



Universidad Rey Juan Carlos

# ACalorados

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Contest (1)

template.cpp	14 lines
#include <bits/stdc++.h> using namespace std;  #define rep(i, a, b) for(int i = a; i < (b); ++i) #define all(x) begin(x), end(x) #define sz(x) (int)(x).size() typedef long long ll; typedef pair<int, int> pii; typedef vector<int> vi;  int main() { cin.tie(0)->sync_with_stdio(0); cin.exceptions(cin.failbit); }  .bashrc	
alias c='g++ -Wall -Wconversion -Wfatal-errors -g -std=c++14 \ -fsanitize=undefined,address' xmodmap -e 'clear lock' -e 'keycode 66=less greater' #caps =◇	
.vimrc	6 lines
set cin aw ai is ts=4 sw=4 tm=50 nu noeb bg=dark ru cul sy on   im jk <esc>   im kj <esc>   no ; : " Select region and then type :Hash to hash your selection. " Useful for verifying that there aren't mistypes. ca Hash w !cpp -dD -P -fpreprocessed \  tr -d '[:space:]' \ \  md5sum \  cut -c-6  hash.sh	
# Hashes a file, ignoring all whitespace and comments. Use for # verifying that code was correctly typed. cpp -dD -P -fpreprocessed   tr -d '[:space:]'   md5sum  cut -c-6	
troubleshoot.txt	52 lines
Pre-submit: Write a few simple test cases if sample is not enough. Are time limits close? If so, generate max cases. Is the memory usage fine? Could anything overflow? Make sure to submit the right file.  Wrong answer: Print your solution! Print debug output, as well. Are you clearing all data structures between test cases? Can your algorithm handle the whole range of input? Read the full problem statement again. Do you handle all corner cases correctly? Have you understood the problem correctly? Any uninitialized variables? Any overflows? Confusing N and M, i and j, etc.? Are you sure your algorithm works? What special cases have you not thought of? Are you sure the STL functions you use work as you think? Add some assertions, maybe resubmit. Create some testcases to run your algorithm on. Go through the algorithm for a simple case. Go through this list again. Explain your algorithm to a teammate. Ask the teammate to look at your code. Go for a small walk, e.g. to the toilet. Is your output format correct? (including whitespace)	

Rewrite your solution from the start or let a teammate do it.

Runtime error:  
Have you tested all corner cases locally?  
Any uninitialized variables?  
Are you reading or writing outside the range of any vector?  
Any assertions that might fail?  
Any possible division by 0? (mod 0 for example)  
Any possible infinite recursion?  
Invalidated pointers or iterators?  
Are you using too much memory?  
Debug with resubmits (e.g. remapped signals, see Various).

Time limit exceeded:  
Do you have any possible infinite loops?  
What is the complexity of your algorithm?  
Are you copying a lot of unnecessary data? (References)  
How big is the input and output? (consider scanf)  
Avoid vector, map. (use arrays/unordered\_map)  
What do your teammates think about your algorithm?

Memory limit exceeded:  
What is the max amount of memory your algorithm should need?  
Are you clearing all data structures between test cases?

Mathematics (2)

2.1 Equations

ax^2 + bx + c = 0 => x = (-b +/- sqrt(b^2 - 4ac)) / 2a

The extremum is given by x = -b/2a.

ax + by = e, cx + dy = f => x = (ed - bf) / (ad - bc), y = (af - ec) / (ad - bc)

In general, given an equation Ax = b, the solution to a variable xi is given by

xi = (det Ai') / det A

where Ai' is A with the i'th column replaced by b.

2.2 Recurrences

If an = c1an-1 + ... + ck an-k, and r1, ..., rk are distinct roots of x^k - c1x^{k-1} - ... - ck, there are d1, ..., dk s.t.

an = d1r1^n + ... + dk r\_k^n.

Non-distinct roots r become polynomial factors, e.g.

an = (d1n + d2)r^n.

2.3 Trigonometry

sin(v + w) = sin v cos w + cos v sin w

cos(v + w) = cos v cos w - sin v sin w

tan(v + w) = (tan v + tan w) / (1 - tan v tan w)

sin v + sin w = 2 sin((v + w)/2) cos((v - w)/2)

cos v + cos w = 2 cos((v + w)/2) cos((v - w)/2)

(V + W) tan(v - w)/2 = (V - W) tan(v + w)/2

where V, W are lengths of sides opposite angles v, w.

a cos x + b sin x = r cos(x - phi)

a sin x + b cos x = r sin(x + phi)

where r = sqrt(a^2 + b^2), phi = atan2(b, a).

2.4 Geometry

2.4.1 Triangles

Side lengths: a, b, c

Semiperimeter: p = (a + b + c) / 2

Area: A = sqrt(p(p - a)(p - b)(p - c))

Circumradius: R = (abc) / (4A)

Inradius: r = A / p

Length of median (divides triangle into two equal-area triangles):

ma = 1/2 sqrt(2b^2 + 2c^2 - a^2)

Length of bisector (divides angles in two):

sa = sqrt(bc [1 - (a/(b + c))^2])

Law of sines: sin alpha / a = sin beta / b = sin gamma / c = 1 / (2R)

Law of cosines: a^2 = b^2 + c^2 - 2bc cos alpha

Law of tangents: (a + b) / (a - b) = (tan((alpha + beta)/2)) / (tan((alpha - beta)/2))

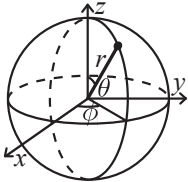
2.4.2 Quadrilaterals

With side lengths a, b, c, d, diagonals e, f, diagonals angle theta, area A and magic flux F = b^2 + d^2 - a^2 - c^2:

4A = 2ef · sin theta = F tan theta = sqrt(4e^2 f^2 - F^2)

For cyclic quadrilaterals the sum of opposite angles is 180°, ef = ac + bd, and A = sqrt((p - a)(p - b)(p - c)(p - d)).

2.4.3 Spherical coordinates



$$x = r \sin \theta \cos \phi$$
$$y = r \sin \theta \sin \phi$$
$$z = r \cos \theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \operatorname{acos}(z/\sqrt{x^2 + y^2 + z^2})$$
$$\phi = \operatorname{atan2}(y, x)$$

2.5 Derivatives/Integrals

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \tan x = 1 + \tan^2 x$$
$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

$$\int \tan ax = -\frac{\ln|\cos ax|}{a}$$
$$\int x \sin ax = \frac{\sin ax - ax \cos ax}{a^2}$$

$$\int e^{-x^2} = \frac{\sqrt{\pi}}{2} \operatorname{erf}(x)$$
$$\int x e^{ax} dx = \frac{e^{ax}}{a^2} (ax - 1)$$

Integration by parts:

$$\int_a^b f(x)g(x)dx = [F(x)g(x)]_a^b - \int_a^b F(x)g'(x)dx$$

2.6 Sums

$$c^a + c^{a+1} + \cdots + c^b = \frac{c^{b+1} - c^a}{c - 1}, c \neq 1$$

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(2n+1)(n+1)}{6}$$
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$
$$1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

2.7 Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, (-\infty < x < \infty)$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, (-1 < x \leq 1)$$
$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{2x^3}{32} - \frac{5x^4}{128} + \dots, (-1 \leq x \leq 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, (-\infty < x < \infty)$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, (-\infty < x < \infty)$$

2.8 Probability theory

Let  $X$  be a discrete random variable with probability  $p_X(x)$  of assuming the value  $x$ . It will then have an expected value (mean)  $\mu = \mathbb{E}(X) = \sum_x x p_X(x)$  and variance  $\sigma^2 = V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \sum_x (x - \mathbb{E}(X))^2 p_X(x)$  where  $\sigma$  is the standard deviation. If  $X$  is instead continuous it will have a probability density function  $f_X(x)$  and the sums above will instead be integrals with  $p_X(x)$  replaced by  $f_X(x)$ .

Expectation is linear:

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$$

For independent  $X$  and  $Y$ ,

$$V(aX + bY) = a^2V(X) + b^2V(Y).$$

2.8.1 Discrete distributions

Binomial distribution

The number of successes in  $n$  independent yes/no experiments, each which yields success with probability  $p$  is  $\operatorname{Bin}(n, p)$ ,  $n = 1, 2, \dots$ ,  $0 \leq p \leq 1$ .

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mu = np, \sigma^2 = np(1-p)$$

$\operatorname{Bin}(n, p)$  is approximately  $\operatorname{Po}(np)$  for small  $p$ .

First success distribution

The number of trials needed to get the first success in independent yes/no experiments, each which yields success with probability  $p$  is  $\operatorname{Fs}(p)$ ,  $0 \leq p \leq 1$ .

$$p(k) = p(1-p)^{k-1}, k = 1, 2, \dots$$

$$\mu = \frac{1}{p}, \sigma^2 = \frac{1-p}{p^2}$$

Poisson distribution

The number of events occurring in a fixed period of time  $t$  if these events occur with a known average rate  $\kappa$  and independently of the time since the last event is  $\operatorname{Po}(\lambda)$ ,  $\lambda = t\kappa$ .

$$p(k) = e^{-\lambda} \frac{\lambda^k}{k!}, k = 0, 1, 2, \dots$$

$$\mu = \lambda, \sigma^2 = \lambda$$

2.8.2 Continuous distributions

Uniform distribution

If the probability density function is constant between  $a$  and  $b$  and 0 elsewhere it is  $\operatorname{U}(a, b)$ ,  $a < b$ .

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = \frac{a+b}{2}, \sigma^2 = \frac{(b-a)^2}{12}$$

Exponential distribution

The time between events in a Poisson process is  $\operatorname{Exp}(\lambda)$ ,  $\lambda > 0$ .

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\mu = \frac{1}{\lambda}, \sigma^2 = \frac{1}{\lambda^2}$$

Normal distribution

Most real random values with mean  $\mu$  and variance  $\sigma^2$  are well described by  $\mathcal{N}(\mu, \sigma^2)$ ,  $\sigma > 0$ .

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

If  $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$  and  $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$  then

$$aX_1 + bX_2 + c \sim \mathcal{N}(\mu_1 + \mu_2 + c, a^2\sigma_1^2 + b^2\sigma_2^2)$$

2.9 Markov chains

A *Markov chain* is a discrete random process with the property that the next state depends only on the current state. Let  $X_1, X_2, \dots$  be a sequence of random variables generated by the Markov process. Then there is a transition matrix  $\mathbf{P} = (p_{ij})$ , with  $p_{ij} = \Pr(X_n = i | X_{n-1} = j)$ , and  $\mathbf{p}^{(n)} = \mathbf{P}^n \mathbf{p}^{(0)}$  is the probability distribution for  $X_n$  (i.e.,  $p_i^{(n)} = \Pr(X_n = i)$ ), where  $\mathbf{p}^{(0)}$  is the initial distribution.

$\pi$  is a stationary distribution if  $\pi = \pi \mathbf{P}$ . If the Markov chain is *irreducible* (it is possible to get to any state from any state), then  $\pi_i = \frac{1}{\mathbb{E}(T_i)}$  where  $\mathbb{E}(T_i)$  is the expected time between two visits in state  $i$ .  $\pi_j/\pi_i$  is the expected number of visits in state  $j$  between two visits in state  $i$ .

For a connected, undirected and non-bipartite graph, where the transition probability is uniform among all neighbors,  $\pi_i$  is proportional to node  $i$ 's degree.

A Markov chain is *ergodic* if the asymptotic distribution is independent of the initial distribution. A finite Markov chain is ergodic iff it is irreducible and *aperiodic* (i.e., the gcd of cycle lengths is 1).  $\lim_{k \rightarrow \infty} \mathbf{P}^k = \mathbf{1}\pi$ .

A Markov chain is an **A**-chain if the states can be partitioned into two sets **A** and **G**, such that all states in **A** are absorbing ( $p_{ii} = 1$ ), and all states in **G** leads to an absorbing state in **A**. The probability for absorption in state  $i \in \mathbf{A}$ , when the initial state is  $j$ , is  $a_{ij} = p_{ij} + \sum_{k \in \mathbf{G}} a_{ik} p_{kj}$ . The expected time until absorption, when the initial state is  $i$ , is  $t_i = 1 + \sum_{k \in \mathbf{G}} p_{ki} t_k$ .

## Data structures (3)

### OrderStatisticTree.h

**Description:** A set (not multiset!) with support for finding the n'th element, and finding the index of an element. To get a map, change null\_type. **Time:**  $\mathcal{O}(\log N)$

```
#include <bits/extc++.h>
using namespace __gnu_pbds;

template<class T>
using Tree = tree<T, null_type, less<T>, rb_tree_tag,
                 tree_order_statistics_node_update>;

void example() {
    Tree<int> t, t2; t.insert(8);
    auto it = t.insert(10).first;
    assert(it == t.lower_bound(9));
    assert(t.order_of_key(10) == 1);
    assert(t.order_of_key(11) == 2);
    assert(*t.find_by_order(0) == 8);
    t.join(t2); // assuming T< T2 or T> T2, merge t2 into t
}
```

### HashMap.h

**Description:** Hash map with mostly the same API as unordered\_map, but ~3x faster. Uses 1.5x memory. Initial capacity must be a power of 2 (if provided).

```
#include <bits/extc++.h>
// To use most bits rather than just the lowest ones:
struct chash { // large odd number for C
    const uint64_t C = 1l(4e18 * acos(0)) | 7l;
    ll operator()(ll x) const { return __builtin_bswap64(x*C); }
};
__gnu_pbds::gp_hash_table<ll,int, chash> h({},{},{},{},{1<16});
```

### SegmentTree.h

**Description:** Zero-indexed max-tree. Bounds are inclusive to the left and exclusive to the right. Can be changed by modifying T, f and unit. **Time:**  $\mathcal{O}(\log N)$

```
struct Tree {
    typedef int T;
    static constexpr T unit = INT_MIN;
    T f(T a, T b) { return max(a, b); } // (any associative fn)
    vector<T> s; int n;
    Tree(int n = 0, T def = unit) : s(2*n, def), n(n) {}
    void update(int pos, T val) {
        for (s[pos += n] = val; pos /= 2; )
            s[pos] = f(s[pos * 2], s[pos * 2 + 1]);
    }
    T query(int b, int e) { // query [b, e)
        T ra = unit, rb = unit;
        for (b += n, e += n; b < e; b /= 2, e /= 2) {
            if (b % 2) ra = f(ra, s[b++]);
            if (e % 2) rb = f(s[--e], rb);
        }
    }
};
```

```
        return f(ra, rb);
    }
};
```

### LazySegmentTree.h

**Description:** Segment tree with ability to add or set values of large intervals, and compute max of intervals. Can be changed to other things. Use with a bump allocator for better performance, and SmallPtr or implicit indices to save memory. **Usage:** Node\* tr = new Node(v, 0, sz(v)); **Time:**  $\mathcal{O}(\log N)$ .

```
"../various/BumpAllocator.h" 34ecf5, 50 lines

const int inf = 1e9;
struct Node {
    Node *l = 0, *r = 0;
    int lo, hi, mset = inf, madd = 0, val = -inf;
    Node(int lo, int hi):lo(lo),hi(hi){} // Large interval of -inf
    Node(vi& v, int lo, int hi) : lo(lo), hi(hi) {
        if (lo + 1 < hi) {
            int mid = lo + (hi - lo)/2;
            l = new Node(v, lo, mid); r = new Node(v, mid, hi);
            val = max(l->val, r->val);
        }
        else val = v[lo];
    }
    int query(int L, int R) {
        if (R <= lo || hi <= L) return -inf;
        if (L <= lo && hi <= R) return val;
        push();
        return max(l->query(L, R), r->query(L, R));
    }
    void set(int L, int R, int x) {
        if (R <= lo || hi <= L) return;
        if (L <= lo && hi <= R) mset = val = x, madd = 0;
        else {
            push(), l->set(L, R, x), r->set(L, R, x);
            val = max(l->val, r->val);
        }
    }
    void add(int L, int R, int x) {
        if (R <= lo || hi <= L) return;
        if (L <= lo && hi <= R) {
            if (mset != inf) mset += x;
            else madd += x;
            val += x;
        }
        else {
            push(), l->add(L, R, x), r->add(L, R, x);
            val = max(l->val, r->val);
        }
    }
    void push() {
        if (!l) {
            int mid = lo + (hi - lo)/2;
            l = new Node(lo, mid); r = new Node(mid, hi);
        }
        if (mset != inf)
            l->set(lo,hi,mset), r->set(lo,hi,mset), mset = inf;
        else if (madd)
            l->add(lo,hi,madd), r->add(lo,hi,madd), madd = 0;
    }
};
```

### UnionFindRollback.h

**Description:** Disjoint-set data structure with undo. If undo is not needed, skip st, time() and rollback(). **Usage:** int t = uf.time(); ...; uf.rollback(t); **Time:**  $\mathcal{O}(\log(N))$

```
de4ad0, 21 lines
```

```
struct RollbackUF {
    vi e; vector<pii> st;
    RollbackUF(int n) : e(n, -1) {}
    int size(int x) { return -e[find(x)]; }
    int find(int x) { return e[x] < 0 ? x : find(e[x]); }
    int time() { return sz(st); }
    void rollback(int t) {
        for (int i = time(); i --> t;)
            e[st[i].first] = st[i].second;
        st.resize(t);
    }
    bool join(int a, int b) {
        a = find(a), b = find(b);
        if (a == b) return false;
        if (e[a] > e[b]) swap(a, b);
        st.push_back({a, e[a]});
        st.push_back({b, e[b]});
        e[a] += e[b]; e[b] = a;
        return true;
    }
};
```

### SubMatrix.h

**Description:** Calculate submatrix sums quickly, given upper-left and lower-right corners (half-open). **Usage:** SubMatrix<int> m(matrix); m.sum(0, 0, 2, 2); // top left 4 elements **Time:**  $\mathcal{O}(N^2 + Q)$

```
template<class T>
struct SubMatrix {
    vector<vector<T>>> p;
    SubMatrix(vector<vector<T>>& v) {
        int R = sz(v), C = sz(v[0]);
        p.assign(R+1, vector<T>(C+1));
        rep(r,0,R) rep(c,0,C)
            p[r+1][c+1] = v[r][c] + p[r][c+1] + p[r+1][c] - p[r][c];
    }
    T sum(int u, int l, int d, int r) {
        return p[d][r] - p[d][l] - p[u][r] + p[u][l];
    }
};
```

### Matrix.h

**Description:** Basic operations on square matrices. **Usage:** Matrix<int, 3> A; A.d = {{{{1,2,3}}, {4,5,6}}, {{7,8,9}}}}; vector<int> vec = {1,2,3}; vec = (A^N) \* vec;

```
template<class T, int N> struct Matrix {
    typedef Matrix M;
    array<array<T, N>, N> d{};
    M operator*(const M& m) const {
        M a;
        rep(i,0,N) rep(j,0,N)
            rep(k,0,N) a.d[i][j] += d[i][k]*m.d[k][j];
        return a;
    }
    vector<T> operator*(const vector<T>& vec) const {
        vector<T> ret(N);
        rep(i,0,N) rep(j,0,N) ret[i] += d[i][j] * vec[j];
        return ret;
    }
    M operator^(ll p) const {
        assert(p >= 0);
        M a, b(*this);
        rep(i,0,N) a.d[i][i] = 1;
        while (p) {
```

```
        if (p&1) a = a*b;
        b = b*b;
        p >>= 1;
    }
    return a;
}
};
```

LineContainer.h

**Description:** Container where you can add lines of the form  $kx+m$ , and query maximum values at points  $x$ . Useful for dynamic programming (“convex hull trick”).  
**Time:**  $\mathcal{O}(\log N)$

