MATHEMATICS

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MATRIX

* Representation of matrix

$$\begin{array}{c|c} & \begin{array}{ccc} C_1 & C_2 & \cdots \leftarrow \text{column} \end{array} \\ A = \begin{array}{c} R_1 \\ R_2 \\ \vdots \\ \uparrow \\ \text{Row} \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \end{bmatrix}_{\text{row} \times \text{column}} \end{array}$$

- $A_{\text{row} \times \text{column}}$

- a_{ij} — i: row, j: column

- ${}_pA^q$ — $q \leftarrow$ column, p: row

* TYPES OF MATRIX :

(i) column matrix : which has only single column. ($A_{n \times 1}$)

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}_{3 \times 1}$$

Transpose : Row are converted into column and columns are converted into row.

(ii) Row matrix : Matrix with single row ($A_{1 \times n}$)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}_{1 \times 3}$$

(iii) Rectangular matrix :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times3}$$

- order/size ⟶ row ≠ column
  $i \neq j$
- we can not determine the 'Eigenvalue' and 'Determinent' of rectangular matrix.
- But 'Rank' can be determine.

(iv) Square matrix :

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}_{2\times2}$$

- order/size ⟶ row = column.
  $i = j$
- we can calculate, 'Determinent', 'eigen values', and 'Rank' of a square matrix.

(v) Diagonal matrix :

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 7 & 0 \\ 5 & 0 & 0 \end{bmatrix}$$

secondary diagonal

leading diagonal/ Primary diagonal/main diagonal/ Principle diagonal

* Primary diagonal elements may or may not be zero.

- Matrix in which primary diagonal elements may be or may not be zero. But apart from that other elements should be compulsory zero.

- Trace of matrix = sum of principle Diagonal element (valid only for square matrix)

Benifits : i. $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & 11 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 13 \end{bmatrix}$

ii. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} * \begin{bmatrix} 10 & 0 \\ 0 & 11 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 22 \end{bmatrix}$

Inverse iii. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1^{-1} & 0 \\ 0 & 2^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

Determinent iv. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ = Product of diagonal elements is determinent.

$= 1 \times 2 = 2$

(v) Eigen values are it self diagonal elements.

- Minimum Number of zero in Diagonal matrix:

$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}_{2\times2}$ = 2 (no. of zero)

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 8 \end{bmatrix}_{3\times3}$ = 6 (no. of zero)

∴ No. of zero = Total element $(n \times n)$ − Primary dig. element (n)

$\therefore$ min. no. of zeros $= n^2 - n = n(n-1)$

- Inverse : $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}^{-1}$ $\rightarrow$ can not be found.

(vi) Unit matrix: Primary diagonal elements are only 1.

$I = [1]_{1\times1}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$

(A) $I \times I \times I \times \text{------} = I^n = I$

(B) $|I| = 1$

(C) $A \cdot I = I \cdot A = A$

(vii) Null Matrix / zero matrix.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2}$

(viii) Upper triangular matrix :

$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ — elements may / may not be zero

elements must be zero.

ix. Lower Triangular matrix :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 8 & 9 \end{bmatrix}$$

→ elements must be zero.

elements may / may not be zero.

- Determinant of UTM & LTM = Product of diagonal elements.

- Eigen values of UTM & LTM = diagonal elements itself.

x. Scalar matrix :

Matrix which can be represented as (K·I)

$$A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 5I$$

xi. Symmetrix matrix :

(i) $A = A^T$ (ii) $a_{ij} = a_{ji}$; $i \neq j$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 0 \end{bmatrix}$$

$\therefore A = A^T$

XII. skew - symmetric matrix :

- $A = -A^T$
- $a_{ij} = -a_{ji}$

$$A = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & 6 \\ +3 & -6 & 0 \end{bmatrix}$$

Diagonal elements must be zero for skew-symm matrix.

xiii. Hermitian matrix :

- $a_{ij} = \overline{a_{ji}}$

Conjugate

$$A = \begin{bmatrix} 5 & 2+3i \\ 2-3i & 2 \end{bmatrix}$$

Necessary : Primary diagonal elements is Real.

xiv. skew - Hermitian matrix :

- $a_{ij} = -\overline{a_{ji}}$

$$A = \begin{bmatrix} 0 & -2-3i \\ 2-3i & 0 \end{bmatrix}$$

Diagonal elements should be zero / purely imaginary

XV. Orthogonal Matrix :

(i) if $|A| = \pm 1$; then, A = may/may not be orthogonal.

Necessary :

(i) $A \cdot A^T = I$

(ii) $A^T \cdot A = I$

Trick [∴ for a orthogonal matrix ; $A^{-1} = A^T$]

- Rotational matrix : subset of orthogonal matrix.
  - $|A| = 1$ only.
  - $AA^T = A^T \cdot A = I$

XVI. Singular matrix :

$|A| = 0$

XVII. Non-singular matrix.

$|A| \neq 0$

$$\boxed{A^{-1} = \frac{\text{adj. } A}{|A|}}$$

Exist only if $|A| \neq 0$; A = Non-singular matrix.
or
Invertible matrix.

Does not exist if $|A| = 0$; A = Singular matrix.
or
Non-invertible.

XVIII. Unitary matrix :

- $A A^{\theta} = A^{\theta} . A = I$

- A^{θ} = Transpose of conjugate

  $\rightarrow A^*$

  $\rightarrow (\bar{A})' = (\bar{A})^T$

  $\rightarrow (\overline{A'}) = (\overline{A^T})$

Trick : $A A^{\theta} = I$, $A A^{-1} = I$ $\therefore \boxed{A^{-1} = A^{\theta}}$

* Special matrix :

XIX. Involutary matrix : $A \cdot A = I$; I_n

XX. Idempotent matrix : $A \cdot A = A$

XXI. Nilpotent matrix : $A^K = 0$; K - class/index.

Example : $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow$ Idempotent matrix ✓

Involotary matrix ✓

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow$ Involutary matrix ✗

Idemptotent matrix ✓

(XXII) Equal matrix :

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

* Important point

i. $A = \underbrace{\tfrac{1}{2}(A + A^T)}_{\text{symm.}} + \underbrace{\tfrac{1}{2}(A - A^T)}_{\text{skew-symm.}}$

A : Any square matrix

symm. → Always symm

skew-symm. → Always symm.

ii. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ← Any real square matrix.

$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ $\quad AA^T = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix}$

Always symm.

$A^T A = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix}$

Alway symm.

(III) symm. matrix.

$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ $\quad B = \begin{bmatrix} d & e \\ e & f \end{bmatrix}$

- $A + B = \begin{bmatrix} a+d & b+e \\ b+e & c+f \end{bmatrix}$

- $A - B = \begin{bmatrix} a-d & b-e \\ b-e & c-f \end{bmatrix}$

Always symm.

- $AB = \begin{bmatrix} ad+be & ac+bf \\ bd+ce & bc+cf \end{bmatrix}$ may or may not be symm.

