HODGE THOERY AND COMPLEX ALGEBRA GEOMETRY

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1. Overview

In this course, we will introduce two parts:

I Kähler manifold and Hodge decomposition;

II Hodge theory in algebra geometry.

For the first part, if X is a compact complex manifold, we can consider the following structures:

(1) Topology: $H^*_B(X,\mathbb{Z})$, singular cohomology, where B means "Betti".
(2) C^∞-structure: $H^*_{dR}(X,\mathbb{R}) = H^*(X,\Omega^{\cdot}_{X,\mathbb{R}})$, de Rham cohomology. In fact, de Rham theorem implies that
$$H^*_{dR}(X,\mathbb{R}) \cong H^*_B(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$$
(3) Complex structure: For $x \in X$, $T_{X,x}$ is tangent space at x, its real dimension is $2n$, that is $\dim_{\mathbb{R}} T_{X,x} = 2n$. And there is a linear map $J_x : T_{X,x} \to T_{X,x}$, such that $J_x^2 = -\operatorname{id}$. If we complexificate $T_{X,x}$, then we can decompose it into
$$T_{X,x} \otimes \mathbb{C} = T^{1,0}_{X,x} \oplus T^{0,1}_{X,x}$$
where $T^{1,0}_{X,x}$ is the characteristic subspace belonging to eigenvalue $\sqrt{-1}$, and $T^{0,1}_{X,x}$ is the characteristic subspace belonging to eigenvalue $-\sqrt{-1}$.

If we consider its dual, we will get bundle/sheaf of differential forms, and we can also decompose them as follows
$$\Omega^1_{X,\mathbb{C}} = \Omega^1_{X,\mathbb{R}} \otimes \mathbb{C} = \Omega^{1,0}_X \oplus \Omega^{0,1}_X$$
and
$$\Omega^k_{X,\mathbb{C}} = \Omega^k_{X,\mathbb{R}} \otimes \mathbb{C} = \bigoplus_{p+q=k} \Omega^{p,q}_X$$
where $\Omega^{p,q}_X = \wedge^p \Omega^{1,0}_X \otimes \wedge^q \Omega^{0,1}_X$

Since we have such decomposition for differential forms, it's natural to ask if there is a similar decomposition for de Rham cohomology? that is, do we have
$$H^k_{dR}(X,\mathbb{R}) = \bigoplus_{p+q=k} H^{p,q}(X)$$
such that $\overline{H^{p,q}(X)} = H^{q,p}(X)$?

The Hodge decomposition says it's true for compact Kähler manifolds. It's a very beautiful result, connecting "Topology" and "Complex geometry", since de Rham cohomology reflects the topology information and

$$H^{p,q}(X) \cong H^q_{Dol}(X, \Omega^p_X)$$

where "Dol" means Dolbealt cohomology.

Here is some examples of Kähler manifolds, in fact, almost every interesting manifold is Kähler manifold:

Example 1.1. *Riemanian Surfaces, complex torus, projective manifolds are all Kähler manifolds.*

We also need to know an example that is not Kähler manifold:

Example 1.2 (Hopf surface)**.** *Consider $\mathbb{Z}$ acts on $\mathbb{C}^2\backslash\{0\}$ by $mz = \lambda^m z, m \in \mathbb{Z}$ for some $\lambda \in (0,1)$, then we define Hopf surface as follows*

$$S = \mathbb{C}^2\backslash\{0\}/\mathbb{Z}$$

As we can see, S is diffeomorphic to $S^3 \times S^1$,then $\dim_{\mathbb{C}} H^1(S,\mathbb{C}) = 1$, so S can not be a Kähler manifold by Hodge's decomposition, since for a Kähler manifold, $\dim_{\mathbb{C}} H^1$ must be even.

Remark 1.3. *Since a projective manifold X is a Kähler manifold, then by Chow's theorem/GAGA, X can be defined by polynomial equations, i.e., X is a projective variety, so here comes the forth structure, and that's the second part of this course, we want to apply Hodge theory in algebraic geometry.*

(4) Algebraic structure.

2. Complex manifold

Definition 2.1. *A complex manifold consists of $(X, \{U_i, \phi_i\}_{i\in I})$, where X is a connected, Hausdorff topological space, $\{U_i\}_{i\in I}$ is an open cover of X such that the index set I is countable, and ϕ_i is a homeomorphism from U_i to an open subset V_i of $\mathbb{C}^n$, such that*

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is biholomorphic.