Sec1c7, 30 lines

```
struct Line {
    mutable ll k, m, p;
    bool operator<(const Line& o) const { return k < o.k; }
    bool operator<(ll x) const { return p < x; }
};

struct LineContainer : multiset<Line, less<>> {
    // (for doubles, use inf = 1/.0, div(a,b) = a/b)
    static const ll inf = LLONG_MAX;
    ll div(ll a, ll b) { // floored division
        return a / b - ((a ^ b) < 0 && a % b); }
    bool isect(iterator x, iterator y) {
        if (y == end()) return x->p = inf, 0;
        if (x->k == y->k) x->p = x->m > y->m ? inf : -inf;
        else x->p = div(y->m - x->m, x->k - y->k);
        return x->p >= y->p;
    }
    void add(ll k, ll m) {
        auto z = insert({k, m, 0}), y = z++, x = y;
        while (isect(y, z)) z = erase(z);
        if (x != begin() && isect(--x, y)) isect(x, y = erase(y));
        while ((y = x) != begin() && (--x->p >= y->p)
            isect(x, erase(y)));
    }
    ll query(ll x) {
        assert(!empty());
        auto l = *lower_bound(x);
        return l.k * x + l.m;
    }
};
```

Treap.h

**Description:** A short self-balancing tree. It acts as a sequential container with log-time splits/joins, and is easy to augment with additional data.  
**Time:**  $\mathcal{O}(\log N)$

9556fc, 55 lines

```
struct Node {
    Node *l = 0, *r = 0;
    int val, y, c = 1;
    Node(int val) : val(val), y(rand()) {}
    void recalc();
};

int cnt(Node* n) { return n ? n->c : 0; }
void Node::recalc() { c = cnt(l) + cnt(r) + 1; }

template<class F> void each(Node* n, F f) {
    if (n) { each(n->l, f); f(n->val); each(n->r, f); }
}

pair<Node*, Node*> split(Node* n, int k) {
    if (!n) return {};
    if (cnt(n->l) >= k) { // "n->val" >= k" for lower_bound(k)
        auto pa = split(n->l, k);
        n->l = pa.second;
```

```
        n->recalc();
        return {pa.first, n};
    } else {
        auto pa = split(n->r, k - cnt(n->l) - 1); // and just "k"
        n->r = pa.first;
        n->recalc();
        return {n, pa.second};
    }
}

Node* merge(Node* l, Node* r) {
    if (!l) return r;
    if (!r) return l;
    if (l->y > r->y) {
        l->r = merge(l->r, r);
        l->recalc();
        return l;
    } else {
        r->l = merge(l, r->l);
        r->recalc();
        return r;
    }
}

Node* ins(Node* t, Node* n, int pos) {
    auto pa = split(t, pos);
    return merge(merge(pa.first, n), pa.second);
}

// Example application: move the range [l, r) to index k
void move(Node*& t, int l, int r, int k) {
    Node *a, *b, *c;
    tie(a,b) = split(t, l); tie(b,c) = split(b, r - l);
    if (k <= l) t = merge(ins(a, b, k), c);
    else t = merge(a, ins(c, b, k - r));
}
```

FenwickTree.h

**Description:** Computes partial sums  $a[0] + a[1] + \dots + a[pos - 1]$ , and updates single elements  $a[i]$ , taking the difference between the old and new value.  
**Time:** Both operations are  $\mathcal{O}(\log N)$ .

e62fac, 22 lines

```
struct FT {
    vector<ll> s;
    FT(int n) : s(n) {}
    void update(int pos, ll dif) { // a[pos] += dif
        for (; pos < sz(s); pos |= pos + 1) s[pos] += dif;
    }
    ll query(int pos) { // sum of values in [0, pos)
        ll res = 0;
        for (; pos > 0; pos &= pos - 1) res += s[pos-1];
        return res;
    }
    int lower_bound(ll sum) { // min pos st sum of [0, pos] >= sum
        // Returns n if no sum is >= sum, or -1 if empty sum is.
        if (sum <= 0) return -1;
        int pos = 0;
        for (int pw = 1 << 25; pw; pw >>= 1) {
            if (pos + pw <= sz(s) && s[pos + pw-1] < sum)
                pos += pw, sum -= s[pos-1];
        }
        return pos;
    }
};
```

FenwickTree2d.h

**Description:** Computes sums  $a[i,j]$  for all  $i < I, j < J$ , and increases single elements  $a[i,j]$ . Requires that the elements to be updated are known in advance (call `fakeUpdate()` before `init()`).  
**Time:**  $\mathcal{O}(\log^2 N)$ . (Use persistent segment trees for  $\mathcal{O}(\log N)$ .)

157f07, 22 lines

```
"FenwickTree.h"
struct FT2 {
    vector<vi> ys; vector<FT> ft;
    FT2(int limx) : ys(limx) {}
    void fakeUpdate(int x, int y) {
        for (; x < sz(ys); x |= x + 1) ys[x].push_back(y);
    }
    void init() {
        for (vi& v : ys) sort(all(v)), ft.emplace_back(sz(v));
    }
    int ind(int x, int y) {
        return (int)(lower_bound(all(ys[x]), y) - ys[x].begin()); }
    void update(int x, int y, ll dif) {
        for (; x < sz(ys); x |= x + 1)
            ft[x].update(ind(x, y), dif);
    }
    ll query(int x, int y) {
        ll sum = 0;
        for (; x; x &= x - 1)
            sum += ft[x-1].query(ind(x-1, y));
        return sum;
    }
};
```

RMQ.h

**Description:** Range Minimum Queries on an array. Returns  $\min(V[a], V[a + 1], \dots V[b - 1])$  in constant time.  
**Usage:** `RMQ rmq(values);`  
`rmq.query(inclusive, exclusive);`  
**Time:**  $\mathcal{O}(|V| \log |V| + Q)$

510c32, 16 lines

```
template<class T>
struct RMQ {
    vector<vector<T>> jmp;
    RMQ(const vector<T>& V) : jmp(1, V) {
        for (int pw = 1, k = 1; pw * 2 <= sz(V); pw *= 2, ++k) {
            jmp.emplace_back(sz(V) - pw * 2 + 1);
            rep(j, 0, sz(jmp[k]))
                jmp[k][j] = min(jmp[k - 1][j], jmp[k - 1][j + pw]);
        }
    }
    T query(int a, int b) {
        assert(a < b); // or return inf if a == b
        int dep = 31 - __builtin_clz(b - a);
        return min(jmp[dep][a], jmp[dep][b - (1 << dep)]);
    }
};
```

MoQueries.h

**Description:** Answer interval or tree path queries by finding an approximate TSP through the queries, and moving from one query to the next by adding/removing points at the ends. If values are on tree edges, change step to add/remove the edge  $(a, c)$  and remove the initial add call (but keep in).  
**Time:**  $\mathcal{O}(N\sqrt{Q})$

a12ef4, 49 lines

```
void add(int ind, int end) { ... } // add a[ind] (end = 0 or 1)
void del(int ind, int end) { ... } // remove a[ind]
int calc() { ... } // compute current answer

vi mo(vector<pii> Q) {
    int L = 0, R = 0, blk = 350; // ~N/sqrt(Q)
    vi s(sz(Q)), res = s;
    #define K(x) pii(x.first/blk, x.second ^ -(x.first/blk & 1))
    iota(all(s), 0);
```

```
sort(all(s), [&](int s, int t){ return K(Q[s]) < K(Q[t]); });
for (int qi : s) {
    pii q = Q[qi];
    while (L > q.first) add(--L, 0);
    while (R < q.second) add(R++, 1);
    while (L < q.first) del(L++, 0);
    while (R > q.second) del(--R, 1);
    res[qi] = calc();
}
return res;
}

vi moTree(vector<array<int, 2>> Q, vector<vi>& ed, int root=0){
    int N = sz(ed), pos[2] = {}, blk = 350; // ~N/sqrt(Q)
    vi s(sz(Q)), res = s, I(N), L(N), R(N), in(N), par(N);
    add(0, 0), in[0] = 1;
    auto dfs = [&](int x, int p, int dep, auto& f) -> void {
        par[x] = p;
        L[x] = N;
        if (dep) I[x] = N++;
        for (int y : ed[x]) if (y != p) f(y, x, !dep, f);
        if (!dep) I[x] = N++;
        R[x] = N;
    };
    dfs(root, -1, 0, dfs);
#define K(x) pii I[x[0]] / blk, I[x[1]] ^ -(I[x[0]] / blk & 1))
    iota(all(s), 0);
    sort(all(s), [&](int s, int t){ return K(Q[s]) < K(Q[t]); });
    for (int qi : s) rep(end,0,2) {
        int &a = pos[end], b = Q[qi][end], i = 0;
#define step(c) { if (in[c]) { del(a, end); in[a] = 0; } \
                    else { add(c, end); in[c] = 1; } a = c; }
        while (!(L[b] <= L[a] && R[a] <= R[b]))
            I[i++] = b, b = par[b];
        while (a != b) step(par[a]);
        while (i--) step(I[i]);
        if (end) res[qi] = calc();
    }
    return res;
}
```

Fib-dp.py21 lines

```
def fib(n):

    # base case
    if n <= 1:
        return n

    # if fib(n) has already been computed
    # we do not do further recursive calls
    # and hence reduce the number of repeated
    # work
    if term[n] != 0:
        return term[n]

    else:

        # store the computed value of fib(n)
        # in an array term at index n to
        # so that it does not needs to be
        # precomputed again
        term[n] = fib(n - 1) + fib(n - 2)
        return term[n]
```

Laberinto.py29 lines

```
import copy
from math import inf
```

```
def esFactible(laberinto,f,c):
    if f>=0 and f<len(laberinto) and c>=0 and c<len(laberinto
        [0]):
        return laberinto[f][c] == 0
    else:
        return False

def esSolucion(laberinto,f,c):
    return f == len(laberinto)-1 and c == len(laberinto)-1

def esMejor(sol1,sol2):
    return sol1[len(sol1)-1][len(sol1[0])-1] < sol2[len(sol2)
        -1][len(sol2[0])-1]

def salirDelLaberinto(laberinto,mejorSol,f,c,k):
    if esSolucion(laberinto,f,c):
        if esMejor(laberinto,mejorSol):
            mejorSol = copy.deepcopy(laberinto)

    else:
        desplazamientos=[[1,0],[0,1],[-1,0],[0,-1]]
        i = 0
        while not esSol and i < len(desplazamientos):
            if esFactible(laberinto,f+desplazamientos[i][0],c+
                desplazamientos[i][1]):
                laberinto[f+desplazamientos[i][0]][c+
                    desplazamientos[i][1]] = k
                mejorSol = salirDelLaberinto(laberinto,mejorSol
                    ,f+desplazamientos[i][0],c+desplazamientos
                        [i][1],k+1)
                laberinto[f + desplazamientos[i][0]][c +
                    desplazamientos[i][1]] = 0

                i += 1
        return mejorSol
```

Coloreado-de-Grafo.py33 lines

```
def inicializarSolucion(grafo):
    sol = [0] * grafo['n']
    return sol

def esFactible(grafo,sol,nodo,color):
    factible = True
    adyacenciaNodo = grafo['adyacencia'][nodo]
    i = 0
    while factible and i < len(adyacenciaNodo):
        if adyacenciaNodo[i] < nodo:
            factible = color != sol[adyacenciaNodo[i]]

            i += 1
        return factible

def esSolucion(nodo, grafo):
    return nodo >= grafo['n']

def coloreadoVA(grafo, m, sol, nodo):
    #if nodo >= grafo['n']:
    if esSolucion(nodo, grafo):
        esSol = True
    else:
        esSol = False
        color = 1
        while not esSol and color <= m:
            if esFactible(grafo, sol, nodo, color):
                sol[nodo] = color
                [sol, esSol] = coloreadoVA(grafo, m, sol, nodo
                    + 1)

                if not esSol:
                    sol[nodo] = 0
                    color += 1
```

```
return sol, esSol
```

Hamiltonian-Cycles.py12 lines

```
def hamiltonian_cycles(current, graph, visited, solution):
    for adjacent in graph[current]:
        if adjacent not in visited: # esFactible
            visited.add(adjacent)
            solution.append(adjacent)
            if len(solution) == len(graph): # esCasoBase
                if 0 in graph[adjacent]: # esSolucion
                    print(solution)
            else:
                hamiltonian_cycles(adjacent, graph, visited,
                    solution)
            solution.pop()
            visited.remove(adjacent)
```

Mochila.py10 lines

```
def mochilaVA(solucion, mejorSol, datos, k):
    if esSolucion(solucion, datos):
        mejorSol = mejor(mejorSol, solucion)
    else:
        for i in range(k, datos['N']):
            if esFactible(solucion, datos, i):
                solucion = asignar(solucion, i, datos)
                mejorSol = mochilaVA(solucion, mejorSol, datos,
                    i)
                solucion = borrar(solucion, i, datos)
        return mejorSol
```

Prefijo-Comun-Mas-Largo.py18 lines

```
def commonPrefixUtil(str1, str2):
    result = ""
    n1, n2 = len(str1), len(str2)
    i, j = 0, 0
    while i <= n1 - 1 and j <= n2 - 1:
        if str1[i] != str2[j]:
            break
        result += str1[i]
        i, j = i + 1, j + 1
    return result
def commonPrefix(v, low, high):
    if low == high:
        return arr[low]
    if high > low:
        mid = (low + high) // 2
        str1 = commonPrefix(v, low, mid)
        str2 = commonPrefix(v, mid + 1, high)
        return commonPrefixUtil(str1, str2)
```

Numerical (4)

4.1 Polynomials and recurrences

Polynomial.hc9b7b0, 17 lines

```
struct Poly {
    vector<double> a;
    double operator()(double x) const {
        double val = 0;
        for (int i = sz(a); i--;) (val *= x) += a[i];
        return val;
    }
    void diff() {
        rep(i,1,sz(a)) a[i-1] = i*a[i];
```



```
        a.pop_back();
    }
    void divroot(double x0) {
        double b = a.back(), c; a.back() = 0;
        for(int i=sz(a)-1; i--;) c = a[i], a[i] = a[i+1]*x0+b, b=c;
        a.pop_back();
    }
};
```

**PolyRoots.h**  
**Description:** Finds the real roots to a polynomial.  
**Usage:** polyRoots({{2,-3,1}},-1e9,1e9) // solve x<sup>2</sup>-3x+2 = 0  
**Time:**  $\mathcal{O}(n^2 \log(1/\epsilon))$

"Polynomial.h"b00bfe, 23 lines

```
vector<double> polyRoots(Poly p, double xmin, double xmax) {
    if (sz(p.a) == 2) { return {p.a[0]/p.a[1]}; }
    vector<double> ret;
    Poly der = p;
    der.diff();
    auto dr = polyRoots(der, xmin, xmax);
    dr.push_back(xmin-1);
    dr.push_back(xmax+1);
    sort(all(dr));
    rep(i,0,sz(dr)-1) {
        double l = dr[i], h = dr[i+1];
        bool sign = p(l) > 0;
        if (sign ^ (p(h) > 0)) {
            rep(it,0,60) { // while (h - l > 1e-8)
                double m = (l + h) / 2, f = p(m);
                if ((f <= 0) ^ sign) l = m;
                else h = m;
            }
            ret.push_back((l + h) / 2);
        }
    }
    return ret;
}
```

**PolyInterpolate.h**  
**Description:** Given  $n$  points  $(x[i], y[i])$ , computes an  $n-1$ -degree polynomial  $p$  that passes through them:  $p(x) = a[0] * x^0 + ... + a[n-1] * x^{n-1}$ . For numerical precision, pick  $x[k] = c * \cos(k/(n-1) * \pi), k = 0 \dots n-1$ .  
**Time:**  $\mathcal{O}(n^2)$

08bf48, 13 lines

```
typedef vector<double> vd;
vd interpolate(vd x, vd y, int n) {
    vd res(n), temp(n);
    rep(k,0,n-1) rep(i,k+1,n)
        y[i] = (y[i] - y[k]) / (x[i] - x[k]);
    double last = 0; temp[0] = 1;
    rep(k,0,n) rep(i,0,n) {
        res[i] += y[k] * temp[i];
        swap(last, temp[i]);
        temp[i] -= last * x[k];
    }
    return res;
}
```

**BerlekampMassey.h**  
**Description:** Recovers any  $n$ -order linear recurrence relation from the first  $2n$  terms of the recurrence. Useful for guessing linear recurrences after brute-forcing the first terms. Should work on any field, but numerical stability for floats is not guaranteed. Output will have size  $\leq n$ .  
**Usage:** berlekampMassey({0, 1, 1, 3, 5, 11}) // {1, 2}  
**Time:**  $\mathcal{O}(N^2)$

"../number-theory/ModPow.h"96548b, 20 lines

```
vector<ll> berlekampMassey(vector<ll> s) {
    int n = sz(s), L = 0, m = 0;
```

```
vector<ll> C(n), B(n), T;
C[0] = B[0] = 1;

ll b = 1;
rep(i,0,n) { ++m;
    ll d = s[i] % mod;
    rep(j,1,L+1) d = (d + C[j] * s[i - j]) % mod;
    if (!d) continue;
    T = C; ll coef = d * modpow(b, mod-2) % mod;
    rep(j,m,n) C[j] = (C[j] - coef * B[j - m]) % mod;
    if (2 * L > i) continue;
    L = i + 1 - L; B = T; b = d; m = 0;
}

C.resize(L + 1); C.erase(C.begin());
for (ll& x : C) x = (mod - x) % mod;
return C;
}
```

**LinearRecurrence.h**  
**Description:** Generates the  $k$ 'th term of an  $n$ -order linear recurrence  $S[i] = \sum_j S[i-j-1]tr[j]$ , given  $S[0 \dots \geq n-1]$  and  $tr[0 \dots n-1]$ . Faster than matrix multiplication. Useful together with Berlekamp-Massey.  
**Usage:** linearRec({0, 1}, {1, 1}, k) //  $k$ 'th Fibonacci number  
**Time:**  $\mathcal{O}(n^2 \log k)$

f4e444, 26 lines

```
typedef vector<ll> Poly;
ll linearRec(Poly S, Poly tr, ll k) {
    int n = sz(tr);

    auto combine = [&](Poly a, Poly b) {
        Poly res(n * 2 + 1);
        rep(i,0,n+1) rep(j,0,n+1)
            res[i + j] = (res[i + j] + a[i] * b[j]) % mod;
        for (int i = 2 * n; i > n; --i) rep(j,0,n)
            res[i - 1 - j] = (res[i - 1 - j] + res[i] * tr[j]) % mod;
        res.resize(n + 1);
        return res;
    };

    Poly pol(n + 1), e(pol);
    pol[0] = e[1] = 1;

    for (++k; k; k /= 2) {
        if (k % 2) pol = combine(pol, e);
        e = combine(e, e);
    }

    ll res = 0;
    rep(i,0,n) res = (res + pol[i + 1] * S[i]) % mod;
    return res;
}
```

## 4.2 Optimization

**GoldenSectionSearch.h**  
**Description:** Finds the argument minimizing the function  $f$  in the interval  $[a, b]$  assuming  $f$  is unimodal on the interval, i.e. has only one local minimum and no local maximum. The maximum error in the result is  $eps$ . Works equally well for maximization with a small change in the code. See Ternary-Search.h in the Various chapter for a discrete version.  
**Usage:** double func(double x) { return 4+x+.3\*x\*x; }  
double xmin = gss(-1000,1000,func);  
**Time:**  $\mathcal{O}(\log((b-a)/\epsilon))$

31d45b, 14 lines

```
double gss(double a, double b, double (*f)(double)) {
    double r = (sqrt(5)-1)/2, eps = 1e-7;
    double x1 = b - r*(b-a), x2 = a + r*(b-a);
    double f1 = f(x1), f2 = f(x2);
    while (b-a > eps)
```

```
    if (f1 < f2) { //change to > to find maximum
        b = x2; x2 = x1; f2 = f1;
        x1 = b - r*(b-a); f1 = f(x1);
    } else {
        a = x1; x1 = x2; f1 = f2;
        x2 = a + r*(b-a); f2 = f(x2);
    }
    return a;
}
```

**HillClimbing.h**  
**Description:** Poor man's optimization for unimodal functions.

Seeaef, 14 lines

```
typedef array<double, 2> P;

template<class F> pair<double, P> hillClimb(P start, F f) {
    pair<double, P> cur(f(start), start);
    for (double jmp = 1e9; jmp > 1e-20; jmp /= 2) {
        rep(j,0,100) rep(dx,-1,2) rep(dy,-1,2) {
            P p = cur.second;
            p[0] += dx*jmp;
            p[1] += dy*jmp;
            cur = min(cur, make_pair(f(p), p));
        }
    }
    return cur;
}
```

**Integrate.h**  
**Description:** Simple integration of a function over an interval using Simpson's rule. The error should be proportional to  $h^4$ , although in practice you will want to verify that the result is stable to desired precision when epsilon changes.