* Operation on Matrices.

(i) Addition : $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

- Commutative : $A+B = B+A$

Associative : $A + (B+C) = (A+B)+C$

(ii) Substraction : $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

Not commutative : $A - B \neq B - C$

Not Associative : $(A-B) - C \neq A - (B-C)$

(III) scalar multiplication :

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot 2$$

$$= K \cdot A \quad = A \cdot K$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \neq 2\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

* scalar multiplicatⁿ ⟷ ✗ Determinant.

$$\begin{vmatrix} 2 & 4 \\ 6 & 8 \end{vmatrix} \neq 2\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \neq 2(4-6) \neq -4$$

But,

$$\begin{vmatrix} \boxed{2 \quad 4} \\ 6 \quad 8 \end{vmatrix} = 2\begin{vmatrix} 1 \quad 2 \\ \boxed{6 \quad 8} \end{vmatrix} = 2.2\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2 \cdot 2 \cdot (-2) = -8$$

Also,

$$\begin{vmatrix} \boxed{\begin{matrix} 2 \\ 6 \end{matrix}} & \begin{matrix} 4 \\ 8 \end{matrix} \end{vmatrix} = 2\begin{vmatrix} \begin{matrix} 1 \\ 3 \end{matrix} & \boxed{\begin{matrix} 4 \\ 8 \end{matrix}} \end{vmatrix} = 2.2\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2 \cdot 2 \cdot (-2) = -8$$

- $K(A+B) = KA + KB$
- $(k+g)A = KA + gA$

(iv) MATRIX MULTIPLICATION.

$A_{2\times2} \cdot B_{3\times2} \neq$ can not Perform.

- To perform matrix multiplication : colum of first matrix = Row of second matrix.

- $A_{2\times2} \cdot B_{2\times3} = C_{2\times3}$

- $A_{3\times3} \cdot B_{3\times3} = C_{3\times3}$

* Important Point

(1) * In general → $A.B \neq B.A$

* Special case : $AB = BA$

(i) $B = I$ → $A \cdot I = I \cdot A = A$

(ii) $B = 0$ → $A \cdot 0 = 0 \cdot A = 0$

(iii) $B = A^{-1}$ → $A \cdot A^{-1} = A^{-1}A = I$

(2). if AB exist then B.A exist only when both are square matrix of same size. (wrong statement)

Example : (i) $A_{3\times3} \cdot B_{3\times3}$ = AB exist

$B_{3\times3} \cdot A_{3\times3}$ = BA exist

(ii) $A_{3\times4} \cdot B_{4\times3}$ = AB exist

$B_{4\times3} \cdot A_{3\times4}$ = BA exist.

∴ if AB exist, then BA exist.

if, row of A = column of B.

and.

its not necessary that both of them should be square matrix.

Commutative : $AB \neq BA$

Associative : $A(BC) = (AB)C$

Distributive : $A(B+C) = AB + AC$

- if $A \cdot B = 0$, then it is not necessary for any matrix to be zero to get $AB = 0$.

$\boxed{|A| = |B| = 0}$

Example :

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- $AB = AC$
  - $\rightarrow$ A^{-1} exist $\rightarrow$ $B = C$ (A is Non-singular) $|A| \neq 0$
  - $\rightarrow$ A^{-1} does not exist $\rightarrow$ $B \neq C$ (A is singular) $|A| = 0$

- $(P+Q)(P+Q) = P \cdot P + P \cdot Q + Q \cdot P + Q \cdot Q$

DETERMINANT

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} a_{21}$$

MINOR : $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Minor of a_{21} : $M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12} \cdot a_{33} - a_{13} \cdot a_{32}$

Minor of a_{32} : $M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = a_{11} \cdot a_{23} - a_{13} \cdot a_{21}$

COFACTOR : $(-1)^{i+j} \times M_{ij}$

cofactor of a_{21} : $C_{21} = (-1)^{2+1} \cdot M_{21}$

$C_{21} = (-1)^{3} \cdot M_{21}$

$C_{21} = -M_{21}$

Ques.

$$\begin{array}{cc} & \begin{matrix} C_1 & C_2 & C_3 \end{matrix} \\ R_1 \rightarrow & \begin{vmatrix} 1 & 2 & 0 \\ -1 & 6 & 1 \\ 2 & 0 & 2 \end{vmatrix} \end{array}$$

$$= 1(-1)^{1+1} \begin{vmatrix} 6 & 1 \\ 0 & 2 \end{vmatrix} + 2(-1)^{1+2} \begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} + 0(-1)^{1+3} \begin{vmatrix} -1 & 6 \\ 2 & 0 \end{vmatrix}$$

$$= 1 \times 1 \times 12 + 2(-1) \times (-4) + 0 = 20$$ Ans

SARRUS METHOD : To calculate Determinant. (only for 3x3)

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{matrix} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{matrix}$$

$$(a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3) - (a_3 b_2 c_1 + b_3 c_2 a_1 + c_3 a_2 b_1)$$

* Properties of Determinant

(i) if atleast one Row or column is zero then, $\Delta = 0$

if atleast two row or column are same then, $\Delta = 0$

(ii) Determinant of lower triangle matrix and upper triangular matrix.
then, Δ = product of principle diagonal elements

(iii) for a Diagonal matrix, Δ = product of diagonal elements

Ques. $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$ then, $|A^T A^{-1}| = ?$

- $|X \cdot Y| = |X| \cdot |Y|$

- $A \cdot A^{-1} = I$

$|A A^{-1}| = I$

$|A|\,|A^{-1}| = I$ $\qquad \therefore |A^{-1}| = \dfrac{1}{|A|}$

• $|A^T| = |A|$

$\therefore \; |A^T A^{-1}| = |A^T|\,|A^{-1}| = |A| \times \frac{1}{|A|} = 1$ Ans

(iv) $|B| = (-1)^n \cdot |A|$; n = no. of times Row/column interchanged.

Example : $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2$

$\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = -2$

(V) when, $B = KA$ then, $|B| = |KA| \neq K|A|$

$|B| = K^n |A|$; n = order of matrix.

(VI) Operations : $R_i \rightarrow R_i \pm K \cdot R_j$

$C_i \rightarrow C_i \pm K C_j$

Row or column operation does not affect the value of Determinant only if applied in above manner.

Ques. $\begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix} \longrightarrow C_1 \rightarrow C_1 + C_2 + C_3 + C_4$

$$\begin{vmatrix} 5 & 1 & 1 & 1 \\ 5 & 2 & 1 & 1 \\ 5 & 1 & 2 & 1 \\ 5 & 1 & 1 & 2 \end{vmatrix}$$

$$= 5 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix} \longrightarrow \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \\ R_4 \rightarrow R_4 - R_1 \end{array}$$

$$5 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$\therefore$ Ans = 5.

