NA 568 - Winter 2022

Matrix Lie Groups for Robotics

Maani Ghaffari

January 25, 2022

UNIVERSITY OF MICHIGAN

Some Motivation

1. How to learn features that are invariant to the image orientation?

2. How to model a rigid motion of an object or articulated robot?
3. What about 3D reconstruction using a monocular camera?
4. How about modeling and integrating data from an IMU (gyroscope + accelerometer)?

A group is a nonempty set $\mathcal{G}$ together with a binary group operation $\cdot$, e.g., $g \cdot h$ where $g,h \in \mathcal{G}$, that satisfies the following properties:

1. **Closure:** if $g,h \in \mathcal{G}$ then also $g \cdot h \in \mathcal{G}$;
2. **Associativity:** for all $g,h,l \in \mathcal{G}$, $(g \cdot h) \cdot l = g \cdot (h \cdot l)$;
3. **Identity:** there exist a unique identity element $e \in \mathcal{G}$ such that $e \cdot g = g \cdot e = g$ for all $g \in \mathcal{G}$;
4. **Inverse:** if $g \in \mathcal{G}$ there exists an element $g^{-1} \in \mathcal{G}$ such that $g^{-1} \cdot g = g \cdot g^{-1} = e$.

Check if $(\mathbb{R},+)$ and $(\mathbb{R}\backslash\{0\},\cdot)$ are groups.

1. **Closure:**

2. **Associativity:**

3. **Identity:**

4. **Inverse:**

- A matrix group is a group of invertible matrices.

In general we can work with the set of all m by n matrices with entries in $\mathbb{R}$ denoted $\mathrm{M}_{m,n}(\mathbb{R})$.

Definition

The *general linear group* over $\mathbb{R}$ is:

$$\mathrm{GL}_n(\mathbb{R}) = \{A \in \mathrm{M}_n(\mathbb{R}) : \det(A) \neq 0\}.$$

- The n-dimensional *affine group* over $\mathbb{R}$ is
$$\mathrm{Aff}_n(\mathbb{R}) = \left\{ \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} : A \in \mathrm{GL}_n(\mathbb{R}),\ t \in \mathbb{R}^n \right\}.$$
- If we identify $x \in \mathbb{R}^n$ with $\begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathbb{R}^{n+1}$, then as a consequence of the formula
$$\begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Ax + t \\ 1 \end{bmatrix}$$
we obtain an action of $\mathrm{Aff}_n(\mathbb{R})$ on $\mathbb{R}^n$.

- The vector space $\mathbb{R}^n$ itself can be viewed as the *translation subgroup* of $\mathrm{Aff}_n(\mathbb{R})$,

$$\mathrm{Trans}_n(\mathbb{R}) = \left\{ \begin{bmatrix} I_n & t \\ 0 & 1 \end{bmatrix} : t \in \mathbb{R}^n \right\} \subseteq \mathrm{Aff}_n(\mathbb{R}),$$

  and this is a closed subgroup.

Definition

The *orthogonal group* over $\mathbb{R}$ is denoted $\mathrm{O}(n)$ and defined as:

$$\mathrm{O}(n) = \{A \in \mathrm{GL}_n(\mathbb{R}) : A \cdot A^\mathsf{T} = I_n\},$$

where "$\cdot$" denotes the standard matrix multiplication as the group operation and is dropped hereafter, i.e., AA^T.

- Looking closer into the orthogonal group, we see that $\det(AA^\mathsf{T}) = \det(A)^2 = \det(I_n) = 1$;
- therefore, $\det(A) = \pm 1$. Thus we have $\mathrm{O}(n) = \mathrm{O}(n)^+ \cup \mathrm{O}(n)^-$ where

$$\mathrm{O}(n)^+ = \{A \in \mathrm{O}(n) : \det(A) = 1\},$$

$$\mathrm{O}(n)^- = \{A \in \mathrm{O}(n) : \det(A) = -1\}.$$

- Notice that $\mathrm{O}(n)^+ \cap \mathrm{O}(n)^- = \varnothing$, so $\mathrm{O}(n)$ is the *disjoint union* of the subsets $\mathrm{O}(n)^+$ and $\mathrm{O}(n)^-$.

- The important subgroup $\mathrm{SO}(n) = \mathrm{O}(n)^+ \leq \mathrm{O}(n)$ is the $n \times n$ *special orthogonal group*.

