{Math GATE}

- Matrices :- (M x n).

* TYPES OF MATRIX :-

① Coloumn Matrix :-
→ only one coloumn.

② Row matrix :-
→ only one Row.

③ Rectangular matrix :-
→ $M \neq n$
- $|A| \neq$ not possible
- $|A - \lambda I| = 0$ not possible.

④ Square Matrix :-
→ $M = n$.

⑤ Diagonal matrix :-
→ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ where $a_{ij} = 0$.
↳ primary diagonal (or) principle dia. (May (or) may not be zero).

- $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} 10 & 0 \\ 0 & 11 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 22 \end{bmatrix}$
- $\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/2 \end{bmatrix}$

* Concept of minm no. of zero :-
→ $2 \times 2 = 2$
$3 \times 3 = 6$
$4 \times 4 = 12$
⋮
$\boxed{n \times n = n(n-1)}$

⑥ Unit Matrix :-
→ $I = [1]$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- $I^{-1} = I$ $IA = A$.
$|I| = 1$
$A \cdot I = A$

⑦ Upper Triangular Matrix :-
→ $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 7 \end{bmatrix}$
→ Eigen value $= \lambda_1, \lambda_2, \lambda_3$
$= 1, 2, 7.$
→ $|A|$ = Product of dia. elements

⑧ Lower Triangular matrix :-
→ $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix}$
→ other same as UTM.

⑨ Scalar Matrix :-
→ $\begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} = KI$
↳ same

⑩ Symmetric matrix :-
→ $A = A^T$
$a_{ij} = a_{ji}$

⑪ Skew-symm Matrix :-
→ $A = -A^T$
$a_{ij} = -a_{ji}$
→ Diagonal elements are zero.

⑫ Hermitian matrix :-
→ $\boxed{a_{ij} = \overline{a_{ji}}}$
→ $A = \begin{bmatrix} 5 & 2+3i \\ 2-3i & 2 \end{bmatrix}$

⑬ Skew-Her. matrix :-
→ $\boxed{a_{ij} = -\overline{a_{ji}}}$
→ Dia = 0 (or) it is purely imaginary.

⑭ Orthogonal matrix :-
→ $\boxed{AA^T = A^T A = I}$
→ $\boxed{A^{-1} = A^T}$

⑮ Singular matrix :-
→ $\boxed{|A| = 0}$ (Non-Invertible)
→ $\boxed{|A| \neq 0}$ (Invertible).
↳ Non-Singular.

⑯ Unitary matrix :-
→ $A \cdot A^\theta = A^\theta A = I$
↳ Transpose of conjugate.
→ $\boxed{A^\theta = A^* = (\bar{A})^T = (\overline{A^T})}$

* $\left.\begin{aligned} A \cdot A^\theta &= I \\ A A^{-1} &= I \end{aligned}\right\} A^{-1} = A^\theta$

⑱ Special matrix :-
- Involutary matrix
$\boxed{A^2 = I}$
- Idempotent Matrix
$\boxed{A^2 = A}$
- Nilpotent Matrix.
$\boxed{A^K = 0}$
→ $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -6 & -1 & -3 \end{bmatrix}$ ⇒ $A^1 \neq 0$, $A^2 \neq 0$, $A^3 = 0$
↓
'A' is N.M. of ~~order~~ index/class 3.

* Matrix Multiplication :-

→ Associative $(AB)C = A(BC)$
Not commutative.

* $A_{2 \times 2} \times B_{2 \times 3} = C_{2 \times 3}$
(equal)

→ $A_{3 \times 3} \times B_{4 \times 4} \neq$ not possible

→ $AB = BC$ → $\boxed{B = C}$
when 'A' is non-singular.

* A is real square matrix then
$A + A'$ → Always symmetric
$A - A'$ → Always skew symm

$\boxed{A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')}$
↓ symmetric   ↓ skew-symm

* If A is symmetric matrix (or) Real square matrix then AA^T is Always symm matrix

* $(P+Q)^2 = (P+Q)(P-Q)$
$= P \cdot P + Q \cdot Q + PQ + QP$.

* DETERMINANTS *

→ $\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$.

* Minors and co-factors :-
→ $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

* Minor of $a_{21} = M_{21} = a_{12}a_{33} - a_{13}a_{32}$

* $\boxed{\text{Co-factor} = A_{ij} = (-1)^{i+j} \cdot M_{ij}}$

* The det. of UTM, LTM & diagonal matrix is its multiplication of diagonal elements.

* Properties :-
① $\boxed{|A^{-1}| = \frac{1}{|A|}}$
② $\boxed{|A^T| = |A|}$
③ If two rows (or) coloums are interchanged then 'A' multiplied by '-1'.
④ If $|A|_{n \times n}$ matrix then
$\boxed{|KA| = K^n |A|}$

* If we have give $|A|$ and we perform operation $R_i : R_i \pm KR_j$ (or) $C_i : C_i + KC_j$ then obtained $|B|$ remains unchanged

* Trick :-
$A = \begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$

$|A| = (abcd)\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$

* $\boxed{|\text{Adj} A| = |A|^{n-1}}$

* $|\text{adj}(\text{adj}(A))| = |A|^{(n-1)^2}$

* Inverse of Matrix *

→ $A^{-1} = \frac{\text{adj} A}{|A|}$
→ $\text{adj} A = [\text{Co-factor}]^T$
→ $\boxed{A \cdot A^{-1} = A^{-1} \cdot A = I}$
→ $(AB)^{-1} = B^{-1} A^{-1}$

* Trick :-
$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then
$A^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{|A|}$

* 3x3 matrix :-
Ex:- $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $A^{-1} = ?$

* $|A| = -1 - 1 = -2$.

Step-I] Minors of $A = \begin{bmatrix} -1 & 0 & -1 \\ -1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

Step-II] Co-factors of $A = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & -1 \\ -1 & -2 & 1 \end{bmatrix}$

Step-III] adj A
$= \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$

* $\boxed{(A^{-1})^{-1} = A}$ **

* RANK OF MATRIX *

⇒ Rank of matrix = No. of non-zero rows + Echeleon Matrix.

- Note :-
$A = \begin{bmatrix} 5 & 10 & 10 \\ 1 & 0 & 2 \\ 3 & 6 & 6 \end{bmatrix}$

Step-I] $|A|_{3 \times 3} = 0 \rightarrow \rho(A) \neq n$
$\neq 0 \rightarrow \rho(A) = n$.

