

DIFFERENTIAL FORMS IN ALGEBRAIC TOPOLOGY

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ABSTRACT. Dedicated to my youth.

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1. INTRODUCTION

The most intuitively evident topological invariant of a space is the number of connected pieces into which it falls. But we wonder is there any high-dimensional analogues. These are the homotopy and cohomology groups of the space.

The evolution of the higher homotopy groups from the component concept is deceptively simple and essentially unique. To describe it, let $\pi_0(X)$ denote the set of path connected components of X and if p is a point of X , we let $\pi_0(X)$ denote the set $\pi_0(X)$ with the path connected component of p singled out. Also, corresponding to such a point p , let $\Omega_p X$ denote the space of maps¹ of the unit circle which send 1 to p , made into a topological space via the compact open topology². The path components of this so called loop space $\Omega_p X$ are now taken to be the elements of $\pi_1(X, p)$:

$$\pi_1(X, p) = \pi_0(\Omega_p X, \bar{p})$$

The composition of loops induces a group structure on $\pi_1(X, p)$ in which the constant map \bar{p} of the circle to p plays the role of identity; so endowed, $\pi_1(X, p)$ is called the fundamental group of X at p . In general, it's not abelian.

To get general case, all the higher homotopy groups $\pi_k(X, p)$ for $k \geq 2$ can now be defined through the inductive formula

$$\pi_{k+1}(X, p) = \pi_k(\Omega_p X, p)$$

For Riemannian surface, the higher $\pi_k, k \geq 2$ are all trivial, and it is in part for this reason that $\pi_1(X)$ is sufficient to classify them. The group π_k for $k \geq 2$ turn out to be abelian and therefore do not seem to have been taken seriously until the 1930's when W.Hurewicz defined them and showed that, far from being trivial, they constituted the basic ingredients needed to describe the homotopy-theoretic properties of a space.

Unfortunately, the great drawback of these easy defined invariants of a space is quite difficult to calculate. To this day not all the homotopy groups of 2-sphere have been computed. Nonetheless, by now much is known concerning the general properties of the homotopy groups, largely due to the formidable algebraic techniques to which the "cohomological extension" of the component concept lends itself, and the relations between homotopy and cohomology which have been discovered over the years.

This cohomological extension starts with the dual point of view in which a component is characterized by the property that on it every locally constant function is global constant. Such a component is sometimes called a connected component, to distinguish it from a path component. Thus, if we

¹In this note, map means continuous function

²The subbasis of compact open topology is $M_{K,U} = \{f : X \rightarrow Y \mid f(K) \subset U\}$, for compact set $K \subset X$ and open set $U \subset Y$.

define $H^0(X)$ to be the vector space of real-valued locally constant functions on X , then $\dim H^0(X)$ tells us the number of connected components of X . Note that on reasonable spaces where path connected component and connected component agree, we therefore have the formula

$$|\pi_0(X)| = \dim H^0(X)$$

Still the two concepts are dual to each other, the first using map of the unit interval into X to test for connectedness and the second using map of X into \mathbb{R} for the same purpose. One further difference is that the cohomology group has a natural \mathbb{R} -module structure.

So what is the proper high-dimensional analogues of $H^0(X)$? Unfortunately there is no decisive answer here. Many plausible definitions of $H^k(X)$ for $k > 0$ have been proposed, all with slightly different properties but all isomorphic on “reasonable spaces”. Furthermore, in the realm of differential manifolds, all these theories coincide with the de Rham theory, in some sense the most perfect example of a cohomology theory. The de Rham theory is also unique in that it stands at the crossroads of topology, analysis, and physics, enriching all three disciplines.

The gist of the “de Rham extension” is comprehended most easily when M is an open set in \mathbb{R}^n , with coordinate x_1, \dots, x_n . Then amongst the C^∞ functions on M the locally constant ones are precisely those whose gradient

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

vanishes identically. Thus here $H^0(M)$ appears as the space of solutions of the differential equation $df = 0$. This suggests that $H^1(M)$ should also appear as the space of solutions of some natural differential equations on the manifold M .

Consider 1-form on M :

$$\theta = \sum a_i dx_i$$

where the a_i 's are C^∞ functions on M . Such an expression can be integrated along a smooth path γ , so that we may think of θ as a function on path γ :

$$\gamma \mapsto \int_\gamma \theta$$

It then suggests itself to seek those θ which give rise to locally constant function of γ . Stokes' theorem tells us that these line integrals are characterized by

$$\frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i} = 0$$

On the other hand, the fundamental theorem of calculus implies that $\int_\gamma df = f(Q) - f(P)$, where P and Q are the endpoints of γ , so that the gradients are trivially locally constant.

One is here irresistibly led to the definition of $H^1(M)$ as the vector space of locally constant line integrals modulo the trivially constant ones. Similarly

the higher cohomology groups are defined by simply replacing line integrals with their high-dimensional analogues, the k -volume integrals.

2. DE RHAM THEORY

2.1. The Mayer-Vietoris Argument. The Mayer-Vietoris sequences relates the cohomology of a union to those of the subsets. Together with the Five lemma, this gives a method of proof which proceeds by induction on the cardinality of an open cover, called the Mayer-Vietoris argument. As evidence of its power and versatility, we derive from it the finite dimensionality of the de Rham cohomology, Poincaré duality, the Künneth formula, the Leray-Hirsch theorem, and the Thom isomorphism, all for manifold with finite good cover.

2.1.1. Existence of Good Cover. Let M be a manifold of dimension n . An open cover $\mathfrak{U} = \{U_\alpha\}$ of M is called a good cover if all nonempty finite intersections $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ are diffeomorphic to \mathbb{R}^n . A manifold which has a finite good cover is said to be of finite type.

Theorem 2.1.1. *Every manifold has a good cover. If the manifold is compact, then the cover may be chosen to be finite.*

Proof. Endow M with a Riemannian structure. Now we quote the theorem in differential geometry that every point in a Riemannian manifold has a geodesically convex neighborhood. The intersection of any two such neighborhood is again geodesically convex. Since a geodesically convex neighborhood in a Riemannian manifold of dimension n is diffeomorphic to \mathbb{R}^n , an open cover consisting of geodesically convex neighborhoods will be a good cover. \square

Given two covers $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ and $\mathfrak{B} = \{V_\beta\}_{\beta \in J}$, if every V_β is contained in some U_α , we say that \mathfrak{B} is a refinement of \mathfrak{U} and write $\mathfrak{U} < \mathfrak{B}$. In fact, for any open cover on a manifold has a refinement which is a good cover: simply take geodesically convex neighborhoods around each point to be inside some open set of the given cover.

The set of open covers on a manifold is a directed set, since any two open cover always have a common refinement. A subset J of a directed set I is cofinal in I if for every $i \in I$, there is a $j \in J$ such that $i < j$. It is clear that J is also a directed set.

Corollary 2.1.2. *The good covers are cofinal in the set of all covers of a manifold.*

2.1.2. Finite Dimensional of de Rham Cohomology.

Proposition 2.1.3. *If the manifold M has a finite good cover, then its cohomology is finite dimensional*

Proof. From the Mayer-Vietoris sequence

$$\cdots \rightarrow H^{q-1}(U \cap V) \xrightarrow{d^*} H^q(U \cup V) \xrightarrow{r} H^q(U) \oplus H^q(V) \rightarrow \cdots$$

we get

$$H^q(U \cup V) \cong \ker r \oplus \operatorname{im} r \cong \operatorname{im} d^* \oplus \operatorname{im} r$$

Thus, if $H^q(U)$, $H^q(V)$ and $H^{q-1}(U \cap V)$ are finite-dimensional, then so is $H^q(U \cup V)$.