Definition 2.2. *Such $\phi_i \circ \phi_j^{-1}$ is called transition function; n is called the dimension of X, denoted by $\dim_{\mathbb{C}} C$; $\{U_i, \phi_i\}_{i\in I}$ is called complex atlas.*

Definition 2.3. *Two atlas are equivalent, if the union of them is still an atlas.*

Definition 2.4. *A complex structure is an equivalence class of a complex atlas.*

Remark 2.5. *Replace $\mathbb{C}^n$ by $\mathbb{R}^n$, and biholomorphism is replaced by homeomorphism or diffeomorphism, then we get topological manifold or smooth manifold.*

Remark 2.6. *$V_i \subset \mathbb{C}^n$ usually can not be the whole $\mathbb{C}^n$. For example, there is no non-constant holomorphism from $\mathbb{C}$ to unit disk $\mathbb{D}$. More generally, X is called Brody hyperbodic if there is no non-constant holomorphism from $\mathbb{C}$ to X.*

Example 2.7. *Projective space $\mathbb{P}^n$ is a complex manifold. Atlas are $U_i = \{[z] \in \mathbb{P}^n \mid z_j \neq 0\}, 0 \leq i \leq n, \phi_i : U_i \to \mathbb{C}^n$ is defined as follows*

$$[z] \mapsto (\frac{z_0}{z_i}, \dots, \frac{\widehat{z_i}}{z_i}, \dots, \frac{z_n}{z_i})$$

Transition functions are calculated as follows, for $i < j$

$$\phi_i \circ \phi_j^{-1} : (u_1, \dots, u_n) \mapsto (\frac{u_1}{u_i}, \dots, \frac{\widehat{u_i}}{u_i}, \dots, \frac{u_{j-1}}{u_i}, \frac{1}{u_i}, \frac{u_{j+1}}{u_i}, \dots, \frac{u_n}{u_i})$$

In fact, $\mathbb{P}^n$ is a compact complex manifold, since $\mathbb{P}^n$ is diffeomorphic to S^{2n+1}/S^1.

Example 2.8. *Grassmannian manifold*

$$G(r, n) = \{r\text{-dimensional subspace of } \mathbb{C}^n\}$$

Atlas: given $T_i \subset \mathbb{C}^n$ of dimension $n - r$, set $U_i = \{S \subset \mathbb{C}^n \text{of dimension } r \mid S \cap T_i = 0\}$. Choose $S_i \in U_i$, define

$$\phi_i : U_i \to \mathrm{Hom}(S_i, T_i) \cong \mathbb{C}^{r(n-r)}$$

as $S \mapsto f$, such that S is the graph of f.

Example 2.9. *Complex torus is $\mathbb{C}^n/\Lambda$ where Λ is a free abelian subgroup of $\mathbb{C}^n$ with rank $2n$, called a lattice.*

Definition 2.10. *Let X, Y be complex manifolds of dimension n, m, with atlas $(U_i, \phi_i : U_i \to V_i)$ and $(M_j, \psi_j : M_j \to N_j)$ respectively. A continous map $f : X \to Y$ is called holomorphic, if for any two charts, we have*

$$\psi_j \circ f \circ \phi_i^{-1} : V_i \to \psi_j(f(U_i) \cap M_j)$$

is holomorphic.

Definition 2.11. *A holomorphic function on X is a holomorphic map $f : X \to \mathbb{C}$.*

Example 2.12. *Let $S = (\mathbb{C}^2 \backslash \{0\})/\mathbb{Z}$ be Hopf surface, then*

$$f : S \to \mathbb{P}^1 = (\mathbb{C}^2 \backslash \{0\})/\mathbb{C}^*$$

is a holomorphic map. The fibers of f are biholomorphic to 1-dimensional complex torus.

Proposition 2.13. *If X is a compace complex manifold, then every holomorphic function on X is constant.*

Definition 2.14. *A holomorphic map $f : X \to Y$ is called an immersion(resp submersion), if for all $x \in X$, there exists $(x \in U_i, \phi_i), (f(x) \in M_j, \psi_j)$, such that*

$$J_{\psi_j \circ f \circ \phi_i^{-1}}(\phi_i(x))$$

has the max rank $\dim X$(resp $\dim Y$)

Definition 2.15. *$f : X \to Y$ is an embedding, if it is immersion and $f : X \to f(X) \subset Y$ is homeomorphism.*

Definition 2.16. *A closed connected subset Y of X is called a submanifold, if for all $x \in Y$, there exists $x \in U \subset X$ and a holomorphic submersion $f : U \to \mathbb{D}^k$ such that*

$$U \cap Y = f^{-1}(0)$$

where k is the codimension of Y in X.

