Ex. $(\{1, -1, i, -i\}, \times)$ then which is a subgroup:

(A) $(\{-1, 1\}, \times)$

Closure ✓ $\{-1, 1\} \subseteq \{1, -1, i, -i\}$

Assoc ✓ $\therefore$ Subgroup

Identity (1) ✓

Inverse ✓

(B) $(\{1, i\}, \times)$

Closure ✗.

Not Group. $\therefore$ Not Subgroup

(C) $(\{1, 0, i\}, \times)$

Not subset. $\therefore$ Not Subgroup.

(D) $(\{1, -1\}, +)$

Diff operation. $\therefore$ Not subgroup.

- To create subgroup from group $(G, *)$:
  (1) Take identity element.
  (2) If you take a, then take $a^{-1}, a^2, a^3, a^4, \ldots$ as well.
  {Because of closure property}

* Subgroups generated by element.

- For an element $a \in G$, $(G, *)$,
  $\langle a \rangle$ = Smallest subgroup generated by a.
  $\Rightarrow \{a^n \mid n \in \mathbb{Z}\}$

Ex. $(Z, +)$. Find $\langle 2 \rangle$

$e = 0$ must be there.

& will be there.

$2^{-1} = -2$ will be there. $(2^{-1})^2 = -4$

$2 * 2 = 2^2 = 4$

$2^2 * 2 = 2^3 = 6$

$\vdots$

$\therefore \langle 2 \rangle = \{0, 2, 4, 6, 8, -2, -4, -6, -8, \ldots\}$

$\Rightarrow 2Z$

$\Rightarrow \{2^N \mid N \in Z\}$ $\{2^N = 2^{N-1} * 2\}$

Ex. $(\{1, -1, i, -i\}, \times)$

$\langle i \rangle = \{1, i, i^2, i^{-1}, i^3, \ldots\}$

$\Rightarrow \{1, i, -1, -i\}$

$\Rightarrow$

$\langle 1 \rangle = \{\overset{e}{1}, 1^2, 1^{-1}, \ldots\}$

$\Rightarrow \{1\}$

$\langle -1 \rangle = \{1, -1, (-1)^2, (-1)^{-1}, \ldots\}$

$\Rightarrow \{1, -1\}$

$\langle -i \rangle = \{1, -i, -i^2, -i^{-1}, \ldots\}$

$= \{1, -i, -1, i\}$

So, $\langle i \rangle$ & $\langle -i \rangle$ is the whole set.

* Generator: An element $a \in G$, where $(G, *)$ is a group is an element such that:

$$\langle a \rangle = G$$

then 'a' is called generator of G.

Ex. $(\{0,1,2,3\}, \oplus_4)$

$\langle 0 \rangle = \{0\}$

$\langle 1 \rangle = \{1^0, 1^1, 1^2, 1^{-1}, 1^3 \ldots\}$

$= \{0, 1, 2, 3\}$

$\langle 2 \rangle = \{2^0, 2^1, 2^2, 2^{-1} \ldots\}$

$= \{0, 2\}$

$\langle 3 \rangle = \{3^0, 3^1, 3^2, 3^3, 3^{-1}, \ldots\}$

$= \{0, 3, 2, 1\}$

$\therefore$ $\langle 1 \rangle$ & $\langle 3 \rangle$ generate whole set, hence 1 & 3 are generators of G.

- A group may/may not have a generator.
- If for a group G, 'a' is a generator, then a^{-1} is also a generator.

$\langle a^{-1} \rangle = \{(a^{-1})^0, (a^{-1})^2, a^{-1}, (a^{-1})^{-1} \ldots\}$

$\Rightarrow \{e, \ldots\ldots a \ldots\}$

$\because$ 'a' is generator it generates rest elements.

Ex. $(G, *)$. $a \in G$, & $a^{-1} = a$, then $\langle a \rangle$?

$\langle a \rangle = \{a^0, a^1, a^2, a^{-1}, \ldots\}$
$= \{e, a, e, a, \ldots\}$
$\Rightarrow \{a, e\}$

Ex. Let G is a group that can be generated by element 'a', which is its own inverse. Comment on G's order.

$\langle a \rangle = \{a^0, a^1, a^2, a^{-1}, \ldots\}$
$\Rightarrow \{e, a, e, a \ldots\}$
$\Rightarrow \{a, e\}$.

But a can be e itself.

$\therefore$ $\langle a \rangle$ can be $\{a, e\}$ & $\{e\}$

$\therefore$ $|G|$ is 1 or 2.

* Order of an element: Order of an element $a \in G$, for $(G, *)$ is the no. of elements in subgroup generated by a, i.e.

$|\langle a \rangle|$

* Let $(G, *)$ is a group, & $a \in G$, $a \neq e$, such that

$$\langle a \rangle = \{a^0 = e, a^1 = a, a^2 = b, a^3 = c\}$$

then,

$$a^4 = a^3 * a = c * a = a$$
$$a^5 = a^4 * a = a * a = a^2 = b$$
$$\vdots$$

{Elements just repeat}

$$a^{-1} * a = e.$$
$$a^2 * a = e$$
$$\therefore a^{-1} = a^2$$
$$a^{-2} = (a^{-1})^2 = (a^2)^2 = a^4 = a$$
$$a^{-3} = (a^{-1})^3 = (a^2)^3 = a^6 = a^5 * a = b * a = e.$$
$$\vdots$$

{Elements just repeat}.

So, for an element a, such that:

$$a^n = e, \quad n \geq 1,$$

after n all elements will repeat.

Ex. $(Z_{10}, \oplus_{10})$

$$\langle 1 \rangle = \{1^0 = 0, 1^1 = 1, 1^2 = 2, 1^3 = 3, 1^4 = 4, 1^5 = 5, 1^6 = 6, 1^7 = 7, 1^8 = 8, 1^9 = 9, 1^{10} = \underline{0}\}$$

$\therefore \langle 1 \rangle$ will repeat starting from $n \geq 10$ & 1^9 is 1^{-1}.

Also 1 is a generator.

$\langle 2 \rangle = \{2^0 = 0, 2^1 = 1, 2^2 = 4, 2^3 = 6, 2^4 = 8, 2^5 = \underline{\underline{0}}\}$

Elements repeat for $n \geq 5$.

2 is not generator.

$|2| = |\langle 2 \rangle| = 5$

$\langle 3 \rangle = \{3^0 = 0, 3^1 = 3, 3^2 = 6, 3^3 = 9, 3^4 = 2, 3^5 = 5, 3^6 = 8, 3^7 = 1,$
$3^8 = 4, 3^9 = 7, 3^{10} = \underline{\underline{0}}\}$

Elements repeat for $n \geq 10$.

3 is a generator since $\langle 3 \rangle = Z_{10}$.

$|3| = |\langle 3 \rangle| = 10$.

$\therefore$ Order of element a can also be defined as the smallest power n such that

$a^n = e$, $n \geq 1$

Ex $(Z, +)$. Check order of 2.

$\langle 2 \rangle = \{2^0 = 0, 2^1 = 2, 2^2 = 4, 2^3 = 6, \ldots\}$

If we keep going in +ve direction, e is never found. So, for ∞ group we have to go in -ve dir. as well.

$|\langle 2 \rangle| = \infty$.

So, Order of $2 = \infty$.

$\therefore$ Order of $a \in G$ where G is ∞ group may / may not be finite.

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* Cyclic Group: A group with at least one generator.

Ex. $(\{1,-1,i,-i\}, \times)$ is Cyclic group with i & $-i$ as generators.

* Theorem: Every cyclic group is abellian.

Proof: Let $(G, *)$ is a cyclic group with generator g.

$G = \{g^n \mid n \in \mathbb{Z}\}$

let $a, b \in G$. $a = g^m$ $b = g^n$

then,

$a * b = g^m * g^n$
$\Rightarrow g^{m+n}$

$b * a = g^n * g^m$
$\Rightarrow g^{m+n}$

$\therefore$ G is abellian.

Ex. Order of smallest group that is not cyclic?
4.

Order 1: $\{e\}$
Order 2: $\{e, a\}$ a is its own inverse & $a * a^{-1} = a^{-1} * a = e$
Order 3: $\{e, a, b\}$ $a^{-1} = b$. $\langle a \rangle = \{a^0 = e, a^1 = a, a^{-1} = b\} = \langle b \rangle$.
Order 4:
(1) $\{e, a, b, c\}$ $a^{-1} = b$. $c^{-1} = c$
$\langle c \rangle = \{c^0 = e, c^1 = c, c^2 = e\} = \{e, c\}$
$\langle a \rangle = \langle b \rangle = \{a^0 = e, a^1 = a, a^{-1} = b, a^2 = c\} = \{e, a, b, c\}$.

(2) $\{e, a, b, c\}$ $a^{-1} = a$ $b^{-1} = b$ $c^{-1} = c$.

No generator.

$\therefore$ Not Cyclic.

Ex. List all the elements generated by 5 in $(U(18), \otimes_{18})$

$U_{18} = \{1, 5, 7, 11, 13, 17\}$

$\langle 5 \rangle = \{5^1, 5^2, \ldots\}$

$\Rightarrow \{5, 7, 17, 13, \underset{5^{-1}}{11}, \underset{e}{1}\}$

$\therefore$ 5 is a generator for $(U(18), \otimes_{18})$.

also, $5^{-1} = 11$, so, 11 is also generator.

Ex. Subgroup of i in $(C^*, \times)$

$C^* =$ Complex Nos. $- \{0\}$
$\{ai + b\}$

$\langle i \rangle = \{i^1 = i, i^2 = -1, i^3 = \underset{i^{-1}}{-i}, i^4 = 1 = e\}$

Although ∞ group, the order is finite, for subgroup

- i is not generator.

- Similarly there is no generator. $\therefore$ $(\mathbb{C}^*, \times)$ is not cyclic group.

Ex. For a finite group G, if $|\langle a \rangle| = |G|$ then can we say a is a generator?

yes, for finite group.

Ex. For an infinite group G, if $|\langle a \rangle| = |G| = \infty$, then can we say a is generator?

No. Ex. $G = (\mathbb{Z}, +)$.

$\langle 2 \rangle = \{2, 4, 6, 8, -2, -4, 0, \ldots\} = 2\mathbb{Z}$.

Ex. The definition of subgroup generated by a is:

(A) $\langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}$
(B) $\langle a \rangle = \{a^n \mid n \in \mathbb{N}\}$
(C) $\langle a \rangle = \{a^n \mid n \in \omega\}$
(D) $\langle a \rangle = \{a^{-n} \mid n \in \mathbb{N}\}$

① for Inf. group.
A

② for finite group only
A, B, C, D.

Ex. For a group G, "g" is some element. If $m, n \in \mathbb{Z}$, $m \neq n$, $g^m = g^n$ then can we say $\langle g \rangle$ is finite?

Yes. $\because$ $g^m = g^n$, $\langle g \rangle$ repeats with $|m-n|$ elements.

$$\left\{ \begin{array}{l} \text{either } g^m = g^n * g^K, \text{ such that } g^K = e \\ \text{or } g^n = g^m * g^K \end{array} \right\}$$

Ex. Let $(G, *)$ be a group. For $g \in G$

$$\langle g \rangle = \{ g^n \mid n \in \mathbb{Z} \}.$$

Which is true?

(A) $\langle g \rangle$ is also a group.
(B) $\langle g \rangle$ is a cyclic subgroup of G.
(C) $\langle g \rangle$ is abellian.
(D) If $H \leq G$ & $g \in H$, then $\langle g \rangle \leq H$.

A: $\langle g \rangle$ is subgroup, hence also a group.
B: Yes, g will be the generator for $\langle g \rangle$.
C: If cyclic then abellian.
D: $\because$ $g \in H$, $\langle g \rangle$ will be subgroup of H.

Ex. Which of the following is/are true about $\langle g \rangle$?

(A) If $\langle g \rangle$ is finite cyclic group, $|\langle g \rangle| = n$, then $\{g^0, g^1, g^2, \ldots g^{n-1}\}$ are distinct.

(B) If $\langle g \rangle$ is ∞ cyclic & if m & n are integers, $m \neq n$, then $g^m \neq g^n$.

(C) Subgroups of cyclic groups are cyclic.

(D) $\langle g \rangle$ is abellian.

A: Since $\langle g \rangle$ is finite group generated by g. $g^0, g^1, \ldots g^{n-1}$ are distinct.

B: $\because \infty$ group, $m \neq n \rightarrow g^m \neq g^n$.

C: Theorem (proof later)

D: $\because \langle g \rangle$ is cyclic, $\langle g \rangle$ is abellian.

Ex. Let G is a cyclic group, then the no. of generators is:

(A) Even

(B) Odd

(C) Could be either.

Ex. $\{e, a\}$ & $\{e, a, b\}$

a is only gen. a, b are gens.

Ex. Every two cyclic groups of same order are isomorphic:

Yes. $G = \{g^0, g^1, g^2 \ldots\}$

$H = \{h^0, h^1, h^2 \ldots\}$

* Lagrange's Theorem

- Order of subgroup $(S, *)$ divides the order of group $(G, *)$. $\{ S \leq G \}$

  i.e. $|S| \mid |G|$

- For a group $(G, *)$, $\forall g \in G$, where G is finite,
  $\langle g \rangle \leq G$
  Therefore $|g|$ divides $|G|$.

- Lagrange Theorem is one way:
  $H \leq G \longrightarrow |H| \mid |G|$.

  But for Abellian Group, it is two way
  $H \leq G \longleftrightarrow |H| \mid |G|$

  i.e.
  If $H \leq G$ then $|H|$ divides $|G|$.
  and.
  If $n \in N$ divides $|G|$, then there exist at least one subgroup $H \leq G$, such that $|H|$ is n.

Ex. If $|G|$ is prime, then comment on order of element a, $a \neq e$.

$|a| = |\langle a \rangle|$.

Since $\langle a \rangle \leq G$, $|\langle a \rangle|$ divides $|G|$.
Since $|G|$ is prime, the only nos. dividing $|G|$ are 1 & $|G|$.

$|\langle a \rangle|$ can't be 1, since at least $\langle a \rangle = \{a^0, a^1\}$
$= \{e, a\}$.

$\therefore |\langle a \rangle| = |G|$

$\therefore \left(\forall a \in G,\ a \neq e, \ \& \ |G| = p\right) \Rightarrow a$ is generator.

$\therefore G, |G| = p \rightarrow G$ is cyclic.

- Theorem: Every subgroup of cyclic group is cyclic
Theorem: For a prime # p, every group of order p^2 is ~~cyclic~~ abellian.

Ex. $|G| = 6$, $H \leq G$ & $1 < |H| < 6$, then which is correct:

(A) Both G & H are always cyclic.
(B) G maynot be cyclic. H is always cyclic.
(C) G is always cyclic. H maynot be.
(D) Both G & H maynot be cyclic.

$|G| = 6$. Maynot be cyclic.

$|H|$ can be 2 or 3, since $|H|$ divides $|G|$. $\because |H|$ is always prime, it is always cyclic.

- Theorem: Order of $a \in Z_n$, in $(Z_n, \oplus_n)$ is $n / \gcd(a, n)$.

Ex. $Z_6 = \{0, 1, 2, 3, 4, 5\}$.

$$|3| = \frac{6}{3} = 2.$$

$$\langle 3 \rangle = \{ 3^1 = 3,\ 3^2 = \underset{\hookrightarrow e}{0} \}.$$

$\therefore |3| = 2.$

* Facts

- Cyclic Group $\rightarrow$ Abellian Group.
- Every cyclic group of order n is isomorphic to the group $(Z_n, \oplus_n)$.
-

(1) Order 1: Only 1 non-isomorphic group. Abellian.
(2) Similarly for order 2, 3, 5, since prime order, hence cyclic & abellian.
(3) Order 4: $4 = 2^2 = p^2$ $\therefore$ Two non-isomorphic groups, abellian. Only one is cyclic.

(4) Order 6: 2 non-isomorphic groups. 1 abellian. 1 non-abellian.

Ex. If G is a group such that $(ab)^2 = a^2b^2$ then check if G is abellian.

To show: $ab = ba$.
Given: $(ab)^2 = a^2b^2$

$$\Rightarrow (ab)(ab) = (aabb)$$
$$\Rightarrow a^{-1}ab\,abb^{-1} = a^{-1}aabbb^{-1}$$
$$\Rightarrow ba = ab$$

This is alternate def. of abellian.

Ex. If in a group every element is its own inverse show that G is abellian.

$\forall a \in G, \; a^{-1} = a.$

w.k.t. in a group.

$$(ab)^{-1} = b^{-1}a^{-1}$$

$$\Rightarrow ab = ba. \qquad (\because a^{-1} = a, b^{-1} = b)$$

Ex. If in a group G, $\forall x, y, z \in G$, $(xy = zx) \rightarrow yz$ then is G abellian?

$$xy = zx \rightarrow y = z$$

$$\equiv \; y \neq z \rightarrow xy \neq zx$$

$$\underbrace{b}_{x}\underbrace{(ab)}_{y} = \underbrace{(ba)}_{z}\underbrace{b}_{x}$$

$$\Rightarrow ab = ba.$$

This is alternate ~~the~~ def. of Abellian group.

Ex. $\forall x, y, z, a, b$, $(xyz = ayb) \rightarrow (xz = ab)$. Check if abellian group.

$$\underbrace{ba}_{x}\underbrace{be}_{y\,z} = \underbrace{eb}_{a\,y}\underbrace{ab}_{b} \Rightarrow (ba)e = e(ab)$$

$$\Rightarrow ba = ab.$$

Ex. Show group G is abellian iff $\forall a, b \in G$, $(ab)^{-1} = a^{-1}b^{-1}$

$$(ab)^{-1} = a^{-1}b^{-1}$$
$$\Rightarrow (ab)^{-1}(ab) = a^{-1}b^{-1}ab$$
$$\Rightarrow e = a^{-1}b^{-1}ab$$
$$\Rightarrow a = aa^{-1}b^{-1}ab$$
$$\Rightarrow ba = bb^{-1}ab$$
$$\Rightarrow ba = ab.$$

* Theorem: Intersection of two subgroups is also subgroup.

RECURENCE RELATIONS

* Sequence: An ordered list of elements.

Ex. 0, 1, 2, 3
1, 0, 2, 3 are diff. sequences.

* Recurence relation happens when some term depends on some previous term(s).

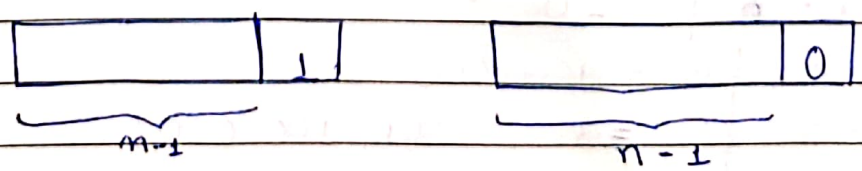
Ex. AP: 1, 6, 11, 16, 21, ...
$a_n = a_{n-1} + 5$
$a_0 = 1$.

Fibonacci Sequence: 1, 1, 2, 3, 5, 8, 13, ...
$a_n = a_{n-1} + a_{n-2}$
$a_0 = a_1 = 1$

* Ex. Find the recurence rel. & give initial conditions for the no. of bits strings of length n that don't have two consecutive 0s. How many such bit strings are there of length 5?

for $n = 1$: 1, 0 , $f(1) = 2$
for $n = 2$: 01, 10, 11, $f(2) = 3$

for n: n^{th} bit can be 1 or 0.



if last bit is 1, $(n-1)^{th}$ bit can be 0 or 1.
So, all $(n-1)$ length strings, without no consecutive 0s are valid with 1

if last bit is 0, $(n-1)^{th}$ bit must be 1.
So, all $(n-2)$ length strings, without consecutive 0s are valid with 0.

$$\therefore f(n) = f(n-1) + f(n-2)$$

Ex. Find recurence rel. for no. of permutations of a set with n elements.

$n=1$: $f(n) = 1$
$n=2$: $f(n) = 2$
for n:

$$f(n) = n \times f(n-1)$$

Since in a perm. of $(n-1)$ elements we have n positions possible for n^{th} pos.
Ex. _1_2_3_ has 4 possibility.

Ex. Recurence rel. for bit strings of length n that contain a pair of consecutive 0s.

$n=1$: $f(n) = 0$
$n=2$: $f(n) = 1$ $\{00\}$
$n=3$: $f(n) = 3$ $\{001, 100, 000\}$

for n:

if n^{th} bit is 1, then all the $(n-1)$ len strings having consecutive 0s are valid.

if n^{th} bit is 0, then

(1) All $(n-2)$ strings with consecutive 0s are valid if end with 1.

(2) All $(n-1)$ strings which end with 0 are also valid.

$$T(n) = T(n-1) + T(n-2) + 2^{n-2}$$

Ex. Recurence rel. for no. of strings that contain 3 consecutive 0s.

$n=1$: $f(n)=0$

$n=2$: $f(n)=0$

$n=3$: $f(n)=1$ $\{000\}$

$n=4$: $f(n)=3$ $\{0001, 1000, 0000\}$

for n:

if n^{th} bit is 1, then all $(n-1)$ len strings having 3 consecutive 0s.

if n^{th} bit is 0:

(1) $\boxed{\sim n-2 \sim | 1 | 0}$ $\Rightarrow f(n-2)$

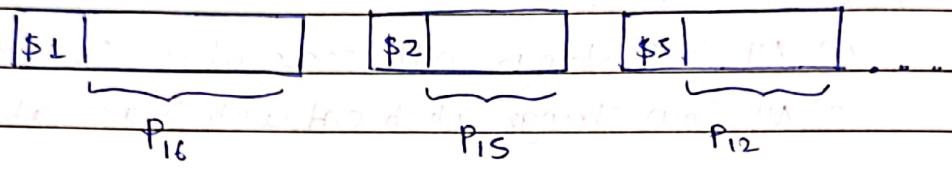
(2) $\boxed{\sim n-3 \sim | 1 | 0 | 0}$ $\Rightarrow f(n-3)$

(3) $\boxed{\sim n-3 \sim | 0 | 0 | 0}$ $\Rightarrow 2^{n-3}$

$$\therefore f(n) = f(n-1) + f(n-2) + f(n-3) + 2^{n-3}$$

Ex. Find no. of ways in which bill of \$17 is paid with coins of value \$1, \$2, \$5, \$10 & bills of value \$5, \$10.

(Order matters).



$$P_{17} = \underset{(\$1)}{P_{16}} + \underset{(\$2)}{P_{15}} + \underset{(\$5)}{P_{12}} + \underset{(\$10)}{P_{7}} + \underset{\substack{(\$5)\\ \text{Bill}}}{P_{12}} + \underset{\substack{(\$10)\\ \text{Bill}}}{P_{7}}$$

COMBINATORICS

- Combinatorics allows us to count without counting.

* Sum Rule

① If set A & B are disjoint, then

$$|A \cup B| = |A| + |B|$$

If not disjoint, then

$$|A \cup B| < |A| + |B|$$

② Say you have to choose one fruit among 3 apples & 2 oranges.

Task T_1 (apples)	Task T_2 (oranges)
3	2.

Total no. of ways = 3 + 2 = 5 ways.

Note: 3 ways of doing T_1 are disjoint from 2 ways of T_2.

Summation rules says if for a task T there are subtasks $T_1, T_2, \ldots T_n$, any one of which satisfies T.

T_1 can be done in m_1 ways, T_2 in m_2 & so on.

If all m_1 ways are disjoint from rest $m_2, m_3 \ldots$ ways, m_2 ways are disjoint from $m_1, m_3, m_4, \ldots$ & so on, then

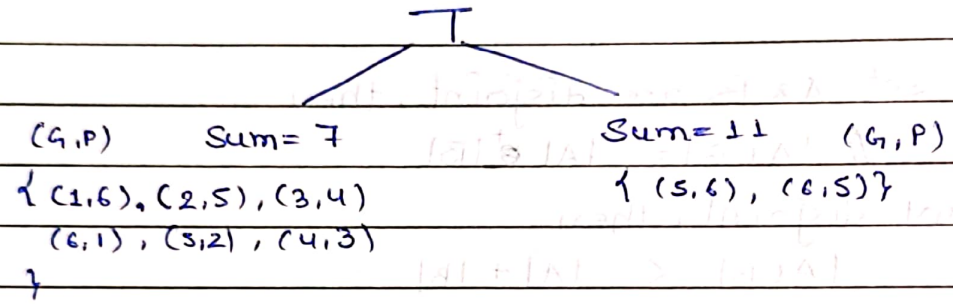
$$T \Rightarrow T_1 \text{ or } T_2 \text{ or } T_3 \ldots \text{ or } T_n$$

$$\#\text{ways}(T) = \underbrace{m_1 + m_2 + m_3 \ldots + m_n}_{\text{disjoint}}.$$

Note: OR & Disjoint are req. for sum rule to be applied.

Ex. If two dice are rolled: green & purple. How many ways to get a sum of 7 or 11.

(Green & purple dice are distinct)



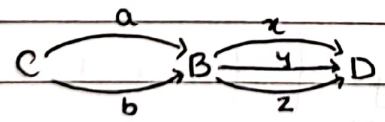
$\therefore$ # ways (T) = # ways (7) + # (11)

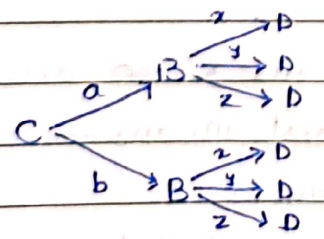
$\Rightarrow 6 + 2$

$\Rightarrow 8$

* Product Rule

Ex. Say a task $C \to D$ requires $C \to B$ and $B \to D$





So, for 2 roads from $C \to B$, there are 3 roads from $B \to D$.

$\therefore$ # ways $(C \to D)$

= # ways $(C \to B) \times$ # ways $(B \to D)$

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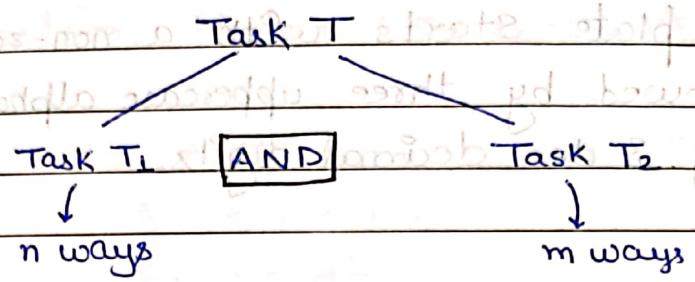
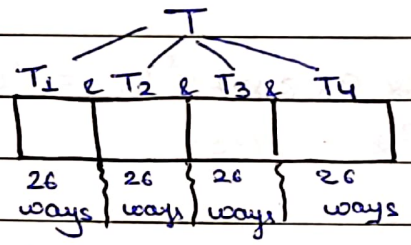


Task T requires both T_1 & T_2 done. Once T_1 is done in any one of n ways, there are m ways to do T_2, so ways to get task T done are

$$n \times m$$

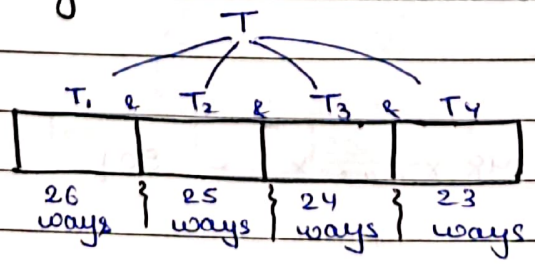
Ex. 4 letter english words.



$$\therefore \#T = \#T_1 \times \#T_2 \times \#T_3 \times \#T_4$$

$$\Rightarrow 26^4$$

Ex. 4 letter english words without repetition.



Once T_1 is done, only 25 ways for T_2 possible, since one of the ways is used for T_1 & repitition not allowed, & so on.

Ex. License plate starts with a non-zero digit, followed by three uppercase alphabets, followed by 3 ~~dec~~ decimal digits.

9	26	26	26	10	10	10

$\Rightarrow 9 \times 26^3 \times 10^3$

- Say no repetition of digit & alphabet allowed.

9	26	25	24	9	8	7

$\Rightarrow 9 \times 26 \times 25 \times 24 \times 9 \times 8 \times 7.$

Ex. A travelling salesman wants to travel all 50 state capitals in US.

Consider calendar with 50 days d1 to d50.
D_1 has 50 choices, D_2 has 49 choices, D_3 48, ...
D_{50} has 1 choice.

$50 \times 49 \times 48 \times \ldots \times 1 = 50!$

* Subtraction Rules (Inclusion - Exclusion)

Ex. Say Set A & B are <u>not disjoint</u>, say

$A = \{a, b, c\}$

$B = \{c, d\}$

then, $\#(A \cup B) = \#A + \#B$ is wrong.

we have counted common elements twice.

So, we remove one of two times.

$\therefore \#(A \cup B) = \#A + \#B - \#(A \cap B)$

Ex. No. of len 8 bit string, either start with 1 or end with 00.

$\#(\text{start } 1) = 2^7$ $\quad$ $\#(\text{ending } 00) = 2^6$

$\#(\text{both}) = 2^5$

$\therefore \#\text{total} = \underbrace{2^7}_{\boxed{1\ldots}} + \underbrace{2^6}_{\boxed{\ldots 00}} - \underbrace{2^5}_{\boxed{1\ldots\ldots 00}}$

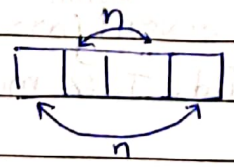
Ex. Len 8 bit string, starting with 00 or 000.

Not disjoint

$\#\text{total} = \underbrace{2^6}_{\boxed{00\ldots}} + \underbrace{2^5}_{\boxed{000\ldots}} - \underbrace{2^5}_{\boxed{000\ldots}}$

Ex. How many k length palindrome can be created from charset of size n.

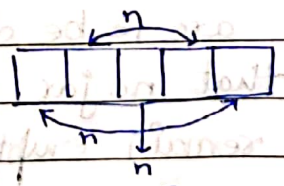
k=4 :



$\Rightarrow n^2$

$\therefore n^{\lceil \frac{k}{2} \rceil}$

k=5 :



$\Rightarrow n^3$

* Counting By Case

When dividing a task T into subtask (or cases) $T_1, T_2, \ldots T_n$, ensure that the cases are mutually exclusive (disjoint) & exhaustive (i.e. all the elements of T are covered by the cases).

∴ All element of T, say a, should belong to <u>exactly one</u> subcase T_i ($i \in [1, n]$).

Ex. Stu Id is made of 3 letters followed by two ~~Id~~ digits.
No. of Ids with even no. of A's.

0 A's

25	25	25	10	10

$\Rightarrow 25^3 \times 10^2$

2 A's

→ A A		1	25×10^2
→ A	A	1	25×10^2
→	A A	1	25×10^2

$\Rightarrow 3 \times 25 \times 10^2$

∴ total $= 25^3 \times 10^2 + 3 \times 25 \times 10^2$

Ex. A & B have three offsprings a, b, c. The five are to be distributed among seven distinct jars such that no jar gets both parent & child. Jars may remain empty. Count ways.

C1: Both Parents Together

$\boxed{A\&B}$ a b c

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

7 6 × 6 × 6

$\Rightarrow 7 \times 6^3$

C2: Parents Separate

A B a b c

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

7 6 5 5 5

$\Rightarrow 7 \times 6 \times 5^3$

$$\text{Total} = 7 \times 6^3 + 7 \times 6 \times 5^3$$

* Complement Rule

Desired Cases = Total Cases − Undeserved Cases

Ex. # of non-injective funcs from A to B. $|A| = m$ $|B| = n$.

$$\#\text{non-injective} = \#\text{total} - \#\text{injective}$$
$$\Rightarrow n^m - {}^nP_m$$

Ex. A computer password is 6 to 8 letters long. Each char is either uppercase English alpha or digit. Each password must contain atleast 1 digit. Count passwords.

C1: len 6: ${}^6C_1 \times 10 \times 36^5$ ✗ or $(36)^6 - (26)^6$ ✓ (all pass − no digit pass)

C2: len 7: ${}^7C_1 \times 10 \times 36^6$ $\quad 36^7 - 26^7$

C3: len 8: ${}^8C_1 \times 10 \times 36^7$ $\quad 36^8 - 26^8$

$\Downarrow$

This is wrong.

Consider len 2 pass. i.e. ${}^2C_1 \times 10 \times 36^1$

C1: $\underset{10}{__}\ \underset{\times 36}{__}$ C2: $\underset{36}{__}\ \underset{\times 10}{__}$

4 A 4 z 4 1 → 42 43 44

A 4 z 4 1 4 2 4 → 34 44

∴ 44 appears in both subcases. Hence not mutually exclusive.

