TOPCIS IN COMPLEX ALGEBRAIC GEOMETRY

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Contents

0. Preface

0.1. **Introduction.** In this lecture, the object we're most interested in is the complex variety.

Definition 0.1 (complex variety)**.** A complex algebraic variety or simply a complex variety is a quasi-projective[1] variety over $\mathbb{C}$.

Definition 0.2 (non-singular)**.** A complex variety X is non-singular if the sheaf of Kähler differentials $\Omega_{X/\mathbb{C}}$ is locally free.

Given any non-singular projective complex variety X, one can show that $X \subseteq \mathbb{CP}^n$ is a submanifold by using inverse function theorem. Conversely, Chow showed that

Theorem 0.1 (Chow)**.** Any compact complex submanifold[2] of complex projective space must be a complex variety.

Chow's theorem implies that there is a deep connection between complex manifolds and complex varieties, and thus techniques from complex differential geometry may be used to solve some questions in algebraic geometry, such as corollaries of Calabi-Yau theorem. On the other hand, motivated by Chow's theorem, it's natural to ask whether a compact complex manifold can be (holomorphically) embedded into complex projective space or not.

Theorem 0.2 (Riemann)**.** Any compact Riemann surface can be embedded into $\mathbb{CP}^N$.

Theorem 0.3 (Kodaira)**.** A compact complex manifold with a positive holomorphic line can be embedded into $\mathbb{CP}^N$.

Remark 0.1. In fact, Riemann's result can be obtained from Kodaira's embedding. Given a Hermitian holomorphic line bundle (L, h), its Chern curvature $\sqrt{-1}\Theta_h$ represents the first Chern class $c_1(L)$, and $\partial\overline{\partial}$-lemma shows that any real $(1,1)$-form which represents $c_1(L)$ can be realized as the Chern curvature of some Hermitian metric h. Thus if we want to see whether a holomorphic line bundle is positive or not, it suffice to compute its first Chern class, and there always exists holomorphic line with positive first Chern class[3].

The Kähler manifold is an important object in the complex differential geometry, which lies in the intersection of Riemannian manifold, complex

[1] The set $X \subseteq \mathbb{CP}^n$ is a projective variety if it's the zero-locus of some (finite) family of homogeneous polynomials, that generate a prime ideal, and it's called quasi-projective if it's an open subset of a projective variety.

[2] In fact, "submanifold" can be replaced by analytic subvariety, that is, we allow some singularities.

[3] For holomorphic line bundle L over Riemann surface, the "positivity" of first Chern class is determined by its degree, that is, holomorphic line bundle with positive degree has positive first Chern class.

manifold and symplectic manifold, and has many elegant properties. One of the most profound results is the Hodge decomposition.

Theorem 0.4 (Hodge)**.** Let (X, ω) be a compact Kähler manifold. Then there is a decomposition

$$H^n(X) \cong \bigoplus_{p+q=n} H^{p,q}(X),$$

where $H^{p,q}(X)$ is the Dolbeault cohomology of X.

Remark 0.2. The Hodge decomposition is independent of the choice of Kähler form ω, but for the proof, we need to use theory of harmonic forms and Kähler identities.

The Hodge decomposition has lots of consequences in algebraic geometry. Let X be a non-singular projective complex variety. The algebraic de Rham complex is defined by

$$\Omega^\bullet_{X/\mathbb{C}} \colon \mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega_{X/\mathbb{C}} \xrightarrow{\mathrm{d}} \ldots \xrightarrow{\mathrm{d}} \Omega^n_{X/\mathbb{C}},$$

where $n = \dim X$, and the algebraic de Rham cohomology is defined by the hypercohomology of above complex as follows

$$H^k_{alg}(X) = \mathbb{H}^k(\Omega^\bullet_{X/\mathbb{C}}),$$

where $k \in \mathbb{Z}_{\geq 0}$. Note that there is a natural filteration on algebraic de Rham complex

$$\Omega^\bullet_{X/\mathbb{C}} = F^0\Omega^\bullet_{X/\mathbb{C}} \supseteq F^1\Omega^\bullet_{X/\mathbb{C}} \supseteq \cdots \supseteq F^n\Omega^\bullet_{X/\mathbb{C}} = \{0\},$$

where

$$F^p\Omega^\bullet_{X/\mathbb{C}} \colon 0 \to \cdots \to 0 \to \Omega^p_{X/\mathbb{C}} \to \cdots \to \Omega^n_{X/\mathbb{C}}.$$

This filteration gives the Hodge to de Rham spectral sequence.

Theorem 0.5 (E_1-degeneration)**.** Let X be a non-singular projective complex variety. The Hodge to de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_{X/\mathbb{C}}) \Longrightarrow H^{p+q}_{alg}(X)$$

degenerates at E_1-page, and

$$\dim_{\mathbb{C}} H^p(X, \Omega^q_{X/\mathbb{C}}) = \dim_{\mathbb{C}} H^q(X, \Omega^p_{X/\mathbb{C}}).$$

Remark 0.3. The inequality

$$\dim H^k_{alg}(X) \leq \sum_{p+q=k} H^q(X, \Omega^p_{X/\mathbb{C}})$$

always holds, and the equality holds if and only if the Hodge to de Rham spectral sequence degenerates at E_1-page.

There are several important developments in Kähler geometry after Hodge and Kodaira, such as the solution to Calabi conjecture given by Shing-Tung Yau, and the connection between stable vector bundles and Hermitian-Yang-Mills metrics proved by Uhlenbeck-Yau.

Theorem 0.6 (Calabi-Yau)**.** Let (X,ω) be a compact Kähler manifold and χ be a real $(1,1)$-form that represents the first Chern class. Then there exists a unique $\omega_h \in [\omega]$ such that $\mathrm{Ric}(\omega_h) = \chi$.

Corollary 0.1. There exists a unique Ricci-flat Kähler metric on compact Kähler manifold with vanishing first Chern class.

Now let's introduce some algebraic consequence of Calabi-Yau theorem.

Theorem 0.7. Let X be a non-singular projective complex variety with ample canonical bundle K_X. Then

$$(-1)^n\left(c_1^n(X) - \frac{2(n+1)}{n}c_1^{n-2}(X)c_2(X)\right) \leq 0.$$

Moreover, the equality holds if and only if X is a locally symmetric variety of ball type.

Corollary 0.2. If X is a locally symmetric variety of ball type, then X^σ is again a locally symmetric variety of ball type for any $\sigma \in \mathrm{Aut}(\mathbb{C})$.

Theorem 0.8. Let X be a non-singular projective complex variety with $c_1(X) = 0$. Then for any ample line bundle L on X,

$$c_2(X) \cdot L^{n-2} \geq 0.$$

Moreover, the equality holds if and only if X is an abelian variety.

To state Uhlenbeck-Yau's theorem, we need the following preparations.

Definition 0.3 (slope)**.** Let (X,ω) be a compact Kähler and E be a holomorphic vector bundle. The slope of E with respect to ω is defined by

$$\mu_\omega(E) = \frac{\deg_\omega(E)}{\mathrm{rk}\, E},$$

where

$$\deg_\omega(E) = \int_X c_1(E) \cdot \omega^{n-1}.$$

Definition 0.4 (stability)**.** Let (X,ω) be a compact Kähler and E be a holomorphic vector bundle.

(1) E is μ_ω-stable if for all subbundle $F \subseteq E$, one has

$$\mu_\omega(F) < \mu_\omega(E).$$

(2) E is μ_ω-semistable if for all subbundle $F \subseteq E$, one has

$$\mu_\omega(F) \leq \mu_\omega(E).$$

Definition 0.5 (Hermitian-Yang-Mills metric)**.** Let (X,ω) be a compact Kähler and E be a holomorphic vector bundle. A Hermitian metric h on E is called Hermitian-Yang-Mills if

$$\wedge_\omega \Theta_h = \lambda\, \mathrm{id}_E,$$

where $\lambda \in \mathbb{R}$.

Theorem 0.9 (Uhlenbeck-Yau)**.** Let (X, ω) be a compact Kähler and E be a μ_ω-stable holomorphic bundle. Then there exists a unique Hermitian-Yang-Mills metric on E.

It also has lots of algebraic consequences.

Theorem 0.10. Let X be a non-singular projective complex variety and H be a line bundle. Let E be a μ_H-semistable vector bundle. Then

$$\left(c_1^2(E) - \frac{\mathrm{rk}(E)+1}{\mathrm{rk}(E)} c_2(E)\right) \cdot H^{n-2} \geq 0$$

Theorem 0.11. Let X be a non-singular projective complex variety and H be a line bundle. Let E be a μ_H-semistable vector bundle. Then

$$H^p(X, \Omega_X^q \otimes E \otimes H) = 0$$

for all $p + q > \dim X$.

In particular, above vanishing theorem generalizes the classical Kodaira vanishing theorem, which is a consequence of Hodge theory.

0.2. **Outlines.**

0.2.1. *Part I: Hodge theory.*

(1) Existence of harmonic forms.
(2) Kähler condition and Hodge package.
(3) Kodaira's vanishing theorem.
(4) Cartier descent theorem.
(5) De Rham decomposition theorem of Deligne-Illusie's theorem.
(6) Hodge symmetry.

0.2.2. *Part II: Non-abelian Hodge theory.*

(1) Existence of Hermitian-Yang-Mills metrics.
(2) Higgs bundle and the variant.
(3) Non-abelian Hodge theory.
(4) Ogus-Vologodsky theorem.
(5) Higgs-de Rham flow.

Part 1. Hodge theory

1. Existence of harmonic forms

Let (M, g) be an oriented compact Riemannian manifold. Then there is a L^2-metric on $\mathcal{A}^n(M) := \Gamma(M, \Omega^n_M)$ defined by

$$\begin{aligned}(\text{-},\text{-})_{L^2} \colon \mathcal{A}^n(M) \times \mathcal{A}^n(M) &\to \mathbb{C} \\ (\omega, \tau) &\mapsto \int_X \langle \omega, \tau \rangle_g \operatorname{vol}_g .\end{aligned}$$

The vector space of harmonic forms is defined by

$$\mathcal{H}^n(M) := \{\omega \in \mathcal{A}^n(M) \mid \Delta_g(\omega) = 0\}.$$

Remark 1.1. A differential form ω is harmonic if and only if $\mathrm{d}\omega = 0$ and $\mathrm{d}^*\omega = 0$.

Theorem 1.1 (Hodge)**.** Let (M, g) be an oriented compact Riemannian manifold. Then $\mathcal{H}^n(M) \cong H^n_{dR}(M)$.

Remark 1.2. The Hodge theorem gives a split of following exact sequence.

$$\begin{tikzcd} 0 \arrow[r] & B^n(M) \arrow[r] & Z^n(M) \arrow[r] & H^n_{dR}(M) \arrow[r] & 0, \\ & & \mathcal{H}^n(M) \arrow[u, hook] \arrow[ur, "\cong"] & & \end{tikzcd}$$

and one of advantages of above split is that we can regard elements of de Rham cohomology as a closed forms with certain properties, not an equivalent class.

1.1. Elliptic operators.

Let E, F be smooth complex vector bundles over a smooth manifold M of dimension d.

1.1.1. *Differential operators.*

Definition 1.1.1 (differential operator)**.** A differential operator of order k is a $\mathbb{C}$-linear map

$$P \colon \Gamma(M, E) \to \Gamma(M, F)$$

such that on every local trivialization U of E, F with local coordinate $\{x^1, \dots, x^d\}$, the map P is given by a matrix (p_{ij}), where

$$p_{ij} = \sum_{|I| \leq k} P_{I,ij} \frac{\partial}{\partial x^I},$$

and $P_{I,ij} \in \mathcal{A}^0(U)$.

Notation 1.1.1. The set of all differential operators from E to F of order k is denoted by $\mathrm{Diff}_k(E, F)$, and the set of all differential operators from E to E of order k is denoted by $\mathrm{Diff}_k(E)$ for convenience.

Example 1.1.1. Let M be a smooth manifold of dimension 1 and $E = F = M \times \mathbb{C}$ be the trivial bundle of rank one. For convenience, we assume the trivialization of M is given by two charts as follows

$$\begin{aligned} x\colon U &\to \mathbb{R} \\ y\colon V &\to \mathbb{R}\,. \end{aligned}$$

Let P be a differential operator of order k locally given by ∂_x^k on the trivialization U. By chain rule and Leibniz's rule one has

$$\partial_y^k \equiv \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^k \partial_x^k \quad \text{(mod lower order terms)}.$$

Remark 1.1.1. More generally, let $P \in \operatorname{Diff}_k(E, F)$ be a differential operator of order k, which is locally given by the matrix (p_{ij}). If we define

$$P_{ij}^k = \sum_{|I|=k} P_{I,ij} \frac{\partial}{\partial x^I}.$$

Then by chain rule and Leibniz's rule one can see the coefficients of matrix P^k are transformed like the sections of the $\operatorname{Sym}^k TM$ and similarly by a change of trivialization of the bundles E and F, the matrix transforms like a section of $\operatorname{Hom}(E, F)$. It gives a section of $\operatorname{Sym}^k TM \otimes \operatorname{Hom}(E, F)$, which is called the **symbol** of P, and denoted by σ_P.

Definition 1.1.2. A differential operator $P \in \operatorname{Diff}_k(E, F)$ is said to be **elliptic** if for all $x \in M$ and $0 \neq \omega \in \Omega_{M,x}$, the homomorphism

$$\sigma_P(\omega)\colon E_x \to F_x$$

is an isomorphism.

1.1.2. *Adjoint operator.* Let (M, g) be an oriented compact Riemannian manifold and $P \in \operatorname{Diff}_k(E, F)$ be a differential operator from smooth vector bundles E to F. A **formal adjoint** of P, denoted by P^*, is a differential operator from F to E such that

$$(\alpha, P\beta)_{L^2} = (P^*\alpha, \beta)_{L^2}$$

holds for all $\alpha \in \Gamma(M, F)$ and $\beta \in \Gamma(M, E)$. The symbol of P^* equals to the adjoint of the symbol of P, that is,

$$\sigma_{P^*,\omega} = (\sigma_{P,\omega})^*.$$

In particular, if E and F are of equal rank, then P is elliptic if and only if its adjoint is elliptic.

Definition 1.1.3. Let $P \in \operatorname{Diff}_k(E)$ be a differential operator. Then P is called **self-adjoint** if $P = P^*$.

Example 1.1.2. The Laplacian operator $\Delta_{\mathrm{d}} = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$ is a self-adjoint elliptic operator of order 2.

1.1.3. *Fundamental decomposition theorem for self-adjoint elliptic operator.* Let (X, g) be a Riemannian manifold and E be a Hermitian vector bundle on (X, g).

Theorem 1.1.1. Let $L \in \mathrm{Diff}_k(E)$ be self-adjoint and elliptic. Then there exist linear mappings $H_L, G_L\colon \Gamma(X, E) \to \Gamma(X, E)$ such that

(1) $H_L(\Gamma(X, E)) = \mathcal{H}_L$, and $\dim \mathcal{H}_L < \infty$, where
$$\mathcal{H}_L = \{\alpha \in \Gamma(X, E) \mid L\alpha = 0\}.$$
(2) $L \circ G_L + H_L = G_L \circ L + H_L = \mathrm{id}_E$.
(3) The following decomposition is orthogonal with respect to L^2-norm
$$\begin{aligned}\Gamma(X, E) &= \mathcal{H}_L \oplus G_L \circ P(\Gamma(X, E)) \\ &= \mathcal{H}_L \oplus L \circ G_L(\Gamma(X, E)).\end{aligned}$$

Proof. See Theorem 4.12 in [Wel80]. □

Remark 1.1.2. The above theorem was first proved by Hodge for the case $E = \Omega^p_M$ and L is the Laplacian operator with respect to the Riemannian metric on (X, g). In this case, $\mathcal{H}_L$ is exactly the vector spaces of harmonic forms

1.1.4. *Elliptic complex and generalized Laplacian operator.* In this section, we study a generalization of elliptic operators to be called elliptic complexes. Suppose there is a sequence of differential operators,

$$\Gamma(M, E_0) \xrightarrow{L_0} \Gamma(M, E_1) \xrightarrow{L_1} \Gamma(M, E_2) \xrightarrow{L_2} \dots \xrightarrow{L_{N-1}} \Gamma(M, E_N), \tag{1.1}$$

where L_i is a differential operator of order k such that $L_i \circ L_{i-1} = 0$ and $E_0, E_1, \dots, E_N$ are a sequence of smooth complex vector bundles on smooth manifold M.

Definition 1.1.4. The complex (1.1) is called a **elliptic complex** if for all $x \in X$ and $0 \neq \omega \in \Omega_{M,x}$, the sequence

$$(E_0)_x \xrightarrow{\sigma_{L_0}(\omega)} (E_1)_x \xrightarrow{\sigma_{L_1}(\omega)} (E_2)_x \xrightarrow{\sigma_{L_2}(\omega)} \dots \xrightarrow{\sigma_{L_{N-1}}(\omega)} (E_{N-1})_x$$

is exact.

Example 1.1.3.

$$\mathcal{A}^0(M) \xrightarrow{\mathrm{d}} \mathcal{A}^1(M) \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \mathcal{A}^n(M)$$

is an elliptic complex.

Definition 1.1.5. The **Laplacian operator** of the complex (1.1) is defined by

$$\Delta_i = L_i^* \circ L_i + L_{i-1} \circ L_{i-1}^*,$$

which is a differential operator from E_i to E_i.

Lemma 1.1. Let U, V, W be finite-dimensional vector spaces. If we have a diagram of finite-dimensional vector spaces

$$\begin{tikzcd} U \arrow[r, "A"] \arrow[d, no head] & V \arrow[r, "B"] \arrow[d, no head] & W \arrow[d, no head] \\ U & V \arrow[l, "A^*"] & W, \arrow[l, "B^*"] \end{tikzcd}$$

which is exact at V. Then

(1) $V = \operatorname{im} A \oplus \operatorname{im} B^*$.
(2) AA^* is injective on $\operatorname{im} A$ and is zero on $\operatorname{im} B^*$.
(3) BB^* is injective on $\operatorname{im} B^*$ and is zero on $\operatorname{im} A$.
(4) $AA^* + BB^* \colon V \to V$ is an isomorphism.

Corollary 1.1.1. If the complex (1.1) is elliptic, then Δ_i is elliptic for all i.

Theorem 1.1.2. Let (E, L) be an elliptic complex equipped with an inner product, where $E = \bigoplus_{i=0}^N E_i$ and $L = \bigoplus_{i=0}^N L_i$.