For a manifold which is diffeomorphic to \mathbb{R}^n , the finite dimensionality of $H^*(M)$ follows from Poincaré lemma. We now proceed by induction on the cardinality of a good cover. Suppose the cohomology of any manifold having a good cover with at most p open sets is finite dimensional. Consider a manifold having a good cover $\{U_0, \dots, U_p\}$ with $p+1$ open sets. Now $(U_0 \cup \cdots \cup U_{p-1}) \cap U_p$ has a good cover with p open sets. By hypothesis, the q -th cohomology of $U_0 \cup \cdots \cup U_{p-1}$, U_p and $(U_0 \cup \cdots \cup U_{p-1}) \cap U_p$ are finite-dimensional, so is the q -th cohomology of $U_0 \cup \cdots \cup U_p$. This completes the induction. \square

2.1.3. Poincaré Duality on an Orientable Manifold. Consider the following pair

$$\int : H^q(M) \otimes H_c^{n-q}(M) \rightarrow \mathbb{R}$$

given by the integral of the wedge product of two forms. Our first version of Poincaré duality asserts that this pairing is nondegenerate whenever M is a orientable and has a finite good cover, or equivalently

$$H^q(M) \cong (H_c^{n-q})^*$$

Note that both of these groups are finite-dimensional.

Lemma 2.1.4. *The two Mayer-Vietoris sequences may be paired together to form a sign-commutative diagram*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^q(U \cup V) & \xrightarrow{\text{restriction}} & H^q(U) \oplus H^q(V) & \xrightarrow{\text{difference}} & H^q(U \cap V) \xrightarrow{d^*} H^{q+1}(U \cup V) \\ & & \otimes & & \otimes & & \otimes & & \otimes \\ & & & & & & & & \\ \cdots & \longleftarrow & H_c^{n-q}(U \cup V) & \xleftarrow{\text{sum}} & H_c^{n-q}(U) \oplus H_c^{n-q}(V) & \longleftarrow & H_c^{n-q}(U \cap V) \xleftarrow{d_*} H_c^{n-q-1}(U \cup V) \\ & & \downarrow \int_{U \cup V} & & \downarrow \int_U + \int_V & & \downarrow \int_{U \cap V} & & \downarrow \int_{U \cup V} \\ & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} \end{array}$$

Here sign-commutativity means, for instance, that

$$\int_{U \cap V} \omega \wedge d_* \tau = \pm \int_{U \cup V} (d^*) \wedge \tau$$

for $\omega \in H^q(U \cap V), \tau \in H_c^{n-q-1}(U \cap V)$. This lemma is equivalent to saying that the pairing induces a map from the upper exact sequence to the dual of the lower exact sequence such that the following diagram is sign-commutative:

$$\begin{array}{ccccccc}
\longrightarrow & H^q(U \cup V) & \longrightarrow & H^q(U) \oplus H^q(V) & \longrightarrow & H^q(U \cap V) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & \\
\longleftarrow & H_c^{n-q}(U \cup V)^* & \longleftarrow & H_c^{n-q}(U)^* \oplus H_c^{n-q}(V)^* & \longleftarrow & H_c^{n-q}(U \cap V)^* & \longleftarrow
\end{array}$$

Proof. The first two squares are in fact commutative as is straightforward to check. We will show the sign-commutativity of the third square.

In fact, we need to make how d_* or d^* map clearly. First, $d^*\omega$ is a form in $H^{q+1}(U \cup V)$ such that

$$\begin{aligned}
d^*\omega|_U &= -d(\rho_V\omega) \\
d^*\omega|_V &= d(\rho_U\omega)
\end{aligned}$$

and $d_*\tau$ is a form in $H_c^{n-q}(U \cap V)$ such that

$$(-(\text{extension by 0 of } d_*\tau \text{ to } U), (\text{extension by 0 of } d_*\tau \text{ to } V)) = (d(\rho_U\tau), d(\rho_V\tau))$$

Note that $d(\rho_V\tau) = (d\rho_V)\tau$ because τ is closed; similarly, $d(\rho_V\omega) = (d\rho_V)\omega$.

$$\int_{U \cap V} \omega \wedge d_*\tau = \int_{U \cap V} \omega \wedge (d\rho_V)\tau = (-1)^{\deg \omega} \int_{U \cap V} (d\rho_V)\omega \wedge \tau.$$

Since $d^*\omega$ has support in $U \cap V$,

$$\int_{U \cup V} d^*\omega \wedge \tau = - \int_{U \cap V} (d\rho_V)\omega \wedge \tau$$

Therefore,

$$\int_{U \cap V} \omega \wedge d_*\tau = (-1)^{\deg \omega + 1} \int_{U \cup V} (d^*) \wedge \tau$$

□

By the Five lemma if Poincaré duality holds for U, V and $U \cap V$, then it holds for $U \cup V$. We now proceed by induction on the cardinality of a good cover. For M is diffeomorphic to \mathbb{R}^n , Poincaré duality holds from Poincaré lemmas. Next suppose Poincaré duality holds for any manifold having a good cover with at most p open sets, and consider a manifold having a good cover $\{U_0, \dots, U_p\}$ with $p+1$ open sets. Now $(U_0 \cup \dots \cup U_{p-1}) \cap U_p$ has a good cover with p open sets. By hypothesis, Poincaré duality holds for $U_0 \cup \dots \cup U_{p-1}, U_p, (U_0 \cup \dots \cup U_{p-1}) \cap U_p$, so it holds for $U_0 \cup \dots \cup U_p$ as well. This induction argument proves Poincaré duality for any orientable manifold having a finite good cover.

Remark 2.1.5. *The finiteness assumption on the good cover is in facts not necessary. The statement is as follows: If M is a orientable manifold of dimension n , whose cohomology is not necessarily finite dimensional, then*

$$H^q(M) \cong (H_c^{n-q}(M))^*, \quad \text{for any integer } q$$

However, the reverse implication $H_c^q(M) \cong (H^{n-q}(M))^*$ is not always true.

2.1.4. *The Künneth formula and the Leray-Hirsch theorem.* The Künneth formula states that the cohomology of the product of two manifolds M and F is the tensor product

$$H^*(M \times F) = H^*(M) \otimes H^*(F)$$

This means

$$H^n(M \times F) = \bigoplus_{p+q=n} H^p(M) \otimes H^q(F)$$

More generally we are interesting in the cohomology of a fiber bundle.

Definition 2.1.6. Let G be a topological group which acts effectively on a space F on the left. A surjection $\pi : E \rightarrow B$ between topological space is a fiber bundle with fiber F and structure group G if B has an open cover $\{U_\alpha\}$ such that there are fiber-preserving homeomorphisms

$$\phi_\alpha : E|_{U_\alpha} = \pi^{-1}(U_\alpha) \cong U_\alpha \times F$$

and the transition functions are continuous with values in G :

$$g_{\alpha\beta}(x) = \phi_\alpha \phi_\beta^{-1}|_{\{x\} \times F} \in G$$

Sometimes the total space E is referred to as the fiber bundle. A fiber bundle with structure group G is also called a G -bundle. If $x \in B$, the set $E_x = \pi^{-1}(x)$ is called the fiber at x .

The following proof of the Künneth formula assumes that M has a finite good cover. This assumption is necessary for the induction argument.

The two natural projections

$$\begin{array}{ccc} M \times F & \xrightarrow{\rho} & F \\ \downarrow \pi & & \\ M & & \end{array}$$

give rise to a map on forms

$$\omega \otimes \phi \mapsto \pi^* \omega \wedge \rho^* \phi$$

which induces a map in cohomology

$$\psi : H^*(M) \otimes H^*(F) \rightarrow H^*(M \times F)$$

If $M = \mathbb{R}^n$, then this is simply the Poincaré lemma. In the following we will regard $M \times F$ as a product bundle over M . Let U and V be open sets in M and n a fixed integer. From the Mayer-Vietoris sequence

$$\cdots \rightarrow H^p(U \cup V) \rightarrow H^p(U) \oplus H^p(V) \rightarrow H^p(U \cap V) \rightarrow \cdots$$

we get an exact sequence by tensoring $H^{n-p}(F)$

$$\cdots \rightarrow H^p(U \cup V) \otimes H^{n-p}(F) \rightarrow (H^p(U) \oplus H^p(V)) \otimes H^{n-p}(F) \rightarrow H^p(U \cap V) \otimes H^{n-p}(F) \rightarrow \cdots$$

since tensoring with a vector space preserves exactness. Summing over all integers p yields the exact sequence

$$\begin{aligned} \dots &\rightarrow \bigoplus_{p=0}^n H^p(U \cup V) \otimes H^{n-p}(F) \\ &\rightarrow \bigoplus_{p=0}^n (H^q(U) \otimes H^{n-p}(F)) \oplus (H^q(V) \otimes H^{n-p}(F)) \\ &\rightarrow \bigoplus_{p=0}^n H^p(U \cap V) \otimes H^{n-p}(F) \rightarrow \dots \end{aligned}$$