Ex. No. of ways to pick two cards, with at least one ace. one after another

Total − No ace

$\Rightarrow 52 \times 51 - 48 \times 47$

or

C1:	Ace	Non Ace	4×48
C2:	Non Ace	Ace	48×4
C3:	Ace	Ace	4×3

Ex. $n > 2$

① No. of $n \times n$ matrices with each entry from set $\{a, b, c\}$

② Set of all funcs. from $A = \{0, 1, 2, \ldots, n^2 - 1\}$ to $B = \{0, 1, 2\}$

(1) $3^{n \times n}$

(2) 3^{n^2}

Ex. 4 digit nums, not starting with 0, no-repeat & odd.

Ends with: 1, 3, 5, 7, 9.

8	8	7	5

$\therefore 8^2 \times 7 \times 5$

Ex. Same as last but even no.

Ends with: 0, 2, 4, 6, 8.

8 or 9	8	7	5

C1: Ends with 0

			0
9	8	7	1

$\Rightarrow 9 \times 8 \times 7$

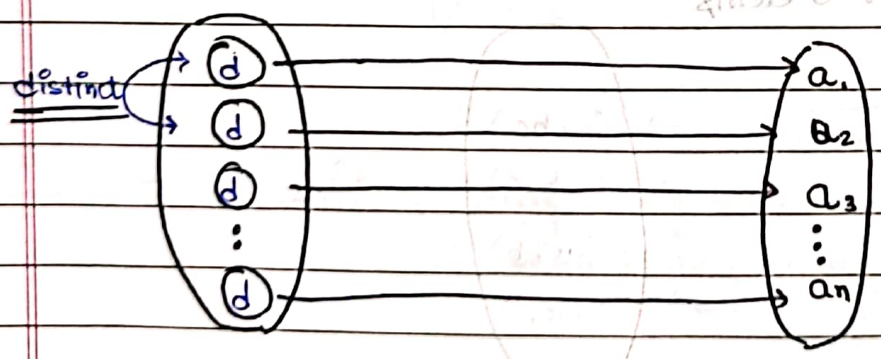
C2: Ends with non-zero

8	8	7	4

$\Rightarrow 8 \times 8 \times 7 \times 4$

* Division Rule.

"To count cows, count the legs & divide by 4"



$$n = \frac{m \text{ (\# elems)}}{d \text{ (\# elem per } a)}$$

Ex. Each cow has distinct $d = 4$ legs. If 100 legs are there, with each how having distinct 4 legs, then

$$n = \frac{100}{4} = 25 \text{ cows.}$$

- Distinct & equal then division rule.

Ex. Divide n people in two group ~~of 2~~.

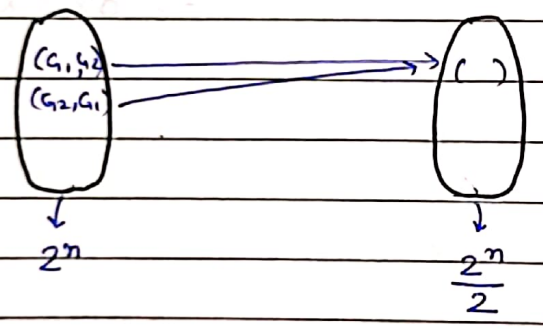
Here order {side/group} doesn't matter, i.e. there is no distinction b/w the two groups.

So, say with 3 elems:

① $\{\{1\},\{2,3\}\}$ & $\{\{2,3\},\{1\}\}$ are same.

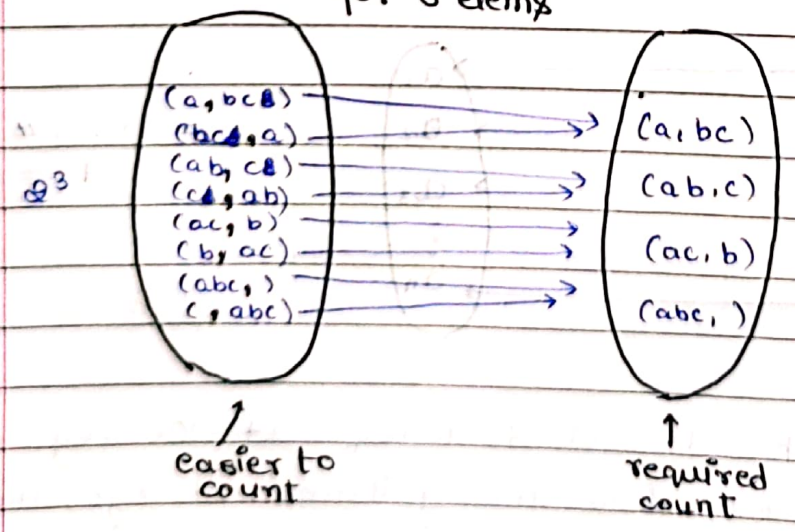
② $\{\{1,2\},\{3\}\}$ & $\{\{3\},\{1,2\}\}$ are same

& so on.



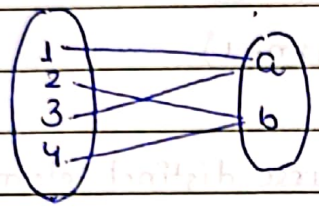
labelled groups — unlabelled groups

for 3 elems



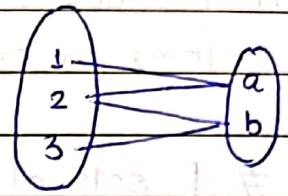
- Division rule is based on K to 1 correspondance, i.e.
  1. K to 1 mapping
  2. Onto. i.e. ~~for~~ $\forall y \in$ Codomain there exist K preimg in domain

Ex.



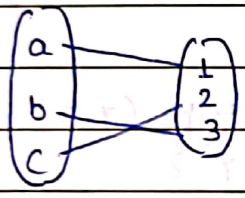
2 to 1 correspondance

Ex.



This isn't even a func.

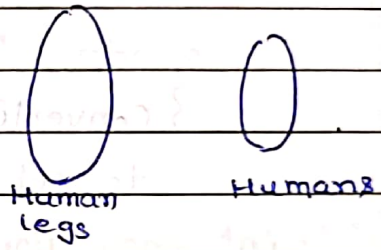
Ex.



One-One & Onto

$\therefore$ 1 to 1 correspondance (or bijection)

Ex.



∞:1 correspondance.

For a K to 1 correspondance ($f: A \to B$),

$$|A| = K \times |B|$$

Ex. In how many ways can we place two identical rooks on 8x8 chessboard so that they occupy different rows & columns.

- Rooks are identical
- If rooks were distinguishable, then #ways $= 8^2 \times 7^2$
- Since identical $= \frac{8^2 \times 7^2}{2}$

{ Ex. Positions $\{(2,5), (3,4)\}$ & $\{(3,4), (2,5)\}$ are same. }

Ex. How many subsets of exactly 2 elements are there for a set of n elements?

# of ordered pairs where the two elems aren't same = $n \times (n-1)$

# of sets of size 2 (of course distinct elems)

$$\Rightarrow \frac{n \times (n-1)}{2}$$

{ Converting ordered pair of type (x, x) to set will yeild size 1, $\{a\}$ }

Ex. How many linear orders of size 6 from elements a, b, c, d, e, f such that a is to the left side of b (not immediately)?

We know total no. of linear orders

$$\Rightarrow 6 \times 5 \times 4 \times 3 \times 2 \times 1.$$

Also a can be on left of b or right, i.e. the total # of linear orders can be divided into two parts.

$\therefore$ $6! / 2$

{ Ex. a c d e b f
b c d e a f }

- **Subsequence:** A subseq. of a string is a new string formed by removing 0 or more chars from the string.

  Ex. "acd" is subseq. of "abcd"
  "a" is subseq. of "bcad"

Substring: A consecutive subsequence.
Ex. "abc" is substr. of "abcd"
"bc" is substr. of "abcde"

Ex. Linear order of a, b, c, d, e, f such that "abc" is subsequence.

for every arrangement of d, e, f we can get 6 strings. Ex.

e _ f _ d _

the three vacant places can have 6 ~~combinati~~ arrangements of a, b & c.

(abc), (acb)
(bac), (bca)
(cab), (cba)

Out of every 6 we only want one.

$$\therefore \frac{6!}{6} \text{ or } \frac{6!}{3!}$$

Ex. No. of linear arrangements of a, b, c, d, e, f such that abcd is subsequence.

$$\frac{6!}{24} \text{ or } \frac{6!}{4!}$$

* Factorial

$$n! = \begin{cases} n \times (n-1)! & , n > 0 \\ 1 & , n = 0 \end{cases}$$

* Permutation | Combination

$$^nP_r = \frac{n!}{(n-r)!} \qquad ^nC_r = \frac{n!}{(n-r)!\, r!}$$

$$^nP_r = r!\, ^nC_r$$

^nP_r Gives order or arrangements.

^nC_r Gives combination or selection.

Permutation	Combination
① Picking captain, pitcher & keeper from team	① Picking three players from the team.
② Picking 2 colors in order from brochure.	② Picking two colors from brochure.
③ Picking 1st, 2nd & 3rd winners.	③ Picking three winners.

Ex. ${}^{n}C_{r}$, when $r > n = 0$.

Say $|A| = n$ where A is a set. What is the no. of subsets of size $(n+1)$ of A?

$${}^{n}C_{n+1} = 0$$

Ex. 9 Maths faculty. 11 CS faculty. No. of ways to make a comittee such that 3 from Maths & 4 from CS faculty.

T

Select 3 from Math & Select 4 from CS

$${}^{9}C_{3} \times {}^{11}C_{4}$$

Ex. Hand in poker = Set of randomly chosen 5 cards out of 52.

Full House: Hand such that 3 are of same value x & rest 2 have same value y.

Ex. $\boxed{5\diamondsuit}$ $\boxed{5\spadesuit}$ $\boxed{5\heartsuit}$ $\boxed{2\diamondsuit}$ $\boxed{2\spadesuit}$

No. of full house possible?

FH

Select 3 of same val & Select 2 of a diff. value

Select 1 of 13 value & Select 3 of 4 cards

Select 1 of 12 remaining value & Select 2 of 4 cards

$$\left({}^{13}C_{1} \times {}^{4}C_{3}\right) \times \left({}^{12}C_{1} \times {}^{4}C_{2}\right)$$

Ex. How many bit strings of len 10 has:

① Exactly 3 0s.

$^{10}C_3$

② More 0s than 1s.

$^{10}C_6 + {}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}$

{6 0s} {7 0s} {8 0s} {9 0s} {10 0s}

③ At least 7 1s.

$^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}$

④ At least 1 1s.

$2^{10} - {}^{10}C_0$

all 10 len bit string

all 10 len bit ~~string~~ string w/ exactly 0 1s

Ex. 4 len bit string with at least 2 1s.

Thought 1: We reseve any 2 places for 1s, rest places can be 0 or 1.

$^4C_2 \times 2^2$

This is wrong. Say we reserve 1st & 3rd pos.

1 _ 1 _ then one of the possible string for this reservation is 1111.

Now we reseve 2nd & 3rd. _ 1 1 _. Again a possible string for this reservation is 1111.

Same string counted multiple times for different reservation.

Therefore each reservation isn't mutually exclusive

$\therefore$ Overcounting.

Thought: Create cases for reserving 2 places, 3 & 4 places. non-reserved will only be 0.

$$^4C_2 + {}^4C_3 + {}^4C_4$$

There is no string which belong to more than one case or no case.

$\therefore$ Each string counted exactly once.

Ex. 10 len b.string with at least 3 1s.

$$2^{10} - ({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2)$$

Ex. No. of perm. of ABCDEFGH such that ABC is a substring.

$\boxed{ABC}$ DEFGH

$6!$

② Contains substr BA & GF.

$\boxed{BA}$ C D E $\boxed{GF}$ H

$6!$

③ Has ABE together (not necessarily in that order).

$\boxed{ABE}$ C D G F H

$6! \times 3!$ → inner perm. of ABE

④ Contains substr $\boxed{ABC}$ & $\boxed{CDE}$

Only possible when ABCDE is together.

$\boxed{ABCDE}$ F G H

$4!$

Ex. 7 Ws & 9 Ms. No. of ways to make a comittee of 5, such that at least one W & one M.

Three negative cases:

① No W & No M: 0

② No W & all M: ${}^{9}C_5$

③ All W & no M: ${}^{7}C_5$

$\therefore \; {}^{16}C_5 - (0 + {}^{9}C_5 + {}^{7}C_5)$

Ex. No. of 3 digit nos from 2 diff digits.

One of the two digits will appear twice.

${}^{10}C_2 \times {}^{2}C_1 \times {}^{3}C_2$

choose which digit appears twice

choose 2 places where they appear

* Combinatorial Arguments

① $^nC_r = {}^nC_{n-r}$

Choose r out of n & reject $(n-r)$

Choose $(n-r)$ out of n & reject r.

② $^{n+1}C_r = {}^nC_{r-1} + {}^nC_r$ (Pascal Identity)

LHS: $\dfrac{(n+1)!}{(n+1-r)!\,r!}$

RHS: $\dfrac{n!}{(n-r+1)!\,(r-1)!} + \dfrac{n!}{(n-r)!\,r!}$

$$\Rightarrow \frac{n!}{(n-r+1)(n-r)!(r-1)!} + \frac{n!}{(n-r)!\,r\times(r-1)!}$$

$$\Rightarrow \frac{n!\,(\cancel{r}+n-\cancel{r}+1)}{(n-r+1)\times r\times(n-r)!\times(r-1)!}$$

$$\Rightarrow \frac{(n+1)(n!)}{(n-r+1)!\;r!}$$

$$\Rightarrow {}^{n+1}C_r$$

LHS: Choose r from $n+1$ & discard $(n+1-r)$ elements

RHS: Choose $(r-1)$ from n with the $(n+1)^{th}$ element already chosen OR choose all r elements from n only & discard $(n+1)^{th}$ element.

Variations:

① $^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$

② $^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-2}C_{r-1} + {}^{n-2}C_r$

③ $^{n}C_r = {}^{n-1}C_r + {}^{n-2}C_{r-2} + {}^{n-2}C_{r-1}$

Pascal identity can be thought of as recursive definition of $^{n}C_r$ {Binomial Coeff}.

$$\text{BinCoeff}(n,r) = \begin{cases} \text{BinCoeff}(n-1,r) + \text{BinCoeff}(n-1,r-1), & n>k>0 \\ 1, & k=0 \text{ or } k=n \\ 0, & k>n \end{cases}$$

③ $$^{n+m}C_r = {}^{n}C_0\,{}^{m}C_r + {}^{n}C_1\,{}^{m}C_{r-1} + {}^{n}C_2\,{}^{m}C_{r-2} + \dots + {}^{n}C_r\,{}^{m}C_0$$

$$\Rightarrow \sum_{k=0}^{r} {}^{n}C_k\,{}^{m}C_{r-k} \Rightarrow \sum_{k=0}^{r} {}^{n}C_{n-k}\,{}^{m}C_{m-(r-k)}$$

Ex. $$\binom{30}{12} = \sum_{k=0}^{12} \binom{16}{k}\binom{14}{12-k} \checkmark$$

$$\neq \sum_{k=0}^{12} \binom{20}{k}\binom{10}{12-k} \times$$

$m=20$ $n=10$

n is less than r

$\therefore$ wrong

$$= \sum_{k=0}^{12} \binom{15}{k}\binom{15}{\underbrace{3+k}_{15-(12-k)}} \checkmark$$

Ex. Justify: $\binom{2n}{2} = 2\binom{n}{2} + n^2$

LHS: Select 2 elems from $2n$. $\Rightarrow {}^{2n}C_2$

RHS: Divide the elems in two equal sets of size n.

① Either choose 2 from set 1. : ${}^{n}C_2$

② OR Choose 2 from set 2. : ${}^{n}C_2$

③ OR Select 1 from set 1 & other from S2 : ${}^{n}C_1 \times {}^{n}C_1$

$\therefore \; {}^{n}C_2 + {}^{n}C_2 + n^2$

$\Rightarrow 2\,{}^{n}C_2 + n^2$

Note: When asked to check LHS = RHS, create such story.

~~Binomial Theorem~~

Ex. Justify: $2^n = \sum_{r=0}^{n} {}^{n}C_r$

LHS: All comb. of 0, 1, 2, 3, ... n elements

RHS: All comb. of 0 items + All comb. of 1 item + ... All comb. of n items.

Ex. Justify: $n\,2^{n-1} = \sum_{r=0}^{n} r\,{}^{n}C_r$

LHS: Pick one of n people. Out of remaining $(n-1)$ people, create comittee of any size. i.e. ${}^{n}C_1 \times 2^{n-1} = n \times 2^{n-1}$

RHS: Create a comittee of size r & make one of them president. i.e. ${}^{n}C_r \times {}^{r}C_1 = r \times {}^{n}C_r$

Ex. Justify: ${}^{n}C_r + {}^{n-1}C_r + {}^{n-2}C_r + \dots + {}^{r}C_r$
$= {}^{n+1}C_{r+1}$

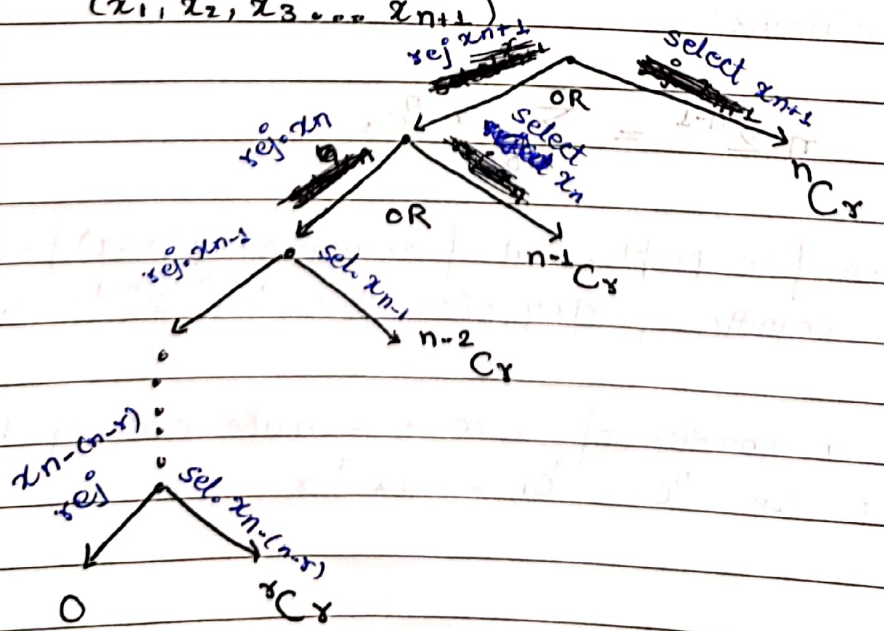
$$\Rightarrow \sum_{k=0}^{n-r} {}^{n-k}C_r = {}^{n+1}C_{r+1}$$

LHS: Either pick all r people from n
OR ~~pick~~ discard n^{th} person & select r people from $n-1$
OR discard n^{th} & $(n-1)^{th}$ person & select r people from $n-2$
⋮

RHS: Select $(r+1)$ people from $(n+1)$ people.

$${}^{n+1}C_{r+1} = {}^{n}C_r + {}^{n}C_{r+1} \qquad \text{(Pascal Identity)}$$
$$\Rightarrow {}^{n}C_r + {}^{n-1}C_r + {}^{n-1}C_{r+1}$$
$$\Rightarrow {}^{n}C_r + {}^{n-1}C_r + {}^{n-2}C_r + {}^{n-2}C_{r+1}$$
$$\vdots$$
$$\Rightarrow {}^{n}C_r + {}^{n-1}C_r + {}^{n-2}C_r + \dots {}^{n-(n-r)}C_r + \cancel{{}^{n-(n-r)}C_{r+1}} \to 0$$

$(x_1, x_2, x_3 \dots x_{n+1})$



Since less than r elements remaining

* Binomial Theorem

$$(a+b)^n = {}^nC_0 a^0 b^n + {}^nC_1 a^1 b^{n-1} + {}^nC_2 a^2 b^{n-2} + \dots + {}^nC_{n-1} a^{n-1} b^1 + {}^nC_n a^n b^0$$

$$\Rightarrow \sum_{k=0}^{n} {}^nC_k a^k b^{n-k} \qquad \{n+1 \text{ terms}\}$$

Ex. Justify: $\sum_{k=0}^{n} {}^nC_k = 2^n$

w.K.t.

$$(a+b)^n = \sum_{k=0}^{n} {}^nC_k a^k b^{n-k}$$

Put $a, b = 1$

$$\Rightarrow 2^n = \sum_{k=0}^{n} {}^nC_k$$

Ex. $(x+y)^{25}$. What is coeff of:

① $x^{12}y^{13}$

$(12+13) = 25$ ✓

So, ${}^{25}C_{12}$ or ${}^{25}C_{13}$

② $x^{12}y^{14}$

$(12+14) \neq 25$

$\therefore 0$

Ex. $(2x-3y)^9$. Coeff of:

① x^5y^4

$(5+4) = 9$ ✓

$\therefore {}^9C_5 \times (2)^5 \times (-3)^4$

② x^4y^5

$(5+4) = 9$ ✓

$\therefore {}^9C_5 \times (2)^4 \times (-3)^5$

Ex. $(x^3 + 2y^2)^{10}$. Coeff. of $2x^{12}y^{12}$

$$\Rightarrow \sum_{k=0}^{10} {}^{10}C_k (x^3)^k (2y^2)^{10-k}$$

$x^{12}y^{12}$ possible for $k = 4$.

$$\Rightarrow {}^{10}C_4 \, x^{12} (2y^2)^6 = 2^5 \times {}^{10}C_4 \times 2x^{12}y^{12}$$

$$\therefore \text{Coeff} = 2^5 \times {}^{10}C_4$$

* Bijective Proof

A Bijective proof in combinatronics just means that you map a difficult counting problem to another counting problem which is easier by showing that there is a bijection or one-to-one mapping from hard problem to easier one.

Basically,
If Bijection from Hard to Easy counting problem then no. of sol. to easy problem = no. of sol. to hard problem.

Ex. $n=4$.

P1: Select 1 person
P2: Reject 3 people
} They have 1-to-1 mapping.

Ex. P1: Subsets from set of size n.
P2: Bit strings of size n.

* Permutation with repetition

Ex. How many strings of len r can be created using uppercase English letters

$_____ \ldots ____$

$26 \; 26 \; 26 \ldots 26$

$\therefore \; 26^r$

- No. of r-permutation with repetition from n distinct elements:

$$n^r$$

* Permutation with repeated elements

Number of r-permutation with n (non-distinct necessirily) elements:

Say m_1 As, 1B, 1C, m_2 Ds such that $m_1+1+1+m_2=n$.

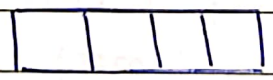
$$\Rightarrow \frac{n!}{m_1!\, m_2!}$$

Ex. No. of 3 digit number from {1,2,2}.

① (1 2 2) ② (2 1 2) ③ (2 2 1) $\frac{3!}{2!} = 3$.

Ex. No. of permutations of word "COOK".

M1:



① Select two places for "O"s. Only single arrangement possible there.
② AND in remaining places arrange 'C' & 'K'.

$$({}^4C_2 \times 1!) \times \underset{\times 2!}{({}^2C_2)} = 6 \times 2 = 12$$

M2: Division Rule.

Total no. of perms. = 4!

But this counts every position of "Os" twice.

$$\therefore \frac{4!}{2} = 12$$

{ Ex. $O_1 C O_2 K$, $O_2 C O_1 K$ → OCOK

M3:

Unique elements : C, O, K.

for C we have to assign 1 pos : ${}^4C_1$

then, for O we have to assign 2 pos. : ${}^3C_2$

then, for K we have to assign 1 pos : ${}^1C_1$

$\therefore 4 \times 6 \times 1 = 12.$

M4: Think of every letter as distinct, then possible perms = 4!

Now, arrangement b/w the 2 Os don't matter.

CO_1KO_2 & CO_2KO_1 are same.

No. of possible perms of 2Os = 2!

So. for ~~every~~ COKO we've overcounted as CO_1KO_2 & CO_2KO_1.

$\therefore \frac{4!}{2!}$

Ex. "SUCCESS"

M1: Select 3 pos. for S & then permute them : ${}^7C_3 \times 1$

Then select 2 pos. for C & per. them : ${}^4C_2 \times 1$

Then in remaining pos permute rest : ${}^2C_2 \times 2!$

$\Rightarrow ({}^7C_3) \times ({}^4C_2) \times (2!)$

$\Rightarrow 35 \times 6 \times 2$

$\Rightarrow 420$

M3: Assign 1 pos. for U : ${}^7C_1$

then, 1 pos. for E : ${}^6C_1$

then, 3 pos. for S : ${}^5C_3$

then, 2 pos for C : ${}^2C_2$

$\therefore \quad 7 \times 6 \times 10 \times 1 = 420$

M4: $\dfrac{7!}{\underset{S}{3!} \times \underset{C}{2!} \times \underset{U}{1!} \times \underset{E}{1!}} = 420.$

~~Distributing~~

Ex. Justify: $K \, {}^nC_K = n \, {}^{n-1}C_{K-1}$

LHS: First select K people from n people, then select 1 leader from K people.

$${}^nC_K \times {}^KC_1 = K \, {}^nC_K.$$

RHS: First select 1 person as leader from n people, then select K-1 people from remaining (n-1) people.

$${}^nC_1 \times {}^{n-1}C_{K-1}$$

$$\Rightarrow n \, {}^{n-1}C_{K-1}$$

Ex. In how many ways MAMMAL can be permuted?

3 spots for M $\Rightarrow {}^6C_3$

then 2 spots for A $\Rightarrow {}^3C_2$

then 1 spot for L $\Rightarrow {}^1C_1$

$\Rightarrow 20 \times 3 \times 1$

$\Rightarrow 60.$

Ex. In how many ways letters of "PERMUTATIONS" be arranged if there are always 4 letters b/w P & S.

M1:

- (P, S) can be arranged in (manual counting) $7 \times 2! = 14$
- For remaining 10 letters $= \frac{10!}{2!}$

M2: THIS IS WRONG.

- Choose two pos. for P & S: $2!$

_ _ _ _ _ _ _ P _ _ _ _ S _ _ _ _ _ _ _

_ _ _ _ _ _ _ S _ _ _ _ P _ _ _ _ _ _ _

- Now, the 4 cells b/w P & S will always be taken.
- From rest 12, we've to acquire 6: ${}^{12}C_6$.
- $(6+4) = 10$ pos., arrange 10 letters: $\frac{10!}{2!}$

Wrong in step 3. ${}^{12}C_6$ would include acquisitions such as: III III III _ _ _ P _ _ _ _ S _ _ III IIII _ III

Such acquisitions are wrong, with space b/w letters

Ex. No. of perm. A, B, C, D, E, F such that ~~they~~ A, B stand together.

$\boxed{AB}$ C D E F

$5! \times 2!$

↳ (internal perm. of A, B)

Ex. No. of perm. of "WONDERING" with all 3 vowels consecutively appearing.

$\frac{7! \times 3!}{2!}$ → vowel perm. → (2 Ns)

Ex. Arrange A, B, C, D, E, F such that A & B are never together.

M1: _ C _ D _ E _ F _

$(4!) \times ({}^5C_2 \times 2!) = 480.$

M2: $(6!) - (5! \times 2!)$

↓ (All perms.) ↓ (A, B together)

Ex. "WONDERING" with exactly 2 consecutive vowels.

_ W _ N _ D _ R _ N _ G _

$6!/2! \times {}^3C_2 \times {}^7P_2 \times 2!$

rest letters — Select 2 vowels to be together — (2 vowel) (1 vowel) perm. in 7 places — 2 vowel perm.

Ex. Perm. of ABCDEF, such that ABC is subseq.

M1: $6!/3!$ (Division Rule)

M2: • Select 3 out of 6 pos for A, B, C. = ${}^6C_3 \times 1$

• Permute rest chars : $3!$

$\therefore {}^6C_3 \times 3!$

Ex. Perm. of ABCDEFG, such that AB & CD are :

① subseq.

$$\underset{AB}{{}^7C_2} \times \underset{CD}{{}^5C_2} \times \underset{rest}{3!}$$

② Substr

① $\boxed{AB}\ \boxed{CD}\ E\ F\ G$

$5!$

② $_\ E\ _\ F\ _\ G\ _$

$3! \times {}^4C_2 \times 2!$ (when $\boxed{AB}$, $\boxed{CD}$ not together)

$+\ 3! \times {}^4C_1 \times 2!$ (when $\boxed{ABCD}$ together)

Ex. No. of perm. of [5, 2, 1, 8, 9, 7] such that:

① 5 must come first

② 8 must come before 9 & 7.

③ 2 must come before 1.

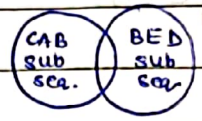
5 _ _ _ _ _

$$\underset{\text{5 at 1st place}}{1} \times \underset{\text{Choose 2 place for 2, 1.}}{({}^5C_2 \times 1)} \times (1 \times 2!)$$

→ 8 will always be at 1st pos out of rem. 3 pos.

9 & 7 can permute.

Ex. How many ~~subseq~~ perm. of ABCDEFGH contains the subseq. CAB or BED.



Not mutually exclusive.

Ex. C F A G B H E D

$\therefore$ Subseq. CAB + Subseq. BED − Subseq. CABED.

$$\Rightarrow \frac{8!}{3!} + \frac{8!}{3!} - \frac{8!}{5!}$$

* Distribution of Objects in Boxes

① Diff. Boxes, Diff. Objects

Ex. Obj: a, b, c, d, e

Box: B_1, B_2, B_3

Each obj has 3 choices.

$\therefore 3 \times 3 \times 3 \times 3 \times 3$

$\Rightarrow 3^5$

Ex. How many ways to distribute hands of 5 cards to each of 4 players, from deck of 52 cards.

$$^{52}C_5 \times {}^{47}C_5 \times {}^{42}C_5 \times {}^{37}C_5$$

$$\Rightarrow \frac{52!}{5! \times 5! \times 5! \times 5! \times 32!}$$

Note: Most Common Mistake

Say, Obj: x, y, z, a, b, c

Box: B_1, B_2, B_3

& have to give 2 obj to each box.

- Select one box & give 2 obj: $^3C_1 \times {}^6C_2$
- Then select another box & give 2 obj: $^2C_1 \times {}^4C_2$
- for last box: $^1C_1 \times {}^2C_2$

This is wrong. We are overcounting.

Say,

B_1	B_2	B_3
xy	za	bc

B_2	B_1	B_3
za	xy	bc

⋮

B_3	B_2	B_1
bc	za	xy

• 3! times we've counted same dist.

∴ Selecting the boxes was a mistake.
Since the boxes are diff., there already is an order, we don't have to choose boxes.

• Select any 2 obj. for B_1 : ${}^6C_2$
then select 2 obj. for B_2 : ${}^4C_2$
then for B_3 : ${}^2C_2$
} follows B_1, B_2, B_3 order.

• If you have n distinct objects & K distinct boxes & have a distribution such that B_1 has n_1 objects, B_2 has n_2, ... B_K has n_K & $n_1 + n_2 + \dots + n_K = n$, then

No. of ways to make the distribution are

$$\Rightarrow {}^nC_{n_1} \times {}^{n-n_1}C_{n_2} \times \dots {}^{n-n_1-n_2-\dots-n_{K-1}}C_{n_K}$$

Ex. 8 objects, 4 people. Everyone should get at least 2 obj.

$${}^8C_2 \times {}^6C_2 \times {}^4C_2 \times {}^2C_2$$

Ex. 8 objects & 3 people. Everyone should get at least 2 objects.

- After distributing 2 objects to each, the remaining 2 objs can be distributed in two ways:

(1)	2	2+1	2+1
(2)	2	2+2	2

C_1: ${}^3C_1 \times {}^8C_2 \times {}^6C_3 \times {}^3C_3$

Choose person who gets 2 obj

C_2: ${}^3C_1 \times {}^8C_4 \times {}^4C_2 \times {}^4C_2$

Choose person who gets 4 objs.

② Diff. Boxes Identical Objects

Analogy: Distributing candy to children.

- Star Bar problem

We have r * & n bars (|), then the no. of permutations:

$$\frac{(n+r)!}{r!\,n!} = {}^{n+r}C_r$$

DBIO == Star Bar problem

Ex. 5 identical has to be distributed among 3 children.

For the children only the number matters. They don't differentiate b/w the candies.

					Child 1 2 3
| *	| *	*	*	*	(0, 1, 4)
*	* |	*	* |	*	(2, 2, 1)
*	*	* |	* |	*	(3, 1, 1)

So to distribute b/w 3 children we need 2 bars.

$$\therefore \#\text{distributions} = \frac{(5+2)!}{5! \times 2!}$$

Ex. 4 chocolates. 3 children.

$$\#\text{dist.} = \frac{(4+2)!}{4!\,2!}$$

A	B	C	
0	0	4	||****
0	1	3	|*|***
0	2	2	|**|**
1	1	2	*|*|**
⋮			⋮

Note: To distribute among n boxes, (n-1) bars are needed.