Example 2.17. *Let X, Y be complex manifold with dimension n, m, $y \in Y$ such that rank$J_{f(x)}$ reaches maximum m for all $x \in f^{-1}(y)$, then $f^{-1}(y)$ is a submanifold of codimension m.*

Definition 2.18. *A projective manifold X is a submanifold of $\mathbb{P}^N$ of the form*

$$X = \{[z] \in \mathbb{P}^N \mid f_1(z) = \cdots = f_m(z) = 0\}$$

where f_i is a homogenous polynomial in $\mathbb{C}[z_0, \dots, z_n]$

Remark 2.19. *Here we always assume $(f_1, \dots, f_m) \subset \mathbb{C}[z_0, \dots, z_n]$ is a prime ideal. So there are no following cases:*

1. $f^2 = 0$
2. $f_1 f_2 = 0$
3. *X has a singular point.*

Definition 2.20. *Let $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{P}^n$ be the projection, then X is a submanifold of codimension k if and only if*

$$J = (\frac{\partial f_i}{\partial z_j})_{\substack{1 \leq i \leq m \\ 0 \leq j \leq N}}$$

has rank k, for all $x \in \pi^{-1}(X)$.

Then X is called a complete intersection, if $m = k$.

Example 2.21. *Consider $C \subset \mathbb{P}^n$ defined by*

$$xw - yz = y^2 - xz = z^2 - yw = 0$$

is not a complete intersection, called twisted cubic.

Example 2.22. *Plücker embedding*

$$\Phi : G(r, V) \hookrightarrow \mathbb{P}(\wedge^r V)$$

defined by $S \subset V$ with basis $s_1, \dots, s_r$ is mapped to $[s_1 \wedge \cdots \wedge s_r]$. Check Φ is well-defined embedding.

In fact, this embedding is an explict construction of Chow theorem in the case of Grassmannian manifold.

Theorem 2.23 (Chow/GAGA). *Every compact complex manifold X admitting embedding $X \hookrightarrow \mathbb{P}^n$ is defined by homogenous polynomials.*

3. Vector bundle

Definition 3.1 (complex vector bundle). *Let X be a differential manifold, E is a complex vector bundle of rank r on X*

1. (*Via total space*) *E is a differential manifold with surjective map $\pi : E \to X$, such that*
   (1) *For all $x \in X$, fiber E_x is a $\mathbb{C}$-vector space of dimension r.*
   (2) *For all $x \in X$, there exists $x \in U \subset X$ and $\pi^{-1}(U)$ is diffeomorphic to $U \times \mathbb{C}^r$ via h, such that*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\pi"] \arrow[dr, "h"'] & & U \\ & U \times \mathbb{C}^r \arrow[ur, "p_1"'] \arrow[r, "p_2"] & \mathbb{C}^r \end{tikzcd}$$

   and for all $y \in U$, $E_y \xrightarrow{p_2 \circ h} \mathbb{C}^r$ is a $\mathbb{C}$- vector space isomorphism.
   Such (U, h) is called a local trivialization.

Remark 3.2. *Consider two local trivialization $(U_\alpha, h_\alpha), (U_\beta, h_\beta)$, then $h_\alpha \circ h_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$, this induces*

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{\mathit{diff}} \mathrm{GL}(r, \mathbb{C})$$

such $g_{\alpha\beta}$ are called transition function, such that

$$g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = \mathrm{id} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma$$
$$g_{\alpha\alpha} = \mathrm{id} \quad \text{on } U_\alpha$$

2. (*Via transition function*) *E is the data of*
   (1) *open covering $\{U_\alpha\}$*
   (2) *transition functions $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{\mathit{diff}} \mathrm{GL}(r, \mathbb{C})\}$, satisfies*

$$g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = \mathrm{id} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma$$
$$g_{\alpha\alpha} = \mathrm{id} \quad \text{on } U_\alpha$$

Definition 3.3 (holomorphic vector bundle). *X is a complex manifold, $\pi : E \to X$ is a complex vector bundle, given by $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{C})$, E is called holomorphic if $g_{\alpha\beta}$ is holomorphic.*