- One of the main reasons for the study of the orthogonal groups $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ is their relationships with *isometries*, where an isometry of $\mathbb{R}^n$ is a distance-preserving bijection $f : \mathbb{R}^n \to \mathbb{R}^n$, i.e.,

$$\|f(x) - f(y)\| = \|x - y\| \qquad x,y \in \mathbb{R}^n$$

- If such an isometry fixes the origin, 0, then it is a *linear transformation*, often referred to as *linear isometry*, and so with respect with the standard basis it corresponds to a matrix $A \in \mathrm{GL}_n(\mathbb{R})$

Remark

The special orthogonal group $\mathrm{SO}(n)$ is the simultaneous rotation of n perpendicular planes!

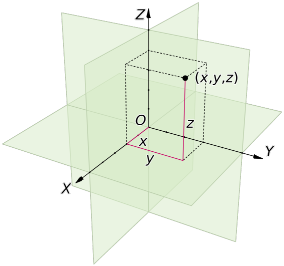


Figure: For example, $\mathrm{SO}(3)$ is the rotation group of $\mathbb{R}^3$ and defines the simultaneous rotation of three perpendicular planes which construct the three-dimensional (3D) Euclidean space.

- Elements of $\mathrm{SO}(n)$ often called *direct isometries* or *rotations*, while elements of $\mathrm{O}(n)^-$ are sometimes called *indirect isometries*.

- We define the full *isometry group* of $\mathbb{R}^n$ as

$$\mathrm{Isom}_n(\mathbb{R}) = \{f : \mathbb{R}^n \to \mathbb{R}^n : f \text{ is an isometry}\},$$

  which clearly contains the subgroup of translations.

- In fact, $\mathrm{Isom}_n(\mathbb{R}) \subseteq \mathrm{Aff}_n(\mathbb{R})$ and is actually a closed subgroup, hence is a matrix subgroup,

$$\mathrm{Isom}_n(\mathbb{R}) = \left\{ \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} : A \in \mathrm{O}(n),\ t \in \mathbb{R}^n \right\}.$$

- The *special Euclidean group* is the isometry group that requires A to be a valid right-handed rotation matrix:

$$\mathrm{SE}(n) = \left\{ \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} : A \in \mathrm{SO}(n),\ t \in \mathbb{R}^n \right\}.$$

- This is the group of valid rigid body transformations of $\mathbb{R}^n$.

Flat Earth Society

How many of you are members!?

The
FLAT EARTH
SOCIETY

We can “chop” up manifold M into pieces that each look like $\mathbb{R}^n$.

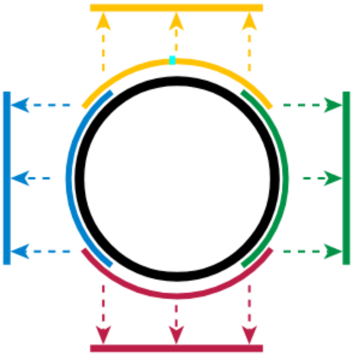
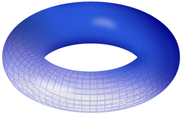


Figure: https://en.wikipedia.org/wiki/Manifold

Many common objects are manifolds.

1. Every Euclidean space, $\mathbb{R}^n$.
2. The 2-sphere, S^2

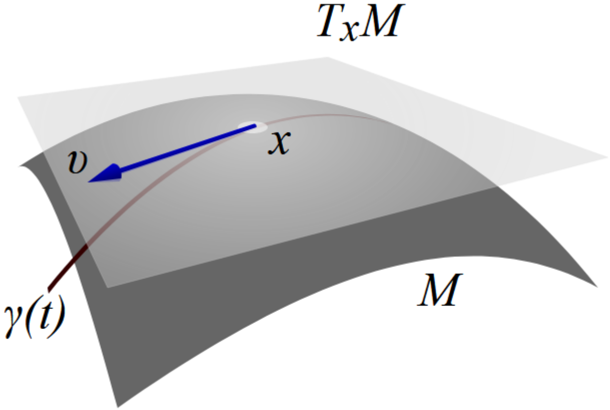
3. The Torus T^2



https://www.jpl.nasa.gov/edu/teach/activity/ocean-world-earth-globe-toss-game/

https://en.wikipedia.org/wiki/Torus

To study the geometry of a manifold, we need the notion of a tangent space. Let γ be some curve in some manifold M, then its derivative $\dot{\gamma}$ is a *tangent vector*.