TRICK : $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$

$$\Delta = (abcd)\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$$

* $|Adj\, A| = |A|^{n-1}$

$$A^{-1} = \frac{adj.A}{|A|} \longrightarrow |adj.A| = \left| |A|\, A^{-1} \right|$$

$$|adj.A| = |A|^n . |A^{-1}| \quad \because |KA| = K^n |A|$$

$$|Adj.A| = |A|^n . \frac{1}{|A|}$$

$$\therefore |Adj\, A| = |A|^{n-1}$$

$\because n$ - size/order.

* $|Adj\,(Adj\, A)| = |A|^{(n-1)^2}$

* INVERSE OF MATRIX.

(i) $A^{-1} = \dfrac{adj.A}{|A|}$, $|A| \neq 0$

(ii) A^{-1} exist when $|A| \neq 0$; {A is non-singular}

A^{-1} does not exist when $|A| = 0$; {A is singular}

(iii) $(AB)^{-1} = B^{-1}.A^{-1}$

(iv) $(ABC)^{-1} = C^{-1}.B^{-1}.A^{-1}$

(v) $(A^{-1})^{-1} = A$

(vi) $X^{-1}I = IX^{-1} = X^{-1}$

(vii) $(A.B)^{T} = B^{T}A^{T}$

Ques. $A = \begin{bmatrix} -3 & 5 \\ 2 & 1 \end{bmatrix}$

TRICK : $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

adj. $A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

$A^{-1} = \dfrac{adj.A}{|A|}$

$A^{-1} = \dfrac{\begin{bmatrix} 1 & -2 \\ -5 & -3 \end{bmatrix}}{(-3-10)}$

$A^{-1} = \dfrac{-1}{13}\begin{bmatrix} +1 & -2 \\ -5 & -3 \end{bmatrix}$

Ques. $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(i) $|A| = -2$

(ii) adj A : (i) Minors of all elements of 'A'.
(ii) $(-1)^{i+j}$
(iii) Transpose

$$\text{adj. } A = \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{adj. } A}{|A|} = -\frac{1}{2} \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

* Special matrix :

(i) Diagonal matrix : $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ then, $A^{-1} = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}$

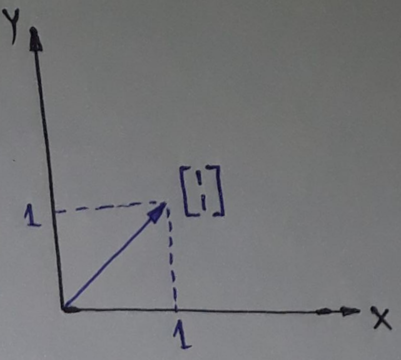
(ii) Orthogonal matrix : $AA^T = I$ then, $A^{-1} = A^T$

EIGEN VALUE'S AND EIGEN VECTORS

Representation :

eingen vector : $\hat{i} + \hat{j}$

$$= \begin{bmatrix} \hat{i} & \hat{j} \end{bmatrix}_{1\times 2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{2\times 1}$$

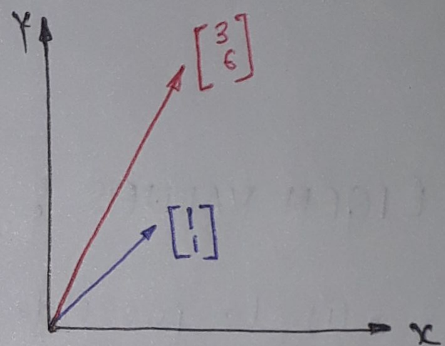


Example :

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$AX = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}_{2\times 2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{2\times 1}$$

$$= \begin{bmatrix} 3 \\ 6 \end{bmatrix}_{2\times 1} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{2\times 1}$$

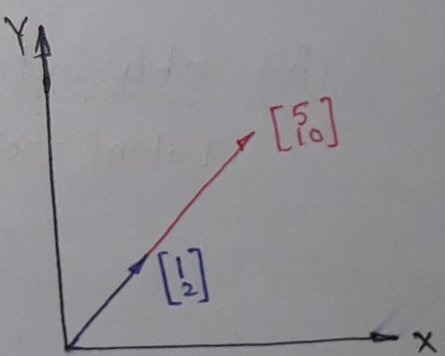


Rotation + scaled.
(Not in spam)

$\therefore$ x is not eigen vector.

Example :

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}_{2\times 2} \quad X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



$$AX = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 5X$$

so, $AX = 5X$

$\therefore$ X is a eigen vector.

Conclusion : (i) AX is parallel to X

(ii) $AX = \lambda X$ → Eigen vector

λ → Eigen values

(iii) $AX = \lambda X$

$AX - \lambda X = 0$

$AX - \lambda \cdot IX = 0$

$\therefore (A - \lambda I) X = 0$

* EIGEN VALUES :

(i) Is possible only for square matrix.

(ii) characteristic polynomial ; $|A - \lambda I|$

(iii) characteristic equation ; $|A - \lambda I| = 0$

(iv) characteristic roots ; $\lambda = \lambda_1, \lambda_2, \lambda_3, \ldots$

Latent roots, Eigen values, proper values, characteristic values.

(v) $A_{2\times2} \longrightarrow \lambda = 2$ values

$A_{3\times3} \longrightarrow \lambda = 3$ values

$A_{n\times n} \longrightarrow \lambda = n$ values.

Example : (2×2) $A = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}$

Method 1 : charct. eqn : $|A - \lambda I| = 0$

(standard method)

$$\begin{vmatrix} -1-\lambda & 4 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$a\lambda^2 + b\lambda + C = 0 \longleftarrow \lambda^2 + 2\lambda - 15 = 0$

TRICK :

(a) $\lambda = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

+ −	+ + (qud. eq.)		
+ −	+ + (factor)	(b)	$(\lambda+5)(\lambda-3) = 0$
− +	− −		
− −	− +		$\therefore \lambda = 3, -5$

Method 2 : $A = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}$

General eqn : $\boxed{\lambda^2 - S_1\lambda + |A| = 0}$

sum of diagonal elements

$|A| = 1 - 16 = -15$

$\therefore S_1 = -1 + (-1) = -2$

$\lambda^2 - (-2)\lambda + (-15) = 0$

$\lambda^2 + 2\lambda - 15 = 0$

$\therefore \lambda = 3, -5$

Method 3 :

$$A = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix} \begin{array}{l} \longrightarrow R_1 = -1+4 = 3 \\ \longrightarrow R_2 = 4-1 = 3 \end{array}$$

$$\boxed{\Sigma R_{min} \leq \lambda \leq \Sigma R_{max}}$$

$3 \leq \lambda \leq 3$

so, $\lambda_1 = 3$.

- sum of Eigen value = sum of Primary diagonal elements.

- Product of eigen values = determinent of matrix.

