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Matrix Lie Groups for Robotics

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Matrix Groups

- A matrix group is a group of invertible matrices.
- This is a simple and purely algebraic definition.

- Some application examples of matrix groups are:
  - **Computer graphics** where matrix groups are used for three-dimensional rotation and translation of objects.
  - The theory of **differential equations** relies on matrix groups and, in particular, matrix exponentiation.
  - The **shape of the universe**. Cosmologists find it far easier to model the space as a *simply connected* space under the action of a matrix group.
  - **Quantum computing** is based on the group of *unitary matrices*.
  - In **linear algebra** the theory of matrix group provides a uniform tools which are essential in disciplines ranging from topology and geometry to discrete math and statistics.
  - **Riemannian geometry** relies heavily on matrix groups.

A group is a nonempty set $\mathcal{G}$ together with a binary group operation $\cdot$, e.g., $g \cdot h$ where $g,h \in \mathcal{G}$, that satisfies the following properties:

1. **Closure:** if $g,h \in \mathcal{G}$ then also $g \cdot h \in \mathcal{G}$;
2. **Associativity:** for all $g,h,l \in \mathcal{G}$, $(g \cdot h) \cdot l = g \cdot (h \cdot l)$;
3. **Identity:** there exist a unique identity element $e \in \mathcal{G}$ such that $e \cdot g = g \cdot e = g$ for all $g \in \mathcal{G}$;
4. **Inverse:** if $g \in \mathcal{G}$ there exists an element $g^{-1} \in \mathcal{G}$ such that $g^{-1} \cdot g = g \cdot g^{-1} = e$.

- In general we can work with the set of all m by n matrices with entries in $\mathbb{K}$ denoted $\mathrm{M}_{m,n}(\mathbb{K})$, where $\mathbb{K} = \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.
- While it is possible to work with *complex numbers* $\mathbb{C}$ and *quaternions* $\mathbb{H}$ as entries of the matrix, we shall limit our attention to the field of reals $\mathbb{R}$.
- The reason is twofold: first, in most of the problems we encounter we deal with real numbers; second, it turns out that all matrix groups are real matrix groups using the following theorem.

Definition

The *general linear group* over $\mathbb{K}$ is:

$$\mathrm{GL}_n(\mathbb{K}) = \{\mathbf{A} \in \mathrm{M}_n(\mathbb{K}) : \det(\mathbf{A}) \neq 0\}.$$

Theorem

1. $\mathrm{GL}_n(\mathbb{C})$ *is isomorphic to a subgroup of* $\mathrm{GL}_{2n}(\mathbb{R})$.
2. $\mathrm{GL}_n(\mathbb{H})$ *is isomorphic to a subgroup of* $\mathrm{GL}_{2n}(\mathbb{C})$.

It follows that $\mathrm{GL}_n(\mathbb{H})$ is isomorphic to a subgroup of $\mathrm{GL}_{4n}(\mathbb{R})$.

- The n-dimensional *affine group* over $\mathbb{R}$ is

$$\mathrm{Aff}_n(\mathbb{R}) = \left\{ \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} : \mathbf{A} \in \mathrm{GL}_n(\mathbb{R}),\ \mathbf{t} \in \mathbb{R}^n \right\} \subseteq \mathrm{GL}_{n+1}(\mathbb{R}).$$

- This is a closed subgroup of $\mathrm{GL}_{n+1}(\mathbb{R})$. If we identify $\mathbf{x} \in \mathbb{R}^n$ with $\begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} \in \mathbb{R}^{n+1}$, then as a consequence of the formula

$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}\mathbf{x} + \mathbf{t} \\ 1 \end{bmatrix}$$

we obtain an action of $\mathrm{Aff}_n(\mathbb{R})$ on $\mathbb{R}^n$.

- Transformations of $\mathbb{R}^n$ with the form $\mathbf{x} \mapsto \mathbf{A}\mathbf{x} + \mathbf{t}$ with $\mathbf{A}$ invertible is called *affine transformation* and preserves lines. The associated geometry is *affine geometry*.
- Examples of affine transformations include translation, scaling, similarity transformation, reflection, rotation, shear mapping, and compositions of them in any combination and sequence.

- The vector space $\mathbb{R}^n$ itself can be viewed as the *translation subgroup* of $\mathrm{Aff}_n(\mathbb{R})$,

$$\mathrm{Trans}_n(\mathbb{R}) = \left\{ \begin{bmatrix} \mathbf{I}_n & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} : \mathbf{t} \in \mathbb{R}^n \right\} \subseteq \mathrm{Aff}_n(\mathbb{R}),$$

  and this is a closed subgroup.

- There is also the closed subgroup

$$\left\{ \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} : \mathbf{A} \in \mathrm{GL}_n(\mathbb{R}) \right\} \subseteq \mathrm{Aff}_n(\mathbb{R}),$$

  which will be identified with $\mathrm{GL}_n(\mathbb{R})$.

Definition

The *orthogonal group* over $\mathbb{R}$ is denoted $\mathrm{O}(n)$ and defined as:

$$\mathrm{O}(n) = \{\mathbf{A} \in \mathrm{GL}_n(\mathbb{R}) : \mathbf{A} \cdot \mathbf{A}^\mathsf{T} = \mathbf{I}_n\},$$

where "$\cdot$" denotes the standard matrix multiplication as the group operation and is dropped hereafter, i.e., $\mathbf{A}\mathbf{A}^\mathsf{T}$.

- Looking closer into the orthogonal group, we see that $\det(\mathbf{A}\mathbf{A}^\mathsf{T}) = \det(\mathbf{A})^2 = \det(\mathbf{I}_n) = 1$, and therefore, $\det(\mathbf{A}) = \pm 1$. Thus we have $\mathrm{O}(n) = \mathrm{O}(n)^+ \cup \mathrm{O}(n)^-$ where

$$\mathrm{O}(n)^+ = \{\mathbf{A} \in \mathrm{O}(n) : \det(\mathbf{A}) = 1\},$$

$$\mathrm{O}(n)^- = \{\mathbf{A} \in \mathrm{O}(n) : \det(\mathbf{A}) = -1\}.$$

- Notice that $\mathrm{O}(n)^+ \cap \mathrm{O}(n)^- = \varnothing$, so $\mathrm{O}(n)$ is the *disjoint union* of the subsets $\mathrm{O}(n)^+$ and $\mathrm{O}(n)^-$.

