MATHS
BY
SHRENIK JAIN SIR

BY: SHASHI KANT SINGH

PROBABILITY

Experiment:

we have thrown a dice : $S = \{1, 2, 3, 4, 5, 6\}$

All possible outcome of experiment is sample space (S)

- $n(S) = 6$

Total no. of possible outcomes.

Event (E):

E: Getting Even number on die.

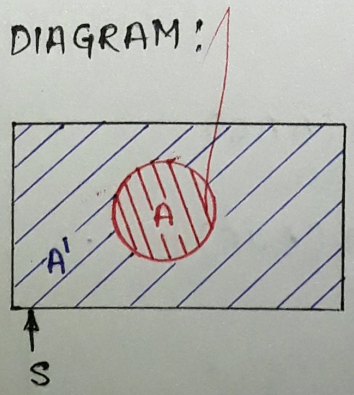
$\{2, 4, 6\}$

$n(E) = 3$

then, probability of Event: P(E

$$\boxed{\text{Probability} = \frac{\text{Required}}{\text{Total}}}$$

VENN DIAGRAM:



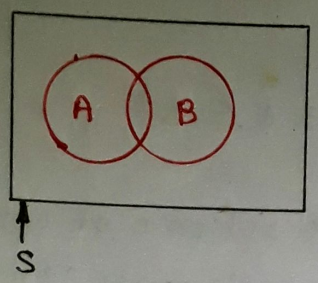
$S = A + A'$ ← complementary event

$$\frac{n(S)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(A')}{n(S)}$$

$$1 = P(A) + P(\bar{A})$$

$$\boxed{P(A) = 1 - P(A')}$$

Law of Addition :



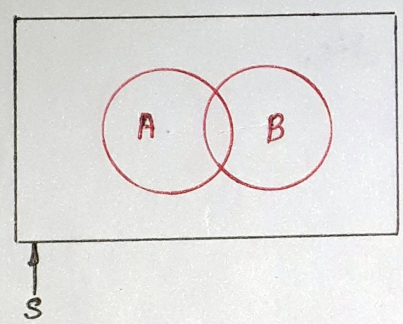
$A \cup B$: Either only A or only B or A intersection B.

$$A \cup B = A + B - A \cap B$$

$$\frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} + \frac{n(A \cap B)}{n(S)}$$

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

- Neither A nor B



$\overline{A \cup B}$

$$n(\overline{A \cup B}) = n(S) - n(A \cup B)$$

$$P(\overline{A \cup B}) = 1 - P(A \cup B)$$

Demorgan's law : $P(\overline{A \cup B}) = P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$

$$P(\overline{A \cap B}) = P(\bar{A} \cup \bar{B}) = 1 - P(A \cap B)$$

Distribution law:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Associative law:

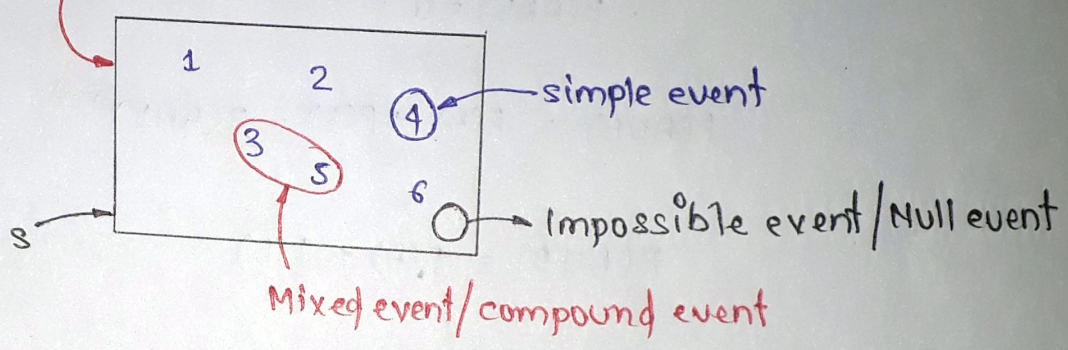
- $A \cap (B \cap C) = (A \cap B) \cap C$
- $A \cup (B \cup C) = (A \cup B) \cup C$

Commutative law:

- $A \cup B = B \cup A$
- $A \cap B = B \cap A$

EVENTS:

Sure event = P(sure event) = P(S) = 1



Equally likely event:

A: Getting even No.

B: Getting odd No.

$\boxed{P(A) = P(B)}$

Exhaustive Events:

A: Even No. $\{2, 4, 6\}$

B: Odd No. $\{1, 3, 5\}$

$A \cup B = \{\underbrace{1, 3, 5, 2, 4, 6}_{S}\}$

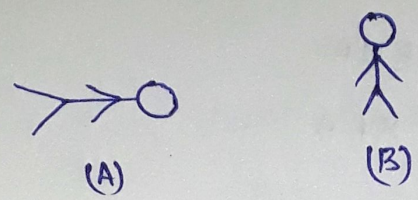
$\boxed{A \cup B = S}$ — Exhaustive event

'OR' • $A \cup B \cup C \ldots\ldots = S$ ← Exhaustive event

$P(A \cup B \cup C \ldots\ldots) = P(S) = 1$

• Mutually Exclusive event :

↑ law of Addition.



$A \cap B$ = happening A & B → ϕ (Null)

$P(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B) - \cancel{P(A \cap B)} \to 0$

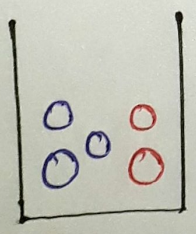
$P(A \cup B) = P(A) + P(B)$

• Independent event :

↑ law of multiplication.

$P(A \cap B) = P(A) \cdot P(B)$

*



(i) P (2 Red from this bag with replacement)

P (1st Attempt = Red Ball & 2nd Attempt = Red Ball)

independent case → $P(E_1) \times P(E_2) = \frac{2}{5} \times \frac{2}{5} = \frac{4}{25}$.

(ii) P (2 Red Balls without replacement)

P(Red Ball in 1st Attempt & Red Ball in 2nd Attempt)

$P(E_1) \cdot P(E_2) =$

$P(A \cap B) = P(A) \times P(B/A)$ ← Dependent case.

$= \frac{2}{5} \times \frac{1}{4} = \frac{2}{20} = \frac{1}{10}$

(iii) P (1 Red Ball & 1 Blue Ball with replacement)

= (Red & Blue) or (Blue Red)

$= \frac{2}{5} \times \frac{3}{5} + \frac{3}{5} \times \frac{2}{5}$

$= \frac{6}{25} + \frac{6}{25} = \frac{12}{25}$ Ans

NOTE!

And → ×

or → +

Conditional Probability:

$P(A \cap B) = P(A) \cdot P(B/A)$

$P(A \cap B) = P(B) \cdot P(A/B)$

A & B are independent

$P(A) \cdot P(B) = P(A) \cdot P(B/A)$

$\boxed{P(B) = P(B/A)}$

$\boxed{P(A) = P(A/B)}$

Ques: X and Y are two random independent events. it is known that $P(X) = 0.40$ and $P(X \cup Y^c) = 0.7$ which one of the following is the value of $P(X \cup Y)$?

(A) 0.7 (B) 0.5 (C) 0.4 (D) 0.3

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$$

$$= P(X) + P(Y) - \underbrace{P(X) \cdot P(Y)}_{\text{independent events}}$$

$$= 0.4 + x - 0.4 \times x$$

$$= 0.4 + 0.5 - 0.4 \times 0.5 = 0.7 \text{ Ans}$$

$$P(X \cup Y^c) = P(X) + P(Y^c) - P(X \cap Y^c)$$

$$0.7 = 0.4 + (1-x) - P(X) \cdot P(Y^c)$$

$$0.7 = 0.4 + (1-x) - 0.4 \times (1-x)$$

$$x = 0.5$$

Ques: The probability that it will rain today is 0.5. The probability that it will rain tomorrow is 0.6. The probability that it will rain either today or tomorrow ($\cup$) is 0.7. what is the probability that it will rain today and ($\cap$) tomorrow?

(A) 0.3 (B) 0.25 (C) 0.35 (D) 0.4

$P(A) = 0.5$ & $P(B) = 0.6$

$P(A \cup B) = 0.7$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$0.7 = 0.5 + 0.6 - P(A \cap B)$$

$$P(A \cap B) = 0.4$$

Ques. E_1 and E_2 are events in probability space satisfying the following constraints $P(E_1) = P(E_2)$; $P(E_1 \cup E_2) = 1$; E_1 and E_2 are independent then $P(E_1)$ is

(A) 0 (B) $1/4$ (C) $1/2$ (D) 1

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1) \cdot P(E_2)$$

$$1 = x + x - x \cdot x$$

$$1 = 2x - x^2$$

$$x^2 - 2x + 1 = 0$$

$$(x-1)^2 = 0$$

$$x = 1$$ Ans.

Ques. Let $P(E)$ denote the probability of an event E. Given $P(A) = 1$, $P(B) = 1/2$ the values of $P(A/B)$ and $P(B/A)$ respectively are

(A) $\frac{1}{4}, \frac{1}{2}$ (B) $\frac{1}{2}, \frac{1}{4}$ (C) $\frac{1}{2}, 1$ (D) $1, \frac{1}{2}$

$$P(A \cap B) = P(B) \cdot P(A/B)$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \cdot \cancel{P(B)}}{\cancel{P(B)}} = 1$$

$$P(A \cap B) = P(A) \cdot P(B/A)$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A) \cdot P(B)}{P(A)} = \frac{1}{2}$$

Ques. The chance of a student passing an exam is 20% (0.2). The chance of a student passing the exam and ($\cap$) getting above 90% in it is 5% (0.05). Given that a student passes the examination the probability that the student gets above 90% marks is.

(A) $1/18$ (B) $1/4$ (C) $2/9$ (D) $5/18$

$P(A) = 0.2$

$P(A \cap B) = 0.05$

$P(B/A) = ?$

$P(A \cap B) = P(A) \cdot P(B/A)$

$0.05 = 0.2 \times x$

$x = 1/4$ Ans

Ques. An examination consist of two papers, Paper 1 and Paper 2. The probability of failing in Paper 1 is 0.3 and then in Paper 2 is 0.2. Given that a student has failed in paper 2, the probability of failing in Paper 1 is 0.6. The Probability of a student failing in both the paper is

$\cap$

(A) 0.5 (B) 0.18 (C) 0.12 (D) 0.06

$P(A) = 0.3$; $P(B) = 0.2$; $P(A/B) = 0.6$

$P(A \cap B) = P(B) \cdot P(A/B)$

$P(A \cap B) = 0.2 \times 0.6 = 0.12$

COINS :

- 1 coin : ○ $\longrightarrow$ $S = \{H, T\}$ $\longrightarrow$ $n(S) = 2$

- 2 coins : ○ ○ $\longrightarrow$ $S = \{HH, HT, TH, TT\}$ $\longrightarrow$ $n(S) = 4$

- 3 coins : ○○○ $\longrightarrow$ $S = \{$ HHH HTH THH TTH
  HHT HTT THT TTT $\}$ $\longrightarrow$ $n(S) = 8$

- $\boxed{n \text{ times} : n(S) = 2^n}$

- 4 coins : $n(S) = 2^4 = 16$

$\{$ HHHH HTHH THHH TTHH
HHHT HTHT THHT TTHT
HHTH HTTH THTH TTTH
HHTT HTTT THTT TTTT $\}$

Ques. Two coins are simultaneously tossed. The probability of two heads simultaneously appearing is

(A) $1/8$ (B) $1/6$ (C) $1/2$ (D) $1/4$

$S = \{HH, HT, TH, TT\}$ $n(S) = 4$

$$P(HH) = \frac{n(E)}{n(S)} = \frac{1}{4}$$

Ques. if three coins are tossed simultaneously the probability of getting at least one head is

(A) $1/8$ (B) $3/8$ (C) $1/2$ (D) $7/8$

M1: $S = \{HHH\ HTH\ THH\ TTH\ HHT\ HTT\ THT\ TTT\}$ $n(S) = 8$

$n(E) = 7$

$$P(\text{at least one head}) = \frac{n(E)}{n(S)} = \frac{7}{8}$$

M2: $P(\text{at least one head}) = 1 - P(\text{no. head})$

$$= 1 - \frac{1}{8}$$

$$= 7/8$$

Ques: A fair coin is tossed independently four times. the probabability of the event " the number of times heads show up is more than the number of times tails show up" is

(A) $1/16$ (B) $1/8$ (C) $1/4$ (D) $5/16$

M1: $n(S) = 2^4$

$$P = \frac{n(H) > n(T)}{n(S)} = \frac{5}{16}$$

M2 : Combinations

case 1 : (3H, 1T) $\longrightarrow$ (HHHT, HHTH, HTHH, THHH)

case 2 : (4H, 0T) $\longrightarrow$ (HHHH)

$$\therefore P = \frac{5}{16}$$

Ques. A coin is tossed 4 times. what is the probability of getting heads exactly 3 times?

(A) 1/4 (B) 3/8 (C) 1/2 (D) 3/4

$n(S) = 2^4 = 16$

$n(E) = 4$

$P = \frac{4}{16} = \frac{1}{4}$ Ans

DIE :

n times $\Rightarrow n(S) = 6^n$

- 1 Die : $\{1, 2, 4, 3, 5, 6\} \rightarrow n(S) = 6$
- 2 Dice : $\{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6)$
  $(2,1)\ (2,2)\ (2,3)\ (2,4)\ (2,5)\ (2,6)$
  $(3,1)\ (3,2)\ (3,3)\ (3,4)\ (3,5)\ (3,6)$
  $(4,1)\ (4,2)\ (4,3)\ (4,4)\ (4,5)\ (4,6)$
  $(5,1)\ (5,2)\ (5,3)\ (5,4)\ (5,5)\ (5,6)$
  $(6,1)\ (6,2)\ (6,3)\ (6,4)\ (6,5)\ (6,6)\}$

$n(S) = 6^2 = 36$

Ques. A fair dice is rolled twice. the probability that an odd number will follow an even number is

(A) $1/2$ (B) $1/6$ (C) $1/3$ (D) $1/4$

M1: Generating sample space

$n(S) = 6^2 = 36$ $n(E) = 9$; $\{(2,1)\ (2,3)\ (2,5)\ (4,1)\ (4,3)\ (4,5)\ (6,1)\ (6,3)\ (6,5)\}$

$$P = \frac{n(S)}{n(E)} = \frac{9}{36} = \frac{1}{4} \text{ Ans}$$

M2: Combination

Die 1 : 1 2 3 4 5 6

Die 2 : 1 2 3 4 5 6

$$\therefore P = \frac{9}{36} = \frac{1}{4}$$

M3 :

= P(odd no. follows an even no.)

= P (Getting even no on Die 1 & Getting odd no. on Die2)

$= P(E) \cdot P(O)$

$= \frac{3}{6} \cdot \frac{3}{6}$

$= \frac{9}{36} = \frac{1}{4}$ Ans

Ques. Two dice are thrown. what is the probability that the sum of the numbers on the two dice is eight?

(A) 5/36 (B) 5/18 (C) 1/4 (D) 1/3

M1 : Sample space

M2 : Combination.

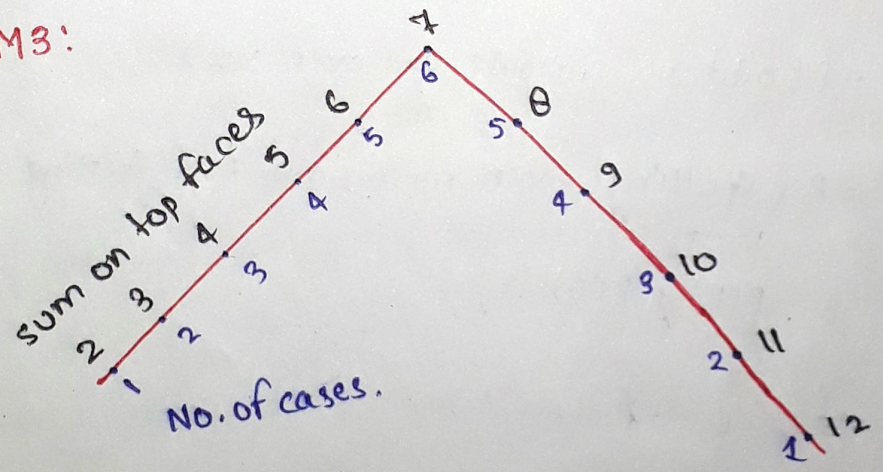
Die 1 : 1 2 3 4 5 6

Die 2 : 1 2 3 4 5 6

Die 1 + Die 2 = 8 $\{(2,6)(3,5)(4,4)(5,3)(6,2)\}$

$$P = \frac{n(E)}{n(S)} = \frac{5}{6^2} = 5/36$$

M3:



Ques. A Box contains 5 Black and 5 Red Balls. Two balls are randomly picked one after another from the box, without replacement. The probability for both balls being red is:

(A) $1/90$ (B) $1/2$ (C) $19/90$ (D) $2/9$

M1: Prob. $= \frac{5}{10} \times \frac{4}{9} = \frac{2}{9}$ Ans.