4756fc, 7 lines

```
template<class F>
double quad(double a, double b, F f, const int n = 1000) {
    double h = (b - a) / 2 / n, v = f(a) + f(b);
    rep(i,1,n*2)
        v += f(a + i*h) * (i&1 ? 4 : 2);
    return v * h / 3;
}
```

**IntegrateAdaptive.h**  
**Description:** Fast integration using an adaptive Simpson's rule.  
**Usage:** double sphereVolume = quad(-1, 1, [](double x) { return quad(-1, 1, [&](double y) { return quad(-1, 1, [&](double z) { return x\*x + y\*y + z\*z < 1; }},{});});

92dd79, 15 lines

```
typedef double d;
#define S(a,b) (f(a) + 4*f((a+b) / 2) + f(b)) * (b-a) / 6

template <class F>
d rec(F& f, d a, d b, d eps, d S) {
    d c = (a + b) / 2;
    d S1 = S(a, c), S2 = S(c, b), T = S1 + S2;
    if (abs(T - S) <= 15 * eps || b - a < 1e-10)
        return T + (T - S) / 15;
    return rec(f, a, c, eps / 2, S1) + rec(f, c, b, eps / 2, S2);
}

template<class F>
d quad(d a, d b, F f, d eps = 1e-8) {
    return rec(f, a, b, eps, S(a, b));
}
```

## Simplex.h

**Description:** Solves a general linear maximization problem: maximize  $c^T x$  subject to  $Ax \leq b, x \geq 0$ . Returns -inf if there is no solution, inf if there are arbitrarily good solutions, or the maximum value of  $c^T x$  otherwise. The input vector is set to an optimal  $x$  (or in the unbounded case, an arbitrary solution fulfilling the constraints). Numerical stability is not guaranteed. For better performance, define variables such that  $x = 0$  is viable.

**Usage:** vvd A = {{1,-1}, {-1,1}, {-1,-2}};  
vd b = {1,1,-4}, c = {-1,-1}, x;  
T val = LPSolver(A, b, c).solve(x);  
**Time:**  $\mathcal{O}(NM * \# \text{pivots})$ , where a pivot may be e.g. an edge relaxation.  $\mathcal{O}(2^n)$  in the general case.

aa8530, 68 lines

```
typedef double T; // long double, Rational, double + modkP>...
typedef vector<T> vd;
typedef vector<vd> vvd;
```

```
const T eps = 1e-8, inf = 1/.0;
#define MP make_pair
#define ltj(X) if(s == -1 || MP(X[j],N[j]) < MP(X[s],N[s])) s=j
```

```
struct LPSolver {
    int m, n;
    vi N, B;
    vvd D;
```

```
    LPSolver(const vvd& A, const vd& b, const vd& c) :
        m(sz(b)), n(sz(c)), N(n+1), B(m), D(m+2, vd(n+2)) {
        rep(i,0,m) rep(j,0,n) D[i][j] = A[i][j];
        rep(i,0,m) { B[i] = n+i; D[i][n] = -1; D[i][n+1] = b[i]; }
        rep(j,0,n) { N[j] = j; D[m][j] = -c[j]; }
        N[n] = -1; D[m+1][n] = 1;
    }
```

```
    void pivot(int r, int s) {
        T *a = D[r].data(), inv = 1 / a[s];
        rep(i,0,m+2) if (i != r && abs(D[i][s]) > eps) {
            T *b = D[i].data(), inv2 = b[s] * inv;
            rep(j,0,n+2) b[j] -= a[j] * inv2;
            b[s] = a[s] * inv2;
        }
        rep(j,0,n+2) if (j != s) D[r][j] *= inv;
        rep(i,0,m+2) if (i != r) D[i][s] *= -inv;
        D[r][s] = inv;
        swap(B[r], N[s]);
    }
```

```
    bool simplex(int phase) {
        int x = m + phase - 1;
        for (;;) {
            int s = -1;
            rep(j,0,n+1) if (N[j] != -phase) ltj(D[x]);
            if (D[x][s] >= -eps) return true;
            int r = -1;
            rep(i,0,m) {
                if (D[i][s] <= eps) continue;
                if (r == -1 || MP(D[i][n+1] / D[i][s], B[i])
                    < MP(D[r][n+1] / D[r][s], B[r])) r = i;
            }
            if (r == -1) return false;
            pivot(r, s);
        }
    }
```

```
T solve(vd &x) {
    int r = 0;
    rep(i,1,m) if (D[i][n+1] < D[r][n+1]) r = i;
    if (D[r][n+1] < -eps) {
        pivot(r, n);
        if (!simplex(2) || D[m+1][n+1] < -eps) return -inf;
    }
```

```
    rep(i,0,m) if (B[i] == -1) {
        int s = 0;
        rep(j,1,n+1) ltj(D[i]);
        pivot(i, s);
    }
}
bool ok = simplex(1); x = vd(n);
rep(i,0,m) if (B[i] < n) x[B[i]] = D[i][n+1];
return ok ? D[m][n+1] : inf;
};
```

### 4.3 Matrices

#### Determinant.h

**Description:** Calculates determinant of a matrix. Destroys the matrix.  
**Time:**  $\mathcal{O}(N^3)$

bd5cec, 15 lines

```
double det(vector<vector<double>>& a) {
    int n = sz(a); double res = 1;
    rep(i,0,n) {
        int b = i;
        rep(j,i+1,n) if (fabs(a[j][i]) > fabs(a[b][i])) b = j;
        if (i != b) swap(a[i], a[b]), res *= -1;
        res *= a[i][i];
        if (res == 0) return 0;
        rep(j,i+1,n) {
            double v = a[j][i] / a[i][i];
            if (v != 0) rep(k,i+1,n) a[j][k] -= v * a[i][k];
        }
    }
    return res;
}
```

#### IntDeterminant.h

**Description:** Calculates determinant using modular arithmetics. Modulos can also be removed to get a pure-integer version.  
**Time:**  $\mathcal{O}(N^3)$

3313dc, 18 lines

```
const ll mod = 12345;
ll det(vector<vector<ll>>& a) {
    int n = sz(a); ll ans = 1;
    rep(i,0,n) {
        rep(j,i+1,n) {
            while (a[j][i] != 0) { // gcd step
                ll t = a[i][i] / a[j][i];
                if (t) rep(k,i,n)
                    a[i][k] = (a[i][k] - a[j][k] * t) % mod;
                swap(a[i], a[j]);
                ans *= -1;
            }
        }
        ans = ans * a[i][i] % mod;
        if (!ans) return 0;
    }
    return (ans + mod) % mod;
}
```

#### SolveLinear.h

**Description:** Solves  $A * x = b$ . If there are multiple solutions, an arbitrary one is returned. Returns rank, or -1 if no solutions. Data in  $A$  and  $b$  is lost.  
**Time:**  $\mathcal{O}(n^2 m)$

44c9ab, 38 lines

```
typedef vector<double> vd;
const double eps = 1e-12;

int solveLinear(vector<vd>& A, vd& b, vd& x) {
    int n = sz(A), m = sz(x), rank = 0, br, bc;
    if (n) assert(sz(A[0]) == m);
    vi col(m); iota(all(col), 0);
```

```
    rep(i,0,n) {
        double v, bv = 0;
        rep(r,i,n) rep(c,i,m)
            if ((v = fabs(A[r][c])) > bv)
                br = r, bc = c, bv = v;
        if (bv <= eps) {
            rep(j,i,n) if (fabs(b[j]) > eps) return -1;
            break;
        }
        swap(A[i], A[br]);
        swap(b[i], b[br]);
        swap(col[i], col[bc]);
        rep(j,0,n) swap(A[j][i], A[j][bc]);
        bv = 1/A[i][i];
        rep(j,i+1,n) {
            double fac = A[j][i] * bv;
            b[j] -= fac * b[i];
            rep(k,i+1,m) A[j][k] -= fac*A[i][k];
        }
        rank++;
    }

    x.assign(m, 0);
    for (int i = rank; i--;) {
        b[i] /= A[i][i];
        x[col[i]] = b[i];
        rep(j,0,i) b[j] -= A[j][i] * b[i];
    }
    return rank; // (multiple solutions if rank < m)
}
```

#### SolveLinear2.h

**Description:** To get all uniquely determined values of  $x$  back from SolveLinear, make the following changes:

08e495, 7 lines

```
"SolveLinear.h"
rep(j,0,n) if (j != i) // instead of rep(j,i+1,n)
// ... then at the end:
x.assign(m, undefined);
rep(i,0,rank) {
    rep(j,rank,m) if (fabs(A[i][j]) > eps) goto fail;
    x[col[i]] = b[i] / A[i][i];
fail;; }
```

#### SolveLinearBinary.h

**Description:** Solves  $Ax = b$  over  $\mathbb{F}_2$ . If there are multiple solutions, one is returned arbitrarily. Returns rank, or -1 if no solutions. Destroys  $A$  and  $b$ .  
**Time:**  $\mathcal{O}(n^2 m)$

fa2d7a, 34 lines

```
typedef bitset<1000> bs;

int solveLinear(vector<bs>& A, vi& b, bs& x, int m) {
    int n = sz(A), rank = 0, br;
    assert(m <= sz(x));
    vi col(m); iota(all(col), 0);
    rep(i,0,n) {
        for (br=i; br<n; ++br) if (A[br].any()) break;
        if (br == n) {
            rep(j,i,n) if(b[j]) return -1;
            break;
        }
        int bc = (int)A[br]._Find_next(i-1);
        swap(A[i], A[br]);
        swap(b[i], b[br]);
        swap(col[i], col[bc]);
        rep(j,0,n) if (A[j][i] != A[j][bc]) {
            A[j].flip(i); A[j].flip(bc);
        }
        rep(j,i+1,n) if (A[j][i]) {
```



$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} d_0 & p_0 & 0 & 0 & \cdots & 0 \\ q_0 & d_1 & p_1 & 0 & \cdots & 0 \\ 0 & q_1 & d_2 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{n-3} & d_{n-2} & p_{n-2} \\ 0 & 0 & \cdots & 0 & q_{n-2} & d_{n-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{pmatrix}.$$

```

Time:  $\mathcal{O}(N \log N)$ 
ced03d, 33 lines


---



```

const ll mod = (119 << 23) + 1, root = 62; // = 998244353
// For p < 2^30 there is also e.g. 5 << 25, 7 << 26, 479 << 21
// and 483 << 21 (same root). The last two are > 10^9.
typedef vector<ll> vl;
void ntt(vl &a) {
    int n = sz(a), L = 31 - __builtin_clz(n);
    static vl rt(2, 1);
    for (static int k = 2, s = 2; k < n; k *= 2, s++) {
        rt.resize(n);
        ll z[] = {1, modpow(root, mod >> s)};
        rep(i, k, 2*k) rt[i] = rt[i / 2] * z[i & 1] % mod;
    }
}

```


```

```
vi rev(n);
rep(i,0,n) rev[i] = (rev[i / 2] | (i & 1) << L) / 2;
rep(i,0,n) if (i < rev[i]) swap(a[i], a[rev[i]]);
for (int k = 1; k < n; k *= 2)
    for (int i = 0; i < n; i += 2 * k) rep(j,0,k) {
        ll z = rt[j + k] * a[i + j + k] % mod, &ai = a[i + j];
        a[i + j + k] = ai - z + (z > ai ? mod : 0);
        ai += (ai + z >= mod ? z - mod : z);
    }
}
vl conv(const vl &a, const vl &b) {
    if (a.empty() || b.empty()) return {};
    int s = sz(a) + sz(b) - 1, B = 32 - __builtin_clz(s), n = 1
        << B;
    int inv = modpow(n, mod - 2);
    vl L(a), R(b), out(n);
    L.resize(n), R.resize(n);
    ntt(L), ntt(R);
    rep(i,0,n) out[-i & (n - 1)] = (ll)L[i] * R[i] % mod * inv %
        mod;
    ntt(out);
    return {out.begin(), out.begin() + s};
}
```

### FastSubsetTransform.h

**Description:** Transform to a basis with fast convolutions of the form  $c[z] = \sum_{z=x\oplus y} a[x] \cdot b[y]$ , where  $\oplus$  is one of AND, OR, XOR. The size of  $a$  must be a power of two.  
**Time:**  $\mathcal{O}(N \log N)$

464cf3, 16 lines

```
void FST(vi& a, bool inv) {
    for (int n = sz(a), step = 1; step < n; step *= 2) {
        for (int i = 0; i < n; i += 2 * step) rep(j,i,step) {
            int &u = a[j], &v = a[j + step]; tie(u, v) =
                inv ? pii(v - u, u) : pii(v, u + v); // AND
                inv ? pii(v, u - v) : pii(u + v, u); // OR
                pii(u + v, u - v); // XOR
        }
    }
    if (inv) for (int& x : a) x /= sz(a); // XOR only
}
vi conv(vi a, vi b) {
    FST(a, 0); FST(b, 0);
    rep(i,0,sz(a)) a[i] *= b[i];
    FST(a, 1); return a;
}
```

## Number theory (5)

### 5.1 Modular arithmetic

#### ModularArithmetic.h

**Description:** Operators for modular arithmetic. You need to set mod to some number first and then you can use the structure.

"euclid.h"35bfea, 18 lines

```
const ll mod = 17; // change to something else
struct Mod {
    ll x;
    Mod(ll xx) : x(xx) {}
    Mod operator+(Mod b) { return Mod((x + b.x) % mod); }
    Mod operator-(Mod b) { return Mod((x - b.x + mod) % mod); }
    Mod operator*(Mod b) { return Mod((x * b.x) % mod); }
    Mod operator/(Mod b) { return *this * invert(b); }
    Mod invert(Mod a) {
        ll x, y, g = euclid(a.x, mod, x, y);
        assert(g == 1); return Mod((x + mod) % mod);
    }
}
```

```
Mod operator^(ll e) {
    if (!e) return Mod(1);
    Mod r = *this ^ (e / 2); r = r * r;
    return e&1 ? *this * r : r;
}
};
```

#### ModInverse.h

**Description:** Pre-computation of modular inverses. Assumes  $\text{LIM} \leq \text{mod}$  and that mod is a prime.

6f684f, 3 lines

```
const ll mod = 1000000007, LIM = 200000;
ll* inv = new ll[LIM] - 1; inv[1] = 1;
rep(i,2,LIM) inv[i] = mod - (mod / i) * inv[mod % i] % mod;
```

#### ModPow.h

b83e45, 8 lines

```
const ll mod = 1000000007; // faster if const

ll modpow(ll b, ll e) {
    ll ans = 1;
    for (; e; b = b * b % mod, e /= 2)
        if (e & 1) ans = ans * b % mod;
    return ans;
}
```

#### ModLog.h

**Description:** Returns the smallest  $x > 0$  s.t.  $a^x = b \pmod m$ , or  $-1$  if no such  $x$  exists.  $\text{modLog}(a,1,m)$  can be used to calculate the order of  $a$ .  
**Time:**  $\mathcal{O}(\sqrt{m})$

c040b8, 11 lines

```
ll modLog(ll a, ll b, ll m) {
    ll n = (ll) sqrt(m) + 1, e = 1, f = 1, j = 1;
    unordered_map<ll, ll> A;
    while (j <= n && (e = f = e * a % m) != b % m)
        A[e * b % m] = j++;
    if (e == b % m) return j;
    if (__gcd(m, e) == __gcd(m, b))
        rep(i,2,n+2) if (A.count(e = e * f % m))
            return n * i - A[e];
    return -1;
}
```

#### ModSum.h

**Description:** Sums of mod'ed arithmetic progressions.  
 $\text{modsum}(\text{to}, c, k, m) = \sum_{i=0}^{\text{to}-1} (ki + c) \% m$ .  $\text{divsum}$  is similar but for floored division.  
**Time:**  $\log(m)$ , with a large constant.

5c5bc5, 16 lines

```
typedef unsigned long long ull;
ull sumsq(ull to) { return to / 2 * ((to-1) | 1); }

ull divsum(ull to, ull c, ull k, ull m) {
    ull res = k / m * sumsq(to) + c / m * to;
    k %= m; c %= m;
    if (!k) return res;
    ull to2 = (to * k + c) / m;
    return res + (to - 1) * to2 - divsum(to2, m-1 - c, m, k);
}

ll modsum(ull to, ll c, ll k, ll m) {
    c = ((c % m) + m) % m;
    k = ((k % m) + m) % m;
    return to * c + k * sumsq(to) - m * divsum(to, c, k, m);
}
```

#### ModMulLL.h

**Description:** Calculate  $a \cdot b \bmod c$  (or  $a^b \bmod c$ ) for  $0 \leq a, b \leq c \leq 7.2 \cdot 10^{18}$ .  
**Time:**  $\mathcal{O}(1)$  for  $\text{modmul}$ ,  $\mathcal{O}(\log b)$  for  $\text{modpow}$

bbbd8f, 11 lines

```
typedef unsigned long long ull;
ull modmul(ull a, ull b, ull M) {
    ll ret = a * b - M * ull(1.L / M * a * b);
    return ret + M * (ret < 0) - M * (ret >= (1l)M);
}
ull modpow(ull b, ull e, ull mod) {
    ull ans = 1;
    for (; e; b = modmul(b, b, mod), e /= 2)
        if (e & 1) ans = modmul(ans, b, mod);
    return ans;
}
```

#### ModSqrt.h

**Description:** Tonelli-Shanks algorithm for modular square roots. Finds  $x$  s.t.  $x^2 = a \pmod p$  ( $-x$  gives the other solution).  
**Time:**  $\mathcal{O}(\log^2 p)$  worst case,  $\mathcal{O}(\log p)$  for most  $p$

"ModPow.h"19a793, 24 lines

```
ll sqrt(ll a, ll p) {
    a %= p; if (a < 0) a += p;
    if (a == 0) return 0;
    assert(modpow(a, (p-1)/2, p) == 1); // else no solution
    if (p % 4 == 3) return modpow(a, (p+1)/4, p);
    // a^(n+3)/8 or 2^(n+3)/8 * 2^(n-1)/4 works if p % 8 == 5
    ll s = p - 1, n = 2;
    int r = 0, m;
    while (s % 2 == 0)
        ++r, s /= 2;
    while (modpow(n, (p - 1) / 2, p) != p - 1) ++n;
    ll x = modpow(a, (s + 1) / 2, p);
    ll b = modpow(a, s, p), g = modpow(n, s, p);
    for (; r = m) {
        ll t = b;
        for (m = 0; m < r && t != 1; ++m)
            t = t * t % p;
        if (m == 0) return x;
        ll gs = modpow(g, 1LL << (r - m - 1), p);
        g = gs * gs % p;
        x = x * gs % p;
        b = b * g % p;
    }
}
```

### 5.2 Primality

#### FastEratosthenes.h

**Description:** Prime sieve for generating all primes smaller than LIM.  
**Time:**  $\text{LIM}=1\text{e}9 \approx 1.5\text{s}$

6b2912, 20 lines

```
const int LIM = 1e6;
bitset<LIM> isPrime;
vi eratosthenes() {
    const int S = (int)round(sqrt(LIM)), R = LIM / 2;
    vi pr = {2}, sieve(S+1); pr.reserve((int)(LIM/log(LIM)*1.1));
    vector<pii> cp;
    for (int i = 3; i <= S; i += 2) if (!sieve[i]) {
        cp.push_back({i, i * i / 2});
        for (int j = i * i; j <= S; j += 2 * i) sieve[j] = 1;
    }
    for (int L = 1; L <= R; L += S) {
        array<bool, S> block{};
        for (auto &[p, idx] : cp)
            for (int i=idx; i < S+L; idx = (i+=p)) block[i-L] = 1;
        rep(i,0,min(S, R - L))
            if (!block[i]) pr.push_back((L + i) * 2 + 1);
    }
    for (int i : pr) isPrime[i] = 1;
    return pr;
}
```

MillerRabin.h

**Description:** Deterministic Miller-Rabin primality test. Guaranteed to work for numbers up to  $7 \cdot 10^{18}$ ; for larger numbers, use Python and extend A randomly.

