10/03/20 Q) $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = e^{2x}$

Soln AE: $m^2 - 7m + 6 = 0$

$\Rightarrow (m-6)(m-1) = 0$

$\Rightarrow m = 6, m = 1$

CF = $c_1 e^{6x} + c_2 e^{x}$

PI = $\frac{e^{2x}}{(D-6)(D-1)} = \frac{e^{2x}}{-4 \times 1} = -\frac{e^{2x}}{4}$

$\therefore y = c_1 e^{6x} + c_2 e^{x} - \frac{e^{2x}}{4}$

Q) $\frac{d^2y}{dt^2} - y = \sin t, \; t \geq 0$

Soln AE: $m^2 - 1 = 0$

$m = \pm 1$

CF = $c_1 e^{t} + c_2 e^{-t}$

PI = $\frac{\sin t}{D^2 - 1} = -\frac{\sin t}{2}$

$\therefore y = c_1 e^{t} + c_2 e^{-t} - \frac{\sin t}{2}$

Q) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 5\cos x$. Find particular solution.

a) $0.5\cos x + 1.5\sin x$

b) $1.5\cos x + 0.5\sin x$

c) $1.5\sin x$

d) $0.5\cos x$

Soln M1 PI = $\frac{5\cos x}{D^2 + 3D + 2} = \frac{5\cos x}{3D + 1} = \frac{5\cos x\,(1 - 3D)}{(1 - 9D^2)} = \frac{5\cos x + 15\sin x}{10}$

$\Rightarrow$ PI = $0.5\cos x + 1.5\sin x$ $\Rightarrow$ option a.

M2 We can inspect the options & try elimination technique.

By direct observation here we can eliminate option c & d.

Note Always be cool in your exam [From S.J. Sir]

***

Q) $\frac{d^4v}{dx^4} + 4\lambda^4 v = 1 + x + x^2$

Soln: AE: $m^4 + 4\lambda^4 = 0$

$\Rightarrow (m^2 + 2\lambda^2)^2 - 4m^2\lambda^2 = 0$ $\Rightarrow (m^2 + 2\lambda^2 + 2m\lambda)(m^2 + 2\lambda^2 - 2m\lambda) = 0$

$$m^2 + 2\lambda^2 + 2\lambda m = 0 \qquad \text{OR} \qquad m^2 + 2\lambda^2 - 2\lambda m = 0$$

$$\Rightarrow (m+\lambda)^2 + \lambda^2 = 0 \qquad \Rightarrow (m-\lambda)^2 + \lambda^2 = 0$$

$$\Rightarrow (m+\lambda)^2 - (i\lambda)^2 = 0 \qquad \Rightarrow (m-\lambda)^2 - (i\lambda)^2 = 0$$

$$\Rightarrow (m+\lambda+i\lambda)(m+\lambda-i\lambda) = 0 \qquad \Rightarrow (m-\lambda+i\lambda)(m-\lambda-i\lambda) = 0$$

$$\Rightarrow m = -\lambda - i\lambda, \ m = -\lambda + i\lambda \qquad \Rightarrow m = \lambda - i\lambda, \ m = \lambda + i\lambda$$

$$\therefore CF = e^{-\lambda x}[c_1\cos(\lambda x) + c_2\sin(\lambda x)] + e^{\lambda x}[c_3\cos(\lambda x) + c_4\sin(\lambda x)]$$

$$PI = \frac{1+x+x^2}{D^4 + 4\lambda^4} = \frac{1+x+x^2}{4\lambda^4\left(1+\frac{D^4}{4\lambda^4}\right)} = \frac{1}{4\lambda^4}\left(1+\frac{D^4}{4\lambda^4}\right)^{-1}(1+x+x^2)$$

$$\Rightarrow PI = \frac{1}{4\lambda^4} \times (1+x+x^2)$$

$$\therefore y = e^{-\lambda x}[c_1\cos(\lambda x) + c_2\sin(\lambda x)] + e^{\lambda x}[c_3\cos(\lambda x) + c_4\sin(\lambda x)] + \frac{1}{4\lambda^4}(1+x+x^2).$$

Q) $\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} - \frac{u}{x^2} = 8x$, If $u=0$ at $x=0$, $u=2$ at $x=1$, Find u at $x = 1/2$.

Soln $\quad x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} - u = 8x^3$

$x = e^z$

$x^2\frac{d^2u}{dx^2} = D(D-1)u$

$x\frac{du}{dx} = Du$

$[D(D-1) + D - 1]u = 8e^{3z}$

$\Rightarrow [D^2-1]u = 8e^{3z}$

AE: $m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

$CF = c_1 e^z + c_2 e^{-z}$

$PI = \frac{8e^{3z}}{D^2-1} = \frac{8e^{3z}}{8} = e^{3z}$

$$\therefore u = c_1 e^z + c_2 e^{-z} + e^{3z} \quad \Rightarrow \quad u = c_1 x + \frac{c_2}{x} + x^3$$

$u(0) = 0: \quad 0 = 0 + \frac{c_2}{0} + 0 \quad \Rightarrow \quad c_2 = 0$

$u(1) = 2: \quad 2 = c_1 + 1 \quad \Rightarrow \quad c_1 = 1$

$\therefore u = x + x^3 \quad$ i.e $\quad u(1/2) = 1/2 + 1/8 = 5/8$

Scanned with CamScanner

Q) $\frac{d^4y}{dx^4} - y = 15\cos 2x$

Soln AE: $m^4 - 1 = 0$

$\Rightarrow m^4 = 1$

$\Rightarrow m = +1, -1, +i, -i$

$CF = c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x$

$PI = \frac{15\cos 2x}{D^4 - 1} = \frac{15\cos 2x}{16-1} = \cos 2x$

$\therefore y = c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x + \cos 2x$

HW

Q) $\frac{d^2y}{dt^2} + \lambda^2 y = \cos\omega t + k, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = 0$, λ, ω, k are const.

Soln AE: $(m^2 + \lambda^2) = 0$

$m^2 = -\lambda^2$

$m = \pm\lambda i$

$CF = c_1\cos\lambda t + c_2\sin\lambda t$

$PI = \frac{\cos\omega t + k}{D^2 + \lambda^2} = \frac{\cos\omega t}{D^2+\lambda^2} + \frac{k}{D^2+\lambda^2}$

$\Rightarrow PI = \frac{\cos\omega t}{\lambda^2 - \omega^2} + \frac{k}{\lambda^2}$

$\therefore y = c_1\cos\lambda t + c_2\sin\lambda t + \frac{\cos\omega t}{\lambda^2-\omega^2} + \frac{k}{\lambda^2}$

$y(0) = 0$: $0 = c_1 + \frac{1}{\lambda^2-\omega^2} + \frac{k}{\lambda^2} \Rightarrow c_1 = -\left[\frac{1}{\lambda^2-\omega^2} + \frac{k}{\lambda^2}\right]$

$\frac{dy}{dt} = -\lambda c_1\sin\lambda t + \lambda c_2\cos\lambda t + \frac{\omega}{\omega^2-\lambda^2}\sin\omega t$

$\frac{dy}{dt}(0) = 0$: $0 = c_2\lambda \Rightarrow c_2 = 0$

$\therefore y = \left[\frac{1}{\omega^2-\lambda^2} - \frac{k}{\lambda^2}\right]\cos\lambda t + \frac{\cos\omega t}{\lambda^2-\omega^2} + \frac{k}{\lambda^2}$

$\therefore y = \left(\frac{\ln x}{x}\right)^2$ | $\therefore y(e) = \left(\frac{1}{e}\right)^2 = \frac{1}{e^2}$ $\Rightarrow$ option d

Q) $x''(t) + 3x'(t) + 2x(t) = 5$ · Solution of DE when $t \to \infty$

Soln AE: $(m^2 + 3m + 2 = 0)$ $\Rightarrow m = -2, -1$ | CF: $c_1 e^{-2t} + c_2 e^{-t}$

PI $= \dfrac{5e^{0t}}{D^2 + 3D + 2} = 5/2$

$\therefore x = c_1 e^{-2t} + c_2 e^{-t} + 5/2$

$\therefore x(\infty) = 5/2$

Q) $(D^2 - 4D + 4)y = 0$

a) $c_1 e^{2x}$ b) $c_1 e^{2x} + c_2 e^{-2x}$

c) $c_1 e^{2x} + c_2 e^{2x}$ d) $c_1 e^{2x} + c_2 x e^{2x}$

Soln M1 Homogeneous HODE

M2 option a is straight away eliminated
by observation we can get the roots as 2, 2.
$\therefore$ option d is the ans.

Q) $\dfrac{d^2y}{dx^2} + k^2 y = 0$ BC i) $y = 0, x = 0$ ii) $y = 0, x = a$

The form of non zero soln. (where 'm' varies over all integer) are

a) $y = \sum_m A_m \sin\left(\frac{m\pi x}{a}\right)$ b) $y = \sum_m A_m \cos\left(\frac{m\pi x}{a}\right)$

c) $y = \sum_m A_m x^{m\pi/a}$

Soln M1 AE: $m^2 + k^2 = 0$ $\Rightarrow m = \pm ik$ | CF $= c_1 \cos kx + c_2 \sin kx$.

$\therefore y = c_1 \cos kx + c_2 \sin kx$

$y(0) = 0$: $0 = c_1$ | $y(a) = 0$: $0 = c_2 \sin ak$.

For non. zero soln, $c_2 \neq 0$ &

$\sin ak = 0$
$\Rightarrow ak = m\pi$
$\Rightarrow k = \frac{m\pi}{a}$.

$\therefore y = c_2 \sin\left(\frac{m\pi x}{a}\right)$ $\Rightarrow$ option a

M2 Try to satisfy the boundary conditions with the options $\Rightarrow$ only option a satisfies.

11/03/... Q) $\frac{d^2y}{dx^2} = 3x-2$; $y(0)=2$, $y'(1)=3$

a) $y' = 9\frac{x^3}{3} - \frac{x^2}{2} + 3x - 6$

b) $y = 3x^3 - \frac{x^2}{2} - 5x + 2$

c) $y = \frac{x^3}{2} - x^2 - \frac{5x}{2} + 2$

d) $y = x^3 - \frac{x^2}{2} + 5x + 3/2$

Soln M1 Double Integration

M2 Non-homo HODE

Trick M3 Differentiating options twice then we need to get $3x-2$ as ans.
Since the first term is $3x$, the only option that gives the first term as $3x$ is option c i.e. $\frac{d^2}{dx^2}\left(\frac{x^3}{2}\right) = \frac{d}{dx}\left(\frac{3}{2}x^2\right) = 3x$.

$\therefore$ option c is the Ans.

Q) $\frac{d^2y}{dx^2} + y = x$; i) $x=0, y=1$ ii) $x=\frac{\pi}{2}, y=\frac{\pi}{2}$

Soln AE: $m^2+1=0$
$\Rightarrow m = \pm i$

CF $= c_1\cos x + c_2\sin x$

PI $= \frac{xe^{0x}}{D^2+1} = \frac{x}{1} = x$

$\therefore y = c_1\cos x + c_2\sin x + x$.

$y(0)=1$: $1 = c_1$

$y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$: $\frac{\pi}{2} = 0 + c_2 1 \frac{\pi}{2} \Rightarrow c_2 = 0$

$\therefore y = \cos x + x$

Q) If $x^2\frac{dy}{dx} + 2xy = \frac{2\ln(x)}{x}$; $y(1)=0$, $y(e)=?$

a) e b) 1 c) 1/e d) $1/e^2$

Soln First order Linear DE (Leibnitz)

$\frac{dy}{dx} + \frac{2}{x}y = \frac{2\ln x}{x^3}$

IF $= e^{\int P dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

$\therefore y \times x^2 = \int x^2 \times \frac{2\ln x}{x^3}dx \Rightarrow yx^2 = 2\int \frac{\ln x}{x}dx$

$\Rightarrow yx^2 = 2\frac{(\ln x)^2}{2} + c \Rightarrow y = \left(\frac{\ln x}{x}\right)^2 + \frac{c}{x^2}$.

$y(1)=0$: $0 = 0 + c \Rightarrow c = 0$

Q) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3e^{2x}$

Soln AE: $m^2 + 4m + 3 = 0$

$\Rightarrow m = -3, -1$

$CF = c_1e^{-3x} + c_2e^{-x}$

$PI = \frac{3e^{2x}}{D^2+4D+3} = \frac{3e^{2x}}{4+8+3} = \frac{1}{5}e^{2x}$

$\therefore y = c_1e^{-3x} + c_2e^{-x} + \frac{1}{5}e^{2x}$

12/03/20

LAPLACE TRANSFORM: (Application in IIODE)

$x(t) \xrightarrow{LT} X(s)$

$$X(s) = \int_0^\infty x(t)\, e^{-st}\, dt$$

[In general limits range from $-\infty$ to ∞ but in Eng. maths limits range from 0 to ∞]

Some Imp. Formulae ($t = 0$ to ∞)

$f(t)$	$\rightarrow$	$F(s)$
a	$\rightarrow$	a/s
e^{at}	$\rightarrow$	$\frac{1}{s-a}$
$\cos\omega t$	$\rightarrow$	$\frac{s}{s^2+\omega^2}$
$\cosh\omega t$	$\rightarrow$	$\frac{s}{s^2-\omega^2}$
e^{-at}	$\rightarrow$	$\frac{1}{s+a}$
$\sin\omega t$	$\rightarrow$	$\frac{\omega}{s^2+\omega^2}$
$\sinh\omega t$	$\rightarrow$	$\frac{\omega}{s^2-\omega^2}$
t^n	$\rightarrow$	$\frac{n!}{s^{n+1}}$

Note: For fractions $\Rightarrow n! = n\Gamma n$

Q) $f(t) = \sqrt{t}$ $\quad F(s) = ?$

Soln $F(s) = \frac{1/2\,!}{s^{1/2+1}} = \frac{\frac{1}{2}\Gamma\frac{1}{2}}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$

Shifting Property: $L\{e^{-at}f(t)\} = F(s+a)$

Q) $L\{e^{-at}\sin\omega t\} = ?$

Soln $L\{e^{-at}\sin\omega t\} = \frac{\omega}{(s+a)^2+\omega^2}$

Q) $L\{1\} = ?$

Soln M1 $L\{1\} = L\{e^{0t}\} = \frac{1}{s}$ | M2 $L\{1\} = L\{t^0\} = \frac{1}{s}$

Q) $X(s) \xrightarrow{LT} x(t)$? : $L^{-1}\left[\frac{2}{s-3} - \frac{3s}{s^2+16}\right]$

Soln $= L^{-1}\left\{\frac{2}{s-3}\right\} - L^{-1}\left\{\frac{3s}{s^2+16}\right\} = 2e^{3t} - 3\cos(4t)$, $t > 0$

Q) $L^{-1}\left[\frac{1}{s^2-2s+5}\right] = ?$

Soln $= L^{-1}\left[\frac{1}{(s-1)^2+2^2}\right] = \frac{1}{2} L^{-1}\left[\frac{2}{(s-1)^2+2^2}\right] = \frac{1}{2} e^{+t} \sin 2t$, $t > 0$

Q) $L^{-1}\left[\frac{2s-1}{s^2-5s+6}\right] = ?$

Soln $f(t) = L^{-1}\left[\frac{2s-1}{(s-3)(s-2)}\right]$

$= L^{-1}\left[\frac{5}{s-3}\right] - L^{-1}\left[\frac{3}{s-2}\right]$

$\Rightarrow f(t) = 5e^{3t} - 3e^{2t}$, $t > 0$

Partial fractions:

$\frac{2s-1}{(s-3)(s-2)} = \frac{A}{(s-3)} + \frac{B}{(s-2)}$

$A = \left.\frac{2s-1}{(s-2)}\right|_{s=3} = 5$

$B = \left.\frac{(2s-1)}{s-3}\right|_{s=2} = -3$

Q) $y'' + y = \sin 3t$, $y(0) = 0$, $y'(0) = 0$.

Soln Imp

$$\boxed{\begin{aligned} L[y] &= s^0 Y(s) \\ L[y'] &= s\,Y(s) - s^0 Y(0) \\ L[y''] &= s^2 Y(s) - \cancel{s Y(0)}\ s^1 Y(0) - s^0 Y'(0) \\ L[y'''] &= s^3 Y(s) - s^2 Y(0) - s^1 Y'(0) - s^0 Y''(0) \end{aligned}}$$

Note: If initial conditions are not given, we consider $y(0) = y'(0) = y''(0) = 0$.

$y'' + y = \sin 3t$

$\Rightarrow \quad s^2 Y(s) - s^1 \cancel{y(0)}^{0} - s^0 \cancel{y'(0)}^{0} + s^0 Y(s) = \frac{3}{s^2+3^2}$

$\Rightarrow \quad (s^2+1) Y(s) = \frac{3}{s^2+3^2}$

$\Rightarrow \quad Y(s) = \frac{3}{(s^2+1)(s^2+3^2)} = \frac{3}{8}\left[\frac{(s^2+9)-(s^2+1)}{(s^2+1)(s^2+3^2)}\right] = \frac{3}{8}\left[\frac{1}{s^2+1} - \frac{1}{s^2+3^2}\right]$

$\therefore \quad y(t) = \frac{3}{8} L^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{3}{8} L^{-1}\left\{\frac{1}{s^2+3^2}\right\} = \frac{3}{8}\left[\sin t - \frac{\sin 3t}{3}\right], \ t>0$

Q) $y'' + 2y' + 5y = e^{-x}\sin x \quad , y(0)=0 \quad , y'(0) = 1$

Soln $\quad s^2 Y(s) - s^1 \cancel{y(0)}^{0} - s^0 \cancel{y'(0)}^{1} + 2[sY(s) - s^0 \cancel{y(0)}^{0}] + 5 s^0 Y(s) = \frac{1}{(s+1)^2+1}$

$\Rightarrow \quad (s^2+2s+5) Y(s) = \frac{1}{(s+1)^2+1} + 1 \quad \Rightarrow \quad Y(s) = \frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$

$p = s^2+2s$

$\Rightarrow \quad Y(s) = \frac{p+3}{(p+5)(p+2)}$

Partial fraction

$\frac{(p+3)}{(p+5)(p+2)} = \frac{A}{p+5} + \frac{B}{p+2}$

$A = \left.\frac{(p+3)}{(p+2)}\right|_{p=-5} = \frac{-2}{-3} = 2/3$

$B = \left.\frac{(p+3)}{(p+5)}\right|_{p=-2} = \frac{1}{3}$

$\Rightarrow Y(s) = \frac{2}{3(p+5)} + \frac{1}{3(p+2)}$

$\Rightarrow Y(s) = \frac{2}{3} \times \frac{1}{(s^2+2s+5)} + \frac{1}{3(s^2+2s+2)}$

$\Rightarrow Y(s) = \frac{2}{3} \times \frac{1}{(s+1)^2+2^2} + \frac{1}{3}\left[\frac{1}{(s+1)^2+1}\right]$

$\therefore \cancel{y(t) = \frac{2}{3} e^{-t} \times \frac{1}{2}\sin 2t + \frac{1}{3} e^{-t} \sin t \Rightarrow y(t)}$

$\therefore y(x) = \frac{\cancel{2}}{3} e^{-x} \frac{1}{\cancel{2}} \sin 2x + \frac{1}{3} e^{-x} \sin x \quad \Rightarrow \quad y(x) = \frac{1}{3} e^{-x}[\sin x + \sin 2x]$

Q) $\frac{dx}{dt} + y = 0$; $\frac{dy}{dt} - x = 0$; $x(0) = 1$, $y(0) = 0$

Soln

$sX(s) - x(0) + Y(s) = 0 \Rightarrow sX(s) - 1 + Y(s) = 0$ — ①

$sY(s) - y(0) - X(s) = 0 \Rightarrow sY(s) - X(s) = 0$ — ②

From ② $\Rightarrow X(s) = sY(s)$

$\therefore s^2 Y(s) + Y(s) = 1 \Rightarrow Y(s)[s^2+1] = 1 \Rightarrow Y(s) = \frac{1}{s^2+1}$

$\therefore y(t) = \sin t \Rightarrow x = \frac{dy}{dt} = \cos t$

13/03/20

Q) $\ddot{y} + 2\dot{y} + 101y = (10.4)e^x$, $y(0) = 1.1$, $\dot{y}(0) = -0.9$

P: General soln. of HODE

Q: PI

R: Total solution satisfying B.C.

~~Match the foll.~~

1: $0.1e^x$

2: $e^{-x}[A\cos 10x + B\sin 10x]$

3: $e^{-x}\cos 10x + 0.1e^x$.

Soln By observation we can say P-2, Q-1, R-3

Q) ~~GATE~~ $\ddot{x} + 3x = 0$, $x(0) = 1$, $\dot{x}(0) = 0$, $x(1) = ?$

Soln AE: $m^2 + 3 = 0$

$\Rightarrow m = \pm i\sqrt{3}$

$x = CF = c_1 \cos\sqrt{3}t + c_2 \sin\sqrt{3}t$

$\frac{dx}{dt} = -\sqrt{3}c_1 \sin\sqrt{3}t + \sqrt{3}c_2 \cos\sqrt{3}t$

$x(0) = 1 \Rightarrow 1 = c_1$

$\dot{x}(0) = 0 \Rightarrow 0 = \sqrt{3}c_2 \Rightarrow c_2 = 0$

$\therefore x = c_1 \cos\sqrt{3}t \Rightarrow \therefore x(1) = \cos\sqrt{3} = -0.16$

Q) G.S. of $\frac{d^2y}{dx^2} + y = 0$

a) $y = p\cos x + q\sin x$

b) $y = p\cos x$

c) $y = p\sin x$

d) $y = p\sin^2 x$

Soln option a

Q) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$, $y(0)=1$, $y'(0) = -2$

Soln AE: $m^2 + 2m + 1 = 0$ $\Rightarrow m = -1, -1$

$y = CF = (c_1 t + c_2)e^{-t}$

$y' = -c_1 t e^{-t} + c_1 e^{-t} - c_2 e^{-t}$

$y(0)=1 \Rightarrow 1 = c_2$ | $y'(0) = -2$: $\Rightarrow -2 = c_1 - \cancel{c_2}^{1} \Rightarrow c_1 = -1$

$\therefore y = (-t+1)e^{-t}$

Q) $y'' + 2y' + y = 0$, $y(0) = 0$, $y(1) = 0$, $y(0.5) = ?$

Soln AE: $(m^2 + 2m + 1 = 0)$ $\Rightarrow m = -1, -1$

$y = (c_1 x + c_2)e^{-x}$

$y(0) = 0 \Rightarrow 0 = c_2$

$y(1) = 0 \Rightarrow 0 = c_1 e^{-1} \Rightarrow c_1 = 0$

$\therefore y = 0 \Rightarrow y(0.5) = 0$

***

Q) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$

a) $e^{-(1+i)x}$, $e^{-(1-i)x}$
b) $e^{(1+i)x}$, $e^{(1-i)x}$
c) $e^{-(1+i)x}$, $e^{+(1-i)x}$
d) $e^{(1+i)x}$, $e^{-(1-i)x}$

Soln AE: $m^2 + 2m + 2 = 0$

$m = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

$\therefore y = c_1 e^{(-1+i)x} + c_2 e^{(-1-i)x}$

$\Rightarrow y = c_1 e^{-x(1-i)} + c_2 e^{-x(1+i)}$ $\Rightarrow$ option a

Q) $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = r$. Homogeneous part of DE has real distinct roots if

a) $p^2 - 4q > 0$ b) $p^2 - 4q < 0$ c) $p^2 - 4q = 0$ d) $p^2 - 4q = r$

Soln. Real & distinct roots $\Rightarrow$ $p^2 - 4q > 0$ $\Rightarrow$ option a

Q) $\dfrac{d^2 n(x)}{dx^2} - \dfrac{n(x)}{L^2} = 0$ where L is constant. BC: $n(0) = k$, $n(\infty) = 0$

Soln. of DE = ?

a) $n(x) = k \exp(x/L)$

b) $n(x) = k \exp(-x/\sqrt{L})$

c) $n(x) =$ ~~k²/L~~ $k^2 \exp(-x/L)$

d) $n(x) = k \exp(-x/L)$

Soln

AE: $m^2 - \dfrac{1}{L^2} = 0$

$\Rightarrow m = \pm \dfrac{1}{L}$

$n = CF = c_1 e^{x/L} + c_2 e^{-x/L}$

$n(0) = k$: $k = c_1 + c_2$

$n(\infty) = 0$: $0 = c_1 e^{\infty} + 0 \Rightarrow c_1 = 0$.

$\therefore c_2 = k$

$\therefore n = k e^{-x/L}$

Q) $\dfrac{d^2 x}{dt^2} + 6\dfrac{dx}{dt} + 8x = 0$; $x(0) = 1$; $\left.\dfrac{dx}{dt}\right|_{t=0} = 0$ Soln. is ?

a) $x(t) = 2e^{-6t} - e^{-2t}$

b) $x(t) = 2e^{-2t} - e^{-4t}$

c) $x(t) = -e^{-6t} + 2e^{-4t}$

d) $x(t) = e^{-2t} + 2e^{-4t}$

Soln

AE: $m^2 + 6m + 8 = 0$

$\Rightarrow (m+4)(m+2) = 0$

$\Rightarrow m = -4, m = -2$

$x = CF = c_1 e^{-4t} + c_2 e^{-2t}$

$x(0) = 1 \Rightarrow 1 = c_1 + c_2$ —(1)

$\dfrac{dx}{dt} = -4c_1 e^{-4t} - 2c_2 e^{-2t}$

$\left.\dfrac{dx}{dt}\right|_{t=0} = 0 \Rightarrow 0 = -4c_1 - 2c_2 \Rightarrow c_2 = -2c_1$ —(2)

$\therefore 1 = c_1 - 2c_1 \Rightarrow c_1 = -1$ // & $c_2 = 2$ //

$\therefore x = -e^{-4t} + 2e^{-2t}$ $\Rightarrow$ option b

Q) $\ddot{y} + 2\dot{y} + y = 0$; $y(0) = 1$, $y(1) = 0$, $y(2) = ?$

a) -1

b) $-e^{-1}$

c) $-e^{-2}$

d) e^2

Soln

AE: $m^2 + 2m + 1 = 0$

$\Rightarrow m = -1, -1$

$y = CF = (c_1 x + c_2) e^{-x}$

$y(0) = 1$: $\Rightarrow 1 = c_2$

$y(1) = 0$: $\Rightarrow 0 = (c_1 + 1) e^{-1}$

$\Rightarrow c_1 = -1$

$\therefore y = (-x + 1) e^{-x}$ $\Rightarrow y(2) = -e^{-2}$ $\Rightarrow$ option c

14/03/25 Q) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 9x+6$

a) $y = (c_1x+c_2)e^{-3x}$

b) $y = c_1e^{3x} + c_2e^{-3x}$

c) $y = (c_1x+c_2)e^{-3x} + x$

d) $y = (c_1x+c_2)e^{3x} + x$

Soln AE: $m^2+6m+9=0$

$\Rightarrow m = -3, -3$

$\Rightarrow CF = (c_1x+c_2)e^{-3x}$

$PI = \frac{9x+6}{(D+3)^2} = \frac{1}{9(D/3+1)^2}(9x+6)$

$\Rightarrow PI = \frac{1}{9}\left[1 - 2D/3\right](9x+6) = \frac{1}{9}[9x+\cancel{6}-\cancel{6}]$

$\Rightarrow PI = x$

$\therefore y = (c_1x+c_2)e^{-3x} + x$

$\Downarrow$

option c

Q) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0$, $y(0)=0$, $y(1)=1$, Complete soln of DE = ?

a) x^2 b) $\sin\left(\frac{\pi}{2}x\right)$ c) $e^x\sin\left(\frac{\pi}{2}x\right)$ d) $e^{-x}\sin\left(\frac{\pi}{2}x\right)$

Soln M1 option c & d does not satisfy the Boundary conditions.

Putting option (a) in the DE $\Rightarrow$ $x^2(2) + x(2x) - 4x^2 = 0$

$\Rightarrow LHS = RHS \Rightarrow$ option a

M2 Cauchy's form

Q) The max. value of soln $y(t)$ of the DE $y(t) + \ddot{y}(t) = 0$ with initial conditions $\dot{y}(0)=1$, $y(0)=1$, $t \geq 0$. What is the max. value?

a) 1 b) 2 c) π d) $\sqrt{2}$

Soln AE: $m^2+1=0$

$\Rightarrow m = \pm i$

$y = CF = c_1\cos t + c_2\sin t$

$\dot{y} = -c_1\sin t + c_2\cos t$

$y(0)=1 \Rightarrow 1 = c_1$

$\dot{y}(0)=1 \Rightarrow 1 = c_2$

$\therefore y = \cos t + \sin t \Rightarrow$ Max. value occurs at $t = \pi/4$

i.e $y_{max} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} \Rightarrow$ option d

11/03/20 Q) The Soln to the DE $\frac{d^2u}{dx^2} - k\frac{du}{dx} = 0$ where k is const

B C: $u(0) = 0$, & $u(L) = U$ is

a) $u = U\frac{x}{L}$ b) $u = U\left[\frac{1-e^{kx}}{1-e^{kL}}\right]$ c) $u = U\left[\frac{1-e^{-kx}}{1-e^{-kL}}\right]$ d) $u = U\left[\frac{1+e^{kx}}{1+e^{kL}}\right]$

Soln M1 option d does not satisfy the Boundary conditions.

AE: $m^2 - km = 0$ | $u = c_1 + c_2 e^{kx}$ (Here k is +ve & hence option c is eliminated)

$\Rightarrow m = 0,\ m = k$

Now by observation;

option a does not satisfy the DE.

ie $\frac{d^2u}{dx^2} - k\frac{du}{dx} = 0 - k\frac{U}{L} \neq 0$

$\Rightarrow LHS = RHS$.

M2 Use B.C to find c_1 & c_2. Hence option b is the ans

Q) $\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + y = 0$ has 2 equal roots, then the value of α are: a) ± 1 b) 0, 0 c) $\pm j$ d) $\pm 1/2$.

Soln 2 equal roots $\Rightarrow$ $b^2 - 4ac = 0$ $\Rightarrow$ $4\alpha^2 = 4$. $\Rightarrow$ $\alpha = \pm 1$

(option a)

Q) If a & b are constants, most general soln of DE, $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$ is ?

a) ae^{-t} b) $ae^{-t} + bte^{-t}$

c) $ae^{t} + bte^{-t}$ d) ae^{-2t}

Soln AE: $m^2 + 2m + 1 = 0$ | $x = (at + b)e^{-t}$ $\Rightarrow$ option b

$\Rightarrow m = -1, -1$

Q) Soln of DE $\frac{d^2x}{dt^2} = -9x$, $x(0) = 1$, $\left.\frac{dx}{dt}\right|_{t=0} = 1$

a) $t^2 + t + 1$ b) $\sin 3t + \frac{1}{3}\cos 3t + \frac{2}{3}$ c) $\frac{1}{3}\sin 3t + \cos 3t$ d) $\cos 3t + t$

Soln AE: $m^2 + 9 = 0$ | $x = CF = c_1\cos 3t + c_2\sin 3t$ ($\because$ option a & d is eliminated)

$\Rightarrow m = \pm 3i$ | $\dot{x} = -3c_1\sin 3t + 3c_2\cos 3t$

$x(0) = 1$: $1 = c_1$ | $\dot{x}(0) = 1$: $1 = 3c_2$ $\Rightarrow c_2 = 1/3$.

$\therefore$ $x = \cos 3t + \frac{1}{3}\sin 3t$ $\Rightarrow$ option c

Q) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$. Soln of DE = ?

Soln

$x = e^z$

$x^2\frac{d^2y}{dx^2} = D(D-1)y$

$x\frac{dy}{dx} = Dy$

$D(D-1)y + Dy - 1 = 0$

$\Rightarrow (D^2-1)y = 0$

AE: $m^2 - 1 = 0$

$m = \pm 1$

$y = CF = c_1e^z + c_2e^{-z}$

$\Rightarrow y = c_1e^{\ln x} + c_2e^{-\ln x}$

i.e $y = c_1x + \frac{c_2}{x}$

Q) $y = f(x)$ is the soln. of DE: $\frac{d^2y}{dx^2} = 0$ with BC,
$y = 5$ at $x = 0$; $\frac{dy}{dx} = 2$ at $x = 10$; $f(15) =$ ____ ?

Soln

AE: $m^2 = 0$

$\Rightarrow m = 0, 0$

$y = xc_1 + c_2$

$y(0) = 5$: $5 = c_2$ | $\frac{dy}{dx} = c_1 \Rightarrow c_1 = 2$

$\therefore y = 2x + 5 \Rightarrow f(15) = y|_{x=15} = 35$

Consider two solns

Q) $x(t) = x_1(t)$ & $x(t) = x_2(t)$ of DE $\frac{d^2x(t)}{dt^2} + x(t) = 0, t > 0$
such that $x_1(0) = 1$, $\left.\frac{dx_1(t)}{dt}\right|_{t=0} = 0$; $x_2(0) = 0$, $\left.\frac{dx_2(t)}{dt}\right|_{t=0} = 1$

The Wronskian $W(t)$ at $t = \frac{\pi}{2}$ is ____

a) 1 b) -1 c) 0 d) $\pi/2$

Soln

AE: $m^2 + 1 = 0$

$\Rightarrow m = \pm i$

$x = CF = c_1\cos t + c_2\sin t$

$\therefore x_1 = \cos t$ (satisfies the B.C of x_1)

$x_2 = \sin t$ (satisfies the B.C of x_2)

$$\therefore W(t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

$\therefore W(\pi/2) = 1 \Rightarrow$ option a

Q) $y(0) = y'(0) = 1$. Soln of DE $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$ at $x = 1$ is ____

Soln

AE: $m^2 + 4m + 4 = 0$

$m = -2, -2$

$y = CF = (c_1x + c_2)e^{-2x}$

$y' = -2(c_1x + c_2)e^{-2x} + c_1e^{-2x}$

B.C

$1 = c_2$

$1 = c_1 - 2c_2$

$\Rightarrow c_1 = 3$

$\therefore y = (3x + 1)e^{-2x} \Rightarrow y|_{x=1} = 4e^{-2} = 0.541$

9) $\frac{d^2y}{dx^2} = -12x^2 + 24x - 20$, BC: $x=0, y=5$ & $x=2, y=21$; y at $x=1$ is ____

Soln $y = \iint -12x^2 + 24x - 20 \Rightarrow y = \int (-4x^3 + 12x^2 - 20x + c_1)\,dx$

$\Rightarrow y = -x^4 + 4x^3 - 10x^2 + c_1x + c_2$

$y(0)=5$: $5 = c_2$ | $y(2)=21$: $21 = -2^4 + 4\cdot 2^3 - 10\cdot 2^2 + 2c_1 + 5$

$\Rightarrow c_1 = 20$.

$\therefore y = -x^4 + 4x^3 - 10x^2 + 20x + 5 \Rightarrow y(1) = 18$

HW
Q. $\frac{d^2y}{dx^2} = y$. Soln of DE that passes through the origin & point $(\ln 2, 3/4)$ is ?

Soln: $2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}$

$\Rightarrow \int 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}dx = \int 2y\frac{dy}{dx}dx$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = y^2 + c$.

$\Rightarrow \frac{dy}{dx} = \sqrt{y^2+c}$

$\frac{dy}{dx} = \sqrt{y^2+c}$

$\Rightarrow \int \frac{dy}{\sqrt{y^2+k^2}} = \int dx$

$\Rightarrow \log\left[y + \sqrt{y^2+k^2}\right] = x + c$.

$\Rightarrow y + \sqrt{y^2+k^2} = e^{x+c}$

Curve passes through origin

$\Rightarrow y(0)=0$: $k = e^c \Rightarrow c = \ln k$

$y(\ln 2) = 3/4$: $\frac{3}{4} + \sqrt{9/16 + k^2} = e^{\ln 2 + c}$

$\Rightarrow \frac{3}{4} + \sqrt{(3/4)^2 + e^{2c}} = 2\cdot e^c$.

$\Rightarrow$

HW
Q) $\frac{d^2y}{dx^2} = y$. Soln of DE that passes through the origin & point $(\ln 2, 3/4)$ is ?

Soln: AE: $m^2 - 1 = 0$

$m = \pm 1$

$y = CF = c_1e^x + c_2e^{-x}$

$y(0) = 0$: $0 = c_1 + c_2$ —①

$y(\ln 2) = 3/4$: $3/4 = 2c_1 + c_2/2$ —②

Scanned with CamScanner

$3/4 = 2c_1 - \frac{c_1}{2} = \frac{3}{2}c_1 \Rightarrow c_1 = 1/2$ & $c_2 = -1/2$

$y = \frac{e^x}{2} - \frac{e^{-x}}{2} = \frac{e^x - e^{-x}}{2}$

$\Rightarrow y = \sinh x$

18/3/20 9) $\frac{d^2y}{dx^2} = y$ passes through $(0,0)$ & $(\ln 2, 3/4)$

Soln M2 $\int 2\frac{dy}{dx}\frac{d^2y}{dx^2}dx = \cancel{\int y\,dx}$ $\int 2\frac{dy}{dx}y\,dx$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 2\frac{y^2}{2} + c \Rightarrow \left(\frac{dy}{dx}\right) = \sqrt{y^2 + c_1^2}$

$\int \frac{dy}{\sqrt{y^2+c_1^2}} = \int dx \Rightarrow \ln[y + \sqrt{y^2+c_1^2}] = x + c_2$

Point $(0,0)$: $\ln c_1 = c_2$

Point $(\ln 2, 3/4)$: $\ln[3/4 + \sqrt{(3/4)^2 + c_1^2}] = \ln 2 + \ln c_1$

$\Rightarrow \frac{3}{4} + \sqrt{9/16 + c_1^2} = 2c_1$

$\Rightarrow \cancel{\frac{9}{16}} + c_1^2 = 4c_1^2 + \cancel{\frac{9}{16}} - 3c_1$

$\Rightarrow 3c_1^2 - 3c_1 = 0$

$\Rightarrow c_1 = 0, \ c_1 = 1 \qquad \Rightarrow c_2 = \ln 1 = 0$

Not possible [$\because \ln 0$ does not exist]

$\therefore \ln[y + \sqrt{y^2+1}] = x$

$\Rightarrow y + \sqrt{y^2+1} = e^x$

$\Rightarrow \cancel{y^2} + 1 = e^{2x} + \cancel{y^2} - 2ye^x$

$\Rightarrow y = \frac{e^{2x} - 1}{2e^x} \Rightarrow y = \frac{e^x - e^{-x}}{2}$

Q) $\frac{d^2x(t)}{dt^2} + 3\frac{dx(t)}{dt} + 2x(t) = 0$; $x(0) = 20$, $x(1) = 10/e$, $e = 2.718$. $x(2)$ is ______

Soln AE: $m^2 + 3m + 2 = 0$

$(m+2)(m+1) = 0$

$m = -2, m = -1$

$x = c_1 e^{-2t} + c_2 e^{-t}$

$x(0) = 20$: $20 = c_1 + c_2$

$x(1) = \frac{10}{e}$: $\frac{10}{e} = c_1 e^{-2} + c_2 e^{-1}$

$\Rightarrow 10 = c_1 e^{-1} + c_2$

$\Rightarrow 10 = c_1 e^{-1} + 20 - c_1$

$\Rightarrow -10 = c_1 [e^{-1} - 1]$

$\Rightarrow c_1 = \frac{10}{1-e^{-1}} = \frac{10e}{e-1}$

$\Rightarrow c_1 = \underline{15.82}$

$c_2 = 20 - \frac{10e}{e-1} = \frac{10e - 20}{e-1} = \underline{4.18}$

$\therefore x = 15.82 e^{-2t} + 4.18 e^{-t}$

$x(2) = 15.82 e^{-4} + 4.18 e^{-2} = \underline{0.855}$

Q) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$, $y(0) = 2$, $y(1) = -\left(\frac{1-3e}{e^3}\right)$, $\frac{dy(0)}{dt} =$ ______

Soln AE: $m^2 + 5m + 6 = 0$

$(m+3)(m+2) = 0$

$m = -3, -2$

$y = c_1 e^{-3t} + c_2 e^{-2t}$

$y(0) = 2$: $2 = c_1 + c_2$

$y(1) = \frac{3e-1}{e^3}$: $\frac{3e-1}{e^3} = c_1 e^{-3} + c_2 e^{-2}$

$\Rightarrow 3e - 1 = c_1 + c_2 e$

$\Rightarrow 3e - 1 = 2 - c_2 + c_2 e = 2 + c_2[e-1]$

$\Rightarrow \frac{3e-3}{e-1} = c_2$

$\Rightarrow c_2 = \underline{3} \Rightarrow c_1 = -1$

$\therefore y = -e^{-3t} + 3e^{-2t}$

$\dot{y} = 3e^{-3t} - 6e^{-2t}$

$\dot{y}(0) = \underline{-3}$

Q) $x^2 y'' + xy' - y = 0$ is

a) $y = c_1 x^2 + c_2 x^{-3}$ b) $y = c_1 + c_2 x^{-2}$ c) $y = c_1 x + c_2/x$ d) $y = c_1 x + c_2 x^4$

Soln $D(D-1)y + Dy - y = 0$

$(D^2 - 1)y = 0$

$m = \pm 1$

$y = c_1 e^z + c_2 e^{-z} = \underline{c_1 x + c_2/x} \Rightarrow$ option c

Q) $y(0) = 1$, $y(1) = 3e^{-1}$, $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$, $y(2) = ?$ a) $5e^{-1}$ b) $7e^{-1}$ c) $5e^{-2}$ d) $7e^{-2}$

Soln AE: $m^2 + 2m + 1 = 0$

$m = -1, -1$

$y = (c_1 t + c_2)e^{-t}$

$y(0) = 1$: $\underline{1 = c_2}$

$y(1) = 3e^{-1}$: $3\cancel{e^{-1}} = (c_1 + 1)\cancel{e^{-1}}$

$\Rightarrow c_1 = 2$

$y = (2t+1)e^{-t}$

$y(2) = \underline{5e^{-2}} \Rightarrow$ option c.

Q) $y = f(t)$, $y'' + 9y = 0$, $y(0) = 0$, $y(\pi/2) = \sqrt{2}$, $y(\pi/4) = ?$

Soln: AE: $m^2 + 9 = 0$

$m = \pm 3i$

$y = c_1 \cos 3t + c_2 \sin 3t$

$y(0) = 0$: $\underline{0 = c_1}$

$y(\pi/2) = \sqrt{2}$: $\sqrt{2} = c_2 \sin \frac{3\pi}{2}$

$\Rightarrow c_2 = \underline{-\sqrt{2}}$

$y = -\sqrt{2} \sin 3t$

$y(\pi/4) = -\sqrt{2} \times \sin \frac{3\pi}{4}$

$= -\sqrt{2} \times \frac{1}{\sqrt{2}}$

$y(\pi/4) = \underline{-1}$

HW

Q) $\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 36y = 0$, $y(0) = 3$, $\left.\frac{dy}{dx}\right|_{x=0} = -36$, Soln = ?

Soln AE: $m^2 + 12m + 36 = 0$

$\Rightarrow (m+6)^2 = 0$

$\Rightarrow m = -6, -6$

$y = (c_1 x + c_2) e^{-6x}$

$y(0) = 3$: $\underline{3 = c_2}$

$y' = -6(c_1 x + c_2) e^{-6x} + c_1 e^{-6x}$

$y'(0) = -36$: $-36 = -6c_2 + c_1$

$\Rightarrow c_1 = \underline{-18}$

$\therefore y = \underline{(-18x + 3) e^{-6x}}$

19/03/20 Q) $(D^2 + 12D + 36)y = 0$, $y(0) = 3$, $\left.\frac{dy}{dx}\right|_{x=0} = -36$ [same as above problem]

Q) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$, $y(0) = 0$, $\left.\frac{dy}{dx}\right|_{x=0} = 1$, $y(1) =$ ______

Soln: AE: $m^2 - 4m + 4 = 0$

$m = 2, 2$

$y = (c_1 x + c_2) e^{2x}$

$y' = 2(c_1 x + c_2) e^{2x} + c_1 e^{2x}$

$y(0) = 0$: $\underline{0 = c_2}$

$y'(0) = 1$: $\underline{1 = c_1}$

$\therefore y = x e^{2x}$ $\Rightarrow$ $\underline{y(1) = e^2}$

Q) $\frac{d^4y}{dx^4} + 3\frac{d^2y}{dx^2} = 108x^2$

Soln AE: $m^4 + 3m^2 = 0$

$m^2(m^2 + 3) = 0$

$m = 0, 0, \pm\sqrt{3}i$

CF $y = (c_1 x + c_2) + c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x$ //

$PI = \frac{108x^2}{D^2(D^2+3)} = \frac{108x^2}{3D^2(1 + \frac{D^2}{3})} = \frac{1}{3D^2}\left(1 - \frac{D^2}{3}\right)(108x^2)$

$\Rightarrow PI = \frac{1}{3D^2}[108x^2 - 72] = \iint (36x^2 - 24)\,dx\,dx$

$$\Rightarrow PI = \int \left(\frac{36}{3}x^3 - 24x\right)dx = \frac{12}{4}x^4 - \frac{24}{2}x^2 = 3x^4 - 12x^2$$

$$\therefore y = (c_1 x + c_2) + c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x + 3x^4 - 12x^2$$

Q) $\frac{d^2y}{dx^2} + y = 0$, $y|_{x=0} = 5$ & $\frac{dy}{dx}\Big|_{x=0} = 10$?

a) $y = 5 + 10\sin x$ b) $y = 5\cos x - 5\sin x$ (c) $y = 5\cos x + 10x$

d) $y = 5\cos x + 10x$

Soln AE: $m^2 + 1 = 0$ $\Rightarrow m = \pm i$ | $y = c_1\cos x + c_2 \sin x$; $y' = -c_1 \sin x + c_2\cos x$ | $y(0) = 5$: $5 = c_1$; $y'(0) = 10$: $10 = c_2$

$\therefore$ $y = 5\cos x + 10\sin x$ => No options matching (NOTA)

5) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 5y = 0$

Soln AE: $m^2 + 2m - 5 = 0$

$$m = \frac{-2 \pm \sqrt{4+20}}{2}$$

$$\Rightarrow m = \frac{-2 \pm 2\sqrt{6}}{2}$$

$$\Rightarrow m = -1 \pm \sqrt{6}$$

$$\therefore y = k_1 e^{(-1+\sqrt{6})x} + k_2 e^{(-1-\sqrt{6})x}$$

Q) $y'' - 4y' + 3y = 2t - 3t^2$ PI = ?

a) $-2 - 2t - t^2$ b) $-2t - t^2$ c) $2t - 3t^2$ d) $-2 - 2t - 3t^2$

Soln M1 $PI = \frac{2t - 3t^2}{(D-3)(D-1)} = \frac{1}{(3-D)}(1 + D + D^2)(2t - 3t^2)$

$$\Rightarrow PI = \frac{1}{3[1 - D/3]}(2t - 3t^2) + (2 - 6t) - (6) = \frac{1}{3}[1 - D/3]^{-1}[-3t^2 - 4t - 4]$$

$$\Rightarrow PI = \frac{1}{3}\left[1 + D/3 + \frac{D^2}{9}\right][-3t^2 - 4t - 4] = \frac{1}{3}\left[-3t^2 - 4t - 4 - 2t - \frac{4}{3} - \frac{2}{3}\right]$$

$$\Rightarrow PI = \frac{[-3t^2 - 6t - 6]}{3} = -t^2 - 2t - 2 \Rightarrow \text{option a}$$

M2 If we carefully observe, option c & option d are eliminated.
Reason: There is $-3t^2$ in RHS : To obtain this y should consist of only $-t^2$ term but not $-3t^2$ since our DE consists of 3y term.
Now put either option a or b in the DE & verify for the ans.

Q) $\frac{d^2y}{dx^2} + 16y = 0$, $\left.\frac{dy}{dx}\right|_{x=0} = 1$, $\left.\frac{dy}{dx}\right|_{x=\pi/2} = -1$ has what

a) No solution b) Exactly 2 soln. c) Exactly 1 soln d) Infinitely many soln.

Soln AE: $m^2 + 16 = 0$
$m = \pm 4i$

$y = c_1 \cos 4x + c_2 \sin 4x$

$y' = -4c_1 \sin 4x + 4c_2 \cos 4x$

$y'(0) = 1: \quad 1 = 4c_2 \Rightarrow c_2 = 1/4$

$y'(\pi/2) = -1: \quad -1 = 4c_2 \Rightarrow c_2 = -1/4$

$\Rightarrow$ No solution

Q) $3y''(x) + 27y(x) = 0$, $y(0) = 0$, $y'(0) = 2000$, $y(1) =$ ______

Soln AE: $3m^2 + 27 = 0$
$\Rightarrow m^2 = -9$
$\Rightarrow m = \pm 3i$

$y = c_1 \cos 3x + c_2 \sin 3x$

$y' = -3c_1 \sin 3x + 3c_2 \cos 3x$

$y(0) = 0: \quad 0 = c_1$

$y'(0) = 2000: \quad 2000 = 3c_2$
$\Rightarrow c_2 = \frac{2000}{3}$

$\therefore y = \frac{2000}{3}\sin(3x) \Rightarrow y(1) = \frac{2000}{3} \times \sin 3^c = 94.08$

Q) $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

a) $y = (c_1 - c_2 x)e^x + c_3 \cos x + c_4 \sin x$

b) $y = (c_1 + c_2 x)e^x - c_3 \cos x + c_4 \sin x$

c) $y = (c_1 + c_2 x)e^x + c_3 \cos x - c_4 \sin x$

d) $y = (c_1 + c_2 x)e^x + c_3 \cos x + c_4 \sin x$

Soln M1 By direct observation we can say the ans is option d.

M2 AE: $m^4 - 2m^3 + 2m^2 - 2m + 1 = 0$

$m = 1$

1	1	-2	2	-2	1
		1	-1	1	-1
	1	-1	1	-1	0

$\Rightarrow m^3 - m^2 + m - 1 = 0$

$\Rightarrow m = 1$

1	1	-1	1	-1
		1	0	1
	1	0	1	0

$\Rightarrow m^2 + 1 = 0 \Rightarrow m = \pm i$

$\therefore m = 1, 1, \pm i \Rightarrow y = (c_1 x + c_2)e^x + c_3 \cos x + c_4 \sin x$

20/03/20 Q) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 3e^{2x}$, $y(0)=0$, $y'(0)=-2$

a) $y = e^{-x} - e^{2x} + xe^{2x}$ b) $y = e^{x} - e^{-2x} - xe^{2x}$ c) $y = e^{-x} + e^{2x} + xe^{2x}$ d) $y = e^{x} - e^{-2x} + xe^{2x}$

Soln option c is directly eliminated by $y(0)=0$

AE: $m^2 - m - 2 = 0 \Rightarrow (m-2)(m+1)=0 \Rightarrow m=2, m=-1$

$\therefore$ CF $= c_1e^{2x} + c_2e^{-x} \Rightarrow$ option a

Q) $\frac{d^2y}{dt^2} + y = 0$, $y=1$, $\frac{dy}{dt}=0$, when $t=0$: $y = ?$

a) $\sin t$ b) $\cos t$ c) $\tan t$ d) $\cot t$

Soln $y(0)=1 \Rightarrow$ option b

Q) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$, $y(0)=1$, $\frac{dy}{dx}(0)=-1$, $y(1)=?$

Soln AE: $m^2+2m+1=0$
$\Rightarrow (m+1)^2=0$
$\Rightarrow m=-1,-1$

$y = (c_1x+c_2)e^{-1x}$
$y' = -(c_1x+c_2)e^{-x} + c_1e^{-x}$

$y(0)=1$: $1=c_2$
$y'(0)=-1$: $-1=-1+c_1$
$\Rightarrow c_1=0$.

$\therefore y = e^{-x} \Rightarrow y(1) = e^{-1}$

5) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$, $y(0)=0$, $\frac{dy}{dx}(0)=1$, $y(1) =$ ______

Soln AE: $m^2+m-6=0$
$\Rightarrow (m+3)(m-2)=0$
$\Rightarrow m=2,-3$

$y = c_1e^{2x} + c_2e^{-3x}$
$y' = 2c_1e^{2x} - 3c_2e^{-3x}$

$y(0)=0$: $0=c_1+c_2 \Rightarrow c_1=-c_2$
$y'(0)=1$: $1=2c_1-3c_2$
$\Rightarrow 1=2c_1+3c_1$
$\Rightarrow c_1=1/5$, $c_2=-1/5$

$\therefore y = \frac{1}{5}[e^{2x}-e^{-3x}] \Rightarrow y(1) = \frac{1}{5}[e^2-e^{-3}]$

$\Rightarrow y(1) = 1.4678$

5) $\frac{d^2y}{dt^2} = -\frac{dy}{dt} - \frac{5y}{4}$, $y(0)=1$, $\frac{dy}{dt}(t=0)=0$, $y(\pi)$ is ______

Soln AE. $m^2+m+\frac{5}{4}=0$.

$m = \frac{-1 \pm \sqrt{1-5}}{2} = -\frac{1}{2} \pm i$

$y = e^{-1/2t}[c_1\cos t + c_2\sin t]$

$y' = e^{-1/2t}[-c_1\sin t + c_2\cos t] - \frac{1}{2}e^{-1/2t}[c_1\cos t + c_2\sin t]$

$y(0)=1$: $\quad 1 = c_1 \quad ; \quad y'(0)=0 \quad : \quad 0 = c_2 - 1/2 \Rightarrow c_2 = 1/2$

$\therefore y = e^{-1/2 t}\left[\cos t + 1/2 \sin t\right] \; : \; \Rightarrow y(\pi) = -e^{-1/2 \pi} = -0.208$

9) $\dfrac{dy}{dx} = \dfrac{y\phi'(x) - y^2}{\phi(x)}$

a) $y = \dfrac{x}{\phi(x)+c}$ b) $y = \dfrac{\phi(x)}{x} + c$

c) $y = \dfrac{\phi(x)+c}{x}$ d) $y = \dfrac{\phi(x)}{x+c}$

Soln M1

$$\frac{dy}{dx}\phi(x) = y\phi'(x) - y^2$$

$$\frac{1}{y^2}\frac{dy}{dx}\phi(x) - \frac{1}{y}\phi'(x) = -1 \quad \text{[Bernoulli Baba]}$$

$-\dfrac{1}{y} = t$

$\Rightarrow +\dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{dt}{dx}$

$$\frac{dt}{dx}\phi(x) + t\phi'(x) = -1$$

$$\Rightarrow \frac{dt}{dx} + t\frac{\phi'(x)}{\phi(x)} = -\frac{1}{\phi(x)}$$

$$IF = e^{\int \frac{\phi'(x)}{\phi(x)}dx} = e^{\log \phi(x)} = \phi(x)$$

$$t\phi(x) = \int \phi(x)\left(-\frac{1}{\phi(x)}\right)dx + c \quad \Rightarrow t\phi(x) = -x + c$$

$$\Rightarrow -\frac{1}{y} = \frac{-x+c}{\phi(x)}$$

$=$

$$\Rightarrow y = \frac{\phi(x)}{x+c} \Rightarrow \text{option d}$$

M2: Ninja Technique   Let $\phi(x) = 1 \Rightarrow \phi'(x) = 0$.

$$\therefore \frac{dy}{dx} = -y^2 \Rightarrow \int -y^{-2}\,dy = \int dx \Rightarrow y^{-1} = x + c.$$

$$\Rightarrow y = \frac{1}{x+c} \Rightarrow \text{option d}$$

9) $-\operatorname{cosec}^2(x+y)\,dy = dx$

a) $y - c = \sin(x+y)$ b) $x - c = \sin(x+y)$

c) $y - c = \tan(x+y)$ d) NOTA

Soln

$$-\frac{dy}{dx} = \sin^2(x+y)$$

$x + y = t$

$1 + \dfrac{dy}{dx} = \dfrac{dt}{dx} \Rightarrow -\dfrac{dy}{dx} = 1 - \dfrac{dt}{dx}$

$$1 - \frac{dt}{dx} = \sin^2 t$$

$$\Rightarrow \frac{dt}{dx} = \cos^2 t \Rightarrow \int \sec^2 t\,dt = \int dx$$

$$\Rightarrow \tan t = x + c \Rightarrow \tan(x+y) = x + c. \Rightarrow \text{option c}$$

Q) $f(x) = \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}$, $f'(x) = ?$

a) $\frac{1}{1-2f(x)}$ b) $\frac{1}{2f(x)-1}$ c) $\frac{1}{1+2f(x)}$ d) $\frac{1}{2+f(x)}$

Soln

$$f(x) = \sqrt{x + f(x)}$$

$$\{f(x)\}^2 = x + f(x) \Rightarrow 2f(x)f'(x) = 1 + f'(x)$$

$$\Rightarrow f'(x)[2f(x)-1] = 1 \Rightarrow f'(x) = \frac{1}{2f(x)-1}$$

$\Downarrow$ option b

23/03/20 Q) $(x+y)(dx - dy) = dx + dy$

Soln

$$(x+y)\left(1 - \frac{dy}{dx}\right) = \left(1 + \frac{dy}{dx}\right)$$

$x + y = t$

$1 + \frac{dy}{dx} = \frac{dt}{dx}$

$$\Rightarrow t\left[1 - \frac{dt}{dx} + 1\right] = \frac{dt}{dx}$$

$$\Rightarrow 2t - t\frac{dt}{dx} = \frac{dt}{dx} \Rightarrow \frac{dt}{dx}[1+t] = 2t \Rightarrow \int \frac{dt(1+t)}{2t} = \int dx$$

$$\Rightarrow \frac{1}{2}\ln t + \frac{1}{2}t = x + c \Rightarrow \ln t + t = 2x + c'$$

$$\Rightarrow \ln(x+y) + x + y = 2x + c'$$

$$\Rightarrow y - x + \ln(x+y) = c'$$ ans

Q) Rate of growth of bacteria is proportional to no. of bacteria present at that time. If x is the no. of bacteria present at any instant t, then which one of the foll is correct? $(k=1)$

a) $x = \ln t$ b) $x = ce^t$ c) $e^x = t$ d) $x = \sqrt{t}$

Soln

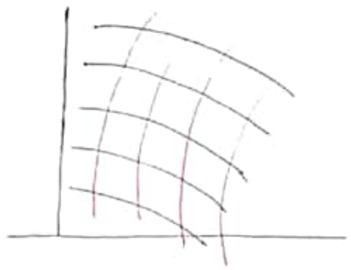
$$\frac{dx}{dt} = kt \Rightarrow \frac{dx}{dt} = x$$

$$\therefore \int \frac{dx}{x} = \int dt \Rightarrow \ln x = t + c \Rightarrow x = e^{t+c}$$

i.e. $x = c_1 e^t$ $\Rightarrow$ option b

Scanned with CamScanner

Orthogonal Trajectories: 2 family of curves such that every member of their family cut each member of the other family at right angles.



