YANG-MILLS EQUATIONS ON RIEMANN SURFACE

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Abstract.

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1. The Yang-Mills equations

In this section we assume G is a compact Lie group, and (M, g) is an oriented compact Riemannian manifold.

1.1. **The Yang-Mills functional.**

Let P be a principal G-bundle with local trivializations $\{U_\alpha\}$ and $\rho\colon G \to \mathrm{GL}(V)$ be a linear representation of G. Note that any $s \in C^\infty(M, \Omega^k_M(P \times_\rho V))$ is given by data $\{(s_\alpha) \in \prod C^\infty(U_\alpha, \Omega^k_M(V)) \mid s_\alpha = \rho(g_{\alpha\beta})s_\beta\}$, so if we want to construct an inner product on $\Omega^k_M(P \times_\rho V)$, it suffices to construct a ρ-invariant inner product $\langle \text{-}, \text{-} \rangle$ on V since we already have a Riemannian metric g.

The case that captivates our utmost interest is when $V = \mathfrak{g}$, as the curvature of a connection is a section of $\Omega^2_M(\mathrm{Ad}\,\mathfrak{g})$. In order to construct an inner product on $\Omega^k_M(\mathrm{Ad}\,\mathfrak{g})$, we need an inner product on $\mathfrak{g}$ which is invariant under the adjoint action. Since G is compact, its Killing form is a non-degenerate inner product, that's what we're looking for! Thus we have a pointwise inner product on the bundle $\Omega^k_M(\mathrm{Ad}\,\mathfrak{g})$, denoted by $\langle \text{-}, \text{-} \rangle$, and define a global inner product on $\Omega^k_M(\mathrm{Ad}\,\mathfrak{g})$ as

$$(\alpha, \beta) := \int_M \langle \alpha, \beta \rangle \,\mathrm{vol}$$

where $\alpha, \beta \in C^\infty(M, \Omega^k_M(\mathrm{Ad}\,\mathfrak{g}))$.

Definition 1.1.1 (Hodge star operator)**.** The Hodge star operator

$$\begin{aligned} \star\colon C^\infty(M, \Omega^k_M(\mathrm{Ad}\,\mathfrak{g})) &\to C^\infty(M, \Omega^{n-k}_M(\mathrm{Ad}\,\mathfrak{g})) \\ \beta &\mapsto \star\beta \end{aligned}$$

where $\star\beta$ is given by

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \,\mathrm{vol}, \quad \forall \alpha \in C^\infty(M, \Omega^k_M(\mathrm{Ad}\,\mathfrak{g}))$$

Having established these preliminary results, we now proceed to introduce the Yang-Mills functional.

Definition 1.1.2 (Yang-Mills functional)**.** The Yang-Mills functional is the map $YM\colon \mathcal{A}(P) \to \mathbb{R}$ given by

$$YM(\omega) := \|F_\omega\|^2 = \int_M \langle F_\omega, F_\omega \rangle \,\mathrm{vol}$$

where F_ω is the curvature of connection ω.

Remark 1.1.1. By using Hodge star operator, Yang-Mills functional can be rewritten as follows

$$YM(\omega) = \int_M F_\omega \wedge \star F_\omega$$

The advantages of writing Yang-Mills functional in this way is that we can use some properties of Hodge operator to simplify our computations.

Proposition 1.1.1. Yang-Mills functional is gauge invariant, that is for any gauge transformation $\Phi \in \mathcal{G}(P)$, one has $YM(\Phi^*\omega) = YM(\omega)$ holds for connection ω.

Proof. On each local trivialization U_α, the curvature of $\Phi^*\omega$ is given by $\mathrm{Ad}(\phi^{-1}) \circ F_\alpha$, where ϕ is given by $\Phi|_{U_\alpha}(x, g) = (x, \phi(x)g)$. Thus Yang-Mills functional is gauge invariant since inner product $\langle \text{-}, \text{-} \rangle$ is adjoint invariant. $\square$

Definition 1.1.3 (Yang-Mills connection)**.** A Yang-Mills connection is a connection $A \in \mathcal{A}(P)$ which is a local extremum of Yang-Mills functional.

Notation 1.1.1. $\mathcal{A}_{YM}(P)$, or briefly $\mathcal{A}_{YM}$ denotes the set of all Yang-Mills connections.

1.2. **The variational problem.** Let's see how to use a second-order partial differential equation to characterize Yang-Mills connection. Recall that $\mathcal{A}(P)$ is an affine space modelled on $\Omega^1_M(\mathrm{Ad}\,\mathfrak{g})$, so the tangent space to $\mathcal{A}(P)$ at any point is isomorphic to $\Omega^1_M(\mathrm{Ad}\,\mathfrak{g})$.

Given $\omega \in \mathcal{A}(P)$ and $\tau \in C^\infty(M, \Omega^1_M(\mathrm{Ad}\,\mathfrak{g}))$, the directional derivative of Yang-Mills functional at ω in the direction τ is given by

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} YM(\omega + t\tau)$$

And Yang-Mills condition states that this vanishes for all τ. In order to see what this means, firstly we need the following lemma.

