MATHEMATICS

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CALCULUS

* LIMITS *

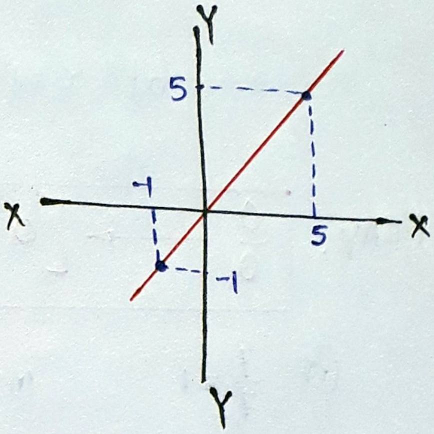
- Dependent / Independent variable

  - $y = x$ ← independent variable

dependent variable
or
function of dependent variable

Example: $x = 5 \rightarrow y = 5$

$x = 1 \rightarrow y = 1$



- $y = x \longrightarrow f(x) = x$

- $f(x,y) = x + y$ ⟶ in this case we are in 3-dimensions.

dependent. independent variable.

- Exact value:

(i) $f(x) = 2 + x$

at $x = 0 \longrightarrow f(0) = 2$

at $x = 1 \longrightarrow f(1) = 1$

at $x = 1 \longrightarrow f(1) = 3$

Points

Amplitude ⟶ it is the exact value.

Ques. $f(x) = \frac{x-1}{x-1}$ Exact value = ?

at $x = 0 \longrightarrow f(0) = 1$

at $x = 10 \longrightarrow f(10) = 1$

at $x = -1 \longrightarrow f(-1) = 1$

Exact value.

at $x = 1 \longrightarrow f(1) = \frac{0}{0}$ (undefined) [Approaching]

⇓

LIMIT.

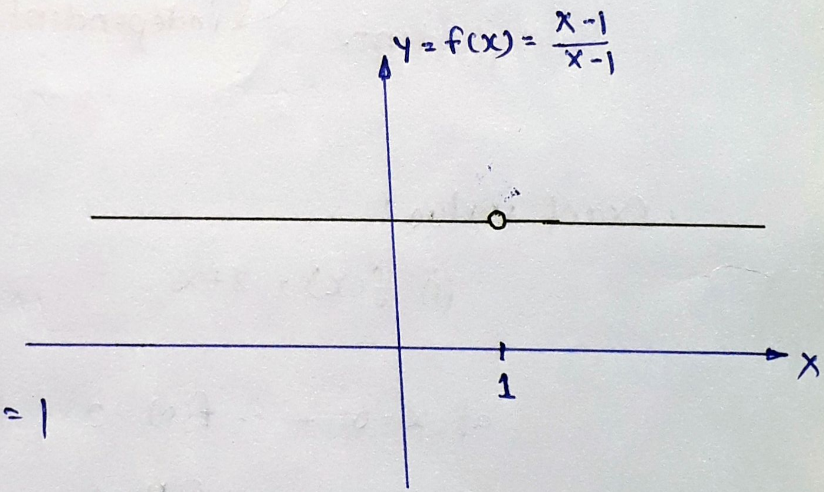
PLAY: $\boxed{\frac{0}{0} \longrightarrow [0 \text{ or } \infty] \therefore \text{it is undefined/indeterminate form.}}$

(i) $\frac{7}{7} = 1$

(ii) $\frac{N=7}{7} = \frac{0}{7} = 0$

LIMIT: Approaching to exact value of function.

(iii) $\frac{7}{D=7} = \frac{7}{0.0000....1} = 7 \times 10^{Big} = \frac{7}{0} = \infty$



$LHL = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x-1}{x-1} = 1$

$RHL = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x-1}{x-1} = 1$

$\therefore$ LHL = RHS — limit exist.

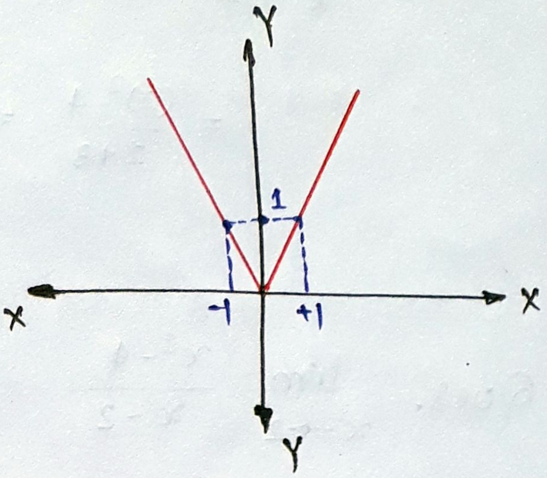
and limiting value is 1.

Ques. $f(x) = |x|$

$$f(x) = |x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

at $x = 1 \longrightarrow f(x) = 1$

at $x = -1 \longrightarrow f(-1) = 1$

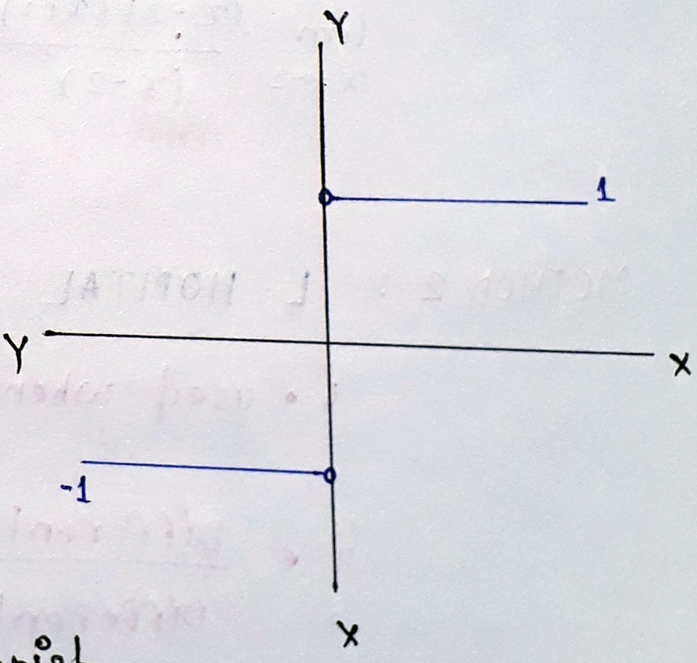


Ques. $f(x) = \lim_{x \to 0} \frac{|x|}{x}$

$$f(x) = \begin{cases} \frac{x}{x}, & x > 0 \\ -\frac{x}{x}, & x < 0 \end{cases}$$

$$LHL = \lim_{x \to 0^-} \frac{-x}{x} = -1$$

$$RHL = \lim_{x \to 0^+} \frac{x}{x} = 1$$



Here, $LHL \neq RHL$

so, limit does not exist.

* DIRECT SUBSTITUTION

Ques. $\lim_{x \to 2} \frac{x^2-4}{x+3}$

$= \frac{(2)^2-4}{2+3} = \frac{0}{5} = 0$

Ques. $\lim_{x \to 2} \frac{x^2-4}{x-2}$

$\lim_{x \to 2} \frac{(2)^2-4}{2-2} = \frac{0}{0}$ form.

Method-1 : Rationalization

$\lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)} = \lim_{x \to 2} (x+2) = 4$

METHOD 2 : L-HOPITAL

i. used when $\frac{0}{0}$ or $\frac{\infty}{\infty}$

ii. $\frac{\text{Differentiate Numerator}}{\text{Differentiate Denominator}}$

iii. Direct substitution
- finit answer
- again $\frac{0}{0}$ or $\frac{\infty}{\infty}$ $\downarrow$ then repeat from step 2.

$$\lim_{x \to 2} \frac{x^2-4}{x-2} \longrightarrow \text{(i) } \frac{0}{0} \quad \text{(iii) } \lim_{x \to 2} \frac{d(x^2-4)}{d(x-2)}$$

$$\lim_{x \to 2} 2x = 2 \times 2 = 4$$

Ques. $\lim_{x \to 8} \frac{x^{1/3}-2}{x-8}$

i. $\frac{0}{0}$ form.

L. HOPITAL, ii. $\lim_{x \to 8} \frac{\frac{1}{3}x^{\frac{1}{3}-1}}{1-0} = \frac{1}{3} \times (8)^{-2/3} = \frac{1}{12}$

$$\boxed{\text{FORMULA: } \lim_{x \to a} \frac{x^n - a^n}{x-a} = n a^{n-1}}$$

* $\frac{\infty}{\infty}$ form : indeterminate form.

i. $\frac{\infty}{K} = \infty$

ii. $\frac{K}{\infty} = 0$

iii. $\frac{N}{D} = \frac{\infty}{\infty} = [\infty \text{ or } 0]$

$\therefore$ undefined.

- $(\infty + \infty)$ form

not a number but a never ending thing.
idea of

$\therefore \infty + \infty = \infty$

- $(\infty - \infty)$ form : undefined/indeterminate form.

$[-\infty \text{ or } \infty]$

Ques. $\lim\limits_{n\to\infty} \dfrac{n}{\sqrt{n^2+n}}$

idea → variable $= \infty$ → $\frac{1}{V} = \frac{1}{\infty} = 0$ (used when $\frac{\infty}{\infty}$ form)

$$\lim_{n\to\infty} \frac{n/n}{\sqrt{n^2+n}/\sqrt{n^2}} = \lim_{n\to\infty} \frac{1}{\sqrt{\frac{n^2+n}{n^2}}}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = \frac{1}{\sqrt{1+\frac{1}{\infty}}} = 1$$

- ∞^0 form : Indeterminate form.

(i) $\infty^K = \infty$

(ii) $(K)^0 = 1$

(iii) $p^q = [\infty \text{ or } 1] \therefore$ it is undefined.

Ques. $\lim_{x \to \infty} x^{\frac{1}{x}}$

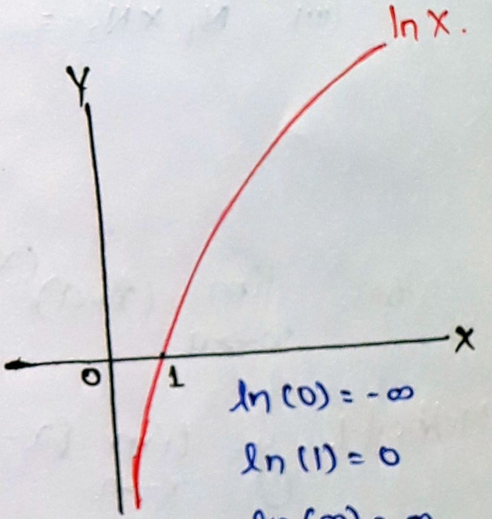
idea : $(\text{Variable})^{\text{Variable}}$ $\longrightarrow$ logarthmic technique.

i. $y = \lim_{x \to \infty} x^{\frac{1}{x}}$

ii. $\ln y = \ln \left(\lim_{x \to \infty} x^{\frac{1}{x}} \right) = \lim_{x \to \infty} \left(\ln x^{\frac{1}{x}} \right)$

$\ln y = \lim_{x \to \infty} \left(\frac{1}{x} \ln x \right)$

$\ln y = \lim_{x \to \infty} \left(\frac{\ln x}{x} \right)$ $\left(\frac{\infty}{\infty} \text{ form} \right)$



$\ln (0) = -\infty$

$\ln (1) = 0$

$\ln (\infty) = \infty$

$\ln y = \lim_{x \to \infty} \frac{1/x}{1} = \frac{1}{\infty} = 0$

$\ln y = 0$

$y = e^0 = 1$ $\therefore y = 1$ Ans

$$\boxed{\text{FORMULA: } \lim_{x \to \infty} x^{\frac{1}{x}} = 1}$$

- 0^0 form : Inderminate form.

i. $0^a = 0$

ii. $a^0 = 1$

iii. $p^q = [0 \text{ or } 1]$ – undefined.

- $(0 \times \infty)$ form : Indeterminate form.

(i) $0 \times K = 0$

(ii) $K \times \infty = \infty$

(iii) $N_1 \times N_2 = [0 \text{ or } \infty]$ undefined.

Ques. $\lim\limits_{x \to a} (x-a)^{(x-a)}$

Method-1. $y = \lim\limits_{x \to a} (x-a)^{(x-a)}$

$$\ln y = \lim_{x \to a} \ln (x-a)^{(x-a)}$$

$$\ln y = \lim_{x \to a} (x-a) \ln (x-a)$$

$$\ln y = \lim_{x \to a} \frac{\ln(x-a)}{1/(x-a)}$$

$$\ln y = \lim_{x \to a} \frac{1/(x-a)}{-1/(x-a)^2}$$

$$\ln y = \lim_{x \to a} -(x-a)$$

$$\ln y = 0 \qquad \therefore y = e^0 \longrightarrow y = 1 \ \underline{\text{Ans.}}$$

Method 2 :

we know that, $\lim_{y \to 0} y^y = 1$

$$\lim_{x \to a} (x-a)^{(x-a)}$$

$$\lim_{(x-a) \to 0} (x-a)^{(x-a)} = 1 \text{ Ans.}$$

- 1^∞ form : ideterminate form.

(i) $1 \times 1 = 1$

$1 \times 1 \times 1 \times \ldots\ldots = 1$

(ii) $(1+h)^\infty$

$h = 0.001 \longrightarrow (1+0.001)^\infty = (1.001)^\infty = \infty$

$h = -0.001 \longrightarrow (1-0.001)^\infty = (0.999)^\infty = 0$

Ques. $\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{2n}$

$$\boxed{\text{Formula: } \lim_{n\to\infty} \left(1+\frac{a}{n}\right)^{n} = e^{a} \qquad \lim_{n\to\infty} (1+an)^{1/n} = e^{a}}$$

$$\lim_{n\to\infty} \left\{\left(1-\frac{1}{n}\right)^{n}\right\}^{2}$$

$$= \left\{\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{n}\right\}^{2} = (e^{-1}) = e^{-2} \text{ Ans.}$$

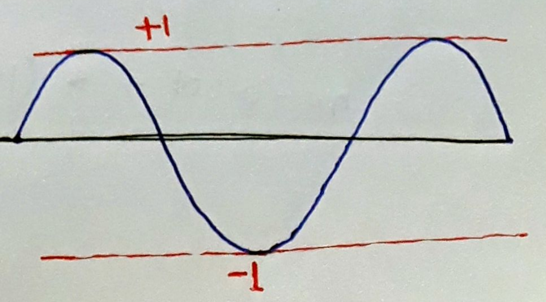
• $f(x) = \frac{\sin x}{x}$

(i) at $x=0 \longrightarrow f(0) = \frac{\sin 0}{0} = \frac{0}{0}$ (Indermi nate form)

(ii) $\lim_{x\to 0} \frac{\sin x}{x}$ $\left(\frac{0}{0} \text{ form}\right)$

$$\lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1 \text{ Ans}$$

(iii) $\lim_{x\to\infty} \frac{\sin x}{x} = \frac{\sin \infty}{\infty}$



$$= \frac{[+1,-1]}{\infty} = \frac{\text{finite}}{\infty}$$

$$= 0 \text{ Ans.}$$

iv. $\lim_{x \to -\infty} \frac{\sin x}{x} = 0$ Ans.

Ques. $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$

M-1 : $\lim_{x \to 0} \frac{\sin(1/x)}{(1/x)}$

put, $\frac{1}{x} = t \to$ $x \to 0$, $t \to \infty$

$\lim_{t \to \infty} \frac{\sin t}{t} = 0$ Ans.

M-2 :

$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$

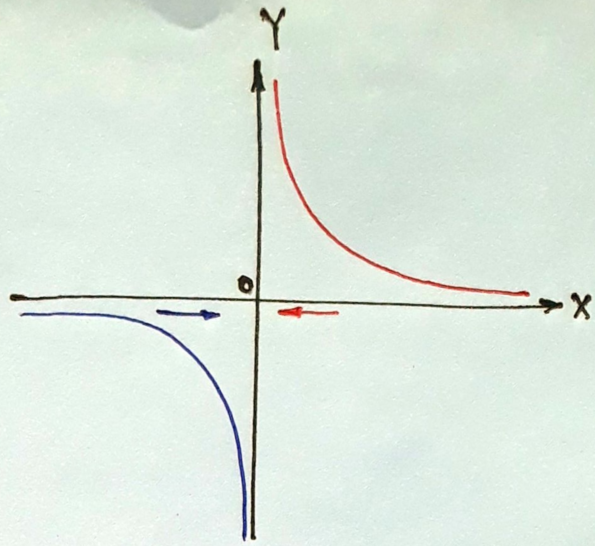
$= 0 \sin\left(\frac{1}{0}\right)$

$= 0 \cdot [-1, +1]$

$= 0 \times$ finite value

$= 0$ Ans.

* $\frac{1}{0}$: $(-\infty$ or $+\infty)$ undefined.



CASE 1 : $\frac{1}{0.1} = 10^1$

$\frac{1}{0.001} = 10^3$

$\frac{1}{0.00.......1} = 10^{\text{very big}} = \infty$

$\therefore \boxed{\frac{1}{0} = \infty}$

CASE 2 : $\frac{1}{-0.1} = -10^1$

$\frac{1}{-0.001} = -10^3$

$\therefore \boxed{\frac{1}{0} = -\infty}$

$\frac{1}{-0.00.....1} = -10^{\text{very big}} = -\infty$

Ques. $\lim\limits_{x \to 0} \left(\frac{e^{5x}-1}{x} \right)^2$

$= \left(\lim\limits_{x \to 0} \frac{e^{5x}-1}{x} \right)^2$ $\left(\frac{0}{0} \text{ form}\right)$

$= \left(\lim\limits_{x \to 0} \frac{5e^{5x}}{1} \right)^2$

$= (5 \cdot e^{5 \times 0})^2 = (5 \cdot 1)^2 = 25$ Ans.