- The important subgroup $\mathrm{SO}(n) = \mathrm{O}(n)^+ \leq \mathrm{O}(n)$ is the $n \times n$ *special orthogonal group.*

- One of the main reasons for the study of the orthogonal groups $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ is their relationships with *isometries*, where an isometry of $\mathbb{R}^n$ is a distance-preserving bijection $f : \mathbb{R}^n \to \mathbb{R}^n$, i.e.

$$\|f(\mathbf{x}) - f(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\| \qquad \mathbf{x},\mathbf{y} \in \mathbb{R}^n$$

- If such an isometry fixes the origin, $\mathbf{0}$, then it is a *linear transformation*, often referred to as *linear isometry*, and so with respect with the standard basis it corresponds to a matrix $\mathbf{A} \in \mathrm{GL}_n(\mathbb{R})$

Remark

The special orthogonal group $\mathrm{SO}(n)$ *is the simultaneous rotation of* n *perpendicular planes! For example,* $\mathrm{SO}(3)$ *is the rotation group of* $\mathbb{R}^3$ *and defines the simultaneous rotation of three perpendicular planes which construct the three-dimensional (3D) Euclidean space.*

- Elements of $\mathrm{SO}(n)$ often called *direct isometries* or *rotations*, while elements of $\mathrm{O}(n)^-$ are sometimes called *indirect isometries*.

- We define the full *isometry group* of $\mathbb{R}^n$ as

$$\mathrm{Isom}_n(\mathbb{R}) = \{f : \mathbb{R}^n \to \mathbb{R}^n : f \text{ is an isometry}\},$$

which clearly contains the subgroup of translations.

- In fact, $\mathrm{Isom}_n(\mathbb{R}) \subseteq \mathrm{Aff}_n(\mathbb{R})$ and is actually a closed subgroup, hence is a matrix subgroup,

$$\mathrm{Isom}_n(\mathbb{R}) = \left\{ \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} : \mathbf{A} \in \mathrm{O}(n),\ \mathbf{t} \in \mathbb{R}^n \right\}.$$

- The *special Euclidean group* is the isometry group that requires $\mathbf{A}$ to be a valid right-handed rotation matrix:

$$\mathrm{SE}(n) = \left\{ \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} : \mathbf{A} \in \mathrm{SO}(n),\ \mathbf{t} \in \mathbb{R}^n \right\}.$$

- This is the group of valid rigid body transformations of $\mathbb{R}^n$.

Formally, a (smooth) manifold is a topological space M such that there is a collection of pairs $(U_\alpha, \varphi_\alpha)$ where

1. U_α forms an open cover of M, i.e.

$$\bigcup_\alpha U_\alpha = M,$$

2. For each α

$$\varphi_\alpha : U_\alpha \to \mathbb{R}^n$$

   is a diffeomorphism onto an open set,

3. If $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha \left(U_\alpha \cap U_\beta \right) \to \varphi_\beta \left(U_\alpha \cap U_\beta \right),$$

   is smooth.

This definition means we can "chop" up M into pieces U_α that each look like $\mathbb{R}^n$.

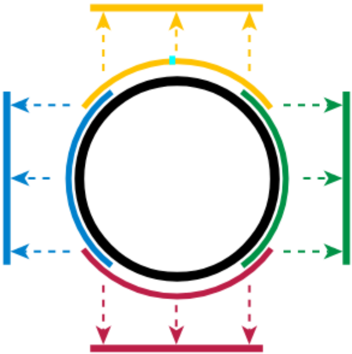
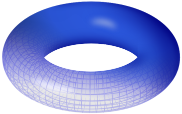


Figure: https://en.wikipedia.org/wiki/Manifold

Many common objects are manifolds.

1. Every Euclidean space, $\mathbb{R}^n$.
2. The 2-sphere, S^2

3. The Torus T^2



https://www.jpl.nasa.gov/edu/teach/activity/ocean-world-earth-globe-toss-game/

https://en.wikipedia.org/wiki/Torus

Just about everything falls under the category of manifolds. For example, everything that can be described as a level-set of a function counts.

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. The set $f^{-1}(y) = \{x \in \mathbb{R}^n : f(x) = y\}$ is called a level-set of f. Moreover, y is called a *regular value* of f if for all $x \in f^{-1}(y)$, $\nabla f \neq 0$.

Theorem (Preimage theorem)

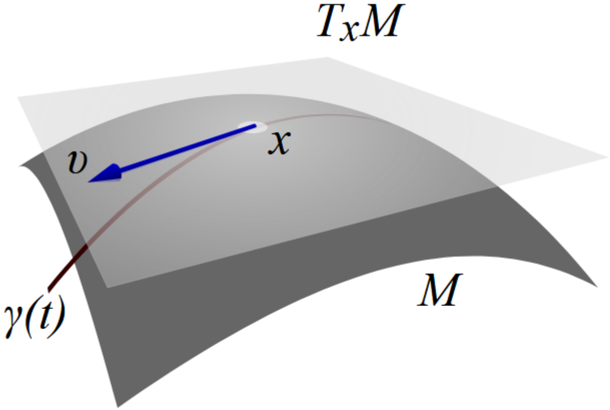
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function and y be a regular value. Then $f^{-1}(y)$ is a manifold.

The preimage theorem immediately shows that the sphere is a manifold. Let $f(x,y,z) = x^2 + y^2 + z^2$. Then $\nabla f = [2x,2y,2z]^\mathsf{T} \neq 0$ and so the set

$$S^2 = f^{-1}(1) = \left\{x^2 + y^2 + z^2 = 1\right\}$$

is a manifold.

In order to study the geometry of a manifold, we need the notion of a tangent space. Let γ be some curve in some manifold M, then its derivative $\dot{\gamma}$ is a *tangent vector*.



If $x \in M$ is a point in the manifold, then the space of *all possible* tangent vectors is called the *tangent space* and is denoted by T_xM.

It is important to point out that T_xM is a vector space and

$$\dim T_xM = \dim M.$$

- A matrix group is an algebraic object. However, it can also be seen as a geometric object since it is a subset of a Euclidean space:

$$\mathcal{G} \subset \mathrm{GL}_n(\mathbb{R}) \subset \mathrm{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}.$$

- In addition, looking at a matrix group as a subset of a Euclidean space means we can discuss its tangent space.

Definition (Tangent space)

Let $\mathcal{G} \subset \mathbb{R}^m$ be a subset, and let $g \in \mathcal{G}$. The *tangent space* to $\mathcal{G}$ at g is:

$$T_g\mathcal{G} = \{\gamma'(0) : \gamma : (-\epsilon, \epsilon) \to \mathcal{G} \text{ is differentiable with } \gamma(0) = g\}\,.$$

$T_g\mathcal{G}$ means the set of initial velocity vectors of differentiable paths through g in $\mathcal{G}$. The term *differentiable* means that, when we consider γ as a path in $\mathbb{R}^m$, the m components of γ are differentiable functions from $(-\epsilon, \epsilon)$ to $\mathbb{R}$.

