INTRODUCTION

- Probability can be defined as: (layman's def^n)

$$\frac{\#\text{favourable cases}}{\#\text{total cases}}$$

- Sample Space: Every probablistic model involves an underlying process, called the experiment which produces exactly one of several outcomes. (Ω)

The set of all possible outcomes is called the Sample Space of the experiment.

Ex. Experiment : Coin toss
Sample Space : $\{H, T\}$

- Event : Event is a set of possible outcomes out of the sample space, such that
Event $\subseteq$ Sample Space.

Ex. Exp: Rolling a dice.
Sample Space: $\{1, 2, 3, 4, 5, 6\}$
Event:
- Prime outcome : $\{2, 3, 5\}$
- Outcome div. by 3: $\{3, 6\}$
- Outcome ≥ 4: $\{4, 5, 6\}$
- Outcome < 1: $\{\}$ or $\emptyset$

$\therefore$ #possible ~~outcomes~~ events $= 2^6$

We are generally interested in probability of an event.

So, Probability is a func. that maps events to the $\mathbb{R}$ value in range $[0,1]$.

$$P: 2^{\Omega} \rightarrow [0,1]$$

So, whenever we write $P(X)$, X must be an event.

* Properties of Sample Space (Ω):

- It is list of all possible outcomes
- Each outcome has be to be mutually exclusive w/ all other outcomes.
- Sample space has to be collectively exhaustive set of outcomes.

Ex. Say 7 horses $H_1, H_2 \ldots H_7$ run in a race.

(1) Order in which horses finish:
Ω : all 7! permutation of Horses
(2) Which horse came first:
Ω : all 7 horses.
(3) Hores at 1st & 2nd positions:
Ω : 7×6 possibilities.

So, sample space depends on the desired outcome of the experiment as well.

* Axioms

- Non-negativity: $P(E) \geq 0$
- Normalisation: $P(\Omega) = 1$
- Exhaustivity: If two events A & B are such that

$$A \cap B = \phi$$

then

$$P(A \cup B) = P(A) + P(B)$$

We can extend it as such:

$\Omega = \{ S_1, S_2, S_3, \ldots S \}$

$\Rightarrow P(S_1 \cup S_2 \cup S_3 \ldots) = P(S_1) + P(S_2) + P(S_3) + \ldots + P(S_n)$

* If $\Omega = \{S_1, S_2, S_3, \ldots S_n\}$ & $P(S_1) = P(S_2) = P(S_3) \ldots = P(S_n)$

then for some event $E = \{S_K\}$

$$P(E) = \frac{\text{No. of outcomes in } E}{\text{Total No. of outcomes}}$$

&

$P(S_1) + P(S_2) + \ldots + P(S_n) = P(\Omega)$

$\Rightarrow n \times P(S_1) = 1$

$\Rightarrow P(S_1) = \frac{1}{n}$

Should be written as $P(\{S_1\})$ but abuse of notation

Ex. Two dices rolled. What is the prob. that sum of upturned faces are 7?

possible outcomes = $|\Omega| = 36$

favourable outcomes = $|\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}|$

$\Rightarrow 6$

$\therefore P(Sum = 7) = \frac{6}{36} = \frac{1}{6}$

Ex. Following is probability mass distribution for a dice roll.

O:	1	2	3	4	5	6
P(O):	1/6	2/6	1/6	1/6	0	1/6

Find $P(\{1, 2, 5\})$

first check if distribution follows axioms or not: $\sum P(O_i) = 1/6 \times 4 + 2/6 = 1$

Checks.

$\therefore P(\{1, 2, 5\}) = P(\{1\}) + P(\{2\}) + P(\{5\})$

(∵ Mutually exclusive)

$\Rightarrow \frac{3}{6} = \frac{1}{2}$

Ex. Sample Space = $\mathbb{N} = \{1, 2, 3, \ldots\}$

$\forall i \in \mathbb{N}, P(\{i\}) = 2^{-i}$

Find P(outcome is even).

- Events are just set of outcomes, so opⁿ valid on sets are valid on events as well.

First checking axioms:

$P(\{i\}) \geq 0$ ✓

$\sum_{i \in \Omega} P(\{i\}) = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$

$\Rightarrow \frac{1/2}{1 - 1/2} = 1.$

∴ Valid distb.

So, P(outcome is even) $= P(2) + P(4) + P(6) + \dots$

$\Rightarrow \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \dots$

$\Rightarrow \frac{1/2^2}{1 - 1/4} = \frac{2^2}{3} \times \frac{1}{4} = \frac{1}{3}$

* Inclusion Exclusion Principle

For two events, E & F, such that $E \cap F \neq \emptyset$

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Similarly,

$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E \cap F) - P(E \cap G) - P(F \cap G) + P(E \cap F \cap G).$$

* De Morgan

- $P(\overline{(E \cup F)}) = P(\bar{E} \cap \bar{F})$
- $P(\overline{(E \cap F)}) = P(\bar{E} \cup \bar{F})$

* Complement $P(\bar{E}) = 1 - P(E)$

* Conditional Probability

"Calculation of probability based on some partial information"

Ex. After typing "I", autocomplete suggests "am" & "will" in that order

This means "given the typing of I, the probability of next word being "am" is higher than "will"."

$$P(\text{am} \mid I) > P(\text{will} \mid I)$$

Ex. Given that after rolling two dice the sum is 9, what is the probability that ~~sum~~ first dice had 6 ?

$$P(\text{First dice } 6 \mid \text{Sum } 9) = \frac{1}{4}$$

(6, 3), (3, 6), (4, 5), (5, 4).

Since we already know sum is 9, we reduce our sample space to only those outcomes which produce known truth (sum 9) & then calc. the probability.

Ex. Family has two children. We're interested in their genders. Our sample space is

$$\Omega = \{(G,G), (G,B), (B,G), (B,B)\}$$

Assume all outcomes are equally likely.

(a) Find prob. of both child girls, if we know 2nd child girl.

$$\Omega' = \{(G,G), (B,G)\}$$

$$P(\text{Both Girls} \mid 2^{nd} \text{ girl}) = \frac{1}{2}$$

(b) Prob. of both child girls, given at least one child is girl.

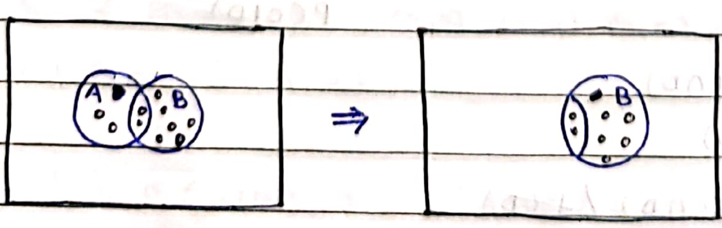
$$\Omega' = \{(G,G), (B,G), (G,B)\}$$

$$P(\text{Both girls} \mid \text{atleast one girl}) = \frac{1}{3}$$

- So we can observe each time, we reduce sample space based on given fact & recalculate the favourable outcomes based on it.

- The conditional probability of an event A, given the happening of event B, and $P(B) > 0$,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$



$P(\circ) = \frac{1}{10}$

$8 \times P(\circ) = \frac{8}{10}$ but $\Sigma P(\circ)$ should be 1

$\Rightarrow 8 \times P(\circ) \times \frac{10}{8} = 1$

$\therefore P(\circ) = 1/8$

* Multiplication Rule

From Cond. Probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow P(A \cap B) = P(A|B)\,P(B)$$

$$\Rightarrow P(B|A)\,P(A)$$

For $P(A \cap B \cap C) \equiv P(A, B, C)$

$$P(A \cap B \cap C) = P(A|B \cap C) \cdot P(B \cap C)$$

$$\Rightarrow P(A \cap B|C) \cdot P(C)$$

$$\Rightarrow P(B|C) \cdot P(A|B \cap C) \cdot P(C)$$

$$\therefore P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B)$$

Similarly,

$$P(A, B, C, D) = P(A) \cdot P(B|A) \cdot P(C|A, B) \cdot P(D|A, B, C)$$

Ex. Check if $P(A, B|C, D) = \dfrac{P((A, B, C)|D)}{P(C|D)}$

LHS: $\dfrac{P(A \cap B \cap C \cap D)}{P(C \cap D)}$

RHS: $\dfrac{P(A \cap B \cap C \cap D) / \cancel{P(D)}}{P(C \cap D) / \cancel{P(D)}}$

$\therefore$ Yes.

Ex. Check if : $P(A,B,C \mid X,Y,Z) = \dfrac{P(A,B,C,X \mid Y,Z)}{P(X \mid Y,Z)}$

RHS: $\dfrac{P(A,B,C,X,Y,Z) \,/\, \cancel{P(Y,Z)}}{P(X,Y,Z) \,/\, \cancel{P(Y,Z)}}$

$\Rightarrow P(A,B,C \mid X,Y,Z)$

Ex. Same prob. can be written in diff. ways:

$$P(A,B,C \mid X,Y,Z) = \frac{P(A,B,C,X,Y,Z)}{P(X,Y,Z)}$$

$$\Rightarrow \frac{P(A,B,C,X \mid Y,Z)}{P(X \mid Y,Z)}$$

$$\Rightarrow \frac{P(A,B,C,Y,Z \mid X)}{P(Y,Z \mid X)}$$

Ex. Two dies are rolled. Let A be event of 3 on first dice.
Let B be event, $\min(D_1, D_2) = 2$.
Find $P(A \mid B)$.

$|\Omega| = 36$

$A = \{(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)\}$

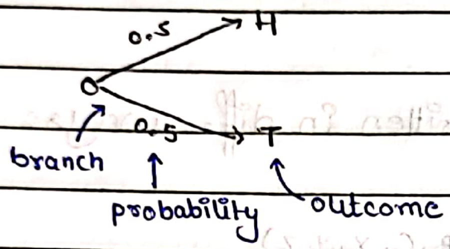
$B = \{(2,2), (2,3), (2,4), (2,5), (2,6), (3,2), (4,2), (5,2), (6,2)\}$

$A \cap B = \{(3,2)\}$

$$\therefore P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{9/36} = \frac{1}{9}$$

* Tree Method (for Conditional Prob).

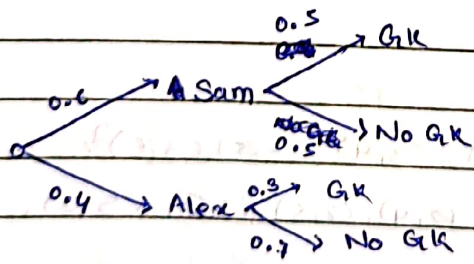
If there is a sequence of events, then a tree can be created for the experiment.



From a node, all the outgoing branches' sum must be 1.
Here 2 branches.
Sum = 0.5 + 0.5 = 1

Ex. You becoming goalkeeper today depends on the coach. Sam makes you goalk. with 0.5 prob & Alex makes you GK with 0.3 prob.
Prob. of Sam being coach today is 0.6.

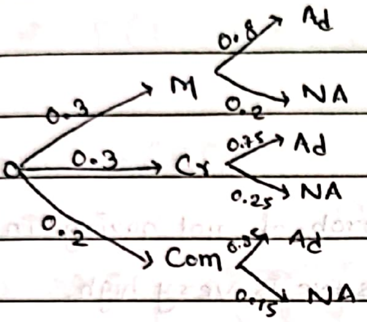
Sequence of events: choose coach today, then makes or not GK.



What is prob. you'll be GK today?
$0.6 \times 0.5 + 0.4 \times 0.3 = 0.3$

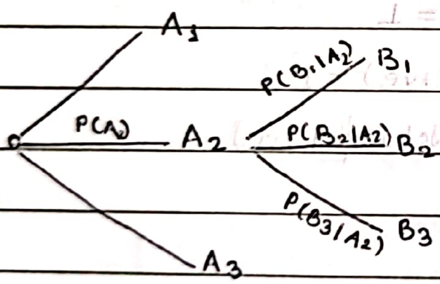
Ex. On a Sunday, 30% watch M, 30% watch Cricket, & 20% watch Comedy. % people skipping ads are 20%, 25% & 15% respectively.

P(watch cricket & see ad)?



$\therefore$ P(watch C & see ad) = 0.3×0.75

$\Rightarrow 0.225$

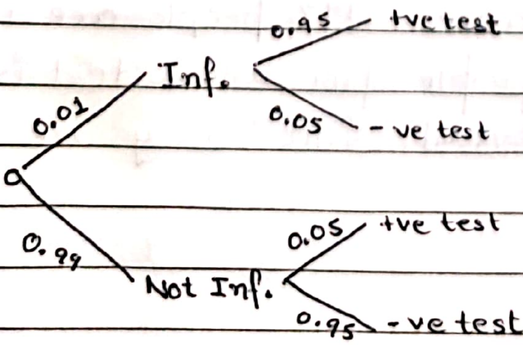


$\Rightarrow P(A_2) \cdot P(B_1|A_2) = P(A_2 \cap B_1)$

$\Rightarrow P(A_2) \cdot P(B_2|A_2) = P(A_2 \cap B_2)$

$\Rightarrow P(A_2) \cdot P(B_3|A_2) = P(A_2 \cap B_3)$

Ex. Suppose a medical diagnosis is 95% accurate, i.e. there's 0.95 prob. that test shows +ve & you're actually infected, and 0.95 prob. test shows −ve & you are not infected. Suppose there are only 1% infected people.



