TORIC VARIETY

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Contents

Part 1. Basic theories of toric varieties

1. Preliminaries

1.1. Affine semigroups.

Definition 1.1.1 (affine semigroup)**.** An affine semigroup S is a semigroup group such that

(1) The binary operation on S is communicative.
(2) The semigroup is finitely generated.
(3) The semigroup can be embedded in a lattice M.

Example 1.1.1. $\mathbb{N}^n \subseteq \mathbb{Z}^n$ is an affine semigroup.

Example 1.1.2. Given a finite set $\mathcal{A}$ of a lattice M, $\mathbb{N}\mathcal{A} \subseteq M$ is an affine semigroup.

Definition 1.1.2 (semigroup algebra)**.** Let $S \subseteq M$ be an affine semigroup. The semigroup algebra $\mathbb{C}[S]$ is the vector space over $\mathbb{C}$ with S as basis and multiplication is induced by the semigroup structure.

Remark 1.1.1. To make this precise, we write

$$\mathbb{C}[S] = \{\sum_{m \in S} c_m \chi^m \mid c_m \in C \text{ and } c_m = 0 \text{ for all but finitely many } m\}$$

with multiplication given by

$$\chi^m \cdot \chi^{m'} = \chi^{m+m'}.$$

If $S = \mathbb{N}\mathcal{A}$ for $\mathcal{A} = \{m_1, \dots, m_s\}$, then $\mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, \dots, \chi^{m_s}]$.

Example 1.1.3. The affine semigroup $\mathbb{N}^n \subseteq \mathbb{Z}^n$ gives the polynomial ring

$$\mathbb{C}[\mathbb{N}^n] = \mathbb{C}[x_1, \dots, x_n]$$

where $x_i = \chi^{e_i}$ and $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{Z}^n$.

Example 1.1.4. If $e_1, \dots, e_n$ is a basis of a lattice M, then M is generated by $\mathcal{A} = \{\pm e_1, \dots, \pm e_n\}$ as an affine semigroup, and the semigroup algebra gives the Laurent polynomial ring

$$\mathbb{C}[M] = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

where $x_i = \chi^{e_i}$.

1.2. Strongly convex rational polyhedral ones.

From now on, unless otherwise specified, we always assume M, N are dual lattices with associated $\mathbb{R}$-vector spaces $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, and the pairing between M and N is denoted by $\langle \text{-}, \text{-} \rangle$.

1.2.1. *Convex polyhedral cones.*

Definition 1.2.1 (convex polyhedral cone)**.** Let $S \subseteq N_{\mathbb{R}}$ be a finite subset. A convex polyhedral cone in $N_{\mathbb{R}}$ generated by S is a set of the form

$$\sigma = \operatorname{Cone} S = \{\sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0\} \subseteq N_{\mathbb{R}}.$$

Notation 1.2.1. $\operatorname{Cone}(\varnothing) = \{0\}$.

Remark 1.2.1. A convex polyhedral cone is convex, that is $x, y \in \sigma$ implies $\lambda x + (1-\lambda) y \in \sigma$ for all $0 \leq \sigma \leq 1$, and it's a cone, that is $x \in \sigma$ implies $\lambda x \in \sigma$ for all $\lambda \geq 0$. Since we will only consider convex cones, the cones satisfying Definition 1.2.1 will be called polyhedral cone for convenience.

Definition 1.2.2 (dimension)**.** The dimension of a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is the dimension of the smallest subspace $W \subseteq N_{\mathbb{R}}$ containing σ, and such W is called the span of σ.

Definition 1.2.3 (dual cone)**.** Let $\sigma \subseteq N_{\mathbb{R}}$ be a polyhedral. The dual cone is defined by

$$\sigma^\vee := \{u \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}.$$

Definition 1.2.4 (hyperplane)**.** Given $m \in M_{\mathbb{R}}$, the hyperplane given by m is defined by

$$H_m := \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle = 0\} \subseteq N_{\mathbb{R}},$$

and the closed half-space given by m is defined by

$$H_m^+ := \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle \geq 0\} \subseteq N_{\mathbb{R}}.$$

Definition 1.2.5 (supporting hyperplane)**.** The supporting hyperplane of a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is a hyperplane H_m such that $\sigma \subseteq H_m^+$, and H_m^+ is called a supporting half-space.