Exercise 3.4. *Show that the total space of a holomorphic vector bundle E is a complex manifold.*

Definition 3.5 (morphism between vector bundles). *ϕ is a diffeomorphic/holomorphic morphism of vector bundle on X, if $\phi : E \to F$ is diffeomorphic/holomorphic map such that*

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi_1"'] & \arrow[d, phantom, "\circlearrowleft"] & F \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

Example 3.6. *X is a differential/complex manifold, then $X \times \mathbb{C}^r$ is the trivial rank r complex/holomorphic vector bundle on X.*

Example 3.7. *E, F are complex/holomorphic vector bundles on X, then $E \oplus F, E \otimes F, \mathrm{Hom}(E, F).E^* = \mathrm{Hom}(E, \mathbb{C}), \mathrm{Sym}^k E, \wedge^k E, \det E$ are complex/holomorphic vector bundles.*

Example 3.8. *holomorphic line bundle L is a rank 1 vector bundle, i.e.,*

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{holo} \mathbb{C}^*$$

Exercise 3.9. *L is a trivial line bundle $X \times \mathbb{C}$ if and only if up to refinement, there exists $s_\alpha : U_\alpha \to \mathbb{C}^*$, such that $g_{\alpha\beta} = s_\alpha / s_\beta$ on $U_\alpha \cap U_\beta$*

Definition 3.10 (picard group)**.** *X is a complex manifold, then*

$$\mathrm{Pic}(X) = (\{\textit{holomorphic line bundles on } X\}/\textit{isomorphism}, \otimes)$$

called the Picard group of X.

Example 3.11. *Line bundle on $\mathbb{P}^n$*

$$\begin{tikzcd} L = \{([l], x) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid x \in l\} \subset \mathbb{P}^n \times \mathbb{C}^{n+1} \arrow[d, "\pi"] \\ \mathbb{P}^n \end{tikzcd}$$

is called tautological line bundle.

Consider open covers

$$U_i = \{[l] = [l_1, \ldots, l_n] \in \mathbb{P}^n \mid l_i \neq 0\}$$

there is a map $U_i \to \pi^{-1}(U_i)$, defined as

$$[l] \mapsto ([l], (\frac{l_0}{l_i}, \ldots, 1, \ldots, \frac{l_n}{l_i}))$$

and local trivialization $h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ defined as

$$([l], x) \mapsto ([l], \lambda)$$

where

$$x = \lambda(\frac{l_0}{l_i}, \ldots, 1, \ldots, \frac{l_n}{l_i})$$

so we can calculate transition function

$$\begin{aligned} h_i \circ h_j^{-1} : (U_i \cap U_j) \times \mathbb{C} &\longrightarrow (U_i \cap U_j) \times \mathbb{C} \\ ([l], \lambda_j) &\mapsto ([l], \lambda_j(\frac{l_0}{l_j}, \ldots, \frac{l_n}{l_j})) \mapsto ([l], \lambda_i) \end{aligned}$$

such that

$$\lambda_j(\frac{l_0}{l_j},\ldots,\frac{l_n}{l_j}) = \lambda_i(\frac{l_0}{l_i},\ldots,\frac{l_n}{l_i})$$

which implies

$$\lambda_i = \lambda_j \frac{l_i}{l_j}$$

so we can see transition function $g_{ij} = l_i/l_j \in \mathbb{C}^*$*.*

The line bundle L *will be denoted by* $\mathscr{O}_{\mathbb{P}^n}(-1)$

Definition 3.12. *We can define*

$$\mathscr{O}_{\mathbb{P}^n}(-k) = \mathscr{O}_{\mathbb{P}^n}(-1)^{\otimes k}, \quad k \in \mathbb{N}^+$$
$$\mathscr{O}_{\mathbb{P}^n}(k) = (\mathscr{O}_{\mathbb{P}^n}(-k))^*, \quad k \in \mathbb{N}^+$$
$$\mathscr{O}_{\mathbb{P}^n}(0) = \mathbb{P}^n \times \mathbb{C}, \quad \textit{trivial line bundle.}$$

Remark 3.13. *In fact, line bundle listed above contain all possible line bundles over* $\mathbb{P}^n$*.*

Definition 3.14 (section)**.** $\pi : E \to X$ *is a complex/holomorphic vector bundle. A* (*global*) *section of* E *is a differential/holomorphic map* $s : X \to E$*, such that* $\pi \circ s = \mathrm{id}_X$*, denoted by* $C^\infty(X, E)$ / $\Gamma(X, E)$*.*