Q) Find the other orthogonal trajectories of the curve $xy = c$.

Soln: $xy = c$

Step 1: Diff wrt x: $x\frac{dy}{dx} + y = 0$

Step 2: Replace $\frac{dy}{dx} = -\frac{dx}{dy}$ $[\because m_1 m_2 = -1]$: $x\left(-\frac{dx}{dy}\right) + y = 0$

$\Rightarrow -x\,dx + y\,dy = 0$

$\Rightarrow x^2 = y^2 + c \Rightarrow \underline{x^2 - y^2 = c}$

Q) Find the other orthogonal trajectories of the curve $r = a(1 - \cos\theta)$

Soln: Step 1: Diff wrt θ: $\frac{dr}{d\theta} = a\sin\theta$

Step 2: Replace $\frac{dr}{d\theta} = -r^2\frac{d\theta}{dr}$: $-r^2\frac{d\theta}{dr} = a\sin\theta$

$\Rightarrow -r^2\frac{d\theta}{dr} = \frac{r}{1-\cos\theta} \times \sin\theta$

$\Rightarrow \int \frac{\cos\theta - 1}{\sin\theta}\,d\theta = \int \frac{dr}{r}$

$\Rightarrow \int -\frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}\,d\theta = \ln r + c \Rightarrow -\int \tan\theta/2\,d\theta = \ln r + c$

$\Rightarrow$ ~~-8~~ $-2\log|\sec\theta/2| = \log r + c$

$\Rightarrow \log(r\sec^2\theta/2) = \log c \Rightarrow \underline{r\sec^2\theta/2 = c}$

$\Rightarrow r = c\cos^2\theta/2$

$\Rightarrow r = c\left[\frac{1+\cos\theta}{2}\right]$

$\Rightarrow r = c.(1+\cos\theta)$

Q) Find the Integrating factor of $x^2(x^2-1)\frac{dy}{dx} + x(x^2+1)y = x^2-1$

Soln

$$\frac{dy}{dx} + \frac{x(x^2+1)}{x^2(x^2-1)}y = \frac{x^2-1}{x^2(x^2-1)}.$$

$$IF = e^{\int \frac{x(x^2+1)}{x^2(x^2-1)}dx} = e^{I}.$$

$$I = \int \frac{x(x^2+1)}{x^2(x^2-1)}dx$$

$$\frac{x(x^2+1)}{x^2(x^2-1)} = \frac{A}{x^2} + \frac{B}{x^2-1}$$

$$\Rightarrow x(x^2+1) = A(x^2-1) + Bx^2$$

$\Rightarrow$

M1

$$I = \int \frac{x(x^2+1)}{x^2(x^2-1)}dx = \int \frac{x^2+1}{x(x+1)(x-1)}dx$$

$$\frac{x^2+1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x-1)} \Rightarrow x^2+1 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$$

$A+B+C = 1 \Rightarrow 2B = 2 \Rightarrow B = C = 1$

$-B+C = 0 \Rightarrow B = C$

$-A = 1 \Rightarrow A = -1$

$$\therefore \frac{x^2+1}{x(x+1)(x-1)} = \frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-1}$$

$$\therefore I = -\ln x + \ln(x+1) + \ln(x-1) = \ln\left[\frac{x^2-1}{x}\right] = \ln\left[x - 1/x\right]$$

$$\therefore IF = e^{I} = x - \frac{1}{x}$$

M2

$$I = \int \frac{x^2+1}{x(x^2-1)}dx = \int \frac{1+1/x^2}{(x-\frac{1}{x})}dx = \ln(x - 1/x)$$

$$\therefore IF = e^{I} = x - 1/x$$

Q) $\frac{dy}{dx} + (3x^2\tan^{-1}y - x^3)(1+y^2) = 0$ $\quad I.F = ?$

Soln $\quad \tan^{-1}y = t \Rightarrow \frac{1}{1+y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$$\frac{1}{(1+y^2)}\frac{dy}{dx} = x^3 - 3x^2\tan^{-1}y \Rightarrow \frac{dt}{dx} = x^3 - 3x^2t \Rightarrow \frac{dt}{dx} + 3x^2t = x^3$$

$$IF = e^{\int 3x^2dx} = e^{x^3}$$

Q) $\dot{x}(t) + 2x(t) = \delta(t)$, $x(0) = 0$.

Soln $sX(s) - \underset{0}{x(0)} + 2X(s) = 1$ $[\because L(\delta(t)) = 1]$

$\Rightarrow X(s)[s+2] = 1 \Rightarrow X(s) = \frac{1}{s+2} \Rightarrow x(t) = e^{-2t}u(t)$

OR

$e^{-2t}, t > 0$

Q) $y'' + \lambda y = 0$; $y'(0) = y'(\pi) = 0$ will have non zero solution when λ is not equal to

a) $0, \pm 1, \pm 2, \ldots$
b) $1, 2, 3, \ldots$
c) $1, 4, 9, \ldots$
d) $1, 9, 25, \ldots$

Soln AE: $m^2 + \lambda = 0$

$\Rightarrow m = \pm i\sqrt{\lambda}$

$y = CF = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$.

$y' = -\sqrt{\lambda}c_1 \sin\sqrt{\lambda}x + \sqrt{\lambda}c_2 \cos\sqrt{\lambda}x$

$y'(0) = 0$: $0 = c_2\sqrt{\lambda} \Rightarrow c_2 = 0$.

$y'(\pi) = 0$: $0 = -\sqrt{\lambda}c_1 \cdot \sin\sqrt{\lambda}\pi$. $\Rightarrow \sin\sqrt{\lambda}x = 0$.

$\Rightarrow \sin\sqrt{\lambda}\pi = \sin n\pi$

$\Rightarrow \lambda = n^2 \Rightarrow$ option c.

Q) $(x^2 - x)\frac{dy}{dx} = (2x-1)y$, $y(x_0) = y_0$ has a unique solution if (x_0, y_0) equals a) $(2,1)$ b) $(1,1)$ c) $(0,0)$ d) $(0,1)$

Soln $\frac{dy}{dx} + \left(\frac{1-2x}{x^2-x}\right)y = 0$.

$IF = e^{-\int \frac{(2x-1)}{x^2-x}dx} = e^{-\ln(x^2-x)} = \frac{1}{x^2-x}$.

$y\frac{1}{x^2-x} = c$ | $y(x_0) = y_0$: $\frac{y_0}{x_0^2 - x_0} = c$

$\therefore y = \frac{y_0}{x_0^2 - x_0}(x^2 - x)$

$\therefore \frac{y_0}{x_0^2 - x_0}$ exists only for option a

28/04/20

VECTOR CALCULUS

starting point

Ending / Terminal point

A B

Ray

$\overrightarrow{AB}$ → Starting from A & ending at B

$\overrightarrow{BA}$ / $-\overrightarrow{AB}$ → Starting from B & ending at A

Note.

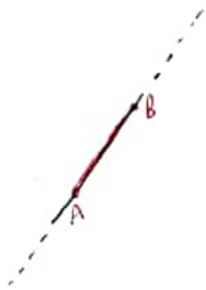
*  → Line

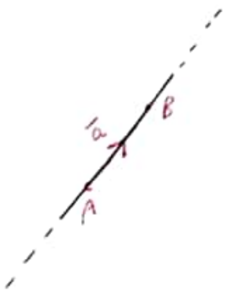
 → Ray

 → Segment

* A line includes many segment

Directed line: → 

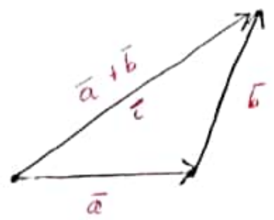
Directed line segment → 

↳ 

Here $\bar{a}$ has 2 things

$\bar{a} = \overrightarrow{AB}$ (Direction)

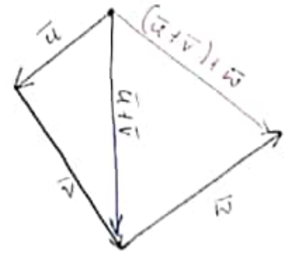
$|\vec{a}| = |\overrightarrow{AB}|$ (Magnitude / Euclidean Norm)

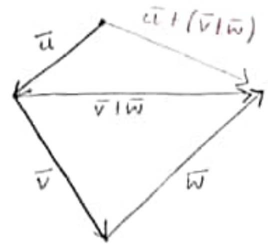
↳ 

Vector addition

$\bar{c} = \bar{a} + \bar{b}$

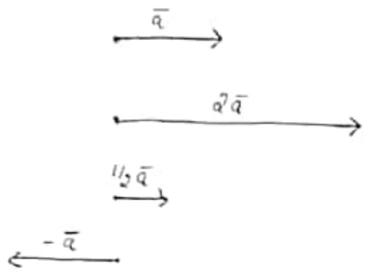
eg.





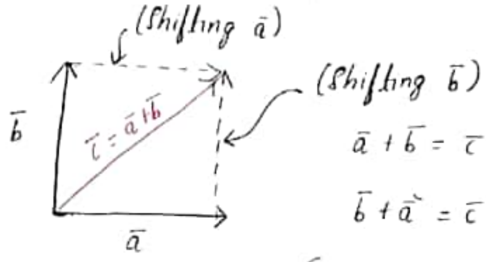
$\therefore (\bar{u}+\bar{v})+\bar{w} = \bar{u}+(\bar{v}+\bar{w})$ (Associative property)

eg.



(Scaling)

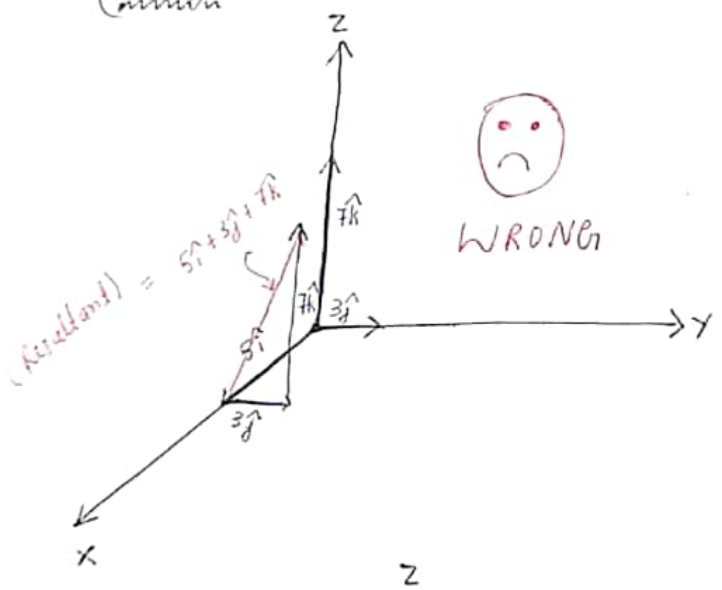
Law of Parallelogram



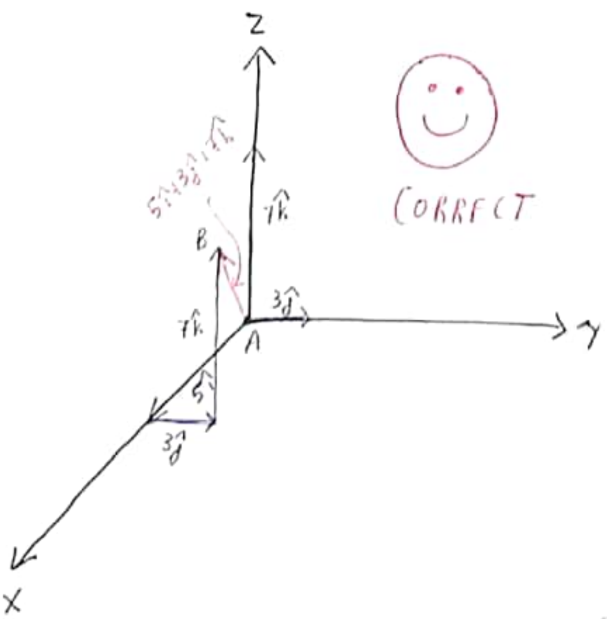
$\bar{a}+\bar{b} = \bar{c}$

$\bar{b}+\bar{a} = \bar{c}$

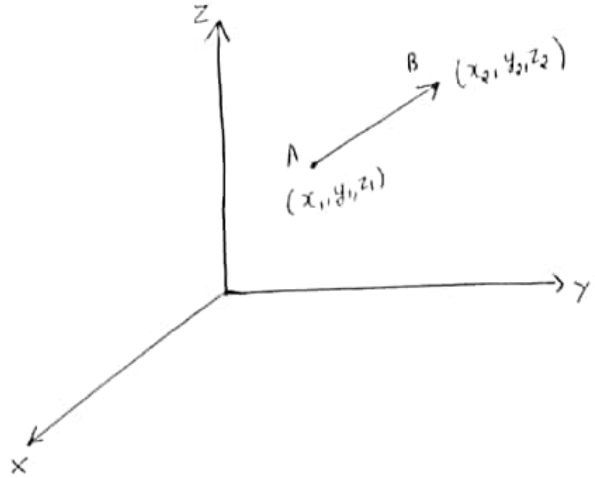
(Commutative property)



eg.



eg.



$$\vec{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

Component of Vectors :

X : $(x_2 - x_1) = a_1$

Y : $(y_2 - y_1) = a_2$

Z : $(z_2 - z_1) = a_3$

$$\therefore \vec{AB} = [a_1 \quad a_2 \quad a_3]$$

* In the previous example, since $(x_1, y_1, z_1) = (0,0,0)$

$\vec{AB} = 5\hat{i} + 3\hat{j} + 7\hat{k}$ & $\vec{AB}$ is called position vector

* $$\text{Magnitude of } \vec{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Note

 → Vectors cannot be a curve.

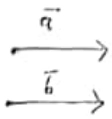
 → Zero vector

 → Vector loop

→ Coinitial vector

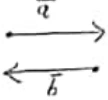
Unit Vector. $\hat{i}, \hat{j}, \hat{k}$ → Magnitude is 1

Equal Vectors.



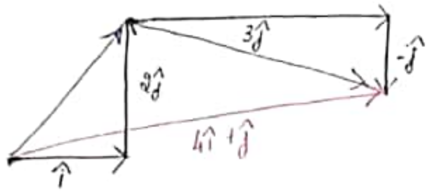
Same direction & same magnitude

Note



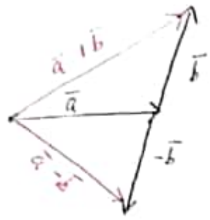
→ Not equal vectors

eg. $(\hat{i} + 2\hat{j}) + (3\hat{i} - \hat{j}) = 4\hat{i} + \hat{j}$



$\therefore (a_1\hat{i} + a_2\hat{j}) + (b_1\hat{i} + b_2\hat{j}) = (a_1+b_1)\hat{i} + (a_2+b_2)\hat{j}$

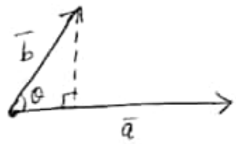
eg.



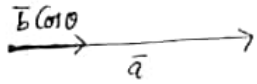
$\bar{a} - \bar{b} = \bar{a} + (-\bar{b})$

Vector Multiplication:

Dot product: $\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos\theta$



$\bar{a}\cdot\bar{b}$



Here the direction is not of much importance

Hence dot product is a scalar product

Note

* If $\bar{a}$ & $\bar{b}$ are $\perp$er to each other then $\bar{a}\cdot\bar{b} = 0$

* $(\hat{i}\cdot\hat{j}) = (\hat{j}\cdot\hat{k}) = (\hat{k}\cdot\hat{i}) = 0$

* If $\bar{b} = \bar{a}$ then $\bar{a}\cdot\bar{a} = |\bar{a}|^2$

* $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\therefore \bar{a}\cdot\bar{b} = a_1b_1 + a_2b_2 + a_3b_3$

HW

Q) If $\bar{a}$ is unit vector $(\bar{x}-\bar{a})\cdot(\bar{x}+\bar{a}) = 8$, then $|\bar{x}|$ is ?

Soln

$(\bar{x} - \bar{a})\cdot(\bar{x} + \bar{a}) = 8$

$\Rightarrow$ $|\bar{x}|^2 - |\bar{a}|^2 = 8$

$\Rightarrow |\bar{x}|^2 = 9 \Rightarrow |\bar{x}| = 3$

Scanned with CamScanner

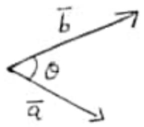
Cauchy Schwartz Inequality:

$$\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos\theta$$

$$\Rightarrow \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} = \cos\theta \qquad [\cos\theta \le 1]$$

Hence $\boxed{\bar{a}\cdot\bar{b} \le |\bar{a}||\bar{b}|}$

Cross Product:



$$\boxed{\begin{aligned}\bar{a}\times\bar{b} &= |\bar{a}||\bar{b}|\sin\theta \ \hat{n} \\ |\bar{a}\times\bar{b}| &= |\bar{a}||\bar{b}|\sin\theta\end{aligned}}$$

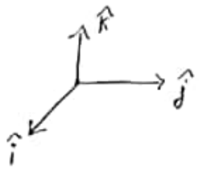
, $\hat{n}$ - unit normal vector (⊥r to the plane of paper)

* $\bar{a}\times\bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

* $\bar{a}\times\bar{b} = -(\bar{b}\times\bar{a})$

* If $\bar{a}\times\bar{b} = \bar{0}$ $\Rightarrow$ $\bar{a}$ & $\bar{b}$ are parallel vectors

*



$$\boxed{\begin{aligned}&\hat{i}\times\hat{j} = \hat{k} && \hat{j}\times\hat{i} = -\hat{k} \\ &\hat{j}\times\hat{k} = \hat{i} && \hat{k}\times\hat{j} = -\hat{i} \\ &\hat{k}\times\hat{i} = \hat{j} && \hat{i}\times\hat{k} = -\hat{j}\end{aligned}}$$

Going by anticlockwise $\rightarrow$ +ve

clockwise $\rightarrow$ -ve

* $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \bar{0}$ $\Rightarrow$ Zero vector

Q) If $\bar{a}$ & $\bar{b}$ are arbitrary vectors with magnitude a & b, then $|\bar{a}\times\bar{b}|^2 = ?$

a) $a^2b^2 - (\bar{a}\cdot\bar{b})^2$ c) $a^2b^2 + (\bar{a}\cdot\bar{b})^2$

b) $ab - \bar{a}\cdot\bar{b}$ d) $ab + \bar{a}\cdot\bar{b}$

Soln

$$\begin{aligned}|\bar{a}\times\bar{b}|^2 &= (ab\sin\theta)^2 \\ &= a^2b^2(1-\cos^2\theta) \\ &= a^2b^2 - (\bar{a}\cdot\bar{b})^2\end{aligned}$$

$\Rightarrow$ option a.

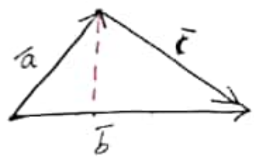
Scalar Triple Product

$$\bar{a}\cdot(\bar{b}\times\bar{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = [a \;\; b \;\; c]$$

* $[\bar{a}\;\bar{b}\;\bar{c}] = [\bar{b}\;\bar{c}\;\bar{a}] = [\bar{c}\;\bar{a}\;\bar{b}]$ [cyclic]
* $[\bar{a}\;\bar{b}\;\bar{c}] = -[\bar{b}\;\bar{a}\;\bar{c}]$
* If $[\bar{a}\;\bar{b}\;\bar{c}] = 0 \Rightarrow$ Coplanar vectors

Vector Triple Product:

$$\boxed{\begin{aligned} \bar{a}\times(\bar{b}\times\bar{c}) &= (\bar{a}\cdot\bar{c})\,\bar{b} - (\bar{a}\cdot\bar{b})\,\bar{c} \\ (\bar{a}\times\bar{b})\times\bar{c} &= (\bar{a}\cdot\bar{c})\,\bar{b} - (\bar{b}\cdot\bar{c})\,\bar{a} \end{aligned}}$$

* 

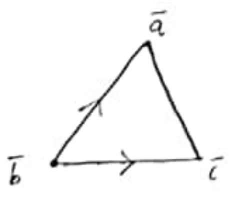
$$\boxed{\text{Area of } \Delta = \frac{1}{2}\,|\bar{a}\times\bar{b}|}$$ when 2 adjacent sides are known

Proof:

height $= |\bar{a}|\sin\theta$

Base $= |\bar{b}|$

$\therefore$ Area $= \frac{1}{2}\,|\bar{a}||\bar{b}|\sin\theta = \frac{1}{2}\,|\bar{a}\times\bar{b}|$

Q) 

Area of Δ = ? , $\bar{a}, \bar{b}$ & $\bar{c}$ are position vectors.

Soln:

$\overrightarrow{BA} = (\bar{a} - \bar{b})$

$\overrightarrow{BC} = (\bar{c} - \bar{b})$

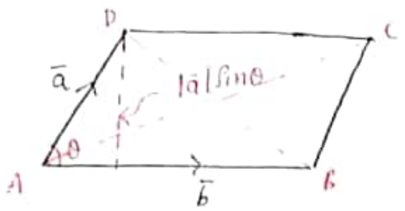
$\therefore$ Area $= \frac{1}{2}\,\left|(\bar{c}-\bar{b})\times(\bar{a}-\bar{b})\right|$

$= \frac{1}{2}\,\left|(\bar{c}\times\bar{a}) - (\bar{c}\times\bar{b}) - (\bar{b}\times\bar{a}) + \underbrace{(\bar{b}\times\bar{b})}_{0}\right|$

***

$$\boxed{\text{Area} = \frac{1}{2}\,\left|(\bar{c}\times\bar{a}) + (\bar{b}\times\bar{c}) + (\bar{a}\times\bar{b})\right|}$$

Area of Parallelogram:



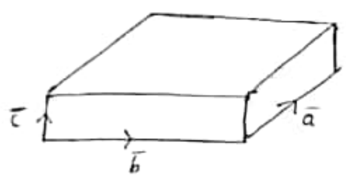
In terms of side:

$$\boxed{\text{Area}(\square) = |\bar{a} \times \bar{b}|}$$

In terms of diagonal:

$$\boxed{\text{Area}(\square) = \frac{1}{2}|\overline{AC} \times \overline{BD}|}$$

Parallelopiped



$$\boxed{\text{Volume} = |\bar{a} \cdot (\bar{b} \times \bar{c})| = |[\bar{a} \ \bar{b} \ \bar{c}]|}$$

HW

Q) $\bar{a} = [1 \ \ 2 \ \ 3]$ $\quad \bar{b} = [-1 \ \ 1 \ \ 2]$ $\quad \bar{c} = [2 \ \ 1 \ \ 4]$

Find the volume of parallelopiped.

Soln

$$\text{Volume} = [\bar{a} \ \ \bar{b} \ \ \bar{c}] = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix}$$

$$= (4-2) - 2(-4-4) + 3(-1-2)$$

$$= 2 + 16 - 9$$

$$= 9$$

HW

Q) $P[0.866 \ \ 0.500 \ \ 0]$, $Q_1 = [0.259 \ \ 0.966 \ \ 0]$ Angle b/w P & Q = ?

Soln $\bar{P} \cdot \bar{Q} = |\bar{P}||\bar{Q}| \cos\theta$

$\Rightarrow (0.866)(0.259) + (0.5)(0.966) + 0 = (0.9999)(1.0001) \cos\theta$

$\Rightarrow \cos\theta = 0.7073$

$\Rightarrow \theta = \underline{44.98^\circ}$ or $\underline{0.785^c}$

HW

Q) $\bar{a} = [1 \;\; 4 \;\; 2]$ $\quad \bar{b} = [3 \;\; -2 \;\; 7]$ $\quad (\bar{a} \times \bar{b}) = ?$

Soln

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix} = \underline{\underline{32\hat{i} - \hat{j} - 14\hat{k}}}$$

Note * If $\bar{b} = \lambda \bar{a}$ then we can say $\bar{a}$ & $\bar{b}$ are parallel vectors.

HW

Q) Check whether $\bar{a} = [3 \;\; -2 \;\; 4]$ & $\bar{b} = [-6 \;\; 4 \;\; -8]$ are parallel vectors? Find the value of λ

Soln

$$\bar{b} = -6\hat{i} + 4\hat{j} - 8\hat{k} = -2\left(3\hat{i} + (-2\hat{j}) + 4\hat{k}\right) = -2\bar{a}$$

$\therefore$ $\bar{b}$ & $\bar{a}$ are parallel vectors & $\underline{\underline{\lambda = -2}}$

Q) $[\bar{a}+\bar{b} \quad \bar{b}+\bar{c} \quad \bar{c}+\bar{a}] = 2[\bar{a} \;\; \bar{b} \;\; \bar{c}]$ True / False ?

Soln

$$[\bar{a}+\bar{b} \quad \bar{b}+\bar{c} \quad \bar{c}+\bar{a}] = \begin{vmatrix} \bar{a}+\bar{b} & \bar{b}+\bar{c} & \bar{c}+\bar{a} \end{vmatrix}$$

$$[\bar{a}+\bar{b} \quad \bar{b}+\bar{c} \quad \bar{c}+\bar{a}] = \begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ c_1+a_1 & c_2+a_2 & c_3+a_3 \end{vmatrix}$$

$$= \begin{vmatrix} 2a_1+2b_1+2c_1 & 2a_2+2b_2+2c_2 & 2a_3+2b_3+2c_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ c_1+a_1 & c_2+a_2 & c_3+a_3 \end{vmatrix}$$

$$= 2\begin{vmatrix} a_1+b_1+c_1 & a_2+b_2+c_2 & a_3+b_3+c_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ c_1+a_1 & c_2+a_2 & c_3+a_3 \end{vmatrix}$$

$$= 2\begin{vmatrix} a_1+b_1+c_1 & a_2+b_2+c_2 & a_3+b_3+c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\Rightarrow [\bar{a}+\bar{b} \quad \bar{b}+\bar{c} \quad \bar{c}+\bar{a}] = 2\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

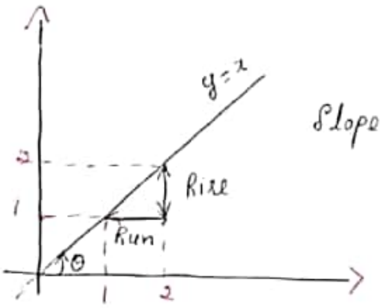
$$= 2\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\Rightarrow \boxed{[\bar{a}+\bar{b} \quad \bar{b}+\bar{c} \quad \bar{c}+\bar{a}] = 2[\bar{a} \;\; \bar{b} \;\; \bar{c}]}$$

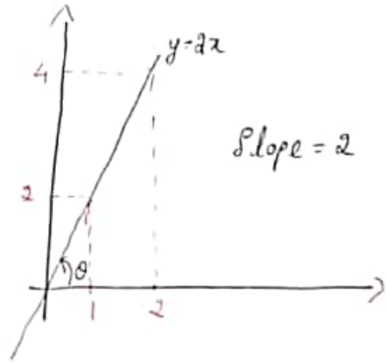
$\Downarrow$

Hence True

29/04/20



Slope = 1

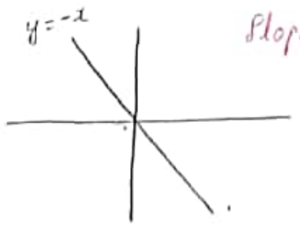


Slope = 2

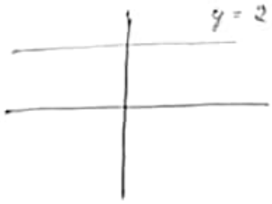
$$\text{Steepness} = \text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$ = Avg. rate of change of y wrt x

$$\hookrightarrow \frac{dy}{dx} = \tan\theta$$

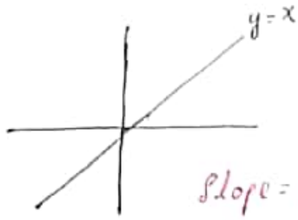
For linear graph:



Slope = -1



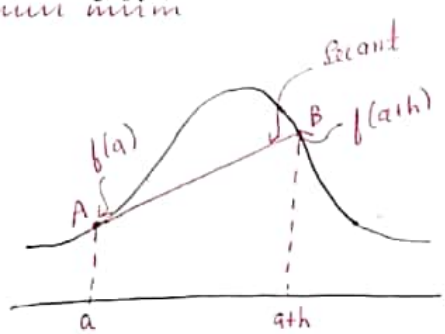
Slope = 0



Slope = +1

Slope = ∞ / Not defined

For a Curve:



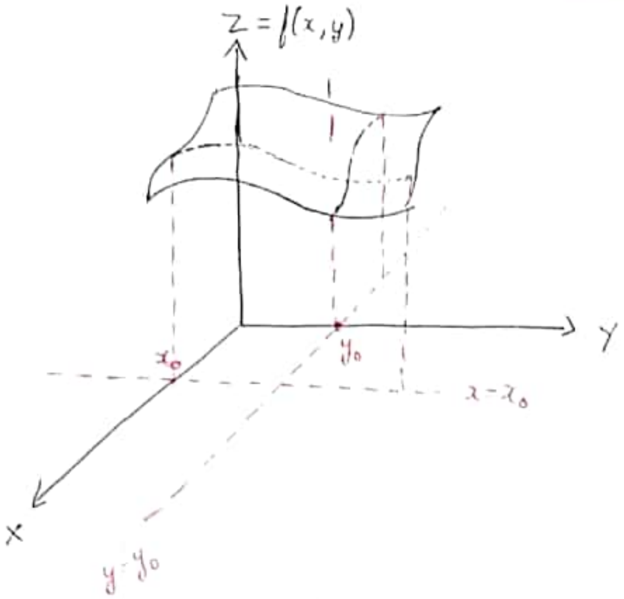
$$\boxed{\therefore \text{Slope} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}}$$

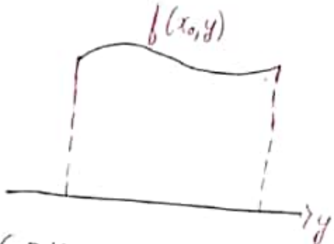
Instantaneous

Note Slope can also be said as the limiting case of secant (i.e. tangent)

$$\therefore \text{Slope} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{\cancel{a}+h-\cancel{a}} = f'(a)$$

$$\boxed{\therefore \left.\frac{dy}{dx}\right|_{x=a} = f'(x=a)}$$

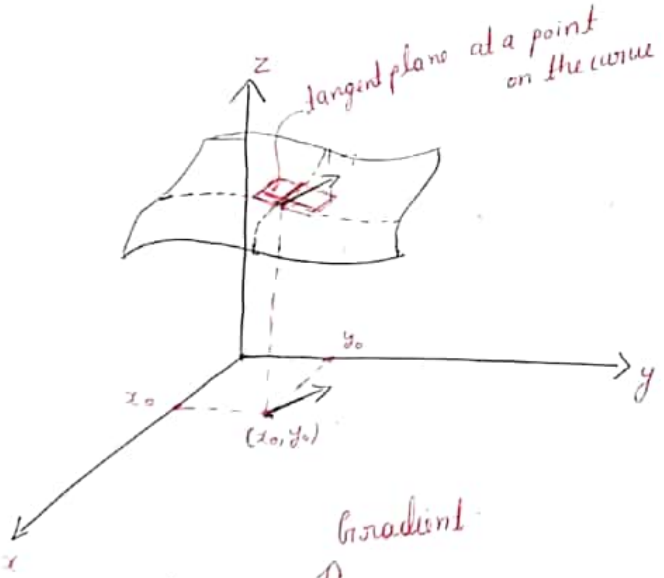




$\left(\frac{\partial f}{\partial y}\right)_{x=x_0}$ ⇒ Differentiating f wrt y treating x as constant.

⟹

IIIly $\left(\frac{\partial f}{\partial x}\right)_{y=y_0}$ ⇒ Differentiating of f wrt x treating y as constant



<u>Note</u> To find a tangent plane in a particular direction we use gradient.

* ∇ = del/Nebula

$= \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

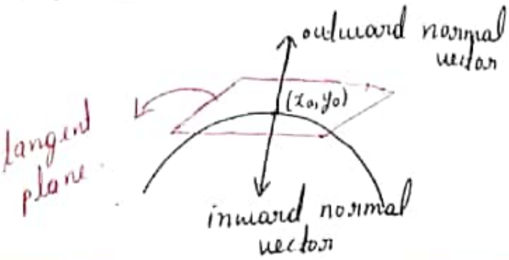
Gradient

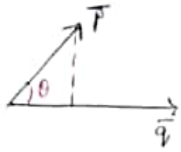
Directional ~~Gradient~~: Derivative

$$\boxed{\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}}$$

Directional Derivative

~~Gradient~~ of f is a vector normal to the tangent plane. It can be both outward normal vector as well as inward normal vector.





Component of $\bar{p}$ in direction of $\bar{q}$ $= |\bar{p}| \cos\theta$

[Let $\bar{p}$ be the normal vector to the tangent plane i.e. ∇f.

Let $\bar{q}$ be some vector in the direction of unit vector $\hat{a}$]

$$= |\bar{p}| \frac{(\bar{p} \cdot \bar{q})}{|\bar{p}|\,|\bar{q}|}$$

$$= \bar{p} \cdot \hat{q}$$

$$= \nabla f \cdot \hat{a} \Rightarrow \text{unit vector}$$

$\therefore$ The component of directional derivative vector in a particular direction $= \nabla f \cdot \hat{a} \Rightarrow$ Scalar.

If the directional $(\hat{a})$ vector is in the direction of directional derivative vector (∇f) then we will be having maximum directional derivative

i.e. $\quad$ Max D.D $= \nabla f \cdot \hat{\nabla f}$

$$= \nabla f \cdot \frac{\nabla f}{|\nabla f|}$$

$$= \frac{|\nabla f|^2}{|\nabla f|}$$

i.e. Max. Directional Derivative $= |\nabla f|$

Note: Gradient of a scalar function will give you a normal.

Q) $f = xyz^3$ at $(1,0,2)$ Find the magnitude of the gradient.

Soln.

$$\nabla f = \frac{\partial}{\partial x}(xyz^3)\,\hat{i} + \frac{\partial}{\partial y}(xyz^3)\,\hat{j} + \frac{\partial}{\partial z}(xyz^3)\,\hat{k}$$

$$\nabla f = yz^3\,\hat{i} + xz^3\,\hat{j} + 3z^2xy\,\hat{k}$$

$$\therefore \nabla f\big|_{(1,0,2)} = 0\hat{i} + 8\hat{j} + 0\hat{k} = 8\hat{j}$$

$$\therefore |\nabla f|_{(1,0,2)} = 8$$

Q) $x^2yz = 1$ Find the normal vector at $(1,1,1)$

Soln

$$\nabla f = \frac{\partial}{\partial x}(x^2yz)\,\hat{i} + \frac{\partial}{\partial y}(x^2yz)\,\hat{j} + \frac{\partial}{\partial z}(x^2yz)\,\hat{k}$$

$$\Rightarrow \nabla f = 2xyz\,\hat{i} + x^2z\,\hat{j} + x^2y\,\hat{k}$$

$$\therefore \nabla f\Big|_{(1,1,1)} = 2\hat{i} + \hat{j} + \hat{k}$$

Since they have asked only a normal vector we must consider both outward normal $[2\hat{i} + \hat{j} + \hat{k}]$ & inward normal $[-2\hat{i} - \hat{j} - \hat{k}]$

$$\therefore \nabla f\Big|_{1,1,1} = \text{Normal vector}\Big|_{(1,1,1)} = \pm[2\hat{i} + \hat{j} + \hat{k}]$$

Note To find unit normal vector:

$$\hat{\nabla} f\Big|_{1,1,1} = \pm\frac{(2\hat{i} + \hat{j} + \hat{k})}{\sqrt{4+1+1}} = \pm\frac{(2\hat{i} + \hat{j} + \hat{k})}{\sqrt{6}}$$

Q) $f = 2x^2 + 3y^2 + z^2$ at Point $(1,2,3)$ in the direction $\bar{a} = \hat{i} - 2\hat{k}$

Directional derivative = ?

Soln

$$\nabla f \cdot \hat{a} = \cancel{\frac{\partial}{\partial x}(2x^2)\hat{i}}\left[\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right] \cdot \left[\frac{\hat{i}}{\sqrt{5}} - \frac{2}{\sqrt{5}}\hat{k}\right]$$

$$\Rightarrow \nabla f \cdot \hat{a} = [4x\,\hat{i} + 6y\,\hat{j} + 2z\,\hat{k}] \cdot \left[\frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{k}\right]$$

$$\Rightarrow \nabla f \cdot \hat{a}\Big|_{(1,2,3)} = \frac{4}{\sqrt{5}} \times (1) + 0 + \frac{2 \cdot (-2) \cdot 3}{\sqrt{5}}$$

$$\Rightarrow \nabla f \cdot \hat{a}\Big|_{(1,2,3)} = \frac{4-12}{\sqrt{5}} = \frac{-8}{\sqrt{5}}$$

Q) Find the Directional Derivative. $f(x,y) = x^2 + y^2$

$\bar{a} = 4\hat{i} + 3\hat{j}$ at $(1,2)$

Soln

$$\nabla f \cdot \hat{a} = [2x\,\hat{i} + 2y\,\hat{j}] \cdot \left[\frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}\right]$$

$$\therefore \nabla f \cdot \hat{a}\Big|_{(1,2)} = \left(\frac{8}{5}\right)(1) + \frac{6}{5}(2) = 4$$

Divergence

If Divergence of a vector : $\nabla \cdot \bar{a} > 0 \Rightarrow +ve \Rightarrow$ Source

If Divergence of a vector : $\nabla \cdot \bar{a} < 0 \Rightarrow -ve \Rightarrow$ Sink

If Divergence of a vector : $\nabla \cdot \bar{a} = 0 \Rightarrow$ No source/sink

or

Solenoidal

or

No Divergence

Q) $\bar{V} = 5xy\,\hat{i} + 2y^2\hat{j} + 3yz^2\hat{k}$

Divergence of $\bar{V}$ at $(1,1,1)$

Soln

$$\text{Divergence of } \bar{V} = \nabla\cdot\bar{V} = \frac{\partial}{\partial x}(5xy) + \frac{\partial}{\partial y}(2y^2) + \frac{\partial}{\partial z}(3yz^2)$$

$$= 5y + 4y + 6yz$$

$$\nabla\cdot\bar{V}\Big|_{(1,1,1)} = 5+4+6 = 15$$

Gradient	Divergence
Scalar fn. f	$\bar{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$
$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$	$\nabla\cdot\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

HW

Q) $\bar{V} = -(x\cos xy + y)\hat{i} + (y\cos xy)\hat{j} + (\sin z^2 + x^2+y^2)\hat{k}$

$\nabla\cdot\bar{V} = ?$

Soln

$$\nabla\cdot\bar{V} = \frac{\partial}{\partial x}(-x\cos xy - y) + \frac{\partial}{\partial y}(y\cos xy) + \frac{\partial}{\partial z}(\sin z^2 + x^2+y^2)$$

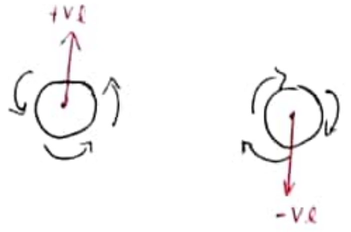
$$= -[xy(\cancel{-}\sin xy) + \cancel{\cos xy}] + [\cancel{\cos xy} + xy(-\cancel{\sin xy})]$$

$$+ (2z)\cos z^2$$

$$\nabla\cdot\bar{V} = 2z\cos z^2$$

Scanned with CamScanner

{ **Curl** $(\nabla \times \bar{v})$



I| $\nabla \times \bar{v} = +ve \Rightarrow$ Anticlockwise

$\nabla \times \bar{v} = -ve \Rightarrow$ Clockwise

$\nabla \times \bar{v} = 0 \Rightarrow$ Irrotational

Q) $\bar{F} = y\hat{i} - x\hat{j}$. Curl of $\bar{F}$

Soln
$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + (-1-1)\hat{k}$$

$\therefore \quad \underline{\nabla \times \bar{F} = -2\hat{k}} \quad \equiv$

Q) $\bar{F} = x\hat{i} - y\hat{j}$ Comment on Divergence & Rotation

Soln
$$\nabla \cdot \bar{F} = \frac{\partial x}{\partial x} + \frac{\partial (-y)}{\partial y} = 1 - 1 = 0 \Rightarrow \text{Divergence Free}$$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & -1 & 0 \end{vmatrix} = \bar{0} \Rightarrow \text{Irrotational}$$

Note. * $\bar{r}$ = position vector $= x\hat{i} + y\hat{j} + z\hat{k}$

$|\bar{r}| = r = \sqrt{x^2+y^2+z^2}$

$\nabla \cdot \bar{r} = 3$

$\nabla \times \bar{r} = 0$

Q) $\bar{r}$ is a position vector, $|\bar{r}| = r$, $\phi = \ln r$ $\quad \nabla\phi = ?$

a) $\bar{r}$

b) $\bar{r}/|\bar{r}|$

c) $\bar{r}/\bar{r}\cdot\bar{r}$

d) $\bar{r}/r^3$

Soln $\cancel{\nabla\phi = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2}\,\hat{i} + \frac{\partial}{\partial y}\sqrt{x^2+y^2+z^2}}$

$\cancel{\nabla\phi = \frac{\partial}{\partial x}\frac{1}{2}\log(x^2+\ldots}$

Soln M1

$$\nabla\phi = \frac{\partial}{\partial x}\left[\tfrac{1}{2}\log(x^2+y^2+z^2)\right]\hat{i} + \frac{\partial}{\partial y}\left[\tfrac{1}{2}\log(x^2+y^2+z^2)\right]\hat{j} + \frac{\partial}{\partial z}\left[\tfrac{1}{2}\log(x^2+y^2+z^2)\right]\hat{k}$$

$$\Rightarrow \nabla\phi = \frac{x}{x^2+y^2+z^2}\hat{i} + \frac{y}{x^2+y^2+z^2}\hat{j} + \frac{z}{x^2+y^2+z^2}\hat{k}$$

$$\Rightarrow \nabla\phi = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{x^2+y^2+z^2} = \frac{\bar{r}}{\bar{r}\cdot\bar{r}}$$ $\Rightarrow$ option c

M2

$$\cancel{\nabla f(r)} = \cancel{f'(r)\cdot\nabla r}$$

$$\boxed{\nabla f(r) = f'(r)\,\nabla r = f'(r)\,\frac{\bar{r}}{r}}$$ ***

$$\nabla\phi = \nabla(\ln r) = \frac{1}{r}\cdot\frac{\bar{r}}{r} = \frac{\bar{r}}{r^2}$$ $\Rightarrow$ option c

$$= \frac{\bar{r}}{\bar{r}\cdot\bar{r}}$$

Q) $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $|\bar{r}| = r$

$\text{Div}(r^2\nabla(\ln r)) = ?$

Soln

$$\nabla(\ln r) = \frac{1}{r}\frac{\bar{r}}{r} = \frac{\bar{r}}{r^2}$$

$$\therefore \text{Div}\left(r^2\nabla(\ln r)\right) = \text{Div}\left(\cancel{r^2}\left(\frac{\bar{r}}{\cancel{r^2}}\right)\right) = \text{Div}(\bar{r})$$

$$\Rightarrow \text{Div}\left(r^2\nabla(\ln r)\right) = 3$$

Note * $\bar{r} \neq y\hat{i} + x\hat{j} + z\hat{k}$

Here $\nabla\cdot\bar{r} = 1$ [Hence this is not a position vector]

Scanned with CamScanner

30/04/20

Formulas:

I.

1) $\text{grad}(fg) = f\,\text{grad}(g) + g\,\text{grad}(f)$

i.e $\nabla(fg) = f\nabla g + g\nabla f$

2) $\text{div}(f\bar{G}) = (\text{grad}(f))\cdot\bar{G} + f(\text{div}\,\bar{G})$

i.e $\nabla\cdot(f\bar{G}) = (\nabla f)\cdot\bar{G} + f(\nabla\cdot\bar{G})$

3) $\text{Curl}(f\bar{G}) = (\text{grad} f)\times\bar{G} + f(\text{curl}\,\bar{G})$

i.e $\nabla\times(f\bar{G}) = \nabla f\times\bar{G} + f(\nabla\times\bar{G})$

4) $\nabla(\bar{A}\cdot\bar{B}) = (\bar{B}\cdot\nabla)\bar{A} + (\bar{A}\cdot\nabla)\bar{B} + \bar{B}\times(\nabla\times\bar{A}) + \bar{A}\times(\nabla\times\bar{B})$

5) $\nabla\cdot(\bar{A}\times\bar{B}) = \bar{B}\cdot(\nabla\times\bar{A}) - \bar{A}\cdot(\nabla\times\bar{B})$

6) $\nabla\times(\bar{A}\times\bar{B}) = (\bar{B}\cdot\nabla)\bar{A} - \bar{B}(\nabla\cdot\bar{A}) - (\bar{A}\cdot\nabla)\bar{B} + A(\nabla\cdot\bar{B})$

II.

1) $\text{div}(\text{grad} f) = \nabla\cdot(\nabla f) = \nabla^2 f$

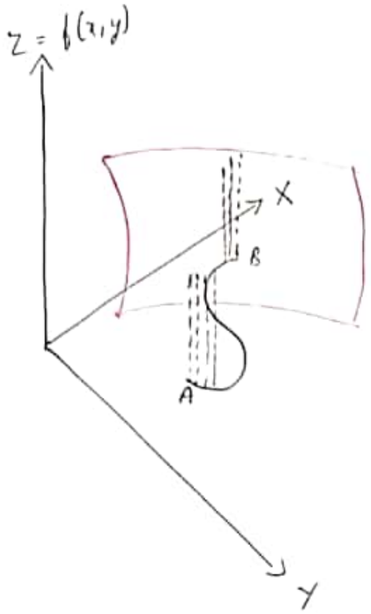
2) $\text{curl}(\text{grad} f) = \nabla\times(\nabla f) = \bar{0}$

3) $\text{div}(\text{curl}\,\bar{F}) = \nabla\cdot(\nabla\times\bar{F}) = 0$

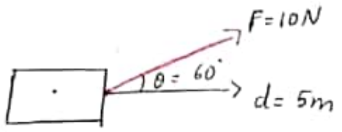
4) $\text{curl}(\text{curl}\,\bar{F}) = \text{grad}(\text{div}\,\bar{F}) - \nabla^2\bar{F}$

i.e $\nabla\times(\nabla\times\bar{F}) = \nabla(\nabla\cdot\bar{F}) - \nabla^2\bar{F}$

Line Integration



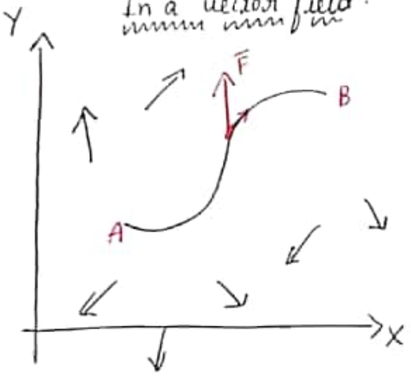
$$\boxed{\int_A^B f(x,y)\,ds} \Rightarrow \text{Line Integration of a Scalar Function}$$



Work Done $= \bar{F}\cdot\bar{d} = Fd\cos\theta$

$\Rightarrow \quad W = 10\times5\times\cos60^\circ = \underline{\underline{25J}}$

In a vector field:



$$\therefore \quad \boxed{W = \int dW = \int_A^B \bar{F}\cdot d\bar{r}}$$

$\int_A^B \bar{F}\cdot d\bar{r} \rightarrow$ Line integral along a path

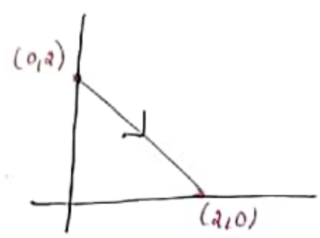
$$\boxed{\int_c \bar{F}\cdot d\bar{r} = \int_c (F_1\,dx + F_2\,dy + F_3\,dz)}$$

Q) Integration of $\bar{F} = (x^2+xy)\hat{i} + (y^2+xy)\hat{j}$ over a straight line from (0,2) to (2,0) = ?

Soln: $I = \int \bar{F}\cdot d\bar{r} = \int_c (x^2+xy)\,dx + (y^2+xy)\,dy$

Eq. of line is : $\quad x+y=2 \quad \Rightarrow y = 2-x$

$\Rightarrow dy = -dx$



$$\therefore \quad I = \int_{x=0}^{2} \left[\left(x^2 + x(2-x)\right) - (2-x)^2 - x(2-x)\right] dx = \int_{x=0}^{2} \left[\left(\cancel{x^2} + 2x - \cancel{x^2}\right) - (2-x)\left[2 - \cancel{x} + \cancel{x}\right]\right] dx$$

$$\Rightarrow I = \int_{x=0}^{2} (2x - 4 + 2x)\, dx = \frac{4}{2}\left[x^2\right]_0^2 - 4\left[x\right]_0^2 = 8 - 8 = 0$$

6) $\bar{V} = 2xyz\,\hat{i} + x^2z\,\hat{j} + x^2y\,\hat{k}$ Line integral from $(0,0,0)$ to $(1,1,1)$ over a straight line.

Soln

$$I = \int_c \bar{V}\cdot d\bar{r} = \int_c (2xyz)\,dx + (x^2z)\,dy + (x^2y)\,dz$$

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\Rightarrow x = y = z = t$$

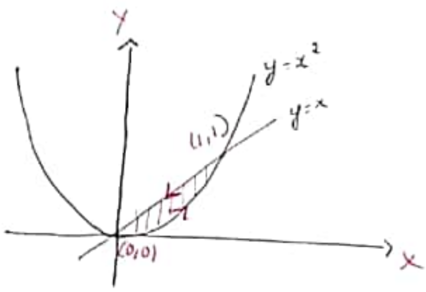
$$\therefore dx = dy = dz = dt$$

$$\therefore I = \int_{t=0}^{1} \left[2t^3\,dt + t^3\,dt + t^3\,dt\right] = \int_{t=0}^{1} 4t^3\,dt$$

$$\Rightarrow I = 1$$

5) $\displaystyle\int_{\text{closed curve}} (xy + y^2)\,dx + x^2\,dy$ bounded by $y = x$ & $y = x^2$. Line Integral = ?

Soln



$$I = \int_{x=0}^{1} \left(x\cdot(x^2) + x^4\right)dx + x^2\cdot(2x\,dx)$$

$$+ \int_{x=1}^{0} (x^2 + x^2)\,dx + (x^2\,dx)$$

$y = x^2$

$dy = 2x\,dx$

$y = x$

$dy = dx$

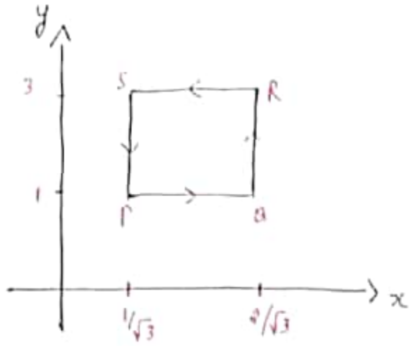
$$\Rightarrow I = \int_{x=0}^{1} (3x^3 + x^4)\,dx + \int_{x=1}^{0} 3x^2\,dx$$

$$= \frac{3}{4} + \frac{1}{5} + [0 - 1]$$

$$= \frac{15 + 4 - 20}{20}$$

$$\Rightarrow I = -1/20$$

Q)



$$\oint \bar{A} \cdot d\bar{l} = ?$$

$$\bar{A} = xy\hat{i} + x^2\hat{j}$$

Soln M1

Eq. of PQ: $y = 1 \Rightarrow dy = 0$

Eq. of QR: $x = 2/\sqrt{3} \Rightarrow dx = 0$

Eq. of RS: $y = 3 \Rightarrow dy = 0$

Eq. of SP: $x = 1/\sqrt{3} \Rightarrow dx = 0$

M2 We can also solve quickly using Green's Theorem

$$I = \oint \bar{A} \cdot d\bar{l} = \oint xy\,dx + x^2 dy = \int_{PQ} xy\,dx + x^2 dy + \int_{QR} xy\,dx + x^2 dy$$

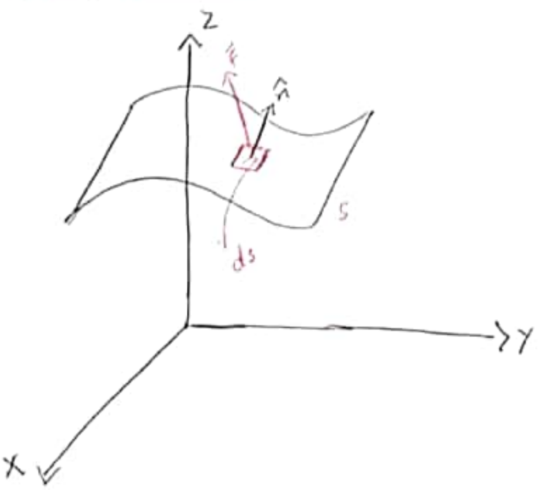
$$+ \int_{RS} xy\,dx + x^2 dy + \int_{SP} xy\,dx + x^2 dy$$

$$\Rightarrow I = \int_{x=1/\sqrt{3}}^{2/\sqrt{3}} x\,dx + \int_{y=1}^{3} \frac{4}{3}\,dy + \int_{x=2/\sqrt{3}}^{1/\sqrt{3}} 3x\,dx + \int_{y=3}^{1} \frac{dy}{3}$$

$$= \int_{x=1/\sqrt{3}}^{2/\sqrt{3}} -2x\,dx + \int_{y=1}^{3} 1\,dy$$

$$\Rightarrow I = (-1)\left[4/3 - 1/3\right] + 2 = 1$$

Surface Integral:



$$\boxed{\text{Surface Integral} = \iint \bar{F} \cdot \hat{n}\, ds}$$

Stokes Theorem:

$$\boxed{\int_C \bar{F}\cdot d\bar{r} = \iint (\nabla\times\bar{F})\cdot\hat{n}\, ds}$$

$\hat{n}$ - outward unit normal vector

Note: By default, we are integrating in anticlockwise direction.
If the question is asking for clockwise direction, do the entire calculation & put negative sign to the final ans

$$\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}, \quad d\bar{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

For surface in XY plane $\Rightarrow$ $ds = dxdy$ & $\hat{n} = \hat{k}$

For surface in YZ plane $\Rightarrow$ $ds = dydz$ & $\hat{n} = \hat{i}$

For surface in ZX plane $\Rightarrow$ $ds = dzdx$ & $\hat{n} = \hat{j}$

Note: Stoke's Theorem deals with 3D

$\rightarrow$ If $\bar{F} = F_1\hat{i} + F_2\hat{j}$

Then $$\nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} = -\cancel{\frac{\partial F_2}{\partial z}}^{0}\hat{i} + \cancel{\frac{\partial F_1}{\partial z}}^{0}\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$$\therefore \nabla\times\bar{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$\therefore$ From Stoke's theorem: $$\int_C \bar{F}\cdot d\bar{r} = \iint (\nabla\times\bar{F})\cdot\hat{k}\, ds$$

$$= \iint \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}\cdot\hat{k}\, dxdy$$

GREEN'S THEOREM (2D)

$$\boxed{\underset{\text{Anticlockwise}}{\oint} \bar{F}\cdot d\bar{r} = \iint \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy}$$

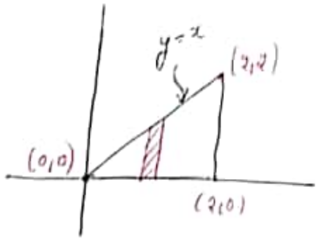
$$\boxed{\underset{\text{Clockwise}}{\oint} \bar{F}\cdot d\bar{r} = -\iint \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy}$$

Note: Green's & Stoke's theorem are applicable only for closed loop.

a) $\oint \vec{F}\cdot d\vec{r} = ?$, $\vec{F} = 2y^2\hat{i} + 3x^2\hat{j} + (2x+z)\hat{k}$ & C is boundary of Δ whose vertices are $(0,0,0)$, $(2,0,0)$ & $(2,2,0)$

Soln

$$\oint \vec{F}\cdot d\vec{r} = \iint (\nabla\times\vec{F})\cdot\hat{k}\,ds$$



$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y^2 & 3x^2 & (2x+z) \end{vmatrix}$$

$$= (0-0)\hat{i} - (2-0)\hat{j} + (6x-4y)\hat{k}$$

$$\Rightarrow \nabla\times\vec{F} = -2\hat{j} + (6x-4y)\hat{k}$$

Since the surface is in XY plane, $ds = dx\,dy$

$$\therefore \oint \vec{F}\cdot d\vec{r} = \iint_S \{-2\hat{j} + (6x-4y)\hat{k}\}\cdot dx\,dy\,\hat{k}$$

$$\Rightarrow \oint \vec{F}\cdot d\vec{r} = \iint_S (6x-4y)\,dx\,dy = \int_{x=0}^{2}\int_{y=0}^{x} (6x-4y)\,dx\,dy$$

$$\Rightarrow \oint \vec{F}\cdot d\vec{r} = \int_{x=0}^{2} (6x^2 - 2x^2)\,dx = 2[x^3]_0^2 - \frac{2}{3}[x^3]_0^2$$

$$\Rightarrow \oint \vec{F}\cdot d\vec{r} = 16 - \frac{16}{3} = 32/3$$

a) $\oint y\,dx - x\,dy = ?$ along a circle $x^2+y^2 = 1/4$ traversed in a counter clockwise direction.