Ex. 30 identical objects. 3 distinct boxes. At least 5 to everyone.

Since all chocolates are same give 5 to each child upfront.
Remaining: 15 chocolates

$$\therefore \frac{(15+2)!}{15! \times 2!}$$

Ex. In how many ways can a deck of 52 cards be distributed among 4 people equally, such that the oldest gets queen of spade.

- Just assign 4 to oldest upfront.
- Distribute remaining 51.

$$^{51}C_{12} \times {}^{39}C_{13} \times {}^{26}C_{13} \times {}^{13}C_{13}$$

Ex. How many ways to distribute 15 distinct objects in 5 distinguishable boxes such that the boxes get 2, 2, 3, 7, 1 objects.
Since it isn't specified which specific box should get how many, multiple configs. are possible.

B_1	B_2	B_3	B_4	B_5
2	2	1	3	7
2	1	3	2	7
3	1	2	7	2
		⋮		

$5!/2!$ ← (2 is repeting)

Total ~~of~~ configs are possible.

For each config, the objects can be distributed as:

$$^{15}C_2 \times {}^{13}C_2 \times {}^{11}C_1 \times {}^{10}C_7 \times {}^{3}C_3$$

$\therefore$ final answer $= \left({}^{15}C_2 \times {}^{13}C_2 \times {}^{11}C_1 \times {}^{10}C_7 \times {}^{3}C_3 \right) \times \frac{5!}{2!}$

Ex. Same ques: 2, 2, 3, 4, 4.

No. of configs for boxes $= \frac{5!}{2!\,2!}$

No. Distribution of objects $= {}^{15}C_2 \times {}^{13}C_2 \times {}^{11}C_3 \times {}^{8}C_4 \times {}^{4}C_4$.

Ex. How many ways can n objects can be put in r boxes, such that total no. of balls in each box can be anything.

first obj has r boxes
second obj has r boxes
⋮
n^{th} obj has r boxes

$\therefore r^n$

- Combinition w/ Repetition

Ex. You want 4 fruits. You have Apples, Mangoes & Oranges.

So all 4 can be Apples.
all 4 can be Oranges.
2 can be A, 1 M, 1 O.
1 A, 2M, 1 O
2A, 2M, 0 or.

Again, this can be thought of as Star-Bar problem.

Objects → Stars → Every fruit selection
Boxes → Bar → 2 bars since 3 types.

A	M	O	
0	2	2	|**|**
1	1	2	*|*|**
2	0	2	**||**
3	1	0	***|*|

⋮

∴ CwR ≡ S-B Problem ≡ DBIO

Ex. Cookie shop has 4 diff. kind of cookies & you want to choose 6 cookies. No. of ways?

****** | | |

$$\frac{(6+3)!}{6!\,3!} = \frac{9!}{6!\,3!} = {}^9C_3.$$

* : Choice of cookie = 6.

| : Division among 4 kinds = 3.

Ex. $10 = x+y+z$. No. of non-neg. int solution.

* : 10 1s.

| : Divison among x, y & z = 2.

1 1 1 1 1 1 1 1 1 1 | |

$$\therefore \frac{(10+2)!}{10!\,2!} = {}^{12}C_2.$$

Ex. $10 = x+y+z$, $x \geq 2$, $y \geq 1$, $z > 1$. No. of int solutions.

Min. $x = 2$

Min. $y = 1$

Min. $z = 2$.

$\therefore$ * : $10-2-1-2 = 5$

| : 2

$$\therefore \frac{(5+2)!}{5!\,2!} = {}^7C_2$$

Ex. $x+y+z \leq 8$. No. of non-neg. int sol.

we have 8 1s. But we may distribute less than 8 to x, y & z. Remaining of 8 would go to trash.

$$x+y+z+T=8$$

$\therefore$ $* : 8$

$| : 3$

So, $\frac{(8+3)}{8!\,3!}$

Ex. $x_1+x_2+x_3+x_4+x_5=20$. No. of pos int sols, such that $x_3 \leq 3$.

$\therefore$ $x_i > 0$ & $x_3 \leq 3$.

- Give the min. 1 chocolate to all children.
- x_3 can have value 1, 2, 3, but since we've given 1 chocolate already so, 0, 1, 2 can be given more.

C1: $x_3 = 1$ $\quad * = 15$, $| = 3$ $\{\because x_3$ is gone$\}$

$^{18}C_3$

C2: $x_3 = 2$ $\quad * = 20-1-1-2-1-1 = 14$ $\quad | = 3$

$^{17}C_3$

C3: $x_3 = 3$ $\quad * = 20-1-1-3-1-1 = 13$ $\quad | = 3$

$^{16}C_3$

M2: Complement

Total Cases: $x_i \geq 1$

$\Rightarrow \frac{(15+4)!}{15!\,4!} = {}^{19}C_4$

Neg. Cases: $x_i \geq 1$ & $x_3 > 3$

$\Rightarrow \frac{(12+4)!}{12!\,4!} = {}^{16}C_4$

$\therefore {}^{19}C_4 - {}^{16}C_4$.

Ex. How many 3 digit nums sum to 9.

for a no. to be 3 digit 100s place must have at least 1.

$\geq 1 \quad \geq 0 \quad \geq 0$

$x_{100} + x_{10} + x_0 = 9$.

$\Rightarrow x'_{100} + x_{10} + x_0 = 8$.

$\therefore \frac{(8+2)!}{8!\,2!} = {}^{10}C_2$

Ex. Multiset: A set with repitition of values allowed.

Ex. $\langle 1,2 \rangle \neq \langle 1,1,2 \rangle$ & $\langle 1,1,2 \rangle = \langle 1,2,1 \rangle$.

How many multisets can be created of size 3 from elements 1 & 2.

*: No. of seats in multiset = 3

|: two diff elements $\Rightarrow 1$

$\langle 1,1,1 \rangle$

$\langle 1,2,1 \rangle$

$\langle 1,2,2 \rangle$

$\langle 2,2,2 \rangle$

$\therefore \frac{(3+1)!}{3!\,1!} = {}^4C_1 = 4$

Ex. Given a set S of size n:

① No. of lists (order matters) of size k.

(a) No repeat

$n \times (n-1) \times (n-2) \times \dots (n-k+1)$

$\Rightarrow {}^{n}P_{k}$

(b) With repeat

$n \times n \times n \times \dots \times n$

$\Rightarrow n^{k}$.

② No. of subsets (order doesn't matter) of size k.

(a) No repeat

${}^{n}C_{k}$

(b) With repeat

$*$: all k seats in subset

$|$: since n types of values, so $(n-1)$

$\therefore \dfrac{(n+k-1)!}{(k)!(n-1)!} = {}^{n+k-1}C_{k}$

• Non-decreasing int sequence

Ex. From set $\{1,2,3\}$ create 4 digit nums such that the digits are non-decreasing from left to right.

Ex: 3333

3332

3322

⋮

- If we select:

①	②	③		Possible 4 digit nums
2	2	0	⇒	2 2 1 1
2	1	1	⇒	3 2 1 1
1	2	1		3 2 2 1
0	2	2	⇒	3 3 2 2

for every selection we get exactly 1 ~~num~~ 4 digit num.

*: 4 digits
|: 3 type of nums ⇒ 2

$$\therefore \frac{(4+2)!}{2!\,4!} = {}^6C_4 = 15$$

Ex. How many seq. of len r from set $\{1, 2, \ldots n\}$ are possible such that subsequent digits are non-decr.

*: len of sequence $= r$
|: $(n-1)$ ∵ (n) value types are there.

$$\frac{(r+n-1)!}{(r)!\,(n-1)!} = {}^{n+r-1}C_r$$

Ex. How many n digit natural nos, such that digits are:
① non-decreasing.

Natural nos: ≥ 1 $\quad$ n digit: i.e. 10^{n-1} th place (Leftmost dig) (LMD) can't be 0.

Since at min LMD is 1 & all other digits have to be in non-decr. order ∴ Options: $\{1, 2, \ldots 9\}$.

∴ *: n
|: $9-1 = 8$

$$\Rightarrow \frac{(n+8)!}{n!\,8!} = {}^{(n+8)}C_8$$

② non-increasing.

Options: $\{0, 1, 2, \ldots 9\}$

Total: All n len seq.

$*$: n

$|$: $10 - 1 = 9$.

$\Rightarrow {}^{n+9}C_9$

Neg: All n len seq. starting with 0s

Only 1 such string:

000...0 {n times}

{since non-incr.}

$\therefore$

$\Rightarrow {}^{n+9}C_9 - 1$

Ex. All 5 digit nums such that digits are:

① Strictly increasing.

~~Total: ${}^{10}C_5$~~

~~Neg: 0 at Leftmost pos ${}^{9}C_4$~~

${}^{9}C_5$

~~$\therefore {}^{10}C_5 - {}^{9}C_4$~~

② Strictly decreasing

${}^{10}C_5$

Sequence : Ordered, repetition allowed by default.

Ex. How many 6 len sequence from {0, 1, 2, 3} such that:

① Non-decreasing

*: 6 | : 4-1 = 3

$^{6+3}C_3 = {}^{9}C_3$ {Sequence not int}

② Non-increasing

*: 6 | : 4-1 = 3

$^{9}C_3$

Ex. Same as above but integer.

① Non-decreasing.

0 can't be chosen. Any choice with 0 would make the int. less than 6 digit.

Ex. 108, 218, 128 & 238

would make 011233

which isn't 6 digit.

∴ *: 6 | : 3-1 = 2 {only 1, 2 & 3}

$^{8}C_2$

② Non-increasing

All combo are valid except 000000.

*: 6 | : 4-1 = 3

$^{9}C_3 - 1$

• Integer Composition

A composition of an integer n is a way of writing n as the sum of a sequence of positive ~~numbers~~ integers. Order matters.

Ex. $n = 4$

(4)	(1, 1, 2)
(1, 3)	(1, 2, 1)
(3, 1)	(2, 1, 1)
(2, 2)	(1, 1, 1, 1)

$\therefore$ 8 compositions.

Ex. Set of composition of 6 in 3 parts.

3 parts $\Rightarrow$ $a+b+c = 6$, $a, b, c \geq 1$

$\therefore$ $*$: $6-3$ $\Rightarrow 3$ | : $3-1=2$

$\Rightarrow$ ${}^{5}C_{2}$.

Ex. No. of composition of n in K parts.

$x_1 + x_2 + \dots x_K = n$, $x_1, x_2, \dots x_K \geq 1$.

$*$: $n-K$ | : $K-1$

$\Rightarrow$ ${}^{n-K+K-1}C_{K-1}$

$\Rightarrow$ ${}^{n-1}C_{K-1}$

Note: Whenever you see structure like:

$$B_1 + B_2 + \dots B_K = n, \quad B_1, B_2, B_3 \dots \geq S$$

think of TODB template.

Ex. Given a +ve int. n, how many compositions are there in total?

w.k.t for K part comp:

$${}^{n-1}C_{K-1}$$

& $K \in [1, n]$

$$\boxed{\therefore \sum_{K=1}^{n} {}^{n-1}C_{K-1}}$$

Say, $K - 1 = x$

$K = 1 \Rightarrow x = 0$

$K = n \Rightarrow x = n - 1$

$$\Rightarrow \sum_{x=0}^{n-1} {}^{n-1}C_x = 2^{n-1}$$

(3) Distinct Object Identical Boxes

Analogy: • Friends trip to haunted hotel with same rooms

• No. of equivalence classes of a set

Ex. 5 friend. Two groups, size 2 & size 3.

$$^5C_3 \times {}^2C_2$$

Ex. 4 friends. Two groups, size 2 each.

$$^4C_2 \times {}^2C_2$$

WRONG.

Say we have a, b, c, d.

We choose ab, then cd another group.

choose ad, then bc —— " ——

choose cd, then ab —— " ——

So, (ab, cd) is counted twice.

$$\therefore \frac{^4C_2}{2!}$$

Ex. 6 friends. 3 groups, 2 each.

$$\frac{^6C_2 \times {}^4C_2 \times {}^2C_2}{3!}$$

(ab, cd, ef)
(cd, ab, ef)
(ab, ef, cd)
(cd, ef, ab)
(ef, ab, cd)
(ef, cd, ab)

(ab, cd, ef)

Ex. 8 friends. Three groups, 2, 2 & 4.

$$^8C_4 \times \frac{^4C_2 \times {}^2C_2}{2!}$$

Ex. 6 friends. 4 groups of size 1, 1, 2, 2.

$$\frac{^6C_1 \times {}^5C_1 \times {}^4C_2 \times {}^2C_2}{2! \times 2!}$$

(a, b, cd, ef)
(b, a, cd, ef)
(a, b, ef, cd)
(b, a, ef, cd)

(a, b, cd, ef)

Think first that order matters, then invalidate the order b/w similar sized groups, since the groups (boxes) are indistinguishable.

Ex. 16 friends. 6 groups of size, 2, 2, 2, 3, 3, 4.

$$\frac{^{16}C_2 \times {}^{14}C_2 \times {}^{12}C_2}{3!} \times \frac{^{10}C_3 \times {}^7C_3}{2!} \times {}^4C_4$$

Ex. 3 friends. How many partitions?
Say, a, b & c.

(abc)
(a, bc)
(ab, c)
(ac, b)
(a, b, c)

Ex. 4 friends. Count partitions.

1 Part: (4) $\Rightarrow$ 1

2 Parts: (1, 3) $\Rightarrow$ ${}^4C_3 = 4$

(2, 2) $\Rightarrow$ $\frac{{}^4C_2}{2!} = 3$

3 Parts: (1, 1, 2) $\Rightarrow$ $\frac{{}^4C_2 \times {}^2C_1 \times {}^1C_1}{2!} = 6$

4 Parts: (1, 1, 1, 1) $\Rightarrow$ $\frac{{}^4C_1 \times {}^3C_1 \times {}^2C_1 \times {}^1C_1}{4!} = 1$.

$\therefore$ # partitions = 15.

OR

1				$B_1 = 1$
1	2			$B_2 = 2$
2	3	5		$B_3 = 5$
5	7	10	15	$B_4 = 15$

$\therefore$ # partitions of 4 sized set = 15.

Ex. 4 employees. 3 indistinguishable offices, each office can contain any no. of employees. All office should have employee.

3 offices $\Rightarrow$ (1, 1, 2)

$\Rightarrow \frac{{}^4C_2 \times {}^2C_1 \times {}^1C_1}{2!} = 6.$

Ex. How many partitions of size 5 set.

1

1 2

2 3 5

5 7 10 15

15 20 27 37 52

$\therefore$ 52.

(4) Identical Objects, Identical Boxes

- Integer Partition: Same as Integer Composition but order doesn't matter.

Ex. $n = 4$

Composition		Partition
(4)	$\longrightarrow$	(4)
(1,3) (3,1)	$\longrightarrow$	(1,3)
(2,2)	$\longrightarrow$	(2,2)
(1,1,2) (1,2,1) (2,1,1)	$\longrightarrow$	(1,1,2)
(1,1,1,1)	$\longrightarrow$	(1,1,1,1)

Ex. Objects: a, a, a
Boxes: B, B.

1 part: (aaa,)
2 part: (aa, a)

$\therefore$ 2 ways

Ex. 6 copies of same book & identical 4 boxes.

1 part: (aaaaaa, , , ,) $\Rightarrow 1$
2 part: (a, aaaaa, ,)
(aa, aaaa, ,)
(aaa, aaa, ,) $\Rightarrow 3$
3 part: (a, a, aaaa), (a, aa, aaa)
(aa, aa, aa,) $\Rightarrow 3$
4 part: (1, 1, 1, 3), (1, 1, 2, 2) $\Rightarrow 2$
$\therefore$ 9 ways

Ex. 7 identical balls, 4 identical bins.

1 part: (7, 0, 0, 0)
2 part: (6, 1, 0, 0), (5, 2, 0, 0), (4, 3, 0, 0)
3 part: (5, 1, 1, 0), (4, 2, 1, 0), (3, 2, 2, 0), (3, 3, 1, 0)
4 part: (4, 1, 1, 1), (3, 2, 1, 1), (2, 2, 2, 1)

∴ 11 ways.

Note: No formula for TOTB. Count systematically.

* Inclusion Exclusion Principle

A tool to find union of finite no. of sets.
It gives the count of unique elements in union of finite no. of sets.

- For 2 sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- For 3 sets

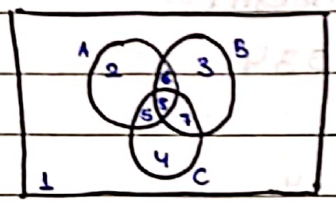
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cup B \cup C|$$

- For n sets

$$|A_1 \cup A_2 \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i \neq j} |A_i \cap A_j| + \sum_{i \neq j \neq k} |A_i \cap A_j \cap A_k|$$

$$- \sum_{i \neq j \neq k \neq l} |A_i \cap A_j \cap A_k \cap A_l| \ldots$$

Ex.



$A \cup B \Rightarrow 2, 3, 6, 8, 5, 7$

Only in A OR Only in B $\Rightarrow 2, 3$.

$A - C \Rightarrow 2, 6$

$A \cap \bar{B} \cap \bar{C} \Rightarrow 2$

- How big / small is union of finitely many finite sets :

$$\max(|A|, |B|) \leq |A \cup B| \leq |A| + |B|$$

one is subset of other

$A \cap B = \emptyset$

Ex. How many integers b/w 1 to 40 multiple of 6.

$$\left\lfloor \frac{40}{6} \right\rfloor = 6.$$

Ex. How many ints b/w 530 to 1033 mult. of 6.

least int ≥ 530 div. by 6 $\Rightarrow$ 534

$$\left\lfloor \frac{1033 - 534 + 1}{6} \right\rfloor = \frac{250}{3} = 83$$

This is wrong. It doesn't count the first val: 534.

Say, $a = 530$ $b = 540$

least val $\geq a$ div. by 6 $\Rightarrow$ 534.

$534 \xrightarrow{+6} 540$

$\therefore$ 2 nos in range

but,

$$\left\lfloor \frac{540 - 534 + 1}{6} \right\rfloor = \left\lfloor \frac{7}{6} \right\rfloor = 1$$

Correct Approach:

$a = 530 \quad b = 1033$

Greatest val $< a$ div by 6 $\Rightarrow 528$

$$\Rightarrow \left\lfloor \frac{1033 - 528 + 1}{6} \right\rfloor = 84$$

$a = 534 \quad b = 540$

Gr. val $< a \Rightarrow 528$

$$\Rightarrow \left\lfloor \frac{540 - 528 + 1}{6} \right\rfloor = 2.$$

- No. of ints b/w a & b div. by d

$$\Rightarrow \left\lfloor \frac{b}{d} \right\rfloor - \left\lfloor \frac{a-1}{d} \right\rfloor$$

Ex. How many nos. in range 100 & 200 div by 2 or 3.

Div by only 2 $\Rightarrow \left\lfloor \frac{200}{2} \right\rfloor - \left\lfloor \frac{99}{2} \right\rfloor = 51$

Div by only 3 $\Rightarrow \left\lfloor \frac{200}{3} \right\rfloor - \left\lfloor \frac{99}{3} \right\rfloor = 33$

Div by both $\Rightarrow \left\lfloor \frac{200}{6} \right\rfloor - \left\lfloor \frac{99}{6} \right\rfloor = 17$

$\therefore 51 + 33 - 17 = 67.$

Ex. 1807 students. 453 students take CS. 567 students take Maths. 299 take both. How many students not take either CS or Maths.

$$1807 - (567 + 453 - 299) = 1086.$$

Ex. How many strings over {A, B, C} of len 6:

① Start with C or start with B.

C _ _ _ _ _ B _ _ _ _ _

$3^2 + 3^2 - 0 = 18$

② St. with C or End with B.

C _ _ _ _ B C B

$3^2 + 3^2 - 3^1 = 15$

③ St. with C & End with B

C _ _ _ _ B

3

Ex. How many 5 card hands from 52 deck have exactly one King or exactly 1 Queen (or both)?

1 K, 1 Q, 1 K, 1 Q

$\underbrace{{}^4C_1}_{K} \times \underbrace{{}^{48}C_4}_{\text{No K}} + \underbrace{{}^4C_1}_{K} \times \underbrace{{}^{48}C_4}_{\text{No Q}} - \underbrace{{}^4C_1}_{K} \times \underbrace{{}^4C_1}_{Q} \times \underbrace{{}^{44}C_3}_{\text{No K, No Q}}$

Ex. Line up of 7. Mother, Father, 3 sons, 2 daughter. How many lineup such that M is next to atleast one of her son.

_ _ _ _ _ _ _

M1: Fixed position of mother.

C1: M at 1st pos. ⇒ 1
2nd pos ⇒ $^3C_1$
5 pos ⇒ 5!

C2: M at last pos ⇒ 1
2nd last pos ⇒ $^3C_1$
5 pos ⇒ 5!

C3: M at any of the 5 middle pos ⇒ $^5C_1$
M's left ⇒ $^3C_1$
M's right ⇒ $^2C_1$
Rest pos ⇒ 4!

C4: Same as C3 but only 1 side son ⇒ $^5C_1$
M's one side ⇒ $^3C_1$
M's other side ⇒ $^3C_1$
Rest ⇒ 4!

All cases are exclusive & exhaustive.

$$2 \times 3 \times 5! + 5 \times 3 \times 2 \times 4! + 2 \times 5 \times 3 \times 3 \times 4!$$

$$\Rightarrow 3600$$

M2: IEP

(M, S_1) or (M, S_2) or (M, S_3)

$$\Rightarrow (M,S_1) + (M,S_2) + (M,S_3) - (M,S_1,S_2) - (M,S_1,S_3) - (M,S_2,S_3) + \underbrace{(M,S_1,S_2,S_3)}_{0}$$

$S_1\ M\ S_2$
$S_2\ M\ S_1$

$$\Rightarrow 3 \times (6! \times 2!) - 3 \times (7! \times 2)$$

$(M,S_1), S_2, S_3, F, D_1, D_2$

$(M,S_1,S_2), S_3, F, D_1, D_2$

Ex. How many strings of len 6 over A,B,C,D,E can have 4 consecutive As?

Pos 1-4 : 5^2 Pos 1-5 : 5

Pos 2-5 : 5^2 Pos 2-6 : 5

Pos 3-6 : 5^2 Pos 1-6 : 1

$\Rightarrow 3 \times 5^2 - 5 - 5 - 1 + 1$

$\Rightarrow 75 - 10$

$\Rightarrow 65$

Ex. How many 5 hand have atleast one ace or at least one queen (Inclusive OR)

At least 1 A $\cup$ At least 1 Q = No Ace $\cap$ No Queen

$\Rightarrow$ Total − (No Ace $\cap$ No Queen)

$\Rightarrow {}^{52}C_5 - ({}^{44}C_5)$

Ex. How many +ve ints less than 1000 are :

$a = 1 \quad b = 999$

① Div By 7

$\left\lfloor \frac{999}{7} \right\rfloor = 142$

(b) Are div by 7 but not 11.

Div by 7 = 142

Div by both = 12

$\therefore$ 130

(c) Div by both 7 & 11

$\Rightarrow$ 12.

(d) Div by 7 or 11.

Div by 11 = $\left\lfloor \frac{999}{11} \right\rfloor = 90$

$\therefore$ $142 + 90 - 12 = 220$.

(e) Either by 7 or by 11, not both.

$(7 \cup 11) - (7 \cap 11)$

$\Rightarrow$ $220 - 12$

$\Rightarrow$ 208

(f) Neither by 7, nor 11.

$999 - |7 \cup 11|$

$\Rightarrow$ $999 - 220 = 779$.

(g) have distinct digits

$1 \to 9 \Rightarrow 9$

$10 \to 99 \Rightarrow 9 \times 9 = 81$

$100 \to 999 \Rightarrow 9 \times 9 \times 8 = 648$

$\Rightarrow$ 738.

(h) Have distinct digits & are even.

$1 \to 9 : 4$

$10 \to 99 : 9 \times 1 + 8 \times 4 = 41$

$100 \to 999 : 9 \times 8 \times 1 + 8 \times 8 \times 4 = 328$

$\Rightarrow 373$

Ex. How many sol. does the foll. have:

$$x_1 + x_2 + x_3 = 11.$$

$x_1 \leq 3, \quad x_2 \leq 4, \quad x_3 \leq 6.$

Total $-$ ($x_1 > 3$ or $x_2 > 5$ or $x_3 > 6$)

$A \Rightarrow x_1 > 3$ & rest are unbound.

$$x_1' + x_2 + x_3 = 7$$

$\therefore \; {}^9C_2$

$B \Rightarrow x_2 > 4$

$$x_1 + x_2' + x_3 = 6$$

$\therefore \; {}^8C_2$

$C \Rightarrow x_3 > 6$

$$x_1 + x_2 + x_3' = 4$$

$\therefore \; {}^6C_2$

$A \cap B \Rightarrow x_1 > 3$ & $x_2 > 4 \Rightarrow {}^4C_2$

$A \cap C \Rightarrow x_1 > 3$ & $x_3 > 6 \Rightarrow 1$

$B \cap C \Rightarrow x_2 > 4$ & $x_3 > 6 \Rightarrow 0$

$A \cap B \cap C \Rightarrow x_1 > 3$ & $x_2 > 4$ & $x_3 > 6 \Rightarrow 0$

$$\therefore \; {}^{13}C_2 - ({}^9C_2 + {}^8C_2 + {}^6C_2 - {}^4C_2 - 1 - 0 + 0)$$

① Derangement

A derangement of $1, 2, \ldots n$ ~~elements~~ elements can be thought of as arrangement of n ~~envelopes~~ letters in n envelops, where i^{th} letter belongs to i^{th} envelop, then any permutation of the letters such that no letter is put in its respective envelope is called as a derangement.

Ex. Say L_1, L_2 & L_3.

E_1	E_2	E_3		
L_1	L_2	L_3	✗	{all letter in right env.}
L_1	L_3	L_2	✗	
L_2	L_1	L_3	✗	
L_2	L_3	L_1	✓	$\therefore$ for 3, $D(3) = 2$.
L_3	L_1	L_2	✓	
L_3	L_2	L_1	✗	

Total no. of permutations $= n!$ {we counted the perm. with right assignment too.}

All perm. with exactly one correct assign. $= {}^nC_1 \times (n-1)!$

{we've overcounted cases with more than 2 right assign.}

All perm. with exactly two correct assign. $= {}^nC_2 \times (n-2)!$

$\vdots$

All perm. with all n correct assign. $= {}^nC_n \times (n-n)! = {}^nC_n$

Say A_i = i^{th} letter in correct envelop.

So, we want

$$\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_n = \overline{A_1 \cup A_2 \cup A_3 \ldots \cup A_n}$$

$$\Rightarrow n! - ({}^nC_1 \times (n-1)! - {}^nC_2 \times (n-2)! + {}^nC_3 \times (n-3)! \ldots)$$

$$\Rightarrow n! - \sum_{r=1}^{n} (-1)^r \, {}^nC_r \, (n-r)!$$

$$\therefore D(n) = n! - \sum_{r=1}^{n} (-1)^r \, {}^nC_r \, (n-r)!$$

Ex. D(2), D(3), D(4), D(5)

$D(1) = 1! - 1 = 0$

$D(2) = 2! - {}^2C_1 \times 1! + {}^2C_2 \times 0! = 1$

$D(3) = 3! - {}^3C_1 \times 2! + {}^3C_2 \times 1! - {}^3C_3 \times 0! = 2$

$D(4) = 4! - {}^4C_1 \times 3! + {}^4C_2 \times 2! - {}^4C_3 \times 1! + {}^4C_4 \times 0!$

$\Rightarrow \cancel{24} - \cancel{24} + 12 - 4 + 1$

$\Rightarrow 9.$

$D(5) = 5! - {}^5C_1 \times 4! + {}^5C_2 \times 3! - {}^5C_3 \times 2! + {}^5C_4 \times 1 - {}^5C_5 \times 0!$

$\Rightarrow 5! - 5! + 60 - 20 + 5 - 1$

$\Rightarrow 44.$

$D(6) = 6! - {}^6C_1 \times 5! + {}^6C_2 \times 4! - {}^6C_3 \times 3! + {}^6C_4 \times 2! - {}^6C_5 \times 1! + {}^6C_6 \times 0!$

$\Rightarrow 360 - 120 + 30 - 6 + 1$

$\Rightarrow 265$

Note: • D(0) is taken as 1 by default as convention.

• Recursive formula for D(n)

$$D(n) = \begin{cases} (n-1) \times (D(n-1) + D(n-2)), & n > 1 \\ 1, & n = 0 \\ 0, & n = 1 \end{cases}$$

Ex. Find no. of derangements for:

(1) aab 0 {all arrangements have at least 1 correct assignment}

(2) aabc

_ _ a a $\Rightarrow 2! = 2.$

(3) LIMIT

I I I _ _

- ~~Place~~ Choose 2 of 3 places for Is: $^3C_2$
- 'T' can only be in one of 2 places: $^2C_1$
- L&M can have 2! perm. : 2!

$\therefore {}^3C_2 \times {}^2C_1 \times 2!$

$\Rightarrow 3 \times 2 \times 2$

$\Rightarrow 12.$

Ex. 10 people. $a_1, a_2, \ldots a_{10}$. No. of perm. with no person in home of their own.

D(10)

Ex. Same as above, but, at least 1 person not in their own home.

All perm except the one where everyone is in their own home.

$10! - 1$

Ex. Same as above, but, at least one person in their own home.

All perm − no one in own home.

$\Rightarrow 10! - D(10)$

② Euler Totient Function

- The Euler Totient or ϕ function, for $\forall n \in \mathbb{N}$, gives the count of co-prime numbers from 1 to $(n-1)$ with n.

- Prime no. n has exactly two divisors 1 & n.
- Two natural nos. n, m are co-prime iff their HCF is 1.

- If n is divisible by some composite number, then n is divisible by some prime number as well.
- If n is not divisible by any other prime no., then n is itself prime.
- If n is prime & $n | ab$ then $n | a$ or $n | b$.
- Every natural no. n can be written as product of primes. $n > 1$.

  Ex. $12 = 2 \times 2 \times 3$

  $18 = 3 \times 3 \times 2$

  $34 = 17 \times 2$

  $2014 = 2 \times 19 \times 53$

- How to check if a number n is prime or not:

If n is composite $\rightarrow \exists x \, (x > 1 \wedge x < n \wedge x | n)$

So, $xy = n \rightarrow$ either $x \leq \sqrt{n}$ or $y \leq \sqrt{n}$

$\therefore$ n is composite $\rightarrow \exists x \, (x > 1 \wedge x \leq \sqrt{n} \wedge x | n)$

$\therefore$ If there is no x in range $[2, \sqrt{n}]$ dividing n, then n is prime.

Ex. 2014

- $2 \mid 2014$ & $2014/2 = 1007$
- $19 \mid 1007$ & $1007/19 = 53$
- No prime no. $x \leq \sqrt{53}\ (7)$ divides 53.

  $\therefore$ 53 is prime.

$2014 = 2 \times 19 \times 53$.

Ex. GCD(14, 25)

$\Rightarrow$ GCD(25, 14)

$\Rightarrow$ GCD(14, 11)

$\Rightarrow$ GCD(11, 3)

$\Rightarrow$ GCD(3, 2)

$\Rightarrow$ GCD(2, 1)

$\Rightarrow$ GCD(1, 0)

$\therefore$ GCD(14, 25) = 1

Ex. GCD(2014, 361)

$\Rightarrow$ GCD(361, 209)

$\Rightarrow$ GCD(209, 152)

$\Rightarrow$ GCD(152, 57)

$\Rightarrow$ GCD(57, 38)

$\Rightarrow$ GCD(38, 19)

$\Rightarrow$ GCD(19, 0)

$\therefore$ GCD = 19.

- $$GCD(a, b) = \begin{cases} GCD(b, a\%b), & a \geq b \\ GCD(a, b\%a), & b > a \\ a, & b = 0 \\ b, & a = 0 \end{cases}$$

Ex. for 20, how many co-primes b/w 1 to 20?