Step-II] $|A|_{2 \times 2} = 0 \rightarrow \rho(A) \neq (n-1)$
$\neq 0 \rightarrow \rho(A) = (n-1)$

* Eigen values (λ) :-

→ Characteristic eqⁿ of 'A'

$$\boxed{|A - \lambda I| = 0}$$

where, $\lambda = \lambda_1, \lambda_2, \lambda_3 \ldots$

i) Charact roots/values
ii) latent roots
iii) proper values
iv) eigen values

* For 2x2 matrix:-

→ Characteristic eqn:-

$$\boxed{\lambda^2 - \underbrace{S_1}_{\text{Trace of 'A'}}\lambda + |A| = 0}$$

* For 3x3 matrix:-

$$\boxed{\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0}$$

S_2: sum of minors of dia elements

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{33} \end{vmatrix}$$

* Property:-

① Sum of all eigen values = sum of all dia. elements.

$$\boxed{\Sigma \text{ of EV} = \Sigma \text{ of DE}}$$

② $|A|$ = product of All EV's.

③ The EV of UTM, LTM, dia M, unit M, scalar M is its diagonal elements.

④ Eigen values:-
→ Real no's → Not necessary that they will appear in pair
→ Complex no's → $a \pm bi$ (Always in pair).

* Trace of matrix:-

→ sum of diagonal elements.

* Important:-

① Hermitian matrix → EV are real
② symmetric matrix → EV are real
③ Skew-symm matrix → EV are imaginary (or) zero.
④ unitary matrix → EV are 1 (or) −1.
⑤ orthogonal Matrix → 'λ' is EV of OM then '$\frac{1}{\lambda}$' is one of EV

→ If λ is an EV of 'A' then
$A^{-1} \to \lambda^{-1}$ ($1/\lambda$) | $adj(A) = \frac{|A|}{\lambda}$
$A^n \to \lambda^n$
→ EV of A and A^T are same

* Eigen vectors:-

$$\boxed{AX = \lambda X}$$

$$\boxed{[A - \lambda I]X = 0}$$

↳ To find Eigen vectors.

* Properties:-

→ E values = Real then E vectors = Real
→ E values = complex then E vectors = complex.
→ E vectors of A & A^m is same.
→ E vectors of symm. matrix are orthogonal vectors.
↳ dot product is zero
$u \cdot v = 0$

* Linearly Independent/dependent Vector:-

$$\boxed{k_1x_1 + k_2x_2 + k_3x_3 = 0}$$

when, $k_1 = k_2 = k_3 = 0$ then EV are Linearly independent otherwise it L.D.

* Linearly Independent vector = $n - r$

n → no of variables
r → rank of $[A - \lambda I]$.

* Cayley Hamilton Theorem:-

⇒ Every square matrix satisfies its characteristic equation.
$A \to \lambda$
$\lambda \to A$

* System of Linear equation:-

AX=B → Non-Homoge equation
↳ unique soln
↳ infinitely many soln
↳ No soln.

AX=0 → Homogen. system
↳ Unique soln
↳ Infinitely many soln.

* condition:-

① $\rho(A) \neq \rho(A:B)$ (or) $\rho(A) < \rho(A:B)$
↳ No soln.

② $\rho(A) = \rho(A:B)$
→ ρ = no. of variables = Unique soln
→ ρ < no. of variables = Infinitely many soln

* Trick:-

① $2x+3y=7$, $x-y=1$ } valid distinct ⇒ unique soln.

② $2x+3y=7$, $8x+12y=28$ } valid (same) ⇒ Infinitely many soln (1 eqn 2 variables)

③ $2x+3y=7$, $4x+6y=20$ } Not valid ⇒ No soln.

* For (3x3)

$|A| \neq 0$ (unique soln)
$|A| = 0$ → Infinitely many soln / No soln

* AX=0

$|A| = 0$
- Infin. many soln
- Non-zero soln
- Non-trivial soln

$|A| \neq 0$
- Unique soln
- $x=y=z=0$
- zero soln
- Trivial soln

* Note:-

Row addition

$$\boxed{\Sigma R_{min} \leq \text{Real Eigen value} \leq \Sigma R_{max}}$$

* Diagonalisation of matrix:-

A → (Linearly Independent vectors)

$$\boxed{D = P^{-1}AP}$$

D → Diagonal matrix
A → simple matrix of n×n
P → $[x_1 \ x_2 \ x_3]$ (modal matrix m) Eigen vectors

CALCULUS

① Infinite series:-

→ AGP series:- N^r behaves like AP and D^r behaves like GP

* Taylor's series:-

$$F(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots$$

↳ about $x=a$

* Maclaurian's series about at x=0:-

$$F(0) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \ldots$$

* Some Imp series:-

① $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$
② $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$
③ $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots$
④ $(1-x)^{-1}$ = All +ve' ---
⑤ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$
⑥ $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$
⑦ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$
⑧ $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$
⑨ $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$
⑩ $\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$

② Limits & continuity

* condⁿ:-

$$\lim_{x\to 0^-} f(x) = \lim_{x \to 0^+} f(x)$$

LHL = RHL

* Std. limits:-

① $\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$
② $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\tan x}{x} = 1.$
③ $\lim_{x\to\infty} \frac{\sin x}{x} = 0$
④ $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1.$

* Trick:-

→ use L'Hospital Rule when $\left(\frac{0}{0}\right)\left(\frac{\infty}{\infty}\right)$.

→ (1^∞)

$$\lim_{x\to a} f(x)^{g(x)} = e^{\lim_{x\to a} g(x)[f(x)-1]}$$

* Continuity at point:-

⇒ LHL = RHL = f(a).

* Continuity in an Interval:-

→ A fn said to be continuous in [a,b] if
ⓐ $f(x)$ is cont. $\forall x \in (a,b)$
ⓑ $\lim_{x\to a^+} f(x) = f(a)$
ⓒ $\lim_{x\to a^-} f(x) = f(b)$.

* Differentiability:-

$$\boxed{LHD = RHD}$$

Note:
"All Differentiable fn's are continuous but all continuous fn's are not differentiable."

* Mean value Theorems :-

① Rolle's Theorem :-

→ $f(a)=f(b)$, $\boxed{f'(c)=0}$

② Lagrange's MVT :-

→ $f(a)\neq f(b)$, $\boxed{f'(c)=\frac{f(b)-f(a)}{b-a}}$

③ Cauchy's MVT :-

→ $\boxed{\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}}$

* Maxima and Minima :-

i) When tangent makes acute Angle with x-axis then graph is strictly Increasing.

ii) Tangent make obtuse Angle with x-axis then graph strictly Decreasing.

iii) $f'(x)\geq 0$ ⇒ Increasing graph.

iv) $f'(x)<0$ ⇒ Decreasing graph.

→ Steps to obtain maxima & minima :-

① $f'(x)=0$

→ $x=a,b,c$.

② find $f''(x)$

- If $f''(a)<0$ → maxima.
- If $f''(b)>0$ → Minima.
- If $f''(c)=0$ → Saddle Pt.

→ Absolute maxima & minima :-

- $f'(x)=0$ → $x=a,b$
- evaluate $f(a)$ & $f(b)$.
- Max^m value → Ab. maxima.
- Min^m value → Ab. minima.

→ Greatest value & Least value :- (in interval).

Eg:- $f(x)=2x^2-9x-1$, $[2,5]$

→ $f'(x)=4x-9=0$

$x=9/4$

→ $f''(x)=4>0$.