Lemma 1.2.1. Given $\omega \in \mathcal{A}(P)$ and $\tau \in C^\infty(M, \Omega^1_M(\mathrm{Ad}\,\mathfrak{g}))$, then

$$F_{\omega+\tau} = F_\omega + \mathrm{d}_\omega \tau + \frac{1}{2}\tau \wedge \tau$$

where d_ω is connection induced by ω on $\Omega^1_M(\mathrm{Ad}\,\mathfrak{g})$.

Proof. On local trivialization U_α one has

$$\begin{aligned}
(F_{\omega+\tau})_\alpha &= \mathrm{d}(A_\alpha + \tau_\alpha) + \frac{1}{2}(A_\alpha + \tau_\alpha) \wedge (A_\alpha + \tau_\alpha) \\
&= (F_\omega)_\alpha + \mathrm{d}\tau_\alpha + \frac{1}{2}(A_\alpha \wedge \tau_\alpha + \tau_\alpha \wedge A_\alpha) + \frac{1}{2}\tau_\alpha \wedge \tau_\alpha \\
&\overset{(1)}{=} (F_\omega)_\alpha + \mathrm{d}\tau_\alpha + A_\alpha \wedge \tau_\alpha + \frac{1}{2}\tau_\alpha \wedge \tau_\alpha \\
&\overset{(2)}{=} (F_\omega)_\alpha + \mathrm{d}_\omega \tau_\alpha + \frac{1}{2}\tau_\alpha \wedge \tau_\alpha
\end{aligned}$$

where

(1) holds from both A_α, τ_α are 1-form valued in $\mathfrak{g}$.
(2) holds from the definition of d_ω.

$\square$

Proposition 1.2.1 (first variation formula)**.** Let ω be a Yang-Mills connection. Then

$$\mathrm{d}_\omega^* F_\omega = 0$$

Proof. A direct computation shows

$$\begin{aligned} YM(\omega + t\tau) &= \int_M \langle F_{\omega+t\tau}, F_{\omega+t\tau}\rangle \,\mathrm{vol} \\ &= \int_M \langle F_\omega + \frac{t^2}{2}(\tau\wedge\tau) + t\mathrm{d}_\omega\tau, F_\omega + \frac{t^2}{2}(\tau\wedge\tau) + t\mathrm{d}_\omega\tau\rangle \,\mathrm{vol} \end{aligned}$$

The coefficient of linear term is

$$\int_M \langle F_\omega, \mathrm{d}_\omega\tau\rangle + \langle \mathrm{d}_\omega\tau, F_\omega\rangle \,\mathrm{vol} = 2\int_M \langle \mathrm{d}_\omega\tau, F_\omega\rangle \,\mathrm{vol}$$

Let d_ω^* be the formal adjoint[1] of d_ω. Then

$$0 = \int_M \langle \mathrm{d}_\omega\tau, F_\omega\rangle \,\mathrm{vol} = \int_M \langle \tau, \mathrm{d}_\omega^* F_\omega\rangle \,\mathrm{vol}$$

holds for arbitrary τ. This shows $\mathrm{d}_\omega^* F_\omega = 0$. □

Definition 1.2.1 (Yang-Mills equations)**.** A connection $\omega \in \mathcal{A}(P)$ is called satisfying Yang-Mills equations, if

$$\begin{cases} \mathrm{d}_\omega F_\omega = 0 \\ \mathrm{d}_\omega^* F_\omega = 0 \end{cases}$$

Remark 1.2.1. In fact, the first equation is exactly the Bianchi identity, which is automatically holds.

Example 1.2.1. In the case that $G = U(1)$, we have that the curvature of a connection A can be identified as a section of Ω_M^2. Indeed, the curvature form takes value in the bundle $\mathrm{Ad}\,\mathfrak{g}$, but here $G = U(1)$ is abelian, thus the adjoint action on $\mathfrak{u}(1)$ is trivial, so

$$\mathrm{Ad}\,\mathfrak{g} = M \times \mathfrak{u}(1) = M \times \mathbb{R}$$

is trivial bundle. Furthermore, ω is a Yang-Mills connection if and only if F_ω is a harmonic 2-form, that is $\Delta F_\omega = 0$, where $\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$. Indeed, thanks to $U(1)$ is abelian again, d_ω can be reduced to d, since for arbitrary form β, we have[2] $\omega\wedge\beta = 0$. This shows the Yang-Mills equations in this case is

$$\begin{cases} \mathrm{d}F_\omega = 0 \\ \mathrm{d}^* F_\omega = 0 \end{cases}$$

[1] In fact, the form adjoint of d_ω can be explicitly written as $\mathrm{d}_\omega^* = (-1)^{2n+1} \star \mathrm{d}_\omega \star$.

[2] This follows from in the definition of wedge product of forms valued in Lie algebra we used Lie bracket, and abelian Lie algebra has trivial Lie bracket.