Soln

$$\oint y\,dx - x\,dy = \iint_S \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right)dx\,dy$$

$$\oint y\,dx - x\,dy = \iint_S (-1-1)\,dx\,dy = -2\iint_S dx\,dy$$

$\Rightarrow \therefore \oint y\,dx - x\,dy = -2 \times \pi (1/2)^2 = -\frac{\pi}{2}$

Note: If the direction is clockwise, then ans would be $\pi/2$

01/05/20 Q) $\bar{F} = z\hat{i} + x\hat{j} + y\hat{k}$ S is a portion of sphere $x^2+y^2+z^2=1$ for $z \geq 0$. Then $\int_S (\nabla \times \bar{F})\, d\bar{s}$ = ______

Soln

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & x & y \end{vmatrix} = 1\hat{i} - (-1)\hat{j} + 1\hat{k}$$

$$\Rightarrow \nabla \times \bar{F} = \hat{i} + \hat{j} + \hat{k}$$

Note: $\hat{n}\, ds = d\bar{s}$

$$\therefore \int_S (\nabla \times \bar{F})\, d\bar{s} = \int_S (\hat{i} + \hat{j} + \hat{k}) \cdot (d\bar{s})$$

$$= \int_S (\hat{i} + \hat{j} + \hat{k}) \cdot (dx\,dy\,\hat{k})$$

$$= \int_S dx\,dy$$

$$= [\pi r^2]\Big|_{r=1}$$

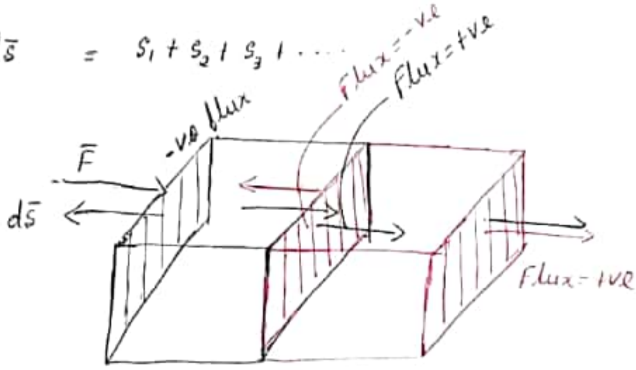
$$= \pi$$

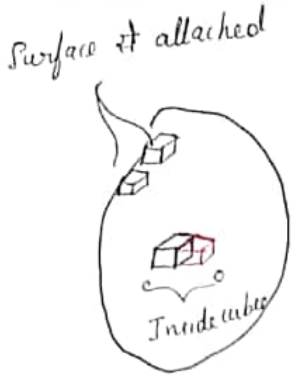
Divergence Theorem

$$\underset{\text{at a point}}{\nabla \cdot \bar{F}\, dv} = \lim_{dv \to 0} \frac{\oiint \bar{F} \cdot d\bar{s}}{dv}$$

$$\Rightarrow \underset{\text{at a point}}{\nabla \cdot \bar{F}\, dv} = \lim_{dv} \oiint \bar{F} \cdot d\bar{s} = s_1 + s_2 + s_3 + \cdots$$

Flux $= \bar{F} \cdot d\bar{s} = |\bar{F}|\,|d\bar{s}| \cos\theta$





$$\cancel{\iiint (\nabla\cdot\bar F)\,dv} = \cancel{\oiint \bar F\cdot\hat n\, d\bar s}$$

$$\boxed{\oiint \bar F\cdot\hat n\, ds = \iiint_V (\nabla\cdot\bar F)\,dv}$$

$\Downarrow$

Gauss - Divergence Theorem

Q) Consider a closed surface S surrounding a volume V. If $\bar r$ is position vector, $\hat n$ = unit normal vector on surface S,

$\iint 5\bar r\cdot\hat n\, ds = ?$ a) V b) 3V c) 15V d) 10V

Soln

$$\iint 5\bar r\cdot\hat n\, ds = \oiint \underbrace{(5x\hat i + 5y\hat j + 5z\hat k)}_{\bar F}\cdot\hat n\, ds$$

$$= \iiint_V (\nabla\cdot\bar F)\,dv$$

$$= \iiint_V (5+5+5)\,dv$$

$$= \underline{15V}$$ $\Rightarrow$ option c

Q) $\oiint \bar F\cdot\hat n\, ds = ?$, S: sphere $x^2+y^2+z^2=9$, $\bar F = (x+y)\hat i + (x+z)\hat j + (y+z)\hat k$

Soln

$$I = \oiint \bar F\cdot\hat n\, ds = \iiint_V (\nabla\cdot\bar F)\,dv = \iiint_V (1+0+1)\,dv$$

$$\Rightarrow I = 2\times\iiint_V dv = 2\times\frac{4}{3}\pi\times(3)^3$$

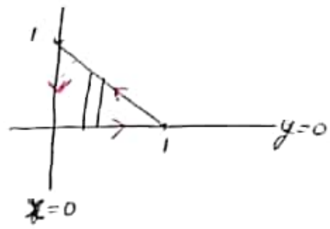
$$\Rightarrow \underline{I = 72\pi}$$

GREEN'S THEOREM	$\oint M dx + N dy$	$=$	$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$
Stokes Theorem	$\oint \bar{F} \cdot d\bar{r}$	$=$	$\iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds$
GAUSS DIVERGENCE THEOREM	$\iint \bar{F} \cdot \hat{n}\, ds$	$=$	$\iiint (\nabla \cdot \bar{F})\, dv$

Q) $\int_C [(3x - 8y^2)dx + (4y - 6xy)dy]$ C is a region bounded by $x=0, y=0, x+y=1$

Soln:

$$I = \int_C [\underbrace{(3x-8y^2)}_{M}dx + \underbrace{(4y-6xy)}_{N}dy]$$



$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint (-6y + 16y)\,dx\,dy$$

$$\Rightarrow I = 10\int_{x=0}^{1}\int_{y=0}^{1-x} y\,dx\,dy = \frac{10}{2}\int_{x=0}^{1} (1-x)^2\,dx$$

$$\Rightarrow I = 5\int_{x=0}^{1} (1+x^2-2x)\,dx = 5\left[x + \frac{x^3}{3} - x^2\right]_0^1 \Rightarrow I = 5/3$$

Q) $\bar{F} = x\hat{i} + y\hat{j} + z\hat{k}$. $\iint_S \frac{1}{4}(\bar{F}\cdot\hat{n})\,dA$, $x^2+y^2+z^2=1$ Value of integral = ?

Soln

$$I = \iint_S \frac{1}{4}(\bar{F}\cdot\hat{n})\,dA = \frac{1}{4}\iiint_V (\nabla\cdot\bar{F})\,dv$$

$$\Rightarrow I = \frac{1}{4}\iiint_V 3\,dv = \frac{3}{4} \times \frac{4}{3}\pi(1)^3 \Rightarrow I = \pi$$

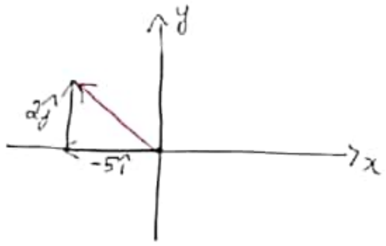
Q) $\bar{F} = y\hat{i} + x\hat{j} + z\hat{k}$, $\oiint \bar{F}\cdot d\bar{s} = ?$ over the closed surface of a cube $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,0,1)$, $(1,1,1)$, $(0,1,1)$, $(1,1,0)$.

Soln

$$I = \oiint \bar{F}\cdot d\bar{s} = \iiint_V (\nabla\cdot\bar{F})\,dv = \iiint_V 1\,dv$$

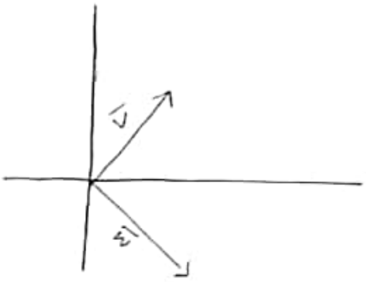
$$\Rightarrow \underline{I = 1}$$

Vector Space:



$$\begin{bmatrix} -5 \\ 2 \end{bmatrix} = \begin{bmatrix} \hat{i} \\ \hat{j} \end{bmatrix}$$

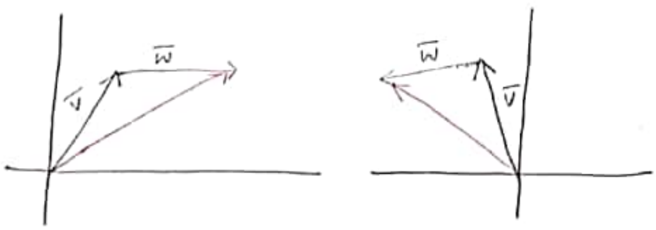
Here the basis is $\hat{i}$ & $\hat{j}$



2 operations:

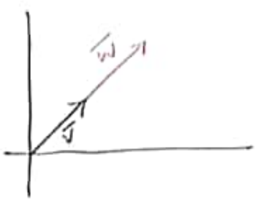
* Scaling
* Addition

} We can capture the entire space using $\bar{v}$ & $\bar{w}$



Linearly dependent vectors

Note * If $\bar{v}$ & $\bar{w}$ are collinear vectors i.e $\bar{w} = \lambda\bar{v}$ then entire space cannot be captured



* If $\bar{v}$ & $\bar{w}$ becomes zero vector then also we cannot capture the entire space

"Span of $\bar{v}$ & $\bar{w}$ is set of all their linear combinations i.e. $a\bar{v} + b\bar{w}$, where $\bar{v}$ & $\bar{w}$ are linearly independent vectors"

R^3 (3D) $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$

If $\bar{v}_1, \bar{v}_2$ & $\bar{v}_3$ are linearly independent vectors $\Rightarrow$ captures entire 3D vector space

If only $\bar{v}_1$ & $\bar{v}_2$ are linearly independent vectors but not $\bar{v}_3$

$\Rightarrow$ captures 2D vector space (plane).

$\Rightarrow$ $\bar{v}_1, \bar{v}_2$ & $\bar{v}_3$ do not span R^3 but it will span R^2 (sub-space of R^3)

Q) Check $v_1 [1 \; 2 \; -1]$ span $R^3 = ?$
$v_2 [2 \; 3 \; 0]$
$v_3 [-1 \; 2 \; 5]$

Soln
M1 $k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$

* If $k_1 = k_2 = k_3 = 0 \Rightarrow$ linearly independent vectors.

* If not the above case $\Rightarrow$ linearly dependent vectors

$k_1 [1 \; 2 \; -1] + k_2 [2 \; 3 \; 0] + k_3 [-1 \; 2 \; 5] = 0$

$\therefore k_1 + 2k_2 - k_3 = 0$

$2k_1 + 3k_2 + 2k_3 = 0$

$-k_1 + 0k_2 + 5k_3 = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \\ -1 & 0 & 5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 4 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 4 \\ 0 & 0 & 12 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $12k_3 = 0 \Rightarrow k_3 = 0$

$-k_2 + 4k_3 = 0 \Rightarrow k_2 = 0$

$\therefore k_1 = 0$

$\Rightarrow k_1 = k_2 = k_3 = 0$

$\Rightarrow$ linearly independent vectors.

M2 Determinant concept

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ -1 & 2 & 5 \end{vmatrix} = -12 \neq 0 \Rightarrow |A|_{3\times3} \neq 0$$

$\Rightarrow \rho(A) = 3$.

i.e $\rho(A) =$ no. of vectors

$\Rightarrow$ Unique soln (Trivial soln)

$\Rightarrow$ Linearly independent vectors

Hence here the basis vectors span R^3.

HW

Q) $[1 \quad 2 \quad 3]$ $[4 \quad 5 \quad 6]$ $[7 \quad 8 \quad 9]$ Does these vectors span R^3 ??

Soln

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1[45-48] - 2[36-42] + 3[32-35]$$

$$= -3 + 12 - 9$$

$$= 0$$

$\Rightarrow \rho(A) <$ no. of vectors

$\Rightarrow$ multiple soln

$\Rightarrow k_1 = k_2 = k_3 = 0$ is not necessary

$\Rightarrow$ Linearly dependent vectors

$\Rightarrow$ cannot span R^3.

PROBABILITY

Basic

Sum of probability = 1

E → Event

$$P(E) = \frac{n(E)}{n(S)} \cdot \frac{\text{no. of elements in } E}{\text{no. of elements in } S} \quad \frac{\text{Requirement}}{\text{Total}}$$

$n(S)$ → Sample space

Types of Events

1) Equally Likely :-

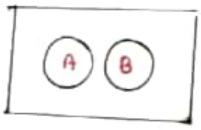
E(A) and E(B) will have equal probability out of whole sample space if only these are the two events possible

eg:- tossing a coin

$P(H) = P(T) = \frac{1}{2}$

2) Mutually Exclusive (ME)

E(A) and E(B) will happen individually seperately cannot happen together



Nothing common ∴ $A \cap B = 0$

3) Independent Event (IE)

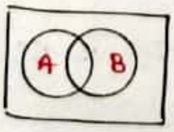
E(A) and E(B) will not depend on each other.

eg:- If a die show 6 what is prob that Head occours on tossing a coin.
Here, coin doesn't care what's die is showing 6, 5, 4 or any the prob. of Head is simple $\frac{1}{2}$ it is not depending on die.

Axioms / Rules of Probability

1) Addition Theorem

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



for ME :- ∵ $A \cap B = 0$

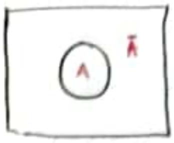
∴ $P(A \cup B) = P(A) + P(B)$

Remember :-

∩ → and → multiply
∪ → or → Add

2) Compliment Rule

i) $\bar{A} = 1 - A$



ii) $P(\overline{A \cup B}) = 1 - P(A \cup B)$

or

$P(\overline{A \cup B}) = P(\bar{A} \cap \bar{B})$ — De morgan's Rule

Use :–

$P(x>2) = P(x=3) + P(x=4) + P(x=5)$ and so on

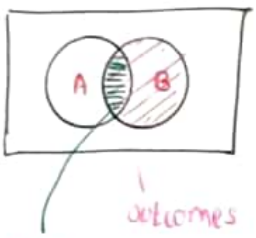
or

$P(x>2) = 1 - \overline{P(x<2)}$ Compliment

$= 1 - P(x=0) + P(x=1)$

3) Conditional Probability

$P(A/B) \xrightarrow{\text{reads as}}$ given that outcomes are from B what is probability of A.



$\because$ outcomes are from B $\therefore$ B becomes our new sample space.

and $A \cap B$ become new event

$\therefore P(A/B) = \dfrac{\text{Event}}{\text{Sample Space}} = \dfrac{P(A \cap B)}{P(B)}$

This probability of A is what they asking when B is sample space.

4) Multiplication theorem

$P(A \cap B) = P(A/B) \cdot P(B)$

→ Prob. of A if B occurs

→ Prob. of B occuring

or

$= P(B/A) \cdot P(A)$

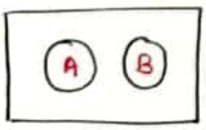
for IE :– $\because$ Events do not depend on each other

$\therefore P(A/B) = P(A)$ since A does not depend on B

A is seperate event $\therefore$ seperate probability

$\therefore P(A \cap B) = P(A) \cdot P(B)$ for IE

for ME :–



$\left.\begin{array}{l} P(A/B) = 0 \\ P(B/A) = 0 \end{array}\right\}$ as there is no intersection $\rightarrow P(A \cap B) = 0$

5) Baye's theorem

Reversed is asked.

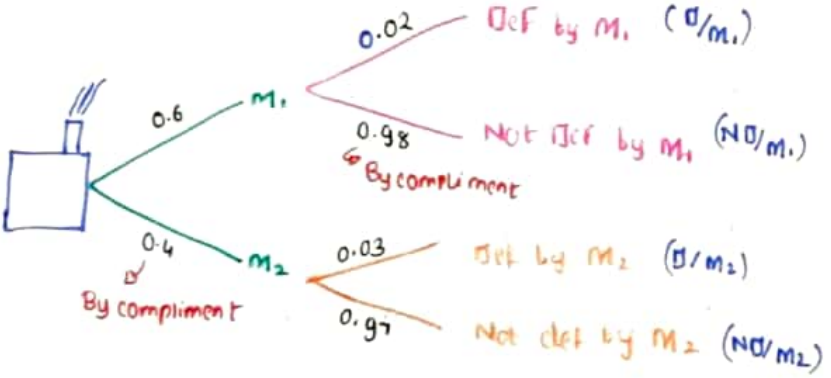
eg:- Two machines M_1 (B) & M_2 (D) produce Def (A) & Non Defective (C) pieces what is prob. of M_1 producing defective pieces → straight way of asking. $P(D/M_1)$

Now, If the component is defective what is probability it is produced by machine M_1 → Reversed is asked. $P(M_1/D)$

$$P(B/A) = \frac{P(A/B)\cdot P(B)}{\boxed{P(A/B)\cdot P(B) + P(C/D)\cdot P(D)}}$$ → Total thm. of probability.

PROBABILITY TREE

eg:- M_1 & M_2 produce total components of which M_1 produce 60%.
2% of M_1 & 3% of M_2 produce defective.
If the comp. is found defective what is prob. it is from M_2.



Now, in questn given that comp. is def. → already given, what is prob. it is from M_2.

∴ $P(M_2/D)$ ∴ Reversed is asked ∴ Use Bay's thm.

(D given that)

$$P(M_2/D) = \frac{P(D/M_2)\cdot P(M_2)}{P(D/M_2)\cdot P(M_2) + P(D/M_1)\, P(M_1)}$$

← Req. — we req. Def only from M_2

← Total from where we can get defective either from M_2 OR (+) from M_1

★ SAMPLING

Randomly drawing object from a set is called Sampling

Without Replacement (By default)

Object that was drawn is put aside

With Replacement

Object was drawn is placed back into the set.

★ COINS $S = 2^n$

Experiment is tossing a coin.

1 coin → S = [H , T] $n(S) = 2^1$

2 coin → S = [HH , HT , TH , TT] $n(S) = 2^2$

3 coin → S = [HHH HTH THH TTH
HHT HTT THT TTT] $n(S) = 2^3$

4 coin → S = [HHHH HTHH THHH TTHH
HHHT HTHT THHT TTHT
HHTH HTTH THTH TTTH
HHTT HTTT THTT TTTT $]_{4\times4}$ $n(S) = 2^4$

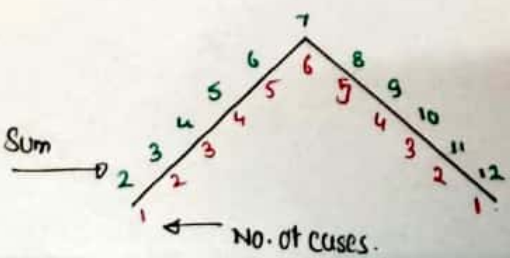
for 5, 6 ... higher coins use BINOMIAL DISTRIBUTION.

★ DIE $S = 6^n$

1 dice → S = [1 2 3 4 5 6] $n(S) = 6^1$

2 die → S = [11 12 13 14 15 16
21 22 23 24 25 26
31 32 33 34 35 36
41 42 43 44 45 46
51 52 53 54 55 56
61 62 63 64 65 66 $]_{6\times6}$ $n(S) = 6^2$

Sum for 2 die



★ CARDS

$S = 52$ cards

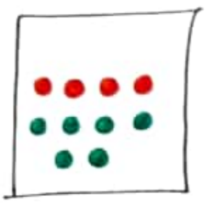
52
- 13 Red ♢ Diamond
- 13 Red ♡ Heart
- 13 Black ♣ Club
- 13 Black ♠ Spade

Number cards: Ace, 2, 3 ... 10 ; Face cards: Jack, King, Queen

★ COMBINATION LOGIC

When the position is not specified then all combinations comes into picture

eg:-



Box has 4 Red & 6 green balls. 3 balls are drawn

Find probability that one is red and other two are green.

→ ∵ Nothing is mention in question ∴ by default consider it a case of without replacement.

Now, Prob. of getting first ball as Red $= \frac{4}{10}$ (4 → Red, 10 → Total) and Second ball green $= \frac{6}{9}$ (Since one red ball is removed making total 9.) and third ball green $= \frac{5}{8}$ (one green already removed)

∴ Probability of set ● ● ● $= \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8}$

$= 1/6$

But the first ball can be green also ∴ this case also possible

		Probability
case (i)	● ● ●	1/6
or (ii)	● ● ●	+ 1/6 (→ or)
or (iii)	● ● ●	+ 1/6 (→ or)

$= 1/2$

If the position was specified in question like 1st ● 2nd & 3rd ● then only the 1st case would have been possible

TRICK :- works for $\boxed{\text{without replacement}}$ only

$$\frac{{}^{4=n}C_{r=1}\ {}^{n=6}C_{r=2}}{{}^{10=n}C_{r=3}} \quad \begin{matrix}\leftarrow \text{Required} \\ \leftarrow \text{Total}\end{matrix}$$

4 red present; 1 red required; 3 balls out of 10 drawn

RANDOM VARIABLE

A funtion which maps sample space to a real no.

eg:- $x \to$ no. of heads when a coin is tosses 3 times

$S = [$ HHH HTH THH TTH
HHT HTT THT TTT $]$

all possible values of x can be 0, 1, 2, 3

x	0	1	2	3	
	TTT	TTH THT HTT	THH HTH HHT	HHH	
$P(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$= 1$

This x is random variable which maps sample space entirely and $P(x)$ is the probability distribution for different values of x

$\therefore$ $P(x)$ is called Probability Density funt^n (pdf) or Prob. mass fun^m (pmf) and is denoted by $f(x)$

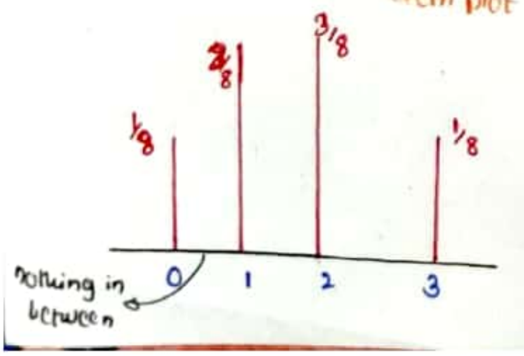
x	0	1	2	3
$f(x)$	$1/8$	$3/8$	$3/8$	$1/8$
$F(x)$	$1/8$	$4/8$	$7/8$	1

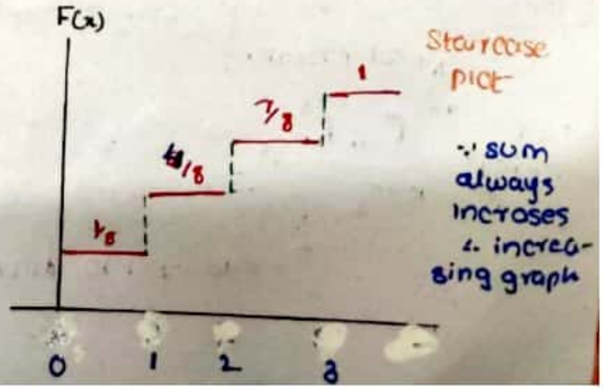
$F(x)$ is the accumulation of the previous $f(x)$ i.e Cumulative distributive funt^m (cdf) $\boxed{0 \leq F(x) \leq 1}$ – range

$\because$ No. of heads are countable therefore x is Discrete Random variable (DRV)

but when the things comes in range the x is Continuous Random Variable (CRV)

• Plot of $f(x)$ pdf & $F(x)$ cdf of DRV





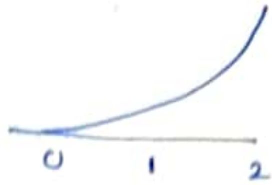
★ CRV

DRV

Here as there was nothing in between ∴ we added individual points. $\sum$

$P(0 \le x \le 1) = P(x=0) + P(x=1)$

$\sum_{-\infty}^{\infty}$ of all Prob $= 1$
$f(x)$



CRV

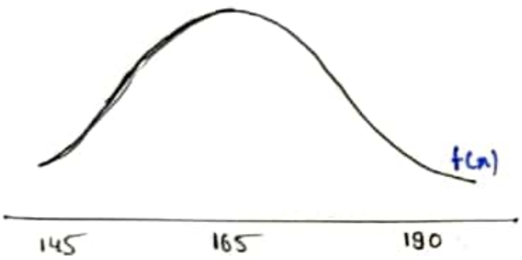
Here there is something in between ∴ we will integrate $\int$

$P(0 < x < 1) = \int_0^1 f(x)\,dx$

$\int_{-\infty}^{\infty} f(x)\,dx = 1$

⊙ CDF for CRV

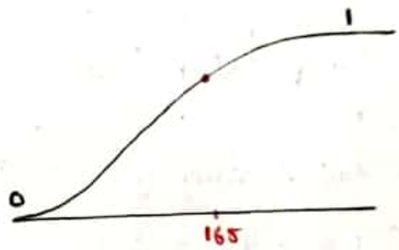
eg: Height of Boys



pdf

$\int_{-\infty}^{\infty} f(x)\,dx = 1$

$P(145 < x < 165) = \int_{145}^{165} f(x)\,dx$



cdf

$\int_{-\infty}^{x} f(x) = F(x)$ at x

$\int_{-\infty}^{165} f(x) = F(x)$ at 165

To get pdf from cdf just derivate it

$\frac{d}{dx}\int f(x) = \frac{d}{dx} F(x)$

$f(x) = \frac{d}{dx} F(x)$

⊙ for splitted domain

$\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_{-\infty}^{x} f(x)_1\,dx + \int_{x}^{\infty} f(x)_2\,dx = 1$

2D RANDOM VARIABLE

★ DISCRETE 2DRV

eg. $P(X = x, Y = y) = \frac{1}{27}(x + 2y)$ for $x = 0, 1, 2$; $y = 0, 1, 2$

→ x & y are countable ∴ DRV (2D) $^{x,}_{y}$

x \ y	0	1	2
0	0	2/27	4/27
1	1/27	3/27	5/27
2	2/27	4/27	6/27

4/27 → $x = 0$ $y = 2$ $= \frac{1}{27}(0 + 2\times2)$

$\boxed{\text{Sum} = 1}$

Now, pdf of $X = x$; as we are seeing x ∴ x will be fix & y can vary.

X	0	1	2
$P(x)$	6/27	9/27	12/27

12/27 → $2/27 + 4/27 + 6/27$

fix $x=0$ & vary y

$0 + \frac{2}{27} + \frac{4}{27} = \frac{6}{27}$

fix $x=1$ & vary y

$\frac{1}{27} + \frac{3}{27} + \frac{5}{27} = \frac{9}{27}$

pdf of $Y = y$; as we are seeing y ∴ y will be fix & x can vary

Y	0	1	2
$P(Y)$	3/27	9/27	15/27

Now, $P(X=1, Y=2)$.

just trace it from table

$P(x=1, y=2) = \frac{5}{27}$

$P(x \leq 1, y > 1)$

x can be 0 and 1

and y can be 2

∴ $P(x = 0, 1 ; Y = 2)$

Now trace and add

$\underset{x=0,y=2}{\frac{4}{27}} + \underset{x=1,y=2}{\frac{5}{27}} = \frac{9}{27}$

Scanned with CamScanner

$P(x=1/y=2)$

Same as conditional Prob. formula

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P(x=1/y=2) = \frac{P(x=1, y=2)}{P(y=2)}$$

$$= \frac{5/27}{4/27 + 5/27 + 6/27}$$ — as y is fix, x can vary

$$= 1/3$$

$P(x+y<3)$

See all the points which are satisfying <3 sum

(0,0) (0,1) (1,0) (1,1)

$\therefore P(x=0, y=0) + P(x=0, y=1) + P(x=1, y=0) + P(x=1, y=1)$

$= 0 + 2/27 + 1/27 + 3/27$

$= 6/27$

★ **CONTINUOUS 2D Random variable** $f(x,y)$ Same as 1D RV just Double integrate

$f(x,y)$ is $\boxed{\text{pdf}}$ then

$$\boxed{\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f(x,y)\,dx\,dy = 1}$$

$F(x,y)$ is $\boxed{\text{cdf}}$ then

$$\boxed{F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)\,dx\,dy}$$

$$f(x) = \int_{y=-\infty}^{\infty} f(x,y)\,dy$$ $\because$ x is fix $\therefore$ y can vary

$$f(y) = \int_{x=-\infty}^{\infty} f(x,y)\,dx$$ $\because$ y is fix $\therefore$ x can vary

If these limits are constant then it is indpendent case of double integrals but if limits depends on each other ex. $x<y<0$ then use strip concept coz it is dependent case of double integral

If $\boxed{f(x,y) = f(x).f(y)}$ then Independent

If cdf is given $F(x,y)$ & they asked pdf $f(x,y)$ then derivate partial derivation as 2 variables involved

$$\boxed{f(x,y) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y} F(x,y)\right]}$$

eg:- $f(x,y) = \frac{1}{8}(6-x-y)$ for $0<x<2$, $2<y<4$ is pdf.

→ Now,

i) $f(x) = \int_{y=2}^{4} \overbrace{\tfrac{1}{8}(6-x-y)}^{f(x,y)}\, dy$

y can vary
x is fix

ii) $f(y) = \int_{x=0}^{2} \overbrace{\tfrac{1}{8}(6-x-y)}^{f(x,y)}\, dx$

x can vary
y is fix

iii) $P(X<1, Y<3)$

$= \int_{x=0}^{1}\int_{y=2}^{3} \overbrace{\tfrac{1}{8}(6-x-y)}^{f(x,y)}\, dy\, dx$

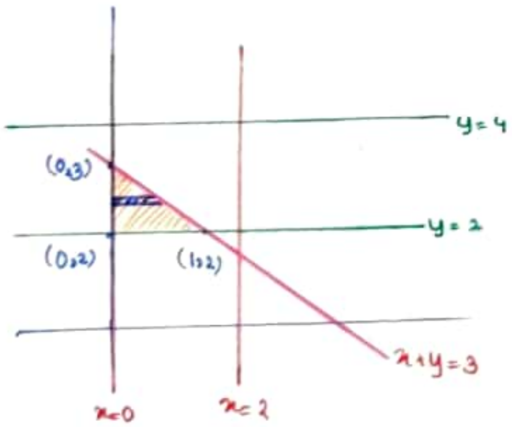
iv) $P(Y<3)$ ∵ Y is the only variable

$= \int_{y=2}^{3} f(y)\, dy$ ∴ f(y) will come

iv) $P(X<1 / Y<3)$ Conditional

$$= \frac{P(X<1, Y<3)}{P(Y<3)}$$

v) $P(x+y<3)$ Now you have to draw graph as x & y depend.



$\therefore P(x+y<3) = \int_{y=2}^{3}\int_{x=0}^{3-y} \overbrace{\tfrac{1}{8}(6-x-y)}^{f(x,y)}\, dx\, dy$

EXPECTATION & VARIANCE

Expectation is average value of x. denoted by $E(x)$

* Expectation of DRV

x	1	2	3	4
$P(x)$	$1/10$	$2/10$	$3/10$	$4/10$

First moment

$$\boxed{E(x) = \sum_{1}^{4} x_i \, p(x_i)}$$

$$= 1 \times 1/10 + 2 \times 2/10 + 3 \times 3/10 + 4 \times 4/10$$

$$= 3$$

Second Moment

$$\boxed{E(x^2) = \sum_{1}^{4} x_i^2 \, p(x_i)}$$

$$= 1^2 \times 1/10 + 2^2 \times 2/10 + 3^2 \times 3/10 + 4^2 \times 4/10$$

$$= 10$$

* VARIANCE of DRV

$$\boxed{Var(x) = E(x^2) - [E(x)]^2}$$

$$= 10 - [3]^2$$

$$Var(x) = 1$$

$$\boxed{E(g(x)) = \sum_{\forall x} g(x) \cdot f(x)}$$ ***

* Expectation of CRV

$$\boxed{E(x) = \int_{-\infty}^{\infty} x \, f(x) \, dx}$$

$$\boxed{E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) \, dx}$$

$$\boxed{E(g(x)) = \int_{-\infty}^{\infty} g(x) \cdot f(x) \, dx}$$ ***

* Variance of CRV

$$\boxed{Var(x) = E(x^2) - [E(x)]^2}$$

* Moment Generating funct^n

$$\boxed{E(x^n) = \frac{d^n}{dt^n} E(e^{xt})}$$ → mgf

$$E(e^{xt}) = \int_{-\infty}^{\infty} e^{xt} \cdot p(x) \, dx$$

$$E(x) = \frac{d}{dt} E(e^{xt}) \Big|_{t=0}$$

DEFINITION OF DIFFERENTIAL EQUATIONS

An equation which consist dependent variable, independent variable and differential coefficient of dependent variable with respect to independent variables is known as differential equation.

Ordinary Differential Equations (ODE)

If there is only one independent variable in a differential equation then it is ordinary differential equation.

Partial Differential Equations (PDE)

If there is more than one independent variable, present in the differential equation then it is called partial differential equation.

Examples : (1) $\left(\frac{dy}{dx}\right)^2 + 5xy = \sin x$ (2) $\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^3 + \log x = 5$

(3) $\left(\frac{dy}{dx}\right)^2 = \left(\frac{dy}{dx}\right)$ (4) $\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} + 5xy = 0$

ORDER AND DEGREE OF ORDINARY DIFFERENTIAL EQUATION

Order : The order of highest derivative occurs in a differential eqn is known as its order.

Degree : Degree of a differential equation is the exponent of highest order derivative when it is made free from fractional notations and Radical signs.

OR

It is the exponent of highest order derivative when it is written as a polynomial in differential coefficient.

Example 1 : Find the order and degree of following; $\rho = \dfrac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

Solution : First we will write above in form of polynomial i.e.

$$\rho \frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$$

Squaring both side :

$$\rho^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3 \Rightarrow \rho^2\left(\frac{d^2y}{dx^2}\right)^2 - \left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = 0$$

ORDER $\rightarrow$ 2 & DEGREE $\rightarrow$ 2

2. $y = x\left(\frac{dy}{dx}\right)+\left(\frac{dy}{dx}\right)^3$ ORDER $\rightarrow$ 1 & DEGREE $\rightarrow$ 3

3. $\frac{d^2y}{dx^2}+\sqrt{1+\left(\frac{dy}{dx}\right)^3}=0$ ORDER $\rightarrow$ 2 & DEGREE $\rightarrow$ 2

4. $\frac{d^2y}{dx^2}=\left(1+\left(\frac{dy}{dx}\right)^3\right)$ ORDER $\rightarrow$ 2 & DEGREE $\rightarrow$ 1

5. $\left(\frac{d^3y}{dx^3}\right)^4-6x^2\left(\frac{dy}{dx}\right)^8=0$ ORDER $\rightarrow$ 3 & DEGREE $\rightarrow$ 4

NON-LINEAR DIFFERENTIAL EQUATION

A differential equation is said to be non-linear if it satisfies any of the following four properties.

1. Degree is more than one.
2. Exponent of dependent variable (i.e. y) is more than one.
3. Exponent of any differential coefficient is more than one.
4. Product containing dependent variable and its any differential coefficient is present.

Linear Differential Equation

If a differential equation does not possess any of the above four properties then it is linear.

Example :

1. $x^2\left(\frac{d^2y}{dx^2}\right)^2+x\left(\frac{dy}{dx}\right)+2=0$ (Non linear) $\because$ degree more than 1.
2. $\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2+2y=5$ (Non linear) (exponent of $\frac{dy}{dx}$ is more than 1)
3. $y'''+yy'+2x^2=7$ Non linear (Property 4)
4. $y'''+(y'')^2+y^2x=\sin x$ Non linear (Property 2 & 3)
5. $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}+\sin x\frac{dy}{dx}=\log x$ (Linear)

SOLUTION OF DIFFERENTIAL EQUATIONS

Relation, between dependent variable and independent variable which satisfies the given differential equation is known as its solution.

Types of Solutions

(i) General solution (ii) Particular solution

(i) **General solution** : If a solution contains same no. of arbitrary constant as the order of differential equation then it is known as general solution.

(ii) **Particular solution** : If we assign particular value to arbitrary constant then it is known as particular solution.

FORMATION OF DIFFERENTIAL EQUATION

By eliminating the given number of arbitrary constants from general solution, we can easily form a differential equation.

Example 2 : Find the differential equation whose solution is, $y = e^x (A\cos x + B\sin x)$ where A and B are arbitrary constant.

Solution : $y' = e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x) = y + e^x(-A\sin x + B\cos x)$

Again differentiating,

$\Rightarrow y'' = y' + e^x(-A\sin x + B\cos x) + e^x(-A\cos x - B\sin x)$

$\Rightarrow \quad y'' = y' + (y' - y) - y \quad \Rightarrow \quad y'' = 2y' - 2y \quad \Rightarrow \quad y'' - 2y' + 2y = 0$

Example 3 : Find a differential equation of which $xy = Ae^x + Be^{-x} + x^2$ is the solution by eliminating A and B.

Solution : $xy = Ae^x + Be^{-x} + x^2$

$\Rightarrow \quad y + xy' = Ae^x - Be^{-x} + 2x \quad \Rightarrow \quad y' + (y' + xy'') = Ae^x + Be^{-x} + 2$

$\Rightarrow \quad 2y' + xy'' - (xy - x^2) - 2 = 0$

Example 6 : Which one of the following differential equations has a solution given by the function

$$y = 5\sin\left(3x + \frac{\pi}{3}\right)$$

(a) $\frac{dy}{dx} - \frac{5}{3}\cos(3x) = 0$ (b) $\frac{dy}{dx} + \frac{5}{3}\cos(3x) = 0$

(c) $\frac{d^2y}{d^2x} + 9y = 0$ (d) $\frac{d^2y}{d^2x} - 9y = 0$

Solution : (c) $\because \frac{dy}{dx} = 15\cos\left(3x + \frac{\pi}{3}\right)$ and $\frac{d^2y}{dx^2} = -45\sin\left(3x + \frac{\pi}{3}\right)$

So only option (c) satisfying it.

Q) $y = 1 + \frac{dy}{dx} + \frac{1}{1\cdot 2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{1\cdot 2\cdot 3}\left(\frac{dy}{dx}\right)^3 + \cdots$

$\Rightarrow e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\Rightarrow y = e^{dy/dx} \Rightarrow \ln y = \frac{dy}{dx} \Rightarrow \boxed{\begin{matrix} O:1 \\ D:1 \end{matrix}}$

Q) $\left(\frac{d^2y}{dx^2}\right)^5 + 4\frac{\left(\frac{d^2y}{dx^2}\right)^3}{\left(\frac{d^3y}{dx^3}\right)} + \frac{d^3y}{dx^3} = x^2 - 1$

$\Rightarrow (y'')^5 + 4\frac{(y'')^3}{(y''')} + y''' = x^2 - 1$

$\Rightarrow (y''')(y'')^5 + 4(y'')^3 + (y''')^2 = (x^2-1)(y''')$

$\Rightarrow \boxed{\begin{matrix} O:3 \\ D:2 \end{matrix}}$

Q) $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y \Rightarrow \frac{dy}{dx} = \sin(x+y) \Rightarrow \boxed{\begin{matrix} O:1 \\ D:1 \end{matrix}}$

Q) D.E which represents family of curve $y = c_1 e^{c_2 x}$.

$\Rightarrow y = c_1 e^{c_2 x}$

$\Rightarrow y' = c_1 c_2 e^{c_2 x}$

$\Rightarrow y' = c_2 y \Rightarrow y'' = c_2 y' \Rightarrow y'' = \left(\frac{y'}{y}\right)\cdot y'$

$\Rightarrow y y'' = (y')^2$

$\Rightarrow \boxed{\begin{matrix} O:2 \\ D:1 \end{matrix}}$

WRONSKIAN

If y_1, y_2 are two solutions of second order differential equation $a_0\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_2y=0$ then their wronskian is defined as; $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

If, y_1, y_2, y_3 are three solutions of third order differential equation then their Wronskian is defined as;

$$W=\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

LINEARLY DEPENDENT (LD) AND LINEARLY INDEPENDENT (LI) SOLUTIONS

Linearly dependent (LD) $W=0$

Let y_1, y_2, y_3 are solutions of given differential equation and if there exist a relation between y_1, y_2, y_3 such that $c_1y_1+c_2y_2+c_3y_3=0$ where c_1, c_2, c_3 not all zero simultaneously then they are linearly dependent.

Linearly independent (LI)

If there does not exist any relation between y_1, y_2, y_3 then they are linearly independent (LI) i.e.
Relation $c_1y_1 + c_2y_2 + c_3y_3 = 0$ exist only when $c_1 = c_2 = c_3 = 0$ $W \neq 0$

NOTE

1. If wronskian of y_1, y_2, y_3 is zero then they are linearly dependent, i.e., $W = 0 \Rightarrow$ LD solutions and $W \neq 0 \Rightarrow$ LI Solutions
2. If y_1, y_2, are two LI solution of a given differential equation then $y = c_1y_1 + c_2y_2$ is the general solution of that equation.

Example 7 : Determine the differential equation whose set of independent solutions are $\{e^x, xe^x, x^2e^x\}$

Solution : Let $y_1 = e^x,\ y_2 = xe^x,\ y_3 = x^2e^x$

$\Rightarrow$ $w \neq 0 \Rightarrow y_1, y_2, y_3$ are LI

then general solution is, $y = c_1y_1 + c_2y_2 + c_3y_3 = c_1e^x + c_2xe^x + c_3x^2e^x$ ...(1)

where c_1, c_2, c_3 are arbitrary constant.

Now,
$$y' = e^xc_1 + (e^xc_2 + xe^xc_2) + (2xe^xc_3 + x^2e^xc_3) = y + e^xc_2 + 2xe^xc_3$$
$$y'' = y' + e^xc_2 + (2e^xc_3 + 2xe^xc_3) = y' + 2e^xc_3 + (y' - y) = 2y' - y + 2e^xc_3$$
$$y''' = 2y'' - y' + 2e^xc_3 = 2y'' - y' + (y'' - 2y' + y) = 3y'' - 3y' + y$$
$$y''' - 3y'' + 3y' - y = 0$$

Example 8 : Prove that 1, x, x^2 are linearly independent and form the differential equation whose roots are $\{1, x, x^2\}$

Solution : $W = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2$ i.e. $W \neq 0$ So, given Solutions are L.I. Hence, general solution is

$$y = c_1 + xc_2 + x^2c_3$$
$$y' = c_2 + 2xc_3 \Rightarrow y'' = 2c_3 = c_2 + 2xc_3 \Rightarrow y'' = 2c_3 \Rightarrow y''' = 0$$

1.* The differential $\frac{d^2y}{dx^2}+\frac{dy}{dx}+\sin y=0$ is

(a) linear (b) non-linear

(c) homogeneous (d) of degree two

[GATE-1993 (ME)]

Sol–1: (b)

Given equation is a non-linear differential equation because y has an exponent more than one.

Sol–2: ([illegible])

3. The differential equation $EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} + ky = 0$ is (where, E, I, P, K are functions of x -only)

(a) Linear of Fourth order

(b) Non-linear of Fourth order

(c) Non-Homogeneous

(d) Linear and Fourth degree

[GATE-1994]

Sol–3: (a)

It is free from all the four properties of non-linear DE.

1. The differential equation
$$y'' + \left(y^3 \sin x\right)^5 \; y' + y = \cos x^3 \text{ is}$$
(a) homogeneous

(b) nonlinear

(c) second order linear

(d) nonhomogeneous with constant coefficients

[GATE-1995]

Sol–7: (b)

$\because$ y has exponent other than 1.

13. $\dfrac{d^2y}{dx^2} + (x^2 + 4x)\dfrac{dy}{dx} + y = x^8 - 8.$

The above equation is a

(a) partial differential equation

(b) nonlinear differential equation

(c) non-homogeneous differential equation

(d) ordinary differential equation

[GATE-1999]

Sol–13: (c) & (d)

$$\frac{d^2y}{dx^2}+(x^2+4x)\frac{dy}{dx}+y = x^8 - 8$$

A homogeneous equation is of the form

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

where, f (x,y) and g(x,y) are homogeneous functions of the same degree in x and y. An ordinary differential equation is that in which all differential coefficients are with respect to a single independent variable. A partial differential equation has two or more independent variable.

A differential equation is said to be linear if the independent variable and its differential coefficient occur only in first degree and not multiplied together.

18. The following differential equation has

$$3\left(\frac{d^2y}{dt^2}\right)+4\left(\frac{dy}{dt}\right)^3+y^2+2=x$$

(a) degree = 2, order = 1

(b) degree = 1, order = 2

(c) degree = 4, order = 3

(d) degree = 2, order = 3

[GATE-2005-EC]

Sol–18: (b)

Given differential equation

$$3\left(\frac{d^2y}{dt^2}\right)+4\left(\frac{dy}{dt}\right)^3+y^2+2 = x.$$

The derivative with highest order = $\frac{d^2y}{dt^2}$

Hence the power of the highest order derivative = 1.

Therefore;

$$\text{degree} = 1$$

$$\text{order} = 2.$$

21. The differential equation

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = C^2\left[\frac{d^2y}{dx^2}\right]^2 \text{ is of}$$

(a) 2nd order and 3rd degree

(b) 3rd order and 2nd degree

(c) 2nd order and 2nd degree

(d) 3rd order and 3rd degree

[GATE-2005 (PI)]

Sol–21: (c)

Given $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = C^2\left[\frac{d^2y}{dx^2}\right]^2$

$\because$ It is free from fractions and radicals. So, order = 2 and degree = 2

25. The degree of the differential equation $\frac{d^2x}{dt^2} + 2x^3 = 0$ is

(a) zero (b) 1

(c) 2 (d) 3

[GATE-2007-CE]

Sol–25: (b)

Given,

$$\frac{d^2x}{dt^2} + 2x^3 = 0 \qquad ...(i)$$

$\because$ The given differential equation is free of radicals.

$\therefore$ Degree of (i) = exponent of highest order derivative

= 1

32. The order of the differential equation $\frac{d^2y}{dt^2}+\left(\frac{dy}{dt}\right)^3+y^4=e^{-t}$ is

(a) 1
(b) 2
(c) 3
(d) 4

[GATE-2009-EC]

Sol–32: (b)

Given differential equation

$$\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^3 + y^4 = e^{-t}$$

The derivative with highest order

$$= \frac{d^2y}{dt^2}$$

Hence order = 2.

Solution:

Given equation is $$\frac{dy}{dx} = e^y(e^x + x^2)$$

or $$e^{-y}\, dy = (e^x + x^2)dx$$

Integrating both sides, $$\int e^{-y}dy = \int (e^x + x^2)dx + c$$

$$-e^{-y} = e^x + \frac{x^3}{3} + c$$

$$3e^{-y} = -3e^x - x^3 + c'$$

(V.T.U.,

Example 11.7. *Solve* $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$.

Solution. Putting $x + y = t$ so that $dy/dx = dt/dx - 1$

The given equation becomes $\frac{dt}{dx} - 1 = \sin t + \cos t$

or $$dt/dx = 1 + \sin t + \cos t$$

Integrating both sides, we get $dx = \int \frac{dt}{1 + \sin t + \cos t} + c.$

or $\int \frac{dt}{2\cos^2 t/2 + 2\sin t/2 \cos t/2}$ $$x = \int \frac{2d\theta}{1 + \sin 2\theta + \cos 2\theta} + c$$ [Putti

$\int \frac{1}{2} \frac{\sec^2 t/2}{1 + \tan t/2}$ $$= \int \frac{2d\theta}{2\cos^2\theta + 2\sin\theta\cos\theta} + c = \int \frac{\sec^2\theta}{1 + \tan\theta} d\theta + c$$

$\ln(1 + \tan t/2)$ $$= \log(1 + \tan\theta) + c$$

Hence the solution is $x = \log\left[1 + \tan\frac{1}{2}(x+y)\right] + c.$

Example 11.8. *Solve $dy/dx = (4x + y + 1)^2$, if $y(0) = 1$.*

Solution. Putting $4x + y + 1 = t$, we get $\frac{dy}{dx} = \frac{dt}{dx} - 4$.

$\therefore$ the given equation becomes $\frac{dt}{dx} - 4 = t^2$ or $\frac{dt}{dx} = 4 + t^2$

Integrating both sides, we get $\int \frac{dt}{4 + t^2} = \int dx + c$

or $$\frac{1}{2}\tan^{-1}\frac{t}{2} = x + c \quad \text{or} \quad \frac{1}{2}\tan^{-1}\left[\frac{1}{2}(4x + y + 1)\right] = x + c.$$

or $$4x + y + 1 = 2\tan 2(x + c)$$

When $x = 0, y = 1 \quad \therefore \quad \frac{1}{2}\tan^{-1}(1) = c$ *i.e.* $c = \pi/8$.

Hence the solution is $4x + y + 1 = 2\tan(2x + \pi/4)$.

Example 11.9. *Solve* $\frac{y}{x}\frac{dy}{dx} + \frac{x^2 + y^2 - 1}{2(x^2 + y^2) + 1} = 0.$

Solution. Putting $x^2 + y^2 = t$, we get $2x + 2y\frac{dy}{dx} = \frac{dt}{dx}$ or $\frac{y}{x}\frac{dy}{dx} = \frac{1}{2x}\frac{dt}{dx} - 1.$

Therefore the given equation becomes $\frac{1}{2x}\frac{dt}{dx} - 1 + \frac{t-1}{2t+1} = 0$

or $$\frac{1}{2x}\frac{dt}{dx} = 1 - \frac{t-1}{2t+1} = \frac{t+2}{2t+1} \quad \text{or} \quad 2x\,dx = \frac{2t+1}{t+2}dt$$

$$\frac{2t}{t+2} + \frac{1}{t+2}$$

$$\frac{2t+4-4}{t+2} + \frac{1}{t+2}$$

or $$2x\,dx = \left(2 - \frac{3}{t+2}\right) dt$$

$$\frac{2(t+2)}{t+2} - \frac{4}{t+2} + \frac{1}{t+2}$$

Integrating, we get $$x^2 = 2t - 3\log(t+2) + c$$

or $$x^2 + 2y^2 - 3\log(x^2+y^2+2) + c = 0 \qquad [\because\ t = x^2 + y^2]$$

which is the required solution.

5. For the differential equation $\frac{dy}{dt} + 5y = 0$ with $y(0) = 1$, the general solution is

(a) e^{5t}

(b) e^{-5t}

(c) $5e^{-5t}$

(d) $e^{\sqrt{-5t}}$

[GATE-1994 (ME)]

Sol–5: (b)

$$\frac{dy}{dt} + 5y = 0 \qquad \ldots(i)$$

$$\Rightarrow \quad \frac{dy}{y} = -5dt$$

$$\Rightarrow \quad \log y = -5t + \log c$$

$$\Rightarrow \quad \log\left(\frac{y}{c}\right) = -5t$$

$$\Rightarrow \quad y = ce^{-5t} \qquad \ldots(ii)$$

Using $y(o) = 1$, we get $c = 1$

$\therefore$ The general solution is $y = e^{-5t}$.

8. If at every point of a certain curve, the slope of the tangent equals $-2x/y$, the curve is _____.

(a) a straight line
(b) a parabola
(c) a circle
(d) an ellipse

[GATE-1995 (CS)]

Sol–8: (d)

Slope of tangent at point is

$$= -\frac{2x}{y}$$

$$\frac{dy}{dx} = \frac{-2x}{y}$$

$$ydy = -2x\, dx$$

on integrating $\frac{y^2}{2} = -x^2 + c$

or $\frac{x^2}{c} + \frac{y^2}{2c} = 1$

which represents ellipse

[GATE-2005 (PI)]

22. The solution of the differential equation $\frac{dy}{dx} + 2xy = e^{-x^2}$ with $y(0) = 1$ is

(a) $(1+x)e^{+x^2}$
(b) $(1+x)e^{-x^2}$
(c) $(1-x)e^{+x^2}$
(d) $(1-x)e^{-x^2}$

[GATE-2006-ME]

an

Sol–22: (b)

$$\frac{dy}{dx} + 2xy = e^{-x^2} \; ; \; y(0) = 1$$

Integrating factor,

$$I.F. = e^{\int (2x)dx} = e^{x^2}$$

General solution,

$$y.e^{x^2} = \int e^{-x^2} e^{x^2} dx + c$$

$$y\, e^{x^2} = x + c$$

$$y = (x + c)\; e^{-x^2}$$

$$y(0) = 1$$

$$\Rightarrow \quad 1 = (0 + c)\, e^{-0}$$

$$\Rightarrow \quad c = 1$$

$$y = (x + 1)\; e^{-x^2}$$

26. The solution for the differential equation $\frac{dy}{dx} = x^2y$ with the condition that y = 1 at x = 0 is

(a) $y = e^{\frac{1}{2x}}$

(b) $\ln(y) = \frac{x^3}{3} + 4$

(c) $\ln(y) = \frac{x^2}{2}$

(d) $y = e^{\frac{x^3}{3}}$

[GATE-2007-CE]

Sol–26: (d)

Given, $$\frac{dy}{dx} = x^2y; \text{ \{variables are separable\}}$$

$$\Rightarrow \quad \frac{dy}{y} = x^2dx$$

Integrating both sides, we get

$$\int \frac{dy}{y} = \int x^2dx + C$$

$$\ln y = \frac{x^3}{3} + C$$

Given, $y(0) = 1$

$$\ln 1 = 0 + C$$

$\Rightarrow$ $C = 0$

$\Rightarrow$ $\ln y = \frac{x^3}{3}$

$$y = e^{(x^3/3)}$$

M2

$x=0, y=1$

C & D

$\frac{dy}{dx} = yx \to$ C

$\frac{dy}{dx} = yx^2 \to$ D

11.7 HOMOGENEOUS EQUATIONS

are of the form $\dfrac{dy}{dx} = \dfrac{f(x, y)}{\phi(x, y)}$

where $f(x, y)$ and $\phi(x, y)$ are homogeneous functions of the same degree in x and y (see page 20

To solve a homogeneous equation (*i*) Put $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$,

(*ii*) Separate the variables v and x, and integrate.

Example 11.10. *Solve* $(x^2 - y^2)\,dx - xy\,dy = 0$.

Solution. Given equation is $\dfrac{dy}{dx} = \dfrac{x^2 - y^2}{xy}$ which is homogeneous in x and y.

Put $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. $\therefore$ (*i*) becomes $v + x\dfrac{dv}{dx} = \dfrac{1 - v^2}{v}$

or
$$x\frac{dv}{dx} = \frac{1 - v^2}{v} - v = \frac{1 - 2v^2}{v}.$$

Scanned by CamScanner

Separating the variables, $\dfrac{v}{1-2v^2}\,dv = \dfrac{dx}{x}$

Integrating both sides, $\displaystyle\int \frac{v\,dv}{1-2v^2} = \int \frac{dx}{x} + c$

$$-\frac{1}{4}\int \frac{-4v}{1-2v^2}\,dv = \int \frac{dx}{x} + c \quad \text{or} \quad -\frac{1}{4}\log(1-2v^2) = \log x + c$$

$$4\log x + \log(1-2v^2) = -4c \quad \text{or} \quad \log x^4(1-2v^2) = -4c$$

$$x^4(1-2y^2/x^2) = e^{-3c} = c'$$

Hence the required solution is $x^2(x^2-2y^2) = c'$.

Example 11.12. *Solve* $(1 + e^{x/y})\,dx + e^{x/y}(1 - x/y)\,dy = 0.$

(V.T.U., 2014 S ; P.T.U., 2006 ; Rajasthan,

Solution. The given equation may be rewritten as

$$\frac{dx}{dy} = -\frac{e^{x/y}(1 - x/y)}{1 + e^{x/y}}$$

which is a homogeneous equation. Putting $x = vy$ so that (*i*) becomes

$$v + y\frac{dv}{dy} = -\frac{e^{v}(1 - v)}{1 + e^{v}} \quad \text{or} \quad y\frac{dv}{dy} = -\frac{e^{v}(1 - v)}{1 + e^{v}} - v = -\frac{v + e^{v}}{1 + e^{v}}$$

Separating the variables, we get

$$-\frac{dy}{y} = \frac{1 + e^{v}}{v + e^{v}}\,dv = \frac{d(v + e^{v})}{v + e^{v}}$$

Integrating both sides, $-\log y = \log(v + e^{v}) + c$

or $$y(v + e^{v}) = e^{-c} \quad \text{or} \quad x + ye^{x/y} = c' \quad \text{(say)}$$

which is the required solution.

70. A curve passes through the point ($x = 1$, $y = 0$) and satisfies the differential equation $\frac{dy}{dx} = \frac{x^2 + y^2}{2y} + \frac{y}{x}$.