$\lambda_1 + \lambda_2 = -2$

$\lambda_1 + 3 = -2$ $\quad \therefore \lambda_1 = -5$

Example : (3×3)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

Method 1 : $|A - \lambda I| = 0$

$$\begin{vmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ -6 & -11 & -6-\lambda \end{vmatrix} = 0$$

on solving determinant.

we get, $\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$

Method ⓐ : for solving eqn.
(Trial & error)

$\lambda = 0 \longrightarrow f(\lambda) \neq 0$

$\lambda = 1 \longrightarrow f(\lambda) \neq 0$

✓ $\lambda = -1 \longrightarrow f(\lambda) = 0$

synthetic division technique :

$\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$

$$
\begin{array}{c|cccc}
-1 & 1 & 6 & 11 & 6 \\
\hline
 & \downarrow & -1 & -5 & -6 \\
 & 1 & 5 & 6 & \boxed{0}
\end{array}
$$

multiply

add^n

so, $1 \cdot \lambda^2 + 5\lambda + 6\lambda^0 = 0$

$\therefore$ eqn : $\lambda^2 + 5\lambda + 6 = 0$

$(\lambda + 3)(\lambda + 2) = 0$

$\lambda = -3, -2$

Method ⓑ : TRICK ! $\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$

(i) $a \times b \times c$ = constant of eqn.

$abc = 6$

$[-1, -2, 3]$ ✗

$[1, 2, 3]$ ✓

$[-1, 2, 3]$ ✗

(ii) $a+b+c$ = coeff. of λ^2

$a+b+c = 6$

(iii) $ab+bc+ca$ = coeff. of λ

$ab+bc+ca = 11$

$\therefore$ factors = $(1,2,3)$

roots = $-1, -2, -3$

$S_1 = -6$

sum of diagonal element

Method 2 :

General eqn : $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

Sum of minors of diagonal element.

$S_2 = M_1 + M_2 + M_3$

$S_2 = 11 + 0 + 0 = 11$

→ Eigen values of , Lower Triangular matrix, upper triangular matrix, Diagonal matrix are its Diagonal elements itself.

→ roots of complex number exist in pair.

i.e, $a \pm ib \rightarrow a+ib,\ a-ib$

* Properties of Eigen values:

- $A \rightarrow \lambda_1, \lambda_2, \lambda_3, \ldots$
- $A^{-1} \rightarrow \lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}, \ldots$
- $A^n \rightarrow \lambda_1^n, \lambda_2^n, \lambda_3^n, \ldots$
- $K \cdot A \rightarrow K\lambda_1, K\lambda_2, K\lambda_3, \ldots$
- $A + KI \rightarrow \lambda_1 + K, \lambda_2 + K, \lambda_3 + K, \ldots$
- Eigen values of $A^T \rightarrow \lambda_1, \lambda_2, \lambda_3, \ldots$
- $\text{Adj. } A = A^{-1} \cdot |A| \longrightarrow \frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}, \ldots$

Ques. Eigen value of 2x2 matrix X are -2 & -3. then eigen values of $(X+I)^{-1}(X+5I)$?

$X \rightarrow \lambda_1, \lambda_2$

then, $(X+I)^{-1}(X+5I) = (+\lambda_1 + I)^{-1}(\lambda_1 + 5I)$

$= (-2+1)^{-1}(-2+5) = -3$

for λ_2 : $(X+I)^{-1}(X+5I) = (\lambda_2+I)^{-1}(\lambda_2+5I)$

$= (-3+1)^{-1}(-3+5) = -1$

$\therefore$ Ans $= -1, -3.$

Ques. $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$ Eigen values are $1, -1, 3$

then, Trace of $(A^3 - 3A^2)$ is ____.

* Trace of matrix = sum of Diagonal elements

= sum of all eigen values.

$= \lambda_1' + \lambda_2' + \lambda_3'$

$A \longrightarrow \lambda_1 = 1, \ \lambda_2 = -1, \ \lambda_3 = 3.$

$A^3 - 3A^2 = \lambda^3 - 3\lambda^2$

$\therefore \lambda_1 = 1 \longrightarrow \lambda_1' = (+1)^3 - 3\times 1^2 = -2$

$\lambda_2 = -1 \longrightarrow \lambda_2' = (-1)^3 - 3\times(-1)^2 = -4$

$\lambda_3 = 3 \longrightarrow \lambda_3' = 0$

$\therefore$ Trac of $(A^3 - 3A^2) = \lambda_1' + \lambda_2' + \lambda_3' = -2 + (-4) + 0$

$= -6$

Important :

Eigen values.

(i) Hermitian matrix | symm. matrix $\longrightarrow$ Real

(ii) skew Hermitian matrix | skew- symm. $\longrightarrow$ zero or purely imaginary ($\pm ib$)

(iii) orthogonal matrix | unitary matrix $\longrightarrow$ $|\lambda| = 1$

Modulus of $\lambda = |a+ib| \rightarrow \sqrt{a^2+b^2}$

Nature of eigen values : $\lambda, \frac{1}{\lambda}$

Ques! $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$AA^T = I$

$\therefore \lambda^2 - S_1\lambda + |A| = 0$

$\lambda^2 - 0 + 1 = 0$

$\lambda^2 + 1 = 0$

$\lambda^2 = -1$

$\boxed{\lambda = \pm i}$

therefor, $|\lambda| = 1$

&

$\lambda = i$

$\frac{1}{\lambda} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$

(iv) Idempotent matrix ⟶ zero or 1

(v) Involutory matrix ⟶ 1 or -1

* Eigen vector *

	MATRIX 'A'	EIGEN VALUE 'λ'	EIGEN VECTOR 'X'
case. i.	✓	?	?
case ii.	✓	?	✓
case iii	✓	✓	?
case iv.	?	✓	✓

CASE.01

$$A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} \begin{matrix} \longrightarrow R_1 = 8+(-4) = 4 \\ \longrightarrow R_2 = 2+2 = 4 \end{matrix}$$

$$\therefore \lambda_1 = 4$$

$$\lambda_1 + \lambda_2 = 10$$

$$4 + \lambda_2 = 10 \qquad \lambda_2 = 6$$

for $\lambda = 4$: $[A - \lambda I]x = 0$

$[A - 4I]X = 0$

$$\begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4x_1 - 4x_2 = 0 \longrightarrow x_1 - x_2 = 0$

$2x_1 - 2x_2 = 0 \longrightarrow x_1 - x_2 = 0$

$> x_1 = x_2 = k$ (let)

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = 6$; $[A - \lambda I]X = 0$

$[A - 6I]X = 0$

$$\begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$2x_1 - 4x_2 = 0 \longrightarrow x_1 = 2x_2$

let, $x_2 = k$

$2x_1 - 4x_2 = 0 \longrightarrow x_1 = 2x_2$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2k \\ k \end{bmatrix} = k \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Case-02 :

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \quad \& \quad X = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

find λ.

$AX = \lambda X$

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \lambda$$

$$\begin{bmatrix} 5 \\ 10 \\ -5 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$5 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$\therefore \lambda = 5$ Ans

CASE 03.

$$A = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \lambda = -2$$

find 'x' vector. ?