→ $f(\frac{9}{4})=\frac{-89}{8}$ → min^m/least value

→ $f(2)=-11$, $f(5)=4$ → greatest value

→ Extremum value = optimum value

↳ either max^m/min^m.

* 2 variable max^m and minima :-

→ $F_x=\frac{\partial F}{\partial x}$ (p), $f_{xx}=\frac{\partial^2 F}{\partial x^2}$ (r)

$F_y=\frac{\partial F}{\partial y}$ (q), $f_{yy}=\frac{\partial^2 F}{\partial y^2}$ (t)

$f_{xy}=\frac{\partial^2 F}{\partial x\cdot\partial y}$ (s).

① Find p, q, r, s, t

② $\left(\frac{\partial F}{\partial x}\right)=0$, $\left(\frac{\partial F}{\partial y}\right)=0$

③ at each stationary pt. find r, s, t.

④ $rt-s^2>0$, $r>0$

↳ minima at st. pt.

⑤ $rt-s^2>0$, $r<0$

↳ maxima at st pt.

⑥ $rt-s^2<0$.

↳ saddle pt.

⑦ $rt-s^2=0$

↳ Undecided.

* Partial differentiation :-

① Homogeneous function :-

$\boxed{f(\lambda x,\lambda y)=\lambda^n[f(x,y)]}$ (n → degree)

② Euler's Homog eqn :-

$\boxed{x\frac{\partial F}{\partial x}+y\frac{\partial F}{\partial y}=n.F}$

$\boxed{x^2\frac{\partial^2 F}{\partial x^2}+2xy\frac{\partial^2 F}{\partial x\cdot\partial y}+y^2\frac{\partial^2 F}{\partial y^2}=n(n-1)f}$

③ Total derivative :-

$\boxed{\frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}}$

* Integration :-

① $\int x^n dx=\frac{x^{n+1}}{n+1}+c$.

② $\int (ax+b)^n dx=\frac{(ax+b)^{n+1}}{a(n+1)}+c$.

③ $\int \frac{1}{x}dx=\ln(x)+c$.

④ $\int a^x dx=\frac{a^x}{\ln a}+c$.

⑤ $\int \ln x\, dx=x(\ln x-1)+c$.

⑥ $\int \sin x\, dx=-\cos x+c$.

⑦ $\int \cos x\, dx=\sin x+c$.

⑧ $\int \tan x\, dx=\ln|\sec x|+c$

⑨ $\int \cot x\, dx=\ln|\sin x|+c$.

⑩ $\int \sec x\, dx=\ln|\sec x+\tan x|+c$.

⑪ $\int \operatorname{cosec} x\, dx=\ln|\operatorname{cosec} x-\cot x|+c$.

⑫ $\int \sec^2 x\, dx=\tan x+c$.

⑬ $\int \sec x.\tan x\, dx=\sec x+c$.

⑭ $\int \operatorname{cosec}^2 x\, dx=-\cot x+c$.

⑮ $\int \operatorname{cosec} x.\cot x\, dx=-\operatorname{cosec} x+c$.

⑯ $\int \frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$.

⑰ $\int \frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\left(\frac{x}{a}\right)+c$

⑱ $\int e^{ax}\sin bx\, dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)$.

⑲ $\int e^{ax}\cos bx\, dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)$.

⑳ $\int \frac{f'(x)}{f(x)}dx=\ln|f(x)|+c$

㉑ $\int \frac{1}{a^2-x^2}dx=\frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)+c$

㉒ $\int \frac{1}{x^2-a^2}dx=\frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)+c$.

㉓ $\int \frac{1}{\sqrt{x^2+a^2}}dx=\sinh^{-1}\left(\frac{x}{a}\right)=\ln(x+\sqrt{x^2+a^2})+c$

㉔ $\int \frac{1}{\sqrt{x^2-a^2}}dx=\cosh^{-1}\left(\frac{x}{a}\right)+c=\ln(x+\sqrt{x^2-a^2})+c$

㉕ $\int \sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$.

* Integration by parts (ILATE) :-

$\boxed{\int uv=u\int v-\int(u'\int v)}$

* Hindu's Method :-

$\boxed{\int x^n f(x)\,dx=x^n\int f(x)-\frac{d(x^n)}{dx}\iint f(x)+\text{----}}$

↳ Trigo/expo

* Definite Integral :-

① $\int_a^b f(x)dx=[F(x)]_a^b=F(b)-F(a)$

② $\int_a^b f(x)dx=\int_a^b f(t)\,dt$

③ $\int_a^b f(x)\,dx=-\int_b^a f(x)\,dx$.

④ $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$, where, $a<c<b$.

⑤ $\int_0^a f(x)dx=\int_0^a f(a-x)dx$.

⑥ $\int_{-a}^a f(x)dx=\begin{cases}2\int_0^a f(x)\,dx, & \text{even fn. } [f(-x)=f(x)]\\ 0, & \text{odd fn. } [f(-x)=-f(x)].\end{cases}$

⑦ $\int_0^{2a} f(x)dx=\begin{cases}2\int_0^a f(x)\,dx, & \text{even fn } [f(2a-x)=f(x)]\\ 0, & \text{odd fn } [f(2a-x)=-f(x)]\end{cases}$

⑧ $\int_0^{na} f(x)\,dx=n\int_0^a f(x)\,dx$.

⑨ $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$.

* $\int_0^{\pi/2}\sin^n x\,dx=\int_0^{\pi/2}\cos^n x\,dx=\left[\frac{(n-1)(n-3)\text{----}2\times 1}{n(n-2)(n-4)\text{---}2\times 1}\right]\times K$.

Where, $K=\begin{cases}\pi/2, & \text{'n' even}\\ 1, & \text{'n' odd.}\end{cases}$

* $\int_0^{\pi/2}\sin^m x.\cos^n x\,dx=\left\{\frac{[(m-1)(m-3)\text{---}\times 2\times 1][(n-1)(n-3)\text{---}2\times 1]}{[(m+n)(m+n-2)(m+n-4)\text{----}]}\right\}\times K$

where, $K=\begin{cases}\pi/2, & \text{m \& n are even}\\ 1, & \text{otherwise}\end{cases}$

* $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx=\frac{b-a}{2}$.

* $\int_0^{\pi/2}\sin^m\theta\,\cos^n\theta\,d\theta=\frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$

*IMP :-

$\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta}=\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}=\frac{2\pi}{\sqrt{a^2-b^2}}$

*Double Integral:-

→ $\int_c^d \int_a^b f(x)\, dx \cdot dy$.

→ Dependent Case:-

↳ (order of Int. matters).

$\int \int f(x,y)\, dx \cdot dy$.

↳ 1 varible

↳ only constants

* Double Integration in Polar form:-

→ $x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$ } (r,θ).