On the other hand, a form is harmonic if and only if it satisfies above equations since

$$
\begin{aligned}
0 &= \int_M \langle \Delta F_\omega, F_\omega \rangle \,\mathrm{vol} \\
&= \int_M \langle \mathrm{d}\mathrm{d}^* F_\omega, F_\omega \rangle + \langle \mathrm{d}^* \mathrm{d} F_\omega, F_\omega \rangle \,\mathrm{vol} \\
&= \int_M \|\mathrm{d}^* F_\omega\|^2 + \|\mathrm{d} F_\omega\|^2 \,\mathrm{vol}
\end{aligned}
$$

2. Kempf-Ness theorem

Recall that the Yang-Mills functional is gauge invariant, so if a connection ω solves the Yang-Mills equations, so does any gauge transformed $\Phi^*\omega$. In other words, the gauge group acts on $\mathcal{A}_{YM}$, so it's natural to consider the quotient $\mathcal{A}_{YM}/\mathcal{G}$. In general it is infinite dimensional, and the topology of this space may be quite bad (It may be neither Hausdorff nor a smooth manifold).

In this section we will show a baby version of Kempf-Ness theorem, and we will see there is an analogue in the setting of Yang-Mills connection and gauge transformation, which is called Narasimhan-Seshadri theorem, and will be introduced in Section 3.

2.1. **Baby version.**

To get a picture of what to expect, we firstly study a finite dimensional analogue. Consider the following setting:

(1) Let V be a complex vector space with a Hermitian inner product.
(2) Let $S^1 \to \mathrm{U}(V)$ be an action of circle by unitary matrices.
(3) Let $\mathbb{C}^* \to \mathrm{GL}(V)$ be the complexification of this action.

Example 2.1.1. Consider $\lambda \in \mathbb{C}^*$ acting on $\mathbb{C}^2$ by $(x, y) \mapsto (\lambda x, \lambda^{-1} y)$. The orbits are

(1) the conics $xy = c \neq 0$,
(2) the axes $y = 0, x \neq 0$ and $x = 0, y \neq 0$,
(3) the origin.

It's clear to see the quotient topology on the orbit space is non-Hausdorff since the axes come arbitrarily close to the origin. But $\mathbb{C}^2 \setminus \{\text{axes}\}/\mathbb{C}^*$ is homeomorphic to $\mathbb{C}$, and thus Hausdorff.

The reason for the orbit space fail to be Hausdorff is that there exists non-closed sets, so generally we want to form a Hausdorff quotients by considering only closed orbits.

Definition 2.1.1 (stable)**.** A point $v \in V$ is stable if its orbit under $\mathbb{C}^*$ is closed.

Theorem 2.1.1 (Kempf-Ness)**.** A point v is stable if and only if the function $\|\cdot\|^2$ restricted to its orbit attains its minimum.

We can think this function as a function $p_v \colon \mathbb{C}^* \to \mathbb{R}$ given by $p_v(g) = \|g(v)\|^2$. Note that since the norm is $\mathrm{U}(V)$-invariant, the function p_v is S^1-invariant and descends to a function defined on $(\mathbb{C}^*/S^1, \times) \xrightarrow{\log} (\mathbb{R}, +)$.

$$p_v(x) = \|e^x(v)\|^2$$

In Example 2.1.1 one has

$$e^x(v_1, v_2) = (e^{-x}v_1, e^x v_2)$$

Thus

$$p_v(x) = \|v_1\|^2 e^{-2x} + \|v_2\|^2 e^{2x}$$

By taking derivative one has

$$\frac{\mathrm{d}p_v}{\mathrm{d}x} = -2x\|v_1\|^2 e^{-2x} + 2x\|v_2\|^2 e^{2x}$$

If both v_1 and v_2 are non-zero, then it obtains its minimum at

$$\frac{1}{2}\left(\log(\|v_1\|) - \log(\|v_2\|)\right)$$

and if $v = 0$, then it obtains its minimum at $x = 0$. Furthermore, the minimum is not obtained along two punctured axes, so in Example 2.1.1 the stable points are $\{(x, y) \mid xy \neq 0\} \cup \{(0, 0)\}$, and definitely the orbit space of stable points are Hausdorff.

Proof of Theorem 2.1.1.

Lemma 2.1.1. If v is not stable, then p_v does not attains its minimum.

Proof. □

□

To understand the space of stable points it's therefore important to understand the critical points of p_v. Suppose $p_v(x)$ is written as $\sum_{m=1}^{n} \|v_m\|^2 e^{2j_m x}$. Then

$$\frac{\mathrm{d}p_v}{\mathrm{d}x} = 2\sum_{m=1}^{n} j_m \|v_m\|^2 e^{2j_m x}$$

Suppose v is stable and that the minimum occurs at $x = x_0$. Without lose of generality we may assume $x = 0$ since by replacing v by $v' = e^{-x_0}v$ one has

$$\frac{\mathrm{d}p_{v'}}{\mathrm{d}x} = \sum_{m=1}^{n} j_m \|e^{-j_m x_0} v_m\|^2 e^{2j_m x_0} = \sum_{m=1}^{n} j_m \|v_m\|^2$$

Therefore the orbit of a stable vector contains a zero of the function

$$\mu = \sum_{m=1}^{n} j_m \|v_m\|^2$$

In fact it contains a whole S^1 since μ is S^1-invariant, and this shows the following correspondence.