- If $x \in M$ is a point in the manifold, then the space of *all possible* tangent vectors is called the *tangent space* and is denoted by $T_x M$.

- It is important to point out that $T_x M$ is a vector space and

$$\dim T_x M = \dim M.$$

- A matrix group is an algebraic object. However, it can also be seen as a geometric object since it is a subset of a Euclidean space:

$$\mathcal{G} \subset \mathrm{GL}_n(\mathbb{R}) \subset \mathrm{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}.$$

- In addition, looking at a matrix group as a subset of a Euclidean space means we can discuss its tangent space.

Definition (Tangent space)

Let $\mathcal{G} \subset \mathbb{R}^m$ be a subset, and let $g \in \mathcal{G}$. The *tangent space* to $\mathcal{G}$ at g is:

$$T_g\mathcal{G} = \{\gamma'(0) : \gamma : (-\epsilon, \epsilon) \to \mathcal{G} \text{ is differentiable with } \gamma(0) = g\}.$$

$T_g\mathcal{G}$ means the set of initial velocity vectors of differentiable paths through g in $\mathcal{G}$. The term *differentiable* means that, when we consider γ as a path in $\mathbb{R}^m$, the m components of γ are differentiable functions from $(-\epsilon, \epsilon)$ to $\mathbb{R}$.

Definition (Lie algebra)

The *Lie algebra* of a matrix group $\mathcal{G} \subset \mathrm{GL}_n(\mathbb{R})$ is the tangent space to $\mathcal{G}$ at the identity e. It is denoted $\mathfrak{g} = \mathfrak{g}(\mathcal{G}) = T_e\mathcal{G}$.

- Note that the choice of identity is due to the fact that all groups contain at least the identity element.

Let us consider the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$, denoted $\mathfrak{gl}_n(\mathbb{R})$.

Proposition

$\mathfrak{gl}_n(\mathbb{R}) = \mathrm{M}_n(\mathbb{R})$. *In particular,* $\dim(\mathrm{GL}_n(\mathbb{R})) = n^2$.

Proof.

Let $A \in \mathrm{M}_n(\mathbb{R})$. The path $\gamma(t) = I + t.A$ in $\mathrm{M}_n(\mathbb{R})$ satisfies $\gamma(0) = I$ and $\gamma'(0) = A$. Also, γ restricted to sufficiently small interval $(-\epsilon, \epsilon)$ lies in $\mathrm{GL}_n(\mathbb{R})$. To justify this, notice $\det(\gamma(0)) = 1$. Since the determinant function is continuous, $\det(\gamma(t))$ is close to 1 (and is therefore non-zero) for t close to 0. This demonstrates that $A \in \mathfrak{gl}_n(\mathbb{R})$. $\square$

Example: Lie Algebras of $\mathrm{SO}(n)$

The set $\mathfrak{so}(n) = \{A \in \mathrm{M}_n(\mathbb{R}) : A + A^{\mathsf{T}} = 0\}$ is denoted $\mathfrak{so}(n)$ and called *skew-symmetric* matrices.

Corollary

$\dim(\mathrm{SO}(n)) = \frac{n(n-1)}{2}$.

Proof.

Skew-symmetric matrices have zeros on the diagonal, arbitrary real numbers above, and entries below determined by those above, so $\dim(\mathfrak{so}(n)) = \frac{n(n-1)}{2}$. □

We know that $\mathfrak{so}(n) = T_e\mathrm{SO}(n)$ is the space of all skew-symmetric matrices:

$$\mathfrak{so}(n) = \left\{ A \in M_n(\mathbb{R}) : A^{\mathsf{T}} = -A \right\}.$$

Q. What is $T_g\mathrm{SO}(n)$ for $g \neq e$?

A. Let γ be a curve in $\mathrm{SO}(n)$ such that $\gamma(0) = g$. Then we know that $\dot{\gamma}(0) \in T_g\mathrm{SO}(n)$.

Consider $\gamma(t) = g\exp(At)$ where A is skew. Then we know that γ lies in $\mathrm{SO}(n)$ and $\gamma(0) = g$.