**Time:** 7 times the complexity of  $a^b \bmod c$ .

```
"ModMulLL.h"
60dcd1, 12 lines

bool isPrime(ull n) {
    if (n < 2 || n % 6 % 4 != 1) return (n | 1) == 3;
    ull A[] = {2, 325, 9375, 28178, 450775, 9780504, 1795265022},
        s = __builtin_ctzll(n-1), d = n >> s;
    for (ull a : A) { // ^ count trailing zeroes
        ull p = modpow(a%n, d, n), i = s;
        while (p != 1 && p != n-1 && a % n && i--)
            p = modmul(p, p, n);
        if (p != n-1 && i != s) return 0;
    }
    return 1;
}
```

Factor.h

**Description:** Pollard-rho randomized factorization algorithm. Returns prime factors of a number, in arbitrary order (e.g. 2299 -> {11, 19, 11}).

**Time:**  $\mathcal{O}(n^{1/4})$ , less for numbers with small factors.

```
"ModMulLL.h", "MillerRabin.h"
a33cf6, 18 lines

ull pollard(ull n) {
    auto f = [n](ull x) { return modmul(x, x, n) + 1; };
    ull x = 0, y = 0, t = 30, prd = 2, i = 1, q;
    while (t++ % 40 || __gcd(prd, n) == 1) {
        if (x == y) x = ++i, y = f(x);
        if ((q = modmul(prd, max(x,y) - min(x,y), n))) prd = q;
        x = f(x), y = f(f(y));
    }
    return __gcd(prd, n);
}

vector<ull> factor(ull n) {
    if (n == 1) return {};
    if (isPrime(n)) return {n};
    ull x = pollard(n);
    auto l = factor(x), r = factor(n / x);
    l.insert(l.end(), all(r));
    return l;
}
```

5.3 Divisibility

euclid.h

**Description:** Finds two integers  $x$  and  $y$ , such that  $ax + by = \gcd(a, b)$ . If you just need gcd, use the built in `__gcd` instead. If  $a$  and  $b$  are coprime, then  $x$  is the inverse of  $a \pmod b$ .

```
33ba8f, 5 lines

ll euclid(ll a, ll b, ll &x, ll &y) {
    if (!b) return x = 1, y = 0, a;
    ll d = euclid(b, a % b, y, x);
    return y -= a/b * x, d;
}
```

CRT.h

**Description:** Chinese Remainder Theorem.

`crt(a, m, b, n)` computes  $x$  such that  $x \equiv a \pmod m, x \equiv b \pmod n$ . If  $|a| < m$  and  $|b| < n$ ,  $x$  will obey  $0 \leq x < \text{lcm}(m, n)$ . Assumes  $mn < 2^{62}$ .

**Time:**  $\log(n)$

```
"euclid.h"
04d93a, 7 lines

ll crt(ll a, ll m, ll b, ll n) {
    if (n > m) swap(a, b), swap(m, n);
    ll x, y, g = euclid(m, n, x, y);
    assert((a - b) % g == 0); // else no solution
    x = (b - a) % n * x % n / g * m + a;
    return x < 0 ? x + m*n/g : x;
}
```

}

5.3.1 Bézout’s identity

For  $a \neq 0, b \neq 0$ , then  $d = \gcd(a, b)$  is the smallest positive integer for which there are integer solutions to

$$ax + by = d$$

If  $(x, y)$  is one solution, then all solutions are given by

$$\left(x + \frac{kb}{\gcd(a,b)}, y - \frac{ka}{\gcd(a,b)}\right), \quad k \in \mathbb{Z}$$

phiFunction.h

**Description:** Euler’s  $\phi$  function is defined as  $\phi(n) := \#$  of positive integers  $\leq n$  that are coprime with  $n$ .  $\phi(1) = 1, p$  prime  $\Rightarrow \phi(p^k) = (p-1)p^{k-1}$ ,  $m, n$  coprime  $\Rightarrow \phi(mn) = \phi(m)\phi(n)$ . If  $n = p_1^{k_1}p_2^{k_2}\dots p_r^{k_r}$  then  $\phi(n) = (p_1-1)p_1^{k_1-1}\dots(p_r-1)p_r^{k_r-1}$ .  $\phi(n) = n \cdot \prod_{p|n} (1-1/p)$ .  $\sum_{d|n} \phi(d) = n, \sum_{1 \leq k \leq n, \gcd(k,n)=1} k = n\phi(n)/2, n > 1$

**Euler’s thm:**  $a, n$  coprime  $\Rightarrow a^{\phi(n)} \equiv 1 \pmod n$ .

**Fermat’s little thm:**  $p$  prime  $\Rightarrow a^{p-1} \equiv 1 \pmod p \forall a$ .

```
cf7d6d, 8 lines

const int LIM = 5000000;
int phi[LIM];

void calculatePhi() {
    rep(i,0,LIM) phi[i] = i&1 ? i : i/2;
    for (int i = 3; i < LIM; i += 2) if(phi[i] == i)
        for (int j = i; j < LIM; j += i) phi[j] -= phi[j] / i;
}
```

5.4 Fractions

ContinuedFractions.h

**Description:** Given  $N$  and a real number  $x \geq 0$ , finds the closest rational approximation  $p/q$  with  $p, q \leq N$ . It will obey  $|p/q - x| \leq 1/qN$ .

For consecutive convergents,  $p_{k+1}q_k - q_{k+1}p_k = (-1)^k$ .  $(p_k/q_k)$  alternates between  $> x$  and  $< x$ .) If  $x$  is rational,  $y$  eventually becomes  $\infty$ ; if  $x$  is the root of a degree 2 polynomial the  $a$ ’s eventually become cyclic.

**Time:**  $\mathcal{O}(\log N)$

```
dd6c5e, 21 lines

typedef double d; // for N ~ 1e7; long double for N ~ 1e9
pair<ll, ll> approximate(d x, ll N) {
    ll LP = 0, LQ = 1, P = 1, Q = 0, inf = LLONG_MAX; d y = x;
    for (;;) {
        ll lim = min(P ? (N-LP) / P : inf, Q ? (N-LQ) / Q : inf),
            a = (ll)floor(y), b = min(a, lim),
            NP = b*P + LP, NQ = b*Q + LQ;
        if (a > b) {
            // If b > a/2, we have a semi-convergent that gives us a
            // better approximation; if b = a/2, we *may* have one.
            // Return {P, Q} here for a more canonical approximation.
            return (abs(x - (d)NP / (d)NQ) < abs(x - (d)P / (d)Q)) ?
                make_pair(NP, NQ) : make_pair(P, Q);
        }
        if (abs(y = 1/(y - (d)a)) > 3*N) {
            return {NP, NQ};
        }
        LP = P; P = NP;
        LQ = Q; Q = NQ;
    }
}
```

FracBinarySearch.h

**Description:** Given  $f$  and  $N$ , finds the smallest fraction  $p/q \in [0, 1]$  such that  $f(p/q)$  is true, and  $p, q \leq N$ . You may want to throw an exception from  $f$  if it finds an exact solution, in which case  $N$  can be removed.

**Usage:** `fracBS([<f>](Frac f) { return f.p>=3*f.q; }, 10);` // {1,3}

```
27ab3e, 25 lines

struct Frac { ll p, q; };

template<class F>
Frac fracBS(F f, ll N) {
    bool dir = 1, A = 1, B = 1;
    Frac lo{0, 1}, hi{1, 1}; // Set hi to 1/0 to search (0, N]
    if (f(lo)) return lo;
    assert(f(hi));
    while (A || B) {
        ll adv = 0, step = 1; // move hi if dir, else lo
        for (int si = 0; step; (step *= 2) >= si) {
            adv += step;
            Frac mid{lo.p * adv + hi.p, lo.q * adv + hi.q};
            if (abs(mid.p) > N || mid.q > N || dir == !f(mid)) {
                adv -= step; si = 2;
            }
        }
        hi.p += lo.p * adv;
        hi.q += lo.q * adv;
        dir = !dir;
        swap(lo, hi);
        A = B; B = !adv;
    }
    return dir ? hi : lo;
}
```

5.5 Pythagorean Triples

The Pythagorean triples are uniquely generated by

$$a = k \cdot (m^2 - n^2), \quad b = k \cdot (2mn), \quad c = k \cdot (m^2 + n^2),$$

with  $m > n > 0, k > 0, m \perp n$ , and either  $m$  or  $n$  even.

5.6 Primes

$p = 962592769$  is such that  $2^{21} \mid p-1$ , which may be useful. For hashing use 970592641 (31-bit number), 31443539979727 (45-bit), 3006703054056749 (52-bit). There are 78498 primes less than 1 000 000.

Primitive roots exist modulo any prime power  $p^a$ , except for  $p = 2, a > 2$ , and there are  $\phi(\phi(p^a))$  many. For  $p = 2, a > 2$ , the group  $\mathbb{Z}_{2^a}^\times$  is instead isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_{2^{a-2}}$ .

5.7 Estimates

$\sum_{d|n} d = O(n \log \log n)$ .

The number of divisors of  $n$  is at most around 100 for  $n < 5e4$ , 500 for  $n < 1e7$ , 2000 for  $n < 1e10$ , 200 000 for  $n < 1e19$ .

5.8 Mobius Function

$$\mu(n) = \begin{cases} 0 & n \text{ is not square free} \\ 1 & n \text{ has even number of prime factors} \\ -1 & n \text{ has odd number of prime factors} \end{cases}$$

Mobius Inversion:

$$g(n) = \sum_{d|n} f(d) \Leftrightarrow f(n) = \sum_{d|n} \mu(d)g(n/d)$$

Other useful formulas/forms:

$$\sum_{d|n} \mu(d) = [n = 1] \text{ (very useful)}$$
$$g(n) = \sum_{n|d} f(d) \Leftrightarrow f(n) = \sum_{n|d} \mu(d/n)g(d)$$
$$g(n) = \sum_{1 \leq m \leq n} f(\lfloor \frac{n}{m} \rfloor) \Leftrightarrow f(n) = \sum_{1 \leq m \leq n} \mu(m)g(\lfloor \frac{n}{m} \rfloor)$$

## Combinatorial (6)

### 6.1 Permutations

#### 6.1.1 Factorial

<i>n</i>	1	2	3	4	5	6	7	8	9	10
<i>n</i> !	1	2	6	24	120	720	5040	40320	362880	3628800
<i>n</i>	11	12	13	14	15	16	17			
<i>n</i> !	4.0e7	4.8e8	6.2e9	8.7e10	1.3e12	2.1e13	3.6e14			
<i>n</i>	20	25	30	40	50	100	150	171		
<i>n</i> !	2e18	2e25	3e32	8e47	3e64	9e157	6e262	>DBL_MAX		

IntPerm.h  
**Description:** Permutation -> integer conversion. (Not order preserving.)  
Integer -> permutation can use a lookup table.  
**Time:**  $\mathcal{O}(n)$

```
044568, 6 lines
int permToInt(vi& v) {
    int use = 0, i = 0, r = 0;
    for(int x:v) r = r * ++i + __builtin_popcount(use & -(1<<x)),
        use |= 1 << x; // (note: minus, not ~!)
    return r;
}
```

#### 6.1.2 Cycles

Let  $g_S(n)$  be the number of  $n$ -permutations whose cycle lengths all belong to the set  $S$ . Then

$$\sum_{n=0}^{\infty} g_S(n) \frac{x^n}{n!} = \exp\left(\sum_{n \in S} \frac{x^n}{n}\right)$$

#### 6.1.3 Derangements

Permutations of a set such that none of the elements appear in their original position.

$$D(n) = (n-1)(D(n-1)+D(n-2)) = nD(n-1) + (-1)^n = \left\lfloor \frac{n!}{e} \right\rfloor$$

#### 6.1.4 Burnside’s lemma

Given a group  $G$  of symmetries and a set  $X$ , the number of elements of  $X$  *up to symmetry* equals

$$\frac{1}{|G|} \sum_{g \in G} |X^g|,$$

where  $X^g$  are the elements fixed by  $g$  ( $g.x = x$ ).

If  $f(n)$  counts “configurations” (of some sort) of length  $n$ , we can ignore rotational symmetry using  $G = \mathbb{Z}_n$  to get

$$g(n) = \frac{1}{n} \sum_{k=0}^{n-1} f(\gcd(n, k)) = \frac{1}{n} \sum_{k|n} f(k) \phi(n/k).$$

### 6.2 Partitions and subsets

#### 6.2.1 Partition function

Number of ways of writing  $n$  as a sum of positive integers, disregarding the order of the summands.

$$p(0) = 1, \; p(n) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k+1} p(n - k(3k - 1)/2)$$

$$p(n) \sim 0.145/n \cdot \exp(2.56\sqrt{n})$$

<i>n</i>	0	1	2	3	4	5	6	7	8	9	20	50	100
<i>p</i> ( <i>n</i> )	1	1	2	3	5	7	11	15	22	30	627	~2e5	~2e8

#### 6.2.2 Lucas’ Theorem

Let  $n, m$  be non-negative integers and  $p$  a prime. Write  $n = n_k p^k + \dots + n_1 p + n_0$  and  $m = m_k p^k + \dots + m_1 p + m_0$ . Then  $\binom{n}{m} \equiv \prod_{i=0}^k \binom{n_i}{m_i} \pmod{p}$ .

#### 6.2.3 Binomials

multinomial.h  
**Description:** Computes  $\binom{k_1 + \dots + k_n}{k_1, k_2, \dots, k_n} = \frac{(\sum k_i)!}{k_1! k_2! \dots k_n!}$ .

```
a0a312, 6 lines
ll multinomial(vi& v) {
    ll c = 1, m = v.empty() ? 1 : v[0];
    rep(i, 1, sz(v)) rep(j, 0, v[i])
        c = c * ++m / (j+1);
    return c;
}
```

### 6.3 General purpose numbers

#### 6.3.1 Bernoulli numbers

EGF of Bernoulli numbers is  $B(t) = \frac{t}{e^t - 1}$  (FFT-able).  
 $B[0, \dots] = [1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \dots]$

Sums of powers:

$$\sum_{i=1}^n n^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k \cdot (n+1)^{m+1-k}$$

Euler-Maclaurin formula for infinite sums:

$$\begin{aligned} \sum_{i=m}^{\infty} f(i) &= \int_m^{\infty} f(x) dx - \sum_{k=1}^{\infty} \frac{B_k}{k!} f^{(k-1)}(m) \\ &\approx \int_m^{\infty} f(x) dx + \frac{f(m)}{2} - \frac{f'(m)}{12} + \frac{f'''(m)}{720} + O(f^{(5)}(m)) \end{aligned}$$

#### 6.3.2 Stirling numbers of the first kind

Number of permutations on  $n$  items with  $k$  cycles.

$$c(n, k) = c(n-1, k-1) + (n-1)c(n-1, k), \; c(0, 0) = 1$$
$$\sum_{k=0}^n c(n, k) x^k = x(x+1) \dots (x+n-1)$$

$$c(8, k) = 8, 0, 5040, 13068, 13132, 6769, 1960, 322, 28, 1$$
$$c(n, 2) = 0, 0, 1, 3, 11, 50, 274, 1764, 13068, 109584, \dots$$

#### 6.3.3 Eulerian numbers

Number of permutations  $\pi \in S_n$  in which exactly  $k$  elements are greater than the previous element.  $k$   $j$ :s s.t.  $\pi(j) > \pi(j+1)$ ,  $k+1$   $j$ :s s.t.  $\pi(j) \geq j$ ,  $k$   $j$ :s s.t.  $\pi(j) > j$ .

$$E(n, k) = (n-k)E(n-1, k-1) + (k+1)E(n-1, k)$$

$$E(n, 0) = E(n, n-1) = 1$$

$$E(n, k) = \sum_{j=0}^k (-1)^j \binom{n+1}{j} (k+1-j)^n$$

#### 6.3.4 Stirling numbers of the second kind

Partitions of  $n$  distinct elements into exactly  $k$  groups.

$$S(n, k) = S(n-1, k-1) + kS(n-1, k)$$

$$S(n, 1) = S(n, n) = 1$$

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n$$

#### 6.3.5 Bell numbers

Total number of partitions of  $n$  distinct elements.  $B(n) = 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \dots$ . For  $p$  prime,

$$B(p^m + n) \equiv mB(n) + B(n+1) \pmod{p}$$

#### 6.3.6 Labeled unrooted trees

# on  $n$  vertices:  $n^{n-2}$   
# on  $k$  existing trees of size  $n_i$ :  $n_1 n_2 \dots n_k n^{k-2}$   
# with degrees  $d_i$ :  $(n-2)! / ((d_1-1)! \dots (d_n-1)!)$

#### 6.3.7 Catalan numbers

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!n!}$$

$$C_0 = 1, \; C_{n+1} = \frac{2(2n+1)}{n+2} C_n, \; C_{n+1} = \sum C_i C_{n-i}$$

$$C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \dots$$

- sub-diagonal monotone paths in an  $n \times n$  grid.
- strings with  $n$  pairs of parenthesis, correctly nested.
- binary trees with with  $n+1$  leaves (0 or 2 children).

- ordered trees with  $n + 1$  vertices.
- ways a convex polygon with  $n + 2$  sides can be cut into triangles by connecting vertices with straight lines.
- permutations of  $[n]$  with no 3-term increasing subseq.