Definition (Lie algebra)

The *Lie algebra* of a matrix group $\mathcal{G} \subset \mathrm{GL}_n(\mathbb{R})$ is the tangent space to $\mathcal{G}$ at the identity e. It is denoted $\mathfrak{g} = \mathfrak{g}(\mathcal{G}) = T_e\mathcal{G}$.

- In particular, $\mathfrak{g}$ is a subspace of the Euclidean space $\mathrm{M}_n(\mathbb{R})$ which shows that matrix groups are "nice" sets.
- Note that the choice of identity is due to the fact that all groups contain at least the identity element.

Proposition

The Lie algebra $\mathfrak{g}$ *of a matrix group* $\mathcal{G} \subset \mathrm{GL}_n(\mathbb{R})$ *is a real subspace of* $\mathrm{M}_n(\mathbb{R})$*.*

Definition

The *dimension* of a matrix group $\mathcal{G}$ means the dimension of its Lie algebra.

Let us consider the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$, denoted $\mathfrak{gl}_n(\mathbb{R})$.

Proposition

$\mathfrak{gl}_n(\mathbb{R}) = \mathrm{M}_n(\mathbb{R})$. *In particular,* $\dim(\mathrm{GL}_n(\mathbb{R})) = n^2$.

Proof.

Let $\mathbf{A} \in \mathrm{M}_n(\mathbb{R})$. The path $\gamma(t) = \mathbf{I} + t.\mathbf{A}$ in $\mathrm{M}_n(\mathbb{R})$ satisfies $\gamma(0) = \mathbf{I}$ and $\gamma'(0) = \mathbf{A}$. Also, γ restricted to sufficiently small interval $(-\epsilon, \epsilon)$ lies in $\mathrm{GL}_n(\mathbb{R})$. To justify this, notice $\det(\gamma(0)) = 1$. Since the determinant function is continuous, $\det(\gamma(t))$ is close to 1 (and is therefore non-zero) for t close to 0. This demonstrates that $\mathbf{A} \in \mathfrak{gl}_n(\mathbb{R})$. $\square$

Example: Lie Algebras of (Real) Orthogonal Group

The set $\mathfrak{o}(n) = \{\mathbf{A} \in \mathrm{M}_n(\mathbb{R}) : \mathbf{A} + \mathbf{A}^\mathsf{T} = \mathbf{0}\}$ is denoted $\mathfrak{so}(n)$ and called *skew-symmetric* matrices. We wish to show that $\mathfrak{so}(n)$ is the Lie algebra of $\mathrm{O}(n)$.

Theorem

The Lie algebra of $\mathrm{O}(n)$ *equals* $\mathfrak{so}(n)$.

Example: Lie Algebras of (Real) Orthogonal Group

Proof.

Suppose $\gamma : (-\epsilon, \epsilon) \to \mathrm{O}(n)$ is differentiable with $\gamma(0) = \mathbf{I}$. Using the product rule to differentiate both sides of

$$\gamma(t) \cdot \gamma(t)^{\mathsf{T}} = \mathbf{I}$$

gives $\gamma'(0) + \gamma'^{\mathsf{T}}(0) = \mathbf{0}$, so $\gamma'(0) \in \mathfrak{so}(n)$. This demonstrates that $\mathfrak{g}(\mathrm{O}(n)) \subset \mathfrak{so}(n)$.

Proving the other inclusion means explicitly constructing a path in $\mathrm{O}(n)$ in the direction of any $\mathbf{A} \in \mathfrak{so}(n)$. It is simpler and sufficient to do so for all $\mathbf{A}$ in a basis of $\mathfrak{so}(n)$.

Proof.

The natural basis of $\mathfrak{so}(n)$ is the set

$$\{E_{ij} - E_{ji} : 1 \leq i < j \leq n\}$$

where E_{ij} denotes the matrix with ij-entry 1 and other entries zero. For example,

$$\begin{aligned}\mathfrak{so}(3) &= \operatorname{span}\{E_{12} - E_{21}, E_{13} - E_{31}, E_{23} - E_{32}\} \\ &= \left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}\end{aligned}$$

Proof.

The path

$$\gamma_{ij}(t) \triangleq \mathbf{I} + \sin(t)E_{ij} - \sin(t)E_{ji} + (-1 + \cos(t))(E_{ii} + E_{jj})$$

lies in $\mathrm{SO}(n)$, has $\gamma_{ij}(0) = \mathbf{I}$, and has initial direction

$$\gamma'_{ij}(0) = E_{ij} - E_{ji}.$$

$R_{\gamma_{ij}(t)} : \mathbb{R}^n \to \mathbb{R}^n$ rotates the subspace $\mathrm{span}\{e_i, e_j\}$ by an angle t and does nothing to the other basis vectors. For example, the path

$$\gamma_{13}(t) = \begin{bmatrix} \cos(t) & 0 & \sin(t) \\ 0 & 1 & 0 \\ -\sin(t) & 0 & \cos(t) \end{bmatrix}$$

in $\mathrm{SO}(3)$ satisfies $\gamma'_{13}(0) = E_{13} - E_{31}$. □

Example: Lie Algebras of (Real) Orthogonal Group

Remark

Suppose $\mathbf{x}$ *is a row vector.* $R_{\gamma(t)}(\mathbf{x}) \triangleq \mathbf{x} \cdot \gamma(t)$ *is the right multiplication of matrix* $\gamma(t)$ *with vector* $\mathbf{x}$*. Similarly,* $L_{\gamma(t)} \triangleq (\gamma(t) \cdot \mathbf{x}^\mathsf{T})^\mathsf{T}$ *is the left multiplication and* $L_{\gamma(t)} = R_{\gamma^\mathsf{T}(t)}$*. Unlike the previous example, we often work with column vectors and left multiplication, i.e.,* $L_{\gamma(t)} = \gamma(t) \cdot \mathbf{x}$*.*

Example: Lie Algebras of (Real) Orthogonal Group

Corollary

$\dim(\mathrm{SO}(n)) = \frac{n(n-1)}{2}$.

Proof.

