

A BRIEF INTRODUCTION TO HODGE THEORY

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ABSTRACT. In this talk we give a brief introduction to Hodge theory as preliminaries for [Fil16], such as (polarized) Hodge structures, variation of Hodge structures and differential geometry of Hodge bundles.

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1. HODGE STRUCTURES

1.1. Hodge structures. Let (X, ω) a compact Kähler manifold. The classical Hodge theory says that there is a decomposition on the k -th cohomology as follows

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where $H^{p,q}(X)$ is the Dolbeault cohomology. The Hodge structure generalized this structure.

1.1.1. *Objects.*

Definition 1.1.1. An **(effective) \mathbb{Z} -Hodge structure** of weight k consists of the following data:

1. a finitely generated abelian group $V_{\mathbb{Z}}$;
2. a decomposition

$$V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$;

3. $V^{p,q} = 0$ unless $p, q \geq 0$.

Definition 1.1.2. The **Deligne torus** \mathbb{S} is the real algebraic group

$$\mathbb{S} = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a^2 + b^2 \neq 0 \right\}.$$

Its real points are naturally isomorphic to \mathbb{C}^* and its complex points are isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$.

Proposition 1.1.1. A Hodge structure on $V_{\mathbb{Z}}$ is the same as an algebraic representation of the Deligne torus \mathbb{S} on $V_{\mathbb{Z}}$.

Definition 1.1.3. Let $(V_{\mathbb{Z}}, V^{p,q})$ be a \mathbb{Z} -Hodge structure of weight k . The **Hodge filtration** F^p is defined by

$$F^p = \bigoplus_{p' \geq p} V^{p',q}.$$

It's a decreasing filtration which satisfies

$$(1.1) \quad V_{\mathbb{C}} = F^p \oplus \overline{F^{k-p+1}}.$$

Remark 1.1.1. Let $V_{\mathbb{Z}}$ be a finitely generated abelian group and F^p be a filtration satisfies (1.1). Then it determines a Hodge structure by

$$V^{p,q} = F^p \cap \overline{F^q}.$$

In other words, a Hodge structure of weight k is equivalent to a filtration F^p satisfying (1.1).

Example 1.1.1. Let V, W be two Hodge structures of weight k and l respectively. Then

- (1) V^* is a Hodge structure of weight $-k$;
- (2) $V \otimes W$ is a Hodge structure of weight $k + l$;
- (3) $\text{Hom}(V, W)$ is a Hodge structure of weight $-k + l$;
- (4) $V^{\otimes n}$, $\text{Sym}^n V$ and $\bigwedge^n V$ are Hodge structures of weight nk .

1.1.2. Morphisms.

Definition 1.1.4. Let $(V_{\mathbb{Z}}, V^{p,q}), (W_{\mathbb{Z}}, W^{p,q})$ be two Hodge structures of weight k and $k + 2r$ and $\phi: V_{\mathbb{Z}} \rightarrow W_{\mathbb{Z}}$ be a morphism of abelian groups. Then ϕ is called a **morphism of Hodge structure of type (r, r)** , if its \mathbb{C} -linear extension $\phi_{\mathbb{C}}$ satisfies

$$\phi_{\mathbb{C}}(V^{p,q}) \subseteq W^{p+r, q+r}.$$

Proposition 1.1.2. Let ϕ be a morphism between Hodge structures. Then $\ker \phi$, $\text{im } \phi$ and $\text{coker } \phi$ are Hodge structures.

Example 1.1.2. Let $f: X \rightarrow Y$ be a holomorphic map between compact Kähler manifolds. Then $f^*: H^k(Y, \mathbb{Z}) \rightarrow H^k(X, \mathbb{Z})$ is a morphism of Hodge structure of type $(0, 0)$.

Example 1.1.3. Let X, Y be two compact Kähler manifolds such that $\dim X = n, \dim Y = m$ and $m = n + r$. Then

$$\begin{array}{ccc} H^k(X, \mathbb{Z}) & \longrightarrow & H^{k+2r}(Y, \mathbb{Z}) \\ \downarrow & & \uparrow \\ H_{2n-k}(X, \mathbb{Z}) & \xrightarrow{f_*} & H_{2n-k}(Y, \mathbb{Z}). \end{array}$$

This gives a morphism of Hodge structure of type (r, r) , which is called **Gysin pushforward**.

1.2. Polarization. Let (X, ω) be a complex Kähler n -manifold. There is an intersection form Q on $H^k(X, \mathbb{R})$ given by

$$Q(\alpha, \beta) = (-1)^{\frac{k(k-1)}{2}} \int_X \omega^{n-k} \wedge \alpha \wedge \beta.$$

The induced Hermitian form $H(\alpha, \beta) := Q(\alpha, \bar{\beta})$ on $H^k(X, \mathbb{C})$ satisfies the following properties:

- (1) The Hodge decomposition is orthogonal with respect to H .
- (2) $(\sqrt{-1})^{p-q} H(\alpha, \bar{\alpha}) > 0$ for $0 \neq \alpha \in H^{p,q}(X)$.