Theorem 2.1.2. Let V^s denote the space of stable vectors under the action of $\mathbb{C}^*$. Then

$$V^s/\mathbb{C}^* = \mu^{-1}(0)/S^1$$

2.2. **Review of symplectic geometry.**

2.2.1. *Symplectic manifold.* Let M be a smooth manifold admitting a Lie group G action, such manifold is often called a G-manifold. There is a one to one correspondence

$$\{\text{action of } \mathbb{R} \text{ on } M\} \longleftrightarrow \{\text{complete vector fields over } M\}$$

given by $\psi \mapsto X_p = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \psi(t,p)$. In particular, let X be an element of Lie algebra $\mathfrak{g}$, there is a complete vector field given by

$$\sigma(X) := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \exp(-tX)p$$

which is called fundamental field of X.

Definition 2.2.1 (symplectic manifold)**.** A symplectic manifold M is an even-dimensional manifold with a non-degenerate closed 2-form ω, which is called symplectic form.

Definition 2.2.2 (symplectomorphic)**.** A diffeomorphism between two symplectic manifolds $f\colon (M,\omega_M) \to (N,\omega_N)$ is called a symplectomorphic if

$$f^*\omega_N = \omega_M$$

Notation 2.2.1. The group consisting of symplectomorphic of (M,ω) is denoted by $\mathrm{Sympl}(M,\omega)$, and it's a subgroup of $\mathrm{Diff}(M)$.

Example 2.2.1 (standard symplectic manifold)**.** For $\mathbb{R}^{2n}$, there is a natural symplectic form given by

$$\omega_{\mathbb{R}^{2n}} = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}y^i$$

$(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$ is called standard symplectic manifold, and it's clear $(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$ is symplectomorphic to $(\mathbb{C}^n, \omega_{\mathbb{C}^n})$, where $\omega_{\mathbb{C}^n} = \sqrt{-1}\sum_{i=1}^n \mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i$.

Theorem 2.2.1 (Darboux)**.** Let (M,ω) be a symplectic $2n$-manifold. For every $x \in M$, there exists a local coordinate $(x^1,\dots,x^n,y^1,\dots,y^n)$, which is sometimes caleed Darboux coordinate, such that

$$\omega = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}y^i$$

that is (M,ω) is locally symplectomorphic to the standard symplectic manifold $(\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$.

Definition 2.2.3 (Hamiltonian vector field)**.** Let f be a smooth function over symplectic manifold (M,ω). The Hamiltonian vector field of f (denoted by X_f) is a vector field defined as follows

$$\mathrm{d}f = \iota_{X_f}\omega$$

Remark 2.2.1 (local form). Suppose $(x^1,\dots,x^n,y^1,\dots,y^n)$ is the Darboux coordinate. Then

$$\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x^i}\mathrm{d}x^i + \sum_{i=1}^n \frac{\partial f}{\partial y^i}\mathrm{d}y^i$$
$$X_f = \sum_{i=1}^n \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i} - \sum_{i=1}^n \frac{\partial f}{\partial y^i}\frac{\partial}{\partial y^i}$$

2.2.2. *Hamiltonian action.*

Definition 2.2.4 (symplectic action)**.** A symplectic action of a Lie group G on a symplectic manifold (M,ω) is a Lie group action which preserves ω.

Remark 2.2.2. If X is the vector field given rise from this action, then it's symplectic if and only if $\mathcal{L}_X\omega = 0$.

Example 2.2.2. Let V be a complex vector space equipped with a Hermitian product $\langle \text{-}, \text{-}\rangle$. A natural symplectic form on V is given by its fundamental form, that is

$$\omega = -\mathrm{Im}\langle \text{-}, \text{-}\rangle$$

and it's clear (V,ω) is symplectomorphic to $(\mathbb{C}^n, \omega_{\mathbb{C}^n})$. If a compact Lie group K acts on V by unitary matrices, then the action of K is symplectic.

For symplectic manifold (M,ω), the non-degeneracy of ω gives an isomorphism $T_pM \to T_pM^*$ for each $p \in M$, so we have the following one to one correspondence

$$C^\infty(M, TM) \longleftrightarrow C^\infty(M, \Omega^1_M)$$
$$X \mapsto \iota_X\omega$$

Cartan's formula says $\mathcal{L}_X\omega = \iota_X\mathrm{d}\omega + \mathrm{d}(\iota_X\omega)$, so by closedness of ω one has $\iota_X\omega$ is closed if and only if $\mathcal{L}_X\omega = 0$. This yields the well-defineness of following definition.

Definition 2.2.5 (symplectic vector field)**.** A vector field X on a symplectic manifold (M,ω) is symplectic if the following equivalent conditions are satisfied

(1) its associated 1-form is closed;
(2) its associated $\mathbb{R}$-action is symplectic;
(3) $\mathcal{L}_X\omega = 0$.