* Triple Integration:-

$$\int_x \int_y \int_z f(x,y,z)\, dx\,dy\,dz$$

* Application of Integration:-

Case-I] $y=f(x)$, $a \le x \le b$.

→ $ds = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$.

→ Length of curve, $L = \int_a^b ds$

Case-II] $x=f(y)$, $a \le y \le b$.

→ $ds = \int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}$

Case-III] $x=f(t)$, $y=\phi(t)$.

→ $L=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$.

Case-IV] $r=f(\theta)$.

→ $L = \int_{\theta=\alpha}^{\beta} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$.

Case-V] $\theta=f(r)$.

→ $L = \int_{r=a}^{b} \sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}\, dr$.

* Volume of Revolution:-

① Rev. around x-axis →

$V = \int_a^b \pi y^2\, dx$.

② Around y-axis:-

$V = \int_a^b \pi x^2\, dy$.

③ Parametric form:-

$\theta=0$, $V = \int_\alpha^\beta \frac{2\pi}{3} r^3 \sin\theta\, d\theta$.

$\theta=\pi/2$, $V = \int_\alpha^\beta \frac{2\pi}{3} r^3 \cos\theta\, d\theta$.

* PROBABILITY:-

$$\boxed{P = \frac{n(E)}{n(S)} = \frac{\text{Req. Case}}{\text{Total Case}}}$$

① Equally Likely Events:-

→ All elementary events of 'S' have same chance of occuring.

② Mutually exhaustive events:-

→ $\boxed{A \cup B = S}$

→ A' = complement event.

$\boxed{S = A + A'}$

$\therefore \boxed{P(A) + P(A') = 1}$

Note:-

$P(X>2) = P(X=3) + P(X=4) + \dots$

$\therefore \boxed{P(X>2) = 1 - P(X \le 2)}$

→ $\boxed{P(A\cup B) = P(A) + P(B) - P(A\cap B)}$

Either A/or B/or Both. Common to both A & B.

→ $(A\cup B)'$ = Neither A nor B.

$\boxed{P(A\cup B)' = 1 - P(A\cup B)}$

* Rules of Probability:-

① $0 \le P \le 1$

② $P(E) + P(E') = 1$

③ $P(A'\cap B') = P(A\cup B)' = 1 - P(A\cup B)$.

④ $P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(A\cap C) + P(A\cap B\cap C)$.

⑤ Mutually exclusive events:-

→ Two event can't happen together.

→ $A \cap B = \phi$.

→ $\boxed{P(A\cup B) = P(A) + P(B)}$.

↳ when A & B mutually exclusive

④ Independent event:- (with replacement).

→ A & B can happen together.

→ occurance of A & B won't affect prob. of each other.

→ $\boxed{P(A\cap B) = P(A) \cdot P(B)}$.

* Coins:-

1 coin → $S = \{H,T\}$, $n(S) = 2^1$

2 coin → $S = \{HH, HT, TH, TT\}$, $n(S) = 2^2$.

3 coin = $S = \{HHH, HTH, THH, TTH, HHT, HTT, THT, TTT\}$

$n(S) = 8 = 2^3$

$\boxed{n \text{ coins} = 2^n}$

* Dice:-

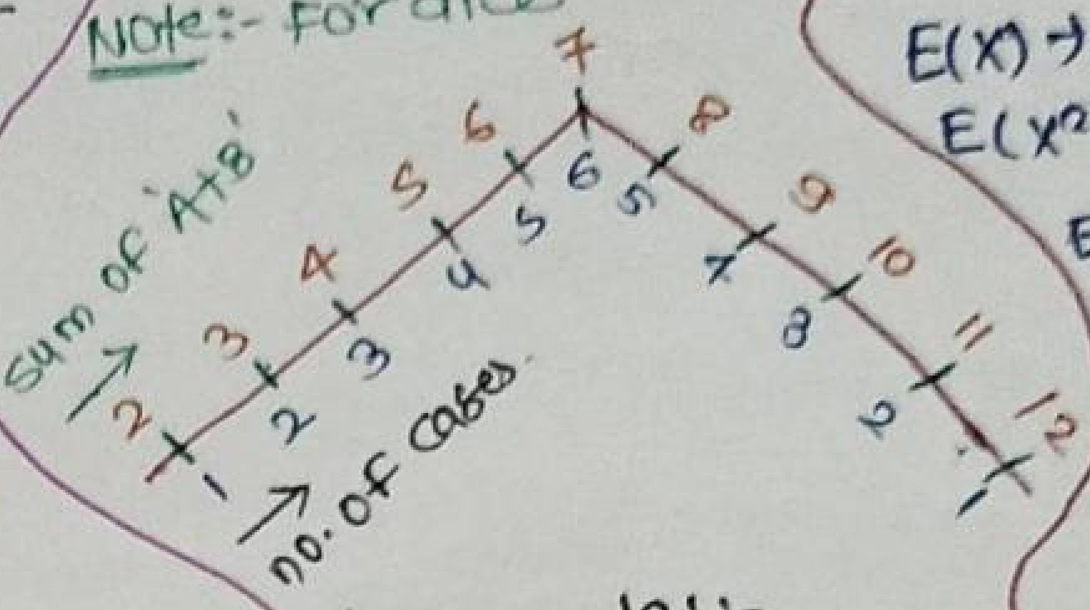
1 die → $S = \{1,2,3,4,5,6\}$

$n(S) = 6$

2 die → $n(S) = 36 = 6^2$

$\boxed{n \text{ die} \to n(S) = 6^n}$

Note:- For dice



* Formulas:-

① $^nP_r = \frac{n!}{(n-r)!}$ (without repetation)

② $^nP_r = n^r$ (with R)

③ $^nC_r = \frac{n!}{r!\,(n-r)!}$ (without repet)

④ $^nC_r = \frac{(n+r-1)!}{r!\,(n-r)!}$ (with R)

* $^nP_n = n!$, $^nP_1 = n$, $^nC_1 = n$

$^nP_0 = 1$, $^nC_0 = {}^nC_n = 1$, $^nC_r = \frac{^nP_r}{r!}$

* Bayo's Theorem:-

$$\boxed{P(A/E) = \frac{P(E/A)\cdot P(A)}{P(E/A)\cdot P(A) + P(E/B)\cdot P(B)}}$$

* RANDOM Variables:-

Discrete RV

→ $\Sigma P(x) = 1$

→ Finite

→ Countable numbers.

Continuous RV

→ Infinite no of possible value

→ usually measurements.

→ $P(x)$ = Prob. mass fn (or) Prob. density fn (Pmf (or) Pdf).

→ $F(x)$ = Commulative distribution fn (Cdf).

$= \sum_{-\infty}^{x} P(x)$.