(1) The following decomposition is orthogonal

$$\Gamma(X, E) = \mathcal{H}(E) \oplus LL^*G(\Gamma(X, E)) \oplus L^*LG(\Gamma(X, E)).$$

(2) The following commutation relations are valid

$$\begin{aligned} H \circ G = G \circ H = H \circ \Delta = \Delta \circ H = 0 \\ L \circ H = H \circ L = L^* \circ H = H \circ L^* = 0 \\ L \circ \Delta = \Delta \circ L, L^* \circ G = G \circ L^* \\ L \circ G = G \circ L, L^* \circ G = G \circ L^*. \end{aligned}$$

(3) The dimension of $\mathcal{H}(E)$ is finite, and there is a canonical isomorphism

$$\mathcal{H}(E_i) \cong H^i(E)$$

for each i.

Proof. See Theorem 5.2 in [Wel80]. □

1.2. **Heat equation.** Let (M, g) be a compact Riemannian manifold and $\omega(t) \colon \mathbb{R}_{\geq 0} \to \mathcal{A}^n(M)$. The heat equation is given by

$$\begin{cases} (\frac{\partial}{\partial t} + \Delta)(\omega(t)) = 0, \\ \omega(0) = \omega_0. \end{cases} \tag{1.2}$$

Now let's explain why the heat equation can be used to prove Hodge theorem. The idea is to use heat equation to flow what we have to something we desired, and this philosophy is frequently used in solving other problems, such as Ricci flow, Kähler-Einstein flow and so on.

Suppose $\omega(t)$ is a solution defined on $\mathbb{R}$ for heat equation (1.2). Roughly speaking we desire

$$\lim_{t \to \infty} \frac{\partial}{\partial t} \omega(t) = 0,$$

and thus $\omega_\infty := \lim_{t \to \infty} \omega(t)$ is expected to be a harmonic form. On the other hand, we desire the flow doesn't change the cohomology class, that is,

$[\omega_0] = [\omega(t)]$. If so, for $\alpha \in H^n_{dR}(M)$, we pick an arbitary representative ω_0 and consider the heat equation with initial value ω_0. If there exists a unique solution $\omega(t)$ defined on $\mathbb{R}$, then ω_∞ is the unique harmonic representative of α. This proves the Hodge theorem.

Before we come into deep theories about partial differential equations, let's consider a baby example.

Example 1.2.1. Let $M = S^1$ equipped with the metric induced from $\mathbb{R}^2$. Then the heat equation is

$$(\partial_t - \partial_\theta^2) f(t,\theta) = 0,$$

where θ is the coordinate on S^1. Let $f_0(\theta)$ be the initial value with Fourier expansion

$$f_0(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{\sqrt{-1}n\theta},$$

and

$$f(t,\theta) = \sum_{n=\infty} a_n(t) e^{\sqrt{-1}n\theta}.$$

Then the heat equation is given by

$$(\partial_t - (\partial_\theta)^2)(f(t,\theta)) = \sum_{n=\infty}^{\infty} (a_n'(t) + a_n(t)n^2) e^{\sqrt{-1}n\theta}.$$

This shows

$$a_n(t) = a_n e^{-n^2 t}.$$

Thus the solution is given by

$$\lim_{t\to\infty} f(t,\theta) = a_0.$$

1.2.1. *Existence and uniqueness of the heat equation.*

Theorem 1.2.1. For any $\omega_0 \in \mathcal{A}^n(M)$, there exists a unique smooth map

$$\omega(t)\colon \mathbb{R}_{\geq 0} \to \mathcal{A}^n(M)$$

such that

$$\begin{cases} (\partial_t + \Delta_g)(\omega(t)) = 0 \\ \omega(0) = 0 \end{cases}$$

Definition 1.2.1. For $t \in [0,\infty)$, let $H_t\colon \mathcal{A}^n(M) \to \mathcal{A}^n(M)$ be the **heat-equation-solver operator**, which takes a form ω_0 to its solution under the heat flow (1.2) at time t.

Lemma 1.2.1.

(1) $H_0 = \mathrm{id}$.
(2) $H_{t+s} = H_s \circ H_t = H_t \circ H_s$.
(3) The operator H_t commutes with both Δ and d.

Proposition 1.2.1. Let $W_0(\mathcal{A}^n(M))$ be the L^2-complete of $\mathcal{A}^n(M)$. Then the operator H_t extends to compact self-adjoint operator

$$H_t\colon W_0(\mathcal{A}^n(M)) \to W_0(\mathcal{A}^n(M)).$$

Theorem 1.2.2 (spectral decomposition)**.** Let $T\colon H \to H$ be a compact self-adjoint operator on countable dimensional Hilbert space. Then there is an orthogonal eigenvalue decomposition

$$H = \bigoplus_{k=0}^{\infty} \langle v_n \rangle$$

such that

(1) each eigenspace $\langle v_n \rangle$ is finite dimensional.
(2) if $Tv_k = \gamma_n v_k$, then $\lim_{k\to\infty} \gamma_k = 0$.

Proposition 1.2.2. There is an orthogonal decomposition $W_0(\mathcal{A}^n(M)) = \bigoplus_{k=0}^{\infty} \langle \omega_k \rangle$ such that

$$\Delta(\omega_k) = \lambda_k \omega_k$$
$$H_t(\omega_k) = e^{-\lambda_k t} \omega_k$$

with $\lambda_k \geq 0$ with finite multiplicity.

Proof. By spectral decomposition theorem (Theorem 1.2.2), for each H_t with $t > 0$, there is an orthogonal decomposition

$$W_0(\mathcal{A}^n(M)) = \bigoplus_{k=0}^{\infty} \langle \omega_n(t) \rangle$$

with $H_t(\omega_k(t)) = \gamma_k(t)\omega_k(t)$. On the other hand, since $H_t \circ H_s = H_s \circ H_t$ for all t, s, one has $\{H_t\}_{t>0}$ are simultaneously diagonalizable. As a result, $\{\omega_k(t)\}_{t>0}$ is independent on t.

In the following let's prove the eigenvalue of H_t are strictly positive for $t > 0$.

(1) Note that
$$\gamma_k(t) = \langle H_t(\omega_k), \omega_k \rangle_{L^2} = \langle H_{\frac{t}{2}}(\omega_k), H_{\frac{t}{2}}(\omega_k) \rangle_{L^2} = \| H_{\frac{t}{2}}(\omega_k) \|^2_{L^2} \geq 0$$
This shows $\gamma_k(t) \geq 0$.
(2) Now let's prove $\gamma_n(t) \neq 0$. Suppose $H_t(\omega) = 0$ for some t and ω. Then
$$0 = \langle H_t(\omega), \omega \rangle_{L^2} = \langle H_{\frac{t}{2}}(\omega), H_{\frac{t}{2}}(\omega) \rangle_{L^2} = \| H_{\frac{t}{2}}(\omega) \|^2_{L^2}$$
implies that $H_{\frac{t}{2}}(\omega) = 0$. Repeating above process one has $H_{\frac{t}{2^m}}(\omega) = 0$ for all $m \in \mathbb{N}$. Then
$$0 = \lim_{m\to\infty} H_{\frac{t}{2^m}}(\omega) = \lim_{t\to 0} H_t(\omega) = \omega.$$
Thus H_t is injective for each $t > 0$.

□

Now let's use heat flow to give a proof of Hodge theorem.

Proof of Theorem 1.1. Consider the following linear map

$$I\colon \mathcal{H}^n(M) \to H^n_{dR}(M)$$
$$\omega \mapsto [\omega].$$

It's well-defined, since every harmonic form is closed.

(1) If $[\omega]=0$, then $\omega = \mathrm{d}\tau$ for some τ, and thus

$$0 = \langle \mathrm{d}^*\omega . \tau\rangle = \langle \mathrm{d}^*\mathrm{d}\tau, \tau\rangle = \|\mathrm{d}\tau\|^2.$$

This shows $\omega = \mathrm{d}\tau = 0$, that is, the map I is injective.

(2) Let ω be a closed n-form. Note that

$$\begin{aligned} H_t\omega - \omega &= H_t\omega - H_0\omega \\ &= \int_0^t \partial_t H_t\omega \\ &= -\int_0^t \Delta H_t\omega \\ &= -\int_0^t H_t(\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})\omega \\ &= -\int_0^t H_t \mathrm{d}\mathrm{d}^*\omega \\ &= \mathrm{d}\{-\int_0^t H_t\mathrm{d}^*\omega\}. \end{aligned}$$

This shows the heat flow preseves the cohomology class of ω, that is, $[H_t\omega] = [\omega]$ for all t. By using the spectral decomposition of Δ, one has

$$\omega = \sum_k a_k\omega_k,$$

so

$$H_t\omega = \sum_k a_k H_t\omega_k = \sum_k a_k e^{-\lambda_k t}\omega_k.$$

Thus

$$\omega_\infty := \lim_{t\to\infty} H_t\omega = \sum_k a_k\omega_k(\lim_{t\to\infty} e^{-\lambda_k t}) = \sum_{n=0}^N a_k\omega_k,$$

where $\{\omega_1, \ldots, \omega_N\}$ spans the kernel of Δ. In other words, the long time limit ω_∞ is harmonic representative of $[\omega]$. This shows the map I is surjective.

□

2. Hodge package

The theory of harmonic forms shall show a much richer structure on the graded $\mathbb{Q}$-algebra $(\bigoplus_{k=0}^{2n} H^k(X,\mathbb{Q}),\cup)$. Before the theory, we basically only know (for any compact complex manifold) the Poincaré duality, that is,

$$H^k(X,\mathbb{Q})\otimes H^{2n-k}(X,\mathbb{Q})\xrightarrow{\cup} H^{2n}(X,\mathbb{Q})\cong\mathbb{Q}$$

is a perfect pairing, and the following diagram commutes

$$\begin{array}{ccc} H^k(X,\mathbb{C})\otimes H^{2n-k}(X,\mathbb{C}) & \xrightarrow{\quad\cup\quad} & H^{2n}(X,\mathbb{C}) \\ {\scriptstyle\cong}\big\downarrow & & \big\downarrow{\scriptstyle\cong} \\ H^k_{dR}(X)\otimes H^{2n-k}_{dR}(X) & \xrightarrow{\quad\wedge\quad} & H^{2n}_{dR}(X). \end{array}$$

But for a compact complex manifold X which is Kählerizable (that is, X admits a Kähler form), the following much finer properties (so called Hodge package) hold.

(1) Hodge star operator and Lefschetz operator;
(2) Kähler identities;
(3) Hodge decomposition;
(4) Lefschetz decomposition;
(5) Hodge metric;
(6) Hodge index.

2.1. **Hodge star operator and Lefschetz operator.**

2.1.1. *Hodge star operator.* Let (X,ω) be a compact Hermitian n-manifold. A (p,q)-form α can be locally written as

$$\alpha=\frac{1}{p!\times q!}\alpha_{i_1\dots i_p\bar{j}_1\dots\bar{j}_q}\mathrm{d}z^{i_1}\wedge\cdots\wedge\mathrm{d}z^{i_p}\wedge\mathrm{d}\overline{z}^{j_1}\wedge\cdots\wedge\mathrm{d}\overline{z}^{j_q}.$$

Then for $\alpha,\beta\in\mathcal{A}^{p,q}(X)$, the local inner product is defined as

$$\langle\alpha,\beta\rangle=\frac{1}{p!\times q!}h^{i_1\bar{k}_1}\dots h^{i_p\bar{k}_p}h^{l_1\bar{j}_1}\dots h^{l_q\bar{j}_q}\alpha_{i_1\dots i_p\bar{j}_1\dots\bar{j}_q}\overline{\beta_{k_1\dots k_p\bar{l}_1\dots\bar{l}_q}},$$

which is a smooth function on X.

Definition 2.1.1. An **inner product on the space of** (p,q)**-form** is defined as

$$(\alpha,\beta):=\int_X\langle\alpha,\beta\rangle\frac{\omega^n}{n!},$$

where $\alpha,\beta\in\mathcal{A}^{p,q}(X)$.

Holding the inner product (-,-), the formal adjoint operator of d is defined as an operator

$$\mathrm{d}^*\colon\mathcal{A}^k(X)\to\mathcal{A}^{k-1}(X)$$

satisfying $(\alpha,\mathrm{d}\beta)=(\mathrm{d}^*\alpha,\beta)$ for α,β with appropriate degrees, similarly one can define ∂^* and $\overline{\partial}^*$. In order to construct these adjoint operators, we need to introduce the well-known Hodge star operator.

Definition 2.1.2. There exists so called **Hodge star operator**

$$\star\colon \mathcal{A}^{p,q}(X) \to \mathcal{A}^{n-q,n-p}(X)$$

such that

$$(\alpha, \beta) = \int_X \alpha \wedge \star\overline{\beta}.$$

Remark 2.1.1. It's well-defined since $\overline{\beta}$ is a (q,p)-form, and thus $\star\overline{\beta}$ is a $(n-p, n-q)$-form.

Lemma 2.1.1.

(1) $\star 1 = \omega^n/n!$;
(2) $\star\omega = \omega^{n-1}/(n-1)!$;
(3) $\overline{\star\psi} = \star\overline{\psi}$;
(4) $\star\star = (-1)^{p+q}$ on $\mathcal{A}^{p,q}(X)$;
(5) $(\star\varphi, \star\psi) = (\varphi, \psi)$.

Proposition 2.1.1. $\mathrm{d}^* = -\star \mathrm{d}\star$.

Proof. For arbitrary $\alpha \in \mathcal{A}^{p+q}(X)$ and $\beta \in \mathcal{A}^{p+q+1}(X)$, then

$$\begin{aligned}
(\mathrm{d}\alpha, \beta) &= \int_X \mathrm{d}\alpha \wedge \star\beta \\
&= \int_X \mathrm{d}(\alpha \wedge \star\beta) - (-1)^{p+q}\alpha \wedge \mathrm{d}\star\beta \\
&= (-1)^{p+q+1} \int_X \alpha \wedge \mathrm{d}\star\beta \\
&\overset{(1)}{=} (-1)^{p+q+1}(-1)^{2n-(p+q+1)+1} \int_X \alpha \wedge \star\star\,\mathrm{d}\star\beta \\
&= -(\alpha, \star\mathrm{d}\star\beta).
\end{aligned}$$

where (1) holds from (4) of Lemma 2.1.1. □

Proposition 2.1.2.

$$\begin{aligned}
\partial^* &= -\star\overline{\partial}\star \\
\overline{\partial}^* &= -\star\partial\star.
\end{aligned}$$

Proof. By the same computations as above. □

2.1.2. *Lefschetz operator.*

Definition 2.1.3. Let (X, ω) be a compact Kähler n-manifold. The **Lefschetz operator** is defined as

$$\begin{aligned}
L\colon \mathcal{A}^{p,q}(X) &\to \mathcal{A}^{p+1,q+1}(X) \\
\alpha &\mapsto \omega \wedge \alpha.
\end{aligned}$$

Lemma 2.1.2. $\Lambda := L^* = (-1)^{p+q}\star L\star$ on (p,q)-forms.

Proof. For $\alpha \in \mathcal{A}^{p,q}(X), \beta \in \mathcal{A}^{p+1,q+1}(X)$, direct computation shows

$$\begin{aligned}(L\alpha, \beta) &= \int_X L\alpha \wedge \star\beta \\ &= \int_X \omega \wedge \alpha \wedge \star\beta \\ &\overset{(1)}{=} \int_X \alpha \wedge \omega \wedge \star\beta \\ &\overset{(2)}{=} \int_X \alpha \wedge (-1)^{p+q} \star\star\omega \wedge \star\beta \\ &= (\alpha, (-1)^{p+q} \star L \star \beta),\end{aligned}$$

where

(1) holds from ω is a 2-form.
(2) holds from (4) of Lemma 2.1.1.

□

2.1.3. *Local computations of adjoint operators.*

Proposition 2.1.3.

$$\langle \mathrm{d}z^i \wedge \alpha, \beta\rangle = \langle \alpha, h^{p\bar{i}}\iota_p\beta\rangle$$

holds for α, β with appropriate bidegrees.

Theorem 2.1.1. Let (X, h) be a Kähler manifold. Then around each point there exists a holomorphic coordinate $(z^1, \dots, z^n)$ such that

$$h_{i\bar{j}}(z) = \delta_{i\bar{j}} - \Theta_{i\bar{j}k\bar{l}}(p)z^k\bar{z}^l + O(|z|^2).$$

Proposition 2.1.4. Let (X, h) be a Kähler manifold. Then locally

$$\begin{cases}\partial = \mathrm{d}z^i \wedge \nabla_i \\ \partial^* = -h^{i\bar{j}}\iota_i \circ \nabla_j = -h^{i\bar{j}}\nabla_j \circ \iota_i\end{cases} \qquad \begin{cases}\bar{\partial} = \mathrm{d}\bar{z}^i \wedge \nabla_{\bar{i}} \\ \bar{\partial}^* = -h^{i\bar{j}}\iota_{\bar{j}} \circ \nabla_i = -h^{i\bar{j}}\nabla_i \circ \iota_{\bar{j}}.\end{cases}$$

Proof. Here we only give the proof of the case ∂ and ∂^*, the proof for the other two cases are same. It suffices to check pointwisely, and at each point we may also choose normal coordinate in Theorem 2.1.1. For (p, q)-form α, locally written as $\alpha = \alpha_{J\overline{K}}\mathrm{d}z^J \wedge \mathrm{d}\bar{z}^K$. Then

$$\partial\alpha = \frac{\partial\alpha_{J\overline{K}}}{\partial z^i}\mathrm{d}z^i \wedge \mathrm{d}z^J \wedge \mathrm{d}\bar{z}^K,$$

and

$$\begin{aligned}\mathrm{d}z^i \wedge \nabla_i\alpha &= \mathrm{d}z^i \wedge \nabla_i(\alpha_{J\overline{K}}\mathrm{d}z^J \wedge \mathrm{d}\bar{z}^K) \\ &= \mathrm{d}z^i \wedge \frac{\partial\alpha_{J\overline{K}}}{\partial z^i}\mathrm{d}z^J \wedge \mathrm{d}\bar{z}^K + \alpha_{J\overline{K}}\nabla_i(\mathrm{d}z^J \wedge \mathrm{d}\bar{z}^K) \\ &\overset{(1)}{=} \frac{\partial\alpha_{J\overline{K}}}{\partial z^i}\mathrm{d}z^i \wedge \mathrm{d}z^J \wedge \mathrm{d}\bar{z}^K,\end{aligned}$$

where (1) holds from our choice of normal coordinate. To see formula of ∂^*, take arbitrary forms α, β with appropriate bidegrees, then

$$\begin{aligned}(\partial\alpha, \beta) &= (\mathrm{d}z^i \wedge \nabla_i\alpha, \beta) \\ &\overset{(2)}{=} (\nabla_i\alpha, h^{p\bar{i}}\iota_p\beta) \\ &\overset{(3)}{=} -(\alpha, h^{p\bar{i}}\nabla_i \circ \iota_p\beta),\end{aligned}$$

where

(2) holds from Proposition 2.1.3.
(3) holds from Stokes' theorem and the fact Chern connection is compatible with metric.