$20 = 2^2 \times 5$

So, all no. b/w 1 & 20 which aren't divisible by 2 or 5

Total nos. − (Divisible by 2 or 5)

$\Rightarrow 20 - \left(\left\lfloor \frac{20}{2} \right\rfloor + \left\lfloor \frac{20}{5} \right\rfloor - \left\lfloor \frac{20}{10} \right\rfloor \right)$

$\Rightarrow 20 - 12$

$\Rightarrow 8.$

- Euler Totient func. gives this value only, i.e.

$$\phi(20) = 8.$$

Ex. Find $\phi(120)$

$120 = 2^3 \times 3 \times 5$

So, $\Rightarrow$ Not Div by 2 $\cap$ Not div by 3 $\cap$ Not div by 5

$\Rightarrow$ Total nos. $-$ (All no. div. by 2 or 3 or 5 b/w 1 to 120)

$$\Rightarrow 120 - \left(\left\lfloor \frac{120}{2} \right\rfloor + \left\lfloor \frac{120}{3} \right\rfloor + \left\lfloor \frac{120}{5} \right\rfloor - \left\lfloor \frac{120}{10} \right\rfloor - \left\lfloor \frac{120}{6} \right\rfloor - \left\lfloor \frac{120}{15} \right\rfloor + \left\lfloor \frac{120}{30} \right\rfloor \right)$$

$\Rightarrow 32.$

Euler Totient func. for $n > 1$ is defined as:

$n = n_1^a \times n_2^b \times \dots n_m^x$, $n_1, n_2, \dots$ are primes

then

$$\boxed{\phi(n) = n \times \left(1 - \frac{1}{n_1}\right) \times \left(1 - \frac{1}{n_2}\right) \dots \times \left(1 - \frac{1}{n_m}\right)}$$

where $\phi(n)$ is no. of coprimes of n b/w 1 & $n-1$.

Ex. $\phi(120)$

$$\Rightarrow 120 \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)$$

$\Rightarrow 32.$

- For a prime no. p

$$\phi(p) = p \times \left(1 - \frac{1}{p}\right) = p - 1$$

- For $m, n \in \mathbb{N}$ such that $GCD(m,n) = 1$ then they have no common prime divisors.

  i.e.

$$m = p_1^{n_1} \times p_2^{n_2} \times \dots$$

$$n = p_3^{n_3} \times p_4^{n_4} \times \dots$$

Say, $m = a^2 \times b^3$ & $n = c^4 d^5$

then, $\phi(mn) = mn \times \left(1 - \frac{1}{a}\right) \times \left(1 - \frac{1}{b}\right) \times \left(1 - \frac{1}{c}\right) \times \left(1 - \frac{1}{d}\right)$

$$\Rightarrow \phi(m) \times \phi(n)$$

- For prime no. p

$$\phi(p^k) = p^k \times \left(1 - \frac{1}{p}\right) = p^k \left(\frac{p-1}{p}\right)$$

$$\Rightarrow p^{k-1}(p-1)$$

③ Sieve of Erathosthenes

Gives the no. of primes that are less than equal to n.

Ex. How many primes ≤ 40.

Total − Composites − 1. → neither prime nor composite.

Any composite no. ≤ 40 should be divisible by some prime $\leq \lfloor \sqrt{40} \rfloor$ i.e. ≤ 6.

∴ All vals. div. by 2 or 3 or 5 b/w 1 & 40 except 2, 3 & 5 themselves.

$$\Rightarrow 40 - \left(\left\lfloor \frac{40}{2} \right\rfloor + \left\lfloor \frac{40}{3} \right\rfloor + \left\lfloor \frac{40}{5} \right\rfloor - \left\lfloor \frac{40}{6} \right\rfloor - \left\lfloor \frac{40}{10} \right\rfloor - \left\lfloor \frac{40}{15} \right\rfloor + \left\lfloor \frac{40}{30} \right\rfloor \right) - 1$$

$$\Rightarrow 40 - 30 - 1 = 9 + 3 = 12$$

$+(3) \rightsquigarrow \{2, 3, 5\}$

④ Number of Onto function

A func. $f: A \to B$ is surjective iff for every element of B. there is at least one preimage in A.

$|A| = m \qquad |B| = n$.

$\#\text{onto} = \text{Total}\ \#\text{funcs} - (\text{Not onto})$

$\Rightarrow \text{Total}\ \#\text{funcs} - (\bar{b}_1 \text{ or } \bar{b}_2 \text{ or } \bar{b}_3 \ldots)$

funcs where b_1 doesn't have preimg

$$|\bar{b}_1 \cup \bar{b}_2 \cup \bar{b}_3 \ldots| = |\bar{b}_1| + |\bar{b}_2| + \ldots - |\bar{b}_1 \cap \bar{b}_2| - |\bar{b}_2 \cap \bar{b}_3| - \ldots + |\bar{b}_1 \cap \bar{b}_2 \cap \bar{b}_3| + \ldots$$

Ex. $A = \{1, 2, 3, 4\} \qquad B = \{a, b\}$

$$|\bar{b}_1 \cup \bar{b}_2| = |\bar{a}| + |\bar{b}| - |\bar{a} \cap \bar{b}|$$

$$\Rightarrow 1^4 + 1^4 - 0^4$$

$$\Rightarrow 2.$$

$\therefore \#\text{onto} = 2^4 - 2 = 14$

Ex. $X = \{1,2,3,4\}$ $\quad Y = \{a, b, c\}$ $\quad f: X \to Y$

$$|\bar{a} \cup \bar{b} \cup \bar{c}| = |\bar{a}| + |\bar{b}| + |\bar{c}| - |\bar{a}\bar{b}| - |\bar{a}\bar{c}| - |\bar{b}\bar{c}| + |\bar{a}\bar{b}\bar{c}|$$

$$\Rightarrow 2^4 + 2^4 + 2^4 - 1^4 - 1^4 - 1^4 + 0^4$$

$$\Rightarrow 48 - 3$$

$$\Rightarrow 45$$

no such func where neither a, nor b, nor c is img for domain, since $\forall x$ in domain there must be an img in codomain for f to be a func.

$$\therefore \# \text{onto} = 3^4 - 45$$

$$= 81 - 45$$

$$\Rightarrow 36$$

Ex. How many ways to assign five diff. jobs to four diff. people if everyone is assigned at least one job.

M1:

We have to remove situations where E_1 doesn't get any job, or E_2 doesn't or E_3 doesn't or E_4 doesn't.

$$|\bar{E}_1 \cup \bar{E}_2 \cup \bar{E}_3 \cup \bar{E}_4| = 4 \times 3^5 - 6 \times 2^5 + 4 \times 1^5 - 0^5$$

$$\Rightarrow 784.$$

$$\# \text{req.} = 4^5 - 784 = 240.$$

M2:

E_1	E_2	E_3	E_4
2	1	1	1
1	2	1	1
1	1	2	1
1	1	1	2.

$$\therefore 4 \times ({}^5C_2 \times {}^3C_1 \times {}^2C_1 \times {}^1C_1)$$

$$\Rightarrow 240$$

M3: Choose 2 of 5 jobs to be in a bag: $\boxed{J_1 J_5}$ J_2 J_3 J_4

$$\Rightarrow {}^5C_2 \times 4!$$

$$\Rightarrow 240.$$

GENERATING FUNCTION

A surprising tool to solve counting problems.

Def^n: A generating func. for a seq. of real nos.: $a_0, a_1, a_2, \ldots a_k, \ldots$ is the infinite series

$$G(x) = a_0 + a_1 x + \ldots + a_k x^k + \ldots$$
$$= \sum_{k=0}^{\infty} a_k x^k$$

{ Sequence $\longrightarrow$ function }

Sequence : Order matters. Repitition allowed.

Ex. $a_0, a_1, a_2, \ldots$
a_0 → 0th term, a_2 → 2nd term.

Ex. $(0,0,1) \neq (0,1)$

- Generating func. only defined for infinite series.

Ex. S: 1, 1, 1, ...

$$G(s) = 1 \times x^0 + 1 \times x^1 + 1 x^2 + \ldots$$
$$\Rightarrow \sum_{k=0}^{\infty} x^k$$

Ex. S: $1, a, a^2, a^3, \ldots$

$$G(s) = 1 \times x^0 + a \times x^1 + a^2 x^2 + \ldots$$
$$= \sum_{k=0}^{\infty} a^k x^k$$

Ex. $G(S) = x^2 - x^4 + x^5 + x^6 + x^7 + \dots$

S: 0, 0, 1, 0, -1, 1, 1, 1, ...

Note: When we use Generating funcs., we are not interested in x, its just a placeholder. We are generally interested in the coeff.

Ex. $1 + x + x^2 + x^3 + \dots$

{Infinite GP} $\Rightarrow \frac{1}{1-x}, -1 < x < 1$

But in Gen. funcs. we don't care about range of x, since we don't care about x.

Ex. [when given finite series]

S: 1, 2, 3

$\Rightarrow$ S: 1, 2, 3, 0, 0, 0, ...

$G(S) = 1 \times x^0 + 2 \times x^1 + 3 \times x^2 + 4 \times x^0 + \dots$

$= 1 + 2x + 3x^2$

Ex. $G(S) = x^6 - 1$

Coeff $x^0 = -1$

$x^1 = 0$

$x^2 = 0$

$\vdots$

$x^6 = 1$

$\therefore S \Rightarrow -1, 0, 0, 0, 0, 0, 1, 0, 0, \dots$

* AP, GP & AGP

- AP: $a, a+d, a+2d, \ldots$

$$a_n = a + (n-1)d$$

$$S_n = \frac{n}{2} \times (a+l) = \frac{1}{2} \times n \times (a + a + (n-1)d)$$

- GP: $a, ar, ar^2, \ldots$

$$a_n = a \times r^{n-1}, \quad n \geq 0$$

$$S_n = a \times \frac{(r^{\#\text{terms}} - 1)}{(r-1)}, \quad \text{\# terms is finite.}$$

$$\Rightarrow \frac{a}{(1-r)}, \quad \text{\# terms is } \infty$$

- AGP: $\underbrace{a}_{AP}, \underbrace{(a+d)}_{AP}r, \underbrace{(a+2d)}_{AP}r^2, \underbrace{(a+3d)}_{AP}r^3, \ldots$ (GP: r, r^2, r^3)

$$a_n = (a + (n-1)d) \times r^{n-1}$$

$$S_\infty = a + (a+d)r + (a+2d)r^2 + (a+3d)r^3 + \ldots$$

$$r \times S_\infty = \quad ar + (a+d)r^2 + (a+2d)r^3 + \ldots$$

$$\Rightarrow (1-r)S_\infty = a + dr + dr^2 + dr^3 + \ldots$$

$$\Rightarrow (1-r)S_\infty = a + d \times \frac{r}{1-r}$$

imp

$$\Rightarrow \boxed{S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}}$$

Sum of ∞ AGP series if GP starts with 1

Ex. $S = \{a_k\}$, $k \geq 0$, $a_k = 3$

$$G(s) = \sum_{k=0}^{\infty} a_k x^k$$

$$\Rightarrow \sum_{k=0}^{\infty} 3 \times x^k$$

$$\Rightarrow 3 \times \sum_{k=0}^{\infty} x^k$$

$$\Rightarrow 3 \times \frac{1}{1-x}$$

$$\Rightarrow \frac{3}{1-x}$$

Ex. $S = \{a_k\}$, $k \geq 0$, $a_k = k+1$.

$$G(s) = \sum_{k=0}^{\infty} a_k x^k$$

$$\Rightarrow \sum_{k=0}^{\infty} (k+1) x^k$$ } Info. AGP Sum }

$$\Rightarrow \frac{1}{1-x} + \frac{1 \times x}{(1-x)^2}$$

$$\Rightarrow \frac{1}{1-x} + \frac{x}{(1-x)^2}$$

$$\Rightarrow \frac{1-x+x}{(1-x)^2}$$

$$\Rightarrow \frac{1}{(1-x)^2}$$

* Differentiation

Ex. S: $a_0, a_1, a_2, a_3, \ldots$

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots a_n x^n \ldots$$

$$G'(x) = 0 + a_1 + 2a_2 x + 3a_3 x^2 + \ldots + n a_n x^{n-1} + \ldots$$

S': $a_1, 2a_2, 3a_3, 4a_4, \ldots, n a_n, \ldots$

We get a new series by differentiating GF of original sequence. This can again be repeated for new sequence.

Ex. S: $1, 1, 1, \ldots$

$$G(x) = 1 + x + x^2 + x^3 \ldots$$

$$\Rightarrow \frac{1}{1-x}$$

$$G'(x) = -1 \times \frac{1}{(1-x)^2} \times (-1) = \frac{1}{(1-x)^2}$$

Ex. S: $0, 1, 2, \ldots$

$$G(x) = 1 \times x + 2 \times x^2 + 3 \times x^3 + \ldots$$

$$\Rightarrow \sum_{k=0}^{\infty} k \times x^k$$

$$\Rightarrow \frac{x}{1-x} + \frac{1 \times x}{(1-x)^2}$$

$$\Rightarrow \frac{2x - x^2}{(1-x)^2} \Rightarrow \frac{x(2-x)}{(1-x)^2}$$

Ex. S: 0, 1, 2, 3, ...

$$G(S) = 0 \times x^0 + 1 \times x^1 + 2x^2 + 3x^3 + \dots$$

$$\Rightarrow \underset{a}{0} + \underset{r}{x^1} + 2x^2 + 3x^3 + \dots \quad d = 1$$

$$G(S) = \frac{0}{(1-x)} + \frac{1 \times x}{(1-x)^2} = \frac{x}{(1-x)^2}$$

Note:

① GF is not a formula, but a function, i.e. don't put values of x.

② Values for x can be put as a shortcut to find terms.

$$G(x) = a_0 + a_1 x^1 + a_2 x^2 + \dots$$

$$G(0) = a_0$$

$$G'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

$$G'(0) = a_1$$

$$G''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + \dots$$

$$G''(0) = 2a_2$$

$$\vdots$$

$$G^n(0) = n! \times a_n$$

Ex. [Determining Sequence of GF]

$$G(x) = (2x-3)^3$$

M1:

$a_0 = G(0) = -27$

$a_1 = G'(0)/1! = 3(2x-3)^2 \times (2)\Big|_0 = 3 \times 9 \times 2$
$\Rightarrow 54$

$a_2 = G''(0)/2! = \dfrac{6 \times 2 \times (2x-3) \times (2)}{2!}\Big|_0 = 12 \times -3$
$\Rightarrow -36$

$a_3 = G'''(0)/3! = \dfrac{24 \times (2)}{3!} = 8$

$a_4 = G''''(0)/4! = 0.$

$\therefore -27, 54, -36, 8, 0, 0, 0 \ldots$

M2:

$G(x) = (2x-3)^3 = 8x^3 - 36x^2 + 54x - 27.$

$\therefore S: -27, 54, -36, 8, 0, 0, 0 \ldots$

Ex. Find GF for seq $\{a_k\}$ $k \geq 0$, $a_k = 2^k$

$G(x) = 2^0x^0 + 2^1x^1 + 2^2x^2 + 2^3x^3 + \ldots$

$\Rightarrow 1 + 2x + 2^2x^2 + 2^3x^3 + \ldots$

$\Rightarrow (2x)^0 + (2x)^1 + (2x)^2 + (2x)^3 + \ldots$

$\Rightarrow \dfrac{1}{1-2x}$ {Inf GP}

Ex. Find GF for finite seq. 2, 2, 2, 2, 2, 2.

S: 2, 2, 2, 2, 2, 2, 0, 0, 0, ...

$$G(x) = 2\times x^0 + 2\times x^1 + 2x^2 + 2x^3 + 2x^4 + 2x^5 + 0x^6 + \dots$$

$$\Rightarrow 2(1 + x + x^2 + x^3 + x^4 + x^5)$$

$$\Rightarrow 2 \times 1 \times \frac{(x^6 - 1)}{(x-1)}$$

$$\Rightarrow \frac{2(x^6 - 1)}{(x-1)}$$

Ex. Find GF for 2, 4, 8, 16, 32, ...

S: $2, 2^2, 2^3, 2^4, \dots$

$$G(x) = 2^1 \times x^0 + 2^2 \times x^1 + 2^3 \times x^2 + \dots$$

$$\Rightarrow 2 + 2^2 x + 2^3 x^2 + 2^4 x^3 + \dots$$

$$\Rightarrow \frac{2}{1 - 2x}$$

Ex. Find Series Sum: $1\times2 + 2\times2^2 + 3\times2^3 + \dots + 100\times2^{100}$

$$S = 1\times2 + 2\times2^2 + 3\times2^3 + \dots + 99\times2^{99} + 100\times2^{100}$$

$$2S = \quad 1\times2^2 + 2\times2^3 + 3\times2^4 \dots + 99\times2^{100} + 100\times2^{101}$$

$$\Rightarrow S(1-2) = 2 + 2^2 + 2^3 + \dots 2^{100} - 100\times2^{101} + (2^0 - 2^0)$$

$$\Rightarrow -S = 2^{101} - 1 - 1 - 100\times2^{101}$$

$$\Rightarrow S = 99\times2^{101} + 2.$$

Ex. Find the sum: $1/2 + 2/4 + 3/8 + \dots$

Can't apply S_∞ formula since GP starts with $1/2$, not 1.

$$S = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots$$

$$\Rightarrow S = \frac{1}{2}\left(1 + \frac{2}{2} + \frac{3}{2^2} + \dots\right)$$

Now GP starts with 1.

$$\Rightarrow S = \frac{1}{2}\left[\frac{1}{1-\frac{1}{2}} + \frac{1\times1/2}{(1-\frac{1}{2})^2}\right]$$

$$\Rightarrow \frac{1}{2}\left[2 + 2\right]$$

$$\Rightarrow 2.$$

Ex. Find Sum: $1/10 + 2/10^2 + 3/10^3 + \dots + n/10^n + \dots$

$S: \frac{0}{10^0} + \frac{1}{10} + \frac{2}{10^2} + \dots$ (So, GP starts with 1)

$$S_\infty = \frac{0}{1-1/10} + \frac{1\times1/10}{(1-\frac{1}{10})^2} = \frac{10}{81}$$

$$\Rightarrow \frac{10}{81}$$

Ex. S: 0, 1, 2, 3, 4, ... Find GF.

$$G(x) = 0 \times x^0 + 1 \cdot x^1 + 2x^2 + 3x^3 + 4x^4 + \dots$$

$$\Rightarrow \frac{0}{1-x} + \frac{1 \times x}{(1-x)^2}$$

$$\Rightarrow \frac{x}{(1-x)^2}$$

Ex. Series $\langle a_k \rangle$, $k \geq 0$, $a_k = C(m, k)$ for some +ve int. m. Find GF.

$$G(x) = {}^mC_0 \times x^0 + {}^mC_1 \times x^1 + {}^mC_2 \times x^2 + \dots$$

$$\Rightarrow (1+x)^m$$

Ex. Series $\langle a_k \rangle$, $k \geq 0$, $a_k = \begin{cases} n+1, & n \text{ is odd} \\ 1, & \text{otherwise.} \end{cases}$

Find GF.

$$G(x) = 1 \times x^0 + (1+1)x^1 + x^2 + (3+1)x^3 + x^4 + (5+1)x^5 + \dots$$

$$\Rightarrow 1 + 2x + x^2 + 4x^3 + x^4 + 6x^5 + \dots$$

$$\Rightarrow (1 + x^2 + x^4 + x^6 \dots) + (2x + 4x^3 + 6x^5 + \dots)$$

$$\Rightarrow \frac{1}{1-x^2} + 2x(1x^0 + 2x^2 + 3x^4 + \dots)$$

$$\Rightarrow \frac{1}{1-x^2} + 2x\left(\frac{1}{1-x^2} + \frac{1 \times x^2}{(1-x^2)^2}\right)$$

$$\Rightarrow \frac{1+2x}{1-x^2} + \frac{2x^3}{(1-x^2)^2}$$

$$\Rightarrow \frac{(1+2x)(1-x^2) + 2x^3}{(1-x^2)^2}$$

$$\Rightarrow \frac{1 - x^2 + 2x - \cancel{2x^3} + \cancel{2x^3}}{(1-x^2)^2}$$

$$\Rightarrow \frac{1}{1-x^2} + \frac{2x}{(1-x^2)^2}$$

$$\Rightarrow \frac{1}{(1-x)(1+x)} + \frac{2x}{(1-x^2)^2}$$

$$\Rightarrow \frac{1}{2}\left[\frac{1}{(1-x)} + \frac{1}{(1+x)}\right] + \frac{2x}{(1-x^2)^2}$$

$$\Rightarrow \frac{1}{1-x} + \frac{1}{2(1+x)} + \frac{2x}{(1-x^2)^2} - \frac{1}{2(1-x)}$$

$$\Rightarrow \frac{1}{1-x} + \frac{2x}{(1-x^2)^2} - \frac{x}{1-x^2}$$

$$\Rightarrow \frac{1}{1-x} + \frac{2x - x(1-x^2)}{(1-x^2)^2}$$

$$\Rightarrow \frac{1}{1-x} + \frac{x + x^3}{(1-x^2)^2}$$

$$\Rightarrow \frac{1}{1-x} + \frac{x(1+x^2)}{(1-x^2)^2}$$

* Extended Binomial Coeff.

We already know, Binomial Coeff. ${}^nC_K$, where $n \in \mathbb{N}$ & $r \in \mathbb{W}$

Extended Binomial Coeff. is defined as :

for Bin. Coeff. say.

${}^5C_2$ we write $\frac{5 \times (5-1)}{2!}$ {2 terms}

${}^{10}C_4$ we write $\frac{10 \times (10-1) \times (10-2) \times (10-3)}{4!}$ {4 terms}

Similarly,

${}^{-2}C_3 = \frac{-2 \times (-2-1) \times (-2-2)}{3!}$ {3 terms}

${}^{1/2}C_2 = \frac{1/2 \times (1/2 - 1)}{2!}$

So, $EBC(n,K) = \frac{n(n-1)(n-2) \ldots (n-K+1)}{K!}$

where $n \in \mathbb{R}$ & $K \in \mathbb{W}$

$$\therefore ECB(n,K) = \begin{cases} 1 & , K = 0 \\ \frac{n \times (n-1) \times \ldots (n-K+1)}{K!} & , K \neq 0 \end{cases}$$

We can extend this to Extended Binomial Theorem

$$(1+x)^n = EBC(n,0)\,x^0 + EBC(n,1)\,x^1 + \dots$$

where, $n \in \mathbb{R}$

Ex. $(1+x)^{-2}$

$$\Rightarrow \frac{-2}{0!} \times x^0 + \frac{-2\times(-2-1)}{1!} \times x^1 + \frac{-2(-3)(-4)}{2!} \times x^2 + \dots$$

$\therefore$ Only $(1+x)^n$ is defined not $(x+y)^n$.

$$(1+x)^n = EBC(n,0)\,x^0 + ECB(n,1)\,x^1 + EBC(n,2)\,x^2 + \dots$$

Ex. $(2+x)^{-2} \Rightarrow 2^{-2}\left(1+x/2\right)^{-2}$

$$\Rightarrow 2^{-2}\left(\frac{-2}{0!}\times\left(\frac{x}{2}\right)^0 + \frac{-2\times-3}{1!}\left(\frac{x}{2}\right)^1 + \frac{-2\times-3\times-4}{2!}\times\left(\frac{x}{2}\right)^2 + \dots\right)$$

Ex. $^{-3}C_4 = \frac{-3\times-4\times-5\times-6}{4!} = \frac{6\times5\times4\times3\times2\times1}{4!\times2\times1} = \frac{6!}{4!\times2!}$

$\Rightarrow {}^6C_4$

Ex. $^{-4}C_3 = \frac{-4\times-5\times-6}{3!} = \frac{6\times5\times4\times3\times2\times1}{3!\times3\times2\times1}\times(-1) = \frac{6!}{3!\times3!}\times(-1)$

$\Rightarrow {}^6C_3$

So,

$$^{-n}C_r = (-1)^r \times {}^{n+r-1}C_r$$

where $n \in \mathbb{N}$

Ex. $G(x) = (1+x)^{-10}$

find 7th term.

$$^{-10}C_7 = (-1)^7 \times {}^{+10+7-1}C_7 = -{}^{16}C_7$$

Ex. Find coeff. of x^{12} in $(x^3 + x^4 + x^5 + x^6 + \dots)^3$

$$\Rightarrow \left(x^3 (1 + x + x^2 + x^3 + x^4 + \dots)\right)^3$$

$$\Rightarrow x^9 (1 + x + x^2 + x^3 + x^4 + \dots)^3$$

$$\Rightarrow x^9 \left(\frac{1}{1-x}\right)^3$$

$$\Rightarrow x^9 \times (1-x)^{-3}$$

$\therefore$ we need coeff. of x^3 in $(1-x)^{-3}$

$$\therefore {}^{-3}C_3 = \frac{-\cancel{3} \times -\cancel{4}^{2} \times -5}{\cancel{6}} = -10 \times (-x)^3$$

$$\Rightarrow \boxed{10}$$

$\therefore 1 \times +10 = +10.$

Ex. Sequence $\langle a_n \rangle$ $n \geq 0$, $G(z) = \dfrac{1+z}{(1-z)^3}$, then,

$a_3 - a_0$.

a_3 is coeff. of x^3 & a_0 is coff. of x^0.

$$a_0 = G(0) = \frac{1}{(1-0)^3} = 1$$

for a_3:

$$G(z) = (1+z)(1-z)^{-3}$$

$$\Rightarrow \underbrace{1 \times (1-z)^{-3}}_{\text{coeff of } z^3} + z \times \underbrace{(1-z)^{-3}}_{\text{coeff. of } z^2}$$

$\Rightarrow {}^{-3}C_3 = (-1)^3 \, {}^5C_3 = -10 \times (-z)^3$ $\Rightarrow 10z^3$

$\Rightarrow {}^{-3}C_2 = (-1)^2 \times {}^4C_2 \times (-z)^2$ $\Rightarrow 6z^2$

$\therefore a_3 = 10 + 6 = 16$.

$\therefore a_3 - a_0 = 16 - 1 = 15$.

Ex. $\langle a_n \rangle$, $n \geq 0$, $a_n = 2n+3$. Find GF.

$$G(x) = 3 + 5x + 7x^2 + 9x^3 + 11x^4 + \dots$$

This is an AGP, & GP starts with 1

$$\therefore G(x) = \frac{3}{(1-x)} + \frac{2x}{(1-x)^2} = \frac{3 - 3x + 2x}{(1-x)^2} = \frac{3-x}{(1-x)^2}$$

Ex. $G(x) = \frac{1}{(1-x)^2}$, what is a_n where

$$G(x) = \sum_{x=0}^{\infty} a_n x^n$$

$$G(x) = (1-x)^{-2}$$

$a_0 =$ coeff. of $x^0 = {}^{-2}C_0 \times (-x)^0 = 1$

$a_1 =$ coeff. of $x^1 = {}^{-2}C_1 \times (-x)^1 = 2$

$a_2 = {}^{-2}C_2 \times (-x)^2 = 3$

$\therefore a_n = n+1$

Ex. Coeff. of x^3 in $(1+x)^3(2+x^2)^{10}$

$(1+3x+3x^2+x^3)(2+x^2)^{10}$

in $(2+x^2)^{10}$:

Coeff. of x^3 : N.A.

Coeff. of x^2 : ${}^{10}C_1 \times (x^2)^1 \times 2^9 = 5120\,x^2$

$\Rightarrow 5120$ or ${}^{10}C_1 \times 2^9$

Coeff. of x^0 : ${}^{10}C_0 \times (x^2)^0 \times 2^{10} = 1 \times 2^{10} = 2^{10}$

$\therefore$ Coeff. of $x^3 = 3 \times 10 \times 2^9 + 2^{10}$

* Exponential GF (not in syllabus)

For a sequence, $a_0, a_1, a_2, a_3, \ldots$ the Exponential GF $G(x)$ is:

$$G(x) = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!}$$

$$\Rightarrow \frac{a_0}{0!} + \frac{a_1 x^1}{1!} + \frac{a_2 x^2}{2!} + \frac{a_3 x^3}{3!} + \ldots$$

also, if $\langle a_k \rangle, k \geq 0, a_k = 1$

then,

$$G(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

Pigeonhole Principle

If there are n holes out of which $(n+1)$ pegions fly out, then we can guarentee that at least two pegions go out of same hole.

Ex. 3 holes (A, B, C). 4 Pegeons (P_i, $i \in [1,4]$).

P_1 goes to A
P_2 goes to C
P_3 goes to B

Now P_4 will have to go out of one of A, B & C & hence use previously used hole.

All possiblity will have at least 2 pegions using same hole.

Ex. Among a collection of 21 digits, what is the max. no. digits that must be same?

d_1 d_{11} d_{21}
d_2 d_{12}
$\vdots$ $\vdots$
d_{10} d_{20}

$\therefore$ At the least, 3 of the 21 digits must be same.

Ex. How many students req. for a course to guarentee at least six will get the same grade. Grades: A, B, C, D, F

A	S_1	S_6	S_{11}	S_{16}	S_{21}	S_{26}
B	S_2	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
C	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
D	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
F	S_5	S_{10}	S_{15}	S_{20}	S_{25}	

$\therefore$ At min. 26 students req.

Ex. What is the min. no. of non-neg. ints that should be chosen to ensure that there are 2 pairs (a,b) & (c,d) in chosen set such that $a \equiv c \mod 3$ & $b \equiv d \mod 2$.

that means, $a\%3 = c\%3$ & $b\%2 = d\%2$

3 ↓ \ 2 →	0	1
0	(0,0)	(0,1)
1	(1,0)	(1,1)
2	(2,0)	(2,1)

After this choosing any pair (x, y) would fall under one of the 6 cases.

$\therefore 6+1 = 7.$

$\Rightarrow 3 \times 2 + 1 = 7.$

GRAPH THEORY

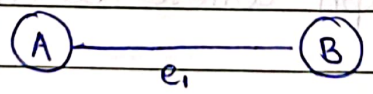
- Euler is the father of Graph Theory & "The Bridges of Konisberg" is the first problem modelled as graph.

- Graph Theory is a field in Mathematics in which we model real world problems using structure of dots & lines.

- Graph consists of two sets :
  1. Set of Vertices
  2. Set of Edges

* Types of Edges

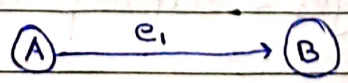
- Undirected : An edge with no direction. This can be represented as set of two vertices.



$e_1 = \{A, B\}$

A & B are endpoints of e_1.

- Directed : An edge with direction. Can be represented as ordered pair of two vertices.

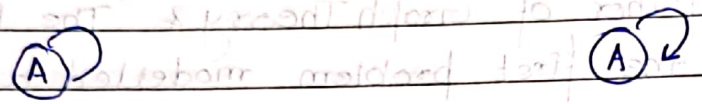


$e_1 = (A, B)$

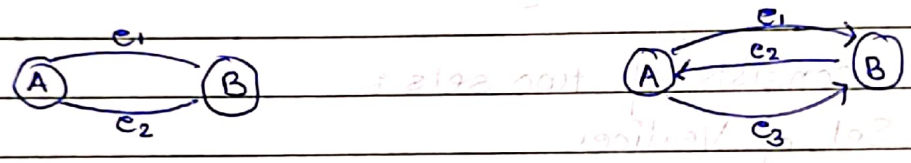
A is source.

B is destination.

③ Self loop: An edge from & to itself.



④ Multiedge / Parallel edge: Two edges are parallel if their source & dest. are same.



$e_1 == e_2$

$\therefore e_1 \parallel e_2$

$e_1 \parallel e_3$ $\quad e_1 \nparallel e_2$

$e_2 \nparallel e_3$

* Types of Graph (Based on Edge type)

① Undirected: A graph where all edges are undirected.

② Directed: A graph where all edges are directed.

③ Multigraph: An undirected graph where parallel edges are allowed, but no self loops.

④ Pseudograph: An undirected graph with both parallel edges & self loop allowed.

⑤ Directed Multigraph: A directed graph where both parallel edges & self loops allowed.

⑥ Simple Graph: Undirected. No self-loop. No multi-edges.

⑦ Simple Directed Graph: Directed. No self-loop. No multiedge.

Ex. A B

$V = \{A, B\}$ $E = \{\{A,B\}, \{A,B\}\}$

∴ In multigraph, E is a multi-set.

⑧ Intersection Graph: It is a graph of collection of sets, $A_1, A_2, \ldots A_n$, such that each set A_i is represented as a vertex, and an edge b/w A_i & A_j represent that the intersection $A_i \cap A_j \neq \emptyset$.

- Undirected since $A_i \cap A_j = A_j \cap A_i$

Note: Unless mentioned, all the following concepts are for simple graph.

* Adjacency: An edge $e = \{a, b\}$ is said to be ~~adjacent~~ incident on a & b, and a & b are said to be adjacent to each other.

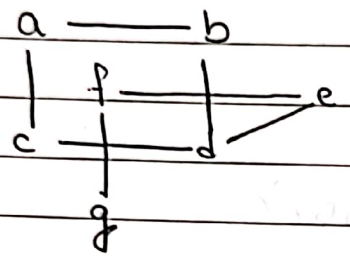
Neighbourhood of vertex: ~~All~~ Set of all adjacent vertices.

Adjacent Edges: Two edges are adjacent if they have at least one common endpoint.

* Degree of Vertex

Number of edges incident on the vertex.

Ex.



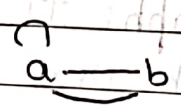
deg(a) = 2 deg(d) = 3
deg(b) = 2 deg(e) = 2
deg(c) = 2 deg(f) = 2
deg(g) = 1.