* property of Pdf in 2D:-

$$\boxed{\int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} f(x,y)\, dx\, dy = 1.}$$ **

* Mean of RV X:-

→ $\boxed{\text{mean} = \mu = E[X]}$

$$\boxed{E[X] = \sum_{i=1}^{4} x_i\, P(x_i)}$$

→ Variance:-

$$\boxed{V(X) = E(X^2) - (E(X))^2}$$

* Moment generating fn:-

$$\boxed{\left[\frac{d^n E(e^{xt})}{dt^n}\right]_{t=0} = E(X^n)}$$

$E(X) \to n=1 \to 1^{st}$ moment.

$E(X^2) \to n=2 \to 2^{nd}$ moment.

$E(X^n) \to n=n \to n^{th}$ moment.

$\boxed{\frac{d^n}{dt^n} m_0(t) = E(X^n)}$

↳ at origin.

* Binomial Distribution:-

$\boxed{P(X) = {}^nC_x\, p^x\, q^{n-x}}$

↳ no. of Bernoulli trials A is an event

↳ prob. of success

→ $q = 1-p$ = prob of failure

* $\boxed{E(X) = np}$

* $\boxed{V(X) = npq}$

* Std. deviation:-

$\boxed{Sd. = \sqrt{npq} = \sqrt{V(X)}}$

* Poisson's Distribution:-

$\therefore \boxed{P(X) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}}$

where $x = 0, 1, 2, ---$

$\boxed{E(X) = \lambda}$

$\boxed{var(X) = \lambda}$

⑤ * Uniform distribution:-

→ A cont RV $X \to U(a,b)$

$\because \boxed{f(x) = \frac{1}{b-a}}$, $a<x<b$.

* $\boxed{\text{Mean} = \frac{b+a}{2}}$

* $\boxed{\text{Variance} = \frac{(b-a)^2}{12}}$

* Exponential distribution:-

→ A CRV $X \sim E(\lambda)$ is said to have exp. distribution.

$\boxed{f(x) = \lambda e^{-\lambda x}}$, $x>0$ & $\lambda>0$.

$\boxed{\text{Mean} = \frac{1}{\lambda}}$

$\boxed{\text{Var} = \frac{1}{\lambda^2}}$

* Normal Distribution:-

→ $\boxed{f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}$

→ $\boxed{\text{Mean} = \text{Median} = \text{Mode}.}$

* $P(z<0) = P(z>0) = \int_0^\infty f(z)\,dz = \int_{-\infty}^0 f(z)\,dz = \frac{1}{2}$

* $\int_{-\infty}^{\infty} f(z)\,dz = 1$

* $P(-1<z<1) = 0.6827$

* $P(-2<z<2) = 0.9545$

* $P(-3<z<3) = 0.9973$.

*Mean, mode & median:-

① Mean:-

$\boxed{\bar{x} = \frac{\Sigma x_i}{n} = \frac{\Sigma x_i f_i}{\Sigma f_i}}$

② Median:-

↳ middle no. → Arrange no's lowest to highest.

↳ No middle value → take '2' middle no's and take Avg. of two.

③ Mode:-

↳ occurs most often.

* When 'N' is even.

$\left(\frac{N}{2}\right)^{th}$ and $\left(\frac{N}{2}+1\right)^{th}$

→ When 'N' is odd.

$\left(\frac{N}{2}\right)^{th}$.

*Median for interval:-

$\boxed{= L + \left(\frac{\frac{N}{2} - PCF}{f}\right) \times h}$

*Relationship:-

$\boxed{\text{Mode} \cong 3\,\text{median} - 2m}$

* Std. deviation:-

$\boxed{\sigma = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}}}$

* Vector Calculus:-

* Scalar (or) Dot product:-

$\boxed{\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta}$

→ $\boxed{\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1}$

→ $\boxed{\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0}$

* Cauchy's schwartz inequality:-

$\boxed{\vec{a}\cdot\vec{b} \le |\vec{a}|\,|\vec{b}|}$

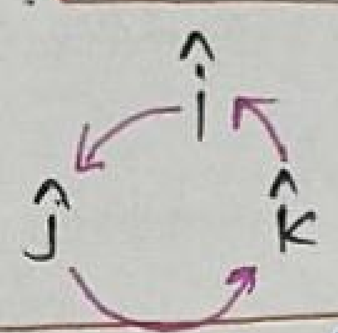
* Cross product:-

$\boxed{\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}}$

$\boxed{|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

*



$\boxed{\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j}}$

$\boxed{\hat{j}\times\hat{i} = -\hat{k},\ \hat{k}\times\hat{j} = -\hat{i},\ \hat{i}\times\hat{k} = -\hat{j}}$

$\boxed{\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0}$

Note:-

Dot product	Cross product
$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$	$\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$
$\vec{a}\cdot\vec{b} = 0\ (\perp)$	$\vec{a}\times\vec{b} = 0\ (\parallel)$

* Scalar Triple product:-

→ $\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

→ $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

→ $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{b}\ \vec{a}\ \vec{c}]$.

→ $[\vec{a}\ \vec{b}\ \vec{c}] = 0$ → coplanar vector.

* Vector Triple product:-

→ $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$.

→ $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\cdot\vec{b} - (\vec{c}\cdot\vec{b})\cdot\vec{a}$

→ $A(\Delta) = \frac{1}{2}|\vec{a}\times\vec{b}|$.

→ $\boxed{\text{Volume} = |\vec{a}\cdot(\vec{b}\times\vec{c})|}$

* Vector differential operator (∇):-

$\boxed{\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}}$

$\boxed{\nabla^2 = \nabla\cdot\nabla = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}}$

$\boxed{\nabla^2\phi = 0 \to \text{Laplace eqn.}}$

* Gradient of scalar fn:-

$\boxed{\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}}$

$\boxed{\text{Unit Normal} = \frac{\nabla\phi}{|\nabla\phi|}}$

* Properties of Gradient:-

① $\nabla(f+g) = \nabla f + \nabla g$

② $\nabla(c_1 f + c_2 g) = c_1\nabla f + c_2\nabla g$

③ $\nabla(fg) = f\,\nabla g + g\,\nabla f$

④ $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$, $g \neq 0$.

* Note:-

① Grad $r = \nabla r = \frac{\vec{r}}{r}$

② $\nabla(r^2) = 2\vec{r}$

③ $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

④ $\nabla(\log r) = \frac{\vec{r}}{r^2}$

⑤ $\nabla(r^n) = n\,r^{n-2}\,\vec{r}$.

⑥ $\nabla(e^{r^2}) = 2e^{r^2}\vec{r}$

* Directional Derivative (DD):-

→ $\boxed{DD = \nabla\phi\cdot\hat{n}}$

where $\hat{n} = \frac{\vec{r}}{|\vec{r}|}$

→ Max^m value of DD $= |\nabla\phi|$.

* Divergence of vector:-

Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

$\boxed{\text{div}(\vec{v}) = \nabla\cdot\vec{v}}$.

$\boxed{\text{div}(\vec{v}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}}$

$\text{div}(\vec{v}) = 0$

↳ solenoidal vector

→ It gives rate at which fluid flows out of unit volume

* Curl of vector:-

Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

$\text{curl}(\vec{v}) = \nabla\times\vec{v}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

→ $\text{curl}(\vec{v}) = \vec{0}$ then $\vec{v}$ → irrotational vector.

