting the sample points will then be an exact simulation.

Eq. 6. yields (after much algebra) the following formulas for the absolute acceleration and relative displacement models.

Absolute acceleration model --

$$\tilde{H}(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - 2C z^{-1} + E^2 z^{-2}} \qquad (7)$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

$$E = e^{-\zeta \omega_n t}$$

$$K = T \omega_d$$

$$C = E \cos K$$

$$S = E \sin K$$

$$S' = S/K = E \sin K / K$$

$$b_0 = 1 - S'$$

$$b_1 = 2(S' - C)$$

$$b_2 = E^2 - S'$$

Relative displacement model --

$$\tilde{H}(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - 2C z^{-1} + E^2 z^{-2}} \qquad (8)$$

$$b_0 = \frac{1}{T\omega_n}\left[2\zeta(C-1) + \frac{(2\zeta^2 - 1)S}{\sqrt{1-\zeta^2}} + T\omega_n\right]$$

$$b_1 = \frac{1}{T\omega_n}\left[-2CT\omega_n + 2\zeta(1 - E^2) - \frac{2(2\zeta^2 - 1)S}{\sqrt{1-\zeta^2}}\right]$$

$$b_2 = \frac{1}{T\omega_n}\left[E^2(T\omega_n + 2\zeta) - 2\zeta C + \frac{(2\zeta^2 - 1)S}{\sqrt{1-\zeta^2}}\right]$$

These models can be converted into a recursive formula of the form

$$\ddot{y}_m = b_0 \ddot{x}_m + b_1 \ddot{x}_{m-1} + b_2 \ddot{x}_{m-2}$$
$$\qquad - a_1 \ddot{y}_{m-1} - a_2 \ddot{y}_{m-2}$$

$$(9)$$

where

$$a_1 = -2C$$
$$a_2 = E^2$$

b_0, b_1, b_2 as given above for the

relative displacement model, and for the absolute acceleration model.

The denominator has the same form as the impulse invariant simulations given in [2].

When the natural frequency is much less than the sample rate, the filter weights become nearly integers i.e.,

$$b_0, b_1, b_2 \to 0$$

$$a_1 \to -2$$

$$a_2 \to 1$$

The output of the filter can be found more accurately if written in the form

$$\ddot{y}_m = b_0 \ddot{x}_m + b_1 \ddot{x}_{m-1} + b_2 \ddot{x}_{m-2} + \ddot{y}_{m-1}$$
$$+ (\ddot{y}_{m-1} - \ddot{y}_{m-2}) - a_1{}' \ddot{y}_{m-1} - a_2{}' \ddot{y}_{m-2}$$

$$(10)$$

where

$$a_1{}' = a_1 + 2 \qquad (11)$$

$$a_2{}' = a_2 - 1 \qquad (12)$$

In this form

$$\ddot{y}_m = \ddot{y}_{m-1} + \text{smaller terms} \qquad (13)$$

Incidentally, the formulas in [2] have the same problem can be improved by writing them in a similar form. When the natural frequency is much less than the sample rate even double precision calculations will not yield accurate filter weights. The following approximations derived from a power series expansion of the formulas can then be used

$$a_1{}' \sim 2\zeta\omega_n T + (\omega_n T)^2 \left(1 - 2\zeta^2\right) \qquad (14)$$

$$a_2{}' \sim -2\zeta\omega_n T + 2\zeta^2 (\omega_n T)^2 \qquad (15)$$

absolute acceleration model

$$b_0 \sim \zeta\omega_n T + (\omega_n T)^2 \left(\frac{1}{6} - \frac{2}{3}\zeta^2\right) \qquad (16)$$

$$b_1 \sim \frac{2}{3}(\omega_n T)^2 \left(1 - \zeta^2\right) \qquad (17)$$

$$b_2 \sim -\zeta\omega_n T + (\omega_n T)^2 \left(\frac{1}{6} + \frac{4}{3}\zeta^2\right) \qquad (18)$$

relative displacement model,

$$b_0 \sim (\omega_n T)^2 / 6 \qquad (19)$$

$$b_1 \sim 2 (\omega_n T)^2 / 3 \qquad (20)$$

$$b_2 \sim (\omega_n T)^2 / 6 \qquad (21)$$

The ramp invariant filters have one more weight filter than the impulse invariant filters have one more weight than the impulse invariant filters. (For the impulse invariant filters b_2=0). The additional weight requires one more multiply add. Round off errors can cause the filter to be unstable when the natural frequency, ω_n, is small and the damping, ζ, is zero. This can be avoided by not using ζ exactly zero when the natural frequency is small compared to the sample rate.

For the proper calculation of the residual shock response spectra, the response must be calculated for a minimum of one full cycle after the input has ended. The number of sample in one cycle is the inverse of the non-dimensional frequency, $f_n T$. When $f_n T$ is small, this can be a large number of samples. An efficient way to avoid this problem is to reduce the sample rate when calculating the residual response for low natural frequencies.

The shock response spectra for the examples as in the impulse invariant section were calculated using the ramp invariant model as shown in Figs. 7 and 8. The spectra was computed with good accuracy over a non-dimensional frequency range, $f_n T$, of 10^{-4} to 2.0.