So the following diagram is commutative³

$$\begin{array}{ccc} \bigoplus_{p=0}^n H^p(U \cup V) \otimes H^{n-p}(F) & \xrightarrow{\psi} & H^n((U \cap V) \times F) \\ \downarrow & & \downarrow \\ \bigoplus_{p=0}^n (H^q(U) \otimes H^{n-p}(F)) \oplus (H^q(V) \otimes H^{n-p}(F)) & \xrightarrow{\psi} & H^n(U \times F) \oplus H^n(V \times F) \\ \downarrow & & \downarrow \\ \bigoplus_{p=0}^n H^p(U \cap V) \otimes H^{n-p}(F) & \xrightarrow{\psi} & H^n((U \cap V) \times F) \end{array}$$

The commutativity is clear except possibly for the square

$$\begin{array}{ccc} \bigoplus_{p=0}^n H^p(U \cap V) \otimes H^{n-p}(F) & \xrightarrow{\psi} & H^n((U \cap V) \times F) \\ \downarrow d^* & & \downarrow d^* \\ \bigoplus_{p=0}^n H^{p+1}(U \cup V) \otimes H^{n-p}(F) & \xrightarrow{\psi} & H^{n+1}((U \cup V) \times F) \end{array}$$

Which we need to check. Let $\omega \otimes \phi$ be in $H^q(U \cap V) \otimes H^{n-p}(F)$. Then

$$\begin{aligned} \psi d^*(\omega \otimes \phi) &= \pi^*(d^*\omega) \wedge \rho^*\phi \\ d^*\psi(\omega \otimes \phi) &= d^*(\pi^*\omega \wedge \rho^*\phi) \end{aligned}$$

Since the pull back $\{\pi^*\rho_U, \pi^*\rho_V\}$ form a partition of unity on $(U \cup V) \times F$ subordinate to the cover $\{U \times F, V \times F\}$, on $(U \cap V) \times F$

$$\begin{aligned} d^*(\pi^*\omega \wedge \rho^*\phi) &= d((\pi^*\rho_U)\pi^*\omega \wedge \rho^*\phi) \\ &= (d\pi^*(\rho_U\omega)) \wedge \rho^*\phi, \quad \text{since } \phi \text{ is closed} \\ &= \pi^*(d^*\omega) \wedge \rho^*\phi \end{aligned}$$

So the diagram is commutative.

By the Five lemma if the theorem is true for U, V and $U \cap V$, then it is also true for $U \cup V$. The Künneth formula now follows by induction on the cardinality of a good cover, as in the proof before.

³Here I used an unusual way to draw this diagram, for there is no enough space when I using the normal way.

2.1.5. Poincaré Dual of a Closed Oriented Submanifold. Let M be a oriented manifold of dimension n and S a closed oriented submanifold of dimension k ; here by “closed” we mean as a subspace of M .

To every closed oriented submanifold $i : S \hookrightarrow M$ of dimension k , one can associated a unique cohomology class $[\eta_S]$ in $H^{n-k}(M)$, called its Poincaré dual, as follows.

Let ω be a closed k -form with compact support on M . Since S is closed in M , $\text{Supp}(\omega|_S)$ is closed not only in S , but also in M . $i^*\omega$ also has compact support on S , so the integral $\int_S i^*\omega$ is defined. By Stokes's theorem integrations over S induces a linear functional on $H_c^k(M)$, so Poincaré duality tells us that the integration over S corresponds to a unique cohomology class $[\eta_S]$ in $H^{n-k}(M)$. By definition, the Poincaré dual of S is the unique cohomology class in $H^{n-k}(M)$ satisfying

$$\int_S i^*\omega = \int_M \omega \wedge \eta_S$$

For any $\omega \in H_c^k(M)$.

Now suppose S is a compact oriented submanifold of dimension k in M . Since a compact subset of a Hausdorff space is closed, S is also a closed oriented submanifold and hence has a Poincaré dual $\eta_S \in H^{n-k}(M)$. This η_S we will call the closed Poincaré dual of S , to distinguish it from the compact Poincaré dual to be defined below.

Because S is compact, one can in fact integrate over S not only k -forms with compact support on M , but any k -form on M . In this way S defines a linear functional on $H^k(M)$ and by Poincaré duality corresponds to a unique cohomology class $[\eta'_S]$ in H_c^{n-k} , the compact Poincaré dual of S . We must assume here that M has a finite good cover; otherwise, the duality $(H^k(M))^* \cong H_c^{n-k}(M)$ does not hold. The compact Poincaré dual $[\eta'_S]$ is uniquely characterized by

$$\int_S i^*\omega = \int_M \omega \wedge \eta'_S$$

For any $\omega \in H^k(M)$. If the above holds for any closed k -form ω , then it certainly holds for any closed k -form with compact support. So as a form, η'_S is also the closed Poincaré dual of S , i.e., the natural map $H_c^{n-k}(M) \rightarrow H^{n-k}(M)$ sends the compact Poincaré dual to closed Poincaré dual.

However, as cohomology class, $[\eta_S] \in H^{n-k}(M)$ and $[\eta'_S] \in H_c^{n-k}(M)$ could be quite different, as the following examples demonstrate.

Example 2.1.7 (Poincaré dual of a point p on \mathbb{R}^n). Since $H^n(\mathbb{R}^n) = 0$, the closed Poincaré dual η_p is trivial and can be represented by any closed n -form on M , but the compact Poincaré dual is the nontrivial class in $H_c^n(\mathbb{R})$ represented by a bump form with total integral 1.

2.2. The Thom Isomorphism. So far we have encountered two kinds of C^∞ invariants of a manifold, de Rham cohomology and compactly supported cohomology. For vector bundles there is another invariant, namely, cohomology with compact support in the vertical direction. The Thom isomorphism is a statement about this last-named cohomology.

2.2.1. Vector Bundles and the Reduction of Structure Groups.

Definition 2.2.1. Let $\pi : E \rightarrow M$ be a surjective map of manifolds whose fiber $\pi^{-1}(x)$ is a vector space for every $x \in M$. The map π is a C^∞ real vector bundle of rank n if there is an open cover $\{U_\alpha\}$ of M and fiber-preserving diffeomorphisms

$$\phi_\alpha : E|_{U_\alpha} = \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$$

which are linear isomorphisms on each fiber. The maps

$$\phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \rightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

are vector-space automorphisms of \mathbb{R}^n in each fiber and hence give rise to maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(n, \mathbb{R})$$

defined as $g_{\alpha\beta}(x) = \phi_\alpha \phi_\beta^{-1}|_{\{x\} \times \mathbb{R}^n}$

Definition 2.2.2. Let U be an open set in M . A map $s : U \rightarrow E$ is a section of the vector bundle E over M if $\pi \circ s$ is the identity on U . The space of all sections over U is written $\Gamma(U, E)$.

Definition 2.2.3. A collection of sections s_1, \dots, s_n over an open set U in M is a frame on U if for every point x in U , $s_1(x), \dots, s_n(x)$ form a basis of the vector space $\pi^{-1}(x)$.

The transition functions $\{g_{\alpha\beta}\}$ of a vector bundle satisfy the cocycle condition. The cocycle $\{g_{\alpha\beta}\}$ depends on the choice of the trivialization.

3. ČECH-DE RHAM COMPLEX

3.1. The Generalized Mayer-Vietoris Principle. Mayer-Vietoris sequence allows us to compute the cohomology of the whole space M using an open covering \mathfrak{U} consisting of two open subsets U, V . And in fact this argument can be generalized to countably many open subsets.