Remark 2.2.3. The symplectic vector field is just like Killing field in Riemannian geometry, and by the same reason one has symplectic vector fields are closed under Lie bracket, since

$$\mathcal{L}_{[X,Y]}\omega = \mathcal{L}_X\mathcal{L}_Y\omega - \mathcal{L}_Y\mathcal{L}_X\omega$$

Definition 2.2.6 (Hamiltonian action)**.** Let G be a Lie group and (M,ω) be a symplectic G-manifold. The action of G is Hamiltonian if there exists a map $\mu\colon M \to \mathfrak{g}^*$ such that

(1) For every $X \in \mathfrak{g}$, if $\mu^X\colon M \to \mathbb{R}$ is given by $\mu^X(p) := \langle \mu(p), X\rangle$, then
$$\iota_{\sigma(X)}\omega = \mathrm{d}\mu^X$$
(2) μ is equivariant with respect to the action of G on M and co-adjoint action of G on $\mathfrak{g}^*$.

The function μ above is called moment map and functions μ^X are called Hamiltonian functions.

Proposition 2.2.1. Let V be a Hermitian vector space equipped symplectic form given by its fundamental form ω and K be a compact Lie group acting on V by unitary matrices. Then the action of K is Hamiltonian.

Proof. If K acts through a group homomorphism $\rho\colon K \to \mathrm{U}(V)$, then the Lie algebra $\mathfrak{k}$ of K acts on V by the differential of ρ. To be explicit, for $\xi \in \mathfrak{k}$
$$\xi v := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\rho(\exp(t\xi))v$$
Now we're going to show the moment map of this action is given by
$$\langle \mu(v), \xi\rangle := \frac{1}{2}\mathrm{Im}\langle v, \xi v\rangle$$
where $v \in V$ and $\xi \in \mathfrak{k}$ as follows:

(1) Direct computation shows
$$\begin{aligned}\mathrm{d}\mu^\xi(v)(w) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathrm{Im}\langle v + tw, \xi v + t\xi w\rangle \\ &= \frac{1}{2}\mathrm{Im}\langle w, \xi v\rangle + \frac{1}{2}\mathrm{Im}\langle v, \xi w\rangle\end{aligned}$$
Note that $\mathfrak{k}$ acts on V as skew-Hermitian matrices, so we have
$$\langle v, \xi w\rangle = -\overline{\langle w, \xi v\rangle}$$
This shows
$$\begin{aligned}\mathrm{d}\mu^\xi(v)(w) &= \frac{1}{2}\mathrm{Im}(\langle w, \xi v\rangle - \overline{\langle w, \xi v\rangle}) \\ &= -\mathrm{Im}\langle \xi v, w\rangle \\ &= \omega_v(\sigma(\xi), w)\end{aligned}$$
(2) To see μ is K-equivariant:
$$\begin{aligned}\langle \mu(gv), \xi\rangle &= \frac{1}{2}\mathrm{Im}\langle \rho(g)v, \mathrm{d}\rho(e)(\xi)\rho(g)v\rangle \\ &= \frac{1}{2}\mathrm{Im}\langle v, \rho(g)^*\mathrm{d}\rho(e)(\xi)\rho(g)v\rangle\end{aligned}$$
Since $\rho(g)$ is unitary, that is $\rho(g)^* = \rho(g)^{-1}$, one has $\rho(g)^*\mathrm{d}\rho(e)(\xi)\rho(g)v = \mathrm{ad}_g(\xi)v$, which impiles
$$\langle \mu(gv), \xi\rangle = \langle \mu(v), \mathrm{ad}_g(\xi)\rho\rangle$$
This completes the proof. $\square$

2.3. **Kempf-Ness theorem.** Consider S^1-action on the Hermitian vector space V which is given by

$$\lambda v = (\lambda^{j_1} v_1, \ldots, \lambda^{j_n} v_n)$$

where $\lambda \in S^1$ and $\lambda_i \in \mathbb{R}$. The Lie algebra of S^1 acts on V as follows:

$$\begin{aligned}\xi v &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{t\xi} v \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (e^{tj_1\xi} v_1, \ldots, e^{tj_n\xi} v_n) \\ &= (j_1 v_1, \ldots, j_n v_n)\end{aligned}$$

where $\xi \in \mathbb{R}$. Then by Proposition 2.2.1 the moment map of this action is given by

$$\langle \mu(v), \xi \rangle = \frac{1}{2}\mathrm{Im}\langle v, \xi v\rangle = \frac{1}{2}\sum_{m=1}^{n} j_m \|v_m\|^2$$

and that's exactly the function μ we encounter in section 2.1. Thus Theorem 2.1.2 can be restated as follows.

Theorem 2.3.1. Let V be a Hermitian vector space and $S^1 \to \mathrm{U}(V)$ be a circle action by unitary matrices with complexification $\mathbb{C}^* \to \mathrm{GL}(V)$. Suppose V^s is the set of stable points under $\mathbb{C}^*$-action and μ is the moment map of S^1-action. Then

$$V^s / \mathbb{C}^* = \mu^{-1}(0)/S^1$$

3. Narasimhan-Seshadri theorem

3.1. **The moment map in Yang-Mills theory.** Let M be a Riemann surface and P be a principal G-bundle over M. Then the space of connections $\mathcal{A}(P)$ has a natural symplectic form.

