Discrete Mathematics Notes [Week 1]

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Date 13/6/23

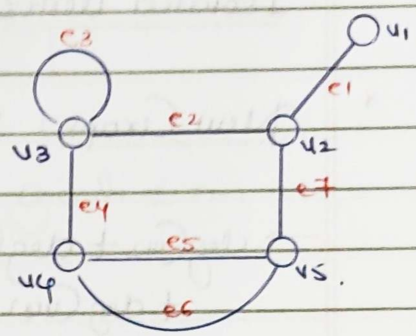
Graph Theory.

Basics of Graph:

* Point/joint/node → vertex
* line/arc/branch → edge

→ Graph 'G' = (V, E)

Set of vertices, Set of edges.



$V = \{v_1, v_2, v_3, v_4, v_5\}$

$E = \{e_1, e_2 \dots e_7\}$. * Each edge must be associated with unordered pair of vertices.

$e_1 \rightarrow (v_1, v_2) \mid (v_2, v_1)$

$G = (V, E, \psi)$

$V = \{\dots\}$

$E = \{\dots\}$ → Each edge must be associated with unordered pair of vertices.

$\psi : E \rightarrow V \times V$.

* **Incident Point:** Meeting point of vertex and edge
* **End vertices:** each edge is associated with unordered pair of vertices called as end vertices.
* **Loop/self-loop:** if end vertices are Same, edge is called loop.
* **Parallel edges:** two or more edges associated with Same end vertices.

  eg: $e_5 \rightarrow (v_4, v_5)$

  $e_6 \rightarrow (v_4, v_5)$

* **Degree/valency** $(d(v_i))$: no. of edges incident with vertex.

* **Pendant vertex**: degree one vertex is called pendent vertex.

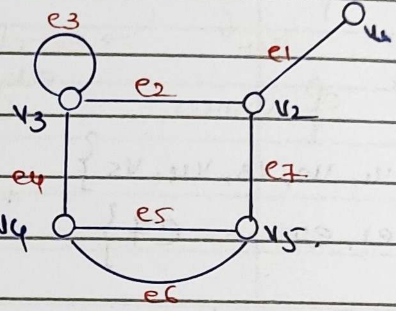
* **Isolated vertex**: degree '0' vertex is called isolated vertex.

* **Null Graph**: Set of isolated vertices.

→ $deg(v_1) + deg(v_2) + deg(v_3) + deg(v_4) + deg(v_5)$.

$= 1+3+4+3+3$

$= 14 = \underbrace{2 \times 7}_{\text{no. of edges}}$

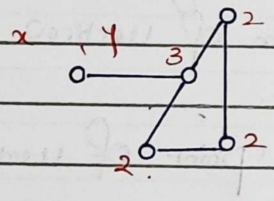


[Theorem1] * Sum of degrees of all vertices is equal to twice the no of edges.

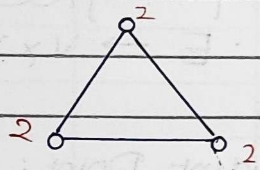
[Theorem5] * Sum of degrees of all vertices will be even.

[Theorem2]: No. of odd degree vertices in a graph will always be even.

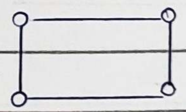
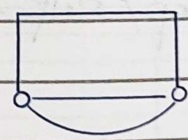
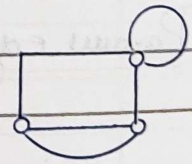
no. of odd degree vertices = 2.



— " — = 2



— " — = 0

	loop.	'll' edges.	
Simple Graph	×	×	
Multi Graph	×	✓	
Pseudo Graph	✓	✓	

[Theorem 3]: Maximum no. of edges in Simple Graph $\leq \frac{n(n-1)}{2}$

classmate

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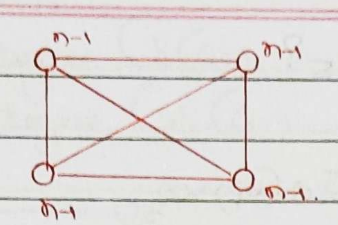
$n=4$



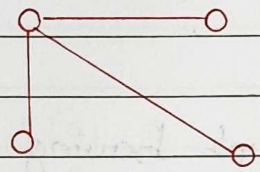
$\Sigma d(v_i) = 22$

$n \times (n-1) = 22$

$$\boxed{e = \frac{n(n-1)}{2}}$$

[Theorem 4]: Maximum degree in Simple graph $\leq n-1$.

$n=4$



Note: if degrees of all vertices are $n-1$, then it will have exactly $\frac{n(n-1)}{2}$ edges.

eg. $n=4$

Maximum edges $= \frac{4.3}{2} = 6$

Total no. of graphs $= 2^6 = 2^{\frac{4.3}{2}}$

n: total vertices

$\rightarrow$ Total no. of Graphs $= 2^{\frac{n(n-1)}{2}}$

Q1. How many graphs are possible with 4 vertices and 1 edge?

Q2. How many graphs possible with n vertices and 'e' edges.

$\rightarrow$ ${}^{\frac{n(n-1)}{2}}C_e$.