This gives a polarization.

Definition 1.2.1. Let $(V_{\mathbb{Z}}, V^{p,q})$ be a \mathbb{Z} -Hodge structure. The **Weil operator** \mathbb{L} associated to $V_{\mathbb{Z}}$ acts by $(\sqrt{-1})^{p-q}$ on $V^{p,q}$.

Definition 1.2.2. A **polarized \mathbb{Z} -Hodge structure** of weight k is \mathbb{Z} -Hodge structure $(V_{\mathbb{Z}}, V^{p,q})$ of weight k together with a morphism of Hodge structure $Q: V_{\mathbb{Z}} \otimes V_{\mathbb{Z}} \rightarrow \mathbb{Z}$ of type $(-k, -k)$ such that

$$\begin{aligned} H: V_{\mathbb{C}} \otimes V_{\mathbb{C}} &\rightarrow \mathbb{C} \\ (\alpha, \beta) &\mapsto Q(\mathbb{L}\alpha, \bar{\beta}) \end{aligned}$$

is a positive definite Hermitian form.

2. VARIATION OF HODGE STRUCTURES

2.1. Local system and flat connection. In this section we always assume X is a complex manifold.

Definition 2.1.1. A sheaf \mathcal{V} on X is called a locally constant sheaf of rank r valued in \mathbb{C} , if for each point $x \in X$, there is an open subset U containing x such that $\mathcal{V}|_U$ is constant sheaf $\underline{\mathbb{C}}^r$.

Remark 2.1.1. In other words, there exists an open covering $\{U_\alpha\}$ such that $\mathcal{V}|_{U_\alpha}$ is isomorphic to constant sheaf $\underline{\mathbb{C}}^r$. Then the local system \mathcal{V} is completely determined by the transition functions $g_{\alpha\beta}: U_{\alpha\beta} \rightarrow \mathrm{GL}_n(\mathbb{C})$, which are locally constant functions.

Definition 2.1.2. Let \mathcal{E} be a locally free sheaf on X . A **connection** is a \mathbb{C} -linear map

$$\nabla: \mathcal{E} \rightarrow \mathcal{A}_X^1 \otimes \mathcal{E}$$

satisfying the following condition

$$\nabla(\varphi \otimes e) = d\varphi \otimes e + \varphi \nabla e$$

for all sections e of \mathcal{E} and φ of \mathcal{O}_X .

Remark 2.1.2. The definition of ∇ extends to $\nabla: \mathcal{A}_X^p \otimes \mathcal{E} \rightarrow \mathcal{A}_X^{p+1} \otimes \mathcal{E}$ by defining

$$\nabla(\omega \otimes e) = d\omega \otimes e + (-1)^p \omega \wedge \nabla e$$

for all sections ω of \mathcal{A}_X^p and sections e of \mathcal{E} .

Remark 2.1.3. Let $\{e_\alpha\}$ be a local frame of \mathcal{E} . For any section $s = s^\alpha e_\alpha$ of \mathcal{E} , one has

$$\nabla(s^\alpha e_\alpha) = ds^\alpha e_\alpha + s^\alpha \nabla e_\alpha.$$

Thus the connection ∇ is completely determined by

$$\nabla e_\alpha = \omega_\alpha^\beta e_\beta,$$

where ω_α^β are 1-forms, which forms a (smooth) 1-form valued matrix ω .

Definition 2.1.3. A connection ∇ is **integrable** if its curvature $\nabla^2: \mathcal{E} \rightarrow \mathcal{A}_X^2 \otimes \mathcal{E}$ vanishes.

Remark 2.1.4. Let $\{e_\alpha\}$ be a local frame of \mathcal{E} . For any section $s = s^\alpha e_\alpha$ of \mathcal{E} , one has

$$\begin{aligned} \nabla^2(s^\alpha e_\alpha) &= \nabla(ds^\alpha \otimes e_\alpha + s^\alpha \omega_\alpha^\beta \otimes e_\beta) \\ &= -ds^\alpha \wedge \omega_\alpha^\beta \otimes e_\beta + d(s^\alpha \omega_\alpha^\beta) \otimes e_\beta - s^\alpha \omega_\alpha^\beta \wedge \omega_\beta^\gamma \otimes e_\gamma \\ &= s^\alpha (d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta) \otimes e_\beta, \\ \nabla^2(e_\alpha) &= \nabla(\omega_\alpha^\beta \otimes e_\beta) \\ &= d\omega_\alpha^\beta \otimes e_\beta - \omega_\alpha^\beta \wedge \nabla e_\beta \\ &= d\omega_\alpha^\beta \otimes e_\beta - \omega_\alpha^\beta \wedge \omega_\beta^\gamma \otimes e_\gamma \\ &= (d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta) \otimes e_\beta. \end{aligned}$$

This shows ∇^2 is a global section of $\mathcal{A}_X^2 \otimes \underline{\text{End}}_{\mathcal{O}_X}(\mathcal{E})$, which is locally given by $d\omega - \omega \wedge \omega$.