M2: only valid for 'without replacement' case

$$P = \frac{\text{Req.}}{\text{Total}} = \frac{{}^5C_2}{{}^{10}C_2} = \frac{\frac{5\times4}{2\times1}}{\frac{10\times9}{2\times1}}$$

$$= \frac{5}{10} \times \frac{4}{9} = \frac{2}{9}$$

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

Ques: A Box contains 4 Reb balls and 6 Black balls. Three balls are selected randomly from the box one after another, without replacement. The probability that the selected set contains one red ball and two black balls is

(A) $1/20$ (B) $1/12$ (C) $3/10$ (D) $1/2$

M1: P(1 Red ball & 2 Black balls)
Combination.

$$P = RBB + BRB + BBR$$

$$P = \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8} + \frac{6}{10} \times \frac{4}{9} \times \frac{5}{8} + \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8}$$

$$P = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}$$

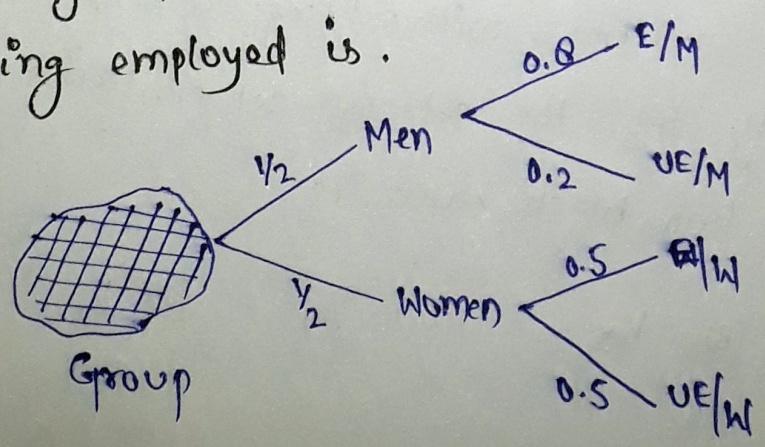
$$P = \frac{1}{2} \text{ Ans}$$

M2: only valid for 'without replacement' case.

$$\text{Prob.} = \frac{\text{Req.}}{\text{Total}} = \frac{{}^4C_1 \times {}^6C_2}{{}^{10}C_3}$$

$$= \frac{\frac{4}{1} \times \frac{6 \times 5}{2 \times 1}}{\frac{10 \times 9 \times 8}{3 \times 2 \times 1}} = \frac{1}{2} \text{ Ans}$$

Ques. A group consists of equal number of men & women. of this group 20% of the men and 50% of the women are unemployed. if a person is selected at random this group, the probability of the selected person being employed is.



Total Probability

P(Person is employed)

$$P(M) \cdot P(E/M) + P(W) \cdot P(E/W)$$

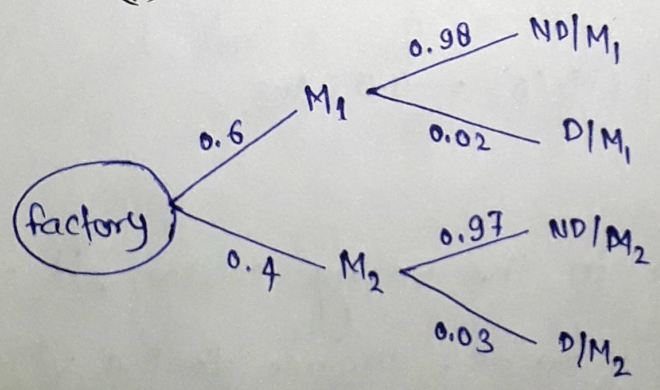
$$= \frac{1}{2} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{1}{2}$$

$$= 0.65 \text{ Ans.}$$

BAYES THEOREM:

Ques. In a factory, two machines M_1 and M_2 manufactore 60% and 40% of the auto components respectively. out of the total production, 2% of M_1 and 3% of M_2 are found to be defeactive. if a randomly drawn auto component from the combined lot is found defective. what is the probability that it was manufactured by M_2?

(A) 0.35 (B) 0.45 (C) 0.5 (D) 0.4



find: $P(M_2/D)$

M1 : Bayes theorem formula.

$$P(M_2/D) = \frac{P(D/M_2) \cdot P(M_2)}{\text{Total Defective} = \Sigma D \text{ (Total Prob.)}}$$

$$P(M_2/D) = \frac{P(D/M_2) \cdot P(M_2)}{P(M_2) \cdot P(D/M_2) + P(M_1) \cdot P(D/M_1)}$$

$$P(M_2/D) = \frac{0.03 \times 0.4}{0.4 \times 0.03 + 0.6 \times 0.02}$$

$$P(M_2/D) = 0.5$$

M2 :

$$\text{Prob.} = \frac{\text{Req.}}{\text{Total.}}$$

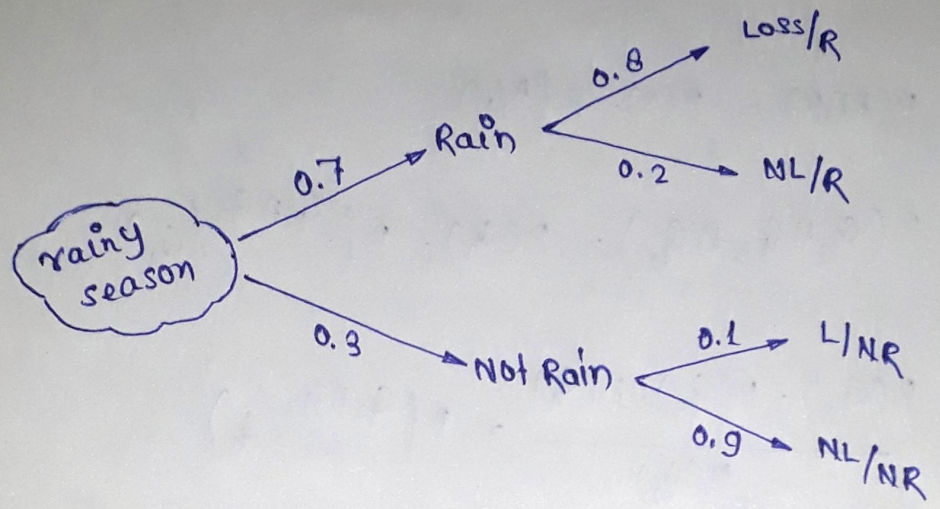
$$P(M_2/D) = \frac{P(M_2) \cdot P(D/M_2)}{P(M_1) P(D/M_1) + P(M_2) \cdot P(D/M_2)}$$

$$P(M_2/D) = 0.5 \text{ Ans.}$$

Ques. In a given day in the rainy season, it may rain 70 % of the time. if it rains, chance that a village fair will make a loss on that day is 80 %. However, if it does not rain, chance that the fair will make a loss on that day is only 10 %. if the fair has not made a loss on a given day in the rainy season, what

is the probability that it has not rained on that day?

(A) 3/10 (B) 9/11 (C) 14/17 (D) 27/41



find : $P(NR/NL)$

$$\text{Prob.} = \frac{\text{Req.}}{\text{Total}}$$

$$P(NR/NL) = \frac{P(NL/NR) \cdot P(NR)}{P(R) \cdot P(NL/R) + P(NR) \cdot P(NL/NR)}$$

$$P(NR/NL) = \frac{0.3 \times 0.9}{0.7 \times 0.2 + 0.3 \times 0.9} = \frac{27}{41}$$

Ques. A hydraulic structure has four gates which operate independently. The probability of failure of each gate is 0.2. Given that gate 1 has failed, the probability that both gates 2 and 3 will fail is

(n → x)

(A) 0.240 (B) 0.200 (C) 0.040 (D) 0.008

M1 : $P(G_1) = 0.2$; $P(G_2) = 0.2$; $P(G_3) = 0.2$; $P(G_4) = 0.2$

then, $P(G_2 \cap G_3 / G_1) = ?$

$$P(A \cap B) = P(B) \cdot P(A/B)$$

$$P(G_2 \cap G_3 \cap G_1) = P(G_1) \cdot P(G_2 \cap G_3 / G_1)$$

$$\frac{P(G_2) \cdot P(G_3) \cdot \cancel{P(G_1)}}{\cancel{P(G_1)}} = P(G_2 \cap G_3 / G_1)$$

$$\therefore P(G_2 \cap G_3 / G_1) = 0.2 \times 0.2 = 0.040$$

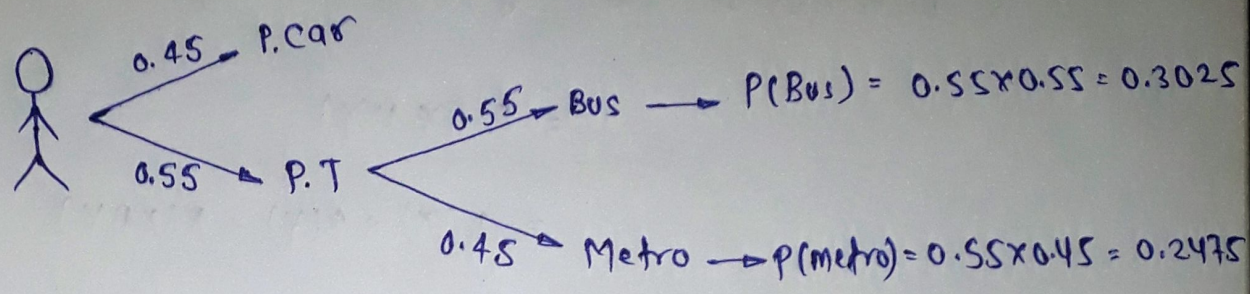
M2 ! All the case are independent.

$$P(G_2 \cap G_3) = P(G_2) \cdot P(G_3) = 0.2 \times 0.2 = 0.04$$

Ques. A person on a trip has a choice between private car and public transport. The probability of using a private car is 0.45. while using the public transport further choices available are bus and metro, out of which the probability of commuting by a bus is 0.55. In such a situation, the probability (rounded up to two decimals) of using a car, bus and metro respectively would be

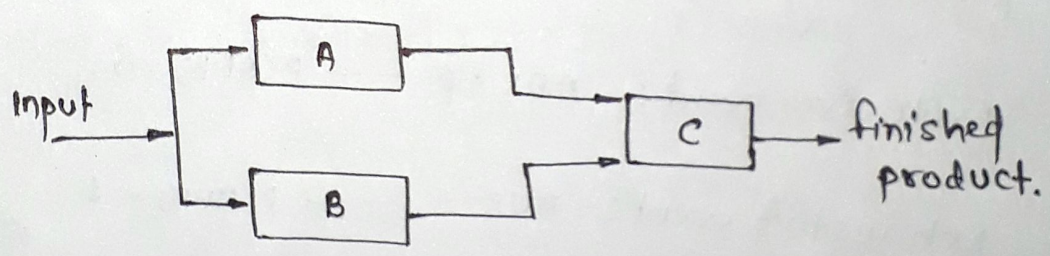
(a) 0.45, 0.30 and 0.25 (B) 0.45, 0.25 and 0.30

(C) 0.45, 0.55 and 0.00 (D) 0.45, 0.35 and 0.20



Ques. The fig. shown the schematic of production process with machine A, B and C. An input job needs to be preprocessed eithe by A or by B before it is fed to C, from which the final finished product comes out. The prob. of failure of machines are given as:

$P_A = 0.15$, $P_B = 0.05$, $P_C = 0.1$



Assuming, independence of failure of the machines, the probability that a given job is successfully processed (upto the three decimal place) is.

failure: $P_A' = 0.15$ $P_B' = 0.05$ $P_C' = 0.1$

success: $P_A = 1-0.15 = 0.85$, $P_B = 1-0.05 = 0.95$

$P_c = 1 - 0.1 = 0.99$

success: $P((A \cup B) \cap C)$

Distributive law: $P[(A \cap C) \cup (B \cap C)]$

$= P(X \cup Y)$

$= P(X) + P(Y) - P(X \cap Y)$

$= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$

(Independent): $= P(A) \cdot P(C) + P(B) \cdot P(C) - P(A) \cdot P(B) \cdot P(C)$

$= 0.85 \times 0.9 + 0.95 \times 0.9 - 0.8 \times 0.95 \times 0.9$

$= 0.894$

Ques: Let S be a sample space and two mutually exclusive events A and B such that $A \cup B = S$. if $P(.)$ denotes the probability of the event, the max. value of $P(A)P(B)$ is —

Exclusive event: $A \cap B = \phi \longrightarrow P(A \cap B) = 0$

Exhaustive event: $A \cup B = S \longrightarrow P(A \cup B) = 1$

find: $[P(A) \cdot P(B)]_{max}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$1 = P(A) + P(B) - 0$

Now, $P(A) = x$

$P(B) = 1 - x$

$$P(A) \cdot P(B) = f(x) = x(1-x)$$

$$f(x) = x - x^2$$

$$f'(x) = 0 \longrightarrow 1 - 2x = 0$$

$$x = 1/2$$

$$f''(x) = -2 \rightarrow -ve \ \{max.\}$$

$\therefore$ max. value : $f(x = 1/2) = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$ Ans.

Ques. Three vendors were asked to supply a very high precision component. The respective probabilities of their meeting the strict design specifications are 0.8, 0.7 and 0.5. Each vendor supplies one component. The prob. that out of total three components supplied by the vendors at least one will meet the design specification is ___.

P(at least one will meet DS) = 1 − P(Nobody will meet DS)

$= 1 - (0.8 \times 0.7 \times 0.5)$

$= 0.97$ Ans.