This shows

$$\partial^* = -h^{i\bar{j}}\nabla_j \circ \iota_i \overset{(4)}{=} -h^{i\bar{j}}\iota_i \circ \nabla_j,$$

where (4) holds from $\iota_i \circ \nabla_j = \nabla_j \circ \iota_i$. ☐

Proposition 2.1.5. Let (X, ω) be a Kähler manifold. Then locally

$$\Lambda = \sqrt{-1}h^{i\bar{j}}\iota_i \circ \iota_{\bar{j}} = -\sqrt{-1}h^{i\bar{j}}\iota_{\bar{j}} \circ \iota_i.$$

Proof. For arbitrary forms α, β with appropriate bidegrees, direct computation shows

$$\begin{aligned}(\omega \wedge \alpha, \beta) &= (\sqrt{-1}h_{i\bar{j}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j \wedge \alpha, \beta) \\ &\overset{(1)}{=} (\sqrt{-1}h_{i\bar{j}}\mathrm{d}\bar{z}^j \wedge \alpha, h^{p\bar{i}}\iota_p\beta) \\ &\overset{(2)}{=} (\sqrt{-1}h_{i\bar{j}}\alpha, h^{p\bar{i}}h^{j\bar{q}}\iota_{\bar{q}} \circ \iota_p\beta) \\ &\overset{(3)}{=} (\alpha, -\sqrt{-1}h_{j\bar{i}}h^{p\bar{i}}h^{j\bar{q}}\iota_{\bar{q}} \circ \iota_p\beta) \\ &= (\alpha, -\sqrt{-1}h^{p\bar{i}}\iota_{\bar{i}} \circ \iota_p\beta),\end{aligned}$$

where

(1) and (2) hold from Proposition 2.1.3.
(3) holds from $h_{i\bar{j}}$ is Hermitian, that is $\overline{h_{i\bar{j}}} = h_{j\bar{i}}$.

This shows

$$\Lambda = -\sqrt{-1}h^{i\bar{j}}\iota_{\bar{j}} \circ \iota_i \overset{(4)}{=} \sqrt{-1}h^{i\bar{j}}\iota_i \circ \iota_{\bar{j}},$$

where (4) holds from $\iota_i \circ \iota_{\bar{j}} = -\iota_{\bar{j}} \circ \iota_i$. ☐

2.2. **Kähler identities.**

Definition 2.2.1. Let A, B be two differential operators. The **commutor** of A, B is defined as

$$[A, B] := AB - (-1)^{\deg A \deg B}BA.$$

Lemma 2.2.1 (Jacobi identity)**.** Let A, B, C be differential operators. Then

$$(-1)^{\deg A \deg C}[A, [B, C]] + (-1)^{\deg B \deg A}[B, [C, A]] + (-1)^{\deg C \deg B}[C, [A, B]] = 0.$$

Proposition 2.2.1 (Kähler identities of 1-st order)**.** If (X, ω) is a compact Kähler manifold, then

$$
\begin{aligned}
[\overline{\partial}^*, L] &= \sqrt{-1}\partial \\
[\partial^*, L] &= -\sqrt{-1} \cdot \overline{\partial} \\
[\Lambda, \overline{\partial}] &= -\sqrt{-1}\partial^* \\
[\Lambda, \partial] &= \sqrt{-1} \cdot \overline{\partial}^*.
\end{aligned}
$$

Proof. By taking conjugates and adjoints, it suffices to prove the first identity, which is a first order identity of differential equation. But by Theorem 2.1.1, locally we have $h_{i\bar{j}} = \delta_{i\bar{j}} + O(|\xi^2|)$. Thus it suffices to check Kähler identity for the case $U \subseteq \mathbb{C}^n$ equipped with standard Hermitian metric.

Suppose (p, q)-form α is locally given by $\alpha = \alpha_{JK} \mathrm{d}z^J \wedge \mathrm{d}\bar{z}^K$, then by Proposition 2.1.4 one has $\overline{\partial}^* \alpha = -\sum_l \iota_{\frac{\partial}{\partial \bar{z}^l}} \frac{\partial \alpha}{\partial z^l}$. Thus

$$
\begin{aligned}
[\overline{\partial}^*, L]\alpha &= \overline{\partial}^*(\omega \wedge \alpha) - \omega \wedge \overline{\partial}^* \alpha \\
&= -\sum_l \iota_{\frac{\partial}{\partial \bar{z}^l}} \frac{\partial}{\partial z^l}(\omega \wedge \alpha) + \omega \wedge \sum_l \iota_{\frac{\partial}{\partial \bar{z}^l}} \frac{\partial \alpha}{\partial z^l} \\
&\overset{(1)}{=} -\sum_l \iota_{\frac{\partial}{\partial \bar{z}^l}} (\omega \wedge \frac{\partial \alpha}{\partial z^l}) + \omega \wedge \sum_l \iota_{\frac{\partial}{\partial \bar{z}^l}} \frac{\partial \alpha}{\partial z^l} \\
&= -\{\sum_l (\iota_{\frac{\partial}{\partial \bar{z}^l}} \omega) \wedge \frac{\partial \alpha}{\partial z^l} + \omega \wedge \iota_{\frac{\partial}{\partial \bar{z}^l}} \frac{\partial \alpha}{\partial z^l}\} + \sum_l \omega \wedge \iota_{\frac{\partial}{\partial \bar{z}^l}} \frac{\partial \alpha}{\partial z^l} \\
&= -\sum_l (\iota_{\frac{\partial}{\partial \bar{z}^l}} \omega) \wedge \frac{\partial \alpha}{\partial z^l} \\
&\overset{(2)}{=} \sqrt{-1} \sum_l \mathrm{d}z^l \wedge \frac{\partial \alpha}{\partial z^l} \\
&= \sqrt{-1}\partial \alpha,
\end{aligned}
$$

where

(1) holds from ω is a closed $(1, 1)$-form.
(2) holds from $\omega = \sqrt{-1}\sum_{i=1}^n \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^i$.

□

Theorem 2.2.1 (Kähler identities of 2-nd order)**.** Let (X, ω) be a compact Kähler manifold. Then

$$\Delta_{\mathrm{d}} = 2\Delta_{\partial} = 2\Delta_{\overline{\partial}}.$$

Proof. Note that

$$\Delta_{\mathrm{d}} = (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}).$$

By the fourth Kähler identity, one has

(1) The first term can be computed as

$$\begin{aligned}(\partial+\overline{\partial})(\partial^*+\overline{\partial}^*)&=(\partial+\overline{\partial})(\partial^*-\sqrt{-1}\Lambda\partial+\sqrt{-1}\partial\Lambda)\\&=\partial\partial^*-\sqrt{-1}\partial\Lambda\partial+\overline{\partial}\partial^*-\sqrt{-1}\cdot\overline{\partial}\Lambda\partial+\sqrt{-1}\cdot\overline{\partial}\partial\Lambda.\end{aligned}$$

(2) The second term can be computed as

$$\begin{aligned}(\partial^*+\overline{\partial}^*)(\partial+\overline{\partial})&=(\partial^*-\sqrt{-1}\Lambda\partial+\sqrt{-1}\partial\Lambda)(\partial+\overline{\partial})\\&=\partial^*\partial+\sqrt{-1}\partial\Lambda\partial+\partial^*\overline{\partial}-\sqrt{-1}\Lambda\partial\overline{\partial}+\sqrt{-1}\partial\Lambda\overline{\partial}.\end{aligned}$$

By the third Kähler identity, one has

$$\partial^*=\sqrt{-1}[\Lambda,\overline{\partial}]=\sqrt{-1}\Lambda\overline{\partial}-\sqrt{-1}\overline{\partial}\Lambda.$$

Then

$$\begin{aligned}\overline{\partial}\partial^*&=\overline{\partial}(\sqrt{-1}\Lambda\overline{\partial}-\sqrt{-1}\cdot\overline{\partial}\Lambda)=\sqrt{-1}\cdot\overline{\partial}\Lambda\overline{\partial}\\\partial^*\overline{\partial}&=(\sqrt{-1}\Lambda\overline{\partial}-\sqrt{-1}\cdot\overline{\partial}\Lambda)\overline{\partial}=-\sqrt{-1}\cdot\overline{\partial}\Lambda\overline{\partial}=-\overline{\partial}\partial^*.\end{aligned}$$

Now we have

$$\begin{aligned}\Delta_{\mathrm{d}}&=\Delta_\partial-\sqrt{-1}\cdot\overline{\partial}\Lambda\partial-\sqrt{-1}\Lambda\partial\overline{\partial}+\sqrt{-1}\overline{\partial}\partial\Lambda+\sqrt{-1}\partial\Lambda\overline{\partial}\\&=\Delta_\partial+\sqrt{-1}(\Lambda\overline{\partial}\partial-\overline{\partial}\Lambda\partial)+\sqrt{-1}(\partial\Lambda\overline{\partial}-\partial\overline{\partial}\Lambda)\\&=\Delta_\partial+\sqrt{-1}[\Lambda,\overline{\partial}]\partial+\sqrt{-1}\partial[\Lambda,\overline{\partial}]\\&=\Delta_\partial+\partial^*\partial+\partial\partial^*\\&=2\Delta_\partial.\end{aligned}$$

□

Corollary 2.2.1. On a compact Kähler manifold, Δ_{d}-harmonic is equivalent to Δ_∂-harmonic, and is equivalent to $\Delta_{\overline{\partial}}$-harmonic.

Corollary 2.2.2. Let (X,ω) be a Kähler manifold and α be a (p,q)-form. Then $\Delta_{\mathrm{d}}\alpha$ is still a (p,q)-form.

Proof. It's clear to see $\Delta_\partial\alpha$ is still a (p,q)-form. □

Proposition 2.2.2 (Kähler identities of 0-th order)**.** Show that for compact Kähler manifold we have

$$\begin{aligned}&[\Delta_{\mathrm{d}},L]=0,\\&[L,\Lambda]=(k-n)\,\mathrm{id}\quad\text{on }\mathcal{A}^k(X).\end{aligned}$$

Proof. For the first identity, we have $\Delta_{\mathrm{d}}=2\Delta_\partial=2(\partial\partial^*+\partial^*\partial)$. Thus

$$[\Delta_{\mathrm{d}},L]=2([\partial\partial^*,L]+[\partial^*\partial,L])=2(\partial[\partial^*,L]+[\partial^*,L]\partial).$$

The last equality holds by the fact that L commutes with ∂ since ω is ∂-closed. Now we use the identity $[\partial^*,L]=-\sqrt{-1}\cdot\overline{\partial}$, which anticommutes with ∂ to conclude.

For the second identity, without lose of generality it suffices to check on $U\subseteq\mathbb{C}^n$ equipped with standard Hermitian metric since we are considering operators of order zero. Suppose $\varphi=\varphi_{IJ}\mathrm{d}z^I\mathrm{d}\overline{z}^J$ is a k-form with type (p,q). A direct computation shows

$$\begin{aligned}
L\Lambda\varphi =&L\left(\sqrt{-1}\sum_{i=1}^{n}\varphi_{IJ}\iota_i\circ\iota_{\overline{i}}(\mathrm{d}z^I\wedge\mathrm{d}\overline{z}^J)\right)\\
=&L\left(\sqrt{-1}\sum_{i=1}^{n}(-1)^p\varphi_{IJ}\iota_i\mathrm{d}z^I\wedge\iota_{\overline{i}}\mathrm{d}\overline{z}^J\right)\\
=&(\sqrt{-1})^2\sum_{i,j=1}^{n}(-1)^{2p-1}\varphi_{IJ}\mathrm{d}z^j\wedge\iota_i\mathrm{d}z^I\wedge\mathrm{d}\overline{z}^j\iota_{\overline{i}}\mathrm{d}\overline{z}^J.\\
\Lambda L\varphi =&\Lambda\left(\sqrt{-1}\sum_{j=1}^{n}(-1)^p\varphi_{IJ}\mathrm{d}z^j\wedge\mathrm{d}z^I\wedge\mathrm{d}\overline{z}^j\wedge\mathrm{d}\overline{z}^J\right)\\
=&(\sqrt{-1})^2\sum_{i,j=1}^{n}(-1)^p\varphi_{IJ}\iota_i\circ\iota_{\overline{i}}(\mathrm{d}z^j\wedge\mathrm{d}z^I\wedge\mathrm{d}\overline{z}^j\wedge\mathrm{d}\overline{z}^J)\\
=&(\sqrt{-1})^2\sum_{i,j=1}^{n}(-1)^{2p+1}\varphi_{IJ}\iota_i\left(\mathrm{d}z^j\wedge\mathrm{d}z^I\wedge\iota_{\overline{i}}(\mathrm{d}\overline{z}^j\wedge\mathrm{d}\overline{z}^J)\right)\\
=&(\sqrt{-1})^2\sum_{i,j=1}^{n}(-1)^{2p+1}\varphi_{IJ}\iota_i\left(\mathrm{d}z^j\wedge\mathrm{d}z^I\wedge\underbrace{(\delta_{\overline{i}}^{\overline{j}}\mathrm{d}\overline{z}^J-\mathrm{d}\overline{z}^j\wedge\iota_{\overline{i}}\mathrm{d}\overline{z}^J)}_{A}\right)\\
=&(\sqrt{-1})^2\sum_{i,j=1}^{n}(-1)^{2p+1}\varphi_{IJ}(\delta_i^j\mathrm{d}z^I\wedge A-\mathrm{d}z^j\wedge\iota_i\mathrm{d}z^I\wedge A).
\end{aligned}$$

Then

$$\begin{aligned}
L\Lambda\varphi-\Lambda L\varphi &= (\sqrt{-1})^2\sum_{j=1}^{n}\varphi_{IJ}\left(\mathrm{d}z^I\wedge\mathrm{d}\overline{z}^J-\mathrm{d}z^I\wedge\mathrm{d}\overline{z}^j\wedge\iota_{\overline{j}}\mathrm{d}z^J-\mathrm{d}z^j\wedge\iota_j\mathrm{d}z^I\wedge\mathrm{d}\overline{z}^J\right)\\
&=(k-n)\varphi.
\end{aligned}$$

□

2.3. **Hodge decomposition.**

Theorem 2.3.1. Let (X,h) be a compact Kähler manifold, $\alpha=\sum_{p+q=k}\alpha^{p,q}$. Then α is harmonic if and only if $\alpha^{p,q}$ is harmonic, that is

$$\mathcal{H}^k(X)\otimes_{\mathbb{R}}\mathbb{C}=\bigoplus_{p+q=k}\mathcal{H}^{p,q}(X)$$

with $\overline{\mathcal{H}^{p,q}(X)}=\mathcal{H}^{q,p}(X)$.

Proof. It follows from Δ_{d} preserves bidegree. □

Theorem 2.3.2 (Hodge decomposition)**.** Let (X,h) be a compact Kähler manifold. Then

$$H^k(X,\mathbb{C})\cong\bigoplus_{p+q=k}H^{p,q}(X)$$

with $\overline{H^{p,q}(X)}=H^{q,p}(X)$.

Proof. It follows from there are natural isomorphisms $H^k(X,\mathbb{C})\cong\mathcal{H}^k(X)\otimes\mathbb{C}$ and $H^{p,q}(X)\cong\mathcal{H}^{p,q}(X)$. $\square$

Corollary 2.3.1. Let (X,h) be a compact Kähler manifold. Then

$$b_k=\sum_{p+q=k}h^{p,q}$$

with $h^{p,q}=h^{q,p}$, where $b_k=\dim H^k(X,\mathbb{C})$ and $h^{p,q}=\dim H^{p,q}(X)$.

Corollary 2.3.2. b_k is even when k is odd.

Corollary 2.3.3. $b_k\neq 0$ when k is even.

Proof. $h^{k,k}\neq 0$ since $0\neq\omega^k\in H^{k,k}(X)$. $\square$

There are many relations between $h^{p,q}$, and we can draw a picture as follows, called Hodge diamond since it has the same symmetry as a diamond.

$$\begin{array}{cccccccccccc}
 & & & & & h^{0,0} & & & & & b_0 \\
 & & & & h^{1,0} & & h^{0,1} & & & & b_1 \\
 & & & h^{2,0} & & h^{1,1} & & h^{0,2} & & & b_2 \\
 & & \iddots & & & \vdots & & & \ddots & & \vdots \\
\text{Hodge}\updownarrow & h^{n,0} & & \cdots & & \underset{\text{Serre}}{\curvearrowleft} & & \cdots & & h^{0,n} & b_n \\
 & & \ddots & & & \vdots & & & \iddots & & \vdots \\
 & & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & & b_{2n-2} \\
 & & & & h^{n,n-1} & & h^{n-1,n} & & & & b_{2n-1} \\
 & & & & & h^{n,n} & & & & & b_{2n} \\
 & & & & & \underset{\text{conjugation}}{\longleftrightarrow} & & & & &
\end{array}$$

Example 2.3.1.

$$H^{p,q}(\mathbb{CP}^n)=\begin{cases}\mathbb{C} & 0\leq p=q\leq n\\ 0 & \text{otherwise}\end{cases}$$

Proof. It's known to all that the singular cohomology of $\mathbb{CP}^n$ with complex coefficient is

$$H^k(\mathbb{CP}^n,\mathbb{C})=\begin{cases}\mathbb{C} & k\text{ is even}\\ 0 & k\text{ is odd}\end{cases}$$

Thus it's clear to compute Dolbeault cohomology of $\mathbb{CP}^n$ using the symmetry of Hodge diamond. $\square$

2.3.1. *Bott-Chern cohomology.* In the proof Hodge decomposition, we used the Kähler metric. A natural question is to consider (in)denpendence of the Kähler metric. In this section we will show our decomposition is independent of the choice of Kähler metric, by using Bott-Chern cohomology.