Definition 2.1.4. A locally free sheaf together with an integrable connection is called a **flat bundle**.

Proposition 2.1.1. Let ∇ be a integrable connection on locally free sheaf \mathcal{E} on X . Then the horizontal section $\mathcal{E}^{\nabla=0}$ is a local system.

Proposition 2.1.2. Let \mathcal{L} be a local system on X . Then the locally free sheaf $\mathcal{E} := \mathcal{O}_X \otimes \mathcal{L}$ together with canonical connection $\nabla_{\text{can}}(f \otimes s) := df \otimes s$ is a flat bundle.

Theorem 2.1.1. The functor $(\mathcal{E}, \nabla) \mapsto \mathcal{E}^{\nabla=0}$ is an equivalence between category of flat bundles and the category of the complex local system with quasi-inverse $\mathcal{L} \mapsto (\mathcal{O}_X \otimes \mathcal{L}, \nabla_{\text{can}})$.

Proposition 2.1.3. Let \mathcal{L} be a local system on X . Then

$$H^*(X, \mathcal{L}) \cong \mathbb{H}^*(X, \mathcal{A}_X^\bullet \otimes \mathcal{L}).$$

Proof. Note the following complex of sheaves

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{A}_X^\bullet \otimes \mathcal{L}$$

gives a resolution of \mathcal{L} by coherent sheaves. \square

2.2. Abstract variation of Hodge structures. In this section we always assume S is a complex manifold.

Definition 2.2.1. A **variation of Hodge structure of weight k** on S consists of the following data:

- (1) a local system $\mathbb{V}_{\mathbb{Z}}$ of finitely generated abelian groups on S ;
- (2) a finite decreasing filtration $\{\mathcal{F}^p\}$ of the holomorphic vector bundle $\mathcal{V} := \mathbb{V}_{\mathbb{Z}} \otimes \mathcal{O}_X$ by holomorphic subbundles (**the Hodge filtration**).

These data should satisfy the following conditions:

- (a) for each $s \in S$ the filtration $\{\mathcal{F}^p(s)\}$ of $\mathbb{V}_s \simeq \mathbb{V}_{\mathbb{Z},s} \otimes_{\mathbb{Z}} \mathbb{C}$ defines a Hodge structure of weight k on the finitely generated abelian group $\mathbb{V}_{\mathbb{Z},s}$;
- (b) the Gauss-Manin connection $\nabla^{GM} : \mathcal{V} \rightarrow \Omega_S^1 \otimes \mathcal{V}$ whose sheaf of horizontal sections is $\mathbb{V}_{\mathbb{C}}$ satisfies the **Griffiths' transversality condition**

$$\nabla^{GM}(\mathcal{F}^p) \subseteq \Omega_S^1 \otimes \mathcal{F}^{p-1}.$$

The notion of a **morphism of variations of Hodge structure** is defined in the obvious way.

Example 2.2.1. Given two variations \mathbb{V}, \mathbb{V}' of Hodge structure over S of weights k and k' , there is an obvious structure of variation of Hodge structure on the underlying local systems of $\mathbb{V} \otimes \mathbb{V}'$ and $\text{Hom}(\mathbb{V}, \mathbb{V}')$ of weights $k + k'$ and $k - k'$ respectively.

Definition 2.2.2. A **polarized variation of Hodge structure** of weight k is a variation of Hodge structures \mathbb{V} of weight k together with a bilinear pairing $I(-, -)$ satisfying

- (1) The pairing is flat, that is, preserved by the Gauss-Manin connection ∇ .
- (2) On each fiber of \mathcal{V} , the pairing induces a polarization of the Hodge structure.

Remark 2.2.1. The intersection pairing $I(-, -)$ on a polarization variation of Hodge structure is flat for the Gauss-Manin connection, but the Hodge metric $Q(-, -)$ may not. Since the Hodge metric Q is expressed in terms of I and the Weil operator \mathbb{C} , the compatibility with the Weil operator implies the compatibility with the Hodge metric.

2.3. Variation of Hodge structures coming from smooth families.

In this section we will explain the motivation of variation of Hodge structures and the Griffiths transversality condition is inspired by the geometric case naturally.

Let $f: X \rightarrow S$ be a family¹ of compact Kähler manifolds. Then $R^k f_* \mathbb{C}$ is a local system on S such that for each point $s \in S$, one has $(R^k f_* \mathbb{C})_s \cong H^k(X_s, \mathbb{C})$. The flat bundle corresponding to the local system $R^k f_* \mathbb{C}$ is the relative de Rham cohomology

$$H_{dR}^k(X/S) := \mathcal{O}_S \otimes R^k f_* \mathbb{C}$$

together with the Gauss-Manin connection ∇^{GM} .

Proposition 2.3.1 ([Del70]). Let $f: X \rightarrow S$ be a family of complex manifolds and \mathbb{V} be a local system of complex vector spaces on X . There is a natural isomorphism

$$\mathcal{O}_S \otimes R^k f_* \mathbb{V} \cong R^k f_* (\Omega_{X/S}^\bullet \otimes \mathbb{V}),$$

where $\Omega_{X/S}^\bullet = \Omega_X^\bullet / f^* \Omega_S^\bullet$ is the relative de Rham complex.