→ If $\vec{v}$ is linear velocity then angular velocity

$\boxed{\vec{\omega} = \frac{1}{2}[\text{curl}(\vec{v})]}$

* Vector Identities:-

① $\text{curl}(\text{grad}\,\phi) = \vec{0}$

② $\text{div}(\text{curl}(\vec{v})) = 0$

③ $\boxed{\text{div}(\nabla\phi) = \nabla^2\phi}$

④ $\text{div}[\phi\vec{v}] = \phi\,\text{div}(\vec{v}) + \nabla\phi\cdot\vec{v}$

⑤ $\text{curl}[\phi\vec{v}] = \phi\,\text{curl}(\vec{v}) + \nabla\phi\times\vec{v}$

⑥ $\text{curl}[\text{curl}\,\vec{v}] = \nabla(\text{div}\,\vec{v}) - \nabla^2\vec{v}$

* Line Integral:

(Integration along curve)

$\boxed{\int_C \vec{F}\cdot d\vec{r}}$

* Stokes theorem:-

$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$

Relation betn line & surface integration.

* Green's Theorem:-

$$\boxed{\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

↳ Relation betn Line & double integration

* Gauss-divergence theorem:-

$$\boxed{\oiint \vec{F}\cdot\hat{n}\,ds = \iiint_V (\nabla\cdot\vec{F})\,dv}$$

↳ Relation betn surface & volume integration.

* Complex Variables :-

* Integer power of iota:-

→ $i = \sqrt{-1}$

→ $i = \sqrt{-1}$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$\boxed{i^{4n} = 1}$

→ $\boxed{i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0.}$

* Complex no. :-

$\boxed{z = a + ib}$

where :- a = real part
b = imaginary part

* Imaginary quantity :-

→ $\boxed{\sqrt{-a} \times \sqrt{-b} = -\sqrt{ab}}$

* Properties :-

① $(z_1 + z_2) = (a+c) + i(b+d)$

② $(z_1 - z_2) = (a-c) + i(b-d)$

③ $z_1 \cdot z_2 = [ac - bd] + i(ad + bc)$

④ $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
↳ Associative

⑤ $z_1 \cdot z_2 = z_2 \cdot z_1$
$z_1 + z_2 = z_2 + z_1$ } commutative

⑥ $\dfrac{z_1}{z_2} = \dfrac{(ac+bd) + i(bc - ad)}{c^2 + d^2}$

* Complex conjugate ($\bar{z}$, z^*) :-

$\boxed{\bar{z} = a - ib}$

* Modulus of z :-

① $z = |z| = \sqrt{a^2 + b^2}$

② $|\bar{z}| = \sqrt{a^2 + b^2}$

③ $\boxed{|z|\,|\bar{z}| = |\bar{z}|^2}$

④ $z^{-1} = \dfrac{\bar{z}}{|z|^2}$

⑤ $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

⑥ $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

⑦ $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

⑧ $z\bar{z} = \{Re(z)\}^2 + \{Im(z)\}^2$

* Argument / Amplitude in the quadrant:-

① when $z = x + iy$, if $x > 0$, $y > 0$.

(I) $\text{Arg} = \tan^{-1}\left|\frac{y}{x}\right|$

$\boxed{\theta = \tan^{-1}\left|\frac{y}{x}\right|}$

② In 2nd quadrant if $x < 0$, $y > 0$.

$\boxed{\theta = \pi - \tan^{-1}\left|\frac{y}{x}\right|}$ (II)

③ when $z = x + iy$, if $x < 0$, $y < 0$

(III) $\boxed{\theta = \pi + \tan^{-1}\left|\frac{y}{x}\right|}$

(or)

$\boxed{\theta = \tan^{-1}\left|\frac{y}{x}\right| - \pi}$

④ when $z = x + iy$, $x > 0$, $y < 0$.

$\boxed{\theta = 2\pi - \tan^{-1}\left|\frac{y}{x}\right|}$

(or) (IV)

$\boxed{\theta = -\tan^{-1}\left|\frac{y}{x}\right|}$

* Polar form of complex no. :-

$\boxed{z = r(\cos\theta + i\sin\theta)}$

* Euler's Form of complex no. :-

$\boxed{\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}}$

$\boxed{\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2}}$

$\boxed{z = r(\cos\theta + i\sin\theta)}$

$\boxed{z = re^{i\theta}}$

* De-Movier's Theorem:-

$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta = e^{in\theta}.}$

$\boxed{(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta = e^{-in\theta}}$

* cube root of unity :-

$z^3 = 1$

∴ roots → $1, \omega, \omega^2$

$\boxed{\omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}}$

$\boxed{\omega^2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}}$

$1 + \omega^n + \omega^{3n} = \begin{cases} 0, & \text{'n' is not multiple of '3'} \\ 3, & \text{multiple of '3'} \end{cases}$

* Analytic function:-

→ derivative of $f(z)$ exists at each & every pt in that region.

→ C-R eqn :-

$\boxed{\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \quad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}}$

$\boxed{u_x = v_y, \quad u_y = -v_x}$

* Analytic fn in polar form:-

$z = re^{i\theta}$

$\boxed{\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}, \quad \dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r}}$

* construction of f(z) :-

① Milne's Thompson method:-

$\boxed{x = z, \quad y = 0}$

② Total derivative concept :-

$\boxed{v = \int (-u_y)\,dx + \int u_x\,dy + c}$

* Complex Integration:-

* Cauchy's Integral theorem

$\boxed{\oint f(z)\,dz = 0}$ — $f(z)$ is analytic; 'c' is closed curve

* Cauchy's Integral Formula:-

$\boxed{\oint \dfrac{f(z)}{z-a}\,dz = 2\pi i\, f(a)}$

Generalized C-I formula

$\oint \dfrac{f(z)}{(z-a)^{n+1}} = 2\pi i\, \dfrac{f^n(a)}{n!}$

* Laurent's series :-

→ when $f(z)$ is not analytic

$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \dfrac{b^n}{(z - z_0)^n}$

* Singularities :-

① Isolated singularity. (Not in that circle)

$\boxed{D^r = 0}$

② Non-Isolated singularity (In given circle)

$\boxed{D^r = 0}$

③ Removable singularity :-

$f(z) = \dfrac{z^2 - 3z + 2}{(z-1)(z+2)}$

$z = 1$ is removable sing and $z = -2$ is simple pole

$f(z) = \dfrac{(z-2)}{(z+2)}$.