(a) $P(\text{No Inf} \mid +\text{ve test})$

$$\Rightarrow \frac{P(\text{No Inf} \cap +\text{ve test})}{P(+\text{ve test})}$$

$$\Rightarrow \frac{P(+\text{ve test} \mid \text{not inf}) \times P(\text{not inf})}{P(+\text{ve test})}$$

$$\Rightarrow \frac{0.99 \times 0.05}{0.01 \times 0.95 + 0.99 \times 0.05}$$

$$\Rightarrow \frac{99}{118}$$

This means that prob. of not having inf. when test shows +ve is very high.

(b) $P(\text{Inf} \mid +\text{ve test})$

w.k.t

$$P(\text{Inf}) + P(\text{Not Inf}) = 1$$

$$\Rightarrow P(\text{Inf} \mid +\text{ve}) + P(\text{Not Inf} \mid +\text{ve}) = 1$$

$$\Rightarrow P(\text{Inf} \mid +\text{ve}) = 1 - P(\text{Not Inf} \mid +\text{ve})$$

$$\Rightarrow 1 - \frac{99}{118}$$

$$\Rightarrow \frac{19}{118}$$

(c) Why is it so that, even though test is 95% accurate, but $P(\text{Not Inf} \mid +\text{ve test})$ is so high?

Due to data imbalance, 99% people are not infected. So, from the pool of people for which test is +ve majority will be non-infected, since they are larger in count.

Ex. Check the following:

(1) $P(E|F) = 1 - P(E^c|F)$

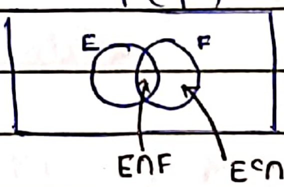
M1:

$P(E) + P(E^c) = 1$

$\Rightarrow P(E|F) + P(E^c|F) = 1$

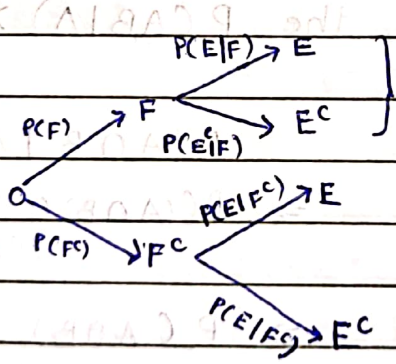
M2: $\dfrac{P(E \cap F) + P(E^c \cap F)}{P(F)}$.

$\Rightarrow$



$\Rightarrow \dfrac{P(F)}{P(F)} = 1$

M3:



$\Rightarrow P(E|F) + P(E^c|F) = 1$

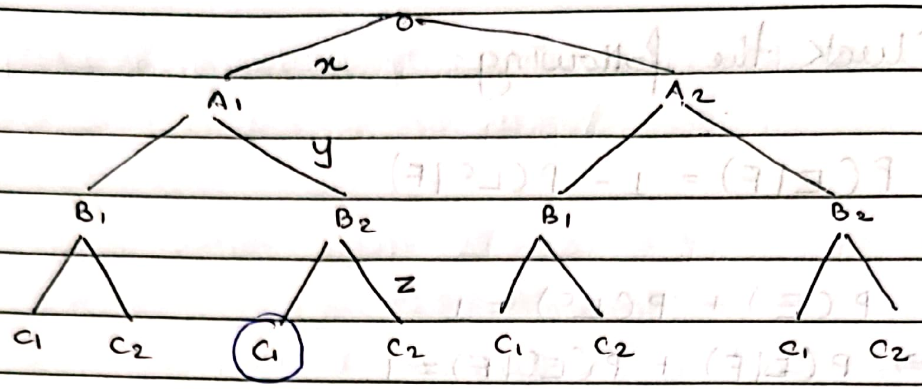
(2) $P(E,F|F) - P(E,F|E)$

LHS $\Rightarrow \dfrac{P(E \cap F \cap F)}{P(F)} = P(E|F)$

$\therefore$ Not Equal.

RHS $\Rightarrow \dfrac{P(E \cap F \cap E)}{P(E)} = P(F|E)$

Ex.



x: $P(A_1)$

y: $P(B_2 | A_1)$

z: $P(C_2 | A_1 \cap B_2)$

Circle Represents: $P(C_1 \cap B_2 \cap A_1)$

Ex. Show that if $P(A) > 0$, the $P(A, B | A) \geq P(A, B | A \cup B)$

$P(A \cap B | A)$

$\Rightarrow \dfrac{P(A \cap B)}{P(A)}$

$P(A \cap B | A \cup B)$

$\Rightarrow \dfrac{P((A \cap B) \cap (A \cup B))}{P(A \cup B)}$

$\Rightarrow \dfrac{P(A \cap B)}{P(A \cup B)}$

Since, $P(A) \leq P(A \cup B)$

$\Rightarrow \dfrac{1}{P(A)} \geq \dfrac{1}{P(A \cup B)}$

$\Rightarrow \dfrac{P(A \cap B)}{P(A)} \geq \dfrac{P(A \cap B)}{P(A \cup B)}$

- Just like Probability Axioms, Conditional Probability also satisfies the three axioms.

(1) Non-negativity

$$P(X) \geq 0$$

So,

$$P(X \mid Y) \geq 0$$

(2) Normalization

$$P(\Omega) = 1$$

So,

$$P(\Omega \mid X) = 1$$

(3) Exclusivity

Given $A \cap B = \phi$

$$P(A \cup B) = P(A) + P(B)$$

So,

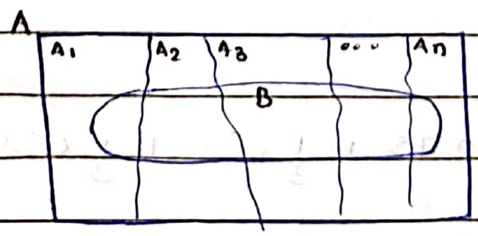
$$P(A \cup B \mid X) = P(A \mid X) + P(B \mid X)$$

(4) IEP: $P(A \cup B \mid X) = P(A \mid X) + P(B \mid X) + P(A \cap B \mid X)$

(5) Demorgan: $P(\overline{A \cap B} \mid X) = 1 - P((\bar{A} \cup \bar{B}) \mid X)$

* Divide & Conquer

Suppose event A has been partitioned into: $A_1, A_2, A_3, \ldots, A_n$.



(A ~~portion~~ partition of set A, $\langle A_1, A_2 \ldots \rangle$ has $\bigcup_i A_i = A$, pairwise exclusive & no $A_i = \phi$)

We can write event B as,

$$B = (A_1 \cap B) \cup (A_2 \cap B) \cup \ldots (A_n \cap B)$$

Then we can write $P(B)$ as.

$$\Rightarrow P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \ldots + P(A_n \cap B)$$
$$- P(A_1 \cap B \cap A_2 \cap B) \ldots$$

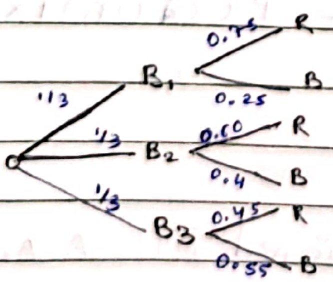
0 since all A_i are pairwise exclusive

$$\Rightarrow P(B) = P(B \cap A_1) + P(B \cap A_2) + \ldots + P(B \cap A_n)$$

Ex. I have 3 bags, each with 100 marble.

B_1:	75R	25 B
B_2:	60R	40 B
B_3:	45R	55 B

I choose one bag at random & then choose a marble from it. What is the prob. that marble is red?



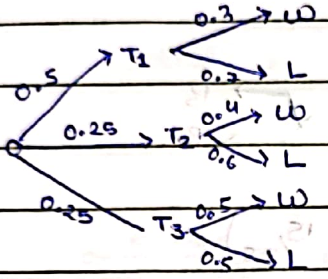
(∵ Bag chosen at random, hence uniform prob. distribution)

$$P(\text{Red}) = \frac{1}{3} \times 0.75 + \frac{1}{3} \times 0.6 + \frac{1}{3} \times 0.45$$

$$\Rightarrow \frac{1}{3} \times \left(\frac{9}{5}\right) = \frac{3}{5}$$

Ex. You enter a chess tournament, where chance of winning against half the players is 0.3, against a quater of player is 0.4 & against the remaining quater, 0.5.

→ Choose a player → { Win, Loose.

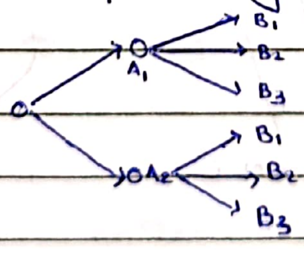


$P(W) = 0.5 \times 0.3 + 0.25 \times 0.4 + 0.25 \times 0.5$

$P(L) = 0.5 \times 0.7 + 0.25 \times 0.6 + 0.25 \times 0.5$

• Baye's Theorem

Consider the following event chain:



Say we know that event B_1 happend. Now we want to find the prob. of B_1 happening ~~afte~~ through A_1.

i.e. we have

$$P(A_1),\ P(B_1 | A_1),\ P(A_2),\ P(B_1 | A_2)$$

& we want: $P(A_1 | B_1)$

$$\Rightarrow P(A_1|B_1) \Rightarrow \frac{P(A_1 \cap B_1)}{P(B_1)} = \frac{P(A_1) \times P(B_1|A_1)}{P(A_1)P(B_1|A_1) + P(A_2)P(B_1|A_2)}$$

Ex. 2 Bags. B_1 : 3 R 7 B

B_2 : 10 R 15 B

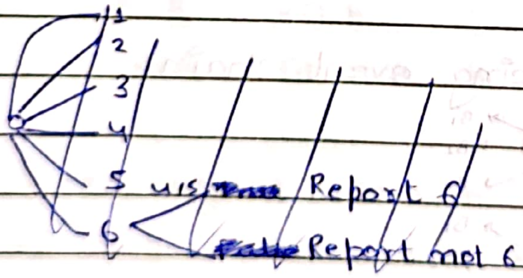
A bag is chosen at random & then a ball is picked. The ball is found to be Blue. What is the prob. that B_1 was chosen?

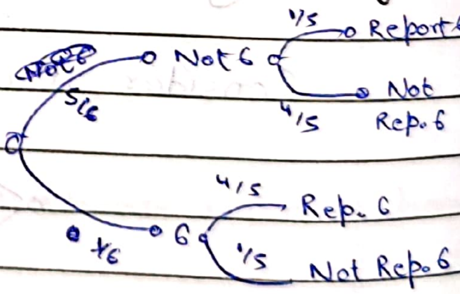
Chain of event: Select Bag $\longrightarrow$ B / R

$$P(B_1 \mid B) = \left(\frac{1}{2}\times\frac{7}{10}\right) \Big/ \left(\frac{1}{2}\times\frac{7}{10} + \frac{1}{2}\times\frac{15}{25}\right)$$

$$\Rightarrow \frac{7}{13}$$

Ex. X speaks truth 4/5 time. He reports that there is a six on die roll. What is the prob. that it actually is?

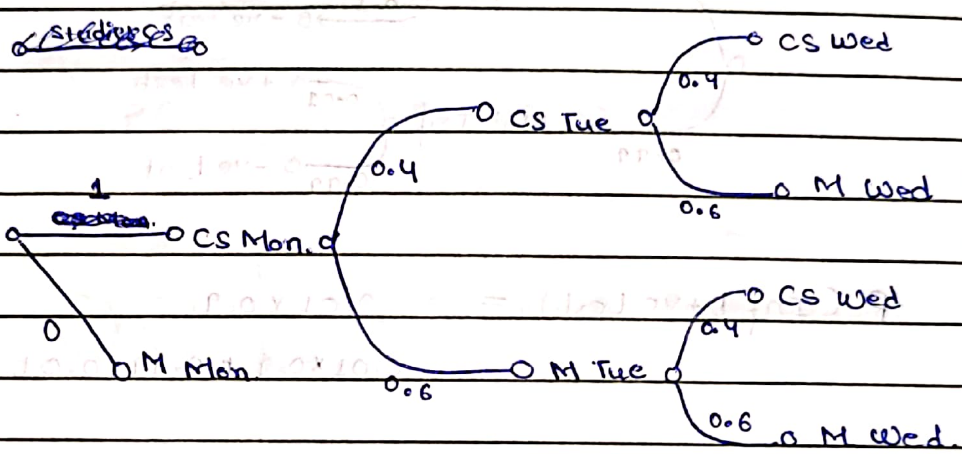




$P(6 \mid \text{Report 6}) = \cancel{\frac{1/6 \times 4/5}{1/6 \times 4/5 \times 6}} = \cancel{1/6}$ $\quad \frac{\frac{1}{6}\times\frac{4}{5}}{\frac{4}{5}\times\frac{1}{6}+\frac{5}{6}\times\frac{1}{5}}$

$$\Rightarrow \frac{4}{9}$$

Ex. X studies either CS or M on a day. If X studies CS today, then prob. of studying M next day is 0.6. If X studies M today, then prob. of studying CS next day is 0.4. Given X studies CS on Monday what is the prob. that she studies CS on Wednesday?