$AX = \lambda X$ or $[A - \lambda I] X = 0$

$[A + 2I] X = 0$

$$\begin{bmatrix} 5 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5x_1 - 2x_2 + 2x_3 = 0$

$x_3 = 0$

$3x_3 = 0$

$5x_1 - 2x_2 + 0 = 0$

$x_1 = \frac{2}{5} x_2$ let, $x_2 = K$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2x_2}{5} \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2/5\,k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 2/5 \\ 1 \\ 0 \end{bmatrix}$$

for $k = 5 \rightarrow X = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

CASE-4 : $\lambda = -1, -2$ & $X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ respectively

then, find matrix 'A'.

(i) $AX = \lambda X$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$a - b = -1$ ——— (1)

$c - d = 1$ ——— (2)

(ii) $AX = \lambda X$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$a - 2b = -2$ ——— (3)

$c - 2d = 4$ ——— (4)

from eq. (1) & (3) : $b = 1, a = 0$

$\therefore A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

from eq. (2) & (4) : $d = -3, c = -2$

* Important : (i) Eigen value are Real
then,
Eigen vectors are also Real.

(ii) E. values are Imaginary
then.
Eigen vector are also imaginary.

(III) Matrix ——— Eigen values ——— E. vectors.

A ——— λ_1, λ_2 ——— X_1, X_2

A^m ——— λ_1^m, λ_2^m ——— X_1, X_2 (same)

(iv) Eigen vectors of symmetric matrix are orthogonal vectors. (i.e, dot product is zero)

Ques. E. values of A is $1, -2$ with E. vector X_1, X_2.
then eigen values & E. vector of $A^2 - 3A + 4I$?

E. values : $\lambda^2 - 3\lambda + 4$

$\lambda = 1$; $1^2 - 3 \times 1 + 4 = 2 \rightarrow \lambda_1'$

$\lambda = -2$; $(-2)^2 - 3 \times (-2) + 4 = 14 \rightarrow \lambda_2'$

E. vectors : remains same, i.e, X_1, X_2

* Cayley Hamilton Theorem

(i). Every square matrix satisfies its characteristic equation.

(ii) A can be replaced by λ

or

λ can be replaced by A

Ques. $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ charac. eqn: $\lambda^2 - S_1\lambda + |A| = 0$

$\lambda^2 - 3\lambda - 10 = 0$

replace $\lambda \to A$

$A^2 - 3A - 10I = 0$

Ques. $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ find $A^5 = ?$

ch. eqn: $\lambda^2 - S_1\lambda + |A| = 0$

$\lambda^2 - 3\lambda - 10 = 0$

$\lambda \to A$: $A^2 - 3A - 10I = 0$

$A^2 = 3A + 10I$

$A^4 = A^2 \cdot A^2 = (3A + 10I)(3A + 10I)$

$A^4 = 87A + 190I$

$$A^5 = A^4 . A = (87A + 190I) A$$

$$= 87A^2 + 190A$$

$$= 87(3A + 10I) + 190A$$

$$A^5 = 451A + 870I$$ Ans

Ques. $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ find $A^{-1} = ?$

charc. eqn: $\lambda^2 - S_1\lambda + |A| = 0$

$$\lambda^2 - 3\lambda - 10 = 0$$

$\lambda \to A$: $A^2 - 3A - 10I = 0$

$$A^2 = 3A + 10I$$

$$(A^{-1}A)A = 3A^{-1}A + 10A^{-1}I$$

$$A = 3I + 10A^{-1}$$

$$10A^{-1} = A - 3I$$

$$\therefore A^{-1} = \frac{A - 3I}{10}$$

Ques: the product of the non-zero eigen values of the matrix.

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} = ?$$

Method. 01 : $[A - \lambda I] = 0$

$$R_1 \rightarrow \begin{bmatrix} 1-\lambda & 0 & 0 & 0 & 1 \\ 0 & 1-\lambda & 1 & 1 & 0 \\ 0 & 1 & 1-\lambda & 1 & 0 \\ 0 & 1 & 1 & 1-\lambda & 0 \\ 1 & 0 & 0 & 0 & 1-\lambda \end{bmatrix} \rightarrow (1-\lambda) \begin{vmatrix} 1-\lambda & 1 & 1 & 0 \\ 1 & 1-\lambda & 1 & 0 \\ 1 & 1 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix}$$

$$+ 1 \begin{vmatrix} 0 & 1-\lambda & 1 & 1 \\ 0 & 1 & 1-\lambda & 1 \\ 0 & 1 & 1 & 1-\lambda \\ 1 & 0 & 0 & 0 \end{vmatrix} = 0$$

$$(1-\lambda)(1-\lambda) \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\underbrace{((1-\lambda)(1-\lambda)-1)} \underbrace{\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix}} = 0$$

$(1-\lambda)^2 - 1 = 0$

$\lambda(\lambda-2) = 0$

$\lambda = 0, 2$

$(-\lambda^3)\left(1 - \frac{1}{\lambda} - \frac{1}{\lambda} - \frac{1}{\lambda}\right) = 0$

$\lambda^2(\lambda-3) = 0$

$\lambda = 3, 0, 0$

$\therefore$ Ans $= 3 \times 2 = 6$ //

Method. 2. $AX = \lambda X$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$x_1 + x_5 = \lambda x_1 \quad \text{——} (1)$$

$$x_2 + x_3 + x_4 = \lambda x_2 \quad \text{——} (2)$$

$$x_2 + x_3 + x_4 = \lambda x_3 \quad \text{——} (3)$$

$$x_2 + x_3 + x_4 = \lambda x_4 \quad \text{——} (4)$$

$$x_1 + x_5 = \lambda x_5 \quad \text{——} (5)$$

from eq. (1) & (5)

$$2(x_1 + x_5) = \lambda(x_1 + x_5)$$

$$\therefore \lambda = 2$$

from eq. (2), (3) & (4)

$$3(x_2 + x_3 + x_4) = \lambda(x_2 + x_3 + x_4)$$

$$\therefore \lambda = 3.$$

$\therefore$ Ans. $= \lambda_1 \times \lambda_2 = 3 \times 2 = 6$

Ques. Consider a non-singular 2x2 square matrix 'A'. if trace (A) = 4 and trace $(A^2) = 5$. The determinant of matrix 'A' is ______ (upto 1 decimal place)

Method-1

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A \cdot A = A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$

$a+d = 4$ —— ①

$a^2+bc+bc+d^2 = 5$

$|A| = ad - bc$ —— ③

$a^2+2bc+d^2 = 5$ —— ②

$a+d = 4$

$(a+d)^2 = 16 \rightarrow a^2+2ad+d^2 = 16$

$5 - 2bc + 2ad = 16$

$5 - 2(ad - bc) = 16$

$ad - bc = \frac{11}{2} = 5.5$

$\therefore |A| = ad - bc = 5.5$ Ans

METHOD-2.