## Graph (7)

dfs-bfs.py41 lines

```
from collections import deque

def dfs(g, visited, v):
    q = deque()
    q.append(v)
    while q:
        aux = q.pop()
        if not visited[aux]:
            visited[aux] = True
        for adj in reversed(g[aux]):
            if not visited[adj]:
                q.append(adj)

#Recursivo
def dfsRec(g, visited, v):
    visited[v] = True
    for adj in g[v]:
        if not visited[adj]:
            dfsRec(g,visited,adj)

def dfs(g):
    n = len(g)
    visited = [False]*n
    for v in range(1,n):
        if not visited[v]:
            dfsRec(g,visited,v)

def bfs(g, visited, v):
    q = deque()
    q.append(v)
    while q:
        aux = q.popleft()
        if not visited[aux]:
            print(aux, end=" ")
            visited[aux] = True
        for adj in g[aux]:
            if not visited[adj]:
                q.append(adj)
```

Ordenar listas con dos keys:minus means decreasing way  
s = sorted(s, key = lambda x: (-x[0], x[1], x[2]))

DetectarCiclos-Dirigido.py32 lines

```
def isCyclicUtil(g, v, visited, recStack):

    # Mark current node as visited and
    # adds to recursion stack
    visited[v] = True
    recStack[v] = True

    # Recur for all neighbours
    # if any neighbour is visited and in
    # recStack then graph is cyclic
    for neighbour in g[v]:
        if visited[neighbour] == False:
            if isCyclicUtil(g, neighbour, visited, recStack
                ) == True:
                return True
```

```
        elif recStack[neighbour] == True:
            return True

        # The node needs to be popped from
        # recursion stack before function ends
        recStack[v] = False
        return False

# Returns true if graph is cyclic else false
def isCyclic(g):
    n=len(g)
    visited = [False] * n
    recStack = [False] * n
    for node in range(0,n):
        if visited[node] == False:
            if isCyclicUtil(g, node, visited, recStack) ==
                True:
                    return True
            return False
```

DetectarCiclos-NoDirigido.py27 lines

```
def isCyclicUtil(g, v, visited, parent):

    # Mark current node as visited and
    visited[v] = True
    # Recur for all neighbours
    # if any neighbour is visited and in
    for neighbour in g[v]:
        if visited[neighbour] == False:
            if isCyclicUtil(g, neighbour, visited, v) ==
                True:
                    return True
            elif parent != i:
                return True

    # The node needs to be popped from
    # recursion stack before function ends
    recStack[v] = False
    return False

# Returns true if graph is cyclic else false
def isCyclic(g):
    n=len(g)
    visited = [False] * n
    for node in range(0,n):
        if visited[node] == False:
            if isCyclicUtil(g, node, visited, -1) == True:
                return True
            return False
```

TopSort.py20 lines

Alternative Topological Sort algorithm  
from collections import deque

```
def altTopSort(g):
    inEdges = [0 for _ in range(len(g))]
    for v in range(len(g)):
        for a in g[v]:
            inEdges[a] = inEdges[a] + 1
    startNodes = []
    for v in range(len(g)):
        if inEdges[v] == 0:
            startNodes.append(v)
    path = []
    while startNodes:
        origin = startNodes.pop(0)
        path.append(origin)
```

```
for a in g[origin]:
    inEdges[a] = inEdges[a] - 1
    if inEdges[a] == 0:
        startNodes.append(a)
```

Kruskal.py52 lines

```
def find(parent, i):
    if parent[i] != i:
        parent[i] = find(parent, parent[i])
    return parent[i]

def union(parent, rank, x, y):

    if rank[x] < rank[y]:
        parent[x] = y
    elif rank[x] > rank[y]:
        parent[y] = x

    else:
        parent[y] = x
        rank[x] += 1
```

```
def KruskalMST(g,nodes):

    result = []
    i = 0
    e = 0

    # sort by weight
    g = sorted(g, key=lambda item: item[2])
```

```
parent = []
rank = []

# Create V subsets with single elements
for node in range(0, nodes):
    parent.append(node)
    rank.append(0)
```

# Number of edges to be taken is less than to V-1  
while e < nodes - 1:

```
    u, v, w = g[i]
    i = i + 1
    x = find(parent, u)
    y = find(parent, v)

    if x != y:
        e = e + 1
        result.append([u, v, w])
        union(parent, rank, x, y)
    # Else discard the edge
```

```
minimumCost = 0
for u, v, weight in result:
    minimumCost += weight
```

Prim.py37 lines

```
def select_min(shortest_edges, visited):
    vertex = None
    weight = float('inf')

    #shortest_edges = [inf, inf, inf, 1, 2, inf, inf, 6]
    for i in range(1, len(shortest_edges)):
        if not visited[i] and shortest_edges[i] < weight:
            vertex = i
            weight = shortest_edges[i]
```



```
        return vertex, weight

def prim(g):
    initial = randint(1, len(g) -1)
    visited = [False] * len(g)

    mst = 0
    visited[initial] = True
    shortest_edges = [float('inf')] * len(g)

    for start, end, weight in g[initial]:
        shortest_edges[end] = weight

    for i in range (2, len(g)):
        next_node, cost = select_min(shortest_edges, visited)
        if cost < float('inf'):
            mst += cost
            visited[next_node] = True
            #para el nodo 3, [(3,1,1), (3,4,3), (3,7,5)]
            for edge in g[next_node]:
                start, end, weight = edge
                # shortest_edges = [inf, inf, inf, 1, 2, inf,
                    inf, 5]
                if not visited[end]:
                    shortest_edges[end] = min(shortest_edges[
                        end], weight)

    return mst
```

Dijkstra.py

30 lines

```
def select_min(distances, visited):
    min_dist = float('inf')
    index = 0
    for i in range(1, len(distances)):
        if not visited[i] and distances[i] < min_dist:
            min_dist = distances[i]
            index = i
    return index

def dijkstra(g, origin):
    distances = [float('inf')] * len(g)
    visited = [False] * len(g)

    distances[origin] = 0
    visited[origin] = True

    #si origin es el nodo 1
    #[(1,2,5), (1,4,3)]

    for start, end, weight in g[origin]:
        distances[end] = weight

    #distances = [inf, 0, 5, inf, 3, inf]
    for i in range(2, len(g)):
        next_node = select_min(distances, visited)
        visited[next_node] = True
        for start, end, weight in g[next_node]:
            distances[end] = min(distances[end], distances[
                start]+weight)

    return distances
```

FuertementeConexos.py

48 lines

```
time=0
def SCCUtil(g, u, low, disc, stackMember, st):

    # Initialize discovery time and low value
    disc[u] = time
```

```
    low[u] = time
    time += 1
    stackMember[u] = True
    st.append(u)

    # Go through all vertices adjacent to this
    for v in g[u]:

        # If v is not visited yet, then recur for it
        if disc[v] == -1:

            SCCUtil(v, low, disc, stackMember, st)

        low[u] = min(low[u], low[v])

    elif stackMember[v] == True:

        '''Update low value of 'u' only if 'v' is still
            in stack
            (i.e. it's a back edge, not cross edge).
            Case 2 (per above discussion on Disc and Low
            value) '''
        low[u] = min(low[u], disc[v])

    # head node found, pop the stack and print an SCC
    w = -1 # To store stack extracted vertices
    if low[u] == disc[u]:
        while w != u:
            w = st.pop()
            stackMember[w] = False

    # The function to do DFS traversal.
    # It uses recursive SCCUtil()

    def SCC(g):
        n=len(g)
        disc = [-1] * n
        low = [-1] * n
        stackMember = [False] * n
        st = []

        for i in range(0,n):
            if disc[i] == -1:
                SCCUtil(g, i, low, disc, stackMember, st)
```

Camino-Euler-Indirecto.py

51 lines

```
def isConnected(g):
    n=len(g)
    # Mark all the vertices as not visited
    visited = [False]*(n)

    # Find a vertex with non-zero degree
    for i in range(n):
        if len(g[i]) != 0:
            break

    # If there are no edges in the graph, return true
    if i == n-1:
        return True

    # Start DFS traversal from a vertex with non-zero
    degree
    DFSUtil(i, visited)

    # Check if all non-zero degree vertices are visited
    for i in range(n):
        if visited[i] == False and len(g[i]) > 0:
            return False
```

```
        return True

'''The function returns one of the following values
0 -> If graph is not Eulerian
1 -> If graph has an Euler path (Semi-Eulerian)
2 -> If graph has an Euler Circuit (Eulerian) '''

def isEulerian(g):
    n=len(g)
    # Check if all non-zero degree vertices are connected
    if isConnected() == False:
        return 0
    else:
        # Count vertices with odd degree
        odd = 0
        for i in range(n):
            if len(g[i]) % 2 != 0:
                odd += 1

        '''If odd count is 2, then semi-eulerian.
        If odd count is 0, then eulerian
        If count is more than 2, then graph is not Eulerian
        Note that odd count can never be 1 for undirected
        graph'''
        if odd == 0:
            return 2
        elif odd == 2:
            return 1
        elif odd > 2:
            return 0
```

Camino-Euler-Directo.py

49 lines

```
def findpath(graph, n):
    numofadj = []
    # Find out number of edges each
    # vertex has
    for i in range(n):
        numofadj.append(sum(graph[i]))
    # Find out how many vertex has
    # odd number edges
    startpoint, numofodd = 0, 0
    for i in range(n - 1, -1, -1):
        if (numofadj[i] % 2 == 1):
            numofodd += 1
            startpoint = i
    # If number of vertex with odd number of edges
    # is greater than two return "No Solution".
    if (numofodd > 2):
        print("No Solution")
        return
    # If there is a path find the path
    # Initialize empty stack and path
    # take the starting current as discussed
    stack = []
    path = []
    cur = startpoint
    # Loop will run until there is element in the
    # stack or current edge has some neighbour.
    # while (len(stack) > 0 or sum(graph[cur])!= 0):
    # If current node has not any neighbour
    # add it to path and pop stack set new
    # current to the popped element
    if (sum(graph[cur]) == 0):
        path.append(cur)
        cur = stack[-1]
        del stack[-1]
    # If the current vertex has at least one
```



```
# neighbour add the current vertex to stack,
# remove the edge between them and set the
# current to its neighbour.
else:
    for i in range(n):
        if (graph[cur][i] == 1):
            stack.append(cur)
            graph[cur][i] = 0
            graph[i][cur] = 0
            cur = i
            break
# Print the path
for ele in path:
    print(ele, end = " -> ")
```

Puntos-Articulacion.py 37 lines

```
def checkAP(u, v, children, p, ap, low, d):
    # u es la r a z del rbol y tiene al menos dos hijos
    if p[u] == -1 and children > 1:
        ap[u] = True
    # u no es r a z del rbol y tiene un hijo v
    if p[u] != -1 and low[v] >= d[u]:
        ap[u] = True
def artPointsRec(g, visited, v, ap, d, low, parent, t):
    visited[v] = True
    t += 1
    d[v] = t
    low[v] = t
    children = 0
    for adj in g[v]:
        if not visited[adj]:
            children += 1
            parent[adj] = v
            artPointsRec(g, visited, adj, ap, d, low, parent, t)
        if low[adj] < low[v]:
            low[v] = low[adj]
            checkAP(v, adj, children, parent, ap, low, d)
    elif adj != parent[v]:
        if low[v] > d[adj]:
            low[v] = d[adj]
def artPoints(g):
    n = len(g)
    visited = [False] * n
    d = [-1] * n # Iteraci n de descubrimiento
    low = [-1] * n # Descubrimiento m n i m o
    parent = [-1] * n # Padre de cada nodo
    ap = [False] * n # Nodos que son AP
    t = 0 # Iteraci n actual
    for v in range(1, n):
        if not visited[v]:
            artPointsRec(g, visited, v, ap, d, low, parent, t)
    return ap
```

Geometry (8)

8.1 Geometric primitives

**Point.h**  
**Description:** Class to handle points in the plane. T can be e.g. double or long long. (Avoid int.)

```
template <class T> int sgn(T x) { return (x > 0) - (x < 0); }
template<class T>
struct Point {
    typedef Point P;
    T x, y;
```

```
explicit Point(T x=0, T y=0) : x(x), y(y) {}
bool operator<(P p) const { return tie(x,y) < tie(p.x,p.y); }
bool operator==(P p) const { return tie(x,y)==tie(p.x,p.y); }
P operator+(P p) const { return P(x+p.x, y+p.y); }
P operator-(P p) const { return P(x-p.x, y-p.y); }
P operator*(T d) const { return P(x*d, y*d); }
P operator/(T d) const { return P(x/d, y/d); }
T dot(P p) const { return x*p.x + y*p.y; }
T cross(P p) const { return x*p.y - y*p.x; }
T cross(P a, P b) const { return (a-*this).cross(b-*this); }
T dist2() const { return x*x + y*y; }
double dist() const { return sqrt((double)dist2()); }
// angle to x-axis in interval [-pi, pi]
double angle() const { return atan2(y, x); }
P unit() const { return *this/dist(); } // makes dist()==1
P perp() const { return P(-y, x); } // rotates +90 degrees
P normal() const { return perp().unit(); }
// returns point rotated 'a' radians ccw around the origin
P rotate(double a) const {
    return P(x*cos(a)-y*sin(a),x*sin(a)+y*cos(a)); }
friend ostream& operator<<(ostream& os, P p) {
    return os << "(" << p.x << ", " << p.y << ")"; }
};
```

lineDistance.h

**Description:**  
Returns the signed distance between point p and the line containing points a and b. Positive value on left side and negative on right as seen from a towards b. a==b gives nan. P is supposed to be Point<T> or Point3D<T> where T is e.g. double or long long. It uses products in intermediate steps so watch out for overflow if using int or long long. Using Point3D will always give a non-negative distance. For Point3D, call .dist on the result of the cross product.

```
template<class P>
double lineDist(const P& a, const P& b, const P& p) {
    return (double) (b-a).cross(p-a)/(b-a).dist(); }
}
"Point.h" f6bf6b, 4 lines
```

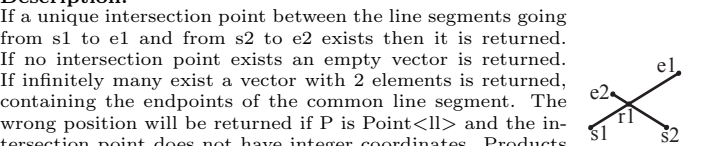
SegmentDistance.h

**Description:**  
Returns the shortest distance between point p and the line segment from point s to e.  
**Usage:** Point<double> a, b(2,2), p(1,1);  
bool onSegment = segDist(a,b,p) < 1e-10;  
**Point.h** 5c88f4, 6 lines

```
typedef Point<double> P;
double segDist(P& s, P& e, P& p) {
    if (s==e) return (p-s).dist();
    auto d = (e-s).dist2(), t = min(d,max(.0, (p-s).dot(e-s)));
    return ((p-s)*d-(e-s)*t).dist()/d; }
}
```

SegmentIntersection.h

**Description:**  
If a unique intersection point between the line segments going from s1 to e1 and from s2 to e2 exists then it is returned. If no intersection point exists an empty vector is returned. If infinitely many exist a vector with 2 elements is returned, containing the endpoints of the common line segment. The wrong position will be returned if P is Point<ll> and the intersection point does not have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow if using int or long long.



```
Usage: vector<P> inter = segInter(s1,e1,s2,e2);
if (sz(inter)==1)
    cout << "segments intersect at " << inter[0] << endl;
"Point.h", "OnSegment.h" 9d57f2, 13 lines
template<class P> vector<P> segInter(P a, P b, P c, P d) {
    auto oa = c.cross(d, a), ob = c.cross(d, b),
        oc = a.cross(b, c), od = a.cross(b, d);
    // Checks if intersection is single non-endpoint point.
    if (sgn(oa) * sgn(ob) < 0 && sgn(oc) * sgn(od) < 0)
        return {(a * ob - b * oa) / (ob - oa)};
    set<P> s;
    if (onSegment(c, d, a)) s.insert(a);
    if (onSegment(c, d, b)) s.insert(b);
    if (onSegment(a, b, c)) s.insert(c);
    if (onSegment(a, b, d)) s.insert(d);
    return {all(s)};
}
```

lineIntersection.h

**Description:**  
If a unique intersection point of the lines going through s1,e1 and s2,e2 exists {1, point} is returned. If no intersection point exists {0, (0,0)} is returned and if infinitely many exists {-1, (0,0)} is returned. The wrong position will be returned if P is Point<ll> and the intersection point does not have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow if using int or ll.  
**Usage:** auto res = lineInter(s1,e1,s2,e2);  
if (res.first == 1)  
 cout << "intersection point at " << res.second << endl;  
**Point.h** a01f81, 8 lines

```
template<class P>
pair<int, P> lineInter(P s1, P e1, P s2, P e2) {
    auto d = (e1 - s1).cross(e2 - s2);
    if (d == 0) // if parallel
        return {(s1.cross(e1, s2) == 0), P(0, 0)};
    auto p = s2.cross(e1, e2), q = s2.cross(e2, s1);
    return {1, (s1 * p + e1 * q) / d}; }
}
```

sideOf.h

**Description:** Returns where p is as seen from s towards e. 1/0/-1 ⇔ left/on line/right. If the optional argument eps is given 0 is returned if p is within distance eps from the line. P is supposed to be Point<T> where T is e.g. double or long long. It uses products in intermediate steps so watch out for overflow if using int or long long.  
**Usage:** bool left = sideOf(p1,p2,q)==1;  
**Point.h** 3af81c, 9 lines

```
template<class P>
int sideOf(P s, P e, P p) { return sgn(s.cross(e, p)); }
```

```
template<class P>
int sideOf(const P& s, const P& e, const P& p, double eps) {
    auto a = (e-s).cross(p-s);
    double l = (e-s).dist()*eps;
    return (a > l) - (a < -l); }
}
```

OnSegment.h

**Description:** Returns true iff p lies on the line segment from s to e. Use (segDist(s,e,p)<=epsilon) instead when using Point<double>.  
**Point.h** c597e8, 3 lines

```
template<class P> bool onSegment(P s, P e, P p) {
    return p.cross(s, e) == 0 && (s - p).dot(e - p) <= 0; }
}
```

### linearTransformation.h

**Description:**

Apply the linear transformation (translation, rotation and scaling) which takes line p0-p1 to line q0-q1 to point r.

"Point.h"	03a306, 6 lines
<pre>typedef Point&lt;double&gt; P; P linearTransformation(const P&amp; p0, const P&amp; p1,     const P&amp; q0, const P&amp; q1, const P&amp; r) {     P dp = p1-p0, dq = q1-q0, num(dp.cross(dq), dp.dot(dq));     return q0 + P((r-p0).cross(num), (r-p0).dot(num))/dp.dist2(); }</pre>	

### Angle.h

**Description:** A class for ordering angles (as represented by int points and a number of rotations around the origin). Useful for rotational sweeping. Sometimes also represents points or vectors.

**Usage:** vector<Angle> v = {w[0], w[0].t360() ...}; // sorted  
int j = 0; rep(i,0,n) { while (v[j] < v[i].t180()) ++j; }  
// sweeps j such that (j-i) represents the number of positively oriented triangles with vertices at 0 and i

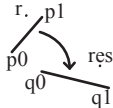
<pre>struct Angle {     int x, y;     int t;     Angle(int x, int y, int t=0) : x(x), y(y), t(t) {}     Angle operator-(Angle b) const { return {x-b.x, y-b.y, t}; }     int half() const {         assert(x    y);         return y &lt; 0    (y == 0 &amp;&amp; x &lt; 0);     }     Angle t90() const { return {-y, x, t + (half() &amp;&amp; x &gt;= 0)}; }     Angle t180() const { return {-x, -y, t + half()}; }     Angle t360() const { return {x, y, t + 1}; } };  bool operator&lt;(Angle a, Angle b) {     // add a.dist2() and b.dist2() to also compare distances     return make_tuple(a.t, a.half(), a.y * (1ll)b.x) &lt;         make_tuple(b.t, b.half(), a.x * (1ll)b.y); }  // Given two points, this calculates the smallest angle between // them, i.e., the angle that covers the defined line segment. pair&lt;Angle, Angle&gt; segmentAngles(Angle a, Angle b) {     if (b &lt; a) swap(a, b);     return (b &lt; a.t180() ?         make_pair(a, b) : make_pair(b, a.t360())); }  Angle operator+(Angle a, Angle b) { // point a + vector b     Angle r(a.x + b.x, a.y + b.y, a.t);     if (a.t180() &lt; r) r.t--;     return r.t180() &lt; a ? r.t360() : r; }  Angle angleDiff(Angle a, Angle b) { // angle b - angle a     int tu = b.t - a.t; a.t = b.t;     return {a.x*b.x + a.y*b.y, a.x*b.y - a.y*b.x, tu - (b &lt; a)}; }</pre>	
---	--

## 8.2 Circles

### CircleIntersection.h

**Description:** Computes the pair of points at which two circles intersect. Returns false in case of no intersection.

"Point.h"	84d6d3, 11 lines
<pre>typedef Point&lt;double&gt; P; bool circleInter(P a,P b,double r1,double r2,pair&lt;P, P&gt;* out) {     if (a == b) { assert(r1 != r2); return false; }     P vec = b - a;     double d2 = vec.dist2(), sum = r1+r2, dif = r1-r2,</pre>	



### CircleTangents.h

**Description:** Finds the external tangents of two circles, or internal if r2 is negated. Can return 0, 1, or 2 tangents - 0 if one circle contains the other (or overlaps it, in the internal case, or if the circles are the same); 1 if the circles are tangent to each other (in which case .first = .second and the tangent line is perpendicular to the line between the centers). .first and .second give the tangency points at circle 1 and 2 respectively. To find the tangents of a circle with a point set r2 to 0.

"Point.h"	b0153d, 13 lines
<pre>template&lt;class P&gt; vector&lt;pair&lt;P, P&gt;&gt; tangents(P c1, double r1, P c2, double r2) {     P d = c2 - c1;     double dr = r1 - r2, d2 = d.dist2(), h2 = d2 - dr * dr;     if (d2 == 0    h2 &lt; 0) return {};     vector&lt;pair&lt;P, P&gt;&gt; out;     for (double sign : {-1, 1}) {         P v = (d * dr + d.perp() * sqrt(h2) * sign) / d2;         out.push_back({c1 + v * r1, c2 + v * r2});     }     if (h2 == 0) out.pop_back();     return out; }</pre>	

### CirclePolygonIntersection.h

**Description:** Returns the area of the intersection of a circle with a ccw polygon.

**Time:**  $\mathcal{O}(n)$

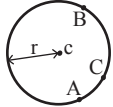
"../content/geometry/Point.h"	a1ee63, 19 lines
<pre>typedef Point&lt;double&gt; P; #define arg(p, q) atan2(p.cross(q), p.dot(q)) double circlePoly(P c, double r, vector&lt;P&gt; ps) {     auto tri = [&amp;](P p, P q) {         auto r2 = r * r / 2;         P d = q - p;         auto a = d.dot(p)/d.dist2(), b = (p.dist2()-r*r)/d.dist2();         auto det = a * a - b;         if (det &lt;= 0) return arg(p, q) * r2;         auto s = max(0., -a-sqrt(det)), t = min(1., -a+sqrt(det));         if (t &lt; 0    1 &lt;= s) return arg(p, q) * r2;         P u = p + d * s, v = p + d * t;         return arg(p,u) * r2 + u.cross(v)/2 + arg(v,q) * r2;     };     auto sum = 0.0;     rep(i,0,sz(ps))         sum += tri(ps[i] - c, ps[(i + 1) % sz(ps)] - c);     return sum; }</pre>	

### circumcircle.h

**Description:**

The circumcirle of a triangle is the circle intersecting all three vertices. ccRadius returns the radius of the circle going through points A, B and C and ccCenter returns the center of the same circle.