Ques. P and Q are considering to apply for a job. The Probability the P applies for the job is 1/4. The probability that P applies for the job given that Q applies for the job is 1/2, and the probability that Q applies for the job given that P applies for

the job is $1/3$. Then the probability that P does not apply for the job given that Q does not apply for the job is _

(A) $4/5$ (B) $5/6$ (C) $7/8$ (D) $11/12$

A = P applies for job

B = Q applies for job

$P(A) = 1/4$ $P(A/B) = 1/2$ $P(B/A) = 1/3$

$$P(\bar{A}/\bar{B}) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{B})} = \frac{P(\overline{A \cup B})}{P(\bar{B})} = \frac{1 - P(A \cup B)}{1 - P(B)}$$

$$\therefore P(\bar{A}/B) = \frac{1 - 1/3}{1 - 1/6} = \frac{2}{3} \times \frac{6}{5} = \frac{4}{5} \text{ Ans}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \quad \text{——— (1)}$$

$$P(A/B) = \frac{P(A \cap B)}{P(A)} \quad \text{——— (2)}$$

from (1) & (2) ; $P(B) = 1/6$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{4} + \frac{1}{6} - \underbrace{\left\{ P(A/B) \cdot P(B) \right\}}_{(1/12)}$$

$$= 1/3$$

CARDS :

- 52 Card
  - BLACK 26
    - Spades ♠ 13
    - club ♣ 13
  - RED 26
    - Hearts ♥ 13
    - Diamond ♦ 13

suits

13 Cards :

Ace ← A 2 3 4 5 6 7 8 9 10 J Q K

Jack, Queen, King

NON-FACE CARDS → FACE CARD

IN TOTAL :

How many face cards : $3 \times 4 = 12$

Non-face cards : $10 \times 4 = 40$

Ques. Two cards are drawn at random in succession with replacement from a deck of 52 well shuffled card probability of getting both 'Aces' is

(a) $1/169$ (b) $2/169$ (c) $1/13$ (d) $2/13$

sol:

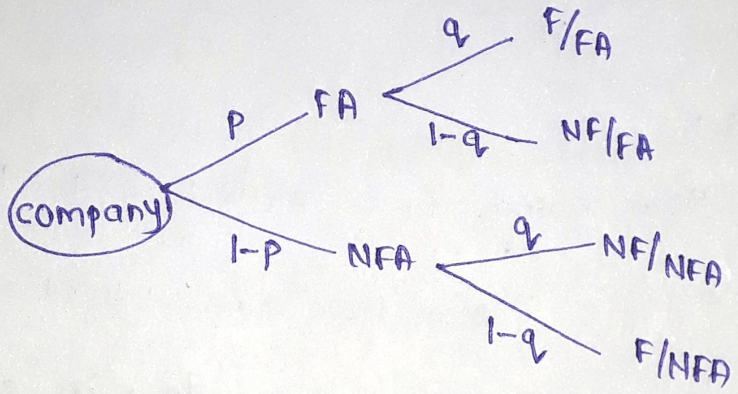
$$P(\text{Both Aces}) = \frac{4}{52} \times \frac{4}{52}$$

$$= \frac{1}{13} \times \frac{1}{13}$$

$$= \frac{1}{169} \text{ Ans.}$$

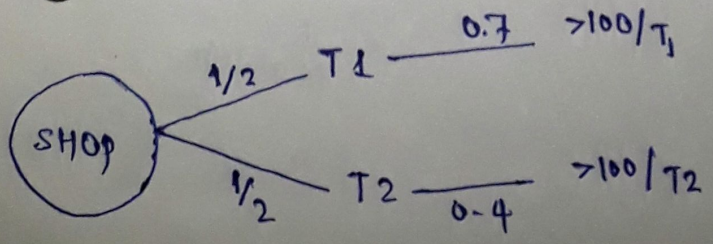
Ques. Consider a company that assembles computers. The probability of faulty assemble of any company is P. The company therefore subjected each computer for a testing process. The testing process gives the correct result for any computer with a probability of q. what is the prob. of a computer being declared faulty?

(A) $pq + (1-p)(1-q)$ (B) $(1-q)p$ (C) $(1-p)q$ (D) pq



$\therefore$ P(computer being declared faulty) = $pq + (1-p)(1-q)$

Ques: Suppose that a shop has an equal number of LED bulbs of two different kinds/Types. The probability of an LED BULB lasting more than 100 hours given that it is of type 1 is 0.7 and given that it is of type 2 is 0.4. The Probability that an LED bulb chosen uniformly at random lasts more than 100 hours is –



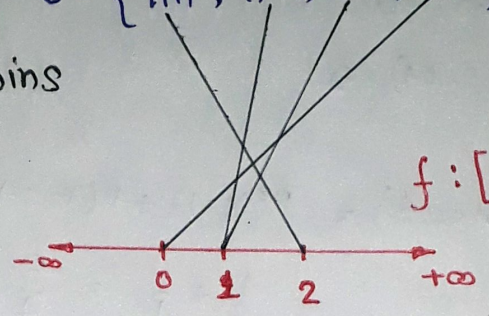
$\therefore$ Probability $= 0.5 \times 0.7 + 0.5 \times 0.4 = 0.55$ Ans

RANDOM VARIABLE:

Experiment: $S = \{HH, HT, TH, TT\}$

Tossing 2 coins

$f : [X = x = \text{no. of heads}]$



- $X = x$ (value)

(i) X → countable → Discrete Random variable

X → uncountable (Range) → continuous Random variable

(ii) $0 \leq P \leq 1$

(iii) $X =$ '-ve' OR '0' OR +ve

Example: 2 coins are tossed.

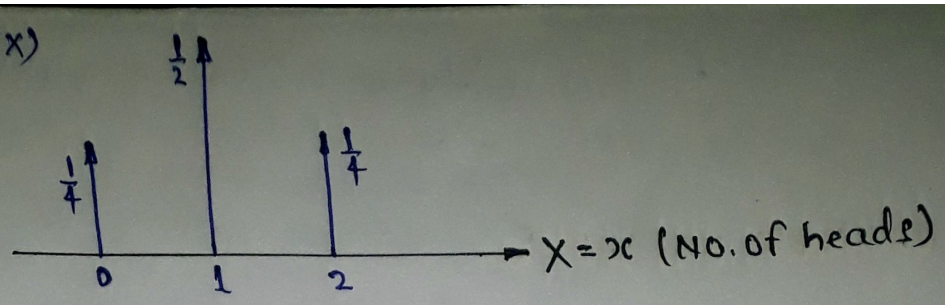
$S = \{HH, HT, TH, TT\}$

$X =$ no. of heads.

$X = x$	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

Prob. mass function (Pmf) / Prob. density function (Pdf)

Pdf → (stum plot)



$P(X=1.5)=0$

$P(x=0.99)=0$

$P(\text{Atleast 1 head}) = P(X=1)+P(X=2)$

$= \frac{1}{2}+\frac{1}{4} = \frac{3}{4}$

Cumulative Distribution function (cdf):

$X=x$	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$F(X=x)$	$\frac{1}{4}$	$\frac{3}{4}$	1

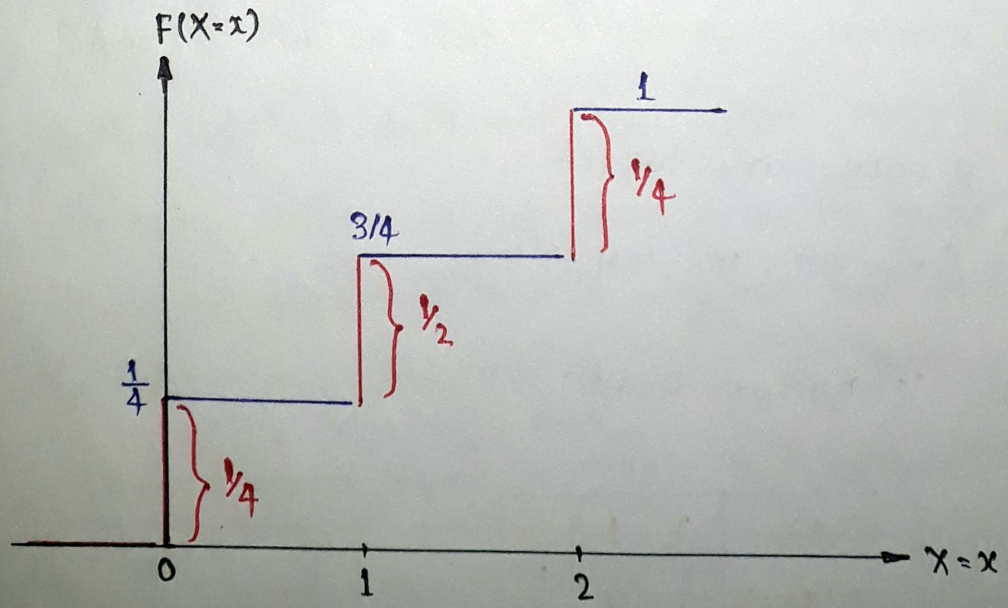
cdf

$$F(X=x) = \sum_{-\infty}^{x} P(X=x)$$

$F(X=0) = 1/4$

$F(X=1) = 3/4 \quad \because \{\frac{1}{4}+\frac{1}{2}\}$

$F(X=2) = 1$



stair-case plot

- $F(X=2) = 1$

$\downarrow$

$P(X \leq 2) = 1$

$P(X=0) + P(X=1) + P(X=2) = 1$

- $F(X=1) = 3/4$

$\downarrow$

$P(X \leq 1) = 3/4$

$P(X=0) + P(X=1) = 3/4$

$$\therefore P(X=2) = 1 - \frac{3}{4} = \frac{1}{4} \text{ Ans}$$

Ques. find : $K = ?$

$X =$	0	1	2
$P(X) =$	$\frac{K}{4}$	$\frac{2K}{4}$	$\frac{K}{4}$

$$\sum P(X) = 1$$

$$\frac{K}{4} + \frac{2K}{4} + \frac{K}{4} = 1$$

$$\therefore K = 1$$

$\uparrow$ Normalizing factor / Normalization constant.

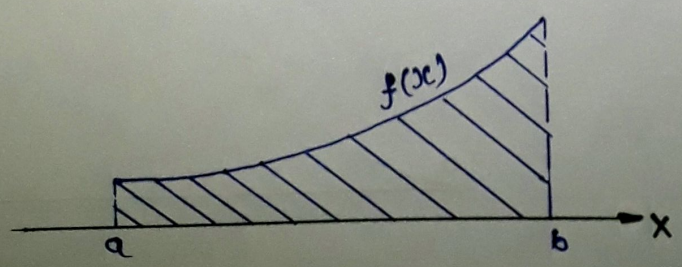
Continuous Random variable:

$$[X = x] \quad \forall \; x \in (a, b)$$

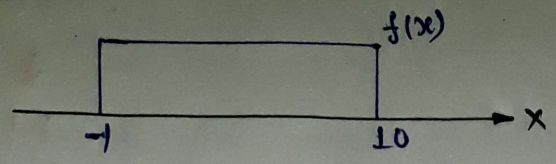
$f(x) =$ pdf

- $\int_a^b f(x)\,dx = 1$

- $\int_{-\infty}^{+\infty} f(x)\,dx = 1$



Ex. :



M1 : check : $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_{-1}^{10} (10)\,dx \neq 1$ $\therefore$ f(x) is not pdf.

M2 : Area under curve.

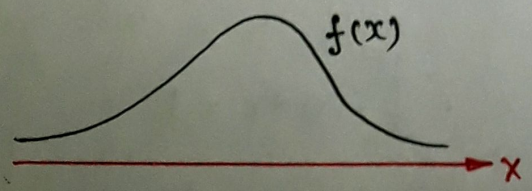
* $P(a < x < b) = \int_a^b f(x)\,dx$

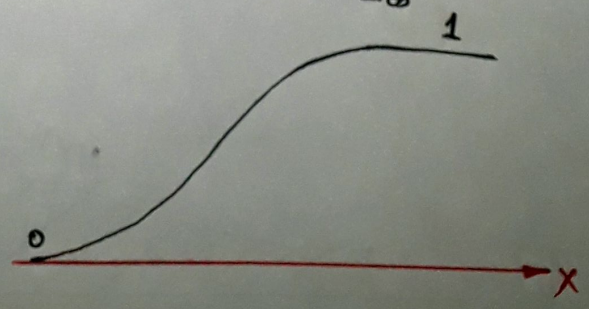
- $P(0 < x < 1) = \int_0^1 f(x)\,dx$
- $P(0 \leq x \leq 1) = \cancel{P(x=0)}^{0} + P(0 < x < 1) + \cancel{P(x=1)}^{0}$
- $P(0 \leq x < 1) = P(0 < x < 1)$
- $P(0 < x \leq 1) = P(0 < x < 1)$

cdf :

F(x) = cdf

$F(x) = \int_{-\infty}^{x} f(x)\,dx$





Lebinet's Rule : $\boxed{\frac{dF(x)}{dx} = 1 \cdot f(x)}$ — pdf

Ques. $f(x) = Kx^2$; $0 < x < 2$

(i) find K :

• check : continuous R.V. {∵ x is in range}

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{0}^{2} Kx^2\,dx = 1 \longrightarrow K \cdot \left(\frac{x^3}{3}\right)_0^2 = 1$$

$$K = 3/8$$

(ii) find : $P(0.2 < x < 0.5)$

$$\int_{0.2}^{0.5} f(x)\,dx = \int_{0.2}^{0.5} \frac{3}{8} x^2\,dx$$

$$= \frac{3}{8}\left(\frac{x^3}{3}\right)_{0.2}^{0.5} \approx 0.014 \text{ Ans}$$

(iii) find : cdf (F(x))

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{0} (0)\,dx + \int_{0}^{x} \frac{3}{8} x^2\,dx$$

$$= \frac{3}{8} \times \frac{x^3}{3} = \frac{x^3}{8} \; ; \; 0 < x < 2$$

Ques: $f(x) = 6x(1-x)$; $0<x<1$

Find, b if $P(x \leq b) = P(x>b)$

M1: $\int_{-\infty}^{b} f(x)\,dx = \int_{b}^{\infty} f(x)\,dx$

$$\int_{0}^{b} 6x(1-x)\,dx = \int_{b}^{1} 6x(1-x)\,dx$$

$$6b^2 - 4b^3 = 1$$

$$b^2(6-4b) = 1$$

$$b = 1/2 \text{ Ans}$$

M2: $P(X \leq b) = P(x>b)$

$$= 1 - P(x \leq b)$$

$$2P(x \leq b) = 1$$

$$\int_{0}^{b} 6x(1-x)\,dx = 1/2$$

$$\therefore b = 1/2 \text{ Ans}$$

Ques: $F(x) = 1 - e^{-x^2/b}$, $b>0$, $x>0$

cdf

Find: $f(x)$

pdf

sol. $\frac{d\,F(x)}{dx} = f(x)$

$$\frac{d}{dx}\left(1 - e^{-x^2/b}\right) = f(x)$$

$$\therefore f(x) = \frac{2x}{b} e^{-x^2/b} \quad ; \quad x > 0$$

Ques: Splitted Domain

$$f(x) = \begin{cases} K(1+x) & , -1 < x \leq 0 \\ K(1-x) & , 0 < x \leq 1 \\ 0 & , \text{elsewhere.} \end{cases}$$

then, (i) find K.

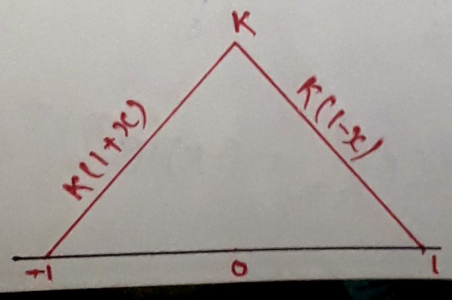
$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-1}^{0} K(1+x)\,dx + \int_{0}^{1} K(1-x)\,dx = 1$$

$$K\left(x + \frac{x^2}{2}\right)_{-1}^{0} + K\left(x - \frac{x^2}{2}\right)_{0}^{1} = 1$$

$$\therefore K = 1$$

(ii) Plot f(x) :



Area under $f(x) = 1$

$\frac{1}{2} \times 2 \times K = 1$

$K = 1$

(iii) cdf : $F(x<-1)=0$

$F(x>+1)=1$

$-1<x\leq 0$: $F(-1<x\leq 0) = \int_{-1}^{x} f(x)dx$

$$= \int_{-1}^{x} (1+x)dx = x+\frac{x^2}{2}+\frac{1}{2}$$

$0<x\leq 1$: $F(0<x\leq 1) = \int_{0}^{x} f(x)\,dx$

$$= \int_{-1}^{0} (1+x)dx + \int_{0}^{x} (1-x)\,dx.$$

$$= x-\frac{x^2}{2}+\frac{1}{2}, \quad 0<x\leq 1$$

Ques: $f(x) = Ke^{-|x|}$; $x\in(-\infty,\infty)$

- continuous random variable

$$\text{pdf} \rightarrow f(x) = \begin{cases} Ke^{-(+x)}, & x>0 \\ Ke^{+x}, & x<0 \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)dx = 1 \rightarrow \int_{-\infty}^{0} Ke^{x}dx + \int_{0}^{\infty} Ke^{-x}dx = 1$$