- In pseduograph degree of vertex is the number of times edges are incident on vertex.

Ex.

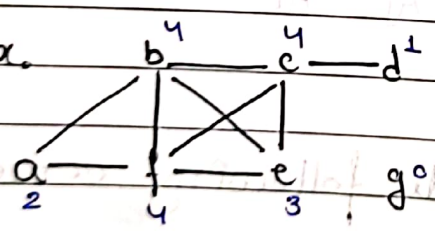
deg(a) = 2

Ex.



deg(a) = 4
deg(b) = 2.

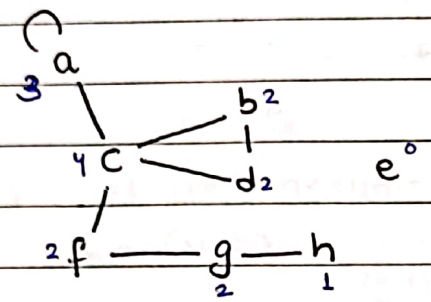
Ex.



- Isolated Vertex: 0 degree vertex.
- Pendant Vertex: 1 degree vertex.

* Order of Graph : Number of vertices.
Size of Graph : Number of edges.

Ex.



~~Min-deg~~

Min-deg(G) = δ = 0

Max-deg (G) = Δ = 4

Total deg = 16

Avg deg = $\frac{16}{8}$ = 2.

Order (G) = 8

Size (G) = 8 :

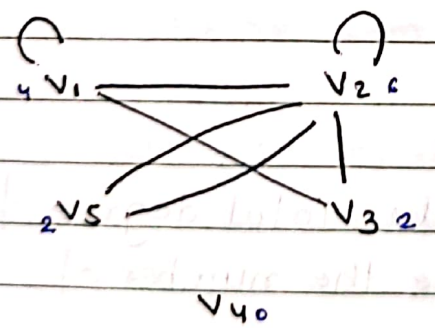
#even deg vertices = 6

odd deg vertices = 2.

Degree Sequence (degree's in descending order)

4 3 2 2 2 2 1 0

Ex.



Order (G) = 5 size (G) = 7

δ(G) = 0 Δ(G) = 6

Total deg = 14.

(∵ Undir., Selfloop, Multiedge ⇒ Pseudograph)

* • Total deg $= \sum_{v_i} \deg(v_i)$

$\delta \leq$ Avg. degree $\leq \Delta$

$\Rightarrow n\delta \leq$ ~~Avg.~~ Total degree $\leq n\Delta$

Ex.

A B

C D

	Total deg	Num Edges
S1: Add {A, B}	2	1
S2: Add {C, D}	4	2
S3: Add {A, A}	6	3
S4: Add {A, B}	8	4

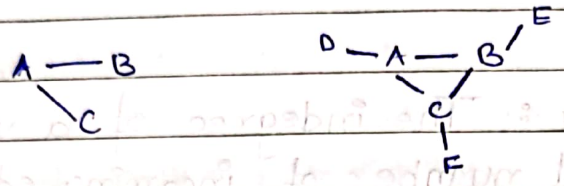
∴ Every edge contributes +2 to total degree.

• Handshaking Lemma

"In an undirected graph the total degree of graph is equal to twice the number of edges"

$$\sum_{v_i} \deg(v_i) = 2 \times |E|$$

Ex. Can you create a graph with odd no. of odd degree vertices?



Such a graph is not possible.

Proof: Let us assume that it is possible.

$G = (V, E)$ $\quad |V| = n \quad |E| = m$

Say there are $(2K+1)$ odd deg. vertices, i.e. deg of type $(2n_i + 1)$, where n_i is some whole no.

So, deg. sum:

$(2n_1+1) + (2n_2+1) + \dots (2n_{(2K+1)}+1)$ $\quad$ {2K+1 terms. Each odd}

$+ 2l_1 + 2l_2 + \dots + 2l_{n-(2K+1)}$ $\quad$ {n-(2K+1) remaining terms}

$\Rightarrow \left[2(n_1 + n_2 + \dots + n_{2K+1}) + (2K+1)\right] + 2(l_1 + l_2 + \dots l_{n-(2K+1)})$

$\Rightarrow 2(n_1 + n_2 + \dots + n_{2K+1} + l_1 + l_2 + \dots l_{n-(2K+1)} + K) + 1$

$\therefore$ Degree sum is odd.

But by Handshaking Lemma: deg. sum $= 2 \times E$

$\therefore$ Degree sum is even.

$\therefore$ Proved by contradiction.

$\therefore$ There must be even no. of odd degree vertices

* The following concepts are defined for directed graph only.

- **Indegree of Vertex :** The indegree of a vertex is the total number of incoming edges.

- **Outdegree of Vertex :** The outdegree of a vertex is the total number of outgoing edges.

- **Handshaking Lemma :**
  "In directed graphs, the total sum of indegrees, is equal to the total sum of outdegrees, is equal to the number of edges."
  i.e.

$$\sum_{v_i} \text{indeg}(v_i) = \sum_{v_i} \text{outdeg}(v_i) = |E|$$

* [Simple Graph]

Walk : A traversal of graph in which repetition of both vertices & edges are allowed.

Trail : A traversal of graph in which repetition of only vertices is allowed.

Path : A traversal of graph in which repetition of neither edge, nor vertex allowed.

Open Walk: $v_1 - v_2 - \ldots - v_n$

$v_1 \neq v_n$

Closed Walk: $v_1 - v_2 - \ldots - v_n$

$v_1 = v_n$

Length of Walk: No. of edges in-between.

Closed Trail / Circuit: A trail where source & destination is same.

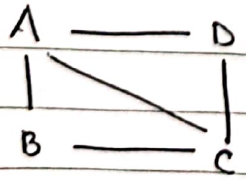
Cycle: A path with same source & destination.
{ Cycle is not path since vertex repeats }

- No vertex repetition → No edge repetition.
- The term cycle always implies the presence of at least three vertices.

Ex. is not cycle.

Ex. is not cycle.

Ex. is a cycle.

Ex. 

3 len cycle = 2 {ABC, ADC}

4 len cycle = 1

- If there is a walk from v to w, then there is a path from v to w.

- For a directed graphs all the definitions hold with the following exception:

Cycle implies presence of atleast 2 vertices.

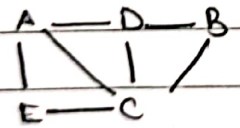
Ex. $A \rightleftarrows B$ is a cycle.

* [Simple Graph]

- ~~Distance~~
- Connected Graph: A graph is called connected iff there is a path from every vertex v_i to every other vertex v_j.

- Distance b/w two vertex: The min. length among all paths from v_i to v_j is called the "distance b/w v_i & v_j".

Ex.



Say, A & E.

① A—E

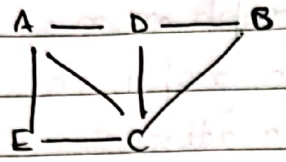
② A—C—E

③ A—D—C—E

④ A—D—B—C—E

$\therefore$ dist (A, E) = 1

- Diameter: The max. of all possible distances for all v_i & v_j is called "Diameter of a graph".

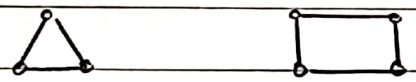
Ex. 

dist(A, E) = 1 dist(A, D) = 1
dist(A, B) = 2 dist(A, C) = 1
dist(B, C) = 1 dist(B, D) = 1
dist(B, E) = 1 dist(C, D) = 1
dist(C, E) = 2 dist(D, E) = 2.

$\therefore$ diam. (G) = 2.

- Diameter tells us, "from any vertex v_i to any vertex v_j the distance would be less than equal to diam(G)".
- If graph is not connected, then diameter is ∞.

* • Regular Graph: Every vertex has same degree.

Ex. 

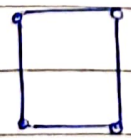
* K-Regular Graph: Every vertex has the degree 'K'.

- w.k.t. $\sum_{v_i} \deg(v_i) = 2E$

$\Rightarrow n \times K = 2E \Rightarrow E = \frac{nK}{2}$

- $\delta = K$ $\Delta = K$ Avg = K
- Graph can be connected or disconnected.

Ex. $\therefore$ 2-Regular graph.

 2-Regular graph.

* **Complete Graph:** A complete graph on n vertices, denoted by K_n is a simple graph in which every pair of vertices are adjacent.

- Since every vertex is adjacent to others:
  $\delta = (n-1)$ $\quad \Delta = n-1$ $\quad$ Avg. $= n-1$.
  $\therefore$ $(n-1)$- Regular Graph.
- Since n vertices & deg. of every vertex is $(n-1)$:

$$n \times (n-1) = 2E$$

$$\Rightarrow E = \frac{n \times (n-1)}{2}$$

- Diameter $= 1$.

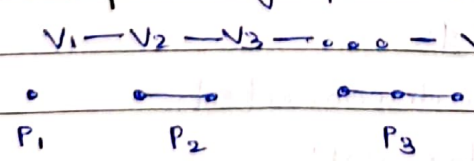
* **Empty / Null / Edgeless Graph:** A graph on n vertices without any edge. (E_n)

- $\delta = 0$ $\quad \Delta = 0$ $\quad$ Avg $= 0$
- 0-regular graph
- Connected iff $n = 1$.

* **Path Graph:** A graph which looks like a path.

P_n : $v_1 - v_2 - v_3 - \cdots - v_n$

Ex.



- Deg Seq $= \begin{cases} 0, & P_1 \\ 1\ 1, & P_2 \\ 1\ 2 \underbrace{\cdots}_{n-2 \text{ times}} 1\ 1, & P_{n \geq 3} \end{cases}$

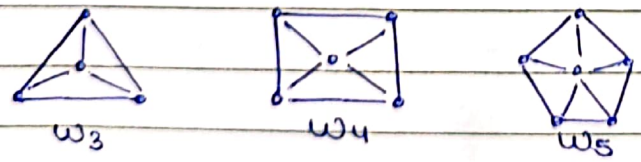
- Diameter $= \begin{cases} 0, & P_1 \\ n-1, & P_n \end{cases}$

- $|E| = n-1$
- ~~Diame~~ Total deg $= 2(n-1)$
- Connected.

* **Cycle Graph:** A graph which is cycle on n vertices. C_n.

- $n \geq 3$
- $\delta = 2$ $\quad \Delta = 2$ $\quad$ Avg $= 2$ $\quad$ Total $= 2n$
- Cycle Graph is 2-Regular
- $|E| = \frac{2n}{2} = n$
- Diameter: $\left\lfloor \frac{n}{2} \right\rfloor$
- Connected only.

* **Wheel Graph:** A graph of the following form:



W_n is obtained by adding 1 vertex (~~connected~~ adjacent to all other vertices) to C_n.

$\therefore |V| = n+1$.

- $n \geq 3$
- $\delta = 3$ $\quad \Delta = n$ $\quad$ Avg $= \frac{n \times 3 + (n)}{n+1} = \frac{4n}{n+1}$
- Diam $= 2$ $\quad n \geq 4$ $\quad$ $|E| = \frac{4n}{2} = 2n$

  1 $\quad n = 3$

* Hypercube Graph / n-Cube Graph

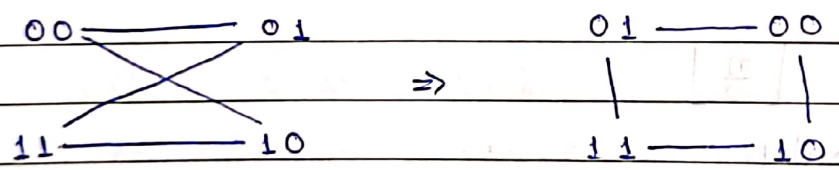
- Hamming Distance b/w two strings of equal length is equal to the number of positions where symbols differ.

- A Hypercube Graph Q_n consists of all bit strings of length n. There is an edge b/w two vertices if their Hamming Distance is 1.

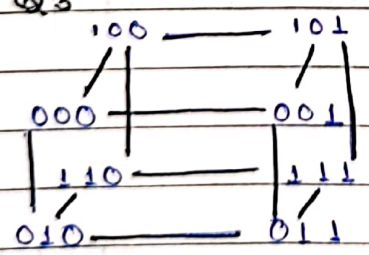
  i.e.

  $$(v_i, v_j) \in E \longrightarrow HD(v_i, v_j) = 1$$

Ex. Q_2



Ex. Q_3



- $|V| = 2^n$, $n \geq 0$
- Degree of each vertex $= n$
- $\delta = n$ $\Delta = n$ Avg $= n$
- Total deg $= 2^n \times n$
- $|E| = \frac{2^n \times n}{2} = 2^{n-1} \times n$

- Diameter = n (since at max 2 strings of len n can differ at n pos.)

- n- Regular

* w.K.t.

$$\sum_{v_i} \deg(v_i) = 2 \times |E|$$

- Say $|V| = n$ & avg. deg $= \Omega$ then

$$n\Omega = 2|E|$$

- $n\delta \leq 2|E|$
- $n\Delta \geq 2|E|$

Ex. G is an undir. graph with n vert. & 25 edges, such that deg. of each vertex is at least 3, then max. possible value of n is?

w.K.t.

$$\delta \times n \leq 2 \times |E|$$

$$\Rightarrow 3n \leq 50$$

$$\Rightarrow n \leq \frac{50}{3}$$

$$\therefore n \leq 16$$

Ex. Graph G satisfies $|E| \leq 3|V| - 6$. Which of the following can't be δ

(A) 3

(B) 4

(C) 5

(D) 6

$\Rightarrow 3|V| - 6 \geq |E|$

$\Rightarrow 3(|V|-2) \geq |E|$

$\Rightarrow 6(|V|-2) \geq 2|E|$

w.k.t. $\delta \times |V| \leq 2|E|$

$\therefore \delta \times |V| \leq 6(|V|-2)$

$\delta \leq \frac{6|V|-12}{|V|} \Rightarrow \frac{12}{(6-\delta)} \leq |V|$

A: $\delta = 3$

$4 \leq |V|$

B: $\delta = 4$

$6 \leq |V|$

C: $\delta = 5$

$12 \leq |V|$

D: $\delta = 6$

$\frac{12}{0} \leq |V|$

$\therefore$ δ can't be 6.

* • If a vertex is deleted, all edges incident ~~of~~ on the vertex are also deleted.

• If an edge is deleted, then only the edge is gone.

* Subgraph

Subgraph of a Graph G is obtained by: ~~either~~

(1) Either deleting 0 or more vertices of G

(2) OR deleting 0 or more edges of G

(3) OR both.

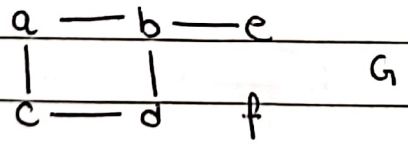
∴ If H is subgraph of G then

$V(H) \subseteq V(G)$

$E(H) \subseteq E(G)$

• Every graph G is a subgraph of itself.

Ex.



G

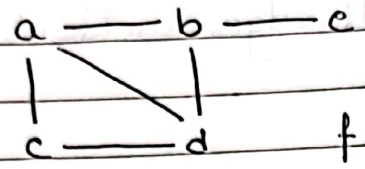
✓(a) G ✗(c) a—e ✓(e) a—b—e

✓(b) f ✓(d) a e ✗(f) a h

• Spanning Subgraph: Vertex deletion not allowed.

• Induced Subgraph: Vertex & its incident edges deletion allowed, otherwise edge deletion not allowed.

Ex.

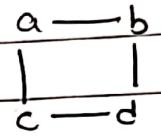


G

(A) a — d

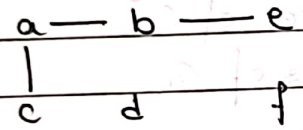
Induced
(by a, d)

(B)



Neither

(C)

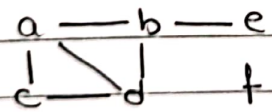


Spanning

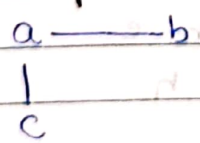
(d) G Spanning & Induced.

For any graph G, there is only one subgraph which is both induced & spanning. G itself.

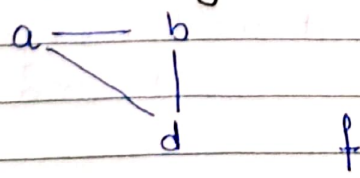
Ex.



(1) Graph induced by a, b, c



(2) Induced by a, b, d, f



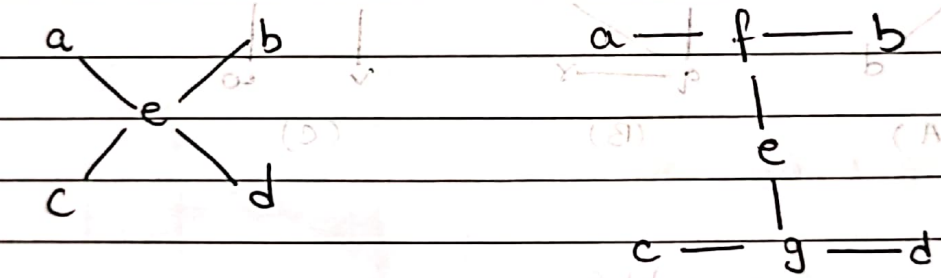
So, $G \xrightarrow{\text{Remove } S} H$ & $V(H) = V - S$

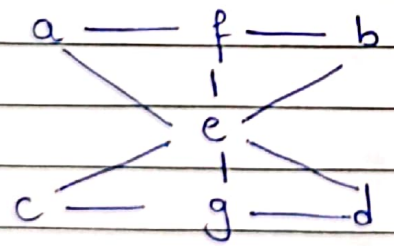
$S \subseteq V$

* Graph Union

The union of two simple graph $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$ is a simple graph with vertex set $V_1 \cup V_2$ & edge set $E_1 \cup E_2$.

Ex.





Ex. The no. of simple graphs with n distinct vertices?

Total no. of possible edges $= \frac{n \times (n-1)}{2}$

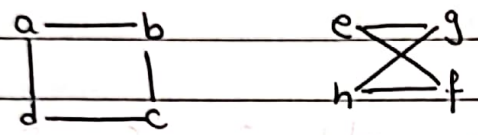
Each ~~vert~~ edge can / can't appear

$\therefore 2^{\frac{n(n-1)}{2}}$

* Graph Isomorphism

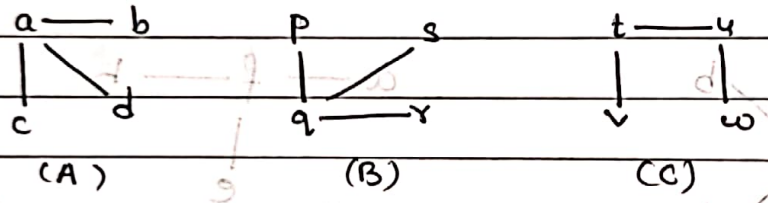
If in abstract view, two graphs are same then they are Isomorphic. ($G \equiv H$)

Ex.



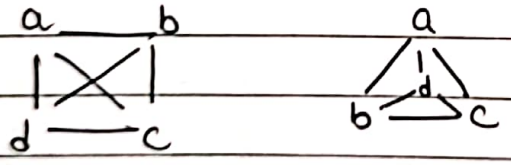
These are isomorphic.

Ex.



A & B are isomorphic.

Ex.



Isomorphic.

- Formal Def^n: Two graphs ~~G(V1, V2)~~ $G(V_1, E_1)$ & $H(V_2, E_2)$ are isomorphic iff there is a bijection b/w V_1 & V_2 which preserve edges.

i.e.

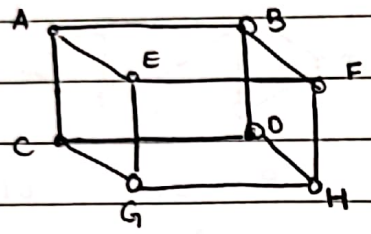
iff $(u,v) \in E_1$ then $(f(u), f(v)) \in E_2$.

- Simple checks for bijection:

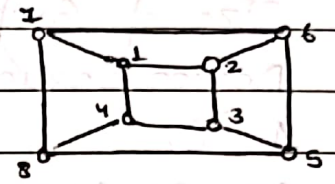
(1) If no. of vertices aren't same, then no bijection.

(2) If no. of edges aren't same, then no edge preservence

Ex.

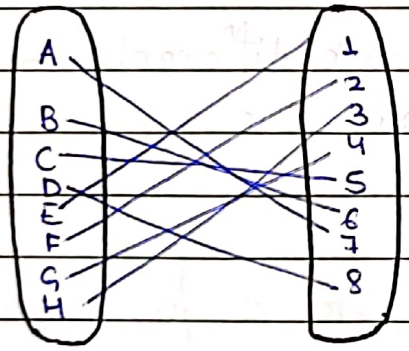


&

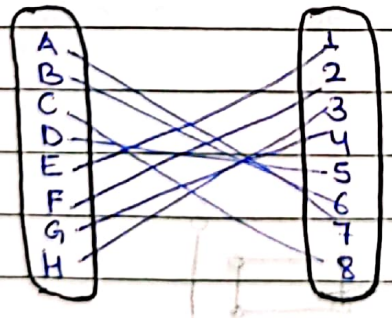


The no. of edges & vertices are same.

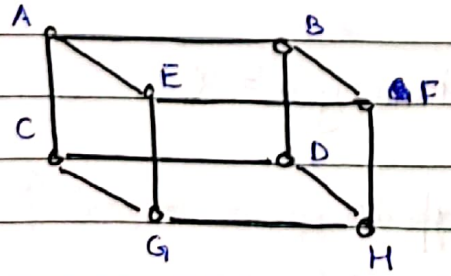
Total no. of bijections = 8!

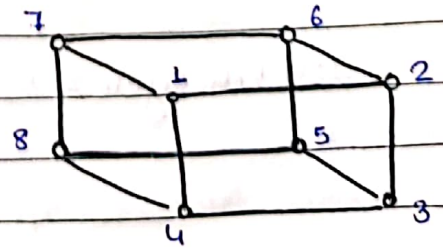


This doesn't preserve edges.



This bijection preserves edges.



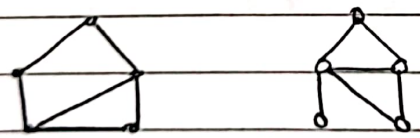


- Showing two graphs are non-isomorphic can be done using "Graph Invariants":

① Order of Graph
② Size of Graph
③ Degree Sequence
④ Δ, δ,
⑤ Connectedness
⑥ Diameter
⑦ Number of 3-len cycle.
⑧ Number of K-len cycle.

If any of these properties are different for two graph, then they aren't isomorphic.

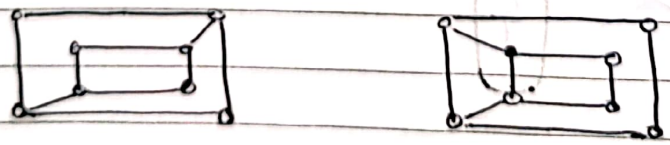
Ex.



$\Delta = 3$ $\Delta = 4$

$\therefore$ Non-isomorphic.

Ex.

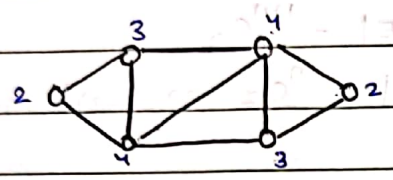


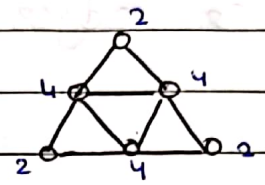
#4 len cycle = 2 #4 len cycle = 3

$\therefore$ Non-isomorphic

- Graph Invariants are the properties which are same for Isomorphic graphs.

Ex.





$|V| = 6 \quad |E| = 9$

$\delta = 2 \quad \Delta = 4$

#3 len cycle = 4

4 4 3 3 2 2

$|V| = 6 \quad |E| = 9$

$\delta = 2 \quad \Delta = 4$

#3 len cycle = 4.

4 4 4 2 2 2

$\therefore$ Non isomorphic.

* Complement of a Graph

- Every graph on n vertices is a subgraph of K_n.
- The complement of graph $G(V, E)$ is given as $\bar{G}(V, \bar{E})$ where $\bar{E} = E(K_n) - E$.

$\therefore \bar{G}(V, \bar{E})$ has $\bar{E} = E(K_n) - E$

$$|\bar{E}| = {}^{n}C_{2} - |E|$$

- Self Complementary Graph is a graph G such that G & $\bar{G}$ are isomorphic, i.e.

$$G \equiv \bar{G}$$

A self complementary graph would contain half the total no. of edges possible with n vertices, i.e.

$\frac{1}{2} \times {}^{n}C_2$ edges or $|E| = {}^{|V|}C_2 - |E|$

$\Rightarrow 2|E| = {}^{|V|}C_2$

$\Rightarrow |E| = \frac{{}^{n}C_2}{2} = \frac{n(n-1)}{4}$

Ex. A cycle on n vertices is isomorphic to its complement. Find n.

No. of edges in $C_n = n$.

No. of edges in $K_n = {}^{n}C_2 = \frac{n(n-1)}{2}$

$\Rightarrow \quad n = \frac{n(n-1)}{2} - n$

$\Rightarrow \quad 4n = n(n-1)$

$\Rightarrow \quad 4 = n-1 \qquad \{\text{with } n \neq 0\}$

$\Rightarrow \quad n = 5$

$\therefore \quad n = 5$ or $\cancel{n = 0}$ not feasible for cycle.

Hence $n = 5$

Ex. Is there any graph with 6 vertices which is self complementary?

No. $|E| = \frac{6 \times (6-1)}{4} \notin \mathbb{N}$

- w.k.t for self complementary graph:

$|E| = \frac{n(n-1)}{4}$, so either $4 \mid n$ or $4 \mid (n-1)$

$\therefore \quad 4K = n \qquad$ or $\qquad 4K = n-1$

$\Rightarrow 4K+1 = n$

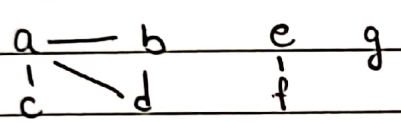
$\therefore$ If G is self-compl. then $n \% 4 = 0$ or 1.

* Components in a Graph

A maximal connected subgraph is component of that graph.

A graph consists of more than one component if there exists a cut of the graph such that there is no edge b/w the two sets of the cut.

Ex.



3 components.

- Checking for components

$\{a\}$ is conn.
$\{a,b\}$ is conn.
$\{a,b,c\}$ is conn.
$\{a,b,c,d\}$ is conn.
$\{a,b,c,d,e\}$ is disc.
$\{a,b,c,d,f\}$ is disc.
$\{a,b,c,d,g\}$ is disc.

$\therefore$ $\{a,b,c,d\}$ is 1 component.

$\{e\}$ is conn.
$\{e,f\}$ is conn.
$\{e,f,g\}$ is disc.

$\therefore$ $\{e,f\}$ is 1 component

$\{g\}$ is conn.

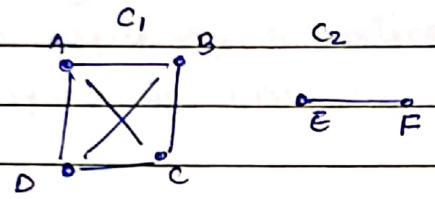
$\therefore$ $\{g\}$ is 1 component.

- A graph is connected if it has a single components.

Ex. Create a disconnected graph whose compl. is also disconnected.

Not possible. If a graph is disconnected graph then its complement is connected.

Say there are two components



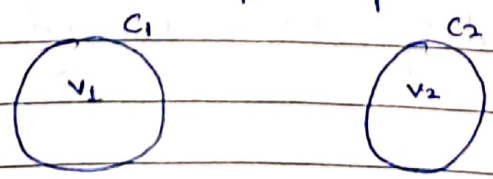
then its compl. will have all vertices from C_1 adjacent to C_2 vertices.
Therefore there is a path from ~~s~~ to C_1 vertices through C_2 vertices.

Ex. If there are exactly two odd degree vertices in the graph, then there exists a path b/w the two.

~~Since~~ If there is a path b/w two vertices, then they belong to same component.

$\therefore$ If there are exactly 2 odd deg. vertices then they are in same component.

Say they are in diff. components :



But then each component (which also is a graph) has odd (1) number of vertices.

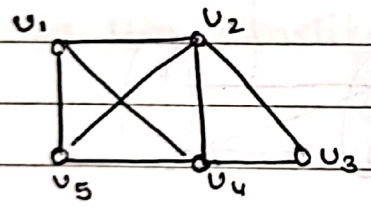
$\therefore$ By contradiction, this can't be true.

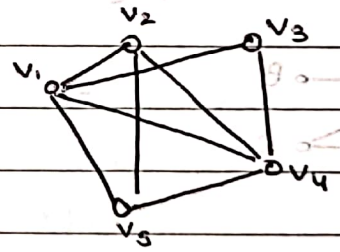
Ex. No. of edges possible for n vertex directed simple graph.

Edges are of type: (A, B)

$n \times (n-1)$

Ex. Are these isomorphic?





4 4 3 3 2

4 4 3 3 2

$u_1 \to v_1$, $u_2 \to v_2$, $u_4 \to v_5$, $u_5 \to v_4$, $u_3 \to v_3$

* Bipartite Graph

A graph G is called bipartite iff there exists a bipartition of the graph (A_1, A_2) such that there is no edge $(u,v) \in E$ such that both u & v belong to the same set.

Graph G is bipartite (X, Y) iff:

(1) $X \cap Y = \phi$

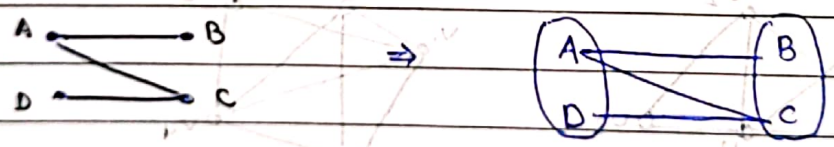
(2) $\forall a,b \in X,\ (a,b) \notin E$

(3) $\forall a,b \in Y,\ (a,b) \notin E$

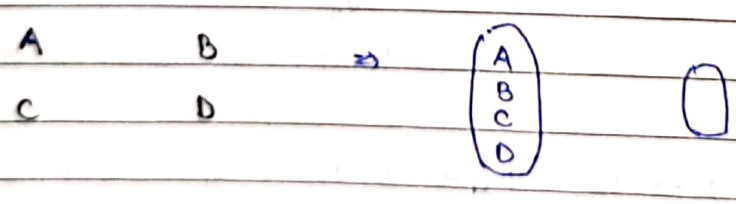
(4) $X \cup Y = V$

{ X & Y can be empty as well }

Ex.

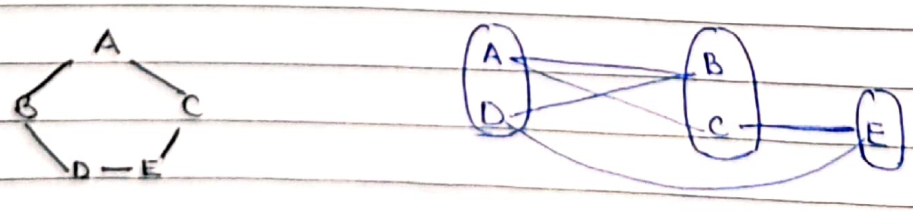


Ex.



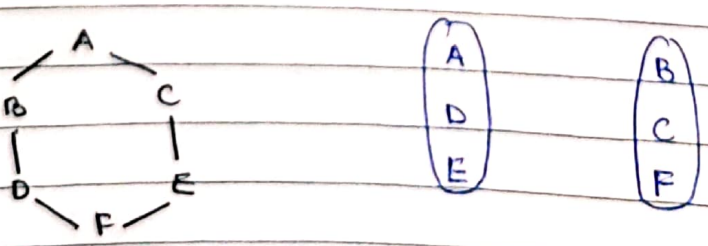
is also a bipartite

Ex.



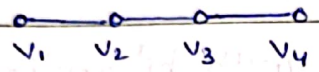
∴ Not bipartite

Ex.



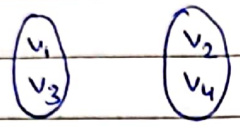
∴ Bipartite.

Ex. Is P_n bipartite?



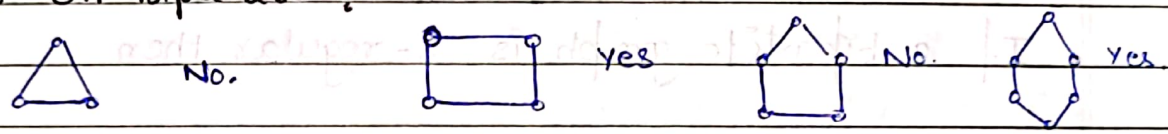
$X = \{v_i \mid 2 \mid i\}$

$Y = \{v_i \mid 2 \nmid i\}$



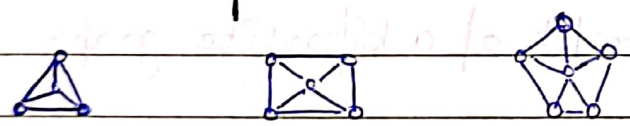
Yes, for all values of n.