Corollary 2.3.1. Let $f: X \rightarrow S$ be a family of complex manifolds. Then

$$H_{dR}^k(X/S) \cong R^k f_* \Omega_{X/S}^\bullet.$$

Remark 2.3.1. In this viewpoint, the Hodge filtration on $R^k f_* \Omega_{X/S}^\bullet$ is described as follows

$$\mathcal{F}^p := \text{Im} \left\{ R^k f_* \sigma^{\geq p} \Omega_{X/S}^\bullet \rightarrow R^k f_* \Omega_{X/S}^\bullet \right\},$$

where $\sigma^{\geq p}$ is the stupid filtration.

Proposition 2.3.2. Let $f: X \rightarrow S$ be a family of compact Kähler manifolds. The Gauss-Manin connection ∇^{GM} satisfies the Griffiths transversality, that is,

$$\nabla^{GM}(\mathcal{F}^p) \subseteq \Omega_S^1 \otimes \mathcal{F}^{p-1}.$$

¹In other words, f is a proper holomorphic submersion between complex manifolds such that every fiber of f is a compact Kähler manifold.

Proof. See Corollary 10.31 in [PS08]. □

Corollary 2.3.2. Let $f: X \rightarrow S$ be a family of compact Kähler manifolds. Then the local system $R^k f_* \underline{\mathbb{C}}$ underlies a variation of Hodge structures of weight k .

3. DIFFERENTIAL GEOMETRY OF HODGE BUNDLES

3.1. General setting. In this section, we consider a special connection on a holomorphic vector bundle E on a complex manifold X equipped with a Hermitian metric. Recall that we can define connection for any (smooth) vector bundle over X , which is a \mathbb{C} -linear map between (smooth) sections satisfying the Leibniz rules.

There is a canonical connection, called Chern connection, on Hermitian vector bundle E , which is uniquely defined by the following two conditions:

- (1) It's compatible with the Hermitian metric.
- (2) It's compatible with the holomorphic structure.

3.1.1. Hermitian vector bundle.

Definition 3.1.1. Let E be a complex vector bundle. A **Hermitian metric** h on E is a smooth section of $E^* \otimes \overline{E}^*$.

Remark 3.1.1. Let $\{e_\alpha\}$ be a local frame of E . Then a (positive definite) Hermitian metric is determined by a (positive definite) Hermitian matrix $(h_{\alpha\bar{\beta}})$, that is

$$h = h_{\alpha\bar{\beta}} e^\alpha \otimes \bar{e}^\beta,$$

where $h_{\alpha\bar{\beta}} = h(e_\alpha, \bar{e}_\beta)$.

Definition 3.1.2. A complex vector bundle E together with a Hermitian metric h is called a **Hermitian vector bundle** (E, h) .

Remark 3.1.2. Let L be a Hermitian line bundle. A Hermitian metric h is locally given by $e^{-2\varphi}$, where φ is a smooth function, which is called **metric weight**. Suppose $\{g_{\alpha\beta}\}$ is the transition function of L with respect to open covering $\{U_\alpha\}$. Then h is given by a collection $\{h_\alpha \in C^\infty(U_\alpha)\}$ such that $h_\alpha = |g_{\alpha\beta}|^{-2} h_\beta$. In other words, a Hermitian metric is a collection of metric weights $\{\varphi_\alpha \in C^\infty(U_\alpha)\}$ such that

$$\varphi_\alpha = \varphi_\beta + \log |g_{\alpha\beta}|.$$

Definition 3.1.3. For a Hermitian vector bundle (E, h) over complex manifold X , there is a **sesquilinear map**

$$\begin{aligned} \mathcal{A}_X^p(E) \times \mathcal{A}_X^q(E) &\rightarrow \mathcal{A}_X^{p+q} \\ (s, t) &\mapsto \{s, t\}, \end{aligned}$$

which is locally given by

$$\{s^\alpha e_\alpha, t^\beta e_\beta\} = h_{\alpha\bar{\beta}} s^\alpha \wedge \bar{t}^\beta.$$

Definition 3.1.4. A connection ∇ on a Hermitian vector bundle (E, h) is **compatible with the metric**, if

$$d\langle s, t \rangle = \{\nabla s, t\} + \{s, \nabla t\},$$

where s, t are smooth sections of E .