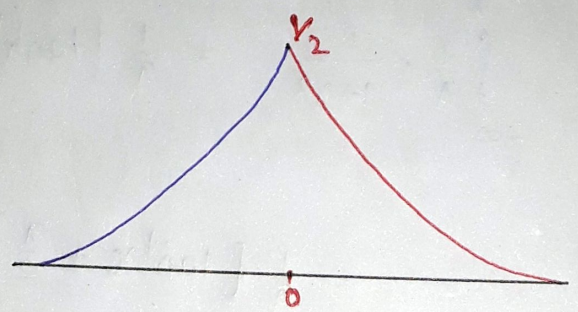
$$K(e^{x})_{-\infty}^{0} + (-K)(e^{-x})_{0}^{\infty} = 1$$

$$K\left\{e^{0}-e^{-\infty}-(e^{-\infty}-e^{0})\right\}=1$$

$$K\left\{1-0-0+1\right\}=1 \qquad \therefore K=\tfrac{1}{2} \text{ Ans.}$$

Plot $f(x)$:

$$f(x)=\begin{cases}\frac{1}{2}e^{-x}, & x>0\\ \frac{1}{2}e^{x}, & x<0\end{cases}$$



cdf:

$$F(x<0)=\int_{-\infty}^{x} f(x)\,dx$$

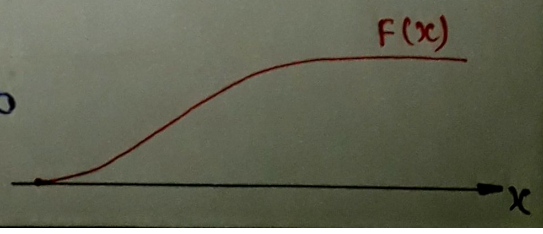
$$=\int_{-\infty}^{x}\frac{1}{2}e^{x}\,dx \quad =\frac{1}{2}e^{x}, \quad x<0$$

$$F(x>0)=\int_{-\infty}^{x} f(x)\,dx$$

$$=\int_{-\infty}^{0}\frac{1}{2}e^{x}\,dx+\int_{0}^{x}\frac{1}{2}e^{-x}\,dx$$

$$=\frac{1}{2}+\frac{1}{2}\left(\frac{e^{-x}}{-1}\right)_{0}^{x}$$

$$=1-\frac{1}{2}e^{-x}, \quad x>0$$



Ques: $f(x) = K \cdot \frac{1}{1+x^2}$, $-\infty < x < \infty$

(i) find: K

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} K \cdot \frac{1}{1+x^2}\,dx = 1 \longrightarrow K = \frac{1}{\pi}$$

(ii) find: cdf

$$\int_{-\infty}^{x} \frac{1}{\pi} \cdot \frac{1}{1+x^2} = \frac{1}{\pi}\left(\tan^{-1}x\right)_{-\infty}^{x}$$

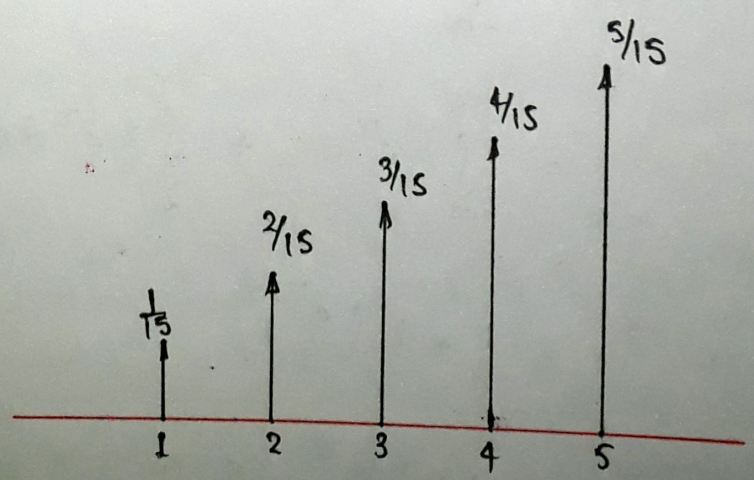
$$= \frac{1}{\pi}\left[\tan^{-1}x + \frac{\pi}{2}\right] \text{ Ans.}$$

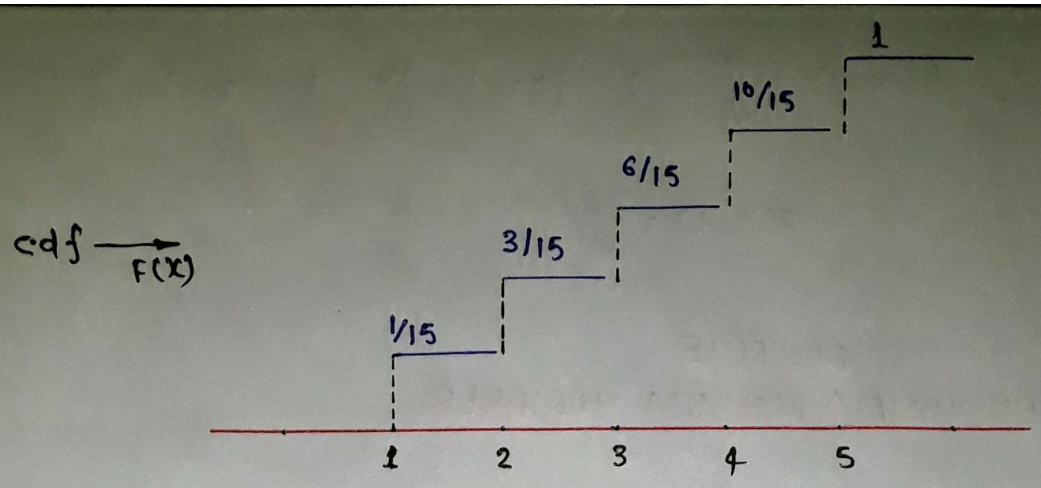
Ques. $f(x) = \frac{x}{15}$ $\forall$ $x = 1, 2, 3, 4, 5$

(i) Pdf :

$X = x$	1	2	3	4	5
$P(X=x)$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$
$F(X=x)$	$\frac{1}{15}$	$\frac{3}{15}$	$\frac{6}{15}$	$\frac{10}{15}$	$\frac{15}{15} = 1$

Pdf $\xrightarrow{f(x)}$





- $P\left(\frac{1}{2} < x < \frac{3}{2}\right) = P(0.5 < x < 1.5) = 1/15$

- $P\left(\frac{1}{2} \leq x \leq 5/2\right) = P(0.5 \leq x \leq 2.5) = P(x=1) + P(x=2)$

$$= \frac{1}{15} + \frac{2}{15} = \frac{3}{15} \text{ Ans.}$$

Ques: Random variable: $x = 1, 2, 3, 4$

$$2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$$

sol^n: • check :- Discrete Random variable

$$2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = K$$

$$P(X=1) = \frac{K}{2}$$

$$P(X=2) = \frac{K}{3}$$

$$P(X=3) = K$$

$$P(X=4) = \frac{K}{5}$$

$$\Sigma P(X) = 1$$

$$K\left(\frac{1}{2} + \frac{1}{3} + 1 + \frac{1}{5}\right) = 1 \qquad \therefore K = \frac{30}{61}$$

$\therefore \; P(X=1) = \frac{15}{61} \; ; \; P(X=2) = \frac{10}{61} \; ; \; P(X=3) = \frac{30}{61}$

$P(X=4) = \frac{6}{61}$

DISCRETE

*TWO DIMENSIONAL^ RANDOM VARIABLE

$X_1 \backslash X_2$	0	1
0	1/4	1/4
1	1/4	1/4

$x_1 = 0, 1 \;\&\; x_2 = 0, 1$

$P(X_1=0, X_2=0) = 1/4$

$P(X_1=0, X_2=1) = 1/4$

$P(X_1=1, X_2=0) = 1/4$

$P(X_1=1, X_2=1) = 1/4$

X_1	0	1
$P(X_1)$	1/2	1/2

$P(X_1=0) = P(X_1=0, X_2=)$

$= P(X_1=0, X_2=0) + P(X_1=0, X_2=1)$

$= \sum_{x_1=-\infty}^{+\infty} P(X_1=0, X_2=x_2)$

Ques: $P(X_1 = x_1, X_2 = x_2) = \frac{1}{27}(x_1 + 2x_2)$

$x_1 = 0, 1, 2$

$x_2 = 0, 1, 2$

X_1 \ X_2	0	1	2
0	0	$2/27$	$4/27$
1	$1/27$	$3/27$	$5/27$
2	$2/27$	$4/27$	$6/27$

(i) Pdf of x_1 ⟶ Eliminate X_2

M1: x_1 : 0 1 2

$P(X_1)$: $6/27$ $3/9$ $4/9$

M2: $P(X_1) = \sum_{X_2=0}^{2} P(X_1, X_2)$

$= P(X_1, 0) + P(X_1, 1) + P(X_1, 2)$

$= \frac{1}{27}(X_1 + 0) + \frac{1}{27}(X_1 + 2) + \frac{1}{27}(X_1 + 4)$

$= \frac{2 + X_1}{9}$; $X_1 = 0, 1, 2$

(ii) Pdf of X_2 : Eliminate X_1

M1: X_2 : 0 1 2

$P(X_2)$: $1/9$ $9/27$ $5/9$

M2:

$$P(X_2) = \sum_{x_1=0}^{2} P(x_1, x_2)$$

$$= P(0, X_2) + P(1, X_2) + P(2, X_2)$$

$$= \frac{1}{27}(0+2X_2) + \frac{1}{27}(1+2X_2) + \frac{1}{27}(2+2X_2)$$

$$= \frac{1+2X_2}{9} \quad ; \quad x = 0, 1, 2$$

Ques: $P(X, Y) = \begin{cases} c(x^2+2y) & ; \ x = 0,1,2 \ \& \ y = 1,2,3,4 \\ 0 & ; \text{elsewhere} \end{cases}$

(i) Find 'c':

X \ Y	1	2	3	4
0	2C	4C	6C	8C
1	3C	5C	7C	9C
2	6C	8C	10C	12C

$$\sum\sum P(X,Y) = 1$$

$$80C = 1 \qquad \therefore \ C = 1/80 \ \text{Ans}$$

(ii) $P(X=2, Y=3)$

$$P(X=2, Y=3) = 10C = 10 \times \frac{1}{80}$$

$$= \frac{1}{8} \ \text{Ans.}$$

(iii) $P(X \leq 1, Y > 2)$

$$P(X=0,1,\ Y=3,4) = P(X=0, Y=3) + P(X=0, Y=4) + P(X=1, Y=4) + P(X=1, Y=3)$$

$$= 3/8 \text{ Ans.}$$

(iv) Pdf of X :

X :	0	1	2
P(X) :	$20 \cdot \frac{1}{80}$	$24 \cdot \frac{1}{80}$	$36 \cdot \frac{1}{80}$

(v) Pdf of Y :

Y :	1	2	3	4
P(Y) :	$11 \cdot \frac{1}{80}$	$17 \cdot \frac{1}{80}$	$23 \cdot \frac{1}{80}$	$29 \cdot \frac{1}{80}$

Ques :

Y \ X	1	2	3
1	1/12	1/6	0
2	0	1/9	1/5
3	1/18	1/4	2/15

(i) $P(X=1/Y=2) = \dfrac{P(X=1, Y=2)}{P(Y=2)} = 0$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

(ii) $P(Y=3/X=3)$

$$= \frac{P(Y=3, X=3)}{P(X=3)} = \frac{2/15}{1/5 + 2/15} = \frac{2}{5}$$

(iii) $P(X=2/Y=2)$

$$= \frac{P(X=2, Y=2)}{P(Y=2)} = \frac{1/9}{1/9 + 1/5} = 5/14$$

(iv) $P(X+Y<4)$

$X = 1, 2, 3$

$Y = 1, 2, 3$

$$= P(x=1, y=1) + P(x=1, y=2) + P(x=2, y=1)$$

$$= \frac{1}{12} + 0 + \frac{1}{6}$$

$$= \frac{1+2}{12} = \frac{1}{4}$$ Ans.

NOTE!

	one Dimensional	Two Dimensional
Discrete Random variable :	$\Sigma P(X)=1$	$\Sigma\Sigma P(X_1, X_2)=1$
continuous Random variable:	$\int f(x)dx=1$	$\iint f(x,y)\,dx\,dy=1$

TWO DIMENSIONAL CONTINUOUS RANDOM VARIABLE:

Ques: $f(x,y) = \begin{cases} x^3y^3/16 \; ; & 0<x<2 \;\&\; 0<y<2 \\ 0 \; ; & \text{otherwise} \end{cases}$

(i) Find: $f(x)$; x = constant
y = eliminated

$$f(x) = \int_{y=0}^{2} f(x,y)\,dy = \int_{y=0}^{2} \frac{x^3y^3}{16}\,dy$$

$$= \frac{x^3}{16}\left(\frac{y^4}{4}\right)_0^2 = \frac{x^3}{4} \; ; \; 0<x<2$$

(ii) find: $f(y)$; y = constant
x = eliminated

$$f(y) = \int_{x=0}^{2} f(x,y)\,dx = \int_{x=0}^{2} \frac{x^3y^3}{16}\,dx$$

$$= \frac{y^3}{16} \times \left(\frac{x^4}{4}\right)_0^2 = \frac{y^3}{4} \; ; \; 0<y<2$$

(iii) x and y are independent variable

$f(x,y) = f(x) \cdot f(y)$ ← for independency

$$\frac{x^3y^3}{16} = \frac{x^3}{4} \cdot \frac{y^3}{4}$$

$\therefore$ x and y are independent variable.

Ques: $f(x,y) = \begin{cases} x+y & ; 0<x<1 \ \& \ 0<y<1 \\ 0 & ; \text{otherwise} \end{cases}$

check, $f(x,y)$ is pdf ?

sol. $\int_{x=0}^{1}\int_{y=0}^{1} f(x,y)\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1} (x+y)\,dx\,dy$

$$= \int_{x=0}^{1} \left(xy + \frac{y^2}{2}\right)_0^1 dx$$

$$= \int_{x=0}^{1} \left(x + \frac{1}{2}\right) dx = \left(\frac{x^2}{2} + \frac{1}{2}x\right)_0^1 = 1$$

Hence, $f(x,y)$ is pdf.

Ques. $f(x,y) = \begin{cases} \frac{1}{8}(6-x-y) & ; 0<x<2 \ \& \ 2<y<4 \\ 0 & ; \text{otherwise} \end{cases}$

(i) $P(x<1, y<3)$

$$\int_{x=0}^{1}\int_{y=2}^{3} f(x,y)\,dx\,dy$$

$$\int_{x=0}^{1}\int_{y=2}^{3} \frac{1}{8}(6-x-y)\,dx\,dy$$

$= 3/8$ Ans.