"Point.h"	1caa3a, 9 lines
<pre>typedef Point&lt;double&gt; P; double ccRadius(const P&amp; A, const P&amp; B, const P&amp; C) {     return (B-A).dist()*(C-B).dist()*(A-C).dist()/         abs((B-A).cross(C-A))/2; }</pre>	



<pre>P ccCenter(const P&amp; A, const P&amp; B, const P&amp; C) {     P b = C-A, c = B-A;     return A + (b*c.dist2()-c*b.dist2()).perp()/b.cross(c)/2; }</pre>	
---	--

### MinimumEnclosingCircle.h

**Description:** Computes the minimum circle that encloses a set of points.

**Time:** expected  $\mathcal{O}(n)$

"circumcircle.h"	09dd0a, 17 lines
<pre>pair&lt;P, double&gt; mec(vector&lt;P&gt; ps) {     shuffle(all(ps), mt19937(time(0)));     P o = ps[0];     double r = 0, EPS = 1 + 1e-8;     rep(i,0,sz(ps)) if ((o - ps[i]).dist() &gt; r * EPS) {         o = ps[i], r = 0;         rep(j,0,i) if ((o - ps[j]).dist() &gt; r * EPS) {             o = (ps[i] + ps[j]) / 2;             r = (o - ps[i]).dist();             rep(k,0,j) if ((o - ps[k]).dist() &gt; r * EPS) {                 o = ccCenter(ps[i], ps[j], ps[k]);                 r = (o - ps[i]).dist();             }         }     }     return {o, r}; }</pre>	

## 8.3 Polygons

### InsidePolygon.h

**Description:** Returns true if p lies within the polygon. If strict is true, it returns false for points on the boundary. The algorithm uses products in intermediate steps so watch out for overflow.

**Usage:** vector<P> v = {P{4,4}, P{1,2}, P{2,1}};  
bool in = inPolygon(v, P{3, 3}, false);  
**Time:**  $\mathcal{O}(n)$

"Point.h", "OnSegment.h", "SegmentDistance.h"	2bf504, 11 lines
<pre>template&lt;class P&gt; bool inPolygon(vector&lt;P&gt; &amp;p, P a, bool strict = true) {     int cnt = 0, n = sz(p);     rep(i,0,n) {         P q = p[(i + 1) % n];         if (onSegment(p[i], q, a)) return !strict;         //or: if (segDist(p[i], q, a) &lt;= eps) return !strict;         cnt ^= ((a.y&lt;p[i].y) - (a.y&lt;q.y)) * a.cross(p[i], q) &gt; 0;     }     return cnt; }</pre>	

### PolygonArea.h

**Description:** Returns twice the signed area of a polygon. Clockwise enumeration gives negative area. Watch out for overflow if using int as T!

"Point.h"	f12300, 6 lines
<pre>template&lt;class T&gt; T polygonArea2(vector&lt;Point&lt;T&gt;&gt;&amp; v) {     T a = v.back().cross(v[0]);     rep(i,0,sz(v)-1) a += v[i].cross(v[i+1]);     return a; }</pre>	

### PolygonCenter.h

**Description:** Returns the center of mass for a polygon.

**Time:**  $\mathcal{O}(n)$

"Point.h"	9706dc, 9 lines
<pre>typedef Point&lt;double&gt; P; P polygonCenter(const vector&lt;P&gt;&amp; v) {     P res(0, 0); double A = 0;</pre>	

```

for (int i = 0, j = sz(v) - 1; i < sz(v); j = i++) {
    res = res + (v[i] + v[j]) * v[j].cross(v[i]);
    A += v[j].cross(v[i]);
}
return res / A / 3;
}

```

## PolygonCut.h

### Description:

Returns a vector with the vertices of a polygon with everything to the left of the line going from s to e cut away.

**Usage:** vector<P> p = ...;  
p = polygonCut(p, P(0,0), P(1,0));

"Point.h", "lineIntersection.h" f2b7d4, 13 lines

```

typedef Point<double> P;
vector<P> polygonCut(const vector<P>& poly, P s, P e) {
    vector<P> res;
    rep(i,0,sz(poly)) {
        P cur = poly[i], prev = i ? poly[i-1] : poly.back();
        bool side = s.cross(e, cur) < 0;
        if (side != (s.cross(e, prev) < 0))
            res.push_back(lineInter(s, e, cur, prev).second);
        if (side)
            res.push_back(cur);
    }
    return res;
}

```

## ConvexHull.h

### Description:

Returns a vector of the points of the convex hull in counter-clockwise order. Points on the edge of the hull between two other points are not considered part of the hull.

**Time:**  $\mathcal{O}(n \log n)$

"Point.h" 310954, 13 lines

```

typedef Point<ll> P;
vector<P> convexHull(vector<P> pts) {
    if (sz(pts) <= 1) return pts;
    sort(all(pts));
    vector<P> h(sz(pts)+1);
    int s = 0, t = 0;
    for (int it = 2; it--; s = --t, reverse(all(pts)))
        for (P p : pts) {
            while (t >= s + 2 && h[t-2].cross(h[t-1], p) <= 0) t--;
            h[t++] = p;
        }
    return {h.begin(), h.begin() + t - (t == 2 && h[0] == h[1])};
}

```

## HullDiameter.h

**Description:** Returns the two points with max distance on a convex hull (ccw, no duplicate/collinear points).

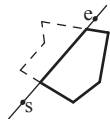
**Time:**  $\mathcal{O}(n)$

"Point.h" c571b8, 12 lines

```

typedef Point<ll> P;
array<P, 2> hullDiameter(vector<P> S) {
    int n = sz(S), j = n < 2 ? 0 : 1;
    pair<ll, array<P, 2>> res({0, {S[0], S[0]}});
    rep(i,0,j)
        for (; j = (j + 1) % n) {
            res = max(res, {(S[i] - S[j]).dist2(), {S[i], S[j]}});
            if ((S[(j + 1) % n] - S[j]).cross(S[i + 1] - S[i]) >= 0)
                break;
        }
    return res.second;
}

```



## PointInsideHull.h

**Description:** Determine whether a point t lies inside a convex hull (CCW order, with no collinear points). Returns true if point lies within the hull. If strict is true, points on the boundary aren't included.

**Time:**  $\mathcal{O}(\log N)$

"Point.h", "sideOf.h", "OnSegment.h" 71446b, 14 lines

```

typedef Point<ll> P;

bool inHull(const vector<P>& l, P p, bool strict = true) {
    int a = 1, b = sz(l) - 1, r = !strict;
    if (sz(l) < 3) return r && onSegment(l[0], l.back(), p);
    if (sideOf(l[0], l[a], l[b]) > 0) swap(a, b);
    if (sideOf(l[0], l[a], p) >= r || sideOf(l[0], l[b], p) <= -r)
        return false;
    while (abs(a - b) > 1) {
        int c = (a + b) / 2;
        (sideOf(l[0], l[c], p) > 0 ? b : a) = c;
    }
    return sgn(l[a].cross(l[b], p)) < r;
}

```

## LineHullIntersection.h

**Description:** Line-convex polygon intersection. The polygon must be ccw and have no collinear points. lineHull(line, poly) returns a pair describing the intersection of a line with the polygon:  $\bullet(-1, -1)$  if no collision,  $\bullet(i, -1)$  if touching the corner  $i$ ,  $\bullet(i, i)$  if along side  $(i, i+1)$ ,  $\bullet(i, j)$  if crossing sides  $(i, i+1)$  and  $(j, j+1)$ . In the last case, if a corner  $i$  is crossed, this is treated as happening on side  $(i, i+1)$ . The points are returned in the same order as the line hits the polygon. extrVertex returns the point of a hull with the max projection onto a line.

**Time:**  $\mathcal{O}(\log n)$

"Point.h" 7cf45b, 39 lines

```

#define cmp(i,j) sgn(dir.perp().cross(poly[(i)%n]-poly[(j)%n]))
#define extr(i) cmp(i + 1, i) >= 0 && cmp(i, i - 1 + n) < 0
template <class P> int extrVertex(vector<P>& poly, P dir) {
    int n = sz(poly), lo = 0, hi = n;
    if (extr(0)) return 0;
    while (lo + 1 < hi) {
        int m = (lo + hi) / 2;
        if (extr(m)) return m;
        int ls = cmp(lo + 1, lo), ms = cmp(m + 1, m);
        (ls < ms || (ls == ms && ls == cmp(lo, m)) ? hi : lo) = m;
    }
    return lo;
}

```

```

#define cmpl(i) sgn(a.cross(poly[i], b))
template <class P>
array<int, 2> lineHull(P a, P b, vector<P>& poly) {
    int endA = extrVertex(poly, (a - b).perp());
    int endB = extrVertex(poly, (b - a).perp());
    if (cmpl(endA) < 0 || cmpl(endB) > 0)
        return {-1, -1};
    array<int, 2> res;
    rep(i,0,2) {
        int lo = endB, hi = endA, n = sz(poly);
        while ((lo + 1) % n != hi) {
            int m = ((lo + hi + (lo < hi ? 0 : n)) / 2) % n;
            (cmpl(m) == cmpl(endB) ? lo : hi) = m;
        }
        res[i] = (lo + !cmpl(hi)) % n;
        swap(endA, endB);
    }
    if (res[0] == res[1]) return {res[0], -1};
    if (!cmpl(res[0]) && !cmpl(res[1]))
        switch ((res[0] - res[1] + sz(poly) + 1) % sz(poly)) {
            case 0: return {res[0], res[0]};
            case 2: return {res[1], res[1]};
        }
}

```

```

}
return res;
}

```

## 8.4 Misc. Point Set Problems

### ClosestPair.h

**Description:** Finds the closest pair of points.

**Time:**  $\mathcal{O}(n \log n)$

"Point.h" ac41a6, 17 lines

```

typedef Point<ll> P;
pair<P, P> closest(vector<P> v) {
    assert(sz(v) > 1);
    set<P> S;
    sort(all(v), [](P a, P b) { return a.y < b.y; });
    pair<ll, pair<P, P>> ret{LLONG_MAX, {P(), P()}};
    int j = 0;
    for (P p : v) {
        P d{1 + (ll)sqrt(ret.first), 0};
        while (v[j].y <= p.y - d.x) S.erase(v[j++]);
        auto lo = S.lower_bound(p - d), hi = S.upper_bound(p + d);
        for (; lo != hi; ++lo)
            ret = min(ret, {( *lo - p).dist2(), { *lo, p } });
        S.insert(p);
    }
    return ret.second;
}

```

## kdTree.h

**Description:** KD-tree (2d, can be extended to 3d)

"Point.h" bac5b0, 63 lines

```

typedef long long T;
typedef Point<T> P;
const T INF = numeric_limits<T>::max();

```

```

bool on_x(const P& a, const P& b) { return a.x < b.x; }
bool on_y(const P& a, const P& b) { return a.y < b.y; }

```

```

struct Node {
    P pt; // if this is a leaf, the single point in it
    T x0 = INF, x1 = -INF, y0 = INF, y1 = -INF; // bounds
    Node *first = 0, *second = 0;
}

```

```

T distance(const P& p) { // min squared distance to a point
    T x = (p.x < x0 ? x0 : p.x > x1 ? x1 : p.x);
    T y = (p.y < y0 ? y0 : p.y > y1 ? y1 : p.y);
    return (P(x,y) - p).dist2();
}

```

```

Node(vector<P>&& vp) : pt(vp[0]) {
    for (P p : vp) {
        x0 = min(x0, p.x); x1 = max(x1, p.x);
        y0 = min(y0, p.y); y1 = max(y1, p.y);
    }
    if (vp.size() > 1) {
        // split on x if width >= height (not ideal...)
        sort(all(vp), x1 - x0 >= y1 - y0 ? on_x : on_y);
        // divide by taking half the array for each child (not
        // best performance with many duplicates in the middle)
        int half = sz(vp)/2;
        first = new Node({vp.begin(), vp.begin() + half});
        second = new Node({vp.begin() + half, vp.end()});
    }
}
};

```

```

struct KDTree {
    Node* root;
    KDTree(const vector<P>& vp) : root(new Node({all(vp)})) {}
}

```

```
pair<T, P> search(Node *node, const P& p) {
    if (!node->first) {
        // uncomment if we should not find the point itself:
        // if (p == node->pt) return {INF, P()};
        return make_pair((p - node->pt).dist2(), node->pt);
    }

    Node *f = node->first, *s = node->second;
    T bfirst = f->distance(p), bsec = s->distance(p);
    if (bfirst > bsec) swap(bsec, bfirst), swap(f, s);

    // search closest side first, other side if needed
    auto best = search(f, p);
    if (bsec < best.first)
        best = min(best, search(s, p));
    return best;
}

// find nearest point to a point, and its squared distance
// (requires an arbitrary operator< for Point)
pair<T, P> nearest(const P& p) {
    return search(root, p);
}
};
```

FastDelaunay.h

**Description:** Fast Delaunay triangulation. Each circumcircle contains none of the input points. There must be no duplicate points. If all points are on a line, no triangles will be returned. Should work for doubles as well, though there may be precision issues in 'circ'. Returns triangles in order {t[0][0], t[0][1], t[0][2], t[1][0], ... }, all counter-clockwise.

**Time:**  $\mathcal{O}(n \log n)$

```
"Point.h" eefdf5, 88 lines

typedef Point<ll> P;
typedef struct Quad* Q;
typedef __int128_t lll; // (can be ll if coords are < 2e4)
P arb(LLONG_MAX, LLONG_MAX); // not equal to any other point

struct Quad {
    Q rot, o; P p = arb; bool mark;
    P& F() { return r()->p; }
    Q& r() { return rot->rot; }
    Q prev() { return rot->o->rot; }
    Q next() { return r()->prev(); }
} *H;

bool circ(P p, P a, P b, P c) { // is p in the circumcircle?
    lll p2 = p.dist2(), A = a.dist2()-p2,
        B = b.dist2()-p2, C = c.dist2()-p2;
    return p.cross(a,b)*C + p.cross(b,c)*A + p.cross(c,a)*B > 0;
}

Q makeEdge(P orig, P dest) {
    Q r = H ? H : new Quad{new Quad{new Quad{new Quad{0}}}};
    H = r->o; r->r()->r() = r;
    rep(i,0,4) r = r->rot, r->p = arb, r->o = i & 1 ? r : r->r();
    r->p = orig; r->F() = dest;
    return r;
}

void splice(Q a, Q b) {
    swap(a->o->rot->o, b->o->rot->o); swap(a->o, b->o);
}

Q connect(Q a, Q b) {
    Q q = makeEdge(a->F(), b->p);
    splice(q, a->next());
    splice(q->r(), b);
    return q;
}
```

FastDelaunay PolyhedronVolume Point3D 3dHull

```
pair<Q,Q> rec(const vector<P>& s) {
    if (sz(s) <= 3) {
        Q a = makeEdge(s[0], s[1]), b = makeEdge(s[1], s.back());
        if (sz(s) == 2) return { a, a->r() };
        splice(a->r(), b);
        auto side = s[0].cross(s[1], s[2]);
        Q c = side ? connect(b, a) : 0;
        return {side < 0 ? c->r() : a, side < 0 ? c : b->r() };
    }

#define H(e) e->F(), e->p
#define valid(e) (e->F().cross(H(base)) > 0)
    Q A, B, ra, rb;
    int half = sz(s) / 2;
    tie(ra, A) = rec({all(s) - half});
    tie(B, rb) = rec({sz(s) - half + all(s)});
    while ((B->p.cross(H(A)) < 0 && (A = A->next())) ||
        (A->p.cross(H(B)) > 0 && (B = B->r()->o)));
    Q base = connect(B->r(), A);
    if (A->p == ra->p) ra = base->r();
    if (B->p == rb->p) rb = base;

#define DEL(e, init, dir) Q e = init->dir; if (valid(e)) \
    while (circ(e->dir->F(), H(base), e->F())) { \
        Q t = e->dir; \
        splice(e, e->prev()); \
        splice(e->r(), e->r()->prev()); \
        e->o = H; H = e; e = t; \
    }
    for (;;) {
        DEL(LC, base->r(), o); DEL(RC, base, prev());
        if (!valid(LC) && !valid(RC)) break;
        if (!valid(LC) || (valid(RC) && circ(H(RC), H(LC))))
            base = connect(RC, base->r());
        else
            base = connect(base->r(), LC->r());
    }
    return { ra, rb };
}
```

```
vector<P> triangulate(vector<P> pts) {
    sort(all(pts)); assert(unique(all(pts)) == pts.end());
    if (sz(pts) < 2) return {};
    Q e = rec(pts).first;
    vector<Q> q = {e};
    int qi = 0;
    while (e->o->F().cross(e->F(), e->p) < 0) e = e->o;
#define ADD { Q c = e; do { c->mark = 1; pts.push_back(c->p); \
    q.push_back(c->r()); c = c->next(); } while (c != e); }
    ADD; pts.clear();
    while (qi < sz(q)) if (!(e = q[qi++])->mark) ADD;
    return pts;
}
```

8.5 3D

PolyhedronVolume.h

**Description:** Magic formula for the volume of a polyhedron. Faces should point outwards.

```
3058c3, 6 lines

template<class V, class L>
double signedPolyVolume(const V& p, const L& trilst) {
    double v = 0;
    for (auto i : trilst) v += p[i.a].cross(p[i.b]).dot(p[i.c]);
    return v / 6;
}
```

Point3D.h

**Description:** Class to handle points in 3D space. T can be e.g. double or long.

```
8058ae, 32 lines

template<class T> struct Point3D {
    typedef Point3D P;
    typedef const P& R;
    T x, y, z;
    explicit Point3D(T x=0, T y=0, T z=0) : x(x), y(y), z(z) {}
    bool operator<(R p) const {
        return tie(x, y, z) < tie(p.x, p.y, p.z); }
    bool operator==(R p) const {
        return tie(x, y, z) == tie(p.x, p.y, p.z); }
    P operator+(R p) const { return P(x+p.x, y+p.y, z+p.z); }
    P operator-(R p) const { return P(x-p.x, y-p.y, z-p.z); }
    P operator*(T d) const { return P(x*d, y*d, z*d); }
    P operator/(T d) const { return P(x/d, y/d, z/d); }
    T dot(R p) const { return x*p.x + y*p.y + z*p.z; }
    P cross(R p) const {
        return P(y*p.z - z*p.y, z*p.x - x*p.z, x*p.y - y*p.x);
    }
    T dist2() const { return x*x + y*y + z*z; }
    double dist() const { return sqrt((double)dist2()); }
    //Azimuthal angle (longitude) to x-axis in interval [-pi, pi]
    double phi() const { return atan2(y, x); }
    //Zenith angle (latitude) to the z-axis in interval [0, pi]
    double theta() const { return atan2(sqrt(x*x+y*y),z); }
    P unit() const { return *this/(T)dist(); } //makes dist()==1
    //returns unit vector normal to *this and p
    P normal(P p) const { return cross(p).unit(); }
    //returns point rotated 'angle' radians ccw around axis
    P rotate(double angle, P axis) const {
        double s = sin(angle), c = cos(angle); P u = axis.unit();
        return u.dot(u)*(1-c) + (*this)*c - cross(u)*s;
    }
};
```

3dHull.h

**Description:** Computes all faces of the 3-dimension hull of a point set. \*No four points must be coplanar\*, or else random results will be returned. All faces will point outwards.