The equation that describes the curve is

(a) $\ln\left(1+\frac{y^2}{x^2}\right) = x - 1$

(b) $\frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) = x - 1$

(c) $\ln\left(1+\frac{y}{x}\right) = x - 1$

(d) $\frac{1}{2}\ln\left(1+\frac{y}{x}\right) = x - 1$

[GATE 2018 (EC)]

Sol–70 : (a)

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2y} + \frac{y}{x} \qquad \text{...(1)}$$

Put $y = vx$

$\Rightarrow$ $$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2vx} + v$$

$$x\frac{dv}{dx} = \frac{x}{2v} + \frac{vx}{2}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{1}{2}\left[\frac{1}{v} + v\right]$$

$$\left(\frac{2v}{1+v^2}\right)dv = dx$$

$$\Rightarrow \quad \log(1 + v^2) = x + c$$

$$\log\left[1 + \frac{y^2}{x^2}\right] = x + c$$

Now $\quad y(1) = 0$

$$\Rightarrow \quad C = -1$$

so $\quad \log\left(1 + \frac{y^2}{x^2}\right) = x - 1$

67. The solution of the differential equation

$y\sqrt{1-x^2}dy + x\sqrt{1-y^2}dx = 0$ is

(a) $\sqrt{1-x^2} = c$

(b) $\sqrt{1-y^2} = c$

(c) $\sqrt{1-x^2}+\sqrt{1-y^2}=c$

(d) $\sqrt{1+x^2}+\sqrt{1+y^2}=c$

[ESE 2017 (EE)]

Sol–67: (c)

$$y\sqrt{1-x^2}\,dy + x\sqrt{1-y^2}\,dx = 0$$

Using variable separable,

$$\frac{y.dy}{\sqrt{1-y^2}} = -\frac{xdx}{\sqrt{1-x^2}}$$

On integrating, $$\int\frac{ydy}{\sqrt{1-y^2}} = -\int\frac{xdx}{\sqrt{1-x^2}}$$

Put $1 - y^2 = u^2$ and $1 - x^2 = v^2$

i.e.,$ydy = -udy$ and $xdx = -vdv$

so, $$\int -\frac{udu}{u} = -\int -\frac{vdv}{v}$$

$\Rightarrow$ $$-u = v + c$$

$$-\sqrt{1-y^2} = \sqrt{1-x^2} + c$$

or $$\sqrt{1-x^2} + \sqrt{1-y^2} = c$$

$$\int \frac{y}{\sqrt{1-y^2}}$$

$$\sin t = y \Rightarrow \cos t \, dt = dy.$$

$$\int \frac{\sin t}{\cancel{\cos t}} \cancel{\cos t} \, dt$$

$$= -\cos t = -\sqrt{1-y^2} = \sqrt{1-x^2}+C$$

$$=$$

$$\int \frac{f'(x)}{\sqrt{f(x)}} = 2\sqrt{f(x)}$$

$$-\frac{1}{2}\int \frac{-2y}{\sqrt{1-y^2}} = -\frac{1}{2}\, 2\sqrt{1-y^2}$$

$$= -\sqrt{1-y^2}$$

Scanned by CamScanner

64.* Consider the differential equation $(t^2 - 81)\frac{dy}{dt} + 5ty = \sin(t)$ with $y(1) = 2\pi$. There exists a unique solution for this differential equation when t belongs to the interval

(a) $(-2, 2)$

(b) $(-10, 10)$

(c) $(-10, 2)$

(d) $(0, 10)$

[GATE-2017 EE Session-I]

Sol–64: (a)

Given differential equation is

$$\left(t^2-81\right)\frac{dy}{dt}+5ty = \sin t \qquad \text{...(1)}$$

initial condition $y(1) = 2\pi$

Converting the given equation into standard form

$$\frac{dy}{dt}+\left(\frac{5t}{t^2-81}\right)y = \frac{\sin t}{t^2-81} \qquad \text{...(2)}$$

This is of the form

$$\frac{dy}{dt}+py = Q$$

where

$$P = \frac{5t}{t^2-81}, Q=\frac{\sin t}{t^2-81}$$

We know integrating factor

$$(\text{I.F}) = e^{\int \rho dt}$$

$$= e^{\int \frac{5t}{t^2-81}dt}$$

$$= e^{\int \frac{5}{2}\left(\frac{2t}{t^2-81}\right)}$$

$$= e^{\frac{5}{2}\ln\left(t^2-81\right)} \qquad \left[\because e^{\ln x} = x\right]$$

$$\text{I.F} = \left(t^2-81\right)^{5/2}$$

and sol. of (2) is

$$y(\text{I.F}) = \int Q\,\text{IF}\,dt+c$$

$$y(t^2-81)^{5/2} = \int \frac{\sin t}{\left(t^2-81\right)}\left(t^2-81\right)^{5/2}dt+c$$

$$y\left(t^2-81\right)^{5/2} = \int \sin t\left(t^2-81\right)^{3/2}dt+c$$

$$y = \frac{\int \sin t\cdot\left(t^2-81\right)^{3/2}dt}{\left(t^2-81\right)^{5/2}}+\frac{c}{\left(t^2-81\right)^{5/2}}$$

and solving from the options by verifying initial condition we get unique solution.

If t = ± 9 then solution is not unique hence range (–10, 10), (–10, 2), (0, 10) can be eliminated, then left option is (–2, 2)

Sol–65: (d)

59.* Consider the following differential equation :

$$x(ydx + xdy)\cos\frac{y}{x} = y(xdy - ydx)\sin\frac{y}{x}$$

Which of the following is the solution of the above equation (c is an arbitrary constant) ?

(a) $\frac{x}{y}\cos\frac{y}{x} = c$

(b) $\frac{x}{y}\sin\frac{y}{x} = c$

(c) $xy\cos\frac{y}{x} = c$

(d) $xy\sin\frac{y}{x} = c$

[GATE-2015-CE-SET-I]

Sol–59: (c)

$$x(ydx + xdy)\cos\frac{y}{x} = y\ (xdy - ydx)\ \sin\frac{y}{x}$$

$$\frac{ydx + xdy}{xdy - ydx} = \frac{y}{x}\tan\frac{y}{x}$$

Let

$$y = v \cdot x$$

$$dy = v\ dx + xdv$$

$$\frac{vxdx + vxdx + x^2dv}{vxdx + x^2dv - vxdx} = v\tan v$$

$$\frac{xdv + 2vdx}{xdv} = v\tan v$$

$$1 + \frac{2v}{x}\frac{dx}{dv} = v\tan v$$

$$\frac{2v}{x}\frac{dx}{dv} = v\tan v - 1$$

$$2\frac{dx}{x} = \left(\tan v - \frac{1}{v}\right) dv$$

Integrating both sides

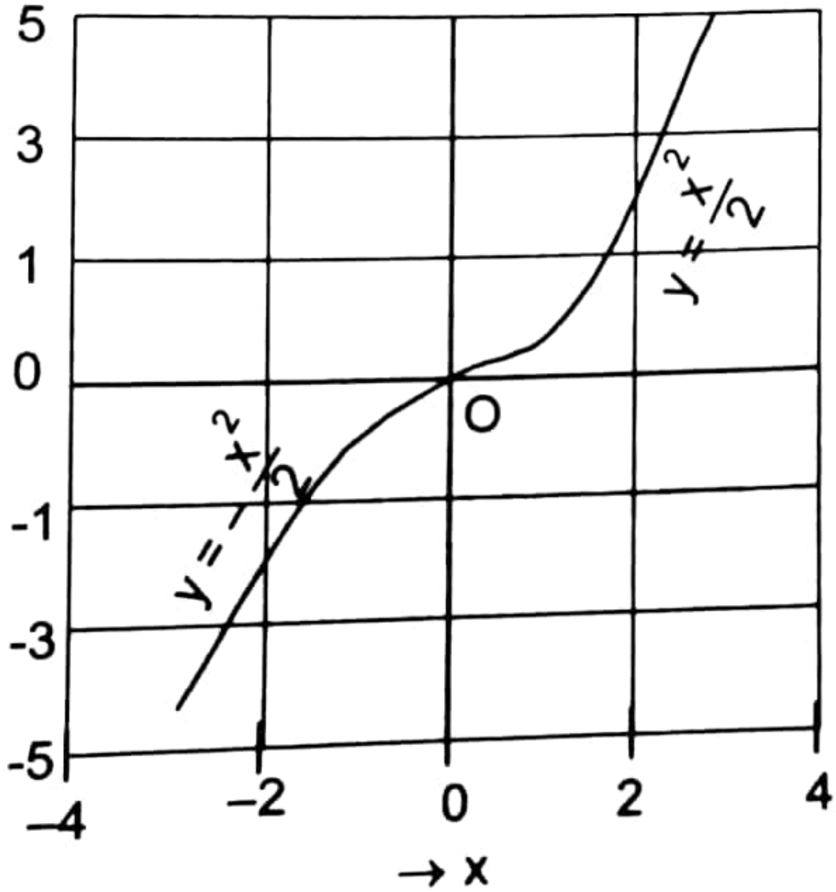
$$2 \log x = \log |\sec v| - \log v + \log c$$

$$\Rightarrow \qquad x^2 = \frac{c \sec v}{v}$$

$$\Rightarrow \qquad x^2 . \frac{y}{x} = c \sec y/x$$

$$\Rightarrow \qquad xy \cos \frac{y}{x} = c$$

54.* The figure shows the plot of y as a function of x. The function shown is the solution of the differential equation (assuming all initial conditions to be zero) is



(a) $\frac{d^2y}{dx^2} = 1$

(b) $\frac{dy}{dx} = -x$

(c) $\frac{dy}{dx} = -x$

(d) $\frac{dy}{dx} = |x|$

[GATE-2014 (IN-Set 1)]

Sol–54: (d)

From the graph

$$y = \begin{cases} \dfrac{x^2}{2}; & x \geq 0 \\ -\dfrac{x^2}{2}; & x \leq 0 \end{cases}$$

$$\Rightarrow \quad \frac{dy}{dx} = \begin{cases} x; & x \geq 0 \\ -x; & x \leq 0 \end{cases} = |x|$$

(By definition of Mod)

56. The general solution of the differential equation $\frac{dy}{dx} = \frac{1+\cos 2y}{1-\cos 2x}$ is

(a) $\tan y - \cot x = c$ (c is a constant)

(b) $\tan x - \cot y = c$ (c is a constant)

(c) $\tan y + \cot x = c$ (c is a constant)

(d) $\tan x + \cot y = c$ (c is a constant)

[GATE-2015-EC-Set-2]

Sol–56: (c)

$$\frac{dy}{dx} = \frac{1+\cos 2y}{1-\cos 2x}$$

$$\frac{dy}{dx} = \frac{2\cos^2 y}{2\sin^2 x}$$

$$\sec^2 y\,dy = \operatorname{cosec}^2 x\,dx$$

Integrating both sides.

$$\tan y = -\cot x + C$$

$$\tan y + \cot x = C$$

66. For the initial value problem

$$\frac{dx}{dt} = \sin(t), x(0) = 0$$

the value of x at $t = \pi/3$, is ______.

[GATE-2017 (CH)]

$$\frac{dx}{dt} = \sin t \Rightarrow \int dx = \int \sin t\, dt \Rightarrow x = -\cos t + C$$

Using $x(0) = 0 \Rightarrow c = 1$

$\therefore \; x = -\cos t + 1$

at $t = \frac{\pi}{3}, x = -\cos\frac{\pi}{3} + 1 = \frac{-1}{2} + 1 = \frac{1}{2} = 0.5.$

63. Which one of the following is the general solution of the first order differential equation

$$\frac{dy}{dx} = (x + y - 1)^2,$$

where x, y are real?

(a) $y = 1 + x + \tan^{-1}(x + c)$, where c is a constant

(b) $y = 1 + x + \tan(x + c)$, where c is a constant

(c) $y = 1 - x + \tan^{-1}(x + c)$, where c is a constant

(d) $y = 1 - x + \tan(x + c)$, where c is a constant

[GATE-2017 EC Session-I]

Given,

$$\frac{dy}{dx} = \frac{dy}{dx} = (x+y-1)^2 \qquad \cdots (1)$$

Let, $x + y - 1 = t$

Then, $1 + \frac{dy}{dx} - 0 = \frac{dt}{dx}$

Or $$\frac{dy}{dx} = \frac{dt}{dx} - 1$$

From equation (1)

$$\frac{dt}{dx} - 1 = t^2$$

Or $$\frac{dt}{1+t^2} = dx$$

Or $$\int \frac{dt}{1+t^2} = \int dx$$

Or $$\tan^{-1} t = x + c$$

Or $$t = \tan(x+c)$$

Or $$x + y - 1 = \tan(x+c)$$

Or $$y = 1 - x + \tan(x+c)$$

53.* The general solution of the different equation $\frac{dy}{dx} = \cos(x+y)$, with c as a constant, is

(a) $y + \sin(x+y) = x + c$

(b) $\tan\left(\frac{x+y}{2}\right) = y + c$

Sol–53: (d)

$$\frac{dy}{dx} = \cos(x + y) \qquad \ldots(1)$$

put, $x + y = t$

$$\Rightarrow \qquad 1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} - 1 = \cos t \qquad \text{[using (1)]}$$

$$\frac{dt}{1+\cos t} = dx$$

$$\frac{dt}{2\cos^2 \frac{t}{2}} = dx$$

$$\frac{1}{2}\sec^2\frac{t}{2}dt = dx$$

Integrating both sides

$$\tan\frac{t}{2} = x + c$$

$$\boxed{\tan\left(\frac{x+y}{2}\right) = x + c}$$

51. Which ONE of the following is a linear non-homogeneous differential equation, where x and y are the independent and dependent variables respectively?

(a) $\frac{dy}{dx} + xy = e^{-x}$ (b) $\frac{dy}{dx} + xy = 0$

(c) $\frac{dy}{dx} + xy = e^{-y}$ (d) $\frac{dy}{dx} + e^{-y} = 0$

[GATE-2014-EC-Set-3]

Sol–51: (a)

(a) $\frac{dy}{dx}+xy=e^{-x}$ is first order non-homogeneous linear eq.

(b) $\frac{dy}{dx}+xy=0$ is first order homogeneous linear eq.

(c) (d) are non linear eq's.

49.* Choose the CORRECT set of functions, which are linearly dependent.

(a) $\sin x$, $\sin^2 x$ and $\cos^2 x$

(b) $\cos x$, $\sin x$, and $\tan x$

(c) $\cos 2x$, $\sin^2 x$ and $\cos^2 x$

(d) $\cos 2x$, $\sin x$ and $\cos x$

[GATE-2013 (ME)]

Sol–49: (c)

Let $\qquad y_1 = \cos 2x,\ y_2 = \cos^2 x,\ y_3 = \sin^2 x$

then, $\qquad \cos 2x = \cos^2 x - \sin^2 x$

$\Rightarrow \qquad \cos 2x = (1)\cos^2 x + (-1)\sin^2 x$

$\Rightarrow \qquad (1)y_1 - (1)y_2 + (1)y_3 = 0$

$\Rightarrow \qquad C_1y_1 + C_2y_2 + C_3y_3 = 0$

So, linear combination exist between y_1, y_2, y_3. So, they are linearly dependent.

40. Which one of the following differential equations has a solution given by the function

$$y = 5\sin\left(3x + \frac{\pi}{3}\right)$$

(a) $\frac{dy}{dx} - \frac{5}{3}\cos(3x) = 0$

(b) $\frac{dy}{dx} + \frac{5}{3}\cos(3x) = 0$

(c) $\frac{d^2y}{d^2x} + 9y = 0$

(d) $\frac{d^2y}{d^2x} - 9y = 0$

[GATE-2010(PI)]

Sol–40: (c)

$$y = 5\sin\left[3x + \frac{\pi}{3}\right]$$

$$y = 5\left[\sin 3x \cdot \cos\frac{\pi}{3} + \cos 3x \cdot \sin\frac{\pi}{3}\right]$$

or $$y = \left(5\cos\frac{\pi}{3}\right)\sin(3x) + \left(5\sin\frac{\pi}{3}\right)\cos(3x)$$

$$y = C_1 \sin 3x + C_2 \cos 3x \text{ (let)}$$

then $$\frac{dy}{dx} = 3C_1 \cos 3x - 3C_2 \sin 3x$$

$$\frac{d^2y}{dx^2} = -9C_1 \sin 3x - 9C_2 \cos 3x$$

$$= -9\left[C_1 \sin(3x) + C_2 \cos 3x\right]$$

$$\frac{d^2y}{dx^2} = -9y \Rightarrow \frac{d^2y}{dx^2} + 9y = 0$$

$$y = 5\sin\left(3x + \frac{\pi}{3}\right)$$

$$\frac{dy}{dx} = 5\cos\left(3x + \frac{\pi}{3}\right) \cdot 3$$

$$\frac{dy}{dx} = 15\cos\left(3x + \frac{\pi}{3}\right)$$

$$\frac{d^2y}{dx^2} = -15\sin\left(3x + \frac{\pi}{3}\right) \cdot 3$$

$$\frac{d^2y}{dx^2} = -9y \,//$$

$$\frac{d^2y}{dx^2} + 9y = 0 \,//$$

.* The Blasius equation, $\frac{d^3f}{d\eta^3}+\frac{f\,d^2f}{2d\eta^2}=0$, is a

(a) second order nonlinear ordinary differential equation

(b) third order nonlinear ordinary differential equation

(c) third order linear ordinary differential equation

(d) mixed order nonlinear ordinary differential equation

[GATE-2010-ME]

Sol–37: (b)

Given,

$$\frac{d^3f}{d\eta^3} + \underbrace{\frac{f}{2}\,\frac{d^2f}{d\eta^2}}_{\text{cross product term}} = 0$$

Hence, non-linear differential equation obviously order is three.

38.* The order and degree of the differential equation $\frac{d^3y}{dx^3} + 4\sqrt{\left(\frac{dy}{dx}\right)^3 + y^2} = 0$ are respectively

(a) 3 and 2

(b) 2 and 3

(c) 3 and 3

(d) 3 and 1

[GATE-2010-CE]

Sol–38: (a)

Making the given differential equation free of radicals, we get

$$\left(\frac{d^3y}{dx^3}\right)^2 = \left(-4\sqrt{\left(\frac{dy}{dx}\right)^3 + y^2}\right)^2$$

$$= 16\left[\left(\frac{dy}{dx}\right)^3 + y^2\right]$$

Order = order of highest order derivative

= 3

Degree = exponent of highest order derivative

= 2.

36. The solution of the differential equation $\frac{d^2y}{dx^2} = 0$ with boundary conditions

(i) $\frac{dy}{dx} = 1 \text{at } x = 0$ (ii) $\frac{dy}{dx} = 1 \text{at } x = 1$ is

(a) $y = 1$

(b) $y = x$

(c) $y = x + c$ where c is an arbitrary constants

(d) $y = C_1x + C_2$ where C_1, C_2 are arbitary constants

[GATE-2009 (PI)]

Sol–36: (c)

Given $$\frac{d^2y}{dx^2} = 0 \qquad \text{...(1)}$$

$$\Rightarrow \quad \frac{dy}{dx} = C_1 \Rightarrow y = C_1x + C_2$$

Using $y'(0) = 1$, we get

$$C_1 = 1$$

so general solution is $\boxed{y = x + C_2}$

27.* The solution of $dy/dx = y^2$ with initial value $y(0) = 1$ bounded in the interval

(a) $-\infty < x < \infty$ (b) $-\infty < x \leq 1$

(c) $x < 1, x > 1$ (d) $-2 \leq x \leq 2$

[GATE-2007-ME]

Sol–27: (c)

$$\frac{dy}{dx} = y^2$$

$$\int \frac{dy}{y^2} = \int dx$$

$$\therefore \quad x = -\frac{1}{y} + c$$

$$\because \quad y(0) = 1$$

$$\therefore \quad c = 1$$

$$\therefore \quad y = \frac{1}{1-x}$$

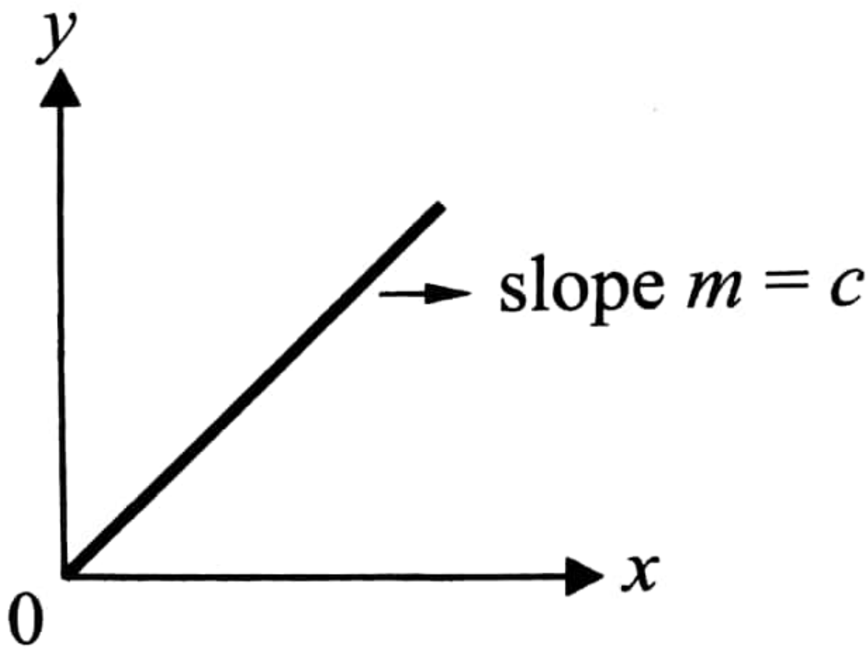
But $\quad 1 - x \neq 0$

$$x \neq 1$$

$$\therefore \quad x \in (-\infty, \infty) - \{1\}$$

or $\quad x > 1,$

$$x < 1$$

(P)	$\frac{dy}{dx}=\frac{y}{x}\Rightarrow\frac{dy}{y}=\frac{dx}{x}$
	$\int\frac{dy}{y}=\int\frac{dx}{x}$ $\log y=\log x+\log c=\log xc$ $y=xc$ This is a straight line. 

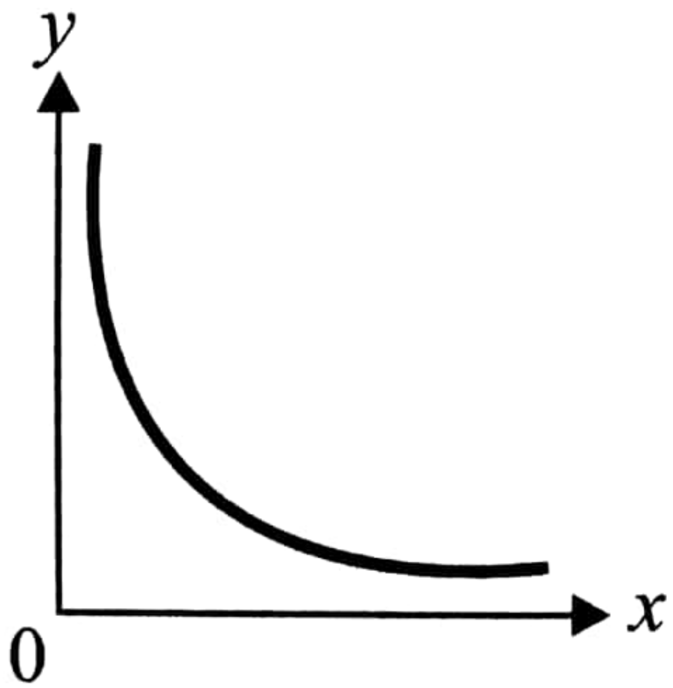
(Q) $\frac{dy}{dx} = \frac{-y}{x} \Rightarrow \frac{dy}{y} = -\frac{dx}{x}$

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\log y = -\log x + \log c$$

$$\log(xy) = \log c \Rightarrow xy = c$$

This is a rectangular hyperbola.

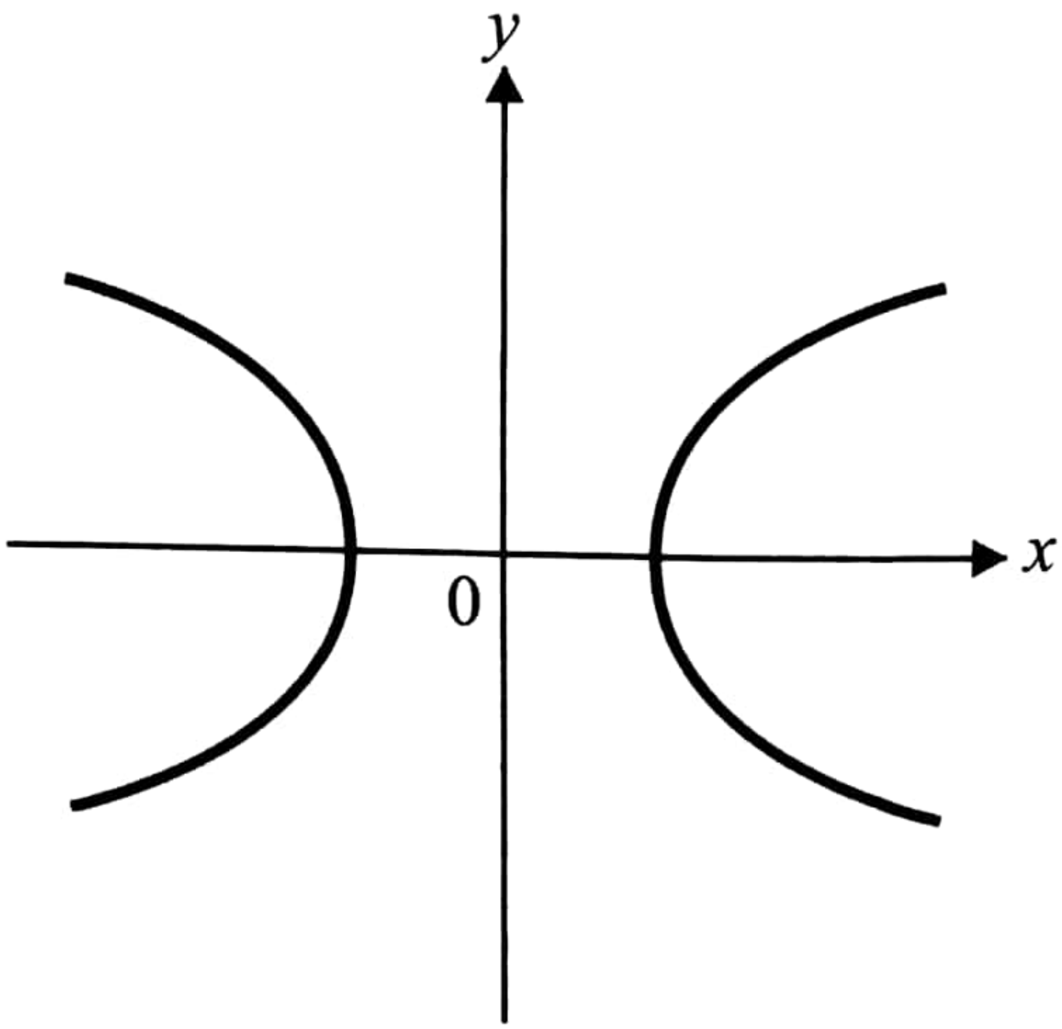


(R) $$\frac{dy}{dx}=\frac{x}{y}\Rightarrow\int y\,dy=\int x\,dx$$

$$\frac{y^2}{2}=\frac{x^2}{2}+c$$

$$\frac{y^2}{2}-\frac{x^2}{2}=c$$

This is also a hyperbola.



(S) $\dfrac{dy}{dx} = \dfrac{-x}{y} \Rightarrow \int y dy = -\int x dx$

$$\frac{y^2}{2} = \frac{-x^2}{2} + c$$

$$x^2 + y^2 = 2c \Rightarrow x^2 + y^2 = r^2$$

3.2.3.4 Leibnitze linear equation

The standard form of a linear equation of the first order, commonly known as **Leibnitz's linear equation**, is

$$\frac{dy}{dx} + Py = Q \text{ where } P, Q, \text{ are arbitrary functions of } x. \qquad \text{... (i)}$$

To solve the equation, multiply both sides by $e^{\int P dx}$ so that we get

$$\frac{dy}{dx} \cdot e^{\int P dx} + y(e^{\int P dx} P) = Qe^{\int P dx} \text{ i.e. } \frac{d}{dx}(ye^{\int P dx}) = Qe^{\int P dx}$$

Integrating both sides, we get $ye^{\int P dx} = \int Qe^{\int P dx} dx + c$ as the required solution.

NOTE

The factor $e^{\int P dx}$ on multiplying by which the left-hand side of (1) becomes the differential coefficient of a single function, is called the **integrating factor (I.F.)** of the linear equation (i).
So remember the following:

$$\text{I.F.} = e^{\int P dx}$$

and the solution is

$$y(\text{I.F.}) = \int Q(\text{I.F.}) dx + c.$$

24. The solution of the differential equation $x^2\frac{dy}{dx}+2xy-x+1=0$ given that at x = 1, y = 0 is

(a) $\frac{1}{2}-\frac{1}{x}+\frac{1}{2x^2}$

(b) $\frac{1}{2}-\frac{1}{x}-\frac{1}{2x^2}$

(c) $\frac{1}{2}+\frac{1}{x}+\frac{1}{2x^2}$

(d) $-\frac{1}{2}+\frac{1}{x}+\frac{1}{2x^2}$

[GATE-2006-CE]

Sol–24: (a)

$$x^2\frac{dy}{dx}+2xy-x+1 = 0 \qquad ...(i)$$

$$\Rightarrow \qquad \frac{dy}{dx}+\frac{2}{x}y = \left(\frac{x-1}{x^2}\right)$$

$$\text{I.F.} = e^{\int\frac{2}{x}dx}$$

$$= e^{2\ln x} = x^2$$

$\therefore$ Solution of (i) is given by

$$y\,(\text{I.F.}) = \int\frac{(x-1)}{x^2}(\text{I.F.})\,dx+c$$

$$yx^2 = \int(x-1)\,dx+c$$

$$yx^2 = \frac{x^2}{2}-x+c$$

Given at

$$x = 1$$

$$y = 0$$

$$0 = \frac{1}{2}-1+c \quad \Rightarrow \quad \boxed{c=\frac{1}{2}}$$

33. The solution of $x\frac{dy}{dx}+y=x^4$ with the condition $y(1)=\frac{6}{5}$ is

(a) $y=\frac{x^4}{5}+\frac{1}{x}$

(b) $y=\frac{4x^4}{5}+\frac{4}{5x}$

(c) $y=\frac{x^4}{5}+1$

(d) $y=\frac{x^5}{5}+1$

[GATE-2009-ME]

Sol–33: (a)

The differential equation is

$$x\frac{dy}{dx}+y = x^4$$

$$\Rightarrow \quad \frac{dy}{dx}+\left(\frac{1}{x}\right)y = x^3 \qquad ...(1)$$

$$\text{I.F.} = e^{\int \frac{1}{x}dx} = e^{\ln x}$$

$$\text{I.F.} = x$$

Hence, solution of (1) is

$$y.(x) = \int x^3.x\,dx+c = \frac{x^5}{5}+c$$

$$\boxed{y = \frac{x^4}{5}+\frac{c}{x}}$$

Given

$$y(1) = 6/5$$

$$\Rightarrow \quad \frac{6}{5} = \frac{1}{5}+\frac{c}{1} \Rightarrow c=1$$

$$y = \frac{x^4}{5}+\frac{1}{x}$$

47. With initial condition $x(1) = 0.5$, the solution of the differential equation, $t\frac{dx}{dt} + x = t$ is

(a) $x = t - \frac{1}{2}$ (b) $x = t^2 - \frac{1}{2}$

(c) $x = \frac{t^2}{2}$ (d) $x = \frac{t}{2}$

[GATE-2012-EC, EE, IN]

Sol–47: (d)

$$t\frac{dx}{dt}+x = t,\ x(1) = \frac{1}{2}$$

$$\frac{dx}{dt}+\frac{1}{t}x = 1$$

which is a linear differential equation.

$$\text{I.F.} = e^{\int\frac{1}{t}dt} = e^{\log t} = t$$

Sol is $\quad x.(\text{I.F.}) = \int 1.(\text{IF})dt + C$

$$x \times t = \int t\cdot 1 dt + c$$

$$tx = \frac{t^2}{2} + c$$

$$x(1) = \frac{1}{2}$$

$\Rightarrow \quad 1\left(\frac{1}{2}\right) = \frac{1}{2} + c \quad \Rightarrow \quad c = 0$

So, $\quad tx = \frac{t^2}{2} \quad \Rightarrow \quad x = \frac{t}{2}$

3.2.3.5 Bernoulli's Equation

The equation $\frac{dy}{dx} + Py = Qy^n$ ... (i)

where P, Q are functions of x, is reducible to the Leibnitz's linear and is usually called the Bernoulli's equation.

To solve (i), divide both sides by y^n, so that $y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$ ... (ii)

Put $y^{1-n} = z$ so that $(1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ Eq. (ii) becomes $\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$

or $$\frac{dz}{dx} + P(1-n)z = Q(1-n),$$

which is Leibnitz's linear in z and can be solved easily.

Example:

Solve $\dfrac{dy}{dx} + y = 4y^3$

Solution:

Dividing throughout by y^3,

$$y^{-3}\frac{dy}{dx} + y^{-2} = 4$$

Put $y^{-2} = z$, so that $-2y^{-3}\dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ Eq. (i) becomes $-\dfrac{1}{2}\dfrac{dz}{dx} + z = 4$

or
$$\frac{dz}{dx} - 2z = -8$$

which is Leibnitz's linear in z.

$$\text{I.F.} = e^{\int -2dx} = e^{-2x}$$

$\therefore$ The solution of (ii) is $z\,(\text{I.F.}) = \int (-8)(\text{I.F.})dx + c$

$$ze^{-2x} = \int (-8)e^{-2x}dx + c$$

$$\Rightarrow \quad y^{-2}\,e^{-2x} = 4e^{-2x} + c$$

$$\Rightarrow \quad y^{-2} = 4 + ce^{2x}$$

$$\Rightarrow \quad y = (4 + ce^{2x})^{-1/2}$$

20.* Transformation to linear form by substituting $v = y^{1-n}$ of the equation $\frac{dy}{dt}+p(t)y=q(t)y^n$; $n > 0$ will be

(a) $\frac{dv}{dt}+(1-n)pv=(1-n)q$

(b) $\frac{dv}{dt}+(1-n)pv=(1+n)q$

(c) $\frac{dv}{dt}+(1+n)pv=(1-n)q$

(d) $\frac{dv}{dt}+(1+n)pv=(1+n)q$

[GATE-2005-CE]

$$\Rightarrow \quad y^{-n}\frac{dy}{dt} + p(t)y^{1-n} = q(t) \qquad \text{...(i)}$$

Putting $\qquad y^{(1-n)} = v$

$$\Rightarrow \quad (1-n)y^{-n}\frac{dy}{dt} = \frac{dv}{dt} \qquad \text{...(ii)}$$

Putting these values in (i) m, we get,

$$\frac{1}{(1-n)}\frac{dv}{dt} + p(t)v = q(t)$$

$$\frac{dv}{dt} + (1-n)pv = (1-n)q$$

11.11 EXACT DIFFERENTIAL EQUATIONS

(1) Def. *A differential equation of the form* $\mathbf{M}(x, y)\, \mathbf{dx} + \mathbf{N}(x, y)\, \mathbf{dy} = 0$ *is said to be* ***exact*** *if its left hand member is the exact differential of some function* $u(x, y)$ *i.e.,* $du = Mdx + Ndy = 0$. Its solution, therefore, is $u(x, y) = c$.

(2) Theorem. The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial \mathbf{M}}{\partial \mathbf{y}} = \frac{\partial \mathbf{N}}{\partial \mathbf{x}}$$

Condition is necessary :

The equation $Mdx + Ndy = 0$ will be exact, if

$$Mdx + Ndy \equiv du \quad \text{...(1)}$$

where u is some function of x any y.

But $$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \quad \text{...(2)}$$

$\therefore$ equating coefficients of dx and dy in (1) and (2), we get $M = \frac{\partial u}{\partial x}$ and $N = \frac{\partial u}{\partial y}$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

But $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \quad \text{(Assumption)}$$

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ which is the necessary condition for exactness.

Condition is sufficient : i.e., if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then $Mdx + Ndy = 0$ is exact.

Let $\int Mdx = u$, where y is supposed constant while performing integration.

Then $$\frac{\partial}{\partial x}\left(\int Mdx\right) = \frac{\partial u}{\partial x}, \quad i.e., \; M = \frac{\partial u}{\partial x} \quad \text{...(3)}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{or} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \qquad \left\{ \begin{array}{l} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ (given)} \\ \text{and } \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \end{array} \right.$$

Interating both sides w.r.t. x (taking y as constant).

$$N = \frac{\partial u}{\partial y} + f(y), \text{ where } f(y) \text{ is a function of } y \text{ alone.} \quad \text{...(4)}$$

$$\therefore \quad Mdx + Ndy = \frac{\partial u}{\partial x} dx + \left\{\frac{\partial u}{\partial y} + f(y)\right\} dy \quad \text{[By (3) and (4)]}$$

$$= \left\{\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right\} + f(y)\, dy = du + f(y)\, dy = d\left[u + \int f(y) dy\right] \quad \text{...(5)}$$

which shows that $Mdx + Ndy = 0$ is exact.

(3) Method of solution. By (5), the equation $Mdx + Ndy = 0$ becomes $d[u + \int f(y) dy] = 0$

Integrating $\quad u + \int f(y) dy = 0.$

But $\quad u = \int_{y \text{ constant}} Mdx$ and $f(y)$ = terms of N not containing x.

$\therefore$ *The solution of* $Mdx + Ndy = 0$ *is*

$$\int_{(y \text{ cons.})} \mathbf{M\, dx} + \int (\textbf{terms of N not containing x})\, \mathbf{dy} = \mathbf{c}$$

provided $\quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$

Example 11.25. *Solve* $(y^2 e^{xy^2} + 4x^3)\,dx + (2xy\,e^{xy^2} - 3y^2)\,dy = 0.$ *(V.T.U., 2006)*

Solution. Here $\quad M = y^2 e^{xy^2} + 4x^3$ and $N = 2xy\,e^{xy^2} - 3y^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 2y\,e^{xy^2} + y^2\,e^{xy^2}\,.\,2xy = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y\text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

i.e.,
$$\int_{(y\text{ const.})} (y^2 e^{y^2 x} + 4x^3)\,dx + \int (-3y^2)\,dy = c \quad \text{or} \quad e^{xy^2} + x^4 - y^3 = c.$$

Example 11.26. *Solve* $\left\{y\left(1 + \frac{1}{x}\right) + \cos y\right\} dx + (x + \log x - x \sin y)\,dy = 0.$

(Marathwada, 2008 S ; V.T.U., 2006)

Solution. Here $M = y\,(1 + 1/x) + \cos y$ and $N = x + \log x - x \sin y$

$$\therefore \quad \frac{\partial M}{\partial y} = 1 + 1/x - \sin y = \frac{\partial N}{\partial x}$$

Then the equation is exact and its solution is

$$\int_{(y\text{ const})} M dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

$$\int_{(y\text{ const})} \left\{\left(1 + \frac{1}{x}\right) y + \cos y\right\} dx = c \quad \text{or} \quad (x + \log x)\,y + x \cos y = c.$$

Example 11.27. *Solve* $(1 + 2xy \cos x^2 - 2xy)\,dx + (\sin x^2 - x^2)\,dy = 0.$

Solution. Here $M = 1 + 2xy \cos x^2 - 2xy$ and $N = \sin x^2 - x^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 2x \cos x^2 - 2x = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y\text{ const})} M dx + \int (\text{terms of } N \text{ not containing } x) = c$$

i.e.,
$$\int_{(y\text{ const})} (1 + 2xy \cos x^2 - 2xy)\,dx = c \quad \text{or} \quad x + y\left[\int \cos x^2\,.\,2x dx - \int 2x\,dx\right] = c$$

or
$$x + y \sin x^2 - yx^2 = c.$$

Example 11.28. *Solve* $\dfrac{dy}{dx} + \dfrac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0.$ *(Andhra, 2016 ; V.T.U., 2016 ; Rohtak, 2011)*

Solution. Given equation can be written as

$$(y \cos x + \sin y + y)\,dx + (\sin x + x \cos y + x)\,dy = 0.$$

Here, $M = y \cos x + \sin y + y$ and $N = \sin x + x \cos y + x$.

$$\therefore \quad \frac{\partial M}{\partial y} = \cos x + \cos y + 1 = \frac{\partial N}{\partial x}.$$

Thus the equation is exact and its solution is

$$\int_{(y\text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

i.e.,
$$\int_{(y\text{ const.})} (y \cos x + \sin y + y)\,dx + \int (0)\,dx = c \quad \text{or} \quad y \sin x + (\sin y + y)\,x = c.$$

10 : Solve; $(1 + e^{x/y})dx + e^{x/y}(1 - x/y)dy = 0$ by using the concept of homogeneous DE method

Putting $x = vy \Rightarrow dx/dy = v + ydv/dy$

$$\Rightarrow \quad (1 + e^{v})dx + e^{v}(1 - v)dy = 0 \quad \Rightarrow \quad \frac{dx}{dy} = \frac{-e^{v}(1-v)}{(1+e^{v})} \quad \Rightarrow \quad v + y\frac{dv}{dy} = \frac{-e^{v}(1-v)}{(1+e^{v})}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{-e^{v}(1-v)}{(1+e^{v})} - v \quad \Rightarrow \quad y\frac{dv}{dy} = \frac{-e^{v} + e^{v}v - v - ve^{v}}{(1+e^{v})} \quad \Rightarrow \quad y\frac{dv}{dy} = \frac{-(e^{v} + v)}{(1+e^{v})}$$

$$\Rightarrow \quad \int \frac{(1+e^{v})}{(e^{v}+v)} dv = \int \frac{-dy}{y} + \log c \quad \Rightarrow \quad \log(v+e^{v}) = -\log y + \log c$$

$$\Rightarrow \quad v + e^{v} = \frac{c}{y} \quad \Rightarrow \quad \frac{x}{y} + e^{x/y} = \frac{c}{y}$$

ample 14 : $(1 + e^{x/y})dx + e^{x/y}(1 - x/y)dy = 0$

lution : Comparing with $Mdx + Ndy = 0$

$$M = (1 + e^{x/y}),\ N = e^{x/y}(1 - x/y)$$

$$\frac{\partial M}{\partial y} = e^{x/y}\left(\frac{-x}{y^2}\right)$$

$$\frac{\partial N}{\partial x} = e^{x/y}\left(\frac{1}{y}\right)\left(1-\frac{x}{y}\right)+e^{x/y}\left(\frac{-1}{y}\right)=e^{x/y}\left(\frac{1}{y}\right)\left(1-\frac{x}{y}-1\right)=e^{x/y}\left(\frac{-x}{y^2}\right)$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Hence, equation is in exact form. So, solution is

$$\int Mdx+\int(\text{those terms of N which are free from x})\,dy = C$$

$$\int_{y=\text{constant}} \left(1+e^{\frac{x}{y}}\right)dx+\int 0dx = c$$

$$x+\int e^{\frac{x}{y}}dx+\int 0dy = c \Rightarrow x+\int e^{t}ydt = c \Rightarrow x + ye^{t} = c \Rightarrow x+ye^{\frac{x}{y}} = c$$

2. The necessary and sufficient for the differential equation of the from $M(x,y)\,dx + N(x,y)\,dy = 0$ to be exact is

(a) $M = N$

(b) $\dfrac{\partial M}{\partial x} = \dfrac{\partial N}{\partial y}$

(c) $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

(d) $\dfrac{\partial^2 M}{\partial x^2} = \dfrac{\partial^2 M}{\partial y^2}$

[GATE-1994]

Sol–2: (c)

$\because$ It is the standard result.

10.* For the differential equation, $f(x,y)\frac{dy}{dx}+g(x,y)=0$ to be exact,

(a) $\frac{\partial f}{\partial y}=\frac{\partial g}{\partial x}$

(b) $\frac{\partial f}{\partial x}=\frac{\partial g}{\partial y}$

(c) $f = g$

(d) $\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 g}{\partial y^2}$

[GATE-1997-CE]

Sol–10: (b)

$$f(x, y)\frac{dy}{dx} + g(x, y) = 0$$

$$\Rightarrow \quad g(x, y)\, dx + f(x, y)\, dy = 0$$

It is an exact differentiable equation

if $$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}$$

24.* The solution of the differential equation $x^2 \frac{dy}{dx} + 2xy - x + 1 = 0$ given that at x = 1, y = 0 is

(a) $\frac{1}{2} - \frac{1}{x} + \frac{1}{2x^2}$

(b) $\frac{1}{2} - \frac{1}{x} - \frac{1}{2x^2}$

(c) $\frac{1}{2} + \frac{1}{x} + \frac{1}{2x^2}$

(d) $-\frac{1}{2} + \frac{1}{x} + \frac{1}{2x^2}$

[GATE-2006-CE]

$$\frac{1}{2} \quad \frac{1}{x} \quad \frac{1}{2x^2}$$

Method II :

Given

$$(2xy - x + 1)dx + x^2dy = 0 \qquad ...(1)$$

On comparison with

$$Mdx + Ndy = 0$$

we get $\quad M = 2xy - x + 1$

and $\quad N = x^2$

Here $M_y = 2x = N_x$ so equation (1) is exact differential equation.

$\therefore$ Solution of given equation is

$$\int_{y=\text{const}} (2xy - x + 1)dx + \int (0)dy = C$$

[$\because$ Standard Result is $\int_{y=\text{const.}} (Mdx) + \int$ (those terms of N which are from x) dy = C]

$$\Rightarrow \qquad x^2y - \frac{x^2}{2} + x = C$$

Using $y(1) = 0$, $\quad C = \frac{1}{2}$

so solution is

$$x^2y - \frac{x^2}{2} + x = \frac{1}{2}$$

i.e., $\quad y = \frac{1}{2x^2} + \frac{1}{2} - \frac{1}{x}$

Example 11.31. *Solve* $(x^2y - 2xy^2)\, dx - (x^3 - 3x^2y)\, dy = 0$.

Solution. This equation is homogeneous in x and y.

$$\therefore \qquad \text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{(x^2y - 2xy^2)\,x - (x^3 - 3x^2y)y} = \frac{1}{x^2y^2}$$

Multiplying throughout by $1/x^2y^2$, the equation becomes

$$\left(\frac{1}{y} - \frac{2}{x}\right) dx - \left(\frac{x}{y^2} - \frac{3}{y}\right) dy = 0, \text{ which is exact.}$$

$\therefore$ the solution is $\int_{(y\ \text{const})} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$ or $\dfrac{x}{y} - 2 \log x + 3 \log y = c$.

(3) I.F. for an equation of the type $f_1(xy)ydx + f_2(xy)xdy = 0$.

If the equation $Mdx + Ndy = 0$ be of this form, then $1/(Mx - Ny)$ is an integrating factor $(Mx - Ny \neq 0)$.

$$(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$$

$$\rightarrow \frac{dx}{dy} = \frac{(x^3 - 3x^2y)}{(x^2y - 2xy^2)} = \frac{(x/y)^3 - 3(x/y)^2}{\frac{x^2}{y^2} - 2\frac{x}{y}}$$

$$\rightarrow \boxed{\begin{array}{l} \frac{x}{y} = t \\ \therefore\ x = yt \\ \frac{dx}{dy} = y\frac{dt}{dy} + t \end{array}}$$

$$\therefore\ y\frac{dt}{dy} + t = \frac{t^3 - 3t^2}{t^2 - 2t}$$

$$\therefore\ y\frac{dt}{dy} = \frac{t^2 - 3t}{t-2} - t$$

$$y\frac{dt}{dy} = \frac{\cancel{t^2} - 3t - \cancel{t^2} + 2t}{t-2}$$

$$y\frac{dt}{dy} = \frac{-t}{t-2}$$

$$\int \frac{t-2}{t}\,dt = \int -\frac{dy}{y}$$

$$t - 2\ln t = -\ln y + c$$

$$\frac{x}{y} - 2\ln\left(\frac{x}{y}\right) = -\ln(y) + c.$$

$$\frac{x}{y} + \ln y - \ln\frac{x^2}{y^2} = C$$

$$\frac{x}{y} + \ln\left[y/x^2/y^2\right] = c$$

$$\frac{x}{y} + \ln\left(\frac{y^3}{x^2}\right) = c \Rightarrow \boxed{\frac{x}{y} + 3\ln y - 2\ln x = C}$$

Scanned by CamScanner

(3) I.F. for an equation of the type $f_1(xy)ydx + f_2(xy)xdy = 0$.

If the equation $Mdx + Ndy = 0$ be of this form, then $1/(Mx - Ny)$ is an integrating factor $(Mx - Ny \neq 0)$.

Example 11.32. *Solve* $(1 + xy)\, ydx + (1 - xy)\, xdy = 0$. (C.S.V.T.U., 2014 ; V.T.U., 2014)

Solution. The given equation is of the form $f_1(xy)\, ydx + f_2(xy)\, xdy = 0$

Here $$M = (1 + xy)\, y,\ N = (1 - xy)\, x.$$

$\therefore$ $$\text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{(1+xy)yx - (1-xy)xy} = \frac{1}{2x^2y^2}$$

Multiplying throughout by $1/2x^2y^2$, it becomes

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0, \text{ which is an exact equation.}$$

$\therefore$ the solution is $\int_{(y\text{ const})} Mdx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$

$$\frac{1}{2y}\left(-\frac{1}{x}\right) + \frac{1}{2}\log x - \frac{1}{2}\log y = c \quad \text{or} \quad \log\frac{x}{y} - \frac{1}{xy} = c'.$$

(4) In the equation Mdx + Ndy = 0,

(a) if $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ *be a function of x only* $= f(x)$ *say, then* $e^{\int f(x)dx}$ *is an integrating factor.*

(b) if $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ *be a function of y only* $= F(y)$ *say, then* $e^{\int F(y)dy}$ *is an integrating factor.*

Example 11.33. *Solve* $(xy^2 - e^{1/x^3})dx - x^2 ydy = 0.$ (C.S.V.T.U., 2009 ; M

Solution. Here $M = xy^2 - e^{1/x^3}$ and $N = -x^2y$

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2y} = -\frac{4}{x} \text{ which is a function of } x \text{ only.}$$

$\therefore$ $$\text{I.F.} = e^{\int \frac{-4}{x} dx} = e^{-4 \log x} = x^{-4}$$

Multiplying throughout by x^{-4}, we get $\left(\dfrac{y^2}{x^3} - \dfrac{1}{4^4} e^{1/x^3}\right) dx - \dfrac{y}{x^2}\, dy = 0$

vhich is an exact equation.

$\therefore$ the solution is $\int_{(y \text{ const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x)\, dy = c.$

r $$\int \left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{1/x^3} \right) dx + 0 = c$$

r $$-\frac{y^2 x^{-2}}{2} + \frac{1}{3} \int e^{x^{-3}} (-3x^{-4})\, dx = c \text{ or } \frac{1}{3} e^{x^{-3}} - \frac{1}{2} \frac{y^2}{x^2} = c.$$

$$(xy^2 - e^{1/x^3})\,dx = x^2 y\,dy$$

$$\cancel{\frac{dy}{dy}} \quad \cancel{x^2 y} \quad \frac{dy}{dx} = \frac{y}{x} - \frac{e^{1/x^3}}{x^2 y}$$

$$\frac{dy}{dx} - \left(\frac{1}{x}\right) y = -\frac{e^{1/x^3}}{x^2 y}$$

$$y\frac{dy}{dx} - \left(\frac{1}{x}\right) y^2 = -\frac{e^{\frac{1}{x^3}}}{x^2}$$

$$\rightarrow \quad y^2 = t$$

$$2y\frac{dy}{dx} = \frac{dt}{dx}$$

$$\therefore \quad \frac{1}{2}\frac{dt}{dx} - \left(\frac{1}{x}\right) t = -\frac{e^{1/x^3}}{x^2}$$

$$\therefore \quad \boxed{\frac{dt}{dx} - \left(\frac{2}{x}\right) t = \frac{-2e^{1/x^3}}{x^2}}$$

$$IF = e^{\int -2/x\,dx} = e^{-2\ln x} = e^{\ln x^{-2}} = 1/x^2$$

$$\underline{Sol^n}: \quad t \cdot \frac{1}{x^2} = \int \frac{1}{x^2} \cdot \frac{-2e^{1/x^3}}{x^2}\,dx + c$$

$$\frac{1}{x^3} = z$$

$$-3x^{-4}\,dx = dz$$

$$\frac{t}{x^2} = \int -2e^z\left(\frac{-1}{3}\right)dz + c$$

$$\frac{t}{x^2} = \int \frac{2}{3} e^z\,dz + c$$

$$\frac{t}{x^2} = \frac{2}{3}(e^z) + c$$

$$\frac{t}{x^2} = \frac{2}{3} e^{1/x^3} + c$$

$$\left(\frac{y^2}{x^2}\right) - \frac{2}{3} e^{1/x^3} = c.$$

$$\Rightarrow \quad \frac{1}{3} e^{1/x^3} - \frac{1}{2}\frac{y^2}{x^2} = -c = c' \;//$$

Example 11.34. *Solve* $(xy^3 + y)\,dx + 2\,(x^2y^2 + x + y^4)\,dy = 0.$ (*V.T.U., 2015 ; P.T.U., 2014 ; D.T.U., 2013*)

Solution. Here $M = xy^3 + y$, $N = 2(x^2y^2 + x + y^4)$

$$\therefore \quad \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y(xy^2+1)}(4xy^2 + 2 - 3xy^2 - 1) = \frac{1}{y}, \text{ which is a function of } y \text{ alone.}$$

$$\therefore \quad \text{I.F.} = e^{\int 1/y\,dy} = e^{\log y} = y$$

Multiplying throughout by y, it becomes $(xy^4 + y^2)\,dx + (2x^2y^3 + 2xy + 2y^5)\,dy = 0$, which is an exact equation.

$$\therefore \quad \text{its solution is } \int_{(y\text{ const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x)\,dy = 0$$

or
$$\int_{(y\text{ const})} (xy^4 + y^2)dx + \int 2y^5\,dy = c \quad \text{or} \quad \frac{1}{2}x^2y^4 + xy^2 + \frac{1}{3}y^6 = c.$$

Example 11.35. *Solve $(y \log y)\, dx + (x - \log y)\, dy = 0$*

Solution. Here $M = y \log y$ and $N = x - \log y$

$$\therefore \qquad \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y \log y}(1 - \log y - 1) = -\frac{1}{y}, \text{ which is a function of } y \text{ alone.}$$

$$\therefore \qquad \text{I.F.} = e^{-\int \frac{1}{y} dy} = e^{-\log y} = \frac{1}{y}$$

Multiplying the given equation throughout by $1/y$, it becomes

$$\log y\, dx + \frac{1}{y}(x - \log y)\, dy = 0$$

which is an exact equation $\left[\because \frac{\partial}{\partial y}(\log y) = \frac{\partial}{\partial x}\left(\frac{x - \log y}{y}\right)\right.$

$\therefore$ its solution is $\int_{(y \text{ const})} (M dx) + \int (\text{terms of } N \text{ not containing } x)\, dy = c$

or $\quad \log y \int dx + \int \left(\frac{-\log y}{y}\right) dy = c \qquad$ or $\quad x \log y - \frac{1}{2}(\log y)^2 = c.$

$$xdy + ydx = d\,(xy)$$

$$\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right);\ \frac{xdy - ydx}{xy} = d\left[\log\left(\frac{y}{x}\right)\right]$$

$$\frac{xdy - ydx}{y^2} = -\,d\left(\frac{x}{y}\right);\ \frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$$

$$\frac{xdy - ydx}{x^2 - y^2} = d\left(\frac{1}{2}\log\frac{x+y}{x-y}\right).$$

Example 11.30. *Solve* $y\,(2xy + e^x)\,dx = e^x\,dy$.

Solution. It is easy to note that the terms ye^xdx and e^xdy should be pu

$\therefore$ $$(ye^xdx - e^xdy) + 2xy^2\,dx = 0$$

Now we observe that the term $2xy^2\,dx$ should not involve y^2. This suggests that $1/y^2$ may be I.F. Multiplying throughout by $1/y^2$, it follows

$$\frac{ye^x\,dx - e^x\,dy}{y^2} + 2xdx = 0 \quad \text{or} \quad d\left(\frac{e^x}{y}\right) + 2xdx = 0$$

Integrating, we get $\dfrac{e^x}{y} + x^2 = c$ which is the required solution.

3.2.4.3 Newton's Law of Cooling

Definitions

The temperature of a body changes at a rate which is proportional to the difference in temperature between that of the surrounding medium and that of the body itself.

The differential equation is $$\frac{d\theta}{dt} = -k(\theta - \theta_s)$$

by variable separable $$\int \frac{d\theta}{\theta - \theta_s} = \int -k\,dt$$

$$\Rightarrow \qquad \log(\theta - \theta_s) = -kt + \log c$$

$$\Rightarrow \qquad \theta - \theta_s = ce^{-kt}$$

is the solution of Newton's law of cooling.

Example 3.

A body originally at 80°C cools down to 60°C in 20 minutes, the temperature of air being 40°C. What will be the temperature of body after 40 minutes from the original?