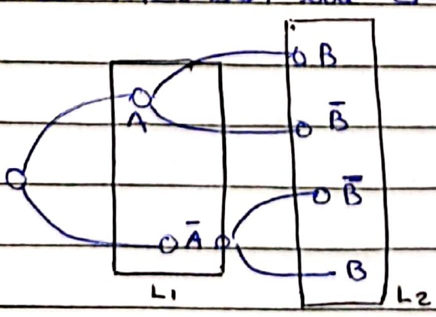


$P(CS\ Wed \mid CS\ Mon) = 0.4 \times 0.4 + 0.6 \times 0.4$

$\Rightarrow 0.4$

(The prob. of studying CS or M next day, depends on only the subject studied today.)

Note: In Tree method, the sum of all prob. of a level is 1.



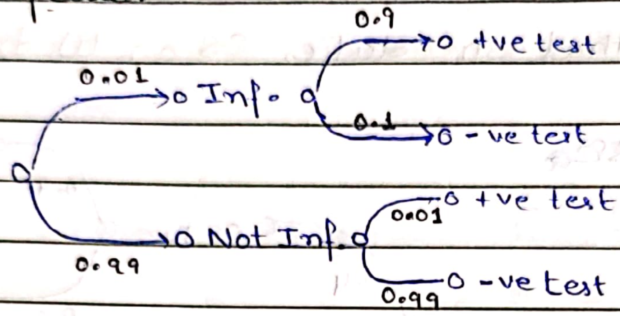
L1: $P(A) + P(\bar{A}) = 1$

L2: $P(B \cap A) + P(\cancel{A}\bar{B} \cap A) + P(\bar{B} \cap \bar{A}) + P(B \cap \bar{A})$

$\Rightarrow P(A) + P(\bar{A})$

$\Rightarrow 1$

Ex. 1% people infected. Test is accurate for 90% infected people & 99% accurate for non-inf. people. Given test is +ve, what is prob. that a person is actually infected.



$$P(\text{Inf} \mid \text{+ve test}) = \frac{0.01 \times 0.9}{0.01 \times 0.9 + 0.99 \times 0.01}$$

$$\Rightarrow 0.476$$

* Independent Events

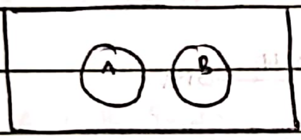
Sometimes two events are not dependent on each other, or two events aren't conditional on another.

i.e.

$$P(A|B) = P(A)$$

So,

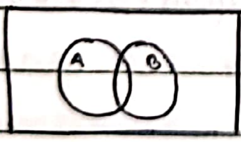
$$P(A \cap B) = P(A) \times P(B)$$

Ex. 

Are A & B independent

$$P(A \cap B) = 0 \quad \text{since} \quad (A \cap B) = \phi.$$

So. given that A has occured prob. of B happening 0, but $P(B) > 0$, so ~~P~~ prob. of ~~A~~ B happening depends on happening of ~~B~~ A.

Ex. 

Are A & B independent?

$P(A) = 0.3$ $\quad P(B) = 0.4$

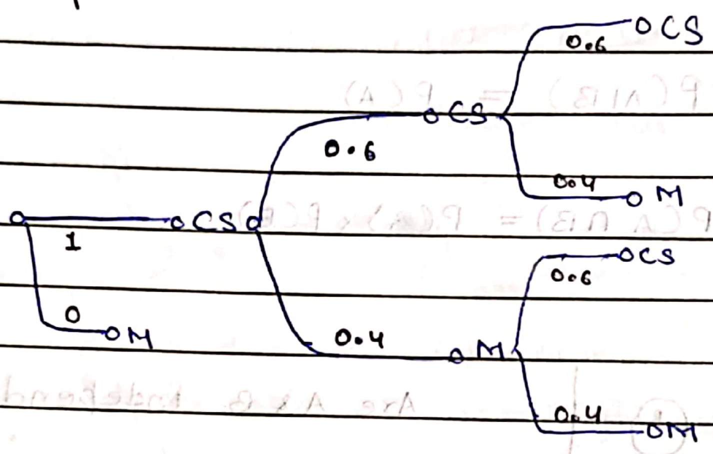
$P(A \cap B) = 0.12.$

$$P(A \cap B) = P(A)\,P(B)$$

$\therefore$ Yes, independent.

Another way of thinking of event A & event B being independent is, that if both happen simultaneously then all combinations of ~~out~~ their outcomes are possible.

Ex. There are two subjects: CS & M. Prob. of studying CS on a day is 0.6. Given that I study CS today what is prob. of studying CS again day after?



Given that I study CS or M today has no affect on what I'd study tommorow.

OR Studying CS today, allows for all possibilites with same probability.

$\therefore$ P(CS day after | CS today) = $1 \times (0.6 \times 0.6 + 0.4 \times 0.6)$

$\Rightarrow 0.6$

- Can two mutually exclusive events be independent ever?

For that to happen

$P(A \cap B) = P(A) \cdot P(B)$

LHS: $P(A \cap B) = 0$

$\therefore$ If either one or both of $P(A)$ & $P(B)$ are 0, then yes.

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- If A & B are independent then A & B^c are also independent.

Given: $P(A \cap B) = P(A) \cdot P(B)$ To Prove: $P(A \cap \bar{B}) = P(A) \cdot P(\bar{B})$

w.k.t $A = (A \cap B^c) \cup (A \cap B)$

$\Rightarrow P(A) = P(A \cap B^c) + P(A \cap B) - P(A \cap B^c \cap A \cap B)$

$\Rightarrow P(A) = P(A \cap B^c) + P(A) \cdot P(B)$

$\Rightarrow P(A \cap B^c) = P(A)\,[1 - P(B)]$

$\Rightarrow P(A \cap B^c) = P(A) \cdot P(B^c)$

So. If A & B indp. then A & B^c indp.

then A^c & B indp.

then A^c & B^c indp.

- For three events A, B, C,

if

$P(A \cap B) = P(A) P(B)$

$P(A \cap C) = P(A) \cdot P(C)$

$P(B \cap C) = P(B) \cdot P(C)$

& $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

then,

A & B & C are independent.

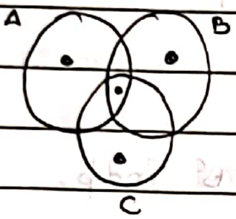
$\therefore$ for $A_1, A_2, A_3, \ldots A_n$ to be independent all combinations of length $2, 3, 4, \ldots n$ has to be independent.

Ex. (1) If a collection of n events are independent, then they are pairwise independent ?

(2) If a collection of n events are pairwise independent then they are independent ?

1 is true. 2 is false.

Ex.



Are they pairwise indp.?
Are they indp.?

$P(A) = 2/4 = 1/2$ $\quad P(B) = 1/2$ $\quad P(C) = 1/2$

$P(A \cap B) = \frac{1}{4}$ $\qquad P(A)P(B) = \frac{1}{4}$

$P(A \cap C) = \frac{1}{4}$ $\qquad P(A).P(C) = \frac{1}{4}$

$P(B \cap C) = \frac{1}{4}$ $\qquad P(B)P(C) = \frac{1}{4}$

$\therefore$ Yes, pairwise indp.

$P(A \cap B \cap C) = \frac{1}{4}$ $\qquad P(A).P(B).P(C) = \frac{1}{8}$

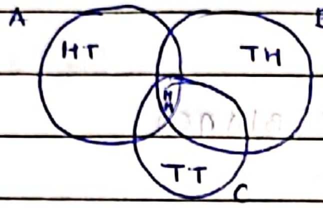
$\therefore$ No, not independence.

Ex. Two Coins tossed.

A: 1^{st} toss is Head

B: 2^{nd} toss is Head

C: Both tosses have same face.



Same as last example. Pairwise independent, but not independent.

- k-wise independence: For a collection of n events $(n \geq k)$, they are k-wise independent if every combination of k events satisfies.

$$P(E_1 \cap E_2 \cap E_3 \cap \dots \cap E_k)$$
$$\Rightarrow P(E_1).P(E_2).P(E_3)\dots P(E_k)$$

For two values of k, $2 \leq k_1 \leq n$ & $2 \leq k_2 \leq n$, & $k_1 \neq k_2$

(1) k_1-way indp. doesn't imply k_2-way indp.

(2) k_2-way indp. doesn't imply k_1-way indp.

n events are called independent if for all k, $2 \leq k \leq n$, the events are k-way independent.

* Conditional Independence.

Two events, A & B are independent given C iff:

$$P(A \cap B | C) = P(A|C) \cdot P(B|C)$$

w.k.t.

$$P(A \cap B | C) = P(A|C) \times P(B|A \cap C)$$

So, for A & B to be independent given C

$$\Rightarrow P(B|A \cap C) = P(B|C)$$

Ex. Given $P(A \cap B|C) = P(A|C) \cdot P(B|C)$ can we say $P(A|B) = P(A)$

$$P(A|B) = P(A) \implies P(A \cap B) = P(B) \cdot P(A)$$

$\therefore$ No, A B are indp. given C doesn't imply A & B are indp.

Ex. Given $P(A,B|E) = P(A|E)\,P(B|E)$ which is equivalent

(A) $P(A|B) = P(A)$

(B) $P(A|BE) = P(A)$

(C) $P(A|BE) = P(A|E)$

w.r.t A isn't.

$$P(A|BE) = \frac{P(AB|E)}{P(B|E)} = \frac{P(A|E)\,\cancel{P(B|E)}}{\cancel{P(B|E)}}$$

$$\Rightarrow P(A|E)$$

$\therefore$ (C)

- Given A & B are indp. given C, then A & B^c are also indp given C.
- $[P(A|B) = P(A)P(B)] \not\Leftrightarrow [P(AB|C) = P(A|C).P(B|C)]$
  Neither implies the other.

Ex. Given $P(A,B,C) = P(C)\,P(A|C)\,P(B|C)$

can we say A & B are indp. given C?

$$P(A\cap B|C) = \frac{P(A\cap B\cap C)}{P(C)}$$

$$\Rightarrow P(A|C)\,P(B|C)$$

$\therefore$ Yes.

Ex. Which are equivalent to "A & B are indp. given C"

(A) $P(A\cap B\cap C) = P(A|C)\,P(B|C)\,P(C)$ ✓

(B) $P(A\cap B\cap C) = P(A|B,C)\,P(B|C)\,P(C)$ ✓ Def^n. Not of Independence

(C) $P(A\cap B) = P(A)\,P(B)$

(D) $P(A,B|C) = P(A|C)\,P(B|C)$ ✓

Ex. If A, B, C are ~~conditionally~~ independent, ~~then~~ & $P(C) > 0$, then A&B given C are independent?

for A&B given C to be indp.

$P(A \cap B | C) = P(A|C)\ P(B|C)$

LHS: $\frac{P(A \cap B \cap C)}{P(C)} \Rightarrow \frac{P(A)\,P(B)\,P(C)}{P(C)} = P(A)\,P(B)$

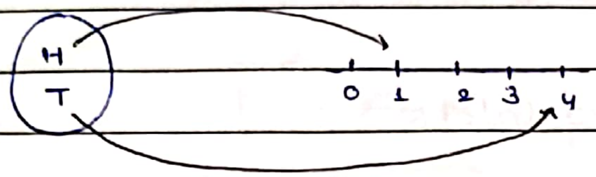
RHS: $\frac{P(A \cap C)}{P(C)} \times \frac{P(B \cap C)}{P(C)} = \frac{P(A).P(C).P(B).P(C)}{P(C)\,P(C)} = P(A)\,P(B)$

$\therefore$ True.

Random Variable

* Random Variable is a deterministic function which takes outcomes from sample space and maps it to a real number line.

Ex. Coin Toss



So the RV $X: \Omega \rightarrow Q$, $X(H) = 1$, $X(T) = 4$

"Random Variable" is a misnomer. Just for convinence we write $X = 1$, instead of $X(H) = 1$.

$X = 1$ is a set such that, $\{\omega \mid X(\omega) = 1\}$.
Similarly, $X < 6$: $\{\omega \mid X(\omega) < 6\} = \{H, T\}$

Any condition on a random variable is a set.

So, $P(X=1)$ would mean $P(\{\omega \mid X(\omega) = 1\})$
$\Rightarrow P(\{H\})$

$\therefore$ Conditions on Random Variable are events.

Ex. Dice Roll (Fair die)

Outcome	P(o)	X
1	1/6	1
2	1/6	4
3	1/6	7
4	1/6	1
5	1/6	9
6	1/6	7

$\therefore X = 1 \Rightarrow \{1, 4\}$
$P(X=1) = P(\{1, 4\}) = 2/6$
$P(X > 4) = P(\{3, 5, 6\}) = 3/6$
$P(X < 1) = P(\phi) = 0$

Ex. Unfair Die Roll.