(ii) $P(x<1/y<3)$

$$P(x<1/y<3) = \frac{P(x<1, y<3)}{P(y<3)} = \frac{3/8}{5/8} = 3/5 \text{ Ans.}$$

$P(y<3)$: $\quad P(y<3) = \int_{y=2}^{3} f(y)\,dy$ (OR) $\int_{y=2}^{3}\left[\int_{x=0}^{2} f(x,y)\,dx\right]dy$

$$P(y<3) = \int_{y=2}^{3} \frac{1}{4}(5-y)\,dy \qquad \because f(y) = \int_{x=0}^{2} f(x,y)\,dx$$

$$P(y<3) = \frac{1}{4}\left(5y - \frac{y^2}{2}\right)_2^3 \qquad f(y) = \int_{x=0}^{2} (6-x-y)\frac{1}{8}\,dx$$

$$P(y<3) = 5/8 \qquad f(y) = \frac{1}{8}\left(6x - \frac{x^2}{2} - yx\right)_0^2$$

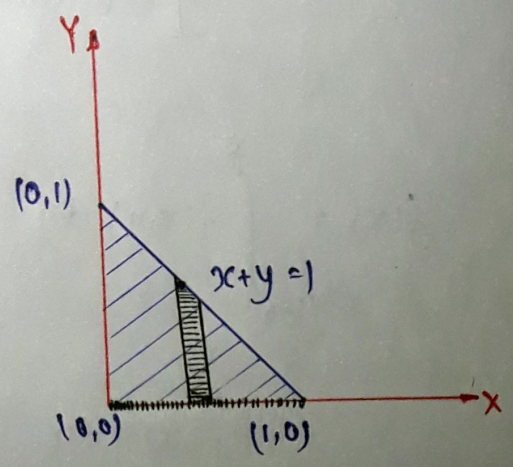
$$f(y) = \frac{1}{4}(5-y) \;;\; 2<y<4$$

Ques: $f(x,y) = \begin{cases} e^{-(x+y)} & ; x \geq 0, y \geq 0 \\ 0 & ; \text{elsewhere} \end{cases}$

(i) $P(x+y<1)$

$$\int_{x=0}^{1}\int_{y=0}^{1-x} f(x,y)\,dx\,dy$$

$$\int_{x=0}^{1}\int_{y=0}^{1-x} e^{-x}\cdot e^{-y}\,dx\,dy$$



$$= \int_0^1 e^{-x} \left(\frac{e^{-y}}{-1}\right)_0^{1-x} dx$$

$$= -\int_0^1 e^{-x} \left(e^{-(1-x)} - e^0\right) dx = -\int_0^1 e^{-x} \left(e^{-1} \cdot e^{x} - 1\right) dx$$

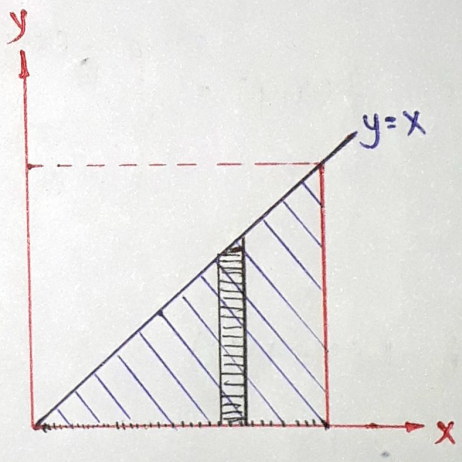
$$= -\int_0^1 \left(e^{-1} - e^{-x}\right) dx = -\left(x e^{-1} - \frac{e^{-x}}{-1}\right)_0^1$$

$$= -\left(e^{-1} x + e^{-x}\right)_0^1 = \left(-e^{-x} - e^{-1} x\right)_0^1$$

$$= -e^{-1} - e^{-1} - (-e^0 - 0)$$

$$= 1 - 2e^{-1} \quad \text{Ans.}$$

Ques. $f(x,y) = \begin{cases} 8xy, & 0<x<1 \ \& \ 0<y<x \\ 0, & \text{elsewhere} \end{cases}$



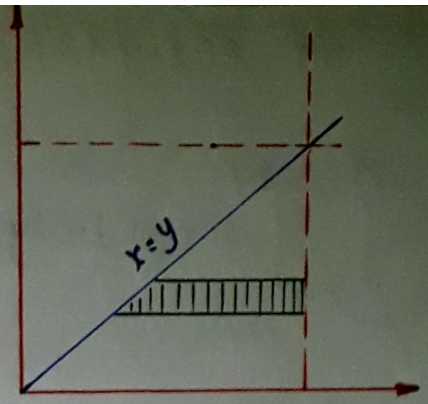
(i) $f(x)$: x = constant
y = eliminate

$$f(x) = \int_{y=0}^{x} f(x,y)\, dy$$

$$= \int_{y=0}^{x} 8xy\, dy = 8x \left(\frac{y^2}{2}\right)_0^x = 8x \cdot \frac{x^2}{2}$$

$$= 4x^3 \text{ Ans.} ; \ 0<x<1$$

(ii) $f(y)$: y = constant
x = eliminated



$$f(y) = \int_{x=y}^{1} f(x,y)\,dx = \int_{x=y}^{1} 8xy\,dx$$

$$8y\left(\frac{x^2}{2}\right)_y^1 = 8y \times \frac{1}{2} \times (1-y^2)$$

$$= 4y(1-y^2) \; ; \; 0y<1$$

Ques: $f(x,y) = \begin{cases} K(6-x-y) & ; \; 0<x<2 \;\&\; 2<y<4 \\ 0 & ; \; \text{elsewhere} \end{cases}$

(i) K :

$$\int_{x=0}^{2}\int_{y=2}^{4} K(6-x-y)\,dx\,dy = 1$$

$$K\int_{x=0}^{2}\left(6y - xy - \frac{y^2}{2}\right)_2^4 dx = 1$$

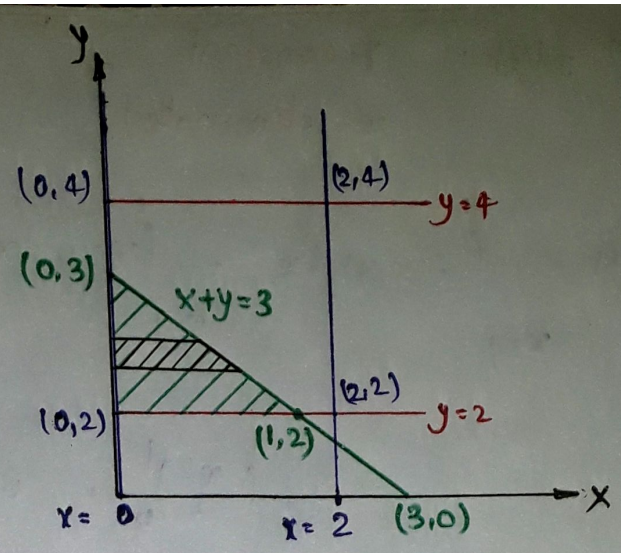
$$K\int_{x=0}^{2}\left(6\times4 - 4x - \frac{\cancel{16}^{8}}{\cancel{2}} - 6\times2 + 2x + \frac{\cancel{4}^{2}}{\cancel{2}}\right)dx = 1$$

$$K\int_0^2 (12-2x-6)\,dx = K\int_0^2 (6-2x)\,dx = 1$$

$$K\left(6x - \cancel{2}\cdot\frac{x^2}{\cancel{2}}\right)_0^2 = K\left(6x - x^2\right)_0^2 = 1$$

$$K \times (12-4) = 1 \quad \therefore K = 1/8 \text{ Ans.}$$

(ii) $P(x+y<3)$



$$\int_{y=2}^{3}\int_{x=0}^{3-y} f(x,y)\,dx\,dy$$

$$\int_{y=2}^{3}\int_{x=0}^{3-y} \frac{1}{8}(6-x-y)\,dx\,dy$$

$$\int_{y=2}^{3} \frac{1}{8}\left(6x-\frac{x^2}{2}-xy\right)_0^{3-y} dy = \int_{y=2}^{3} \frac{1}{8}\left(6(3-y)-\frac{(3-y)^2}{2}-(3-y)y\right) dy$$

$$\frac{1}{8}\int_{y=2}^{3}\left(18-6y-\left(\frac{9+y^2-6y}{2}\right)-(3y-y^2)\right) dy$$

$$\frac{1}{8}\left(18y-6y^2/2-\frac{9}{2}y-\frac{y^3}{6}+\cancel{\frac{3y^2}{2}}-\cancel{\frac{3y^2}{2}}+\frac{y^3}{3}\right)_2^3$$

$$\frac{1}{8}\left(\left(\frac{y^3}{3}-\frac{y^3}{6}\right)-3y^2+13.5y\right)_2^3 = 5/24 \quad \text{Ans.}$$

Ques: $f(x,y) = \begin{cases} \frac{1}{2}xe^{-y} ; & 0<x<2 \ \& \ y>0 \\ 0 ; & \text{otherwise} \end{cases}$

find: cdf ($F(x)$)

$$F(x,y) = \int_{v=-\infty}^{y}\int_{u=-\infty}^{x} f(u,v)\,du\,dv$$

$$\int_{v=0}^{y} \int_{u=0}^{x} \frac{1}{2} u e^{-v} \, du \, dv$$

$$\int_{v=0}^{y} \frac{1}{2} e^{-v} \left(\frac{u^2}{2} \right)_0^x dv$$

$$\frac{1}{2} \cdot \frac{x^2}{2} \int_{v=0}^{y} e^{-v} dv = \frac{x^2}{4} \cdot \left(\frac{e^{-v}}{-1} \right)_0^y$$

$$= \frac{-x^2}{4} \left(e^{-y} - e^{0} \right)$$

$$= \frac{x^2}{4} \left(1 - e^{-y} \right) \quad ; \; 0 < x < 2, \; y > 0$$

Ques: $F(x,y) = (1-e^{-x})(1-e^{-y}) \; ; \; x>0 \; \& \; y>0$

find: $P(1<x<3, 1<y<2)$

Pdf, $f(x,y) = \frac{\partial}{\partial x}\left\{ \frac{\partial}{\partial y} F(x,y) \right\} = \frac{\partial}{\partial y}\left\{ \frac{\partial}{\partial x} F(x,y) \right\}$

$$f(x,y) = \frac{\partial}{\partial x}\left\{ \frac{\partial}{\partial y}\left((1-e^{-x})(1-e^{-y}) \right) \right\}$$

$$= \frac{\partial}{\partial x}\left\{ (1-e^{-x})(0+e^{-y}) \right\}$$

$$= e^{-y} \cdot (0+e^{-x}) = e^{-x} \cdot e^{-y}$$

$$\cdot P(1<x<3, 1<y<2) = \int_{x=1}^{3} \int_{y=1}^{2} f(x,y) \, dx \, dy$$

$$= \int_{x=1}^{3} \int_{y=1}^{2} e^{-x} \cdot e^{-y} \, dx \, dy = \frac{(1-e^{2})}{e^{3}} \cdot \frac{(1-e)}{e^{2}}$$

AVERAGE | EXPECTATION : DISCRETE RANDOM VARIABLE

Example: Data set : $\{1, 2, 2, 3, 3, 3, 4, 4, 4\}$

M1:

$$\text{Average} = \frac{\text{Total sum}}{\text{Total count}} = (1+2+2+3+3+3+4+4+4)/10$$

Average = 3 ans.

M2:

x_i	f_i	$x_i f_i$
1	1	1
2	2	4
3	3	9
4	4	16
	$\Sigma f_i = 10$	$\Sigma x_i f_i = 30$

$$\therefore \text{Average} = \frac{\Sigma x_i f_i}{\Sigma f_i} = \frac{30}{10} = 3$$

$$\left.\begin{array}{l} 1 \text{ — } \frac{1}{10} \\ 2 \text{ — } \frac{2}{10} \\ 3 \text{ — } \frac{3}{10} \\ 4 \text{ — } \frac{4}{10} \end{array}\right\} P(x) = \frac{x}{10} \; ; \; x = 1, 2, 3, 4$$

x	1	2	3	4
$P(x)$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$

Average = Expectation = weighted Average = $E(X)$

$$E(X) = \sum X \cdot P(X)$$

$$E(X) = 1 \cdot \frac{1}{10} + 2 \cdot \frac{2}{10} + 3 \cdot \frac{3}{10} + 4 \cdot \frac{4}{10}$$

$$E(X) = 3$$

* Expectation $E(X)$ in Random variable

- Does not gives us value of X, we can expect in single trial.
- it is same as average i.e, Number which we get in outcome is close to average.

Ques: Person throws 2 coins over and over again will get one head per toss on average.

$$S = \{HH, HT, TH, TT\}$$

X = No. of heads.

X :	0	1	2
$P(X)$:	$1/4$	$2/4$	$1/4$

$$E(X) = \sum X \cdot P(X)$$

$$= 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4}$$

$$= 1$$

Ques: A fair coin is tossed till head appears. what is the expectation of No. of tosses required?

E (No. of tosses required for getting head)

$E(X) = \sum x \cdot P(X)$

Event :	H	TH	TTH	....
x :	1	2	3	....
P (x) :	$\frac{1}{2}$	$\frac{1}{2} \cdot \frac{1}{2}$	$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$	....

$$E(X) = \sum x \cdot P(x)$$

$$= 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2}^2 + 3 \cdot \frac{1}{2}^3 + \cdots$$

$$= \frac{1}{2} \left\{ 1 + 2 \cdot \left(\frac{1}{2}\right) + 3 \cdot \left(\frac{1}{2}\right)^2 + \cdots \right\}$$

$$= \frac{1}{2} \left\{ \left(1 - \frac{1}{2}\right)^{-2} \right\}$$

$$= \frac{1}{2} \times \frac{1}{\left(\frac{1}{2}\right)^2} = \frac{1}{2} \times 4 = 2 \text{ Ans.}$$

Average / Expectation : Continuous Random variable.

$$E(X) = \int_{-\infty}^{+\infty} x \cdot f(x)\, dx$$

Ques: $f(x) = \begin{cases} K(2-x)x & ; 0<x<2 \\ 0 & ; \text{otherwise} \end{cases}$

(i) find, K:

$$\int_0^2 K(2x-x^2)\,dx = 1$$

$$K\left(2\cdot\frac{x^2}{2} - \frac{x^3}{3}\right)_0^2 = 1$$

$$K\left(4 - \frac{8}{3}\right) = 1 \qquad \therefore K = 3/4$$

(ii) find, E(X):

$$E(X) = \int_0^2 x\cdot f(x)\,dx$$

$$= \int_0^2 x\cdot\frac{3}{4}(2x-x^2)\,dx$$

$$= \int_0^2 \frac{3}{4}(2x^2 - x^3)\,dx$$

$$= \frac{3}{4}\left(2\cdot\frac{x^3}{3} - \frac{x^4}{4}\right)_0^2 = 1 \text{ Ans.}$$

Expectation of function of Random variable

- Discrete Random variable:

$$E(X) = \sum x\cdot P(X)$$

$$E(g(x)) = \sum g(x)\cdot p(x)$$

$$E(x^2) = \sum x^2\cdot (x)$$

continuous random variable:

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$E(g(x)) = \int_{-\infty}^{\infty} g(x) \cdot f(x)\,dx$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx$$

[DRV]

Ques:

x :	0	1	2	3
$P(x)$:	0.1	0.3	0.5	0.1

(i) if $y = x^2 + 2x$ then, $P(Y)$

x :	0	1	2	3
$y = x^2 + 2x$:	0	3	8	15
$P(x) = P(Y)$:	0.1	0.3	0.5	0.1

(ii) Find: $E(Y)$

M1: $E(y) = \sum y \cdot P(y)$

$= 0 \times 0.1 + 3 \times 0.3 + 8 \times 0.5 + 15 \times 0.1$

$= 6.4$ Ans.

M2: $E(y) = E(X^2 + 2X)$

$= E(X^2) + E(2X)$

$= E(X^2) + 2E(X)$ $\quad \because E(2X) = \sum (2X) \cdot P(X)$

$E(2X) = 2 \sum X P(X)$

$\therefore E(2X) = 2E(X)$

X :	0	1	2	3
X^2 :	0	1	4	9
$P(X)$:	0.1	0.3	0.5	0.1

$E(X^2) = \sum X^2 \cdot P(X)$

$= 0 \times 0.1 + 1 \times 0.3 + 4 \times 0.5 + 9 \times 0.1$

$= 3.2$

$E(X) = \sum X \cdot P(X)$

$= 0 \times 0.1 + 1 \times 0.3 + 2 \times 0.5 + 3 \times 0.1$

$= 1.6$

$\therefore E(y) = E(X^2) + 2E(X)$

$E(y) = 3.2 + 2 \times 1.6 = 6.4$

Ques: (CRV) $f(x) = \frac{2}{9} x \left(2 - \frac{x}{2}\right) \; ; \; 0 < x < 3$

if $y = (x+1)^2$ then find $E(y) = ?$

$E(y) = E(y = (x+1)^2) = \int_0^3 (x+1)^2 f(x)\, dx = 8.1$ Ans.

* $E(c) = c$ $\{\because c = \text{constant}\}$

DRV : $E(c) = \sum c\, P(x)$

$E(c) = c \sum P(x)$

$E(c) = c$

CRV : $E(c) = \int_{-\infty}^{\infty} c \cdot f(x)\,dx$

$E(c) = c \cdot \int_{-\infty}^{\infty} f(x)\,dx$

$E(c) = c$

* PROPERTIES OF EXPECTATION [E(X)]

- $E(aX) = a\,E(X)$
- $E(aX^n) = a\,E(X^n)$
- $E(X^2) \neq [E(X)]^2$
- $E(1/X) \neq 1/E(X)$
- $E(aX+b) = E(aX) + E(b)$

  $= a\,E(X) + b$
- $E(X \pm Y) = E(X) \pm E(Y)$
- if X, Y are independent Random variable

  then $E(XY) = E(X) \cdot E(Y)$

Ques: $E(X_1) = 4$, $E(X_2) = -2$ then , $E(2X_1 + X_2 - 3) = ?$

$E(2X_1 + X_2 - 3) = E(2X_1) + E(X_2) - E(3)$

$= 2\,E(X_1) + E(X_2) - 3$

$= 2 \times 4 + (-2) - 3$

$= 3$ Ans.