To make this generalization, we first reformulate the Mayer-Vietoris sequence in language of double complex $C^*(\mathfrak{U}, \Omega^*) = \bigoplus K^{p,q} = \bigoplus C^p(\mathfrak{U}, \Omega^q)$, where

$$K^{0,q} = \Omega^q(U) \oplus \Omega^q(V)$$

$$K^{1,q} = \Omega^q(U \cap V)$$

$$K^{2,q} = 0, \quad p \geq 2$$

Remark 3.1.1. If you regard \mathfrak{U} as a open covering consisting of countably many open subsets as the following way

$$\mathfrak{U} = \{U, V, \emptyset, \emptyset, \dots\}$$

And you can guess that we can generalize above double complex by considering the intersection of more open subsets, since here intersection of any three or more open subsets must be empty.

Note that there is two differential operators on double complex $C^*(\mathfrak{U}, \Omega^*)$, d in the vertical direction and difference operator δ in horizontal direction. Clearly d, δ commute with each other.

It's necessary for us to consider a more general double complex, since it's crucial ingredient of spectral sequences we will discuss later. For a double complex $K^{*,*}$ we means it's a complex with two differential operators d, δ in vertical and horizontal direction, and we can make it into a singly graded complex K^{*4} in the following way: Consider $K^n = \bigoplus_{p+q=n} K^{p,q}$ and differential operator is defined by

$$D = \delta + (-1)^p d, \quad \text{on } K^{p,q}$$

Let's check D is a differential operator. Indeed, since K^n is a direct sum of $K^{p,q}$, so it suffice to take $\alpha^{p,q} \in K^{p,q}$ and check $D^2(\alpha^{p,q})$:

$$\begin{aligned} D^2(\alpha^{p,q}) &= D(\delta\alpha^{p,q} + (-1)^p d\alpha^{p,q}) \\ &= (-1)^{p+1} d\delta\alpha^{p,q} + (-1)^p \delta d\alpha^{p,q} \\ &= 0 \end{aligned}$$

Here we use the fact $\delta\alpha^{p,q} \in K^{p+1,q}$, $d^2 = \delta^2 = 0$ and d commutes with δ .

Remark 3.1.2. $(-1)^p$ is crucial here, otherwise we won't get $D^2 = 0$.

Now Mayer-Vietoris sequence is reformulated as following

Theorem 3.1.3. *The double complex $C^*(\mathfrak{U}, \Omega^*)$ commutes the de Rham cohomology of M .*

⁴Sometimes call it total complex.

Proof. The hallmark of proof is we need to note the following thing: A q -cochain α in double complex may have two components

$$\alpha = \alpha_0 + \alpha_1, \quad \alpha_0 \in K^{0,q}, \alpha_1 \in K^{1,q-1}$$

And we need to show that it's D -cohomologous to a cochain with only top components. And this fact holds mainly rely on the exactness of Mayer-Vietoris sequence:

$$0 \rightarrow \Omega^*(U \cup V) \xrightarrow{r} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{\delta} \Omega^*(U \cap V) \rightarrow 0$$

By surjectivity of δ we can take $\beta \in \Omega^*(U) \oplus \Omega^*(V)$ such that $\delta\beta = \alpha$, then $\alpha - D\beta$ only contains top components. \square

Now we generalize our ideal as what we have mentioned in Remark 3.1.1. Fix an open covering $\mathfrak{U} = \{U_\alpha\}_{\alpha \in J}$ of M . And use $U_{\alpha\beta}$ to denote the intersection $U_\alpha \cap U_\beta$, similarly for $U_{\alpha\beta\gamma}$. Clearly there is a natural inclusion $\partial_\alpha : U_{\alpha\beta\gamma} \rightarrow U_{\beta\gamma}$. Thus we will have the following sequence of differential forms:

$$\Omega^*(M) \xrightarrow{r} \prod \Omega^*(U_{\alpha_0}) \xrightarrow[\delta_1]{\delta_0} \prod_{\alpha_0 < \alpha_1} \Omega^*(U_{\alpha_0\alpha_1}) \xrightarrow[\delta_2]{\delta_1} \prod_{\alpha_0 < \alpha_1 < \alpha_2} \Omega^*(U_{\alpha_0\alpha_1\alpha_2}) \rightarrow \dots$$

Recall difference operator δ we defined in Mayer-Vietoris is $\delta = \delta_0 - \delta_1$, here we also define generalized difference operator in a similar way: For $\omega \in \prod \Omega^q(U_{\alpha_0 \dots \alpha_p})$, we define

$$(\delta\omega)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \dots \widehat{\alpha_i} \dots \alpha_{p+1}}$$

We also need this alternating signal in order to make $\delta^2 = 0$, that is making above sequence a complex.

Remark 3.1.4. It's a little weird in our definition about difference operator δ we require indices in $\omega_{\alpha_0 \dots \alpha_p}$ are arranged in an increasing order, since by definition $U_{\alpha\beta} = U_{\beta\alpha}$. More generally we will allow indices in any order⁵, by setting

$$\omega_{\dots\alpha\dots\beta\dots} = -\omega_{\dots\beta\dots\alpha\dots}$$

But we need to check it's consistent with definition of difference operator, that is Exercise 8.4 in Bott-Tu, and we check this as follows

Proof. We use the definition formula of difference operator to compute $(\delta\omega)_{\dots\beta\dots\alpha\dots}$. For terms $\omega_{\dots\beta\dots\widehat{\alpha_i}\dots\alpha\dots}$, it's clearly that it equals to $-\omega_{\dots\alpha\dots\widehat{\alpha_i}\dots\beta\dots}$ by definition. So it suffice to show

$$(-1)^i \omega_{\dots\widehat{\beta}\dots\alpha\dots} + (-1)^j \omega_{\dots\beta\dots\widehat{\alpha}\dots} = (-1)\{(-1)^i \omega_{\dots\widehat{\alpha}\dots\beta\dots} + (-1)^j \omega_{\dots\alpha\dots\widehat{\beta}\dots}\}$$

⁵And we should do this, since sometimes it's difficult to ask these indices to be arranged in an increasing order, we will see this in later.

But by definition we can see

$$\begin{aligned}\omega_{\dots\widehat{\alpha}\dots\beta\dots} &= (-1)^{j-i-1}\omega_{\dots\beta\dots\widehat{\alpha}\dots} \\ \omega_{\dots\alpha\dots\widehat{\beta}\dots} &= (-1)^{j-i-1}\omega_{\dots\widehat{\beta}\dots\alpha\dots}\end{aligned}$$

This completes the proof. \square

As what we have seen in Mayer-Vietoris sequence, the exactness of Mayer-Vietoris plays an important role in proving isomorphism between cohomology of total complex and de Rham cohomology of the whole space. Here we still desire our generalized Mayer-Vietoris sequence is also exact.

Proposition 3.1.5 (The generalized Mayer-Vietoris sequence). *The following sequence is exact*

$$\Omega^*(M) \xrightarrow{r} \prod \Omega^*(U_{\alpha_0}) \xrightarrow{\delta} \prod \Omega^*(U_{\alpha_0\alpha_1}) \xrightarrow{\delta} \prod \Omega^*(U_{\alpha_0\alpha_1\alpha_2}) \rightarrow \dots$$

Proof. The proof here is quite similar to what we have done in Mayer-Vietoris sequence, that is using partition of unity to construct coboundaries we desired.

But if we regard the construction process as a homotopy operator, things become interesting. We construct an operator

$$K : \prod \Omega^*(U_{\alpha_0\dots\alpha_p}) \rightarrow \prod \Omega^*(U_{\alpha_0\dots\alpha_{p-1}})$$

And we showed that

$$\delta K + K\delta = 1$$

In other words, we showed that identity map is homotopic to zero map, but homotopic chain maps induce the same map between cohomology groups. So cohomology of this complex is isomorphic to trivial group, that is it's exact. \square

By the same method, we can show

Proposition 3.1.6 (Generalized Mayer-Vietoris Principle). *The double complex $C^*(\mathfrak{U}, \Omega^*)$ computes the cohomology of M . Furthermore, restriction map $r : \Omega^*(M) \rightarrow C^*(\mathfrak{U}, \Omega^*)$ induces an isomorphism in cohomology.*

Remark 3.1.7. The philosophy here is that if every row of an augmented double complex is exact, then the cohomology of total complex will compute the cohomology of the first column. We will revisit this ideal in spectral sequences.

Note that the rows and columns are symmetric, so we may desire if we make this double complex into a column augmented double complex, and all columns are exact, then the cohomology of total complex will reflect the cohomology of the first row.