**Time:**  $\mathcal{O}(n^2)$

```
"Point3D.h" 5b45fc, 49 lines

typedef Point3D<double> P3;

struct PR {
    void ins(int x) { (a == -1 ? a : b) = x; }
    void rem(int x) { (a == x ? a : b) = -1; }
    int cnt() { return (a != -1) + (b != -1); }
    int a, b;
};

struct F { P3 q; int a, b, c; };

vector<F> hull3d(const vector<P3>& A) {
    assert(sz(A) >= 4);
    vector<vector<PR>> E(sz(A), vector<PR>(sz(A), {-1, -1}));
#define E(x,y) E[f.x][f.y]
    vector<F> FS;
    auto mf = [&](int i, int j, int k, int l) {
        P3 q = (A[j] - A[i]).cross((A[k] - A[i]));
        if (q.dot(A[l]) > q.dot(A[i]))
            q = q * -1;
        F f{q, i, j, k};
        E(a,b).ins(k); E(a,c).ins(j); E(b,c).ins(i);
        FS.push_back(f);
    };
    rep(i,0,4) rep(j,i+1,4) rep(k,j+1,4)
        mf(i, j, k, 6 - i - j - k);
};
```



```
rep(i,4,sz(A)) {
    rep(j,0,sz(FS)) {
        F f = FS[j];
        if(f.q.dot(A[i]) > f.q.dot(A[f.a])) {
            E(a,b).rem(f.c);
            E(a,c).rem(f.b);
            E(b,c).rem(f.a);
            swap(FS[j--], FS.back());
            FS.pop_back();
        }
        int nw = sz(FS);
        rep(j,0,nw) {
            F f = FS[j];
#define C(a, b, c) if (E(a,b).cnt() != 2) mf(f.a, f.b, i, f.c);
            C(a, b, c); C(a, c, b); C(b, c, a);
        }
        for (F& it : FS) if ((A[it.b] - A[it.a]).cross(
            A[it.c] - A[it.a]).dot(it.q) <= 0) swap(it.c, it.b);
        return FS;
    };
};
```

sphericalDistance.h

**Description:** Returns the shortest distance on the sphere with radius radius between the points with azimuthal angles (longitude) f1 ( $\phi_1$ ) and f2 ( $\phi_2$ ) from x axis and zenith angles (latitude) t1 ( $\theta_1$ ) and t2 ( $\theta_2$ ) from z axis (0 = north pole). All angles measured in radians. The algorithm starts by converting the spherical coordinates to cartesian coordinates so if that is what you have you can use only the two last rows. dx\*radius is then the difference between the two points in the x direction and d\*radius is the total distance between the points.

```
double sphericalDistance(double f1, double t1,
    double f2, double t2, double radius) {
    double dx = sin(t2)*cos(f2) - sin(t1)*cos(f1);
    double dy = sin(t2)*sin(f2) - sin(t1)*sin(f1);
    double dz = cos(t2) - cos(t1);
    double d = sqrt(dx*dx + dy*dy + dz*dz);
    return radius*2*asin(d/2);
}
```

Strings (9)

KMP.h

**Description:** pi[x] computes the length of the longest prefix of s that ends at x, other than s[0...x] itself (abacaba -> 0010123). Can be used to find all occurrences of a string.

**Time:**  $\mathcal{O}(n)$

```
vi pi(const string& s) {
    vi p(sz(s));
    rep(i,1,sz(s)) {
        int g = p[i-1];
        while (g && s[i] != s[g]) g = p[g-1];
        p[i] = g + (s[i] == s[g]);
    }
    return p;
}

vi match(const string& s, const string& pat) {
    vi p = pi(pat + '\0' + s), res;
    rep(i,sz(p)-sz(s),sz(p))
        if (p[i] == sz(pat)) res.push_back(i - 2 * sz(pat));
    return res;
}
```

Zfunc.h

**Description:** z[x] computes the length of the longest common prefix of s[i:] and s, except z[0] = 0. (abacaba -> 0010301)

**Time:**  $\mathcal{O}(n)$

```
vi Z(const string& S) {
    vi z(sz(S));
    int l = -1, r = -1;
    rep(i,1,sz(S)) {
        z[i] = i >= r ? 0 : min(r - i, z[i - 1]);
        while (i + z[i] < sz(S) && S[i + z[i]] == S[z[i]])
            z[i]++;
        if (i + z[i] > r)
            l = i, r = i + z[i];
    }
    return z;
}
```

Manacher.h

**Description:** For each position in a string, computes p[0][i] = half length of longest even palindrome around pos i, p[1][i] = longest odd (half rounded down).

**Time:**  $\mathcal{O}(N)$

```
array<vi, 2> manacher(const string& s) {
    int n = sz(s);
    array<vi,2> p = {vi(n+1), vi(n)};
    rep(z,0,2) for (int i=0,l=0,r=0; i < n; i++) {
        int t = r-i+!z;
        if (i<r) p[z][i] = min(t, p[z][l+t]);
        int L = i-p[z][i], R = i+p[z][i]-!z;
        while (L>=1 && R+1<n && s[L-1] == s[R+1])
            p[z][i]++, L--, R++;
        if (R>r) l=L, r=R;
    }
    return p;
}
```

MinRotation.h

**Description:** Finds the lexicographically smallest rotation of a string.

**Usage:** rotate(v.begin(), v.begin()+minRotation(v), v.end());

**Time:**  $\mathcal{O}(N)$

```
int minRotation(string s) {
    int a=0, N=sz(s); s += s;
    rep(b,0,N) rep(k,0,N) {
        if (a+k == b || s[a+k] < s[b+k]) {b += max(0, k-1); break;}
        if (s[a+k] > s[b+k]) {a = b; break;}
    }
    return a;
}
```

SuffixArray.h

**Description:** Builds suffix array for a string. sa[i] is the starting index of the suffix which is i'th in the sorted suffix array. The returned vector is of size n + 1, and sa[0] = n. The lcp array contains longest common prefixes for neighbouring strings in the suffix array: lcp[i] = lcp(sa[i], sa[i-1]), lcp[0] = 0. The input string must not contain any zero bytes.

**Time:**  $\mathcal{O}(n \log n)$

```
struct SuffixArray {
    vi sa, lcp;
    SuffixArray(string& s, int lim=256) { // or basic_string<int>
        int n = sz(s) + 1, k = 0, a, b;
        vi x(all(s)+1, y(n), ws(max(n, lim)), rank(n);
        sa = lcp = y, iota(all(sa), 0);
        for (int j = 0, p = 0; p < n; j = max(1, j * 2), lim = p) {
            p = j, iota(all(y), n - j);
            rep(i,0,n) if (sa[i] >= j) y[p++] = sa[i] - j;
            fill(all(ws), 0);
        }
    }
};
```

```
rep(i,0,n) ws[x[i]]++;
rep(i,1,lim) ws[i] += ws[i - 1];
for (int i = n; i--;) sa[--ws[x[y[i]]]] = y[i];
swap(x, y), p = 1, x[sa[0]] = 0;
rep(i,1,n) a = sa[i - 1], b = sa[i], x[b] =
    (y[a] == y[b] && y[a + j] == y[b + j]) ? p - 1 : p++;
}
rep(i,1,n) rank[sa[i]] = i;
for (int i = 0, j; i < n - 1; lcp[rank[i++]] = k)
    for (k && k--, j = sa[rank[i] - 1];
        s[i + k] == s[j + k]; k++);
}
};
```

SuffixTree.h

**Description:** Ukkonen's algorithm for online suffix tree construction. Each node contains indices [l, r) into the string, and a list of child nodes. Suffixes are given by traversals of this tree, joining [l, r) substrings. The root is 0 (has l = -1, r = 0), non-existent children are -1. To get a complete tree, append a dummy symbol – otherwise it may contain an incomplete path (still useful for substring matching, though).

**Time:**  $\mathcal{O}(26N)$

```
struct SuffixTree {
    enum { N = 200010, ALPHA = 26 }; // N ~ 2*maxlen+10
    int toi(char c) { return c - 'a'; }
    string a; // v = cur node, q = cur position
    int t[N][ALPHA], l[N], r[N], p[N], s[N], v=0, q=0, m=2;

    void ukkadd(int i, int c) { suff:
        if (r[v]<=q) {
            if (t[v][c]==-1) { t[v][c]=m; l[m]=i;
                p[m++]=v; v=s[v]; q=r[v]; goto suff; }
            v=t[v][c]; q=l[v];
        }
        if (q==-1 || c==toi(a[q])) q++; else {
            l[m+1]=i; p[m+1]=m; l[m]=l[v]; r[m]=q;
            p[m]=p[v]; t[m][c]=m+1; t[m][toi(a[q])]=v;
            l[v]=q; p[v]=m; t[p[m]][toi(a[l[m]])]=m;
            v=s[p[m]]; q=l[m];
            while (q<r[m]) { v=t[v][toi(a[q])]; q+=r[v]-l[v]; }
            if (q==r[m]) s[m]=v; else s[m]=m+2;
            q=r[v]-(q-r[m]); m+=2; goto suff;
        }
    }

    SuffixTree(string a) : a(a) {
        fill(r,r+N,sz(a));
        memset(s, 0, sizeof s);
        memset(t, -1, sizeof t);
        fill(t[1],t[1]+ALPHA,0);
        s[0] = 1; l[0] = l[1] = -1; r[0] = r[1] = p[0] = p[1] = 0;
        rep(i,0,sz(a)) ukkadd(i, toi(a[i]));
    }
};
```

```
// example: find longest common substring (uses ALPHA = 28)
pii best;
int lcs(int node, int i1, int i2, int olen) {
    if (l[node] <= i1 && i1 < r[node]) return 1;
    if (l[node] <= i2 && i2 < r[node]) return 2;
    int mask = 0, len = node ? olen + (r[node] - l[node]) : 0;
    rep(c,0,ALPHA) if (t[node][c] != -1)
        mask |= lcs(t[node][c], i1, i2, len);
    if (mask == 3)
        best = max(best, {len, r[node] - len});
    return mask;
}

static pii LCS(string s, string t) {
    SuffixTree st(s + (char)('z' + 1) + t + (char)('z' + 2));
```

```
    st.lcs(0, sz(s), sz(s) + 1 + sz(t), 0);
    return st.best;
}
};
```

Hashing.h

**Description:** Self-explanatory methods for string hashing. 2d2a67, 44 lines

```
// Arithmetic mod 2^64-1. 2x slower than mod 2^64 and more
// code, but works on evil test data (e.g. Thue-Morse, where
// ABBA... and BAAB... of length 2^10 hash the same mod 2^64).
// "typedef ull H;" instead if you think test data is random,
// or work mod 10^9+7 if the Birthday paradox is not a problem.
typedef uint64_t ull;
struct H {
    ull x; H(ull x=0) : x(x) {}
    H operator+(H o) { return x + o.x + (x + o.x < x); }
    H operator-(H o) { return *this + ~o.x; }
    H operator*(H o) { auto m = (__uint128_t)x * o.x;
        return H((ull)m) + (ull)(m >> 64); }
    ull get() const { return x + !~x; }
    bool operator==(H o) const { return get() == o.get(); }
    bool operator<(H o) const { return get() < o.get(); }
};
static const H C = (1ll)1e11+3; // (order ~ 3e9; random also ok)
```

```
struct HashInterval {
    vector<H> ha, pw;
    HashInterval(string& str) : ha(sz(str)+1), pw(ha) {
        pw[0] = 1;
        rep(i,0,sz(str))
            ha[i+1] = ha[i] * C + str[i],
            pw[i+1] = pw[i] * C;
    }
    H hashInterval(int a, int b) { // hash [a, b)
        return ha[b] - ha[a] * pw[b - a];
    }
};
```

```
vector<H> getHashes(string& str, int length) {
    if (sz(str) < length) return {};
    H h = 0, pw = 1;
    rep(i,0,length)
        h = h * C + str[i], pw = pw * C;
    vector<H> ret = {h};
    rep(i,length,sz(str)) {
        ret.push_back(h = h * C + str[i] - pw * str[i-length]);
    }
    return ret;
}
```

```
H hashString(string& s){H h{}; for(char c:s) h=h*C+c;return h;}
```

AhoCorasick.h

**Description:** Aho-Corasick automaton, used for multiple pattern matching. Initialize with AhoCorasick ac(patterns); the automaton start node will be at index 0. find(word) returns for each position the index of the longest word that ends there, or -1 if none. findAll(–, word) finds all words (up to  $N\sqrt{N}$  many if no duplicate patterns) that start at each position (shortest first). Duplicate patterns are allowed; empty patterns are not. To find the longest words that start at each position, reverse all input. For large alphabets, split each symbol into chunks, with sentinel bits for symbol boundaries.

**Time:** construction takes  $\mathcal{O}(26N)$ , where  $N$  = sum of length of patterns. find(x) is  $\mathcal{O}(N)$ , where  $N$  = length of x. findAll is  $\mathcal{O}(NM)$ . f35677, 66 lines

```
struct AhoCorasick {
    enum {alpha = 26, first = 'A'}; // change this!
    struct Node {
        // (nmatches is optional)
```

Hashing AhoCorasick Merge-Sort Quick-Sort

```
    int back, next[alpha], start = -1, end = -1, nmatches = 0;
    Node(int v) { memset(next, v, sizeof(next)); }
};
vector<Node> N;
vi backp;
void insert(string& s, int j) {
    assert(!s.empty());
    int n = 0;
    for (char c : s) {
        int& m = N[n].next[c - first];
        if (m == -1) { n = m = sz(N); N.emplace_back(-1); }
        else n = m;
    }
    if (N[n].end == -1) N[n].start = j;
    backp.push_back(N[n].end);
    N[n].end = j;
    N[n].nmatches++;
}
AhoCorasick(vector<string>& pat) : N(1, -1) {
    rep(i,0,sz(pat)) insert(pat[i], i);
    N[0].back = sz(N);
    N.emplace_back(0);

    queue<int> q;
    for (q.push(0); !q.empty(); q.pop()) {
        int n = q.front(), prev = N[n].back;
        rep(i,0,alpha) {
            int &ed = N[n].next[i], y = N[prev].next[i];
            if (ed == -1) ed = y;
            else {
                N[ed].back = y;
                (N[ed].end == -1 ? N[ed].end : backp[N[ed].start])
                    = N[y].end;
                N[ed].nmatches += N[y].nmatches;
                q.push(ed);
            }
        }
    }
}
vi find(string word) {
    int n = 0;
    vi res; // ll count = 0;
    for (char c : word) {
        n = N[n].next[c - first];
        res.push_back(N[n].end);
        // count += N[n].nmatches;
    }
    return res;
}
vector<vi> findAll(vector<string>& pat, string word) {
    vi r = find(word);
    vector<vi> res(sz(word));
    rep(i,0,sz(word)) {
        int ind = r[i];
        while (ind != -1) {
            res[i - sz(pat[ind]) + 1].push_back(ind);
            ind = backp[ind];
        }
    }
    return res;
}
};
```

Algoritmos (10)

10.1 Sorting Algorithms

Merge-Sort.py 33 lines

```
def merge(left, right, v):
    k = l = r = 0
    while l < len(left) and r < len(right) and k < len(v):
        if left[l] <= right[r]:
            v[k] = left[l]
            l += 1
        else:
            v[k] = right[r]
            r += 1
        k += 1

    while l < len(left):
        v[k] = left[l]
        l += 1
    while r < len(right):
        v[k] = right[r]
        r += 1
        k += 1

def mergeSort(v):
    if len(v) == 1:
        return

    mid = len(v) // 2

    left = v[:mid]
    right = v[mid:]

    mergeSort(left)
    mergeSort(right)

    merge(left, right, v)
```

Quick-Sort.py 23 lines

```
def partition(arr, left, right):
    i = (left-1)
    pivot = arr[right]

    for j in range(left, right):
        if arr[j] <= pivot:
            i = i+1
            arr[i], arr[j] = arr[j], arr[i]

    arr[i+1], arr[right] = arr[right], arr[i+1]
    return (i+1)

def quicksort(arr):
    quicksort_rec(arr, 0, len(arr)-1)

def quicksort_rec(arr, left, right):
    if len(arr) == 1:
        return arr
    if left < right:
        pi = partition(arr, left, right)

        quicksort_rec(arr, left, pi-1)
        quicksort_rec(arr, pi+1, right)
```



Merge-Sort.java

Quick-Sort.java

## 10.2 Searching Algorithms

Binary-Search.py16 lines

```
def rec_binarysearch(e, elements):
    return __rec_bs__(e, 0, len(elements) - 1, elements)

def __rec_bs__(e, low, high, elements):
    # if not found
    # this element could be inserted at -index-1
    if low > high:
        return -low - 1
    mid = (low + high) // 2
    if elements[mid] == e:
        return mid # found
    elif e < elements[mid]:
        return __rec_bs__(e, low, mid - 1, elements)
    else:
        return __rec_bs__(e, mid + 1, high, elements)
```

Ternary-Search.py17 lines

```
def ternarySearch(l, r, key, ar):
    while r >= l:
        mid1 = l + (r-l) // 3
        mid2 = r - (r-l) // 3

        if key == ar[mid1]:
            return mid1
        if key == ar[mid2]:
            return mid2
        if key < ar[mid1]:
            r = mid1 - 1
        elif key > ar[mid2]:
            l = mid2 + 1
        else:
            l = mid1 + 1
            r = mid2 - 1
    return -1
```

Greedy.py22 lines

```
import random, sys
# Devuelve el mejor valor de los candidatos
def getBest(candidates, elemSet):
    bestValue = sys.maxsize
    best = 0
    for c in candidates:
        value = elemSet[c]
        if value < bestValue:
            bestValue = value
            best = c
    return best
# Algoritmo voraz
def greedy(elemSet):
    candidates = set()
    n = len(elemSet)
    for i in range(n):
        candidates.add(i)
    sol = []
    while candidates:
        best = getBest(candidates, elemSet)
        candidates.remove(best)
        sol.append(best)
```

WindowSliding.py10 lines

```
def maxSum(arr, n, k):
    max_sum = -0x3f3f3f
    min_sum = 0x3f3f3f
    for i in range(n - k + 1):
        current_sum = 0
        for j in range(k):
            current_sum += arr[i + j]
        max_sum = max(current_sum, max_sum)
        min_sum = max(current_sum, min_sum)
    return max_sum
```

## Various (11)

### 11.1 Intervals

IntervalContainer.h

**Description:** Add and remove intervals from a set of disjoint intervals. Will merge the added interval with any overlapping intervals in the set when adding. Intervals are [inclusive, exclusive).  
**Time:**  $\mathcal{O}(\log N)$

edge47, 23 lines

```
set<pii>::iterator addInterval(set<pii>& is, int L, int R) {
    if (L == R) return is.end();
    auto it = is.lower_bound({L, R}), before = it;
    while (it != is.end() && it->first <= R) {
        R = max(R, it->second);
        before = it = is.erase(it);
    }
    if (it != is.begin() && (--it)->second >= L) {
        L = min(L, it->first);
        R = max(R, it->second);
        is.erase(it);
    }
    return is.insert(before, {L,R});
}

void removeInterval(set<pii>& is, int L, int R) {
    if (L == R) return;
    auto it = addInterval(is, L, R);
    auto r2 = it->second;
    if (it->first == L) is.erase(it);
    else (int&)it->second = L;
    if (R != r2) is.emplace(R, r2);
}
```

IntervalCover.h

**Description:** Compute indices of smallest set of intervals covering another interval. Intervals should be [inclusive, exclusive). To support [inclusive, inclusive], change (A) to add || R.empty(). Returns empty set on failure (or if G is empty).  
**Time:**  $\mathcal{O}(N \log N)$