Solution:

According to Newton's law of cooling

$$\frac{d\theta}{dt} = -k(\theta - 40)$$

$$\int \frac{d\theta}{\theta - 40} = -\int k\,dt$$

$\Rightarrow$ $$\log(\theta - 40) = -kt + \log c$$

$\Rightarrow$ $$\theta - 40 = ce^{-kt} \quad \text{...(i)}$$

Put $t = 0,\ \theta = 80°$ in equation (i)

We get, $c = 40$

Put, $t = 20$ min, $\theta = 60°$

Then, $$k = \frac{1}{20}\log 2$$

By equation (i), $$\theta = 40 + 40e^{\left(-\frac{1}{2}\log 2\right)t}$$

Put, $t = 40$ min, then $\theta = 50°C$

3.2.4.4 Law of Growth

The rate of change amount of a substance with respect to time is directly proportional to the amount of substance present.

i.e.

$$\frac{dx}{dt} \propto x$$

$$\frac{dx}{dt} = kx \ (k > 0)$$

$$\int \frac{dx}{x} = \int k dt$$

$$\Rightarrow \quad \log x = kt + \log c$$

$$\Rightarrow \quad x = ce^{kt} \text{ is solution of law of growth}$$

Example 4.

The number N of a bacteria in a culture of grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour. What would be the value of N after $1\frac{1}{2}$ hours?

Solution:

According to law of growth, $\frac{dN}{dt} \propto N$

Solution is $N = ce^{kt}$ ...(i)

Put N = 100 and t = 0 in equation (i)

We get, $c = 100$

Then, $N = 100\, e^{kt}$ ...(ii)

Put N = 332, t = 1 in equation (ii)

$$332 = 100e^{k}$$

$$e^{k} = 3.32$$

Put $t = \frac{3}{2}$ in equation (ii)

Then, $N = 100e^{\frac{3}{2}k} = 100(3.32)^{3/2} \simeq 605$

3.2.4.5 Law of Decay

Definitions

The rate of change of amount of substance is directly proportional to the amount of substance present.

i.e. $$\frac{dx}{dt} \propto x$$

The differential equation is

$$\frac{dx}{dt} = -kx \quad (k > 0)$$

$$\int \frac{dx}{x} = -\int kdt$$

$$\Rightarrow \quad \log x = -kt + \log c$$

$$\Rightarrow \quad x = ce^{-kt}$$ is solution of law of decay.

Example 5.

If 30% of radio active substance dissappeared in 10 days. How long will take for 90% of it to disappear?

Solution:

According to law of decay

$$x = ce^{-kt} \quad \text{...(i)}$$

Put $x = 100,\ t = 0$

$$100 = ce^{-k(0)}$$

We get,

$$c = 100$$

Then,

$$x = 100e^{-kt} \quad \text{...(ii)}$$

Put $x = 70$, $t = 10$ in equation (ii)

Then,

$$70 = 100e^{10k}$$

$$k = \frac{1}{10} \ln\left[\frac{7}{10}\right]$$

$\therefore$ Equation (ii) becomes $$x = 100\, e^{\frac{1}{10} ln\left(\frac{7}{10}\right)t}$$

Put, $x = 10$ in equation (iii)

$$10 = 100\, e^{\frac{1}{10} ln\left(\frac{7}{10}\right)t}$$

$$\frac{t}{10} ln(0.7) = ln\left(\frac{1}{10}\right)$$

$$t = \frac{-10 ln 10}{ln(0.7)} = 64.5 \text{ days}$$

Example 11.29. *Solve* $(2x^2 + 3y^2 - 7)\, xdx - (3x^2 + 2y^2 - 8)\, ydy = 0.$

Solution. Given equation can be written as

$$\frac{ydy}{x\,dx} = \frac{2x^2 + 3y^2 - 7}{3x^2 + 2y^2 - 8}$$

or $$\frac{y\,dy + x\,dx}{y\,dy - x\,dx} = \frac{5(x^2 + y^2 - 3)}{-x^2 + y^2 + 1}$$ [By

or $$\frac{x\,dx + y\,dy}{x^2 + y^2 - 3} = 5 \cdot \frac{x\,dx - y\,dy}{x^2 - y^2 - 1}$$

Integrating both sides, we get

$$\int \frac{2xdx + 2ydy}{x^2 + y^2 - 3} = 5 \int \frac{2xdx - 2ydy}{x^2 - y^2 - 1} + c$$

or $$\log (x^2 + y^2 - 3) = 5 \log (x^2 - y^2 - 1) + \log c'$$

or $$x^2 + y^2 - 3 = c' (x^2 - y^2 - 1)^5$$

which is the required solution.

Example 11.37. *Solve* $\dfrac{dy}{dx} - \dfrac{dx}{dy} = \dfrac{x}{y} - \dfrac{y}{x}$. (V.T.U., 2016 ; P.T.U.

Solution. Given equation is $p - \dfrac{1}{p} = \dfrac{x}{y} - \dfrac{y}{x}$ where $p = \dfrac{dy}{dx}$ or $p^2 + p\left(\dfrac{y}{x} - \dfrac{x}{y}\right) - 1 = 0$.

Factorising $(p + y/x)(p - x/y) = 0$.

Thus we have $\quad p + y/x = 0 \quad ...(i) \quad$ and $\quad p - x/y = 0$

From (i), $$\frac{dy}{dx} + \frac{y}{x} = 0 \text{ or } xdy + ydx = 0$$

$$d(xy) = 0. \text{ Integrating, } xy = c.$$

From (ii), $$\frac{dy}{dx} - \frac{x}{y} = 0 \text{ or } xdx - ydx = 0$$

Integrating, $x^2 - y^2 = c$. Thus $xy = c$ or $x^2 - y^2 = c$, constitute the required solution.

Otherwise, combining these into one, the required solution can be written as

$$(xy - c)(x^2 - y^2 - c) = 0.$$

Example 19 :

$$xy^3(ydx + 2xdy) + (3ydx + 5xdy) = 0$$

Solution : Given equation is $xy^4dx + 2x^2y^3dy + 3ydx + 5xdy = 0$

$$(xy^4 + 3y)\,dx + (2x^2y^3 + 5x)\,dy = 0 \qquad ...(1)$$

Compairing (1) with Mdx + Ndy = 0 we have $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ (1) is not exact equation. Let x^hy^k be an I.F., of (1). [Here we are using Rule 5] so we can multiply it in (1)

$$(x^{h+1}y^{k+4} + 3y^{k+1}x^h)\,dx + (2x^{2+h}y^{3+k} + 5x^{1+h}y^k)\,dy = 0 \qquad ...(2)$$

Now it is an exact equation

$$\frac{\partial M}{\partial y} = x^{h+1}(k+4)\,y^{k+3} + 3x^h(k+1)\,y^k$$

$$\frac{\partial N}{\partial x} = 2x^{h+1}(h+2)\,y^{k+3} + 5(1+h)\,x^hy^k$$

Since, (2) is an exact eqn so we have $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Comparing coefficients of x and y

$$(k + 4) = 2(h + 2) \Rightarrow k - 2h = 0]$$

$$3(k + 1) = 5(h + 1) \Rightarrow 3k - 5h = 0$$

By solving, we get h = 2, k = 4

So, $\qquad$ I.F. $= x^2y^4$

Putting value of h and k in eqn. (2)

$$(x^3y^8 + 3y^5x^2)\,dx + (2x^4y^7 + 5x^3y^4)\,dy = 0 \qquad ...(3)$$

Now, eqn. (3) is an exact eqn. so its solution is

$$\int M\cdot dx + \int (\text{those terms of N which are free from x})\,dy = c$$

$$\int_{y\,\text{const}} (x^3y^8 + 3y^5x^2)\,dx + \int 0\cdot dy = c$$

$$\frac{x^4}{4}y^8 + 3y^5\frac{x^3}{3} = c$$

$$\frac{x^4}{4}y^8 + x^3y^5 = c$$

The equations of the form $\dfrac{dy}{dx} = \dfrac{ax + by + c}{a'x + b'y + c'}$ ...(1)

can be reduced to the homogeneous form as follows :

Case I. When $\dfrac{\mathbf{a}}{\mathbf{a'}} \neq \dfrac{\mathbf{b}}{\mathbf{b'}}$

Putting $x = X + h,\ y = Y + k,$ (h, k being constants)

so that $dx = dX,\ dy = dY,$ (1) becomes

$$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')} \quad ...(2)$$

Choose h, k so that (2) may become homogeneous.

Put $ah + bk + c = 0,$ and $a'h + b'k + c' = 0$

so that
$$\frac{h}{bc' - b'c} = \frac{k}{ca' - c'a} = \frac{1}{ab' - ba'}$$

or
$$h = \frac{bc' - b'c}{ab' - b'a},\ k = \frac{ca' - c'a}{ab' - ba'} \quad ...(3)$$

Thus when $ab' - ba' \neq 0$, (2) becomes $\dfrac{dY}{dX} = \dfrac{aX + bY}{a'X + b'Y}$ which is homogeneous in X, Y and can be solved by putting $Y = vX$.

Example 11.13. *Solve* $\dfrac{dy}{dx} = \dfrac{y + x - 2}{y - x - 4}$.

Solution. Given equation is $\dfrac{dy}{dx} = \dfrac{y + x - 2}{y - x - 4}$ $\left[\text{Case } \dfrac{a}{a'} \neq \dfrac{b}{b'}\right]$

Putting $x = X + h$, $y = Y + k$, (h, k being constants) so that $dx = dX$, $dy = dY$, (i) becomes

$$\frac{dY}{dX} = \frac{Y + X + (k + h - 2)}{Y - X + (k - h - 4)}$$

Put $k + h - 2 = 0$ and $k - h - 4 = 0$ so that $h = -1$, $k = 3$.

$\therefore$ (ii) becomes $\dfrac{dY}{dX} = \dfrac{Y + X}{Y - X}$ which is homogeneous in X and Y.

$\therefore$ put $Y = vX$, then $\dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

$\therefore$ (iii) becomes $\quad v + X\dfrac{dv}{dX} = \dfrac{v+1}{v-1}$ or $X\dfrac{dv}{dX} = \dfrac{v+1}{v-1} - v = \dfrac{1 + 2v - v^2}{v - 1}$

or $$\frac{v-1}{1+2v-v^2}\,dv = \frac{dX}{X}.$$

Integrating both sides, $\quad -\dfrac{1}{2}\displaystyle\int \dfrac{2 - 2v}{1 + 2v - v^2}\,dv = \int \dfrac{dX}{X} + c.$

or $$-\frac{1}{2}\log(1 + 2v - v^2) = \log X + c$$

or $$\log\left(1 + \frac{2Y}{X} - \frac{Y^2}{X^2}\right) + \log X^2 = -2c$$

or $$\log(X^2 + 2XY - Y^2) = -2c \quad \text{or} \quad X^2 + 2XY - Y^2 = e^{-2c} = c'$$

Putting $X = x - h = x + 1$, $Y = y - k = y - 3$, (iv) becomes

$$(x + 1)^2 + 2(x + 1)(y - 3) - (y - 3)^2 = c'$$

or $\quad x^2 + 2xy - y^2 - 4x + 8y - 14 = c'$ which is the required solution.

$(y+x-2)\,dx - (y-x-4)\,dy = 0.$

$M_y \quad\quad N_x$

$1 = 1$

Soln:

$$\int (y+x-2)\,dx + \int -y+4 \; dy = 0$$

$$yx + \frac{x^2}{2} - 2x - \frac{y^2}{2} + 4y = c$$

$$x^2 + 2xy - y^2 + 8y - 4x = c$$

Case II. When $\mathbf{\frac{a}{a'} = \frac{b}{b'}}$.

i.e., $ab' - b'a = 0$, the above method fails as h and k become infinite or indeterminate.

Now $$\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m} \text{ (say)}$$

$\therefore$ $$a' = am,\ b' = bm \text{ and (1) becomes}$$

$$\frac{dy}{dx} = \frac{(ax+by)+c}{m(ax+by)+c'} \qquad \ldots$$

Put $ax + by = t$, so that $a + b\dfrac{dy}{dx} = \dfrac{dt}{dx}$

or $$\frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right) \quad \therefore \quad \text{(4) becomes } \frac{1}{b}\left(\frac{dt}{dx} - a\right) = \frac{t+c}{mt+c'}$$

or $$\frac{dt}{dx} = a + \frac{bt+bc}{mt+c'} = \frac{(am+b)t + ac' + bc}{mt+c'}$$

so that the variables are separable. In this solution, putting $t = ax + by$, we get the required solution of (1).

Example 11.14. *Solve* $(3y + 2x + 4)\, dx - (4x + 6y + 5)\, dy = 0.$ (V.

Solution. Given equation is $\dfrac{dy}{dx} = \dfrac{(2x + 3y) + 4}{2(2x + 3y) + 5}$

Putting $2x + 3y = t$ so that $2 + 3\dfrac{dy}{dx} = \dfrac{dt}{dx}$ $\therefore$ (*i*) becomes $\dfrac{1}{3}\left(\dfrac{dt}{dx} - 2\right) = \dfrac{t + 4}{2t + 5}$

or
$$\frac{dt}{dx} = 2 + \frac{3t + 12}{2t + 5} = \frac{7t + 22}{2t + 5} \quad \text{or} \quad \frac{2t + 5}{7t + 22}\, dt = dx$$

Integrating both sides, $\displaystyle\int \frac{2t + 5}{7t + 22}\, dt = \int dx + c$

or
$$\int \left(\frac{2}{7} - \frac{9}{7} \cdot \frac{1}{7t + 22}\right) dt = x + c \quad \text{or} \quad \frac{2}{7}t - \frac{9}{49} \log (7t + 22) = x + c$$

Putting $t = 2x + 3y$, we have $14(2x + 3y) - 9 \log (14x + 21y + 22) = 49x + 49c$

or $21x - 42y + 9 \log (14x + 21y + 22) = c'$ which is the required solution.

Example 11.16. *Solve* $\left(\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} - \dfrac{y}{\sqrt{x}}\right)\dfrac{dx}{dy} = 1.$

Solution. Given equation can be written as $\dfrac{dy}{dx} + \dfrac{y}{\sqrt{x}} = \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$$\therefore \qquad \text{I.F.} = e^{\int x^{1/2}\, dx} = e^{2\sqrt{x}}$$

Thus solution of (*i*) is y (I.F.) $= \displaystyle\int \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$ (I.F.) $dx + c$

or $$ye^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \cdot e^{2\sqrt{x}}\, dx + c$$

or $$ye^{2\sqrt{x}} = \int x^{-1/2}\, dx + c \quad \text{or} \quad ye^{2\sqrt{x}} = 2\sqrt{x} + c.$$

Example 11.17. *Solve* $3x(1-x^2)\,y^2\,\dfrac{dy}{dx}+(2x^2-1)\,y^3=ax^3$ *(Rajasthan, 200*

Solution. Putting $y^3 = z$ and $3y^2\dfrac{dy}{dx}=\dfrac{dz}{dx}$, the given equation becomes

$$x(1-x^2)\frac{dz}{dx}+(2x^2-1)\,z=ax^3, \quad \text{or} \quad \frac{dz}{dx}+\frac{2x^2-1}{x-x^3}\,z=\frac{ax^3}{x-x^3}$$

which is Leibnitz's equation in z

$$\therefore \qquad \text{I.F.} = \exp\left(\int \frac{2x^2-1}{x-x^3}\,dx\right)$$

Now, $\displaystyle\int \frac{2x^2-1}{x-x^3}\,dx=\int\left(-\frac{1}{x}-\frac{1}{2}\,\frac{1}{1+x}+\frac{1}{2}\cdot\frac{1}{1-x}\right)dx=-\log x-\frac{1}{2}\log(1+x)-\frac{1}{2}\log(1-x)$

$$= -\log\,[x\sqrt{(1-x^2)}]$$

$$\therefore \qquad \text{I.F.} = e^{-\log[x\sqrt{(1-x^2)}]}=[x\sqrt{(1-x^2)}]^{-1}$$

Thus the solution of (i) is

$$z(\text{I.F.})=\int \frac{ax^3}{x-x^3}\,(\text{I.F.})\,dx+c$$

or
$$\frac{z}{[x\sqrt{(1-x^2)}]}=a\int\frac{x^3}{x(1-x^2)}\cdot\frac{1}{x\sqrt{(1-x^2)}}\,dx+c=a\int x(1-x^2)^{-3/2}\,dx$$

$$=-\frac{a}{2}\int(-2x)(1-x^2)^{-3/2}\,dx+c=a\,(1-x^2)^{-1/2}+c$$

Hence the solution of the given equation is

$$y^3=ax+cx\,\sqrt{(1-x^2)}\,. \qquad [\because \ z=$$

Example 11.20. *Solve $r \sin \theta \, d\theta + (r^3 - 2r^2 \cos \theta + \cos \theta) \, dr = 0$.*

Solution. Given equation can be rewritten as

$$\sin \theta \frac{d\theta}{dr} + \frac{1}{r}(1 - 2r^2) \cos \theta = - r^2$$

Put $\cos \theta = y$ so that $- \sin \theta \, d\theta/dr = dy/dr$

Then (i) becomes $-\dfrac{dy}{dr} + \left(\dfrac{1}{r} - 2r\right) y = - r^2$ or $\dfrac{dy}{dr} + \left(2r - \dfrac{1}{r}\right) y = r^2$

which is a Leibnitz's equation $\therefore$ I.F. $= e^{\int (2r - 1/r) dr} = e^{r^2 - \log r} = \dfrac{1}{r} e^{r^2}$

Thus its solution is $y \left(\dfrac{1}{r} e^{r^2}\right) = \int r^2 \, . \, e^{r^2} \, . \dfrac{1}{r} \, dr + c$

or
$$y e^{r^2} / r = \frac{1}{2} \int e^{r^2} 2r \, dr + c = \frac{1}{2} e^{r^2} + c$$

or
$$2 e^{r^2} \cos \theta = r e^{r^2} + 2 cr \quad \text{or} \quad r (1 + 2 ce^{-r^2}) = 2 \cos \theta.$$

10/04/20

PROBABILITY & STATISTICS

Rolling a Die: $\longrightarrow$ Experiment

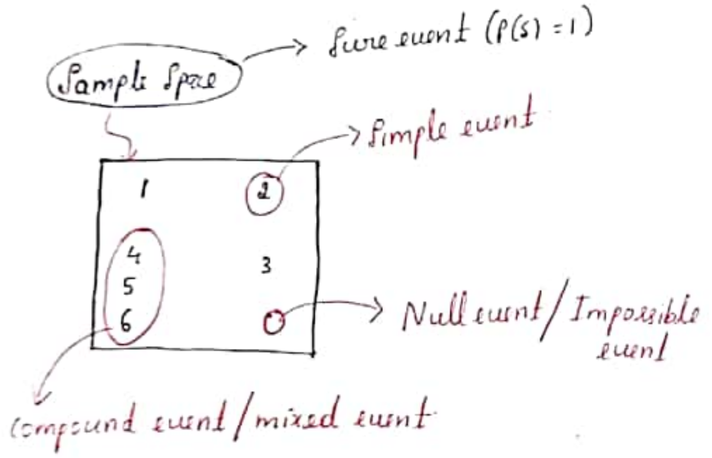
$\{1, 2, 3, 4, 5, 6\}$ $\longrightarrow$ All possible outcomes of an experiment = Sample Space = S

Event : E = Even numbers on die = $\{2, 4, 6\}$

$n(S) = 6$, $n(E) = 3$

$P[\text{Even no}] = \dfrac{n(E)}{n(S)} = \dfrac{3}{6} = \dfrac{1}{2}$

Note Basic Definition of Probability : $\boxed{\text{Probability} = \dfrac{\text{Required}}{\text{Total}}}$



Null event: It is an event whose outcomes are not included in the sample space.

Simple event: It is an event in which only one outcome is included in the sample space

Compound event: It is an event in which multiple outcomes are included in the sample space.

Sure event. It is an event in which the outcomes includes the whole sample space

Equally likely events:

Let experiment = Rolling of a die

A : Odd number ; B: Even number | A & B are events

Here A & B are said to be equally likely events since either event A or event B can occur while rolling a die ~~with~~ with equal chance of occurance for both events.

Here $P(A) = \frac{3}{6} = 1/2$ & $P(B) = 1/2$

i.e. $\boxed{P(A) = P(B)}$

Note * If A & B are equally likely events then $P(A) = P(B)$ but the converse is not true

eg. Let P: Event of occurance of 1 or 2

Q: Event of occurance of 3 or 4

Here $P(P) = \frac{2}{6} = 1/3$ & $P(Q) = \frac{2}{6} = 1/3$

i.e $P(P) = P(Q)$ But the events are not equally likely because while rolling a die the number 5 or 6 is also included in the sample space

There fore If A & B are equally likely events then it must satisfy 2 conditions : ① $P(A) = P(B)$

② $A \cup B = S$

Mutually Exhaustive Events:

Let A: Even numbers ; B: Odd numbers.

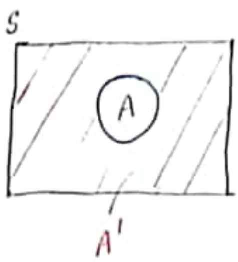
Here A & B are said to be mutually exhaustive events because only one of A or B is possible & $A \cup B = S$

i.e While rolling a die we can have either even number or odd number as output & both A & B events have no common elements & their union will be a samplespace

Note * If A & B are mutually exhaustive events then, $\boxed{P(A \cup B) = 1}$

Complement Events

$(A' / \bar{A} / A^c)$



A' = Complement of A

$A' = S - A$

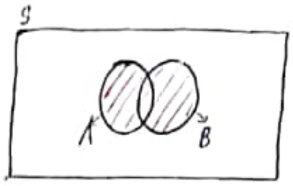
eg: E: Even nos & O: Odd numbers

$S - E = O$ & $S - O = E$

$P(E) + P(O) = 1$

Note * $\boxed{P(A) + P(A') = 1}$

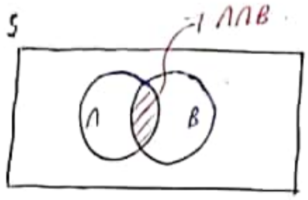
→ **Union of Events:**



$A \cup B \Rightarrow$ Either A or B or Both

Note * $\{P(A \cup B) = P(A) + P(B) - P(A \cap B)\}$

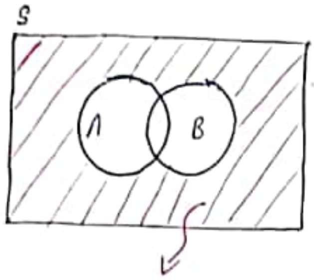
Intersection of Events:



Event A has occured.
Event B has occured

$A \cap B \Rightarrow$ Event which contains common elements of A & B

→



Neither A nor B

Neither A nor B = $(A \cup B)'$

Note * $\{P(\bar{A} \cap \bar{B}) = P(A \cup B)' = 1 - P(A \cup B)\}$

→ Note * $\cap$ → and → Multiplication
* $\cup$ → or → Addition

Rules of Probability:

1) $0 \leq P \leq 1$

2) $P(E) + P(E') = 1$

3) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

4) $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$

Mutually Exclusive Event: * Events cannot happen together

* $A \cap B = \phi \Rightarrow P(A \cap B) = 0$

Independent Events:

* A & B can happen together
* Happening of A & B won't affect the probability of each other
* $P(A \cap B) = P(A) \cdot P(B)$

eg: Let a bag contains 2 balls of Blue & Green colour.
Let the experiment be picking up of 2 balls with replacement
Let E_1: First ball is picked & it is of Blue colour
E_2: Second ball is picked & it is of Green colour

Here since we are replacing the ball after event E_1 then event E_2 has nothing to do with event E_1. Hence we can say event E_1 & E_2 are independent events.

Case ②: In the same bag, let the experiment be picking up of 2 balls without replacement
Let E_1: Picking Blue Ball in first attempt.
E_2: Picking Green Ball in Second attempt.

Here both events E_1 & E_2 are dependent on each other. Infact event E_2 is completely dependent on E_1. If event E_1 does not occur i.e. if Green Ball is picked in first attempt then Event E_2 can never occur. Here E_1 & E_2 are Dependent events.

Conditional Probability: It is the probability of event E_1 when event E_2 has already occured & E_1 & E_2 are dependent events.

Mathematically $\boxed{P(E_1/E_2) = \dfrac{P(E_1 \cap E_2)}{P(E_2)}}$

Q) : We have 2 events E_1 & E_2 in a probability space satisfying the following constraints $P(E_1) = P(E_2)$, $P(E_1 \cup E_2) = 1$, E_1 & E_2 are independent then $P(E_1) =$ ______

a) 0   b) 1/4   c) 1/2   d) 1

Soln:

$$P(E_1 \cup E_2) = 1 \Rightarrow P(E_1) + P(E_2) - \cancel{P(E_1)} \, P(E_1 \cap E_2) = 1$$

$$\Rightarrow P(E_1) + P(E_2) - P(E_1) \cdot P(E_2) = 1$$

[$\because E_1$ & E_2 are independent events]

$$\Rightarrow 2P(E_1) - P(E_1)^2 = 1$$

$$\Rightarrow P(E_1)^2 - 2P(E_1) + 1 = 0$$

$$\Rightarrow \underline{P(E_1) = 1} \Rightarrow \text{option d}$$

Q) X & Y are independent events $P(X) = 0.4$ & $P(X \cup Y^c) = 0.7$, $P(X \cup Y) = ?$

Soln:

$$P(X \cup Y') = 0.7 \Rightarrow P(X) + P(Y') - P(X \cap Y') = 0.7.$$

$$\Rightarrow P(X) + \cancel{P(Y)} \, 1 - P(Y) - P(X) \cdot (1 - P(Y)) = 0.7$$

$$\Rightarrow \cancel{P(X)} + P(X)P(Y) + 1 - P(Y) = 0.7.$$

$$\Rightarrow P(Y) = \frac{0.7 - 1}{0.4 - 1} = \frac{-0.3}{-0.6} = \frac{1}{2}$$

$$\therefore P(X \cup Y) = P(X) + P(Y) - P(X) \cdot P(Y) = 0.4 + 0.5 - (0.4 \times 0.5)$$

$$\Rightarrow \underline{P(X \cup Y) = 0.7} \Rightarrow \text{option a}$$

a) 0.7   b) 0.5   c) 0.4   d) 0.3

Coin Experiment:

1 Coin toss. $S = \{H, T\} \Rightarrow n(S) = 2 = 2^1$

2 Coin toss: $S = \{HH, HT, TH, TT\} \Rightarrow n(S) = 4 = 2^2$

3 Coin toss: $S = \{HHH, HTH, THH, TTH, HHT, HTT, THT, TTT\} \Rightarrow n(S) = 8 = 2^3$

4 coin toss: $\{$ HH HH   HT HH   TH HH   TT HH

HH HT   HT HT   TH HT   TT HT $\Rightarrow n(S) = 2^4 = 16$

HH TH   HT TH   TH TH   TT TH

HH TT   HT TT   TH TT   TT TT $\}_{4\times4}$

Q) A coin is tossed 4 times. What is probability of getting heads exactly 3 times?

Soln $n(S) = 2^4 = 16$, E = $\{$ HHHT HHTH HTHH THHH $\}$

$n(E) = 4$

$\therefore$ Probability of E = $P(E) = \frac{n(E)}{n(S)} = \frac{4}{16} = \frac{1}{4}$

Q) 3 coins are tossed simultaneously. Probability of getting atleast one head is?

Soln P(atleast 1 head) = 1 − P(no head) = $1 - \frac{1}{8}$

$\Rightarrow$ P(atleast 1 head) = $7/8$

Die Experiment:

Rolling 1 die. $S = \{1, 2, 3, 4, 5, 6\}$ $\Rightarrow n(S) = 6$

Rolling 2 die. S = $\{$

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

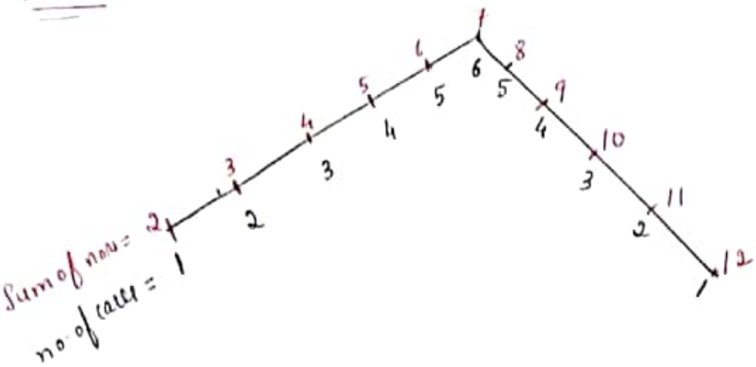
$\}$ $\Rightarrow n(S) = 6^2 = 36$

Q) Two dice are thrown & what is the probability that sum of numbers on 2 dice is 8?

Soln E = $\{$ 26 62 35 53 44 $\}$ $\Rightarrow n(E) = 5$

$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{5}{36}$

Note TRICK



Q) 2 dice are thrown. P(sum of no. of 2 dice is neither 8 nor 9) = ?

Soln $P(\text{Required}) = 1 - P(\text{sum is } 8 \text{ or } 9)$

$= 1 - \frac{9}{36}$

$\Rightarrow P(\text{Required}) = \frac{27}{36} = \frac{3}{4}$

13/04/20 Q)

Permutation & Combination Formulae:

1) ${}^nP_r = \frac{n!}{(n-r)!}$ [without repeatition]

2) ${}^nP_r = n^r$ [with repeatition]

3) ${}^nC_r = \frac{n!}{r!\,(n-r)!}$ [without repeatition]

4) ${}^nC_r = \frac{(n+r-1)!}{r!\,(n-1)!}$ [with repeatition]

5) ${}^nP_n = n!$ ${}^nC_0 = {}^nC_n = 1$

${}^nP_0 = 1$ ${}^nC_1 = n$

${}^nP_1 = n$ ${}^nC_r = \frac{{}^nP_r}{r!}$

Q) A box contains 5 black & 5 red balls. Two balls are randomly
#① picked one after another from the box, without replacement. The probability for both balls being red is :

a) 1/90 b) 1/2 c) 19/90 d) 2/9

Soln: <u>Without replacement</u>:

A : First Ball Red
B : Second Ball Red

$$P(\text{Both balls red}) = P(A \cap B)$$
$$= P(A) \cdot P(B/A)$$
$$= \frac{5}{10} \times \frac{4}{9}$$
$$= 2/9 \Rightarrow \text{option d.}$$

Note: $\boxed{P(A \cap B) = P(A) \cdot P(B/A)}$

<u>Scenario</u>: <u>With replacement</u>:

A & B are independent events $\Rightarrow P(A \cap B) = P(A) \cdot P(B)$
$$= \frac{5}{10} \cdot \frac{5}{10}$$
$$\Rightarrow P(A \cap B) = \frac{1}{4}.$$

Q) A box contains 4 red balls & 6 black balls. Three balls are selected
#② randomly from the box one after another, without replacement. The probability that the selected set contains one red ball & two blacks is

a) 1/20 b) 1/12 c) 3/10 d) 1/2

Soln: <u>Without replacement</u>:

$$P(1R \,\&\, 2B) = P[RBB] + P[BRB] + P[BBR].$$
$$= 3 \times P[RBB] \qquad [\because P[RBB] = P[BRB] = P[BBR]]$$
$$= 3 \times \frac{\cancel{4}}{\cancel{10}\,2} \times \frac{\cancel{6}^{\,3}}{9} \times \frac{\cancel{5}}{\cancel{8}\,2}$$
$$\Rightarrow P(1R \,\&\, 2B) = 1/2$$

Q) A box contains 10 screws, 3 of which are defective. Two screws are drawn at random with replacement. The probability that none of the two screws is defective will be

a) 100%   b) 50%

c) 49%   d) None of these

Soln With replacement: P(None of the two screws defective) $= \frac{7}{10} \times \frac{7}{10} = \underline{0.49} = \underline{49\%}$

⇓

option c

Scenario: 1 Non Defective & 1 Defective

$P(1 \text{ Non Defective } \& \ 1 \text{ Defective}) = \left[\frac{7}{10} \times \frac{3}{10}\right] + \left[\frac{3}{10} \times \frac{7}{10}\right]$

$= 2 \times \frac{21}{100}$

$= \frac{42}{100}$

Note If the position is not specified then we must go for the combination.

Combination Trick: [Valid for without replacement]***

Consider #①:

Soln $P(\text{Both Balls Red}) = \frac{{}^5C_2}{{}^{10}C_2} = \frac{5 \times 4}{(2 \times 1) \times \frac{(10 \times 9)}{(2 \times 1)}} = 2/9$

Consider #②.

Soln $P(1R \ \& \ 2B) = \frac{{}^4C_1 \times {}^6C_2}{{}^{10}C_3} = \frac{4 \times 6 \times 5 \times 3 \times 2}{2 \times 1 \times 10 \times 9 \times 8} = 1/2$

Q) If $P(X) = 1/4$, $P(Y) = 1/3$ & $P(X \cap Y) = 1/12$, the value of $P(Y/X)$ is

a) 1/4   b) 4/25   c) 1/3   d) 29/50

Soln $P(X \cap Y) = P(X) \cdot P(Y/X)$

$\Rightarrow P(Y/X) = \frac{1/12}{1/4} = \underline{1/3}$ ⇒ option c.

Scanned with CamScanner

Q) The chance of a student passing an exam is 20%. The chance of a student passing the exam & getting above 90% in it is 5%. Given that a student passes the examination, the probability that the student gets above 90% marks is

a) 1/18 b) 1/4 c) 2/9 d) 5/18

Soln X : Student passes the exam.
Y : Student gets above 90%.

$P(X \cap Y) = 5/100$

$P(X) = 20/100$

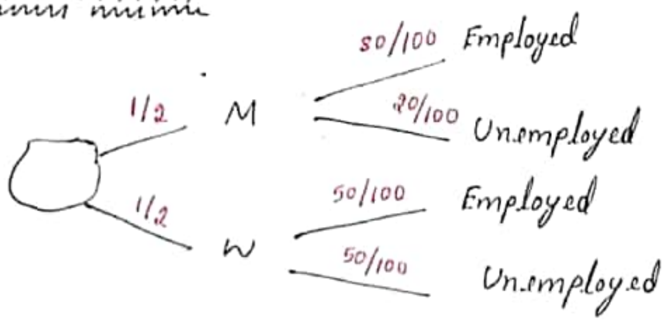
$P(X \cap Y) = P(X) \cdot P(Y/X)$

$\Rightarrow P(Y/X) = \dfrac{5/100}{20/100} = \underline{\underline{1/4}} \Rightarrow$ option b

Q) A group consists of equal number of men & women. Of this group 20% of the men & 50% of the women are unemployed. If a person is selected at random from this group, the probability of the selected person being employed is ______.

Soln $P(M) = P(W) = 1/2$ [∵ M & W are equally likely events]

BAYES THEOREM



~~P[Selected person being employed]~~ $\cancel{= \frac{1}{2} \times \frac{80}{100} + \frac{1}{2} \times \frac{50}{100}}$

$P[\text{Selected person being employed}] = \frac{1}{2} \times \frac{80}{100} + \frac{1}{2} \times \frac{50}{100}$

$= \frac{4}{10} + \frac{2.5}{10}$

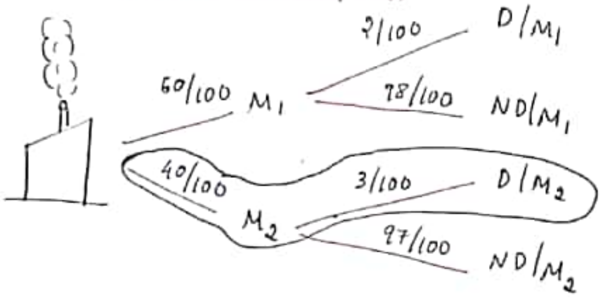
$= \underline{\underline{0.65}}$

Q) In a factory, two machines M_1 & M_2 manufacture 60% & 40% of the auto components resp.ly. Out of the total production, 2% of M_1 & 3% of M_2 are found to be defective. If a randomly drawn auto component from the combined lot is found defective, what is the probability that it was manufactured by M_2?

a) 0.35   b) 0.45   c) 0.5   d) 0.4

Soln:

BAYE'S THEOREM



$$P(M_2/D) = \frac{P(M_2)\cdot P(D/M_2)}{P(M_2)\cdot P(D/M_2) + P(M_1)\cdot P(D/M_1)}$$

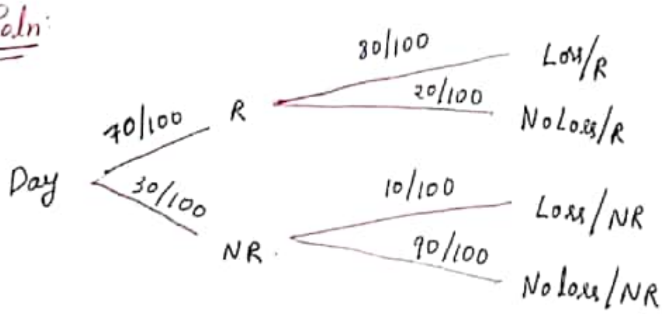
$$= \frac{\frac{40}{100}\cdot\frac{3}{100}}{\frac{40}{100}\cdot\frac{3}{100} + \frac{60}{100}\cdot\frac{2}{100}}$$

$$\Rightarrow P(M_2/D) = 1/2 \Rightarrow \text{option c}$$

Q) In a given day in the rainy season, it may rain 70% of the time. If it rains, chance that a village fair will make a loss on that day is 80%. However, if it does not rain, chance that the fair will make a loss on that day is only 10%. If the fair has not made a loss on a given day in the rainy season, what is the probability that it has not rained on that day?

a) 3/10   b) 9/11   c) 14/17   d) 27/41

Soln:



$$P(NR/\text{No Loss}) = \frac{30/100 \cdot 90/100}{\frac{30}{100}\cdot\frac{90}{100} + \frac{70}{100}\cdot\frac{20}{100}}$$

$$= \frac{3\times 9}{3\times 9 + 7\times 2}$$

$$\Rightarrow P(NR/\text{No loss}) = 27/41 \Rightarrow \text{option d}$$

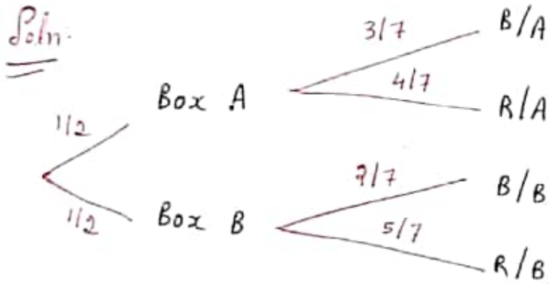
BAYE'S THEOREM

$$P(A/E) = \frac{P(E/A)\cdot P(A)}{P(E/A)\,P(A) + P(E/B)\cdot P(B)}$$

Q) A box contains three blue balls & four red balls. Another identical box contains two blue balls & five red balls. One ball is picked at random from one of the two boxes & it is red. The probability that it came from the first box is

a) $2/3$ b) $4/9$ c) $4/7$ d) $2/7$

Soln.



$$P(A/R) = \frac{1/2 \times 4/7}{1/2 \times 4/7 + 1/2 \times 5/7}$$

$$\Rightarrow P(A/R) = \frac{4}{9} \Rightarrow \text{option b}$$

Q) $P(E) \rightarrow$ Probability of event E

$P(A) = 1$, $P(B) = 1/2$, $P(A/B) = ?$

Soln

$$P(A\cap B) = P(A/B)\cdot P(B) \Rightarrow \cancel{P(B)\neq}\; P(A/B) = \frac{P(A\cap B)}{P(B)}$$

Given $P(A) = 1 \Rightarrow$ A is sure event

$\Rightarrow$ A & B are independent events

$\Rightarrow P(A\cap B) = P(A)\cdot P(B)$

$$\therefore P(A/B) = \frac{P(A)\cdot \cancel{P(B)}}{\cancel{P(B)}} = 1$$

Note * If A & B are independent events then $P(A/B) = P(A)$ & $P(B/A) = P(B)$

14/04/20 Q) Consider two events E_1 & E_2 such that $P(E_1) = 1/2$, $P(E_2) = 1/3$ & $P(E_1 \cap E_2) = 1/5$. Which of the foll. statements is true?

a) $P(E_1 \cup E_2) = 2/3$ b) E_1 & E_2 are independent

c) E_1 & E_2 are not independent d) $P(E_1/E_2) = 4/5$

Soln

$$P(E_1)\cdot P(E_2) = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6} \neq P(E_1 \cap E_2)$$

$\therefore E_1$ & E_2 are not independent $\Rightarrow$ option c

Q) Suppose A & B are two independent events with probabilities $P(A) \neq 0$. Let $\bar{A}$ & $\bar{B}$ be their complements. Which one of the foll. statements is FALSE?

a) $P(A \cap B) = P(A)\cdot P(B)$ b) $P(A/B) = P(A)$

c) $P(A \cup B) = P(A) + P(B)$ d) $P(\bar{A} \cap \bar{B}) = P(\bar{A})\, P(\bar{B})$

Soln: A & B are two independent events

$\therefore P(A \cup B) = P(A) + P(B) - P(A)\cdot P(B)$ $\Rightarrow$ option c

Q) An examination consists of two papers, Paper1 & Paper2. The probability of failing in Paper1 is 0.3 & that in Paper 2 is 0.2. Given that student has failed in Paper 2, the probability of failing in Paper1 is 0.6. The probability of a student failing in both the paper is a) 0.5 b) 0.18 c) 0.12 d) 0.06

Soln: $P(P1) = 0.3$, $P(P_2) = 0.2$, $P(P1/P2) = 0.6$

$$P(P_1 \cap P_2) = P(P_2)\cdot P(P_1/P_2) = \frac{2}{10}\times\frac{6}{10} = 0.12$$ $\Rightarrow$ option c

Q) The probability that a communication system will have a high fidelity is 0.81. The probability that the system will have both high fidelity & high selectivity is 0.18. The probability that a given system with high fidelity will have high selectivity is

a) 0.181 b) 0.191 c) 0.222 d) 0.826

Soln $P(F) = 0.81$, ~~$P(S)$~~ $P(F\cap S) = 0.18$, $P(S/F) = ?$

$$P(S/F) = \frac{P(F\cap S)}{P(F)} = \frac{0.18}{0.81} = 0.222 \Rightarrow \text{option c}$$

Q) A fair coin is tossed independently four times. The probability of the event "the number of times heads show up is more than the number of times tails show up" is

a) 1/16 b) 1/8 c) 1/4 d) 5/16

Soln A : Head > Tail when coin is tossed 4 times

[A = { HHHH, HHHT, HTHH, HHTH, THHH }]

$\therefore P(A) = \frac{5}{16}$ $\Rightarrow$ option d

Q) A fair dice is rolled twice. The probability that an odd number will follow an even number is a) 1/2 b) 1/6 c) 1/3 d) 1/4

Soln: M1 $A = \{ (2,1)\ (2,3)\ (2,5)\ (4,1)\ (4,3)\ (4,5)\ (6,1)\ (6,3)\ (6,5) \}$

$\therefore P(A) = 9/36 = 1/4$ $\Rightarrow$ option d

M2 E : Getting even no. O : Getting odd no. , E & O are independent events.

$\therefore$ P(Odd follows even) $= P(E)\cdot P(O) = 3/6 \cdot 3/6 = 1/4$

Q) A fair coin is tossed three times in succession. If the first toss produces a head, then the probability of getting exactly two heads in three tosses is a) 1/8 b) 1/2 c) 3/8 d) 3/4

Soln M1 First toss is head , A : Getting 2 heads in 3 tosses

$$P(A) = 1\times\tfrac{1}{2}\times\tfrac{1}{2} + 1\times\tfrac{1}{2}\times\tfrac{1}{2} = 1/2 \Rightarrow \text{option b}$$

M2 A : 2 heads in 3 tosses , B : First toss is head

~~P(A|B)~~ $P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{2/8}{1/2} = 1/2$

Q) P & Q are considering to apply for a job. The probability that P applies for the job is $1/4$. The probability that P applies for the job given that Q applies for the job is $1/2$, & the probability that Q applies for the job given that P applies for the job is $1/3$. Then the probability that P does not apply for the job given that Q does not apply for the job is a) $4/5$ b) $5/6$ c) $7/8$ d) $11/12$

Soln:

$$P(P) = 1/4, \quad P(P/Q) = 1/2, \quad P(Q/P) = 1/3$$

$$P(P \cap Q) = P(Q/P) \cdot P(P) = 1/3 \cdot 1/4 = 1/12$$

$$P(P/Q) = \frac{P(P \cap Q)}{P(Q)} \Rightarrow P(Q) = \frac{1/12}{1/2} = 1/6$$

$$P(P \cup Q) = 1/4 + 1/6 - 1/12 = \frac{3+2-1}{12} = 1/3$$

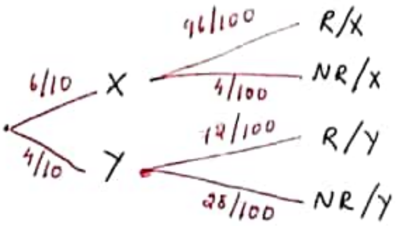
$$\therefore P(P'/Q') = \frac{P(P' \cap Q')}{P(Q')} = \frac{P(\overline{P \cup Q})}{P(\bar{Q})} = \frac{1 - P(P \cup Q)}{1 - P(Q)}$$

$$\Rightarrow P(P'/Q') = \frac{1 - 1/3}{1 - 1/6} = \frac{2/3}{5/6} = 4/5 \Rightarrow \text{option a}$$

Q) An automobile plant contracted to buy shock absorbers from two suppliers X & Y. X supplies 60% & Y supplies 40% of the shock absorbers. All shock absorbers are subjected to a quality test. The ones that pass the quality test are considered reliable. Of X's shock absorbers, 96% are reliable. Of Y's shock absorbers, 72% are reliable. The probability that a randomly chosen shock absorber, which is found to be reliable, is made by Y is

a) 0.288 b) 0.334 c) 0.667 d) 0.720.

Soln



$$P(Y/R) = \frac{4/10 \cdot 72/100}{4/10 \cdot 72/100 + 6/10 \cdot 96/100}$$

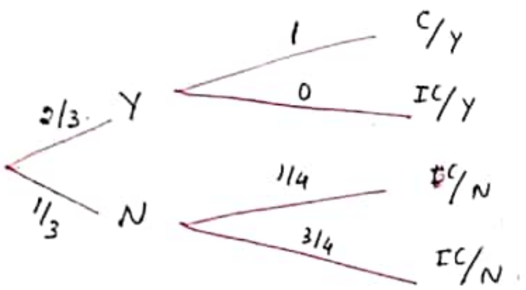
$$\Rightarrow P(Y/R) = 0.334 \Rightarrow \text{option b}$$

Q) The probability that a student knows the correct answer
* to a multiple choice question is $2/3$. If the student does not know the answer, then the student guesses the answer. The probability of the guessed answer being correct is $1/4$. Given that the student has answered the question correctly, the conditional probability that the student knows the correct answer is

a) $2/3$ b) $3/4$ c) $5/6$ d) $8/9$

Soln: Y : Knows the answer C: Correct answer

N : Does not know the answer IC: Incorrect answer



$$P(Y/C) = \frac{2/3 \times 1}{2/3 \times 1 + 1/3 \times 1/4}$$

$$= \frac{2/3}{\frac{8+1}{12}} = \frac{24}{27}$$

$\Rightarrow$ $P(Y/C) = 8/9$ $\Rightarrow$ option d

Q) From a pack of regular playing cards, two cards are drawn at random. What is the probability that both cards will be kings, if first card is NOT replaced?

a) $1/26$ b) $1/52$ c) $1/169$ d) $1/221$

Soln $$P(\text{Required}) = \frac{^4C_2}{^{52}C_2} = \frac{4 \times 3}{52 \times 51} = 1/221$$ $\Rightarrow$ option d

Q) There are 25 calculators in a box. Two of them are defective. Suppose 5 calculators are randomly picked for inspection (i.e., each has the same chance of being selected), what is the probability that only one of the defective calculators will be included in the inspection?

a) 1/2   b) 1/3   c) 1/4   d) 1/5

Soln

$$P(\text{only 1 defective}) = \frac{{}^{2}C_1 \cdot {}^{23}C_4}{{}^{25}C_5} = \frac{2\times \cancel{23\times22\times21}\times 20/4!}{25\times24\times\cancel{23\times22\times21}/5\times4!}$$

$$\Rightarrow P(\text{only 1 defective}) = \underline{\underline{1/3}} \Rightarrow \text{option b}$$

HW

Q) Three cards were drawn from a pack of 52 cards. The probability that they are a king, queen & a jack is

a) $\frac{16}{5525}$ b) $\frac{64}{2197}$ c) $\frac{3}{13}$ d) $\frac{8}{16575}$

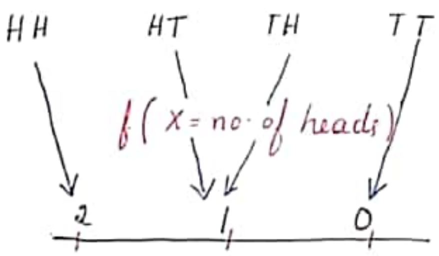
Soln.

$$P(1K, 1Q, 1J) = \frac{{}^{4}C_1 \cdot {}^{4}C_1 \cdot {}^{4}C_1}{{}^{52}C_3} = \underline{\underline{\frac{16}{5525}}} \Rightarrow \text{option a}$$

16/04/20

RANDOM VARIABLES

Exp: Tossing 2 coins



Note: X is a real no. i.e X can be -ve also
eg: (Temperature)

When X is countable → Discrete R.V

When X is in range → Continuous R.V

eg: 2 coins are tossed (Discrete Random Variable)

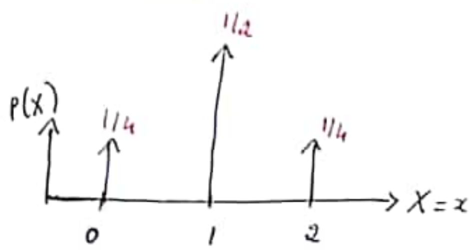
$S = \{HH, HT, TH, TT\}$

X = No. of heads $[X = x]$

X	0	1	2
P(x)	1/4	2/4	1/4

pmf/pdf

Stem Plot



pmf → probability mass function, pdf → probability density function

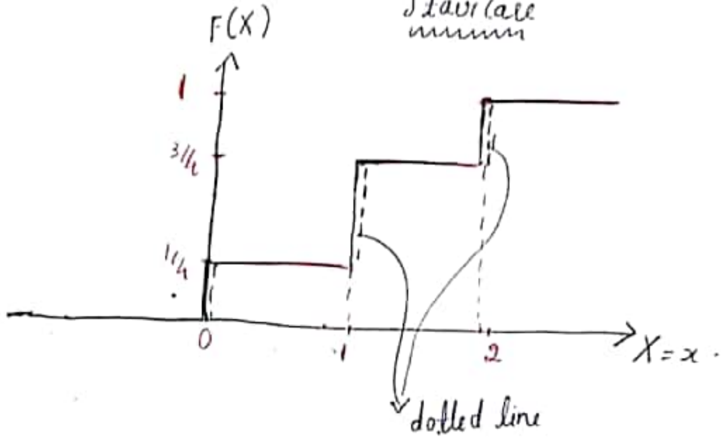
cdf → cummulative distributive function

X	0	1	2
P(x)	1/4	2/4	1/4
F(x)	1/4	3/4	1

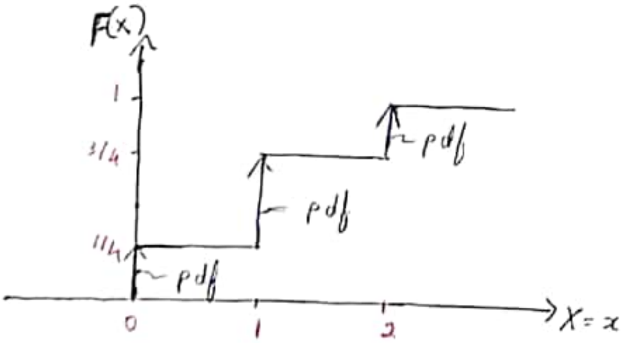
$$F(x) = \sum_{-\infty}^{k} P(x)$$

Note
* $0 \le F(x) \le 1$
* $F(x<a) = 0$
* $F(x>b) = 1$
* $\sum P(x) = 1$

Staircase



To draw pdf from cdf:



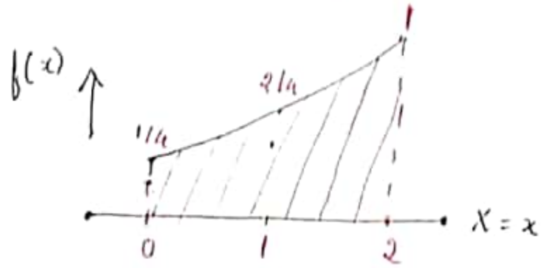
$P(X=2) = F(X=2) - F(X=1) = 1/4$

When X is a continuous Random Variable:

$\cancel{pdf = \int_{-\infty}}$

Note $\int_{-\infty}^{\infty} f(x)\,dx = 1$

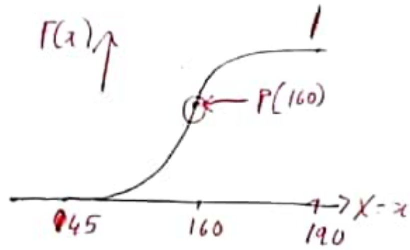
or Area under $f(x) = 1$



Scanned with CamScanner

eg: Height of Boys in Jammu

pdf curve



cdf curve

Q) $f(x) = kx^2, \quad 0 \le x \le 2$, i) $k = ?$

ii) $P(0.2 \le x \le 0.5) = ?$

iii) cdf = ?

Soln i) $\int_0^2 f(x) = 1$

$\Rightarrow \int_0^2 kx^2 = 1 \Rightarrow \frac{k}{3}[x^3]_0^2 = 1 \Rightarrow \frac{8k}{3} = 1 \Rightarrow \underline{k = 3/8}$

ii) $P(0.2 \le x \le 0.5) = \int_{0.2}^{0.5} \frac{3}{8}x^2 = \frac{3}{8} \cdot \left[\frac{x^3}{3}\right]_{0.2}^{0.5} = \underline{0.0146}$

iii) cdf, $F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_0^x 3/8\, x^2\,dx$

$\Rightarrow F(x) = \frac{3}{24} \cdot [x^3]_0^x = \frac{x^3}{8}$

ie $F(x) = \frac{x^3}{8}, \quad 0 \le x \le 2$

Q) $f(x) = 6x(1-x), \quad 0 \le x \le 1$. Find b if $P(x \le b) = P(x > b)$

Soln M1 $P(x \le b) = P(x > b)$

Solving through integration & finding soln.

M2 $P(x \le b) = P(x > b)$

$\Rightarrow P(x \le b) = 1 - P(x \le b)$

$\Rightarrow P(x \le b) = 1/2$

$\Rightarrow 6\int_0^b (x - x^2)\,dx = 1/2$

$\Rightarrow 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^b = 1/2$

$\Rightarrow 6\left[\frac{b^2}{2} - \frac{b^3}{3}\right] = 1/2$

$\Rightarrow 3b^2 - 2b^3 = 1/2$

$\Rightarrow 6b^2 - 4b^3 = 1$

$\Rightarrow \underline{b = 1/2}$ (from inspection)

7) $F(x) = 1 - e^{-x^2/b}$, $f(x) = ?$

Soln

$$f(x) = \frac{d}{dx}F(x) = -e^{-x^2/b}\left(-2x/b\right)$$

$$\Rightarrow f(x) = \frac{2x}{b}e^{-x^2/b}$$

***

8) (Splitted Domain)

$$f(x) = \begin{cases} k(1+x) & , -1 < x \leq 0 \\ k(1-x), & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

i) $k = ?$

Soln

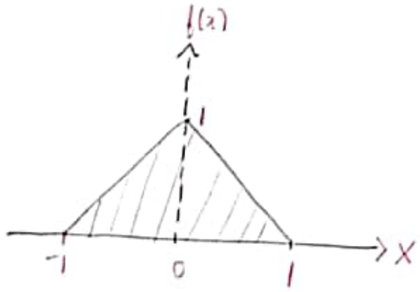
$$\int_{-\infty}^{\infty} f(x) = 1 \Rightarrow \int_{-1}^{0} k(1+x)\,dx + \int_{0}^{1} k(1-x)\,dx + 0 = 1$$

$$\Rightarrow k\left[x + \frac{x^2}{2}\right]_{-1}^{0} + k\left[x - \frac{x^2}{2}\right]_{0}^{1} = 1$$

$$\Rightarrow k\left[-(-1+1/2)\right] + k\left[1 - 1/2\right] = 1$$

$$\Rightarrow k/2 + k/2 = 1 \Rightarrow k = 1$$

ii) Plot pdf:



$$f(x) = \begin{cases} 1+x, & -1 < x \leq 0 \\ 1-x, & 0 < x < 1 \end{cases}$$

Verification:

Area $= \frac{1}{2} \times 2 \times 1 = 1 \Rightarrow$ Hence verified

iii) $F(x)$ {cdf}

$F(x < -1) = 0$

$$F(x < 0) = \int_{-\infty}^{x} f(x)\,dx = \int_{-1}^{x}(1+x)\,dx = \left[x + \frac{x^2}{2}\right]_{-1}^{x} = x + \frac{x^2}{2} + 1/2$$

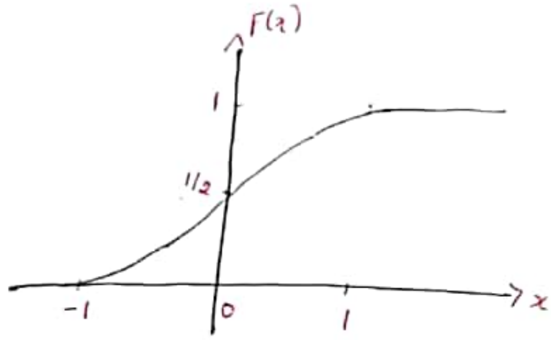
i.e $F(x<0) = x + \frac{x^2}{2} + 1/2$

1.7 . $F(x<1) = \int_{-1}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$

$\Rightarrow F(x<1) = \left[x + x^2/2\right]_{-1}^{0} + \left[x - \frac{x^2}{2}\right]_{0}^{x} = x - \frac{x^2}{2} + 1/2$

i.e $F(x<1) = x - \frac{x^2}{2} + 1/2$

$F(x>1) = 1$



HW.