Remark 3.1.3. If $\{e_\alpha\}$ is a local frame of E , then

$$\begin{aligned} dh_{\alpha\bar{\beta}} &= d\langle e_\alpha, \bar{e}_\beta \rangle \\ &= \{\nabla e_\alpha, \bar{e}_\beta\} + \{e_\alpha, \nabla \bar{e}_\beta\} \\ &= \omega_\alpha^\gamma h_{\gamma\bar{\beta}} + \overline{\omega_\beta^\gamma} h_{\alpha\bar{\gamma}}. \end{aligned}$$

In the matrix notation, we have

$$dh = \omega h + h \bar{\omega}^T.$$

3.1.2. Compatibility with complex structure. Let $E \rightarrow X$ be a complex vector bundle with connection ∇ . Then we can decompose $\nabla = \nabla^{1,0} + \nabla^{0,1}$ by composing the projection as follows

$$\begin{array}{ccc} & & \mathcal{A}_X^{1,0}(E) \\ & \nearrow & \\ \mathcal{A}_X^0(E) & \xrightarrow{\nabla} & \mathcal{A}_X^1(E) \\ & \searrow & \\ & & \mathcal{A}_X^{0,1}(E) \end{array}$$

For convenience, we use $\nabla^{0,1}$ to denote the composition $\mathcal{A}_X^0(E) \xrightarrow{\nabla} \mathcal{A}_X^1(E) \rightarrow \mathcal{A}_X^{0,1}(E)$. On the other hand, there is a natural operator $\bar{\partial}_E: \mathcal{A}_X^0(E) \rightarrow \mathcal{A}_X^{0,1}(E)$, which is locally defined by

$$\bar{\partial}_E(s^\alpha \otimes e_\alpha) = \bar{\partial}s^\alpha \otimes e_\alpha.$$

Definition 3.1.5. A connection ∇ on a holomorphic vector bundle E over a complex manifold X is said to be **compatible with holomorphic structure** if $\nabla^{0,1} = \bar{\partial}_E$.

Remark 3.1.4. Let $\{e_\alpha\}$ be a holomorphic local form of E and denote

$$\nabla e_\alpha = (\Gamma_{i\alpha}^\beta dz^i + \Gamma_{\bar{i}\alpha}^\beta d\bar{z}^i) \otimes e_\beta.$$

Then

$$0 = \nabla^{0,1} e_\alpha = \Gamma_{\bar{i}\alpha}^\beta d\bar{z}^i \otimes e_\beta.$$

This shows ∇ is compatible with holomorphic structure if and only if $\Gamma_{\bar{i}\alpha}^\beta = 0$.

Remark 3.1.5. Let E be a holomorphic vector bundle and ∇ be a connection which is compatible with the holomorphic structure. Then

$$\nabla: \mathcal{E} \rightarrow \Omega_X^1 \otimes \mathcal{E},$$

where \mathcal{E} is the locally free sheaf given by the holomorphic section of E and Ω_X^1 is the locally free sheaf of holomorphic 1-forms.

3.1.3. *Chern connection.*

Theorem 3.1.1. Let X be a complex manifold and (E, h) a Hermitian holomorphic vector bundle. Then there exists a unique connection called **Chern connection** such that it's compatible with holomorphic structure and metric.

Proof. If metric connection ∇ is compatible with holomorphic structure, then the following three equations are equivalent

$$\begin{aligned} dh &= \omega h + h \bar{\omega}^t \\ \partial h &= \omega h \\ \bar{\partial} h &= h \bar{\omega}^t, \end{aligned}$$

since ω is a $(1, 0)$ -valued matrix. This shows the Chern connection is determined by $\omega = (\partial h)h^{-1}$ uniquely. \square

Corollary 3.1.1. Let E be a complex vector bundle on a complex manifold X and h, h' are two Hermitian metrics on E which are same up to a sign. Then the Chern connection of (E, h) is the same as the one of (E, h') .

Remark 3.1.6. The Chern connection is locally determined by

$$\frac{\partial h_{\alpha\bar{\beta}}}{\partial z^i} = \Gamma_{i\alpha}^\gamma h_{\gamma\bar{\beta}}.$$

Definition 3.1.6. Let X be a complex manifold and (E, h) be a Hermitian holomorphic vector bundle. The **Chern curvature** Θ_h of (E, h) is defined as the curvature of Chern connection with respect to h .

Corollary 3.1.2. Let X be a complex manifold and (E, h) a Hermitian vector bundle equipped with Chern connection ∇ locally given by ω . Then

- (1) $\partial\omega = \omega \wedge \omega$.
- (2) $\Theta_h = \bar{\partial}\omega$.
- (3) $\bar{\partial}\Theta_h = 0$.

Proof. For (1). Since $\omega = (\partial h)h^{-1}$, then directly computation shows

$$\begin{aligned} \partial\omega &= -\partial h \wedge \partial(h^{-1}) \\ &= -\partial h \wedge (-h^{-1}\partial h h^{-1}) \\ &= (\partial h)h^{-1} \wedge (\partial h)h^{-1} \\ &= \omega \wedge \omega. \end{aligned}$$

For (2). The Chern curvature Θ_h locally looks like

$$\Theta_h = d\omega - \omega \wedge \omega = d\omega - \partial\omega = \bar{\partial}\omega.$$