Skew-symmetric matrices have zeros on the diagonal, arbitrary real numbers above, and entries below determined by those above, so $\dim(\mathfrak{so}(n)) = \frac{n(n-1)}{2}$. □

1. If G is a Lie group and $e \in G$ is the identity element, then $\mathfrak{g} = T_e G$.

1. If G is a Lie group and $e \in G$ is the identity element, then $\mathfrak{g} = T_e G$.
2. Let $G = \mathrm{SO}(n)$. We know that $\mathfrak{so}_n = T_e\mathrm{SO}(n)$ is the space of all skew-symmetric matrices:

$$\mathfrak{so}_n = \left\{ A \in M_n(\mathbb{R}) : A^{\mathsf{T}} = -A \right\}.$$

What is $T_g\mathrm{SO}(n)$ for $g \neq e$? Let γ be a curve in $\mathrm{SO}(n)$ such that $\gamma(0) = g$. Then we know that $\dot{\gamma}(0) \in T_g\mathrm{SO}(n)$.

Consider $\gamma(t) = g\exp(At)$ where A is skew. Then we know that γ lies in $\mathrm{SO}(n)$ and $\gamma(0) = g$.

Differentiating, we see that $\dot{\gamma}(0) = gA$.

$$T_g\mathrm{SO}_n = \left\{gA : A^{\mathsf{T}} = -A\right\} = g \cdot \mathfrak{so}_n.$$

For a general matrix group G,

$$T_gG = g \cdot \mathfrak{g}.$$

Matrix Exponentiation; Series in $\mathrm{M}_n(\mathbb{R})$

- When the power series of the function $f(x) = \exp(x)$ is applied to a matrix $\mathbf{A} \in \mathrm{M}_n(\mathbb{R})$, the result is called *matrix exponentiation*:

$$\exp(\mathbf{A}) = \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \frac{1}{4!}\mathbf{A}^4 + \cdots .$$

- This series converges for all $\mathbf{A} \in \mathrm{M}_n(\mathbb{R})$.

The "Best" Path in a Matrix Group: One-Parameter Groups

- If $\mathbf{A} \in \mathfrak{gl}_n(\mathbb{R})$, we would like to find the "most natural" path $\gamma(t)$ in $\mathrm{GL}_n(\mathbb{R})$ with $\gamma(0) = \mathbf{I}$ and $\gamma'(0) = \mathbf{A}$.

- We will attempt to choose $\gamma(t)$ such that for all $\mathbf{x} \in \mathbb{R}^n$, the path

$$t \mapsto R_{\gamma(t)}(\mathbf{x})$$

  is an *integral curve* of $R_{\gamma(t)}$.

The “Best” Path in a Matrix Group: One-Parameter Groups

- A *vector field* on $\mathbb{R}^m$ means a continuous function $F : \mathbb{R}^m \to \mathbb{R}^m$. By picturing $F(\mathbf{v})$ as a vector drawn at $\mathbf{v} \in \mathbb{R}^m$, we think of a vector field as associating a vector to each point of $\mathbb{R}^m$.

Definition (Integral curve)

A path $\alpha : (-\epsilon, \epsilon) \to \mathbb{R}^m$ is called an *integral curve* of a vector field $F : \mathbb{R}^m \to \mathbb{R}^m$ if $\alpha'(t) = F(\alpha(t))$ for all $t \in (-\epsilon, \epsilon)$.

The “Best” Path in a Matrix Group: One-Parameter Groups

Proposition

Let $\mathbf{A} \in \mathfrak{gl}_n(\mathbb{R})$. *The path* $\gamma : \mathbb{R} \to \mathrm{M}_n(\mathbb{R})$ *defined as* $\gamma(t) = \exp(t\mathbf{A})$ *is differentiable, and* $\gamma'(t) = \mathbf{A} \cdot \gamma(t) = \gamma(t) \cdot \mathbf{A}$.

The “Best” Path in a Matrix Group: One-Parameter Groups

Proposition

Let $\mathbf{A} \in \mathfrak{gl}_n(\mathbb{R})$. *The path* $\gamma : \mathbb{R} \to \mathrm{M}_n(\mathbb{R})$ *defined as* $\gamma(t) = \exp(t\mathbf{A})$ *is differentiable, and* $\gamma'(t) = \mathbf{A} \cdot \gamma(t) = \gamma(t) \cdot \mathbf{A}$.

Proof.

Each of the n^2 entries of

$$\gamma(t) = \mathbf{I} + t\mathbf{A} + \frac{1}{2}t^2\mathbf{A}^2 + \frac{1}{6}t^3\mathbf{A}^3 + \cdots$$

is a power series in t, which, from familiar real calculus, can be termwise differentiated, giving:

$$\gamma'(t) = \mathbf{0} + \mathbf{A} + t\mathbf{A}^2 + \frac{1}{2}t^2\mathbf{A}^3 + \cdots .$$

This equals $\gamma(t) \cdot \mathbf{A}$ or $\mathbf{A} \cdot \gamma(t)$ depending whether you factor an $\mathbf{A}$ out on the left or right. $\square$

The "Best" Path in a Matrix Group: One-Parameter Groups

Proposition

Let $\mathbf{A} \in \mathrm{M}_n(\mathbb{R})$ *and let* $\gamma(t) = \exp(t\mathbf{A})$.

1. *For all* $\mathbf{x} \in \mathbb{R}^n$, $\alpha(t) = R_{\gamma(t)}(\mathbf{x})$ *is an integral curve of* $R_{\mathbf{A}}$. *Also,* $\alpha(t) = L_{\gamma(t)}(\mathbf{x})$ *is an integral curve of* $L_{\mathbf{A}}$.

2. $\gamma(t)$ *is itself an integral curve of the vector field on* $\mathrm{M}_n(\mathbb{R})$ *whose value at* g *is* $\mathbf{A} \cdot g$, *and is also an integral curve of the vector field whose value at* g *is* $g \cdot \mathbf{A}$.