* LIMITS FORMULA'S

(a) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(b) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

(c) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

(d) $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$

(e) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$

(f) $\lim_{x \to \infty} \frac{\sin x}{x} = 0$

(g) $\lim_{x \to 0} (1+x)^{1/x} = e$

(h) $\lim_{x \to 0} (1+ax)^{1/x} = e^a$

(i) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

(j) $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a$

(k) $\lim_{x \to \infty} (x)^{1/x} = 1$

(l) $\lim_{x \to 0} (x)^x = 1$

* PROPERTIES OF LIMITS

(i) $\lim_{x \to a} [c f(x)] = c \lim_{x \to a} f(x)$

(ii) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

(iii) $\lim_{x \to a} [f(x) g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

(iv) $\lim\limits_{x \to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$ provided $\lim\limits_{x \to a} g(x) \neq 0$

(v) $\lim\limits_{x \to a} [f(x)]^n = \left[\lim\limits_{x \to a} f(x)\right]^n$

(vi) $\lim\limits_{x \to a} \left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim\limits_{x \to a} f(x)}$

* DERIVATIVES *

i. $\dfrac{d}{dx}(x^n) = n x^{n-1}$

ii. $\dfrac{d}{dx}(a^x) = a^x \ln a$

(iii) $\dfrac{d}{dx}(e^x) = e^x$

iv. $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}$

v. $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

vi. $\dfrac{d}{dx}(\sin x) = \cos x$

vii. $\dfrac{d}{dx}(\cos x) = -\sin x$

viii. $\dfrac{d}{dx}(\tan x) = \sec^2 x$

ix. $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

x. $\dfrac{d}{dx}(\sec x) = \sec x \cdot \tan x$

xi. $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$

xii. $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$

xiii. $\dfrac{d}{dx}(\cos^{-1} x) = \dfrac{-1}{\sqrt{1-x^2}}$

xiv. $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$

XV. $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

XVI. $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$

XVII. $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

XVIII. $\frac{d}{dx}(\sinh x) = \cosh x$

XIX. $\frac{d}{dx}(\cosh x) = \sinh x$

XX. Product rule: $\frac{d}{dx}[f(x)\,g(x)] = f(x)\,g'(x) + g(x)\,f'(x)$

XXI. Quotient rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{(g(x))^2}$

XXII. Chain rule: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Ques. $\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$

METHOD-1: $y = mx$; m - any constant

$x \to 0$

$y \to 0$

then, $(mx) \to 0$

$x \to 0$

2 variable $\to$ 1 variable ! $\lim_{x\to 0} \frac{x^2 - x(mx)}{\sqrt{x} - \sqrt{mx}}$

$$\lim_{x \to 0} \frac{x^2(1-m)}{\sqrt{x}\,(1-\sqrt{m})} \Rightarrow \lim_{x \to 0} \frac{x\sqrt{x}\cdot\cancel{\sqrt{x}}\,(1-m)}{\cancel{\sqrt{x}}\,(1-\sqrt{m})}$$

$$\lim_{x \to 0} \frac{x\sqrt{x}\,(1-m)}{(1-\sqrt{m})} = 0 \text{ } \underline{\text{Ans}}.$$

METHOD-2:

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$$

$$\lim_{(x,y)\to(0,0)} \frac{x\,(x-y)}{\sqrt{x}-\sqrt{y}}$$

$$\lim_{(x,y)\to(0,0)} \frac{x\,\cancel{(\sqrt{x}-\sqrt{y})}\,(\sqrt{x}+\sqrt{y})}{\cancel{(\sqrt{x}-\sqrt{y})}}$$

$$= 0 \text{ } \underline{\text{Ans}}$$

Ques. $\lim_{x \to \infty} \left[\sqrt{x^2+x-1} - x\right]$

$$\lim_{x \to \infty} \frac{(\sqrt{x^2+x-1}-x)\,(\sqrt{x^2+x-1}+x)}{\sqrt{x^2+x-1}+x}$$

$$\lim_{x \to \infty} \frac{x-1}{\sqrt{x^2+x-1}+x}$$

$$\lim_{x \to \infty} \frac{(x-1)/x}{(\sqrt{x^2+x-1}+x)/x}$$

$$\lim_{x \to \infty} \frac{(1-1/x)}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1} = \frac{1-0}{\sqrt{1+0+0}+1} = \frac{1}{2} \text{ Ans.}$$

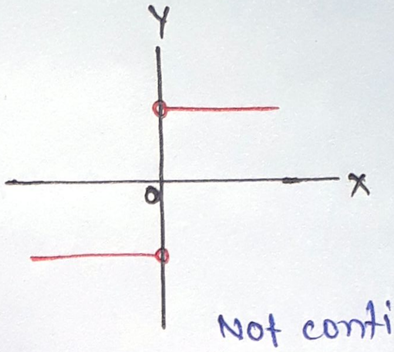
* CONTINUITY *

$[a, b]$: closed interval : a & b are included.

(a, b) : open interval : a & b are not included.

* continuity at a point.

Ex: $f(x) = \frac{|x|}{x}$



Not continuous at $x=0$.

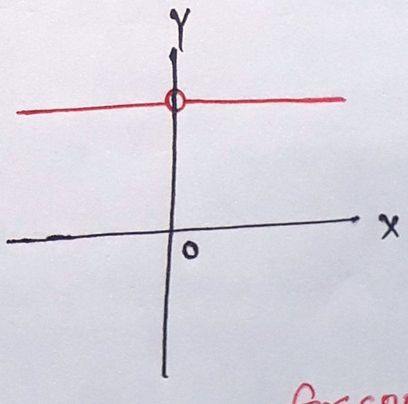
$$LHL = \lim_{x \to 0} \frac{(-x)}{x} = -1$$

$$RHL = \lim_{x \to 0} \frac{+x}{x} = +1$$

$$LHL \neq RHL$$

$\therefore$ Limit does not exist $\to$ Not continuous.

Ques. $f(x) = \frac{x}{x}$



$$LHL = \lim_{x \to 0} \frac{x}{x} = 1$$

$$RHL = \lim_{x \to 0} \frac{x}{x} = 1$$

for continuous

$\boxed{LHL = RHL = f(a)}$

Approach = Exact.

(i) $LHL = RHL$; limit exist

(ii) $LHL = RHL \neq f(0) \to$ Not continuous.

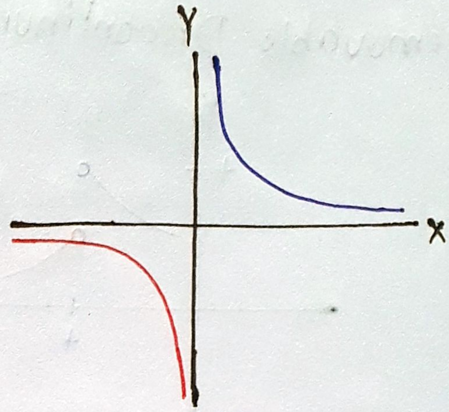
V.Imp. * is '$\infty = \infty$' ?

(i) $\infty \times 1 = \infty$ from (i) & (ii) ; $1 = 2$.

(ii) $\infty \times 2 = \infty$

$\therefore \infty \neq \infty$

Ques. $f(x) = \frac{1}{x}$



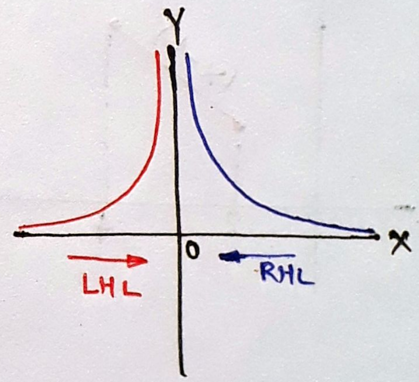
$$LHL = \lim_{x \to 0} \frac{1}{x} = -\infty$$

$$RHL = \lim_{x \to 0} \frac{1}{0} = +\infty$$

$LHL \neq RHL$

$\therefore$ Limit does not exist $\rightarrow$ hence, not continuous.

Ques. $f(x) = \frac{1}{|x|}$



$$LHL = \lim_{x \to 0} \frac{1}{|x|} = \infty$$

$$RHL = \lim_{x \to 0} \frac{1}{|x|} = \infty$$

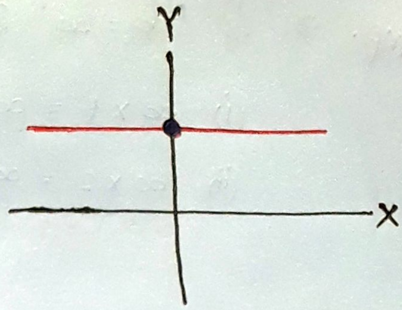
$LHL \neq RHL$; Limit does not exist.

hence, not continuous.

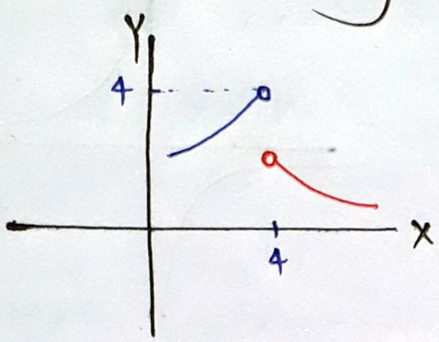
• if you are trying to approach towards zero you get never ending things.

- Removable Discontinuity :

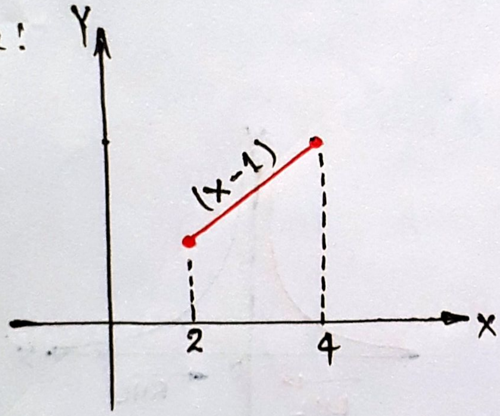
$$f(x) = \frac{x}{x} \longrightarrow f(x) = \begin{cases} \frac{x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$



- Irremovable Discontinuity :



Example :



$f(x) = x-1 \; ; \; 2 \leq x \leq 4$

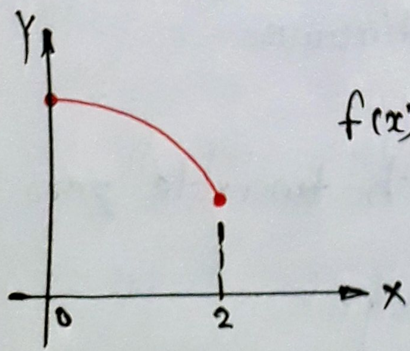
$f(x=4) = 4-1 = 3$

$LHL = \lim_{x\to 4} f(x) = \lim_{x \to 4} (x-1)$

$LHL = 3.$

$\therefore LHL = f(3) \longrightarrow \therefore$ Left continuous

Ex :



$f(x) = -x^2+6$ at $x=0$

$f(0) = 6$

$RHL = \lim_{x \to 0} -x^2+6 = 6$

$RHL = f(0) \longrightarrow$ Right continuous.

Ques. $f(x) = |x|$

$$f(x) = \begin{cases} 0, & x = 0 \\ +x, & x > 0 \\ -x, & x < 0 \end{cases}$$

$$LHL = \lim_{x \to 0} (-x) = 0$$

$$RHL = \lim_{x \to 0} (+x) = 0$$

$$f(0) = 0$$

$\therefore$ $LHL = RHL = f(0)$

Limit exist and continuous.

Ques. $f(x)$ is continuous at $x = \pi/2$

$$f(x) = \begin{cases} \dfrac{\lambda \cos x}{\pi/2 - x}, & x \neq \frac{\pi}{2} \\ 1, & x = \pi/2 \end{cases}$$

find λ.

i. $RHL = LHL = f(a)$

$$LHL = f(\pi/2)$$

$$\lim_{x \to \pi/2} \frac{\lambda \cos x}{\pi/2 - x} = 1$$

$$\lim_{x \to \pi/2} \frac{-\lambda \sin x}{-1} = 1$$

$$\lambda \sin \frac{\pi}{2} = 1$$

$\therefore$ $\lambda = 1$

Ques. $f(x) = \dfrac{x^2 - 3x + 4}{x^2 + 3x - 4}$ is not continuous at $x = ?$

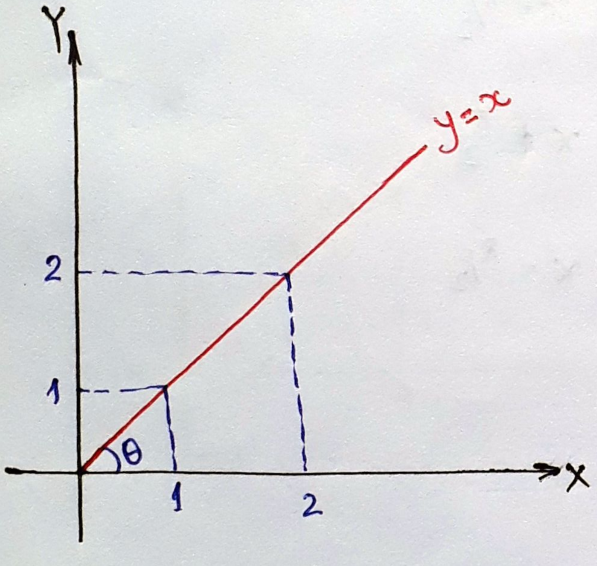
$\dfrac{\text{Numerator(N)}}{\text{Denominator(D)}} \rightarrow$ discontinuous when $D = 0$

$\therefore \quad x^2 + 3x - 4 = 0$

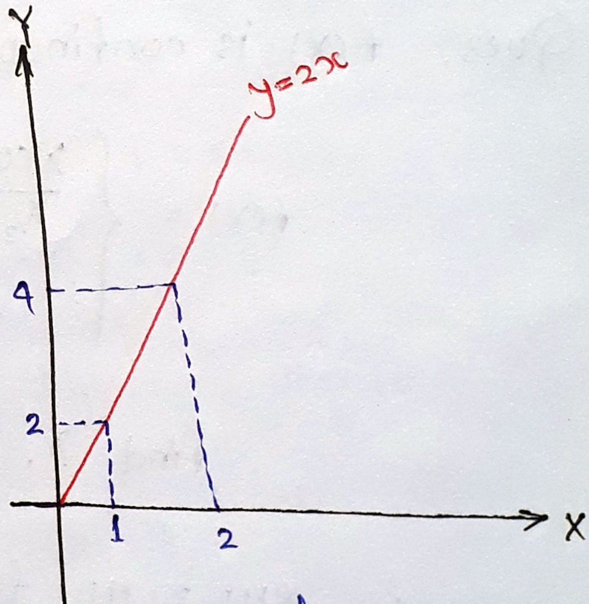
$(x+4)(x-1) = 0$

$x = -4, 1.$

* DERIVATIVE *



slope $= \dfrac{dy}{dx} = 1$

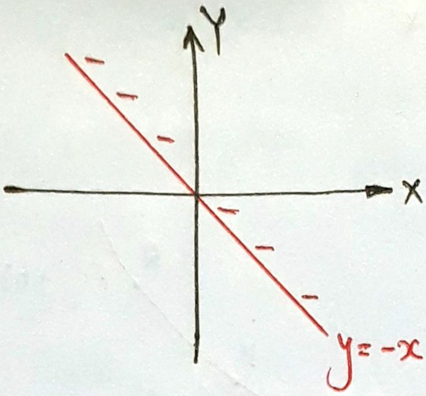


slope $= \dfrac{dy}{dx} = 2$

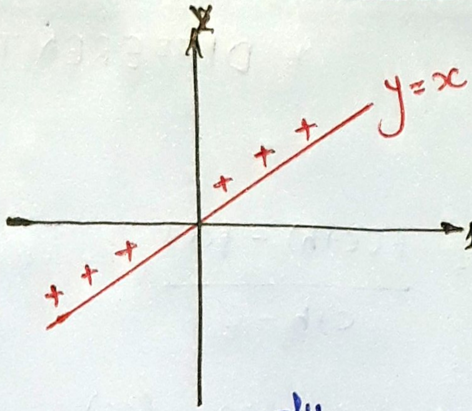
steepness $=$ slope $= \dfrac{dy}{dx} = \dfrac{\text{Rise}}{\text{Run}} = \dfrac{\Delta y}{\Delta x} =$ Average rate of change of y w.r.t x.

$\downarrow$

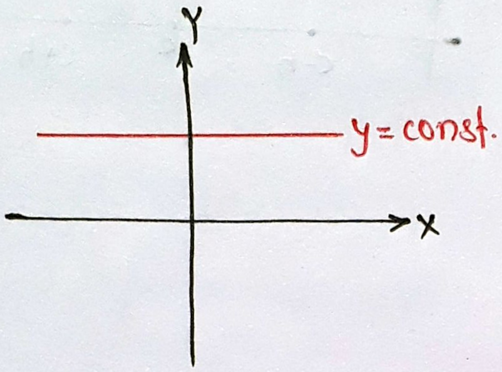
$\tan\theta$



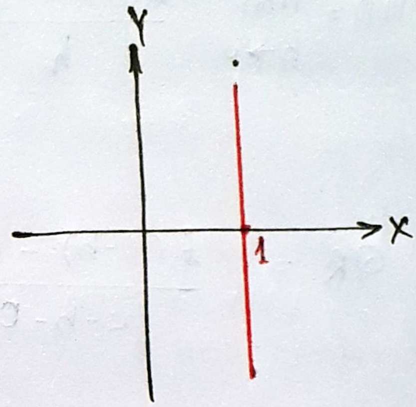
$\frac{dy}{dx} = -1$ (Negative).



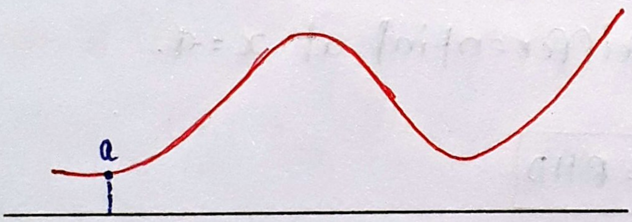
$\frac{dy}{dx} = 1$ (positive)

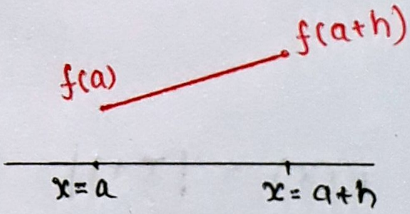


$\frac{dy}{dx} = 0$



$\frac{dy}{dx} = \frac{\Delta y}{0}$ = infinity or Not defined.





$$\text{slope} = \frac{dy}{dx}$$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{a+h-a}$$

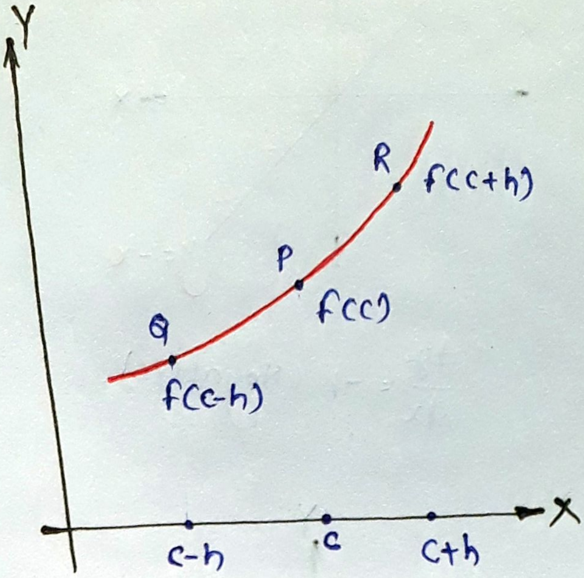
$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

* DIFFERENTIABILITY *

$$PR = \frac{f(c+h) - f(c)}{c+h-c}$$

$$= \frac{f(c+h) - f(c)}{h}$$

$$\therefore RHD = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$



$$QR = \frac{f(c-h) - f(c)}{c-h-c}$$

$$LHD = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h}$$

for, $f(x)$ to be differential at $x = -1$

$$\boxed{LHD = RHD}$$

Ques. $f(x) = |x+1|$ is differentiable at $x = -1$

at $x = -1 \longrightarrow f(x) = 0$

at $x < -1 \longrightarrow f(x) = -(x+1)$

at $x > -1 \longrightarrow f(x) = (x+1)$

$$f(x) = |x+1| = \begin{cases} -(x+1), & x < -1 \\ 0, & x = -1 \\ (x+1), & x > -1 \end{cases}$$

$$LHL = \lim_{x \to -1} -(x+1) = 0$$

$$RHL = \lim_{x \to -1} (x+1) = 0$$

$$f(-1) = 0$$

$\therefore$ LHL = RHL = $f(x)$

Limit exist and continuous.

for $f(x)$ to be differential at $c = -1$

Method 1:

$$LHD = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h}$$ $\because f(x) = -(x+1),\ x < -1$

$$= \lim_{h \to 0} \frac{f(-1-h) - f(-1)}{-h}$$

$$= \lim_{h \to 0} \frac{-[(-1-h)+1] - (0)}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h}$$

$$LHD = -1$$

$$RHD = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

$\because f(x) = x+1 \;, x > -1$

$$= \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$= \lim_{h \to 0} \frac{[(-1+h)+1] - (0)}{h}$$

$$= \lim_{h \to 0} \frac{h}{h}$$

$\therefore$ RHD $= 1$

$\because$ LHD $\neq$ RHD $\therefore$ not differentiable at $x = -1$.