Definition 2.3.1. Let X be a complex manifold. The **Bott-Chern cohomology** is defined as

$$H^{p,q}_{\mathrm{BC}}(X) := \frac{Z^{p,q}_{\mathrm{BC}} := \{\alpha \in \mathcal{A}^{p,q}(X) \mid \mathrm{d}\alpha = 0\}}{\partial\overline{\partial}\mathcal{A}^{p-1,q-1}(X)}.$$

Remark 2.3.1*.* There is a natural map

$$Z^{p,q}_{\mathrm{BC}}(X) \to H^{p+q}(X, \mathbb{C}),$$

which descends to

$$H^{p,q}_{\mathrm{BC}}(X) \to H^{p+q}(X, \mathbb{C}),$$

since $\partial\overline{\partial}\beta = \mathrm{d}\overline{\partial}\beta$. On the other hand, there is also a natural map

$$Z^{p,q}_{\mathrm{BC}}(X) \to H^{p,q}(X),$$

which descends to

$$H^{p,q}_{\mathrm{BC}}(X) \to H^{p,q}(X),$$

since $\partial\overline{\partial}\beta = -\overline{\partial}\partial\beta$. If we can prove there are isomorphisms between

$$H^{p,q}_{\mathrm{BC}}(X) \cong H^{p,q}(X)$$

and

$$\bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X) \cong H^k(X, \mathbb{C}),$$

then Hodge decomposition is independent of choice of Kähler metric since Bott-Chern cohomology is independent of the choice of Kähler metric.

Lemma 2.3.1 ($\partial\overline{\partial}$-lemma)**.** Let (X, ω) be a compact Kähler manifold and α be a d-closed (p, q)-form. If α is $\overline{\partial}$-exact or ∂-exact, then there exists a $(p-1, q-1)$-form such that

$$\alpha = \partial\overline{\partial}\beta.$$

Proof. Suppose α is $\overline{\partial}$-exact. Then $\alpha = \overline{\partial}\gamma$ for some $(p, q-1)$-form γ, and Hodge's theorem implies γ has decomposition

$$\gamma = a + \partial b + \partial^* c,$$

where a is Δ_∂-harmonic, and b, c are forms with appropriate degrees. Direct computation shows

$$\begin{aligned}
\alpha = \overline{\partial}\gamma &= \overline{\partial}a + \overline{\partial}\partial b + \overline{\partial}\partial^* c \\
&= -\partial\overline{\partial}b + \overline{\partial}\partial^* c \\
&= -\partial\overline{\partial}b - \partial^*\overline{\partial}c.
\end{aligned}$$

Now it suffices to show $-\partial^*\overline{\partial}c = 0$. A trick here is to note that

$$0 = \partial\alpha = -\partial\partial^*\overline{\partial}c \implies \partial^*\overline{\partial}c \in \ker\partial \cap \operatorname{im}\partial^* = 0 \implies \partial^*\overline{\partial}c = 0.$$

This shows
$$\alpha = \partial\overline{\partial}(-b)$$
as desired. □

Corollary 2.3.4. Let (X, ω) be a compact Kähler manifold. Then
(1) $H^{p,q}_{\mathrm{BC}}(X) \to H^{p,q}(X)$ is an isomorphism.
(2) $\bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X) \to H^k(X, \mathbb{C})$ is an isomorphism.

Proof. Here we only prove the first isomorphism. From Remark 2.3.1, there is a canonical map $H^{p,q}_{\mathrm{BC}}(X) \to H^{p,q}(X)$, and if we choose a Kähler metric, we have $H^{p,q}(X) \cong \mathcal{H}^{p,q}$, we will show our canonical map is both surjective and injective via this chosen metric.
(1) To see surjectivity: For element in $H^{p,q}(X)$ we choose a $\Delta_{\overline{\partial}}$-harmonic representative. Since Δ_{∂}-harmonic is equivalent to Δ_{d}-harmonic, so this representative is also d-closed.
(2) To see injectivity: Suppose we have $[\alpha] \in H^{p,q}_{\mathrm{BC}}(X)$ such that α is trivial in $H^{p,q}(X)$, that is $\overline{\partial}$-exact. Then Lemma 2.3.1, that is $\partial\overline{\partial}$-lemma implies it's trivial in Bott-Chern cohomology.

□

Corollary 2.3.5. A Hermitian metric ω is Kähler if and only if it can be written locally as
$$\omega = \sqrt{-1}\partial\overline{\partial} f,$$
where f is a real-valued smooth function.

Proof. It's clear if ω is locally written as $\sqrt{-1}\partial\overline{\partial} f$, then it gives a Kähler metric. Conversely, a Kähler metric ω is an element in $H^{1,1}(X)$, and we have already shown that $H^{1,1}(X) = H^{1,1}_{\mathrm{BC}}(X)$, and Dolbeault lemma implies Dolbeault cohomology vanishes on open subset which is sufficiently small, this completes the proof. □

2.4. **Lefschetz decomposition.**

Proposition 2.4.1. Let (X, ω) be a Kähler n-manifold. Then $L^{n-k} \colon \mathcal{A}^k(X) \to \mathcal{A}^{2n-k}(X)$ is an isomorphism for $k \leq n$.

Proof. In fact, we will prove that L^r are injective for all $1 \leq r \leq n-k$. As a consequence, L^{n-k} is an isomorphism since $\Omega^k_{X,\mathbb{R}}$ has the same rank as $\Omega^{2n-k}_{X,\mathbb{R}}$. In Proposition 2.2.2 we have shown that
$$[L, \Lambda]\alpha = (k-n)\alpha,$$
holds for all $\alpha \in \mathcal{A}^k(X)$. Then
$$\begin{aligned}
[L^r, \Lambda] &= L^r\Lambda - \Lambda L^r \\
&= L(L^{r-1}\Lambda - \Lambda L^{r-1}) + L\Lambda L^{r-1} - \Lambda L L^{r-1} \\
&= L[L^{r-1}, \Lambda] + [L, \Lambda]L^{r-1}.
\end{aligned}$$

By induction it's easy to show for all $\alpha \in \mathcal{A}^k(X)$ one has

$$[L^r, \Lambda]\alpha = (r(k-n) + r(r-1))L^{r-1}\alpha.$$

For $\alpha \in \mathcal{A}^k(X)$, if $L^r\alpha = 0, r \leq n-k$, then

$$\begin{aligned} L^r\Lambda\alpha &= [L^r, \Lambda]\alpha \\ &= (r(k-n) + r(r-1))L^{r-1}\alpha. \end{aligned}$$

In other words, we have

$$L^{r-1}(L\Lambda\alpha - (r(k-n) + r(r-1))\alpha) = 0. \tag{2.1}$$

Now let's prove L^r is injective by induction on r: It's clear L is injective, and suppose L^{r-1} is injective. Then by (2.1) one has

$$L\Lambda\alpha = (r(k-n) + r(r-1))\alpha.$$

If we denote $\beta = \Lambda\alpha$, and apply L^r to both side of above equation, then we have

$$L^{r+1}\beta = (r(k-n) + r(r-1))L^r\alpha = 0,$$

where $\beta \in \mathcal{A}^{k-2}(X)$. It's clear $\beta = 0$ if β is a smooth function. Then by induction on k, we have $\beta = 0$, and thus $\alpha = 0$. □

Definition 2.4.1. Let (X, ω) be a Kähler n-manifold. A k-form α is called **primitive** if $L^{n-k+1}\alpha = 0$.

Exercise 2.4.1. A k-form α is primitive if and only if $\Lambda\alpha = 0$.

Proof. For an n-form α, α is primitive if and only if $L\alpha = 0$. On the other hand, the Exercise 2.2.2 implies that

$$[L, \Lambda] = (k-n)\,\mathrm{id}, \quad \text{on } \mathcal{A}^k(X).$$

This shows if $n = k$, then L and Λ commutes. Thus we have α is primitive if and only if $\Lambda\alpha = 0$, since

$$\Lambda\alpha = 0 \iff L\Lambda\alpha = 0 \iff \Lambda L\alpha = 0 \iff L\alpha = 0$$

and the first and last equality we use the fact that L is injective on $\Omega^k_{X,\mathbb{R}}, k \leq n$ and Λ is injective on $\Omega^{n+2}_{X,\mathbb{R}}$. In general case, we have

$$[L^r, \Lambda]\alpha = (r(k-n) + r(r-1))L^{r-1}\alpha$$

and in particular for $r = n-k+1$ where k is the degree of α, we have

$$[L^r, \Lambda]\alpha = 0$$

The argument can be repeated to conclude. □

Proposition 2.4.2. For any k-form α, there exists a unique decomposition

$$\alpha = \sum_r L^r\alpha_r,$$

where α_r is primitive $(k-2r)$-form.

Proof. Firstly let's prove the uniqueness: If $\sum_r L^r\alpha_r = 0$ with primitive α_r, we need to show $\alpha_r = 0$. If not, then take the largest r_m such that $\alpha_{r_m} \neq 0$. By the choice of α_{r_m}, L^{n-k+r_m} kills everything in $\sum_r L^r\alpha_r$ but $L^{r_m}a_{r_m}$. Then

$$0 = L^{n-k+r_m}(\sum_r L^r\alpha_r) = L^{n-k+r_m}(L^{r_m}\alpha_{r_m}) \neq 0,$$

which is a contradiction.

Now let's prove the existence: Since $L^{n-k+2}\colon \mathcal{A}^{k-2}(X) \to \mathcal{A}^{2n-k+2}(X)$ is an isomorphism, then there exists $\beta \in \mathcal{A}^{k-2}(X)$ such that

$$L^{n-k+1}\alpha = L^{n-k+2}\beta.$$

Then $\alpha - L\beta$ is primitive a primitive k-form, that is

$$\alpha = (\alpha - L\beta) + L\beta.$$

By induction on k, we have primitive decomposition for $\beta \in \mathcal{A}^{k-2}(X)$. and this completes the proof. □

Remark 2.4.1. If we define $H = [L, \Lambda]$, then (L, H, Λ) generates an $\mathfrak{sl}_2$-action on $\bigoplus_k \mathcal{A}^k(X)$.

In fact, the Lefschetz operator also defines a map between cohomology groups

$$\begin{aligned} L\colon H^k(X,\mathbb{R}) &\to H^{k+2}(X,\mathbb{R}) \\ [\alpha] &\mapsto [\omega\wedge\alpha]. \end{aligned}$$

Now let's see it's well-defined:

(1) If α is closed, then

$$\mathrm{d}(\omega\wedge\alpha) = \mathrm{d}\omega\wedge\alpha + \omega\wedge\mathrm{d}\alpha = 0.$$

(2) If $\alpha = \mathrm{d}\beta$, then

$$\omega\wedge\mathrm{d}\beta = \mathrm{d}\omega\wedge\beta + \omega\wedge\mathrm{d}\beta = \mathrm{d}(\omega\wedge\beta).$$

Theorem 2.4.1 (hard Lefschetz theorem[4])**.** Let (X,ω) be a compact Kähler n-manifold. Then

$$L^{n-k}\colon H^k(X,\mathbb{R}) \to H^{2n-k}(X,\mathbb{R})$$

is an isomorphism for $1 \leq k \leq n$.

Proof. In Exercise 2.2.2 we have shown $[\Delta_{\mathrm{d}}, L] = 0$, so the Lefschetz operator induces a map between harmonic forms as follows

$$L^{n-k}\colon \mathcal{H}^k(X) \to \mathcal{H}^{2n-k}(X).$$

By Proposition 2.4.1 L^{n-k} is injective and $\mathcal{H}^k(X), \mathcal{H}^{2n-k}(X)$ have the same dimension, we obtain the desired result. □

[4]Though proof of this theorem is quite easy using tools we have, but it's quite hard for Lefschetz, since during his time, there is no Hodge theorem. Here we use L to denote Lefschetz operator, in order to honor Lefschetz.

Definition 2.4.2. Let (X,ω) be a compact Kähler n-manifold. For $[\alpha]\in H^k(X,\mathbb{R})$, it's called **primitive**, if $L^{n-k+1}[\alpha]=0$.

Notation 2.4.1. $H^k(X,\mathbb{R})_{\mathrm{prim}}$ denotes the set of all cohomology classes which are primitive forms.

Corollary 2.4.1 (Lefschetz decomposition)**.** There is the following decomposition

$$H^k(X,\mathbb{R})=\bigoplus_r L^r H^{k-2r}(X,\mathbb{R})_{\mathrm{prim}}.$$

Remark 2.4.2. If $[\omega]\in H^2(X,\mathbb{Z})$, such as ω comes from a positive holomorphic line bundle, then we can state theorem and corollary for $H^k(X,\mathbb{Q})$.

Moreover, we have the following isomorphism

$$L^{n-k}\colon H^{p,q}(X)\to H^{n-q,n-p}(X)$$

for $k=p+q\leq n$.

Corollary 2.4.2. Let (X,ω) be a compact Kähler n-manifold. Then for $2\leq k\leq n$, one has $b_{k-2}\leq b_k$ and $h^{p-1,q-1}\leq h^{p,q}$ with $k=p+q$.

2.5. Hodge metric and Hodge index.

2.5.1. *Surface case.*

Example 2.5.1. For open subset $U\subseteq\mathbb{C}^2$ equipped with canonical Kähler form

$$\omega=\sqrt{-1}\left(\mathrm{d}z^1\wedge\mathrm{d}\overline{z}^1+\mathrm{d}z^2\wedge\mathrm{d}\overline{z}^2\right).$$

The volume form is given by

$$\mathrm{vol}=\frac{\omega^2}{2!}=-\left(\mathrm{d}z^1\wedge\mathrm{d}\overline{z}^1\wedge\mathrm{d}z^2\wedge\mathrm{d}\overline{z}^2\right).$$

Suppose α is a $(2,0)$ form written as

$$\alpha=a\mathrm{d}z^1\wedge\mathrm{d}z^2.$$

Now we're going to compute $\star\overline{\alpha}$, which is also a $(2,0)$-form. Suppose $\star\overline{\alpha}=b\mathrm{d}z^1\wedge\mathrm{d}z^2$. Then by definition, for an arbitrary $(0,2)$-form β one has

$$\langle\beta,\alpha\rangle\,\mathrm{vol}=\beta\wedge\star\overline{\alpha}.$$

In particular, if we choose $\beta=\mathrm{d}\overline{z}^1\wedge\mathrm{d}\overline{z}^2$, then

$$\begin{aligned}\beta\wedge\star\overline{\alpha}&=-b\mathrm{d}z^1\wedge\mathrm{d}\overline{z}^1\wedge\mathrm{d}z^2\wedge\mathrm{d}\overline{z}^2\\ \{\beta,\alpha\}\,\mathrm{vol}&=\{a\mathrm{d}z^1\wedge\mathrm{d}z^2,\mathrm{d}\overline{z}^1\wedge\mathrm{d}\overline{z}^2\}\times-\mathrm{d}z^1\wedge\mathrm{d}\overline{z}^1\wedge\mathrm{d}z^2\wedge\mathrm{d}\overline{z}^2\\ &=-a\mathrm{d}z^1\wedge\mathrm{d}\overline{z}^1\wedge\mathrm{d}z^2\wedge\mathrm{d}\overline{z}^2.\end{aligned}$$

This shows $\alpha=\star\overline{\alpha}$. By the same computation one has $\alpha=\star\overline{\alpha}$ holds for a $(0,2)$-form α. On the other hand, it's clear $(2,0)$-form and $(0,2)$-form are automatically primitive.

Now we're going to see if a $(1,1)$-form α is primitive, what's the relation between α and $\star\overline{\alpha}$. For $(1,1)$-form α, written as

$$\alpha = a_{11}\mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^1 + a_{22}\mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^2 + a_{12}\mathrm{d}z_1 \wedge \mathrm{d}\overline{z}^2 + a_{21}\mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^1.$$

A direct computation shows that

$$\star\overline{\alpha} = a_{22}\mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^1 + a_{11}\mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^2 - a_{12}\mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^2 - a_{21}\mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^1.$$

On the other hand,

$$L\alpha = \omega \wedge \alpha = \sqrt{-1}(a_{11} + a_{22})\mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^2.$$

Then

$$L\alpha = 0 \Longleftrightarrow a_{11} + a_{22} = 0 \Longleftrightarrow \star\overline{\alpha} = -\alpha.$$

Lemma 2.5.1. Let (X,ω) be a Kähler surface. If (p,q)-form α is primitive 2-form, then

$$\star\overline{\alpha} = (-1)^p\alpha.$$

Proof. By taking normal coordinate, it suffices to consider $U \subseteq \mathbb{C}^2$, and that's exactly what we have done in Example 2.5.1. □

Let X be a compact Kähler surface. The Poincaré duality and Stokes theorem imply that we have the following well-defined non-degenerate pairing

$$\begin{aligned} Q\colon H^2(X,\mathbb{R}) \times H^2(X,\mathbb{R}) &\to \mathbb{R} \\ ([\alpha],[\beta]) &\mapsto \int_X \alpha \wedge \beta. \end{aligned}$$

Then we obtain a Hermitian form by considering

$$H([\alpha],[\beta]) = Q([\alpha],\overline{[\beta]}).$$

Lemma 2.5.2. The Lefschetz decomposition $H^2(X,\mathbb{R}) = H^2(X,\mathbb{R})_{\mathrm{prim}} \oplus \mathbb{R}\cdot[\omega]$ is orthonormal with respect to Q.

Proof.

$$Q([\omega],[\alpha]) = \int_X \omega \wedge \alpha = \int_X L\alpha = 0$$

for α is primitive and harmonic. □

Theorem 2.5.1. $H^2(X,\mathbb{C})_{\mathrm{prim}} = \bigoplus_{p+q=2} H^{p,q}(X)_{\mathrm{prim}}$ is orthonormal with respect to H, and $(-1)^pH$ is positive definite on $H^{p,q}(X)_{\mathrm{prim}}$.

Proof. It's clear above decomposition is orthonormal, since if α is not a $(2,2)$-form, one always has

$$\int_X \alpha = 0.$$

To see $(-1)^p H$ is positive definite on $H^{p,q}(X)_{\text{prim}}$, we take a harmonic representative α for any non-zero primitive cohomology class in $H^{p,q}(X)_{\text{prim}}$. Then

$$\begin{aligned}(-1)^p H([\alpha],[\alpha]) &= (-1)^p \int_X \alpha \wedge \overline{\alpha} \\ &= (-1)^{p+q} \int_X \alpha \wedge \star\overline{\alpha} \\ &= \|\alpha\|^2 > 0.\end{aligned}$$

This shows $(-1)^p H$ is positive definite on $H^{p,q}(X)_{\text{prim}}$. □

Corollary 2.5.1 (Hodge index)**.** The index of H defined on $H^2(X,\mathbb{C}) \cap H^{1,1}(X)$ is $(1, h^{1,1}-1)$.