The "Best" Path in a Matrix Group: One-Parameter Groups

The second result from the previous proposition has the following useful implications:

Remark

Suppose a left-translation is given via the smooth map $L_g : \mathcal{G} \to \mathcal{G}$ where $h \mapsto gh$. Its differential gives rise to an isomorphism of tangent spaces, $(L_g)_* : \mathfrak{g} \to T_g\mathcal{G}$, i.e., $T_g\mathcal{G} \cong (L_g)_*\mathfrak{g}$. Then we have:

$$\gamma(t) = g \exp(t\mathbf{A}).$$

A similar argument for the right-translation can be expressed.

$$\gamma(t) = \exp(t\mathbf{A})g.$$

Using the robotics terminology we have the following interpretations:

Remark

In $\gamma(t) = g\exp(t\mathbf{A})$, *the velocity* $\mathbf{A}$ *is considered to be in the body frame, whereas in* $\gamma(t) = \exp(t\mathbf{A})g$ *the velocity* $\mathbf{A}$ *is in the spatial frame (relative to a fixed (inertial) coordinate frame). The relation between the body and spatial velocities motivates us to study the adjoint action in the following.*

Let $\mathcal{G}$ be a matrix group with Lie algebra $\mathfrak{g}$. For all $g \in \mathcal{G}$, the *conjugation map* $C_g : \mathcal{G} \to \mathcal{G}$, define as

$$C_g(a) = gag^{-1},$$

is a smooth isomorphism. The derivative $\mathrm{d}(\mathrm{C}_g)_I : \mathfrak{g} \to \mathfrak{g}$ is a vector space isomorphism, which we denote as Ad_g (adjoint):

$$\mathrm{Ad}_g = \mathrm{d}(\mathrm{C}_g)_I$$

To derive a simple formula for $\mathrm{Ad}_g(\mathbf{B})$, notice that any $\mathbf{B} \in \mathfrak{g}$ can be represented as $\mathbf{B} = b'(0)$, where $b(t)$ is a differentiable path in $\mathcal{G}$ with $b(0) = \mathbf{I}$. The product rule gives:

$$\mathrm{Ad}_g(\mathbf{B}) = \mathrm{d}(\mathrm{C}_g)_I(B) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} g b(t) g^{-1} = g\mathbf{B}g^{-1}.$$

So we learn that (notice the similarity transformation):

$$\mathrm{Ad}_g(\mathbf{B}) = g\mathbf{B}g^{-1}.$$

- If all elements of $\mathcal{G}$ commute with g, then Ad_g is the identity map on $\mathfrak{g}$.
- So in general, Ad_g measures the failure of g to commute with elements of $\mathcal{G}$ near $\mathbf{I}$.
- More specifically, $\mathrm{Ad}_g(\mathbf{B})$ measures the failure of g to commute with elements of $\mathcal{G}$ near $\mathbf{I}$ in the direction of $\mathbf{B}$.

Investigating this phenomena when g is itself close to $\mathbf{I}$ leads one to define:

Definition (Lie bracket)

The Lie bracket of two vectors $\mathbf{A}$ and $\mathbf{B}$ in $\mathfrak{g}$ is:

$$[\mathbf{A}, \mathbf{B}] = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{a(t)}(\mathbf{B}),$$

where $a(t)$ is any differentiable path in $\mathcal{G}$ with $a(0) = \mathbf{I}$ and $a'(0) = \mathbf{A}$.

The following alternative definition is easier to calculate and verifies that the previous definition is independent of the choice of path $a(t)$.

Proposition

For all $\mathbf{A}, \mathbf{B} \in \mathfrak{g}$, $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$.

Proof.

Left as exercise. $\square$

- Notice that $[\mathbf{A}, \mathbf{B}] \in \mathfrak{g}$.
- It measures the failure of elements of $\mathcal{G}$ near $\mathbf{I}$ in the direction of $\mathbf{A}$ to commute with elements of $\mathcal{G}$ near $\mathbf{I}$ in the direction of $\mathbf{B}$.

Proposition

For all $\mathbf{A}, \mathbf{A}_1, \mathbf{A}_2, \mathbf{B}, \mathbf{B}_1, \mathbf{B}_2, \mathbf{C} \in \mathfrak{g}$ *and* $\lambda_1, \lambda_2 \in \mathbb{R}$*,*

1. *(Bilinearity)* $[\lambda_1 \mathbf{A}_1 + \lambda_2 \mathbf{A}_2, \mathbf{B}] = \lambda_1 [\mathbf{A}_1, \mathbf{B}] + \lambda_2 [\mathbf{A}_2, \mathbf{B}]$*.*
2. *(Bilinearity)* $[\mathbf{A}, \lambda_1 \mathbf{B}_1 + \lambda_2 \mathbf{B}_2] = \lambda_1 [\mathbf{A}, \mathbf{B}_1] + \lambda_2 [\mathbf{A}, \mathbf{B}_2]$*.*
3. *(Anticommutativity)* $[\mathbf{A}, \mathbf{B}] = -[\mathbf{B}, \mathbf{A}]$*.*
4. *(Jacobi identity)* $[[\mathbf{A},\mathbf{B}],\mathbf{C}] + [[\mathbf{B},\mathbf{C}],\mathbf{A}] + [[\mathbf{C},\mathbf{A}],\mathbf{B}] = 0$*.*

Example: Lie Bracket on $\mathfrak{so}(3)$ and Cross Product

See `so3_cross_example.m` for numerical examples and details.

$$G_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, G_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, G_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[G_1, G_2] = G_3,\ [G_2, G_3] = G_1,\ [G_3, G_1] = G_2$$

$$e_1 \times e_2 = e_3,\ e_2 \times e_3 = e_1,\ e_3 \times e_1 = e_2$$

- Let $\mathcal{G} \subset \mathrm{GL}_n(\mathbb{R})$ be matrix group of dimension d, with Lie algebra $\mathfrak{g}$. For every $g \in \mathcal{G}$, $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is a vector space isomorphism.
- Once we choose a basis $\mathcal{B}$ of $\mathfrak{g}$, this isomorphism can be represented as $L_{\mathbf{A}}$ for some $\mathbf{A} \in \mathrm{GL}_d(\mathbb{R})$ (left matrix multiplication).
- In other words, after fixing a basis of $\mathfrak{g}$, we can regard the map $g \mapsto \mathrm{Ad}_g$ as a function from $\mathcal{G}$ to $\mathrm{GL}_d(\mathbb{R})$.

For $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathfrak{g}$ with sufficiently small norm, the equation $\exp(\mathbf{X})\exp(\mathbf{Y}) = \exp(\mathbf{Z})$ has a power series solution for $\mathbf{Z}$ in terms of repeated Lie bracket of $\mathbf{X}$ and $\mathbf{Y}$. The beginning of the series is:

$$\mathbf{Z} = \mathbf{X} + \mathbf{Y} + \frac{1}{2}[\mathbf{X}, \mathbf{Y}] + \frac{1}{12}[\mathbf{X}, [\mathbf{X}, \mathbf{Y}]] + \frac{1}{12}[\mathbf{Y}, [\mathbf{Y}, \mathbf{X}]] + \cdots .$$

The existence of such a series means that the group operation is completely determined by the Lie bracket operation; the product of $\exp(\mathbf{X})$ and $\exp(\mathbf{Y})$ can be expressed purely in term of repeated Lie brackets of $\mathbf{X}$ and $\mathbf{Y}$.