Proposition 3.1 (Atiyah-Bott)**.** The following bilinear form

$$Q(\alpha,\beta)=\int_M \alpha\wedge\beta$$

where $\alpha,\beta\in C^\infty(M,\Omega^1_M(\operatorname{Ad}P))$, is a symplectic form defined on $\mathcal{A}(P)$.

Proof. Let's check step by step:

(1) It's clear that ω is a 2-form, since $\mathcal{A}(P)$ is affine modelled on $C^\infty(M,\Omega^1_M(\operatorname{Ad}P))$.
(2)
(3)

□

Lemma 3.1.1. For $\phi\in C^\infty(M,\Omega^0_M(\operatorname{Ad}P))$ and connection ∇, $\nabla\phi$ is the Hamiltonian vector field of function $f\colon\nabla\to-\int_M F_\nabla\wedge\phi$.

Proof. By definition we need to check

$$\mathrm{d}f=\iota_{\nabla\phi}Q$$

For arbitrary $\tau\in C^\infty(M,\Omega^1_M(\operatorname{Ad}P))$, integration by parts shows

$$\begin{aligned}Q(\nabla\phi,\tau)&=\int_M\nabla\phi\wedge\tau\\&=-\int_M\phi\wedge\nabla\tau\\&=-\int_M\nabla\tau\wedge\phi\end{aligned}$$

Since $F_{\nabla+\varepsilon\tau}=F_\nabla+\varepsilon\nabla\tau+O(\varepsilon^2)$, one has

$$\begin{aligned}\mathrm{d}f(\tau)&=\lim_{\varepsilon\to0}\frac{-\int_M F_{\nabla+\varepsilon\tau}\wedge\phi+\int_M F_\nabla\wedge\phi}{\varepsilon}\\&=-\int_M\nabla\tau\wedge\phi\end{aligned}$$

This completes the proof. □

Remark 3.1.1. In our case the $(\operatorname{Lie}\mathcal{G})^*=C^\infty(M,\Omega^2_M(\operatorname{Ad}P))$ and the moment map is just

$$\nabla\mapsto-F_\nabla$$

The Yang-Mills functional is just the norm of the moment map.

3.2. **Complexifying the action of gauge group.** Let M be a Riemann surface, our ultimate goal is to relate moduli spaces of holomorphic vector bundles over M to Yang-Mills connections. Firstly, we want to consider $\mathcal{A}(P)$ as a space of holomorphic vector bundles.

Definition 3.2.1 (holomorphic vector bundle)**.** A holomorphic vector bundle is a complex vector bundle $\pi\colon E \to X$ such that the total space E is a complex manifold and π is holomorphic.

Proposition 3.2. If P is a principal $\mathrm{U}(n)$-bundle over a Riemann surface M and ∇ is a $\mathrm{U}(n)$-connection then $\mathrm{Ad}(P)$ inherits the structure of a holomorphic vector bundle over M such that $\nabla^{0,1} = \overline{\partial}$.

4. Stability of holomorphic vector bundles

In thise section, the guiding problem is to classify holomorphic vector bundles on a Riemann surface with genus g, denoted by Σ_g. For the case $g = 0, 1$, there are complete classification results for holomorphic vector bundles on Σ_g, due to Grothendieck for the case of the Riemann sphere [Gro57], and due to Atiyah for the case of elliptic curves [Ati57]. So in the following discussion, we always assume $g \geq 2$.

4.1. **Stable bundle.**

Definition 4.1.1 (degree)**.** Let $\pi\colon E \to X$ be a holomorphic vector bundle, its degree is defined as

$$\deg(E) := \int_X c_1(E)$$

where $c_1(E) \in H^2(X, \mathbb{Z})$ is the first Chern class of E.

Definition 4.1.2 (slope)**.** Let $\pi\colon E \to X$ be a holomorphic vector bundle, its slope is defined as

$$\mu(E) := \frac{\deg(E)}{\mathrm{rk}(E)}$$

Remark 4.1.1. One thing to note is that the slope of a holomorphic vector bundle is independent of the holomorphic structure, since both the degree and rank are topological invariants.

Definition 4.1.3 (slope stability)**.** Let $\pi\colon E \to X$ be a holomorphic vector bundle, it's

(1) stable if for every non-trivial holomorphic subbundle F, $\mu(F) < \mu(E)$;
(2) semi-stable if for every non-trivial holomorphic subbundle F, $\mu(F) \leq \mu(E)$;
(3) unstable if it's not semi-stable.

Remark 4.1.2.

(a) It's clear that all holomorphic line bundles are stable, since they don't have non-trivial subbundles;
(b) A semi-stable vector bundle with coprime rank and degree is actually stable, since
(c) While the slope is a topological invariant, slope stability is not, since here we only consider holomorphic subbundles, which depends on the holomorphic structure.