Q3. How many graphs are possible with 4 vertices and atleast 2 edges

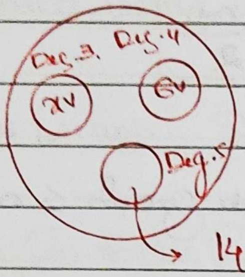
Ans. $m_1 \rightarrow {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_6$

$m_2 \rightarrow 2^6 - {}^6C_0 - {}^6C_1$.

Q4. A certain graph as 'G' has order 14 and size 27. The degree of each vertex of G is 3, 4 or 5. There are 6 vertices of degree 4. How many vertices of G have degree 3 and how

how many have degree 5?

Ans. $n=14$, $E=27$.



$\sum d(v_i) = 2e$

$6\times4 + x\times3 + (8-x)\times5 = 2.27$.

$\underline{x=5}$

$14-6-x$

$(8-x)v$

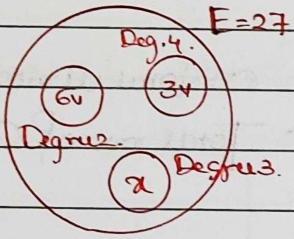
eg. Consider a graph having 27 edges

6v → Degree → 2

3v → Degree 4.

Remaining vertices → Degree 3

Soln:



$\sum d(v_i) = 2e$

$6\times2 + 3\times4 + x\times3 = 2.27$

$12+12+3x=54$

$3x = 54-24 = 30$

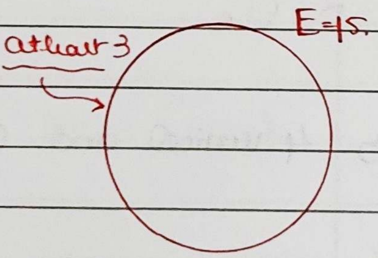
$3x=30$

$\underline{x=10}$

Total vertices $= 6+3+10$

$= \underline{19}$.

eg: Consider a graph having 15 edges, degree of each vertex is atleast 3. What will be the number of vertices?



$\delta(G)=3 \quad n=?$

$\delta(G) \le \frac{2e}{n} \le \Delta(G) \le n-1$.

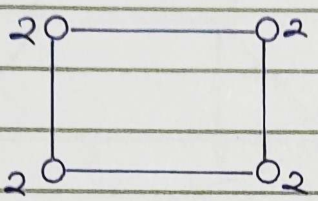
$\delta(G) \le \frac{2e}{n} \Rightarrow 3 \le \frac{2.15}{n} \Rightarrow \underline{n \le 10}$.

Maximum no. of vertices = $\underline{10}$.

Minimum degree ($\delta(G)$)
Maximum degree ($\Delta(G)$)

Case I:

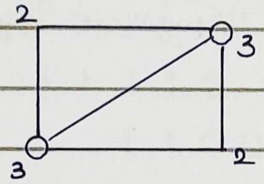
$$\boxed{\delta(G) = \frac{2e}{n} = \Delta(G)}$$



$\delta(G) = 2$
$\Delta(G) = 2$
$\frac{2e}{n} = 2$

avg. degree $= \frac{2+2+2+2}{\text{Total vertices}} = \frac{\sum d(v_i)}{n}$
$= 2$.

eg::



$\delta(G) = 2$
$\Delta(G) = 3$

Case II:

$$\boxed{\delta(G) < \frac{2e}{n} < \Delta(G)}$$

Theorems.

$$\boxed{\delta(G) \leq 2e \leq \Delta(G) \leq n-1}$$

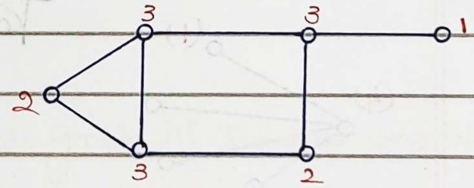
eg:: G is Graph 25 degrees, each vertex is having degree atleast 3. Maximum value of n is 16.

$E = 25 \quad \delta(G) = 3$

$\delta(G) \leq \frac{2e}{n} = 3 \leq \frac{2 \times 25}{n} \Rightarrow n \leq \frac{50}{3} \quad \therefore n = 16$

Degree Sequence:

Writing degrees of all vertices either in increasing or decreasing order is called "Degree Sequence".



→ 3, 3, 3, 2, 2, 1 Or 1, 2, 2, 3, 3, 3.

eg: What will be the no. of edges in DS 5, 2, 2, 2, 1?

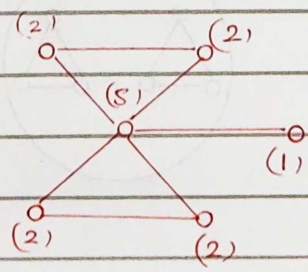
m1: $\sum d(v_i) = 2e$

$5+2+2+2+2+1$

$14 = 2e$

$e = 7$

m2:



$e = 7$

Q2. What will be edges in 2.s of 3,3,3,1?

m1:

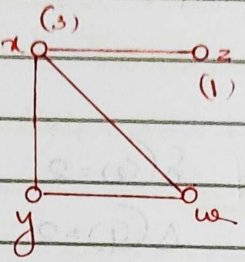
$\sum d(u_i) = 2e$

$3+3+3+1 = 2e$

$10 = 2e$

$e = 5$

m2:



$\boxed{\text{Degree Sequence} \longrightarrow \text{Simple Graph}}$

→ Graphical Sequence

5,2,2,2,2,1 → Graphical Sequence

3,3,3,1 → No Simple Graph.