One important consequence of the Baker-Campbell-Hausdorff series is the following correspondence between Lie algebras and matrix groups.

Theorem (The Lie correspondences theorem)

There is a natural one-to-one correspondences between sub-algebras of $\mathfrak{gl}_n(\mathbb{R})$ and path-connected subgroups of $\mathrm{GL}_n(\mathbb{R})$.

A manifold is a set which is locally diffeomorphic to Euclidean space:

Definition

A subset $\mathcal{X} \subset \mathbb{R}^m$ is called a manifold of dimension n if for all $p \in \mathcal{X}$ there exists a neighborhood $\mathcal{V}$ of p in $\mathcal{X}$ which is diffeomorphic to an open set $\mathcal{U} \subset \mathbb{R}^n$.

Theorem

Any matrix group of dimension n is a manifold of dimension n.

- *Lie groups* are the generalization of matrix groups.
- As such, all of the theory of matrix groups we discussed are also true for Lie groups.

Definition

A *Lie group* is a manifold, $\mathcal{G}$, with a smooth group operation $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$.

- In other words, a Lie group is a manifold that is also a group.
- Since matrix groups are (embedded) manifolds and have a smooth group operation, so matrix groups are Lie groups.
- All important structure of matrix groups carry over to Lie groups.

- Group of 3D rotation matrices, $\mathrm{SO}(3)$; it can model rotations without any singularities or ambiguities.
- Group of direct spatial isometries (3D Rigid Body Transformations), $\mathrm{SE}(3)$.
- Group of K direct isometries, $\mathrm{SE}_K(3)$; for example, it is used for modeling IMU sensors and robot pose plus landmarks and/or contact points.
- Group of 3D similarity transformations, $\mathrm{Sim}(3)$; it is more general than $\mathrm{SE}(3)$ and includes a scale factor and used in monocular vision where the scale is not known.

Let $\boldsymbol{\phi} \in \mathbb{R}^3$ be a vector that can be identified with an element of the Lie algebra, $\boldsymbol{\phi}^\wedge \in \mathfrak{so}(3)$. The corresponding rotation matrix, $\mathbf{R} \in \mathrm{SO}(3)$, can be computed using the group's exponential map:

$$\mathbf{R} = \exp(\boldsymbol{\phi}^\wedge) = \mathbf{I}_3 + \left(\frac{\sin\theta}{\theta}\right)\boldsymbol{\phi}^\wedge + \left(\frac{1-\cos\theta}{\theta^2}\right)(\boldsymbol{\phi}^\wedge)^2 \quad \in \mathrm{SO}(3)$$

where $\theta \triangleq ||\boldsymbol{\phi}||$.

The inverse operation maps the rotation matrix back to a 3×3 skew-symmetric matrix.

$$\boldsymbol{\phi}^\wedge = \log(\mathbf{R}) = \frac{\theta}{2\sin\theta}(\mathbf{R} - \mathbf{R}^\mathsf{T}) \quad \in \mathfrak{so}(3)$$

where $\theta \triangleq \cos^{-1}\left(\frac{\mathsf{tr}(\mathbf{R}) - 1}{2}\right)$

The matrix representation for the adjoint for $\mathrm{SO}(3)$ is simply the rotation matrix itself.

$$\mathrm{Ad}_{\mathbf{R}} = \mathbf{R}$$

The left Jacobian of $\mathrm{SO}(3)$ and its inverse can be calculated using:

$$\mathbf{J}_l(\boldsymbol{\phi}) = \mathbf{I}_3 + \left(\frac{1-\cos\theta}{\theta^2}\right)\boldsymbol{\phi}^\wedge + \left(\frac{\theta-\sin\theta}{\theta^3}\right)(\boldsymbol{\phi}^\wedge)^2$$

$$\mathbf{J}_l^{-1}(\boldsymbol{\phi}) = \mathbf{I}_3 - \frac{1}{2}\boldsymbol{\phi}^\wedge + \left(\frac{1}{\theta^2} - \frac{1+\cos\theta}{2\theta\sin\theta}\right)(\boldsymbol{\phi}^\wedge)^2$$

where $\theta \triangleq ||\boldsymbol{\phi}||$.

Let $\boldsymbol{\xi} \in \mathbb{R}^6$ be a vector that can be identified with an element of the Lie algebra, $\boldsymbol{\xi}^\wedge \in \mathfrak{se}(3)$. The corresponding pose, $\mathbf{H} \in \mathrm{SE}(3)$, can be computed using the exponential map:

$$\boldsymbol{\xi}^\wedge = \begin{bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\rho} \end{bmatrix}^\wedge = \begin{bmatrix} \boldsymbol{\phi}^\wedge & \boldsymbol{\rho} \\ \mathbf{0}_{1\times 3} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{H} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

$$\mathbf{H} = \exp(\boldsymbol{\xi}^\wedge) = \begin{bmatrix} \exp(\boldsymbol{\phi}^\wedge) & \mathbf{J}_l(\boldsymbol{\phi})\boldsymbol{\rho} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \in \mathrm{SE}(3),$$

where $\exp(\boldsymbol{\phi}^\wedge)$ and $\mathbf{J}_l(\boldsymbol{\phi})$ are the exponential map and the left Jacobian of $\mathrm{SO}(3)$.

The logarithm maps the pose back to the tangent space.

$$\boldsymbol{\xi}^\wedge = \log(\mathbf{H}) = \begin{bmatrix} \log(\mathbf{R}) & \mathbf{J}_l^{-1}\left(\log(\mathbf{R})\right)\mathbf{p} \\ 0_{1\times 3} & 0 \end{bmatrix} \in \mathfrak{se}(3)$$

The matrix representation for the adjoint of $\mathrm{SE}(3)$ is given by:

$$\mathrm{Ad}_{\mathbf{H}} = \begin{bmatrix} \mathbf{R} & \mathbf{0}_{3\times 3} \\ \mathbf{p}^\wedge \mathbf{R} & \mathbf{R} \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