④ Simple pole:-

$\boxed{D^r = 0}$

$f(z) = \dfrac{1}{(z-3)(z-7)}$

poles = 3, 7

⑤ Poles of order 'n'

$f(z) = \dfrac{1}{(z+2)^2 (z-5)}$

→ $z = -2$ is pole of order '2'

→ $z = 5$ is pole of order '1'

⑥ Essential singularity :-

$f(z) = \sin\left(\dfrac{1}{z-1}\right)$

$= \dfrac{1}{z-1} - \dfrac{1}{(z-1)^3\, 3!} + \cdots$

∴ $z = 1$ is essential singularity.

* Residues :-

$\int_C f(z)\,dz = 2\pi i \times$ (sum of residues at the singular points within 'c').

* Calculation of Residues:-

① If $z = a$ is simple pole of $f(z)$ then,

$[\text{Res}\, f(z)]_{z=a} = \lim_{z\to a} (z-a)\, f(z)$

② If $z = a$ is pole of order 'n' of $f(z)$ then

$[\text{Res}\, f(z)]_{z=a} = \lim_{z\to a} \dfrac{1}{(n-1)!} \dfrac{d^{n-1}}{dz^{n-1}} \cdot \left((z-a)^n \cdot f(z)\right)$

7 * Differential Eqn:-

* Order and Degree:-

$\left(\frac{d^n y}{dx^n}\right)$ → 'n' is highest

$\left(\frac{d^n y}{dx^n}\right)^P$ → power of highest order.

Note:- For Transcendal eqn's degree is not defined. (Trigo, expo., logr etc)

* Linear & Non-Linear D.E:

→ $P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots P_n y = Q$

where, $P_0, P_1, P_2, \ldots, P_n$ & Q = constant/f(x)

y power = 1.

Ex:- $\frac{d^4 y}{dx^4} + P\frac{d^2 y}{dx^2} + Ky = 0$ (LDE)

$\frac{d^2 y}{dx^2} + \frac{dy}{dx} + \sin y = 0$ (NLDE)

* Formation & soln of D.E:

$y = F(x) \underset{\text{soln of DE}}{\overset{\text{Formation}}{\rightleftarrows}} y'' + Py = Q$

* Leibnitz's Linear equation:-

→ $\frac{dy}{dx} + Py = Q$

where, P & Q = constants/f(x)

→ $IF = e^{\int P\,dx}$

→ $y(IF) = \int Q(IF)\,dx + c.$

* Bernoulis equation:-

→ $\frac{dy}{dx} + P.y = Q.y^n$

→ Divide both side by 'y^n'.

→ put $y^{1-n} = t$.

→ multiply both side by '$n-1$'.

→ $\frac{dt}{dx} + P(n-1)t = Q(n-1)$

→ $I.F = e^{\int (n-1) P\,dx}$

→ $y(IF) = \int (1-n) Q (IF)\,dx + c.$

* Exact D.E:

Form:- $M\,dx + N\,dy = 0$, where M and N are functions of (x,y) and (x,y)

Condition:- $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Solution:-

$\int_{y=\text{const}} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$

* Homogenous D.E:

→ Conversion of Non Exact D.E to exact

→ $IF = \frac{1}{Mx + Ny}$

→ Then follow Exact DE.

* Note:-

① $x\,dy + y\,dx = d(xy)$

② $\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$

③ $\frac{x\,dy - y\,dx}{xy} = d\left(\ln\left(\frac{y}{x}\right)\right)$

* Newton's Cooling law:-

→ $\frac{d\theta}{dt} = -k(\theta - \theta_s)$.

→ $\theta = \theta_s + C.e^{-kt}$.

* Higher order Linear D.E:

→ $\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots a_n y = Q$

→ put $\frac{d}{dx} = D$

→ when,

$Q = F(x) = 0$ → Homogenous eqn.

$Q = f(x) \neq 0$ → Non-Homo. eqn.

→ Soln:- $y = C.F + P.I.$

→ Homogenous eqn, $\boxed{y = CF}$

* Wronskain: W

(For check Linearly dependent (or) independent)

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$|W| = 0$ → Linearly dependent

$|W| \neq 0$ → Linearly independent.

* For Non-Homogenous D.E:

$\boxed{y = C.F + P.I}$ $\boxed{F(x) \neq 0}$

→ To find P.I.:

i) When $X = e^{ax}$

$P.I = \frac{1}{F(D)} e^{ax}$

$= \frac{1}{F(a)} e^{ax}$, $F(a) \neq 0$

→ If $F(a) = 0$.

$P.I. = x.\frac{1}{F'(a)} e^{ax}$

ii) When $X = \sin(ax+b)$ (or) $\cos(ax+b)$

$P.I. = \frac{1}{F(D^2)} \sin(ax+b)$

Replace 'D^2' by '$-a^2$'.

iii) When $X = x^n$

$P.I = \frac{1}{F(D)} . x^n$

$= [F(D)]^{-1} x^n$

↳ expand.

iv) When $X = e^{ax} V$, where V is fn of 'x'

$P.I. = \frac{1}{F(D)} e^{ax} V = e^{ax} \frac{1}{F(D+a)} V$

and then evaluate $\frac{1}{F(D+a)} . V$ as in (i)(ii)(iii).

Roots of Auxiliary eqn (A.E)	Complementary function (C.F).
① Real & different roots. $m_1, m_2, m_3, \ldots$	$C_1 e^{m_1 x} + C_2 e^{m_2 x} + \cdots$
② Real & equal roots. i) $m_1, m_1, m_3, \ldots$	$(C_1 + C_2 x) e^{m_1 x} + C_3 e^{m_3 x} + \cdots$
ii) $m_1, m_1, m_1, m_4, \ldots$	$(C_1 + C_2 x + C_3 x^2) e^{m_1 x} + C_4 e^{m_4 x} + \cdots$
③ Complex roots. $\alpha + i\beta, \alpha - i\beta, m_3, \ldots$	$e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) + C_3 e^{m_3 x} + \cdots$
$\alpha \pm i\beta, \alpha \pm i\beta, m_3, \ldots$	$e^{\alpha x}[(C_1 + C_2 x)\cos\beta x + (C_3 + C_4 x)\sin\beta x + C_5 e^{m_3 x} + \cdots]$

* Partial Differential Equations:

* Lagrange's linear equation:-

→ $\boxed{Pp + Qq = R}$

→ $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

↳ solve $u = a$, $v = b$.

→ $u = f(v)$.

* Higher order partial D.E:

* Rules for finding CF:

Case-I] When the roots are different say $m_1, m_2, m_3, \ldots$

$C.F = f_1(y + m_1 x) + f_2(y + m_2 x) + \cdots$

Case-II] Roots are real & equal

$CF = f_1(y + mx) + x f_2(y + mx) + x^2 f_3(y + mx) + \cdots$

* Rules for P.I:

$P.I = \frac{1}{F(D, D')} F(x,y)$

i) When $F(x,y) = e^{ax+by}$

$P.I = \frac{1}{F(D, D')} e^{ax+by}$

Put $D = a$ and $D' = b$.

ii) When $F(x,y) = \sin(mx+ny)$ (or) $\cos(mx+ny)$

$P.I = \frac{1}{F(D^2, DD', D'^2)} \sin(mx+ny)$

Put $\boxed{D^2 = -m^2,\ DD' = -mn,\ D'^2 = -n^2}$

* $\frac{1}{D-a} X = e^{ax} \int X e^{-ax}\,dx$

* $\frac{1}{D+a} X = e^{-ax} \int e^{ax} X\,dx$.