Method-2 :

$$LHD = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \quad \& \quad RHD = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

$$LHD = \lim_{x \to 1^-} \frac{f(x) - f(-1)}{x - (-1)}$$

$$= \lim_{x \to 1^-} \frac{-(x+1) - (0)}{x+1}$$

$$= \lim_{x \to 1^-} \frac{-(x+1)}{x+1}$$

LHD $= -1$

$$RHD = \lim_{x \to 1^+} \frac{f(x) - f(c)}{x - c}$$

$$= \lim_{x \to 1^+} \frac{[x+1] - [0]}{x - (-1)}$$

$$= \lim_{x \to 1^+} \frac{x+1}{x+1} \qquad RHD = 1$$

$\because$ LHD $\neq$ RHD $\therefore$ Not differentiable at $x = -1$.

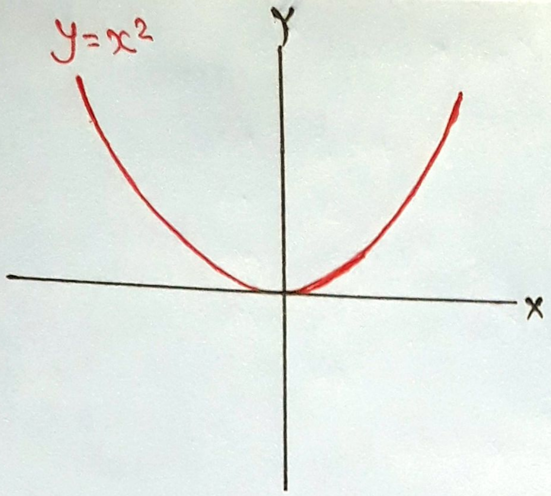
Method - 3.

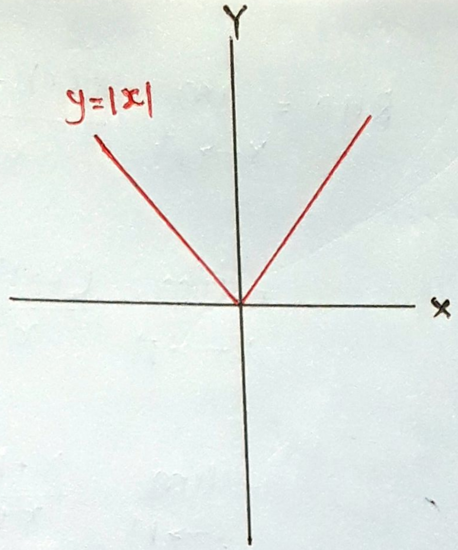
at $x = -1$, $f(x) = |x+1| = \begin{cases} -(x+1) , & x < -1 \\ (x+1) , & x > -1 \end{cases}$

at $x = -1$, $f'(x) = \begin{cases} -1 , & x < -1 \rightarrow LHD = f'(-1^-) = -1 \\ +1 , & x > -1 \rightarrow RHD = f'(-1^+) = 1 \end{cases}$

$\therefore$ LHD $\neq$ RHD

NOT Differentiable at $x = -1$.





- CONTINUOUS
- DIFFERENTIABLE

(smooth curve/edge)

- CONTINIOUS
- NOT DIFFERENTIABLE.

(At shape edge)

$$* \quad y=|x|=\begin{cases} -x, & x<0 \\ +x, & x>0 \end{cases}$$

$$y' = \begin{cases} -1, & x<0 \longrightarrow \text{LHD} = f'(0^-) = -1 \\ +1, & x>0 \longrightarrow \text{RHD} = f'(0^+) = 1 \end{cases}$$

$\because$ LHD $\neq$ RHD

NOT DIFFERENTIABLE.

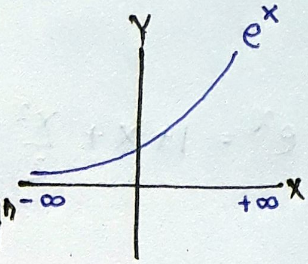
Important point :

- if $f(x)$ is continuous $\longrightarrow$ may / may not be differentiable.
- if $f(x)$ is differentiable $\longrightarrow$ always continuous.

Taylor series: Used to convert 'NON-POLYNOMIAL FUNCTION' into 'POLYNOMIAL FUNCTION' (only in terms of x)

LHS $\cong$ RHS ⇝ Approximation.

Example: $f(x) = e^x$



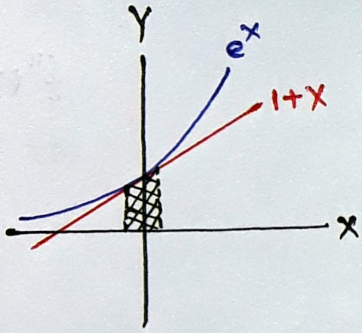
$e^x \cong c_0 + c_1 x$ — linear appoximatⁿ

i. put $x=0$, $e^0 = c_0 + c_1(0)$

$c_0 = 1$

ii. $\frac{d}{dx}$, $e^x = 0 + c_1$

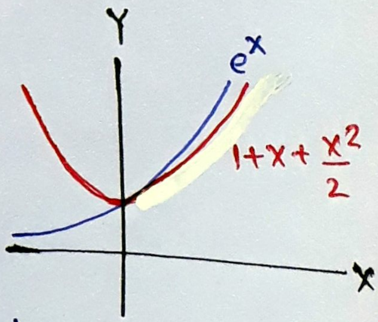
put $x=0$, $e^0 = c_1$ $\therefore c_1 = 1$



linear approximation : $e^x = 1 + x$

$e^x = c_0 + c_1 x + c_2 x^2$

$\because c_0 = 1$, $c_1 = 1$

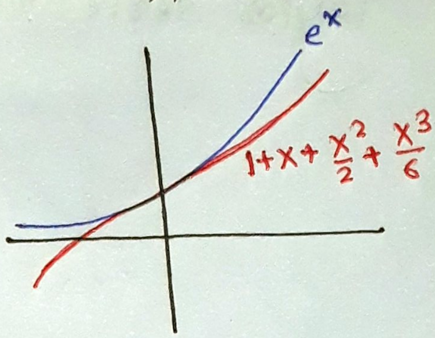


(iii) $\frac{d^2}{dx^2}$: $e^x = 2 \cdot c_2$

put $x=0$, $e^0 = 2 \cdot c_2$ $\therefore c_2 = \frac{1}{2}$

$e^x = 1 + x + \frac{x^2}{2}$ ⇝ Quadratic Approximation.

$e^x = c_0 + c_1 x + c_2 x^2 + c_3 x^3$ — cubic approximation



(i) $c_0 = 1$

(ii) $c_1 = 1$

(iii) $c_2 = 1/2$

(iv) $\frac{d^3}{dx^3}$: $e^x = 6c_3$

put $x=0$, $e^0 = 6c_3$ $\therefore c_3 = 1/6$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$ ⟶ cubic approximation.

$\therefore$ In general :

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

$$f'(x) = c_1 + c_2 x + c_3 x^2 + \cdots + n c_n x^{n-1}$$

$$f''(x) = 2c_2 + 3 \cdot 2 c_3 x + \cdots + n(n-1) c_n x^{n-2}$$

$$\vdots$$

$$f^n(x) = n! c_n$$

for maclaurin series : put $x=0$

$\therefore c_0 = f(0)$, $c_1 = f'(0)$, $c_2 = \frac{f''(0)}{2!}$

$$c_n = \frac{f^n(0)}{n!}$$

$\therefore f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$

$$\boxed{f(x) = f(0) + f'(0) \cdot x + \frac{f''(0) \cdot x^2}{2!} + \cdots + \frac{f^n(0)}{n!} x^n}$$

for Taylor series : at $x = a$ $\forall a \neq 0$

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^n(a)$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a) \cdot (x-a)^n}{n!}$$

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

- $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$

Ques. The series $f(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$ is the expansion of

(a) $\cosh x$ (b) $\sin x$ (c) $\sinh x$ (d) $\cos x$.

By default : $x = 0$; $f(0) = 0 + \frac{(0)^3}{3!} + \frac{(0)^5}{5!} + \cdots$

$$f(0) = 0$$

let $f(x) = \sin x$

Maclaurine series : $f(x) = f(0) + \frac{f'(0)}{1!} + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!} + \cdots$

$\therefore f'(x) = \cos x$ $\qquad f(x) = 0 + \frac{1x}{1!} + 0 + \frac{(-1)}{3!} x^3 + \cdots$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

- $\sin x = +\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

- $\sinh x = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

$f(x) = \sin x$ then, $f(-x) = \sin(-x) = -\sin x$

$$\boxed{f(x) = -f(-x)}$$

$\therefore$ odd function.

- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

- $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$

$f(x) = \cos x$ then, $f(-x) = \cos(-x) = \cos x$

$\boxed{f(x) = f(-x)}$ even function.

Ques. for $|x| << 1$, $\coth(x)$ can be approximated as

(a) x (b) x^2 (c) $\frac{1}{x}$ (d) $\frac{1}{x^2}$

(i) Maclaurin / Taylor

(ii) $x = 0$

(iii) $\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{x^0 + \boxed{\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots} \to 0}{\frac{x^1}{1!} + \boxed{\frac{x^3}{3!} + \frac{x^5}{5!} + \cdots} \to 0}$ $\because x << 1$, $x^2 <<< 1$, $x^3 <<<< 1$

$\therefore \coth'(x) = \frac{1}{x}$

Complex variable :

$$e^{ix} = \cos x + i\sin x \quad \text{--------} ①$$

$$e^{-ix} = \cos x - i\sin x \quad \text{--------} ②$$

from eq. ① & ②

- $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ $\rightsquigarrow$ $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$

- $\sin x = \dfrac{e^{ix} - e^{-ix}}{2}$ $\rightsquigarrow$ $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$

Ques. The series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges to

(a) $2 \ln 2$ (b) $\sqrt{2}$ (c) 2 (d) e ✓

Sol. $e^x = \dfrac{x^0}{0!} + \dfrac{x^1}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$

put $x = 1$; $e^1 = \dfrac{1}{0!} + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots$

$\therefore$ Ans $= e$

Ques. The third term in Taylor series expansion of e^x about $x = a$ would be

(a) $e^a(x-a)$ (b) ✓ $\dfrac{e^a}{2}(x-a)^2$ (c) $\dfrac{x^2}{2}$ (d) $\dfrac{e^a}{6}(x-a)^3$

M-1

third term $= \dfrac{f''(a) \cdot (x-a)^2}{2!}$

$= \dfrac{e^a}{2}(x-a)^2$

M-2

Convert Taylor series $f(x), x=a$ ⟶ Maclaurine series

$x-a=0$

put, $x-a=t$

$t=0$

$f(t+a), t=0$

$\therefore f(x)=e^x, x=a$ ⟶ Taylor series

(i) $x=a$

$(x-a)=0$

let, $(x-a)=t$, $t=0$

$\therefore x=(t+a)$

(ii) $f(t+a)=e^{t+a}$, $t=0$

$$f(t+a)=e^t.e^a$$

$$=e^a\left[\frac{t^0}{0!}+\frac{t^1}{1!}+\frac{t^2}{2!}+\cdots\right]$$

$\therefore$ third term $=e^a \times \frac{t^2}{2!} = e^a \times \frac{(x-a)^2}{2!}$

M-3: Maclaurine series:

$x=a$ ⟶ $(x-a)=0$

$$f(x)=e^{x-a+a}=e^{(x-a)+a}=e^{(x-a)}.e^a$$

$$\therefore f(x)=e^a\left[\frac{(x-a)^0}{0!}+\frac{(x-a)^1}{1!}+\frac{(x-a)^2}{2!}+\cdots\right]$$

Hence, third term $=\frac{e^a.(x-a)^2}{2!}$

Ques. for the function e^{-x}, the linear approximation around $x=2$ is

(a) $(3-x)e^{-2}$ (b) $1-x$ (c) $[3+2\sqrt{2}-(1+\sqrt{2})x]e^{-2}$ (d) e^{-2}

M.1 Taylor series

$$f(x) = f(a) + f'(a)\frac{(x-a)^1}{1!}$$

$\because f(x) = e^{-x}$, $x = a = 2$

$$\therefore e^{-x} = e^{-2} + (-e^{-2})(x-2)$$

$$= e^{-2}(1-x+2) = (3-x)e^{-2}$$

M.2. Maclaurine series.

$$f(x) = f(0) + \frac{x^1}{1!}f'(0)$$

$f(x) = e^{-x}$ at $x=2$

$$f(x) = e^{-x+2-2} \quad , (x-2)=0$$

$$= e^{-(x-2)} \cdot e^{-2}$$

$$= e^{-2}\left(1 - \frac{(x-2)^1}{1!} + \cdots\right)$$

$$= e^{-2}(1-x+2)$$

$$= e^{-2}(3-x)$$

Ques. The Taylor series expansion of $\frac{\sin x}{x-\pi}$ at $x=\pi$ is given by

(a) $1+\frac{(x-\pi)^2}{3!}+\cdots$ (b) $-1-\frac{(x-\pi)^2}{8!}+\cdots$

(c) $1-\frac{(x-\pi)^2}{3!}+\cdots$ (d) $-1+\frac{(x-\pi)^2}{3!}+\cdots$

i. Maclaurine series:

$$\frac{\sin(x+\pi-\pi)}{x-\pi}, \quad x-\pi=0$$

$$= \frac{1}{x-\pi}\sin((x-\pi)+\pi)$$

$$= \frac{1}{x-\pi}\left[-\sin(x-\pi)\right]$$

$$= \frac{-1}{x-\pi}\left[+\frac{(x-\pi)^1}{1!}-\frac{(x-\pi)^3}{3!}+\cdots\right]$$

$$= -1+\frac{(x-\pi)^2}{3!}+\cdots$$

Ques: The Quadratic approximation of $f(x)=x^3-3x^2-5$ at the point $x=0$ is.

(a) $3x^2-6x-5$ (b) $-3x^2-5$ (c) $-3x^2+6x+5$ (d) $3x^2-5$

$$f(x)=f(0)+f'(0)\cdot x^1+f''(0)\cdot\frac{x^2}{2!}+\cdots$$

$\{ f(0)=-5$

$f'(0)=3x^2-6x=0$

$f''(0)=6x-6=-6$

$$f(x)=-5+0\cdot x^1+(-6)\cdot\frac{x^2}{2}$$

$$=-3x^2-5$$

Binomial expansion:

$$(1+x)^n = 1 + nx^1 + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots\cdots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots\cdots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + \cdots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \cdots$$

Ques. The coefficient of x^{12} in $(x^3 + x^4 + x^5 + x^6 \cdots)^3$ is.

$$= (x^3 + x^4 + x^5 + x^6 + \cdots)^3$$

$$= x^9 (1 + x + x^2 + x^3 + \cdots)^3$$

$$= x^9 . ((1-x)^{-1})^3$$

$$= x^9 . (1-x)^{-3} = x^9 \left[1 + (-3)(-x) + \frac{(-3)(-4)}{2!}(-x)^2 + \frac{(-3)(-4)(-5)}{3!}(-x)^5 + \cdots\right]$$

coefficient of $x^{12} = \frac{3 \times 4 \times 5}{6} = 10$ Ans

* SUM OF INFINITE SERIES *

Ques. The sum of infinite series, $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ is

(a) π (b) infinity (divergent) (c) 4 (d) $\frac{\pi^2}{4}$

- $\log(1-x) = -\left[x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right]$

put $x=1$; $\log(1-1) = -\left[1+\frac{1}{2}+\frac{1}{3}+\cdots\right]$

$-\log(0) \Rightarrow \log(0)^{-1} = \left[1+\frac{1}{2}+\frac{1}{3}+\cdots\right]$

$\log\left(\frac{1}{0}\right)$ $\therefore$ Ans $= \log(\infty) = \infty$

Ques. for $x=\frac{\pi}{6}$, the sum of the series

$\sum_1^\infty (\cos x)^{2n} = \cos^2 x + \cos^4 x + \cos^6 x + \cdots$ is

(a) π (b) 3 (d) 1 (c) ∞

- $\cos^2 x + \cos^4 x + \cos^6 x + \cdots$ (GP series)
- common ratio, $r = \frac{\cos^4 x}{\cos^2 x} = \cos^2 x$
- sum of infinite series, $S_\infty = \frac{a}{1-r}$ a — first term.