VARIANCE : $[V(x) / \sigma^2 / s^2]$

- it measures how far a set of number is spread out from their average value.

- variance is expectation of squared deviation of a random variable from this mean.

$$\therefore \quad V(X) = E[(X - E(X))^2]$$

$$V(X) = E[X^2 - 2XE(X) + (E(X))^2]$$

$$V(X) = E(X^2) - 2E(X)E(X) + (E(X))^2$$

$$V(X) = E(X^2) - (E(X))^2$$

Ques.

y :	0	3	8	15
$p(y)$:	0.1	0.3	0.5	0.1

Find : $V(y) = ?$

$$V(y) = E(y^2) - (E(y))^2$$

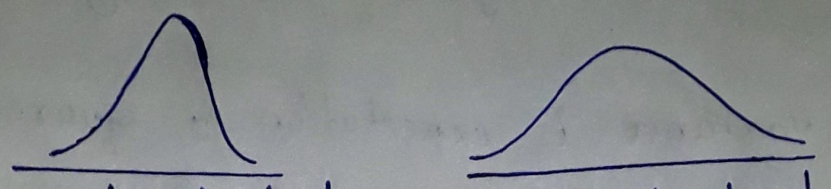
$$V(y) = \Sigma y^2 \cdot p(y) - [\Sigma y \cdot p(y)]^2$$

$$V(y) = (0 \times 0.1 + 9 \times 0.3 + 64 \times 0.5 + (15)^2 \times 0.1) - (0 \times 0.1 + 3 \times 0.3 + 8 \times 0.5 + 15 \times 0.1)^2$$

$$V(y) = 16.24$$

NOTE : σ^2 = Variance

$\sigma = \sqrt{\text{variance}}$ = standard Deviation



low standard Deviation

High standard Deviation

* Properties of variance :

- $V(X) = E(X^2) - (E(X))^2$
- $V(c) \neq c$
- $V(c) = 0$
- $V(aX) = a^2 V(X)$
- $V(-X) = +V(X)$
- if x and y are Random variable

$V(X+Y) = V(X) + V(Y) + 2\,Cov(X,Y)$

$V(X-Y) = V(X) + V(Y) - 2\,Cov(X,Y)$

- if x and y are independent Random variable.

$V(X+Y) = V(X) + V(Y)$

$V(X-Y) = V(X) + V(Y)$

- $e^x = x^0 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$e^{tx} = (tx)^0 + \frac{(tx)}{1!} + \frac{(tx)^2}{2!} + \frac{(xt)^3}{3!} + \cdots$$

$$E(e^{tx}) = 1 + E(tx) + \frac{E(tx)^2}{2!} + \frac{E(tx)^3}{3!} + \cdots$$

$$E(e^{tx}) = 1 + t\,E(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \cdots$$

$$\frac{d\,E(e^{tx})}{dt} = 0 + E(x) + 2t\,E(x^2) + \frac{t^2}{2}E(x^3) + \cdots$$

at $t=0$;

$$\frac{d\,E(e^{tx})}{dt} = E(x)$$

$$\frac{d^2\,E(e^{tx})}{dt^2} = E(x^2)$$

$$\frac{d^3\,E(e^{tx})}{dt^3} = E(x^3)$$

$\because E(e^{xt}) = M_0(t)$

Moment generating function of time about origin.

similarly,

$$\frac{d^n\,E(e^{tx})}{dt^n} = E(x^n)$$

- $n=1$; 1st moment about origin $= E(x^1)$
- $n=2$; 2nd moment about origin $= E(x^2)$
- $n=3$; 3rd moment about orgin $= E(x^3)$

Example : DRV

X :	1	2	3	4	5	6
$P(X)$:	1/6	1/6	1/6	1/6	1/6	1/6

- $E(X) = \sum X P(X)$

$$= 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + 4\cdot\frac{1}{6} + 5\cdot\frac{1}{6} + 6\cdot\frac{1}{6} = 3.5$$

- $M_0(t) = E(e^{Xt}) = \sum_{X=1}^{6} e^{Xt}\cdot P(X)$

$$= \frac{e^{t} + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}}{6}$$

$$E(X) = \frac{d}{dt}\left(M_0(t)\right)\Big|_{t=0} = \frac{1}{6}\left\{e^{t} + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}\right\}\Big|_{t=0}$$

$$= \frac{1}{6}\times\left(1+2+3+4+5+6\right) = \frac{7}{2} = 3.5 \text{ Ans}$$

$$E(X^2) = \frac{d^2}{dt}\left(M_0(t)\right)\Big|_{t=0} = \frac{1}{6}\left|e^{t} + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t}\right|_{t=0}$$

$$= \frac{1}{6}\left(1+4+9+16+25+36\right)$$

$$= 15.166 \text{ Ans.}$$

Example: $f(x) = \begin{cases} K \cdot e^{-Kx} & ; x > 0 \ \& \ K > 0 \\ 0 & ; \text{elsewhere.} \end{cases}$

Find (i) $E(X)$ (ii) $M_0(t)$

Ques: An examination paper has 150 multiple choice questions of one mark each, with each question having four choices. Each incorrect answer fetches -0.25 marks. Suppose 1000 students choose all their answers randomly with uniform probability. The sum total of the expected marks obtained all these students is

(a) 0 (b) 2550 (c) 7525 (d) 9375

- $E(X) = \sum x \cdot P(x)$

- X : $+1$, -0.25

  $P(X)$: $1/4$ $3/4$

- $E(X) = \sum x \cdot P(x) = 1 \times \frac{1}{4} - 0.25 \times \frac{3}{4} = \frac{1}{16}$ marks/ques

$E(X) = 150 \times \frac{1}{16} = 9.375$ mark/student

$E(X) = 150 \times \frac{1}{16} \times 1000 = 9375$ marks

Ques: Three coins are tossed. Find Prob. of exactly two heads.

$$S = \left\{ \begin{matrix} HHH & HTH & THH & TTH \\ HHT & HTT & THT & TTT \end{matrix} \right\}$$

P(exactly two heads) = 3/8

OR

P(exactly two heads) = P(HHT) + P(HTH) + P(THH)

X = two No. of Heads.

$= P(H)\,P(H)\cdot P(T) + P(H)\cdot P(T)\cdot P(H) + P(T)\,P(H)\,P(H)$

• we need heads:

so let, P(H) = P(success) + for one trial

P(T) = P(failure)

∴ P(exactly 2 heads) = PPq + PqP + qPP

$= 3\,p^2\,q$

$\downarrow \quad \downarrow \quad \downarrow$

$^nC_x \quad p^x \quad q^{n-x}$

$^3C_2 \quad p^2 \quad q^{3-2}$

X ⟵ P

(No. of heads) (Getting head for 1 trial)

BINOMIAL DISTRIBUTION !

Valid for : (i) Independent event
(ii) Discrete Random variable
(iii) should be only two prob.
(iv) X is always +ve.

$$P(X) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$$

$\because$ n = no. of trials

x = no. of favourable event

p = Prob. of success (sharing idea with x) for 1 trial.

q = Prob. of failure = 1-p

Ques: The prob. of a defective piece being produced in a manufacturing process is 0.01. The Prob. that out of 5 successive pieces, only one is defective, is

(a) $(0.99)^2 (0.01)$

(b) $(0.99)(0.01)^4$

(c) $5 \times (0.99)(0.01)^4$

(d) $5 \times (0.99)^4 (0.01)$

n = 5
x = 1 (Defective)

p = 0.01

q = 1-p = 1-0.01 = 0.99

M1 : $P(X) = {}^{5}C_{1}\, p\, q^{4}$

$= 5 \times (0.01)(0.99)^4$

M2! $P(X) = 5 \times (0.01)(0.99)^4$

0.01	0.99	0.99	0.99	0.99
D	ND	ND	ND	ND
ND	D	ND	ND	ND
ND	ND	D	ND	ND
ND	ND	ND	D	ND
ND	ND	ND	ND	D

Ques: A lot has 10 % defective items. Ten items are choosen randomly from this lot. The prob. that exactly 2 of the choosen items are defective is

(a) 0.0036 ✓(b) 0.1937 (c) 0.2234 (d) 0.3874

$n = 10$, $x = 2$ (Defective)

$p = 0.1$, $q = 1 - 0.1 = 0.9$

$$P(x) = {}^nC_x \, p^x q^{n-x}$$

$$P(\text{two defective}) = {}^{10}C_2 \times (0.1)^2 (0.9)^8$$

$$= \frac{10 \times 9}{2 \times 1} \times (0.1)^2 (0.9)^8$$

$$= 0.1937$$

Ques: if 20 per cent managers are technocrats, the probability that a random committee of 5 managers consist of exactly 2 technocrats is

(a) 0.2048 (b) 0.4000 (c) 0.4096 (d) 0.9421

$n = 5$ $p = 0.2$

$x = 2$ $q = 0.8$

$$P(X=2 \text{ technocrates}) = {}^nC_x\, p^x q^{n-x}$$

$$= {}^5C_2 \times (0.2)^2 \times (0.8)^3$$

$$= \frac{5\times4}{2\times1} \times (0.2)^2 \times (0.8)^3$$

$$= 0.2048$$

Binomial Distribution:

$$P(x) = {}^nC_x\, p^x q^{n-x}$$

$$P(x=0) = {}^nC_0\, p^0 q^{n-0}$$

$$P(x=1) = {}^nC_1\, p^1 q^{n-1}$$

$$\vdots \qquad \vdots$$

sum of all cases:

$$\sum P = 1$$

$${}^nC_0\, p^0 q^{n-0} + {}^nC_1\, p^1 q^{n-1} + \cdots = 1$$

$$\sum_{x=0}^{n} {}^nC_x\, p^x q^{n-x} = 1$$

$$q^n + {}^nC_1\, p\, q^{n-1} + {}^nC_2\, p^2 q^{n-2} = 1$$

$$\boxed{(p+q)^n = 1}$$

Example: Three coins are tossed.

No. of heads; x :	0	1	2	3
$P(x)$:	$1/8$	$3/8$	$3/8$	$1/8$

(i) $E(x) = \sum x \cdot P(x) = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8}$

$= 3/2$

(ii) $E(x)$ in Binomial Distribution $= n \times p$
(1st moment) $= 3 \times \frac{1}{2} = 3/2$

(iii) $V(x) = E(x^2) - (E(x))^2 = 3/4$

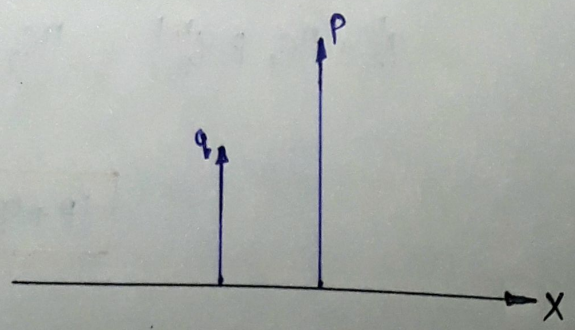
(iv) By Binomial: $V(x) = npq = 3 \times \frac{1}{2} \times \frac{1}{2} = 3/4$

Bernoulli Distribution:

x :	0	1
$P(x)$:	q	p

(i) $E(x) = \sum x P(x) = p$

(ii) $V(x) = pq$



Ques: In a manufacturing plant, the probability of making a defective bolt is 0.1. The mean and standard deviation of defective bolts in a total of 900 bolts are respectively.

(a) 90 and 9 (b) 9 and 90 (c) 30 and 3 (d) 3 and 30

$P = 0.1$

$n = 900$

(i) Mean, $E(X) =$

(ii) Variance, $V(X) = npq = 900 \times 0.1 \times 0.9 = 81$

$SD = \sqrt{V(X)} = \sqrt{81} = 9$

Ques: 10 Markes on table, 6 are defective and 4 are not defective. if 3 are randomly taken from lot, find, P(exactly 1 marker is defective) is ---.

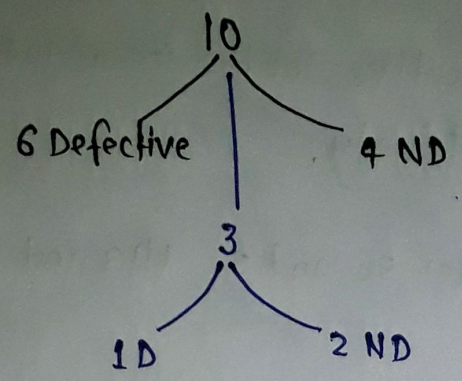
M1: D ND ND $\longrightarrow \frac{6}{10} \times \frac{4}{9} \times \frac{3}{8} = \frac{1}{10}$

ND D ND $\longrightarrow \frac{4}{10} \times \frac{6}{9} \times \frac{3}{8} = \frac{1}{10}$

ND ND D $\longrightarrow \frac{4}{10} \times \frac{3}{9} \times \frac{6}{8} = \frac{1}{10}$

$3 \times \frac{1}{10} = 3/10$ Ans

M2: Hypergeometric Distribution



$$\therefore \frac{{}^6C_1 \times {}^4C_2}{{}^{10}C_3}$$

$$= \frac{{}^{3}\cancel{6} \times \frac{\cancel{4}\times\cancel{3}}{\cancel{2}\times\cancel{1}}}{\frac{10\times\cancel{9}\times\cancel{8}_2}{\cancel{3}\times\cancel{2}\times\cancel{1}}} = 3/10 \text{ Ans}$$

Ques: 10 Markers on the table, 6 are defective and 4 are not defective. if 3 are randomly taken from lot with replacement. then Prob. of exactly 1 marker is defectives is —.

M1:

D ND ND $\longrightarrow \frac{6}{10} \times \frac{4}{10} \times \frac{4}{10} = 0.096$

ND D ND $\longrightarrow \frac{4}{10} \times \frac{6}{10} \times \frac{4}{10} = 0.096$

ND ND D $\longrightarrow \frac{4}{10} \times \frac{4}{10} \times \frac{6}{10} = 0.096$

0.288 Ans

M2: Binomical Distribution

$$P(X) = {}^nC_x \, p^x q^{n-x}$$

$$= {}^3C_1 \times (0.6)^1 (0.4)^2$$

$$= 0.288$$

As. $P(x) = {}^nC_x \, p^x q^{n-x}$

$$P(x) = {}^nC_x \cdot p^x \cdot \frac{q^n}{q^x}$$

$$P(x) = {}^nC_x \cdot \left(\frac{p}{q}\right)^x \cdot q^n$$

* $p + q = 1 \longrightarrow q = 1 - p$

$$P(x) = {}^nC_x \cdot \left(\frac{p}{1-p}\right)^x \cdot (1-p)^n$$

* mean, $m = n \cdot p$

$$\frac{m}{n} = p$$

$$P(x) = {}^nC_x \cdot \left(\frac{m/n}{1 - m/n}\right)^x \cdot (1 - m/n)^n$$

$$P(x) = \lim_{n \to \infty} \left(\frac{{}^nC_x}{n^x}\right) \left(\frac{m^x}{(1 - m/n)^x}\right) (1 - m/n)^n$$

* 'n' is very large

* $p = m/n$ is very small

(i) $\lim_{n \to \infty} \frac{{}^nC_x}{n^x} = \lim_{n \to \infty} \frac{n!/x!\,(n-x)!}{n^x}$

$$\lim_{n\to\infty} \frac{n(n-1)(n-2)\cdots(n-x+1)\cancel{(n-x)!}}{x!\; n^x \cancel{(n-x)!}}$$

$$\lim_{n\to\infty} \frac{\frac{n}{n}\times\left(\frac{n-1}{n}\right)\times\left(\frac{n-2}{n}\right)\times\cdots\frac{(n-x+1)}{n}}{x!}$$

$$\lim_{n\to\infty} \frac{1\times\cancel{\left(1-\frac{1}{n}\right)}^{0}\left(1-\cancel{\frac{2}{n}}\right)^{0}\cdots\left(1-\cancel{\frac{(x-1)}{n}}\right)^{0}}{x!}$$

$$= 1/x!$$

(ii) $$\lim_{n\to\infty} \frac{m^x}{\left(1-\frac{m}{n}\right)^x} = m^x$$

(iii) $$\lim_{n\to\infty} \left(1-\frac{m}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{(-m)}{n}\right)^n = e^{-m}$$

POISSION'S DISTRIBUTION!

$$\therefore\; P(x) = \frac{e^{-m}\cdot m^x}{x!} = \frac{e^{-\lambda}\cdot\lambda^x}{x!}$$

Valid for:

(i) independent variable

(ii) Discrete random variable

(iii) x is always +ve.