Example 3.15. *Global holomorphic sections of trivial holomorphic vector bundle are exactly holomorphic functions* $f : X \to \mathbb{C}^r$*.*

Remark 3.16. *In fact, global holomorphic sections are very rare, as we can seen from the above example, if* X *is a compact complex manifold, then all global holomorphic functions are only constant.*

Exercise 3.17.

$$\Gamma(\mathbb{P}^n, O_{\mathbb{P}^n}(k)) = \begin{cases} 0, & k < 0 \\ \mathbb{C}, & k = 0 \\ \textit{homogeneous polynomials in } n+1 \textit{ varibles of deg } k, & k > 0 \end{cases}$$

Definition 3.18 (subbundle)**.** $\pi : E \to X$ *is a complex/holomorphic vector bundle.* $F \subset E$ *is called a subbundle of rank* s*, if*

1. *For all* $x \in X$*,* $F \cap E_x$ *is a subvector space of dimension* s*.*
2. $\pi|_F : F \to X$ *induces a complex/holomorphic vector bundle.*

Example 3.19. $\mathscr{O}_{\mathbb{P}^n}(-1) \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$*, is a subbundle.*

4. Sheaves

Why we need sheaves here? As we have seen in the last section, the global sections of holomorphic vector bundle are very rare, but there are many local sections, we need to keep these information and learn the connection between global and local systemically. Sheaf is a power language for us to manage global and local at the same time. However, there is nothing more that sheaf can give, it's just a different language, as we can see in Exercise 4.5.

Definition 4.1 (sheaf). *X is a topological space. A sheaf* of abelian group $\mathscr{F}$ on X is the data of:*

1. *For any open subset U of X, $\mathscr{F}(U)$ is an abelian group.*
2. *If $U \subset V$ are two open subsets of X, then there is a group homomorphism $r_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$, such that*
   (1) $\mathscr{F}(\varnothing) = 0$
   (2) $r_{UU} = \mathrm{id}$
   (3) *If $W \subset U \subset V$, then $r_{UW} = r_{VW} \circ r_{UV}$*
   (4) *$\{V_i\}$ is an open covering of $U \subset X$, and $s \in \mathscr{F}(U)$. If $s|_{V_i} := r_{UV_i}(s) = 0, \forall i$, then $s = 0$.*
   (5) *$\{V_i\}$ is an open covering of $U \subset X$, and $s_i \in \mathscr{F}(V_i)$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there exists $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$.*

Example 4.2 (sheaf of sections of a holomorphic vector bundle). *If $\pi : E \to X$ is a holomorphic vector bundle, then define*

$$\mathscr{F}(U) = \Gamma(U, E|_U), \quad \forall U \subset_{open} X$$

This $\mathscr{F}$ will be denoted by $\mathscr{O}_X(E)$. In particular, E is a trivial vector bundle, then $\mathscr{O}_X(E) = \mathscr{O}_X$, the sheaf of holomorphic function, also called the structure sheaf of X.

Definition 4.3 (morphism of sheaves on X). *$\phi : \mathscr{F} \to \mathscr{G}$ is called a morphism of sheaves, if for any open subset U of X, there is a group homomorphism $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$, such that if $U \subset V$ are two open subsets of X, the the following diagram commutes*

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\phi(U)} & \mathscr{G}(U) \\ \downarrow{\scriptstyle r_{UV}} & & \downarrow{\scriptstyle r_{UV}} \\ \mathscr{F}(V) & \xrightarrow{\phi(V)} & \mathscr{G}(V) \end{array}$$

Example 4.4 (locally free sheaves). *A sheaf is called locally free, if there exists covering $\{U_\alpha\}$ such that $\mathscr{F}|_{U_\alpha} \cong \mathscr{O}_{U_\alpha}^{\oplus r}$ of rank r.*

For $r = 1$, it is called invertible sheaf.

Exercise 4.5. *There are correspondences:*

$$\{\textit{holomorphic vector bundles}\} \xleftrightarrow{1-1} \{\textit{locally free sheaves}\}$$

$$\{\textit{holomorphic line bundles}\} \xleftrightarrow{1-1} \{\textit{invertible sheaves}\}$$

*A sheaf which fails to meet $(4), (5)$ is called a presheaf.

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