A BRIEF INTRODUCTION TO HODGE THEORY

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ABSTRACT. In this talk we give a brief introduction to Hodge theory as preliminaries for [Fil16], such as (polarized) Hodge structures, variation of Hodge structures and differential geometry of Hodge bundles.

CONTENTS

1. Hodge structures

1.1. **Hodge structures.**

Let (X,ω) a compact Kähler manifold. The classical Hodge theory says that there is a decomposition on the k-th cohomology as follows

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where $H^{p,q}(X)$ is the Dolbeault cohomology. The Hodge structure generalized this structure.

1.1.1. *Objects.*

Definition 1.1.1. An **(effective) $\mathbb{Z}$-Hodge structure** of weight k consists of the following data:

1. a finitely generated abelian group $V_{\mathbb{Z}}$;
2. a decomposition
$$V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}$$
such that $\overline{V^{p,q}} = V^{q,p}$;
3. $V^{p,q} = 0$ unless $p,q \geq 0$.

Definition 1.1.2. The **Deligne torus** $\mathbb{S}$ is an algebraic group over $\mathbb{R}$, defined by

$$\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} G_{m,\mathbb{C}}.$$

By definition, its real points are naturally isomorphic to $\mathbb{C}^*$ and its complex points are isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$.

Proposition 1.1.1. A Hodge structure on $V_{\mathbb{Z}}$ is the same as an algebraic representation of the Deligne torus $\mathbb{S}$ on $V_{\mathbb{Z}}$.

Definition 1.1.3. Let $(V_{\mathbb{Z}}, V^{p,q})$ be a $\mathbb{Z}$-Hodge structure of weight k. The **Hodge filtration** F^p is defined by

$$F^p = \bigoplus_{p' \geq p} V^{p',q}.$$

It's a decreasing filtration which satisfies

$$V_{\mathbb{C}} = F^p \oplus \overline{F^{k-p+1}}. \tag{1.1}$$

Remark 1.1.1. Let $V_{\mathbb{Z}}$ be a finitely generated abelian group and F^p be a filtration satisfies (1.1). Then it determines a Hodge structure by

$$V^{p,q} = F^p \cap \overline{F^q}.$$

In other words, a Hodge structure of weight k is equivalent to a filtration F^p satisfying (1.1).

Example 1.1.1. *Let V,W be two Hodge structures of weight k and l respectively. Then*

(1) *V^* is a Hodge structure of weight $-k$;*

(2) *$V \otimes W$ is a Hodge structure of weight $k+l$;*
(3) *$\mathrm{Hom}(V,W)$ is a Hodge structure of weight $-k+l$;*
(4) *$V^{\otimes n}, \mathrm{Sym}^n V$ and $\bigwedge^n V$ are Hodge structures of weight nk.*

1.1.2. *Morphisms.*

Definition 1.1.4. Let $(V_{\mathbb{Z}}, V^{p,q}), (W_{\mathbb{Z}}, W^{p,q})$ be two Hodge structures of weight k and $k+2r$ and $\phi\colon V_{\mathbb{Z}} \to W_{\mathbb{Z}}$ be a morphism of abelian groups. Then ϕ is called a **morphism of Hodge structure of type** (r,r), if its $\mathbb{C}$-linear extension $\phi_{\mathbb{C}}$ satisfies

$$\phi_{\mathbb{C}}(V^{p,q}) \subseteq W^{p+r,q+r}.$$

Proposition 1.1.2. Let ϕ be a morphism between Hodge structures. Then $\ker\phi, \mathrm{im}\,\phi$ and $\mathrm{coker}\,\phi$ are Hodge structures.

Example 1.1.2. *Let $f: X \to Y$ be a holomorphic map between compact Kähler manifolds. Then $f^*: H^k(Y,\mathbb{Z}) \to H^k(X,\mathbb{Z})$ is a morphism of Hodge structure of type $(0,0)$.*

Example 1.1.3. *Let X, Y be two compact Kähler manifolds such that $\dim X = n, \dim Y = m$ and $m = n+r$. Then*

$$\begin{tikzcd} H^k(X,\mathbb{Z}) \arrow[r] \arrow[d] & H^{k+2r}(Y,\mathbb{Z}) \\ H_{2n-k}(X,\mathbb{Z}) \arrow[r, "f_*"] & H_{2n-k}(Y,\mathbb{Z}). \arrow[u] \end{tikzcd}$$

*This gives a morphism of Hodge structure of type (r,r), which is called **Gysin push-forward**.*

1.2. **Polarization.** Let (X,ω) be a complex Kähler n-manifold. There is an intersection form Q on $H^k(X,\mathbb{R})$ given by

$$Q(\alpha,\beta) = (-1)^{\frac{k(k-1)}{2}} \int_X \omega^{n-k} \wedge \alpha \wedge \beta.$$

The induced Hermitian form $H(\alpha,\beta) := Q(\alpha,\overline{\beta})$ on $H^k(X,\mathbb{C})$ satisfies the following properties:

(1) The Hodge decomposition is orthogonal with respect to H.
(2) $(\sqrt{-1})^{p-q} H(\alpha,\overline{\alpha}) > 0$ for $0 \neq \alpha \in H^{p,q}(X)$.

