Uniqueness of the Kähler structure of $\mathbb{CP}^n$

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- In this talk we mainly focus on the following two theorems, which show the uniqueness of the Kähler structure of $\mathbb{CP}^n$ in different categories.

Theorem (Hirzebruch, Kodaira,1957; Yau,1977)

If a Kähler manifold M is homeomorphic to $\mathbb{CP}^n$, then M is biholomorphic to it.

Theorem (Yau,1977)

If a compact complex surface M is homotopy equivalent to $\mathbb{CP}^2$, then M is biholomorphic to it.

- To prove these two theorems, the following lemma motivates us it suffices to construct a holomorphic line bundle with some properties.

Lemma (Kobayashi,Ochiai,1973)

If M is a compact Kähler n-manifold and L is a positive holomorphic line bundle over M with $\int_M c_1(L)^n = 1$ and $\dim H^0(M, L) = n + 1$, then M is biholomorphic to $\mathbb{CP}^n$.

Rough idea of proof

- If M is a compact Kähler manifold whose cohomology groups are the same as the ones of $\mathbb{CP}^n$, then
  - $c_1: \operatorname{Pic}(M) \to H^2(M, \mathbb{Z})$ is an isomorphism, which allows us to construct a (unique) holomorphic line bundle L with a given cohomology class as its first Chern class.

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- By using Kodaira vanishing theorem one can conclude $H^k(M, L) = 0$ for $k > 0$. In particular, one has $\dim H^0(M, L) = n + 1$, as desired.

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- Choose a Kähler form $\widetilde{\omega}$ on M. Its cohomology class lies in $H^2(M, \mathbb{R}) \cong \mathbb{R}$, so we can rescale $\widetilde{\omega}$ to get another Kähler form ω whose cohomology class generates $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$. In particular, $\int_M \omega^n = 1$.

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- Since c_1 is an isomorphism, there exists a (unique) holomorphic line bundle L whose first Chern class is $[\omega]$.

Lemma

For any holomorphic line bundle L over M we have

$$\chi(M, L) = \int_M e^{c_1(L) + \frac{c_1(M)}{2}} \left(\frac{\omega/2}{\sinh(\omega/2)} \right)^{n+1}.$$

Corollary

$c_1(M)$ equals either $(n+1)[\omega]$ or $-(n+1)[\omega]$, with the latter only possibly occuring when n is even.

Proof.

Since $[\omega]$ is a generator of $H^2(M, \mathbb{Z})$, we may write $c_1(M) = \lambda[\omega]$. The reduction mod 2 of $c_1(M)$ is the second Stiefel-Whitney class $w_2(M) \in H^2(M, \mathbb{Z})$, which is a topological invariant. Hence it is equals to $w_2(\mathbb{CP}^n)$ which equals $c_1(\mathbb{CP}^n) \equiv n + 1 \pmod 2$. This shows $c_1(M) = (n + 1 + 2s)[\omega]$ for some $s \in \mathbb{Z}$.

Continuation.

By Lemma 4 one has

$$\chi(M,\mathcal{O}) = \int_M e^{\frac{n+1+2s}{2}\omega}\left(\frac{\omega/2}{\sinh(\omega/2)}\right)^{n+1} = \int_M e^{s\omega}\left(\frac{\omega}{1-e^{-\omega}}\right)^{n+1}.$$

By residue theorem a direct computation shows

$$\int_M e^{s\omega}\left(\frac{\omega}{1-e^{-\omega}}\right)^{n+1} = \binom{n+s}{n}.$$

Since $\chi(M,\mathcal{O}) = 1$, one has $\binom{n+s}{n} = 1$, which can be rewritten as

$$n! = (s+n)\dots(s+1).$$

So if n is ood this implies $s = 0$, while if n is even, s is either 0 or $-n-1$. This completes the proof. □

Proof of Theorem 1.

Case I: Assume first $c_1(M) = (n+1)[\omega]$, which implies that M is a Fano manifold. Then $c_1(K_M) = -c_1(M) = -(n+1)c_1(L)$ and so $K_M = -(n+1)L$ since c_1 is an isomorphism. Then Serre duality gives $H^k(M, L) = H^{n-k}(M, K_M - L)$ and $K_M - L = -(n+2)L$ is negative, so $H^k(M, L) = 0$ if $k > 0$ by Kodaira vanishing. Hence one has

$$\dim H^0(M, L) = \chi(M, L) = \int_M e^{c_1(L) + \frac{c_1(M)}{2}} \left(\frac{\omega/2}{\sinh(\omega/2)}\right)^{n+1} = n+1,$$

and Lemma 3 implies M is biholomorphic to $\mathbb{CP}^n$.

Case II: Assume $c_1(M) = -(n+1)[\omega] < 0$, it suffices to show the following identity

$$\left(2(n+1)c_2(M) - nc_1^2(M)\right)[\omega]^{n-2} = 0.$$

Continuation.