Proof. Note that there is the following decomposition

$$H^2(X,\mathbb{C}) \cap H^{1,1}(X) = H^{1,1}(X)_{\text{prim}} \oplus \mathbb{C}[\omega],$$

and we have already shown that H is negative definite on $H^{1,1}(X)_{\text{prim}}$. Then the index for H on $H^2(X,\mathbb{C}) \cap H^{1,1}(X)$ is $(1, h^{1,1}-1)$. □

2.5.2. *General case.* In this section we will introduce a more general case: Let (X,ω) be a compact Kähler n-manifold. Then by Lefschetz decomposition we have

$$H^k(X,\mathbb{R}) = \bigoplus_r L^r H^{k-2r}(X,\mathbb{R})_{\text{prim}}, \quad k \leq n,$$

and by Hodge decomposition we have a more explicit decomposition

$$H^k(X,\mathbb{C})_{\text{prim}} = \bigoplus_{p+q=k} H^{p,q}(X)_{\text{prim}}.$$

As we have seen in the case of surface, H will be positive definite or negative definite in these (p,q) components. Now we introduce some symbols, in order to get a neater result.

Consider

$$\begin{aligned} Q\colon H^k(X,\mathbb{R}) \times H^k(X,\mathbb{R}) &\to \mathbb{R} \\ ([\alpha],[\beta]) &\mapsto (-1)^{\frac{k(k-1)}{2}} \int_X \omega^{n-k} \wedge \alpha \wedge \beta. \end{aligned}$$

Then Q is a bilinear form, and it is symmetric when k is even and anti-symmetric when k is odd.

Definition 2.5.1. The **Weil operator** $\mathsf{C}\colon H^k(X,\mathbb{C}) \to H^k(X,\mathbb{C})$ is defined by $\mathsf{C}|_{H^{p,q}(X)} \mapsto \sqrt{-1}^{p-q}\,\mathrm{id}$.

Remark 2.5.1. The Weil operator C maps $H^k(X,\mathbb{R})$ to $H^k(X,\mathbb{R})$ in fact:

$$\mathsf{C}|_{\overline{H^{p,q}(X)}} = \mathsf{C}_{H^{q,p}(X)} = \sqrt{-1}^{q-p}\,\mathrm{id} = \overline{\sqrt{-1}^{p-q}}\,\mathrm{id} = \overline{\mathsf{C}|_{H^{p,q}(X)}}.$$

Now we define

$$
\begin{aligned}
H\colon H^k(X,\mathbb{C}) \times H^k(X,\mathbb{C}) &\to \mathbb{C} \\
([\alpha],[\beta]) &\mapsto Q(\mathsf{C}[\alpha], \overline{[\beta]}).
\end{aligned}
$$

In other words, we have

$$
H([\alpha],[\beta]) = (-1)^{\frac{k(k-1)}{2}} \sqrt{-1}^{p-q} \int_X \omega^{n-k} \wedge \alpha \wedge \overline{\beta}, \quad \alpha,\beta \in H^{p,q}(X).
$$

Exercise 2.5.1. H is a Hermitian form on $H^{p,q}(X)$.

Proof. For $[\alpha],[\beta] \in H^{p,q}(X)$, one has

$$
\begin{aligned}
\overline{H([\alpha],[\beta])} &= (-1)^{\frac{k(k-1)}{2}} (-1)^{p-q} \sqrt{-1}^{p-q} \int_X \omega^{n-k} \wedge \overline{\alpha} \wedge \beta \\
&= (-1)^{\frac{k(k-1)}{2}} (-1)^{p-q} \sqrt{-1}^{p-q} (-1)^{(p+q)^2} \int_X \omega^{n-k} \wedge \beta \wedge \overline{\alpha}.
\end{aligned}
$$

Note that

$$
(p+q)^2 - p - q = 2pq + p(p-1) + q(q-1)
$$

is always even, this completes the proof. □

Lemma 2.5.3. Let α be a primitive (p,q)-form with $p+q=k$. Then

$$
\star\alpha = (-1)^{\frac{k(k+1)}{2}} \sqrt{-1}^{p-q} \frac{L^{n-k}\alpha}{(n-k)!}.
$$

Theorem 2.5.2 (Hodge-Riemann bilinear relations)**.**
(1) $H^k(X,\mathbb{R}) = \bigoplus_r L^r H^{k-2r}(X,\mathbb{R})_{\mathrm{prim}}$ is orthonormal with respect to Q.
(2) $H^k(X,\mathbb{C})_{\mathrm{prim}} = \bigoplus_{p+q=k} H^{p,q}(X)_{\mathrm{prim}}$ is orthonormal with respect to H.
(3) H is positive definite on $H^{p,q}(X)_{\mathrm{prim}}$.

Proof. For (1). For $r < s$, note that

$$
\omega^{n-k} \wedge L^r\gamma \wedge L^s\delta = (L^{n-k+r+s}\gamma) \wedge \delta = 0
$$

since $L^{n-k+2r+1}\gamma = 0$ and $r < s$.

For (2). If α is a (p,q)-form, and β is (p',q')-form, and $(p,q) \neq (p',q')$, then $\omega^{n-k} \wedge \alpha \wedge \overline{\beta}$ is not a (n,n)-form.

For (3). To see H is positive definite on $H^{p,q}(X)_{\mathrm{prim}}$, we take a harmonic representative α for any non-zero primitive cohomology class in $H^{p,q}(X)_{\mathrm{prim}}$. Then

$$
H([\alpha],[\alpha]) = (-1)^{\frac{k(k-1)}{2}} \sqrt{-1}^{p-q} \int_X \omega^{n-k} \wedge \alpha \wedge \overline{\alpha}
$$

By Lemma 2.5.3 one has

$$
\begin{aligned}
\star\overline{\alpha} &= (-1)^{\frac{k(k+1)}{2}} \sqrt{-1}^{q-p} \frac{L^{n-k}\overline{\alpha}}{(n-k)!} \\
&= (-1)^{\frac{k(k-1)}{2}} \sqrt{-1}^{p-q} \frac{L^{n-k}\overline{\alpha}}{(n-k)!}.
\end{aligned}
$$

Then
$$H([\alpha],[\alpha]) = (n-k)!\int_X \alpha\wedge\star\overline{\alpha} = (n-k)!\|\alpha\|^2 > 0.$$
□

Corollary 2.5.2 (Hodge index theorem)**.** Let X be a compact Kähler n-manifold with n is even[5]. Then $\int_X \alpha\wedge\beta$ on $H^n(X,\mathbb{R})$ is of signature
$$\sum_{p,q}(-1)^p h^{p,q}$$
where summation runs over all p,q.

Proof. Note that the signature of $\int_X \alpha\wedge\beta$ on $H^n(X,\mathbb{R})$ is the same as the signature of $\int_X \alpha\wedge\overline{\beta}$ on $H^n(X,\mathbb{C})$. We write
$$H^n(X,\mathbb{C}) = \bigoplus_{\substack{p+q+2r=n\\ r,p,q\in\mathbb{Z}_{\geq 0}}} L^r H^{p,q}(X)_{\mathrm{prim}}$$
Then Hodge-Riemann bilinear theorem implies that $\int_X \alpha\wedge\overline{\beta}$ is $(-1)^p$-definite on $L^rH^{p,q}(X)_{\mathrm{prim}}$, where we used the fact n is even. Then we have the signature is
$$\sum_{p+q+2r=n}(-1)^p h^{p,q}_{\mathrm{prim}}.$$
But $h^{p,q}_{\mathrm{prim}} = h^{p,q} - h^{p-1,q-1}$, so
$$\sum_{p+q+2r=n}(-1)^p(h^{p,q} - h^{p-1,q-1}).$$
Note that $p+q=n$ counted once and $p+q<n$ counted twice, so rewrite it as
$$\sum_{p+q \text{ even}}(-1)^p h^{p,q},$$
since $h^{p,q} = h^{n-p,n-q}$. And this is also equivalent to sum all p,q, since
$$\sum_{p+q \text{ odd}}(-1)^p h^{p,q} = 0$$
This completes the proof. □

Example 2.5.2. For surface, we have
$$H^2(X,\mathbb{C}) = H^{2,0}(X)\oplus H^{1,1}(X)_{\mathrm{prim}}\oplus\mathbb{C}[\omega]\oplus H^{0,2}(X)$$
Then this corollary implies
$$h^{0,0}+h^{2,0}-h^{1,1}+h^{0,2}+h^{2,2} = h^{2,0}+h^{0,2}+(1-(h^{1,1}-1)),$$
which recovers what we have done in the case of surface.

[5] In this case $\int_X \alpha\wedge\beta$ is symmetric on $H^n(X,\mathbb{R})$.

3. Kodaira vanishing

Theorem 3.1 (Kodaira-Akizuki-Nakano vanishing)**.** Let X be a non-singular projective complex variety with dimension n and L be an ample line bundle on X. Then

$$H^q(X, \Omega_X^p \otimes L) = 0$$

for all $p + q > n$.

Theorem 3.2 (Kodaira vanishing)**.** Let X be a non-singular projective complex variety and L be an ample line bundle on X. Then

$$H^q(X, K_X \otimes L) = 0$$

for all $q > 0$.

In birational geometry, the following vanishing theorem is also extremely useful.

Theorem 3.3 (Kawamata-Viehweg vanishing)**.** Let X be a non-singular projective complex variety and $D = \sum_i a_i D_i$ be an effective $\mathbb{Q}$-divisor, where $a_i \in [0, 1) \cap \frac{1}{N}\mathbb{Z}$ for some integer $N \geq 1$. For the line bundle L such that $L^{\otimes N} \otimes \mathcal{O}_X(-ND)$ is ample[6], it holds that

$$H^q(X, K_X \otimes L) = 0$$

for all $q > 0$.

3.1. Differential geometry method.

Let (X, ω) be a compact Kähler manifold and (E, h) be a Hermitian holomorphic vector bundle over X equipped with Chern connection ∇_h. For the following operators

$$\begin{aligned}
\partial_E &\colon \mathcal{A}^{p,q}(X, E) \to \mathcal{A}^{p+1,q}(X, E) \\
\overline{\partial}_E &\colon \mathcal{A}^{p,q}(X, E) \to \mathcal{A}^{p,q+1}(X, E) \\
L &\colon \mathcal{A}^{p,q}(X, E) \to \mathcal{A}^{p+1,q+1}(X, E) \\
\Lambda &\colon \mathcal{A}^{p,q}(X, E) \to \mathcal{A}^{p-1,q-1}(X, E),
\end{aligned}$$

there are also Kähler identities

$$\begin{aligned}
[\overline{\partial}_E^*, L] &= \sqrt{-1}\partial_E \\
[\partial_E^*, L] &= -\sqrt{-1} \cdot \overline{\partial}_E \\
[\Lambda, \overline{\partial}_E] &= -\sqrt{-1}\partial_E^* \\
[\Lambda, \partial_E] &= \sqrt{-1} \cdot \overline{\partial}_E^*,
\end{aligned}$$

and

$$[L, \Lambda] = (p + q - n)\,\mathrm{id}$$

holds on E-valued (p, q)-forms.

---

[6] In fact, nef and big.

Proposition 3.1.1 (Bochner-Kodaira-Nakano identity)**.** Let (X,ω) be a compact Kähler manifold and (E,h) be a Hermitian holomorphic vector bundle. Then

$$\Delta_{\overline{\partial}_E} = [\sqrt{-1}\Theta_h, \Lambda] + \Delta_{\partial_E}.$$

Proof. Direct computation shows

$$\begin{aligned}
\Delta_{\overline{\partial}_E} &= [\overline{\partial}_E, \overline{\partial}^*_E] \\
&= -\sqrt{-1}[\overline{\partial}_E, [\Lambda, \partial_E]] \\
&= -\sqrt{-1}[\Lambda, [\partial_E, \overline{\partial}_E]] - \sqrt{-1}[\partial_E, [\overline{\partial}_E, \Lambda]] \\
&= -\sqrt{-1}[\Lambda, \Theta_h] - \sqrt{-1}[\partial_E, \sqrt{-1}\partial^*_E] \\
&= [\sqrt{-1}\Theta_h, \Lambda] + \Delta_{\partial_E}.
\end{aligned}$$

□

Corollary 3.1.1 (Bochner-Kodaira-Nakano inequality)**.** Let (X,ω) be a compact Kähler manifold and (E,h) a Hermitian holomorphic vector bundle. Then for $\alpha \in \mathcal{A}^{p,q}(X,E)$, one has

$$([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) \leq (\Delta_{\overline{\partial}_E}\alpha, \alpha)$$

In particular, if α is $\Delta_{\overline{\partial}_E}$-harmonic, then $([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) \leq 0$.

Proof. Direct computation shows

$$\begin{aligned}
(\Delta_{\overline{\partial}_E}\alpha, \alpha) - ([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) &= (\Delta_{\partial_E}\alpha, \alpha) \\
&= \|\partial_E\alpha\|^2 + \|\partial^*_E\alpha\|^2 \geq 0.
\end{aligned}$$

□

Theorem 3.4 (Kodaira-Akizuki-Nakano vanishing)**.** Let X be a compact n-manifold, (L,h) a positive Hermitian holomorphic line bundle. Then

$$H^{p,q}(X,L) = 0$$

for $p+q > n$.

Proof. Let X be endowed with the Kähler metric ω given by Chern curvature of L. Then there is an isomorphism $H^{p,q}(X,L) \cong \mathcal{H}^{p,q}(X,L)$. For $\alpha \in \mathcal{H}^{p,q}(X,L)$, by Corollary 3.1.1 one has

$$[\sqrt{-1}\Theta_h, \Lambda]\alpha \leq 0.$$

On the other hand,

$$([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) = 2\pi(p+q-n)\|\alpha\|^2 \geq 0.$$

Thus if $p+q > n$, one has $\alpha = 0$. This completes the proof. □

Corollary 3.1.2 (Kodaira vanishing)**.** Let X be a compact n-manifold and (L,h) be a positive holomorphic line bundle over X. Then

$$H^q(X, K_X \otimes L) = 0$$

for $q > 0$.

3.2. **Algebraic geometry method.** An algebraic geometry proof of Kodaira's vanishing theorem uses the cyclic cover tricks and logarithmic differential forms. Here we give a brief introduction about these techniques, and a good reference is [EV92].

For convenience, X always denotes a non-singular projective complex variety of dimension n unless otherwise stated.

3.2.1. *Logarithmic differential forms.* Let D be a reduced normal crossing divisor[7] in X. The sheaf of differential k-forms with logarithmic singularities along D, denoted by $\Omega_X^k(\log D)$, is the subsheaf of $\Omega_X^k(*D)$[8] defined by the following condition:

- If α is a meromorphic differential forms on U, holomorphic on $U \setminus D \cap U$, then $\alpha \in \Omega_X^k(\log D)|_U$ if α admits a pole of order at most 1 along D, and the same holds for $\mathrm{d}\alpha$.

Lemma 3.1. Let $\{z_1, \dots, z_n\}$ be a local coordinate on an open subset U of X, in which $D \cap U$ is defined by the equation $z_1 \dots z_r = 0$. For convenience we denote

$$\delta_j = \begin{cases} \mathrm{d}z_j/z_j & j \leq r \\ \mathrm{d}z_j & j > r, \end{cases}$$

and for $I = \{j_1, \dots, j_k\} \subseteq \{1, \dots, n\}$ with $j_1 < \dots < j_s$, we denote

$$\delta_I = \delta_{j_1} \wedge \dots \wedge \delta_{j_k}.$$

Then $\Omega_X^k(\log D)|_U$ is a sheaf of free $\mathcal{O}_U$-modules with basis $\{\delta_I\}_{|I|=k}$.

Proof. See Proposition 2.2 in [EV92]. □

Corollary 3.2.1.

(1) $\Omega_X^k(\log D) = \bigwedge^k \Omega_X^1(\log D)$.
(2) The sheaves $\Omega_X^k(\log D)$ are sheaves of locally free $\mathcal{O}_X$-modules.

Notations for local frames of logarithmic differential forms as Lemma 3.1 will be used along the way. For example, one can defined the following several maps by using local frames.

(1) The first one is

$$\alpha\colon \Omega_X^1(\log D) \to \bigoplus_{j=1}^{r} \mathcal{O}_{D_j}$$

which is locally defined by $\sum_{j=1}^n a_j\delta_j \mapsto \bigoplus_{j=1}^r a_j|_{D_j}$.

[7] A divisor $D = \sum_{j=1}^r D_j$ is called a reduced normal crossing divisor, if locally there exists coordinate $\{z_1, \dots, z_n\}$ on X such that D is defined by the equation $z_1 \dots z_r = 0$ for an integer r which naturally depends on the considered open set.

[8] $\Omega_X^k(*D)$ is the sheaf of meromorphic forms on X, holomorphic on $X \setminus D$.

(2) For $k \geq 1$, one has

$$\beta_1 \colon \Omega_X^k(\log D) \to \Omega_{D_1}^{k-1}(\log(D - D_1)|_{D_1})$$

which is given by: For local section

$$\varphi = \varphi_1 + \varphi_2 \wedge \frac{\mathrm{d}z_1}{z_1},$$

where φ_1 lies in the span of the δ_I with $1 \notin I$ and $\varphi_2 = \sum_{1 \in I} a_I \delta_{I \setminus \{1\}}$, we define

$$\beta_1(\varphi) = \sum a_I \delta_{I \setminus \{1\}}|_{D_1}.$$

(3) Finally the natural restriction gives

$$\gamma_1 \colon \Omega_X^k(\log(D - D_1)) \to \Omega_{D_1}^k(\log(D - D_1)|_{D_1}).$$

Note that $\{z_1 \cdot \delta_I \mid 1 \in I\} \cup \{\delta_I \mid 1 \notin I\}$ gives a local frame of $\Omega_X^k(\log(D - D_1))$. Then γ_1 can be described as

$$\gamma_1(\sum_{1 \in I} z_1 a_I \delta_I + \sum_{1 \notin I} a_I \delta_I) = \sum_{1 \notin I} a_I \delta_I|_{D_1}.$$

Remark 3.2.1. Similarly, β_i and γ_i are the corresponding map for the i-th component D_i.

Proposition 3.2.1. The following sequences of sheaves are exact.

(1)

$$0 \to \Omega_X^1 \to \Omega_X^1(\log D) \xrightarrow{\alpha} \bigoplus_{j=1}^r \mathcal{O}_{D_j} \to 0$$

(2)

$$0 \to \Omega_X^k(\log(D - D_1)) \to \Omega_X^k(\log D) \to \Omega_{D_1}^{k-1}(\log(D - D_1)|_{D_1}) \xrightarrow{\beta_1} 0$$

(3)

$$0 \to \Omega_X^k(\log D)(-D_1) \to \Omega_X^k(\log(D - D_1)) \to \Omega_{D_1}^k(\log(D - D_1)|_{D_1}) \xrightarrow{\gamma_1} 0$$

Proof. It follows from the definition of α, β_1 and γ_1. □

Definition 3.2.1. Let $\mathcal{E}$ be a locally free coherent sheaf on X. A **logarithmic connection** is a $\mathbb{C}$-linear map $\nabla \colon \mathcal{E} \to \Omega_X^1(\log D) \otimes \mathcal{E}$ satisfying the Leibniz rule, that is

$$\nabla(f \cdot e) = f \cdot \nabla e + \mathrm{d}f \otimes e,$$

and it extends to

$$\nabla \colon \Omega_X^k(\log D) \otimes \mathcal{E} \to \Omega_X^{k+1}(\log D) \otimes \mathcal{E}$$

by the rule

$$\nabla(\omega \otimes e) = \mathrm{d}\omega \otimes e + (-1)^k \omega \wedge \nabla e.$$

Definition 3.2.2. A logarithmic connection ∇ is called **flat**[9] if it's curvature is zero, that is, $\nabla^2 = 0$.