The left Jacobian of $\mathrm{SE}(3)$ and its inverse are:

$$\mathbf{J}_l(\boldsymbol{\xi}) = \begin{bmatrix} \mathbf{J}_l(\boldsymbol{\phi}) & \mathbf{0}_{3\times 3} \\ \mathbf{Q}(\boldsymbol{\phi},\boldsymbol{\rho}) & \mathbf{J}_l(\boldsymbol{\phi}) \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

$$\mathbf{J}_l^{-1}(\boldsymbol{\xi}) = \begin{bmatrix} \mathbf{J}_l^{-1}(\boldsymbol{\phi}) & \mathbf{0}_{3\times 3} \\ -\mathbf{J}_l^{-1}(\boldsymbol{\phi})\mathbf{Q}(\boldsymbol{\phi},\boldsymbol{\rho})\mathbf{J}_l^{-1}(\boldsymbol{\phi}) & \mathbf{J}_l^{-1}(\boldsymbol{\phi}) \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

$$\begin{aligned}\mathbf{Q}(\boldsymbol{\phi},\boldsymbol{\rho}) \triangleq\ & \frac{1}{2}\boldsymbol{\rho}^\wedge + \frac{\theta - \sin\theta}{\theta^3}\left(\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge + \boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge + \boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\right) \\ & - \frac{1 - \frac{\theta^2}{2} - \cos\theta}{\theta^4}\left(\boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge + \boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge - 3\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\right) \\ & - \frac{1}{2}\left(\frac{1 - \frac{\theta^2}{2} - \cos\theta}{\theta^4} - 3\,\frac{\theta - \sin\theta - \frac{\theta^3}{6}}{\theta^5}\right)\left(\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge + \boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\right)\end{aligned}$$

Let $\boldsymbol{\xi} \in \mathbb{R}^{3(K+1)}$ be a vector identified with an element of the Lie algebra, $\boldsymbol{\xi}^\wedge \in \mathfrak{se}_K(3)$. The corresponding group element, $\mathbf{X} \in \mathrm{SE}_K(3)$, is computed using the exponential map:

$$\boldsymbol{\xi}^\wedge = \begin{bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\rho}_1 \\ \vdots \\ \boldsymbol{\rho}_K \end{bmatrix}^\wedge = \begin{bmatrix} \boldsymbol{\phi}^\wedge & \boldsymbol{\rho}_1 & \cdots & \boldsymbol{\rho}_K \\ \mathbf{0}_{1\times 3} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{1\times 3} & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{X} = \begin{bmatrix} \mathbf{R} & \mathbf{p}_1 & \cdots & \mathbf{p}_K \\ \mathbf{0}_{1\times 3} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{1\times 3} & 0 & \cdots & 1 \end{bmatrix}$$

$$\mathbf{X} = \exp(\boldsymbol{\xi}^\wedge) = \begin{bmatrix} \exp(\boldsymbol{\phi}^\wedge) & \mathbf{J}_l(\boldsymbol{\phi})\boldsymbol{\rho}_1 & \cdots & \mathbf{J}_l(\boldsymbol{\phi})\boldsymbol{\rho}_K \\ \mathbf{0}_{1\times 3} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{1\times 3} & 0 & \cdots & 1 \end{bmatrix} \in \mathrm{SE}_K(3),$$

where $\exp(\boldsymbol{\phi}^\wedge)$ and $\mathbf{J}_l(\boldsymbol{\phi})$ are the exponential map and the left Jacobian of $\mathrm{SO}(3)$.

$$\boldsymbol{\xi}^\wedge = \log(\mathbf{H}) = \begin{bmatrix} \log(\mathbf{R}) & \mathbf{J}_l^{-1}\left(\log(\mathbf{R})\right)\mathbf{p}_1 & \cdots & \mathbf{J}_l^{-1}\left(\log(\mathbf{R})\right)\mathbf{p}_K \\ \mathbf{0}_{1\times 3} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{1\times 3} & 0 & \cdots & 0 \end{bmatrix} \in \mathfrak{se}_K(3)$$

The matrix representation for the adjoint of $\mathrm{SE}_K(3)$ is given by:

$$\mathrm{Ad}_{\mathbf{X}} = \begin{bmatrix} \mathbf{R} & \mathbf{0}_{3\times3} & \cdots & \mathbf{0}_{3\times3} \\ {\mathbf{p}_1}^{\wedge}\mathbf{R} & \mathbf{R} & \cdots & \mathbf{0}_{3\times3} \\ \vdots & \vdots & \ddots & \vdots \\ {\mathbf{p}_K}^{\wedge}\mathbf{R} & \mathbf{0}_{3\times3} & \cdots & \mathbf{R} \end{bmatrix} \in \mathbb{R}^{3(K+1)\times3(K+1)}$$

The left Jacobian of $\mathrm{SE}_K(3)$ is:

$$\mathbf{J}_l(\boldsymbol{\xi}) = \begin{bmatrix} \mathbf{J}_l(\boldsymbol{\phi}) & \mathbf{0}_{3\times3} & \cdots & \mathbf{0}_{3\times3} \\ \mathbf{Q}(\boldsymbol{\phi}, \boldsymbol{\rho}_1) & \mathbf{J}_l(\boldsymbol{\phi}) & \cdots & \mathbf{0}_{3\times3} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{Q}(\boldsymbol{\phi}, \boldsymbol{\rho}_K) & \mathbf{0}_{3\times3} & \cdots & \mathbf{J}_l(\boldsymbol{\phi}) \end{bmatrix} \in \mathbb{R}^{3(K+1)\times3(K+1)}$$

First-Order Approximation using BCH

- The adjoint representation of a Lie group is a linear map that captures the non-commutative structure of the group.

- The following properties from adjoint representation and BCH formula for the first order approximation are useful.

$$
\begin{aligned}
&\mathbf{X}\exp(\boldsymbol{\xi}^{\wedge})\mathbf{X}^{-1} = \exp((\mathrm{Ad}_{\mathbf{X}}\boldsymbol{\xi})^{\wedge}) \\
&\Longrightarrow \mathbf{X}\exp(\boldsymbol{\xi}^{\wedge}) = \exp((\mathrm{Ad}_{\mathbf{X}}\boldsymbol{\xi})^{\wedge})\mathbf{X} \\
&\Longrightarrow \exp(\boldsymbol{\xi}^{\wedge})\mathbf{X} = \mathbf{X}\exp((\mathrm{Ad}_{\mathbf{X}^{-1}}\boldsymbol{\xi})^{\wedge})
\end{aligned}
$$

- In the above equations $\mathbf{X} \in \mathcal{G}$ and $\boldsymbol{\xi}^{\wedge} \in \mathfrak{g}$.