$P(X > 4) = 0$

$P(X \leq 2) = P(\{1,3,4\})$

$\Rightarrow \frac{1}{2} + \frac{1}{24} + \frac{1}{24}$

$\Rightarrow \frac{7}{12}$

Outcome	P(·)	X
1	1/2	1
2	1/3	4
3	1/24	1
4	1/24	2
5	1/24	3
6	1/24	4

We can hide the outcomes ~~befind~~ behind the random variable

X	P(X)
1	$\frac{1}{2} + \frac{1}{24} = \frac{13}{24}$
2	$1/3 + 1/24 = \frac{9}{24}$
3	$1/24$
4	$1/3 + 1/24 = 9/24$

- Since RV is a func. $X: \Omega \to Q$, for two values a & $b \in Q$, the set $X = a$ & $X = b$ are always disjoint.
  Since RV is a func. every value in the domain (Ω) has to map to only one $x \in Q$.
  $\therefore \{\omega \mid \omega \in \Omega \wedge X(\omega) = a\}$ & $\{\omega \mid \omega \in \Omega \wedge X(\omega) = b\}$ are disjoint.

- If RV $X: \Omega \to Q$ takes values $k_1, k_2, \ldots k_n$, then the set union $\bigcup_{i=1}^{n} (X = k_i) = \Omega$.

$P(X \cap Y) = P(X, Y) = P(XY)$

- Since $X=a$ & $X=b$ are exclusive, so,

$$P(X=a \cup X=b) = P(X=a) + P(X=b)$$

- $P(Y=\alpha) = \sum_{K} P(Y=\alpha, X=K)$

where K is every value X can possibly take

This comes from ~~con~~ total probability only.

Ex.

$P(Y=\alpha) = P(Y=\alpha \cap X=a) + P(Y=\alpha \cap X=b) + \ldots + P(Y=\alpha, X=k)$

Ex. Two Coin Toss

We define following RV

X = number of heads after both toss

X_1 = # heads on first flip.

X_2 = # heads on second flip.

$P(X<2) = P(X=1) + P(X=0)$

$= P(\{HT, TH\}) + P(\{TT\})$

$\Rightarrow 2/4 + 1/4 = 3/4$

$P(X_1+X_2<2) = P(X_1+X_2=0) + P(X_1+X_2=1)$

lets define $Y = X_1 + X_2$

Outcome	X_1	X_2	Y
HH	1	1	2
HT	1	0	1
TH	0	1	1
TT	0	0	0

$\therefore P(X_1+X_2=0) = P(\{TT\}) = 1/4$

$P(X_1+X_2=1) = P(\{HT, TH\}) = 2/4$

$\therefore P(X_1+X_2<2) = 3/4.$

- Whenever an expression is given on RVs, define it as a new RV.

Ex. Two Die Throw

X = Sum of the two die scores.

X	2	3	4	5	6	7	8	9	10	11	12
$P(x)$	$1/36$	$2/36$	$3/36$	$4/36$	$6/36$	$6/36$	$6/36$	$4/36$	$4/36$	$2/36$	$1/36$

Ex.

Outcome	X
S_1	1
S_2	0
S_3	1
S_4	0
S_5	4
S_6	1

Everything is equally probable & equal to $1/6$

(a) $P(X < 1) = P(0) = 2 \times 1/6 = 1/3$

(b) $P(X < 2) = P(0) + P(1) = 5/6$

(c) $P(1 \leq X < 2) = P(1) = 2/6 = 1/3$

Ex. $P(X = x) = \begin{cases} 1/9, & x \text{ is an int in range } [-4, 4] \\ 0, & \text{otherwise.} \end{cases}$

Let $Y = |X|$. Find $P(Y = 2)$

$P(Y = 2) = P(\{-2, 2\}) = 2 \times 1/9 = 2/9.$

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Ex. Let X & Z are two RVs and $Y = X \oplus Z$. X & Z are indp.

x	0	1
$P(X=x)$	p	$(1-p)$

z	0	1
$P(Z=z)$	ϵ	$1-\epsilon$

Find $P(Y=y)$

(xz) Outcomes	X	Z	y
00	p	ϵ	0
01	p	$1-\epsilon$	1
10	$(1-p)$	ϵ	1
11	$(1-p)$	$1-\epsilon$	0

$P(Y=0) = P(X=0, Z=0) + P(X=1, Z=1)$

$\Rightarrow P(X=0 \cap Z=0) + P(X=1 \cap Z=1)$

$\Rightarrow P(X=0) \cdot P(Z=0) + P(X=1) \cdot P(Z=1)$ {$\because$ indp.}

$\Rightarrow p\epsilon + (1-p)(1-\epsilon)$

Similarly

$P(Y=1) = P(X=0, Z=1) + P(X=1, Z=0)$

$= p(1-\epsilon) + (1-p)\epsilon$

y	0	1
$P(Y=y)$	$p\epsilon + (1-p)(1-\epsilon)$	$p(1-\epsilon) + (1-p)\epsilon$

Ex. Consider two RV X & Y, where below table gives the joint prob. dist. $P(X=x, Y=y)$

X ↓ Y →	0	1	2
0	$1/6$	$1/4$	$1/8$
1	$1/8$	$1/6$	$1/6$

(1) $P(X=0, Y=1) = 1/4$

(2) $P(X=0, Y \leq 1) = 1/6 + 1/4$

(3) $P(X=0) = \sum_{y} (P(X=0, Y=y)) = 1/6 + 1/8 + 1/4$

(4) $P(Y=2) = \sum_{x} P(X=x, Y=2) = 1/8 + 1/6$

(5) $P(Y=1 \mid X=0) = \dfrac{P(X=0, Y=1)}{P(X=0)}$

(6) Are X & Y ~~are~~ independent

To prove X & Y are indp., we'll have to show

$\forall x,y \; P(X=x, Y=y) = P(X=x)\, P(Y=y)$

$P(X=0) = 13/24$ $\quad P(Y=0) = 7/24$ $\quad P(Y=2) = 7/24$.

$P(X=1) = 11/24$ $\quad P(Y=1) = 5/12$

$P(X=0, Y=0) = 1/6$ $\quad P(X=0) \times P(Y=0) = 91/576$.

$\therefore$ No. X & Y aren't indp.

Ex. X = No. of heads on two coin toss

Y = No. of heads on other two coin toss

Find

$P(X<2, Y>1)$

X & Y are indp. here since happening of one doesn't affect the happening of other.

$\therefore P(X<2, Y>1)$

$\Rightarrow P(X<2) \cdot P(Y>1)$

$\Rightarrow 3/4 \cdot 1/4 = 3/8$

Ex. Suppose n random var. are all indp. given random variable C.

Find $P(C, X_1, X_2, \ldots X_n)$

Given : $\cancel{P(X_1, X_2, X_3, \ldots X_n) = P(C)}$

$$P(X_1, X_2, \ldots X_n \mid C) = P(X_1 \mid C) \cdot P(X_2 \mid C) \ldots P(X_n \mid C)$$

$$\Rightarrow \frac{P(X_1, X_2, X_3, \ldots X_n, C)}{P(C)} = P(X_1 \mid C) \cdot P(X_2 \mid C) \ldots P(X_n \mid C)$$

$$\Rightarrow P(X_1, X_2, \ldots X_n, C) = P(C)\, P(X_1 \mid C)\, P(X_2 \mid C) \ldots P(X_n \mid C)$$

$$\Rightarrow P(C) \prod_{i=1}^{n} P(X_i \mid C)$$

Ex. [Binary Symmetric Channels]

$$Z \in \{0,1\}$$

$$X \in \{0,1\} \longrightarrow \oplus \longrightarrow Y \in \{0,1\}$$

$P(X=0) = p$

$P(Z=0) = \epsilon$

$Y = X \oplus Z$ & X & Z are indp. Find $P(Y=y)$

X	Z	Y
0	0	0
0	1	1
1	0	1
1	1	0

$\therefore P(Y=0) = P(X=0, Z=0) + P(X=1, Z=1)$

$P(Y=1) = P(X=0, Z=1) + P(X=1, Z=0)$

y	0	1
$P(Y=y)$	$p\epsilon + (1-p)(1-\epsilon)$	$p(1-\epsilon) + (1-p)\epsilon$

. Find $P(Y=y \mid X=x)$

$$P(Y=0 \mid X=0) = \frac{P(Y=0, X=0)}{P(X=0)} =$$

So now we have to find $P(X=0, Y=0)$

$$P(X=0, Y=0) = P(X=0, X \oplus Z=0)$$

$$\Rightarrow P(X=0, 0 \oplus Z=0)$$

$$\Rightarrow P(X=0, Z=0)$$

$$\Rightarrow P(X=0) \cdot P(Z=0)$$

$$\Rightarrow \alpha \epsilon$$

$$\therefore P(Y=0 \mid X=0) = \frac{P(Y=0 \cap X=0)}{P(X=0)} = \frac{\alpha \epsilon}{\alpha} = \epsilon$$

M2: we have to find $\frac{P(Y=0, X=0)}{P(X=0)}$.

x	Z	y	$P(Y=y)$
0	0	0	$\alpha \epsilon$
0	1	1	$\alpha(1-\epsilon)$
1	0	1	$(1-\alpha)\epsilon$
1	1	0	$(1-\alpha)(1-\epsilon)$

Since we know $X=0$ has happened we eliminate & reduce sample space

Now, $\alpha\epsilon + \alpha(1-\epsilon)$ should be 1

but $\alpha\epsilon + \alpha(1-\epsilon) = \alpha$.

$\therefore \quad \epsilon + (1-\epsilon) = 1$

$\therefore$ New Prob.

X	Z	Y	$P(Y=y)$
0	0	0	ϵ
0	1	1	$(1-\epsilon)$

$$\therefore \quad P(Y=0 \mid X=0) = \epsilon$$

- Find $P(X \neq Y)$

$\Rightarrow P(X=0, Y=1) + P(X=1, Y=0)$

$\Rightarrow P(X=0, 0 \oplus Z=1) + P(X=1, 1 \oplus Z=0)$

$\Rightarrow P(X=0, Z=1) + P(X=1, Z=1)$

$\Rightarrow P(Z=1) = (1-\epsilon)$

* ~~Identically Distributed RV~~

~~A RV X is called IDD when $\forall x\ P(X=x) = n$, where $n \in [0,1]$ ~~or~~ $\cancel{\forall x\ P(X=x)}$ where x is all the possible values of RV X or $x \in \text{Range}(X)$.~~

Ex. Suppose X_i for $i=1,2,3$ are indep. & identically distributed RV. whose prob. mass func. are $P(X_i=0) = P(X_i=1) = 1/2$ for $i=1,2,3$. Define another RV $Y = X_1X_2 \oplus X_3$. Then $P(Y=0 \mid X_3=0)$?

X_1	X_2	X_3	Y	$P(Y=y)$
0	0	0	0	1/8
0	0	1	1	1/8
0	1	0	0	1/8
0	1	1	1	1/8
1	0	0	0	1/8
1	0	1	1	1/8
1	1	0	1	1/8
1	1	1	0	1/8

$P(Y=0 \mid X_3=0)$

$\Rightarrow \dfrac{P(Y=0, X_3=0)}{P(X_3=0)}$

$\Rightarrow P(X_1X_2 \oplus 0 = 0, X_3=0) / P(X_3=0)$

$\Rightarrow P(X_1X_2=0, X_3=0) / P(X_3=0)$

$\Rightarrow P(X_1X_2=0)\ P(X_3=0) / P(X_3=0)$

$\Rightarrow P(X_1X_2=0)$

$\Rightarrow P(X_1=0, X_2=0) + P(X_1=0, X_2=1) + P(X_1=1, X_2=0)$

$\Rightarrow \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 3/4$

* If RVs, $X_1, X_2, \ldots X_n$ are independent then any func. defined on them, ex. $X_1 X_2$, $X_1 + X_2$, $X_1 \oplus X_2$, $\bar{X}_1$, etc, are also independent with $X_1, X_2, \ldots X_n$ & other func.

* Types of Random Variables

- Discrete

RV X can take finite or countably infinite value.

- Continuous

RV X can take infinitely uncountable values.

Ex. Ω = Set of student

G: $\Omega \rightarrow$ Grades A, B, C, D, F   Discrete
H: $\Omega \rightarrow$ Height [50kg to 120kg]   Continuous

Ex. $\Omega = \{1, 2, \ldots, \ldots\}$
$P(n) = 2^{-n}$

$\Sigma P(n) = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots = \frac{1/2}{1 - 1/2} = 1$

X: n for val n.
$P(X \geq 2) = \frac{1}{2^2} + \frac{1}{2^3} + \ldots = \frac{1}{3}$

(Discrete RV, with countably inf. values) classmate

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* Probability Mass Function

Listing down probability of each possible value for a discrete RV is PMF

Ex. Check if the given func. is PMF for RV X :

$$P(X=x) = \frac{x+2}{25} \quad \text{for } x = 1, 2, 3, 4, 5.$$

$P(X=1) = \frac{3}{25}$ $P(X=2) = \frac{4}{25}$ $P(X=3) = \frac{5}{25}$ $P(X=4) = \frac{6}{25}$ $P(X=5) = \frac{7}{25}$

$\forall x \; P(X=x) \geq 0$

$\therefore \sum_x P(X=x) = 1 \quad \therefore$ PMF is valid.

Steps :

1. Given a RV X, for each outcome find X & P(X).
2. For each possible value of X, collect P(X) of all the outcomes ω, such that $X(\omega) = x$.

Ex. Two coin toss. X : No. of heads.