Definition 3.2.3. For a flat logarithmic connection

$$\nabla\colon \mathcal{E} \to \Omega^1_X(\log D) \otimes \mathcal{E},$$

the **residue map** along D_1 is defined to be the composed map

$$\mathrm{Res}_{D_1}(\nabla)\colon \mathcal{E} \xrightarrow{\nabla} \Omega^1_X(\log D) \otimes \mathcal{E} \xrightarrow{\beta_1 \otimes \mathrm{id}} \mathcal{O}_{D_1} \otimes \mathcal{E}.$$

3.2.2. *Cyclic covers.* Let $\mathcal{L}$ be an invertible sheaf, $D = \sum_{j=1}^r \alpha_j D_j$ be an effective divisor and N be a positive natural number such that $\mathcal{L}^N = \mathcal{O}_X(D)$. Let $s \in H^0(X, \mathcal{L}^N)$ be a section whose zero divisor is D. Then the dual of $s\colon \mathcal{O}_X \to \mathcal{L}^N$ (that is, $s^\vee\colon \mathcal{L}^{-N} \to \mathcal{O}_X$) gives a $\mathcal{O}_X$-algebra structure on

$$\mathcal{A}' = \bigoplus_{i=0}^{N-1} \mathcal{L}^{-i}$$

Remark 3.2.2. In fact,

$$\mathcal{A}' = \bigoplus_{i=0}^{\infty} \mathcal{L}^{-i}/I,$$

where I is the ideal sheaf generated locally by

$$\{s^\vee(l) - l \mid l \text{ is local section of } \mathcal{L}^{-N}\}.$$

Definition 3.2.4. Let $Y' = \operatorname{Spec} \mathcal{A}' \to X$ be the spectrum of the $\mathcal{O}_X$-algebra $\mathcal{A}'$ and $\pi\colon Y \to X$ be the finite morphism obtained by normalizing $Y' \to X$. Then Y is called **cyclic cover** obtained by taking n-th root out of D.

Amazingly, $\pi\colon Y \to X$ can be quite explicit, if we assume some technical conditions.

Lemma 3.2.1. In the case $(N, \mathrm{char} k) = 1$, if $\mathcal{L}^{(i)^{-1}} = \mathcal{L}^{-i} \otimes \mathcal{O}_X([\frac{i}{N} D])$, then

$$\mathcal{A} = \bigoplus_{i=0}^{N-1} \mathcal{L}^{(i)-1}$$

is naturally a sheaf of $\mathcal{O}_X$-algebra such that the normalization $\sigma\colon Y \to Y'$ is given by the natural inclusion of $\mathcal{O}_X$-algebra

$$\mathcal{A}' \hookrightarrow \mathcal{A}.$$

Moreover, the cyclic group G of order N acts naturally on Y as automorphism over X such that $Y/G = X$ and

$$\pi_* \mathcal{O}_Y = \bigoplus_{i=0}^{N-1} (\mathcal{L}^{(i)})^{-1}$$

is an eigenspace decomposition.

[9]Sometimes is also called integrable.

Proof. See page 23-25 of [EV92]. □

From now on, we shall focus on a particularly simple case, that is, the divisor D is reduced and non-singular. In this case, $\mathcal{A}' = \mathcal{A}$ and Y is even non-singular.

Proposition 3.2.2. Let $\pi\colon Y \to X$ be as above, where the divisor D is non-singular and reduced. Then

(1) $\pi^*\Omega_X^k(\log D) = \Omega_Y^k(\log(\pi^*D)_{\mathrm{red}})$.
(2) For each $1 \le i \le N-1$, $\mathrm{d}\colon \mathcal{O}_Y \to \Omega_Y$ induces a flat logarithmic connection
$$\nabla^i\colon \mathcal{L}^{-i} \to \mathcal{L}^{-i} \otimes \Omega_X^1(\log D)$$
such that
$$\bigoplus_{i=1}^{N-1} \nabla^i = \pi_*\mathrm{d}\colon \pi_*\mathcal{O}_Y \to \pi_*(\Omega_Y^1(\log \pi^*D))$$
is an eigen-decomposition with respect to the action of cyclic group G of order N.
(3) $\pi_*\Omega_Y^k = \Omega_X^k \oplus \bigoplus_{i=1}^{N-1} \Omega_X^k(\log D) \otimes \mathcal{L}^{-i}$ is an eigenspace decomposition so that in (2), the G-invariant eigenspace is just $\mathrm{d} = \nabla_0\colon \mathcal{O}_X \to \Omega_X$.

Proof. Here we only give a proof with the dimension of X is 2 and $N = 2$, and the general case is similar. □

3.2.3. *E_1 degegeration of Hodge to de Rham spectral of logarithmic de Rham complex.*

Proposition 3.2.3. The Hodge to de Rham spectral sequence associated to the logarithmic de Rham complex for $1 \le i \le N-1$
$$\mathcal{L}^{-i} \xrightarrow{\nabla^i} \mathcal{L}^{-i} \otimes \Omega_X^1(\log D) \xrightarrow{\nabla^i} \mathcal{L}^{-i} \otimes \Omega_X^2(\log D) \to \dots$$
degenerates at E_1-page.

3.2.4. *An Algebraic proof of Kodaira's vanishing theorem.* In this section we present an algebraic proof of Kodaira's vanishing theorem by Serre vanishing, E_1-degeneration and cyclic covers.

Algebraic proof of Theorem 3.2. Take sufficiently large integer N such that there exists $0 \ne s \in H^0(X, \mathcal{L}^N)$ with $\mathrm{div}(s) = D$ is integral and non-singular. Let $\pi\colon Y \to X$ be the N-th cyclic cover associated to s. Then by E_1-degegeration (Proposition 3.2.3) one has
$$E_1^{0,p} = H^p(X, \mathcal{L}^{-1}) \xrightarrow{\nabla_1} H^p(X, \mathcal{L}^{-1} \otimes \Omega_X(\log D)) = E_1^{1,p}$$
is a zero map. Hence

$$
\begin{tikzcd}
H^p(X,\mathcal{L}^{-1}) \arrow[rr, "0"] \arrow[dr, "\nabla_1"'] & & H^p(D,\mathcal{L}^{-1}|_D) \\
& H^p(X,\mathcal{L}^{-1}\otimes\Omega_X(\log D)) \arrow[ur, "\mathrm{Res}"'] &
\end{tikzcd}
$$

Thus the long exact sequence of

$$0 \to \mathcal{L}^{-1}\otimes\mathcal{O}_X(-D) \to \mathcal{L}^{-1} \to \mathcal{L}^{-1}|_D \to 0$$

reads

$$\cdots \to H^p(X,\mathcal{L}^{-1}\otimes\mathcal{O}_X(-D)) \to H^p(X,\mathcal{L}^{-1}) \xrightarrow{0} H^p(D,\mathcal{L}^{-1}|_D) \to \cdots.$$

Note that $\mathcal{L}^{-1}\otimes\mathcal{O}_X(-D)=\mathcal{L}^{-N-1}$ and by Serre vanishing theorem

$$H^p(X,\mathcal{L}^{-N-1})=0$$

holds for $p<n$ and sufficiently large N. As a consequence, one has

$$H^p(X,\mathcal{L}^{-1})=0$$

for $p<n$ as desired. $\square$

4. Cartier descent theorem

In this section we assume k is an algebraically closed field with positive characteristic p, $F_k \colon k \to k$ is the Frobenius map and X is a non-singular variety over k. The Frobenius map F_k induces so called **absolute Frobenius map** $F_X \colon X \to X$, which is the identity on the underlying space of X and the p-th power on the structure sheaf.

Remark 4.1. For convenience, we will abbreviate F_X by F if the reference to X is clear, and likewise we will denote the associated map

$$F_X^\sharp \colon \mathcal{O}_X \to F_*\mathcal{O}_X$$

by $F^\sharp$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then $F_*\mathcal{F}$ equals $\mathcal{F}$ as sheaves of abelian groups, but the $\mathcal{O}_X$-module structure on $F_*\mathcal{F}$ is given by $f \cdot s = f^p \cdot s$ for any local sections f of $\mathcal{O}_X$ and s of $\mathcal{F}$.

Let $X^{(p)}$ be the base change of X given by the Frobenius map F_k, that is, there is the following commutative diagram

$$\begin{tikzcd} X^{(p)} \arrow[r, "\pi"] \arrow[d] & X \arrow[d] \\ k \arrow[r, "F_k"] & k. \end{tikzcd}$$

Then there exists a morphism $F_{X/k} \colon X \to X^{(p)}$ such that the following diagram commutes, which is called **relative Frobenius map**, since F_X satisfies the following commutative diagram

$$\begin{tikzcd} X \arrow[dr, dotted] \arrow[drr, "F_X"] \arrow[ddr, "\alpha"'] & & \\ & X^{(p)} \arrow[r, "\pi"'] \arrow[d, "\alpha'"] & X \arrow[d, "\alpha"] \\ & k \arrow[r, "F_k"] & k. \end{tikzcd}$$

Definition 4.1. A ***k*-connection** on X is a pair $(\mathcal{E}, \nabla)$, which consists of the following data:

(1) $\mathcal{E}$ is a (quasi)-coherent $\mathcal{O}_X$-module.
(2) $\nabla \colon \mathcal{E} \to \mathcal{E} \otimes \Omega_{X/k}$ is k-linear such that

$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s$$

Definition 4.2. For a k-connection $(\mathcal{E}, \nabla)$ on X/k, the **curvature** of k-connection $(\mathcal{E}, \nabla)$ is defined by

$$\begin{aligned} \Theta_\nabla \colon \bigwedge^2 T_{X/k} &\to \mathrm{End}_k(\mathcal{E}) \\ D_1 \wedge D_2 &\mapsto [D_1, D_2] - \nabla_{[D_1, D_2]}, \end{aligned}$$

where the Lie bracket on $T_{X/k}$ is defined by $[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$.

Remark 4.2. For a k-connection $(\mathcal{E}, \nabla)$ on X, it can be regarded as a "quasi-representation" as follows

$$\nabla\colon T_{X/k} \to \mathrm{End}_k(\mathcal{E}),$$

and the curvature measures the failure of ∇ to be a Lie algebra representation. Moreover, one can show that $\Theta_\nabla\colon \bigwedge^2 T_{X/k} \to \mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$.

In the case of positive characteristic, there is so called p-curvature which doesn't appear in the zero characteristic case, since the p-th power map $D \mapsto D^p := \underbrace{D \circ \cdots \circ D}_{p \text{ times}}$ gives a map between $T_{X/k} \to T_{X/k}$.

Definition 4.3. The **p-curvature** of a k-connection $(\mathcal{E}, \nabla)$ over X/k is defined by

$$\begin{aligned} \Psi_\nabla\colon T_{X/k} &\mapsto \mathrm{End}_k(\mathcal{E}) \\ D &\mapsto (\nabla_D)^p - \nabla_{D^p}. \end{aligned}$$

Proposition 4.1. For any $D \in T_{X/k}$, the p-curvature $\Psi_\nabla(D)$ is $\mathcal{O}_X$-linear. In other words,

$$\Psi_\nabla\colon T_{X/k} \to \mathrm{End}_{\mathcal{O}_X}(\mathcal{E}).$$

Proof. For any $f \in \mathcal{O}_X$ and $s \in \mathcal{E}$, one has

$$(\nabla_D)^p(fs) = \sum_{i=0}^{p} \binom{p}{i} D^i(f)(\nabla_D)^{p-i}(s) = D^p(f)s + f\nabla_D^p(s).$$

On the other hand, it's clear

$$\nabla_{D^p}(fs) = \nabla^p(f)s + f\nabla_{D^p}(s).$$

Thus it follows

$$\Psi_\nabla(D)(fs) = f((\nabla_D)^p - \nabla_{D^p})(s) = f\Psi_\nabla(D)(s).$$

□

Proposition 4.2. Let $(\mathcal{E}, \nabla)$ be a k-connection over X/k. If the curvature Θ_∇ vanishes, then

(1) $\Psi_\nabla\colon T_{X/k} \to \mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$ is additive.
(2) Ψ_∇ is F_X-linear, that is,

$$\Psi_\nabla(fD) = f^p\Psi_\nabla(D).$$

(3) Ψ_∇ is integrable, that is, $\Psi_\nabla \wedge \Psi_\nabla = 0$.

It's a highly non-trivial fact, whose proof relies on the following algebra result.

Lemma 4.1. Let R be an associated ring with positive characteristic p. For $a, b \in R$,

(1) $(a+b)^p = a^p + b^p + \sum_{i=1}^{p-1} s_i(a,b)$, where
$$(\mathrm{ad}(ta+b))^p(a) = \sum_{i=0}^{p-1} i s_i(a,b) t^i.$$
(2) If $\{a^{(n)}\}_{n\geq 1}$ are mutually commutative, then
$$(ab)^p = a^p b^p + a(a^{p-1})^{(p-1)} b,$$
where
$$a^{(n)} := (\mathrm{ad}\, b)^n(a).$$

Proof. See [Kat70]. □

Now let's begin the proof of Proposition 4.2.

Proof of Proposition 4.2. For (1). For arbitary $D_1, D_2 \in T_{X/k}$, by using (1) of Lemma 4.1 one has
$$\begin{aligned}
(\nabla_{D_1+D_2})^p &= (\nabla_{D_1} + \nabla_{D_2})^p \\
&= (\nabla_{D_1})^p + (\nabla_{D_2})^p + \sum_i s_i(D_1, D_2) \\
\nabla_{(D_1+D_2)^p} &= \nabla_{(\nabla_1^p + \nabla_2^p + \sum_i s_i(D_1,D_2))} \\
&= \nabla_{D_1^p} + \nabla_{D_2^p} + \sum_i \nabla_{s_i(D_1,D_2)}.
\end{aligned}$$
Then
$$\Psi_\nabla(D_1+D_2) = (\nabla_{D_1+D_2})^p - \nabla_{(D_1+D_2)^p} = \Psi_\nabla(D_1) + \Psi_\nabla(D_2).$$

For (2). For arbitary $f \in \mathcal{O}_X$ and $D \in T_{X/k}$, by using (2) of Lemma 4.1, one has
$$\begin{aligned}
(fD)^p &= f^p D^p + f(\mathrm{ad}(D))^{p-1}(f^{p-1})D \\
&= f^p D^p + f(D^{p-1}(f^{p-1}))D,
\end{aligned}$$
since $\mathrm{ad}(D)(f^{p-1}) = D \circ f^{p-1} - f^{p-1}D = D(f^{p-1})$. Thus
$$\nabla_{(fD)^p} = f^p \nabla_{D^p} + f(D^{p-1}(f^{p-1}))\nabla_D.$$
Applying (2) of Lemma 4.1 again, one has
$$\begin{aligned}
(\nabla_{fD})^p = (f\nabla_D)^p &= f^p(\nabla_D)^p + f(\mathrm{ad}(\nabla_D))^{p-1}(f^{p-1})\nabla_D \\
&= f^p(\nabla_D)^p + f(D^{p-1}(f^{p-1}))\nabla_D.
\end{aligned}$$
This completes the proof of (2).

For (3). Let's check this by local computations. Suppose $\{z_1, \dots, z_n\}$ be a local coordinate and write
$$\begin{aligned}
D_1 &= \sum_i a_i \partial_i \\
D_2 &= \sum_j b_j \partial_j,
\end{aligned}$$

where $\partial_i = \frac{\partial}{\partial z_i}$. Note that

$$\begin{aligned}\Psi_\nabla(D_1) &= \Psi_\nabla(\sum_i a_i\partial_i) \\ &= \sum_i a_i^p \Psi_\nabla(\partial_i) \\ &= \sum_i a_i^p \left((\nabla_{\partial_i})^p - \nabla_{\partial_i^p}\right) \\ &= \sum_i a_i^p (\nabla_{\partial_i})^p.\end{aligned}$$

Then one has

$$\begin{aligned}[\Psi_\nabla(D_1), \Psi_\nabla(D_2)] &= \left[\sum_i a_i^p(\nabla_{\partial_i})^p, \sum_j b_j^p(\nabla_{\partial_j})^p\right] \\ &= \sum_{ij} a_i^p b_j^p \left[(\nabla_{\partial_i})^p, (\nabla_{\partial_j})^p\right] \\ &= 0,\end{aligned}$$

since $\nabla_{\partial_i}\nabla_{\partial_j} = \nabla_{\partial_j}\nabla_{\partial_i}$ since the curvature of ∇ vanishes. □

Holding notations as above, now we can state the main theorem of this section, which is a very basic theorem in geometry over field k with characteristic p.

Theorem 4.1 (Cartier descent theorem)**.** There is a natural equivalence of categories between the category of (quasi)-coherent $\mathcal{O}_{X^{(p)}}$-module and the category of flat k-connections $(\mathcal{E}, \nabla)$ on X with vanishing p-curvatures. To be precise, the correspondence is given by

(1) For (quasi)-coherent $\mathcal{O}_{X^{(p)}}$-module $\mathcal{E}$, it corresponds to the k-connection $(F^*_{X/k}\mathcal{E}, \nabla_{\mathrm{can}})$.
(2) For flat k-conenction $(\mathcal{V}, \nabla)$ with vanishing p-curvature, the corresponding (quasi)-coherent $\mathcal{O}_X$-module is the $\mathcal{O}_{X^{(p)}}$-submodule $\mathcal{V}^{\nabla=0} \subseteq \mathcal{V}$.

Remark 4.3.

(1) For (quasi)-coherent $\mathcal{O}_{X^{(p)}}$-module $\mathcal{E}$ and $e \in \mathcal{E}$, we write $e \otimes 1 \in F^*_{X/k}\mathcal{E} = \mathcal{E} \otimes_{\mathcal{O}_{X^{(p)}}} \mathcal{O}_X$. Then the **canonical connection** $\nabla_{\mathrm{can}}\colon F^*_{X/k}\mathcal{E} \to F^*_{X/k}\mathcal{E} \otimes \Omega_X$ is defined by

$$\nabla_{\mathrm{can}}(e \otimes f) = e \otimes \mathrm{d}f.$$

In other words, all pull-back section of $\mathcal{E}$ are flat with respect to ∇_{can}.