Q) $f(x) = k e^{-|x|}$

i) $k = ?$
ii) $f(x) \to$ Plot
iii) $F(x) \to$ Plot

Soln: i)

$$f(x) = \begin{cases} k e^{+x}, & x < 0 \\ k e^{-x}, & x > 0 \end{cases}$$

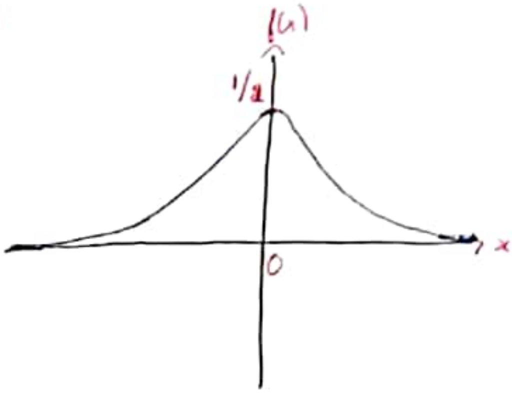
$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow k\int_{-\infty}^{0} e^{x}\,dx + k\int_{0}^{\infty} e^{-x}\,dx = 1$$

$$\Rightarrow k \cdot \left[e^{x}\right]_{-\infty}^{0} + k\left[-e^{-x}\right]_{0}^{\infty} = 1$$

$$\Rightarrow k\left[1 - 0\right] + k\left[0 - (-1)\right] = 1$$

$$\Rightarrow 2k = 1 \Rightarrow k = 1/2$$

ii)



$$f(x) = \begin{cases} 1/2\, e^{x}, & x < 0 \\ 1/2\, e^{-x}, & x > 0 \end{cases}$$

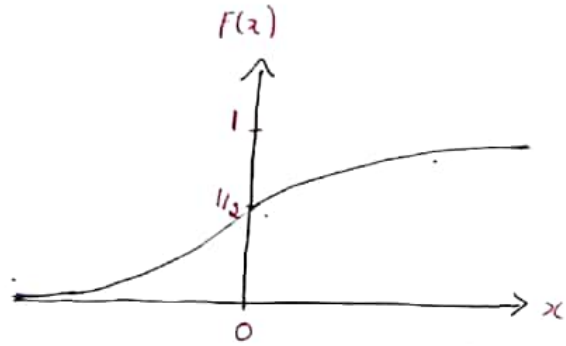
Scanned with CamScanner

iii) F(x) (cdf):

$$F(x<0) = \int_{-\infty}^{x} \tfrac{1}{2} e^{x} dx \quad \tfrac{1}{2}\left[e^{x}\right]_{-\infty}^{x} = \tfrac{1}{2} e^{x}$$

$$F(x>0) = \int_{-\infty}^{0} \tfrac{1}{2} e^{x} dx + \int_{0}^{x} \tfrac{1}{2} e^{-x} dx = \tfrac{1}{2}\left[e^{x}\right]_{-\infty}^{0} + -\tfrac{1}{2}\left[e^{-x}\right]_{0}^{x}$$

$$\Rightarrow F(x>0) = \tfrac{1}{2} - \tfrac{1}{2}\left[e^{-x} - 1\right] = -\tfrac{1}{2} e^{-x} + 1$$



HW

Q) $f(x) = \dfrac{k}{1+x^2}$, $-\infty < x < \infty$ Find k & cdf

$= 0$, otherwise

Soln:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{\infty} \frac{k}{1+x^2}\,dx = 1$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{k}{1+x^2}\,dx = 1 \Rightarrow 2k\left[\tan^{-1}x\right]_{0}^{\infty} = 1$$

$$\Rightarrow 2k\left[\pi/2 - 0\right] = 1$$

$$\Rightarrow k = 1/\pi$$

$$F(x) = \int_{-\infty}^{x} \frac{1}{\pi(1+x^2)}\,dx$$

$$= \frac{1}{\pi}\left[\tan^{-1}(x)\right]_{-\infty}^{x}$$

$$\Rightarrow F(x) = \frac{\tan^{-1}x}{\pi} - \frac{1}{\pi} \times -\frac{\pi}{2} = \frac{\tan^{-1}x}{\pi} + \frac{1}{2} \qquad -\infty < x < \infty$$

17/04/20

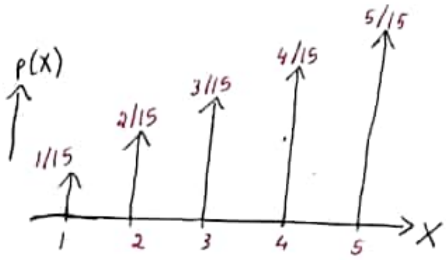
Q) $P(x) = \frac{x}{15}$, $x = 1, 2, 3, 4, 5$

i) pdf
ii) cdf
iii) plot
iv) $P(1/2 < x < 5/2)$

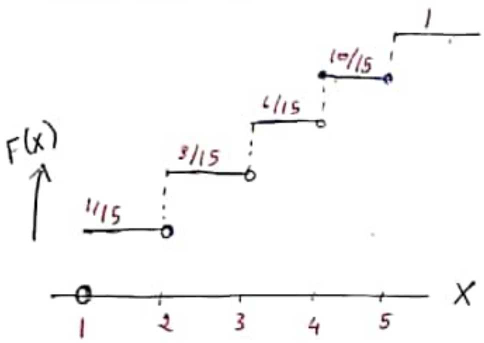
Soln: i) & ii)

X	1	2	3	4	5
P(X)	1/15	2/15	3/15	4/15	5/15
F(X)	1/15	3/15	6/15	10/15	1

iii)



iii)



iv) $P(1/2 < x < 5/2)$

$= P(1) + P(2)$

$= 1/15 + 2/15$

$= 3/15$

Q) R.V $\rightarrow$ $X = 1, 2, 3, 4$

$2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$. Find the pdf/pmf

Soln: M1 $2P(1) = 3P(2) = P(3) = 5P(4) = k$ (say)

$\therefore P(1) = k/2$, $P(2) = k/3$, $P(3) = k$, $P(4) = k/5$

$\therefore P(1) + P(2) + P(3) + P(4) = 1 \Rightarrow k\left[\frac{1}{2} + \frac{1}{3} + 1 + \frac{1}{5}\right] = 1$

$\Rightarrow k\left[\frac{15+10+30+6}{30}\right] = 1$

$\Rightarrow k = 30/61$

X	1	2	3	4
P(X)	15/61	10/61	30/61	6/61

M2 It is simpler to assume LCM times k

i.e $2P(1) = 3P(2) = P(3) = 5P(4) = 30k$.

2D Random Variable

Eg ①: $P(X_1 = x_1, X_2 = x_2) = \frac{1}{27}(x_1 + 2x_2)$, $x_1 = 0,1,2$; $x_2 = 0,1,2$ i) Pdf of x_1 ii) Pdf of x_2

[i) 2 D Discrete Random Variable]

Soln

$X_1 \backslash X_2$	0	1	2
0	0	2/27	4/27
1	1/27	3/27	5/27
2	2/27	4/27	6/27

Verification

$\Sigma P(X_1, X_2) = 1$

$\Rightarrow \frac{0+2+4+1+3+5+2+4+6}{27} = 1$

$\Rightarrow 1 = 1$

$\Rightarrow$ LHS = RHS (Verified)

M1

i) $P(X_1) = \sum_{x_2=0}^{2} P(x_1, x_2)$

$= P(x_1, 0) + P(x_1, 1) + P(x_1, 2)$

$= \frac{x_1}{27} + \frac{x_1+2}{27} + \frac{x_1+4}{27}$

$\Rightarrow P(X_1) = \frac{3x_1+6}{27} = \frac{x_1+2}{9}$

X_1	0	1	2
$P(X_1)$	2/9	3/9	4/9

Verification

LHS $= \frac{2+3+4}{9} = 1$

$\therefore$ LHS = RHS

ii) $P(X_2) = \sum_{x_1=0}^{2} P(x_1, x_2) = P(0, x_2) + P(1, x_2) + P(2, x_2)$

$\Rightarrow P(X_2) = \frac{2x_2}{27} + \frac{1+2x_2}{27} + \frac{2+2x_2}{27} = \frac{6x_2+3}{27}$

$\Rightarrow P(X_2) = \frac{1+2x_2}{9}$

X_2	0	1	2
$P(X_2)$	1/9	3/9	5/9

Verification

LHS $= \frac{1+3+5}{9} = 1$

$\therefore$ LHS = RHS

M2 From the table

$X_1 \backslash X_2$	0	1	2	$P(X_1)$
0	0	2/27	4/27	6/27
1	1/27	3/27	5/27	9/27
2	2/27	4/27	6/27	12/27
$P(X_2)$	3/27	9/27	15/27	

eg ①: $f(x,y) = \begin{cases} c(x^2+2y), & x = 0,1,2 \ ; \ y = 1,2,3,4 \\ 0, & \text{elsewhere} \end{cases}$ i) Find c.

Soln i) Finding c

$x \backslash y$	1	2	3	4	$P(x)$
0	2c	4c	6c	8c	20c
1	3c	5c	7c	9c	24c
2	6c	8c	10c	12c	36c
$P(y)$	11c	17c	23c	29c	

$\Sigma P(x,y) = 1 \Rightarrow 20c + 24c + 36c = 1 \Rightarrow c = 1/80$

ii) $P(x=2, y=3) = ?$

$P(x=2, y=3) = 10c = \frac{10}{80} = 1/8$

iii) $P(x \le 1, y > 2) = ?$

$P(x \le 1, y>2) = P(0,3) + P(0,4) + P(1,3) + P(1,4)$

$= 6c + 8c + 7c + 9c$

$= 30c$

$P(x \le 1, y > 2) = 3/8$

iv) pdf of x & pdf of y

x	0	1	2
$P(x)$	20/80	24/80	36/80

y	1	2	3	4
$P(y)$	11/80	17/80	23/80	29/80

Note

If $P(x,y) = P(x)\cdot P(y) \Rightarrow$ x & y are independent r.v

If $P(x,y) \neq P(x)\cdot P(y) \Rightarrow$ x & y are dependent r.v.

eg ③

Y \ X	1	2	3
1	1/12	1/6	0
2	0	1/9	1/5
3	1/18	1/4	2/15

i) $P(X=1/Y=2) = ?$

ii) $P(X=2/Y=2) = ?$

iii) $P(Y=3/X=3) = ?$

iv) $P(X+Y<4) = ?$

v) $P(Y \leq 2) = ?$

Soln

i) $P(X=1/Y=2) = \dfrac{P(X=1 \cap Y=2)}{P(Y=2)} = \dfrac{\cancel{1/6}\cdot 0}{P(Y=2)} = 0$

ii) $P(X=2/Y=2) = \dfrac{P(X=2 \cap Y=2)}{P(Y=2)} = \cancel{\dfrac{1/9}{1/6+1/9+1/4}} = \cancel{\dfrac{1/9}{\frac{6+4+9}{36}}} = \cancel{\dfrac{36/4}{9\times 19}}$

$\Rightarrow \cancel{P(X=2/Y=2) = 4/19}$

$\Rightarrow P(X=2/Y=2) = \dfrac{1/9}{0+1/9+1/5} = \dfrac{1/9}{\frac{14}{45}}$

$\Rightarrow P(X=2/Y=2) = 5/14$

iii) $P(Y=3/X=3) = \dfrac{P(Y=3 \cap X=3)}{P(X=3)} = \dfrac{2/15}{0+1/5+2/15} = \dfrac{2/15}{5/15}$

$\therefore P(Y=3/X=3) = 2/5$

iv) : $P(X+Y<4) = P(X=1,Y=1) + P(X=1,Y=2) + P(X=2,Y=1)$

$= \frac{1}{12} + 0 + 1/6$

$\Rightarrow P(X+Y<4) = \frac{3}{12} = 1/4$

v) $P(Y \le 2) = P(Y=1) + P(Y=2)$

$= [1/12 + 1/6 + 0] + [0 + 1/9 + 1/5]$

$= 1/4 + \frac{14}{45}$

$P(Y \le 2) = \frac{45+56}{180} = \frac{101}{180}$

ii) 2-D continuous Random Variable

eg ①: $f(x,y) = \begin{cases} x^3y^3/16, & 0 \le x \le 2 \text{ \& } 0 \le y \le 2 \\ 0, & \text{elsewhere} \end{cases}$

i) $f(x) = ?$ ii) $f(y) = ?$

Soln i) $f(x) = \int_{y=0}^{2} f(x,y)\, dy = x^3 \int_0^2 y^3/16\, dy = \frac{x^3}{16 \times 4} [y^4]_0^2 = \frac{x^3}{4}$

ii) $f(y) = \int_{x=0}^{2} f(x,y)\, dx = \frac{y^3}{16} \int_0^2 x^3\, dx = y^3/4$

Note Here $f(x)\, f(y) = \frac{x^3y^3}{16} = f(x,y) \Rightarrow x$ & y are independent rv.

eg ②: $f(x,y) = \begin{cases} x+y, & 0<x<1 \text{ \& } 0<y<1 \\ 0, & \text{elsewhere} \end{cases}$ Check whether $f(x,y)$ is pdf or not.

Soln $\boxed{\int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} f(x,y)\, dx\, dy = 1}$

$$\therefore \int_{x=0}^{1}\int_{y=0}^{1}(x+y)\,dx\,dy = \int_{x=0}^{1}\left\{x[1-0]+\tfrac{1}{2}[1-0]\right\}dx = \int_{x=0}^{1}\left(x+\tfrac{1}{2}\right)dx$$

$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{1}(x+y)\,dx\,dy = \tfrac{1}{2}[1-0]+\tfrac{1}{2}[1-0] = 1$$

$$\therefore \boxed{\int_{x=0}^{1}\int_{y=0}^{1}(x+y)\,dx\,dy = 1} \Longrightarrow f(x,y) \text{ is pdf}$$

H.W.

eg: ③: $f(x,y) = k\,xy\,e^{-(x^2+y^2)}$, $x>0$, $y>0$ — pdf

i) Find k

ii) $f(x,y) \rightarrow x$ & y are independent ??

Soln i) $$\int_{x=0}^{\infty}\int_{y=0}^{\infty} k\cdot xy\,e^{-(x^2+y^2)}\,dx\,dy = 1$$

$$\Rightarrow k\cdot\int_{x=0}^{\infty} x e^{-x^2}dx \cdot \int_{y=0}^{\infty} y e^{-y^2}dy = 1$$

$$\Rightarrow k\cdot I^2 = 1$$

$$\Rightarrow k\cdot\frac{1}{4} = 1$$

$$\Rightarrow \underline{k = 4}$$

Let $\int_{x=0}^{\infty} x e^{-x^2}dx = I$

$x^2 = t \Rightarrow x\,dx = \frac{dt}{2}$

$$\therefore I = \frac{1}{2}\int_{0}^{\infty} e^{-t}\,dt$$

$$= -\frac{1}{2}\left[e^{-t}\right]_0^{\infty}$$

$$\Rightarrow I = -\frac{1}{2}\times[0-1] = 1/2$$

ii) $$f(x) = 4\int_{y=0}^{\infty} xy\,e^{-(x^2+y^2)}dy = 4xe^{-x^2}\int_{y=0}^{\infty} y e^{-y^2}dy = 4xe^{-x^2}\cdot I$$

$$\Rightarrow f(x) = \underline{2xe^{-x^2}}$$

III^{ly} $f(y) = \underline{2ye^{-y^2}}$

$$\therefore f(x)\cdot f(y) = 4xy\,e^{-(x^2+y^2)} = f(x,y)$$

$\therefore x$ & y are independent r.v

Q. ④ $f(x,y) = \begin{cases} \frac{1}{8}(6-x-y) & , 0<x<2 \ ; \ 2<y<4 \\ 0 & , \text{otherwise} \end{cases}$

i) $P(X<1, Y<3)$ ii) $P(X<1 / Y<3)$

Soln i) $P(x<1, y<3) = \int_{x=0}^{1}\int_{y=2}^{3} \frac{1}{8}(6-x-y)\,dx\,dy$

$= \frac{1}{8}\int_{x=0}^{1}\left[6y - xy - \frac{y^2}{2}\right]_2^3 dx = \frac{1}{8}\int_{x=0}^{1}\left(6 - x - \frac{5}{2}\right)dx$

$= \frac{1}{8}\int_{x=0}^{1}\left(\frac{7}{2} - x\right)dx = \frac{1}{8}\left[\frac{7}{2} - \frac{1}{2}\right] = 3/8$

$\Rightarrow P(x<1, y<3) = 3/8$

ii) $P(y<3) = ??$

$f(y) = \int_{x=0}^{2} \frac{1}{8}(6-x-y)\,dx = \frac{1}{8}\left[6\times 2 - \frac{1}{2}[4] - y(2)\right]$

$\Rightarrow f(y) = \frac{1}{8}[12 - 2 - 2y] = \frac{10-2y}{8} = \frac{5-y}{4}$

$\therefore P(y<3) = \int_{y=2}^{3} \frac{5-y}{4}\,dy = \frac{1}{4}\left[5y - \frac{1}{2}y^2\right]_2^3$

$\Rightarrow P(y<3) = \frac{1}{4}\left[5 - \frac{5}{2}\right] = 5/8$

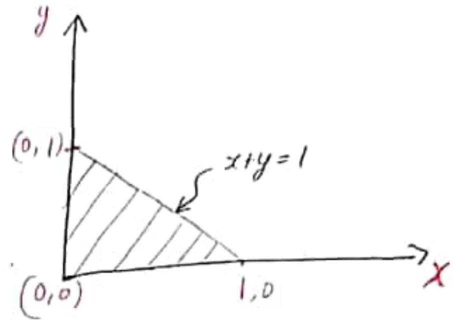
$\therefore P(x<1 / y<3) = \frac{P(x<1, y<3)}{P(y<3)} = \frac{3/8}{5/8} = 3/5$

eg. ⑤: $f(x,y) = e^{-(x+y)}$, $x \geq 0$, $y \geq 0$ $\quad P(x+y<1) = ?$

$\quad = 0$, otherwise

Soln

$$P(x+y<1) = \int_{x=0}^{1}\int_{y=0}^{1-x} e^{-(x+y)}\,dx\,dy$$



$$= \int_{x=0}^{1} e^{-x}\cdot\frac{\left[e^{-y}\right]_0^{1-x}}{(-1)}\,dx$$

$$= -\int_{x=0}^{1} e^{-x}\left[e^{-(1-x)}-1\right]dx = -\int_{x=0}^{1}\left(e^{-1}-e^{-x}\right)dx$$

$$\Rightarrow P(x+y<1) = \int_0^1 \left(e^{-x} - e^{-1}\right)dx = (-1)\left[e^{-x}\right]_0^1 - e^{-1}(1)$$

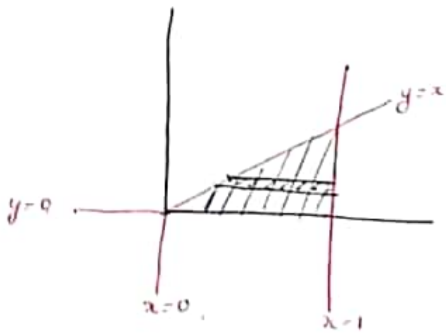
$$\Rightarrow P(x+y<1) = -\left[e^{-1}-1\right] - e^{-1} = \underline{1-2e^{-1}}$$

20/04/20 ⑥ $f(x,y) = 8xy$, $0<x<1$, $0<y<x$ $\quad$ i) $f(x)=?$ $\quad$ ii) $f(y)=?$

$\quad = 0$

Soln i) $f(x) = 8x\int_0^x y\,dy = \frac{8x}{2}\left[\frac{y^2}{2}\right]_0^x = 4x^3$

ii) $f(y) = 8y\int_{x=y}^{1} x\,dx = \frac{8y}{2}\left[x^2\right]_y^1$



$\Rightarrow f(y) = \frac{8y}{2}\left[1-y^2\right] = \underline{4y(1-y^2)}$

⑦ $f(x,y) = k(6-x-y)$, $0<x<2$, $2<y<4$ $\quad$ i) $k=?$

$\quad = 0$

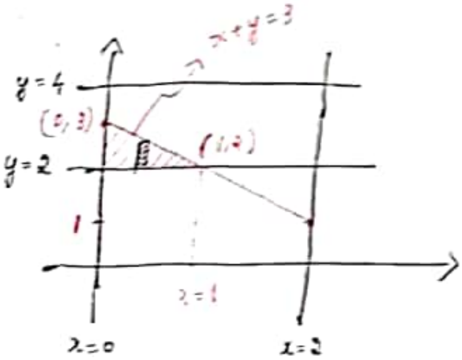
ii) $P(x+y<3) = ?$

Soln i) $k\int_{x=0}^{2}\int_{y=2}^{4}(6-x-y)=1 \Rightarrow k\int_{x=0}^{2}\left[6y-xy-\frac{1}{2}y^2\right]_2^4 dx=1$

$$\Rightarrow k\int_{x=0}^{2}[12-2x-6]dx=1 \Rightarrow k\left[6x-x^2\right]_0^2=1$$

$$\Rightarrow k[12-4]=1 \Rightarrow k=1/8$$

ii) $P[(x+y)<3]=?$



$$P(x+y<3)=\frac{1}{8}\int_{x=0}^{1}\int_{y=2}^{3}(6-x-y)\,dx\,dy$$

$$=\frac{1}{8}\int_{x=0}^{1}\left[6y-xy-\frac{1}{2}y^2\right]_2^3 dx$$

$$=\frac{1}{8}\int_{x=0}^{1}\left[6-x-5/2\right]dx = \frac{1}{8}\cdot\left[3.5x-\frac{x^2}{2}\right]_0^1 = \frac{[3.5-0.5]}{8}$$

$$\therefore P(x+y<3)=3/8$$

$$P(x+y<3)=\frac{1}{8}\int_{x=0}^{1}\int_{y=2}^{3-x}(6-x-y)\,dx\,dy = \frac{1}{8}\int_{x=0}^{1}\left[6y-xy-\frac{y^2}{2}\right]_2^{3-x}dx$$

$$\Rightarrow P(x+y<3)=\frac{1}{8}\int_{x=0}^{1}\left\{6[1-x]-x[1-x]-\frac{1}{2}\left[9+x^2-6x-4\right]\right\}dx$$

$$=\frac{1}{8}\int_{x=0}^{1}\left\{6-6x-x+x^2-\frac{x^2}{2}+3x-\frac{5}{2}\right\}dx = \frac{1}{8}\int_{x=0}^{1}\left[\frac{x^2}{2}-4x+\frac{7}{2}\right]dx$$

$$\Rightarrow P(x+y<3)=\frac{1}{8}\left[\frac{1}{6}[1-0]-2[1-0]+\frac{7}{2}\right]$$

$$\Rightarrow P(x+y<3)=5/24$$

9) $f(x,y) = \frac{1}{2} x e^{-y}$, $0 < x < 2$, $y > 0$ i) cdf = ?

$= 0$, otherwise

Soln i)

$$F(x,y) = \int_{u=-\infty}^{x} \int_{v=-\infty}^{y} f(u,v)\, du\, dv = \int_{u=0}^{x} \int_{v=0}^{y} \frac{1}{2} u e^{-v}\, du\, dv$$

$$\Rightarrow F(x,y) = \frac{1}{2} \cdot \left[\frac{u^2}{2}\right]_0^x (-1)\left[e^{-v}\right]_0^y = -\frac{1}{4}(x^2)\left[e^{-y} - 1\right]$$

$$\therefore F(x,y) = \frac{x^2}{4}(1 - e^{-y})$$

8) $F(x,y) = (1 - e^{-x})(1 - e^{-y})$, $x > 0, y > 0$; $P(1 < x < 3,\ 1 < y < 2) = ?$

Soln

$$f(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$$

$$\Rightarrow f(x,y) = \frac{\partial}{\partial x}\left[(1 - e^{-x})(e^{-y})\right] = e^{-y} e^{-x} = e^{-(x+y)}$$

$$\therefore P[1 < x < 3,\ 1 < y < 2] = \int_{x=1}^{3} \int_{y=1}^{2} e^{-x} \cdot e^{-y}\, dx\, dy$$

$$= (-1)\left[e^{-x}\right]_1^3 (-1)\left[e^{-y}\right]_1^2$$

$$= \left[e^{-3} - e^{-1}\right] \cdot \left[e^{-2} - e^{-1}\right]$$

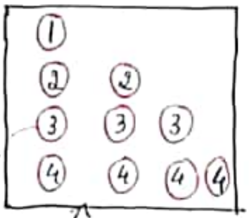
$$= \left[\frac{1}{e^3} - \frac{1}{e}\right]\left[\frac{1}{e^2} - \frac{1}{e}\right]$$

$$\therefore P[1 < x < 3,\ 1 < y < 2] = \left[\frac{1 - e^2}{e^3}\right]\left[\frac{1 - e}{e^2}\right]$$

PART - 2.2

Concept of Average:

i)
$$\boxed{\text{Mean} = \frac{\sum x_i}{n}}$$



Balls with numbers written on it

Box.

$$= \frac{1+(2+2)+(3+3+3)+(4+4+4+4)}{10}$$

$$= 3 \Longrightarrow$$ 3 is the average here

$\Downarrow$

Chance of picking 3 or close to 3 is high

ii)

x_i	f_i	$x_i f_i$
1	1	1
2	2	4
3	3	9
4	4	16
	$\sum f_i = 10$	$\sum x_i f_i = 30$

$$\boxed{\text{Mean} = \frac{\sum f_i x_i}{\sum f_i}} = \frac{30}{10} = 3$$

iii) Using the concept of Expectation:

X	1	2	3	4
P(X)	1/10	2/10	3/10	4/10

$$\boxed{E(X) = \sum_{i=l}^{h} x_i P(x_i)}$$

$h \rightarrow$ higher limit

$l \rightarrow$ lower limit

$$\therefore E(X) = \sum_{i=1}^{4} x_i P(x_i) = (1)(1/10) + (2)(2/10) + (3)(3/10) + (4)(4/10)$$

$$\Rightarrow E(X) = 3$$

Note

eg:

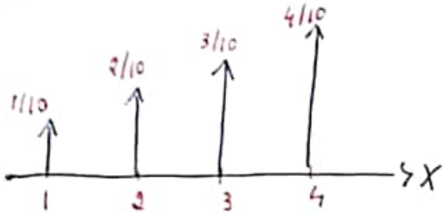
X	1	2	3	4
P(X)	1/10	3/10	3/10	3/10

$E(X) = ?$

$$E(X) = 1 \cdot \frac{1}{10} + 2 \cdot 3/10 + 3 \cdot 3/10 + 4 \cdot 3/10 = 2.8$$

$\therefore$ Hence $E(X)$ value need not always be an exact integer.

Variance:



$$\boxed{V(x) = (x-\bar{x})^2 \, P(x)}$$

$$= E\left((x-\bar{x})^2\right)$$

$$= E\left[\{x - E(x)\}^2\right]$$

$$= E\left[x^2 + \{E(x)\}^2 - 2x\,E(x)\right]$$

$$= E(x^2) + \{E(x)\}^2 - 2\,E(x)\,E(x)$$

$$\boxed{\therefore V(x) = E(x^2) - \{E(x)\}^2} \; ***$$

Note: $\boxed{\begin{array}{l} * \; E(ax) = a\,E(x) \\ * \; E(E(x)) = E(x) \end{array}}$

Here $E(x^2) \rightarrow$ Mean square value

$E(x) \rightarrow$ Mean value

Find the variance = ?

X	1	2	3	4
X^2	1	4	9	16
$P(x)$	1/10	2/10	3/10	4/10

$E(x) = 3.$

$E(x^2) = \sum x^2 P(x)$

$= 1/10 + 8/10 + 27/10 + 64/10$

$\Rightarrow E(x^2) = \cancel{100}\ 10.$

$\therefore V(x) = E(x^2) - \{E(x)\}^2 = 10 - 3^2 = 1$

Note: Variance signifies the spread of the curve.

Q)

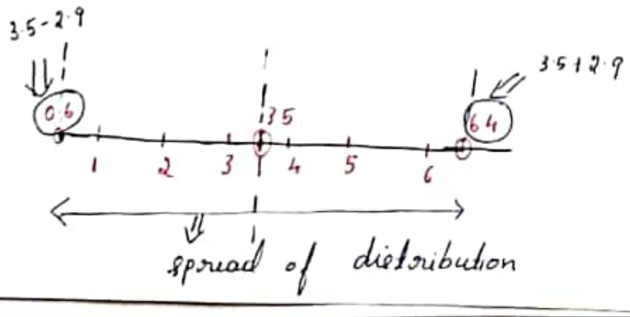
X	1	2	3	4	5	6
$P(x)$	1/6	1/6	1/6	1/6	1/6	1/6

$\Rightarrow E(x) = \bar{x} = \frac{21}{6} = 3.5$

$E(x^2) = \frac{1}{6}\left[1+4+9+16+25+36\right]$

$E(x^2) = 15.17$

$\therefore V(x) = E(x^2) - \{E(x)\}^2 = 2.92$



Scanned with CamScanner

∴ Standard Deviation: $\boxed{S.D = \sqrt{\text{Variance}}}$

eg:

X	-2	-1	0	1	2	3
P(X)	0.2	k	0.1	2k	0.1	2k

i) $k = ?$
ii) Mean = ?
iii) Variance = ?

Soln

i) $\Sigma P(x) = 1 \Rightarrow 0.4 + 5k = 1 \Rightarrow k = \frac{0.6}{5} = 0.12$

ii) Mean $= E(x)$

$= -2(0.2) + (-1)(0.12) + 0 + (1)(0.24) + (2)(0.1) + (3)(0.24)$

Mean $= 0.64$

iii) $V(x) = E(x^2) - \{E(x)\}^2$

$= 3.72 - (0.64)^2$

$\Rightarrow V(x) = 3.31$

$E(x^2) = (4)(0.2) + (1)(0.12) + 0 + (1)(0.24) + (4)(0.1) + (9)(0.24)$

$E(x^2) = 3.72$

eg: $f(x) = \begin{cases} kx(2-x) & , 0 < x < 2 \\ 0 & , \text{otherwise} \end{cases}$

i) $k = ?$
ii) Mean = ?

Soln: i) $k\int_{x=0}^{2} x(2-x)dx = 1 \Rightarrow k\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1$

$\Rightarrow k\left[4 - 8/3\right] = 1$

$\Rightarrow k = 3/4$

ii) $\boxed{E(x) = \int_{x=0}^{2} x f(x)\, dx} = 3/4\int_0^2 [2x^2 - x^3]dx = 3/4\left[2/3\, x^3 - \frac{x^4}{4}\right]_0^2$

$\Rightarrow E(x) = \frac{3}{4}\left[\frac{16}{3} - 4\right] = 1$

Q)

X	0	1	2	3
$P(X)$	0.1	0.3	0.5	0.1

i) $P(Y) = ?$ where $Y = X^2 + 2X$

ii) $E(Y) = ?$

Soln

i) $P(Y)$ is same as $P(X)$

X	0	1	2	3
Y	0	3	8	15
$P(Y)$	0.1	0.3	0.5	0.1

ii) $E(Y) = Y P(Y) = 0 + (0.3)3 + (0.5)8 + (0.1)15$

$\Rightarrow E(Y) = 6.4$

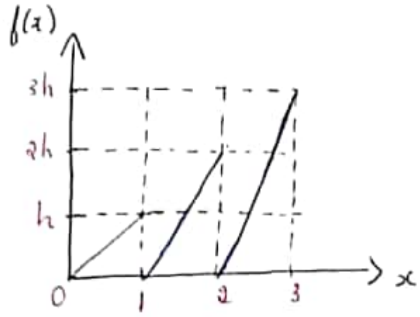
HW

Q) $f(x) = \begin{cases} x & , 0 < x < 1 \\ 2 - x & , 1 < x < 2 \\ 0 & , \text{else where} \end{cases}$ Find $E(Y)$, $Y = (x-1)^2$

Soln

21/04/20

21/04/20 Q) The graph of a function $f(x)$ is shown in fig



For $f(x)$ to be a valid probability density function, the value of h is

a) $1/3$ b) $2/3$ c) 1 d) 3

Soln

Area = 1 $\Rightarrow \frac{1}{2}[(1\times h) + (1\times 2h) + (1\times 3h)] = 1$

$\Rightarrow 6h = 2$

$\Rightarrow h = 1/3$ $\Rightarrow$ option a

Q) For the function $f(x) = a+bx$, $0 \le x \le 1$, to be a valid probability density function, which one of the foll. statements is correct?

a) $a=1, b=4$ b) $a=0.5, b=1$ c) $a=0, b=1$ d) $a=1, b=-1$

Soln

$\int_0^1 f(x)\,dx = 1 \Rightarrow a[1-0] + \frac{b}{2}[1-0] = 1$

$\Rightarrow a + b/2 = 1$

$\Rightarrow a = 0.5, b = 1$ $\Rightarrow$ option b

Q) A six-face fair dice is rolled a larger number of times. The mean value of the outcomes is ______

a) 2 b) 2.5

c) 3 d) 3.5

Soln

X	1	2	3	4	5	6
P(X)	1/6	1/6	1/6	1/6	1/6	1/6

$E(x) = \Sigma x p(x) = \frac{1}{6}[1+2+3+4+5+6]$

$\Rightarrow E(x) = 3.5$

Scanned with CamScanner

(5) A person decides to toss a fair coin repeatedly until he gets a head. He will make at most 3 tosses. Let the random variable Y denotes the number of heads. The value of var(Y), where var(·) denotes the variance, equal

a) 7/8 b) 49/64 c) 7/64 d) 105/64

Soln

Y	0	1
P(Y)	1/8	7/8

1st toss: H T

2nd toss: TH TT

3rd toss: TTH TTT

$\therefore E(Y) = 0 + 7/8 = 7/8$

$E(Y^2) = 0 + 7/8 = 7/8$

$\therefore Var(Y) = 7/8 - 49/64$

$= 7/64$ => option c

6) A probability density function on the interval [a, 1] is given by $1/x^2$ & outside this interval the value of the function is zero. The value of a is ______

Soln

$$\int_a^1 \frac{1}{x^2}\,dx = 1 \Rightarrow \left[\frac{x^{-1}}{(-1)}\right]_a^1 = 1 \Rightarrow \left[\frac{1}{x}\right]_1^a = 1$$

$$\Rightarrow \frac{1}{a} - 1 = 1 \Rightarrow a = 1/2$$

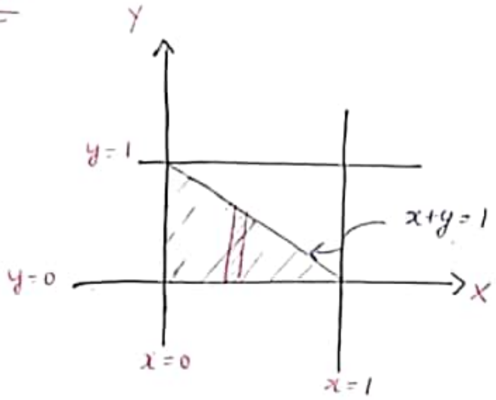
Q) Two random variables X & Y are distributed according to

$$f_{X,Y}(x,y) = \begin{cases} (x+y), & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

The probability $P(X+Y \le 1)$ is ______

Scanned with CamScanner

Soln



$$P(x+y \le 1) = \int_{x=0}^{1} \int_{y=0}^{1-x} (x+y)\, dx\, dy$$

$$= \int_{x=0}^{1} \left\{ x(1-x) + \tfrac{1}{2}(1-x)^2 \right\} dx$$

$$= \int_{x=0}^{1} \left[\cancel{x} - x^2 + \tfrac{1}{2} + \tfrac{1}{2}x^2 - \cancel{x} \right] dx$$

$$\Rightarrow P(x+y \le 1) = \int_{x=0}^{1} \left[\tfrac{1}{2} - \tfrac{1}{2}x^2 \right] dx$$

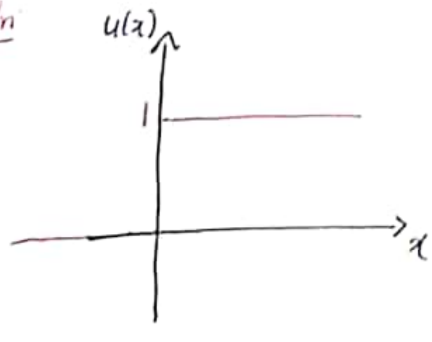
#

$$\Rightarrow P(x+y \le 1) = \tfrac{1}{2}[x]_0^1 - \tfrac{1}{6}[x^3]_0^1 \qquad \tfrac{1}{2} - \tfrac{1}{6}$$

$$\Rightarrow P(x+y \le 1) = 1/3$$

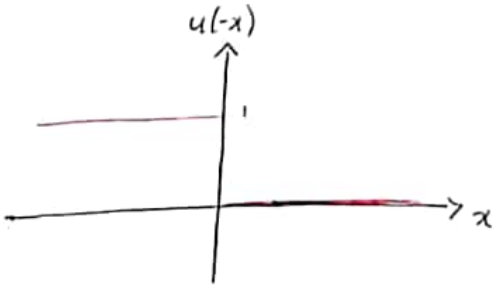
(9) Let the probability density function of a random variable X, be given as: $f_X(x) = \frac{3}{2} e^{-3x} u(x) + a e^{4x} u(-x)$ where $u(x)$ is the unit step function. Then the value of 'a' & prob$\{x \le 0\}$, resp^{ly} are.

Soln



$u(x) = 1, \; x > 0$

$\quad = 0, \; x < 0$



$u(-x) = 1, \; x < 0$

$\quad = 0, \; x > 0$

$$\therefore f(x) = \begin{cases} \frac{3}{2} e^{-3x} & , x > 0 \\ a e^{4x} & , x < 0 \end{cases}$$

$$\therefore \int_{-\infty}^{0} a e^{4x} dx + \int_{0}^{\infty} 3/2 e^{-3x} = 1 \quad \Rightarrow \quad \frac{a}{4}\left[e^{4x}\right]_{-\infty}^{0} + \frac{3}{2} \times -\frac{1}{3} \left[e^{-3x}\right]_{0}^{\infty} = 1$$

$$\Rightarrow \quad \frac{a}{4}[1-0] + \left(-\frac{1}{2}\right)\cdot[0-1] = 1 \quad \Rightarrow \quad \frac{a}{4} + 1/2 = 1$$

$$\Rightarrow \quad \frac{a}{4} = 1/2 \quad \Rightarrow \quad a = 2$$

$$P(x \leq 0) = 2\int_{-\infty}^{0} e^{4x} = \frac{2}{4}\left[e^{4x}\right]_{-\infty}^{0} = \frac{1}{2}[1-0] = 1/2$$

Q) The probability density function of a random variable, x is

$$f(x) = \frac{x}{4}(4-x^2) \text{ for } 0 \leq x \leq 2$$
$$= 0 \qquad \text{otherwise}$$

The mean, μ_x of the random variable is ____.

Soln:

$$\mu_x = \int_a^b x f(x)\, dx = \int_0^2 \left(x^2 - \frac{x^4}{4}\right) dx = \frac{1}{3}[8] - \frac{1}{20}[32]$$
$$= \frac{160-96}{60}$$

$$\Rightarrow \mu_x = \frac{64}{60} = 16/15$$

Q) Consider the foll. probability mass function of a random variable X.

$$p(x, q) = \begin{cases} q & \text{if } X = 0 \\ 1-q & \text{if } X = 1 \\ 0 & \text{otherwise} \end{cases}$$

If $q = 0.4$, the variance of X is ____

Soln: For $q = 0.4$, $p(x) = \begin{cases} 0.4 & , X = 0 \\ 0.6 & , X = 1 \\ 0 & , \text{otherwise} \end{cases}$

X	0	1
$P(X)$	0.4	0.6

$E(X) = 0.6$
$E(X^2) = 0.6$

$$\therefore V(X) = 0.6 - 0.6^2 = 0.24$$

Q) Let $f(x) = 2/3 \quad , x = 1$

$\quad\quad = 1/3 \quad , x = 2$

Now $E(x) = x_1 f(x_1) + x_2 f(x_2)$

$= (1)(2/3) + (2)(1/3)$

$= 4/3$

Let $y = x^2 + 1$

$\therefore f(y) = 2/3, \quad y = 2 \; ; x = 1$

$\quad\quad = 1/3, \quad y = 5 ; x = 2$

$\therefore E(y) = g(x_1) f(x_1) + g(x_2) f(x_2)$

$= 2(2/3) + 5(2/3)$

$= \frac{14}{3}$

$\therefore$ Discrete RV: $\boxed{E(Y) = E(g(x)) = \sum g(x) f(x)}$

eg: $f(x) = x^2/3 \quad , 0 < x < 1 \quad ; \quad y = \sqrt{x}, \quad E(y) = ?$

M1 $$E(y) = \int_{-\infty}^{\infty} y f(y)\, dy = \int_0^1 y f(y)\, dy$$

Note For increasing/decreasing function: $\boxed{f(y) = f(x) \left| \frac{dx}{dy} \right|}$

$y = \sqrt{x} \Rightarrow x = y^2 \Rightarrow \frac{dx}{dy} = 2y$

$f(x) = \frac{x^2}{3} = \frac{y^4}{3}$

$\therefore f(y) = \frac{2}{3} y^5$

$\therefore E(y) = \int_0^1 y \cdot \frac{2}{3} y^5 dy = \frac{2}{3} \times \frac{1}{7} [y^7]_0^1$

$\Rightarrow E(y) = 2/21$

Continuous RV: $\boxed{E(y) = E(g(x)) = \int_{x=-\infty}^{\infty} g(x) f(x)\, dx}$

M2 $$E(y) = \int_0^1 (\sqrt{x}) \cdot \frac{x^2}{3} \cdot dx = \frac{1}{3} \left[\frac{x^{7/2}}{7/2} \right]_0^1 = \frac{2}{21}$$

Q) The probability density function of a random variable X is $P_x(x) = e^{-x}$ for $x \geq 0$ & 0 otherwise. The expected value of the function $g_x(x) = e^{3x/4}$ is ____.

Soln Let $y = g(x) = e^{3x/4}$

$$\therefore E(y) = \int_0^{\infty} e^{3x/4} \cdot e^{-x} dx = \int_0^{\infty} e^{-x/4} dx = -4\left[e^{-x/4}\right]_0^{\infty}$$

$$\Rightarrow E(y) = -4[0-1] = 4$$

Q) The variance of the random variable X with probability density function $f(x) = \frac{1}{2}|x| e^{-|x|}$ is ____

Soln

$$f(x) = \frac{1}{2} x e^{-x} \quad , x > 0$$

$$= -\frac{1}{2} x e^{x} \quad , x < 0$$

$$V(x) = E(x^2) - \{E(x)\}^2$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_{-\infty}^{\infty} x^2 \frac{1}{2} |x| e^{-|x|} dx \quad [\text{even fn}]$$

$$\Rightarrow E(x^2) = 2\int_0^{\infty} x^2 \cdot \frac{1}{2} x e^{-x} dx = \int_0^{\infty} x^3 e^{-x} dx = \Gamma 4 = 3!$$

$$\Rightarrow E(x^2) = 6$$

$$E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{2} |x| e^{-|x|} dx \quad [\text{odd fn}]$$

$$\Rightarrow E(x) = 0$$

$$V(x) = E(x^2) - \{E(x)\}^2 = 6 - 0 \Rightarrow V(x) = 6$$

22/04/20

Moment Generating functions

For CRV: $E(X) = \int x f(x)\, dx$

$$e^{x} = x^0 + \frac{x^1}{1!} + \frac{x^2}{2!} + \dots$$

$$\Rightarrow e^{tx} = (tx)^0 + \frac{(tx)^1}{1!} + \frac{(tx)^2}{2!} + \dots$$

$$E(e^{tx}) = 1 + t E(x) + \frac{t^2}{2} E(x^2) + \dots$$

$$\frac{d E(e^{tx})}{dt} = 0 + E(x) + E(x^2) \cdot t + \dots$$

Putting $t = 0$;

$$\left.\frac{d E(e^{tx})}{dt}\right|_{t=0} = E(x)$$

$\therefore$ In general, $$\boxed{\left[\frac{d^n}{dt^n} E(e^{xt})\right]_{t=0} = E(X^n)}$$ $\Rightarrow$ Moment generating function

$E(X) \Rightarrow n = 1 \rightarrow 1^{st}$ moment

$E(X^2) \Rightarrow n = 2 \rightarrow 2^{nd}$ moment

$E(X^n) \Rightarrow n = n \rightarrow n^{th}$ moment

Std. result: $$\boxed{\left[\frac{d^n}{dt^n} (m_0(t))\right]_{t=0} = E(X^n)}$$

$$\boxed{\begin{aligned} m_0(t) = E(e^{xt}) &= \sum e^{xt} f(x) \rightarrow (DRV) \\ &= \int e^{xt} f(x)\, dx \rightarrow (CRV) \end{aligned}}$$

Q)

X	1	2	3	4	5	6
$P(x)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

i) $E(x) = ?$

ii) $E(x^2) = ?$

iii) $V(x) = ?$

Soln i) M1 Std method.

M2 Using moment generating function:

$$m_0(t) = \sum e^{xt} f(x)$$

$$= \sum_{x=1}^{6} e^{xt}(1/6) = 1/6\left[e^{t} + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}\right]$$

$$\therefore \left\{\frac{d}{dt}(m_0(t))\right\}_{t=0} = E(x)$$

$$\therefore E(x) = \frac{1}{6}\left[e^{t} + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}\right]_{t=0}$$

$$= \frac{1}{6}\left[1+2+3+4+5+6\right]$$

$$\Rightarrow E(x) = 21/6 = 3.5$$

$$\Downarrow$$

$(1^{st}$ moment $= \mu_1')$

ii)

$$E(x^2) = \mu_2' = \frac{d^2}{dt^2}\left[m_0(t)\right] = \frac{1}{6}\left[e^{t} + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t}\right]_{t=0}$$

$$\Rightarrow E(x^2) = \frac{1}{6}\left[1+4+9+16+25+36\right] = 91/6$$

iii)

$$V(x) = \mu_2' - (\mu_1')^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = 35/12$$

Q)

$$f(x) = ke^{-kx} \quad , x>0 \ , k>0$$

$$= 0 \quad , \text{elsewhere}$$

i) $\mu_1' = ?$

Soln:

$$m_0(t) = E(e^{xt}) = \int_0^\infty e^{xt} f(x)\,dx$$

$$\Rightarrow m_0(t) = \int_0^\infty e^{xt}\, ke^{-kx}\,dx = k\int_0^\infty e^{(t-k)x}\,dx$$

$$\Rightarrow m_0(t) = \frac{k}{t-k}\left[e^{(t-k)x}\right]_0^\infty = \frac{k}{k-t}$$

$$m_0(t) = \frac{k}{k-t} = \frac{1}{1-t/k} = \left(1-\frac{t}{k}\right)^{-1} = 1+\frac{t}{k}+\frac{t^2}{k^2}+\frac{t^3}{k^3}+\cdots$$

$$\therefore \mu_1' = E(X) = \left[\frac{d}{dt}m_0(t)\right]_{t=0} = \frac{1}{k}+\frac{2t}{k^2}+\frac{3t^2}{k^3}+\cdots$$

$$\Rightarrow \mu_1' = 1/k$$

$$\therefore \mu_2' = E(X^2) = \frac{2}{k^2}$$

$$V(X) = \mu_2' - (\mu_1')^2 = \frac{2}{k^2} - \frac{1}{k^2} = 1/k^2$$

PART - 3

BINOMIAL DISTRIBUTION:

3 coins are tossed ; $P(2 \text{ heads}) = ?$

$S = 2^3 = \{$ HHH, HTH, THH, TTH, HHT, HTT, THT, TTT $\}$

$\therefore P(2 \text{ heads}) = \frac{3}{8}$

Here favourable events are:

HTH → sfs ⇒ s^2f

THH → fss ⇒ s^2f

HHT → ssf ⇒ s^2f

s - success

f - failure

$3s^2f$

required no. of times of success

Combination of fav events ← 3

s → success prob.

f → failure prob.

→ req. no of times of failure

∴ In General,

$$\boxed{P(x) = {}^nC_x\, p^x q^{n-x}} \Rightarrow \text{Binomial Distribution}$$

* $p + q = 1$
* Only success or failure is possible
* $x_{Base} \rightarrow P_{Base}$
* $x = 0, 1, 2, 3, \ldots \infty$ (practical)
* $P(x=0) = {}^nC_0 p^0 q^{n-0} = q^n$

$P(x=1) = {}^nC_1 p^1 q^{n-1} = {}^nC_1 p q^{n-1}$

$\vdots$

$P(x=n) = {}^nC_n p^n q^{n-n} = p^n$

Adding

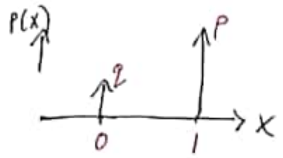
$$\sum_{x=0}^{n} P(X=x) = q^n + {}^nC_1 p q^{n-1} + {}^nC_2 p^2 q^{n-2} + \ldots + p^n$$

$$\Rightarrow 1 = (p+q)^n$$

$$\therefore \boxed{(p+q)^n = 1}$$

X	1	0
P(x)	p	q

$\Rightarrow$ Bernaulli Distribution ($x = 0, 1$)



* X is countable $\Rightarrow$ Discrete Random Variable.

eg:

X	0	1	2	3
P(x)	1/8	3/8	3/8	1/8

$X \rightarrow$ No. of heads obtained when a fair coin is tossed thrice

$E(X) = 0 + 3/8 + 6/8 + 3/8 = 3/2$ //

We can also write $E(X) = (3)(1/2) = \frac{3}{2}$

prob of getting head in a fair coin toss

no. of times a coin is tossed

$\boxed{\therefore E(x) = np} \Rightarrow$ First moment of B.D

$V(X) = E(X^2) - (E(X))^2$ | $E(X^2) = 0 + 3/8 + 12/8 + 9/8 = 3$

$\therefore V(X) = 3 - (1.5)^2 = 3/4$

$\boxed{V(X) = npq} = 3 \cdot 1/2 \cdot 1/2 = 3/4$

<u>Second moment of B.D</u>: $E(X^2) = V(X) + \{E(X)\}^2$

$\Rightarrow E(X^2) = npq + p^2q^2$

$\Rightarrow \boxed{E(X^2) = pq(n + pq)}$

<u>For Bernaulli Distribution</u>:

$\boxed{\begin{array}{l} E(X) = p \\ V(X) = pq \end{array}}$ <u>Note</u>: $n = 1$ for Bernaulli Distribution.

Q) The probability of a defective piece being produced in a manufacturing process is 0·01. The probability that out of 5 successive piece, only one is defective is

a) $(0.99)^2 (0.01)$ b) $(0.99)(0.01)^4$

c) $5 \times (0.99)(0.01)^4$ d) $5 \times (0.99)^4 (0.01)$

<u>Soln</u> $p = 0.01$, $n = 5$, $x = 1$

$\therefore P(x=1) = {}^nC_x p^x q^{n-x} = {}^5C_1 (0.01)^1 (0.99)^4 = \underline{5(0.01)(0.99)^4} \Rightarrow$ (option d)

Q) If 20% managers are technocrats, the probability that a random committee of 5 managers consists of exactly 2 technocrats is

a) 0·2048 b) 0·4000 c) 0·4096 d) 0·9421

<u>Soln</u> $p = 0.2$, $n = 5$, $x = 2$

$P(x=2) = {}^5C_2 (0.2)^2 (0.8)^3 = 10 \times (0.2)^2 (0.8)^3 = \underline{0.2048} \Rightarrow$ option a

Q) A fair die is rolled four times. Find the probability that six shows up twice. a) 1/2 b) 16/325 c) 1/36 d) 25/216

Soln $p = 1/6, \quad n=4, \quad x=2$

$$P(x=2) = {}^4C_2 \, (1/6)^2 \, (5/6)^2 = 6 \times \frac{1}{36} \times \frac{25}{36} = \frac{25}{216}$$ => option d

Q) In an experiment, positive & negative values are equally likely to occur. The probability of obtaining at most one negative in five trials is

a) 1/32 b) 2/32 c) 3/32 d) 6/32

Soln $p = 1/2, \quad n=5, \quad x=1$

$$P(x \leq 1) = P(x=0) + P(x=1)$$
$$= (1/2)^5 + {}^5C_1 \, (1/2)^4 \, (1/2)^4$$
$$= \frac{1}{32} + \frac{5}{32}$$
$$= 6/32$$ => option d

Q) The probability that a screw manufactured by a company is defective is 0.1. The company sells screws in packets containing 5 screws & gives a guarantee of replacement if one or more screws in the packet are found to be defective. The probability that a packet would have to be replaced is ____.

Soln $p = 0.1$, x - no. of defective screws in a packet of 5

$n = 5$

P (packet would have to be replaced) $= P(x > 0)$

$$= 1 - P(x=0)$$
$$= 1 - {}^5C_0 \, p^0 q^5$$
$$= 1 - (0.9)^5$$
$$= 0.40951$$

5) A batch of one hundred bulbs is inspected by testing four randomly chosen bulbs. The batch is rejected if even one of the bulb is defective. A batch typically has five defective bulbs. The probability that the current batch accepted is ____

Soln $p = 5/100 = 0.05$, x = no. of defective bulbs , $n = 4$

$$P(\text{batch accepted}) = P(x=0)$$
$$= {}^4C_0 (p)^0 (q)^4$$
$$= (0.95)^4$$
$$= \underline{0.8145}$$

23/04/20 Q) An unbiased coin is tossed an infinite no. of times. The probability that the fourth head appears at the tenth toss is

a) 0.067 b) 0.073 c) 0.082 d) 0.091

Soln $P(4^{th}$ Head at tenth toss$)$ = $P($3 Head in 9 toss & 1 Head at tenth toss$)$

$$= P(\text{3 Head in 9 toss}) \cdot P(\text{Head})$$
$$= {}^9C_3 \cdot (1/2)^3 (1/2)^6 \cdot 1/2$$
$$= (1/2)^{10} \cdot {}^9C_3$$
$$= \underline{0.082} \Rightarrow \text{option c}$$

Q) There are 10 markers on a table of which 6 are defective & 4 are non defective. If 3 are randomly taken from lot, what is the probability that exactly 1 marker is defective

Soln. $p = 6/10$, $n = 3$, $x = 1$

$$P(x) = {}^3C_1 (6/10)^1 (4/10)^2 = 3 \times \frac{6}{10} \times \frac{16}{100} = \underline{0.288}$$

WRONG

This is not a binomial distribution

Without Replacement

3 Trials: ① D ND ND → $6/10 \quad 4/9 \quad 3/8$

② ND D ND

③ ND ND D

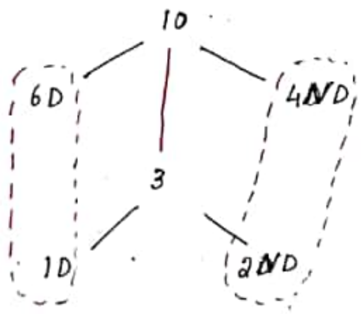
*** **Note** * "We cannot use the concept of Binomial Distribution for without Replacement case."

* Binomial Distribution is applied for independent events only

$\therefore$ P(Exactly 1 marker is defective) $= \dfrac{{}^6C_1 \cdot {}^4C_2}{{}^{10}C_3} = \dfrac{\frac{3\,\cancel{6}\times\cancel{4}\times\cancel{3}}{\cancel{2}}}{\frac{10\times\cancel{9}\times\cancel{8}\,\cancel{4}}{\cancel{3}\times\cancel{2}}}$

$\Rightarrow$ P(Exactly 1 marker is defective) $= 3/10$

Using the concept of Hyper Geometric Distribution.



$\equiv \dfrac{{}^6C_1 \cdot {}^4C_2}{{}^{10}C_3} = 3/10$

Note If the scenario is for with replacement: then we can use binomial distribution

POISSON DISTRIBUTION

Binomial Distribution: $P(x) = {}^nC_x\, p^x q^{n-x}$.

As $n \to \infty$ (very large) ${}^nC_x$ poses problems.

Hence for very large n we use poisson distribution.