For (3). It follows from (2) directly. \square

Remark 3.1.7. The Chern curvature can be expressed in terms of Christoffel symbol as follows

$$\Theta_h = \Theta_{i\bar{j}\alpha}^\gamma dz^i \wedge d\bar{z}^j \otimes e^\alpha \otimes e_\gamma,$$

where $\Theta_{i\bar{j}\alpha}^\gamma = -\partial\Gamma_{i\alpha}^\gamma/\partial\bar{z}^j$. In other type one has

$$\begin{aligned}\Theta_{i\bar{j}\alpha\bar{\beta}} &= h_{\gamma\bar{\beta}}\Theta_{i\bar{j}\alpha}^\gamma \\ &= -h_{\gamma\bar{\beta}}\partial_{\bar{j}}(h^{\gamma\bar{\delta}}\frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i}) \\ &= -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^i\partial\bar{z}^j} + h^{\gamma\bar{\delta}}\frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i}\frac{\partial h_{\gamma\bar{\beta}}}{\partial\bar{z}^j}.\end{aligned}$$

3.1.4. Second variation formula.

Lemma 3.1.1. Let (E, h) be a Hermitian holomorphic vector bundle and ∇ be the Chern connection with Chern curvature Θ . Suppose ϕ is a holomorphic section of E . Then we have the formula

$$\bar{\partial}\partial\log\|\phi\|^2 = \frac{\langle\Theta\phi, \phi\rangle}{\|\phi\|^2} + \frac{\langle\nabla\phi, \phi\rangle\langle\phi, \nabla\phi\rangle - \|\phi\|^2\langle\nabla\phi, \nabla\phi\rangle}{\|\phi\|^4}.$$

Proof. Firstly note that $\partial\|\phi\|^2$ is the $(1, 0)$ -part of $d\|\phi\|^2$, but on the other hand, by the compatibility with the metric, one has

$$d\langle\phi, \phi\rangle = \langle\nabla\phi, \phi\rangle + \langle\phi, \nabla\phi\rangle.$$

Then

$$\partial\log\|\phi\|^2 = \frac{\partial\|\phi\|^2}{\|\phi\|^2} = \frac{\langle\nabla\phi, \phi\rangle}{\|\phi\|^2}.$$

Next, we apply the chain rule

$$\bar{\partial}(\partial\log\|\phi\|^2) = \underbrace{\left(\bar{\partial}\frac{1}{\|\phi\|^2}\right)\langle\nabla\phi, \phi\rangle}_{\text{part I}} + \underbrace{\frac{1}{\|\phi\|^2}\bar{\partial}\langle\nabla\phi, \phi\rangle}_{\text{part II}}.$$

For part I, one has

$$\text{part I} = \frac{-1}{\|\phi\|^4}\bar{\partial}\|\phi\|^2 = -\frac{\langle\phi, \nabla\phi\rangle}{\|\phi\|^4}.$$

For part II, note that $\bar{\partial}\langle\nabla\phi, \phi\rangle$ is the $(1, 1)$ -part of $d\langle\nabla\phi, \phi\rangle$. By the compatibility with the metric, we have

$$d\langle\nabla\phi, \phi\rangle = \langle\Theta\phi, \phi\rangle - \langle\nabla\phi, \nabla\phi\rangle.$$

Combining the above result, this completes the proof. \square

3.1.5. Second fundamental form. Let (E, h) be a Hermitian holomorphic vector bundle over complex manifold X with rank r and S be a holomorphic subbundle of E with rank s . Then there is an exact sequence of holomorphic vector bundles

$$0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0,$$

where Q is the holomorphic quotient bundle, which is isomorphic to S^\perp as complex vector bundle.

Suppose ∇^E is the Chern connection on E and define $\nabla^S := \pi_S \circ \nabla^E$, where $\pi_S: E \rightarrow S$ is the orthogonal projection.

- (1) It's clear ∇^S is compatible with holomorphic structure of S since ∇^E is the Chern connection of E , and S is a holomorphic subbundle of E .
- (2) For sections s, t of S , one has

$$\begin{aligned} dh(s, t) &= h(\nabla^E s, t) + h(s, \nabla^E t) \\ &\stackrel{(a)}{=} h(\pi_S \circ \nabla^E s, t) + h(s, \pi_S \circ \nabla^E t) \\ &= h(\nabla^S s, t) + h(s, \nabla^S t), \end{aligned}$$

where (a) holds from π_S is orthogonal projection.

This shows that ∇^S is the Chern connection of S with respect to Hermitian metric induced by the one on E .

Definition 3.1.7. The **second fundamental form** of the subbundle S of E is defined as

$$B = \nabla^E - \nabla^S: \mathcal{A}^0(S) \rightarrow \mathcal{A}^{1,0}(Q).$$

In other words, the second fundamental form $B \in \mathcal{A}^{1,0}(\text{Hom}(S, Q))$.