This gives a polarization.

Definition 1.2.1. Let $(V_{\mathbb{Z}}, V^{p,q})$ be a $\mathbb{Z}$-Hodge structure. The **Weil operator** $\mathtt{C}$ associated to $V_{\mathbb{Z}}$ acts by $(\sqrt{-1})^{p-q}$ on $V^{p,q}$.

Definition 1.2.2. A **polarized $\mathbb{Z}$-Hodge structure** of weight k is $\mathbb{Z}$-Hodge structure $(V_{\mathbb{Z}}, V^{p,q})$ of weight k together with a morphism of Hodge structure $Q: V_{\mathbb{Z}} \otimes V_{\mathbb{Z}} \to \mathbb{Z}$ of type $(-k,-k)$ such that

$$H: V_{\mathbb{C}} \otimes V_{\mathbb{C}} \to \mathbb{C}$$
$$(\alpha,\beta) \mapsto Q(\mathtt{C}\alpha,\overline{\beta})$$

is a positive definite Hermitian form.

2. Variation of Hodge structures

2.1. **Local system and flat connection.**

In this section we always assume X is a complex manifold.

Definition 2.1.1. A sheaf $\mathcal{V}$ on X is called a locally constant sheaf of rank r valued in $\mathbb{C}$, if for each point $x \in X$, there is an open subset U containing x such that $\mathcal{V}|_U$ is constant sheaf $\underline{\mathbb{C}^r}$.

Remark 2.1.1. In other words, there exists an open covering $\{U_\alpha\}$ such that $\mathcal{V}|_{U_\alpha}$ is isomorphic to constant sheaf $\underline{\mathbb{C}^r}$. Then the local system $\mathcal{V}$ is completely determined by the transition functions $g_{\alpha\beta}\colon U_{\alpha\beta} \to \mathrm{GL}_n(\mathbb{C})$, which are locally constant functions.

Definition 2.1.2. Let $\mathscr{E}$ be a locally free sheaf on X. A **connection** is a $\mathbb{C}$-linear map

$$\nabla\colon \mathscr{E} \to \mathscr{A}^1_X \otimes \mathscr{E}$$

satisfying the following condition

$$\nabla(\varphi \otimes e) = \mathrm{d}\varphi \otimes e + \varphi \nabla e$$

for all sections e of $\mathscr{E}$ and φ of $\mathscr{O}_X$.

Remark 2.1.2. The definition of ∇ extends to $\nabla\colon \mathscr{A}^p_X \otimes \mathscr{E} \to \mathscr{A}^{p+1}_X \otimes \mathscr{E}$ by defining

$$\nabla(\omega \otimes e) = \mathrm{d}\omega \otimes e + (-1)^p \omega \wedge \nabla e$$

for all sections ω of $\mathscr{A}^p_X$ and sections e of $\mathscr{E}$.

Remark 2.1.3. Let $\{e_\alpha\}$ be a local frame of $\mathscr{E}$. For any section $s = s^\alpha e_\alpha$ of $\mathscr{E}$, one has

$$\nabla(s^\alpha e_\alpha) = \mathrm{d}s^\alpha e_\alpha + s^\alpha \nabla e_\alpha.$$

Thus the connection ∇ is completely determined by

$$\nabla e_\alpha = \omega^\beta_\alpha e_\beta,$$

where ω^β_α are 1-forms, which forms a (smooth) 1-form valued matrix ω.

Definition 2.1.3. A connection ∇ is **integrable** if its curvature $\nabla^2\colon \mathscr{E} \to \mathscr{A}^2_X \otimes \mathscr{E}$ vanishes.

Remark 2.1.4. Let $\{e_\alpha\}$ be a local frame of $\mathscr{E}$. For any section $s = s^\alpha e_\alpha$ of $\mathscr{E}$, one has

$$\begin{aligned}\nabla^2(s^\alpha e_\alpha) &= \nabla(\mathrm{d}s^\alpha \otimes e_\alpha + s^\alpha \omega_\alpha^\beta \otimes e_\beta)\\ &= -\mathrm{d}s^\alpha \wedge \omega_\alpha^\beta \otimes e_\beta + \mathrm{d}(s^\alpha \omega_\alpha^\beta) \otimes e_\beta - s^\alpha \omega_\alpha^\beta \wedge \omega_\beta^\gamma \otimes e_\gamma\\ &= s^\alpha(\mathrm{d}\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta) \otimes e_\beta,\\ \nabla^2(e_\alpha) &= \nabla(\omega_\alpha^\beta \otimes e_\beta)\\ &= \mathrm{d}\omega_\alpha^\beta \otimes e_\beta - \omega_\alpha^\beta \wedge \nabla e_\beta\\ &= \mathrm{d}\omega_\alpha^\beta \otimes e_\beta - \omega_\alpha^\beta \wedge \omega_\beta^\gamma \otimes e_\gamma\\ &= (\mathrm{d}\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta) \otimes e_\beta.\end{aligned}$$

This shows ∇^2 is a global section of $\mathscr{A}_X^2 \otimes \underline{\mathrm{End}}_{\mathscr{O}_X}(\mathscr{E})$, which is locally given by $\mathrm{d}\omega - \omega \wedge \omega$.