Graphical?

(a) 5, 4, 3, 2, 1

Not Graphical

Reason 1: Total vertices = 5, n=5

$\Delta(G) \leq n-1$

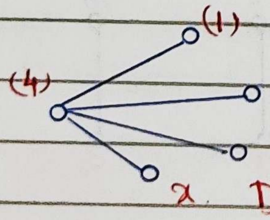
$\Delta(G) \leq 4$.

Reason 2: Theorem 2

5, 4, 3, 2, 1.

(b) 4,4, 3, 2, 1. Not Graphical.

Total vertices = 5



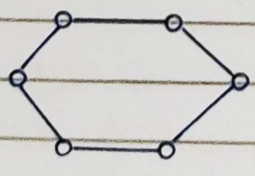
Demand : 3.

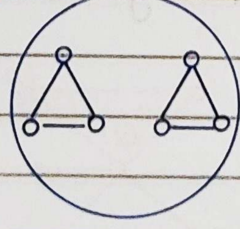
Total vertices = n

→ n-1, n-1, ... 1.

→ not graphical (2. (n-1) and a pendant cannot be graphical).

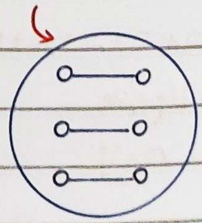
(c) 2, 2, 2, 2, 2, 2. Graphical





(d) 1, 1, 1, 1, 1, 1.

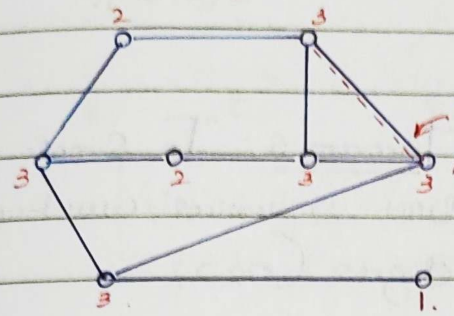
Graphical



Hauel-Hakimi

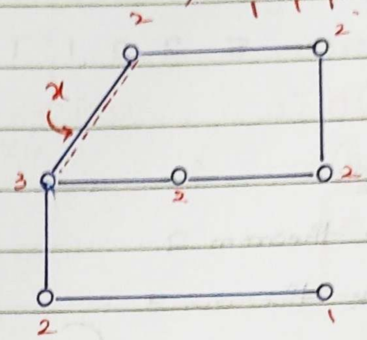
$\cancel{3}\ 3\ 3\ 3\ 3\ 2\ 2\ 1$

2 2 2



$\cancel{3}\ 2\ 2\ 2\ 2\ 2\ 2\ 1$

1 1 1



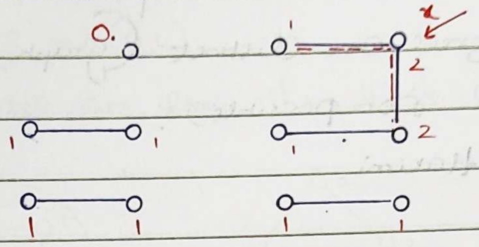
x

(removing that vertex).

$\cancel{2}\ 2\ 1\ 1\ 1\ 1$

$\cancel{1}\ 1\ 1\ 1\ 0$

$1\ 1\ 0\ 0$



$\cancel{3},\ \underset{2}{3}\ \underset{2}{3}\ \underset{2}{3}\ 3\ \ 2\ \ 2\ \ 1.$

$\cancel{3}\ \underset{1}{2}\ \underset{1}{2}\ \underset{1}{2}\ 2\ 2\ 2\ 1.$ (ordering)

$\cancel{2}\ \underset{1}{2}\ \underset{0}{1}\ 1\ 1\ 1.$ (ordering)

$\cancel{1}\ \underset{0}{1}\ 1\ 1\ 0$ (ordering)

$1\ 1\ 0\ 0$

↳ if this can be a simple graph, it's previous ones also can.

eg. Gate PYQ:

a. 7 6 5 (4 4) 3 2 1 ✓

b. 6 6 6 6 (3 3) 2 2 ×

c. 7 (6 6) (4 4) 3 (2 2) ✓

d. 8 (7 7) 6 4 2 (1 1) ×

(a) $\cancel{7}\ 6\ 5\ 4\ 4\ 3\ 2\ 1$

$\cancel{6}\ 4\ 3\ 3\ 2\ 1\ 0$

$\cancel{3}\ 2\ 2\ 1\ 0\ 0$

$1\ 1\ 0\ 0\ 0.$ ✓

(b). $\cancel{6}\ 6\ 6\ 6\ 3\ 3\ 2\ 2$

$\cancel{5}\ 5\ 5\ 2\ 2\ 2\ 1$

→ $\cancel{4}\ 4\ 1\ 1\ 1\ 1$

$3\ 0\ 0\ 0\ 1.$ } no Simple Graph

(c). $\cancel{7}$ 6 6 4 4 3 2 2
$\cancel{5}$ 5 3 3 2 1 1
5 2 2 1 1 0 ×

(d) 8 7 7 6 4 2 1 1 ×
$n = 8$
$\Delta(G) \le 7$.