Ex. Is C_n bipartite?



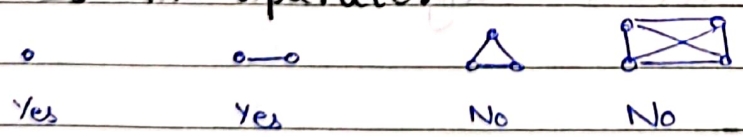
$\therefore$ C_n is bipartite for even n.

Ex. Is W_n bipartite?



$\therefore$ No.

Ex. Is K_n bipartite?



$\therefore$ Only for $n = 1, 2$.

Ex. Is Null graph bipartite?

Yes for all values of $|V|$.

Ex. Is Q_n bipartite?

Yes.

Imp

- **Theorem:** A graph is bipartite iff it doesn't have any odd length cycle.

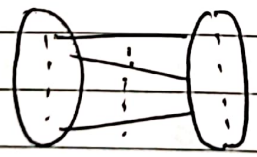
- **Theorem:** If a graph G is bipartite with bipartition (X, Y) then

$$\sum_{v \in X} \deg(v) = \sum_{v \in Y} \deg(v)$$

- If a bipartite graph is K-regular then

$$K \times |A| = K \times |B|$$
$$\Rightarrow |A| = |B|$$

- **Theorem:** Every subgraph of a bipartite graph is also bipartite.



Deletion of any vertex or edge doesn't disturb the bipartition.

- Complete Bipartite Graph ($K_{m,n}$) is a graph G with bipartion (A, B) such that

$$\forall u \in A \ \forall v \in B \quad (u, v) \in E$$

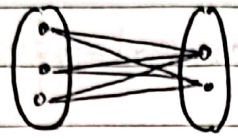
$\therefore$ $|E| = |A| \times |B|$

$|V| = |A| + |B|$

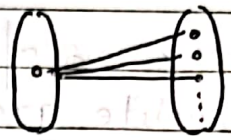
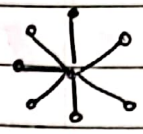
Deg. Seq. $\Rightarrow$ $|A|, |A|, \ldots, |B|, |B|, \ldots$

δ: min $(|A|, |B|)$ $\quad$ Δ: max $(|A|, |B|)$

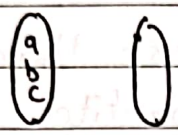
$$\text{Diam} = \begin{cases} 1, & K_{1,1} \\ 2, & K_{m,n} \quad m > 1 \text{ or } n > 1 \\ \infty, & m = 0 \text{ or } n = 0 \end{cases}$$

Ex. 

$K_{3,2}$

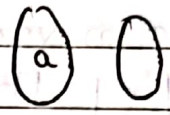
 or 

$K_{1,n}$

Ex. 

$K_{3,0}$

Diam $= \infty$

Ex. 

$K_{1,0}$

Diam $= 1$.

Ex. Max. no. of edges possible in bipartite graph of 4 vertices.

$(0,4)$: 0

$(1,3)$: $3 \times 1 = 3$ $\quad \therefore 4$

$(2,2)$: $2 \times 2 = 4$.

Ex. Same ques, 5 vertices.

$(0,5)$: 0

$(1,4)$: 4 $\quad \therefore 6$.

$(2,3)$: 6.

Ex. Max. no. of edges possible in bipartite graph of n vertices.

$$\max_{i=0}^{n} \left(i \times (n-i) \right) \Rightarrow K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$$

to get this,

$f(i) = i \times (n-i)$ $\qquad f''(i) = 0 - 2 < 0$

$\therefore$ maxima

$f'(i) = n - 2i = 0$ $\qquad \therefore$ max edges with $K_{\frac{n}{2}, \frac{n}{2}}$

$n = 2i$

$\Rightarrow i = n/2$

$\therefore$ max. no. of edges in n vertex Bipartite graph

$\Rightarrow \lceil \frac{n}{2} \rceil \times \lfloor \frac{n}{2} \rfloor$

If a n vertex graph has more than $\lceil \frac{n}{2} \rceil \times \lfloor \frac{n}{2} \rfloor$ then it's not bipartite.

- Theorem: A graph is bipartite iff all its components are bipartite.

Ex. Is Q_n bipartite ?

Q_n is graph on 2^n vertices.
Bipartition can be created based on parity.

odd parity even parity.

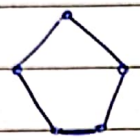
Any string with odd parity can only be changed to even parity by single bit change.

100101 $\rightarrow$ 100100
$\hookrightarrow$ 110101
⋮

* Cyclic Graph

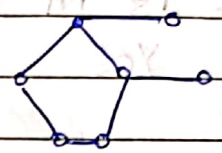
A graph which contains at least one cycle.

Ex.



Cyclic Graph
Cycle Graph (C_5)

Ex.



Cyclic Graph

- Cycle: There is some vertex, starting a trail on which could end up on the same vertex.

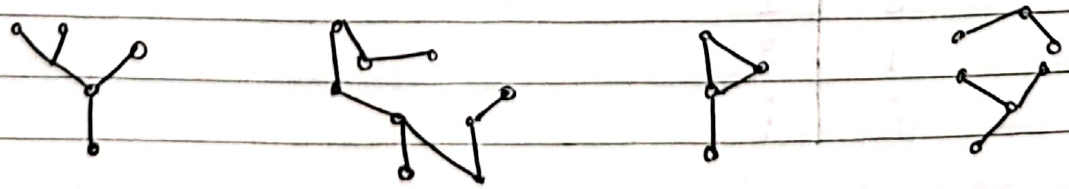
- A graph is cyclic iff there are at least two vertices which have more than one path b/w them.

* Tree

- An undirected connected acyclic graph, is called a Tree.

- Forest: Undirected acyclic graph.

Ex.



Tree
Forest

Tree
Forest

Neither

Forest

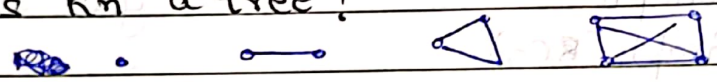
- Every tree is a forest. Tree $\rightarrow$ Forest

- In a tree, a vertex of degree 1 is called leaf.

Ex. Is P_n a tree?

Yes

Ex. Is K_n a tree?



Yes Yes No No

Only for K_1 & K_2.

Ex. Is C_n a tree?

No.

- Say a tree has $|V| = n$, then:

(1) Order of graph $= n$.

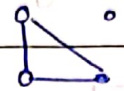
(2) Size of graph:

n	$\lvert E \rvert$
1	0
2	1
3	2
4	3
⋮	⋮

$\therefore \quad |E| = n-1$

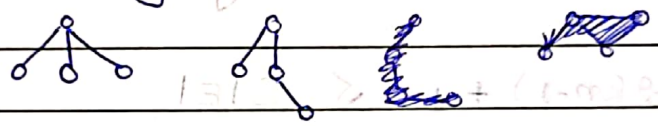
Ex. Does a graph with n vertices & $n-1$ edges imply tree?

No. $|V| = 4$ $|E| = 4-1 = 3$



Ex. Does a connected graph with n vertices & $n-1$ edges imply tree?

Yes. Say $|V| = 4$ then $|E| = 3$



So we always get a tree.

- **Maximally Acyclic:** A graph is maximally acyclic iff adding even a single edge would make the graph cyclic.

- Maximally Acyclic graph on n vertices is a <u>tree</u>.
- In forest, b/w two vertices, there's at most one path.
- In tree, b/w two vertices, there's exactly one path.

- **Minimally Connected:** A graph is minimally connected iff removing a single edge would disconnect it.

- Minimally Connected graph is <u>tree</u>.

- **Theorem:** Every tree with more than 1 vertex has at least two degree 1 vertex.

Proof:

C1: No vertex with deg 1, i.e. $\delta = 2$.

w.k.t.

$$\delta \times n \leq 2|E|$$

$$\Rightarrow 2n \leq 2(n-1)$$

Contradiction.

C2: Only 1 vertex with deg 1.

$$2(n-1) + 1 \leq 2|E|$$

$$\Rightarrow 2n-1 \leq 2n-2$$

Contradiction.

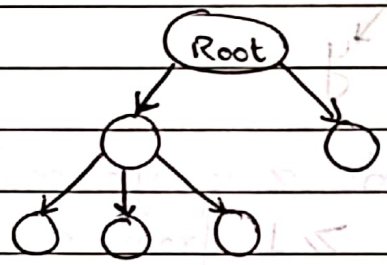
$\therefore$ At least 2 vertex with deg 2.

* Directed Acyclic Graph

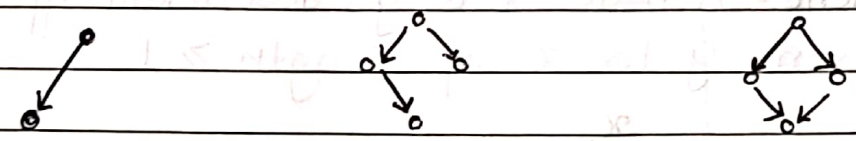
A directed graph with no cycle.

* Rooted Tree

Special type of DAG with a root vertex & a unique path from root to any vertex.



Ex.



✓ ✓ ✗ DAG but not RT

(Not unique path)

- Rooted Tree is defined by:
  (1) Should be DAG
  (2) Unique Root vertex.
  (3) Unique path from root to every node.

Terminology:

(1) Children: A node x is called y's child iff there is a 1-length path from y to x.

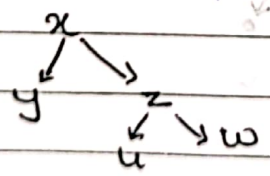
$$y \to x$$

(2) Parent: A node x is called y's parent iff there's a 1-length path from x to y.

$$x \to y$$

(3) Ancestors: A node x is y's ancestor iff there's a path of length ≥ 1 from x to y.

(4) Descendent: A node x is y's descendent iff there's a path from y to x of length ≥ 1.



$Desc(w) = \{\}$

$Ansc(w) = \{x, z\}$

$Ansc(x) = \{\}$

$Desc(x) = \{y, z, u, w\}$

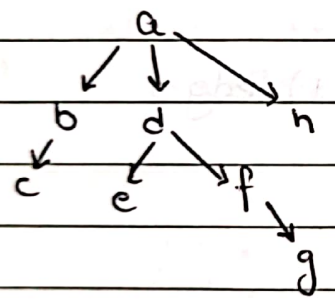
(5) Siblings: Two nodes, x & y are siblings iff they have same parent.

- A node with no parent is Root.
  A node with no children is Leaf.
  All other nodes are ~~Tea~~ Internal.

- Height of a node: No. of edges b/w a node & its farthest descendent, i.e. max no. of edges b/w the node & its descendent.

- Depth of a node: ~~Dista~~ No. of edges b/w root & the node.

Ex.



Node	Height	Depth
a	3	0
b	1	1
c	0	2
d	2	1
e	0	2
f	1	2
g	0	3
h	0	1

- Height of the Rooted Tree: Height of root node.

Ex. What is max. no. of edges in a forest with n vertices?

$n-1$. Maximally acyclic (Tree)

Ex. Max. no. of edges in a forrest which isn't tree?

Would need to have at least 2 comp.

Say,

K nodes $\qquad$ $(n-K)$ nodes.

$\Rightarrow (K-1)$ edges $\qquad$ $\Rightarrow (n-K-1)$ edges

$\therefore (n-K-1) + (K-1)$

$\Rightarrow n-2$.

Ex. Min. no. of edges in a connected graph with n vertices?

$n-1$. Minimally connected (Tree).

Ex. Min. no. of edges in connected cyclic graph?

$(n-1)+1 = n$.

Ex. Is it possible? A tree with 9 vert. & sum of deg 18?

$\Sigma deg = 2|E| = 2 \times 8 = 16$

$\therefore$ No.

Ex. Is it possible: A graph with 9 vert. & 9 edges & no cycle?
forest.

In a forrest, max. no. of edges $= n-1 = 8 < 9$.

$\therefore$ Not possible.

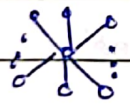
Ex. Is this necessarily a tree: Graph with n vert., one vertex of deg. $n-1$ & rest of vert. of deg 1?

deg. sum $= (n-1) + (n-1) \times 1 = 2n-2$

and, for tree, deg. sum $= 2 \times |E| = 2n-2$.

since this is def. of tree.

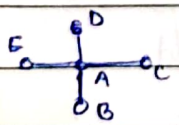
$\therefore$ Yes



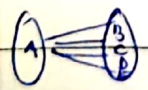
Ex. A Cyclic Graph must have at least 3 vert. of deg. at least 2?

True. Smallest cycle possible is: △ with 3 vert. each of deg 2, therefore for any cycle in a graph three nodes of deg. at least 2 would be req.

Ex. Consider a graph where at least one vertex v is connected to all other vertex. Such graph can be bipartite?



True.



Ex. G is a graph with at least 2 vert & no 1 deg vertex. G can never be tree?

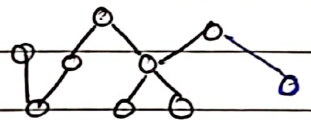
True. A tree has at least 2 vert. of deg. 2 if $|V| \geq 2$.

Ex. To a tree if we add an edge b/w existing vertices, then we get exactly one cycle?

True.

Ex. Tree + 1 vertex of deg 1 = Tree?

Yes.



is still tree.

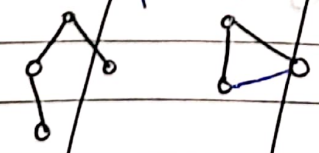
[Prim's Algo uses this]

Ex. If G has no cycles & adding a single edge b/w any two existing vertices creates cycle then G must be tree?

~~False~~.

~~G is acyclic ≡ G is forrest.~~

~~Counterexample:~~



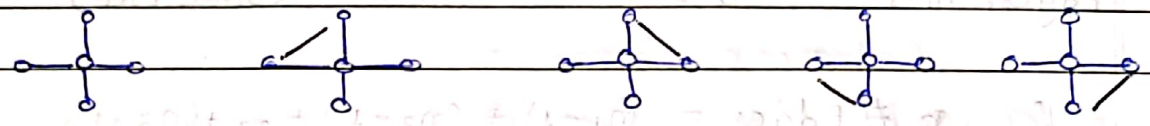
~~Creates a cycle but G isn't tree.~~

True.

G is acyclic ≡ G is forest.

By adding a single edge b/w any two arbitrary vertices should create cycle.

This is only possible when G is a tree.



∴ Adding edge b/w any 2 vertices creates cycle.

Ex. A finite graph with every vertex with deg ≥ 2 has a cycle?

True.

Say n verts. For maximally acyclic, (n-1) edges

∴ $\Sigma deg \leq 2n-2$ for ~~maximally~~ acyclic.

but here every vert. deg ≥ 2.

∴ $\Sigma deg. \geq 2n$

∴ Cyclic ∵ $2n > 2n-2$

Ex. Removing exactly 1 edge from tree results in how many trees?

Exactly 2.

Ex. G is a forest with n verts & K comps then how many edges does G have?

Say K-1 components each have single vertex ⇒ 0 edges.

No. of edges with (n-K+1) verts = (n-K+1)-1

⇒ n-K edges.

OR

Say for $i = 1 \ldots K$ each ith component has n_i verts. such that

$$\sum_{i=1}^{K} n_i = n.$$

	C_1	C_2	$\ldots$	C_K	
Verts:	n_1	n_2		n_K	
Edges:	$n_1 - 1$	$n_2 - 1$		$n_K - 1$	(since tree)

$\therefore$ # Edges $= (n_1 - 1) + (n_2 - 1) + \ldots + (n_K - 1)$

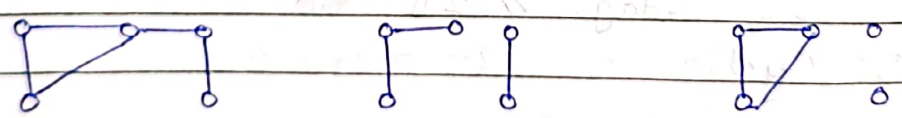
$\Rightarrow n - K.$

Ex. G is a simple connected graph with n verts & n edges. Removing any two ~~verts~~ edges would disconnect the graph?

True.

Either you remove one edge from cycle & other from non-cycle.

OR both from non cycle.



(A) (B)

Disconnected in both cases.

* Rooted Tree (contd.)

- Degree of node: No. of its children.
- Degree of tree: Max. degree among all nodes.

- Binary Tree: A rooted tree with at most two children for every node.

- In Binary Tree, if L is no. of leaf nodes, D'' is no. of degree 2 nodes then:

$$L = D'' + 1$$

Proof 1:

BT $\longrightarrow$ Rooted Tree $\longrightarrow$ Directed $\rightarrow$ Total Indeg = Total Outdeg

Total Indeg = Total Outdeg = $|E| = n-1$.

(1) W.K.T for every node except root, indeg = 1 (from parent)

$\therefore$ Total indeg = $n-1$

(2) We have 3 types of nodes in BT: L, D'', D'

Outdeg. of L & : 0 D' : 1 D'' : 2.

Total outdeg = $L \times 0 + D' \times 1 + D'' \times 2$.

$\Rightarrow D' + 2D''$

$\therefore \quad n-1 = D' + 2D''$

$\Rightarrow n = D' + 2D'' + 1$

but, $n = L + D'' + D'$

$\therefore \quad L + \cancel{D'} + D'' = \cancel{D'} + 2D'' + 1$

$\therefore \quad L = D'' + 1$

Proof 2:

Considering Binary Tree as Tree.
Let's say no. of children in BT node is called Order.
& Degree means, no. of incident edges.

Total ~~ord~~ degree $= 2 \times (n-1)$
w.k.t. degree of leaf $= 1$.
degree of order-1 nodes $= 2$
degree of order-2 nodes $= 3$

$\therefore (O_0 \text{ or } L) \times 1 + O_1 \times 2 + O_2 \times 3 - 1 =$ Total deg.

$2(n-1) = L + 2O_1 + 3O_2 - 1$
$\Rightarrow 2(L + O_1 + O_2 - 1) = L + 2O_1 + 3O_2 - 1$
$\Rightarrow L = O_2 + 1$

- Full Binary Tree: Every node has zero or two children (2-ary Tree)

(1) Total no. of nodes $= D_0 + D_2 + D_1$
$\Rightarrow L + D'' + 0$
$\Rightarrow D'' + 1 + D''$
$\Rightarrow 2D'' + 1$
or $\Rightarrow 2L - 1$

(2) No. of Internal nodes $= D''$

Ex. let f be a func. defined on Graph $G(V,E)$, as $f(G) = \sum_d i_d \times d$, where i_d is the no. of vertex. in G with deg. d. If S & T are two diff. trees with $f(T) = f(S)$ then:

(A) $|S| = 2|T|$

(B) $|S| = |T| - 1$

(C) $|S| = |T|$

(D) $|S| = |T| + 1$

f is basically degsum of G.

$\therefore$ degsum (T) = deg (S)

$\Rightarrow 2 \times (nE(T)) = 2 \times (nE(S))$

$\Rightarrow \cancel{2} \times (|T| - 1) = \cancel{2} \times (|S| - 1)$

$\Rightarrow |T| = |S|$

Ex. What is max. & min. no. of edges in undirected connected graph on n vert.?

Min: $n-1$ {Minimally Connected}

Max: ${}^nC_2$ {K_n}

Ex. Max. no. of edges in disconnected graph?

(1) Disconnected graph $\rightarrow$ at least 2 comp.

(2) For max. no. of edges, we'll have 2 comps (bare min.)

(3) Cases: $(n/2, n/2)$: ${}^{n/2}C_2 \times 2 = \frac{\frac{n}{2} \times (\frac{n}{2} - 1)}{\cancel{2}} \times \cancel{2} = \frac{n(n-2)}{4}$

C2: $(n-1, 1)$: ${}^{n-1}C_2 + 0 = \frac{(n-1)(n-2)}{2}$

C2 > C1

$\therefore$ Max edges $= \frac{(n-1)(n-2)}{2}$

Ex. How many edges must a graph on n verts have to guarentee that its connected?

[Pegion hole Principle]

Lets fully connect $(n-1)$ verts. Adding a single edge would mean that we'll have to connect to n^{th} vert. ∴ Connected guarenteed.

$$\therefore {}^{n-1}C_2 + 1$$

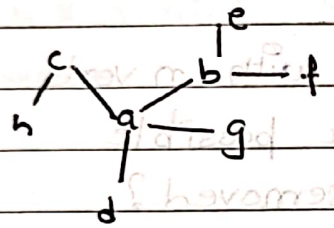
Ex. What is max. no. of edges for a simple graph with n verts & K components?

We'll make $K-1$ single vert. components & the 1 component with $n-(K-1)$ nodes will be fully connected.

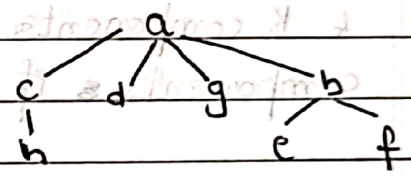
$$\therefore {}^{n-K+1}C_2$$

* Tree	Rooted Tree
(1) Undirected Acyclic Connected	(1) Directed Acyclic with root & unique path b/w root to every node.
(2) No concept of child, parent, height, depth, ancestor, etc.	(2) Well defined, all such concepts.

Ex.

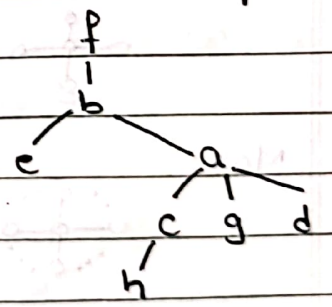


Ex. Rooted tree with 'a' as root



height = 2.

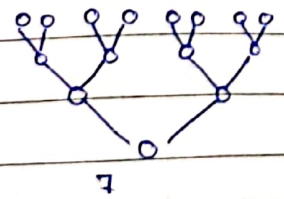
Ex. Rooted tree with 'f' as root



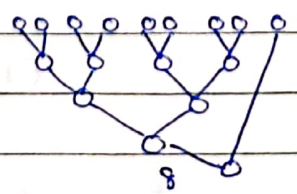
height = 4

Ex. Suppose 1000 people play in tournament played two at at a time, then no. of matches to determine winner?

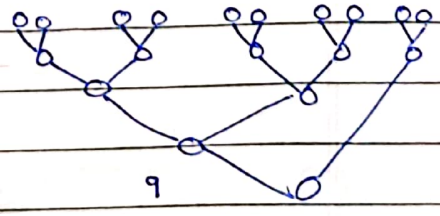
say 8 people



Say 9 people



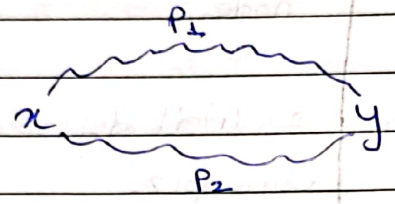
Say 10 people.



∴ everytime full BT.

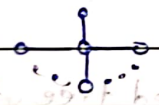
∴ No. of matches = No. of Internal nodes = 1000 − 1 ⇒ 999

Ex. Prove that in a simple graph there are two diff paths by verts. x & y then graph is cyclic.

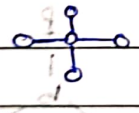


Ex. Let G be an arbitrary graph with n verts & K components then min & max possible components if one vert is removed?

Min:

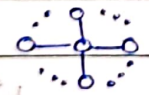


K comps.

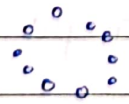


K-1 comps.

Max:



remove central



$\therefore$ $n-1$ comps.

$\therefore$ $K-1$ & $n-1$.

* **Maximal:** A set is maximal w.r.t. property P iff nothing else can be added to the set without violating the property.

Maximum: A set S is maximum w.r.t to property P if for all sets X satisfying property P, S is the largest.

Ex. $\{1, 2, 3, 4, 5\}$. Subsets such that no two elements are consecutive.

$\{1, 2\}$ ✗

$\left.\begin{array}{l}\{1, 3, 5\} \\ \{2, 4\} \\ \{2, 5\} \\ \{3, 5\}\end{array}\right\}$ all are maximal but only $\{1, 3, 5\}$ is maximum.

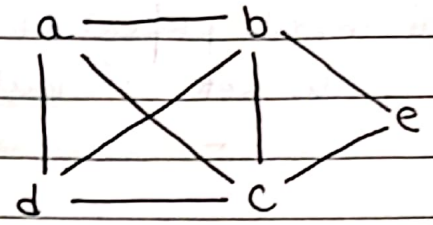
- Every maximum is maximal.
- Maximal may / maynot be unique.
- Maximum may / maynot be unique.

* **Clique:** For a graph $G(V, E)$ is a subset $X \subseteq V$, such that $\forall x \in X, y \in X \ (x, y) \in E$.

- **Independent Set:** For a graph $G(V, E)$ is a subset $X \subseteq V$, such that $\forall x \in X, y \in X \ (x, y) \notin E$
(a.k.a. Stable Set)

- **Clique Number:** Size of maximum clique. $\omega(G)$
- **Independence Number:** Size of maximum Ind. set. $\alpha(G)$

Ex.



Cliques: {a,b}, {a,c}, {a,d}, {b,c}, {b,d}, {b,e}, {c,d}, {c,e}
{a,b,c}, {a,b,d}, {a,c,d}, {b,c,d}, {b,c,e},
{a,b,c,d}.

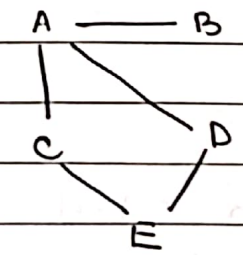
Clique No. : 4

Indp. Sets: {a,e}, {d,e} {a} {b} {c} {d} {e}
Stability No. : 2.

(Underlined are maximal)

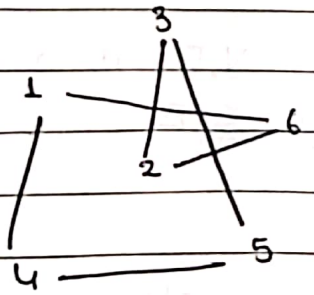
- $\omega(G)$ = Clique Number
  = Size of largest complete subgraph.

Ex.



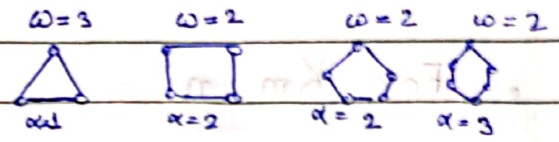
$\omega(G) = 2$ {A, B} or {A, D}...
$\alpha(G) = 3$ {B, C, D}

Ex.



$\alpha(G) = 3$ {1, 2, 5}
$\omega(G) = 2$

- For C_n, $n \geq 3$.



$$\alpha(C_n) = \begin{cases} \frac{n-1}{2}, & n \text{ odd} \\ n/2, & n \text{ even.} \end{cases}$$

$$\therefore \alpha(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$$

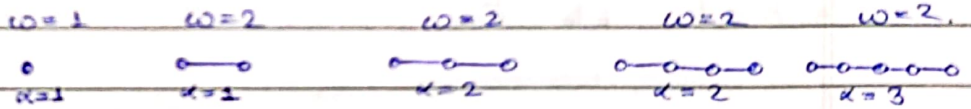
$$\omega(C_n) = \begin{cases} 3, & n = 3 \\ 2, & n > 3. \end{cases}$$

- For K_n

$$\alpha(K_n) = 1.$$

$$\omega(K_n) = n.$$

- For P_n

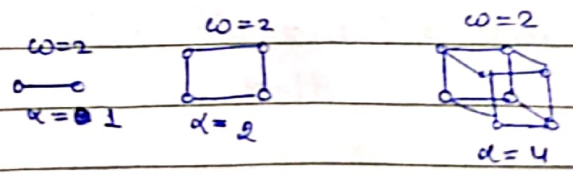


$$\alpha(P_n) = \begin{cases} \frac{n+1}{2}, & n \text{ odd} \\ n/2, & n \text{ even.} \end{cases}$$

$$\therefore \alpha(P_n) = \lceil n/2 \rceil$$

$$\omega(P_n) = \begin{cases} 1, & n = 1 \\ 2, & n > 1 \end{cases}$$

- For Q_n



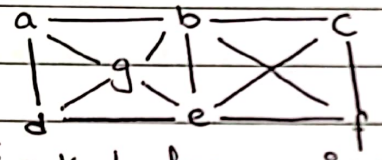
$\alpha(Q_n) = 2^{n/2} = 2^{n-1}$ { Since Bipartite $K_{2^{n/2}, 2^{n/2}}$ }

~~$\omega(Q_n)$~~

- For $K_{m,n}$

$$\alpha(K_{m,n}) = \max(m,n)$$

Ex. A clique in a simple undir. graph is a complete subgraph that isn't contained in any larger complete subgraph. How many cliques are there in below graph?



{asked for maximal clique}

{b, c, e, f} {a, b, g} {a, d, g} {d, e, g} {b, g, e}

$\therefore$ 5.

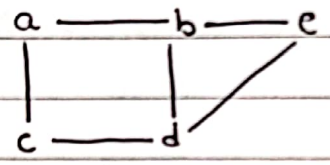
- A clique in $G(V,E)$ would become independent set in complement $\bar{G}(V,\bar{E})$.

- If S is maximum independent set in, then S will become maximum clique in $\bar{G}(V,\bar{E})$. & vice-versa.

* Vertex Cover & Edge Cover

- Vertex Cover : Set of vertices covering all edges.
- Edge Cover : Set of edges which cover all vertices.

Ex.

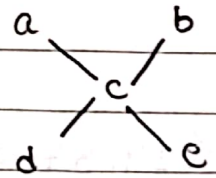


Vertex	Covered Edges
a	ab, ac
b	ab, bd, be
c	ac, cd
d	cd, bd, de
e	be, de.

Edge	Covered Vertex
ab	a, b
ac	a, c
bd	b, d
be	b, e
cd	c, d
de.	d, e.

∴ Min Vertex Cover = {a, d, e}
& Min edge Cover = {ac, be, cd}

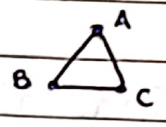
Ex.



Min. vertex cover = {c}
Min. edge cover = {ac, bc, dc, ec}

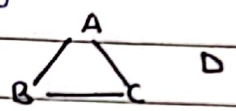
{a, b, c, d, e} is also VC, but its not minimal.
{a, b, e, d} & {c} are both minimal since nothing can be removed.

- An edge can cover 0 or 1 or 2 new vertices.



AB covers A & B. (2)
then, BC covers C only. (1)
then, AC covers no new vert. (0)

- Edge cover exists iff no isolated (0-deg) vertex.

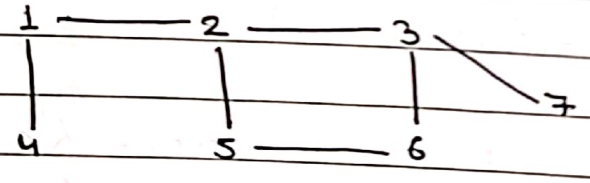


No edge cover exists.

- Size of min. Vertex Cover = Vertex Covering No.
$\beta(G)$

- Size of min. Edge Cover = Edge Covering No.
$= \beta'(G)$

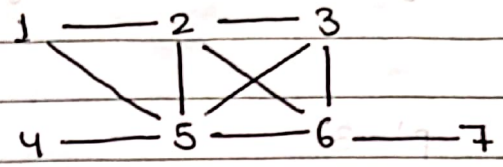
Ex.



MVC = {1, 5, 3}
$\beta = 3$

MEC = {14, 37, 25, 56}
$\beta' = 4$

Ex.



MVC (min) $= \{2, 5, 6\}$

$\beta = 3$

MEC (min) $= \{12, 23, 45, 67\}$

$\beta' = 4$.

Max. Indp. Set $= \{1, 3, 4, 7\}$

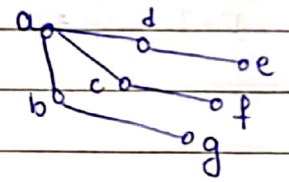
$\alpha = 4$.

Max. Clique $= \{2, 3, 5, 6\}$

$\omega = 4$

Ex. Max. degree vertex must be in every min. vertex cover ?

No. Counterexample



Min. VC $= \{b, c, d\}$

- Every edge at max. can cover 2 vertices. Even if each edge in an n-vertex graph covers two unique vertices, min. edge cover size would be $n/2$.

$\therefore$ Minimum size of edge cover for n-vertex graph is $\geq \frac{n}{2}$ (if edge cover exists).