Scanned with CamScanner

Proof: $P(x) = {}^nC_x\, p^x q^{n-x} = {}^nC_x\, p^x \frac{q^n}{q^x} = {}^nC_x \left(\frac{p}{q}\right)^x q^n$

$\Rightarrow P(x) = {}^nC_x \left(\frac{p}{1-p}\right)^x (1-p)^n$

$= {}^nC_x \left(\frac{\lambda/n}{1-\lambda/n}\right)^x (1-\lambda/n)^n$

Mean $= \lambda = np$

$\Rightarrow p = \lambda/n$

$\Rightarrow P(x) = \frac{{}^nC_x}{n^x} \frac{\lambda^x}{(1-\lambda/n)^x} (1-\lambda/n)^n$

As $n \to \infty$

$\Rightarrow P(x) = \lim_{n\to\infty} \frac{{}^nC_x}{n^x} \cdot \frac{\lambda^x}{(1-\lambda/n)^x} (1-\lambda/n)^n$

$= \frac{1}{x!} \cdot \lambda^x \cdot e^{-\lambda}$

$\therefore \boxed{P(x) = \frac{\lambda^x e^{-\lambda}}{x!}}$, $x = 0,1,2,3,\ldots$

$\frac{{}^nC_x}{n^x} = \frac{\frac{n!}{(n-x)!\,x!}}{n^x}$

$= \frac{n\cdot(n-1)(n-2)\ldots\cancel{(n-x)!}}{\cancel{(n-x)!}\; x!\; n^x}$

$= \frac{\frac{n}{n}\cdot\frac{(n-1)}{n}\frac{(n-2)}{n}\cdots\frac{(n-(x-1))}{n}}{x!}$

$\Rightarrow \frac{{}^nC_x}{n^x} = \frac{1\cdot(1-1/n)(1-2/n)\ldots\left(1-\frac{(x-1)}{n}\right)}{x!}$

$\therefore \lim_{n\to\infty} \frac{{}^nC_x}{n^x} = \frac{1}{x!}$

Note

* x should be a DRV, $x = 0,1,2,3,\ldots$
* $\lambda = np$, $p+q=1$
* Events should be independent
* ~~(x)/graph~~ $(x)_{\text{base/idea}} \rightarrow (\lambda)_{\text{base/idea}}$
* $\boxed{\begin{array}{l} E(X) = \lambda = np \\ V(X) = E(X) = \lambda = np \end{array}}$

Q) Consider a poisson distribution for the tossing of a biased coin. The mean for this distribution is μ. The standard deviation for this distribution is given by:

a) $\sqrt{\mu}$ b) μ^2 c) μ d) $1/\mu$

Ans: option a

Q) The poisson distribution is given by $P(r) = \frac{m^r}{r!} e^{-m}$. The first moment about the origin for the distribution is

a) 0 b) m c) 1/m d) m^2

Ans: option b

Q) The number of accidents occuring in a plant in a month follows poisson distribution with mean as 5.2. The probability of occurrence of less than 2 accidents in the plant during a randomly selected month is ____.

Soln $\lambda = 5.2$

$$P(x<2) = P(x=0) + P(x=1)$$

$$= \frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^1}{1!} = e^{-5.2}[1+5.2]$$

$$\Rightarrow P(x<2) = \underline{0.0342}$$

5) A traffic officer imposes on an average 5 number of penalties daily on traffic violators. Assume that the number of penalties on different days is independent & follows a poisson distribution. The probability that there will be less than 4 penalties in a day is ____

Soln $\lambda = 5$

$$P(x<4) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$$

$$= e^{-5}\left[1 + 5 + \frac{5^2}{2!} + \frac{5^3}{3!}\right]$$

$$= e^{-5}\left[6 + \frac{25}{2} + \frac{125}{6}\right]$$

$$= \underline{0.2650}$$

Q) If a random variable X satisfies the poisson distribution with a mean value of 2 then the prob that $x \geq 2$ is

a) $2e^{-2}$ b) $1-2e^{-2}$ c) $3e^{-2}$ d) $1-3e^{-2}$

Soln

$$\lambda = 2$$

$$P(x \geq 2) = 1 - P(x<2) = 1 - [P(0) + P(1)]$$

$$\Rightarrow P(x \geq 2) = 1 - e^{-\lambda}[1+\lambda] = \underline{1-3e^{-2}} \Rightarrow \text{option d}$$

Q) An observer counts 240 vehicles/hour at a specific highway location. Assume that the vehicle arrival at the locations is poisson distributed, the probability of having one vehicle arriving over a 30 second time interval is ______.

Soln

$$\lambda = 240 \text{ vehicles/hour} = 0.0667 \text{ vehicles/sec} = 2 \text{ vehicles/30 sec}$$

$$P(x=1) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-2}\cdot(2)^1}{1!} = \underline{0.2707}$$

Q) In a manufacturing plant, the probability of making a defective bolt is 0.1. The mean & standard deviation of defective bolts in a total of 900 bolts are resp.ly

a) 90 & 9 b) 9 & 90 c) 30 & 3 d) 3 & 30.

Soln.

$p = 0.1$, $n = 900$ (very large n → Poisson)

$$\therefore \lambda = np = 90 \quad \& \quad S.D = \sqrt{90} = 9.48 \doteqdot 9$$

∴ option a

Binomial Distribution

$$\lambda = np = 90$$

$$V(x) = npq = 90 \times 0.9 = 81$$

$$\therefore SD = \sqrt{81} = 9$$

Q) Passengers try repeatedly to get a seat reservation in any train running between two stations until they are successful. If there is 40% chance of getting reservation in any attempt by a passenger, then the average number of attempts that passengers need to make to get a seat reserved is ____.

Soln x : no. of attempts

x	~~0~~	~~1~~	~~2~~	~~3~~
~~$P(x)$~~		~~4/10~~	~~4/10 6/10~~	~~4/10~~

M1

x	1	2	3	4	.....
$P(x)$	$4/10$	$6/10 \cdot 4/10$	$(6/10)^2 4/10$	$(6/10)^3 4/10$	.....

$$E(x) = 1\cdot 4/10 + 2\cdot 6/10 \cdot 4/10 + 3\cdot(6/10)^2 4/10 + 4(6/10)^3 \cdot 4/10 + \cdots$$

$$= 4/10\left[1 + 2(6/10) + 3(6/10)^2 + 4(6/10)^3 + \cdots\right]$$

$$= 4/10\left[1 - 6/10\right]^{-2}$$

$$\Rightarrow E(x) = \frac{4}{10} \times \left[\frac{4}{10}\right]^{-2} = \frac{10}{4} = 2.5$$

M2 Geometric Distribution:

Geometric Dist^n.

$$\boxed{P(x) = p\,q^{x-1}}, \quad x = 1, 2, 3, \ldots$$

No. of attempts		$P(x)$
$x = 1$	$\longrightarrow$	p
$x = 2$	$\longrightarrow$	qp
$x = 3$	$\longrightarrow$	qqp
$x = 4$	$\longrightarrow$	$qqqp$
$\vdots$		
∞		

$$E(x) = p + 2pq + 3pq^2 + 4pq^3 + \cdots$$

$$= p\left[1 + 2q + 3q^2 + 4q^3 + \cdots\right]$$

$$= p\left[1 - q\right]^{-2}$$

$$= \frac{p}{p^2}$$

$$\Rightarrow \boxed{E(x) = 1/p}$$

Note $\boxed{V(x) = \frac{q}{p^2}}$

Since the given distribution is a Geometric distribution,

$$E(x) = 1/p = 1/\tfrac{40}{100} = 2.5$$

24/04/20

Uniform Distribution:

For Discrete Random Variable:

eg: X = no. of points obtained when you throw a die.

X	1	2	3	4	5	6
$P(x)$	1/6	1/6	1/6	1/6	1/6	1/6

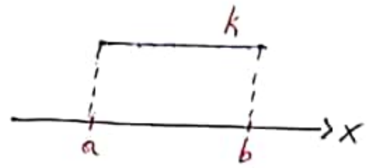
$$\therefore \boxed{\begin{aligned} P(x) &= 1/n, \quad x = 1,2,3,\dots,n. \\ &= 0, \quad \text{otherwise} \end{aligned}}$$

$$E(X) = \sum x P(x) = \frac{1}{6}[1+2+3+4+5+6] = 7/2 = \left(\frac{\overset{6}{n}+1}{2}\right)$$

$$\therefore \boxed{\begin{aligned} E(X) &= \frac{n+1}{2} \\ \text{IIIly} \quad V(X) &= \frac{n^2-1}{12} \end{aligned}}$$

For Continuous Random Variable:

$$f(x) = \begin{cases} \overset{1/b-a}{k}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$



Area under curve = 1

$(b-a)k = 1$

$\Rightarrow k = 1/b-a$

$$\therefore \boxed{f(x) = \begin{cases} 1/(b-a), & a < x < b \\ 0, & \text{otherwise} \end{cases}}$$

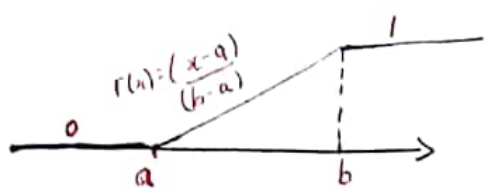
CDF

When $x < a$, $F(x) = 0$.

$a < x < b$, $F(x) = \int_a^x \frac{1}{b-a}\, dx = \frac{1}{(b-a)}(x-a) = \frac{(x-a)}{(b-a)}$

$x > b$, $F(x) = 1$

$$\therefore \boxed{F(x) = \begin{cases} 0, & x < a \\ \frac{(x-a)}{(b-a)}, & a < x < b \\ 1, & x > b \end{cases}}$$



8) A random variable is uniformly distributed over the interval 2 to 10. Its variance will be

a) $16/3$ b) 6 c) $\frac{256}{9}$ d) 36

Soln: M1 $E(x^2) = \int_2^b x^2 \left(\frac{1}{b-a}\right) dx = \frac{1}{8} \cdot \frac{1}{3} [10^3 - 2^3] = \frac{124}{3}$ //

$$E(x) = \frac{1}{8} \int_2^{10} x \, dx = \frac{1}{16} [100-4] = 6 //$$

$$\therefore v(x) = \frac{124}{3} - 6^2 = 16/3 // \Rightarrow \text{option a}$$

M2 For CRV:

$$E(x) = \int_a^b x f(x) \, dx = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{1}{b-a} \cdot \frac{1}{2} (b^2 - a^2) = \frac{b+a}{2}$$

i.e $\boxed{E(x) = \frac{a+b}{2}}$

$$E(x^2) = \int_a^b x^2 \frac{1}{(b-a)} dx = \frac{1}{3(b-a)} (b^3 - a^3) = \ldots\ldots = \frac{a^2 + ab + b^2}{3}$$

$$\therefore v(x) = E(x^2) - \{E(x)\}^2 = \frac{(b-a)^2}{12}$$

$$\therefore \boxed{v(x) = \frac{(b-a)^2}{12}}$$

$$\therefore \text{Here } v(x) = \frac{(10-2)^2}{12} = \frac{64}{12} = 16/3 // \Rightarrow \text{option a}$$

9) X is uniformly distributed random variable that take values between 0 & 1. The value of $E(x^3)$ will be,

a) 0 b) $1/8$ c) $1/4$ d) $1/2$

Soln:

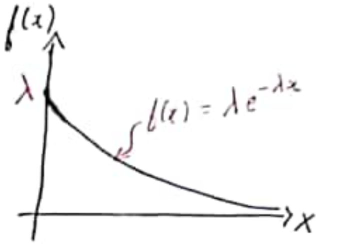
$$f(x) = \frac{1}{b-a} = \frac{1}{1-0} = 1 //$$

$$\therefore E(x^3) = \int_0^1 x^3 \cdot 1 \cdot dx = \frac{1}{4} // \Rightarrow \text{option c}$$

Exponential Distribution

eg: Waiting time in queues for railway tickets

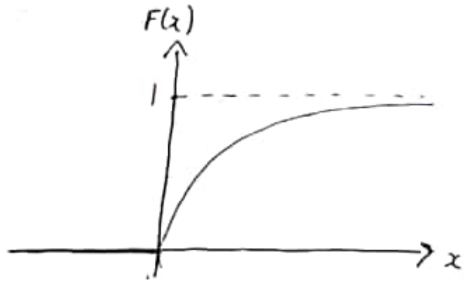
$$\boxed{\begin{aligned} f(x) &= \lambda e^{-\lambda x}, \quad x > 0 \ \& \ \lambda > 0 \\ &= 0, \quad \text{otherwise} \end{aligned}}$$



cdf: For $x < 0$, $F(x) = 0$

$x > 0$, $F(x) = \int_0^x \lambda e^{-\lambda x}\,dx = \lambda \left[\dfrac{e^{-\lambda x}}{(-\lambda)}\right]_0^x = -(e^{-\lambda x} - 1)$

$\Rightarrow F(x) = \underline{1 - e^{-\lambda x}}$



$$\therefore \boxed{\begin{aligned} F(x) &= 0, \quad x < 0 \\ &= 1 - e^{-\lambda x}, \quad x > 0 \end{aligned}}$$

Q) Assume that the duration in minutes of a telephone conversation follows the exponential distribution.

$f(x) = \frac{1}{5} e^{-x/5}$, $x \geq 0$. The probability that the conversation will exceed five minutes is

a) $1/e$ b) $1 - 1/e$ c) $1/e^2$ d) $1 - 1/e^2$

Soln

$P(x > 5) = 1 - P(x < 5)$

$= 1 - \int_0^5 \frac{1}{5} e^{-x/5}\,dx$

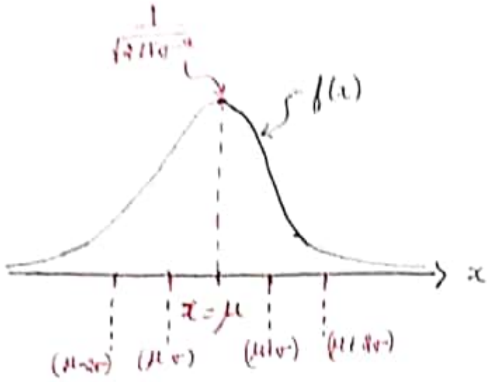
$\Rightarrow P(x > 5) = 1 - \frac{1}{5} \cdot \dfrac{\left[e^{-x/5}\right]_0^5}{(-1/5)} = 1 + 1(e^{-1} - 1)$

$\Rightarrow P(x > 5) = 1/e$ $\Rightarrow$ option a

Normal Distribution / Gaussian Distribution

pdf:

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty, \quad -\infty < \mu = m < \infty, \quad \sigma^2 > 0}$$



Area under curve

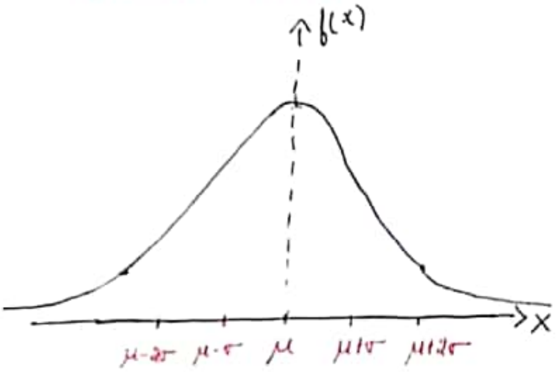
$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$$

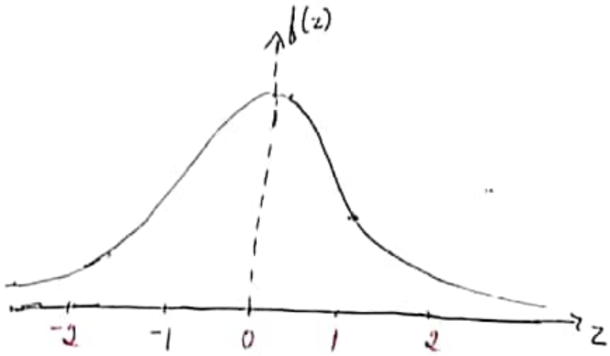
Max value of $f(x)$ occurs at $x=\mu$

$$\therefore \boxed{f(\mu) = \frac{1}{\sqrt{2\pi\sigma^2}}}$$

* $\boxed{\text{Mean} = \text{Mode} = \text{Median} = \mu}$

Z-Distribution / Std. Normal Distribution:



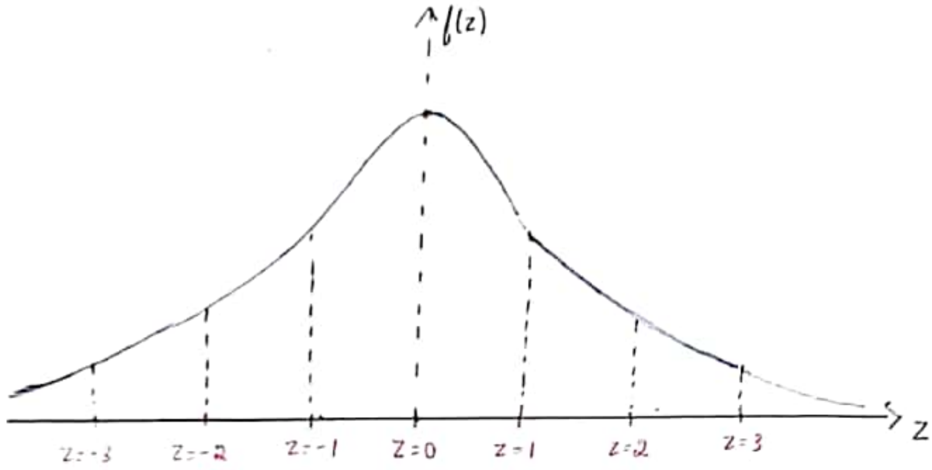


$\boxed{\text{Here } \mu = 0 \text{ \& } \sigma = 1}$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$f(x = \mu + \sigma z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left[z^2\right]}$$

$$\therefore \boxed{f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}}, \quad \mu = 0, \sigma = 1$$



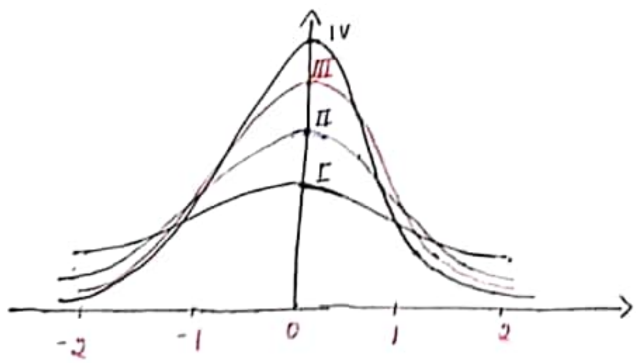
* $\int_{z=0}^{\infty} f(z)\,dz = \int_{-\infty}^{0} f(z)\,dz = 1/2$

* $\int_{-\infty}^{\infty} f(z) = 1$

* $P(-1 < z < 1) = 0.6827$
  $P(-2 < z < 2) = 0.9545$
  $P(-3 < z < 3) = 0.9973$

* $P(0 < z < 1) = \frac{0.6827}{2} = 0.341$

6) Below, which one has the lowest variance?



a) I b) II
c) III d) IV

Soln Ans (option d) $f(x) \propto \frac{1}{\text{Variance}}$

Since f(x) in IV is high ⇒ variance has to be low

9) The annual precipitation data of a city is normally distributed with mean & S.D as 1000 mm & 200 mm, resp.ly The probability that the annual precipitation will be more than 1200mm is

a) $< 50\%$ b) 50% c) 75% d) 100%

Soln $\mu = 1000\ mm$, $\sigma = 200\ mm$

$$P(x > 1200) = P\left(\frac{x-\mu}{\sigma} > \frac{1200 - 1000}{200}\right)$$

$$= P(z > 1)$$

$$= 0.5 - \frac{1}{2} P(-1 < z < 1)$$

$$= 0.5 - \frac{1}{2}(0.6827)$$

$$= \underline{\underline{0.15}} \Rightarrow \text{option a}$$

5) $f_x(x) = \frac{1}{\sqrt{5\pi}} e^{-\left\{\frac{(x-1)^2}{5}\right\}}$, $-\infty < x < \infty$, then $\int_1^{\infty} f_x(x)\, dx = ?$

a) 0 b) $1/2$ c) $1 - 1/e$ d) 1

Soln $\sigma^2 = 4 \Rightarrow \sigma = 2$, $\mu = 1$

$$\therefore \int_1^{\infty} f_x(x)dx = P(x>1) = P\left(z > \frac{1-1}{\sigma}\right) = P(z>0)$$

$$\therefore \int_1^{\infty} f(x)\, dx = 1/2 \;// \Rightarrow \text{option b}$$

HW

6) Let X be a zero mean unit variance Gaussian random variable. $E[|x|]$ is equal to ______.

Soln $$E[|x|] = \int_{-\infty}^{\infty} |x| f(x)\, dx = \int_{-\infty}^{\infty} |x| \cdot \frac{1}{\sqrt{2\pi}} e^{-1/2\, x^2} dx \quad [\because \mu = 0, \sigma = 1]$$

$$\Rightarrow E[|x|] = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} x\, e^{-x^2/2} dx$$

$+ x^2/2 = t$

$\Rightarrow x\,dx = dt$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-t}\, t^0\, dt$$

$$= 0.798 \times \sqrt{1}$$

$$\Rightarrow E[|x|] = \underline{\underline{0.798}}$$

HW 5) $\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x^2/8} dx = ?$

Soln M1 Using Gamma Function

$\frac{x^2}{8} = t \Rightarrow \cancel{\frac{x\,dx}{4}}\ \cancel{t} \cdot\ x\,dx = 4\,dt \Rightarrow dx = \frac{4}{\sqrt{8t}}\,dt = \frac{4}{\sqrt{8}} t^{-1/2} dt$

$I = \frac{1}{\sqrt{2\pi}} \cdot \frac{4}{\sqrt{8}} \int_0^\infty t^{-1/2} e^{-t}\,dt = \frac{4}{\sqrt{8 \cdot 2\pi}} \sqrt{1/2} = \frac{4}{4\sqrt{\pi}} \cdot \sqrt{\pi} = 1$

$\therefore I = 1$

M2 Using Normal Distribution

$I = 2 \int_0^\infty \frac{1}{\sqrt{2\pi (2)^2}} e^{-1/2\left(\frac{x-0}{2}\right)^2} dx = 2\,P(x > 0)$

$\Rightarrow I = 2 \cdot P\left[z > \frac{0-0}{2}\right] = 2\,P(z > 0) = 1$

$\therefore I = 1$

HW

Q) If we break a stick into 3 parts, what is the probability that a triangle is formed from these parts?

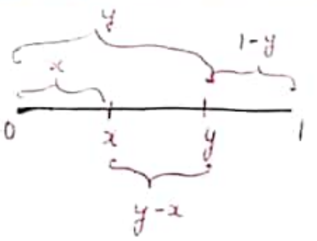
Soln M1 There are only 4 possibilities of breaking the stick into 3 parts:

1. Only first part is greater than half
2. Only second part is greater than half
3. Only third part is greater than half
4. No part is greater than half.

For the formation of triangle, no part should be greater than half. This is satisfied by only the 4^{th} event out of all the 4 event possibilities

$\therefore$ probability $= 1/4$

M2



Case ① Assume $x < y$.

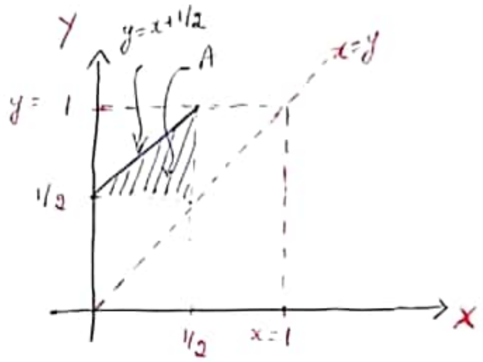
Lengths: x, $y-x$, $1-y$

Necessary condition for a Δ: $a+b > c$

$x + (y-x) > 1-y \Rightarrow y > 1/2$

$y-x + 1-y > x \Rightarrow x < 1/2$

$x + 1-y > y-x \Rightarrow 2y < 1+2x \Rightarrow y < x+1/2$



$A = 1/2 \times 1/2 \times 1/2 = 1/8$

Case ②: Assume $x > y$

Similarly we get $A = 1/8$

$\therefore$ Final ans $= 1/8 + 1/8 = 1/4$

27/04/20 MEAN, MEDIAN & MODE:

$$\boxed{\text{Mean } \bar{x} = \frac{\sum x_i}{n}}$$

Q) No. of visits made by 10 mothers to clinic were 8, 6, 5, 5, 7, 4, 5, 9, 7, 4. Calculate avg. no of visits.

Soln M1 $\bar{x} = \frac{\sum x_i}{n} = \frac{60}{10} = 6$

M2 $\boxed{\bar{x} = \frac{\sum f_i x_i}{\sum f_i}} = \frac{(8\times1)+(6\times1)+(5\times3)+(7\times2)+(4\times2)+(9\times1)}{10}$

$= 6$

Q) The foll data about the flow of liquid was observed in a continuous chemical process plant. Mean flow rate of the liquid is

Flow rate (litres/sec)	7.5 to 7.7	7.7 to 7.9	7.9 to 8.1	8.1 to 8.3	8.3 to 8.5	8.5 to 8.7
Frequency	1	5	35	17	12	10

Soln

$$\bar{x} = \frac{(7.6\times1) + (7.8\times5) + (8\times35) + (8.2\times17) + (8.4\times12) + (8.6\times10)}{80}$$

$$\Rightarrow \bar{x} = 8.16 \text{ litres/sec.}$$

Median.

Q) What is the median of $\{2, 1, 3\}$

Soln Step 1. Arrange in ascending order / Descending order

$\{1, 2, 3\}$

Step 2 Find the middle most value

$\therefore$ Median $= 2$ //

Q) Median of $\{4, 1, 2, 3\}$??

Soln ① $\{1, 2, 3, 4\}$

② Median $= \frac{2+3}{2} = 2.5$ //

eg: $\{1, 2, 3\} \longrightarrow$ Mean $= 2$, Median $= 2$ } sometimes same

$\{1, 2, 3, 6, 8\} \longrightarrow$ Mean $= 4$, Median $= 3$ } sometimes different

Q) The spot speeds (expressed in km/hr) observed at a road section are 66, 62, 45, 79, 32, 51, 56, 60, 53 & 49. The median speed (expressed in km/hr) is ______

Soln Ascending order: 32, 45, 49, 51, 53, 56, 60, 62, 66, 79

$\therefore$ Median speed $=$ ~~53~~ 54.5 km/hr

To find median for even no. of terms:

Step① Find the value of $\left(\frac{N}{2}\right)^{th}$ term [In an appropriate order]

Here $10/2 = 5^{th}$ term is 53

Step② Find the value of $\left(\frac{N}{2}+1\right)^{th}$ term.

Here 6^{th} term is 56

Step③ Find the avg of these values

$\therefore$ Median $= \frac{53+56}{2} = 54.5$

5) 3, 8, 9, 4, 12, 34, 21, 7, 1 Median = ?

Soln Ascending order: 1, 3, 4, 7, 8, 9, 12, 21, 34

Odd number of terms:

Median $= \left(\frac{N+1}{2}\right)^{th}$ term value

$\Rightarrow$ Median $= 5^{th}$ term value $= 8$

6)

x_i	20	29	28	33	42	38	43	25
f_i	6	28	24	15	2	4	1	20

Find the median ???

Soln

x_i	f_i	cf
20	6	6
25	20	26
28	24	50
29	28	78
33	15	93
38	4	97
42	2	99
43	1	100

$100 \Downarrow \Sigma f_i$

$N = 100$ is even

$\therefore$ 50^{th} term is 28

51^{th} term is 29

$\therefore$ Median $= \frac{28+29}{2} = 28.5$

3)

x_i	f_i
65-85	4
85-105	5
105-125	13
125-145	20
145-165	14
165-185	8
185-205	4

cf

Median = ?

Median $M = L + \frac{\{\frac{N}{2} - pcf\}h}{f}$

Soln

x_i	f_i	cf
65-85	4	4
85-105	5	9
105-125	13	22
125-145	20	42
145-165	14	56
165-185	8	64
185-205	4	68

34th term is present here

Total count $N = 68$ (even)

$\therefore \frac{N}{2} = 34^{th}$ term

$\therefore L = 125$, $f = 20$, $pcf = 22$, $h = 20$

$$\therefore M = L + \frac{(N/2 - pcf) \times h}{f} = 125 + \frac{(34 - 22) \times \cancel{20}}{\cancel{20}}$$

$\Rightarrow M = 137$

5) Based on the grouped data below, find the median

Time to travel to work	Frequency
1 - 10	8
11 - 20	14
21 - 30	12
31 - 40	9
41 - 50	7

Soln

x	f	cf
0.5 - 10.5	8	8
10.5 - 20.5	14	22
20.5 - 30.5	12	34
30.5 - 40.5	9	43
40.5 - 50.5	7	50

$\frac{N}{2} = \frac{50}{2} = 25^{th}$ term.

$L = 20.5$

$f = 12$

$h = 10$

$pcf = 22$

$\therefore M = 20.5 + \left[\frac{25 - 22}{12}\right] \times 10$

$\therefore M = 23$

Note: It is very important to arrange the values in ascending/descending order while finding median.

Q) The marks obtained by a set of students are: 38, 84, 45, 70, 75, 60, 48. The mean & median marks, resp.ly are

a) 45 & 75   b) 55 & 48   c) 60 & 60   d) 60 & 70.

Soln Ascending order: 38, 45, 48, 60, 70, 75, 84.

Mean $= \frac{420}{7} = 60$

Median = 4^{th} term value = 60

⇒ option c

MODE:

eg. ① {1, 2, 2, 3, 3, 3} ⟶ Mode = 3 (Unimodal)

② {1, 2, 2, 3, 3} ⟶ Mode = 2 & 3 (Bimodal)

③ {1, 2, 2, 3, 3, 4, 4} ⟶ Mode = 2, 3, 4 (Trimodal)

④ {1, 2, 2, 3, 3, 4, 4, 5, 5} ⟶ Mode = 2, 3, 4 & 5 (Multimodal)

(5) $\{1, 2, 3, 4, 5\} \longrightarrow$ Mode = No mode

6) Marks obtained by 100 students in an examination are given in the table

Sl No.	Marks obtained	Number of students
1	25	20
2	30	20
3	35	40
4	40	20

What would be the mean, median, & mode of the marks obtained by the students?

a) Mean 33; Median 35; Mode 40

b) Mean 35; Median 32.5; Mode 40

c) Mean 33; Median 35, Mode 35

d) Mean 35; Median 32.5, Mode 35

Soln:

$$\text{Mean} = \frac{(25)(20) + (30)(20) + (35)(40) + (40)(20)}{100}$$

$\Rightarrow$ Mean = 33

x	f	cf
25	20	20
30	20	40
35	40	80
40	20	100

$$\text{Median} = \frac{50^{th}\text{ value} + 51^{th}\text{ value}}{2}$$

$$= \frac{35 + 35}{2}$$

$\Rightarrow$ Median = 35

Mode = 35 $\qquad \therefore$ option c

Relation b/w Mean, Mode & Median:

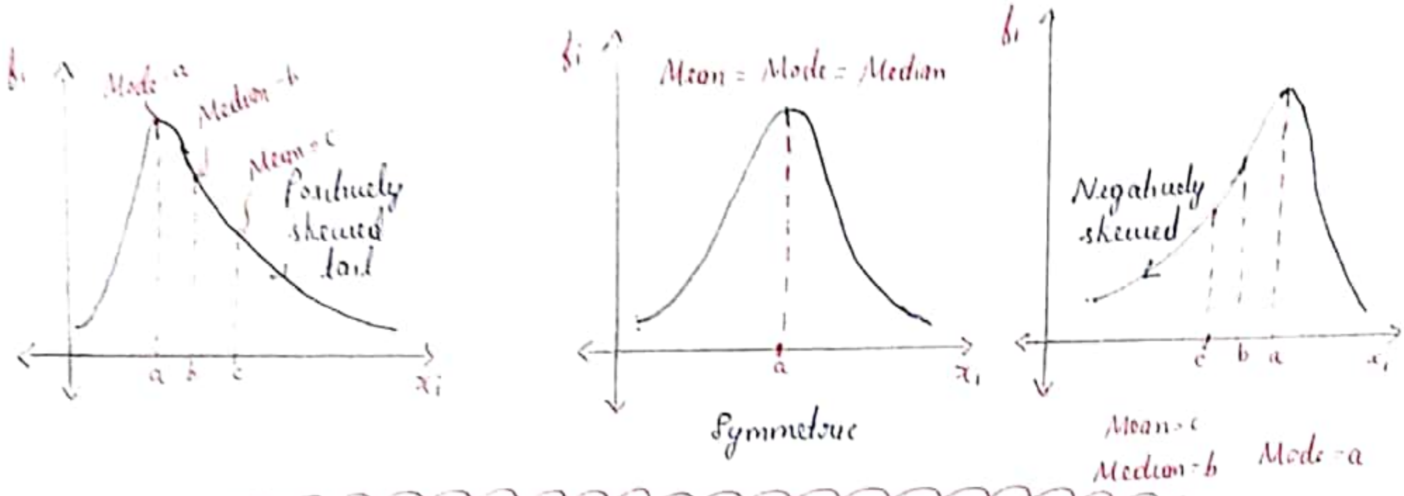
$$\boxed{\text{Mode} + 2\,\text{Mean} \simeq 3\,\text{Median}} \Rightarrow \text{(Approximation)}$$

In previous ques, mode = 35, mean = 33 & median = 35

$\therefore$ Mode = 3 Median − 2 Mean = 3(35) − 2(33) = 39

But mode = 35 $\neq$ 39

Scanned with CamScanner



<u>Note</u>

$$\boxed{\begin{array}{ll} \text{For positively skewed curve:} & \text{Mean} > \text{Median} > \text{Mode} \\ \text{For negatively skewed curve:} & \text{Mean} < \text{Median} < \text{Mode} \\ \text{For symmetric curve:} & \text{Mean} = \text{Mode} = \text{Median} \end{array}}$$

Q) The standard deviation of the data set $\{2, 3, 4, 7, 9\}$ = ?

<u>Soln</u> <u>M1</u>

$$\boxed{\sigma = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}}} = \sqrt{\frac{(2-5)^2+(3-5)^2+(4-5)^2+(7-5)^2+(9-5)^2}{n}}$$

$$= \underline{2.608}$$

<u>Note</u>

$$\boxed{\sigma = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}}$$

<u>M2</u>

$$\sigma^2 = \frac{4+9+16+49+81}{5} - 5^2 = 6.8$$

$$\therefore \sigma = \underline{2.608}$$

Scanned with CamScanner

17/03/20

LINEAR ALGEBRA

Matrix: 2 Dimensional → Rows & Columns.

Representation: $A_{m \times n}$ where m = row & n = column

${}_pA^q$ where p = row & q = column

Types of Matrix:

1) **Column Matrix:** $C = \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_n \end{array} \begin{bmatrix} a \\ b \\ c \\ \vdots \end{bmatrix}_{n \times 1}$

2) **Row Matrix:** $R = \begin{array}{c} \begin{bmatrix} a & b & c & \dots \end{bmatrix}_{1 \times m} \\ \begin{matrix} c_1 & c_2 & \dots & c_m \end{matrix} \end{array}$

3) **Rectangular Matrix:** $\boxed{m \neq n}$ eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3}$

Note * $|A|_{m \times n}$ for a rectangular matrix is not possible

* $|A - \lambda I|_{m \times n} = 0 \Rightarrow$ Eigen values for a rectangular matrix cannot be found.

4) **Square Matrix:** $\boxed{m = n}$ $A = [\ \ ]_{m \times m}$.

5) **Diagonal Matrix:** If the elements are present in primary diagonal & the rest elements are zero, then we can say it is a Diagonal Matrix

Note * primary diag $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ ✓ $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$ ✗ Secondary diag

* For diagonal matrix, $a_{ij} = 0$ but a_{ii} may or may not be 0

eg: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a diagonal matrix.

Benefits of Diagonal Matrix: * $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & 11 \end{bmatrix} = \begin{bmatrix} 11 & 0 \\ 0 & 13 \end{bmatrix}$

* $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 10 & 0 \\ 0 & 11 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 22 \end{bmatrix}$

*** $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/1 & 0 \\ 0 & 1/2 \end{bmatrix}$

* $\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 1 \times 2$

* Min. number of zeroes. $= (n \times n) - n$.

6) Unit Matrix: Always a square matrix where $a_{ij} = 1$ when $i = j$ & others are zero.

* It is a Diagonal matrix
* It is a Identity matrix
* $I^{-1} = I$
* $|I| = 1$
* $AI = A$, $IA = A$.

7) Null Matrix: All the elements should be zero. Also known as zero matrix.

8) Upper Triangular Matrix: → Always a square matrix

eg: $\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 4 & 7 \\ 0 & 0 & 8 \\ 0 & 0 & 8 \end{bmatrix}$

Note * Determinant of Triangular Matrix = Product of Diagonal Elements
* Eigen values of the ~~diagonal~~ triangular matrix are the diagonal elements

9) Lower Triangular Matrix: eg: $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix}$

Lower triangular elements are not zero

10) Scalar Matrix: $= \boxed{kI}$ eg: $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Note: $\begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix}$ is not a scalar matrix

11) Symmetric Matrix: $\boxed{A = A^T}$ eg: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 7 \end{bmatrix}$

Here $\boxed{a_{ij} = a_{ji} \text{ where } i \neq j}$

12) Skew Symmetric Matrix: $\boxed{A = -A^T}$, $\boxed{a_{ij} = -a_{ji} \text{ \& Diagonal elements} = 0}$

eg: $\begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & 6 \\ 3 & -6 & 0 \end{bmatrix}$

Properties: * It is a square matrix

13) Hermition Matrix: $\boxed{a_{ij} = \overline{a_{ji}}}$ ⟹ conjugate

eg: $\begin{bmatrix} 5 & 2+3i \\ 2-3i & 2 \end{bmatrix}$

14) Skew Hermition Matrix: $\boxed{a_{ij} = -\overline{a_{ji}}}$ & $\boxed{\text{Diagonal} = 0 \text{ or purely imaginary}}$

eg: $\begin{bmatrix} 0 & -2-3i \\ 2-3i & 0 \end{bmatrix}$

H.W
Q Check whether $\begin{bmatrix} i & -2-3i & 1 \\ 2-3i & 3i & -1 \\ -1 & 1 & 0 \end{bmatrix}$ is a skew Hermition matrix or not

Soln: * Diagonal elements are either zero or pure imaginary

* Also $a_{ij} = -\overline{a_{ji}}$

∴ It is a skew Hermition matrix

***
15) Orthogonal Matrix: $\boxed{|A| = \pm 1 \text{ \& } AA^T = A^TA = I}$

$AA^T = I \Rightarrow AA^T = AA^{-1} \Rightarrow \boxed{A^T = A^{-1}}$ :: For 100% sureity

Inverse = Transpose

Proof of $|A| = \pm 1$, where A = Orthogonal Matrix

$AA^T = I \Rightarrow |AA^T| = |I| \Rightarrow |AA^T| = 1$

$\Rightarrow |A|\,|A^T| = 1 \Rightarrow |A|^2 = 1 \quad (\because |A^T| = |A|) \Rightarrow \boxed{|A| = \pm 1}$

Note: Converse statement is not true. i.e. If $|A| = \pm 1$ then we cannot say A is an orthogonal matrix (It may or may not be)

16) Rotational Matrix: If $\boxed{A^{-1} = A^T \ \& \ |A| = +1} \Rightarrow$ Rotational matrix

17) Singular Matrix: $\Rightarrow \boxed{|A| = 0}$

Note: * A^{-1} does not exist when A is singular matrix.

* Non singular matrix can also be called Invertible Matrix.

18) Unitary Matrix: $\boxed{AA^\theta = A^\theta \cdot A = I}$, A^θ = Transpose of conjugate

Note: $A^\theta = A^* = (\bar{A})^T = (\overline{A^T})$ [Don't get confuse with the representation]

Q) Check whether $A = \begin{bmatrix} 1+i & -1+i \\ 1-i & -1-i \end{bmatrix}$ is a unitary matrix or not

Soln: $A^\theta = \begin{bmatrix} 1-i & 1+i \\ -1-i & -1+i \end{bmatrix}$

$A \cdot A^\theta = \begin{bmatrix} 1+i & -1+i \\ 1-i & -1-i \end{bmatrix}\begin{bmatrix} 1-i & 1+i \\ -1-i & -1+i \end{bmatrix}$

$\Rightarrow AA^\theta = \begin{bmatrix} 2+2 & 0 \\ 0 & 2+2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4I$

$\therefore$ A is not a unitary matrix

* If $AA^\theta = I \Rightarrow \boxed{A^\theta = A^{-1}} \rightarrow$ TRICK

(19) Special Matrices:

* Involutory Matrix ⇒ $\boxed{A^2 = I}$ eg: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

* Idempotent Matrix ⇒ $\boxed{A^2 = 0}$ eg: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

* Nilpotent Matrix ⇒ $\boxed{A^k = 0}$ ~~eg~~

eg: $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

Here $A^1 \neq 0$, $A^2 \neq 0$.

But $A^3 = 0$ ⇒ A is Nilpotent matrix of class/index 3

Q) If $B = A^T$, $A = ?$ a) Skew symmetric b) Symmetric about Secondary Diagonal c) Always symmetric d) Another general matrix

Soln option d.

Q) A is real square, symmetric. matrix, then AA^T is ?

a) Unsymmetric b) Always symmetric

c) Skew symmetric d) Sometimes symmetric.

Soln Let $AA^T = X$

$\Rightarrow (AA^T)^T = X^T$

$\Rightarrow X^T = (A^T)^T \cdot A^T = AA^T = X$

$\Rightarrow X^T = X$ ⇒ Always symmetric. (option b).

Q) $A^T = A^{-1}$, $A = ?$ a) Normal b) Symmetric c) Hermitian d) Orthogonal

Soln option d.

Q) ~~If~~ B = skew symmetric matrix if a) $B^T = -B$ b) $B^T = B$ c) $B^{-1} = B$ d) $B^{-1} = B^T$

Soln option a.

Operation on Matrices.

1) **Matrix Addition:** $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

* $A + B = B + A$ $\Rightarrow$ Matrix addition is commutative
* $A + (B + C) = (A + B) + C$ $\Rightarrow$ Matrix addition is associative.

2) **Matrix Subtraction:** $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

* $A - B \neq B - A$ $\Rightarrow$ Matrix subtraction is not commutative
* $(A - B) - C \neq A - (B - C)$ $\Rightarrow$ Matrix subtraction is not associative.

3) **~~Mat~~ Scalar Multiplication:** $2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}2$

$\therefore \boxed{kA = [kA] = Ak}$; (k = scalar)

***** Note: * $\begin{bmatrix} 2 & 2 \\ 4 & 5 \end{bmatrix} \neq 2\begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$

* $\begin{vmatrix} 2 & 2 \\ 4 & 5 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ 4 & 5 \end{vmatrix}$

* $k(A + B) = kA + kB$
* $(k + g)A = kA + gA$

4) **Matrix Multiplication:** $A_{m \times p} \times B_{q \times n} = C_{m \times n}$ if $\boxed{p = q}$

* $A_{3\times3} \times B_{4\times4} \neq$ Not possible
* $A_{3\times3} \times B_{3\times4} = C_{3\times4}$

eg: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 7 \end{bmatrix}_{2\times3} \begin{bmatrix} 4 & 8 \\ 6 & 2 \\ 5 & 3 \end{bmatrix}_{3\times2} = \begin{bmatrix} 4+12+15 & 8+4+9 \\ 16+12+35 & 32+4+21 \end{bmatrix} = \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$

*** * $AB \neq BA$ $\Rightarrow$ Matrix multiplication is not commutative.

* If $A \times B$ matrices satisfies the multiplication law then $B \times A$ need not satisfy the law (may or may not)

Scanned with CamScanner

* **Special Cases of $\boxed{AB = BA}$**

Case ①: Let $B = I$ $\Rightarrow$ $AI = IA = A$

Case ②: Let $B =$ Null $\Rightarrow$ $A \cdot 0 = 0 \cdot A = 0$

Case ③: Let $B = A^{-1}$ $\Rightarrow$ $AA^{-1} = A^{-1}A = I$

*** $(A \times B) \times C = A \times (B \times C)$ $\Rightarrow$ Matrix multiplication is associative.

* If $AB = 0$ then A or B need not always be a null matrix.

eg: $\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} \underset{B}{\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

* If $AB = AC \Rightarrow B = C$ need not be always true

Proof:

$$AB = AC$$
$$(A^{-1}A)B = (A^{-1}A)C$$
$$\Rightarrow \underline{B = C} \Rightarrow \text{only when } A^{-1} \text{ exist.}$$

$\therefore$ If $AB = AC$ & $B = C$ $\Rightarrow$ A^{-1} is invertible matrix

H.W

Q If we need to multiply $A \cdot B$, then we should have square matrices only ??

Soln Not necessary eg $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3} \times \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3\times 2}$ is possible.

18/03/20

Let $A =$ real square matrix $= \begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix}$

$$A + A' = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -2 & -1 \\ 2 & 5 & 6 \\ 4 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 10 & 9 \\ 3 & 9 & 6 \end{bmatrix} \Rightarrow \text{Real Symmetric.}$$

$$A - A' = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix} - \begin{bmatrix} 1 & -2 & -1 \\ 2 & 5 & 6 \\ 4 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 5 \\ -4 & 0 & -3 \\ -5 & 3 & 0 \end{bmatrix} \Rightarrow \text{Real skew Symmetric.}$$

$$(A + A') + (A - A') = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 10 & 9 \\ 3 & 9 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 5 \\ -4 & 0 & -3 \\ -5 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 8 \\ -4 & 10 & 6 \\ -2 & 12 & 6 \end{bmatrix}$$

$$\Rightarrow \quad 2A = \begin{bmatrix} 2 & 4 & 8 \\ -4 & 10 & 6 \\ -2 & 12 & 6 \end{bmatrix} \quad \Rightarrow \quad A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix}$$

$$\therefore \quad \boxed{A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)} \quad ***$$

Note * For a given real square matrix A;

$\frac{A + A^T}{2} \Rightarrow$ Symmetric part of A

$\frac{A - A^T}{2} \Rightarrow$ Skew symmetric part of A

* If A = symmetric matrix or any square matrix, then AA^T = Always symmetric matrix.

eg: $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1+4 & 2+10 \\ 2+10 & 4+25 \end{bmatrix} = \begin{bmatrix} 5 & 12 \\ 12 & 29 \end{bmatrix}$

* If A = real symmetric matrix, then also AA^T = Always symmetric matrix

* $(P+Q)^2 \neq P^2 + 2PQ + Q^2$

Proof: $(P+Q)^2 = (P+Q)(P+Q) = P^2 + \underbrace{PQ + QP}_{\text{not same}} + Q^2 \neq P^2 + 2PQ + Q^2$.

Determinants: $\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$

Minors & Cofactors: Consider $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Minor of $a_{21} = M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}a_{33} - a_{13}a_{32}$

Cofactor $= (-1)^{i+j} M_{ij}$

Cofactor of $a_{21} = (-1)^{2+1} M_{21} = a_{13}a_{32} - a_{12}a_{33}$

Q) Find determinant of $\begin{vmatrix} 1 & 2 & 0 \\ -1 & 6 & 1 \\ 2 & 0 & 2 \end{vmatrix}$

Soln

$$\Delta = 1(12) - 2(-2-2) = 12 + 8 = 20$$

Q) Find determinant of $\begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & 1 \end{bmatrix}$

Soln

Select third row:

$$\Delta = (-1)^{3+4} \cdot 1 \cdot \begin{vmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & -2 & 0 \end{vmatrix} = -1 \times [-1(-1)] = -1$$

Some Important Points

① * If $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix} = A$, then $|A| = 0$ $\Rightarrow$ If a matrix contains any row or column with zero elements then its determinant is always zero.

* If A = Lower triangular or Upper triangular or Diagonal matrix, then $|A|$ = Product of the primary diagonal elements.

eg: If $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} = A$, $|A| = 1 \times 4 \times 6 = 24$

② $|A^{-1}| = \dfrac{1}{|A|}$ & $|A^T| = |A|$ *****

Q) If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, then $|A^T A^{-1}| = ?$

Soln

$$|A| = 1 + \tan^2 x = \sec^2 x$$

$$|A^T A^{-1}| = |A^T||A^{-1}| = |A| \times \frac{1}{|A|} = 1$$

③ Q) If $\begin{vmatrix} 1 & 3 & 2 \\ 0 & 5 & -6 \\ 2 & 7 & 8 \end{vmatrix} = 26$, then $\begin{vmatrix} 2 & 7 & 8 \\ 0 & 5 & -6 \\ 1 & 3 & 2 \end{vmatrix} = ?$

Soln

$$\begin{vmatrix} 2 & 7 & 8 \\ 0 & 5 & -6 \\ 1 & 3 & 2 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_3} = -\begin{vmatrix} 1 & 3 & 2 \\ 0 & 5 & -6 \\ 2 & 7 & 8 \end{vmatrix} = -26$$

$\boxed{|B| = (-1)^n |A|}$ where B = modified from A by interchanging positions of either row/column & n is the no. of times the position is changed.

④ Q) If $\begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = -12$, then $\begin{vmatrix} 2 & 6 & 0 \\ 4 & 12 & 8 \\ -2 & 0 & 4 \end{vmatrix} = ?$

Soln
$$\begin{vmatrix} 2 & 6 & 0 \\ 4 & 12 & 8 \\ -2 & 0 & 4 \end{vmatrix} = 2\times2\times2 \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 2^3 \times -12 = -96 //$$

If $B_{n\times n} = k A_{n\times n}$ then $\boxed{|B| = k^n |A|}$

⑤ When matrix operations are performed:

Consider $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} = 1$

Now Perform $R_2 \to R_2 - 3R_1$

ie $\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 //$ (same as $|A|$)

Note If we perform $R_2 \to 3R_1 - R_2$ ie $\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = -1$ $(\because -|A|)$

Hence while finding determinants if we perform matrix operations such as $R_i \to R_i \pm kR_j$ or $C_i \to C_i \pm kC_j$ then the value of the determinant will remain same.

Q) $\begin{bmatrix} 3 & 9 & 45 \\ 7 & 4 & 105 \\ 13 & 2 & 195 \end{bmatrix}$ i) Add R_3 to R_2 ii) Sub. C_3 from C_1 What is the determinant of Resultant matrix?

Soln Let $A = \begin{bmatrix} 3 & 9 & 45 \\ 7 & 4 & 105 \\ 13 & 2 & 195 \end{bmatrix}$ $\therefore |A| = 15 \begin{vmatrix} 3 & 9 & 3 \\ 7 & 4 & 7 \\ 13 & 2 & 13 \end{vmatrix}$

Since the given Matrix operation does not change the value of the resultant determinant

$|\text{Resultant matrix}| = |A| = 15 \times 0 = 0 //$

Note * If a matrix has any 2 rows/column same, then its determinant = zero.

Q) Find the determinant of $\begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}$

Soln M1

$$\Delta = \begin{vmatrix} 5 & 1 & 1 & 1 \\ 5 & 2 & 1 & 1 \\ 5 & 1 & 2 & 1 \\ 5 & 1 & 1 & 2 \end{vmatrix} = 5\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix} = 5\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3 + C_4$

$\Rightarrow \underline{\underline{\Delta = 5}}$

M2

$$\boxed{\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = (abcd)\left[1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right]}$$ *****

Here $a = b = c = d = 1$

$\therefore \Delta = (1 \times 1 \times 1 \times 1)\left[1 + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1}\right] = 5$

***

⑥ * $\boxed{|Adj\,A| = |A|^{n-1}}$, n is the size of the matrix

* $\boxed{|Adj\,(Adj\,(A))| = |A|^{(n-1)^2}}$

Inverse of Matrix: $\boxed{A^{-1} = \frac{adj(A)}{|A|}}$

* When A is non singular then only A^{-1} exists

Properties

* $$\boxed{\begin{aligned} AA^{-1} &= A^{-1}A = I \\ (AB)^{-1} &= B^{-1}A^{-1} \\ (ABC)^{-1} &= C^{-1}B^{-1}A^{-1} \end{aligned}}$$

a) If $A = \begin{bmatrix} -3 & 5 \\ 2 & 1 \end{bmatrix}$, $A^{-1} = ?$

Soln

$\boxed{\text{For } 2\times2 \text{ matrix, if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$

$adj\, A = \begin{bmatrix} 1 & -5 \\ -2 & -3 \end{bmatrix}$ $\quad\therefore A^{-1} = \dfrac{adj(A)}{|A|} = \begin{bmatrix} -1/13 & 5/13 \\ 2/13 & 3/13 \end{bmatrix}$

$|A| = -3 - 10 = -13$

3) If $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $A^{-1} = ?$

Soln

$|A| = -1 + (-1) = -2$

$$\text{Cofactor of } A = \begin{bmatrix} (-1) & 0 & (-1) \\ -(-1) & 0 & -(1) \\ (-1) & -(2) & (1) \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & -1 \\ -1 & -2 & 1 \end{bmatrix}$$

$$adj(A) = [\text{Cofactor of } A]^T = \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} +1/2 & -1/2 & +1/2 \\ 0 & 0 & +1 \\ +1/2 & +1/2 & -1/2 \end{bmatrix}$$

Some Tricks in Inverse:

* Check the first term of the matrix in the options & match it by calculating adj(A) & |A| i.e. $A^{-1} = \dfrac{adj(A)}{|A|}$

* $A \cdot A^{-1} = I$ $\Rightarrow$ Multiply the given matrix with the inverse matrix in the options such that it will give I (first element should be 1)

* $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}^{-1} = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}$ [For Diagonal Matrix]

* If the given matrix is orthogonal i.e. $~~AA^T/I~~$ $AA^T = I$,
then $A^{-1} = A^T$

Q) If $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, then $A^{-1} = ?$

a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 2 & 2 & -2 \\ -2 & 2 & -2 \\ 0 & 2 & 2 \end{bmatrix}$ d) $\begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}$

Soln Calculate the adjoint for the first element

Here $|A| = 2$

$\therefore adj(A)_{11} = \frac{1}{2} \Rightarrow$ option d

Q) $X = \begin{bmatrix} a & 1 \\ -a^2+a-1 & 1-a \end{bmatrix}$, $X^2 - X + I = 0$, $X^{-1} = ?$

Soln $I = X - X^2$

$\Rightarrow X^{-1}I = X^{-1}X - X^{-1}X^2 = I - X$

i.e. $X^{-1} = I - X$

Q) If $A^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & -7 \end{bmatrix}$, then $A = ?$

Soln $(A^{-1})^{-1} = A$

$\therefore A = (A^{-1})^{-1} = \dfrac{\begin{bmatrix} -7 & -2 \\ -4 & -1 \end{bmatrix}}{+7-8} = ~~\begin{bmatrix} -7/15 & 2/15 \\ 4/15 & 1/15 \end{bmatrix}~~$

$\Rightarrow A = \begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix}$

19/08/20
Q) If A & B are 2 matrices & AB exist then BA exist

a) only if A & B are symmetric
b) only if A has as many rows as B has columns
c) only if both A & B are square matrices
d) only if A & B are skew matrices

Soln $A_{m\times p} B_{p\times n} = C_{m\times n}$ | $B_{p\times n} \cdot A_{m\times p} = D_{p\times p} \Rightarrow$ only if $m = n$
[AB exists] $\Downarrow$
option b

Q) Let $A_{n\times n}$ be matrix of order n & I_{12} be the matrix obtained by interchanging the first & second rows of I_n. Then AI_{12} is such that its first.

a) row is the same as its second row

b) row is the same as second row of A

c) column is the same as the second column of A

d) row is a zero row

Soln Let $n=2$ $\quad I_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad I_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \therefore AI_{12} = \begin{bmatrix} b & a \\ d & c \end{bmatrix} \Rightarrow$ option c

Q) The no. of different $n\times n$ symmetric matrices with each elements being either 0 or 1 is

a) 2^n b) 2^{n^2} c) $2^{\frac{n^2+n}{2}}$ d) $2^{\frac{n^2-n}{2}}$

Soln M1 Assume $n=2$ & ~~Set~~

The various possible matrices are : $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$

$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore$ Total no. of possibilities for $n=2$ is 8

By putting $n=2$ in the options only option c satisfies $\therefore$ option c

Q) If A & B are non-zero symmetric matrices. Then $AB=0$ implies

a) $|A|=0$, b) $|B|=0$ c) $|A|=|B|=0$ d) A & B are orthogonal matrices

Soln M1 Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

Here $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow$ Condition satisfies

Also $|A|=0$ & $|B|=0 \Rightarrow$ option c

M.I. Given $AB = 0$.

$A^{-1}AB = A^{-1}0$

$\Rightarrow B = 0$ (But B is a non zero matrix)

$\therefore$ There is only one way to make $AB = 0$

i.e $|A|\,|B| = 0$

$\Rightarrow |A| = 0$ or $|B| = 0$ $\Rightarrow$ option c

Note: * In general if $AB = 0$ $\Rightarrow$ A & B are singular matrices

Q) If A be an invertible matrix & suppose that inverse of $7A$ is $\begin{bmatrix} -1 & 2 \\ 4 & -7 \end{bmatrix}$ then matrix A = ?

a) $\begin{bmatrix} 1 & 2/7 \\ 4/7 & 1/7 \end{bmatrix}$ b) $\begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & -4/7 \\ -2/7 & 1/7 \end{bmatrix}$ d) $\begin{bmatrix} 7 & 4 \\ 2 & 1 \end{bmatrix}$

Soln

$$(7A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & -7 \end{bmatrix}$$

$$\left[(7A)^{-1}\right]^{-1} = \frac{\begin{bmatrix} -7 & -2 \\ -4 & -1 \end{bmatrix}}{-1} = \begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix}$$

$$\therefore \Rightarrow 7A = \begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & 2/7 \\ 4/7 & 1/7 \end{bmatrix}$$

Q) $EF = G$, $E = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $F = ?$

Soln

$EF = G$

$E^{-1}EF = E^{-1}G$

$\Rightarrow F = E^{-1}$ $(G = I)$

$|E| = 1$

$$\text{Cofactors of } E = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ +\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{adj}(E) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore F = E^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q) If $A^+ = (A^TA)^{-1} \cdot A^T$. Which of the foll. is false?

a) $AA^+A = A$ b) $(AA^+)^2 A = AA^+$ c) $A^+A = I$ d) $AA^+A = A^+$

Soln

$$A^+ = (A^TA)^{-1}A^T = A^{-1}(A^T)^{-1}A^T = A^{-1}$$

$\Rightarrow A^+ = A^{-1} \Rightarrow$ option d

Q) Find the value of x such that $\begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix}$ becomes singular

Soln

$$\Delta = -2\,[48 - 12x]$$

$\Rightarrow 0 = -96 + 24x \Rightarrow x = \frac{12}{3} \Rightarrow \underline{x = 4}$

Q) $J_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ which is obtained by reversing the order of column of identity matrix I_6. Let $P = I_6 + \alpha J_6$, α is non-negative real no.