Definition 2.1.4. A locally free sheaf together with an integrable connection is called a **flat bundle**.

Proposition 2.1.1. Let ∇ be a integrable connection on locally free sheaf $\mathscr{E}$ on X. Then the horizontal section $\mathscr{E}^{\nabla=0}$ is a local system.

Proposition 2.1.2. Let $\mathscr{L}$ be a local system on X. Then the locally free sheaf $\mathscr{E} := \mathscr{O}_X \otimes \mathscr{L}$ together with canonical connection $\nabla_{\mathrm{can}}(f \otimes s) := \mathrm{d}f \otimes s$ is a flat bundle.

Theorem 2.1.1. The functor $(\mathscr{E}, \nabla) \mapsto \mathscr{E}^{\nabla=0}$ is an equivalence between category of flat bundles and the category of the complex local system with quasi-inverse $\mathscr{L} \mapsto (\mathscr{O}_X \otimes \mathscr{L}, \nabla_{\mathrm{can}})$.

Proposition 2.1.3. Let $\mathscr{L}$ be a local system on X. Then

$$H^*(X, \mathscr{L}) \cong \mathbb{H}^*(X, \mathscr{A}_X^\bullet \otimes \mathscr{L}).$$

Proof. Note the following complex of sheaves

$$0 \to \mathscr{L} \to \mathscr{A}_X^\bullet \otimes \mathscr{L}$$

gives a resolution of $\mathscr{L}$ by coherent sheaves. □

2.2. **Abstract variation of Hodge structures.** In this section we always assume S is a complex manifold.

Definition 2.2.1. A **variation of Hodge structure of weight** k on S consists of the following data:

(1) a local system $\mathbb{V}_\mathbb{Z}$ of finitely generated abelian groups on S;
(2) a finite decreasing filtration $\{\mathscr{F}^p\}$ of the holomorphic vector bundle $\mathscr{V} := \mathbb{V}_\mathbb{Z} \otimes \mathscr{O}_X$ by holomorphic subbundles (**the Hodge filtration**).

These data should satisfy the following conditions:

(a) for each $s \in S$ the filtration $\{\mathscr{F}^p(s)\}$ of $\mathbb{V}_s \simeq \mathbb{V}_{\mathbb{Z},s} \otimes_\mathbb{Z} \mathbb{C}$ defines a Hodge structure of weight k on the finitely generated abelian group $\mathbb{V}_{\mathbb{Z},s}$;

(b) the Gauss-Manin connection $\nabla^{GM}: \mathcal{V} \to \Omega^1_S \otimes \mathcal{V}$ whose sheaf of horizontal sections is $\mathbb{V}_{\mathbb{C}}$ satisfies the **Griffiths' transversality condition**

$$\nabla^{GM}(\mathscr{F}^p) \subseteq \Omega^1_S \otimes \mathscr{F}^{p-1}.$$

The notion of **a morphism of variations of Hodge structure** is defined in the obvious way.

Example 2.2.1. *Given two variations $\mathbb{V}, \mathbb{V}'$ of Hodge structure over S of weights k and k', there is an obvious structure of variation of Hodge structure on the underlying local systems of $\mathbb{V} \otimes \mathbb{V}'$ and $\mathrm{Hom}(\mathbb{V}, \mathbb{V}')$ of weights $k + k'$ and $k - k'$ respectively.*

Definition 2.2.2. A **polarized variation of Hodge structure** of weight k is a variation of Hodge structures $\mathbb{V}$ of weight k together with a bilinear pairing $I(\text{-},\text{-})$ satisfying

(1) The pairing is flat, that is, preserved by the Gauss-Manin connection ∇.
(2) On each fiber of $\mathcal{V}$, the pairing induces a polarization of the Hodge structure.

Remark 2.2.1. The intersection pairing $I(\text{-},\text{-})$ on a polarization variation of Hodge structure is flat for the Gauss-Manin connection, but the Hodge metric $Q(\text{-},\text{-})$ may not. Since the Hodge metric Q is expressed in terms of I and the Weil operator $\mathtt{C}$, the compatiblity with the Weil operator implies the compatiblity with the Hodge metric.

2.3. **Variation of Hodge structures coming from smooth families.** In this section we will explain the motivation of variation of Hodge structures and the Griffiths transversality condition is inspired by the geometric case naturally.

Let $f: X \to S$ be a family[1] of compact Kähler manifolds. Then $R^k f_* \underline{\mathbb{C}}$ is a local system on S such that for each point $s \in S$, one has $(R^k f_* \underline{\mathbb{C}})_s \cong H^k(X_s, \mathbb{C})$. The flat bundle corresponding to the local system $R^k f_* \underline{\mathbb{C}}$ is the relative de Rham cohomology

$$H^k_{dR}(X/S) := \mathscr{O}_S \otimes R^k f_* \underline{\mathbb{C}}$$

together with the Gauss-Manin connection ∇^{GM}.

