Quantum Mechanical Treatment of the Stern-Gerlach Experiment

1. Introduction

The Stern-Gerlach experiment, first conducted in 1922 by Otto Stern and Walther Gerlach, is one of the most important experiments in quantum mechanics. It demonstrates the quantization of angular momentum and provides direct evidence for the concept of spin. The experiment involves sending a beam of silver atoms through a non-uniform magnetic field and observing discrete splitting on a detector screen, contrary to classical expectations of a continuous distribution.

2. Physical Setup

- A beam of silver atoms is thermally emitted from a hot oven.
- These atoms pass through a collimator and then enter a region with a non-uniform magnetic field.
- The field has a gradient along the z-axis.
- The atoms hit a detection screen after being deflected.

3. Quantum Mechanical Description

3.1 Spin Angular Momentum

For a silver atom, the outermost electron is in a $5s$ orbital with:

$$l = 0, \quad s = \frac{1}{2}$$

Therefore, the total angular momentum is due to spin only.

Spin operators for a spin-$\frac{1}{2}$ particle are represented using Pauli matrices:

$$\hat{S}_x = \frac{\hbar}{2}\sigma_x, \quad \hat{S}_y = \frac{\hbar}{2}\sigma_y, \quad \hat{S}_z = \frac{\hbar}{2}\sigma_z$$

where

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Eigenstates of $\hat{S}_z$:

$$\hat{S}_z \left|+\right\rangle = +\frac{\hbar}{2} \left|+\right\rangle, \quad \hat{S}_z \left|-\right\rangle = -\frac{\hbar}{2} \left|-\right\rangle$$

3.2 Magnetic Moment and Interaction Hamiltonian

The magnetic moment of a spin-$\frac{1}{2}$ particle is given by:

$$\vec{\mu} = -g_s \mu_B \frac{\vec{S}}{\hbar}$$

where:

- $\mu_B = \frac{e\hbar}{2m_e}$ is the Bohr magneton
- $g_s \approx 2$ is the electron spin g-factor

The Hamiltonian for the interaction of the magnetic moment with the magnetic field is:

$$\hat{H} = -\vec{\mu} \cdot \vec{B} = \frac{g_s \mu_B}{\hbar} \hat{S}_z B_z(z)$$

4. Beam Splitting and Force

In the Stern-Gerlach apparatus, the magnetic field has a gradient:

$$\vec{B} = B_z(z)\hat{z}, \quad \frac{dB_z}{dz} \neq 0$$

The force on the magnetic moment is:

$$F_z = \mu_z \frac{dB_z}{dz}$$

Since:

$$\mu_z = \pm \frac{g_s \mu_B}{2}$$

then:

$$F_z = \pm \frac{g_s \mu_B}{2} \frac{dB_z}{dz}$$

This force causes deflection of the atoms in two discrete directions depending on the spin state.

5. Wave Function and Measurement

Let the total wave function be:

$$\Psi(\vec{r}, t) = \psi(\vec{r}, t) \otimes \chi_s$$

where the spinor is:

$$\chi_s = a\,|+\rangle + b\,|-\rangle$$

Due to the spin-dependent potential:

$$V_{\pm}(z) = \mp \frac{g_s \mu_B}{2} B_z(z)$$

the wave packet splits into:

$$\Psi(\vec{r}, t) = \psi_+(\vec{r}, t)\,|+\rangle + \psi_-(\vec{r}, t)\,|-\rangle$$

Upon measurement, the wavefunction collapses to:

- $|+\rangle$ with probability $|a|^2$
- $|-\rangle$ with probability $|b|^2$

6. Conclusion

The Stern-Gerlach experiment is a clear demonstration of:

- Spin quantization
- Measurement postulates in quantum mechanics
- The breakdown of classical ideas about angular momentum

It also establishes the foundational framework for modern quantum information theory and quantum state manipulation.