Proposition 3.1.1.

$${}^S\Theta = {}^E\Theta|_S + B^* \wedge B.$$

Proof. It suffices to check pointwisely. For $p \in X$, suppose $\{e_\alpha\}_{1 \leq \alpha \leq r}$ is a holomorphic local frame of E such that $\{e_\alpha\}_{1 \leq \alpha \leq s}$ is a holomorphic local frame of S , and assume $h_{\alpha\bar{\beta}}(p) = \delta_{\alpha\bar{\beta}}$. By the formula (3.1.6) of Chern connection, for $1 \leq \alpha \leq s$, one has

$$\begin{aligned} \nabla^E e_\alpha(p) &= \sum_{\beta=1}^r \frac{h_{\alpha\bar{\beta}}}{\partial z^i} (p) dz^i \otimes \bar{e}^\beta \\ \nabla^S e_\alpha(p) &= \sum_{\beta=1}^s \frac{h_{\alpha\bar{\beta}}}{\partial z^i} (p) dz^i \otimes \bar{e}^\beta, \end{aligned}$$

and thus

$$B e_\alpha(p) = \sum_{\beta=s+1}^r \frac{h_{\alpha\bar{\beta}}}{\partial z^i} (p) dz^i \otimes \bar{e}^\beta.$$

This shows

$$B(p) = \sum_{\alpha=1}^s \sum_{\beta=s+1}^r \frac{h_{\alpha\bar{\beta}}}{\partial z^i} (p) dz^i \otimes e^\alpha \otimes \bar{e}^\beta,$$

and thus its conjugate transpose is

$$B^*(p) = \sum_{\beta=s+1}^r \sum_{\alpha=1}^s \frac{h_{\beta\bar{\alpha}}}{\partial \bar{z}^j}(p) d\bar{z}^j \otimes e^\beta \otimes e_\alpha.$$

Then

$$\begin{aligned} B^* \wedge B e_\alpha(p) &= B^* \left(\sum_{\gamma=s+1}^r \frac{h_{\alpha\bar{\gamma}}}{\partial z^i}(p) dz^i \otimes e_\gamma \right) \\ &= - \sum_{\beta=1}^s \sum_{\gamma=s+1}^r \frac{h_{\alpha\bar{\gamma}}}{\partial z^i} \frac{h_{\gamma\bar{\beta}}}{\partial \bar{z}^j}(p) dz^i \wedge d\bar{z}^j \otimes e^\beta, \end{aligned}$$

which implies

$$B^* \wedge B(p) = - \sum_{\alpha,\beta=1}^s \left\{ \sum_{\gamma=s+1}^r \frac{\partial h_{\alpha\bar{\gamma}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j}(p) \right\} dz^i \wedge d\bar{z}^j \otimes e^\alpha \otimes \bar{e}^\beta.$$

On the other hand, by using formula (3.1.7), a direct computation shows

$${}^E\Theta|_S(p) - {}^S\Theta(p) = \sum_{\alpha,\beta=1}^s \left\{ \sum_{\gamma=s+1}^r \frac{\partial h_{\alpha\bar{\gamma}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j}(p) \right\} dz^i \wedge d\bar{z}^j \otimes e^\alpha \otimes \bar{e}^\beta.$$

This shows

$${}^S\Theta = {}^E\Theta|_S + B^* \wedge B.$$

□

Proposition 3.1.2.

$${}^Q\Theta = {}^E\Theta|_Q + B \wedge B^*.$$

3.1.6. *Positivity.*

Definition 3.1.8. A real $(1,1)$ -form $\omega = \sqrt{-1}h_{i\bar{j}}dz^i \wedge d\bar{z}^j$ is **positive** if the Hermitian matrix $h_{i\bar{j}}$ is positive definite.

Definition 3.1.9. Let (E, h) be a Hermitian holomorphic vector bundle on X . A form $\Theta \in \mathcal{A}_X^{1,1} \otimes \text{End}(E)$ is **positive** if for any non-zero section e of E , one has $h(\Theta e, e)$ is positive.

Proposition 3.1.3. Let (E, h) be a Hermitian holomorphic vector bundle and $S \subseteq E$ be a holomorphic subbundle. Then $B \wedge B^*$ is positive and $B^* \wedge B$ is negative, where B is the second fundamental form.

Corollary 3.1.3. The curvature decreases in holomorphic subbundles and increases in holomorphic quotient bundles.

3.2. Curvature of Hodge bundles. Consider a variation of polarized Hodge structures of weight k over some fixed complex manifold. This data consists of a flat bundle $H_{\mathbb{C}}$ together with the Gauss-Manin connection ∇^{GM} , and there is a filtration by holomorphic subbundles

$$\dots \subset \mathcal{F}^p \subset \mathcal{F}^{p-1} \subset \dots \subset H_{\mathbb{C}}.$$

Denote the quotient subbundle by

$$\mathcal{H}^{p,q} := \mathcal{F}^p / \mathcal{F}^{p+1},$$

where $p + q = k$. The polarization provides the indefinite forms $I(-, -)$ on $H_{\mathbb{C}}$, and a definite metric

$$Q(-, -) := I(\mathbb{L}(-), -),$$

where \mathbb{L} is the Weil operator. In particular, restricted to $\mathcal{H}^{p,q}$, the definite and indefinite metrics agree up to sign.

Note that ∇^{GM} is the Chern connection on $H_{\mathbb{C}}$ equipped with the indefinite metric $I(-, -)$. On the other hand, viewing $H_{\mathbb{C}}$ as the direct sum of the holomorphic bundles $\mathcal{H}^{p,q}$, each equipped with the definite metric $Q(-, -)$, there is also a Hodge connection ∇^{Hg} , which is defined as the Chern connection of $\bigoplus \mathcal{H}^{p,q}$ equipped with the definite metric.