$= \frac{\cos^2 x}{1-\cos^2 x} = \frac{\cos^2 x}{\sin^2 x}$ at $\frac{\pi}{6}$, $\left[\frac{\cos 30}{\sin 30}\right]^2 = 3$ Ans

Ques. The summation of series $S = 2 + \frac{5}{2} + \frac{8}{2^2} + \frac{11}{2^3} + \cdots + \infty$, is,

(a) 4.5 (b) 6.0 (c) 6.75 (d) 10.0

$$S = 2 + \frac{5}{2} + \frac{8}{2^2} + \frac{11}{2^3} + \cdots \quad \left(\frac{AP}{GP} \rightarrow \text{AGP series}\right)$$

divided by 2 in above eqn.

$$\frac{S}{2} = \frac{2}{2} + \frac{5}{2^2} + \frac{8}{2^3} + \frac{11}{2^4} + \cdots$$

on substracting;

$$S - \frac{S}{2} = 2 + \frac{5-2}{2} + \frac{8-5}{2^2} + \frac{11-8}{2^3} + \cdots$$

$$\frac{S}{2} = 2 + \frac{3}{2}\underbrace{\left[1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right]}_{GP}$$

$$\frac{S}{2} = 2 + \frac{3}{2}\left[\frac{1}{1-\frac{1}{2}}\right]$$

$$\frac{S}{2} = 2 + \frac{3}{2} \times 2 \rightarrow S = 10 \ \underline{\text{Ans}}$$

Ques. The value of $\sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^n$ is.....

- $1 \cdot \frac{1}{2} + 2 \cdot \left(\frac{1}{2}\right)^2 + 3 \cdot \left(\frac{1}{3}\right)^2 + 4 \cdot \left(\frac{1}{4}\right)^2 + \cdots$ (AGP series)

$$(1-x)^{-2} = 1 \cdot x^0 + 2x^1 + 3x^2 + 4x^3 + \cdots$$

$$\frac{1}{2}\left\{1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots\right\}$$

so, $x = \frac{1}{2} \rightarrow \frac{1}{2}\left\{\left(1 - \frac{1}{2}\right)^{-2}\right\} = 2$ $\underline{\text{Ans}}$

* ROLLE'S THEOREM *

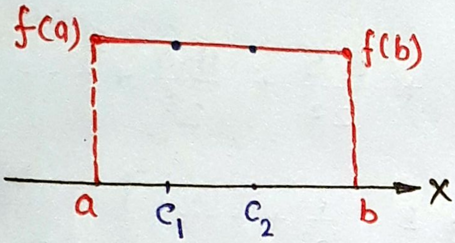
function 'f' ⟶

(i) f is continuous on $[a, b]$

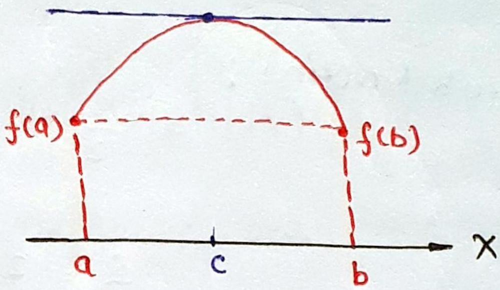
(ii) f is differentiable on (a, b)

(iii) $f(a) = f(b)$

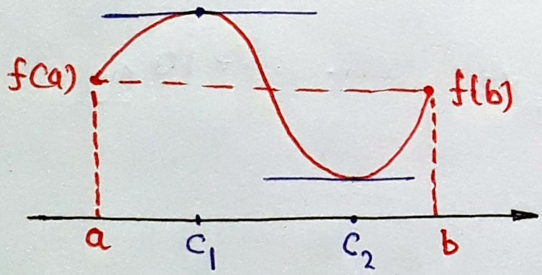
then, atleast a point 'C' in (a, b) such that $f'(c) = 0$

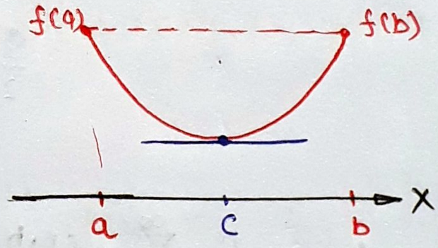


$f'(c) = 0$



$f'(c) = 0$





$f'(c) = 0$

Ques! $f(x) = (x-3)(x+1)^2$ on $[-1, 3]$

(i) $f(x)$ is continuous $[-1, 3]$

(ii) $f(x)$ is differentiable $(-1, 3)$ } Rolle's theorem.

(iii) $f(-1) = 0$ & $f(3) = 0$

$\therefore f'(x)\Big|_c = 0$

$(x-3) \times 2(x+1) + (x+1)^2 \cdot 1 \Big|_c = 0$

$2(c-3)(c+1) + (c+1)^2 = 0$

$(c+1)(3c-5) = 0$

$c = -1$ or $c = 5/3$

$\because c \in (a,b) \longrightarrow c \in (-1,3) \quad \therefore c = 5/3$ Ans//

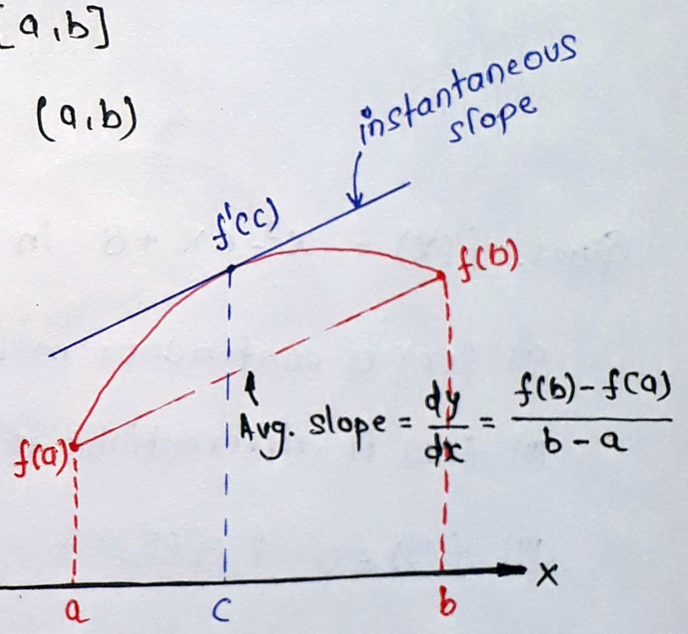
- if function, f is

(i) f is continuous $[a,b]$

(ii) f is differentiable (a,b)

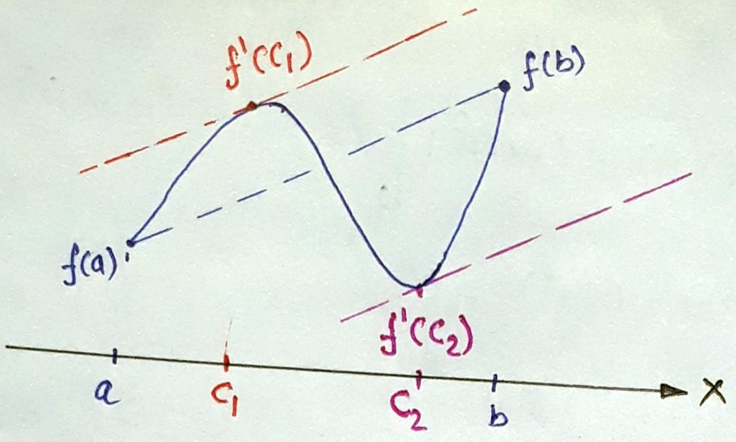
(iii) $f(a) \neq f(b)$

then, instantaneous slope is equal to Average slope.



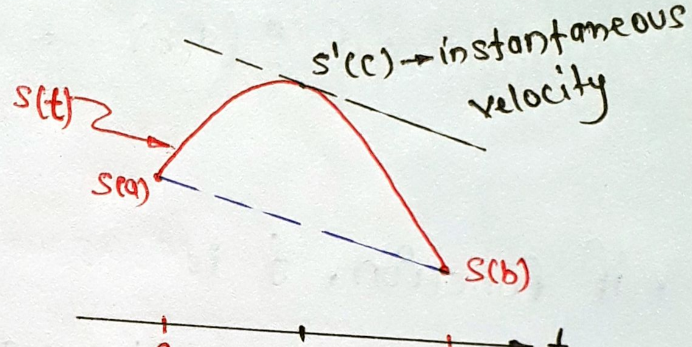
* LAGRANGE MEAN VALUE THEOREM.

$$f'(c) = \frac{f(a) - f(b)}{a - b} \quad \forall\, c \in (a,b)$$



- velocity, $v = \frac{ds}{dt}$

Avg. velocity $= \frac{s(b) - s(a)}{b - a}$

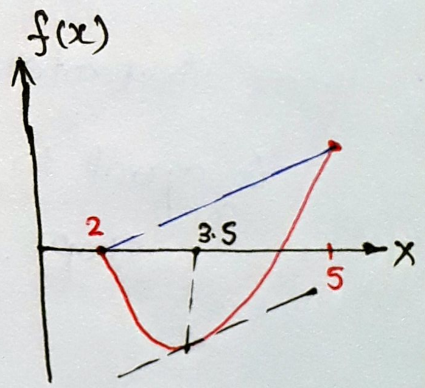


Ques. $f(x) = x^2 - 6x + 8$ in $[2, 5]$

(i) $f(x)$ is continuous in $[2,5]$

(ii) $f(x)$ is differentiable in $(2,5)$

(iii) $f(2) = 0$ & $f(5) = 3$ ∴ $f(2) \neq f(5)$



∴ LMVT ; $f'(c) = \frac{f(5) - f(2)}{5 - 2}$

$$(2x-6)\Big|_{x=c} = \frac{3-0}{5-2}$$

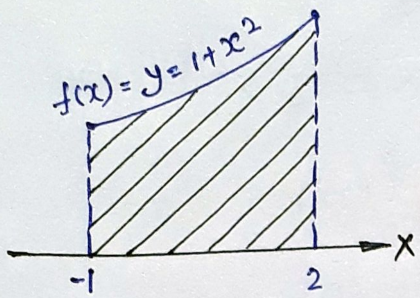
$$2c - 6 = \frac{3}{3}$$

$$c = 3.5 \text{ Ans.}$$

Ques. According to the mean value theorem, for a continuous function $f(x)$ in the interval $[a,b]$, there exists a value ξ in the in the interval such that $\int_a^b f(x)\,dx =$

(a) $f(\xi)(b-a)$ (b) $f(b)(\xi-a)$ (c) $f(a)(b-\xi)$ (d) 0

METHOD-1: Mean value theorem for integrals.



$$\int_{-1}^{2} y\,dx = \int_a^b f(x)\,dx$$

$$\int_{-1}^{2} (1+x^2)\,dx$$

$$\left(x + \frac{x^3}{3}\right)_{-1}^{2} = \frac{18}{3} = 6$$

LHS : $\int_a^b f(x)\,dx$

RHS : $f(\xi)(b-a)$ (consider option A)

LHS = RHS

$6 = f(\xi)(2-(-1))$

$f(\xi) = 2$

$\because f(x) = 1+x^2$

$f(\xi) = 1+\xi^2$

$2 = 1+\xi^2$

$\therefore \xi = \pm 1$

Ans $= \xi = 1$

METHOD : 2 Lagrange mean value theorem.

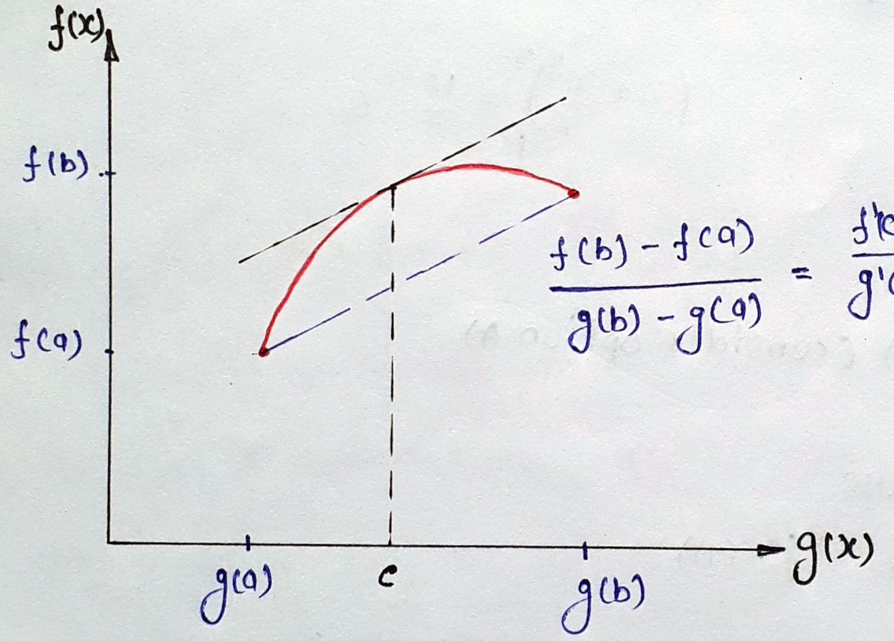
$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

on integrating :

$$f(\xi) = \frac{\int_a^b f(x)\,dx}{b-a}.$$

$$\therefore \int_a^b f(x)\,dx = f(\xi)(b-a)$$

* CAUCHY MEAN VALUE THEOREM



$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Ques: if $f(0) = 2.0$ and $f'(x) = \dfrac{1}{5 - x^2}$, then the lower and upper bounds of $f(1)$ estimated by the mean value theorem are

(a) 1.9, 2.2 ✓(b) 2.2, 2.25 (c) 2.25, 2.5 (d) none of above.

i. interval $\rightarrow [0, 1]$

ii. LMVT : $f'(x) = \dfrac{f(b) - f(a)}{b - a}$

$$\frac{1}{5-x^2} = \frac{f(1) - f(0)}{1 - 0}$$

$$\frac{1}{5-x^2} = f(1) - 2 \longrightarrow f'(c) = f(1) - 2$$

$$f'(1) = f(1) - 2$$

$$f'(0) < f'(c) < f'(1)$$

$$\frac{1}{5} < f'(c) < \frac{1}{4}$$

$$0.2 < f'(c) < 0.25$$

$$0.2 < f(1) - 2 < 0.25$$

$$2.2 < f(1) < 2.25.$$

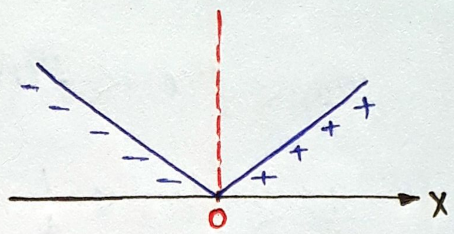
* DERIVATIVE *

- $f(x) = |x|$

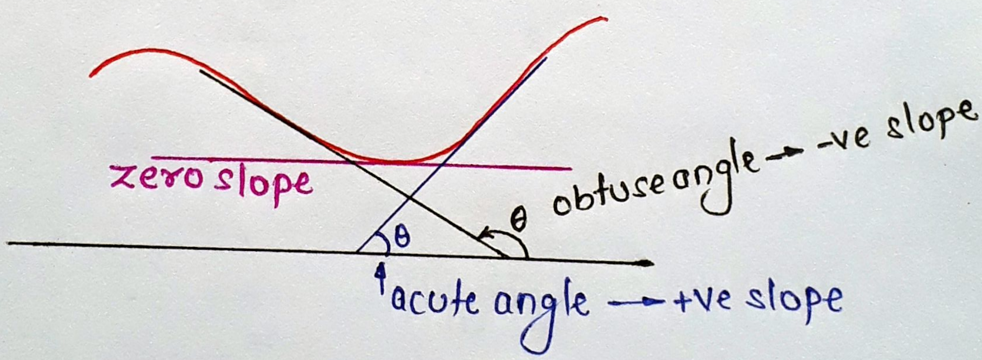
$$\frac{d}{dx} f(x) = \frac{d}{dx}|x| \longrightarrow |x| \text{ is not differentiable at } x=0$$

$$|x| = \frac{x}{|x|} \quad ; \quad x \neq 0$$

$$f(x) = |x| = \begin{cases} +x, & x > 0 \\ -x, & x < 0 \end{cases}$$



$$f'(x) = \frac{d|x|}{dx} = \begin{cases} +1, & x > 0 \\ -1, & x < 0 \end{cases}$$



Ques. $y = \sin^{-1}\left[\frac{2x}{1+x^2}\right]$ find: dy/dx

Method-1.