(i) $A \rightarrow \lambda_1, \lambda_2$

$A^2 \rightarrow \lambda_1^2, \lambda_2^2$

(ii) trace (A) = 4

$\lambda_1 + \lambda_2 = 4$ —— ①

trace $(A^2) = \lambda_1^2 + \lambda_2^2$

$\lambda_1^2 + \lambda_2^2 = 5$ —— ②

from eq. ① & ②

$\lambda_1 \lambda_2 = 5.5$

(iii) $|A| = \lambda_1 \times \lambda_2 = 5.5$ Ans

Vandermonde Matrix : $\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{bmatrix}$

then absolute value of product of eigen values

is given by : $\boxed{(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)(x_3-x_4)}$

Ques. $\begin{bmatrix} 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \\ 1 & 5 & 5^2 & 5^3 \end{bmatrix}$ find Absolute value of product of eigen values ——.

$= (2-3)(2-4)(2-5)(3-4)(3-5)(4-5)$

$= (-1) \times (-2) \times (-3) \times (-1) \times (-2) \times (-1)$

$= 12$ Ans

* RANK OF A MATRIX .

Rank of a matrix is no. of non-zero rows of Echelon Matrix.

Denoted by 'ρ'.

Ques.

Rank of $A = \begin{bmatrix} 1 & -2 & -1 \\ -3 & 3 & 0 \\ 2 & 2 & 4 \end{bmatrix}$

$R_2 \rightarrow R_2 + 3R_1$ & $R_3 \rightarrow R_3 - 2R_1$

Echelon form :

No. of zeros

$R_3 > R_2 > R_1$
(max.) (min)

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & -3 & -3 \\ 0 & 6 & 6 \end{bmatrix}$$

$R_3 \rightarrow R_3 + 2R_2$ $\begin{bmatrix} 1 & -2 & -1 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ rank $(A) = 2$.

* TRICK : "valid for square matrix"

(i) $|A|_{3\times3} = 0 \longrightarrow$ then, $\rho(A) \neq 3$

$|A|_{3\times3} \neq 0 \longrightarrow$ then, $\rho(A) = 3$

if $|A|_{3\times3} = 0$, then check for

(ii) $|A|_{2\times2} = 0 \longrightarrow$ then, $\rho(A) \neq 2$

$|A|_{2\times2} \neq 0 \longrightarrow$ then, $\rho(A) = 2$

Ques. find Rank, $A = \begin{bmatrix} 5 & 10 & 10 \\ 1 & 0 & 2 \\ 3 & 6 & 6 \end{bmatrix}_{3\times3}$.

$|A|_{3\times3} = 0$; so $\rho(A) \neq 3$.

check for $|A|_{2\times2}$: $A_{2\times2} = \begin{vmatrix} 5 & 10 \\ 1 & 0 \end{vmatrix} = -10$

$|A|_{2\times2} \neq 0$

$\therefore \rho(A) = 2$

" NON-SQUARE MATRIX "

Ques.

$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}_{3\times4}.$$

$R_1 \rightarrow R_1 - 2R_3$, $R_2 = R_2 - 3R_3$.

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 4 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

$R_1 \rightarrow R_1 - 4R_2$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank = 2.

* Important Points.

i. Rank of matrix is zero : Null matrix.

ii. Rank of Identity matrix : size/order of matrix.

iii. Rank of singular matrix : Rank < order|size.

$$|A|_{n\times n} = 0 \longrightarrow \rho(A) < n$$

iv. Rank of matrix is never be negative.

v. $\text{Rank}(A) + \text{Rank}(Q) \neq \text{Rank}(P+Q)$

vi. Rank of any $A(m\times n)$ matrix : $\text{Rank}(A) \leq \min.(m,n)$

$A_{m\times n}$: case i $\rightarrow$ $m>n$; then, $\rho(A) \leq n$

case ii $\rightarrow$ $m<n$; then $\rho(A) \leq m$

vii. Ques. if the matrix, ranks equals both no. of rows & no. of column then the matrix is called

(a) Non-singular ✓

(b) Singular

(c) Transpose

(d) Minor

$\rho(A_{n\times n}) = n$

$\downarrow$

$|A|_{n\times n} \neq 0$

$\downarrow$

Non-singular matrix.

viii. Ques. $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ $B = \begin{bmatrix} p^2+q^2 & pr+qs \\ pr+qs & r^2+s^2 \end{bmatrix}$

$\rho(A) = N$ then, $\rho(B) = ?$

(a) $N/2$ (b) $N-1$ ✓(c) N (d) $2N$

put, $p=q=r=s=1$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \longrightarrow \rho(A) = 1$

$B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \longrightarrow \rho(B) = 1$

ix. Rank of 'A' = Rank of 'A^T'

SYSTEM OF LINEAR EQUATION

- $2x+3y=7$
  $x-y=1$
  - valid
  - Distinct
  - unique sol^n.

- $2x+3y=7$
  $8x+12y=28$
  - valid
  - same
  - Infinitely many sol^n

(Consistent system)

- $2x+3y=7$
  $4x+6y=20$
  - Invalid
  - No solution

  (inconsistent system)

- THREE VARIABLE SYSTEM :

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{3\times1}$$

coefficient matrix (A) | column matrix (X) | constant matrix (B)

* FOR NON-HOMOGENEOUS SYSTEM OF LINEAR EQN.

(i) if $\rho(A) \neq \rho(A:B) \longrightarrow$ NO SOL^n

(ii) if $\rho(A) = \rho(A:B)$

- equal to NO. of variable (n) — unique sol^n
- Not equal to NO. of variable (n) — Infinite many sol.

(iii) Determinant of coefficent matrix |A|.

- if |A|
  - equal to zero
    - No sol.
    - Infinite many sol.
  - not equal to zero $\longrightarrow$ unique sol.

- D_x, D_y, D_z
  - equal to zero (All) — infinite many sol
  - not equal to zero (one of them) — No sol.

* CRAMER'S RULE.

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D}$$

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Ques. $2x + 3y + z = 9$

$x + 2y + 3z = 6$

$3x + y + 2z = 8$

Find : (i) Nature of matrix.

(ii) value of x, y, z

(iii) value of x.

i) Gauss elimination method.

$$AX = B$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

Method-1 : Rank concept $\longrightarrow$ To determine Nature of sol?

Augmented matrix : $[A \mid B]$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 9 \\ 1 & 2 & 3 & 6 \\ 3 & 1 & 2 & 8 \end{array}\right] \quad R_1 \leftrightarrow R_2 \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & 3 & 1 & 9 \\ 3 & 1 & 2 & 8 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -1 & -5 & -3 \\ 0 & 0 & 18 & 5 \end{array}\right] \quad R_3 \to R_3 - 5R_2 \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -1 & -5 & -3 \\ 0 & -5 & -7 & -10 \end{array}\right]$$

so, $\rho(A) = \rho(A:B) = 3$ = No. of variable.

$\therefore$ Unique sol.