* Method of variation:-

→ $y'' + Py' + qy = X$

→ $F(D).y = X$

→ CF ⇒ $y = CF = C_1 y_1 + C_2 y_2$.

→ P.I ⇒

$\boxed{-y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx.}$

↳ wronskain matrix.

ii) When $F(x,y) = x^m y^n$

$$P.I = \frac{1}{F(D,D')} x^m y^n$$

$$P.I = [F(D,D')]^{-1} x^m y^n$$

expand in ascending powers of $\frac{D'}{D}$ & operate on $x^m y^n$.

* Classification of PDE:-

A PDE of the form

$$A\frac{\partial^2 z}{\partial x^2} + B\frac{\partial^2 z}{\partial x\,\partial y} + C\frac{\partial^2 z}{\partial y^2} + f(x,y,z,p,q) = 0$$

is said to be,

i) elliptic if $B^2 - 4AC < 0$

ii) parabola if $B^2 - 4AC = 0$

iii) Hyperbola if $B^2 - 4AC > 0$

* Numerical Method *

① Newton-Raphson method:-

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

② Bisection Method:-

$$\boxed{c_1 = \frac{a_0 + b_0}{2}}$$

③ False Position / Regula Falsi method:-

$$\boxed{c = a - \frac{(b-a)}{f(b)-f(a)} \cdot f(a)}$$

↳ use same process for next root. successive iteration

④ Secant Method:-

$$\boxed{x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \cdot f(x_n)}$$

* Trapezoidal Rule:-

$$\int_a^b f(x)\,dx = \frac{h}{2}[(y_0+y_n) + 2(y_1+y_2+\dots+y_{n-1})]$$

where, $h = \frac{b-a}{n}$

* Simpson's method:-

$$\int_a^b f(x)\,dx = \frac{h}{3}[(y_0+y_n) + 2(y_2+y_4+\dots+y_{n-2}) + 4(y_1+y_3+\dots+y_{n-1})]$$

* Simpson's $\frac{3}{8}$th rule:-

$$\int_a^b f(x)\,dx = \frac{3h}{8}[(y_0+y_n) + 2(y_3+y_6+\dots+y_{n-3}) + 3(y_1+y_2+y_4+y_5+\dots+y_{n-1})]$$

* Euler's forward method:-

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$

* Euler's Backward method:-

$$\boxed{y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})}$$

* Runge-Kutta method:-

→ $\frac{dy}{dx} = f(x,y)$

calculate successively,

$k_1 = h f(x_0, y_0)$

$k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$k_3 = h f(x_0 + h, y_0 + \frac{k_2}{2})$

$k_4 = h f(x_0 + h, y_0 + k_3)$

→ order '2'

$$\boxed{K = \frac{k_1 + k_2}{2}}$$

→ order '3'

$$\boxed{K = \frac{1}{6}[k_1 + 4k_2 + k_3]}$$

→ order '4'

$$\boxed{K = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]}$$

* Transform Theory *

* Laplace transformation:-

$$\boxed{L\{f(t)\} = F(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\,dt}$$

* LT of Elementary Functions:-

① $L\{1\} = \frac{1}{s}$

② $L\{e^{at}\} = \frac{1}{s-a}$

③ $L\{\sin at\} = \frac{a}{s^2+a^2}$

④ $L\{\cos at\} = \frac{s}{s^2+a^2}$

⑤ $L\{\sinh at\} = \frac{a}{s^2-a^2}$

⑥ $L\{\cosh at\} = \frac{s}{s^2-a^2}$

⑦ $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$

⑧ $L\{u(t-a)\} = \frac{e^{-as}}{s-a}$

⑨ $L\{t^n\} = \frac{n!}{s^{n+1}}$ +ve integer

* Shifting property:-
(First shifting)

→ $L\{e^{at} f(t)\} = F(s-a)$

$L\{e^{-at} f(t)\} = F(s+a)$.

* Multiplication by t^n:-

$$\boxed{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(F(s))}$$

* Division by 't':-

$L\{f(t)\} = F(s)$ then $\boxed{L\{\frac{1}{t} f(t)\} = \int_s^{\infty} F(s)\,ds}$

* L.T. of periodic function:-

Let $f(t)$ is periodic fn with period 'T'

$$\boxed{L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}}$$

* L.T. of Differential equations:-

→ $L\{f'(t)\} = s^1 F(s) - s^0 f(0)$

$L\{f''(t)\} = s^2 - s^1 f(0) - s^0 f'(0)$

$L\{f'''(t)\} = s^3 F(s) - s^2 f(0) - s f'(0) - s^0 f''(0)$

* $\boxed{L\{\int_0^t f(u)\,du\} = \frac{1}{s} F(s)}$

$\boxed{L\{\int_0^t \int_0^t f(u)\,du\} = \frac{1}{s^2} F(s)}$

* Unit step function:-

$$u(t-a) = \begin{cases} 1, & t > a \\ 0, & t < a. \end{cases}$$

$$\boxed{L\{u(t-a)\} = \frac{e^{-as}}{s}}$$

* Unit impulse function:-

$L\{\delta(t)\} = 1$, $L\{\delta(t+a)\} = e^{as}$.

$L\{\delta(t-a)\} = e^{-as}$

* Division by 's':-

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]\,dt = \int_0^t f(t)\,dt.$$

* Convolution Theorem:-

If $L\{f(t)\} = F(s)$ and $L\{g(t)\} = G(s)$ then

$$L\{f(t) * g(t)\} = L\{\int_0^t f(u) \cdot g(t-u)\,du\} = F(s) \cdot G(s).$$

* Inverse Laplace Transform :-

① $L^{-1}\left\{\frac{1}{s}\right\} = 1$

② $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

③ $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$

④ $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at.$

⑤ $L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at.$

⑥ $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at.$

⑦ $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{\Gamma n} t^{(n-1)}$

⑧ $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{1}{(n-1)!} t^{(n-1)}$

↳ +ve integer.

⑨ $L^{-1}\left\{\frac{e^{-as}}{s-a}\right\} = u(t-a)$

⑩ $L^{-1}\left\{e^{-as}\right\} = \delta(t-a).$

⑪ $L^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \frac{e^{at}\cdot t^{n-1}}{(n-1)!}$

⑫ $L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b} e^{at}\sin bt.$

⑬ $L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = \frac{1}{b} e^{at}\cos bt.$

⑭ $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a} t\sin at.$

⑮ $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{1}{2a^2}(\sin at \cdot at\cos at\}.$

* Scaling property :-

$L\{f(t)\} = F(s)$

$\boxed{L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right).}$