$y \to \theta \to x$

$$\theta = \frac{2x}{1+x^2} \longrightarrow y = \sin^{-1}\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$\frac{dy}{d\theta} = \frac{1}{\sqrt{1-\theta^2}} \quad \& \quad \frac{d\theta}{dx} = \frac{(1+x^2)\cdot 2 - (2x)(2x)}{(1+x^2)^2}$$

$$\frac{d\theta}{dx} = \frac{2(1+x^2) - 4x^2}{(1+x^2)^2}$$

$$\frac{dy}{dx} = \frac{(1+x^2)}{\sqrt{(1+x^2)^2 - 4x^2}} \times \frac{2(1+x^2) - 4x^2}{(1+x^2)^2}$$

$$\frac{dy}{dx} = \frac{2(1-x^2)}{(1-x^2)(1+x^2)}$$

$$\therefore \frac{dy}{dx} = \frac{2}{1+x^2} \quad \underline{\text{Ans}}.$$

Method-2 :

$$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

put : $x = \tan\theta$, $y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$$y = \sin^{-1}(2\sin\theta \cdot \cos\theta)$$

$$y = \sin^{-1}(\sin 2\theta)$$

$$y = 2\theta = 2\tan^{-1}x$$

$$\frac{dy}{dx} = 2/_{1+x^2} \quad \text{Ans}/$$

- Explicit function: $y = f(x)$

Example: $y = e^x + 1$

- Implicit function:

$f(x, y) = 0$

Example: $x^2 + y^2 - 1 = 0$

Ques: $x^2 + y^2 - 1 = 0$ find: dy/dx

$$\frac{d}{dx}(x^2) + \frac{d}{dy}(y^2) \cdot \frac{dy}{dx} - \frac{d}{dx}(1) = 0$$

$$2x + 2y \frac{dy}{dx} - 0 = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Ques. $y = x^x$ find $\frac{dy}{dx}$.

$$\log y = x \log x$$

$$\ln y = x \ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x)$$

$$\frac{1}{y} \frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x \cdot 1$$

$$\frac{dy}{dx} = y(1 + \ln x)$$

$$\frac{dy}{dx} = x^x(1 + \ln x)$$

Ques. $x^y = e^{x-y}$

$\ln x^y = \ln e^{x-y}$

$y \ln x = (x-y) \ln e$

$y \ln x + y = x$

$y(\ln x + 1) = x \Rightarrow y = \frac{x}{\ln x + 1}$

$$\frac{dy}{dx} = \frac{(1+\ln x) \cdot 1 - x\left(\frac{1}{x}\right)}{(1+\ln x)^2} = \frac{\ln x}{(1+\ln x)^2} \text{ Ans}$$

Ques. $x = a(t + \sin t)$

$y = a(1 - \cos t)$ find, $\frac{dy}{dx}$ at $x = \frac{\pi}{2}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{0 + a\sin t}{a + a\cos t}$$

$$\frac{dy}{dx} = \frac{\sin t}{1 + \cos t} \qquad \left.\frac{dy}{dx}\right|_{\text{at } t = \frac{\pi}{2}} = 1$$

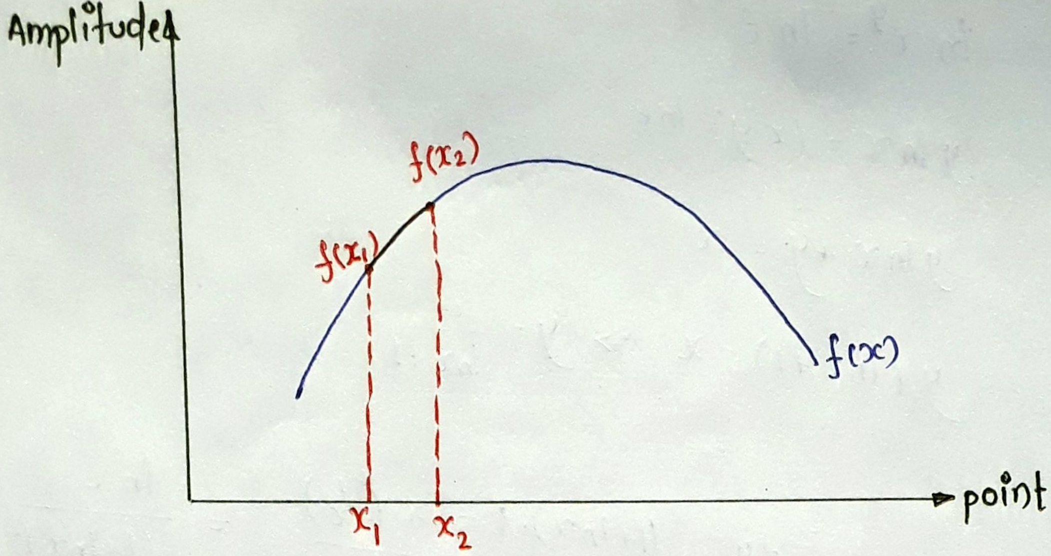
Ques. $y = \frac{x^3}{1-x^3}$, Differentiate y w.r.t x^3. $\left(\frac{dy}{dx^3}\right)$

let, $x^3 = t$ $\qquad \frac{dy}{dx^3} = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

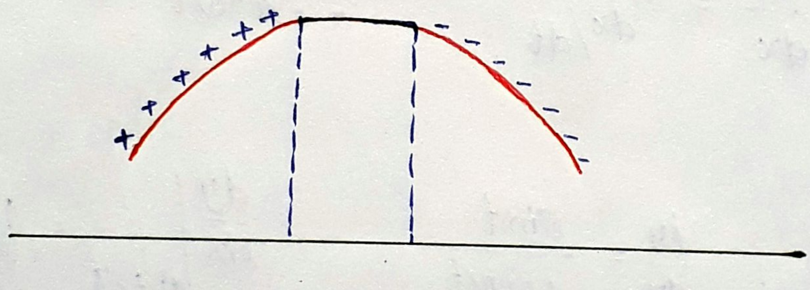
$$= \frac{\frac{(1-x^3)(3x^2) - x^3(-3x^2)}{(1-x^3)^2}}{3x^2}$$

$$\therefore \frac{dy}{dx^3} = \frac{1}{(1-x^3)^2}$$

INCREASING or DECREASING FUNCTION



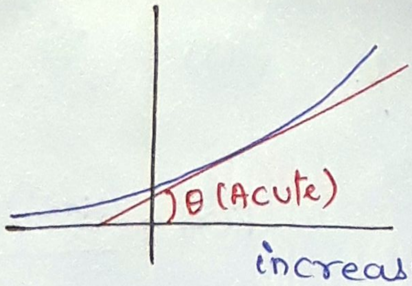
- strictly / Monotonicaly increasing : $f(x_2) > f(x_1)$, $x_2 > x_1$
- strictly / Monotonically decreasing : $f(x_2) < f(x_1)$, $x_2 > x_1$



- $f'(x) > 0$ $\longrightarrow$ for each x $\longrightarrow$ strictly increasing
- $f'(x) = 0$ $\longrightarrow$ for each x $\longrightarrow$ constant
- $f'(x) < 0$ $\longrightarrow$ for each x $\longrightarrow$ strictly decreasing.

Ques. $f(x) = e^x$

$f'(x) = e^x$
$\llcorner$ +ve → increasing

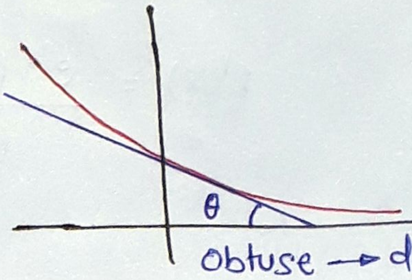


increasing

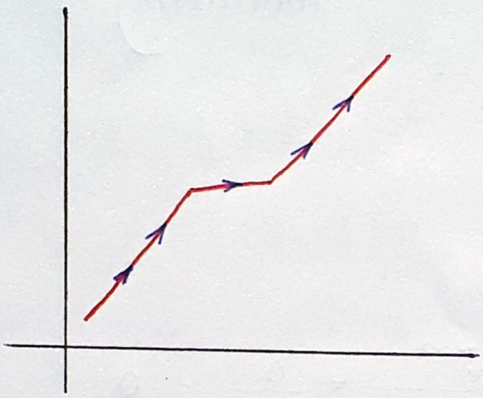
$f(x) = e^{-x}$

$f'(x) = -e^{-x}$

↑
-ve → decreasing



obtuse → decreasing



$f'(x) > 0$
$+$
$f'(x) = 0$ → $f'(x) \geq 0$
increasing function.

$f'(x) < 0$
$+$
$f'(x) = 0$ → $f'(x) \leq 0$
Decreasing funct^n

Ques. As x increased from $-\infty$ to ∞, the function $f(x) = \dfrac{e^x}{1+e^x}$

(a) monotonically increasing ✓

(b) monotonically decreasing

(c) Increases to max. value & then decreases

(d) Decreases to min. value & then increases.

Method-1. $f(x) = \dfrac{e^x}{1+e^x}$

$$f'(x) = \frac{(1+e^x)\,e^x - e^x \cdot e^x}{(1+e^x)^2} \quad ; \; x \in (-\infty, \infty)$$

$$f'(x) = \frac{e^x\,(1+e^x - e^x)}{(1+e^x)^2}$$

$$f'(x) = \frac{e^x}{(1+e^x)^2} > 0$$

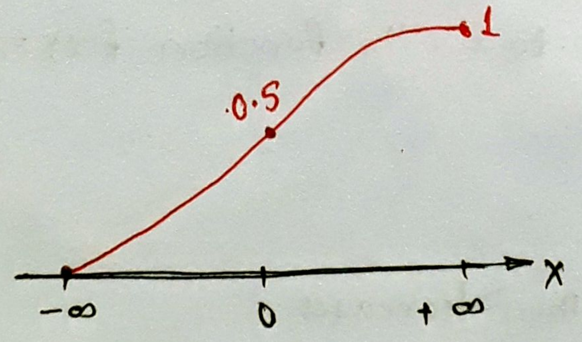
$\therefore$ Monotonically increasing

Method-2: $f(x) = \dfrac{e^x}{1+e^x}$

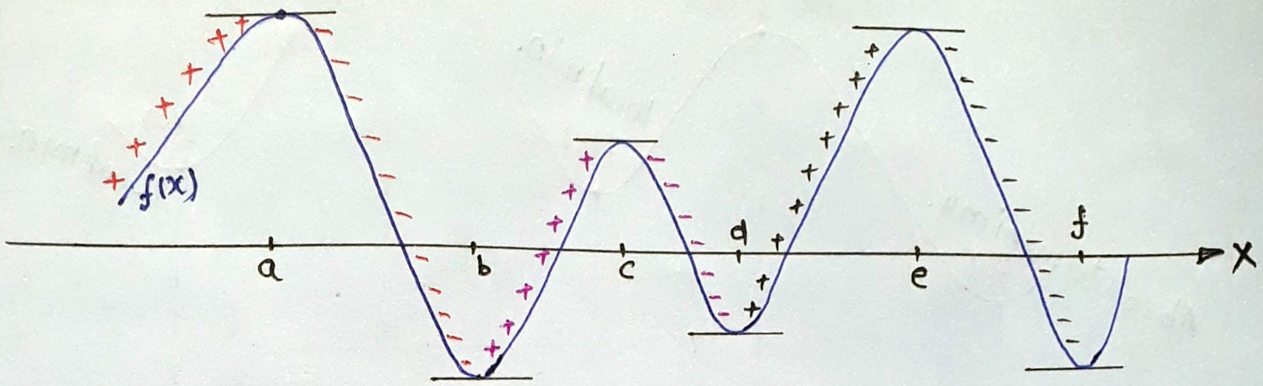
(i) at $x = -\infty$; $f(-\infty) = \dfrac{e^{-\infty}}{1+e^{-\infty}} = \dfrac{0}{1+0} = 0$

(ii) at $x = 0$; $f(0) = \dfrac{e^0}{1+e^0} = \dfrac{1}{2} = 0.5$

(iii) at $x = 0$; $f(\infty) = \dfrac{e^\infty}{1+e^\infty} = \dfrac{1}{\frac{1}{e^\infty}+1} = 1$



MAXIMA & MINIMA.



- if $f(x)$ is in terms of x with power n.

- $f'(x) = 0 \longrightarrow$ critical point. (power $(n-1)$ of x)
  $(n-1)$

i. Turning point or stationary point or critical point.

$f'(x) = 0$ or undefined.

$x = a, b, c, d, e, f$

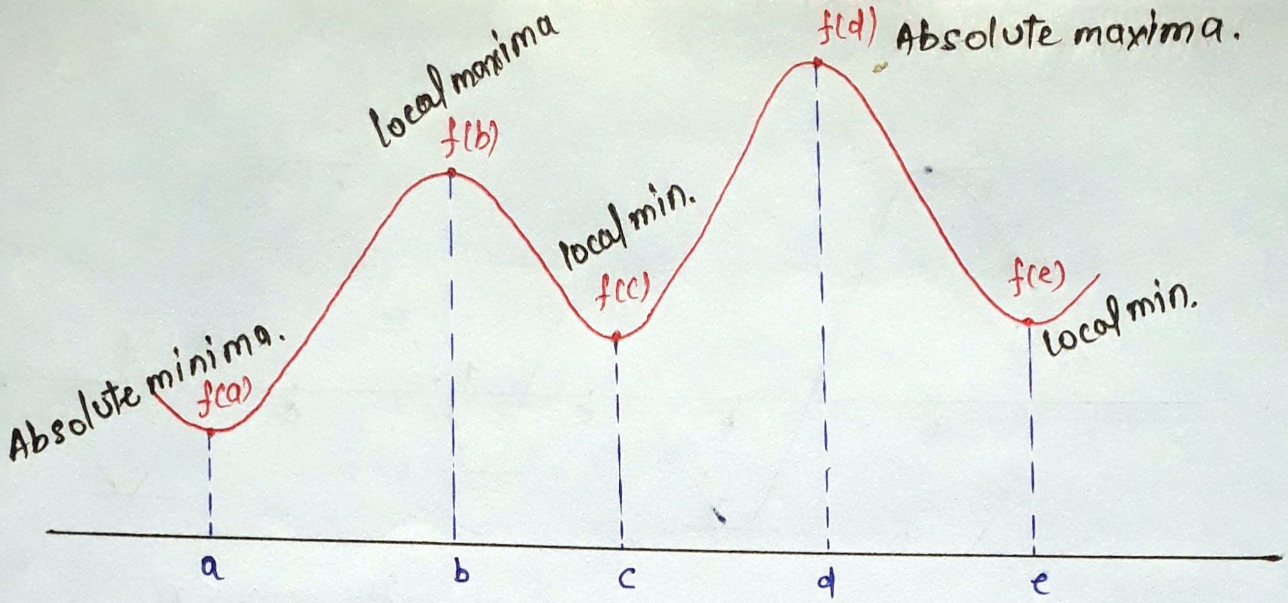
ii. $f''(x)$:

Change

- if $f''(x=a) > 0 \rightarrow$ - to + $\rightarrow$ Minimum

$< 0 \rightarrow$ + to - $\rightarrow$ Maximum.

$= 0 \rightarrow$ neither minima nor maxima



- Min. $\{ f(a), f(c), f(e) \}$
- Max. $\{ f(b), f(d) \}$

- Absolute | Global | ultimate minimum: $f(a)$
- Absolute | Global | ultimate maxima: $f(d)$
- local minima: $f(c), f(e)$
- local maxima: $f(b)$

Ques. $f(x) = x^3 - 3x + 3$ find max/min.

$f'(x) = 3x^2 - 3 = 0$

$x^2 - 1 = 0$

$(x+1)(x-1) = 0$

$x = -1, 1$

stationary point

NOTE: $x^4 - 1 = 0$

$(x^2+1)(x^2-1) = 0$

$(x^2 - i^2)(x^2-1) = 0$

$(x+i)(x-i)(x+1)(x-1) = 0$

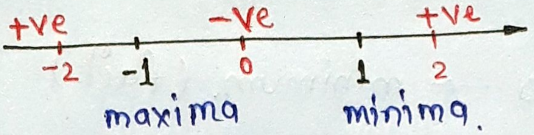
$x = 1, -1, i, -i$

Method 1: first Derivative test.

$$f(x) = x^3 - 3x + 3$$

$$f'(x) = 3x^2 - 3$$

$$f'(-2) = +ve \quad , \quad f'(0) = -ve$$



At $x = -1$; change in slope (+ve to -ve)

$f(x)$ is maxima at $x = -1$

$\therefore$ max. value : $f(-1) = 5$

At $x = +1$: change in slope (-ve to +ve)

$f(x)$. is minima at $x = 1$

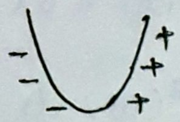
$\therefore$ minimum value : $f(1) = 1$

Method 2: second derivative test.

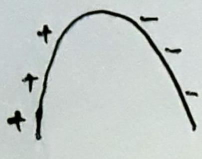
(i) st. points = $-1, 1$

(ii) $f''(x) = 6x$

- $f''(1) = 6 > 0$ (minima)



- $f''(-1) = -6 < 0$ (maxima)



Ques. Consider the function $f(x) = x^2 - x - 2$ The maximum value of $f(x)$ in the closed interval $[-4, 4]$ is

(A) 18 (B) 10 (C) -2.25 (D) Indeterminente

(i) $f'(x) = 2x - 1 = 0 \longrightarrow x = \frac{1}{2}$

$f''(x) = 2 > 0 \rightarrow$ minimum. (fails)

(ii) $x = -4, -3, -2, -1, 0, 1, 2, 3, 4$

$f(x) = 18, 10, 4, 0, -2, -2, 0, 4, 10$

(max)

Ques: At the point $x = 0$, $f(x) = x^3$ has

(a) local maximum

(b) local minimum.

(c) Both local maximum & minimum.

(D) Neither local max. nor local min.

(i) $f(x) = x^3$

$f'(x) = 3x^2 = 0$

$x = 0$ ——— st. point

(ii) $f''(x) = 6x$

$f''(0) = 0$ ——— failure

$f^{even}(x=a) > 0$ (minimum)

< 0 (maximum)

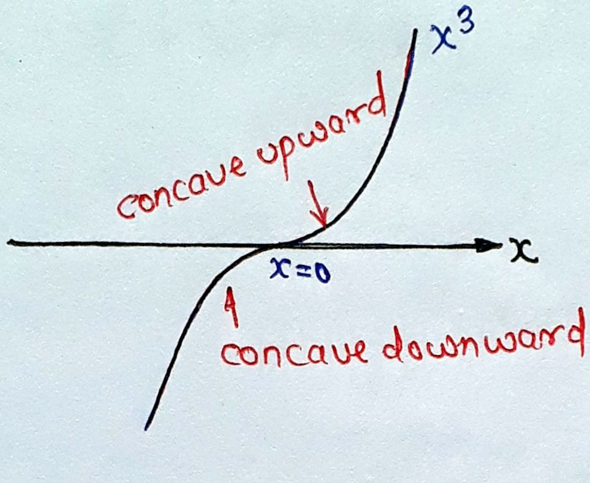
$= 0$ (failure)

$f'''(x) = 6$

$f'''(0) = 6 \neq 0$ —— saddle point / point of inferion.

$f^{odd}(x=a) \neq 0 \longrightarrow$ Point of inflexion

$= 0 \longrightarrow$ failure



PARTIAL DERIVATIVE

- $f_x = \frac{\partial f}{\partial x}$ • $f_y = \frac{\partial f}{\partial y}$

- $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$

- $f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$

$\therefore$ in general : $f_{xy} = f_{yx}$

Ques. $f(x,y) = xy^2 + 5y$ find $\frac{\partial f(x,y)}{\partial x}$ & $\frac{\partial f(x,y)}{\partial y}$

$$\frac{\partial f(x,y)}{\partial x} = y^2 \cdot 1 + 0 = y^2$$

$$\frac{\partial f(x,y)}{\partial y} = x \cdot (2y) + 5 = 2xy + 5$$

Ques. $u = \ln\left(\frac{x^3+y^3}{x^2+y^2}\right)$ find : $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = ?$

$$\frac{\partial u}{\partial x} = \frac{1}{\left(\frac{x^3+y^3}{x^2+y^2}\right)} \cdot \frac{(x^2+y^2)(3x^2) - (x^3+y^3)(2x)}{(x^2+y^2)^2}$$

$$x\frac{\partial u}{\partial x} = x\left(\frac{x^4+3x^2y^2-2xy^3}{(x^3+y^3)(x^2+y^2)}\right)$$

$$y\frac{\partial u}{\partial y} = y\left(\frac{y^4+3x^2y^2-2x^3y}{(x^3+y^3)(x^2+y^2)}\right)$$

$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$$

* Euler's equation / Method :

(i) Homogeneous function

(ii) Degree $= n$

(iii) $f(x,y)$

then, $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

* $a_0y^n + a_1y^{n-1}x^1 + a_2y^{n-2}x^2 + \cdots + a_ny^0x^n = 0$

Degree $= n$

Ques. $f(x,y) = x^2y + xy^2$

$= x^2\left(y + \frac{y^2}{x}\right)$

$= x^3\left(\frac{y}{x} + \left(\frac{y}{x}\right)^2\right)$

$\therefore$ Degree $= 3$

Method 1: $x^{n} f(y/x)$

or

$y^{n} f(x/y)$

$n =$ Degree.

