TOPCIS IN COMPLEX ALGEBRAIC GEOMETRY

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0. Introduction

In this lecture, the object we're most interested in is the complex algebraic variety.

Definition 0.1 (complex algebraic variety)**.** A complex algebraic variety is a quasi-projective[1] variety over $\mathbb{C}$.

Definition 0.2 (non-singular)**.** A complex algebraic variety X is non-singular if $\Omega_{X/\mathbb{C}}$ is locally free.

Given any non-singular projective complex variety X, one can show that $X \subseteq \mathbb{CP}^n$ is a submanifold by using inverse function theorem. Conversely, Chow showed that

Theorem 0.1 (Chow)**.** Any compact complex submanifold[2] of $\mathbb{CP}^n$ must be a complex variety.

Chow's theorem implies that there is a deep connection between complex manifolds and complex varieties, and thus techniques from complex differential geometry may be used to solve some questions in algebraic geometry, such as corollaries of Calabi-Yau theorem. On the other hand, motivated by Chow's theorem, it's natural to ask whether a compact complex manifold can be (holomorphically) embedded into $\mathbb{CP}^n$ or not.

Theorem 0.2 (Riemann)**.** Any compact Riemann surface can be embedded into $\mathbb{CP}^n$.

Theorem 0.3 (Kodaira)**.** A compact complex manifold with a positive holomorphic line can be embedded into $\mathbb{CP}^n$.

Remark 0.1. In fact, Riemann's result can be obtained from Kodaira's embedding. Given a Hermitian holomorphic line bundle (L, h), its Chern curvature $\sqrt{-1}\Theta_h$ represents the first Chern class $c_1(L)$, and $\partial\overline{\partial}$-lemma shows that any real $(1,1)$-form $\sqrt{-1}\chi$ which represents $c_1(L)$ can be realized as the Chern curvature of some Hermitian metric h. Thus if we want to see whether a holomorphic line bundle is positive or not, it suffice to compute its first Chern class, and there always exists holomorphic line with positive first Chern class[3].

Another important conception in complex differential geometry is Kähler. The Kähler manifold lies in the intersection of Riemannian manifold, complex manifold and symplectic manifold, and has many elegant properties.

[1] X is a projective variety if it's the zero-locus of (some) finite family of homogeneous polynomials, that generate a prime ideal, and it's called quasi-projective if it's an open subset of a projective variety.

[2] In fact, "submanifold" can be replaced by analytic subvariety. In other words, we allow singularities.

[3] For holomorphic line bundle L over Riemann surface, the "positivity" of first Chern class is determined by its degree, that is, holomorphic line bundle with positive degree has positive first Chern class.

Theorem 0.4 (Hodge)**.** Let (X, ω) be a compact Kähler manifold. Then there is a decomposition

$$H^n(X) \cong \bigoplus_{p+q=n} H^{p,q}(X),$$

where $H^{p,q}(X)$ is the Dolbeault cohomology of X.

Remark 0.2. The Hodge decomposition is independent of the choice of Kähler form ω, but for the proof, we need to use theory of harmonic forms and Kähler identities.

References

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