Proposition 2.3.1 ([Del70])**.** Let $f: X \to S$ be a family of complex manifolds and $\mathbb{V}$ be a local system of complex vector spaces on X. There is a natural isomorphism

$$\mathscr{O}_S \otimes R^k f_* \mathbb{V} \cong R^k f_*(\Omega^\bullet_{X/S} \otimes \mathbb{V}),$$

where $\Omega^\bullet_{X/S} = \Omega^\bullet_X / f^* \Omega^\bullet_S$ is the relative de Rham complex.

Corollary 2.3.1. Let $f: X \to S$ be a family of complex manifolds. Then

$$H^k_{dR}(X/S) \cong R^k f_* \Omega^\bullet_{X/S}.$$

[1]In other words, f is a proper holomorphic submersion between complex manifolds such that every fiber of f a compact Kähler manifold.

Remark 2.3.1. In this viewpoint, the Hodge filtration on $R^k f_*\Omega^\bullet_{X/S}$ is described as follows
$$\mathscr{F}^p := \operatorname{Im}\left\{R^k f_*\sigma^{\geq p}\Omega^\bullet_{X/S} \to R^k f_*\Omega^\bullet_{X/S}\right\},$$
where $\sigma^{\geq p}$ is the stupid filtration.

Proposition 2.3.2. Let $f : X \to S$ be a family of compact Kähler manifolds. The Gauss-Manin connection ∇^{GM} satisfies the Griffiths transversality, that is,
$$\nabla^{GM}(\mathscr{F}^p) \subseteq \Omega^1_S \otimes \mathscr{F}^{p-1}.$$

Proof. See Corollary 10.31 in [PS08]. □

Corollary 2.3.2. Let $f : X \to S$ be a family of compact Kähler manifolds. Then the local system $R^k f_*\underline{\mathbb{C}}$ underlies a variation of Hodge structures of weight k.

3. DIFFERENTIAL GEOMETRY OF HODGE BUNDLES

3.1. **General setting.** In this section, we consider a speical connection on a holomorphic vector bundle E on a complex manifold X equipped with a Hermitian metric. Recall that we can define connection for any (smooth) vector bundle over X, which is a $\mathbb{C}$-linear map between (smooth) sections satisfying the Leibniz rules.

There is a canonical connection, called Chern connection, on Hermitian vector bundle E, which is uniquely defined by the following two conditions:

(1) It's compatible with the Hermitian metric.
(2) It's compatible with the holomorphic structure.

3.1.1. *Hermitian vector bundle.*

Definition 3.1.1. Let E be a complex vector bundle. A **Hermitian metric** h on E is a smooth section of $E^* \otimes \overline{E}^*$.

Remark 3.1.1. Let $\{e_\alpha\}$ be a local frame of E. Then a (positive definite) Hermitian metric is determined by a (positive definite) Hermitian matrix $(h_{\alpha\overline{\beta}})$, that is

$$h = h_{\alpha\overline{\beta}} e^\alpha \otimes \overline{e}^\beta,$$

where $h_{\alpha\overline{\beta}} = h(e_\alpha, \overline{e}_\beta)$.

Definition 3.1.2. A complex vector bundle E together with a Hermitian metric h is called a **Hermitian vector bundle** (E,h).

Remark 3.1.2. Let L be a Hermitian line bundle. A Hermitian metric h is locally given by $e^{-2\varphi}$, where φ is a smooth function, which is called **metric weight**. Suppose $\{g_{\alpha\beta}\}$ is the transition function of L with respect to open covering $\{U_\alpha\}$. Then h is given by a collection $\{h_\alpha \in C^\infty(U_\alpha)\}$ such that $h_\alpha = |g_{\alpha\beta}|^{-2} h_\beta$. In other words, a Hermitian metric is a collection of metric weights $\{\varphi_\alpha \in C^\infty(U_\alpha)\}$ such that

$$\varphi_\alpha = \varphi_\beta + \log |g_{\alpha\beta}|.$$

Definition 3.1.3. For a Hermitian vector bundle (E,h) over complex manifold X, there is a **sesquilinear map**

$$\begin{aligned} \mathscr{A}^p_X(E) \times \mathscr{A}^q_X(E) &\to \mathscr{A}^{p+q}_X \\ (s,t) &\mapsto \{s,t\}, \end{aligned}$$

which is locally given by

$$\{s^\alpha e_\alpha, t^\beta e_\beta\} = h_{\alpha\overline{\beta}} s^\alpha \wedge \overline{t^\beta}.$$

Definition 3.1.4. A connection ∇ on a Hermitian vector bundle (E,h) is **compatible with the metric**, if

$$\mathrm{d}\langle s,t\rangle = \{\nabla s, t\} + \{s, \nabla t\},$$

where s,t are smooth sections of E.