(2) For (quasi)-coherent $\mathcal{O}_X$-module $\mathcal{V}$ on X, the flat part $\mathcal{V}^{\nabla=0}$ is indeed a $\mathcal{O}_{X^{(p)}}$-module. For $f \in \mathcal{O}_{X^{(p)}}$ and $s \in \mathcal{V}^{\nabla=0}$, one has

$$\begin{aligned} \nabla(f \cdot s) &= \nabla(f^p \cdot s) \\ &= \mathrm{d}(f^p) \cdot s + f^p \cdot \nabla(s) \\ &= 0. \end{aligned}$$

5. De Rham decomposition theorem of Deligne-Illusie

5.1. Introduction.

In this section, unless otherwise specified, k always denotes an algebraically closed field with positive characteristic p. Let X be a non-singular variety over k and $F_{X/k}\colon X \to X^{(p)}$ denote the relative Frobenius map. Then

$$(F_{X/k})_*\Omega^{\bullet}_{X/k}\colon (F_{X/k})_*\mathcal{O}_X \to (F_{X/k})_*\Omega^1_{X/k} \to (F_{X/k})_*\Omega^2_{X/k} \to \dots$$

is a finite complex of coherent $\mathcal{O}_{X^{(p)}}$-module with $\mathcal{O}_{X^{(p)}}$-linear differential.

Theorem 5.1.1 (Deligne-Illusie)**.** Let X be a non-singular variety over k such that X is $W_2(k)$-liftable and $\dim_k X = n < p$. Then there is a quasi-isomorphism

$$(F_*\Omega^{\bullet}_{X/k}, F_*\mathrm{d}) \cong \bigoplus_{i=0}^{n} \Omega^i_{X/k}[-i].$$

Remark 5.1.1.

(1) The condition of $W_2(k)$-liftable (we will introduce later in Definition 5.1.2) cannot be removed, and the first counterexample is given by Michel Raynaud by showing Kodaira's vanishing theorem fails in positive characteristic in [Ray78].
(2) The statement still holds for $\dim_k X = p$, but in [Pet23] the author shows that it fails when $\dim_k X > p$.

5.1.1. *Witt vectors of length two.*

Definition 5.1.1. The **Witt ring** $W_2(k)$ can be interpreted as the set $k\times k$, where the multiplication and addition for $a = (a_0, a_1)$ and $b = (b_0, b_1)$ are defined by

$$ab = (a_0a_1, b_0a_1^p + b_1a_0^p),$$

and

$$a + b = (a_0 + b_0, a_1 + b_1 - \sum_{i=1}^{p-1} p^{-1}\binom{p}{t} a_0^i b_0^{p-i}).$$

Remark 5.1.2. In fact, the operations on $W_2(k)$ makes the ghost polynomial $\Phi(a_0, a_1) = a_0^p + pa_1$ a ring homomorphism.

Example 5.1.1. If $k = \mathbb{Z}/p\mathbb{Z}$, then $W_2(k) = \mathbb{Z}/p^2\mathbb{Z}$.

Proposition 5.1.1. The set $pW_2(k) = \{(0, a) \mid a \in k\}$ is a maximal ideal of $W_2(k)$, and the following sequence is exact

$$0 \to pW_2(k) \to W_2(k) \to k \to 0.$$

Proposition 5.1.2. The ring homomorphism $F_{W_2(k)}\colon W_2(k) \to W_2(k)$ given by $(a_0, a_1) \mapsto (a_0^p, a_1^p)$ reduces to the Frobenius map F_k on k modulo p.

Definition 5.1.2. Let X be a non-singular variety over k. If there exists a flat morphism $\widetilde{X} \to W_2(k)$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\cong"] \arrow[dr] & \widetilde{X} \times_{W_2(k)} k \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] \\ & k \arrow[r] & W_2(k), \end{tikzcd}$$

then X/k is $W_2(k)$**-liftable**.

Remark 5.1.3. Not every non-singular variety X/k is $W_2(k)$-liftable. In fact, there is an obstruction in $\mathrm{ob}(\alpha) \in H^2(X, T_{X/k})$ such that $\mathrm{ob}(\alpha) = 0$ if and only if X/k is $W_2(k)$-liftable.

5.1.2. *Cartier isomorphism.* Before the proof of Deligne-Illusie, there are many evidences, such as Cartier descent theorem.

Theorem 5.1.2. Let X be a non-singular variety over k of dimension n and $F_{X/k}\colon X \to X^{(p)}$ be the relative Frobenius map. Then there is a unique isomorphism of graded $\mathcal{O}_{X^{(p)}}$-algebra

$$C^{-1}\colon \bigoplus_{i=0}^{n} \Omega^i_{X^{(p)}/k} \to \bigoplus_{i=0}^{n} H^i((F_{X/k})_*\Omega^\bullet_{X/k}),$$

which is determined as follows

(1) On the zero degree, the operator $C^{-1}\colon \mathcal{O}_{X^{(p)}} \to H^0((F_{X/k})_*\Omega^\bullet_{X/k})$ is defined by the morphism $(F_{X/k})^*\colon \mathcal{O}_{X^{(p)}} \longrightarrow (F_{X/k})_*\mathcal{O}_X$.
(2) On the first degree, there is the following commutative diagram

$$\begin{tikzcd} C^{-1}\colon \Omega^1_{X^{(p)}/k} \arrow[r] \arrow[dr] & H^1((F_{X/k})_*\Omega^\bullet_{X/k}) \\ & Z^1((F_{X/k})_*\Omega^\bullet_{X/k}) \arrow[u] \end{tikzcd}$$

such that $C^{-1}(\mathrm{d}x') = x^{p-1}\mathrm{d}x \pmod{B^1}$, where $x \in \mathcal{O}_X$ and $x' = \pi^* x \in \mathcal{O}_{X^{(p)}}$.

Proof. The key observation is the global map $\mathcal{O}_{X^{(p)}} \to H^1((F_{X/k})_*\Omega^\bullet_{X/k})$ defined by sending x to $x^{p-1}\mathrm{d}x$ is a derivation. Then it factors throught $\Omega^1_{X^{(p)}}$, that is, the following diagram commutes

$$\begin{tikzcd} \Omega^1_{X^{(p)}/k} \arrow[r, "C^{-1}"] & H^1((F_{X/k})_*\Omega^\bullet_{X/k}) \\ \mathcal{O}_{X^{(p)}} \arrow[u] \arrow[ur] & \end{tikzcd}$$

Now let's prove our observation. For any $x, y \in \mathcal{O}_X$, note that

$$(x+y)^p = x^p + y^p + p\sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} x^i y^{p-i}.$$

Thus

$$(x+y)^{p-1}\mathrm{d}(x+y) = x^{p-1}\mathrm{d}x + y^{p-1}\mathrm{d}y + \mathrm{d}\left(\sum_{i=1}^{p-1}\frac{\binom{p}{i}}{p}x^iy^{p-i}\right),$$

that is,

$$(x+y)^{p-1}\mathrm{d}(x+y) \equiv x^{p-1}\mathrm{d}x + y^{p-1}\mathrm{d}y \pmod{B^1}.$$

On the other hand, a direct computation shows

$$(xy)^{p-1}\mathrm{d}(xy) \equiv x^p(y^{p-1}\mathrm{d}y) + y^p(x^{p-1}\mathrm{d}x) \pmod{B^1}$$

This completes the proof of the observation. To check C^{-1} is an isomorphism, we may assume X is affine, with local coordinate $\{z_1,\dots,z_n\}$. For simplicity we firstly assume $n=2$, then

$$\begin{aligned}
(F_{X/k})_*\mathcal{O}_X &= \mathcal{O}_{X^{(p)}}\{z_1^{i_1}z_2^{i_2} \mid 0\le i_1\le p-1, 0\le i_2\le p-1\}\\
(F_{X/k})_*\Omega^1_X &= \mathcal{O}_{X^{(p)}}\{z_1^{i_1}z_2^{i_2} \mid \dots\}\otimes \mathrm{d}z_1 \oplus \mathcal{O}_{X^{(p)}}\{z_1^{i_1}z_2^{i_2} \mid \dots\}\otimes \mathrm{d}z_2\\
(F_{X/k})_*\Omega^2_X &= \mathcal{O}_{X^{(p)}}\{z_1^{i_1}z_2^{i_2}\}\otimes \mathrm{d}z_1\wedge \mathrm{d}z_2.
\end{aligned}$$

In this case C^{-1} is given by

$$\begin{aligned}
C^{-1}\colon \Omega^1_{X^{(p)}} &\to H^1((F_{X/k})_*\Omega^\bullet_{X/k})\\
\mathrm{d}z_i &\mapsto z_i^{p-1}\mathrm{d}z_i,
\end{aligned}$$

which is clearly an isomorphism. □

5.2. **Deformation theory.**

Definition 5.2.1. Let $f\colon X\to Y$ be a morphism of schemes. Then it's called **locally finite presented** if for all $x\in X$, there exists an affine open neighborhood $x\in \operatorname{Spec} A\subseteq X$, and affine open neighborhood $f(x)\in \operatorname{Spec} B\subseteq Y$ such that $f(u)\in V$ with the following property

$$\begin{tikzcd}
& X \arrow[rr, "f"] & & Y \\
& \operatorname{Spec} A \arrow[u, hook] \arrow[rr] \arrow[d, hook] & & \operatorname{Spec} B \arrow[u, hook] \\
\operatorname{Spec} B[t_1,\dots,t_n], \arrow[urrr] & & &
\end{tikzcd}$$

where $I=\ker(B[t_1,\dots,t_n]\to A)$ is finitely generated.

Definition 5.2.2. Let $f\colon X\to Y$ be a morphism of schemes. Then it's called **smooth/étale/unramified** if for every first order thickening

$$\begin{tikzcd}
T_0 \arrow[r, "g_0"] \arrow[d, hook] & X \arrow[d, "f"] \\
T \arrow[r] \arrow[ur, dashed, "g"] & Y
\end{tikzcd}$$

there exists local lifting/ there exists a unique local lifting(global lifting)/at most one local lifting.

Theorem 5.2.1. Let $f\colon X \to Y$ is smooth morphism. Then

(1) $\Omega_{X/Y}$ is locally free $\mathcal{O}_X$-module of finite type.

(2) Suppose $X \xrightarrow{f} Y \xrightarrow{g} S$. Then

$$0 \to g^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

is exact, locally split

(3)

Corollary 5.2.1. $f\colon X \to Y$ is smooth. Then f is locally of the following type

$$\begin{tikzcd}
X \arrow[r, "etale"] \arrow[d, "f"'] & \mathbb{A}^n_Y = A^n \times_{\mathbb{Z}} Y \arrow[dl, "pr"] \\
Y &
\end{tikzcd}$$

Corollary 5.2.2. If f is smooth and g is also smooth, then for $x \in Z$, there exists $x \in U$, $\{s_1, \dots, s_r\} \subseteq I(U)$ such that $\{(s_1)_x, \dots, (s_r)_x\}$ generates I_x and $\{(\mathrm{d}s_1)(x), \dots, \mathrm{d}s_r)(x)\}$ linearly independent in $\Omega_{X/Y}(x)$.

$$\begin{tikzcd}
Z \arrow[r, hook, "i"] \arrow[d, "g"'] & X \arrow[dl, "f"] \\
Y &
\end{tikzcd}$$

Theorem 5.2.2. Let $f\colon X \to Y$ is smooth. Then

(1) there exists $\mathrm{ob}(g_0) \in \mathrm{Ext}^1(g_0^*\Omega_{X/Y}, I)$, such that $\mathrm{ob}(g_0) = 0$ if and only if there exists a global lifting $g\colon T \to X$.

(2) Assume $\mathrm{ob}(g_0) = 0$. Then the set of all liftings g is an affine space under $\mathrm{Hom}(g_0^*\Omega_{X/Y}, I)$, called torsor.

$$\begin{tikzcd}
T_0 \arrow[r, "g_0"] \arrow[d] & X \arrow[d, "f"] \\
T \arrow[r, "g_2"'] \arrow[d] & Y \\
x \in U_x \arrow[uur, bend left, "g_1"] \arrow[uur, bend right] &
\end{tikzcd}$$

Then we claim

$$(g_1^* - g_2^*)(ab) = (g_1^* - g_2^*)(a)b + a(g_1^* - g_2^*)(b)$$

Then

$$(g_1^* - g_2^*)(ab) = g_1^*(ab) - g_2^*(ab)$$

5.3. **Explicit quasi-isomorphism.** In this section we give the proof of Deligne-Illusie's decomposition theorem.

Proof of Theorem 5.1.1. Firstly let's consider the case that the relative Frobenius map $F\colon X \to X^{(p)}$ lifts over $W_2(k)$. In other words, there exists a morphism $\widetilde{F}\colon \widetilde{X} \to \widetilde{X}'$ such that the following diagram commutes

$$
\begin{tikzcd}
 & X^{(p)} \arrow[rr] \arrow[dd] & & \widetilde{X}' \arrow[dd] \\
X \arrow[ur, "F"] \arrow[rr] \arrow[dr] & & \widetilde{X} \arrow[ur, dashed, "\widetilde{F}"] \arrow[dr] & \\
 & \operatorname{Spec} k \arrow[rr] & & \operatorname{Spec} W_2(k),
\end{tikzcd}
$$

where $\widetilde{X}'$ is the base change of $\widetilde{X}$ given by the Frobenius map of $W_2(k)$. For any $x \in \mathcal{O}_{\widetilde{X}'}$, one has

$$\widetilde{F}^*(x) = x^p + pa$$

since $\widetilde{F}$ is a lifting of F and $F(x) = x^p$.

Then

$$
\begin{aligned}
\mathrm{d}\widetilde{F}(\mathrm{d}x) &= \mathrm{d}(\widetilde{F}^* x) \\
&= \mathrm{d}(x^p + pa) \\
&= p(x^{p-1}\mathrm{d}x + \mathrm{d}a).
\end{aligned}
$$

Therefore $\mathrm{d}\widetilde{F}(\widetilde{F}^*\Omega_{\widetilde{X}'/W_2(k)}) \subseteq p\Omega_{\widetilde{X}/W_2(k)}$, and thus one has

$$0 \to pW_2 \to W_2 \to k \to 0$$
$$0 \to p\mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_X \to 0$$

$$
\begin{tikzcd}
\Omega_{X^{(p)}/k} \arrow[r] \arrow[dr] & (F_{X/k})_*\Omega_{X/k} \arrow[d] \\
 & Z^1(F_{X/k})_*\Omega_{X/k}/B^1(F_{X/k})_*\Omega_{X/k}
\end{tikzcd}
$$

Take any open affine covering $\mathcal{U} = \{U_i\}$ of X such that there exists

$$\widetilde{F}_{U_i}\colon \widetilde{U}_i \to \widetilde{U}_i \hookrightarrow \widetilde{X}'$$

$$
\begin{tikzcd}
\Omega_{X^{(p)}} \arrow[r] \arrow[dr] & (F_{X/k})_*\Omega_{U_{ij}/k} \arrow[d, "(F_{X/k})_* d"] \\
\mathcal{O}_{X^{(p)}} \arrow[u, "d"] \arrow[r, "\frac{1}{[p]}(\widetilde{F}^*_{U_j} - \widetilde{F}^*_{U_i})"'] & (F_{X/k})_*\mathcal{O}_{U_{ij}}
\end{tikzcd}
$$

□

5.4. **Applications of de Rham decomposition.**

5.4.1. *E_1-degeneration.*

Theorem 5.1. Let X/k be a non-singular proper variety such that X/k is $W_2(k)$-liftable and $\dim X < p$. Then the Hodge to de Rham spectral sequence degenerates at E_1.

Proof. Note that one has

$$\dim_k H^n_{dR}(X/k) \leq \sum_{i+j=n} H^j(X, \Omega^i_{X/k}) < \infty.$$

Since the absolute Frobenius $F_X\colon X \to X$ is an identity topologically, one has

$$\mathbb{H}^n(X, \Omega^\bullet_{X/k}) = \mathbb{H}^n(X, (F_X)_*\Omega^\bullet_{X/k}) = \mathbb{H}^n(X, \bigoplus_{i=0}^d \Omega^i_{X/k}[-i]) = \bigoplus_{i=0}^d H^{n-i}(X, \Omega^i_{X/k})$$

□

5.4.2. *Kodaira-Akizuki-Nakano theorem.*

Theorem 5.2. Let X/k be a non-singular projective variety such that X/k is $W_2(k)$-liftable and $\dim_k X < p$. Then for any ample line bundle L on X, one has

$$H^j(X, \Omega^i_{X/k} \otimes L) = 0$$

for all $i + j > d$.

Proof given by M.Raynaud. Note that $F^*L^{-1} = (L^{-1})^p$, then

$$L^{-p} \xrightarrow{\nabla_{\text{can}}} L^{-p} \otimes \Omega_{X/k} \xrightarrow{\nabla_{\text{can}}} \dots.$$

Then we project it

$$F_*L^{-p} \xrightarrow{F_*\nabla_{\text{can}}} F_*(L^{-p} \otimes \Omega^p_{X/k}) \to \dots.$$

By projection formula one has

$$F_*L^{-p} = L^{-1} \otimes F_*\mathcal{O}_X, \quad F_*(L^{-p} \otimes \Omega^p_{X/k}) = L^{-1} \otimes F_*\Omega_{X/k}$$

One can find that $F_*\nabla_{\text{can}} = \mathrm{id} \otimes F_*\mathrm{d}$, and thus above complex is $(F_*\Omega_{X/k}, F_*\mathrm{d}) \otimes L^{-1}$. By de Rham decomposition one has

$$\bigoplus_{i=0}^d \Omega^i_{X/k}[-i] \otimes L^{-1}$$

Then

$$\begin{aligned} \dim \mathbb{H}^n(\bigoplus \Omega^i_{X/k}[-i] \otimes L^{-1}) &= \dim \mathbb{H}^n(X, F_*(L^{-p} \otimes \Omega_{X/k})) \\ &= \dim \mathbb{H}^n(X, L^{-p} \otimes \Omega_{X/k}) \\ &\leq \sum_{i+j=n} \dim H^j(X, L^{-p} \otimes \Omega^i_{X/k}) \\ &\leq \sum_{i+j=n} \dim H^j(X, L^{-Np} \otimes \Omega^i_{X/k}) \end{aligned}$$

Then by Serre vanishing one has

$$H^j(X, \Omega^i_{X/k} \otimes L^{-1}) = 0$$

for all $i + j < d$. □

5.5. **From characteristic p to characteristic 0.**

Lemma 5.1. Let $\{A_i\}_{i\in I}$ be a direct system with direct limit A.