If $|P| = 0$, then $\alpha = ?$

Soln

$$|P| = 0 \Rightarrow \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 1 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 1 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 0.$$

$$\cancel{\Rightarrow \left[(1+\alpha) \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & \alpha \\ 1 & 1 & 0 & 0 & \alpha & 0 \\ 1 & 0 & 1 & \alpha & 0 & 0 \\ 1 & 0 & \alpha & 1 & 0 & 0 \\ 1 & \alpha & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 0 \right.}$$

$\cancel{\Rightarrow 1+\alpha = 0}$ ~~OR~~

$\cancel{\Rightarrow \alpha = -1}$

If $\alpha = 1$ then C_1 & C_6 will be same

$\therefore \underline{\alpha = 1}$

Q) If $|A| = 5$, $|B| = 40$, $|AB| = ?$

Soln $|AB| = |A||B| = 5 \times 40 = 200$

*** Q) No. of terms in expansion of general determinant of order n is

a) n^2 b) $n!$ c) n d) $(n+1)^2$

Soln Let $n = 2$: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \underbrace{ad - bc}_{2 \text{ terms}} \Rightarrow$ (option a & d is eliminated)

$n = 3$: $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a[ei - hf] - b[di - gf] + c[dh - eg]$

$= 6$ terms

$\therefore$ option b

Q) $\begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix} = ?$

Soln $\Delta = -31 - 4(-44) + 9(-17)$

$\Rightarrow \Delta = -8$

Q) If A & B are symmetric matrices, then which of the foll. is true?

a) $AA^T = I$ b) $A = A^{-1}$

c) $AB = BA$ d) $(AB)^T = B^T A^T$

Soln option d

Q) $L = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 4 & 5 \end{bmatrix}$, $M = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, $LM = ?$

Soln $LM = \begin{bmatrix} 6 & 5 \\ 9 & 8 \\ 12 & 13 \end{bmatrix}$

Q) If A is $m \times n$, B is $n \times p$, no. of multiplication operation & addition operation needed to calculate matrix AB resp.ly are?

a) mn^2p, mpn b) mpn, $(n-1)$ c) mpn, $mp(n-1)$ d) mn^2p, $(m+p)n$

Soln: Let $A = \begin{bmatrix} 1 & 2 \end{bmatrix}_{1\times 2}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}_{2\times 2}$ Here $m=1$, $n=2$, $p=2$

$$AB = \begin{bmatrix} (1\times 2)+(2\times 1) & (1\times 1)+(2\times 0) \end{bmatrix} \Rightarrow \text{Multiplication} = 4, \quad \text{Addition} = 2$$

only option (c) satisfies

Rank of a Matrix: No. of non zero rows in a matrix which has reduced to echelon form.

eg: ① $A = \begin{bmatrix} 1 & -2 & -1 \\ -3 & 3 & 0 \\ 2 & 2 & 4 \end{bmatrix} \xrightarrow[R_2 : R_2 + 3R_1]{R_3 : R_3 - 2R_1} \begin{bmatrix} 1 & -2 & -1 \\ 0 & -3 & -3 \\ 0 & 6 & 6 \end{bmatrix} \xrightarrow{R_3 : R_3 + 2R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore \rho(A) = 2$

M1

② $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix} \xrightarrow[R_3 : R_3 - 4R_1]{R_2 : R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & -2 & -4 \end{bmatrix} \xrightarrow{R_3 : R_3 - R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore \rho\,(\text{Rank}) = 2$

M2 Dependency of rows.

$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix}$ Here $R_3 = R_1 + R_2$ $\therefore R_2 \rightarrow R_2 + R_1$ $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 8 \\ 4 & 6 & 8 \end{bmatrix}$

$\downarrow R_3 \rightarrow R_3 - R_2$

$\rho = 2 \Leftarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 8 \\ 0 & 0 & 0 \end{bmatrix}$

Q) Find the rank of $\begin{bmatrix} 1 & 4 & 8 & 7 \\ 0 & 0 & 3 & 0 \\ 4 & 2 & 3 & 1 \\ 3 & 12 & 24 & 2 \end{bmatrix}$

Soln: $R_3 : R_3 - 4R_1$, $R_4 : R_4 - 3R_1$ $\begin{bmatrix} 1 & 4 & 8 & 7 \\ 0 & 0 & 3 & 0 \\ 0 & -14 & -29 & -27 \\ 0 & 0 & 0 & -19 \end{bmatrix} \Rightarrow \text{Rank} = 4$

Q) Find the rank of $A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

Soln $R_4 : R_4 + R_1$ $\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$ [continuation below]

Note
* $\rho(A) = 0 \Rightarrow A$ is a Null matrix
* $\rho(A) \neq -ve$
* For all matrix other than Null matrix, $0 < \rho(A) \leq m$ (no of rows of matrix)

Alternate method to find Rank of a Matrix (TRICK ☺)

Q) $A = \begin{bmatrix} 5 & 10 & 10 \\ 1 & 0 & 2 \\ 3 & 6 & 6 \end{bmatrix}$ Find Rank of A

Soln

Step 1: $|A|_{3\times3} = \begin{cases} 0 \longrightarrow \rho(A) \neq \text{size of the matrix} \\ \text{Not zero} \longrightarrow \rho(A) = \text{size of the matrix } (3) \end{cases}$

Step 2

Step 2: $|A|_{2\times2} = \begin{cases} 0 \longrightarrow \rho(A) \neq \text{size of the minor} \\ \text{Not zero} \longrightarrow \rho(A) = \text{size of the minor } (2) \end{cases}$

Here in this question, $|A| = 5[-12] - 10[6-6] + 10[6] = 0$

$\therefore$ Minor of $\begin{vmatrix} 5 & 10 \\ 1 & 0 \end{vmatrix} = -10 \neq 0 \Rightarrow \rho(A) = 2$

Continuation of the above Question

$|A|_{5\times5} = 1 \times \begin{vmatrix} 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} = 1 \times 0 = 0 \Rightarrow \rho \neq 5$

$|A|_{4\times4} = \begin{vmatrix} 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} = (-1)\begin{vmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (-1)(-1) \neq 0 \Rightarrow \boxed{\rho = 4}$

Note * Rank of a matrix is possible for a non square matrix also

Q) If $A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$, $\rho(A) = ?$

Soln $R_2 : 2R_2 - 3R_1$
$R_3 : 2R_3 - R_1$ $\begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & -1 & -1 \end{bmatrix} \Rightarrow \underline{\rho(A) = 2}$

Q) Rank of $m \times n$, $m < n$ cannot be more than ?

a) m b) n c) mn d) none

Ans option a

Q) If the rank of a matrix = no. of rows & no. of columns then the matrix is

a) non-singular b) singular c) transpose d) minor

Ans option a.

Properties ***

* $\boxed{\rho(AB) \leq \min[\rho(A), \rho(B)]}$

HW
Q) $A = [a_{ij}]$ $1 \leq i, j \leq n$, $n \geq 3$ $a_{ij} = i \times j$, Rank of A = ?

Soln Let $n = 3$, $j = 3$, $i = 1$

$\therefore A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \Rightarrow \underline{\rho(A) = 1} \Rightarrow$ Solution is WRONG

ANS IS CORRECT

SIMPLE NO WORRIES ☺

20/03/20 Eigen Value (λ):

* If A is a square matrix, then characteristic polynomial of A is $|A - \lambda I|$

* Characteristic eq^n of A is : $|A - \lambda I| = 0$

* Here λ = characteristic roots / Latent root / Proper value / characteristic value

Q) If $A = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}$, Find the eigen values

Soln

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} -1-\lambda & 4 \\ 4 & -1-\lambda \end{vmatrix} = 0.$$

$$(-1)^2(1+\lambda)^2 - 16 = 0. \Rightarrow (1+\lambda)^2 = 4^2$$

$$\Rightarrow 1+\lambda = +4 \text{ or } -4$$

$$\Rightarrow \lambda = 3 \text{ or } \lambda = -5$$

Q) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$, find eigen values.

Soln M1

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -6 & -11 & (-6-\lambda) \end{vmatrix} = 0$$

M2 [For 3x3 matrix] *****

Characteristic Eq : $\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0$

$s_1 =$ sum of diagonal elements

$s_2 =$ sum of minors of diagonal elements

$|A| = (-6)1 = -6$

$s_1 = 0+0+(-6) = -6$

$s_2 = \begin{vmatrix} 0 & 1 \\ -11 & -6 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ -6 & -6 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$

$\Rightarrow s_2 = 11$

$\lambda^3 - (-6)\lambda^2 + 11\lambda - (-6) = 0.$

$\Rightarrow \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$

$\lambda = -2,$

-2	1	6	11	6
	↓	-2	-8	-6
	1	4	3	0

$\lambda^2 + 4\lambda + 3 = 0$

$\Rightarrow \lambda = -3, -1$

$$\therefore \lambda_1 = -2, \ \lambda_2 = -3 \ \& \ \lambda_3 = -1$$

M3 Trick to solve cubic eq

Factors of $+6$: $\{1, 2, 3, 6\}$

$abc = +6 \Rightarrow \{1,2,3\}, \{-1,-2,3\}, \{1,-2,-3\}, \{-1,2,-3\}$

By Adding ~~coo~~ (a,b,c) we should get coeff of λ^2 (ie 6) : $1+2+3 = 6 \Rightarrow (a,b,c) = (1,2,3)$

Verification: $ab+bc+$ ~~ab~~ $ca =$ coeff of λ : $(1\times2)+(2\times3)+(1\times3) = 11$

$\therefore$ Factors are $(1, 2, 3)$ $\Rightarrow$ $(\lambda+1)(\lambda+2)(\lambda+3) = 0$

$\Rightarrow \lambda = -1, -2, -3$

Q) Find the eigen values of $A = \begin{bmatrix} 10 & -4 \\ 18 & -12 \end{bmatrix}$

a) 4, 9 b) 6, -8 c) 4, 8 d) -6, 8

Soln M1 Normal method : $|A - \lambda I| = 0$

M2 Properties of Eigen Values:

* Summation of all eigen values of A = Sum of all diagonal elements
* Product of the eigen values of A = Determinant of A

$\therefore$ From options, only option b gives the correct value

i.e. Trace of $A = 10 - 12 = -2$

sum of values of option b $= 6 - 8 = -2$ $\Big\} \Rightarrow$ option b

Q) Eigen values of $A = \begin{bmatrix} 5 & 3 \\ 2 & 9 \end{bmatrix} = ?$

a) 5.13, 9.42
b) 3.85, 2.93
c) 9, 5
d) 10.16, 3.84

Soln. Trace = 14 $\Rightarrow \lambda_1 + \lambda_2 = 14$

Determinant = 39 $\Rightarrow \lambda_1 \lambda_2 = 39$ $\Big\} \Rightarrow$ option d

Q) What are the eigen values of $\begin{bmatrix} 1 & 2 & 34 & 49 \\ 0 & 2 & 43 & 94 \\ 0 & 0 & -2 & 104 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

a) 1, 2, -2, -1
b) -1, -2, -1, -2
c) 1, 2, 2, 1
d) NOTA

Soln Trace = 0 $\Rightarrow$ option c & b are eliminated

$|A| = 4$ $\Rightarrow$ option a

Note * If A = upper triangular or lower triangular or diagonal matrix then its eigen values are its diagonal elements.

Q) Find the eigen values of $\begin{bmatrix} -1 & 3 & 5 \\ -3 & -1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$, $j = i = \sqrt{-1}$

***

a) $3, 3+5j, 6-j$ b) $-6+5j, 3+j, 3-j$ c) $3+j, 3-j, 5+j$ d) $3, -1+3j, -1-3j$

Soln Trace = 1 => option d

Note Complex roots always exists in pairs => only in option d, eigen values are in pairs

Q) Minimum eigen value of matrix, $\begin{bmatrix} 3 & 5 & 2 \\ 5 & 12 & 7 \\ 2 & 7 & 5 \end{bmatrix} = ?$

a) 0 b) 1 c) 2 d) 3

Soln $|A| = \begin{vmatrix} 3 & 5 & 2 \\ 5 & 12 & 7 \\ 2 & 7 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 5 & 2 \\ 5 & 12 & 7 \\ 5 & 12 & 7 \end{vmatrix} = 0$.

=> Min eigen value = 0. => option a

*****

5) Eigen values of 2×2 matrix X are -2 & -3. Eigen values of matrix $(X+I)^{-1}(X+5I)$ are ?

Soln M1 Form a matrix such that its eigen values are -2 & -3. Put that matrix in the expression & then find the ans

M2 $X_{2\times 2} \rightarrow \lambda_1, \lambda_2$

$(X+I)^{-1}(X+5I)$: $\rightarrow (\lambda+1)^{-1}(\lambda+5)$

(Replace X by λ & I by 1)

$\therefore$ Eigen values are : $(\lambda_1+1)^{-1}(\lambda_1+5)$ & $(\lambda_2+1)^{-1}(\lambda_2+5)$

$\Rightarrow (-2+1)^{-1}(-2+5)$ & $(-3+1)^{-1}(-3+5)$

$\therefore$ eigen values are -3 & -1

$\Rightarrow -3$ & -1

***

Q) $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$ Eigen values are $1, -1, 3$. Trace of $(A^3 - 3A^2)$ is ______

Soln Eigen values of matrix $A^3 - 3A^2$ are $(1-3), (-1-3), (27 - 3\times 9) = (-2, -4, 0)$

$\therefore$ Trace of $A^3 - 3A^2 = -2 - 4 + 0 = -6$

Q) Eigen values of $S = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ are 5 & 1. Eigen values of S^2 matrix = ?

Soln Eigen values of $S^2 = 5^2 \& 1^2 = \underline{25 \& 1}$

Q) Some properties:

Hermitian Matrix => Eigen values are real

Symmetric Matrix => Eigen values are real

Skew symmetric => Eigen values are imaginary or zero

Unitary matrix => Eigen values are 1 or -1

Orthogonal matrix => Eigen values are $\lambda, \frac{1}{\lambda}$

Q) If we have square matrix A which is real & symmetric then its eigen values are

a) always real b) always real & +ve,

c) always real & non-negative d) occurs in complex conjugate pair

Soln option a

Note

* If $A \to \lambda$, then $A^{-1} \to \lambda^{-1}$, $A^n \to \lambda^n$
* Eigen values are same for both A & A^T
* $\text{adj } A \to \frac{|A|}{\lambda}$

$A^{-1} = \frac{\text{adj} A}{|A|}$

$\Rightarrow \text{adj} A = \lambda^{-1} |A| = \frac{|A|}{\lambda}$

***

Q) $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$ Product of non zero eigen value of matrix = ?

Soln Suspense Question => Will be covered later

Q) $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, $B = \begin{bmatrix} p^2+q^2 & pr+qs \\ pr+qs & r^2+s^2 \end{bmatrix}$, If $\rho(A) = N$, $\rho(B) = ?$

***

a) $N/2$
b) $N-1$
c) N
d) $2N$

Soln

Let $p = 1, q = 2, r = 0, s = 0$

$\therefore \rho(A) = 1$

$B = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \rho(B) = 1 \Rightarrow$ option c

H W

Q) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix}$, $|A^{-1}| = ?$

Soln

$|A^{-1}| = \frac{1}{|A|} = \frac{1}{4}$

H W

Q) $R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$, top row of R^{-1} is ?

***

Soln

$|R| = (2+3) - 1(4) = 1$

Top row will be first column of [Cofactor matrix of R]/|R|

$\therefore$ Cofactor matrix elements of first column : $5, -3, 1$

$\therefore$ Top row $= 5, -3, 1$

H W

Q) Consider a non-singular square matrix, Trace $(A) = 4$, Trace $(A^2) = 5$, $|A| = ?$

Soln

$\lambda_1 + \lambda_2 = 4$, $\lambda_1^2 + \lambda_2^2 = 5$

$(\lambda_1 + \lambda_2)^2 = 16 \Rightarrow \lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2 = 16$

$\Rightarrow 2\lambda_1\lambda_2 = 16 - 5$

$\Rightarrow |A| = \lambda_1\lambda_2 = 11/2$

H W

Q) $A = \begin{bmatrix} a & 0 & 3 & 7 \\ 2 & 5 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 6 \end{bmatrix}$, $|A| = 100$, Trace $(A) = 14$

$|a-b|$ is ______

Soln $|A| = 100 \Rightarrow$ $R_2: aR_2 - 2R_1$ $\begin{vmatrix} a & 0 & 3 & 7 \\ 0 & 5a & a-6 & 3a-14 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & b \end{vmatrix} = 100$

$\Rightarrow 5a^2 \times \frac{2b}{a} = 100 \Rightarrow a^2 = \frac{5}{3} \Rightarrow a = \pm\sqrt{5/3}$

$\Rightarrow a^2 b = 10$ — ①

Trace $= 14 \Rightarrow a+b+7 = 14 \Rightarrow a+b = 7$ — ②

$\therefore$ From ① & ②,

Soln $|A| = \begin{vmatrix} a & 0 & 3 & 7 \\ 2 & 5 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & b \end{vmatrix} = b \begin{vmatrix} a & 0 & 3 \\ 2 & 5 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 2b(5a) = 10ab$

Given $|A| = 100 \Rightarrow 10ab = 100 \Rightarrow ab = 10$ — ①

Trace $(A) = 14 \Rightarrow a+b+7 = 14 \Rightarrow a+b = 7$ — ②

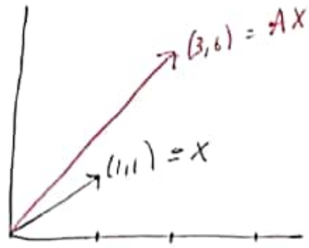
$\therefore a = 5, b = 2 \Rightarrow |a-b| = 3$

23/03/20 **Eigen Vectors**:

Note **Scaling & Rotation**:

eg ① $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

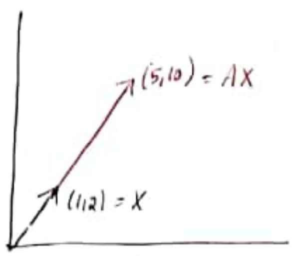
$AX = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$



Here the matrix A after getting multiplied by X, gets scaled & rotated then X is not an eigen vector

eg ②: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$AX = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$



Eigen vectors are such vectors which only scales but donot rotate

$$\underbrace{\boxed{A\underbrace{X}_{\text{Eigen vector}} = \lambda X}}_{\text{Eigen value}}$$

① $A = \checkmark$, $\lambda = ?$, $X = ?$

$$A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$$

Step 1 Find the eigen values: $\left.\begin{array}{l}\lambda_1 + \lambda_2 = 10 \\ \lambda_1 \lambda_2 = 24\end{array}\right\} \Rightarrow \lambda_1 = 6, \lambda_2 = 4$

Step 2 $[A - \lambda I]X = 0$

i) $\lambda = 4$:

$$\begin{bmatrix} 8-4 & -4 \\ 2 & 2-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 4x_1 - 4x_2 = 0$$

$$\Rightarrow x_1 = x_2$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_1 = k, \; x_2 = k$$

Here
- k = arbitrary constant $(k \neq 0)$
- X is a non-zero eigen vector [non zero - non trivial]
- Eigen vector X_1 corresponds to $\lambda = 4$

Note One eigen value can corresponds to many eigen vector

eg Here X_1 can be $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, etc .....
$(\lambda = 4)$

$\therefore$ Here for $\lambda = 4$, there can be infinite solutions.

ii) $\lambda = 6$:

$$\begin{bmatrix} 8-6 & -4 \\ 2 & 2-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 2x_1 - 4x_2 = 0$$

$$\Rightarrow x_1 = 2x_2$$

$$\therefore X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

② A ✓, λ ✓, $X = ?$ / $A = ?$, λ ✓, X ✓ / A ✓ $\lambda = ?$ X ✓

Q) $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$, $X = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\lambda = ?$

Soln

$$AX = \lambda X \Rightarrow \begin{bmatrix} -2+4+3 \\ 2+2+6 \\ -1-4+0 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 \\ 10 \\ -5 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \Rightarrow \lambda = 5$$

Q) $P = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $\lambda = -2$, $X = ?$

Soln

$$[P - \lambda I]X = 0 \Rightarrow \begin{bmatrix} 5 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 5x_1 - 2x_2 + 2x_3 = 0, \quad x_3 = 0, \quad 3x_3 = 0$$

$$\Rightarrow 5x_1 = 2x_2$$

$$\therefore X = \begin{bmatrix} 2/5 \\ 1 \\ 0 \end{bmatrix} \text{ when } x_2 = k = 1$$

Q) $\lambda = -1, -2$, $X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ resp^ly Matrix $A = ?$

Soln

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ | $\lambda = -1$: $AX = \lambda X \Rightarrow \begin{bmatrix} a-b \\ c-d \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\Rightarrow a - b = -1 \quad —①$$
$$c - d = 1 \quad —②$$

$\lambda = -2$: $\begin{bmatrix} a-2b \\ c-2d \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ $\Rightarrow$ $a - 2b = -2$ —③, $c - 2d = 4$ —④

③ $\Rightarrow -1 - b = -2$
$\Rightarrow b = 1$
$\Rightarrow a = 0$

④ $\Rightarrow 1 - d = 4$
$\Rightarrow d = -3$
$\Rightarrow c = -2$

$$\therefore A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Properties.

* { If eigen values are real then eigen vectors are also real }
* { If eigen values are complex then eigen vectors are also complex }

***

(Q) Which one of the foll is eigen vector of $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$

a) $[1\ -2\ 0\ 0]^T$ b) $[0\ 0\ 1\ 0]^T$ c) $[1\ 0\ 0\ -2]^T$ d) $[1\ -1\ 2\ 1]^T$

Soln. ~~Reducing A to~~ $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 5-\lambda & 0 & 0 & 0 \\ 0 & 5-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)(5-\lambda)[2-2\lambda-\lambda+\lambda^2-3] = 0 \Rightarrow (5-\lambda)^2[\lambda^2-3\lambda-1] = 0$$

$$\Rightarrow \lambda = 5, 5, \frac{3 \pm \sqrt{9+4}}{2} \Rightarrow \underline{\lambda = 5, 5, \tfrac{3}{2} \pm \tfrac{\sqrt{13}}{2}}$$

Consider $\lambda = 5$

$$AX = \lambda X \Rightarrow \begin{bmatrix} 5x_1 \\ 5x_2 \\ 2x_3 + x_4 \\ 3x_3 + x_4 \end{bmatrix} = 5\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \Rightarrow \begin{matrix} 2x_3 + x_4 = 5x_3 \Rightarrow 3x_3 = x_4 \\ 3x_3 + x_4 = 5x_4 \Rightarrow 3x_3 = 4x_4 \end{matrix}$$

$$\Rightarrow x_3 = x_4 = 0$$

Here x_1 & x_2 can be anything other than zero

$\therefore X = [k_1\ k_2\ 0\ 0]^T \Rightarrow$ <u>option a</u>

Note * { Eigen vectors of A & A^m are same }

eg: ~~Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \Rightarrow \lambda_1 = 1, \lambda_2 = 2$~~

~~$AX = \lambda X \Rightarrow \begin{bmatrix} x_1 - x_2 \\ x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{matrix} x_1 - x_2 = x_1 \Rightarrow x_2 = 0 \\ x_1 + 2x_2 = x_2 \Rightarrow \end{matrix}$~~

eg Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \lambda_1 = 1, \lambda_2 = 2$

For $\lambda = 2$: $AX = \lambda X \Rightarrow \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} \Rightarrow x_1 = 0$ & $x_2 = k$

$\therefore X_2 = k\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Scanned with CamScanner

Now $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow \lambda = 1, 4$

For $\lambda = 4$: $AX = \lambda X \Rightarrow \begin{bmatrix} x_1 \\ 4x_2 \end{bmatrix} = 4\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow x_1 = 0,\ x_2 = k$

$\Rightarrow X_2 = k\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Hence it is verified that eigen vectors of A & A^m are same

Q) Eigen value of A is $1, -2$ with eigen vectors x_1, x_2. Eigen value & Eigen vector of $A^2 - 3A + 4I$ would be ?

Soln $A^2 - 3A + 4I \rightarrow \lambda^2 - 3A + 4I$

When $\lambda = 1 \Rightarrow 1 - 3 + 4 = 2$

When $\lambda = -2 \Rightarrow 4 + 6 + 4 = 14$

From the above property, eigen vectors of $A^2 - 3A + 4I$ is same as that of A i.e. $\underline{x_1, x_2}$

* Eigen vectors of symmetric matrix are orthogonal vectors.

eg: Let $A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

$\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0$

$s_1 = 8$

$s_2 = -10 + 24 - 10 = 4$, $|A| = -50 + 2 = -48$

$\therefore \lambda^3 - 8\lambda^2 + 4\lambda + 48 = 0$

$\lambda = -2$

-2	1	-8	4	48
	↓	-2	20	-48
	1	-10	24	0

$\lambda^2 - 10\lambda + 24 = 0$

$\Rightarrow (\lambda - 6)(\lambda - 4) = 0$

$\lambda = 6, 4$

$\therefore \lambda = -2, 4, 6$

Corresponding eigen vectors are

$X_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

To check for orthogonal

$\underset{1\times3}{X_1^T} \cdot \underset{3\times1}{X_2} = [0]_{1\times1}$

$[0\ 1\ 0]\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 0$

$X_2^T \cdot X_3 = 0$

$[1\ 0\ -1]\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0$

$X_3^T X_1 = 0$

$\Rightarrow [1\ 0\ 1]\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$

$\therefore X_1^T X_2 = 0,\ X_2^T X_3 = 0,\ X_3^T X_1 = 0$

Linearly Dependent & Independent vectors:

Consider $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ Its eigen values are $\lambda = -3, -3, 5$ (Repeated)

For $\lambda = -3$: $[A - \lambda I]X = 0 \Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_2 : R_2 - 2R_1$
$R_3 : R_3 + R_1$ $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow x_1 + 2x_2 - 3x_3 = 0$

$x_1 + 2x_2 - 3x_3 = 0$: Let $x_2 = k_1$ & $x_3 = k_2$.

$\therefore x_1 + 2k_1 - 3k_2 = 0 \Rightarrow x_1 = 3k_2 - 2k_1$

$\therefore X = \begin{bmatrix} 3k_2 - 2k_1 \\ k_1 \\ k_2 \end{bmatrix} = k_1 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}_{X_1} + k_2 \underbrace{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}_{X_2}$

IIIly for $\lambda = 5 \Rightarrow X = k_3 \underbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}_{X_3}$

To check whether X_1, X_2 & X_3 are linearly dependent/independent.

$$k_1 X_1 + k_2 X_2 + k_3 X_3 = 0$$

* To satisfy the above eq. if $k_1 = k_2 = k_3 = 0 \Rightarrow$ Eigen vectors are linearly independent.

* If any one of k_1, k_2, k_3 is non-zero then eigen vectors are linearly dependent

$$k_1 X_1 + k_2 X_2 + k_3 X_3 = 0$$

$$k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 0$$

$\Rightarrow \begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$
$-2k_1 + 3k_2 + k_3 = 0$
$k_1 + 2k_3 = 0$
$k_2 + k_3 = 0$

$k_2 = -k_3,\ k_1 = -2k_3$: $\therefore\ 4k_3 - 3k_3 + k_3 = 0 \Rightarrow 2k_3 = 0 \Rightarrow k_3 = 0 \Rightarrow k_1 = 0, k_2 = 0$

$\therefore\ \underline{k_1 = k_2 = k_3 = 0} \Rightarrow$ Linearly independent

eg ③: $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}, \quad \lambda = 1, 2, 2$

For $\lambda = 1$: $[A - \lambda I]X = 0 \Rightarrow \begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore X = \underbrace{k\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}_{X_1}$

For $\lambda = 2$: $\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -x_1 + 2x_2 + 2x_3 = 0,\ x_3 = 0 \Rightarrow x_1 = 2x_2$

$\therefore X = \underbrace{k\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{X_2 = X_3}$

$\therefore k_1X_1 + k_2X_2 + k_3X_3 = 0$

$\begin{bmatrix} -1 & 2 & 2 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -k_1 + 2k_2 + 2k_3 = 0 \Rightarrow \underline{k_2 = -k_3}$

$-k_1 + k_2 + k_3 = 0 \Rightarrow \underline{k_2 = -k_3}$

$k_1 = 0$

$\therefore k_1X_1 + k_2X_2 + k_3X_3 = 0$ & $k_2 = -k_3 \neq 0 \Rightarrow$ Linearly Dependent vectors

$$\boxed{\text{No. of Linearly Independent Vectors} = n - r}$$

n = no. of variables

r = Rank of $[A - \lambda I]$.

In this question; for $\lambda = 1$: $n = 3$ & $r = 2$

$\therefore$ 1 linearly indep. vector

IIIly for $\lambda = 2$: $3 - 2 = 1$ indep. vector $\therefore$ There are 2 indep. vectors & 1 dependent vectors.

(Q) Product of non-zero eigen value of $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}_{5\times 5}$

Soln M1

$$\begin{vmatrix} 1-\lambda & 0 & 0 & 0 & 1 \\ 0 & 1-\lambda & 1 & 1 & 0 \\ 0 & 1 & 1-\lambda & 1 & 0 \\ 0 & 1 & 1 & 1-\lambda & 0 \\ 1 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\begin{vmatrix} 1-\lambda & 1 & 1 & 0 \\ 1 & 1-\lambda & 1 & 0 \\ 1 & 1 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 0 & 1-\lambda & 1 & 1 \\ 0 & 1 & 1-\lambda & 1 \\ 0 & 1 & 1 & 1-\lambda \\ 1 & 0 & 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(1-\lambda)\underbrace{\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix}}_{X} + (-1)\underbrace{\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix}}_{X} = 0$$

$$[(1-\lambda)^2 - 1]\, X = 0$$

$\Rightarrow (1-\lambda)^2 - 1 = 0$

$\Rightarrow \cancel{1} + \lambda^2 - 2\lambda - \cancel{1} = 0$

$\Rightarrow \lambda^2 - 2\lambda = 0$

$\Rightarrow \lambda = 0, \lambda = 2$

OR

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 3, \begin{vmatrix} 1 & 1 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda = 3, \lambda^2 = 0 \Rightarrow \lambda = 3, 0, 0$

$\therefore$ Product of non-zero eigen value

$= 2 \times 3 = 6$

M2 $R_1 : R_1 + R_5$
$R_2 : R_2 + R_3 + R_4$

$$\begin{vmatrix} 2-\lambda & 0 & 0 & 0 & 2-\lambda \\ 0 & 3-\lambda & 3-\lambda & 3-\lambda & 0 \\ 0 & 1 & 1-\lambda & 1 & 0 \\ 0 & 1 & 1 & 1-\lambda & 0 \\ 1 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(3-\lambda)\begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1-\lambda & 1 & 0 \\ 0 & 1 & 1 & 1-\lambda & 0 \\ 1 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(3-\lambda)\begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 2, \lambda = 3, \lambda = 0, 0, 0$$

M3

$$AX = \lambda X$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$\Rightarrow$

$x_1 + x_5 = \lambda x_1$ — ①

$x_2 + x_3 + x_4 = \lambda x_2$ — ②

$x_2 + x_3 + x_4 = \lambda x_3$ — ③

$x_2 + x_3 + x_4 = \lambda x_4$ — ④

$x_1 + x_5 = \lambda x_5$ — ⑤

① + ⑤ $\Rightarrow 2(x_1 + x_5) = \lambda(x_1 + x_5)$

$\Rightarrow \underline{\lambda = 2}$

② + ③ + ④ $\Rightarrow 3(x_2 + x_3 + x_4) = \lambda(x_2 + x_3 + x_4)$

$\Rightarrow \underline{\lambda = 3}$

Note TRICK

$$\boxed{\Sigma R_{min} \le \text{Real Eigen Value} \le \Sigma R_{max}}$$

R - Row

1/03/20.

Q) Among the foll, the pair of vectors orthogonal to each other is

a) [3,4,7] , [3,4,7] b) [1,0,0] , [1,1,0]

c) [1,0,2] , [0,5,0] d) [1,1,1] , [-1,-1,-1]

Soln option c: $[1, 0, 2]\begin{bmatrix}0\\5\\0\end{bmatrix} = [0] \Rightarrow$ option c

Q) The matrix $M = \begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & 6\\ -1 & -2 & 0\end{bmatrix}$ has eigen values -3, -3, 5. An eigen vector corresponding to the eigen value 5 is $[1\ 2\ -1]^T$. One of the eigen vector of the matrix M^3 is

a) $[1\ 8\ -1]^T$ b) $[1\ 2\ -1]^T$ c) $[1\ \sqrt[3]{2}\ -1]^T$ d) $[1\ 1\ -1]^T$

Soln. Eigen vector of M^3 will remain same as the eigen vector of M $\therefore$ option b

Q) In the given matrix $\begin{bmatrix}1 & -1 & 2\\ 0 & 1 & 0\\ 1 & 2 & 1\end{bmatrix}$, one of the eigen values is 1. The eigen vectors corresponding to the eigen value 1 are

a) $\{\alpha(4,2,1) \mid \alpha \neq 0, \alpha \in R\}$ b) $\{\alpha(-4,2,1) \mid \alpha \neq 0, \alpha \in R\}$

c) $\{\alpha(\sqrt{2},0,1) \mid \alpha \neq 0, \alpha \in R\}$ d) $\{\alpha(-\sqrt{2},0,1) \mid \alpha \neq 0, \alpha \in R\}$

Soln $AX = \lambda X \Rightarrow \begin{bmatrix}x_1 - x_2 + 2x_3\\ x_2\\ x_1 + 2x_2 + x_3\end{bmatrix} = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} \Rightarrow \begin{matrix}-x_2 + 2x_3 = 0\\ x_1 + 2x_2 = 0\end{matrix}$

$\Rightarrow x_3 = x_2/2, \quad x_1 = -2x_2$

$x_2 = \alpha_1, \quad x_1 = -2\alpha_1, x_3 = \alpha_1/2$ $\therefore X = \begin{bmatrix}-2\alpha\\ \alpha\\ \alpha/2\end{bmatrix} = \alpha_1\begin{bmatrix}-2\\ 1\\ 1/2\end{bmatrix}$

$\Rightarrow X = \alpha\begin{bmatrix}-4\\ 2\\ 1\end{bmatrix} \Rightarrow$ option b

Scanned with CamScanner

Q) The number of linear independent eigen vectors of matrix A

$= \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is ______

Soln ~~$n = 3$, $r = 3$ ⇒ No of linear~~

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & (3-\lambda) \end{vmatrix} = 0 \Rightarrow (2-\lambda)^2(3-\lambda) = 0$

$\Rightarrow \lambda = 2, 2, 3$

For $\lambda = 2$: $[A - \lambda I][X] = 0 \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_2 = 0, \; x_3 = 0, \; x_1 = k$ ∴

Here, no. of indep. eigen vectors $= n - r = 3 - 2 = 1$

For $\lambda = 3$ $[A - \lambda I][X] = 0 \Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Here no. of indep. eigen vectors $= n - r = 3 - 2 = 1$ ⇒ ~~$[-1 \; 1 \; 0]$~~

∴ Total no. of independent eigen vectors of matrix A $= 1 + 1 = 2$

Q) The matrix $\begin{pmatrix} 2 & -4 \\ 4 & -2 \end{pmatrix}$ has

a) Real eigen values & eigen vectors
b) Real eigen values but complex eigen vectors
c) complex eigen values but real eigen vectors
d) complex eigen values & eigen vectors

Soln $\lambda_1 + \lambda_2 = 0$, $\lambda_1 \lambda_2 = 12$

∴ Complex eigen values & eigen vectors ⇒ option d.

Cayley Hamilton Theorem: Every square matrix satisfies its characteristic equations

eg: $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) - 12 = 0$

$\Rightarrow 2 - \lambda - 2\lambda + \lambda^2 - 12 = 0$

$\Rightarrow \lambda^2 - 3\lambda - 10 = 0$

Replace λ by A : $\underline{A^2 - 3A - 10I = 0}$

Q) $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$, $A^5 = ?$ in Eq^n form.

Soln $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = 0 \Rightarrow 2 - \lambda - 2\lambda + \lambda^2 - 12 = 0$

$\Rightarrow \lambda^2 - 3\lambda - 10 = 0$

From CH theorem: $A^2 - 3A - 10I = 0 \Rightarrow A^2 = 3A + 10I$

$\therefore A^3 = 3A^2 + 10A = 3[3A + 10I] + 10A = 19A + 30I$

$\therefore A^5 = A^3 A^2 = (19A + 30I)(3A + 10I) = 57A^2 + 190A + 90A + 300I$

$\Rightarrow A^5 = 57[3A + 10I] + 280A + 300I = 171A + 570I + 280A + 300I$

$\Rightarrow A^5 = \underline{451A + 870I}$

Q) $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$, $A^{-1} = ?$

Soln M1 Normal Inverse method

M2 $\begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = 0 \Rightarrow 2 - \lambda - 2\lambda + \lambda^2 - 12 = 0 \Rightarrow \lambda^2 - 3\lambda - 10 = 0$

CH theorem: $A^2 - 3A - 10I = 0 \Rightarrow A^2 = 3A + 10I$

$(A^{-1}A)A = 3A^{-1}A + 10A^{-1}$

$\Rightarrow A = 3I + 10A^{-1}$

$\Rightarrow A^{-1} = \underline{\dfrac{A - 3I}{10}}$

Q) $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$, $A^9 = ?$

a) $511A + 510I$　　b) $309A + 104I$

c) $154A + 155I$　　d) $55I$

Soln

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow 3\lambda + \lambda^2 + 2 = 0$

CH theorem. $A^2 = -3A - 2I \Rightarrow A^3 = -3A^2 - 2A = +3[3A + 2I] - 2A$

$\Rightarrow A^3 = 9A + 6I - 2A = 7A + 6I$

$A^9 = (7A + 6I)^3 = 343A^3 + 216I + 126AI(7A + 6I)$

$= 343[7A + 6I] + 216I + 882[-3A - 2I] + 756A$

$= 2401A + 2058I + 216I - 2646A - 1764I + 756A$

$\Rightarrow A^9 = \underline{511A + 510I} \Rightarrow$ option a

Q) $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ & $B = A^3 - A^2 - 4A + 5I$, $|B| = ?$

Soln M1

CH: eq of A: $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$.

$S_1 = 1$, $S_2 = -4 - 2 + 2 = -4$, $|A| = (-2)(2) = -4$

$\therefore \lambda^3 - \lambda^2 - 4\lambda + 4 = 0$.

$\Rightarrow A^3 - A^2 - 4A + 4I = 0$

$\therefore B = (A^3 - A^2 - 4A + 4I) + (I) \Rightarrow B = I \Rightarrow \underline{|B| = 1}$

M2

Find the eigen values of A:

CH: $\lambda^3 - \lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda_1 = -2$

Trace $= \lambda_1 + \lambda_2 + \lambda_3 \Rightarrow 1 = -2 + \lambda_2 + \lambda_3 \Rightarrow \lambda_2 + \lambda_3 = 3$

$|A| = \lambda_1\lambda_2\lambda_3 \Rightarrow \frac{-4}{-2} = \lambda_2\lambda_3 \Rightarrow \lambda_2\lambda_3 = 2$

$\left.\right\} \lambda_2 = 1, \lambda_3 = 2$

$\therefore$ Eigen values of A are $\lambda_1 = -2$, $\lambda_2 = 1$, $\lambda_3 = 2$

Eigen values of B: $\lambda_1^3 - \lambda_1^2 - 4\lambda_1 + 5$; $\lambda_2^3 - \lambda_2^2 - 4\lambda_2 + 5$; $\lambda_3^3 - \lambda_3^2 - 4\lambda_3 + 5$

$\Rightarrow 1$; 1 ; 1

$\therefore |B| = \lambda_1\lambda_2\lambda_3 = 1$

System of Linear Equations:

Unique Solution.

$$2x + 3y = 7 \quad x - y = 1 \quad \Rightarrow \quad \begin{array}{r} 2x + 3y = 7 \\ 2x - 2y = 2 \\ \hline 5y = 5 \end{array} \Rightarrow y = 1 ; \; x = 2.$$

Here $x = 2, y = 1$ is the unique solution.

No Solution:

$$\left.\begin{array}{l} x + y = 7 \\ 7x + 7y = 100 \end{array}\right\} \Rightarrow \text{No solution}$$

Infinitely many solution:

$$\left.\begin{array}{l} x + y = 0 \\ 7x + 7y = 0 \end{array}\right\} \Rightarrow \text{Infinitely many solution.}$$

Note: * If the system is in the form $AX = B$ $\Rightarrow$ Non-homogeneous system

eg: $$\underbrace{\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}}_{\text{coefficient matrix}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{variable matrix}} = \underbrace{\begin{bmatrix} 7 \\ 1 \end{bmatrix}}_{\text{constant matrix}}$$

* If the system is in the form $AX = 0$ $\Rightarrow$ Homogeneous system.

* Non homogeneous system consists of no solution, unique solution & Infinitely many solution systems

* Homogeneous system consists of only unique solution & infinitely many solution but not no solution systems.

Q) $$\begin{aligned} 2x + 3y + z &= 9 \\ x + 2y + 3z &= 6 \\ 3x + y + 2z &= 8 \end{aligned}$$ Determine the type of solution

Soln $AX = B \Rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$

i) Augmented Matrix $[A:B]$: $\quad [A:B] = \left[\begin{array}{ccc:c} 2 & 3 & 1 & 9 \\ 1 & 2 & 3 & 6 \\ 3 & 1 & 2 & 8 \end{array}\right]$

ii) Find Rank(A) & Rank[A:B]:

$R_1 \leftrightarrow R_2$: $\begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 2 & 3 & 1 & : & 9 \\ 3 & 1 & 2 & : & 8 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$
$\begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & -1 & -5 & : & -3 \\ 0 & -5 & -7 & : & -10 \end{bmatrix}$

$\therefore$ Rank $(A) = 3$ & Rank $[A:B] = 3$.

Some Conditions

- $*$ If $\rho(A) \neq \rho[A:B] \Rightarrow$ No solution
- $*$ If $\rho(A) = \rho[A:B]$
  - i) $\rho =$ no. of variables $\Rightarrow$ unique solution
  - ii) $\rho <$ no. of variables $\Rightarrow$ IFM solution.

$\therefore$ In the above question : $\rho(A) = \rho[A:B]$ &
$\rho = 3 =$ no. of variables $\Bigg\} \Rightarrow$ unique solution

To find the solution of the system:

$AX = B \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & -5 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ -10 \end{bmatrix}$

$\Rightarrow$
$x_1 + 2y + 3z = 6$
$-y - 5z = -3$
$-5y - 7y = -10$.
$\Bigg\}$ By solving we will get the ans.

Q) $5x + 3y + 7z = 4$
$3x + 26y + 2z = 9$
$7x + 2y + 10z = 5$
Determine the type of solution.

Soln $[A:B] = \begin{bmatrix} 5 & 3 & 7 & | & 4 \\ 3 & 26 & 2 & | & 9 \\ 7 & 2 & 10 & | & 5 \end{bmatrix} \Rightarrow$ $R_2 : 5R_2 - 3R_1$, $R_3 : 5R_3 - 7R_1$ $\begin{bmatrix} 5 & 3 & 7 & | & 4 \\ 0 & 121 & -11 & | & 33 \\ 0 & -11 & 1 & | & -3 \end{bmatrix}$

$\Rightarrow R_3 : 11R_3 + R_2$ $\begin{bmatrix} 5 & 3 & 7 & | & 6 \\ 0 & 121 & -11 & | & 33 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \Rightarrow$ ~~$\rho(A) \neq \rho(A|B)$~~
$\Rightarrow$ ~~No. solution~~

$\therefore \rho(A) = \rho(A:B)$ & $\rho <$ no. of variables
$\Rightarrow$ Infinitely many solutions

Q) $x - y + 2z = 1$
$2x + 2z = 2$ Determine the type of solution
$x - 3y + 4z = 2$

Soln $[A:B] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 0 & 2 & 2 \\ 1 & -3 & 4 & 2 \end{array}\right]$ $\begin{array}{l} R_2 : R_2 - 2R_1 \\ R_3 : R_3 - R_1 \end{array}$ $\Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 1 \end{array}\right]$

$\underset{R_3 : R_3 + R_1}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow$ $\rho(A) = 2$

$\rho(A:B) = 3$

$\therefore$ No solution

Some Tricks for 2 x 2

$2x + 3y = 7$ $x - y = 1$	$2x + 3y = 7$ $8x + 12y = 28$	$2x + 3y = 7$ $4x + 6y = 20$
• Valid • Distinct • Unique solution	• Valid • Same • Infinite soln	• Not valid • No soln.

\# If $|A| \neq 0 \Rightarrow$ Unique solution

If $|A| = 0 \Rightarrow \begin{cases} \text{Infinite soln} \\ \text{No soln.} \end{cases}$

Q) $x + y + z + w = 0$
$x + 3y - 2z + w = 0$ Determine the type of soln.
$2x - 3z + 2w = 0$
$x + y + w = 0$

Soln Homogeneous system $\Rightarrow$ No solution is eliminated

$|A| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 1 & 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & 0 & -1 & 0 \end{vmatrix}$

$\Rightarrow |A| = \begin{vmatrix} 2 & -3 & 0 \\ -2 & -5 & 0 \\ 0 & -1 & 0 \end{vmatrix} = 1 \begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} = 0$

$\Rightarrow |A| = 0 \Rightarrow$ Infinite many soln.

Scanned with CamScanner

Note ***

For a homogeneous system, $AX = 0$

If $|A| \neq 0$

-> unique solution

-> $(0,0)$ is the only soln

-> Zero soln or Trivial soln.

If $|A| = 0$

-> Infinitely many solutions

-> $(0,0)$ & $(0,0)$ k alawa

-> Non Trivial soln

HW

Q)

$$x + y + z + w = 0$$
$$x + 3y - 2z + w = 0$$
$$2x - 3z + 2w = 0$$
$$x + y + w = 0$$

Determine the type of soln

Soln Homogeneous system

$$\therefore |A| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 1 & 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & -3 & 0 \\ 2 & -2 & -5 & 0 \\ 1 & 0 & -1 & 0 \end{vmatrix} = 0$$

$\Rightarrow |A| = 0 \Rightarrow$ Infinite many soln

HW

2)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}$$

It is given that A has only one real eigen value. Find that real eigen value.

Soln ~~$|A-\lambda I|$~~

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 & 4 & 5 \\ 5 & 1-\lambda & 2 & 3 & 4 \\ 4 & 5 & 1-\lambda & 2 & 3 \\ 3 & 4 & 5 & 1-\lambda & 2 \\ 2 & 3 & 4 & 5 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 15-\lambda & 2 & 3 & 4 & 5 \\ 15-\lambda & 1-\lambda & 2 & 3 & 4 \\ 15-\lambda & 5 & 1-\lambda & 2 & 3 \\ 15-\lambda & 4 & 5 & 1-\lambda & 2 \\ 15-\lambda & 3 & 4 & 5 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 15$$

Diagonalization of a Matrix: Diagonalization is possible only if we have linearly independent vectors

Formula: $\{ D = P^{-1} A P \}$, $P \rightarrow$ Matrix of the eigen vectors of A

$P =$ Modal matrix

Diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$ where $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A.

eg. Find the diagonal matrix of $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Soln Step 1: Find the eigen values

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)[(2-\lambda)^2 - 1] = 0$$

$\Rightarrow \lambda = 1$, $(2-\lambda)^2 = 1$

$\Rightarrow 2-\lambda = +1$ or $2-\lambda = -1$

$\Rightarrow \lambda = 1$ $\lambda = 3$

$\therefore \lambda_3 = 1, \lambda_2 = 1, \lambda_1 = 3.$

Step 2: Find the eigen vectors corresponding to the eigen values

$\lambda = 3$:

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -x_1 + x_2 + x_3 &= 0 ; \\ x_1 - x_2 + x_3 &= 0 ; \\ -2x_3 &= 0 . \end{aligned}$$

$\therefore x_3 = 0$; $x_1 = x_2$; $\therefore X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

IIIly $\lambda = 1$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 + x_3 = 0$.

Let $x_3 = k_1$, $x_2 = k_2$

$x_1 = -k_1 - k_2$.

$$\therefore X = \begin{bmatrix} -k_1 - k_2 \\ k_2 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ & $X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Step 3: Create Modal Matrix

$$P = M = [X_1 \; X_2 \; X_3] = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 4 Find P^{-1} or M^{-1}: $P^{-1} = \frac{adj P}{|P|}$; $|P| = 2$.

$$\text{cofactor matrix of } P = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} ; \quad adj(P) = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

Step 5. Perform $D = P^{-1}AP$

$$\therefore D = P^{-1} \times \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow D = P^{-1} \times \begin{bmatrix} 3 & -1 & -1 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 & -1 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow D = \frac{1}{2}\begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q) For the same question find A^4:

Soln

$$D = P^{-1}AP$$

Premultiply by P : $PD = PP^{-1}AP \Rightarrow PD = AP$

Postmultiply by P^{-1}: $PDP^{-1} = APP^{-1} \Rightarrow A = PDP^{-1}$

$$\therefore \boxed{A^n = PD^nP^{-1}} \; ***$$

$$\therefore A^4 = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . P^{-1} = \frac{1}{2}\begin{bmatrix} 81 & -1 & -1 \\ 81 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow A^4 = \frac{1}{2}\begin{bmatrix} 82 & 80 & 80 \\ 80 & 82 & 80 \\ 0 & 0 & 2 \end{bmatrix}$$

Do Little's Method:

Consider
$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$AX = B$$

Writing A in terms of Lower Δ^{lr} matrix & Upper Δ^{lr} matrix.

$$A = L \cdot U = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}}_{U}$$

Now; $AX = B \Rightarrow L(\underbrace{UX}_{Y}) = B$

$\Rightarrow LY = B$; $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

From $LY = B \Rightarrow$ Find y_1, y_2, y_3.

Now: $UX = Y \Rightarrow$ From here find x_1, x_2, x_3.

Q) eg.
$$x + 3y + 8z = 4$$
$$x + 4y + 3z = -2$$
$$x + 3y + 4z = 1$$

Soln

$$\begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

$$A = LU \Rightarrow \begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

From here find all elements of L & U matrices

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & -4 \end{bmatrix}$$

$AX = B \Rightarrow LUX = B \Rightarrow LY = B$, where $UX = Y$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

After solving $y_1 = 4, \; y_2 = -2, \; y_3 = -3$.

$$UX = Y \Rightarrow \begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}$$

$\therefore$ After solving, $x = \frac{-29}{4}, \; y = \frac{7}{4}, \; z = \frac{3}{4}$

Crout's Method: $A = LU \quad [U_{ii} = 1]$

eg: 6)
$$x + y + z = 3$$
$$2x - y + 3z = 16$$
$$3x + y - z = -3$$

Soln

$$A = LU = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ After performing matrix multiplication & solving,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 3 & -2 & -14/3 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$AX = B \Rightarrow (LU)X = B \Rightarrow L(UX) = B \Rightarrow LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 3 & -2 & -14/3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 16 \\ -3 \end{bmatrix} \Rightarrow y_1 = 3, \; y_2 = -10/3, \; y_3 = 4$$

$\cancel{UX = B}$ $\cancel{LY = B}$ $UX = Y$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -10/3 \\ 4 \end{bmatrix} \Rightarrow x_1 = 1, \; y = -2, \; z = 4$$

Q) Necessary condition to diagonalise a matrix M is

a) Eigen values should be distinct

b) Eigen vectors should be linearly indep

c) Eigen values should be real

d) M is non-singular

Ans: option(b)

Q) $A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, A^3 is ?

a) $15A + 12I$ b) $19A + 30I$ c) $17A + 15I$ d) $17A + 21I$

Soln

$$\begin{vmatrix} -5-\lambda & -3 \\ 2 & -\lambda \end{vmatrix} = 0 \Rightarrow 5\lambda + \lambda^2 + 6 = 0$$

$$\Rightarrow A^2 = -5A - 6I$$

$$\Rightarrow A^3 = -5A^2 - 6A = -5(-5A-6I) - 6A$$

$$\Rightarrow A^3 = 19A + 30I \Rightarrow \text{option } b$$

Q)
$x_1 + x_2 + x_3 = 3$
$x_1 - x_2 = 0$
$x_1 - x_2 + x_3 = 1$

a) unique soln c) Infinite soln
b) no soln d) NOTA

Soln M1

$x_1 - x_2 + x_3 = 1$, $x_1 - x_2 = 1 \Rightarrow x_1 = x_2$

$\therefore x_3 = 1$

$x_1 + x_2 + x_3 = 3 \Rightarrow x_1 + x_2 = 2 \Rightarrow x_1 = 1, x_2 = 1$

$\therefore$ unique soln

Q) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ A is an orthonal matrix

$(AA^T)^{-1} = ?$

a) Diagonal Matrix (DM) $1/4, 1/4, 1/2, 1/2$ b) DM $\rightarrow 1,1,1,1$

c) DM $\rightarrow 1/2, 1/2, 0, 1/2$ d) $1/4, 1/4, 1/4, 1/4$

Soln:

$(AA^T)^{-1} = (AA^{-1})^{-1}$ ($\because$ Orthogonal matrix)

$\Rightarrow (AA^T)^{-1} = I^{-1} = I \Rightarrow$ option b

Q) $4x + 2y = 7$ a) unique soln b) No soln

$2x + y = 6$ c) Infinite soln d) Exactly 2 soln

option b

Q) A sequence $x(n)$ is specified as $\begin{bmatrix} x(n) \\ x(n-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, for $n \geq 2$

The initial conditions are $x(0) = 1$, $x(1) = 1$ & $x(n) = 0$ for $n < 0$.

The value of $x(12) =$ ____

Soln

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda + \lambda^2 - 1 = 0$

$A^2 = I + A$; $A^3 = A + A^2 = 2A + I$, $A^4 = A^2 + I + 2AI$

$\Rightarrow A^4 = 3A + 2I$

$\cancel{A^{12} = (2A+I)(3A+2I) = 6A^2 + 4A + 3A + 2I}$

$\cancel{\Rightarrow A^{12} = 6I + 6A + 7A + 2I = 13A + 8I}$

$\cancel{\therefore \begin{bmatrix} x(12) \\ x(11) \end{bmatrix} = \begin{bmatrix} 21 & 13 \\ 13 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 21 \\ 13 \end{bmatrix}}$

$\cancel{\therefore x(12) = 21}$

$A^{12} = (A^4)^3 = (3A)^3 + (2I)^3 + 3(3A)(2I)(3A+2I) = 27A^3 + 8I + 18A(3A+2I)$

$\Rightarrow A^{12} = 27(2A + I) + 8I + 54A^2 + 36A = 54A + 27I + 8I + 54I + 54A + 36A$

$\Rightarrow A^{12} = 144A + 89I$.

$\therefore \begin{bmatrix} x(12) \\ x(11) \end{bmatrix} = \begin{bmatrix} 233 & 144 \\ 144 & 89 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 233 \\ 144 \end{bmatrix}$

$\therefore x(12) = 233$

Q) If $M = \begin{bmatrix} 3/5 & 4/5 \\ x & 3/5 \end{bmatrix}$ & $M^T = M^{-1}$, then $x = ?$

a) $-4/5$
b) $-3/5$
c) $3/5$
d) $4/5$

Soln M1 $M^T = M^{-1} \Rightarrow$ M is orthogonal

$\therefore MM^T = I \Rightarrow \begin{bmatrix} 3/5 & 4/5 \\ x & 3/5 \end{bmatrix} \begin{bmatrix} 3/5 & x \\ 4/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \frac{1}{25} \begin{bmatrix} 25 & 3x'+12 \\ 3x'+12 & (x')^2+9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \because x' = \frac{x}{5}$ ... $\cancel{x'/5}$

$\Rightarrow 3x'+12 = 0 \Rightarrow x' = -\frac{12}{3} = -4$;

$\Rightarrow \cancel{x'/1 / 120}$. $\therefore x = \frac{x'}{5} = \underline{-4/5}$

M2 Since M is orthogonal, $|M| = \pm 1$

$\therefore \begin{vmatrix} 3/5 & 4/5 \\ x & 3/5 \end{vmatrix} = \pm 1 \Rightarrow \frac{9}{25} - \frac{4x}{5} = \pm 1$

$\Rightarrow \frac{9}{25} - 1 = \frac{4x}{5}$ or $\frac{9}{25} + 1 = \frac{4}{5}x$

$\Rightarrow \frac{-16}{25} = \frac{4}{5}x$ or $\underbrace{\frac{34}{25} = \frac{4}{5}x}$

$\Rightarrow x = \underline{-\frac{4}{5}}$ does not match with the options

Q) $A = \begin{bmatrix} k & 2k \\ k^2-k & k^2 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ No. of distinct real values of k for which eq: $AX = 0$ has infinitely many soln is. ____

Soln Given $AX = 0$ has infinitely many soln.

$\therefore |A| = 0 \Rightarrow \begin{vmatrix} k & 2k \\ k^2-k & k^2 \end{vmatrix} = 0 \Rightarrow k^3 - 2k^3 + 2k^2 = 0$

$\Rightarrow k^3 - 2k^2 = 0$

$\Rightarrow k^3 = 2k^2$

$\Rightarrow k = 0, 0 \quad k = 2,$

$\therefore$ No. of distinct real values of $k = \underline{2}$