Proposition 4.1.1. Let $E \to \Sigma_g$ be a holomorphic vector bundle.

(1) It's stable if and only if for every non-trivial holomorphic subbundle F, $\mu(E/F) > \mu(E)$;
(2) It's semi-stable if and only if for every non-trivial holomorphic subbundle F, $\mu(E/F) \geq \mu(E)$.

Proof. Denote r, r', r'' the ranks of $E, F, E/F$ respectively, and d, d', d'' their degrees respectively. From exact sequence
$$0 \to E \to E \to E/F \to 0$$
one has $r = r' + r''$ and $d = d' + d''$, thus
$$\frac{d'}{r'} < \frac{d'+d''}{r'+r''} \iff \frac{d'}{r'} < \frac{d''}{r''} \iff \frac{d'+d''}{r'+r''} < \frac{d''}{r''}$$
and likewise with the case semi-stable. □

A philosophy is that semi-stable bundles don't admit too many subbundles, since any subbundle they may have is of slope no greater than their own. This turns out to have many interesting consequences we're going to show, for example, the category of semi-stable bundles is abelian.

Lemma 4.1.1. If $\varphi\colon E \to E'$ is a non-zero homomorphism of holomorphic vector bundles over Σ_g, then
$$\mu(E/\ker\varphi) \leq \mu(\operatorname{im}\varphi)$$

Proposition 4.1.2. Let E, E' be two semi-stable bundles such that $\mu(E) > \mu(E')$. Then any homomorphism $\varphi\colon E \to E'$ is zero.

Proof. If φ is non-zero, since E is semi-stable, then
$$\mu(\operatorname{im}\varphi) \overset{(1)}{\geq} \mu(E/\ker\varphi) \overset{(2)}{\geq} \mu(E) > \mu(E')$$
where

(1) holds from Lemma 4.1.1;
(2) holds from Proposition 4.1.1.

which contradicts to the semi-stablity of E'. □

Proposition 4.1.3. Let $\varphi\colon E \to E'$ be a non-zero homomorphism of semi-stable holomorphic of slope μ. Then $\ker\varphi$ and $\operatorname{im}\varphi$ are semi-stable bundles of slope μ, and the natural map $E/\ker\varphi \to \operatorname{im}\varphi$ is an isomorphism.

Corollary 4.1.1. The category of semi-stable bundles of slope μ is abelian, and the simple object[3] in this category is the stable bundles of slope μ.

Proof. By Proposition 4.1.3 one has the category of semi-stable bundles of slope μ is abelian. A stable bundle E is simple in this category, since it admits no non-trivial subbundles with slope μ; Conversely, if a semi-stable bundle E is simple, then any non-trivial subbundle F satisfies $\mu(F) \leq \mu(E)$ since E is semi-stable and $\mu(F) \neq \mu(E)$ since E is simple, this shows E is stable. □

Proposition 4.1.4. Let E, E' be two stable vector bundles over Σ_g with same slopes and $\varphi\colon E \to E'$ be a non-zero homomorphism. Then φ is an isomorphism.

[3]Recall a simple object in an abelian category is an object with no non-trivial subobject.

Proof. Since $\varphi\colon E \to E'$ is a non-zero homomorphism between stable bundles with same slopes, then by Proposition 4.1.3 one has $\ker\varphi$ is either 0 or has slope $\mu(E)$, but E is actually stable, then $\ker\varphi$ must be 0, and since φ is strict, this shows φ is injective. Likewise, $\operatorname{im}\varphi \neq 0$ and has slope $\mu(E')$, then it must be E' since E' is stable. Then again by φ is strict, $\operatorname{im}\varphi = E'$ impiles φ is surjective. Therefore φ is an isomorphism. □

Proposition 4.1.5. If E is a stable bundle over Σ_g, then $\operatorname{End} E = \mathbb{C}$. In particular, $\operatorname{Aut} E = \mathbb{C}^*$.

Proof. Let φ be a non-zero endomorphism of E, by Proposition 4.1.4 one has φ is an automorphism, so $\operatorname{End} E$ is a field, which contains $\mathbb{C}$ as its subfield of scalar endomorphisms. For any $\varphi \in \operatorname{End} E$, by Cayley-Hamilton theorem one has φ is algebraic over $\mathbb{C}$, and since $\mathbb{C}$ is algebraic closed, this shows $\operatorname{End} E \cong \mathbb{C}$. □

Corollary 4.1.2. A stable bundle is indecomposable, that is it's not isomorphic to a direct sum of non-trivial subbundles.

Proof. The automorphism group of $E = E_1 \oplus E_2$ contains $\mathbb{C}^* \times \mathbb{C}^*$, so by Proposition 4.1.5 it can't be stable. □

Theorem 4.1.1 (Jordan-Hölder filteration)**.** Any semi-stable bundle of slope μ over Σ_g admits a filteration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_k = E$$

by holomorphic subbundles such that for each $1 \leq i \leq k$, one has

(1) E_i/E_{i-1} is stable;
(2) $\mu(E_i/E_{i-1}) = \mu(E)$.