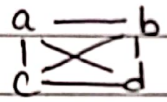
$\therefore \beta' \geq \lceil \frac{n}{2} \rceil$

Ex.

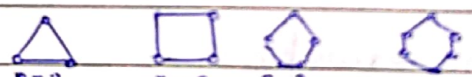
$\beta' = 2$ $\beta' = 3$ Doesn't exist

- For a graph $G(V, E)$, the set V can never be minimum vertex cover.

Even for K_n, β would be $n-1$.
(max. no. of edges possible on n verts.)

Ex. 

$\beta = 3$

- For C_n: 

$\beta = 2$	$\beta = 2$	$\beta = 3$	$\beta = 3$
$\beta' = 2$	$\beta' = 2$	$\beta' = 3$	$\beta' = 3$

$\beta = \left\lceil \frac{n}{2} \right\rceil$

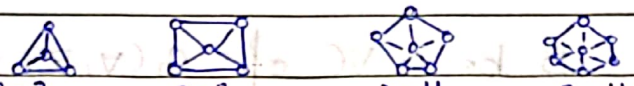
$\beta' = \left\lceil \frac{n}{2} \right\rceil$

- For P_n:

$\beta = 0$	$\beta = 1$	$\beta = 1$	$\beta = 2$	$\beta = 2$	$\beta = 3$
$\beta' = NA$	$\beta' = 1$	$\beta' = 2$	$\beta' = 2$	$\beta' = 3$	$\beta' = 3$

$\beta = \left\lfloor \frac{n}{2} \right\rfloor$ $\beta' = \left\lceil \frac{n}{2} \right\rceil$, $n \geq 2$

- For W_n:



$\beta = 3$, $\beta' = 2$ | $\beta = 3$, $\beta' = 3$ | $\beta = 4$, $\beta' = 3$ | $\beta = 4$, $\beta' = 4$

$$\beta = \left\lceil \frac{n}{2} \right\rceil + 1 \qquad \beta' = \left\lceil \frac{n+1}{2} \right\rceil$$

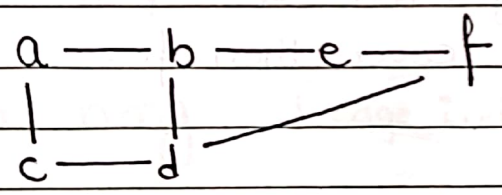
- For K_n:

$$\beta = n-1 \qquad \beta' = \left\lceil \frac{n}{2} \right\rceil$$

- By default Covering Number of a graph refers to vertex covering number.

- Relation b/w Vertex Cover & Ind. Set

Ex.



lets take a VC, say, $\{a, d, e\}$. The remaining vertices $\{b, c, f\}$ form an independent set.

So,

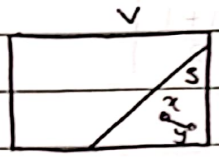
If for a graph $G(V, E)$ S is a vertex cover then $V - S$ is an independent set.

VC should cover all edges, hence it is not possible that there exist any two vertices in $V - S$, which have edge b/w them, otherwise S won't be VC.

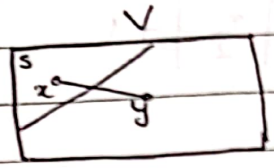
Ex. Let S be a VC of $G(V, E)$ then which of the following isn't possible?

$S \subseteq V$

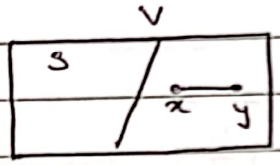
(A)



(B)



(C)



C isn't possible, since S is a VC, two verts x & y having an edge can't be excluded from S.

W.K.t. α is size of largest ind. set & β is size of minimum vertex cover, then for $|V| = n$,

$$|V| = |VC \cup IS|$$

$$\Rightarrow |VC| + |IS| - \underbrace{|VC \cap IS|}_{0 \text{ since no vert. can be in both}}$$

$$\Rightarrow |V| = |VC| + |IS|$$

$$\therefore n = \alpha + \beta$$

$\therefore$ For largest IS, I, $V - I$ is the minimum VC of graph.

~~* Matching~~

Ex. Minimal edge cover of a graph can contain three edges of a triangle?

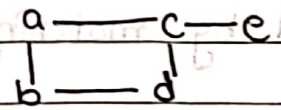
False. No matter the graph, minimal edge cover would only ~~cover~~ have two edges in a triangle.

Ex. Minimal edge cover of a graph can contain a path of length 3?

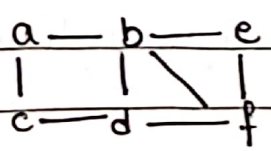
No. For minimal edge cover, only 2 of the three edges are enough.

* Matching (edge independent set)

Matching M of a graph G(V, E) is a set of edges which are pairwise non-adjacent, i.e. no two edges in M are incident on same vertex.

Ex. 

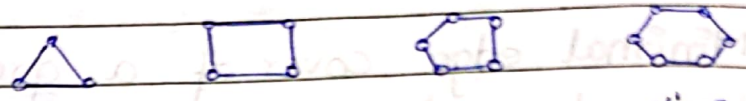
{ab}, {ab, cd}, etc.
maximal & maximum

Ex. 

{ab, cd, ef} maximal & maximum
{bf, ac} maximal.

- Matching No. = $\alpha'(G)$ = Size of largest matching of G.

- For C_n:



$\alpha' = 1$ $\alpha' = 2$ $\alpha' = 2$ $\alpha' = 3$

$\therefore \alpha' = \left\lfloor \frac{n}{2} \right\rfloor$ Perfect Matching when n is even.

- Perfect Matching: Matching which covers all the vertices.

- For K_n:

if n is even: $\alpha' = \frac{n}{2}$ Perfect Matching.
if n is odd: $\alpha' = \frac{n-1}{2}$

$\therefore \alpha' = \left\lfloor \frac{n}{2} \right\rfloor$

- For $K_{m,n}$:

$\alpha' = \min(m, n)$
Perfect matching iff $m = n$

Ex. How many vert. are covered by matching of size 8?

2×8

($\because$ every ~~vert~~ edge covers 2 unique vertices)

Ex. For any graph, $\alpha' \leq \left\lfloor \frac{n}{2} \right\rfloor$?

True. Since every edge covers exactly 2 unique verts, then the max. no. of edges possible are $\left\lfloor \frac{n}{2} \right\rfloor$.

- If Perfect Matching exists then even no. of vertices
- Relationship b/w Matching & Vertex Cover

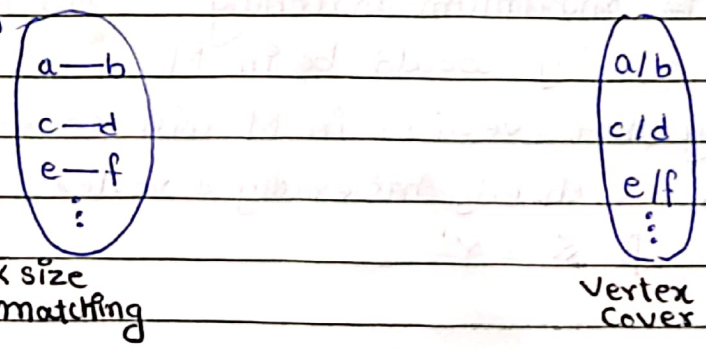
Cardinality of any matching is less than or equal to size of any vertex cover.

Proof: This means that size of largest matching would also be less than or equal to size of smallest vertex cover

$\alpha' \leq \beta$

Say an arbitrary Matching with K edges.
Any vertex cover would have to cover all K edges and then other edges also.

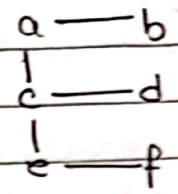
Say



∴ for each edge in matching uv either u or v would be in vertex cover.

∴ size of any matching ≤ size of any vertex cover

Ex.

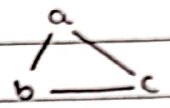


$\alpha' = 3$ $\{ab, cd, ef\}$

$\beta = 3$ $\{a, c, e\}$

$\therefore \alpha' \leq \beta$.

Ex.



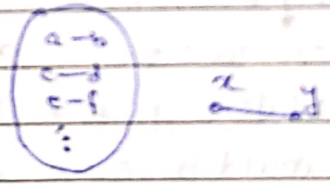
$\alpha' = 1$ $\{ab\}$

$\beta = 2$ $\{a, b\}$

$\therefore \alpha' \leq \beta$

- $\beta(G) \leq 2\alpha'(G)$

If M is the max. matching of G with $\alpha'(G) = |M|$ then this situation isn't possible:



since if there exist such xy edge then it must be in maximum matching. so for all edges xy either x or y would be in M.

We can say that vertices in M will cover all edges, & each edge has exactly 2 vertex,

$\therefore \beta \leq 2\alpha'$

- $\alpha' \leq \beta \leq 2\alpha'$

Ex. Give some graph with $\beta = \alpha'$.

① C_n with even n. $\Rightarrow$ $\frac{n}{2}$

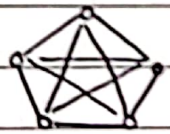
② $K_{m,n}$. $\Rightarrow$ $\min(m,n)$

Ex. Give a graph for which $\beta = \alpha'$

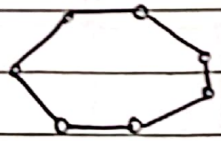
① K_n with odd n. $\beta = n-1$ $\alpha' = \left\lfloor \frac{n}{2} \right\rfloor$

~~C_n with odd n. $\beta = \left\lceil \frac{n}{2} \right\rceil$ $\alpha' =$~~

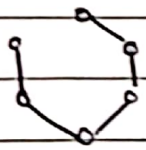
Theorem: If G is bipartite, then $\beta = \alpha'$

Ex. 

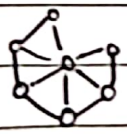
$\omega = 5$ (Clique) $\alpha = 0$ (IS) $\beta' = 3$ (EC) $\alpha' = 2$ (Mat) $\beta = 4$ (VC)

Ex. 

$\omega = 2$ $\alpha = 3$ $\beta = 4$ $\alpha' = 3$ $\beta' = 4$

Ex. 

$\omega = 2$ $\alpha = 3$ $\beta = 3$ $\alpha' = 3$ $\beta' = 3$

Ex. 

$\omega = 3$ $\alpha = 3$ $\beta = 4$ $\alpha' = 3$ $\beta' = 4$

- For any graph G with $\delta > 0$,

$$\alpha \leq \beta'$$

for any largest ind. set S no two vertex have edge b/w them, hence for each one of them there would be a unique edge in smallest edge covering.

$\therefore$ At least $|S|$ no. of edges would be in min. edge covering.

- $$\alpha' + \beta' = |V|$$

Ex. for Q_n

w.k.t Q_n is bipartite (parity bipartition)

So,

$$\alpha = \frac{2^n}{2}$$

$$\beta = \alpha' = \frac{2^n}{2}$$

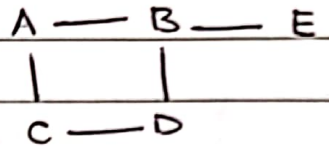
$$\beta' = |V| - \alpha' = 2^n - \frac{2^n}{2}$$

$$\Rightarrow \frac{2^n}{2}$$

* Coloring

- **Vertex Coloring:** A vertex coloring is an assignment of labels to the vertices of a graph such that no two adjacent vertices share same color (labels).

Ex.

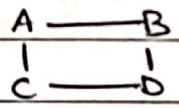


(A: 1, B: 2, C: 3, D: 4, E: 5) is one possible coloring.

(A: 1, B: 2, C: 2, D: 1, E: 1) is the min. possible coloring

- **K-Colorable:** A graph is K-colorable if it can be colored using at most K colors.

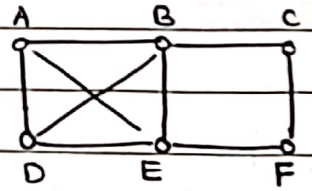
Ex.



G is 2, 3, 4 .... Colorable.
Not 1 colorable.

- **Chromatic No.** $\chi(G)$: The Chromatic No. of graph G is the min. no. of colors required for coloring.

Ex.



Can't do with 1, 2, 3 colors, since ABED is fully connected.
5 is viable.
{A: 1, B: 2, C: 3, D: 4, E: 5, F: 4}

$\therefore \chi(G) = 4$.

4 is also viable.
{A: 1, B: 2, C: 3, D: 4, E: 3, F: 2}

- For K_n :

$$\chi(K_n) = n \quad (\because \text{every 2 verts have edge})$$

- For C_n :

$\chi = 3$ $\quad$ $\chi = 2$ $\quad$ $\chi = 3$ $\quad$ $\chi = 2$ $\quad$ $\chi = 3$

$$\chi(C_n) = \begin{cases} 3, & n \text{ is odd.} \\ 2, & n \text{ is even} \end{cases}$$

- For Null Graph:

$$\chi = 1$$

- For P_n :

1 $\quad$ 2 $\quad$ 2 $\quad$ 2 $\quad$ 2

$$\chi(P_n) = \begin{cases} 1, & n = 1 \\ 2, & n \geq 2 \end{cases}$$

- For Bipartite :

$$\chi = \begin{cases} 1, & \text{if } G \text{ is also Null graph} \\ 2, & \text{else.} \end{cases}$$

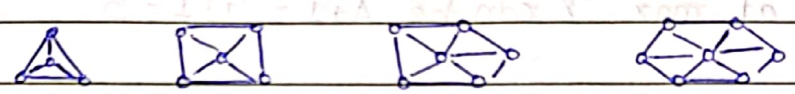
- Every graph with atleast 1 edge is not 1-colorable, i.e. $\chi(G) \geq 2$, if not null graph.

- For Q_n:

Since Q_n is bipartite,

$$\chi(Q_n) = \begin{cases} 1, & n=0 \\ 2, & n \geq 1 \end{cases}$$

- For W_n:



$\chi = 4$ $\quad$ $\chi = 3$ $\quad$ $\chi = 4$ $\quad$ $\chi = 3$

$$\chi(W_n) = \begin{cases} 4, & n \text{ is odd} \\ 3, & n \text{ is even} \end{cases}$$

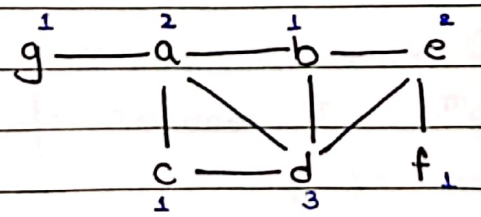
- Greedy Coloring [Doesn't guarentee chromatic No.]
  - ~~Sort in descending order of degree.~~
  - For each vertex assign the minimum label possible.

$$TC = \cancel{T(\text{Sort vertices})} + \sum_{v_i} \left(T(\deg(v_i)) + |V| \right)$$

$$\Rightarrow \cancel{V \log V} + V^2 + E$$

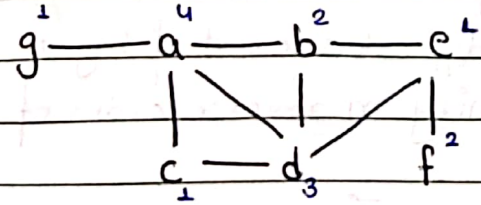
$$\Rightarrow O(V^2 + E)$$

Ex.



for order: $g \to a \to b \to e \to c \to d \to f$

$\therefore$ 3 - Colorable.



for $e \to f \to b \to d \to c \to a \to g$

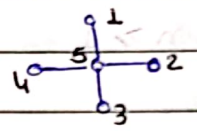
$\therefore$ 4- colorable

Hence doesn't guarentee min coloring.

- If Δ is max degree of vertex in G, then

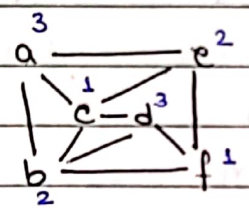
$$\chi(G) \leq \Delta + 1$$

In worst case



at max χ can be $\Delta + 1 = 4 + 1 = 5$

Ex.



Since we have triangle, we can guarentee at least 3 color.
Also we are getting 3-colorable.

$\therefore \quad 3 \leq \chi \leq 3$

$\therefore \chi = 3$

- Relationship b/w χ & ω

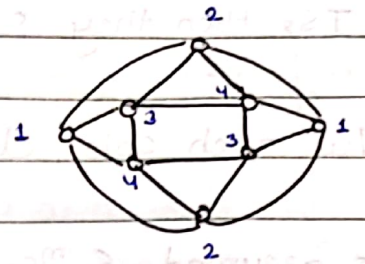
Clique no. tells the size of largest complete subgraph of G. We know K_n req. n colors,

$$\therefore \quad \chi \geq \omega$$

- Guidelines:

(1) Check if standard graph.
(2) Find Clique No. & give diff. colors to vert of largest clique.
(3) Assign colors to remaining in reverse order of degree.

Ex.



We have $\omega = 3$.
Can't be colored in 3 colors.
$\therefore \chi = 4$.

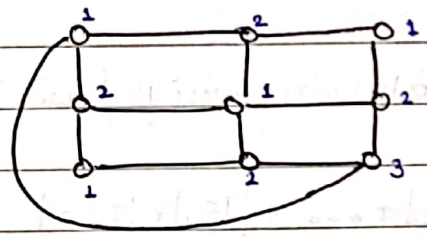
Ex. Find ratio of $\chi(Q_n)$ to $\text{Diam}(Q_n)$.

w.k.t. Q_n is bipartite, so, $\chi = \begin{cases} 1, & n = 0 \\ 2, & n \geq 1 \end{cases}$

also diam $= n$, since two n len bit string can atmost diff. by n positions & has shortest path of len n.

$\therefore \quad \frac{2}{n}, \quad n > 0.$

Ex.



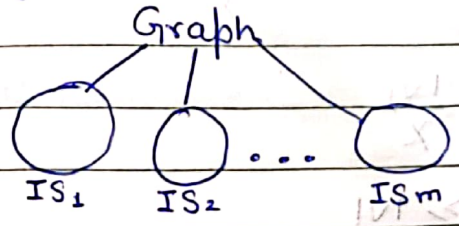
Since C_5 is subgraph $\chi \geq 3$.
$\omega = 2$ & not bipartite.
$\therefore \chi = 3$.

- If G is a properly colored graph then the set of vertices of each color forms an independent set.

- If $\chi(G) = m$, then $|E| \geq {}^mC_2$.

Proof: Assume $|E| < {}^mC_2$ with $|V| = n \geq m$ & $\chi(G) = m$.
Then the graph would have at most $\frac{m(m-1)}{2} - 1$ edges.

(m IS for $\chi = m$)



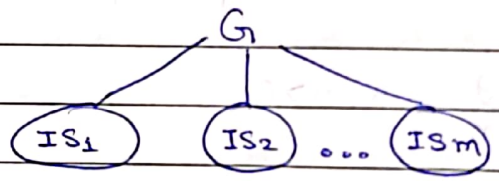
If there is no edge b/w two ISs then they can be merged.

$\therefore$ There is atleast 1 edge b/w each color classes (IS).

$\therefore$ ${}^mC_2$ edges. But we assumed $< {}^mC_2$ edges.

$\therefore$ Contradiction.

- Similarly, to get max. no. of edges with $\chi = m$



we can have full edges between every two IS.

i.e.

$$|IS_1| \times |IS_2| + |IS_1| \times |IS_3| + \dots + |IS_1| \times |IS_m|$$
$$+ |IS_2| \times |IS_3| + \dots + |IS_2| \times |IS_m|$$
$$+ \vdots$$
$$+ |IS_{m-1}| \times |IS_m|$$

- Avg. size of Color class $= \dfrac{|V|}{\chi}$ (Avg no. of vertices in Color classes)

So we can say that there's at least one color class with size $\geq \dfrac{|V|}{\chi}$.

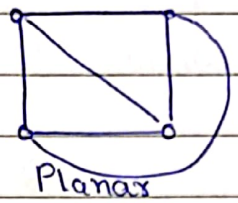
Hence $\alpha \geq \dfrac{|V|}{\chi}$

$\Rightarrow \alpha \chi \geq |V|$

* Planar Graph

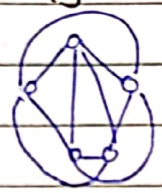
A graph G is called Planar iff it can be drawn on 2D plane (paper) without any two edges crossing (intersecting).

Ex. K_4



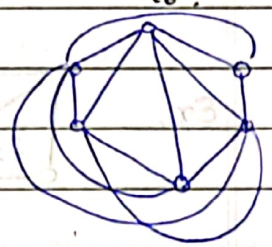
Planar

K_5



Not Planar

K_6.

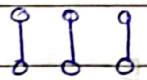


Not Planar.

- The repr. of G which is planar is called Planar Representation of G.

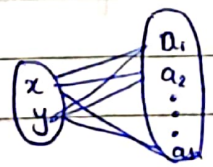
Ex. Is following Planar?

Yes ⇒

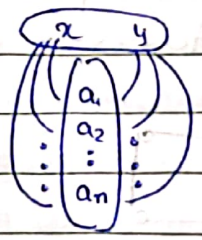


Planar Representation.

- C_n: Yes for all n.
- K_n for K_1, K_2, K_3, K_4: Yes.
- $K_{2,n}$: Yes

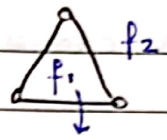


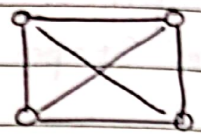
⇒

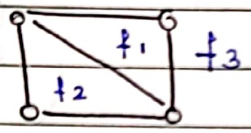


- P_n: Yes for all n.

- **Face / Region :** Any area surrounded by edges in planar representation.

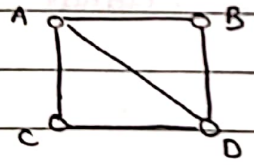
Ex. 

Ex. 
Not Planar Repr.

Ex. 

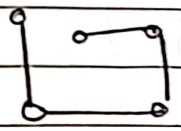
If for a ~~gra~~ planar graph G, P is planar representation then the whole 2D plane is divided into faces/regions of P.

Formally, Region is the maximal area of points in 2D plane in which we can go from any point to any other point without having to cross any vertex or edge of graph.

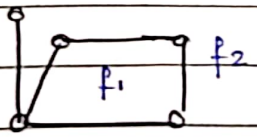
Ex. 

Regions are ABD, ACD & exterior of ABCD.

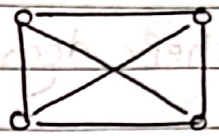
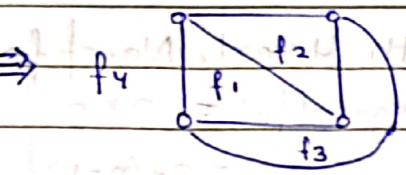
Interior of ABCD isn't region because we'd have to cross AD to go from one point to other.

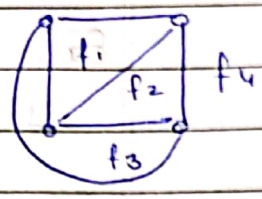
Ex. 

Only 1 face.

Ex. 

2 regions.

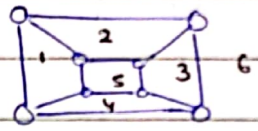
Ex.  $\Rightarrow$ 

OR 

$\therefore$ 4 regions.

- If G is planar its no. of regions doesn't change with diff. planar representation.

Ex. Q3



$\therefore$ 6 regions.

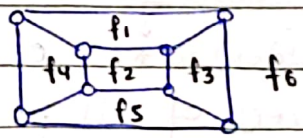
exterior faces = 1.

- Number of exterior regions in any finite planar graph is always one.

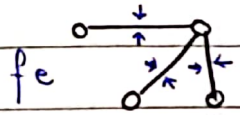
Ex. No. of faces in a tree?

1. Only exterior face.

- Degree of a Faces : Number of edges sides a face is touching.
(each edge has two sides. $\frac{\downarrow}{\uparrow}$)

Ex. 

Deg(f_1) = 4 Deg(f_4) = 4
Deg(f_2) = 4 Deg(f_5) = 4
Deg(f_3) = 4 Deg(f_6) = 4.

Ex. 

Deg(f_e) = 6.

Ex. Tree with 4 vert. No. of faces & their degree?

1 face. $\text{Degree}(f_e) = 2 \times e$

$= 2 \times (n-1)$

$\Rightarrow 6.$

Ex. A connected graph has some face with degree 2. What is this planar graph?

P_2 connected planar

This is the only posibility of a graph with a face of degree 2.

P_1 $\deg(f_e) = 0$

P_1 $\deg(f_e) = 2.$

P_2 $\deg(f_e) = 4.$

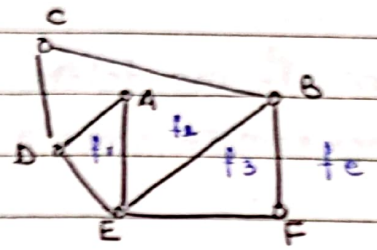
- A connected planar graph G with at least three vertices has every face of degree ≥ 3.

- Total Deg of faces $= \sum_{f_i} \deg(f_i)$

$$\sum_{f_i} \deg(f_i) = 2|E|$$

Since every edge has two sides, $\frac{\downarrow}{\uparrow}$, it would contribute 2 deg to total degree of all faces.

Ex.



deg.

$f_1 \to 3$

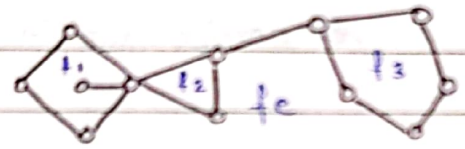
$f_2 \to 5$

$f_3 \to 3$

$f_e \to 5$

Total deg $= 3+5+3+5 = 16. = 2|E|$

Ex.



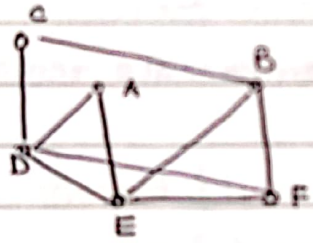
$f_1 \to 6$

$f_2 \to 3$

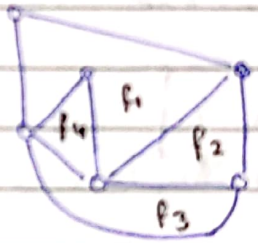
$f_3 \to 5$

$f_e \to 14.$

Ex.



$\Rightarrow$



f_e

$f_1 \to 5$ $\quad f_3 \to 3$ $\quad f_5 \to 4$

$f_2 \to 3$ $\quad f_4 \to 3$

$$\sum_{f \in F} \deg(f) = 18 = 2 \times |E|$$

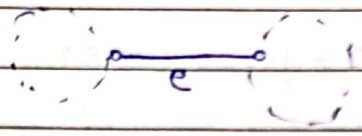
- If G is a planar graph, then all its planar representation has same number of regions & is given for $G(V,E)$ with K components as:

$$\#\text{faces} = |E| - |V| + K + 1$$

Proof: If an edge is deleted from G with K components, then the resultant graph can have either K or K+1 components.

In a planar representation if we delete an edge, the number of faces would either remain same or decrease by one.

C1: e is not part of cycle



delete e.

faces remains same

components increases by one.

C2: e part of cycle.

or

delete e.

faces decreases by 1

components remain same.

∴ Deleting an edge from Planar graph would either reduce no. of faces by one or increase no. of components by one.

Now, we can check Euler's eq^n:

$$f = e - v + k + 1$$

$e=0$: k should be v & face would be 1.

∴ $e=0$, eq^n is true.

let's assume eq^n is true for all values of $e \in [0, e']$.

let's check eq^n for $e = e' + 1$.

{Induction}

So, $f = f(e') = e' - v + k + 1$ is true.

$f = f(e'+1) = e'+1 - v + k + 1$

{deleting 1 edge}

C1: # comp. ~~incr~~ incr. by 1.

$\Rightarrow \quad f = e' - v + (k+1) + 1$. {Remains same}

C2: # faces decrease by 1

$\Rightarrow \quad f - 1 = e' - v + k + 1$

$\Rightarrow \quad f = e' + 1 - v + k + 1$

- Euler formula: $f = e - v + k + 1$

for connected planar: $f = e - v + 2$.

Ex. A connected planar graph has 8 vert, five faces. No. of edges & interior faces?

interior $= 5 - 1 = 4$

edges $= 5 + 8 - 2 = 11$.

- For a planar ~~gra~~ connected graph with n vertices, $n > 2$

$|E| \leq 3n - 6$.

w.k.t. for connected planar graph with $n > 2$, deg. of each face is at least 3. So, $\sum_{f_i} \deg(f_i) \geq 3 \times \#\text{faces}$.

also, $f = e - v + 2$

$\Rightarrow f = e - n + 2$

$\therefore \quad 2e \geq 3 \times (e - n + 2)$

$\Rightarrow \quad e \leq 3n - 6$

- For connected planar graph; $n \geq 3$ & no triangle, ~~min~~ deg of every face is at least 4. since Δ is the only possibility of deg 3.

$$\therefore \quad \sum_{f \in F} \deg(f) \geq 4 \times \#\text{faces}$$
$$\Rightarrow \quad 2e \geq 4 \times (e - n + 2)$$
$$\Rightarrow \quad e \leq 2n - 4.$$

- Theorem [Kuratowski's Graphs]: A graph is not planar iff some subgraph of G is homeomorphic to K_5 or $K_{3,3}$.

- K_5 is the smallest order non-planar graph. (least no. of vertices)
- $K_{3,3}$ is the smallest size non-planar graph. (least no. of edges).

* Coloring (contd.)

- Brook's Theorem: All connected graphs have $\chi(G) < \Delta+1$ except K_n and C_{odd}.

i.e. $\chi(K_n) = \chi(C_{odd}) = \Delta+1$

$\chi(\text{other}) < \Delta+1$

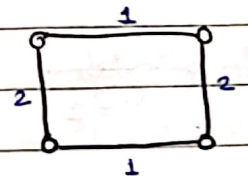
- Say a graph G has K components $C_1, C_2, \ldots C_K$ then the value of

$$\chi(G) = \max_{i=1}^{K} (\chi(C_i))$$

since components are disconnected, so colors can be reused.

- Edge Coloring: No two ~~edges~~ neighbouring edges should have same color.

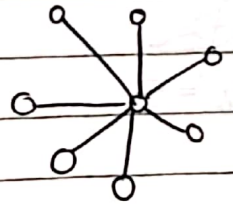
Ex.



$\therefore$ Edge Chromatic No. = 2

- $\chi'(G)$ = Edge Chromatic Number or Chromatic Index.

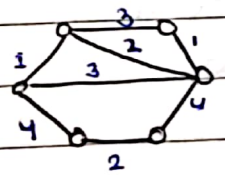
Ex.



$\chi(G) = 2$

$\chi'(G) = 7$

Ex.



since $\Delta = 4$, $\therefore \chi'(G) \geq 4$

- $$\chi'(G) \geq \Delta$$

Since the vertex with degree Δ has Δ edges incident on it, each of which would require different color.

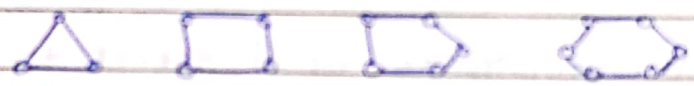
- Vizing Theorem for Edge Coloring :

$$\forall G \quad \chi'(G) = \Delta \text{ or } \Delta+1$$

So every graph can be divided into two classes:

C1: $\chi = \Delta$ C2: $\chi = \Delta + 1$.

- For C_n :



$\chi' = 3$ $\chi' = 2$ $\chi' = 3$ $\chi' = 2$

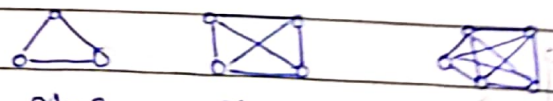
$$\therefore \chi' = \begin{cases} 2 \text{ or } \Delta, & n \text{ even} \\ 3 \text{ or } \Delta+1 & n \text{ odd.} \end{cases}$$

- For P_n :

$\chi' = 0$ $\chi' = 1$ $\chi' = 2$ $\chi' = 2$

$$\chi'(P_n) = \Delta$$

- For K_n :



$\chi' = 3$ $\chi' = 3$ $\chi' = 5$

$$\chi'(K_n) = \begin{cases} \Delta, & n \text{ is even} \\ \Delta+1, & n \text{ is odd} \end{cases}$$

- For Bipartite Graph

Konig's Theorem: All bipartite graphs have $\chi' = \Delta$

So all On. trees, only even-len cycle graphs, etc have $\chi' = \Delta$

imp

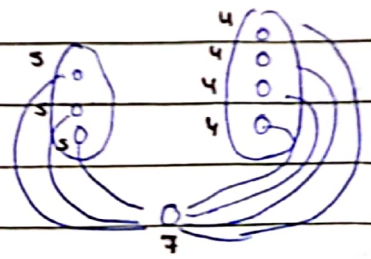
- If G has $\chi'(G) = \Delta + 1$, then G must have at least three vertices with degree $= \Delta$

So, if not at least 3 vert. of deg $= \Delta$ then $\chi'(G) \neq \Delta + 1$
i.e. $\chi' = \Delta$

<3 vert with deg $= \Delta \longrightarrow \chi' = \Delta$

- If G is a regular graph with odd no. of vertices then $\chi' = \Delta + 1$.