Remark 3.1.3. If $\{e_\alpha\}$ is a local frame of E, then

$$\begin{aligned}
\mathrm{d}h_{\alpha\overline{\beta}} &= \mathrm{d}\langle e_\alpha, \overline{e}_\beta\rangle \\
&= \{\nabla e_\alpha, \overline{e}_\beta\} + \{e_\alpha, \nabla \overline{e}_\beta\} \\
&= \omega_\alpha^\gamma h_{\gamma\overline{\beta}} + \overline{\omega_\beta^\gamma} h_{\alpha\overline{\gamma}}.
\end{aligned}$$

In the matrix notation, we have

$$\mathrm{d}h = \omega h + h\overline{\omega}^T.$$

3.1.2. *Compatiblity with complex structure.* Let $E \to X$ be a complex vector bundle with connection ∇. Then we can decompose $\nabla = \nabla^{1,0} + \nabla^{0,1}$ by composing the projection as follows

$$\begin{tikzcd}
 & & \mathscr{A}_X^{1,0}(E) \\
\mathscr{A}_X^0(E) \arrow[r, "\nabla"] & \mathscr{A}_X^1(E) \arrow[ur] \arrow[dr] & \\
 & & \mathscr{A}_X^{0,1}(E)
\end{tikzcd}$$

For convenience, we use $\nabla^{0,1}$ to denote the composition $\mathscr{A}_X^0(E) \xrightarrow{\nabla} \mathscr{A}_X^1(E) \to \mathscr{A}_X^{0,1}(E)$. On the other hand, there is a natural operator $\overline{\partial}_E : \mathscr{A}_X^0(E) \to \mathscr{A}_X^{0,1}(E)$, which is locally defined by

$$\overline{\partial}_E(s^\alpha \otimes e_\alpha) = \overline{\partial} s^\alpha \otimes e_\alpha.$$

Definition 3.1.5. A connection ∇ on a holomorphic vector bundle E over a complex manifold X is said to be **compatible with holomorphic structure** if $\nabla^{0,1} = \overline{\partial}_E$.

Remark 3.1.4. Let $\{e_\alpha\}$ be a holomorphic local form of E and denote

$$\nabla e_\alpha = (\Gamma_{i\alpha}^\beta \mathrm{d}z^i + \Gamma_{\overline{i}\alpha}^\beta \mathrm{d}\overline{z}^i) \otimes e_\beta.$$

Then

$$0 = \nabla^{0,1} e_\alpha = \Gamma_{\overline{i}\alpha}^\beta \mathrm{d}\overline{z}^i \otimes e_\beta.$$

This shows ∇ is compatible with holomorphic structure if and only if $\Gamma_{\overline{i}\alpha}^\beta = 0$.

Remark 3.1.5. Let E be a holomorphic vector bundle and ∇ be a connection which is compatible with the holomorphic structure. Then

$$\nabla : \mathscr{E} \to \Omega_X^1 \otimes \mathscr{E},$$

where $\mathscr{E}$ is the locally free sheaf given by the holomorphic section of E and Ω_X^1 is the locally free sheaf of holomorphic 1-forms.

3.1.3. *Chern connection.*

Theorem 3.1.1. Let X be a complex manifold and (E,h) a Hermitian holomorphic vector bundle. Then there exists a unique connection called **Chern connection** such that it's compatible with holomorphic structure and metric.

Proof. If metric connection ∇ is compatible with holomorphic structure, then the following three equations are equivalent

$$\begin{aligned} \mathrm{d}h &= \omega h + h\overline{\omega}^t \\ \partial h &= \omega h \\ \overline{\partial} h &= h\overline{\omega}^t, \end{aligned}$$

since ω is a $(1,0)$-valued matrix. This shows the Chern connection is determined by $\omega = (\partial h)h^{-1}$ uniquely. □

Corollary 3.1.1. Let E be a complex vector bundle on a complex manifold X and h, h' are two Hermitian metrics on E which are same up to a sign. Then the Chern connection of (E,h) is the same as the one of (E,h').

Remark 3.1.6. The Chern connection is locally determined by

$$\frac{\partial h_{\alpha\overline{\beta}}}{\partial z^i} = \Gamma^{\gamma}_{i\alpha} h_{\gamma\overline{\beta}}.$$

Definition 3.1.6. Let X be a complex manifold and (E,h) be a Hermitian holomorphic vector bundle. The **Chern curvature** Θ_h of (E,h) is defined as the curvature of Chern connection with respect to h.

Corollary 3.1.2. Let X be a complex manifold and (E,h) a Hermitian vector bundle equipped with Chern connection ∇ locally given by ω. Then

(1) $\partial\omega = \omega \wedge \omega$.
(2) $\Theta_h = \overline{\partial}\omega$.
(3) $\overline{\partial}\Theta_h = 0$.

Proof. For (1). Since $\omega = (\partial h)h^{-1}$, then directly computation shows

$$\begin{aligned} \partial\omega &= -\partial h \wedge \partial(h^{-1}) \\ &= -\partial h \wedge (-h^{-1}\partial h h^{-1}) \\ &= (\partial h)h^{-1} \wedge (\partial h)h^{-1} \\ &= \omega \wedge \omega. \end{aligned}$$

For (2). The Chern curvature Θ_h locally looks like

$$\Theta_h = \mathrm{d}\omega - \omega\wedge\omega = \mathrm{d}\omega - \partial\omega = \overline{\partial}\omega.$$