9e9d8d, 19 lines

```
template<class T>
vi cover(pair<T, T> G, vector<pair<T, T>> I) {
    vi S(sz(I)), R;
    iota(all(S), 0);
    sort(all(S), [&](int a, int b) { return I[a] < I[b]; });
    T cur = G.first;
    int at = 0;
    while (cur < G.second) { // (A)
        pair<T, int> mx = make_pair(cur, -1);
        while (at < sz(I) && I[S[at]].first <= cur) {
            mx = max(mx, make_pair(I[S[at]].second, S[at]));
            at++;
        }
        if (mx.second == -1) return {};
        cur = mx.first;
```

```
        R.push_back(mx.second);
    }
    return R;
}

ConstantIntervals.h
Description: Split a monotone function on [from, to) into a minimal set of half-open intervals on which it has the same value. Runs a callback g for each such interval.
Usage: constantIntervals(0, sz(v), [&](int x){return v[x];}, [&](int lo, int hi, T val){...});
Time:  $\mathcal{O}(k \log \frac{n}{k})$ 
```

753a4c, 19 lines

```
template<class F, class G, class T>
void rec(int from, int to, F& f, G& g, int& i, T& p, T q) {
    if (p == q) return;
    if (from == to) {
        g(i, to, p);
        i = to; p = q;
    } else {
        int mid = (from + to) >> 1;
        rec(from, mid, f, g, i, p, f(mid));
        rec(mid+1, to, f, g, i, p, q);
    }
}

template<class F, class G>
void constantIntervals(int from, int to, F f, G g) {
    if (to <= from) return;
    int i = from; auto p = f(i), q = f(to-1);
    rec(from, to-1, f, g, i, p, q);
    g(i, to, q);
}
```

### 11.2 Misc. algorithms

TernarySearch.h

**Description:** Find the smallest i in [a,b] that maximizes f(i), assuming that f(a) < ... < f(i) ≥ ... ≥ f(b). To reverse which of the sides allows non-strict inequalities, change the < marked with (A) to <=, and reverse the loop at (B). To minimize f, change it to >, also at (B).  
**Usage:** int ind = ternSearch(0,n-1,[&](int i){return a[i];});  
**Time:**  $\mathcal{O}(\log(b - a))$

9155b4, 11 lines

```
template<class F>
int ternSearch(int a, int b, F f) {
    assert(a <= b);
    while (b - a >= 5) {
        int mid = (a + b) / 2;
        if (f(mid) < f(mid+1)) a = mid; // (A)
        else b = mid+1;
    }
    rep(i,a+1,b+1) if (f(a) < f(i)) a = i; // (B)
    return a;
}
```

LIS.h

**Description:** Compute indices for the longest increasing subsequence.  
**Time:**  $\mathcal{O}(N \log N)$

2932a0, 17 lines

```
template<class I> vi lis(const vector<I>& S) {
    if (S.empty()) return {};
    vi prev(sz(S));
    typedef pair<I, int> p;
    vector<p> res;
    rep(i,0,sz(S)) {
        // change 0 -> i for longest non-decreasing subsequence
        auto it = lower_bound(all(res), p{S[i], 0});
        if (it == res.end()) res.emplace_back(), it = res.end()-1;
        *it = {S[i], i};
        prev[i] = it == res.begin() ? 0 : (it-1)->second;
```

```
    }
    int L = sz(res), cur = res.back().second;
    vi ans(L);
    while (L--) ans[L] = cur, cur = prev[cur];
    return ans;
}
```

FastKnapsack.h  
**Description:** Given N non-negative integer weights w and a non-negative target t, computes the maximum S <= t such that S is the sum of some subset of the weights.  
**Time:**  $\mathcal{O}(N \max(w_i))$

b20ccc, 16 lines

```
int knapsack(vi w, int t) {
    int a = 0, b = 0, x;
    while (b < sz(w) && a + w[b] <= t) a += w[b++];
    if (b == sz(w)) return a;
    int m = *max_element(all(w));
    vi u, v(2*m, -1);
    v[a+m-t] = b;
    rep(i,b,sz(w)) {
        u = v;
        rep(x,0,m) v[x+w[i]] = max(v[x+w[i]], u[x]);
        for (x = 2*m; --x > m;) rep(j, max(0,u[x]), v[x])
            v[x-w[j]] = max(v[x-w[j]], j);
    }
    for (a = t; v[a+m-t] < 0; a--);
    return a;
}
```

### 11.3 Dynamic programming

KnuthDP.h  
**Description:** When doing DP on intervals:  $a[i][j] = \min_{i < k < j} (a[i][k] + a[k][j]) + f(i, j)$ , where the (minimal) optimal  $k$  increases with both  $i$  and  $j$ , one can solve intervals in increasing order of length, and search  $k = p[i][j]$  for  $a[i][j]$  only between  $p[i][j - 1]$  and  $p[i + 1][j]$ . This is known as Knuth DP. Sufficient criteria for this are if  $f(b, c) \leq f(a, d)$  and  $f(a, c) + f(b, d) \leq f(a, d) + f(b, c)$  for all  $a \leq b \leq c \leq d$ . Consider also: LineContainer (ch. Data structures), monotone queues, ternary search.  
**Time:**  $\mathcal{O}(N^2)$

DivideAndConquerDP.h  
**Description:** Given  $a[i] = \min_{l \circ(i) \leq k < h \circ(i)} (f(i, k))$  where the (minimal) optimal  $k$  increases with  $i$ , computes  $a[i]$  for  $i = L..R - 1$ .  
**Time:**  $\mathcal{O}((N + (hi - lo)) \log N)$

d38d2b, 18 lines

```
struct DP { // Modify at will:
    int lo(int ind) { return 0; }
    int hi(int ind) { return ind; }
    ll f(int ind, int k) { return dp[ind][k]; }
    void store(int ind, int k, ll v) { res[ind] = pii(k, v); }

    void rec(int L, int R, int LO, int HI) {
        if (L >= R) return;
        int mid = (L + R) >> 1;
        pair<ll, int> best(LLONG_MAX, LO);
        rep(k, max(LO, lo(mid)), min(HI, hi(mid)))
            best = min(best, make_pair(f(mid, k), k));
        store(mid, best.second, best.first);
        rec(L, mid, LO, best.second+1);
        rec(mid+1, R, best.second, HI);
    }
    void solve(int L, int R) { rec(L, R, INT_MIN, INT_MAX); }
};
```

### 11.4 Debugging tricks

- signal(SIGSEGV, [](int) { \_Exit(0); }); converts segfaults into Wrong Answers. Similarly one can catch SIGABRT (assertion failures) and SIGFPE (zero divisions). \_GLIBCXX\_DEBUG failures generate SIGABRT (or SIGSEGV on gcc 5.4.0 apparently).
- feenableexcept(29); kills the program on NaNs (1), 0-divs (4), infinities (8) and denormals (16).

### 11.5 Optimization tricks

\_\_builtin\_ia32\_ldmxcsr(40896); disables denormals (which make floats 20x slower near their minimum value).

#### 11.5.1 Bit hacks

- x & -x is the least bit in x.
- for (int x = m; x; ) { --x &= m; ... } loops over all subset masks of m (except m itself).
- c = x&-x, r = x+c; ((r^x) >> 2)/c) | r is the next number after x with the same number of bits set.
- rep(b,0,K) rep(i,0,(1 << K)) if (i & 1 << b) D[i] += D[i^(1 << b)]; computes all sums of subsets.

#### 11.5.2 Pragmas

- #pragma GCC optimize ("Ofast") will make GCC auto-vectorize loops and optimizes floating points better.
- #pragma GCC target ("avx2") can double performance of vectorized code, but causes crashes on old machines.
- #pragma GCC optimize ("trapv") kills the program on integer overflows (but is really slow).

FastMod.h  
**Description:** Compute  $a \% b$  about 5 times faster than usual, where  $b$  is constant but not known at compile time. Returns a value congruent to  $a \pmod b$  in the range  $[0, 2b)$ .

751a02, 8 lines

```
typedef unsigned long long ull;
struct FastMod {
    ull b, m;
    FastMod(ull b) : b(b), m((-1ULL / b) {}
    ull reduce(ull a) { // a % b + (0 or b)
        return a - (ull)((__uint128_t(m) * a) >> 64) * b;
    }
};
```

FastInput.h  
**Description:** Read an integer from stdin. Usage requires your program to pipe in input from file.  
**Usage:** ./a.out < input.txt  
**Time:** About 5x as fast as cin/scanf.

7b3c70, 17 lines

```
inline char gc() { // like getchar()
    static char buf[1 << 16];
    static size_t bc, be;
    if (bc >= be) {
```

```
        buf[0] = 0, bc = 0;
        be = fread(buf, 1, sizeof(buf), stdin);
    }
    return buf[bc++]; // returns 0 on EOF
}
```

```
int readInt() {
    int a, c;
    while ((a = gc()) < 40);
    if (a == '-') return -readInt();
    while ((c = gc()) >= 48) a = a * 10 + c - 48;
    return a - 48;
}
```

BumpAllocator.h  
**Description:** When you need to dynamically allocate many objects and don't care about freeing them. "new X" otherwise has an overhead of something like 0.05us + 16 bytes per allocation.

745db2, 8 lines

```
// Either globally or in a single class:
static char buf[450 << 20];
void* operator new(size_t s) {
    static size_t i = sizeof buf;
    assert(s < i);
    return (void*)&buf[i -= s];
}
void operator delete(void*) {}
```

SmallPtr.h  
**Description:** A 32-bit pointer that points into BumpAllocator memory.

2dd6c9, 10 lines

```
"BumpAllocator.h"
template<class T> struct ptr {
    unsigned ind;
    ptr(T* p = 0) : ind(p ? unsigned((char*)p - buf) : 0) {
        assert(ind < sizeof buf);
    }
    T& operator*() const { return *(T*)(buf + ind); }
    T* operator->() const { return &*this; }
    T& operator[](int a) const { return (&*this)[a]; }
    explicit operator bool() const { return ind; }
};
```

BumpAllocatorSTL.h  
**Description:** BumpAllocator for STL containers.  
**Usage:** vector<vector<int, small<int>>> ed(N);

bb66d4, 14 lines

```
char buf[450 << 20] alignas(16);
size_t buf_ind = sizeof buf;

template<class T> struct small {
    typedef T value_type;
    small() {}
    template<class U> small(const U&) {}
    T* allocate(size_t n) {
        buf_ind -= n * sizeof(T);
        buf_ind &= 0 - alignof(T);
        return (T*)(buf + buf_ind);
    }
    void deallocate(T*, size_t) {}
};
```

SIMD.h

**Description:** Cheat sheet of SSE/AVX intrinsics, for doing arithmetic on several numbers at once. Can provide a constant factor improvement of about 4, orthogonal to loop unrolling. Operations follow the pattern `"_mm(256)?name_(si(128|256)|epi(8|16|32|64)|pd|ps)".` Not all are described here; grep for `_mm_` in `/usr/lib/gcc/*/4.9/include/` for more. If AVX is unsupported, try 128-bit operations, "emmintrin.h" and `#define _SSE_` and `_MMX_` before including it. For aligned memory use `_mm_malloc(size, 32)` or `int buf[N] alignas(32),` but prefer `loadu/storeu`.

```
#pragma GCC target ("avx2") // or sse4.1
#include "emmintrin.h"
```

```
typedef __m256i mi;
#define L(x) _mm256_loadu_si256((mi*)&(x))
```

```
// High-level/specific methods:
// load(u)?_si256, store(u)?_si256, setzero_si256, _mm_malloc
// blendv(epi8|ps|pd) (z?y:x), movemask_epi8 (hibits of bytes)
// i32gather_epi32(addr, x, 4): map addr[] over 32-b parts of x
// sad_epu8: sum of absolute differences of u8, outputs 4xi64
// maddubs_epi16: dot product of unsigned i7's, outputs 16xi15
// madd_epi16: dot product of signed i16's, outputs 8xi32
// extractf128_si256(, i) (256->128), cvtsi128_si32 (128->lo32)
// permute2f128_si256(x,x,1) swaps 128-bit lanes
// shuffle_epi32(x, 3*64+2*16+1*4+0) == x for each lane
// shuffle_epi8(x, y) takes a vector instead of an imm
```

```
// Methods that work with most data types (append e.g. _epi32):
// set1, blend (i8?x:y), add, adds (sat.), mullo, sub, and/or,
// andnot, abs, min, max, sign(1,x), cmp(gt|eq), unpack(lo|hi)
```

```
int sumi32(mi m) { union {int v[8]; mi m;} u; u.m = m;
  int ret = 0; rep(i,0,8) ret += u.v[i]; return ret; }
mi zero() { return _mm256_setzero_si256(); }
mi one() { return _mm256_set1_epi32(-1); }
bool all_zero(mi m) { return _mm256_testz_si256(m, m); }
bool all_one(mi m) { return _mm256_testc_si256(m, one()); }
```

```
ll example_filteredDotProduct(int n, short* a, short* b) {
  int i = 0; ll r = 0;
  mi zero = _mm256_setzero_si256(), acc = zero;
  while (i + 16 <= n) {
    mi va = L(a[i]), vb = L(b[i]); i += 16;
    va = _mm256_and_si256(_mm256_cmpgt_epil6(vb, va), va);
    mi vp = _mm256_madd_epi16(va, vb);
    acc = _mm256_add_epi64(_mm256_unpacklo_epi32(vp, zero),
      _mm256_add_epi64(acc, _mm256_unpackhi_epi32(vp, zero)));
  }
  union {ll v[4]; mi m;} u; u.m = acc; rep(i,0,4) r += u.v[i];
  for (;i<n;++i) if (a[i] < b[i]) r += a[i]*b[i]; // <- equiv
  return r;
}
```

11.6 Kitchen Kit

**KitchenKit.h**  
**Description:** Tricks for the contest  
**Time:** no time

```
<bits/stdc++.h>
line of vectors c++:
using namespace std;
```

```
typedef vector<int> vi;
typedef pair<int,int> pii;
typedef vector<pii> vii;
typedef map<string, int> msi;
#define INF 1000000000
getline(std::cin, line);
istringstream iss(line);
```

KitchenKit FastReader UndefinedInputJava

```
int num;

while (iss >> num) {
    numbers.push_back(num);
}
```

**FastReader.h**  
**Description:** FastReader para inputs en java  
**Time:** no time

```
GNU nano 6.2
```

```
KitchenKit.h

static class FastReader {
    BufferedReader br;
    StringTokenizer st;
```

```
public FastReader() {
    br = new BufferedReader(
        new InputStreamReader(System.in));
}
```

```
String next() {
    while (st == null || !st.hasMoreElements()) {
        try {
            st = new StringTokenizer(br.readLine());
        } catch (IOException e) {
            e.printStackTrace();
        }
    }
    return st.nextToken();
}
```

```
int nextInt() { // 10^9
    return Integer.parseInt(next());
}
```

```
long nextLong() { // 10^18
    return Long.parseLong(next());
}
```

```
double nextDouble() {
    return Double.parseDouble(next());
}
```

```
BigInteger nextBigInteger() {
    return new BigInteger(next());
}
```

```
String nextLine() {
    String str = "";
    try {
        if (st != null && st.hasMoreTokens()) {
            str = st.nextToken("\n");
        } else {
            str = br.readLine();
        }
    } catch (IOException e) {
        e.printStackTrace();
    }
    return str;
}
```

```
Map.Entry<Boolean, Optional<String>> hasNext() {
    try {
        String line = br.readLine();
        return Map.entry(line != null, Optional.
            ofNullable(line));
    }
```

```
    } catch (IOException e) {
        e.printStackTrace();
        return Map.entry(false, Optional.of(""));
    }
}

void close() {
    try {
        br.close();
    } catch (IOException e) {
        e.printStackTrace();
    }
}
```

**UndefinedInputJava.java** 27 lines

```
/**
 * Author: Alberto Dekeno
 * License: CC0
 * Source: Codeforces
 * Description: Para numero de entradas indeterminadas
 * Time: no time
 * Status: no tested
 */
```

```
boolean hayEntrada = true;
while (hayEntrada) {
    Map.Entry<Boolean, Optional<String>> hasNextObject
        = fastReader.hasNext();
    hayEntrada = hasNextObject.getKey();
    if (hayEntrada) {
        //Haciendo hasNextObject.getValue() obtenemos el valor que
        consumio el hasNext
        List<Integer> list = Arrays.stream(
            hasNextObject.getValue().get().split(" "))
            .map(Integer::parseInt).collect(
                Collectors.toList());
    }
    fastReader.close();
}
```

```
while (fastReader.hasNext().getKey()) {
    //En este caso lo consumido por hasNext no nos
    interesa
    List<Integer> list = Arrays.stream(fastReader.
        nextLine().split(" "))
        .map(Integer::parseInt).collect(
            Collectors.toList());
}
fastReader.close();
```

Techniques (A)

techniques.txt	159 lines
Recursion	
Divide and conquer	
Finding interesting points in N log N	
Algorithm analysis	
Master theorem	
Amortized time complexity	
Greedy algorithm	
Scheduling	
Max contiguous subvector sum	
Invariants	
Huffman encoding	
Graph theory	
Dynamic graphs (extra book-keeping)	
Breadth first search	
Depth first search	
* Normal trees / DFS trees	
Dijkstra's algorithm	
MST: Prim's algorithm	
Bellman-Ford	
Konig's theorem and vertex cover	
Min-cost max flow	
Lovasz toggle	
Matrix tree theorem	
Maximal matching, general graphs	
Hopcroft-Karp	
Hall's marriage theorem	
Graphical sequences	
Floyd-Warshall	
Euler cycles	
Flow networks	
* Augmenting paths	
* Edmonds-Karp	
Bipartite matching	
Min. path cover	
Topological sorting	
Strongly connected components	
2-SAT	
Cut vertices, cut-edges and biconnected components	
Edge coloring	
* Trees	
Vertex coloring	
* Bipartite graphs (=> trees)	
* 3^n (special case of set cover)	
Diameter and centroid	
K'th shortest path	
Shortest cycle	
Dynamic programming	
Knapsack	
Coin change	
Longest common subsequence	
Longest increasing subsequence	
Number of paths in a dag	
Shortest path in a dag	
Dynprog over intervals	
Dynprog over subsets	
Dynprog over probabilities	
Dynprog over trees	
3^n set cover	
Divide and conquer	
Knuth optimization	
Convex hull optimizations	
RMQ (sparse table a.k.a 2^k-jumps)	
Bitonic cycle	
Log partitioning (loop over most restricted)	
Combinatorics	

Computation of binomial coefficients
Pigeon-hole principle
Inclusion/exclusion
Catalan number
Pick's theorem
Number theory
Integer parts
Divisibility
Euclidean algorithm
Modular arithmetic
* Modular multiplication
* Modular inverses
* Modular exponentiation by squaring
Chinese remainder theorem
Fermat's little theorem
Euler's theorem
Phi function
Frobenius number
Quadratic reciprocity
Pollard-Rho
Miller-Rabin
Hensel lifting
Vieta root jumping
Game theory
Combinatorial games
Game trees
Mini-max
Nim
Games on graphs
Games on graphs with loops
Grundy numbers
Bipartite games without repetition
General games without repetition
Alpha-beta pruning
Probability theory
Optimization
Binary search
Ternary search
Unimodality and convex functions
Binary search on derivative
Numerical methods
Numeric integration
Newton's method
Root-finding with binary/ternary search
Golden section search
Matrices
Gaussian elimination
Exponentiation by squaring
Sorting
Radix sort
Geometry
Coordinates and vectors
* Cross product
* Scalar product
Convex hull
Polygon cut
Closest pair
Coordinate-compression
Quadtrees
KD-trees
All segment-segment intersection
Sweeping
Discretization (convert to events and sweep)
Angle sweeping
Line sweeping
Discrete second derivatives
Strings
Longest common substring
Palindrome subsequences

Knuth-Morris-Pratt
Tries
Rolling polynomial hashes
Suffix array
Suffix tree
Aho-Corasick
Manacher's algorithm
Letter position lists
Combinatorial search
Meet in the middle
Brute-force with pruning
Best-first (A*)
Bidirectional search
Iterative deepening DFS / A*
Data structures
LCA (2^k-jumps in trees in general)
Pull/push-technique on trees
Heavy-light decomposition
Centroid decomposition
Lazy propagation
Self-balancing trees
Convex hull trick (wcipeg.com/wiki/Convex_hull_trick)
Monotone queues / monotone stacks / sliding queues
Sliding queue using 2 stacks
Persistent segment tree