Outc.	X	P(X)
HH	2	1/4
HT	1	1/4
TH	1	1/4
TT	0	1/4

X	0	1	2
P(X)	1/4	$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$	$\frac{1}{4}$

PMF

$\sum_x P(X=x) = 1 \quad \forall x \; P(X=x) \geq 0.$

* Expectation

Expectation is a way to summarise PMF in one number.

(It can be thought of as weighted avg. of the RV)

Ex. Dice Roll. You get i ruppee for i outcome. How much do you expect to get per trial. How much do you expect in N trials?

Outcome	(x) Rupee	P(x)
1	1	1/6
2	2	1/6
3	3	1/6
4	4	1/6
5	5	1/6
6	6	1/6.

$$E(\text{Rupee}) = \sum_{i} x_i \times P(X = x_i)$$

$$\Rightarrow \frac{1}{6}(1+2+3+\dots+6) = \frac{21}{6} = 3.5$$

$\therefore$ In N trials $= N \times 3.5$

• Average vs Expectation

Ex. Say you are playing soccer matches & these are the goals you score:
1, 2, 1, 5, 2, 3, 2, 6, 5

$$\text{Avg} = \frac{1+2+1+5+2+3+2+6+5}{9} = \frac{1\times 2 + 2\times 3 + 3\times 1 + 5\times 2 + 6\times 1}{9}$$

$$\Rightarrow \frac{2}{9}\times 1 + \frac{3}{9}\times 2 + \frac{1}{9}\times 3 + \frac{2}{9}\times 5 + \frac{1}{9}\times 6.$$

We can kind of see:

$P(1) = \frac{2}{9}$ $P(2) = \frac{3}{9}$ $P(3) = \frac{1}{9}$ $P(5) = \frac{2}{9}$ $P(6) = \frac{1}{9}$

So, average looks a lot like expectation.
What is the difference?

- Average is expectation based on sample probability.
- Expectation is based on real probabilities.

Ex. Say, I toss a fair coin 10 times with following results for RV X: 1 score for H, 2 for T.

H, H, H, T, H, T, T, T, H, H

$\therefore$ Avg. score $= \frac{1 \times 6 + 2 \times 4}{10} = 1.4$

& $P(H) = \frac{6}{10}$ $P(T) = \frac{4}{10}$

But we know the real probabilities for a fair coin:

$P(H) = \frac{1}{2}$ $P(T) = \frac{1}{2}$

X	1	2
P(X)	1/2	1/2

$\therefore E(X) = 1 \times \frac{1}{2} + 2 \times \frac{1}{2} = 1.5$

- $$E(X) = \sum_{x} x \times P(X=x)$$

Ex. Suppose RV $X = n$ with $P(X=n) = \frac{1}{2^{n+1}}$ for $n = 0, 1, 2, \dots$

$E(X)$?

$$E(X) = \sum_{n=0}^{\infty} n \times \frac{1}{2^{n+1}} = \frac{0}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \dots$$

AGP series but GP part doesn't start with 1.

$$\Rightarrow S_\infty = \frac{a}{(1-r)} + \frac{rd}{(1-r)^2}$$

$$\Rightarrow 0 + \frac{1/2 \times 1}{(1-1/2)^2}$$

$$\Rightarrow 1/2 \,/\, 1/4 = 2$$

$$\Rightarrow \left[\left(\frac{-1}{2^0}\right) + \frac{0}{2^1} + \frac{1}{2^2} + \frac{2}{2^3} + \dots\right] - \left(\frac{-1}{2^0}\right)$$

$$\Rightarrow S_\infty = \frac{-1}{(1-\frac{1}{2})} + \frac{\frac{1}{2} \times 1}{(1-1/2)^2} + 1$$

$$\Rightarrow -2 + 2 + 1$$

$$\Rightarrow 1$$

Ex. Suppose we roll a die. Find $E(X^2)$ where X: Outcome of die.

Out.	X	P(X)	X^2	$P(X^2)$
1	1	1/6	1	1/6
2	2	1/6	$2^2=4$	1/6
3	3	1/6	9	1/6
4	4	1/6	16	1/6
5	5	1/6	25	1/6
6	6	1/6	36	1/6

$$E(X^2) = \frac{1}{6}(1+4+9+16+25+36)$$

$$\Rightarrow 15.16$$

Ex.

X	P(X)	X^2
-2	1/5	4
-1	1/5	1
0	1/5	0
1	1/5	1
2	1/5	4

Find $E(X^2)$

X^2	$P(X^2)$
0	1/5
1	2/5
4	2/5

$\therefore E(X^2) = \frac{2}{5} + \frac{8}{5}$

$\Rightarrow 2$

- W.K.T $E(Y) = \sum_y y \times P(y)$

if $Y = f(X)$ where X & Y are both RV,
then

$$E(Y) = \sum_x f(x) \times P(X=x)$$

Ex. Prev. example.

$$E(X^2) = (-2)^2 \times \frac{1}{5} + (-1)^2 \times \frac{1}{5} + 0^2 \times \frac{1}{5} + 1^2 \times \frac{1}{5} + 2^2 \times \frac{1}{5}$$

$\Rightarrow 2.$

It makes intuitive sense. $f(x)$ is just values of Y & for any y $P(Y=y) = \sum_i P(X=x_i)$ where x_i is some of the values of X.

$$E(X+b) = E(X)+b \qquad E(aX) = a \times E(X)$$

$$E(X_1+X_2) = E(X_1)+E(X_2)$$

$$\therefore E(aX+b) = a \cdot E(X)+b$$

we can check using last identity.

$$E(X) = \sum_x P(X=x) \cdot x.$$

$$\Rightarrow E(aX+b) = \sum_x P(X=x) \cdot (aX+b)$$

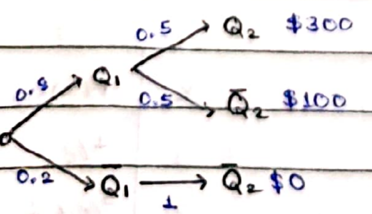
$$\Rightarrow \sum_x aX \cdot P(X=x) + \sum_x P(X=x) \times b$$

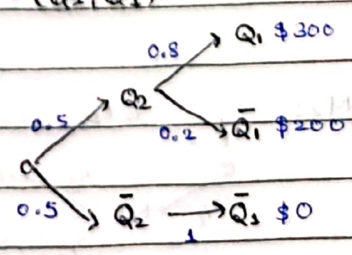
$$\Rightarrow a\sum_x P(X=x) + 1 \times b$$

$$\Rightarrow a E(X)+b.$$

Ex. Consider a quiz game with Q_1 & Q_2. Prob. for answering Q_1 & Q_2 correctly are 0.8 & 0.5 resp. Prize for Q_1 & Q are \$100 & \$200 resp. You have to decide which question to answer first. If first ques. is answered wrong, you don't get to attempt second, and prize is \$0. Choose the order to maximise expected prize.

(Q_1, Q_2) $\qquad\qquad$ (Q_2, Q_1)





$E(\text{Prize}) = 300 \times 0.8 \times 0.5 + 100 \times 0.8 \times 0.5$ $\qquad$ $E(P_2) = 0.5 \times 0.8 \times 300 + 0.5 \times 0.2 \times 200$

$\Rightarrow 160$ $\qquad\qquad$ $\Rightarrow 140$

$\therefore (Q_1, Q_2)$

Imp

Law of Total Expectation

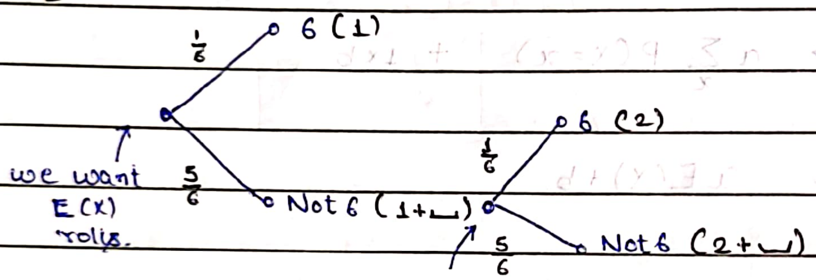
$$E(X) = E_1 \times P(A) + E_2\, P(\bar{A})$$

Ex. What is the expected no. of tries to get 6 on dice roll?

X: No. of tries to get 6.

So we want E(X). Each roll is independent

M1:



we restart.

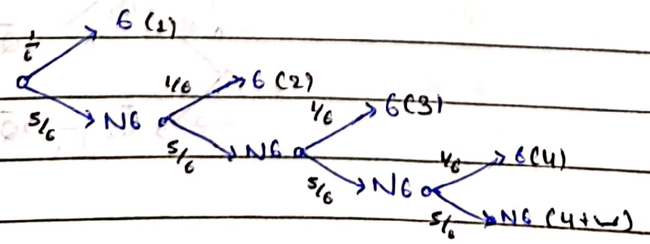
So we still need E(X) rolls.

$$E(X) = \left(\frac{1}{6} \times 1\right) + \frac{5}{6}\left(E(X) + 1\right)$$

$$\Rightarrow \frac{1}{6} E(X) = 1$$

$$\Rightarrow \boxed{E(X) = 6.}$$

M2:



X	P(X)
1	$1/6$
2	$5/6 \times \frac{1}{6}$
3	$(5/6)^2 \times 1/6$
4	$(5/6)^3 \times 1/6$
⋮	

So, $E(x) = 1 \times \frac{1}{6} + 2 \times \frac{5}{6} \times \frac{1}{6} + 3 \times \left(\frac{5}{6}\right)^2 \times \frac{1}{6} + \dots$

$$\Rightarrow \frac{1}{6} \sum_{n=1}^{\infty} n \times \left(\frac{5}{6}\right)^{n-1}$$

∞ AGP with GP starting with 1

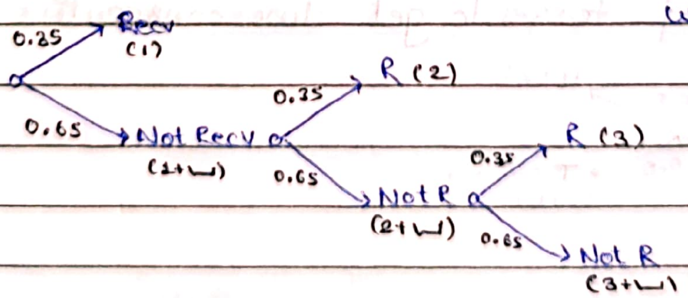
$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

$$\Rightarrow \frac{1}{1-\frac{5}{6}} + \frac{1 \times 5/6}{(1-5/6)^2}$$

$$\Rightarrow 6 + \frac{5/6}{1/36} = 36.$$

$$\therefore E(x) = \frac{1}{6} \times 36 = 6.$$

Ex. Sender sends byte to Reciever with recieving probability 0.35. What is the expected no. of times to ensure successfully sending?



w.r.t two sent ~~bytes~~ attempts are indp.

Each time we fail to send, we start at zero, i.e., we'll have to try Expctd. number of times again.

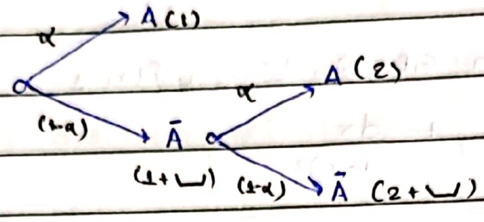
$$E(\text{Send Successfully}) = 0.35 \times 1 + 0.65 \times (1 + E)$$

$$E = 0.35 + 0.65 + 0.65E$$

$$0.35E = 1$$

$$E = \frac{1}{0.35} = 2.85$$

Given that $P(A) = \alpha$ & we want to find the expected no. of trials to ensure A happens and each trial is independent, then



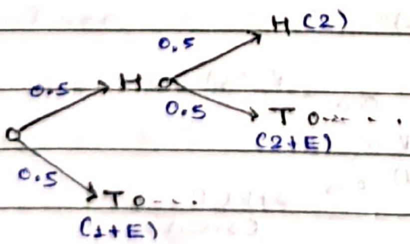
$$E(X{=}A) = E(A) \times P(X=A) + E(\bar{A}) \times P(X \neq A)$$

$$\Rightarrow \quad 1 \times \alpha + (1+E) \times (1-\alpha)$$

$$\therefore E = E(1-\alpha) + 1$$

$$\Rightarrow \boxed{E = \frac{1}{\alpha}}$$

Ex. Consider a expr. with fair coin with $P(H)=P(T)=0.5$. You toss the coin untill you get two heads in a row. Expected no. of tosses to get two consecutive head.



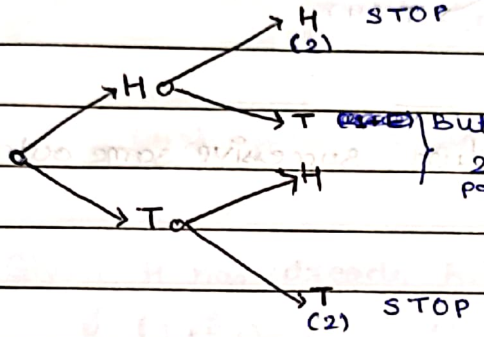
$$E = \underbrace{(0.5)^2 \times 2}_{HH} + \underbrace{(0.5)^2 \times (2+E)}_{HT} + \underbrace{0.5 \times (1+E)}_{T}$$

$$\Rightarrow E = 1.5 + 0.75E$$

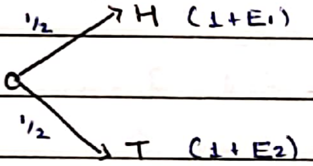
$$\Rightarrow 0.25E = 1.5$$

$$\Rightarrow E = 6.$$

Ex. An unbiased coin is tossed repeatedly until the outcome of two successive tosses is same. Assuming that the trials are independent, the expected number of tosses is ?