Steps:
1. Check theorem 2
2. Check theorem 3.
3. $n-1, n-1 \dots 1$ (not possible)
4. All degrees are distinct (Graph not possible).
5. Havel Hakimi.

Theorem 6: In simple graph at least 2 vertices will have same degree ($n \ge 2$).

Q1. $0 \le x \le 5$
$x, 1, 2, 3, 5, 5$.

for which value of x is it graphical?

5, 5, 3, 2, 1, x.
$n-1$, $n-1$, ... 1.
Total vertices = 6.
} not graphical

∴ for no value it will be graphical.

Q2. $0 \le x \le 7$.
7 6 5 4 3 2 1 x.

for which value of x is it graphical?

x cannot be odd.

$x = 0$ 7 6 5 4 3 2 1 0 × (all distinct)

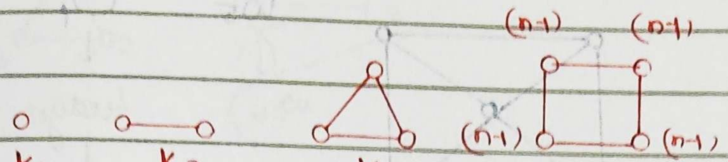
$x = 2$. $\cancel{7}$ 6 5 4 3 2 2 1
$\cancel{5}$ 4 3 2 1 1 0
3 2 1 0 0 0. × (not possible)

$x = 4$ $\cancel{7}$ 6 5 4 4 3 2 1
$\cancel{5}$ 4 3 3 2 1 0
$\cancel{3}$ 2 2 1 0 0
1 1 0 0 0. } Graphical ✓

∴ for $x = 4$ it is graphical

H.W {
$x, 7, 7, 5, 5, 4, 3, 2$
$0 \le x \le 7$
value for x for which it is graphical.
}

Complete Graph : (K_n). ($n \geq 1$)



K_1 K_2 K_3 K_4

* Degree of all vertices are $n-1$.

$\sum d(v) = 2e$

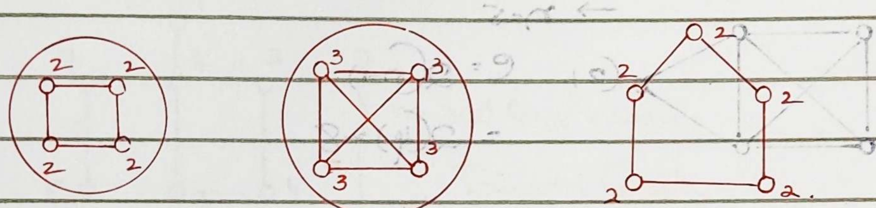
$n(n-1) = 2e$

$$\boxed{e = \frac{n(n-1)}{2}}$$

$$\boxed{\delta(G) = \frac{2e}{n} = \Delta(G) = n-1}$$

Regular Graph ($\delta(G) = \frac{2e}{n} = \Delta(G)$).

→ If degrees of all vertices are same then it is called Regular Graph.



2-Regular 3-Regular. 2-Regular.

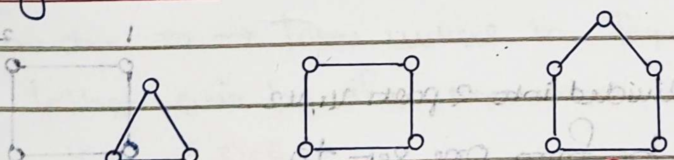
$\delta(G) = \frac{2e}{n} = \Delta(G) = k$

K-Regular Graph

Point about Complete Graph :

* $n-1$ Regular Graph is a Complete Graph
* all K_n are Regular Graph. (True)
* all Regular Graph are K_n. (False).

Cycle Graph : (C_n) ($n \geq 3$)



C_3 C_4 C_5.

$n = e = 3$ $n = 4 = e$ $n = e = 5$.

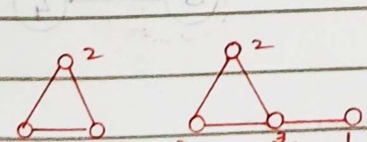
* Degree of all vertices are 2.

$\sum d(v_i) = 2e$

$n \cdot 2 = 2 \cdot e$

* $\underline{n = e}$.

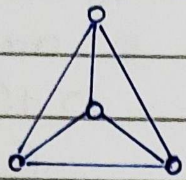
* All Cycle graph are Regular graph
* All regular graph are Cycle graph.



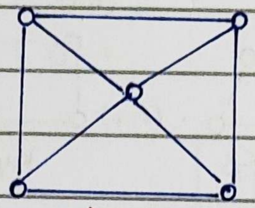
Graph containing cycle

* if Graph is $C_n \to n = e$ (True)
* if $n = e \to$ then it is cycle graph (False)

Wheel Graph (W_n) ($n \geq 4$)



W4.



W5.