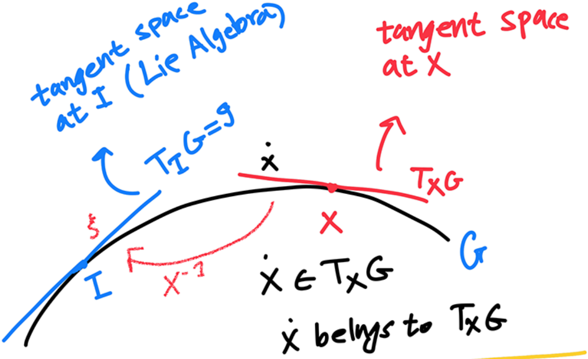
Differentiating, we see that $\dot{\gamma}(0) = gA$.

$$T_g\mathrm{SO}_n = \left\{gA : A^{\mathsf{T}} = -A\right\} = g \cdot \mathfrak{so}_n.$$

For a general matrix group G,

$$T_gG = g \cdot \mathfrak{g}.$$

The “Best” Path in a Matrix Group: One-Parameter Groups



$$X^{-1}\dot{X} = \xi \in \mathfrak{g}$$

Ex.

$$\begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \omega^\wedge & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R\omega^\wedge & Rv \\ 0 & 0 \end{bmatrix}$$

Matrix Exponentiation; Series in $\mathrm{M}_n(\mathbb{R})$

- When the power series of the function $f(x) = \exp(x)$ is applied to a matrix $A \in \mathrm{M}_n(\mathbb{R})$, the result is called *matrix exponentiation*:

$$\exp(A) = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \cdots .$$

- This series converges for all $A \in \mathrm{M}_n(\mathbb{R})$.

The "Best" Path in a Matrix Group: One-Parameter Groups

If $A \in \mathfrak{gl}_n(\mathbb{R})$, we would like to find the "most natural" path $\gamma(t)$ in $\mathrm{GL}_n(\mathbb{R})$ with $\gamma(0) = I$ and $\gamma'(0) = A$.

The “Best” Path in a Matrix Group: One-Parameter Groups

Proposition

Let $A \in \mathfrak{gl}_n(\mathbb{R})$*. The path* $\gamma : \mathbb{R} \to \mathrm{M}_n(\mathbb{R})$ *defined as* $\gamma(t) = \exp(tA)$ *is differentiable, and* $\gamma'(t) = A \cdot \gamma(t) = \gamma(t) \cdot A$*.*

The "Best" Path in a Matrix Group: One-Parameter Groups

Proposition

Let $A \in \mathfrak{gl}_n(\mathbb{R})$. The path $\gamma : \mathbb{R} \to \mathrm{M}_n(\mathbb{R})$ defined as $\gamma(t) = \exp(tA)$ is differentiable, and $\gamma'(t) = A \cdot \gamma(t) = \gamma(t) \cdot A$.

Proof.

Each of the n^2 entries of

$$\gamma(t) = I + tA + \frac{1}{2}t^2A^2 + \frac{1}{6}t^3A^3 + \cdots$$

is a power series in t, which, from familiar real calculus, can be termwise differentiated, giving:

$$\gamma'(t) = 0 + A + tA^2 + \frac{1}{2}t^2A^3 + \cdots .$$

This equals $\gamma(t) \cdot A$ or $A \cdot \gamma(t)$ depending whether you factor an A out on the left or right. $\square$

Let $\mathcal{G}$ be a matrix group with Lie algebra $\mathfrak{g}$. For all $g \in \mathcal{G}$, the *conjugation map* $C_g : \mathcal{G} \to \mathcal{G}$, define as

$$C_g(a) = gag^{-1},$$

is a smooth isomorphism. The derivative $\mathrm{d}(\mathrm{C}_g)_I : \mathfrak{g} \to \mathfrak{g}$ is a vector space isomorphism, which we denote as Ad_g (adjoint):

$$\mathrm{Ad}_g = \mathrm{d}(\mathrm{C}_g)_I$$

To derive a simple formula for $\mathrm{Ad}_g(B)$, notice that any $B \in \mathfrak{g}$ can be represented as $B = b'(0)$, where $b(t)$ is a differentiable path in $\mathcal{G}$ with $b(0) = I$. The product rule gives:

$$\mathrm{Ad}_g(B) = \mathrm{d}(\mathrm{C}_g)_I(B) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g b(t) g^{-1} = gBg^{-1}.$$

So we learn that (notice the similarity transformation):

$$\mathrm{Ad}_g(B) = gBg^{-1}.$$

Definition (Lie bracket)

The Lie bracket of two vectors A and B in $\mathfrak{g}$ is:

$$[A, B] = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{a(t)}(B),$$

where $a(t)$ is any differentiable path in $\mathcal{G}$ with $a(0) = I$ and $a'(0) = A$.