It's natural to argument each column by the kernel of the bottom d, it consists of the locally constant functions defined on $U_{\alpha_0\dots\alpha_p}$, and we denote

it by $C^*(\mathfrak{U}, \mathbb{R})$. We can write this complex explicitly

$$C^0(\mathfrak{U}, \mathbb{R}) \xrightarrow{\delta} C^1(\mathfrak{U}, \mathbb{R}) \xrightarrow{\delta} C^2(\mathfrak{U}, \mathbb{R}) \rightarrow \dots$$

We call the homology of this complex $H^*(\mathfrak{U}, \mathbb{R})$, the Čech cohomology of the cover \mathfrak{U} . Note that it's a purely combinatorial object, since here we only care for the intersections of these open subsets, and it's computable.

But what's the condition for the columns are exact? The failure of p -th column to be exact is measured by

$$\prod H^q(U_{\alpha_0 \dots \alpha_p})$$

So also by Poincaré lemma, if the open covering \mathfrak{U} is a good cover, then all columns are exact, then we get

Theorem 3.1.8 (comparison theorem). *If \mathfrak{U} is a good cover of the manifold M , then the Čech cohomology of the cover \mathfrak{U} computes the de Rham cohomology of M , that is*

$$H_{DR}^*(M) \cong H^*(\mathfrak{U}, \mathbb{R})$$

3.2. The Tic-Tac-Toe Proof of the Künneth formula. Now let's use theory we have developed to revisit Künneth formula, but we prove it in a weaker assumption: We replace M has a finite good cover by F has a finite dimensional cohomology.

Before proving the Künneth formula, let's make some general remarks: Let $\pi : E \rightarrow M$ be a map of manifolds, and \mathfrak{U} is an open covering of M , then $\pi^{-1}(\mathfrak{U})$ is clearly an open covering of E . But in general $U_\alpha \cap U_\beta = \emptyset$ is not equivalent to $\pi^{-1}U_\alpha \cap \pi^{-1}U_\beta = \emptyset$. but one direction $U_\alpha \cap U_\beta = \emptyset \implies \pi^{-1}U_\alpha \cap \pi^{-1}U_\beta = \emptyset$ always holds. Indeed, if not, take $x \in \pi^{-1}U_\alpha \cap \pi^{-1}U_\beta$, then $\pi(x) \in U_\alpha \cap U_\beta$, a contradiction. But the other direction may fail if π is not surjective, since we can't control parts which doesn't lie in the image of π by considering its inverse.

So if π is surjective, then the combinatorial property of \mathfrak{U} and $\pi^{-1}(\mathfrak{U})$ are same. The double complex of $\pi^{-1}\mathfrak{U}$ computes the cohomology of E , which can be related to the cohomology of M . This is a powerful ideal of Čech cohomology.

However, things are not quite easy, since although $\pi^{-1}\mathfrak{U}$ is good cover, \mathfrak{U} may not be a good cover. But for the case of vector bundle $\pi : E \rightarrow M$, the "goodness" of the cover is preserved. So we have

$$H_{DR}^*(E) \cong H_{DR}^*(M)$$

where $E \rightarrow M$ is a vector bundle.

Proposition 3.2.1 (Künneth formula). *If M and F are two manifolds and F has finite dimensional cohomology, then the de Rham cohomology of the product $M \times F$ is*

$$H^*(M \times F) = H^*(M) \otimes H^*(F)$$

Proof. It suffices to show

$$\begin{aligned} \pi_{\mathfrak{U}}^* : H^*(F) \otimes C^*(\mathfrak{U}, \Omega^*) &\rightarrow C^*(\pi^{-1}\mathfrak{U}, \Omega^*) \\ [\omega_\alpha] \otimes \phi &\mapsto \rho^* \omega_\alpha \wedge \pi^* \phi \end{aligned}$$

induces an isomorphism in D -cohomology. It's clear $\pi_{\mathfrak{U}}^*$ induces an isomorphism of d -cohomology of these complexes. And the claim holds from following lemma:

Lemma 3.2.2 (Acyclic Assembly Lemma). *Whenever a homomorphism $f : K \rightarrow K'$ of double complex induces H_d -isomorphism, it also induces H_D -isomorphism.*

Proof. It's an important lemma in homological algebra, but here we can assume only finitely many columns of K and K' are nonzero, since double complexes we concern possess this property. We prove this by induction on the number n of nonzero columns of double complex: If $n = 1$, it's trivial since H_d is the same as H_D . If we have proven for $n < k$, then for $n = k$, consider the following exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1 & \longrightarrow & K & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow f & & \downarrow f & & \\ 0 & \longrightarrow & K'_1 & \longrightarrow & K' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

where C, C' are first columns of K and K' , and K_1, K'_1 are subcomplexes of K, K' obtained by cutting out the first column. Then we get a long exact sequence of cohomology groups:

$$\begin{array}{ccccccccc} \cdots & \rightarrow & H^{i-1}(C') & \longrightarrow & H^i(K'_1) & \longrightarrow & H^i(K') & \longrightarrow & H^i(C') & \longrightarrow & H^i(K'_1) & \rightarrow \cdots \\ & & \uparrow f^* & & \uparrow f^* & & \uparrow f^* & & \uparrow f^* & & \uparrow f^* & \\ \cdots & \rightarrow & H^{i-1}(C) & \longrightarrow & H^i(K_1) & \longrightarrow & H^i(K) & \longrightarrow & H^i(C) & \longrightarrow & H^i(K_1) & \rightarrow \cdots \end{array}$$

And Five lemma will tell the answer. □

□

3.3. Čech cohomology of presheaf. Now let's talk about the philosophy behind what we have done. When we define $C^*(\mathfrak{U}, \Omega^*)$ and difference operator δ , the only two things we used are:

1. For any open subset U , we have a group of differential forms on U ;
2. For two open subsets $V \subset U$, there is a natural restriction $\Omega^*(U) \rightarrow \Omega^*(V)$.

So in fact, we don't need the property of $\Omega^*(U)$ as a differential forms, it can be any group, which implies we can consider same thing for presheaf of abelian groups.

So instead of using Ω^* , we can defined Čech complex for any contravariant presheaf \mathcal{F} as follows

$$0 \rightarrow C^0(\mathfrak{U}, \mathcal{F}) \xrightarrow{\delta} C^1(\mathfrak{U}, \mathcal{F}) \xrightarrow{\delta} C^2(\mathfrak{U}, \mathcal{F}) \rightarrow \dots$$

and the cohomology of this complex is denoted by $H^*(\mathfrak{U}, \mathcal{F})$. Similarly for a covariant presheaf \mathcal{F} , we can define

$$0 \leftarrow C_0(\mathfrak{U}, \mathcal{F}) \xleftarrow{\delta} C_1(\mathfrak{U}, \mathcal{F}) \xleftarrow{\delta} C_2(\mathfrak{U}, \mathcal{F}) \leftarrow \dots$$

and consider its homology $H_*(\mathfrak{U}, \mathcal{F})$.

However, it's not a beautiful definition, since our definition may depend on the choice of open covering \mathfrak{U} . So it's necessary for us to ask what will happen when we change the choice of open covering.

Lemma 3.3.1. *Given $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ an open cover and $\mathfrak{B} = \{V_\beta\}_{\beta \in J}$ a refinement, if ϕ, ψ are two refinement maps $J \rightarrow I$, then there is a homotopy operator between $\phi^\#$ and $\psi^\#$.*

Proof. Define $K : C^q(\mathfrak{U}, \mathcal{F}) \rightarrow C^{q-1}(\mathfrak{B}, \mathcal{F})$ by

$$(K\omega)(V_{\beta_0 \dots \beta_{q-1}}) = \sum (-1)^i \omega(U_{\phi(\beta_0) \dots \phi(\beta_i) \psi(\beta_i) \dots \psi(\beta_{q-1})})$$

And we can check⁶

$$\psi^\# - \phi^\# = \delta K + K \delta$$

as follows: Take a cochain $\omega \in C^q(\mathfrak{U}, \mathcal{F})$, then

□

4. SPECTRAL SEQUENCES AND APPLICATIONS

4.1. The Spectral Sequence of a Filtered Complex.

4.1.1. *exact couple.* A simple way to construct spectral sequence is through exact sequence.