(1) E is a A-module of finite presented, then there exists $i_0 \in I$ and E_{i_0} is a A_{i_0}-module of finite presented such that

$$E_{i_0} \otimes_{A_{i_0}} A \cong E.$$

(2) Let $f\colon X \to S = \operatorname{Spec} A$ is a finite presented morphism (l.f.p., qcqs). Then there exists $i_0 \in I$ and $f_{i_0}\colon X_{i_0} \to \operatorname{Spec} S_{i_0} = \operatorname{Spec} A_{i_0}$ such that

$$\begin{array}{ccc} X & \longrightarrow & X_{i_0} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_{i_0}} \\ S & \longrightarrow & S_{i_0} \end{array}$$

(3) Moreover, if f is smooth/proper/projective, then there exists $i_0 \in I$, and $f_{i_0}\colon X_{i_0} \to S_{i_0}$ such that f_{i_0} is smooth/proper/projective.

Theorem 5.3.

Part 2. Non-abelian Hodge theory

6. Non-abelian Hodge theory

The correspondence between Higgs bundles and local systems can be viewed as a Hodge theorem for non-abelian cohomology.

Carlos T. Simpson

6.1. **Introduction.** Let X be a compact Kähler manifold. The Hodge theory says that the following cohomology groups are isomorphic

$$\underbrace{H^n(X,\mathbb{Z})\otimes\mathbb{C}}_{\text{topological aspect}} \cong \underbrace{H^n_{dR}(X)}_{\text{smooth aspect}} \cong \underbrace{\bigoplus_{p+q=n} H^{p,q}(X)}_{\text{holomorphic aspect}},$$

where

(1) $H^n(X,\mathbb{Z})$ is the n-th singular cohomology;
(2) $H^n_{dR}(X)$ is the n-th de Rham cohomology;
(3) $H^{p,q}(X)=H^q(X,\Omega^p_X)$ is the (p,q)-th Dolbeault cohomology.

In other words, topological, smooth and holomorphic aspects are related to each other closely by the classical Hodge theory.

However, many people thought about "analogue" for cohomology group with coefficients in a non-abelian group for a long time. One of the ideals is to note that

$$H^1(X,\mathbb{C})=\mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{C})=\mathrm{Hom}(\pi_1(X),\mathbb{C}),$$

since the singular homology $H_1(X,\mathbb{Z})$ is the abelianization of the fundamental group $\pi_1(X)$ by Hurwitz theorem. This interesting observation motivates us, instead of considering cohomology groups with coefficients in a non-abelian group G, we may consider $\pi_1(X)$-representations, that is, group homomorphisms from $\pi_1(X)$ to G.

In particular, we're interested in the case $G=\mathrm{GL}(n,\mathbb{C})$, since by Riemann-Hilbert correspondence, the following three objects are same:

(1) $\pi_1(X)$-representations (up to conjugacy).
(2) rank n local systems (up to isomorphism).
(3) smooth flat vector bundles of rank n on X (up to isomorphism).

The first two things are living in the topological world, while the third one stands for the smooth category. Thus, parallel to the classical Hodge theory, the missing piece is some object living in the holomorphic world, and that's Higgs bundle we're going to define.

Definition 6.1.1 (Higgs bundle)**.** A Higgs bundle over a compact Kähler manifold X is a pair (E,θ), where $E\to X$ is a holomorphic vector bundle and $\theta\colon E \to E \otimes \Omega^1$ is an $\mathrm{End}(E)$-valued holomorphic 1-form such that $\theta \wedge \theta = 0$, called Higgs field.

Before we introduce more definitions, let's see a baby example.

Example 6.1.1. Let's consider $n = 1$. Note that

$$\begin{aligned}
\mathrm{Hom}(\pi_1(X), \mathbb{C}^*) &= \mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{C}^*) \\
&\cong \frac{\mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{C})}{\mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{Z})} \\
&\cong \frac{H^0(X, \Omega_X) \oplus H^1(X, \mathcal{O}_X)}{H^1(X, \mathbb{Z})} \\
&\cong \mathrm{Pic}^0(X) \oplus H^0(X, \Omega_X).
\end{aligned}$$

Thus there is a bijection between the following sets

$$\{\rho\colon \pi_1(X) \to \mathbb{C}^*\} \longleftrightarrow \{(L, \theta) \mid L \in \mathrm{Pic}^0(X), \theta \in H^0(X, \Omega_X)\}.$$

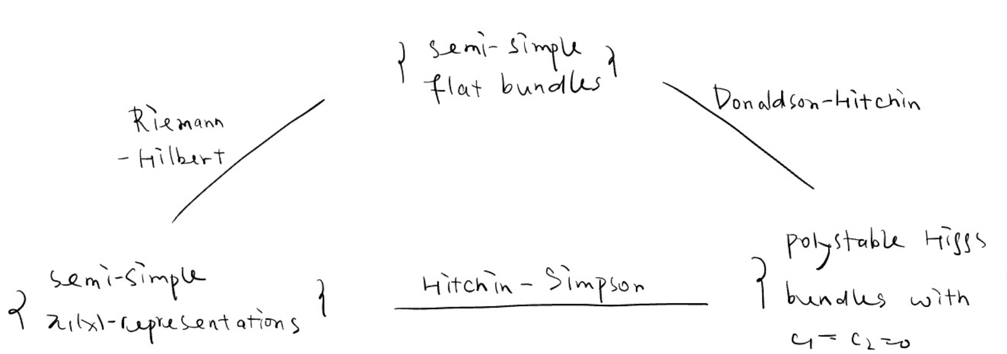
This is exactly the non-abelian Hodge correspondence.

For higher dimension cases, we need to consider polystable Higgs bundles and semi-simple representation to obtain the desired correspondence.

Definition 6.1.2 (stability)**.** Let (E, θ) be a Higgs bundle on a compact Kähler manifold (X, ω). It's called

(1) stable if for every Higgs subbundle[10] F, one has $\mu_\omega(F) < \mu_\omega(E)$.
(2) semi-stable if for every Higgs subbundle F, one has $\mu_\omega(F) \leq \mu_\omega(E)$.
(3) polystable if it's direct sum of stable Higgs bundles, all having the same slope.

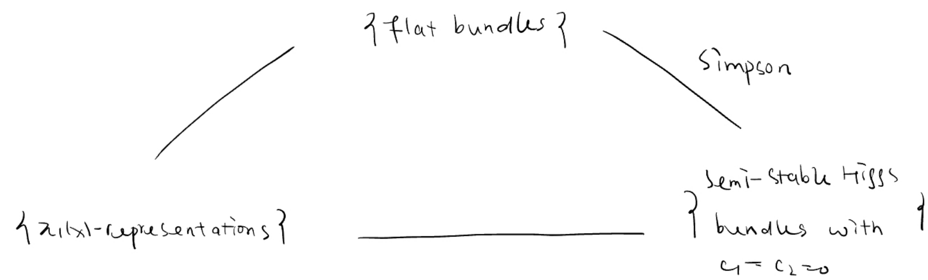
Now we can state the non-abelian Hodge correspondence by drawing the following picture.



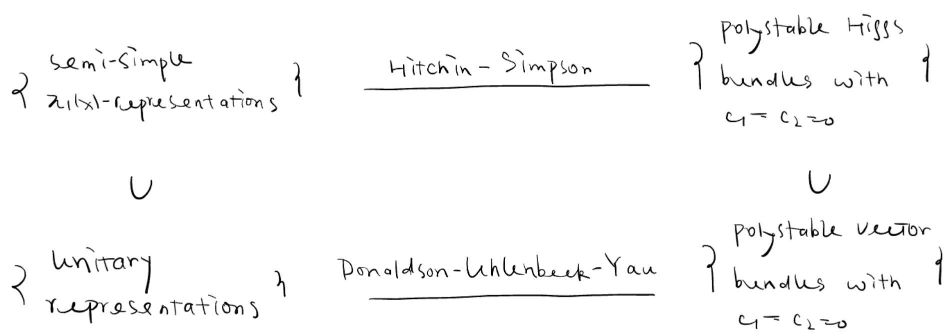
Remark 6.1.1.

(1) If X is projective, then above triangle extends to

[10]A subbundle $F \subseteq E$ is a Higgs subbundle, if $\theta|_F$ is a holomorphic 1-form valued in F.



(2) The Donaldson-Uhlenbeck-Yau correspondence[11] gives the correspondence between unitary representations[12] and polystable vector bundles as follows



6.2. **Harmonic bundle.** In this section, X always denotes a compact complex manifold, V denotes a (smooth) complex vector bundle on X, and E denotes a holomorphic vector bundle on X.

6.2.1. *Smooth Higgs bundle.*

Definition 6.2.1 (Higgs field)**.** A Higgs field on V is a first order differential operator $D''\colon V \to V \otimes \mathcal{A}_X^1$ satisfying the $\overline{\partial}$-Leibnize rule, that is,

$$D''(fs) = \overline{\partial} fs + fD''(s)$$

and integerality $D'' \wedge D'' = 0$.

Definition 6.2.2 (smooth Higgs bundle)**.** A smooth vector bundle V equipped with a Higgs field D'' is called a smooth Higgs bundle.

[11]In particular, when X is a Riemann surface, this correspondence is due to Narasimhan–Seshadri.

[12]A $\pi_1(X)$-representation is said to be unitary, if it factors as $\pi_1(X) \to \mathrm{U}(n) \hookrightarrow \mathrm{GL}(n, \mathbb{C})$.

Remark 6.2.1. Given a Higgs field D'' on V, we may intepret D'' in a different way. Since $V \otimes \mathcal{A}^1_X = V \otimes (\mathcal{A}^{1,0} \oplus \mathcal{A}^{0,1})$, we can write

$$D'' = \theta + \overline{\partial},$$

where $\theta\colon V \to V \otimes \mathcal{A}^{1,0}$ and $\overline{\partial}\colon V \to V \otimes \mathcal{A}^{0,1}$. Then

$$\begin{aligned} D'' \wedge D'' &= (\theta + \overline{\partial}) \wedge (\theta + \overline{\partial}) \\ &= \theta \wedge \theta + (\theta \wedge \overline{\partial} + \overline{\partial} \wedge \theta) + \overline{\partial} \wedge \overline{\partial} \end{aligned}$$

Note that for any $s \in V$, one has

$$(\overline{\partial} \wedge \theta)(a) = \overline{\partial}(\theta a) = \overline{\partial}(\theta)a - \theta \wedge \overline{\partial} a,$$

that is, $\overline{\partial} \wedge \theta + \theta \wedge \overline{\partial} = \overline{\partial}(\theta)$. As a consequence, $D'' \wedge D'' = 0$ is equivalent to the following equations

$$\begin{cases} \theta \wedge \theta = 0 \\ \overline{\partial}(\theta) = 0 \\ \overline{\partial} \wedge \overline{\partial} = 0. \end{cases}$$

The operator $\overline{\partial}$ can be used to define a holomorphic structure on V, that is, $E := V^{\overline{\partial}=0} = \{a \in V \mid \overline{\partial} a = 0\}$ is a holomorphic vector bundle. As $\overline{\partial}$ also satisfies the $\overline{\partial}$-Leibniz rule, $\theta = D'' - \overline{\partial}$ is a zero order differential operator. The condition $\overline{\partial}(\theta) = 0$ means that

$$\theta \in H^0(X, \mathrm{End}(E) \otimes \Omega^1_X).$$

In other words, given a smooth Higgs bundle $(V, D'' = \theta + \overline{\partial})$, it gives a Higgs bundle $(V^{\overline{\partial}=0}, \theta)$.

Lemma 6.2.1. The following map is a bijection.

$$\begin{aligned} \{\text{Higgs bundle}\} &\to \{\text{smooth Higgs bundle}\} \\ (E, \theta) &\mapsto (E \otimes \mathcal{A}^0_X, D'' = \theta + \overline{\partial}) \\ (V^{\overline{\partial}=0}, \theta) &\leftarrow\!\shortmid (V, D''). \end{aligned}$$

6.2.2. *From flat bundle to Higgs bundle.*

Definition 6.2.3 (flat connection)**.** A flat connection on V is a first order differential operator $D\colon V \to V \otimes \mathcal{A}^1_X$ such that $D \wedge D = 0$, and (V, D) is called a flat bundle.

Now we try to use flat connection D to construct a Higgs field on V. Firstly we pick any Hermitian metric h on V and let $D = \mathrm{d}' + \mathrm{d}''$ be the type decomposition. Set δ' and δ'' to be the unique operator of type $(1, 0)$ and $(0, 1)$ such that $\delta' + \mathrm{d}''$ and $\delta'' + \mathrm{d}'$ preseves the metric. Then one has the following four operators

	(1,0)-type	(0,1)-type
1-st order	$\partial = (\mathrm{d}' + \delta')/2$	$\overline{\partial} = (\mathrm{d}'' + \delta'')/2$
0-th order	$\theta = (\mathrm{d}' - \delta')/2$	$\overline{\theta} = (\mathrm{d}'' - \delta'')/2$

Definition 6.2.4 (quasi-Higgs field)**.** A quasi-Higgs field D''_h is defined to be

$$D''_h := \overline{\partial} + \theta = \frac{1}{2}(\mathrm{d}' + \mathrm{d}'' - \delta' + \delta''),$$

and it's a Higgs field if and only if $D''_h \wedge D''_h = 0$.

Definition 6.2.5 (harmonic metric on flat bundle)**.** A Hermitian metric h on a flat bundle is said to be harmonic, if the pseudo-curvature $G_h = D''_h \wedge D''_h$ vanishes.

Remark 6.2.2. In other words, the harmonicity is so-defined such that we obtain a Higgs field out of a flat connection.

6.2.3. *From Higgs bundle to flat bundle.* Given a Higgs bundle (E, θ) equipped with a Hermitian metric h, we set ∂_h to be the Chern connection of Hermitian metric h, that is, $\partial_h + \overline{\partial}_E$ preseves h, and $\overline{\theta}_h$ is the adjoint of θ with respect to h. Thus we have the following four operators

	(1,0)-type	(0,1)-type
1-st order	∂_h	$\overline{\partial}_E$
0-th order	θ	$\overline{\theta}_h$

Now we define

$$D'_h = \partial_h + \overline{\theta}_h$$
$$D'' = \overline{\partial}_E + \theta,$$

and set $D_h = D'_h + D''$, which is called a quasi-flat connection.

Definition 6.2.6 (harmonic metric on Higgs bundle)**.** A Hermitian metric h on Higgs bundle (E, θ) is called harmonic if the quasi-flat connection D_h is flat.

Lemma 6.2.2. The following map is a bijection.

$$\{\text{smooth flat bundle with harmonic metric}\} \to \{\text{smooth Higgs bundle with harmonic metric}\}$$
$$(V, D, h) \mapsto (V, D''_h, h)$$
$$(V, D_h, h) \leftarrow\!\shortmid (V, D'', h).$$

6.2.4. *Harmonic bundle.*

Definition 6.2.7 (harmonic bundle)**.** A harmonic bundle on X is a flat bundle together with a harmonic metric, or a Higgs bundle together with a harmonic metric.

Remark 6.2.3. A choice of a harmonic metric is not a part of defining data for a harmonic bundle.

Lemma 6.2.3 (Kähler identities)**.** Suppose (X,ω) is a Kähler manifold. Then for any Hermitian metric h on Higgs bundle (E,θ),

$$(D'_h)^* = \sqrt{-1}\left[\Lambda_\omega, D''\right]$$
$$(D'')^* = -\sqrt{-1}\left[\Lambda_\omega, D'_h\right]$$

Remark 6.2.4. Consider the trivial Higgs bundle $(E,0)$, that is, a holomorphic vector bundle E equipped with trivial Higgs field $\theta=0$. Then above lemma recovers Kälher identities shown in Theorem **??**.

Analogously, for any Hermitian metric h on smooth flat bundle (V,D), one has the following identities.

Lemma 6.2.4. Suppose (X,ω) is a Kähler manifold. Then for any Hermitian metric h on Higgs bundle (E,θ),

$$(D^c_h)^* = -\sqrt{-1}\left[\Lambda_\omega, D\right]$$
$$D^* = \sqrt{-1}\left[\Lambda_\omega, D^c_h\right],$$

where $D^c_h = D'_h - D''_h$.

Remark 6.2.5. For flat bundle $(V,D)=(\mathcal{A}^0,\mathrm{d})$, $D^c_h=\overline{\partial}-\partial$, and thus

$$(\mathrm{d}^c)^* = -\sqrt{-1}\left[\Lambda_\omega, \mathrm{d}\right]$$
$$\mathrm{d}^* = \sqrt{-1}\left[\Lambda_\omega, \mathrm{d}^c\right].$$

Lemma 6.2.5 (Bianchi identity)**.**

(1) Let (E,θ,h) be a harmonic bundle. Then

$$F_h = (D_h)^2 = D'_h D'' + D'' D'_h$$
$$D'_h F_h = D'' F_h = 0$$

(2) Let (V,D,h) be a harmonic bundle. Then

$$G_h = (D''_h)^2 = \frac{DD^c_h + D^c_h D}{4}$$
$$DG_h = D^c_h G_h = 0$$

Lemma 6.2.6.

(1) Let (E,θ) be a Higgs bundle admitting a Hermitian-Yang-Mills metric h. If $c_1(E)[\omega]^{n-1} = c_2(E)[\omega]^{n-2}=0$, then $F_h=0$.
(2) Let (V,D) be a flat bundle admitting a metric satisfying $\Lambda G_h = 0$. Then $G_h=0$.

Remark 6.2.6. Results in non-linear analysis shows that

(1) A Higgs bundle (E,θ) has a Hermitian-Yang-Mills metric if and only if it's polystable.
(2) A flat bundle (V,D) admits a metric h satisfying $\Lambda G_h = 0$ if and only if it's semi-simple.

6.3. Complex variation of Hodge structure.

Definition 6.3.1 ($\mathbb{C}$-VHS)**.** A (polarized) complex variation of Hodge structure ($\mathbb{C}$-VHS) consists of the following data:

(1) A smooth vector bundle V with a decomposition $V = \bigoplus_{p+q=n} V^{p,q}$.
(2) A flat connection D on V satisfies
$$D\colon V^{p,q} \to \mathcal{A}^{0,1}(V^{p+1,q-1}) \oplus \mathcal{A}^{1,0}(V^{p,q}) \oplus \mathcal{A}^{0,1}(V^{p,q}) \oplus \mathcal{A}^{1,0}(V^{p-1,q+1}),$$
which is called Griffiths transversality.
(3) A parallel Hermitian form Q which makes the decomposition in (1) is orthogonal and which on $V^{p,q}$ is positive definite if p is even and negative definite if p is odd.

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