Trick: $f(x,y)$

if $f(\lambda x, \lambda y) = \lambda^n f(x,y)$

then function is homogeneous.

&

Degree $= n$

Method 2: $f(x,y) = x^2 y + y^2 x$

$$f(\lambda x, \lambda y) = (\lambda x)^2 (\lambda y) + (\lambda x)(\lambda y)^2$$

$$f(\lambda x, \lambda y) = \lambda^3 (x^2 y + x y^2)$$

$$f(\lambda x, \lambda y) = \lambda^3 f(x,y)$$

$\therefore$ Degree $= 3$.

Ques. $u = \ln\left(\dfrac{x^3+y^3}{x^2+y^2}\right)$ find: $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$.

Method 1: Normal / standard method

Method 2:

$$u(x,y) \Rightarrow u(\lambda x, \lambda y) = \ln\left(\frac{\lambda^3 (x^3+y^3)}{\lambda^2 (x^2+y^2)}\right)$$

$$u(\lambda x, \lambda y) \neq \lambda^n u(x,y)$$

$\therefore$ it is not Homogeneous.

$$z = e^u = \frac{x^3+y^3}{x^2+y^2}$$

$$z(x,y) \longrightarrow z(\lambda x, \lambda y) = \lambda\left(\frac{x^3+y^3}{x^2+y^2}\right) = \lambda z(x,y)$$

$$\therefore \; z(x,y) = z(\lambda x, \lambda y) = \lambda[z(x,y)]$$

so, • z is homogeneous function.

• $n = 1$

• Euler's trick

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n.z$$

$$x\frac{\partial (e^u)}{\partial x} + y\frac{\partial}{\partial y}(e^u) = 1.(e^u)$$

$$x\frac{\partial}{\partial x}(e^u).\frac{\partial u}{\partial x} + y.\frac{\partial}{\partial y}(e^u).\frac{\partial u}{\partial y} = 1.e^u$$

$$x e^u\frac{\partial u}{\partial x} + y e^u\frac{\partial u}{\partial y} = e^u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1 \quad \underline{\text{Ans}}$$

Ques: $u = x^2+y^2+z^2$

$x = e^t$ find: $\frac{du}{dt}$

$y = e^t \sin t$

$z = e^t \cos t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$$u \to (x, y, z) \to t$$

$$\frac{du}{dt} = 2x \cdot e^t + 2y\,(e^t \sin t + e^t \cos t) + 2z\,(e^t \cos t - e^t \sin t)$$

$$\frac{du}{dt} = 4e^{2t}$$

Ques. $u = x \ln xy$, $x^3 + y^3 + 3xy = 1$ find: $\frac{du}{dx}$

(i) $\frac{\partial u}{\partial x} = x \cdot \frac{1}{xy}(y) + \ln(xy)$

$$u \to (x, y) \to x$$

(ii) $\frac{\partial u}{\partial y} = x \cdot \frac{1}{xy} \cdot x$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

(iii) $3x^2 + 3y^2\frac{dy}{dx} + 3\left(x\frac{dy}{dx} + y\right) = 0$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$\frac{du}{dx} = 1 + \ln(xy) + \frac{x}{y}\cdot\left(\frac{-3x^2 - 3y}{3y^2 + 3x}\right)$$

$$\frac{du}{dx} = 1 + \ln(xy) - \frac{x}{y}\left(\frac{x^2 + y}{y^2 + x}\right)$$

Ques. Consider a function $f(x,y,z)$ given by

$f(x,y,z) = (x^2+y^2-2z^2)(y^2+z^2)$. The partial derivative of this function w.r.t. x at point, $x=2, y=1$ and $z=3$ is ____.

$$\frac{\partial f}{\partial x} = (2x)(y^2+z^2)$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=2, y=1, z=3} = (2\times2)(1^2+3^2) = 4\times10 = 40 \quad \underline{\text{Ans}}$$

Ques. let $w=f(x,y)$, where x and y are function of t. then according to the chain rule $\frac{dw}{dt}$ is equal to

(a) $\frac{dw}{dx}\cdot\frac{dx}{dt}+\frac{dw}{dy}\cdot\frac{dt}{dt}$ (b) $\frac{\partial w}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t}$

(c) $\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}$ (d) $\frac{dw}{dx}\frac{\partial x}{\partial t}+\frac{dw}{dy}\cdot\frac{\partial y}{\partial t}$

$$w \to f(x,y) \to t$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$$

Ques. if $z = e^{ax+by} F(ax-by)$; the value of $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y}$ is

(a) $2z$ (b) $2a$ (c) $2b$ (d) $2abz$

$$z = e^{ax+by} \cdot F(ax-by)$$

$$b\frac{\partial z}{\partial x} = b\{F(ax-by)\, e^{ax+by} \cdot (a) + e^{ax+by} F'(ax-by) \cdot a\}$$

$+$ $+$

$$a\frac{\partial z}{\partial y} = a\{F(ax-by)\, e^{ax+by} \cdot b + e^{ax+by} F'(ax-by) \cdot (-b)\}$$

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2ab\, e^{ax+by} F(ax-by) = 2abz$$

Ques. let $r = x^2+y-z$ and $z^3-xy+yz+y^3=1$ Assume that x and y are independent variables. At $(x,y,z) = (2,-1,1)$ the

important

value (correct to two decimal place) of $\frac{\partial r}{\partial x}$ is ______.

(i) $\frac{\partial r}{\partial x} = 2x + 0 - \frac{\partial z}{\partial x}$

(ii) $\frac{\partial z}{\partial x} \cdot 3z^2 - y + y\frac{\partial z}{\partial x} + 0 = 0$

$$\frac{\partial r}{\partial x} = 2x - \frac{y}{3z^2+y}$$

$$\frac{\partial z}{\partial x} = \frac{y}{3z^2+y}$$

$$\frac{\partial r}{\partial x} = 2\times 2 - \frac{(-1)}{3\times 1+(-1)} = 4.5 \text{ Ans}$$

TWO VARIABLE MAXIMA / MINIMA

(i) $f(x,y)$: $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

$\Rightarrow (x_0, y_0)$ stationary point

(ii) $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = r$

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = t$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = s$

- $rt - s^2 > 0$ → maxima $(r < 0)$; → minima $(r > 0)$

- $rt - s^2 < 0$ → No maxima / No minima

- $rt - s^2 = 0$ → fails.

Ques. The function $f(x,y) = x^2y - 3xy + 2y + x$ has

(a) No local extremum

(b) one local maxima but no local minimum

(c) one local minimum but no local maximum

(d) one local minimum and one local maximum.

$$f(x,y) = x^2y - 3xy + 2y + x$$

$$\frac{\partial f}{\partial x} = 2xy - 3y + 0 + 1 = 0$$

$$\frac{\partial f}{\partial y} = x^2 - 3x + 2 + 0 = 0$$

$$(x-2)(x-1) = 0$$

$$x = 1, 2$$

$\therefore$ if $x = 1$; then $y = 1$

if $x = 2$; then $y = -1$

$\therefore$ stationary point $(1,1)$ $(2,-1)$

$$r = \frac{\partial^2 f}{\partial x^2} = 2y \qquad t = \frac{\partial^2 f}{\partial y^2} = 0$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 2x - 3$$

$$rt - s^2 = 2y \times 0 - (2x-3)^2$$

$$= -(2x-3)^2.$$

$$= \cancel{+}ve < 0 \longrightarrow \text{No max./No minima.}$$

Ques. find the absolute maxima & minima of the function $f(x,y) = x^2 - xy - y^2 - 6x + 2$ on the rectangular plate $0 \leq x \leq 5$, $-3 \leq y \leq 0$.

(i) $f_x = 2x - y - 6 = 0$

$f_y = -x - 2y = 0$

$\Rightarrow x = \frac{12}{5}, y = \frac{-6}{5}$

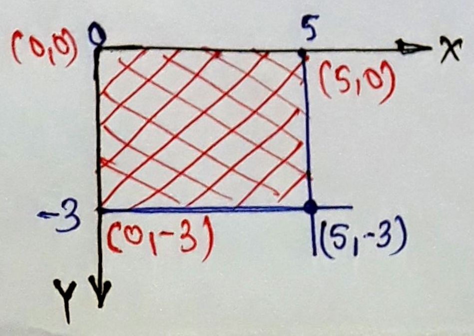
(ii) $r = f_{xx} = 2$ $\quad s = f_{xy} = -1$

$t = f_{yy} = -2$

$\therefore rt - s^2 = 2 \times (-2) - (-1)^2 = -4 - 1 = -5 < 0$

$\therefore$ No max./No min.

(iii) $0 \leq x \leq 5$ & $-3 \leq y \leq 0$



(i) $x=0, y=-3$: $f(0,-3) = -7$ Absolute minima

(ii) $x=0, y=0$: $f(0,0) = 2$

(iii) $x=5, y=0$: $f(5,0) = -3$

(iv) $x=5, y=-3$: $f(5,-3) = 3$ Absolute maxima

Ques. Maximum value of determinent among all 2x2 real symm. matrix with trace 14 is _.

Method-1 : (i) $A = A^T$ (Real symm. matrix)

(ii) $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

(iii) $|A| = \begin{vmatrix} a & b \\ b & c \end{vmatrix} = ac - b^2$

(iv) $a + c = 14 \longrightarrow c = 14 - a$

(v) $|A| = a(14-a) - b^2$

$\therefore f = |A| = 14a - a^2 - b^2$

$f_a = \frac{\partial f}{\partial a} = 14 - 2a = 0$

$f_b = \frac{\partial f}{\partial b} = 0 - 2b = 0$

$a = 7$ & $b = 0$

$\therefore$ stationary point : $(7, 0)$

(2) $r = f_{aa} = -2 \quad t = f_{bb} = -2 \quad s = f_{ab} = 0$

$rt - s^2 = (-2)(-2) - 0 = 4 > 0$ & $r < 0$

$\therefore$ maxima

(3) max. point $(7, 0)$

max. value : $f = |A|_{max} = 14a - a^2 - b^2$

$= 14 \times 7 - 7^2 - 0 = 49$

Method - 2 :

$$A = \begin{bmatrix} x & b \\ b & y \end{bmatrix}$$

$|A| = xy - b^2$

$|A|_{max} = xy - 0 = xy$

$x + y = 14 \rightarrow y = 14 - x$

$|A|_{max} = x(14 - x) = 14x - x^2$

$f = |A|_{max} = 14x - x^2$

$f_x = 14x - \cancel{2x} = 0$

$x = 7$

$f_{xx} = -2 < 0 \longrightarrow$ maxima.

so, $x = 7 \longrightarrow$ max. point

$f(x=7) = |A|_{max} = 14 \times 7 - 7^2 = 49$

Method : 3 :

$$A = \begin{bmatrix} x & b \\ b & y \end{bmatrix}$$

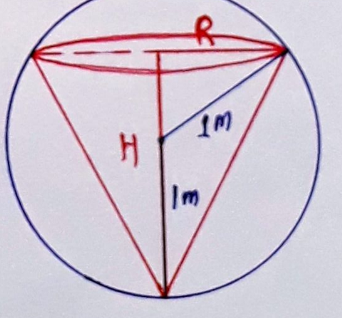
$|A| = xy - b^2$

$|A|_{max} = xy$

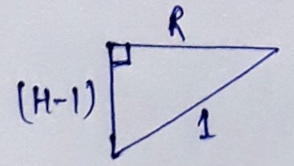
$x + y = 14$

x	y	$x+y$	xy
0	14	14	0
1	13	14	13
2	12	14	24
⋮			
7	7	14	49
8	6	14	48

Ques. Right circular cone of largest volume that can be enclosed by a sphere of 1m radius has height of ___ ?



(i) $V_{cone} = \frac{1}{3}\pi R^2 H$

(ii) 

$R^2 + (H-1)^2 = 1$

$R^2 = 1 - (H-1)^2$

(iii) $V = \frac{1}{3}\pi(2h-h^2)h$

for max. volume.

$\frac{dV}{dh} = 0$; $\frac{1}{3}\pi(4h-3h^2) = 0$

$h = 0, \frac{4}{3}$ — stationary point

$\frac{d^2V}{dh^2} = \frac{\pi}{3}(4-6h)$

at $h=0$; $\frac{d^2V}{dh^2} > 0 \longrightarrow$ minima.

at $h = \frac{4}{3}$; $\frac{d^2V}{dh^2} < 0 \longrightarrow$ maxima.

$\therefore$ height, $h = 4/3$

* INTEGRATION *

$$\frac{d}{dx}(x^2) = 2x$$

$$\frac{d}{dx}(x^2+2) = 2x$$

$$\frac{d}{dx}(x^2-2) = 2x$$

$$\frac{d}{dx}(x^2+5/2) = 2x$$

$$\cancel{\int} \frac{d}{\cancel{dx}}(x^2 \pm c)\cancel{dx} = \int 2x\,dx$$

$$\therefore (x^2+c) = \int 2x\,dx$$

* $\boxed{\int f(x)\,dx = G(x) + c}$ $\longrightarrow$ $\cancel{\frac{d}{dx}}\int f(x)\,\cancel{dx} = \frac{d}{dx}G(x)+c$

$\boxed{f(x) = G'(x)}$

Ques. $\int x^2 dx = ?$

Ques. $\frac{d}{dx}$ of option

(a) x^2+c (b) x^3+c

(c) $\frac{x^3}{3}+c$ (d) $2x+c$

M1: $\int x^2 dx = \frac{x^3}{3}+c$

M2: $\frac{d}{dx}\left(\frac{x^3}{3}+c\right) = \frac{3x^2}{3}+0 = x^2$

* Definite integral.

1. $$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

2. $$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

3. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

4. $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad ; \text{ where } a<c<b$$

V.Imp. 5. $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \longrightarrow \text{Kings formula.}$$

6. $$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

7. $$\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & ; \text{if } f(-x) = f(x) \text{ ; even function} \\ 0 & ; \text{if } f(-x) = -f(x) \text{ ; odd function} \end{cases}$$

8. $$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & ; \text{ if } f(2a-x) = f(x) \\ 0 & ; \text{ if } f(2a-x) = -f(x) \end{cases}$$

* Basic formulae.

1. $\int K\,dx = Kx + C$

2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

3. $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$; for $n \neq -1$

4. $\int \frac{1}{x}\,dx = \ln(x) + C$; for +ve value of x only.

5. $\int \frac{c}{ax+b}\,dx = \frac{c}{a}\ln(ax+b) + C$

* Exponential & log.

1. $\int e^x\,dx = e^x + C$

2. $\int a^x\,dx = \frac{a^x}{\ln a} + C$

3. $\int \ln x\,dx = x\ln x - x + C$

4. $\int \log_a x\,dx = x\log_a x - \frac{x}{\log a} + C$

* Trigonometry.

1. $\int \sin x\,dx = -\cos x + C$

2. $\int \cos x\,dx = \sin x + C$

3. $\int \tan x\,dx = -\ln(\cos x) + C = \ln(\sec x) + C$

4. $\int \cot x\,dx = \ln(\sin x) + C$

5. $\int \sec\,dx = \ln(\sec x + \tan x) + C = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + C$

6. $\int \operatorname{cosec} x \, dx = -\ln(\operatorname{cosec} x + \cot x) + c = \log \tan \frac{x}{2}$

7. $\int \sec^2 x \, dx = \tan x + C$

8. $\int \operatorname{cosec}^2 x \, dx = -\cot x + C$

9. $\int \sec x \tan x \, dx = \sec x + C$

10. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$

* Algebric.

1. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + c$

2. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) = \frac{-1}{a} \cot^{-1}\left(\frac{x}{a}\right)$

3. $\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + C$

4. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log\left(\frac{a+x}{a-x}\right) = \frac{1}{a} \tanh^{-1}\left(\frac{x}{a}\right)$

5. $\int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \cosh^{-1}\left(\frac{x}{a}\right)$

6. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right)$

7. $\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$

8. $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right)+c$

9. $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$

10. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

* Hyperbolic.

1. $\int \sinh x\,dx = \cosh x + c$ 2. $\int \cosh x\,dx = \sinh x + c$

3. $\int \tanh x\,dx = \ln(\cosh x)+c$ 4. $\int \operatorname{cosech} x\,dx = \ln\left(\tanh\frac{x}{2}\right)+c$

5. $\int \operatorname{sech} x\,dx = \tan^{-1}(\sinh x)+c$ 6. $\int \coth x\,dx = \ln(\sinh x)+c$

* Special function

1. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)+c$

2. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)+c$

3. $\int \cos ax\cosh bx\,dx = \frac{1}{a^2+b^2}(a\sin ax\cosh bx + b\cos ax\sinh bx)+c$

4. $\int \sin ax\cosh bx\,dx = \frac{1}{a^2+b^2}(b\sin ax\sinh bx - a\cos ax\cosh bx)+c$

Ques. $\int \tan x \, dx$

$$* \int \frac{f'(x)}{f(x)} dx = \ln(f(x)) + c$$

Method 1 : Substitution Technique.

$$I = \int \frac{\sin x}{\cos x} dx$$

$$I = \int \frac{-dt}{t}$$

put, $\cos x = t$

$-\sin x \, dx = dt$

$$I = -\ln(t) + c$$

$$I = -\ln(\cos x) + c = \ln(\sec x) + c$$

Ques. $\int \frac{dx}{1-\cos x}$

Method - 1 : $\int \frac{dx}{2\sin^2 \frac{x}{2}} = \frac{1}{2} \int \text{cosec}^2 \frac{x}{2} \, dx$

$$= \cancel{\tfrac{1}{2}} \left(- \frac{\cot x/2}{\cancel{1/2}} \right) + c$$

$$= -\cot \frac{x}{2} + c$$

Method - 2 :

$$\int \frac{dx}{1-\cos x} = \int \frac{1+\cos x}{1-\cos^2 x} dx = \int \frac{1+\cos x}{\sin^2 x} dx$$

$$= \int (\operatorname{cosec}^2 x + \cot x \cdot \operatorname{cosec} x)\, dx$$

$$= -\cot x + (-\operatorname{cosec} x) + c$$

$$= \frac{-\cos x}{\sin x} - \frac{1}{\sin x} + c = -\frac{(\cos x + 1)}{\sin x} + c$$

$$= -\frac{2\cos^2 x/2}{2\sin\frac{x}{2}\cos\frac{x}{2}} + c = -\cot\frac{x}{2} + c \quad \text{Ans.}$$

Ques. $\int \frac{\sin^8 x}{\cos^{10} x}\, dx$

$\frac{\text{Numerator}}{\text{Denomenator}} \longrightarrow \tan x \longleftrightarrow \sec x$

$$\int \tan^8 x \cdot \sec^2 x\, dx$$

put, $\tan x = t$

$\sec^2 x\, dx = dt$

$$\int t^8\, dt$$

$$= \frac{t^{8+1}}{8+1} + c$$

$$= \frac{t^9}{9} + c$$

$$= \frac{\tan^9 x}{9} + c$$

* $\int \mathrm{I} \cdot \mathrm{II}\, dx = \mathrm{I} \int \mathrm{II} - \int \left(\frac{d}{dx}\mathrm{I} \left(\int \mathrm{II} \right) \right)$

$\int u \cdot v = u \int v - \int (u' \int v)$

highest priority : I : Inverse
L : Log
A : Algebric
T : Trignometric
E : Exopnential.