Here:
- $\lambda = m =$ mean $= np$
- $n\to\infty$, $p\to 0$
- $p+q=1$
- $x = 0, 1, 2, 3, \ldots$

$\therefore \quad E(X) = m = \lambda = np$

$V(X) = E(X) = \lambda$

Ques: Consider a poission distribution for the tossing of a biased coin. The mean for the distribution is μ. The standard deviation for this distribution is given by.

(a) $\sqrt{\mu}$ (b) μ^2 (c) μ (d) $1/\mu$

mean $= \mu$

standard deviation, $\sigma = \sqrt{V} = \sqrt{\text{mean}}$

$\therefore \sigma = \sqrt{\mu}$

Ques. The poisions Distribution is given by $P(r) = \frac{m^r}{r!} e^{-m}$.

The first moment about origin for the distribution is :

(a) 0 (b) m (c) $1/m$ (d) m^2

$E(X') = \text{mean} = m.$

Ques. The no. of accidents occuring in a plant in a month follows poission distribution with mean as 5.2. The probability of occurance of less than 2 accidents in the plant during a randomly selected month is -

(a) 0.029 (b) 0.034 (c) 0.039 (d) 0.044

Mean = $\lambda = 5.2$

$$P(X<2) = P(X=0) + P(X=1)$$

$$= \frac{e^{-5.2} \times (5.2)^0}{0!} + \frac{e^{-5.2} \times (5.2)^1}{1!}$$

$$= 0.034$$ Ans.

Ques: A traffic officer imposes on an average 5 Number of penalties daily on traffic violators. Assume that the number of penalties on different days is independent and follows poission distribution.

The probability that there will be less than 4 penalties in a day is

(a) 0.16 (b) 0.36 (c) 0.26 (D) 0.46

sol. $\lambda = 5$

$$P(X<4) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= e^{-\lambda}\left\{\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!}\right\}$$

$$= e^{-5} \times \left(1 + 5 + \frac{25}{2} + \frac{125}{6}\right)$$

$$= 0.265$$

Ques. The average no. of accidents occuring monthly on an assembly shop floor is 2. The prob. that there will be atleast one accident in this month is estimated to be -

(a) 0.055 (b) 0.456 (c) 0.865 (d) 0.950

$\lambda = 2$

$$P(X \geq 1) = 1 - P(X<1)$$

$$= 1 - P(X=0) = 1 - \frac{e^{-2} \times 2^0}{0!} = 0.865$$

Ques. if a random variable x satisfies the poission distribution with a mean value of 2 then the prob. that $x \geq 2$ is

(a) $2e^{-2}$ (b) $3e^{-2}$ (c) $1-2e^{-2}$ (d) $1-3e^{-2}$

Sol.

$\lambda = 2$

$$P(x \geq 2) = 1 - P(x < 2)$$

$$= 1 - [P(x=0) + P(x=1)]$$

$$= 1 - \left[\frac{e^{-2} \times 2^0}{0!} + \frac{e^{-2} \times 2^1}{1!}\right]$$

$$= 1 - 3e^{-2} \text{ Ans.}$$

Ques: An observer counts 240 veh/hour at a specific highway location. Assume that the vehicle arrival at the location is poission distribution, the Prob. of having one vehicle arriving over a 30 second time interval is —.

(a) 0.17 (b) 0.37 (c) 0.27 (d) 0.47

mean = λ = 240 vehicle/hour

$$= \frac{240}{60 \times 60} \text{ veh./sec}$$

$$= \frac{240}{60 \times 2} \text{ veh./30 sec.} = 2$$

$$P(x=1) = \frac{e^{-\lambda} \cdot \lambda^1}{1!} = e^{-2} \times 2 = 0.27 \text{ Ans}$$

Ques. it is estimated that the average no. of events during a year is three. what is the prob. of occurrence of not more than two events over two year duration? Assume that the no. of events follow a poission distribution.

(a) 0.052 (b) 0.062 (c) 0.072 (d) 0.082

Avg. no. of event during one year = 3.

$\lambda_{2year} = 6$

$$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= e^{-6} \times \left[\frac{6^0}{0!} + \frac{6^1}{1!} + \frac{6^2}{2!} \right] = 0.062 \text{ Ans.}$$

Ques: Passangers try repeatedly to get a seat reservation in any train running between two stations untill they are successful. if there is 40% chance of getting reservation in any attempt by a passanger then the average no. of attempts that passanger need to make to get a seat reseasved as -

E (No. of attempts needed to get seat reserved)

$$E(X) = \sum x \cdot p(x)$$

M1:

Event :	p	qp	qqp	------
No. of attempt, x :	1	2	3	-----
$P(x)$:	(0.4)	(0.6×0.4)	$(0.6 \times 0.6 \times 0.4)$	------

$$\therefore E(X) = 1 \times \frac{2}{5} + 2 \times \frac{3}{5} \times \frac{2}{5} + 3 \times \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5} + \cdots$$

$$= \frac{2}{5}\left[1 + 2 \times \frac{3}{5} + 3 \times \left(\frac{3}{5}\right)^2 + 4 \times \left(\frac{3}{5}\right)^3 + \cdots\right]$$

$$= \frac{2}{5}\left(1 - 3/5\right)^{-2} = \frac{2}{5} \times \frac{1}{(2/5)^2} = \frac{5}{2}$$

$E(X) = 2.5$ Ans.

M2: GEOMETRIC DISTRIBUTION!

No. of attempts, x :	$1 \rightarrow p$
(seat researved)	$2 \rightarrow qp$
	$3 \rightarrow qqp$
	$\vdots$

$$P(x) = p\,q^{x-1} \quad \forall\, x = 1, 2, 3, \ldots$$

$$E(X) = \sum x \cdot P(x)$$

$$E(X) = 1 \times p + 2 \times q p + 3 \times q q p + \cdots$$

$$E(X) = p\,(1 + 2q + 3q^2 + \cdots)$$

$$E(X) = p\,(1-q)^{-2}$$

$$E(X) = \frac{p}{(1-q)^2} = \frac{p}{p^2} \qquad \because p+q=1$$

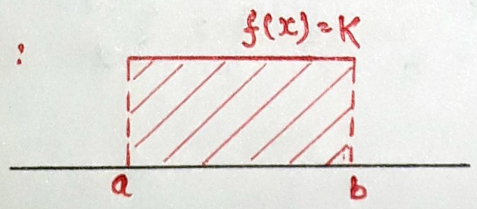
$$** \boxed{E(X) = \frac{1}{p}} = \frac{10}{0.4} = \frac{5}{2} = 2.5 \text{ Ans.}$$

$$** \boxed{V(X) = \frac{q}{p^2}}$$

UNIFORM PROBABILITY DISTRIBUTION : {CONTINUOUS}

$$f(x) = \begin{cases} K, & a \le x \le b \\ 0, & \text{elsewhere} \end{cases}$$

Pdf :



it is a pdf

Area of rectangle = 1

$$K \times (b-a) = 1$$

$$K = 1/(b-a)$$

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{elsewhere} \end{cases}$$

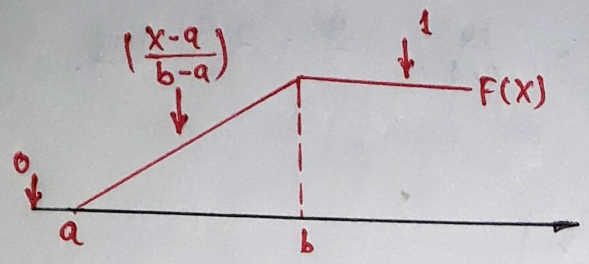
cdf:

$F(x<a) = 0$

$F(x>b) = 1$

$$F(a<x<b) = \int_a^x f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\,(x)_a^x = \frac{x-a}{b-a}$$



Expectation in Uniform Distribution:

$$E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx$$

$$= \int_a^b x \cdot \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left(\frac{x^2}{2}\right)_a^b = \frac{1}{2} \times \frac{b^2-a^2}{b-a}$$

* $E(X) = \frac{a+b}{2}$

* $E(X^2) = \frac{a^2+ab+b^2}{3}$

* $V(X) = \frac{(b-a)^2}{12}$

Ques: A random variable is uniformly distributed over the interval 2 to 10. its variance will be

(a) $\frac{16}{3}$ (b) 6 (c) $\frac{256}{3}$ (d) 36

$$V(X) = \frac{(b-a)^2}{12} = \frac{(10-2)^2}{12} = \frac{64}{12} = \frac{16}{3} \text{ Ans}$$

Ques: X is uniformly distributed random variable that take values between 0 and 1. The value of $f(X^3)$ will be

(a) 0 (b) $1/8$ (c) $1/4$ (d) $1/2$

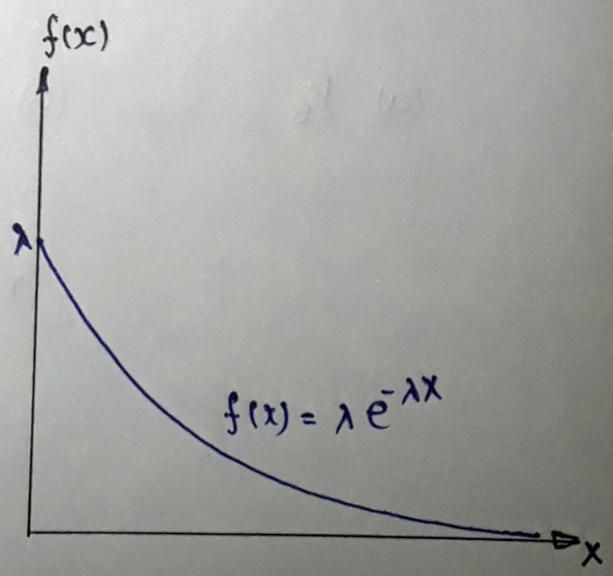
$$E(X^3) = \int_{-\infty}^{\infty} x^3 f(x)\,dx$$

$$= \int_0^1 x^3 \times \frac{1}{1-0}\,dx = \left(\frac{x^4}{4}\right)_0^1 = 1/4 \text{ Ans}$$

Exponential Distribution:

Pdf:

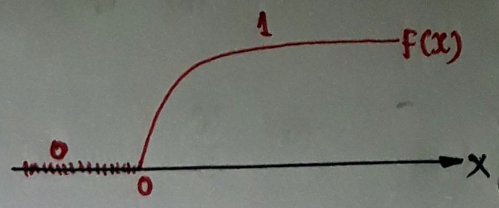
$$f(x) = \begin{cases} \lambda e^{-\lambda x} & ; x \geq 0 \;\&\; \lambda > 0 \\ 0 & ; \text{otherwise} \end{cases}$$



cdf : $F(x<0) = 0$

$$F(x \geq 0) = \int_0^x \lambda e^{-\lambda x}\, dx$$

$$= 1 - e^{-\lambda x} \quad ; \; x \geq 0$$



* $E(X) = 1/\lambda$
* $V(X) = 1/\lambda^2$
* $E(X^n) = \dfrac{n!}{\lambda^n}$

Ques: Assume that the duration in minutes of a telephone conservation follows the exponential distribution.

$$f(x) = \frac{1}{5} e^{-x/5} \quad ; \; x \geq 0$$

The probability that the conversation will exceed five minutes is

(a) $1/e$ (b) $1 - 1/e$ (c) $1/e^2$ (d) $1 - 1/e^2$

$$f(x) = \begin{cases} \frac{1}{5} e^{-x/5} & , \; x \geq 0 \\ 0 & , \; x < 0 \end{cases}$$

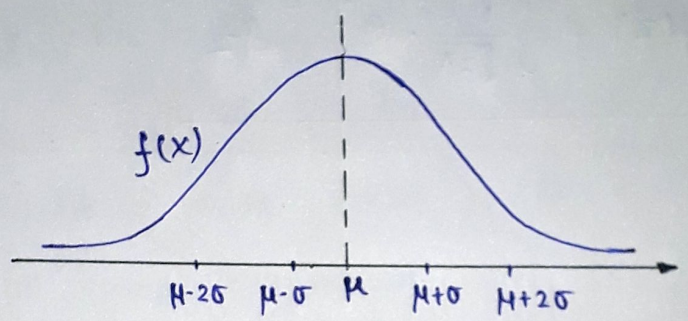
$$P(x>5) = \int_5^\infty f(x)\,dx = \int_5^\infty \frac{1}{5} e^{-x/5}\,dx.$$

$$= \frac{1}{5}\left(\frac{e^{-x/5}}{-1/5}\right)_5^\infty = 1/e \text{ Ans.}$$

* Gaussian Distribution / Normal Distribution / Bell shaped.

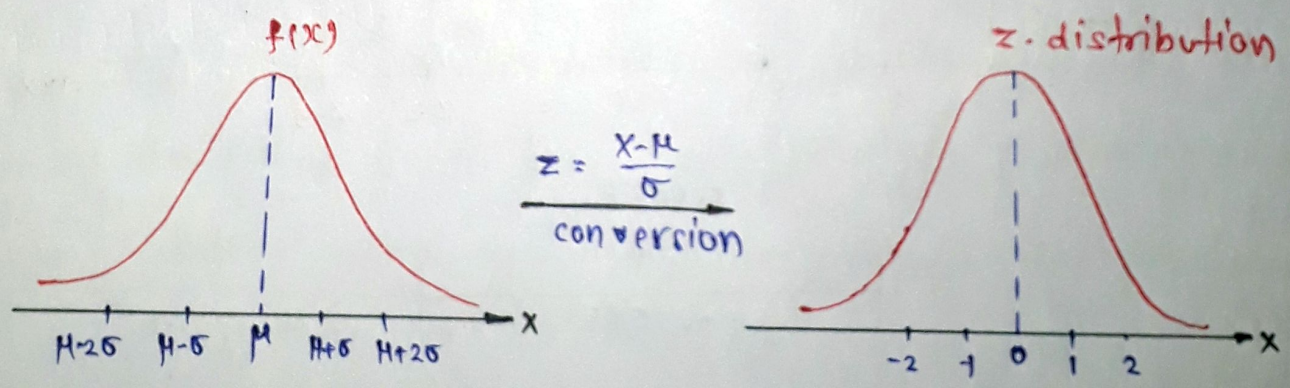
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} * e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad ; \cdot -\infty < x < \infty$$