Proposition

For all $A, B \in \mathfrak{g}$, $[A, B] = AB - BA$.

Proof.

Left as exercise. □

Example: Lie Bracket on $\mathfrak{so}(3)$ and Cross Product

See `so3_cross_example.m` for numerical examples and details.

$$G_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, G_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, G_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[G_1, G_2] = G_3,\ [G_2, G_3] = G_1,\ [G_3, G_1] = G_2$$

$$e_1 \times e_2 = e_3,\ e_2 \times e_3 = e_1,\ e_3 \times e_1 = e_2$$

For $X, Y, Z \in \mathfrak{g}$ with sufficiently small norm, the equation $\exp(X)\exp(Y) = \exp(Z)$ has a power series solution for Z in terms of repeated Lie bracket of X and Y. The beginning of the series is:

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]] + \cdots .$$

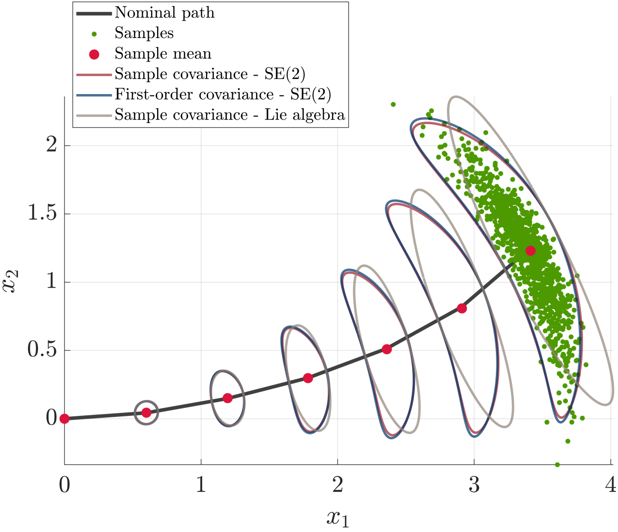
- *Lie groups* are the generalization of matrix groups.
- As such, all of the theory of matrix groups we discussed are also true for Lie groups.

Useful Lie Groups in Robotics

- Group of 3D rotation matrices, $\mathrm{SO}(3)$; it can model rotations without any singularities or ambiguities.
- Group of direct spatial isometries (3D Rigid Body Transformations), $\mathrm{SE}(3)$.
- Group of K direct isometries, $\mathrm{SE}_K(3)$; for example, it is used for modeling IMU sensors and robot pose plus landmarks and/or contact points.
- Group of 3D similarity transformations, $\mathrm{Sim}(3)$; it is more general than $\mathrm{SE}(3)$ and includes a scale factor and used in monocular vision where the scale is not known.

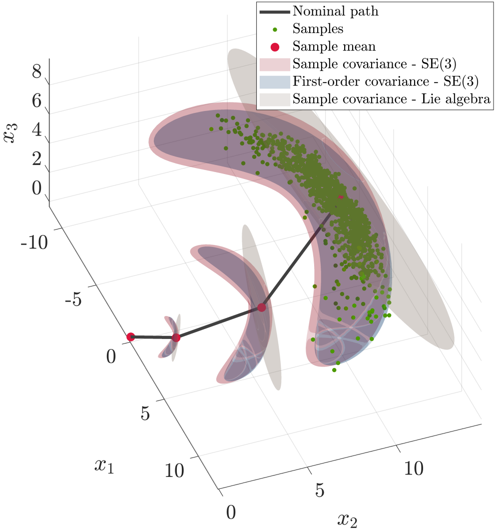
Example: Uncertainty Propagation on $\mathrm{SE}(2)$

See `odometry_propagation_se2.m` for code.



Example: Uncertainty Propagation on $\mathrm{SE}(3)$

See `odometry_propagation_se3.m` for code.



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