For (3). It follows from (2) directly. □

Remark 3.1.7. The Chern curvature can be expressed in terms of Christoffel symbol as follows

$$\Theta_h = \Theta^{\gamma}_{i\bar{j}\alpha} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j \otimes e^{\alpha} \otimes e_{\gamma},$$

where $\Theta^{\gamma}_{i\bar{j}\alpha} = -\partial\Gamma^{\gamma}_{i\alpha}/\partial\bar{z}^j$. In other type one has

$$\begin{aligned}\Theta_{i\bar{j}\alpha\bar{\beta}} &= h_{\gamma\bar{\beta}}\Theta^{\gamma}_{i\bar{j}\alpha}\\ &= -h_{\gamma\bar{\beta}}\partial_{\bar{j}}(h^{\gamma\bar{\delta}}\frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i})\\ &= -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^i\partial\bar{z}^j} + h^{\gamma\bar{\delta}}\frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i}\frac{\partial h_{\gamma\bar{\beta}}}{\partial\bar{z}^j}.\end{aligned}$$

3.1.4. *Second variation formula.*

Lemma 3.1.1. Let (E,h) be a Hermitian holomorphic vector bundle and ∇ be the Chern connection with Chern curvature Θ. Suppose ϕ is a holomorphic section of E. Then we have the formula

$$\bar{\partial}\partial\log\|\phi\|^2 = \frac{\langle\Theta\phi,\phi\rangle}{\|\phi^2\|} + \frac{\langle\nabla\phi,\phi\rangle\langle\phi,\nabla\phi\rangle - \|\phi\|^2\langle\nabla\phi,\nabla\phi\rangle}{\|\phi\|^4}.$$

Proof. Firstly note that $\partial\|\phi\|^2$ is the $(1,0)$-part of $\mathrm{d}\|\phi\|^2$, but on the other hand, by the compatiblity with the metric, one has

$$\mathrm{d}\langle\phi,\phi\rangle = \langle\nabla\phi,\phi\rangle + \langle\phi,\nabla\phi\rangle.$$

Then

$$\partial\log\|\phi\|^2 = \frac{\partial\|\phi\|^2}{\|\phi\|^2} = \frac{\langle\nabla\phi,\phi\rangle}{\|\phi\|^2}.$$

Next, we apply the chain rule

$$\bar{\partial}(\partial\log\|\phi\|^2) = \underbrace{\left(\bar{\partial}\frac{1}{\|\phi\|^2}\right)\langle\nabla\phi,\phi\rangle}_{\text{part I}} + \underbrace{\frac{1}{\|\phi\|^2}\bar{\partial}\langle\nabla\phi,\phi\rangle}_{\text{part II}}.$$

For part I, one has

$$\text{part I} = \frac{-1}{\|\phi\|^4}\bar{\partial}\|\phi\|^2 = -\frac{\langle\phi,\nabla\phi\rangle}{\|\phi\|^4}.$$

For part II, note that $\bar{\partial}\langle\nabla\phi,\phi\rangle$ is the $(1,1)$-part of $\mathrm{d}\langle\nabla\phi,\phi\rangle$. By the compatiblity with the metric, we have

$$\mathrm{d}\langle\nabla\phi,\phi\rangle = \langle\Theta\phi,\phi\rangle - \langle\nabla\phi,\nabla\phi\rangle.$$

Combining the above result, this completes the proof. □

3.1.5. *Second fundamental form.* Let (E,h) be a Hermitian holomorphic vector bundle over complex manifold X with rank r and S be a holomorphic subbundle of E with rank s. Then there is an exact sequence of holomorphic vector bundles

$$0 \to S \to E \to Q \to 0,$$

where Q is the holomorphic quotient bundle, which is isomorphic to $S^{\perp}$ as complex vector bundle.

Suppose ∇^E is the Chern connection on E and define $\nabla^S := \pi_S \circ \nabla^E$, where $\pi_S : E \to S$ is the orthogonal projection.

(1) It's clear ∇^S is compatible with holomorphic structure of S since ∇^E is the Chern connection of E, and S is a holomorphic subbundle of E.
(2) For sections s,t of S, one has

$$\begin{aligned} \mathrm{d}h(s,t) &= h(\nabla^E s,t) + h(s,\nabla^E t) \\ &\overset{(a)}{=} h(\pi_S \circ \nabla^E s,t) + h(s,\pi_S \circ \nabla^E t) \\ &= h(\nabla^S s,t) + h(s,\nabla^S t), \end{aligned}$$

where (a) holds from π_S is orthogonal projection.

This shows that ∇^S is the Chern connection of S with respect to Hermitian metric induced by the one on E.

Definition 3.1.7. The **second fundamental form** of the subbundle S of E is defined as

$$B = \nabla^E - \nabla^S : \mathscr{A}^0(S) \to \mathscr{A}^{1,0}(Q).$$

In other words, the second fundamental form $B \in \mathscr{A}^{1,0}(\mathrm{Hom}(S,Q))$.