W5 C4
↓ ↓
W_n. 4 edges + 4 edges
↓ ↓
n-1 edges + n-1 edges

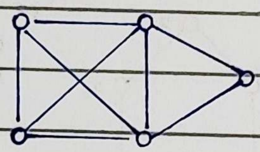
$e(W_n) = n-1 + n-1$

$$\boxed{e(W_n) = 2(n-1)}$$

→ if G is W_n → $e = 2(n-1)$ (True)

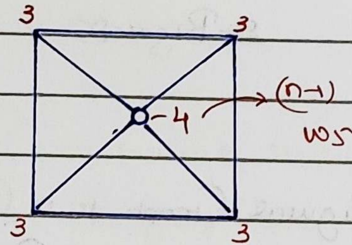
→ if G is having $e = 2(n-1)$ → G is wheel graph. (False)

eg:



→ n=5
$e = 2(n-1)$
$= 2(4) = \underline{8}$

eg::



W5
4, 3, 3, 3, 3

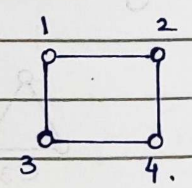
W6
5, $\underbrace{3, 3, 3, 3, 3}_{\text{5 vertices}}$

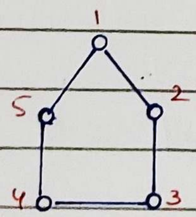
Properties of wheel graph
- all degrees are 3 (corners)
- middle vertex has degree (n-1).

Bipartite Graph.

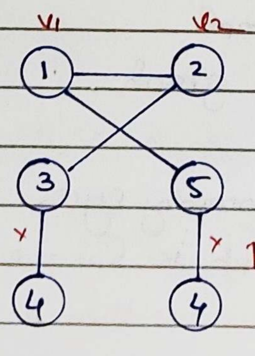
G = (V, E)

* Vertices can be divided into 2 parts V1, V2.
* Each edge must be from one set to another set but not in same set.

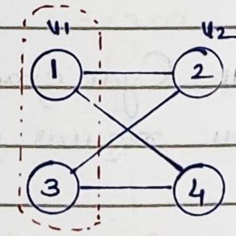




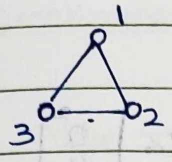
Not Bipartite Graph.

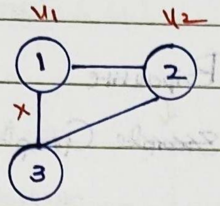


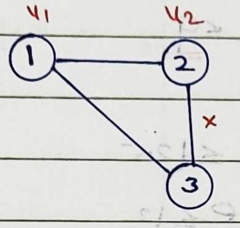
Because 4 is connected to both 3 & 5 so cannot do partition.



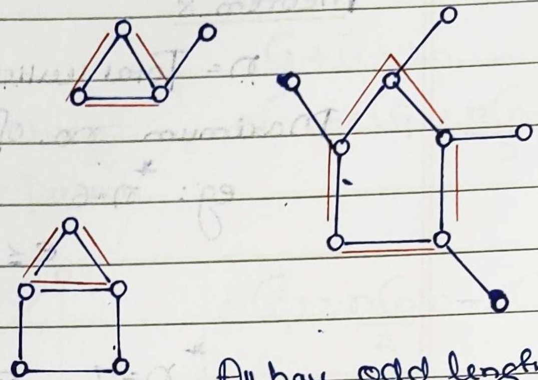
← Bipartite Graph.







Not Bipartite Graph.

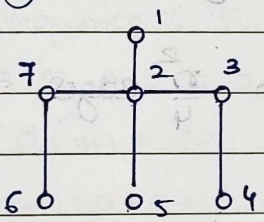


All have odd length cycle. so not Bipartite Graph.

Theorem 7.

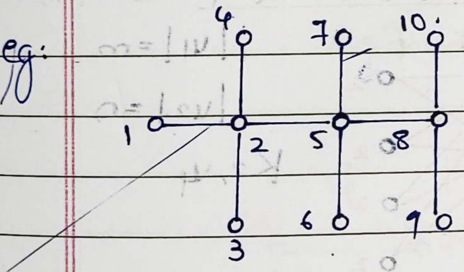
Bipartite Graph does not contain odd length cycle.

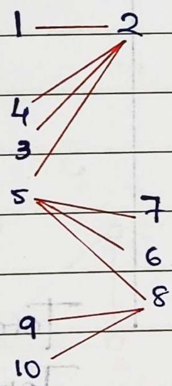
eg:



no odd length cycle ∴ Bipartite Graph

eg:





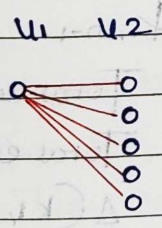
Bipartite Graph.

* Consider $n \rightarrow$ Total vertices in Bipartite Graph

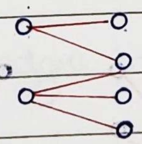
What will be the maximum no. of edges?

$n=6$.

Case 1:

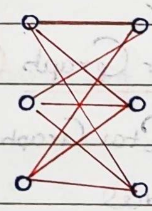


Case 2:



$e=5$.