Definition 4.1.1 (exact couple). *An exact couple is an exact sequence of abelian groups of the form*

$$\begin{array}{ccc} A & \xrightarrow{i} & A \\ & \swarrow k & \searrow j \\ & B & \end{array}$$

where i, j and k are group homomorphisms.

From an exact couple, we can define a homomorphism $d : B \rightarrow B$ by $d = j \circ k$, then $d^2 = 0$, so the homology group $H(B) = \ker d / \operatorname{im} d$ is well-defined.

Furthermore, from this exact couple, we can define a new exact couple, called derived couple,

⁶An exercise you only check once in your whole life.

$$\begin{array}{ccc}
A' & \xrightarrow{i'} & A' \\
& \nwarrow k' \quad \nearrow j' & \\
& B' &
\end{array}$$

by making the following definitions.

1. $A' = i(A)$ and $B' = H(B)$;
2. i' is induced from i , that is $i'(ia) = i(ia)$;
3. For $a' = ia$ for some $a \in A$, then $j'a' = [ja]$. To show j' is well defined, we need to check the following things
 - a. ja is a cycle. Indeed, $d(ja) = jkja = 0$;
 - b. The homology class $[ja]$ is independent of the choice of a . Indeed, if $a' = i\bar{a}$ for some other $\bar{a} \in A$. Then $a - \bar{a} = kb$ for some $b \in B$, since $a - \bar{a} \in \ker i = \text{im } k$. Thus

$$ja - j\bar{a} = jkb = db$$

that is $[ja] = [j\bar{a}]$.

4. k' is induced from k . Let $[b] \in H(B)$, then $db = jkb = 0$ implies $kb \in \ker j = \text{im } i$, so there exists $a \in A$ such that $kb = ia$. Define

$$k'[b] := kb \in i(A) = A'$$

Note that we also need to check k' is well-defined: take another $b' \in [b]$, that is $b' - b = db''$ for some $b'' \in B$. Then

$$kb' = kb + kdb'' = kb + kjkb'' = kb$$

As we have already defined these homomorphisms i', j' and k' , it suffices to check above diagram is an exact sequence. Let's check step by step:

1. $\text{im } j' = \ker k'$: Take $j'a' \in \text{im } j'$, then $k'j'a' = k'j'(ia) = k'[ja] = kjia = 0$; Convesely, if $[b] \in B'$ such that $k'[b] = kb = 0$, that is $b \in \ker k = \text{im } j$. So there exists $a \in A$ such that $b = ja$, so $[b] = [ja] = ja'$, where $a' = ia$.
2. $\text{im } k' = \ker i'$: Take $k'[b] = kb \in \text{im } k'$, then $i'kb = ikb = 0$; Convesely, if $ia \in A'$ such that $i'ia = iia = 0$, so there exists $b \in B$ such that $ia = kb$. Furthermore, such b must be a cycle, since $jkb = jia = 0$. So $ia = kb = k'[b]$.
3. $\text{im } i' = \ker j'$: Take $iia \in \text{im } i'$, then $j'(iia) = [jia] = 0$; Convesely, if $ia \in A'$ such that $j'ia = [ja] = [0]$, that is there exists $b \in B$ such that $db = jkb = ja$, that is $a - kb \in \ker j = \text{im } i$. So there exists $a' \in A$ such that $a - kb = ia'$. So $a - ia' \in \text{im } k = \ker i$, that is $ia = iia'$. This completes the proof.

4.1.2. *The Spectral Sequence of a Filtered Complex.* Let K be a differential complex with a differential operator D . Usually K comes with a grading $K = \bigoplus_{k \in \mathbb{Z}} C^k$ and $D : C^k \rightarrow C^{k+1}$. A subcomplex K' of K is a subgroup such that $DK' \subset K'$. A sequence of subcomplexes

$$K = K_0 \supset K_1 \supset K_2 \supset K_3 \supset \dots$$

is called a filtration on K . This makes K into a filtered complex, with associated graded complex

$$GK = \bigoplus_{p=0}^{\infty} K_p / K_{p+1}$$

Notation 4.1.2. We usually extend the filtration to negative indices by defining $K_p = K$ for $p < 0$.

Now let K be a filtered complex K , and let A be the group

$$A = \bigoplus_{p \in \mathbb{Z}} K_p$$

A is again a differential complex with operator D . Define $i : A \rightarrow A$ to be the inclusion $K_{p+1} \hookrightarrow K_p$ and define B to be the quotient

$$0 \rightarrow A \xrightarrow{i} A \xrightarrow{j} B \rightarrow 0$$

Then it's clear that $B = GK$, the associated graded complex of K . Consider the long exact sequence induced by this short exact sequence

$$\dots \rightarrow H^k(A) \xrightarrow{i_1} H^k(A) \xrightarrow{j_1} H^k(B) \xrightarrow{k_1} H^{k+1}(A) \rightarrow \dots$$

In other words, we can write it as an exact couple as follows

$$\begin{array}{ccc} A_1 & \xrightarrow{i} & A_1 \\ & \swarrow k_1 & \searrow j_1 \\ & B_1 & \end{array}$$

where $A_1 = H(A)$, $B_1 = H(B)$ and $i = i_1$. We suppress the subscript of i_1 to avoid cumbersome notation later. This exact couple gives rise to a sequence of exact couples:

$$\begin{array}{ccc} A_r & \xrightarrow{i} & A_r \\ & \swarrow k_r & \searrow j_r \\ & B_r & \end{array}$$

Example 4.1.3. Let's see a simple example: Consider the filtered complex terminates after K_3 , that is

$$\dots = K_{-1} = K_0 \supset K_1 \supset K_2 \supset K_3 \supset 0$$

Then by definition, A_1 is the direct sum of all terms in the following sequence

$$\dots \xleftarrow{\cong} H(K) \xleftarrow{\cong} H(K) \xleftarrow{i} H(K_1) \xleftarrow{i} H(K_2) \xleftarrow{i} H(K_3) \leftarrow 0$$

And by definition of A_2 , it equals iA_1 , so it's the direct sum of all terms in the following sequence

$$\dots \xleftarrow{\cong} H(K) \xleftarrow{\cong} H(K) \supset iH(K_1) \xleftarrow{i} iH(K_2) \xleftarrow{i} iH(K_3) \leftarrow 0$$

Note that $iH(K_1) \subset H(K)$, and $i : H(K) \rightarrow H(K)$ is identity map, thus $iiH(K_1) = iH(K_1)$. So A_3 is the direct sum of all terms in the following sequence

$$\cdots \xleftarrow{\cong} H(K) \xleftarrow{\cong} H(K) \supset iH(K_1) \supset iiH(K_2) \xleftarrow{i} iiH(K_3) \leftarrow 0$$

Similarly we have A_4 is the sum of

$$\cdots \xleftarrow{\cong} H(K) \xleftarrow{\cong} H(K) \supset iH(K_1) \supset iiH(K_2) \supset iiiH(K_3) \supset 0$$

Since all terms appearing in A_4 is in $H(K)$, then i is identity on A_4 . So A 's are stationary after A_4 and we define

$$A_4 = A_5 = \cdots = A_\infty$$

Furthermore, since $\ker\{i : A_4 \rightarrow A_5\} = \text{im } k_4$, thus $k_4 = 0$. Therefore after the fourth stage all the differential of the exact couple are zero, since $d = jk$. So B 's are also stationary, that is

$$B_4 = B_5 = \cdots = B_\infty$$

In the exact couple

$$\begin{array}{ccc} A_\infty & \xrightarrow{i_\infty} & A_\infty \\ & \nwarrow k_\infty=0 & \nearrow j_r \\ & B_\infty & \end{array}$$

A_∞ is the direct sum of groups

$$\cdots \xleftarrow{\cong} H(K) \xleftarrow{\cong} H(K) \supset iH(K_1) \supset iiH(K_2) \supset iiiH(K_3) \supset 0$$

So if we let above sequence be a filtration of $H(K)$, then B_∞ is the associated graded complex of the filtered complex $H(K)$.