Remark 1.2.2. H_m is a supporting hyperplane of σ if and only if $m \in \sigma^\vee$, and if $m_1, \ldots, m_s$ generates $\sigma^\vee$, then

$$\sigma = H_{m_1}^+ \cap \cdots \cap H_{m_s}^+.$$

Thus every polyhedral cone is an intersection of finitely many closed half-spaces.

Definition 1.2.6 (face)**.** A face of a polyhedral cone σ is $\tau = H_m \cap \sigma$ for some $m \in \sigma^\vee$, written $\tau \preceq \sigma$. Faces $\tau \neq \sigma$ are called proper faces, written $\tau \prec \sigma$.

Definition 1.2.7 (facet and edge)**.** A facet of a polyhedral cone σ is a face of codimension one, and an edge of σ is a face of dimension one.

1.2.2. *Relative interior.*

Definition 1.2.8 (relative interior)**.** The relative interior of a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$, denoted by $\operatorname{Relint}(\sigma)$, is the interior of σ in its span.

1.2.3. *Strongly convex.*

Definition 1.2.9 (strongly convex)**.** A polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is strongly convex if $\{0\}$ is a face of σ.

1.2.4. *Separation.*

Lemma 1.2.1. Let σ_1, σ_2 be two polyhedral cones in $N_{\mathbb{R}}$ that meet along a common face $\tau = \sigma_1 \cap \sigma_2$. Then

$$\tau = H_m \cap \sigma_1 = H_m \cap \sigma_2$$

for any $m \in \operatorname{Relint}(\sigma_1^\vee \cap (-\sigma_2)^\vee)$.

1.2.5. *Rational polyhedral cones.*

Definition 1.2.10 (rational)**.** A polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is rational if $\sigma = \operatorname{Cone}(S)$ for some finite subset $S \subseteq N$.

Definition 1.2.11 (ray generator)**.** Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone and ρ be an edge of σ. The unique generator of semigroup $\rho \cap N$ is called ray generator of ρ, written u_ρ.

Remark 1.2.3. The ray generator is well-defined: Since σ is strongly convex, one has edge of σ is a ray as $\{0\}$ is its face, and since σ is rational, the semigroup $\rho \cap N$ is generated by a unique element, otherwise contradicts to the fact ρ is an edge, that is it's of dimension one.

Definition 1.2.12 (minimal generators)**.** Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone. The ray generators of edges are called minimal generators of σ.

Definition 1.2.13 (smooth and simplicial)**.** Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone.

(1) σ is smooth if its minimal generators form part of a $\mathbb{Z}$-basis of N.
(2) σ is simplical if its minimal generators are linearly independent over $\mathbb{R}$.

2. Fans and toric variety

2.1. Semigroup algebras and affine toric varieties.

Definition 2.1.1 (affine toric variety)**.** An affine toric variety is an irreducible affine variety V containing a torus $T_N \cong (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T_N on itself extends to an algebraic action of T_N on V.

Lemma 2.1.1 (Gordan's lemma)**.** Let $\sigma \subseteq N_{\mathbb{R}}$ be a rational polyhedral cone. The semigroup $S_\sigma := \sigma^\vee \cap M$ is finitely generated and hence is an affine semigroup.

Theorem 2.1.1. Let $\sigma \subseteq N_{\mathbb{R}}$ be a rational polyhedral cone with semigroup $S_\sigma = \sigma^\vee \cap M$. Then

$$U_\sigma := \operatorname{Spec}(\mathbb{C}[S_\sigma])$$

is an affine toric variety, and $\dim U_\sigma = n$ if and only if σ is strongly convex.