CONCLUSION

An efficient (requiring 5 multiply-adds per sample point) recursive formula for calculating the shock response spectra has been derived. The formula will give good results over a wide frequency range with a natural frequency of much less than the sample rate to many times the sample rate. The only requirement is that the input waveform is reasonably well band limited to less than the Nyquist frequency.
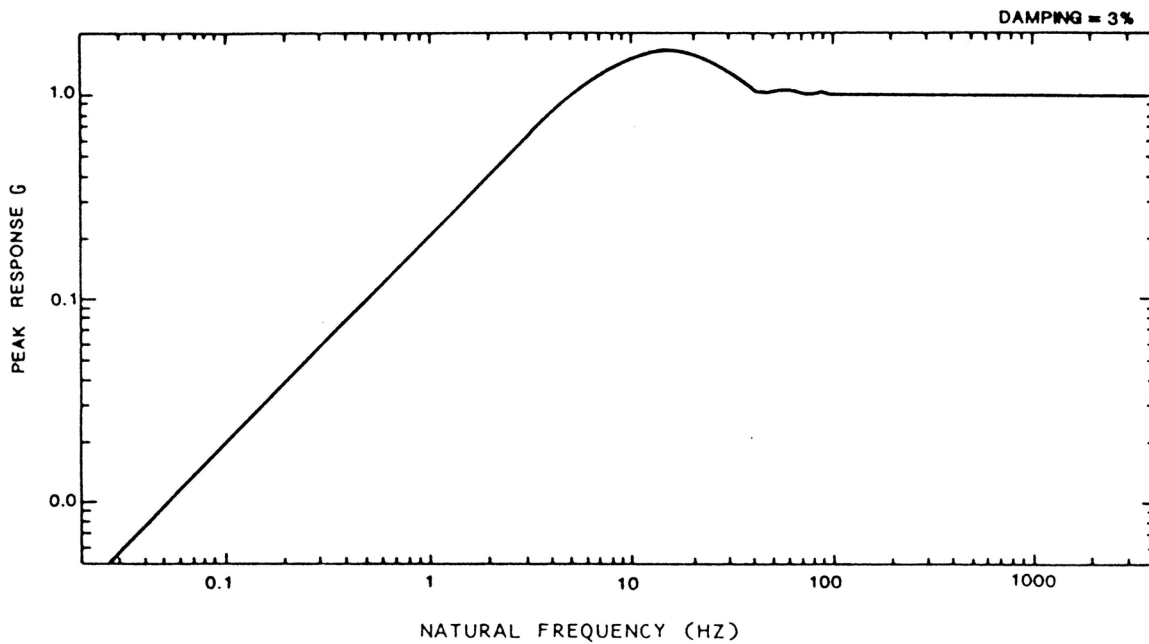
Fig. 7.   Shock Response Spectra of a 64 ms Haversine with Unity Amplitude Sampled 2000 Samples/Sec using a Ramp Invariant Filter
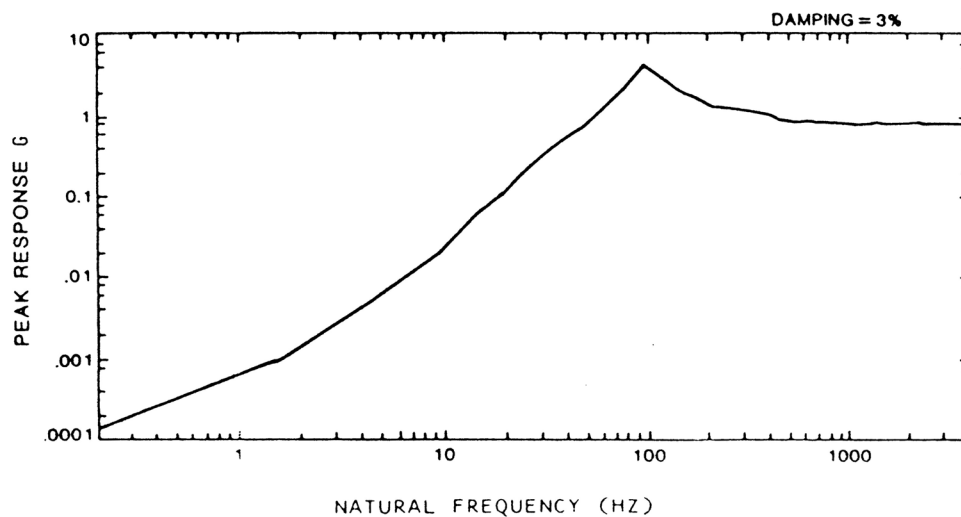

Fig. 8.   Shock Response Spectra of a 100 Hz Decaying Sinusoid Sampled at 2000 Samples/Sec Using a Ramp Invariant Filter

REFERENCES

1. S.D. Stearns, Digital Signal Analysis, Hayden, Rochelle Park, NJ 1975.

2. R. D. Kelly and G. Richman, Principals and Techniques of Shock Data Analysis, SVM-5, Sec. 6.7 Shock and Vibration Information Center, Naval Research Laboratory, Washington, D.C. 1969.

3. D.O. Smallwood and A.R. Nord, "Matching Shock Spectra with Sums of Decaying Sinusoids Compensated for Shaker Velocity and Displacement Limitations," Shock and Vibration Bulletin No. 44, Part 3, pp 43-56, Aug. 1974.

DISCUSSION

Mr. Rubin (The Aerospace Corp): Regarding the residual matter that you just described. An alternative procedure would be to develop the response and its first derivative at the last time your input function and then you can predict what the peak will be and the residual immediately. You don't have to go through decimation or any further calculations at all. All you have to do is get the first derivative of the response. It is an initial value problem of free vibration and you can get the answer.

Mr. Smallwood: I agree. It is an alternate way to do it.