First-Order Approximation using BCH

- The BCH formula can be used to compound two matrix exponentials.
- If both terms are small, by keeping the first two terms ignoring the higher order terms, we have:

$$\mathrm{BCH}(\boldsymbol{\xi}_1{}^{\wedge}, \boldsymbol{\xi}_2{}^{\wedge}) = \boldsymbol{\xi}_1{}^{\wedge} + \boldsymbol{\xi}_2{}^{\wedge} + \mathrm{HOT},$$

$$\exp(\boldsymbol{\xi}_1{}^{\wedge}) \exp(\boldsymbol{\xi}_2{}^{\wedge}) \approx \exp(\boldsymbol{\xi}_1{}^{\wedge} + \boldsymbol{\xi}_2{}^{\wedge}).$$

First-Order Approximation using BCH

- When both terms are not small and assuming $\boldsymbol{\xi}$ is small, by keeping the linear terms in $\boldsymbol{\xi}$, we have:

$$\log(\exp(\mathbf{r}^\wedge)\exp(\boldsymbol{\xi}^\wedge))^\vee \approx \mathbf{r} + \mathbf{J}_r^{-1}(\mathbf{r})\boldsymbol{\xi},$$

$$\log(\exp(\boldsymbol{\xi}^\wedge)\exp(\mathbf{r}^\wedge))^\vee \approx \mathbf{r} + \mathbf{J}_l^{-1}(\mathbf{r})\boldsymbol{\xi}.$$

  where $\mathbf{J}_r$ and $\mathbf{J}_l$ are the right and left Jacobians of the Lie group $\mathcal{G}$, respectively.

- The left and right Jacobians are related through the adjoint map,

$$\mathbf{J}_r(\boldsymbol{\xi}) = \mathrm{Ad}_{\exp(-\boldsymbol{\xi}^\wedge)}\mathbf{J}_l(\boldsymbol{\xi}).$$

Example: Uncertainty Propagation on Matrix Lie Groups

- Suppose a process model on matrix Lie group $\mathcal{G}$ where the state at any two successive keyframes at times-steps $k+1$ and k is related using input such as $\mathbf{U}_k \in \mathcal{G}$. The deterministic process model is as follows.

$$\mathbf{X}_{k+1} = f_{u_i}(\mathbf{X}_k) \triangleq \mathbf{X}_k \mathbf{U}_k$$

- Substituting in the noisy process model we have

$$\mathbf{X}_{k+1} = f_{u_i}(\mathbf{X}_k) \exp({\mathbf{w}_k}^{\wedge})$$

where $\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_{\mathbf{w}_k})$.

- Notice that the noise is defined in the Lie algebra and mapped to the nonlinear error using the Lie exponential map (matrix exponential).

- $\mathbf{w}_k \in \mathbb{R}^n$ is defined in the Euclidean vector space that is isomorphic to the Lie algebra where n is the dimension of the Lie algebra.

Example: Uncertainty Propagation on Matrix Lie Groups

- We use the *left invariant error* to track the covariance of the spatial error as seen in the body-fixed frame: $\boldsymbol{\eta} = \exp(\boldsymbol{\xi}^\wedge) = \mathbf{X}^{-1}\bar{\mathbf{X}}$.

- Then, the state $\mathbf{X}$ is of the following form where $\bar{\mathbf{X}}$ is the mean or estimated value of the true state: $\mathbf{X} = \bar{\mathbf{X}}\exp(-\boldsymbol{\xi}^\wedge)$.

- We substitute $\mathbf{X} = \bar{\mathbf{X}}\exp(-\boldsymbol{\xi}^\wedge)$ in the noisy process model as follows:

$$\bar{\mathbf{X}}_{k+1}\exp(-{\boldsymbol{\xi}_{k+1}}^\wedge) = \bar{\mathbf{X}}_k \exp(-{\boldsymbol{\xi}_k}^\wedge)\mathbf{U}_k \exp({\mathbf{w}_k}^\wedge),$$

- and using the adjoint to shift all noise terms to the right, we get:

$$\bar{\mathbf{X}}_{k+1}\exp(-{\boldsymbol{\xi}_{k+1}}^\wedge) = \bar{\mathbf{X}}_k \mathbf{U}_k \exp((-\mathrm{Ad}_{\mathbf{U}_k^{-1}}\boldsymbol{\xi}_k)^\wedge)\exp({\mathbf{w}_k}^\wedge)$$

Example: Uncertainty Propagation on Matrix Lie Groups

- This previous equation shows the relation between noise terms at timesteps k and $k+1$. Thus,

$$\exp(-\boldsymbol{\xi}_{k+1}{}^{\wedge}) = \exp((-\mathrm{Ad}_{\mathbf{U}_k^{-1}}\boldsymbol{\xi}_k)^{\wedge})\exp(\mathbf{w}_k{}^{\wedge}).$$

- Since all noise terms are small, after applying the BCH formula and keeping the first two terms we arrive at the approximate error dynamics:

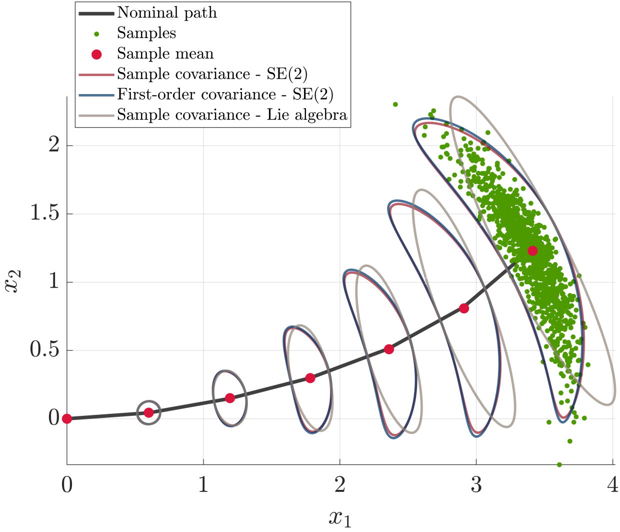
$$\boldsymbol{\xi}_{k+1} = \mathrm{Ad}_{\mathbf{U}_k^{-1}}\boldsymbol{\xi}_k - \mathbf{w}_k.$$

- From here, one can easily show that

$$\boldsymbol{\Sigma}_{k+1} = \mathrm{Ad}_{\mathbf{U}_k^{-1}}\boldsymbol{\Sigma}_k\mathrm{Ad}_{\mathbf{U}_k^{-1}}^{\mathsf{T}} + \boldsymbol{\Sigma}_{\mathbf{w}_k}.$$

Example: Uncertainty Propagation on $\mathrm{SE}(2)$

See `odometry_propagation_se2.m` for code.



Example: Uncertainty Propagation on $\mathrm{SE}(3)$

See `odometry_propagation_se3.m` for code.