(II) Method-2: find x, y, z ?

$$AX = B$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 5 \end{bmatrix}$$

$18z = 5 \longrightarrow z = 5/18$

$-y - 5z = -3 \longrightarrow y = 29/18$

$x + 2y + 3z = 6 \longrightarrow x = 35/18$

(III) Method-3: find x only.

cramer's rule:

$$D = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix} = 18$$

$$D_x = \begin{vmatrix} 7 & 3 & 1 \\ 6 & 2 & 3 \\ 8 & 1 & 2 \end{vmatrix} = 35$$

$$\therefore x = \frac{D_x}{D} = \frac{35}{18}$$

HOMOGENEOUS SYSTEM OF LINEAR EQN.

$AX = 0 \quad \{B = 0\}$

UNIQUE Sol.	INFINIT MANY Sol.
• (0,0) is sol.	• (0,0) & more sol. sets
• zero sol.	• Non zero sol.
• Trivial sol.	• Non-Trivial sol.
• $\|A\|_{n\times n} \neq 0$ (Non-singular)	• $\|A\| = 0$ (singular)

LINEARLY INDEPENDENT / DEPENDENT VECTOR

Ques. $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

i. $[A - \lambda I] = 0 \longrightarrow \lambda = -3, -3, 5$

for $\lambda = -3$; $[A - (-3)I]X = 0$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - 2R_1$ & $R_3 \longrightarrow R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 - 3x_3 = 0$

let, $x_2 = K_1$, $x_3 = K_2$

$x_1 = 3K_2 - 2K_1$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3K_2 - 2K_1 \\ K_1 \\ K_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}_{X_1} K_1 + \underbrace{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}_{X_2} K_2$$

for $\lambda = 5$: $X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} K_3$

To check Linearly dependent / independent vectors

Method-1 : $K_1 X_1 + K_2 X_2 + K_3 X_3 = 0$

$$K_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + K_2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + K_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 0$$

$-2K_1 + 3K_2 + K_3 = 0 \longrightarrow K_3 = 0$

$K_1 + 0K_2 + 2K_3 = 0 \longrightarrow K_1 = -2K_3 \longrightarrow K_1 = 0$

$0K_1 + K_2 + K_3 = 0 \longrightarrow K_2 = -K_3 \longrightarrow K_2 = 0$

$K_1 = K_2 = K_3 = 0$

$\therefore$ X_1, X_2, X_3 are linearly independent.

Method-2. $A = [x_1 \ x_2 \ x_3]$

$$A = \begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_2 \quad A = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 + 2R_1$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 5 \\ 0 & 1 & 1 \end{bmatrix}$$

Rank (A) = No. of vector

$\rho(A) = 3.$

∴ Linearly independent.

- if Rank A = No. of vector → Linearly independent
- if Rank A ≠ No. of vector → Linearly dependent

Method-3:

$|A| = |[x_1 \ x_2 \ x_3]|$ only for square matrix.

- if $|A| \neq 0$ → Non-singluar → $\rho(A) = n$ → L.independ.
- if $|A| = 0$ → singular → $\rho(A) < n$ → L.dependent.

DIAGONALISATION OF MATRIX

$$D = P^{-1} A \cdot P$$

- D = Diagonal matrix $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

- P = Model Matix $[x_1 \ x_2 \ x_3]$

- A = square matrix.

* when, $|P| \neq 0$ then, P is non-singular.
means vectors are linearly independent.

i. STYLE 01 : $\boxed{D = P^{-1} A P}$

ii. STYLE 02 : $D = P^{-1} A P$

Both methods are used when, $|P| \neq 0$ (OR) Non-singular (OR) Lineary indep. vectors.

$$PD = (PP^{-1}) A P$$

$$PD = AP$$

$$PDP^{-1} = A(PP^{-1}) \quad \therefore \boxed{A = PDP^{-1}}$$

$$* \quad A^n = P D^n P^{-1}$$

$$* \quad e^A = P e^D P^{-1}$$

STYL-1 Ques. $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ check : $D = P^{-1}AP$

- $\lambda = 2, 5$
- $\lambda_1 = 2 \longrightarrow x_1 = K_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
- $\lambda_2 = 5 \longrightarrow x_2 = K_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$

LHS.

$P = [x_1 \ x_2] = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \longrightarrow |P| \neq 0$ {vector ar linearly indep. & P is non-singular}

$P^{-1} = \frac{adj A}{|A|} = \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$

$RHS = P^{-1}AP = \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$

$\therefore$ LHS = RHS. {$D = P^{-1}AP$ exists}

STYLE 2 Ques. $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ find $A^{50} = ?$

- $\lambda = 1, 3$
- $\lambda_1 = 1 \longrightarrow x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} K_1$

$\lambda_2 = 3 \longrightarrow x_2 = K_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- $A = PDP^{-1}$

$A^{50} = P D^{50} P^{-1}$

$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^{50} \cdot \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} 1+3^{50} & -1+3^{50} \\ -1+3^{50} & 1+3^{50} \end{bmatrix}$

METHOD-2: When, $|P| \neq 0$

$(2\times2) \cdot \phi(A) = \alpha_1 A + \alpha_0 I$

$(3\times3) \cdot \phi(A) = \alpha_2 A^2 + \alpha_1 A^1 + \alpha_0 I$

$(4\times4) \cdot \phi(A) = \alpha_4 A^4 + \alpha_3 A^3 + \alpha_2 A^2 + \alpha_1 A^1 + \alpha_0 I$

Ques. $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ find $A^{50} = ?$

Method 2: (i) $\phi(A) = \alpha_1 A + \alpha_0 I \longrightarrow (2\times2)$

(ii) $A^{50} = \alpha_1 A + \alpha_0 I$

$\lambda = A \longrightarrow \lambda^{50} = \alpha_1 \lambda + \alpha_0$

(iii) $\lambda = 3, 1$

- if $\lambda = 3 \longrightarrow 3^{50} = \alpha_1(3) + \alpha_0$
- if $\lambda = 1 \longrightarrow 1^{50} = \alpha_1(1) + \alpha_0$

$$\therefore \alpha_1 = \frac{3^{50} - 1^{50}}{2} \quad \& \quad \alpha_0 = 3^{50} - 3\left(\frac{3^{50} - 1^{50}}{2}\right)$$

$$A^{50} = \alpha_1 A' + \alpha_0 I$$

$$A^{50} = \frac{3^{50} - 1}{2}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + 3^{50} - 3\left(\frac{3^{50} - 1}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore A^{50} = \frac{1}{2}\begin{bmatrix} 2(3^{50}-1) & 3^{50}-1 \\ 3^{50}-1 & 2(3^{50}-1) \end{bmatrix} + \frac{1}{2}\left(2 \times 3^{50} - 3 \times 3^{50} + 3\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{50} = \frac{1}{2}\begin{bmatrix} 2(3^{50}-1) & 3^{50}-1 \\ 3^{50}-1 & 2(3^{50}-1) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -3^{50}+3 & 0 \\ 0 & -3^{50}+3 \end{bmatrix}$$

$$A^{50} = \frac{1}{2}\begin{bmatrix} 2 \times 3^{50} - 2 - 3^{50} + 3 & 3^{50}-1 \\ 3^{50}-1 & 2 \times 3^{50} - 2 - 3^{50} + 3 \end{bmatrix}$$

$$A^{50} = \frac{1}{2}\begin{bmatrix} 3^{50}+1 & 3^{50}-1 \\ 3^{50}-1 & +3^{50}+1 \end{bmatrix}$$