Proposition 2.1.1. Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone and σ be a face of σ written as $\tau = H_m \cap \sigma$, where $m \in \sigma^\vee \cap M$. Then the semigroup algebra $\mathbb{C}[S_\tau]$ is the localization of $\mathbb{C}[S_\sigma]$ at $\chi^m \in \mathbb{C}[S_\sigma]$.

2.2. The toric variety of a fan.

Definition 2.2.1 (toric variety)**.** A toric variety is an irreducible variety X containing a torus $T_N \cong (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T_N on itself extends to an algebraic action of T_N on X.

Definition 2.2.2 (fan)**.** A fan Σ in $N_{\mathbb{R}}$ is a finite collection of cones $\sigma \subseteq N_{\mathbb{R}}$ such that

(1) Every $\sigma \in \Sigma$ is strongly convex rational polyhedral cone.
(2) For all $\sigma \in \Sigma$, each face of σ is also in Σ.
(3) For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each.

Notation 2.2.1. $\Sigma(r)$ is the set of r-dimensional cones of Σ.

Definition 2.2.3 (support)**.** The support of a fan σ in $N_{\mathbb{R}}$ is $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma \subseteq N_{\mathbb{R}}$.

Now let's show how the cones in any fan give the combinatorial data necessary to glue a collection of affine toric varieties to yield an abstract toric variety.

(1) Firstly, by Theorem 2.1.1 one has each cone $\sigma \in \Sigma$ gives the affine toric variety

$$U_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma]) = \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap M]).$$

If τ is a face of σ, then there exists some $m \in \sigma^\vee$ such that $\tau = \sigma \cap H_m$, and by Proposition 2.1.1 one has $\mathbb{C}[S_\tau] = (\mathbb{C}[S_\sigma])_{\chi^m}$, which implies

$$U_\tau = (U_\sigma)_{\chi^m}.$$

(2) Secondly, if $\tau = \sigma_1 \cap \sigma_2$, then by Lemma 1.2.1 there exists $m \in \sigma_1^\vee \cap (-\sigma_2)^\vee \cap M$ such that
$$\sigma_1 \cap H_m = \tau = \sigma_2 \cap H_m.$$
This shows
$$U_{\sigma_1} \supseteq (U_{\sigma_1})_{\chi^m} = U_\tau = (U_{\sigma_2})_{\chi^{-m}} \subseteq U_{\sigma_2}.$$
Thus we have an isomorphism
$$g_{\sigma_2\sigma_1} \colon (U_{\sigma_1})_{\chi^m} \cong (U_{\sigma_2})_{\chi^{-m}}.$$

(3) Finally, we use isomorphisms in (2) to glue the collection of affine toric varieties obtained from a fan to construct the toric variety X_Σ associated to the fan Σ.

Theorem 2.2.1. Let Σ be a fan in $N_\mathbb{R}$. The variety X_Σ is normal separated toric variety.

Conversely, any normal separated toric variety comes from a fan, but it's a highly non-trivial fact.

Theorem 2.2.2. Let X be a normal separated toric variety with torus T_N. Then there exists a fan Σ in $N_\mathbb{R}$ such that $X \cong X_\Sigma$.

Proof. See [Sum74] and [Sum75]. □

2.3. Examples.

Example 2.3.1. Consider the fan Σ in $N_\mathbb{R} = \mathbb{R}^2$ in Figure 1, where $N = \mathbb{Z}^2$ has standard basis e_1, e_2.

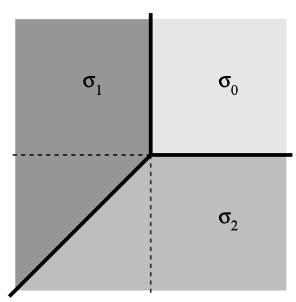

FIGURE 1. The fan Σ for $\mathbb{P}^2$

Example 2.3.2.