Proposition 3.1.1.

$${}^S\Theta = {}^E\Theta|_S + B^* \wedge B.$$

Proof. It suffices to check pointwisely. For $p \in X$, suppose $\{e_\alpha\}_{1\leq\alpha\leq r}$ is a holomorphic local frame of E such that $\{e_\alpha\}_{1\leq\alpha\leq s}$ is a holomorphic local frame of S, and assume $h_{\alpha\overline{\beta}}(p) = \delta_{\alpha\overline{\beta}}$. By the formula (3.1.6) of Chern connection, for $1 \leq \alpha \leq s$, one has

$$\nabla^E e_\alpha(p) = \sum_{\beta=1}^{r} \frac{h_{\alpha\overline{\beta}}}{\partial z^i}(p)\mathrm{d}z^i \otimes \overline{e}^\beta$$

$$\nabla^S e_\alpha(p) = \sum_{\beta=1}^{s} \frac{h_{\alpha\overline{\beta}}}{\partial z^i}(p)\mathrm{d}z^i \otimes \overline{e}^\beta,$$

and thus

$$Be_\alpha(p) = \sum_{\beta=s+1}^{r} \frac{h_{\alpha\overline{\beta}}}{\partial z^i}(p)\mathrm{d}z^i \otimes \overline{e}^\beta.$$

This shows

$$B(p) = \sum_{\alpha=1}^{s} \sum_{\beta=s+1}^{r} \frac{h_{\alpha\overline{\beta}}}{\partial z^i}(p)\mathrm{d}z^i \otimes e^\alpha \otimes \overline{e}^\beta,$$

and thus its conjugate transpose is

$$B^*(p) = \sum_{\beta=s+1}^{r}\sum_{\alpha=1}^{s} \frac{h_{\beta\overline{\alpha}}}{\partial\overline{z}^j}(p)\mathrm{d}\overline{z}^j \otimes e^{\beta} \otimes e_{\alpha}.$$

Then

$$\begin{aligned} B^* \wedge Be_{\alpha}(p) &= B^*(\sum_{\gamma=s+1}^{r} \frac{h_{\alpha\overline{\gamma}}}{\partial z^i}(p)\mathrm{d}z^i \otimes e_{\gamma}) \\ &= -\sum_{\beta=1}^{s}\sum_{\gamma=s+1}^{r} \frac{h_{\alpha\overline{\gamma}}}{\partial z^i}\frac{h_{\gamma\overline{\beta}}}{\partial\overline{z}^j}(p)\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j \otimes \overline{e}^{\beta}, \end{aligned}$$

which implies

$$B^* \wedge B(p) = -\sum_{\alpha,\beta=1}^{s}\left\{\sum_{\gamma=s+1}^{r} \frac{\partial h_{\alpha\overline{\gamma}}}{\partial z^i}\frac{\partial h_{\gamma\overline{\beta}}}{\partial\overline{z}^j}(p)\right\}\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j \otimes e^{\alpha} \otimes \overline{e}^{\beta}.$$

On the other hand, by using formula (3.1.7), a direct computation shows

$${}^{E}\Theta|_{S}(p) - {}^{S}\Theta(p) = \sum_{\alpha,\beta=1}^{s}\left\{\sum_{\gamma=s+1}^{r} \frac{\partial h_{\alpha\overline{\gamma}}}{\partial z^i}\frac{\partial h_{\gamma\overline{\beta}}}{\partial\overline{z}^j}(p)\right\}\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j \otimes e^{\alpha} \otimes \overline{e}^{\beta}.$$

This shows

$${}^{S}\Theta = {}^{E}\Theta|_{S} + B^* \wedge B.$$

□

Proposition 3.1.2.

$${}^{Q}\Theta = {}^{E}\Theta|_{Q} + B \wedge B^*.$$

3.1.6. *Positivity.*

Definition 3.1.8. A real $(1,1)$-form $\omega = \sqrt{-1}h_{i\overline{j}}\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j$ is **positive** if the Hermitian matrix $h_{i\overline{j}}$ is positive definite.

Definition 3.1.9. Let (E,h) be a Hermitian holomorphic vector bundle on X. A form $\Theta \in \mathscr{A}_X^{1,1} \otimes \mathrm{End}(E)$ is **positive** if for any non-zero section e of E, one has $h(\Theta e, e)$ is positive.

Proposition 3.1.3. Let (E,h) be a Hermitian holomorphic vector bundle and $S \subseteq E$ be a holomorphic subbundle. Then $B \wedge B^*$ is positive and $B^* \wedge B$ is negative, where B is the second fundamental form.

Corollary 3.1.3. The curvature decreases in holomorphic subbundles and increases in holomorphic quotient bundles.

3.2. **Curvature of Hodge bundles.** Consider a variation of polarized Hodge structures of weight k over some fixed complex manifold. This data consists of a flat bundle $H_{\mathbb{C}}$ together with the Gauss-Manin connection ∇^{GM}, and there is a filtration by holomorphic subbundles

$$\cdots \subset \mathscr{F}^p \subset \mathscr{F}^{p-1} \subset \cdots \subset H_{\mathbb{C}}.$$

Denote the quotient subbundle by

$$\mathscr{H}^{p,q} := \mathscr{F}^p / \mathscr{F}^{p+1},$$

where $p+q=k$. The polarization provides the indefinite forms $I(\text{-},\text{-})$ on $H_{\mathbb{C}}$, and a definite metric

$$Q(\text{-},\text{-}) := I(\mathbb{C}\text{-},\text{-}),$$

where $\mathbb{C}$ is the Weil operator. In particular, restricted to $\mathscr{H}^{p,q}$, the definite and indefinite metrics agree up to sign.