Indeed, by the equality condition of Chern number inequality of Yau, M has constant holomorphic sectional curvature -1, and thus by uniformization theorem M is biholomorphic to the unit ball in $\mathbb{C}^n$, a contradiction.
To compute $c_2(M)$, note that $p_1(M) = p_1(TM) = -c_2(TM \otimes \mathbb{C})$, $TM \otimes \mathbb{C} \cong TM \oplus \overline{TM}$ and Chern classes satisfy $c_k(\overline{TM}) = (-1)^k c_k(TM)$, so

$$\begin{aligned} p_1(M) &= -c_2(TM \oplus \overline{TM}) \\ &= -c_2(TM) - c_2(\overline{TM}) - c_1(TM)c_1(\overline{TM}) \\ &= -2c_2(M) + c_1^2(M). \end{aligned}$$

On the other hand, $p_1(M) = (n+1)[\omega]^2$. Thus

$$2c_2(M) = (n+1)^2[\omega]^2 - (n+1)[\omega]^2 = n(n+1)[\omega]^2.$$

Proof of Theorem 2.

Let $\tau(M)$ denote the signature of M, that is the signature of its intersection form. The signature is a topological invariant up to sign, and so

$$\tau(M) = \pm\tau(\mathbb{CP}^2) = \pm 1.$$

Hirzebruch's signature theorem gives

$$\tau(M) = \frac{1}{3}\int_M p_1(M) = \frac{1}{3}\int_M (c_1^2(M) - 2c_2(M)) = \pm 1.$$

Chern-Gauss-Bonnet's theorem gives

$$\int_M c_2(M) = \chi(M) = \chi(\mathbb{CP}^2) = 3.$$

As a consequence, $\int_M c_1^2(M) \neq 0$, which implies M is Kähler by Kodaira embeddding.

Continuation.

As before we see that $\chi(M,\mathcal{O})=1$ and Hirzebruch-Riemann-Roch gives

$$\chi(M,\mathcal{O})=\frac{K_M^2+\chi(M)}{12}=\frac{K_M^2+3}{12},$$

which gives $\int_M c_1^2(M)=K_M^2=9$. Let ω be as before, and $c_1(M)=\lambda[\omega]$ for some $\lambda\in\mathbb{Z}$. Then $\lambda=\pm 3$. Here it suffices to show in case $\lambda=3$, $\dim H^0(M,L)=3$, and the case $\lambda=-3$ leads the same contradiction as before. By Hirzebruch-Riemann-Roch formula one has

$$\chi(M,L)=1+\frac{L^2-K_M\cdot L}{2}=3.$$

Serre duality and Kodaira vanishing gives $H^1(M,L)=H^2(M,L)=0$ as before. So $\dim H^0(M,L)=\chi(M,L)=3$. This completes the proof. □

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- A natural question is that whether the Kähler hypothesis is really necessary in Theorem 1. If not, one can also ask whether a complex manifold diffeomorphic to $\mathbb{CP}^n$ must be biholomorphic to it. If it's ture when $n = 3$, then there is no complex structure on S^6.

Lemma

If there exists a compact complex manifold M diffeomorphic to S^6, then there exists a compact complex manifold $\widetilde{M}$ diffeomorphic to $\mathbb{CP}^3$ but not biholomorphic to it.

Proof.

Let M be a compact complex manifold diffeomorphic to S^6 and $\widetilde{M}$ be its blowup at one point $p \in M$. A basic fact is that $\widetilde{M}$ is a compact complex manifold which is diffeomorphic to $S^6 \sharp \overline{\mathbb{CP}^3}$, where $\overline{\mathbb{CP}^3}$ is the smooth manifold obtained from $\mathbb{CP}^3$ by reversing orientation. In particular, $\widetilde{M}$ is diffeomorphic to $\mathbb{CP}^3$. If $\widetilde{M}$ was biholomorphic to $\mathbb{CP}^3$, one has

$$\int_{\widetilde{M}} c_1(\widetilde{M})^3 = \int_{\mathbb{CP}^3} c_1(\mathbb{CP}^3)^3 = 64$$

Continuation.

On the other hand, if we let $\pi\colon \widetilde{M} \to M$ be the blow up map and $E = \pi^{-1}(p)$ be its exceptional divisor, then one has

$$c_1(\widetilde{M}) = \pi^* c_1(M) - 2[E]$$

where $[E]$ is the Poincaré duality of E. Since $b_2(M) = 0$, one has $c_1(M) = 0$. Thus

$$\begin{aligned}
\int_{\widetilde{M}} c_1(\widetilde{M})^3 &= -8 \int_{\widetilde{M}} [E]^3 \\
&= -8 \int_E [E]^2 \\
&= -8 \int_{\mathbb{CP}^2} c_1(\mathcal{O}(-1))^2 = -8
\end{aligned}$$

Therefore $\widetilde{M}$ is not biholomorphic to $\mathbb{CP}^3$, as desired.

Thanks!