2.4. Terminologies.

3. The orbit-cone correspondence

3.1. **Baby example.**

3.2. **The orbit-cone correspondence.**

Theorem 3.2.1 (orbit-cone correspondence)**.** Let X_Σ be the toric variety of the fan Σ in $N_\mathbb{R}$. Then

(1) There is a bijective correspondence

$$\{\text{cones } \sigma \text{ in } \Sigma\} \longleftrightarrow \{T_N\text{-orbits in } X_\Sigma\}$$

$$\sigma \longleftrightarrow O(\sigma) \cong \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*).$$

(2) Let $n = \dim N_\mathbb{R}$. For each cone $\sigma \in \Sigma$, $\dim O(\sigma) = n - \dim \sigma$.

(3) The affine open subsets U_σ is the union of orbits

$$U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau).$$

(4) $\tau \preceq \sigma$ if and only if $O(\sigma) \subseteq \overline{O(\tau)}$, and

$$\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma),$$

where $\overline{O(\tau)}$ denotes the closure in both classical and Zariski topologies.

3.3. **Orbit closure as toric varieties.**

Proposition 3.3.1. Let Σ be a fan in $N_\mathbb{R}$ and $\tau \in \Sigma$. Then the orbit closure $\overline{O(\tau)}$ has a toric variety structure.

4. Divisors on toric variety

4.1. **Weil divisors on toric varieties.**
Let X_Σ be the toric variety of fan in $N_\mathbb{R}$ with $\dim N_\mathbb{R} = n$. In this section we will use torus-invariant prime divisors and characters to give a lovely description of class group of X_Σ.

4.1.1. *The divisor of a character.* By the orbit-cone correspondence, $\rho \in \Sigma(1)$ gives the codimension one orbit $O(\rho)$ whose closure $\overline{O(\rho)}$ admits a codimension one toric subvariety structure by Proposition 3.3.1. Thus $\overline{O(\rho)}$ gives a T_N-invariant prime divisor on X_Σ. To emphasize that $\overline{O(\rho)}$ is a divisor we will denote it by D_ρ for convenience. Then D_ρ gives the DVR $\mathcal{O}_{X_\Sigma, D_\rho}$ with valuation

$$\nu_\rho\colon \mathbb{C}(X_\Sigma)^* \to \mathbb{Z}\,.$$

Recall that any ray $\rho \in \Sigma(1)$ has a minimal generator $u_\rho \in \rho \cap N$, and also note that when $m \in M$, the character $\chi^m\colon T_N \to \mathbb{C}^*$ is a rational function in $\mathbb{C}(X_\Sigma)^*$ since T_N is Zariski open in X_Σ.

Proposition 4.1.1. Let u_ρ be the minimal generator of ray $\rho \in \Sigma(1)$ and χ^m be a character corresponding to $m \in M$. Then

$$\nu_\rho(\chi^m) = \langle m, u_\rho \rangle.$$

Proposition 4.1.2. For $m \in M$, the divisor of character χ^m is given by

$$\operatorname{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho.$$

4.1.2. *Computing the class group.*

Theorem 4.1.1. There is the following exact sequence

$$M \to \operatorname{Div}_{T_N}(X_\Sigma) \to \operatorname{Cl}(X_\Sigma) \to 0,$$

where the first map is $m \mapsto \operatorname{div}(\chi^m)$ and the second sends a T_N-invariant divisor to its divisor class in $\operatorname{Cl}(X_\Sigma)$. Furthermore, one has the following exact sequence

$$0 \to M \to \operatorname{Div}_{T_N}(X_\Sigma) \to \operatorname{Cl}(X_\Sigma) \to 0$$

if and only if $\{u_\rho \mid \rho \in \Sigma(1)\}$ spans $N_\mathbb{R}$.

4.2. **Cartier divisors on toric varieties.**

4.3. **The sheaf of a torus-invariant divisor.**

5. Line bundles on toric variety

6. Canonical divisors of toric variety

References

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