Ques. $A = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$ find $e^A, 4^A$

Method-1:

i. $\lambda = 1, 2$

ii. $\lambda_1 = 1 \longrightarrow x_1 = K_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 2 \longrightarrow X_2 = K_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(iii) $A = PDP^{-1}$

$$e^A = Pe^D P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \times \frac{1}{2}$$

$$\therefore e^A = \frac{1}{2} \begin{bmatrix} e+e^2 & -e+e^2 \\ -e+e^2 & e+e^2 \end{bmatrix}$$

Ques. $A = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$ find $A^{50} = ?$

(i) $\lambda = -1, -1$

(ii) $\phi(A) = \alpha_1 A + \alpha_0 I \longrightarrow A^{50} = \alpha_1 A + \alpha_0 I$

$A = \lambda$! $\quad \lambda^{50} = \lambda^1 \alpha_1 + \alpha_0$ ——— (1)

for $\lambda = -1 \longrightarrow (-1)^{50} = \alpha_1 (-1) + \alpha_0$

$1 = -\alpha_1 + \alpha_0$ ——— (a)

$\therefore \alpha_0 = -49$

differentiate eq.(1) w.r.t. λ

$50 \lambda^{49} = \alpha_1$

for $\lambda = -1 \longrightarrow 50(-1)^{49} = \alpha_1$

$\therefore \alpha_1 = -50$

$$A^{50} = \alpha_1 A + \alpha_0 I = -50 \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix} + (-49) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -149 & 150 \\ 150 & 151 \end{bmatrix}$$

DO LITTLE METHOD
CROUTS METHOD

used to determin solution of system of eqns.

* DO LITTLE METHOD:

$a_{ij} = 1 \ \forall \ i = j$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$AX = B$$

$(L \cdot U)$

$$(LU)X = B$$
$$L(UX) = B$$
$$LY = B \qquad \because Y = UX$$

$$A = L \cdot U = \begin{bmatrix} 1 & 0 & 0 \\ l_{11} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Ques:
$$x + 3y + 8z = 4$$
$$x + 4y + 3z = 2$$
$$x + 3y + 4z = 1$$

solve by DOLITTLE's Method.

(i) $AX = B$

$$\begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

(ii) $A = L \cdot U = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{32} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & -4 \end{bmatrix}$$

(iii) $AX = B$

$L(UX) = B$

$LY = B \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$

$\therefore y_1 = 4, \; y_2 = -2, \; y_3 = -3$

$Y = UX$

$$\begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\therefore x = \frac{-29}{4}, \; y = \frac{7}{4}, \; z = \frac{3}{4}$

* CROUT'S METHOD :

$u_{ij} = 1 \;\forall\; (i=j)$

$$A = LU = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Here, $AX = B$

$(LU)X = B$

$L(UX) = B$

$LY = B \quad \therefore Y = UX$

TRICK :

$A = L \cdot U$

$|A| = |L \cdot U| = |L||U|$

$|L|$: Product of Diagonal elements

$|U|$: Products of Diagonal elements

NO. OF LINEARLY INDEPENDENT SOL^{N}. OF SYSTEM.

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{B}$$

Method · 1

$R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 2R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$AX = 0$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_3 = 0 \longrightarrow x_1 = -2x_3$

$-x_2 - 2x_3 = 0 \longrightarrow x_2 = -2x_3$

let, $x_3 = K$, $x_2 = -2K$, $x_1 = 2K$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = K \begin{bmatrix} -2 \\ -2 \\ +1 \end{bmatrix}$$

Method · 2 :

Linearly independent sol^n

(for Homogeneous sys^n)

unique sol^n $\downarrow$ zero independent

Infinitely many sol. $\downarrow$ ✓

$\therefore$ No. of Linearly independent sol. $= \underset{\text{No. of variable}}{n} - \underset{\text{Rank of A.}}{r}$

No. of. L.I sol. = 3-2 = 1

NULLITY : Dimension of Null Space.

$$\text{Nullity} = n - r$$

Ques. $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

i. $\lambda = 1, 2, 2$

ii. $\lambda = 1 \rightarrow$ No. of linearly indep. eigen vector $\rightarrow$ one.

$\lambda = 2 \rightarrow$ No. of L.I. Eigen vector $\rightarrow$ may be 1 or 2.

$\rightarrow$ Method-1 : solve

Method-2 : No. of linearly indep. eigen vector $= n - r$

$= 3 - 2$

$= 1$

* n = no. of variable
** r = rank of $[A - \lambda I]$

$\Downarrow$

$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Rank} = 2$

Ques. $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ 1 & -2 & 0 \end{bmatrix}$

(i) $\lambda = -3, -3, 5$

(ii) $\lambda = 5$; No. of L.I eigen vector = 1

$\lambda = -3$; No. of L.I eigen vector = $n - r$

$= 3 - 1$

$= 2$

$r = [A - \lambda I] = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ 1 & -2 & 3 \end{bmatrix}$

$\Downarrow$

Rank, $r = 1$

* Algebric multiplicity :

AM : No. of times one eigen value is repeated

Example 1 : $\lambda = 1, 2, 2$

FOR, $\lambda = 1 \longrightarrow AM = 1$

$\lambda = 2 \longrightarrow AM = 2$

Example 2 : $\lambda = 5, 7, 9$

FOR, $\lambda = 5 \longrightarrow AM = 1$

$\lambda = 7 \longrightarrow AM = 1$

$\lambda = 9 \longrightarrow AM = 1$

* Geometric multiplicity :

GM : No. of Linearly indep. eigen vector associated with one eigen value.

$\therefore$ GM = $n - r$

Ques. A sequence $x(n)$ is specified as :

$$\begin{bmatrix} x(n) \\ x(n-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad ; \text{ for } x \geq 2$$

The initial condition are $x(0) = 1$, $x(1) = 1$ and $x(n) = 0$ for $n < 0$. The value of $x(12)$ is ____.

fibonaci series : $c = a + b$

n	$x(n)$
-3, -2, -1	0
0	1
1	1
2	2
3	3
4	5
5	8
6	13
7	21
8	34
9	55

a+b

a+b

a+b

n	$x(n)$
10	89
11	144
12	233 Ans

$\therefore$ $x(12) = 233$.

Ques. $\lim_{x \to \frac{\pi}{4}} \dfrac{\sin 2\left(x - \frac{\pi}{4}\right)}{x - \pi/4}$

$$\lim_{2\left(x-\frac{\pi}{4}\right) \to 0} 2 \times \frac{\sin 2(x - \pi/4)}{2(x - \pi/4)} = 2 \times 1 = 2$$