Case 3:



$e=9$

Theorem 8

n = Total vertices

Maximum no. of edges in ~~Simple~~ Bipartite Graph $e \le \left[\frac{n^2}{4}\right]$

eg: * $n = 6$

$e \le \frac{6^2}{4} = e \le 9$

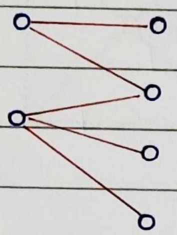
* $n = 7$ $e \le \frac{7^2}{4} = e \le 12.5$

$e \le 12$

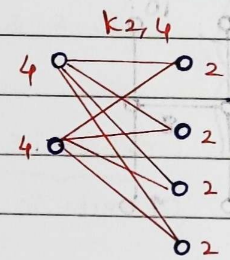
→ Maximum no. of edges in bipartite graph $e \le \frac{n^2}{4}$.

→ If bipartite graph contains more than $\frac{n^2}{4}$ edges then it contains odd length cycle.

Bipartite Graph



Complete Bipartite Graph ($K_{m,n}$)



$|V_1| = m$

$|V_2| = n$

$K_{2,4}$

Total vertices $= 2+4 = 6$ $V = m+n$

Total edges $= 2 \times 4 = 8$ $E = m \cdot n$

$\Delta(K_{2,4}) = 4$ $\Delta(K_{m,n}) = \max(m,n)$

$\delta(K_{2,4}) = 2$ $\delta(K_{m,n}) = \min(m,n)$

Complete Bipartite Graph

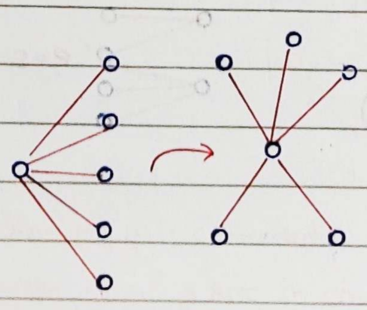
↓

Star Graph ($K_{1,n-1}$)

Draw Star Graph of 6 vertices

$n = 6$

$K_{1,5}$.



$K_{1,n-1}$

Total vertices $= n$

Total edges $= n-1$

$\Delta(K_{1,n-1}) = n-1$

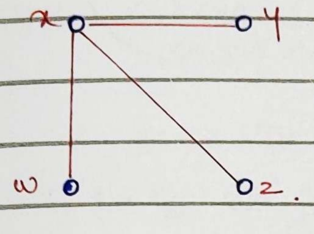
$\delta(K_{1,n-1}) = 1$.

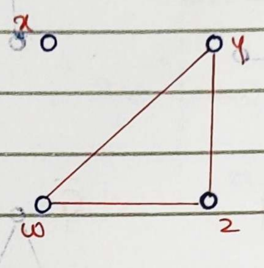
Compliment Graph ($\bar{G}$)

$G + \bar{G} = K_n$

$e(G) + e(\bar{G}) = \frac{n(n-1)}{2}$

G	$\bar{G}$
edge → present	edge → absent.
edge → absent	edge → Present.





$$\boxed{e(\bar{G}) = \frac{n(n-1)}{2} - e(G)}$$

Q1. What will be edges in the compliment of the graph having degree sequence : 5, 2, 2, 2, 2, 1 ?

Ans. $G \to 5, 2, 2, 2, 2, 1$

$\Sigma d(v_i) = 2e$

$5+2+2+2+2+1 = 2e$

$14 = 2e$

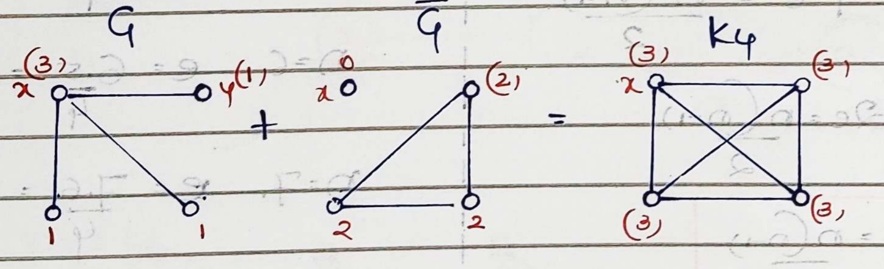
$e = 7.$

$e(\bar{G}) = ?$ $e(G) = 7$

$\Rightarrow e(\bar{G}) + e(G) = \frac{n(n-1)}{2}$ Total vertices $= 6$.

$\Rightarrow 7 + e(\bar{G}) = \frac{6 \cdot 5}{2}$

$\Rightarrow e(\bar{G}) = 15 - 7 = 8$



K_4	3	3	3	3
G	3	1	1	1
$\bar{G}$	0	2	2	2

$n = 4.$

General : Note:

→ K_n	$n-1$	$n-1$	$n-1$	....	$n-1$
G	$d_1,$	$d_2,$	d_3	....	d_n
$\bar{G}$	$n-1-d_1,$	$n-1-d_2$		....	$n-1-d_n$.

eg:: What will be no. of edges in the compliment of the graph having De 5, 2, 2, 2, 2, 1.

→ Total vertices = 6.