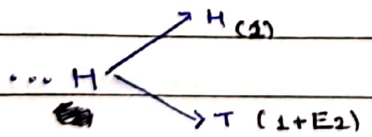
But we can't restart the process of getting 2 same outcomes as we already have one possible outcome.



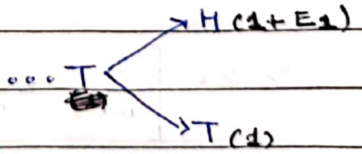
$$E = \frac{1}{2}(1+E_1) + \frac{1}{2}(1+E_2)$$

E_1: Tosses for HH or TT if we already have H.
E_2: Tosses for HH or TT if we already have T

for E_1:



for E_2:



$\therefore E_1 = \frac{1}{2} \times 1 + \frac{1}{2} \times (1+E_2)$

$\Rightarrow \frac{1}{2}(2+E_2)$

$\therefore E_2 = \frac{1}{2}(2+E_1)$

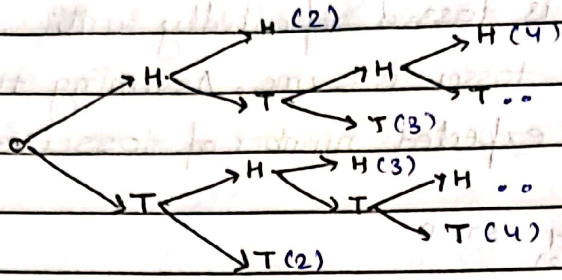
$\Rightarrow 2E_1 = 2 + \frac{1}{2} \times (2+E_1)$

$\therefore E = \frac{6}{2} = 3$

$\Rightarrow 4E_1 = 4 + 2 + E_1$

$\Rightarrow E_1 = 6/3 = 2 \quad E_2 = 2$

M2:



X: No. of tosses for two successive same outcomes.

X	P(X)
2	$(0.5)^2 \times 2$
3	$(0.5)^3 \times 2$
4	$(0.5)^4 \times 2$
⋮	

$$\therefore \quad E(X) = 2\sum_{n=2}^{\infty} n \times (0.5)^n$$

$$\Rightarrow 2\left[\, 2\times(0.5)^2 + 3\times 0.5^3 + \dots \right]$$

∞ AGP but GP doesn't start with 1.

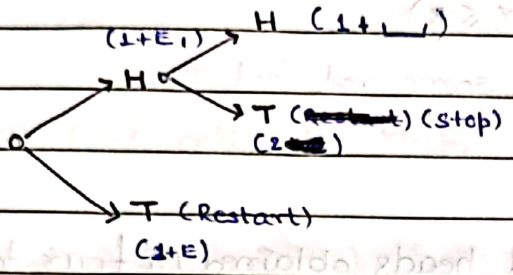
$$\Rightarrow 2\times\left[0\times 0.5^0 + 1\times 0.5^1 + 2\times 0.5^2 + 3\times 0.5^3 + \dots\right] - 2\times(0.5^1)$$

$$\Rightarrow 2\times\left[\frac{0}{1-0.5} + \frac{1\times 0.5}{(1-0.5)^2}\right] - 1$$

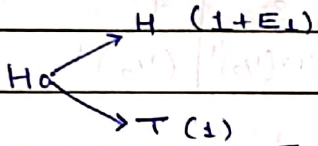
$$\Rightarrow 2\times\left[0 + 2\right] - 1$$

$$\Rightarrow 3.$$

Ex. Same as last but for HT only.



E_1: Given H has already happened. Exp. no. of tosses to get HT.



$$\therefore E_1 = \underbrace{0.5 \times 1}_{T} + \underbrace{0.5 \times (1+E_1)}_{H}$$

$$\Rightarrow E_1 = \frac{1}{1/2} = 2.$$

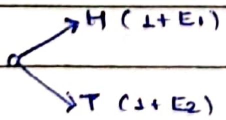
$$\therefore E = \underbrace{(1+E_1) \times 0.5}_{H} + \underbrace{(1+E) \times 0.5}_{T}$$

$$\Rightarrow E = \frac{3}{2} + \frac{1}{2} + \frac{E}{2}$$

$$\Rightarrow \frac{E}{2} = \frac{4}{2}$$

$$\Rightarrow \boxed{E = 4}$$

Ex. Same as last but, either HT or TH.

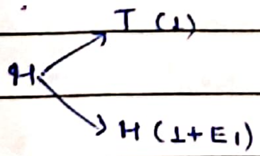


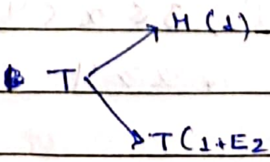
E_1: Given H, exp. to get HT or TH.

E_2: Given T, exp. to get HT or TH.

E_1:

Similarly $E_2 = 2$





$\therefore E = 0.5(1+E_1) + 0.5(1+E_2)$

$\Rightarrow 3$

$E_1 = 0.5 \times 1 + 0.5(1+E_1)$

$\Rightarrow E_1 = 2$

$E_2 = 2.$

* Cumulative Distribution Function

$$F(x) = P(X \leq K)$$

where K is some ~~real~~ real no.

Ex. Find CDF of total heads obtained in four tosses of balanced coin.

PMF

x	0	1	2	3	4
$P(X=x)$	$(1/2)^4$	$4\times(1/2)^4$	$6\times(1/2)^4$	$4\times(1/2)^4$	$(1/2)^4$

CMF

x	0	1	2	3	4
$F(x) = P(X\leq x)$	$(1/2)^4$	$5(1/2)^4$	$11\times(\frac{1}{2})^4$	$15\times(\frac{1}{2})^4$	$16\times(\frac{1}{2})^4$

$$F(X<0) = 0$$
$$F(X \geq 4) = 1$$

- $F(2.5) = P(X \leq 2.5) = 11\times(1/2)^4$

$P(2.5) = 0$

$$F(x) = \begin{cases} 0, & x<0 \\ (1/2)^4, & 0\leq x<1 \\ 5\times(1/2)^4, & 1\leq x<2 \\ 11\times(\frac{1}{2})^4, & 2\leq x<3 \\ 15\times(\frac{1}{2})^4, & 3\leq x<4 \\ 1, & x\geq 4 \end{cases}$$

- CDF is ~~also~~ a non-decreasing function.
  & $0 \leq F(X) \leq 1$

Ex. Given CDF calculate PMF.

$F(1) = 0.25 \quad F(2) = 0.61 \quad F(3) = 0.83 \quad F(4) = 1.$

$$F(x) = \begin{cases} 0, & x < 1 \\ 0.25, & 1 \leq x \leq 2 \\ 0.61, & 2 \leq x < 3 \\ 0.83, & 3 \leq x < 4 \\ 1, & 4 \leq x \end{cases} \qquad P(X) = \begin{cases} 0.25, & x = 1 \\ 0.36, & x = 2 \\ 0.22, & x = 3 \\ 0.17, & x = 4 \end{cases}$$

~~$\therefore P(x_i) \neq (F(x = x_i) - (F($~~

$\therefore P(X = x_i) = F(X = x_i) - F(X = x_{i-1})$

* Variance

Expectation is one way to summarise prob. dist. Variance is just a different way.

Ex. -5 0 5 -1 0 1

$\text{Avg} = \frac{-5+0+5}{3} = 0 \qquad \text{Avg} = \frac{-1+0+1}{3} = 0$

Seems like Exp. gives a similar value for both the situations, so the ~~x~~ summarization has obfuscated the uniqueness of the two situations.

Variance tells how spread the data is.
It is defined as square distance from expectation averaged over all points.

i.e. average of sq. dist. of each point from mean.

Ex.

-5   0   5   -1   0   1

$E(X) = 0$ $E(X) = 0.$

$$V(X) = \frac{\sum_{x_i} (E(x) - x_i)^2}{n}$$

$$V(X) = \frac{(-1-0)^2 + (0-0)^2 + (1-0)^2}{3}$$

$$\Rightarrow \frac{(-5-0)^2 + (0-0)^2 + (5-0)^2}{3}$$

$$\Rightarrow \frac{50}{3} \qquad\qquad \Rightarrow \frac{2}{3}$$

$\therefore$ First data is well spread compared to second.

- Given some data points: $x_1, x_2, \dots x_n$

$$\text{Exp.} = \text{Avg} = \frac{\sum_{i=1}^{n} x_i}{n} \qquad \text{Var.} = \frac{\sum_{i=1}^{n} (\text{Avg} - x_i)^2}{n}$$

- Given probability distribution:

w.r.t

$$E(x) = \sum_{x} X \times P(X=x)$$

$$\boxed{V(X) = E\left([X - E(x)]^2\right)}$$

Ex. Given the PMF, find Mean & Variance.

$$P(x) = \begin{cases} 1/9 & , \text{for } x \in [-4, 4] \text{ \& } x \text{ is int} \\ 0 & , \text{otherwise.} \end{cases}$$

$$E(x) = \sum_x x \cdot p(x)$$

$$\Rightarrow (-4 + -3 + -2 + -1 + \dots 4) \times \frac{1}{9}$$

$$\Rightarrow 0.$$

$$V(x) = E([x - E(x)]^2)$$

$$= \sum_x p(x) \times (x - e)^2$$

$$\Rightarrow \frac{1}{9} \times ((-4)^2 + (-3-0)^2 + (-2-0)^2 + \dots + (4-0)^2)$$

$$\Rightarrow 6.66.$$

- Variance is given by,

$$V(x) = E([X - E(X)]^2)$$

$$\Rightarrow E([X^2 - 2X.E(X) + E(x)^2])$$

$$\Rightarrow E(X^2) - E(2 \times E(x)) + E(E(x)^2)$$

$$\Rightarrow E(x^2) - 2E(X)E(X) + E(X)^2$$

$$\Rightarrow E(x^2) - E(x)^2$$

{ ∵ $E(X)$ & $E(X)^2$ are const. }

$$\therefore V(X) = E(X^2) - E(X)^2$$

Ex. Calc. Mean & Variance for given PMF.

X	1	2	3	4	5
P(X)	1/10	2/10	4/10	2/10	1/10

$E(X) = \sum_{x} x \cdot P(X=x) = \frac{1}{10} \times 1 + 2 \times \frac{2}{10} + 3 \times \frac{4}{10} + 4 \times \frac{2}{10} + 5 \times \frac{1}{10}$

$\Rightarrow 3.$

$E(X^2) = \sum_{x} x^2 P(X=x) = 1^2 \times \frac{1}{10} + 2^2 \times \frac{2}{10} + 3^2 \times \frac{4}{10} + 4^2 \times \frac{2}{10} + 5^2 \times \frac{1}{10}$

$\Rightarrow 10.2$

$V(X) = E(X^2) - E(X)^2$

$\Rightarrow 10.2 - 9$

$\Rightarrow 1.2$

- Since variance is the mean of squared distance its always ≥ 0.

- $V(aX) = a^2 V(X)$
  $V(a+X) = V(X)$

$\therefore V(aX+b) = a^2 V(X)$

$V(X_1 + X_2) = V(X_1) + V(X_2)$ if X_1 & X_2 are indp.

$V(X_1 - X_2) = V(X_1) + V(-X_2)$

$\Rightarrow V(X_1) + (-1)^2 V(X_2)$

$\Rightarrow V(X_1) + V(X_2)$

We can check the property as.

$V(X_1 + X_2)$

$\Rightarrow E((X_1+X_2)^2) - E(X_1+X_2)^2$

$\Rightarrow$

Ex. $V(aX_1 + bX_2 + c)$, where X_1 & X_2 are indp.

$\Rightarrow V(aX_1) + V(bX_2) + V(c)$

$\Rightarrow a^2 V(X_1) + b^2 V(X_2)$

* Standard Deviation

if $V(X) = \sigma^2$

then, Std. Dev. $\sigma = \sqrt{V(X)}$

Discrete RV

There are some distributions that occur frequently in real life, and hence these distr. are named.

Ex Two decision : Coin Toss
choice

$P(H) = 1/2$ $P(T) = 1/2$

So, I can just create a RV.
X: No. of heads in K tosses
& then give it a name
HeadWin RV or
HeadWin Distribution.

* Bernaulli RV

The RV X can take only two values, 1 for success and 0 for failure for an experiment.

Ex. Say a die roll (fair).
we can define X: outcome ≤ 3.
This follows Bernaulli RV pattern.
Success: $X \leq 3$
Failure: $X > 3$
$P(\text{success}) = 3/6 = 1/2$
$P(\text{failure}) = 1 - P(\text{success})$
$\Rightarrow 1 - 1/2 = \frac{1}{2}$

So any RV having two outcomes such that one outcome p & other has outcome $1-p$, we can call it Bernaulli.

$$E(x) = p \times 1 + (1-p) \times 0 = p.$$

$$V(x) = E(x^2) - E(x)^2$$
$$\Rightarrow p - p^2$$
$$\Rightarrow p \times (1-p)$$

$\therefore$ Bernaulli RV

$$P(x) = \begin{cases} p, & x = 1 \\ (1-p), & x = 0. \end{cases}$$

* Binomial RV

Repeated independent trials of Bernaulli.