Ques. $\int \ln x\, dx$

Method 1: $\int \ln x \cdot 1\, dx$

$= \ln x \int 1 dx - \int \left[\left(\frac{d}{dx} \ln x \right) \cdot \int 1 dx \right] dx + c$

$= \ln x \cdot x - \int \left(\frac{1}{x} \cdot x \right) dx + c$

$= x \ln x - x + c$

Method-2 :

put: $\ln x = t$

$x = e^t$

$dx = e^t dt$

$\int \underset{u}{t}\, \underset{v}{e^t} dt$

$= t \int e^t dt - \int \frac{d}{dt}(t) \int e^t dt + c$

$= te^t - e^t + c = \ln x \cdot e^{\ln x} - e^{\ln x} + c = x \ln x - x + c$

Ans

Ques 5. $\int x^2 e^{2x} dx$

Method-1 :

$$= x^2 \cdot \frac{e^{2x}}{2} - \int (2x) \times \frac{e^{2x}}{2} dx + c$$

$$= x^2 \cdot \frac{e^{2x}}{2} - \left[x\frac{e^{2x}}{2} - \int 1 \cdot \frac{e^{2x}}{2} dx + c \right]$$

$$= x^2 \cdot \frac{e^{2x}}{2} - \left[\frac{x}{2} e^{2x} - \frac{e^{2x}}{4} + c \right]$$

Hindu's Method :

$$\int x^n f(x) dx$$

$\hookrightarrow$ Trigno/expon.

$$\therefore \int \underset{u}{x^2} \, \underset{v}{e^{2x}} dx$$

$$= +x^2\left(\frac{e^{2x}}{2}\right) - 2x\left(\frac{e^{2x}}{4}\right) + 2\left(\frac{e^{2x}}{8}\right) - 0\left(\frac{e^{2x}}{16}\right) + c$$

Ques 6. $\int \frac{x^4}{x^2+1} dx$

$$\int \frac{x^4+1-1}{x^2+1} dx = \int \frac{(x^2)^2 - 1^2 + 1}{x^2+1} dx$$

$$\int \frac{(x^2+1)(x^2-1)+1}{x^2+1}\,dx$$

$$\int \left((x^2-1) + \frac{1}{x^2+1}\right) dx$$

$$\int x^2\,dx - \int dx + \int \frac{1}{x^2+1}\,dx$$

$$= \frac{x^3}{3} - x + \tan^{-1} x + C$$

Ques. 7. $$\int \frac{dx}{x^2+2x+2}$$

$$\int \frac{dx}{(x^2+2x+1)+1} = \int \frac{dx}{(x+1)^2+1}$$

put, $x+1 = t$

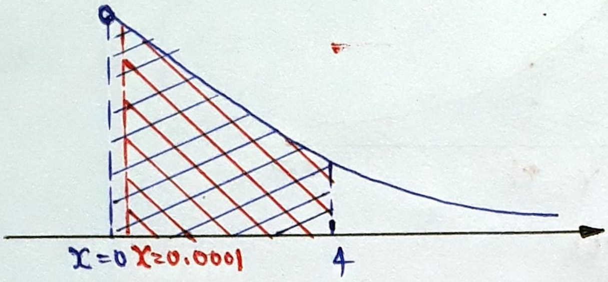
$dx = dt$

$$\int \frac{dt}{t^2+1} = \tan^{-1}(t) + C$$

$$= \tan^{-1}(1+x) + C$$

DEFINITE INTEGRAL

Ques. $\int_0^4 \frac{1}{\sqrt{x}}\,dx$



Method-1 : Improper integral of $\underbrace{\text{2nd kind.}}_{f(x) \text{ is discontinuous.}}$

At $x=0$; $f(x)$ is discontinuous

$$\lim_{h\to 0} \int_{0+h}^{4} \frac{1}{\sqrt{x}}\,dx = \lim_{h\to 0} \left[2\sqrt{x}\right]_h^4$$

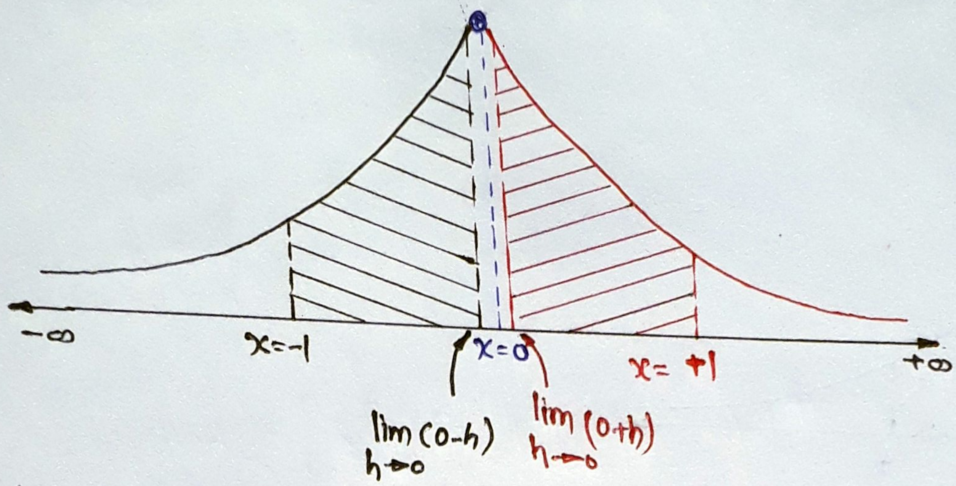
$$= \lim_{h\to 0} \left[2\sqrt{4} - 2\sqrt{h}\right] = 4 - 2(0) = 4 \text{ Ans.}$$

Method 2 : No concept.

$$\int_0^4 \frac{dx}{\sqrt{x}} = \int_0^4 x^{-1/2}\,dx = \left(\frac{x^{-1/2+1}}{-\frac{1}{2}+1}\right)_0^4$$

$$= 2\left(\sqrt{x}\right)_0^4 = 2\left(\sqrt{4} - 0\right) = 4 \text{ Ans}$$

Ques. $\int_{-1}^{1} \frac{dx}{x^2}$



Method 1: Improper integral of second kind.

$$\int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_{0}^{1} \frac{dx}{x^2}$$

$$= \lim_{h\to 0} \int_{-1}^{0-h} \frac{dx}{x^2} + \lim_{h\to 0} \int_{0+h}^{1} \frac{dx}{x^2}$$

$$= \lim_{h\to 0} \left(\frac{x^{-1}}{-1} \right)_{-1}^{-h} + \lim_{h\to 0} \left(\frac{x^{-1}}{-1} \right)_{h}^{1}$$

$$= -1 \left\{ \lim_{h\to 0} \left[\frac{1}{-h} - \frac{1}{(-1)} \right] + \lim_{h\to 0} \left[\frac{1}{1} - \frac{1}{h} \right] \right\}$$

$$= \lim_{h\to 0} \left[\frac{1}{h} - 1 - 1 + \frac{1}{h} \right] = -2 + \left\{ \lim_{h\to 0} \frac{2}{h} \right\} \longrightarrow \infty$$

= Does not exist

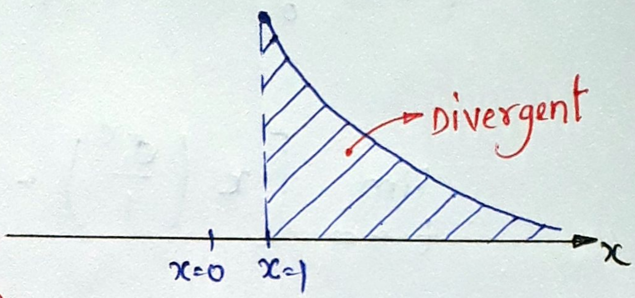
$\therefore$ Divergent

Method-2 : Direct solving. ✗

$$\int_{-1}^{1}\frac{dx}{x^2} = \left[\frac{x^{-2+1}}{-2+1}\right] = -\left(\frac{1}{x}\right)_{-1}^{1} = -(1-(-1)) = -2$$

wrong answer

Ques. $\displaystyle\int_{1}^{\infty}\frac{dx}{x}$



Method 1 : Improper integral of first kind.

↓

limit : ∞ or $-\infty$

$$\int_{1}^{\infty}\frac{dx}{x} = \lim_{t\to\infty}\int_{1}^{t}\frac{dx}{x} = \lim_{t\to\infty}(\ln x)_{1}^{t}$$

$$= \lim_{t\to\infty}(\ln(t)-\ln(1)) = \ln(\infty) - 0 = \infty \text{ Ans}$$

(Divergent)

Method 2 : Direct solving :

$$\int_{1}^{\infty}\frac{dx}{x} = (\ln x)_{1}^{\infty} = \ln\infty - \ln 1$$

$$= \infty - 0 = \infty \text{ Ans}$$

Divergent.

Ques. $\int_0^{\infty} e^{-x} x \, dx$

Method 1: improper integral of first kind.

$$\lim_{t\to\infty} \int_0^t x e^{-x} \, dx$$

$$\lim_{t\to\infty} \left[x\left(\frac{e^{-x}}{-1}\right) - 1\left(\frac{e^{-x}}{+1}\right)\right]_0^t$$

$$= \lim_{t\to\infty} \left[-te^{-t} - e^{-t} - (0 - e^{0})\right]$$

$$= \lim_{t\to\infty} -te^{-t} - \lim_{t\to\infty} e^{-t} + \lim_{t\to\infty} e^{0}$$

$$= -\lim_{t\to\infty} \underbrace{\frac{t}{e^t}}_{(\infty/\infty)\text{ form.}} - e^{-\infty} + 1$$

$$= -\lim_{t\to\infty} \frac{1}{e^t} - 0 + 1$$

$$= -\frac{1}{e^{\infty}} - 0 + 1$$

$$= -0 - 0 + 1 = 1 \text{ Ans}$$

Euler's formula: $\int_0^\infty e^{-x} x^{n-1} \, dx = \Gamma n$

- $\Gamma(n+1) = n \Gamma n = n!$
- $\Gamma 5 = \Gamma(4+1) = 4!$
- $\Gamma \frac{3}{2} = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2} \Gamma \frac{1}{2} = \frac{1}{2}\sqrt{\pi}$

- Reflection formula:

$$\Gamma n \, \Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

(i) put $n = \frac{1}{2}$: $\Gamma \frac{1}{2} \, \Gamma\left(1-\frac{1}{2}\right) = \frac{\pi}{\sin\left(\frac{1}{2}\pi\right)} = \frac{\pi}{1}$

$$\Gamma \frac{1}{2} \, \Gamma \frac{1}{2} = \pi$$

$$\left(\Gamma \frac{1}{2}\right)^2 = \pi \qquad \therefore \boxed{\Gamma \frac{1}{2} = \sqrt{\pi}}$$

(ii) Put $n = \frac{-1}{2}$: $\Gamma \frac{-1}{2} \, \Gamma\left(1-\left(-\frac{1}{2}\right)\right) = \frac{\pi}{\sin\left(-\frac{\pi}{2}\right)} = \frac{\pi}{-1}$

$$\Gamma \frac{-1}{2} \, \Gamma \frac{3}{2} = -\pi$$

$$\Gamma \frac{-1}{2} \, \Gamma\left(\frac{1}{2}+1\right) = -\pi \longrightarrow \Gamma \frac{-1}{2} \cdot \frac{1}{2} \Gamma \frac{1}{2} = -\pi$$

$$\Gamma \frac{-1}{2} = \frac{-2\pi}{\sqrt{\pi}} \qquad \therefore \boxed{\Gamma \frac{-1}{2} = -2\sqrt{\pi}}$$

Method 2: Eulers formula.

$$\int_0^{\infty} e^{-x} x \, dx = \int_0^{\infty} e^{-x} \cdot x^{n-1} dx = \Gamma n$$

$n-1 = 1$

$\therefore n = 2$

$\Gamma 2 = \Gamma 1+1 = 1! = 1$ Ans.

Ques. $\int_0^{\infty} e^{-5x} \cdot x^3 \, dx$

Method-1: (i) substitute: $5x = t$

$5dx = dt$

$dx = \frac{dt}{5}$ $\quad t: 0 \text{ to } \infty$

(ii) $\int_{t=0}^{\infty} e^{-t} \cdot \left(\frac{t}{5}\right)^3 \frac{dt}{5}$

$\frac{1}{5^4} \int_{t=0}^{\infty} e^{-t} t^3 \, dt = \frac{1}{5^4} \times 3!$

$\therefore \frac{6}{5^4}$ Ans.

Method-2: $\int_0^\infty e^{-\lambda x} \cdot x^{n-1}\, dx = \frac{\sqrt{n}}{\lambda^n}$

$$\therefore \int_0^\infty e^{-5x} \cdot x^3\, dx = \frac{\sqrt{4}}{5^4} \text{ Ans.}$$

Ques: $I = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x^2/8}\, dx$

put: $\frac{x^2}{8} = t \longrightarrow x^2 = 8t \longrightarrow x = \sqrt{8t}$

$$\frac{2x}{8}\, dx = dt$$

$$dx = \frac{4}{x}\, dt = \frac{4}{\sqrt{8t}}\, dt = \frac{\sqrt{2}}{\sqrt{t}}\, dt$$

limit: $t \longrightarrow 0$ to ∞

$$I = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t} \cdot \frac{\sqrt{2}}{\sqrt{t}}\, dt$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t} \cdot t^{-1/2}\, dt.$$

$\because n-1 = \frac{-1}{2}$

$n = \frac{-1}{2} + 1 = \frac{1}{2}$

$$= \frac{1}{\sqrt{\pi}} \times \sqrt{\frac{1}{2}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1 \text{ Ans.}$$

Ques. $\int_0^1 x(1-x)^5\,dx$

Method 1: put $1-x=t$ $\qquad x=0 \to t=1$

$-dx = dt \qquad x=1 \to t=0$

$$\int_{t=1}^{0} (1-t)\,t^5\,(-dt) = \frac{1}{42}$$

Method 2: $\int_0^1 (1-x)\left[1-(0+1-x)\right]^5 dx$ {King's formula}

$$\int_0^1 (1-x)\,x^5\,dx = \frac{1}{42}$$

Eulers formula:
(β - function)

(i) $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \beta(m,n) = \beta(n,m)$

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

(ii) $$\int_0^{\pi/2} \sin^m\theta \cdot \cos^n\theta\,d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$$

Method-3: Euler's method.

$m-1 = 1 \rightarrow m = 2$

$n-1 = 5 \rightarrow n = 6$

$$\int_0^1 x(1-x)^5\,dx$$

$$\int_0^1 x^{2-1}(1-x)^{6-1}\,dx = \beta(2,6)$$

$$\beta(2,6) = \frac{\Gamma 2\,\Gamma 6}{\Gamma 2+6} = \frac{1!\,5!}{7!} = \frac{5!}{7\times 6\times 5!} = \frac{1}{42}$$

Ques. $\underbrace{\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}}}_{I_1} \times \underbrace{\int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta}_{I_2}$

$$I_1 = \int_0^{\pi/2} (\sin\theta)^{-1/2}.\cos^0\theta\,d\theta = \beta(-\tfrac{1}{2},0) = \frac{\Gamma\frac{-\frac{1}{2}+1}{2}\,\Gamma\frac{0+1}{2}}{2\,\Gamma\frac{-1/2+0+2}{2}}$$

$$= \frac{\Gamma\frac{1}{4}\,\Gamma\frac{1}{2}}{2\,\Gamma 3/4}$$

$$I_2 = \frac{\Gamma 3/4\,\Gamma 1/2}{2\,\Gamma 5/4}$$

$$\therefore\ I_1\times I_2 = \frac{\Gamma 1/4\times\Gamma 1/2}{2\,\Gamma 3/4}\times\frac{\Gamma 3/4\,\Gamma 1/2}{2\times\frac{1}{4}\Gamma\frac{1}{4}} = \Gamma\tfrac{1}{2}\cdot\Gamma\tfrac{1}{2}$$

$$= \sqrt{\pi}\times\sqrt{\pi} = \pi \text{ Ans.}$$

Double Integration.

Double integeration → LIMITS

- Independent case.
  $\left.\begin{matrix} x \\ y \end{matrix}\right\}$ Both constonts.
  in general : order of integrat^{n} does not matters.
- Dependent case.

case 1 : x : variable
y : constant

case 2 : x : constant
y : variable.

order of integration matters.

Ques. $\displaystyle\int_{x=0}^{1}\int_{y=0}^{x} x \, dy \, dx$

$$\int_{x=0}^{1} x\left(\int_{y=0}^{x} dy\right) dx = \int_{x=0}^{1} x \, (y)_0^x \, dx = \int_0^1 x^2 \, dx.$$

$$= \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3} \text{ Ans}$$

Ques. $\displaystyle\int_{y=0}^{1}\int_{x=1}^{y^2} dx \, dy$

$$\int_{y=0}^{1} (x)_1^{y^2} dy = \int_{y=0}^{1} (y^2-1) \, dy = \left(\frac{y^3}{3}-y\right)_0^1 = \frac{-2}{3} \text{ Ans}$$

Ques. $\int_{x=0}^{y}\int_{y=\sqrt{x}}^{1} xy^2 \, dx \, dy$ ——→ Invalid case.

Ques. $\int_{0}^{\infty}\int_{0}^{\infty} e^{-x^2-y^2} \, dx \, dy$

Method-1 : $\int_{y=0}^{\infty} e^{-y^2} \left(\int_{x=0}^{\infty} e^{-x^2} dx \right) dy$

$$= \int_{y=0}^{\infty} \left(\frac{\sqrt{\pi}}{2}\right) e^{-y^2} dy = \frac{\sqrt{\pi}}{2} \int_{y=0}^{\infty} e^{-y^2} dy$$

$$= \frac{\sqrt{\pi}}{2} \times \frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$$ Ans.

Method-2 : $\int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-x^2-y^2} \, dy \, dx$

$$= \int_{0}^{\infty} e^{-x^2} \left(\int_{0}^{\infty} e^{-y^2} dy \right) dx$$

$$= \frac{\sqrt{\pi}}{2} \int_{0}^{\infty} e^{-x^2} dx$$

$$= \frac{\sqrt{\pi}}{2} \times \frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$$ Ans

Method-3 : $\int_0^{\infty}\int_0^{\infty} e^{-x^2-y^2}\,dx\,dy$.

$$\left(\int_{x=0}^{\infty} e^{-x^2}dx\right)\left(\int_{y=0}^{\infty} e^{-y^2}dy\right)$$

$$= \frac{\sqrt{\pi}}{2} \times \frac{\sqrt{\pi}}{2} = \frac{\pi}{4} \text{ Ans}$$

* Area using Single integration.