- $\sigma^2 > 0$
- $-\infty < \mu < \infty$



Z-Distribution :

$$Z = \frac{x-\mu}{\sigma}$$



$$* \; z = \frac{x-\mu}{\sigma}$$

$$\sigma z = x - \mu$$

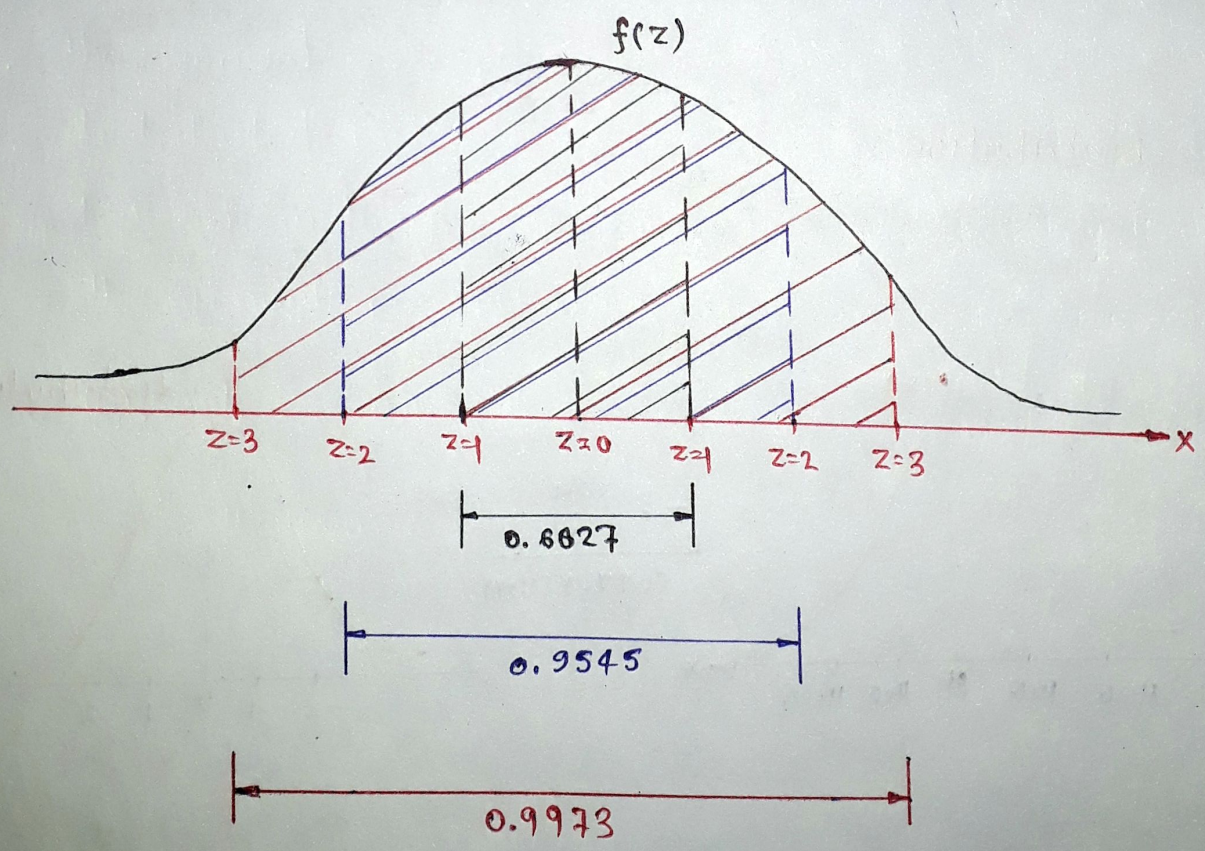
$$x = \mu + \sigma z$$

$$dx = \sigma dz$$

$$f(\mu+\sigma z) = \frac{1}{\sqrt{2\pi\sigma^2}} * e^{-\frac{1}{2}(z)^2}$$

$$\mu = 0 \, , \, \sigma = 1$$

$$\boxed{\int f(z) = \int \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2} z^2} dz}$$



- $\int_{-\infty}^{\infty} f(z)\,dz = 1 \longrightarrow \boxed{P(-\infty < z < \infty) = 1}$

- $\int_{0}^{\infty} f(z)\,dz = \int_{-\infty}^{0} f(z)\,dz = \frac{1}{2} \longrightarrow \boxed{P(z>0) = P(z<0) = 1/2}$

- $P(-1<z<1) = 0.6827$
- $P(-1<z<0) = P(0<z<1) = \frac{0.6827}{2} = 0.3413$
- $P(-2<z<2) = 0.9545$
- $P(-3<z<3) = 0.9973$

(i) Area under curve is 1

(ii) symmetric about mean.

(iii) Mean = median = mode

Ques. A nationalized bank has found that the daily balance available in its savings account follows a normal distribution with a mean of Rs 500 and a standard deviation of Rs 50. The percentage of saving account holders, who maintain an average daily balance more than Rs 500 is 50%.

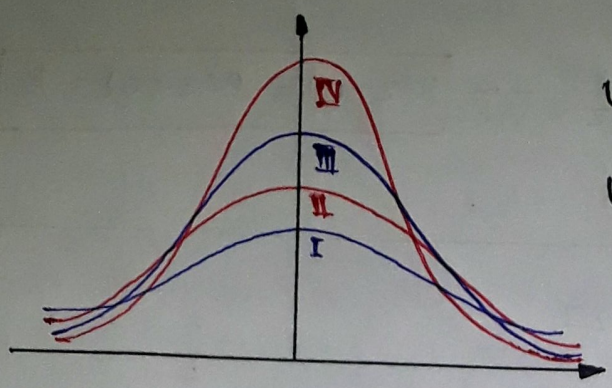
$\mu = 500, \ \sigma = 50$

$P(x > 500) = ?$

$P\left(\frac{x-\mu}{\sigma} > \frac{500-\mu}{\sigma}\right)$

$P(z>0) = 0.5$

Ques: Below, which one has the lowest variance?



(A) I (B) II

(C) III (D) IV

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \times e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$f(x) \propto \frac{1}{\sqrt{\sigma^2}} \propto \frac{1}{\sqrt{V}}$$

· spread of data less = V is less = Highest peak

Ques. The annual precipitation data of a city is normally distributed with mean and standard deviation as 1000 mm and 200 mm, respectively. The probability that the annual precipitation will be more than 1200 mm is

(A) <50 % (B) 50 % (C) 75 % (D) 100 %

$\mu = 1000$, $\sigma = 200$

$$P\left(\frac{X-\mu}{\sigma} > \frac{1200-\mu}{\sigma}\right)$$

$$P(z > 1)$$

$$\therefore P(z > 1) < 50\%$$

$$P(z>1) = P(z>0) - P(0<z<1)$$

$$P(z>1) = 0.5 - \frac{0.68}{2}$$

$$P(z>1) = 0.16$$

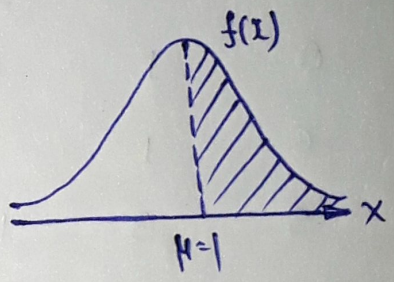
Ques. $f(x) = \frac{1}{\sqrt{\theta\pi}} e^{-\left(\frac{(x-1)^2}{\theta}\right)}$; $-\infty < x < \infty$

then, $\int_1^\infty f(x)\,dx =$

(A) 0 (B) $1/2$ (C) $1-\frac{1}{e}$ (D) 1

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{\sqrt{2\pi(2)^2}} \cdot e^{-\frac{1}{2}\left(\frac{x-1}{2}\right)^2}$$

so, here, $\mu = 1$



Ques: $I = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/8}\,dx$

we know that, $\int_0^\infty f(x)\,dx = 1/2$

when, $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$$\therefore \sqrt{4} \int_0^\infty \frac{1}{\sqrt{2\pi \cdot \sqrt{4}}} \times e^{-\frac{1}{2}\left(\frac{x}{2}\right)^2} = \sqrt{4} \times \frac{1}{2} = 1$$ Ans

Mean:

$$\text{Average} = \frac{\text{sum of all terms}}{\text{Total no. of terms}} = \frac{\Sigma x_i}{n}$$

Example: $\{1, 8, 3, 2, 6\}$

find mean.

Raw | ungrouped data: $\text{Mean} = \frac{1+8+3+2+6}{5} = 4$

Example: Grouped Data.

x_i	f_i	$x_i f_i$
8	1	8
6	1	6
5	3	15
7	2	14
4	2	8
9	1	9
	$\Sigma f_i = 10$	$\Sigma x_i f_i = 60$

M1:

$$\text{Mean} = \frac{8+6+5+5+5+7+7+4+4+9}{10}$$

$$\text{Mean} = 6$$

M2: for Grouped data

$$\text{Mean} = \frac{\Sigma x_i f_i}{\Sigma f_i} = \frac{60}{10}$$

$$\text{Mean} = 6 \text{ Ans.}$$

Ques: The following data about the flow of liquid was observed in a continuous chemical process plant.

Flow rate (litres/sec)	x_i	frequency	$x_i f_i$
7.5 - 7.7	7.6	1	7.6
7.7 - 7.9	7.8	5	39.0
7.9 - 8.1	8.0	35	280
8.1 - 8.3	8.2	17	139.4
8.3 - 8.5.	8.4	12	100.8
8.5 - 8.7	8.6	10	86
		$\Sigma f_i = 80$	$\Sigma x_i f_i = 652.8$

Mean flow rate of the liquid is

(A) 8.00 litres/sec (B) 8.06 lit./sec (C) 8.16 lit./sec.

(D) 8.26 lit./sec.

Sol. $\text{Mean} = \dfrac{\Sigma x_i f_i}{\Sigma f_i} = \dfrac{652.8}{80} = 8.16$

Median :

Example : $\{1, 3, 2\}$

(i) Ascending : 1, 2, 3

OR

Descending : 3, 2, 1

(ii) check that value which divides data set in equal parts

(iii) median = 2 Ans.

Example: $\{1, 2, 3, 4\}$

$$\text{Mean} = \frac{1+2+3+4}{4} = 2.5$$

$$\text{Median} = \frac{2+3}{2} = 2.5.$$

Median formula:

(i) odd data set: n = count

$$\text{Median} = \left(\frac{n+1}{2}\right)^{th} \text{term}$$

(ii) Even data set:

n = count

$$\text{Median} = \frac{\left(\frac{n}{2}^{th} \text{term} + \left(\frac{n}{2}+1\right)^{th} \text{term}\right)}{2}$$

Ques: The spot speeds (expressed in km/hr) observed at a road rection are 66, 62, 45, 79, 32, 51, 56, 60, 53 and 49. The median speed (expressed in km/hr) is.

Ascending order: 32, 45, 49, 51, 53, 56, 60, 62, 66, 79.

$n = 10$

$$\text{median} = \left[\frac{n}{2}^{th} + \left(\frac{n}{2}+1\right)^{th}\right] / 2$$

$$= \frac{53+56}{2} = 54.5$$

Ques. Median = ?

x_i	f_i	cf
20	6	6
25	20	26
28	24	50
29	28	78
33	15	93
38	4	97
42	2	99
43	1	100
	$\Sigma f_i = 100$	

$\therefore \Sigma f_i$ = count = N = 100

$$\text{Median} = \frac{50^{th} + 51^{th}}{2}$$

$$= \frac{28+29}{2} = 28.5$$

Ques!

x_i	f_i	cf
65 – 85	4	4
85 – 105	5	9
105 – 125	13	22
125 – 145	20	42
145 – 165	14	56
165 – 185	8	64
185 – 205	4	68
	$\Sigma f_i = 68$	

$$\text{Median} = L_m + \frac{\left(\frac{N}{2} - pcf\right)}{f_m} \times h$$

$$\text{Median} = 125 + \frac{\left(\frac{68}{2} - 22\right)}{20} \times 20$$

$$= 137 \text{ Ans}$$

MODE :

Ex1 : $\{1, 2, 2, 3, 3, 3\}$

Mode = 3 $\longrightarrow$ Unimodal

Ex 2 : $\{1, 2, 2, 3, 3\}$

Mode = 2, 3 $\longrightarrow$ Bi-modal

Ex 3 : $\{1, 2, 2, 3, 3, 4, 4\}$

Mode = 2, 3, 4 $\longrightarrow$ Tri-modal

Ex 4 : $\{1, 2, 2, 3, 3, 4, 4, 5, 5\}$

Mode : 2, 3, 4, 5 $\longrightarrow$ Multi-modal

Ex 5 : $\{1, 2, 3, 4, 5\}$

NO Mode.

Ques :

x_i	f_i	
0 - 20	10	
20 - 40	35	
40 - 60	52	f_0
60 - 80	61	f_1
80 - 100	38	f_2
100 - 120	29	

$$\text{Mode} = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times h$$

$$\text{Mode} = 60 + \left(\frac{61 - 52}{2 \times 61 - 52 - 38}\right) \times 20$$

$$= 65.625$$

Ques: Marks obtained by 100 students in an examination are given in the table

S. NO	MARK obtained (x_i)	No. of students (f_i)	cf	$x_i f_i$
1	25	20	20	500
2	30	20	40	600
3	35	40	80	1400
4	40	20	100	800

what would be the mean, median and mode of the marks obtained by the students?

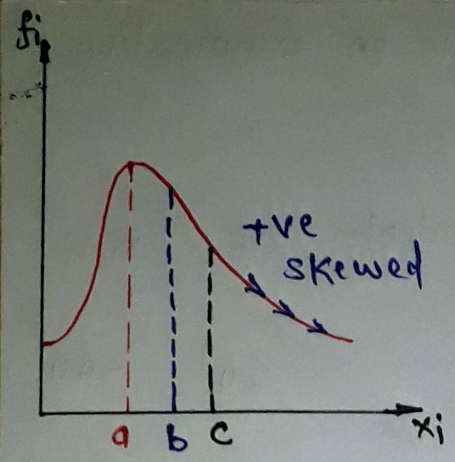
(i) Mean $= \dfrac{\Sigma x_i f_i}{\Sigma f_i} = \dfrac{3300}{100} = 33$

(ii) Median $= \dfrac{\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}}{2}$

$= \dfrac{50^{th} + 51^{th}}{2}$

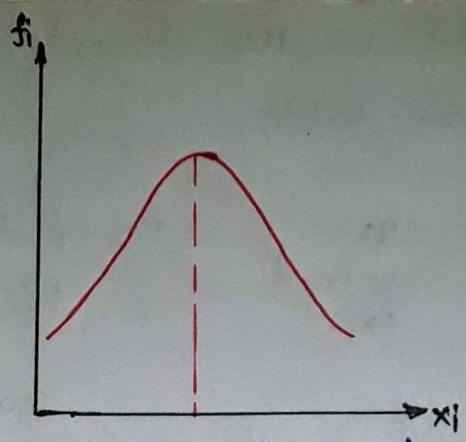
$= \dfrac{35+35}{2} = 35$

(iii) Mode $= 35$

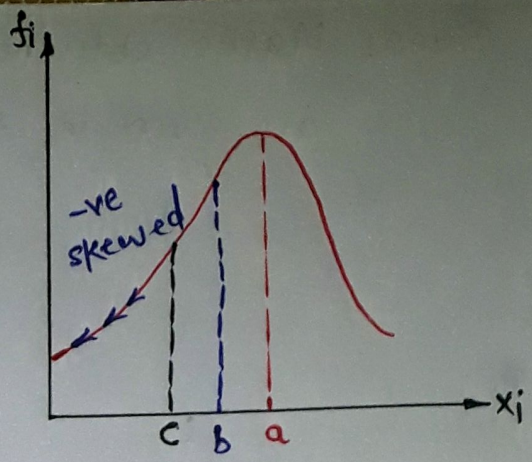


$c > b > a$

Mean > median > mode



Mean = Median = mode



$a > b > c$

Mode > median > mean

* 1 MODE + 2 MEAN = 3 MEDIAN

$\boxed{\text{Dean} = \text{Dian}}$

Ques: $\{2, 3, 4, 7, 9\}$

$$\text{Mean} = \frac{2+3+4+7+9}{05} = 5$$

x	$\bar{x}$	$x-\bar{x}$	$(x-\bar{x})^2$
2	5	-3	9
3	5	-2	4
4	5	-1	1
7	5	2	4
9	5	4	16

M1:

$$\sigma = \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \sqrt{6.8} \cong 2.6$$

$$\because \sigma = \sqrt{\frac{\sum(x-\bar{x})^2}{n}}$$

$$\sigma = \sqrt{\frac{\sum(x^2 - 2x\bar{x} + \bar{x}^2)}{n}}$$

$$= \sqrt{\frac{\sum x^2}{n} - 2\bar{x}\underbrace{\frac{\sum x}{n}}_{\bar{x}} + \frac{\sum \bar{x}^2}{n}}\ n\bar{x}^2$$

$$= \sqrt{\frac{\sum x^2}{n} - \bar{x}^2}$$

$$\therefore \sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$$

M2:

x	x^2
2	4
3	9
4	16
7	49
9	81

$$\sigma = \sqrt{\frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2}$$

$$\sigma = \sqrt{\frac{159}{5} - \left(\frac{25}{5}\right)^2}$$

$$\sigma = \sqrt{6.8} \cong 2.6$$