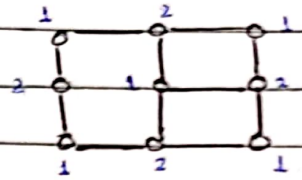
Ex. A graph is obtained by connecting every vertex of $K_{3,4}$ to a new node. Find Chromatic Index.



$\Delta = 7$ & less than 3 verts with deg $= \Delta$

$\therefore \chi' = 7$

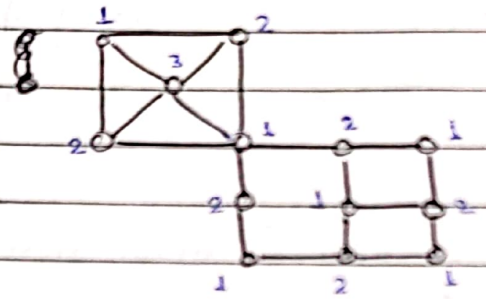
Ex.



$\chi = 2$

$\chi' = 3$ ($\because$ No odd cycle)
($\therefore$ Bipartite)

Ex.



$\chi = 3$

$\chi' = \Delta = 5$

($\because$ ≤ 3 vert deg $= \Delta$)

Ex. Every planar graph has some vertex of degree less than equal to 5.

for $n \geq$

Assume every vertex has deg ≥ 6.

$\therefore \delta = 6$

w.k.t.

$e \leq 3n - 6$

also, $e \geq 3n$ {deg sum $\geq 3n$

$\therefore$ Contradiction.

- The Chromatic Number of any planar graph is at most 4. i.e.

$$\chi(G) \leq 4$$

if G is Planar.

Ex. If G has $\omega \geq 5$, then G is not Planar?

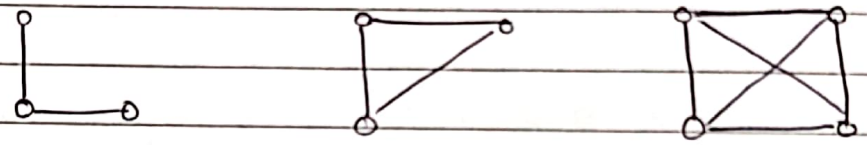
True. $\omega \geq 5$ means K_5 is subgraph of G.

* Euler Problem

Starting from some edge, traverse all the edges exactly once.

- **Euler's Circuit :** In a graph G, Euler circuit exists iff starting from some vertex all the edges can be traversed exactly once & the circuit ends on the same vertex as starting.

Ex.



EC doesn't exist | EC exists | EC doesn't exist.

- A graph G has an Euler Circuit iff its has no odd degree vertex, and G is connected.

Idea: Edges incident on a vertex act as entry/exit to the vertex.
For every entry, there must be an exit & vice-versa. ~~otherwise~~
Hence the degree of each vertex must be even.

Ex. Which of the following is/are Euler?

✗ (1) K_{even}	✓ (4) C_{odd}	✓ (7) Q_{even}	(10) $K_{m,n}$
✓ (2) K_{odd}	✗ (5) P_{even}	✗ (8) Q_{odd}	~~only~~ iff both m & n are even
✓ (3) C_{even}	✗ (6) P_{odd}	✗ (9) W_n	
	✓ (7) P_1		

(11) d-Regular ~~only~~ iff d is even

- **Euler Trail:** A graph G has Euler Trail iff starting from some vertex all the edges can be traversed, i.e., visited exactly once without the restriction of ending the trail on the same vertex as starting.

- For a graph G, an Euler trail exists iff at most two vertices of odd degree in G, and G is connected.

- If exactly two vertices of odd degree exists in G, then they are the endpoints of Euler Trail.

- Every Euler Ckt is also an Euler Trail.

- In Directed Graph Euler Ckt exists iff G is connected & for every vertex, indegree is same as outdegree.

- In Directed Graph Euler Trail exists iff G is connected and for every vertex, indeg is same as outdegree except
  (1) Source : Outdeg = Indeg + 1
  (2) Destination : Indeg = Outdeg + 1

- **Euler Graph :** A graph with Euler circuit.

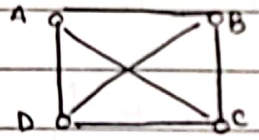
* Hamiltonian Problem

Starting with some vertex, visit every vertex exactly once.

(No vertex repetition → No edge repetition)

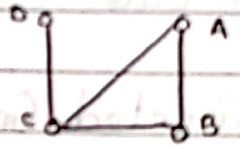
- **Hamiltonian Circuit:** A graph G has a Hamiltonian circuit iff starting with a vertex x, all vertices in the graph can be visited exactly once, and then return to the starting vertex, x.

Ex.



$A \to B \to C \to D \to A$

Ex.



$A \to B \to C \to D \to x$

Not Hamiltonian Graph.

- **Hamiltonian Graph:** Graph with Hamiltonian circuit.

- **Hamiltonian Path:** Graph G has Hamiltonian path if starting from some vertex, all vertices can be visited exactly once.

- There is no efficient algo to check if a graph is Hamiltonian.

- **Theorem:** Every graph with $n \geq 3$ & $\min \deg(S) \geq n/2$ is Hamitonian.

So, $|V| \geq 3 \wedge \delta(G) \geq |V|/2 \longrightarrow$ Hamiltonian.

Ex. Can a path be Hamiltonian & Euler at same time?

Yes. for all P_n is both Hamitonian Path & Euler Trail.

Ex. Let G be an undir. graph with some odd deg. vertices. Add a node v & make it adjacent to all odd deg vertices. The new graph is sure to be:

(1) K_n (3) Euler ✓

(2) Regular (4) Hamiltonian.

There can be even no. of odd deg. vertices. Connecting v to odd deg vertices would make all deg. even.

Ex. Is the complement of C_{25} an Euler graph?

Cond 1: All even deg vertex

In cycle deg. of every vertex is 2.

Max. possible deg. $= n-1 = 24$

Deg. of vertex in $\overline{C_5} = 24 - 2 = 22$ ✓

Cond 2: Connected

$\overline{C_3}$: Disconn.

$\overline{C_4}$: Disconn.

$\overline{C_5}$: To show connected we have to show that there exist a path by every two vertices of $\overline{C_5}$

Let u, v be vertices in C_5.

If $uv \notin E$ then $uv \in \bar{E}$, hence path exists.

If $uv \in E$, then there are 5-4 vertices which aren't adjacent to u or v,

i.e. u, v, u's other neighbour, v's other neighbour

otherthan these 4, all vertices x

$xv \notin E \wedge xu \notin E$

$\therefore xv \in \bar{E} \wedge xu \in \bar{E}$

$\therefore$ Path exists, $u \to x \to v$.

Similarly for all $n \geq 5$, $\bar{C}_n$ is connected.

$\therefore \bar{C}_{25}$ is Euler.

Ex. Are all k-regular with even k Euler?

No. They may/maynot be connected.

Ex. Prove that all vertices in a graph can't have distinct degrees, if $|V| \geq 2$

[Pegionhole Principle].

C1: Connected.

Possible deg $= 1, 2, 3, \ldots n-1$.

verts: $v_1, v_2, v_3, \ldots v_{n-1}, v_n$

$\downarrow 1 \quad \downarrow 2 \quad \downarrow 3 \quad \downarrow (n-1)$

then v_n would have to have one of the possible deg, hence repetition.

C2: Disconnected.

$(n-1)$ isn't possible deg, otherwise the vertex would make graph connected.

Possible deg: $0, 1, 2, \ldots\ldots n-2$

Verts: $v_1, v_2, v_3, \ldots v_{n-1}, v_n$

$\downarrow \quad \downarrow \quad \downarrow \quad \ldots \quad \downarrow$

$0 \quad 1 \quad 2 \quad \ldots \quad n-2$

So, v_n would have to repeat some possible deg.

* Graph Realisation

Given some condition can a graph be realized?

Ex. Deg. Seq.: 4, 3, 2, 1. Can we realise K_n?

No. K_n requires Deg Seq.: $(n-1)$ $(n-1)$ $(n-1)$ ...

Ex. Deg Seq.: 4, 3, 2, 2. Can we create a tree?

In a tree there should be at least 2 deg-1 vert.

Also, in tree,

$$\Sigma deg = 2(n-1)$$

$$4+3+2+2 \neq 6$$

$\therefore$ No.

Ex. Deg. Seq.: 4, 3, 2, 2, 1. Can we have a simple graph?

	4	3	2	2	1
$\Rightarrow$		2	1	1	0
$\Rightarrow$			0	0	0

$\therefore$ Yes.

- For Pseudograph
  - Only check if sum of degree is even or not.

- For all graphs:
  1. Sum of deg. is even
  2. No. of odd deg. vertices is even.

For Multigraph

Given a degree seq. multigraph exists iff:

① $\Sigma \deg(v_i)$ is even

② $\Delta \leq$ Sum of all deg except Δ

Ex. 4, 0

No. $4 > 0$.

Ex. 1, 1, 1, 4, 1, 2.

$4 < 6$ ✓

$\Sigma \deg = 10$ ✓

$\therefore$ Yes.

For Simple Graph

Graphic Sequence: Degree sequence for which graph exists.

Havel - Hakimi Theorem:

(1) Write Degree seq. in non-increasing order.

(2) For current highest degree Δ^c:

(2.1) Delete Δ^c

(2.2) From next Δ^c vertices, reduce 1 each.

(2.3) Reorder in non-increasing order

(2.4) Original Degree Seq. is graphic iff the new sequence is graphic.

Ex. 3, 3, 2, 2, 1, 1, 0, 0

× 2, 1, 1, 1, 1, 0, 0

× 0, 0, 1, 1, 0, 0

⇓

× 1 1 0 0 0 0

× 0 0 0 0 0.

Once all 0s, graphic.

If some deg becomes -ve, not graphic.

- We can also stop early by counting odd deg. vertex as well.

Also, Δ should be $\leq n-1$.

At least two vertices must have same deg.

Ex. 7, 4, 3, 3, 3, 3, 2, 1, 0

Not graphic. Odd no. of odd deg. vertices.

- For Trees:

Given a deg. sequence, a tree exists iff $\delta \geq 1$ and $\Sigma deg = 2 \times (n-1)$ where n is no. of vertices, for at least 2 vertices.

Also, at least 2 vert. should have degree 1.

Ex. 1, 1, 3, 2, 4, 5, 6.

$\delta \geq 1$

$\Sigma deg = 22$ $\qquad 2 \times (n-1) = 2 \times 6 = 12$

$\therefore$ No.

Ex. 6, 1, 1, 1, 1, 1, 1.

$\delta \geq 1$

$\Sigma deg = 12$ $\quad$ $2(n-1) = 12$

$\therefore$ Yes

Ex. 6, 1, 1, 1, 1, 1, 1, 0

$\delta < 1$

$\therefore$ No.

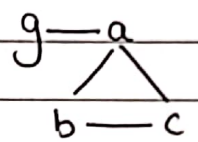
Ex. 0

Yes.

* Cuts & ~~Connectivity~~

- **Articulation Point:** A vertex is called AP iff removal of the vertex and its incident edges in G, disconnects G, i.e., G is disjoint in 2 or more components. (a.k.a Cut Vertex)

Ex.

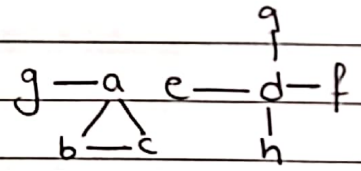


Here a is AP.

Def^n

In any connected or disconnected graph, removal of Articulation Point increases the number of components

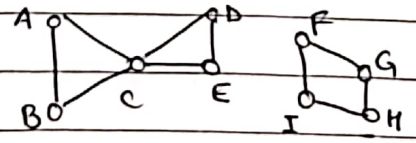
Ex.



Here a & d are cut vertex.

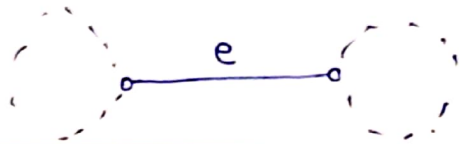
- **Bridge:** An edge in a graph G is a bridge iff removal of the edge results in increase in number of components. (a.k.a Cut Edge)

Ex.



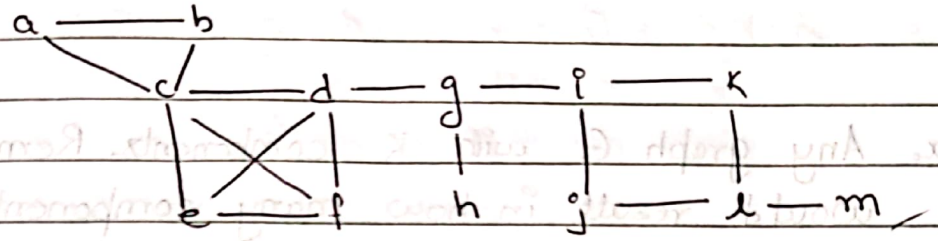
AP: c Bridge: None.

Cut Edge:



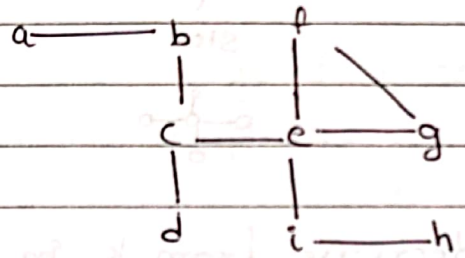
- An edge e in graph G is cut edge iff e is not part of any cycle.

Ex.



Bridges:- dg, gh, gi, lm.

Ex.

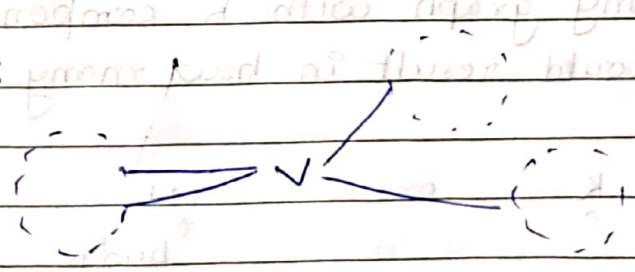


APs: b, c, e, i
(4)

Bridges: ab, bc, cd, ce, ei, ih.
(6)

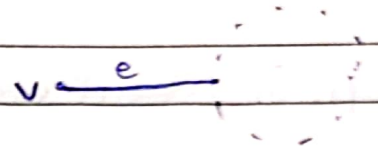
- Theorem: Let G be a graph with a bridge e, incident on the vertex v, then vertex v is cut vertex iff $deg(v) \geq 2$

Cut Vertex:



Proof: Say the edge is as such that the deg. of vertex is <2 & since e is incident on v, $deg(v) \geq 1$

$\therefore deg(v) = 1$



Such a vertex can't be cut vertex

Ex. Any graph G with K components. Removal of one vertex would result in how many components?

$$\underset{\text{isolated vertex}}{k-1} \leq \#\text{components} \leq \underset{\text{star}}{n-1}$$

No. of components can decrease from K in only one case.

Ex. Same ques. with $\delta(G) = 1$ (no isolated vertex)

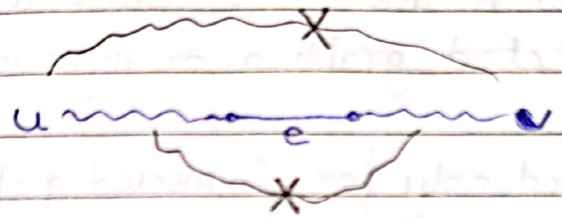
$$k \leq \#\text{components} \leq n-1$$

Ex. From any graph with K components. Removal of 1 edge would result in how many components?

$$\underset{\text{cycle edge.}}{k} \quad \text{or} \quad \underset{\text{bridge}}{k+1}$$

- Every non-pendent vertex of tree is cut vertex.

- e is a cut edge in G iff ~~every~~ there exist two vertices u & v such that every path from u to v goes through e.



Ex. If every edge of a connected graph is a cut edge, then the graph is:

(1) C_n
(2) K_n
(3) P_n
(4) Tree
(5) Forest

Since acyclic & connected, ∴ Tree. & every tree is forest.

- G is a forrest iff every edge is cut edge.

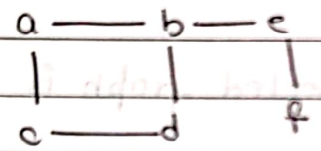
- If G is a graph such that $|V| \geq 3$ & contains a bridge then G contains a cut vertex.

* Connectivity

- **Vertex Cut:** A set of vertices $S \subseteq V$, for graph (separating set) $G(V,E)$ is called a Vertex Cut iff removal of all vertices of S from G disconnects the connected graph G, or single vertex remains.

(Defined only for Connected graphs)

Ex.



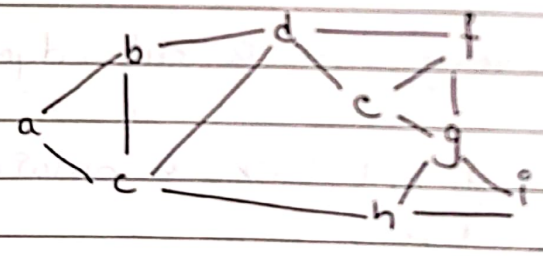
$\{b\}, \{e\}, \{a,d\}, \{c,b\}$
$\{b,e\}$, etc.

Ex.

$\{a\}, \{b\}$, but not $\{a,b\}$.

- **Connectivity Number** (κ): Size of smallest Vertex Cut

Ex.



No size 1 VCs.
$\{b,c\}$ is a VC.

$\therefore \kappa = 2$

Connectivity Number tells how difficult it is to disconnect a graph.

$\kappa = m$ means that removing any combination of $m-1$ vertices, won't disconnect G.

- For every tree $(n \geq 2)$ $K = 1$.
  $n = 1$ $K = 0$
- For K_n, $K = n-1$.
- If G is already disconnected, then $K = 0$.
- For C_n, $K = 2$.
- m-connected: A graph is m connected iff the value of connectivity number is at least m, i.e.
  $K \geq m$.
  OR removal of any m-1 vertices from graph doesn't disconnect graph or leave single vertex in graph.

Ex. Which are 2-connected?

✓(A) C_4 ✓(C) K_6

✓(B) K_4 (D) P_4

Ex. C_4 is $(K = 2)$
1-connected, 2-connected, but not n-connected for $n \geq 3$.

Ex. K_4 is $(K = 3)$
1-conn., 2-conn., 3-conn., but not n-conn. for $n \geq 4$.

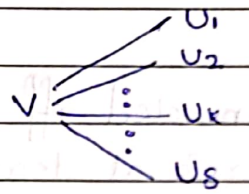
- Separable: A graph is separable if connectivity no. is 1.
- For $K_{m,n}$: $K = \min(m, n)$.
- For W_n: $K = 3$

- **Theorem:** The connectivity of a connected graph is at most its min degree.

  i.e.

$$K(G) \leq \delta(G)$$

Proof: Say $v \in V$ has $\deg(v) = \delta(G)$. Then v is like:



then removing the set $\{u_1, u_2, \ldots u_\delta\}$ then v is sure to be disconnected, since no edge would be incident on v.

$$\therefore K \leq \delta$$

Hence $K \leq n-1$, since $K \leq \delta$ & $\delta \leq (n-1)$.

- **Edge Cut:** A set of edges in graph G is called edge cut iff removing all the edges of the set from graph G disconnects the connected graph into two or more components or leaves a single vertex.

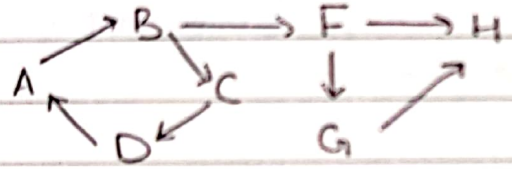
- **Edge Connectivity Number:** Size of smallest Edge Cut. $\lambda(G)$.

- $$\lambda(G) \leq \delta(G)$$

* Connectedness & Components in Directed Graph

- **Strongly Connected:** In a directed graph two nodes, u & v are called strongly connected iff there is a path from u to v & a path from v to u.
  (or just Connected)

Ex.



Strongly Connected: (A,B), (A,C), (A,D), (B,C), (B,D), (C,D)

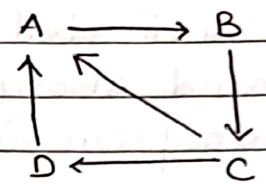
Ex. For any graph G(V,E) define a relation of V as aRb iff a & b are strongly connected, then R is/are:

✓(A) Symmetric
✓(B) Reflexive
✓(C) Transitive
✓(D) Equivalance

Reflexive: Every vertex a has 0 len path to itself.
Symmetric: If A is S.C. to B, then B is S.C to A.
Transitive: ARB implies $A \to\to B$ & $B \to\to A$ &
BRC implies $B \to\to C$ & $C \to\to B$.
$\therefore$ $A \to\to B \to\to C$ & $C \to\to B \to\to A$
$\Rightarrow$ ARC.

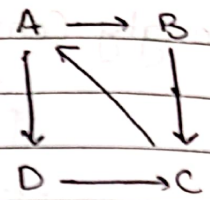
- Since Strongly Connected is Equivalance relation on V, we can find Equivalance Classes.

Ex.



$[A]_R = \{A, B, C, D\}$.

Ex.

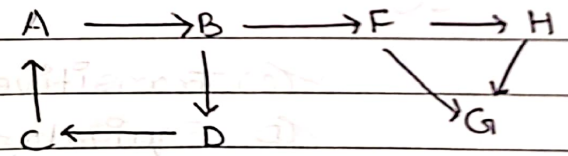


$[A]_R = \{A, B, C\}$
$[D]_R = \{D\}$.

Each equivalance class indicates that each of its elements are pairwise strongly connected.

$\therefore$ Each equivalance class of the "Strongly Connected" relation forms a Strongly Connected Component.

Ex.

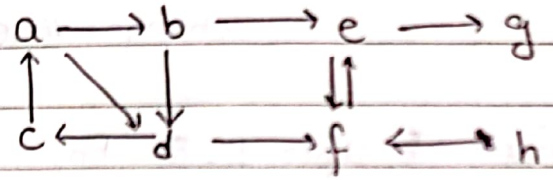


$[A]_R = [B]_R = [C]_R = [D]_R = \{A, B, C, D\}$
$[F]_R = \{F\}$
$[G]_R = \{G\}$
$[H]_R = \{H\}$

- **Strongly Connected Components:** Maximal set of vertices such that every two vertices are pairwise strongly connected.

i.e. Set has to be maximal &
Every two verts have to be strongly connected.

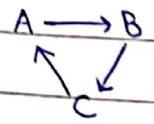
Ex.



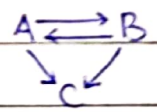
$[a]_{SC} = \{a, b, c, d\}$
$[e]_{SC} = \{e, f\}$
$[g]_{SC} = \{g\}$
$[h]_{SC} = \{h\}$

Ex. Draw 3 vert graph with:

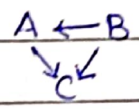
(A) 1 SCC



(B) 2 SCC

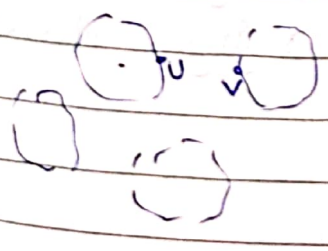


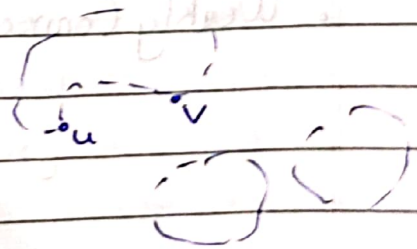
(C) 3 SCC



Ex. In an undir. ~~com~~ graph with K components, add an edge e b/w two existing vertices, then what is the new #components?

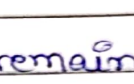
K-1 or K





Ex. In directed graph, with K strongly connected components, adding an edge b/w existing vertices. What happens to #SCC?

- It can't increase. Adding an edge doesn't make graph less connected.

- It can remain same. Ex.  adding A→C won't change #SCC.

- It can even reduce upto 1. Ex. P_n.

P_n had n SCCs but adding an edge makes it to 1 SS.

$\therefore \quad 1 \leq \#SCCs \leq K$

- Strongly Connected Graph: A directed graph is called strongly connected iff #SCC = 1 (or every pair of vertices is strongly connected).

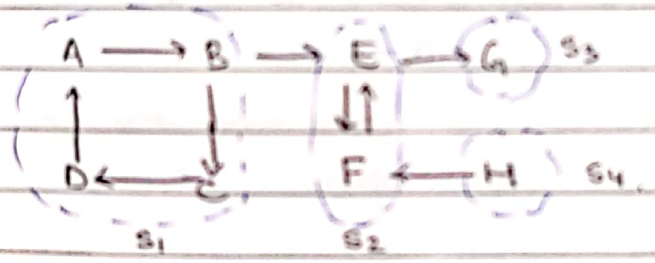
- Weakly Connected Graphs: A directed graph is called weakly connected iff underlying undirected graph is connected.

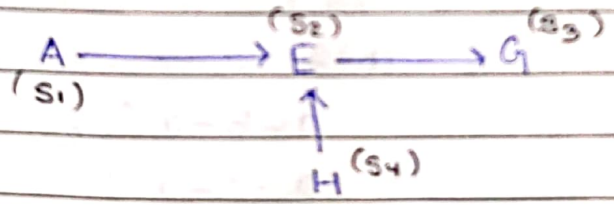
- Every Strongly Connected graph is also weakly connected.

- Associated Directed Acyclic Graph:

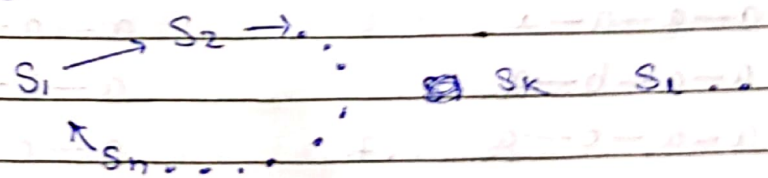
If in a Directed graph G(V, E), if we shrink each of the strongly connected component down to a single node, and draw an edge b/w ~~to~~ two of them if there is an edge b/w some node of the first to some node in the second, we obtain what we call as "Associated DAG".

E.





The obtained graph is alway Acyclic. Suppose we don't get Acyclic, then we'd have graph of ~~the~~ the following type:



But if there is a cycle $(S_1, S_2 .. S_n)$ then these could be merged as one component, hence contradiction with defn of SCC (maximal).

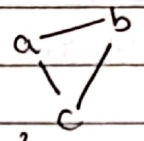
* Power of Adjacency Matrix

- If graph is undirected, AM will be symmetrical.
- If graph has no self-loop, then all diagonal values are 0.

Note: We only consider graphs w/ no multiedges.
& graph can be directed or undirected.

Ex. No. of walks in following graph of len for a to a:

(1)



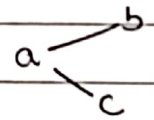
(a) 3 {a— — —a}

a—b—c—a } 2
a—c—b—a

(b) 2 {a— —a}

a—b—a } 2
a—c—a

(2)



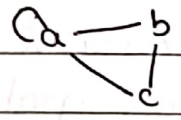
(a) 3

0. {a— — —a

(b) 2

a—b—a } 2
a—c—a

(3)



(a) 3

a—a—a—a
a—a—b—a
a—a—c—a ∴ 7.
a—b—a—a
a—c—a—a
a—b—c—a
a—c—b—a

(b) 2.

a—a—a
a—b—a
a—c—a

For a n vertex undirected connected graph, the ~~dist~~ max. distance b/w any pair of vertices is $\leq n-1$.

So, to check if graph is connected or not, b/w every pair of vertices find a walk of length 1 to $n-1$.

Ex. a — b — c — d.

Suppose in above graph we only check if walks of len $|V|-1 = 4-1 = \boxed{3}$ exist or not.

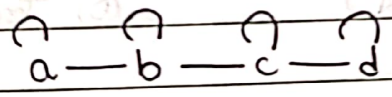
a-b-c-b ✓

a-b-c-d ✓

a- — — c No walk of len 3 exist.

So, we have to check for walks of len 1 to $|V|-1$, i.e. 1, 2 & 3 b/w each pair.

Ex. Suppose in the above graph we make the following modification:



Now, if the original graph had walk of any length b/w 1 to $(n-1)$, then we can get walk of length exactly $(n-1)$ in modified graph:

Original Graph	Modified Graph.
a — b (len = 1)	a-a-a-b (len = 3)
a-b-c (len = 2)	a-b-b-c (len = 3)
a-b-c-d (len = 3)	a-b-c-d (len = 3).

So, in modified graph we can get walk of exactly $(n-1)$ if original graph has walk of len b/w 1 & $(n-1)$

- Power of Adjacency Matrix
- Suppose for a graph we are given an AM M.
This M itself is acting as record of walks of len 1.

- If we do M^2, it will give number of walks of len exactly 2 b/w any pair of vertices.

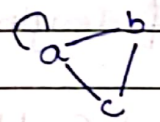
- So, for $K \geq 0$, M^K gives the number of walks of length exactly K b/w any pair of vertices.

- Application 1: Degree of Vertex of Undirected Graph (simple)
M^2's diagonal give $(a-x-a)$ i.e. walk of exactly len 2 from any vertex to itself. This is equal to degree of each vertex.
Also, M^2 trace will give $\sum deg(v_i) = 2|E| = trace(M^2)$

Ex. Undirected Graph which may have self loop. Then, the total deg of graph is given as:

(A) Trace (M^2)
(B) Trace $(M+M^2)$
(C) Trace$(2M+M^2)$
(D) Trace $(M+2M^2)$

Approach 1: Taking small example



$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad M^2 = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$\sum deg(v_i) = 4+2+2$
$\Rightarrow 8.$

So, (B) is correct.

Approach 2: M^2's diagonal give walk of exactly 2. For walks consisting $a-x-a$, $x \neq a$, we have exactly one deg for a. But for edge (a,a) we add 2 deg, but M^2

Note: M^0 will give walks of len 0, $M^0 = I$.

counts only one path. So, we need one extra count of degree for each self loop. So, we can add Trace(M) + Trace(M^2), or $Tr(M + M^2)$.

- Application 2: Find distance b/w two vertices.
  For a pair of vertices v_1 & v_2, the min. K, such that $M^K[v_1, v_2] \neq 0$, gives the min. distance b/w v_1 to v_2.

- Application 3: Checking if undirected graph is connected.
  - Method 1: An undir. graph is connected iff all entries in $(M^1 + M^2 + \dots M^{K-1})$ are non-zero, where K is the no. of vertices.
  - Method 2: For a given graph G, modify it by making all diagonal entries 1 (self loop), or $M \Rightarrow (M + I_K)$. {K is #vert}
    Now iff $(M + I_K)^{K-1}$ has all non-zero entries, then G is connected.
    {$K \geq 3$}

- Application 4: Checking if directed graph is strongly connected
  If G is strongly connected then every vertex is reachable from every vertex, so, b/w every pair of vertices, there is a walk of length atmost "n" for n vertices.
  Ex: $a \to b \to c$, $c \to a$ — a to a has len 3 walk.
  M1:
  So, for a directed graph, it is strongly connected iff $(M^1 + M^2 + \dots + M^n)$ has all non-zero entries.
  M2: For a dir. graph, it is strongly connected iff $(M + I_n)^n$ has all non-zero entries

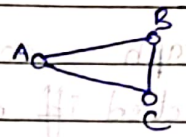
Application 5: Finding Number of 3len cycles in "simple" directed/undirected graph

• Undirected

For a simple graph, M^3 will give at $M^3[i,i]$, all the walks of length 3.

Notice that in simple graph the only possibility of 3len walk is a 3len cycle (since selfloop not allowed)

Ex.



$M^3[A,A]$ would be 2.

① A—B—C—A

② A—C—B—A.

Similarly $M^3[B,B] = M^3[C,C] = 2$.

∴ For a simple graph, a single 3-len cycle contributes 6 3-len walk in Trace(M^3).

So, for a simple undir graph, Trace(M^3)/6 gives the no. of 3-len cycles in the graph.

Also, for any vertex A, $M^3[A,A]/2$ would give the no. of cycles A is involved in.

Note: This idea of Trace(M^k)/2k doesn't work for any k other than 3.

• For directed simple graph Tr(M^3)/2 would also work.

• Directed

In Directed simple graph 3-len cycle contributes 3 3-len walk in Trace(M^3).

So, for simple dir. graph, Trace(M^3)/3 would give the no. of 3len cycles.