So, any RV X such that X is defined as no. of successes while performing n ~~trials~~ indp. trials such that for each trial

$$P(x) = \begin{cases} p, & x = 1 \\ (1-p), & x = 0 \end{cases}$$

can be called a Binomial RV.

Ex. 10 coin tosses. $P(H) = P(T) = 1/2$
X: No. of heads

$$P(X=k) = {}^{10}C_k \times p^k \times (1-p)^{10-k}$$

Say, $n = 5$ & $K = 2$

then $P(H, H, T, T, T) = P($~~A~~$ H$ in 1st Trial $\cap$ H in 2nd $\cap$ T in 3rd ...)

(∵ Each toss is indp.

$\Rightarrow P(H \text{ in } 1^{st}) \cdot P(H \text{ in } 2^{nd}) \cdot P(T \text{ in } 3^{rd}) \dots$

& no. of such exprments where $K=2$ H : ${}^5C_2$

$\therefore P(K=2) = {}^5C_2 \times p \times p \times (1-p) \times (1-p) \times (1-p)$

- For a Binomial RV X, $P(\text{success}) = p$, $P(\text{Failure}) = 1-p$ with n indp. trials :

$$P(X=K) = {}^nC_K \; p^K (1-p)^{n-K}$$

- $E(x) = \sum_{x} x \, P(X=x)$

M1:

let X_i be a Bernaulli RV which ~~denot~~ takes value 1 if success in i^{th} trial & 0 otherwise

& w.k.t X is no. of successes in n trials.

So,

$$X = X_1 + X_2 + \dots + X_n.$$

So,

$$E(X) = E(X_1) + E(X_2) + E(X_3) + \dots E(X_n)$$

$\Rightarrow p + p + p + \dots p$

$\Rightarrow np.$

$$\therefore E(X) = np$$

M2:

$$E(X) = \sum_{k=0}^{n} k \times {}^{n}C_k \times p^k \times (1-p)^{n-k}$$

w.r.t. $k \, {}^{n}C_k = n \, {}^{n-1}C_{k-1}$, for $n=1$

$$\Rightarrow \sum_{k=1}^{n} n \times {}^{n-1}C_{k-1} \, p^k (1-p)^{n-k} + 0 \times \cancel{{}^{n}C_0} \times \ldots$$

let $r = k-1$ $k=1 \Rightarrow r=0$ $k=n \Rightarrow r=n-1$

$$\Rightarrow \left(\sum_{r=0}^{n-1} {}^{n-1}C_r \, p^{r+1} q^{n-(r+1)}\right) \times n$$

$$\Rightarrow \left(\sum_{r=0}^{n-1} {}^{n-1}C_r \, p^{r} q^{(n-1)-r}\right) \times pn$$

$$\Rightarrow (p+q)^{n-1} \times pn$$

$$\Rightarrow np$$

$V(X) = V(X_1) + V(X_2) + V(X_3) + \ldots + V(X_n)$ {$\because X_i$ are indp.}

$\Rightarrow n \times (p \times (1-p))$

$\Rightarrow np(1-p)$

Ex. Given a hash table with n slots, & open-chaining, find expected no. of items per slot, with m items.

Let's say X: No. of items in an arbitrary slot, say S_k.

$E(X)$: no. of items in the slot.

p = prob. of item to go to ~~slot~~ the slot $= 1/n$

X_i: i^{th} item maps to slot k.

$\therefore E(X) = E(X_1) + E(X_2) + \ldots E(X_m)$

$$\Rightarrow \frac{1}{n} + \frac{1}{m} + \frac{1}{n} \ldots = \frac{m}{n} = \frac{\text{no. of items}}{\text{no. of slots}}.$$

Ex. Given a hash table with n slots, find the no. of empty slots for closed chaining, m items.

X : No. of empty slots.
E(X) : Expected no. of empty slot.

$$x_i = \begin{cases} 1 & \text{, if } i^{th} \text{ slot is empty} \\ 0 & \text{, otherwise.} \end{cases}$$

$$X = x_1 + x_2 + \dots + x_n.$$

Prob. that i^{th} slot is empty after m items are inserted.

$$\Rightarrow \left(1 - \frac{1}{n}\right)^m$$

$$E(X) = E(x_1) + E(x_2) + \dots E(x_n)$$

$$\Rightarrow n \times \left(1 - \frac{1}{n}\right)^m$$

- The prob. of getting k out of n successes, the max. prob of success would be

$$P(\text{success}) = \frac{k}{n}$$

Ex. So say we want 2 out of 10 success, so if we have $P(\text{success}) = \frac{2}{10} = 0.2$ then

$P(X=2)$ is maximum.

* Poisson RV

A RV X has Poisson distribution if it is similar to Binomial RV with $n \to \infty$ & $p \to 0$, i.e. very large no. of independent trials & very small probability of success in a trial.

$$P(X=k) = \frac{e^{-\lambda} \lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

where k is the no. of successes,
& λ is parameter (given in question)

- $E(x) = \sum_{k=0}^{\infty} k \times \frac{e^{-\lambda} \times \lambda^k}{k!} = \lambda$

- $V(x) = E(x^2) - (E(x))^2 = \lambda$

- If λ isn't given in question, then $\lambda = np$ since $E(x) = \lambda$.

Ex. On avg. there are two accidents in a day, what is the prob. of 4 accidents on a given day?

X: No. of accidents in a day
Given $E(x) = \lambda = 2$.

$$P(X=4) = \frac{e^{-2}\, 2^4}{4!} = 0.09$$

- To identify a Poisson process look for number of events in a fixed space/interval.

Ex. Suppose that the no. of typos in a book has a Poisson distribution with $\lambda = 0.5$. What is the prob. that there is a typo on a page?

X: No. of errors on a page.

$P(X \geq 1) = 1 - P(X=0)$

$\Rightarrow 1 - \frac{e^{-0.5} \times 0.5^0}{(0)!}$

$\Rightarrow 0.393$

Ex. On avg. 2 accidents per day. Prob. of 4 accidents in 2 days?

Avg. no. of accidents in 2 days $= 2 \times 2 = 4$

$\therefore \lambda = 4$.

X: No. of accidents in 2 days.

$P(X=4) = \frac{e^{-4} \times 4^4}{4!} = 0.19$

Ex. On avg. 6 patients per hour in a hospital.

(a) Find prob. that during a 90 min period no. of patients is:

(i) exactly 7

avg # patient in 60 min = 6

avg # patient in 90 min = $\frac{6}{60} \times 90 = 9$.

$\therefore \quad P(x=7) = \frac{e^{-9} \times 9^7}{7!}$

(ii) At least 3

$$P(x \geq 3) = 1 - \left(P(x=0) + P(x=1) + P(x=2) \right)$$

(b) Say a patient arrives at '11.30 am. Find prob that next patient arrives before 11.45 am.

time period = 15 min

avg # patient in 15 min = $\frac{6}{60} \times 15 = 1.5 = \lambda$

$P(x \geq 1) = 1 - P(x=0)$

$\Rightarrow 1 - \frac{e^{-1.5} \times (1.5)^0}{0!}$

$\Rightarrow 0.776.$

- Make sure that the interval for λ & asked interval matches.

Ex. $\lambda = 2$ coins / hr   asked interval = 30 mins.

Mismatch. Convert λ.

$\lambda = \frac{2}{60} \times 30$ coins / 30 mins = 1 coin / 30 min.

* Uniform Dicrete RV

A RV X has discrete uniform distribution if for each value that X can take, P(x) is same, i.e.

$$\forall x_i \; P(x_i) = K$$

If there are n values in X's range, $x_1, x_2, x_3 \dots x_n$ then

$$P(x_i) = \frac{1}{n}$$

Ex. Rolling a fair die.

X	1	2	3	4	5	6
P(X=x)	1/6	1/6	1/6	1/6	1/6	1/6

Since for all values in X's range has equal prob. that's why Discrete Uniform RV.

- $E(X) = \sum_{x_i} x_i \times P(x_i)$

$$\Rightarrow \frac{1}{n} \sum_{x_i} x_i$$

$$\Rightarrow \frac{\sum_{x_i} x_i}{n}$$

- $V(X) = E(x^2) - (E(X))^2$

$$\Rightarrow \frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2$$

Ex. Suppose X is a disc. Uniform RV on set of $\{1, 2, 3 \ldots N\}$. Find $E(X)$ & $V(X)$.

Since N terms, $P(X = x_i) = \frac{1}{N}$

$$E(x) = \frac{\sum x_i}{N} = \frac{N \times (N+1)}{N} = \frac{N+1}{2}$$

$$E(x^2) = \frac{\cancel{N} \times (N+1) \times (2N+1)}{6 \times \cancel{N}} = \frac{(N+1)(2N+1)}{6}$$

$$V(x) = \frac{(N+1)(2N+1)}{6} - \frac{(N+1)^2}{4}$$

$$\Rightarrow (N+1)\left(\frac{8N+4 - 6N - 6}{24}\right)$$

$$\Rightarrow (N+1)\frac{(2N-2)}{24}$$

$$\Rightarrow \frac{(N+1)(N-1)}{12}$$

$$\Rightarrow \frac{N^2 - 1^2}{12}$$

- Say $X = a, a+1, a+2, \ldots b$ has a Uniform distribution.

#terms $= b - a + 1$

$$\boxed{N = b - a + 1}$$

$$P(X = x_i) = \frac{1}{b-a+1}$$

$$E(x) = \left(\frac{b(b+1)}{2} - \frac{(a-1)a}{2}\right) \times \frac{1}{b-a+1}$$

$$\Rightarrow \boxed{\frac{b+a}{2}}$$

$$\boxed{V(x) = \frac{N^2 - 1}{12}}$$

$$\Rightarrow \frac{(b-a+1)^2 - 1}{12}$$

$$\Rightarrow \frac{(b-a)(b-a+2)}{12}$$

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Continuos RV

Ex. X is RV. $E(X) = 2$ $V(X) = 25$.

$Z = \frac{X-2}{5}$. Find $E(Z)$ & $V(Z)$.

$$E(Z) = E\left(\frac{X-2}{5}\right) = \frac{E(X)}{5} - \frac{2}{5} = \frac{2}{5} - \frac{2}{5} = 0.$$

$$V(Z) = V\left(\frac{X-2}{5}\right) = V\left(\frac{X}{5}\right) = \frac{1}{25} \times V(X) = \frac{25}{25} = 1.$$

Ex. Given RV X, with $E(X) = 3$ $V(X) = 36$, define a new RV Z such that $Z = f(X)$ & $E(Z) = 0$ & $V(Z) = 1$.

we can define $Z = \frac{X-3}{\sqrt{36}} = \frac{X-3}{6}$

i.e. $Z = \frac{X-\mu}{\sigma}$

* Probability Density Function

A func. $f(x)$ for RV X, is called X's PDF if for all values of a & b ∈ Real Nos., the following is satisfied:

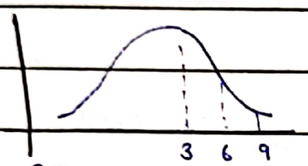
$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

- For a valid PDF
  (i) $P(a \leq X \leq a) = P(X = a) = \int_a^a f(x)\,dx = 0$

(2) $\int_{-\infty}^{\infty} f(x)\,dx = P(-\infty \leq X \leq \infty) = 1$

- PMF was just probability for diff. points collected together.
- PDF doesn't tell the probability directly, but it tells comparitively how much likely it is to find X in a particular range [a,b].

Ex.



PDF

So X is more likely to be found in range [3,6] than [6,9].

- $P(a < X < b) = P(a \leq X < b) = P(a < X \leq b) = P(a \leq X \leq b)$

  for continuous RV X, since point probabilities are 0.

Ex. X is a cont. RV with following PDF.

$$f(x) = \begin{cases} ce^{-x}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases} \qquad c \geq 0.$$

(1) c?

w.r.t. $P(-\infty \leq X \leq \infty) = 1 = \int_{-\infty}^{\infty} ce^{-x}\,dx$

$$\Rightarrow c\int_{-\infty}^{\infty} e^{-x}\,dx = c\,\frac{e^{-x}}{(-1)}\Big|_{-\infty}^{\infty}$$

$$\Rightarrow c = 1$$

(2) $P(1 \leq X \leq 3) = \int_{1}^{3} 1 \times e^{-x}\,dx = -(e^{-3} - e^{-1})$

$$\Rightarrow 0.318$$

Ex. X is RV with PDF

$$f(x) = \begin{cases} 1/x^2 , & x \in [a, 1] \\ 0 , & \text{otherwise.} \end{cases}$$

find a?

$$\int_{-\infty}^{\infty} \frac{1}{x^2} dx = \int_{a}^{1} \frac{1}{x^2} dx = 1$$

$$\Rightarrow \left. \frac{x^{-1}}{-1} \right|_{a}^{1} = 1$$

$$\Rightarrow \left(1 - \frac{1}{a}\right) = -1$$

$$\Rightarrow a = 1/2$$

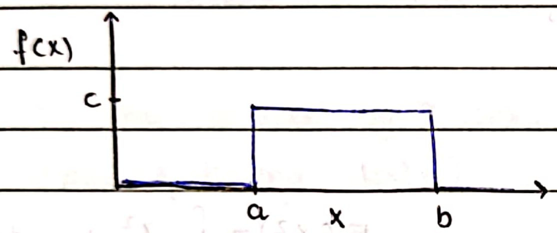
* Uniform Continuous RV

A RV X is having Uniform distribution if it has a PDF

$$f(x) = \begin{cases} c, & a \leq X \leq b \\ 0, & \text{otherwise} \end{cases}$$

where ~~a & b are~~ [a,b] is the range of X & c is some +ve constant.