Now let's come back to general case. The sequence of subcomplexes

$$\cdots = K = K \supset K_1 \supset K_2 \supset K_3 \supset \cdots$$

induces a sequence in cohomology

$$\cdots \xleftarrow{\cong} H(K) \xleftarrow{\cong} H(K) \xleftarrow{i} H(K_1) \xleftarrow{i} H(K_2) \xleftarrow{i} H(K_3) \leftarrow \cdots$$

Note that i are of course no longer inclusions. Let F_p be the image of $H(K_p)$ in $H(K)$. For example, $F_3 = iiiH(K_3)$. There exists a sequence of inclusions

$$H(K) = F_0 \supset F_1 \supset F_2 \supset F_3 \supset \cdots$$

making $H(K)$ into a filtered complex. This filtration is called the induced filtration on $H(K)$.

Definition 4.1.4 (length of filtration). *A filtration K_p on the filtered complex K is said to have length l if $K_l \neq 0$ and $K_p = 0$ for $p > l$.*

So as we can see from simple example we have computed, if the filtration of K has finite length, then A_r and B_r are stationary and the stationary value B_∞ is the associated graded complex $\bigoplus F_p/F_{p+1}$ of the filtered complex $H(K)$.

It's customary to write E_r for B_r , and there is a differential \mathfrak{d}_r on E_r such that $H_{\mathfrak{d}_r}(E_r) = E_{r+1}$, and that's definition of a spectral sequence.

Definition 4.1.5 (spectral sequence). *A sequence of differential complex $\{E_r, \mathfrak{d}_r\}$ in which each E_r is the homology of its predecessor E_r is called a spectral sequence.*

Definition 4.1.6 (convergence of spectral sequence). *A spectral sequence $\{E_r, \mathfrak{d}_r\}$ is said to converge to some filtered group H , if E_∞ is equal to the associated graded group of H .*

Let's summarize what we have done: For a differential complex K and a filtration $\{K_p\}$ of K , if the filtration is finite length, then the spectral sequence we obtained from this filtration will converge to $H(K)$, which may be way to compute $H(K)$, the homology of a given differential complex.

However, it's quit strong requirement for a filtration to be finite length. Suppose filtered complex $K = \bigoplus_n K^n$, then a filtration $\{K_p\}$ on K induces a filtration on K^n for each n , that is $K_p^n := K_p \cap K^n$. And we can prove the same result, only asking $\{K_p^n\}$ to be finite length for each n .

Theorem 4.1.7. *Let $K = \bigoplus_n K^n$ be a graded filtered complex with filtration $\{K_p\}$ and let $H_D^*(K)$ be the cohomology of K with filtration given by $\{K_p\}$. Suppose for each n we have $\{K_p^n\}$ is finite length. Then the short exact sequence of complex*

$$0 \rightarrow \bigoplus K_{p+1} \rightarrow \bigoplus K_p \rightarrow \bigoplus K_p/K_{p+1} \rightarrow 0$$

induces a spectral sequence which converges to $H_D^(K)$.*

Proof. The ideal here is that since it's a convergence between two graded groups, so it suffices to treat the convergence question one dimension at a time, then it's reduced to the ungraded situation.

Since we have the following sequence

$$\dots \xleftarrow{\cong} H(K) \xleftarrow{\cong} H(K) \xleftarrow{i} H(K_1) \xleftarrow{i} H(K_2) \xleftarrow{i} H(K_3) \leftarrow \dots$$

and it induces a filtration $\{F_p\}$ of $H(K)$, where $i^r H(K_p) = F_p$ for all $r \geq p$.

Now fix a number n and consider n -th grade, we have the following sequence

$$\dots \xleftarrow{\cong} H^n(K) \xleftarrow{\cong} H^n(K) \xleftarrow{i} H^n(K_1) \xleftarrow{i} H^n(K_2) \xleftarrow{i} H^n(K_3) \leftarrow \dots$$

Let $\ell(n)$ be the length of $\{K_p^n\}_{p \in \mathbb{Z}}$. Then it's clear that $H^n(K_p) = 0$ for $p \geq \ell(n) + 1$. Thus for all p , if we let $r \geq \ell(n) + 1$, then

$$i^r H^n(K_{p+1}) = F_{p+1}^n$$

and

$$i : i^r H^n(K_{p+1}) \rightarrow i^r H^n(K_p)$$

is an inclusion. It follows that

$$i_r : A_r^n \rightarrow A_r^n$$

is an inclusion and $k_r : B_r^{n-1} \rightarrow A_r^n$ is the zero map. So we have $A_k = A_r$ and $B_k = B_r$ for all $k = r$. More explicitly $A_\infty^n = A_r^n = \bigoplus F_p^n$ and $B_\infty^n = B_r^n = \bigoplus F_p^n / F_{p+1}^n$. Thus

$$B_\infty = \bigoplus_{n,p} F_p^n / F_{p+1}^n = \bigoplus F_p / F_{p+1}$$

that is associated graded complex of $H_D^*(K)$, as desired. \square

4.1.3. The Spectral Sequence of a Double Complex. Now for a double complex $K = \bigoplus K^{p,q}$, we can make it into a complex, called total complex by

$$K = \bigoplus C^k, \quad C^k = \bigoplus_{p+q=k} K^{p,q}$$

Furthermore, there is a natural filtration on K as follows

$$K_p = \bigoplus_{i \geq p, q \geq 0} K^{i,q}$$

The direct sum $A = \bigoplus K_p$ is also a double complex, and we can also make it into a single complex $A = \bigoplus A^k$ by summing the bidegrees. There is an inclusion $i : A^k \rightarrow A^k$ given by

$$i : A^k \cap K_{p+1} \rightarrow A^k \cap K_p$$

The single complex A inherits the differential operator $D = \delta + (-1)^p d$ from K . Similarly for $B = \bigoplus K_p / K_{p+1}$, but differential operator on B is just $(-1)^p d$, since any element in K_p is mapped into K_{p+1} by δ . Therefore

$$E_1 = H_D(B) = H_d(K)$$

From above section, we obtain a spectral sequence which converges $H_D(K)$, since our filtration is finite on each degree n . However, we want to show a more refinement theorem, since in this case our complex comes from a double complex, which has a more subtle structure. In order to do this, we need to compute the explicit formula of d_r .

Now let's compute $k_1 : H(B) \rightarrow H(A)$. Note that

$$B^k = \bigoplus_p B^k \cap (K_p / K_{p+1})$$

So if we want to see $k_1 : H^k(B) \rightarrow H^{k+1}(A)$, it suffices to see

$$H^k(B) \cap (K_p / K_{p+1}) \rightarrow H^{k+1}(A) \cap K_{p+1}$$

Choose $b \in A^k \cap K_p$ represent a cocycle $[b] \in B^k \cap K_p / K_{p+1}$. To get $k_1([b])$, we compute Db and take its inverse under i . Since the differential operator

on B is δ , then $\delta b = 0$. So $Db = \delta b + (-1)^p db = \delta b \in A^{k+1} \cap K_p$. Then $k_1[b] = [\delta b]$. Then differential $d_1 = j_1 k_1$ on E_1 is given by δ . Consequently we have

$$E_2 = H_\delta H_d(K)$$

Now let's compute the differential d_2 on E_2 . For an element of $E_2 = H_\delta H_d(K)$, it can be represented by an element $b \in K$ such that

$$\begin{cases} db = 0 \\ \delta b = -D''c, \quad \text{for some } c \in K \end{cases}$$

where $D'' = (-1)^p d$.

Notation 4.1.8. We will denote the class of b in E_r , if it's well-defined, by $[b]_r$.