Proposition 4.1.6 (Seshadri)**.** Any two Jordan-Hölder filterations

$$S : 0 = E_0 \subset E_1 \subset \cdots \subset E_k = E$$

and

$$S' : 0 = E'_0 \subset E'_1 \subset \cdots \subset E'_l = E$$

of a semi-stable bundle E have same length, and the associated graded objects

$$\operatorname{gr}(S) : 0 = E_1/E_0 \oplus \cdots \oplus E_k/E_{k-1}$$

and

$$\operatorname{gr}(S') : 0 = E'_1/E'_0 \oplus \cdots \oplus E'_k/E'_{k-1}$$

satisfy $E_i/E_{i-1} \cong E'_i/E'_{i-1}$ for all $1 \leq i \leq k$.

Definition 4.1.4 (poly-stable bundle)**.** A holomorphic vector bundle E over Σ_g is called poly-stable if it is isomorphic to a direct sum $E_1 \oplus \cdots \oplus E_k$ of stable bundles of the same slope.

Example 4.1.1. A stable bundle is poly-stable.

Example 4.1.2. The graded object associated to any Jordan-Hölder filteration of a semi-stable bundle E is a poly-stable, and by Proposition 4.1.6, it's unique up to isomorphism, this isomorphic class is denoted by $\mathrm{gr}(E)$.

Definition 4.1.5 (S-equivalence class)**.** The graded isomorphism class $\mathrm{gr}(E)$ associated to a semi-stable bundle E is called the S-equivalence class of E. If $\mathrm{gr}(E) \cong \mathrm{gr}(E')$, E and E' are called S-equivalent, and denoted by $E \sim_S E'$.

Definition 4.1.6. The set $\mathcal{M}_{\Sigma_g}(r,d)$ of S-equivalence classes of semi-stable bundles of rank r and degree d over Σ_g is called its moduli set, it contains the set $\mathcal{N}_{\Sigma_g}(r,d)$ of isomorphism classes of stable bundles of rank r and degree d.

Theorem 4.1.2 (Mumford-Seshadri)**.** Let $g \geq 2, r \geq 1$ and $d \in \mathbb{Z}$.

(1) The set $\mathcal{N}_{\Sigma_g}(r,d)$ admits a structure of smooth, complex quasi-projective variety of dimension $r^2(g-1)+1$;
(2) The set $\mathcal{M}_{\Sigma_g}(r,d)$ admits a structure of complex projective variety of dimension $r^2(g-1)+1$;
(3) $\mathcal{N}_{\Sigma_g}(r,d)$ is an open dense subvariety of $\mathcal{M}_{\Sigma_g}(r,d)$.

In particular, when r and d are coprime, $\mathcal{M}_{\Sigma_g}(r,d) = \mathcal{N}_{\Sigma_g}(r,d)$ is a smooth complex projective variety.

Proof. See [Mum62] and [Ses67]. □

4.2. **The Harder-Narasimhan filteration.**

Theorem 4.2.1 (Harder-Narasimhan)**.** Any holomorphic vector bundle E over Σ_g has a unique filteration

$$0 = E_0 \subset E_1 \subset \dots E_k = E$$

by holomorphic subbundles such that

(1) for all $1 \leq i \leq k$, E_i/E_{i-1} is semi-stable;
(2) the slope $\mu_i := \mu(E_i/E_{i-1})$ of successive quotients satisfies

$$\mu_1 > \mu_2 > \cdots > \mu_k$$

This filteration is called Harder-Narasimhan filteration.

Proof. See [HN75]. □

Remark 4.2.1. If we denote $r = \operatorname{rk} E, d = \deg E, r_i = \operatorname{rk}(E_i/E_{i-1})$ and $d_i = \deg(E_i/E_{i-1})$, one has

$$r_1 + \cdots + r_k = r, \quad d_1 + \cdots + d_k = d$$

The k-tuple

$$\vec{\mu} := (\underbrace{\mu_1, \dots, \mu_1}_{r_1 \text{ times}}, \dots, \underbrace{\mu_k, \dots, \mu_k}_{r_k \text{ times}})$$

is called the Harder-Narasimhan type of E. It's equivalent to the data of the k-tuple $(r_i, d_i)_{1\leq i\leq k}$. In the plane of coordinates (r, d), the polygonal line

$$P_{\vec{\mu}} := \{(0,0), (r_1, d_1), (r_1 + r_2, d_1 + d_2), \dots, (r_1 + \cdots + r_k, d_1 + \cdots + d_k)\}$$

defines a convex polygon called the Harder-Narasimhan polygon of E. The slope of the line from $(0,0)$ to (r_1, d_1) is μ_1, that is the slope of E_1/E_0, and perhaps that's why it's called slope. It's indeed convex, since $\mu_1 > \cdots > \mu_k$. A vector bundle is semi-stable if and only if it is it own Harder-Narasimhan filteration, and if and only if its Harder-Narasimhan filteration is a single line from $(0,0)$ to (r, d).

5. Narasimhan-Seshadri theorem

6. G-equivariant cohomology

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