Probability $\propto$ length of interval $(b-a)$



$$\int_a^b c\,dx = 1$$

$$\Rightarrow cx\Big|_a^b = 1$$

$$\Rightarrow c = \frac{1}{b-a}$$

$$\therefore f(x) = \begin{cases} 1/(b-a), & a \leq X \leq b \\ 0, & \text{otherwise} \end{cases}$$

Ex.

which prob. is bigger $P(a \leq X \leq \alpha)$ or $P(a \leq X \leq \beta)$?

$P(a \leq X \leq \beta)$ since $p \propto$ length of interval

- $$P(\alpha \leq X \leq \beta) = \int_\alpha^\beta \frac{1}{(b-a)}\,dx = \frac{\beta-\alpha}{b-a} \text{ or } \underbrace{(\beta-\alpha)}_{\text{width}} \times \underbrace{c}_{\text{height}}$$

Ex. X is uniformly distributed RV in range (0, 10).

Calc.

(1) $P(X<3)$ $\quad c = 1/(b-a) = 1/10$

$\therefore P(X<3) = (3-0) \times \frac{1}{10} = 0.3$

(2) $P(X>6)$

$P(X>6) = (10-6) \times \frac{1}{10} = 0.4$

(3) $P(5<X<8) = (8-5) \times \frac{1}{10} = 0.3$

- $E(X) = \int_{-\infty}^{\infty} x f(x) dx$

$\Rightarrow \int_a^b x \cdot \frac{1}{b-a} dx$

$\Rightarrow \frac{x^2}{2}\Big|_a^b \times \frac{1}{b-a}$

$\Rightarrow \frac{b+a}{2}$

$E(X^2) = \int_{-\infty}^{\infty} x^2 \frac{1}{b-a} dx$

$\Rightarrow \frac{x^3}{3}\Big|_a^b \times \frac{1}{b-a}$

$\Rightarrow \frac{(b-a)(b^2+ab+a^2)}{3} \times \frac{1}{(b-a)}$

$\Rightarrow \frac{b^2+ab+a^2}{3}$

- $V(X) = E(X^2) - E(X)^2$

$\Rightarrow \frac{b^2+ab+a^2}{3} - \frac{(b^2+2ab+a^2)}{4}$

$\Rightarrow \frac{b^2-2ab+a^2}{12}$

$\Rightarrow \frac{(b-a)^2}{12}$

Ex. A bus is uniformly late b/w 2 to 10 min. How long do you expect to wait?

$$E(X) = \frac{(b+a)}{2} = 6$$

$$V(X) = \frac{(b-a)^2}{12} = 5.3$$

$$\sigma = \sqrt{V(X)} = 2.3$$

$\therefore$ On avg we have to wait 6 min $\pm$ 2.3 minutes.

If bus is late by 7 min. or more you are late. What's the prob. of being late?

$$P(X \geq 7) = (10-7) \times \frac{1}{8} = \frac{3}{8}$$

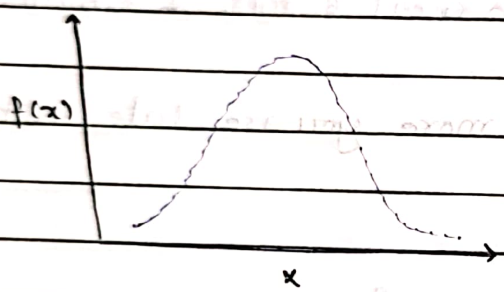
* Normal RV

Arises naturally in nature that's why Normal distribution.

A RV x is having Normal Distribution if it has the following PDF:

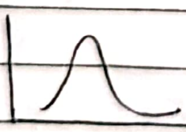
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where $E(X) = \mu$ & $V(X) = \sigma^2$

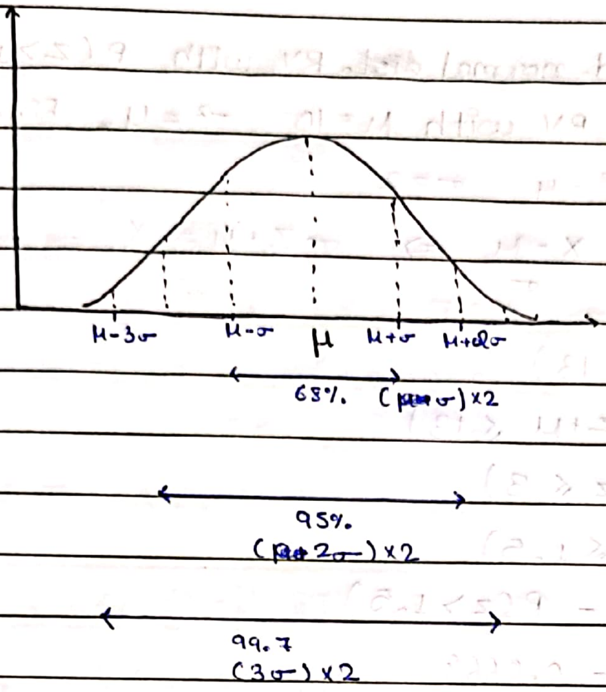


Symmetric about the mean.

- Changing the mean doesn't change shape of curve only move it on x-axis.
- Changing Variance widens the curve, doesn't shift it.

Ex.  is it Normal dist?

No. Not Symmetric about mean.



Standard Normal Distribution

A Normal Distribution with $\mu = 0$ & $\sigma = 1$

Any RV X having Normal Dist. can be converted to Std. Normal dist:

$X,\ E(X) = \mu,\ V(X) = \sigma^2$

$$Z = \frac{X - \mu}{\sigma}$$

$$E(Z) = 0$$

$$V(Z) = 1$$

For Std. Normal Dist, the probabilities are precalculated, that's why we convert X to Z, so that we can directly find prob. of X.

Ex. Let Z is std. normal RV with $P(Z > 1.5) = 0.0668$. Let X is normal RV with $\mu = 10$ $\sigma^2 = 4$. Find $P(X \leq 13)$.

$\mu = 10$ $\sigma^2 = 4$ $\sigma = 2$

$$Z = \frac{X - \mu}{\sigma} \Rightarrow \sigma Z + \mu = X$$

$P(X \leq 13)$

$\Rightarrow P(\sigma Z + \mu \leq 13)$

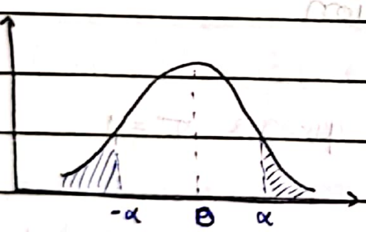
$\Rightarrow P(2Z \leq 3)$

$\Rightarrow P(Z \leq 1.5)$

$\Rightarrow 1 - P(Z > 1.5)$

$\Rightarrow 1 - 0.0668$

$\Rightarrow 0.9332$



For Std. Normal Distribution

$P(Z \leq -\alpha) = P(Z \geq \alpha)$

Since SND curve is symmetrical about the mean 0, then area under curve for range $[-\infty, -\alpha]$ & $[\alpha, \infty]$ must be same, i.e.

$$P(-\infty \leq Z \leq -\alpha) = P(\alpha \leq Z \leq \infty)$$

$$\Rightarrow \boxed{P(Z \leq -\alpha) = P(\alpha \leq Z)}$$

i.e. $P(\alpha \leq Z) = P(Z \leq -\alpha)$

& $P(Z \leq -\alpha) = 1 - P(Z \leq \alpha)$

Ex. Let $X \sim N(70, 100)$, i.e. $\mu = 70$, $\sigma = 10$. Find q such that $P(|X - 70| \geqslant q) = 0.32$. [Given: $P(Z < 1) = 0.84$]

$P((X-70) \geqslant q$ or $(X-70) \leqslant -q)$

$\Rightarrow P(X - 70 \geqslant q) + P(X - 70 \leqslant -q)$

$\Rightarrow P(X \geqslant 70 + q) + P(X \leqslant 70 - q)$

$X = \sigma Z + \mu = 10Z + 70.$

$\Rightarrow P(Z \geqslant q/10) + P(Z \leqslant -q/10)$

(w.r.t. $P(Z \leqslant -\alpha) = P(Z \geqslant \alpha)$)

$\Rightarrow 2 \times P(Z \leqslant -q/10) = 0.32$

$\Rightarrow P(Z \leqslant -q/10) = 0.16. = 1 - P(Z < 1) = P(Z \geqslant 1)$

$\Rightarrow P(Z \geqslant q/10) = P(Z \geqslant 1)$

$\therefore \frac{q}{10} = 1 \Rightarrow q = 10.$

* Exponential Distribution

A continuous RV X has exponential distribution if it has the following PDF:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

where $E(x) = 1/\lambda$ $\quad V(x) = 1/\lambda^2$

Ex. X is exponential RV. $E(x) = 0.5$. Find $E((x+3)^2)$

$E((x+3)^2) = E(x^2 + 6x + 9)$

$\Rightarrow E(x^2) + 6E(x) + E(9)$

w.k.t. $E(x) = \frac{1}{\lambda} = \frac{1}{2} \Rightarrow \lambda = 2$

$V(x) = 1/\lambda^2 = \frac{1}{4} = E(x^2) - E(x)^2$

$\Rightarrow 0.25 + 0.25 = E(x^2)$

$\Rightarrow E(x^2) = 0.5$

$\Rightarrow E((x+3)^2) = 0.5 + 6 \times 0.5 + 9$

$\Rightarrow 12.5.$

Mean, Median, Mode.

* For Data Points

* Mean: ~~Avera~~ Given ~~some~~ n data points $x_1, x_2, x_3 \dots x_n$. their mean is

$$\frac{\sum_{i=1}^{n} x_i}{n}$$

* Mode: Given n data points the most frequent data point.

* Median: Given n data points, arrange the data in ascending order.

If n is odd then $\frac{n+1}{2}$ th number, is median.

If n is even then $\frac{x_{n/2} + x_{(n/2+1)}}{2}$ is median.

* For Probabilistic Distribution

* Mean: $\sum_{x_i} x_i \times p(x_i)$ or $\int_{-\infty}^{\infty} x_i \times p(x_i)$

* Median: The value m_x is called median if:

[Discrete] $\left(P(X \leq m_x) \geq \frac{1}{2} \right) \left(P(X \geq m_x) \geq \frac{1}{2} = 0.5 \right)$

[Continuous] $P(X \leq m_x) = \frac{1}{2} = 0.5$

where X may or maynot take value m_x.

Ex. $P(X=x) = \frac{x^2}{90}$ $x = \{2,3,4,5,6\}$.

Find mean & median.

X	2	3	4	5	6
$P(X=x)$	$4/90$	$9/90$	$16/90$	$25/90$	$36/90$.

(1) Mean $= \sum x \times P(X=x)$

$\Rightarrow \frac{2\times4}{90} + \frac{3\times9}{90} + \dots + \frac{6\times36}{90}$

$\Rightarrow 4.88$

(2) We can draw samples with the given probability

2	3	4	5	6
4	9	16	25	36.

2, 2, 2, 2, 3, 3, ... 3, 4, 4, ... 4, 5, 5, ... 5, 6, 6, 6 ... 6

So, 5 is median.

We can verify.

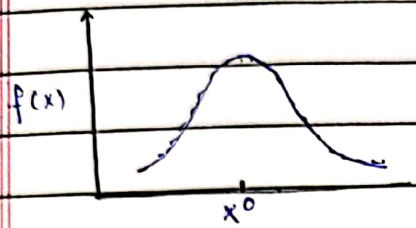
$P(X \leq 5) = \frac{54}{90} \geq 0.5$ $P(X \geq 5) = \frac{61}{90} \geq 0.5$

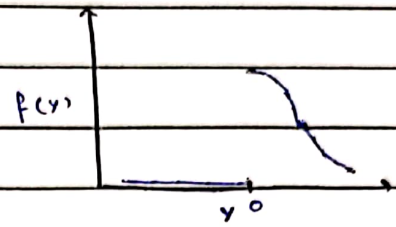
(3) We can check Mode just by looking at the "probabilistically" drawn samples.
6 has highest freq.

$\therefore$ Mode $= 6$.

Ex. Let $X \sim N(0, \sigma^2)$, i.e. $\mu = 0$ $V(X) = \sigma^2$. Let $Y = \max(X, 0)$

The median of Y ?





we can see

$$P(X \leq 0) = P(X \geq 0) = \frac{1}{2}$$

$$P(Y \leq 0) = P(Y = 0) = P(X \leq 0) = \frac{1}{2}$$

$$P(Y \geq 0) = P(X \geq 0) = \frac{1}{2}$$

$\therefore$ Median $Y = 0$.