K_6	5	5	5	5	5	5
G	5	2	2	2	2	1
$\bar{G}$	0,	3,	3,	3,	3,	4

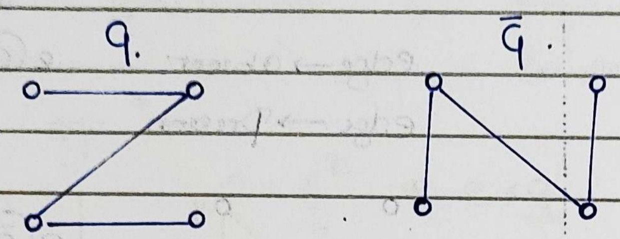
$\Sigma d(v_i) = 2e$

$0+3+3+3+3+4 = 2e$

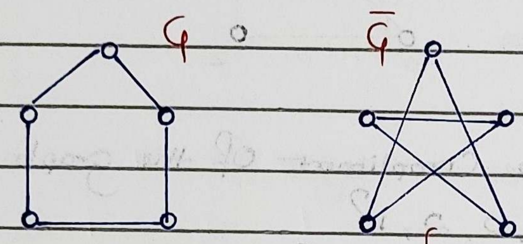
$e = 8$

classmate
Date 18/6/23
Page

Self-Compliment Graph ($G \equiv \bar{G}$)



* Graph is Same as it's Own Compliment.



{ C_5
$n = e = 5$

{ $n = 5 = e$

→ G is Same as $\bar{G}$

$G + \bar{G} = K_n$

$e(G) + e(\bar{G}) = \frac{n(n-1)}{2}$

$e + e = \frac{n(n-1)}{2}$

$2e = \frac{n(n-1)}{2}$

$$\boxed{e = \frac{n(n-1)}{4}}$$

$n = 4 \quad e = \frac{n(n-1)}{4} = \frac{4.3}{4} = 3$ ✓

$n = 5 \quad e = \frac{n(n-1)}{4} = \frac{5.4}{4} = 5$ ✓

$n = 6 \quad e = \frac{6.5}{4} = \frac{15}{2} = 7.5$ ✗

$n = 7. \quad e = \frac{7.6}{4} = \frac{42}{4}$ ✗ Should be divisible by 4.

→ $\underbrace{\frac{n}{4} \text{ or } \frac{n-1}{4}}_{\text{divisible by } 4.}$ then we Get self compliment graph.

→ $a \equiv b \pmod{n}$

↳ $a \% n == b \% n$.

1 and 5 are having Same remainder with respect to 4.

↳ $1 \equiv 5 \pmod{4}$

Or

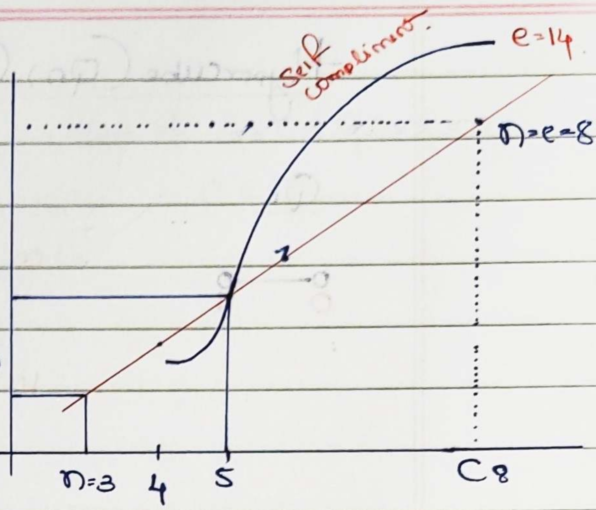
$a \equiv b \pmod{n} = \frac{a-b}{n} \in \mathbb{Z}$.

Cycle graph.

Special. ← C4. C5. C6. C7. C8 C9.... → Selfcompliment e=5 cycle graph.

cycle graph = 8

Self compliment 14.



** Only for C5 we get self compliment graph. ($n=5$).

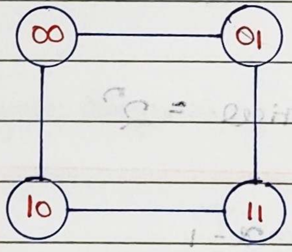
Q1. Graph vertices are represented as a n-bit signal, and 2 vertices are connected, if their bit position changes by 1 bit what will be the total edges?

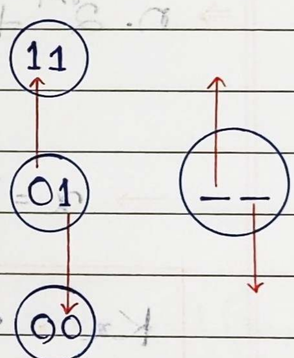
Solⁿ:

$n = 2$ bit

Total vertices $= 2^2$.

00
01
10
11



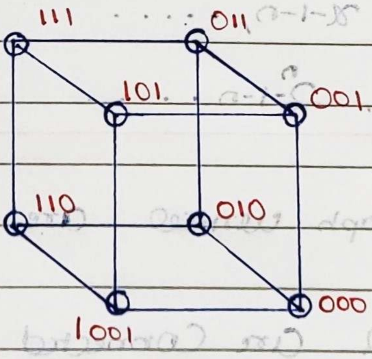


Degree of each vertex will be 2.

$n = 3$ bit.

Total vertices $= 2^3$

000
001
⋮
111



Degree of each vertex will be 3.