So from definition of derived couple, we have

$$d_2[b]_2 = j_2 k_2[b]_2 = j_2 k_1[b]_1$$

To compute $j_2 k_1[b]_1$, by definition we must find $a \in K$ such that $k_1[b]_1 = i[a]_1$. Then $j_2 k_2[b]_2 = j_2 k_1[b]_1 = [j_1 a]_2$. Since $k_1 b \in A^{k+1} \cap K_{p+1}$ and $a \in A^{k+1} \cap K_{p+1}$. To find a we use not b but $b + c$ in $A^k \cap K_p$ to represent $[b]_2$, that's possible since b and $b + c$ have the same image under the projection $K_p \rightarrow K_p/K_{p+1}$, since $c \in K_{p+1}$. Then

$$k_1(b + c) = D(b + c) = \delta b + Dc = \delta b + \delta c + D''c = \delta c$$

So

$$d_2[b]_2 = [\delta c]_2$$

Thus differential d_2 is given by the delta of the tail of the zig-zag which extends b . Now let's check well-defineness:

1. $\delta c \in H_\delta H_d(K)$: $\delta(\delta c) = 0$ is clear; $d\delta c = \delta dc = (-1)^p \delta \delta b = 0$, since $(-1)^p dc = \delta b$.
2. $d_2[b]_2$ is independent of the choice of c : Any two possible c and c' differs something lies in $\ker d$. Assume $c' = c + x$ where $x \in \ker d$, then it suffices to show $[\delta x]_2 = 0$, and that's tautological.

Now let's check when does $d_2[b]_2 = 0$, or in other words, it defines an element in E_3 . For b we have

$$\begin{cases} D''b = 0 \\ \delta b = -D''c_1 \end{cases}$$

And by definition of $d_2[b]_2$ we have $[\delta c_1]_2 = 0$. In other words, $\delta c_1 + d(-1)^p c_2$ for some c_2 , that is

$$\delta c_1 = -D''c_2$$

This completes the proof of Exercise 14.11 of Bott-Tu.

More generally, we say that an element b in K lives to E_r if it represents a cohomology class in E_r , or equivalently, b is a cocycle in E_1, E_2, \dots, E_{r-1} . And we already see there is a zig-zag description for d_1 and d_2 .

In fact, differential d_r on E_r is given by δ of the tail of zig-zag:

$$d_r[b]_r = [\delta c_{r-1}]_r$$

Thus the bidegrees (p, q) of the double complex persist in the spectral sequence

$$E_r = \bigoplus_{p,q} E_r^{p,q}$$

and d_r shifts the bidegrees by $(r, -r + 1)$.

In a summary, we have proven the following refinement:

Theorem 4.1.9. *Given a double complex $K = \bigoplus K^{p,q}$ there is a spectral sequence $\{E_r, d_r\}$ converging to the total cohomology $H_D(K)$ such that E_r has a bigrading with*

$$d_r : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$$

and

$$\begin{aligned} E_1^{p,q} &= H_d^{p,q}(K) \\ E_2^{p,q} &= H_\delta^{p,q} H_d(K) \end{aligned}$$

Furthermore, the associated graded complex of the total cohomology is given by

$$GH_D^n(K) = \bigoplus_{p+q=n} E_\infty^{p,q}(K)$$

Remark 4.1.10. There is another filtration, that is $K_q = \bigoplus_{j \geq q, p \geq 0} K^{p,j}$. This gives a second spectral sequence $\{E'_r, d'_r\}$ converging to the total cohomology $H_D(K)$, but with

$$\begin{aligned} E'_1 &= H_\delta(K) \\ E'_2 &= H_d H_\delta(K) \end{aligned}$$

and

$$d'_r : E'_r^{p,q} \rightarrow E'_r^{p-r+1, q+r}$$

Example 4.1.11 (The extension problem). Since the dimension is the only invariant of a vector space, the associated graded vector space GV of a filtered vector-space V is isomorphic to V itself. In particular, if a double complex K is a vector space, then

$$H_D^n(K) \cong GH_D^n(K) \cong \bigoplus_{p+q=n} E_\infty^{p,q}$$

However, the same thing fails in the realm of abelian groups. For example: the two group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and \mathbb{Z}_4 filtered by

$$\mathbb{Z}_2 \subset \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

and

$$\mathbb{Z}_2 \subset \mathbb{Z}_4$$

have isomorphic associated graded groups, but $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is not isomorphic to \mathbb{Z}_4 . In other words, in a short exact sequence of abelian groups

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

A and C do not determine B uniquely. The ambiguity is called the extension problem.

Proposition 4.1.12. *In a short exact sequence of abelian groups*

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

if C is free, then there exists a homomorphism $s : C \rightarrow B$ such that $g \circ s$ is identity on C .

Proof. Since C is free, then it suffices to define a suitable s on the generators $\{c_i\}$ of C and it automatically extends to C linearly. Take c_i and choose any preimage of c_i , denoted by b_i , then s is defined by $c_i \mapsto b_i$. Clearly $s \circ g$ is identity on C , but note that such s is not unique. \square

Corollary 4.1.13. *Under the hypothesis of the proposition,*

1. *The map $(f, s) : A \oplus C \rightarrow B$ is an isomorphism;*
2. *For any abelian group G the induced sequence*

$$0 \rightarrow \text{Hom}(C, G) \rightarrow \text{Hom}(B, G) \rightarrow \text{Hom}(A, G) \rightarrow 0$$

is exact;

3. *For any abelian group G the sequence*

$$0 \rightarrow A \otimes G \rightarrow B \otimes G \rightarrow C \otimes G \rightarrow 0$$

is exact.

Proof. For (1). Since (f, s) is a group homomorphism, it suffices to check it's both injective and surjective. It's easy to see (f, s) is injective, since f and s are injective; For $b \in B$, if $b \in \text{im } f$, that is $b = f(a)$ for some $a \in A$, then $(a, 0)$ is mapped to b . If $b \notin \text{im } f = \ker g$, then consider $g(b) \in C$. Although $sg(b)$ may not equal to b , we have $sg(b) - b \in \ker g = \text{im } f$, so there exists $a \in A$ such that $f(a) + sg(b) = b$, this completes the proof of surjectivity.

For (2). Since it's known to all $\text{Hom}(-, G)$ is a left exact functor, then it suffices to show $\text{Hom}(B, G) \rightarrow \text{Hom}(A, G)$ is surjective. Take any $k : A \rightarrow G$, then consider the composition of following maps

$$B \xrightarrow{(f, g)^{-1}} A \oplus C \xrightarrow{p_1} A \xrightarrow{k} G$$

it's a map in $\text{Hom}(B, G)$ such that it extends k .

For (3). Since it's known to all $- \otimes G$ is a right exact functor, then it suffices to show $A \otimes G \rightarrow B \otimes G$ is injective, and the proof is quite similar as above. \square

Remark 4.1.14. If you are quite familiar with homological algebra, you will know that the failure of $\text{Hom}(B, G) \rightarrow \text{Hom}(A, G)$ to be exact is measured by $\text{Ext}(C, G)$, and it's zero by the property of Ext , since C is a free abelian group. Similarly, the failure of $A \otimes G \rightarrow B \otimes G$ to be injective is measured by $\text{Tor}(C, G)$, and it's zero for the same reason.

Exercise 4.1.15. Show that if

$$0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow \dots$$

is an exact sequence of free abelian groups and if G is any abelian group, then the two exact sequences

$$0 \leftarrow \text{Hom}(A_1, G) \leftarrow \text{Hom}(A_2, G) \leftarrow \text{Hom}(A_3, G) \leftarrow \dots$$

and

$$0 \rightarrow A_1 \otimes G \rightarrow A_2 \otimes G \rightarrow A_3 \otimes G \rightarrow \dots$$

are both exact.

4.1.4. *The Spectral Sequence of a Fiber Bundle.* Let $\pi : E \rightarrow M$ be a fiber bundle with fiber F over a manifold M . Use spectral sequence we can give a general method of computing the cohomology of E from that of F and M . Given a good cover \mathfrak{U} of M , $\pi^{-1}\mathfrak{U}$ is a cover on E and we can form the double complex

$$K^{p,q} = C^p(\pi^{-1}\mathfrak{U}, \Omega^q) = \prod_{\alpha_0 < \dots < \alpha_p} \Omega^q(\pi^{-1}U_{\alpha_0 \dots \alpha_p})$$

whose E_1 term is

$$E_1^{p,q} = H_d^{p,q} = \prod_{\alpha_0 < \dots < \alpha_p} H^q(\pi^{-1}U_{\alpha_0 \dots \alpha_p}) = C^p(\mathfrak{U}, \mathcal{H}^q)$$

where \mathcal{H}^q is the presheaf $U \mapsto H^q(\pi^{-1}U)$ on M .

4.1.5. *Some Applications.*

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