Consider the second fundamental form (for the indefinite metric)

$$\sigma_p : \mathcal{F}^p \rightarrow \mathcal{H}_{\mathbb{C}} / \mathcal{F}^p \otimes \mathcal{A}_X^{1,0}.$$

The Griffiths transversality condition implies it must in fact map subspaces as follows

$$\sigma_p : \mathcal{H}^{p,q} \rightarrow \mathcal{H}^{p-1,q+1} \otimes \mathcal{A}_X^{1,0}.$$

Proposition 3.2.1.

$$\nabla^{GM} = \nabla^{Hg} + \sigma_{\bullet} + \sigma_{\bullet}^*,$$

where σ_{\bullet} denotes $\bigoplus_p \sigma_p$ and similarly for σ_{\bullet}^* .

Proof. See Proposition 13.1.1 of [CMSP17]. \square

Proposition 3.2.2.

$$\Theta_{\mathcal{H}^{p,q}} = \sigma_p^* \wedge \sigma_p + \sigma_{p+1} \wedge \sigma_{p+1}^*.$$

Proof. Note that the definite and indefinite metrics agree up to sign on $\mathcal{H}^{p,q}$, by Corollary 3.1.1 it suffices to prove the curvature for the indefinite metric. From the exact sequence

$$0 \rightarrow \mathcal{F}^p \rightarrow \mathcal{H}_{\mathbb{C}} \rightarrow \mathcal{H}_{\mathbb{C}} / \mathcal{F}^p \rightarrow 0,$$

we find using Proposition 3.1.1

$$\Theta_{\mathcal{F}^p} = \sigma_p^* \wedge \sigma_p.$$

Next, consider the exact sequence

$$0 \rightarrow \mathcal{F}^{p+1} \rightarrow \mathcal{F}^p \rightarrow \mathcal{H}^{p,q} \rightarrow 0.$$

Again Proposition 3.1.1 yields

$$\begin{aligned}\Theta_{\mathcal{H}^{p,q}} &= \Theta_{\mathcal{F}^p} + \sigma_{p+1} \wedge \sigma_{p+1}^* \\ &= \sigma_p^* \wedge \sigma_p + \sigma_{p+1} \wedge \sigma_{p+1}^*.\end{aligned}$$

This completes the proof. \square

Proposition 3.2.3. Suppose e, e' are two smooth sections of $\mathcal{H}^{p,q}$. Then

$$Q(\Theta_{\mathcal{H}^{p,q}} e, e') = Q(\sigma_p e, \sigma_p e') + Q(\sigma_{p+1}^* e, \sigma_{p+1}^* e').$$

Proof. Note that on $\mathcal{H}^{p,q}$, we have

$$I(-, -) = (\sqrt{-1})^{p-q} Q(-, -) = (\sqrt{-1})^{-k} \times (-1)^p Q(-, -).$$

Then

$$\begin{aligned}(\sqrt{-1})^{-k} Q(\Theta_{\mathcal{H}^{p,q}} e, e') &= (-1)^p I(\Theta_{\mathcal{H}^{p,q}} e, e') \\ &= (-1)^p (I(\sigma_p^* \wedge \sigma_p e, e') + I(\sigma_{p+1} \wedge \sigma_{p+1}^* e, e')) \\ &= (-1)^{p+1} (I(\sigma_p e, \sigma_p e') + I(\sigma_{p+1}^* e, \sigma_{p+1}^* e')) \\ &= (\sqrt{-1})^{-k} \times (-1)^{p+1} ((-1)^{p-1} Q(\sigma_p e, \sigma_p e') + (-1)^{p+1} Q(\sigma_{p+1}^* e, \sigma_{p+1}^* e')) \\ &= (\sqrt{-1})^{-k} Q(\sigma_p e, \sigma_p e') + (\sqrt{-1})^{-k} Q(\sigma_{p+1}^* e, \sigma_{p+1}^* e').\end{aligned}$$

This completes the proof. \square

Corollary 3.2.1. The bundle $\mathcal{H}^{0,k}$ has positive curvature.

Proof. In this case, the second fundamental form σ_0 vanishes, so

$$\Theta_{\mathcal{H}^{0,k}} = \sigma_1 \wedge \sigma_1^*.$$

This completes the proof. \square

REFERENCES

- [CMSP17] James Carlson, Stefan Müller-Stach, and Chris Peters. *Period mappings and period domains*, volume 168 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2017.
- [Del70] Pierre Deligne. *Équations différentielles à points singuliers réguliers*, volume Vol. 163 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1970.
- [Fil16] Simion Filip. Semisimplicity and rigidity of the Kontsevich-Zorich cocycle. *Invent. Math.*, 205(3):617–670, 2016.
- [PS08] Chris A. M. Peters and Joseph H. M. Steenbrink. *Mixed Hodge structures*, volume 52 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2008.