→ n → bit signal

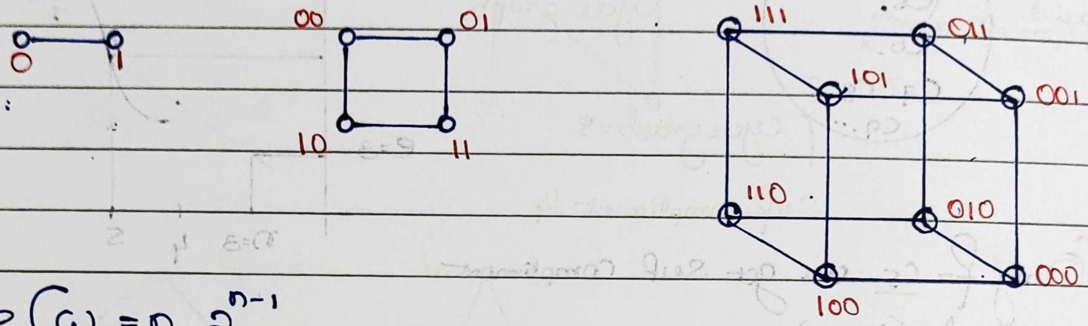
Total vertices $(x) = 2^n$ → $\sum d(v_i) = 2e$

Degree of each vertex $= n$ → $2^n \cdot n = 2e$.

$$\boxed{e = n \cdot 2^{n-1}}$$

Hypercube (Q_n) ($n \geq 1$)

Q_1 $\quad$ Q_2 $\quad$ Q_3



→ $e(G) = n \cdot 2^{n-1}$

$=$ Total vertices $= x = 2^n$

$=$ $e(G) + e(\bar{G}) = Kx$ $\qquad$ $e(\bar{G}) = \dfrac{2^n(2^n-1)}{2} - n \cdot 2^{n-1}$

$=$ $n \cdot 2^{n-1} + e(\bar{G}) = \dfrac{x(x-1)}{2}$

$\Rightarrow n \cdot 2^{n-1} + e(\bar{G}) = \dfrac{2^n(2^n-1)}{2}$

→ $x =$ Total vertices $= 2^n$

$Kx \rightarrow x-1 \quad x-1 \quad x-1$

$G \rightarrow n, \quad n, \quad n \ldots n$

$\bar{G} \rightarrow x-1-n, \; x-1-n, \; \ldots\ldots$

$2^n-1-n, \; 2^n-1-n \ldots\ldots$

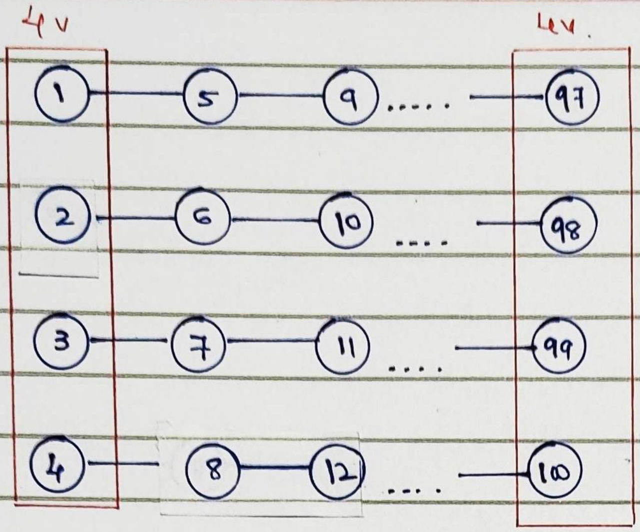
Q2. Consider a graph vertices are represented as a no. from $\{1 \ldots 100\}$

two vertices are connected $|a-b| = 4$ eg $|1-5| = 4$ or $|5-9| = 4$

their difference is 4. like $|1-5| = 4$.

What will be the total edge in this?

Ans :

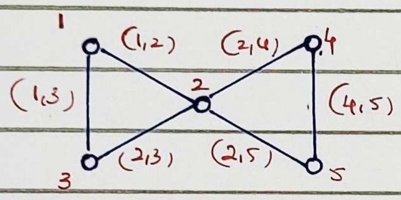


$4\times1 + 4\times1 + 92\times2 = 2e$

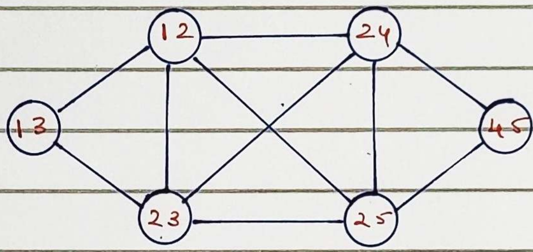
$e = 96.$

Line Graph - L(G)

Line Graph → edge graph.

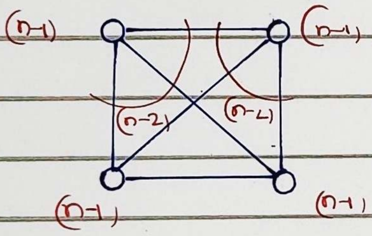


G



L(G).

Q3. What will be the degree sequence of $L(K_n)$?



Degree of each vertex in $L(K_n)$ is $2(n-2)$.