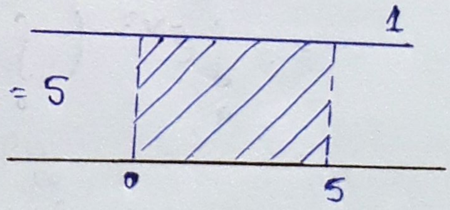
Ques. $\int_0^5 dx$

Method-1 : $(x)_0^5 = 5-0 = 5$ Ans.

Method-2 : $\int_0^5 dx = \int_{x=a}^{b} f(x)\,dx$.

$\because f(x) = 1$ & $a=0, b=5$

Area: $5 \times 1 = 5$



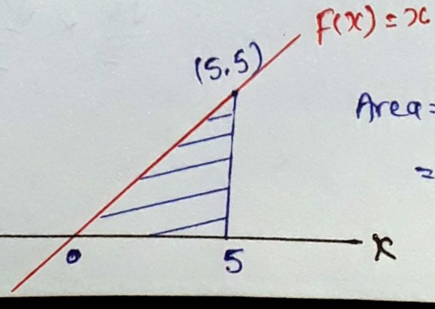
Ques. $\int_0^5 x\,dx$

M-1 : $\left(\frac{x^2}{2}\right)_0^5 = \frac{1}{2}\times 25 = \frac{25}{2}$

M-2 : $f(x) = x$

$a=0$

$b=5$

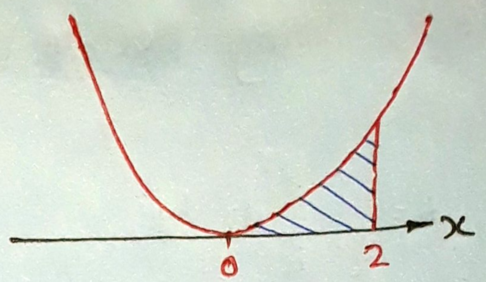


Area $= \frac{1}{2}\times b \times h$

$= \frac{1}{2}\times 5 \times 5$

$= 25/2$

Ques. $\int_0^2 x^2\,dx$



only way → solving.

$\left(\frac{x^3}{3}\right)_0^2 = \frac{8}{3}$ Ans.

• Area → surface Area.

single integratⁿ — Double integration

$\int_a^b f(x)\,dx$ — $\int_x \int_y dx\,dy$

Ques. Area enclosed between $y = x^2$ & $y = x$ is —.

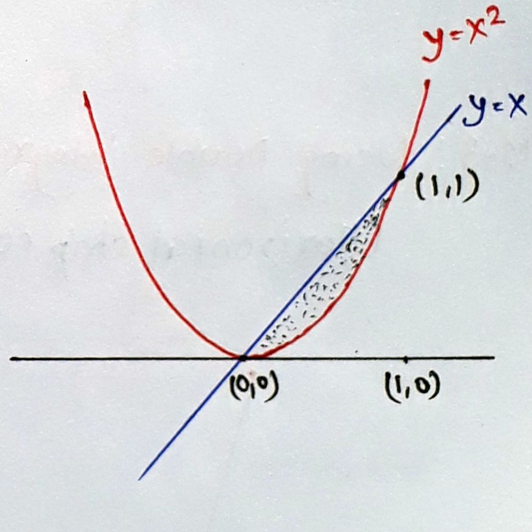
(i) $y = x$ & $y = x^2$

$x = x^2$

$x - x^2 = 0$

$x(1 - x) = 0$

$x = 0, 1$



M-1: Using single integration.

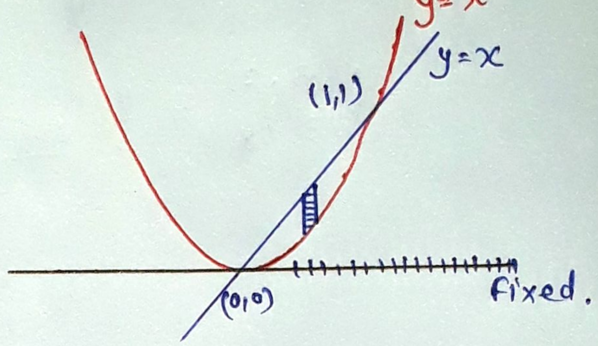
Required = Area of Δ − Area under Parabola

$= \frac{1}{2} \times 1 \times 1 - \int_{x=0}^{1} x^2\,dx$

$= \frac{1}{2} - \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ Ans

M-2 : Using Double Integration (vertical strip concept)

$$A = \int_x \int_y dx\, dy$$

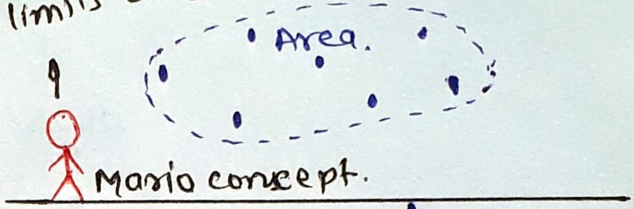


limits : $x = 0$ to 1

$y = x^2$ to x

$$A = \int_{x=0}^{1} \left(\int_{y=x^2}^{x} dy \right) dx$$

y limits change w.r.t. x



this is fixed.

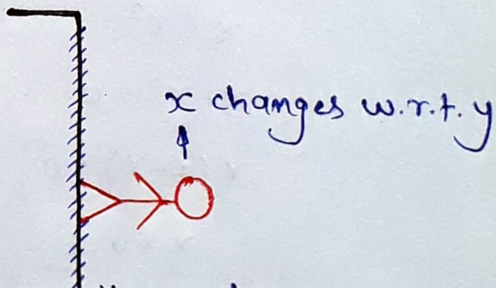
x limits are constant.

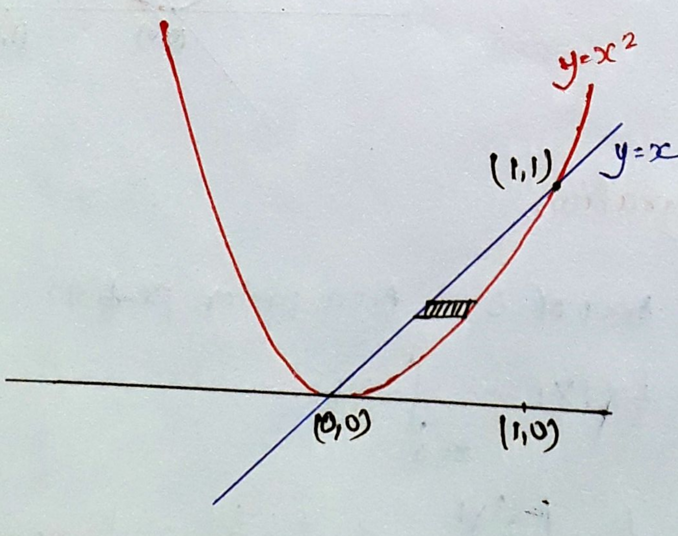
$$A = \int_{x=0}^{1} (y)_{x^2}^{x}\, dx = \int_0^1 (x - x^2)\, dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3} \right)_0^1 = \frac{1}{6} \text{ Ans.}$$

M-3 ! Using Double integration.

(Horizontal strip concept)





limits!

$y = 0$ to 1

$x = y$ to $\sqrt{y}$

$$\int_{y=0}^{1}\left(\int_{x=y}^{\sqrt{y}} dx\right) dy$$

$$= \int_0^1 (x)_y^{\sqrt{y}}\, dy = \int_0^1 (\sqrt{y}-y)\, dy = \left(\frac{y^{3/2}}{3/2} + \frac{y^2}{2}\right)_0^1$$

$$= \frac{2}{3} - \frac{1}{2} = \frac{1}{6} \text{ Ans.}$$

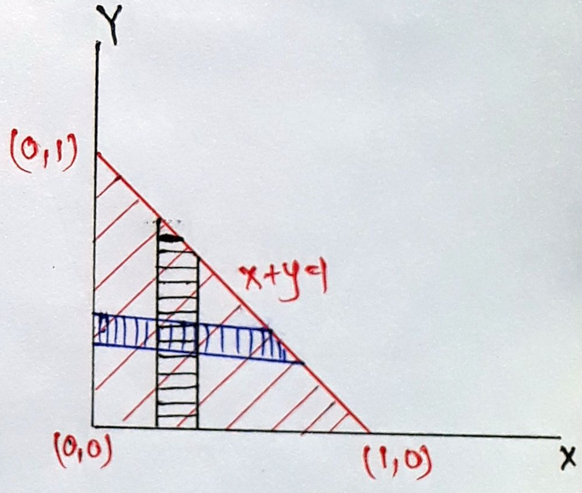
surface Area : $\int f(x)\,dx$

volume : $\int_x \int_y f(x,y)\, dx\, dy$

Ques. ΔABC vertices (0,0) (1,0) (0.1)

find. $\iint 2x\, dx\, dy$

M-1 Horizontal strip.



limits : $y = 0$ to 1

$x = 0$ to $1-y$

$$\int_{y=0}^{1}\int_{x=0}^{1-y} 2x\,dx\,dy$$

$$\int_0^1 \left(2 \times \frac{x^2}{2}\right)_0^{1-y} dy$$

$$\int_0^1 (1-y)^2 dy = \frac{1}{3} \text{ Ans.}$$

straight line.

$$\frac{Y-Y_1}{Y_2-Y_1} = \frac{X-X_1}{X_2-X_1}$$

$x+y=1$

M.2: vertical strip:

limits: $x = 0$ to 1

$y = 0$ to $1-x$

$$\int_{x=0}^{1} \int_{y=0}^{1-x} 2x \, dx \, dy = \frac{1}{3} \text{ Ans.}$$

Ques. $\displaystyle\int_{x=0}^{1} \int_{y=x}^{1} \sin(y^2) \, dx \, dy$

Difficult + Dependent case

$\downarrow$

$\therefore$ Change of order.

TRIPLE INTEGRATION

TYPE 1 : Independent case.

$$\int_0^1\int_0^1\int_0^1 dx\,dy\,dz$$

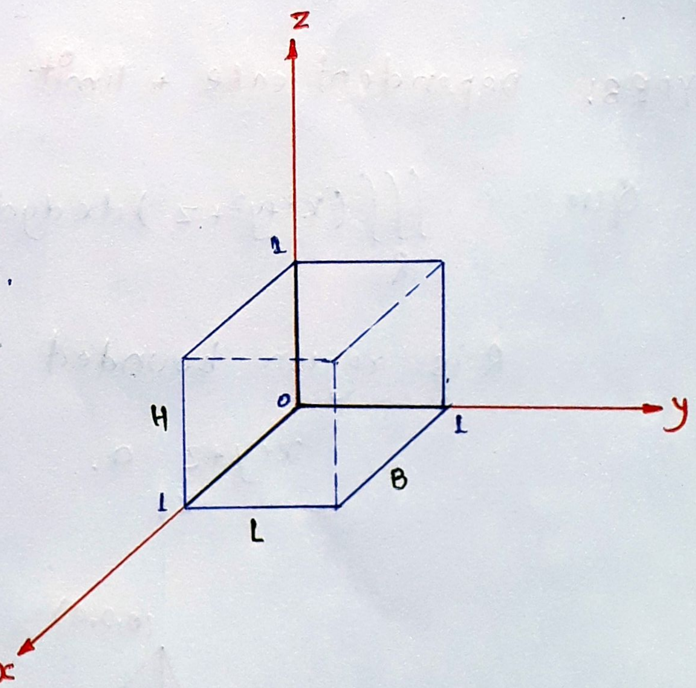
M1 : $\displaystyle\int_{z=0}^{1}\int_{y=0}^{1}(x)_0^1\,dy\,dz = \int_{z=0}^{1}\int_{y=0}^{1} 1\,dy\,dz$

$$\int_{z=0}^{1}(y)_0^1\,dz = \int_{z=0}^{1} dz = (z)_0^1 = 1 \text{ Ans.}$$

M2 : $\displaystyle\int_x\int_y\int_z dx\,dy\,dz = \text{volume.}$

Volume = $L \times B \times H$

$= 1 \times 1 \times 1 = 1$ Ans.



TYPE 2: Dependent case in $\iiint$ + limits are given.

$$\int_{x=0}^{2}\int_{y=0}^{x}\int_{z=0}^{\sqrt{x^2+y^2}} z\,dx\,dy\,dz$$

$$\int_{x=0}^{2}\int_{y=0}^{x}\left(\frac{z^2}{2}\right)_0^{\sqrt{x^2+y^2}} dy\,dz = \int_{x=0}^{2}\int_{y=0}^{x}\frac{x^2+y^2}{2}\,dy\,dz$$

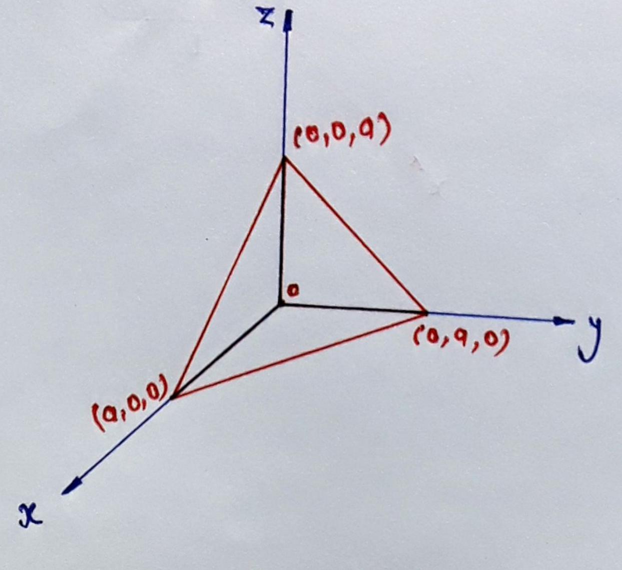
$$\int_{x=0}^{2}\left(\frac{x^2}{2}(y)_0^x + \frac{1}{2}\left(\frac{y^3}{3}\right)_0^x\right)dx$$

$$\int_{x=0}^{2}\left(\frac{x^3}{2} + \frac{x^3}{6}\right)dx = \left(\frac{x^4}{8} + \frac{x^4}{24}\right)_0^2 = 8/3$$

TYPE 3: Dependent case + limits not given.

Ques: $\iiint_R (x^2+y^2+z^2)\,dx\,dy\,dz$

R is region bounded by $x=0,\ y=0,\ z=0$ & $x+y+z=a$.



M1: $x : 0$ to a

$y = 0$ to $a-x$

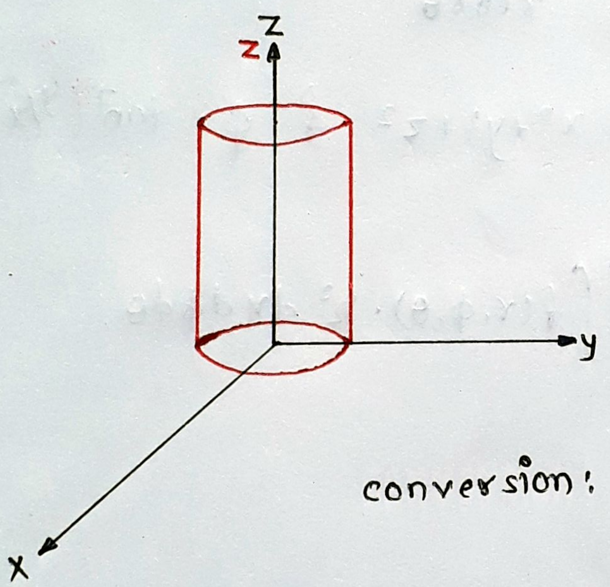
$z = 0$ to $a-x-y$

M2: $x = a$ to b : 1 variable → only constant ∴ $x = 0$ to a

$y = f(x)$: 2nd variable → dependent on prev. variable ∴ $y = 0$ to $a-x$

$z = f(x,y)$: 3rd variable → depend. on previous 2 variable ∴ $z = 0$ to $a-x-y$

* CYLINDRICAL CORDINATE SYSTEM $\{z, \phi, r\}$



Range: $-\infty < z < \infty$

$0 < r < \infty$

$0 < \phi < 2\pi$

conversion: $r = (x^2+y^2)^{1/2}$ & $\phi = \tan^{-1}(y/x)$

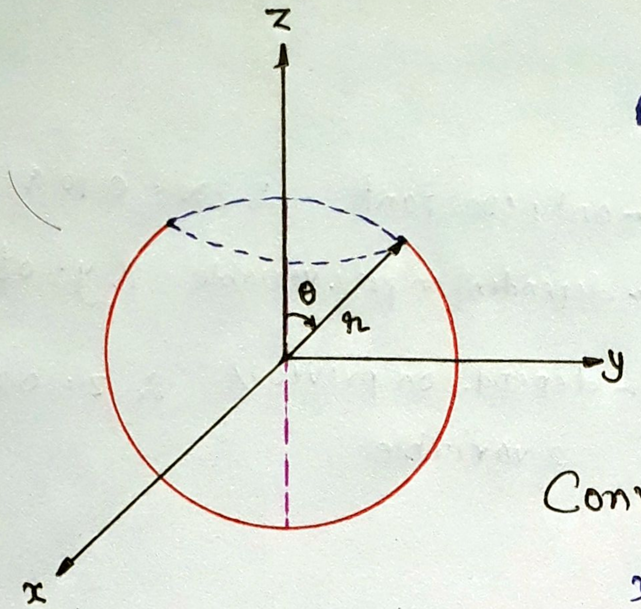
$x = r\cos\phi$

$y = r\sin\phi$

$z = z$

$$\therefore \int_x\int_y\int_z f(x,y,z)\, dx\,dy\,dz \longrightarrow \int_r\int_\phi\int_z f(r,\phi,z)\, r\,dr\,dz\,d\phi$$

* SPHERICAL COORDINATE SYSTEM $\{r, \theta, \phi\}$



Range : $0 < r < \infty$

$0 < \phi < 2\pi$

$0 < \theta < \pi$

Conversion :

$x = r\cos\phi \sin\theta$

$y = r \sin\phi \sin\theta$

$z = r\cos\theta$

$r^2 = x^2 + y^2 + z^2$ & $\phi = \tan^{-1}(y/x)$

$$\therefore \iiint f(x,y,z)\, dx\,dy\,dz = \iiint f(r,\phi,\theta) \cdot r^2 \, dr\, d\phi\, d\theta$$