Note that ∇^{GM} is the Chern connection on $H_{\mathbb{C}}$ equipped with the indefinite metric $I(\text{-},\text{-})$. On the other hand, viewing $H_{\mathbb{C}}$ as the direct sum of the holomorphic bundles $\mathscr{H}^{p,q}$, each equipped with the definite metric $Q(\text{-},\text{-})$, there is also a Hodge connection ∇^{Hg}, which is defined as the Chern connection of $\bigoplus \mathscr{H}^{p,q}$ equipped with the definite metric.

Consider the second fundamental form (for the indefinite metric)

$$\sigma_p : \mathscr{F}^p \to \mathscr{H}_{\mathbb{C}} / \mathscr{F}^p \otimes \mathscr{A}_X^{1,0}.$$

The Griffiths transversality condition implies it must in fact map subspaces as follows

$$\sigma_p : \mathscr{H}^{p,q} \to \mathscr{H}^{p-1,q+1} \otimes \mathscr{A}_X^{1,0}.$$

Proposition 3.2.1.

$$\nabla^{GM} = \nabla^{Hg} + \sigma_\bullet + \sigma_\bullet^*,$$

where $\sigma_\bullet$ denotes $\bigoplus_p \sigma_p$ and similarly for $\sigma_\bullet^*$.

Proof. See Proposition 13.1.1 of [CMSP17]. □

Proposition 3.2.2.

$$\Theta_{\mathscr{H}^{p,q}} = \sigma_p^* \wedge \sigma_p + \sigma_{p+1} \wedge \sigma_{p+1}^*.$$

Proof. Note that the definite and indefinite metrics agree up to sign on $\mathscr{H}^{p,q}$, by Corollary 3.1.1 it suffices to prove the curvature for the indefinite metric. From the exact sequence

$$0 \to \mathscr{F}^p \to \mathscr{H}_{\mathbb{C}} \to \mathscr{H}_{\mathbb{C}} / \mathscr{F}^p \to 0,$$

we find using Proposition 3.1.1

$$\Theta_{\mathscr{F}_p} = \sigma_p^* \wedge \sigma_p.$$

Next, consider the exact sequence

$$0 \to \mathscr{F}^{p+1} \to \mathscr{F}^p \to \mathscr{H}^{p,q} \to 0.$$

Again Proposition 3.1.1 yields

$$\begin{aligned}\Theta_{\mathscr{H}^{p,q}} &= \Theta_{\mathscr{F}^p} + \sigma_{p+1} \wedge \sigma^*_{p+1} \\ &= \sigma^*_p \wedge \sigma_p + \sigma_{p+1} \wedge \sigma^*_{p+1}.\end{aligned}$$

This completes the proof. □

Proposition 3.2.3. Suppose e, e' are two smooth sections of $\mathscr{H}^{p,q}$. Then

$$Q(\Theta_{\mathscr{H}^{p,q}} e, e') = Q(\sigma_p e, \sigma_p e') + Q(\sigma^*_{p+1} e, \sigma^*_{p+1} e').$$

Proof. Note that on $\mathscr{H}^{p,q}$, we have

$$I(\text{-},\text{-}) = (\sqrt{-1})^{p-q} Q(\text{-},\text{-}) = (\sqrt{-1})^{-k} \times (-1)^p Q(\text{-},\text{-}).$$

Then

$$\begin{aligned}(\sqrt{-1})^{-k} Q(\Theta_{\mathscr{H}^{p,q}} e, e') &= (-1)^p I(\Theta_{\mathscr{H}^{p,q}} e, e') \\ &= (-1)^p \left(I(\sigma^*_p \wedge \sigma_p e, e') \right) + I(\sigma_{p+1} \wedge \sigma^*_{p+1} e, e')) \\ &= (-1)^{p+1} \left(I(\sigma_p e, \sigma_p e') \right) + I(\sigma^*_{p+1} e, \sigma^*_{p+1} e')) \\ &= (\sqrt{-1})^{-k} \times (-1)^{p+1} \left((-1)^{p-1} Q(\sigma_p e, \sigma_p e') \right) + (-1)^{p+1} Q(\sigma^*_{p+1} e, \sigma^*_{p+1} e')) \\ &= (\sqrt{-1})^{-k} Q(\sigma_p e, \sigma_p e') + (\sqrt{-1})^{-k} Q(\sigma^*_{p+1} e, \sigma^*_{p+1} e').\end{aligned}$$

This completes the proof. □

Corollary 3.2.1. The bundle $\mathscr{H}^{0,k}$ has negative curvature (with respect to definite metric).

Proof. In this case, the second fundamental form σ_0 vanishes, so

$$Q(\Theta_{\mathscr{H}^{0,k}} e, e') = Q(\sigma^*_1 e, \sigma^*_1 e'),$$

which is negative. □

References

[CMSP17] James Carlson, Stefan Müller-Stach, and Chris Peters. *Period mappings and period domains*, volume 168 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2017.

[Del70] Pierre Deligne. *Équations différentielles à points singuliers réguliers*, volume Vol. 163 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1970.

[Fil16] Simion Filip. Semisimplicity and rigidity of the Kontsevich-Zorich cocycle. *Invent. Math.*, 205(3):617–670, 2016.

[PS08] Chris A. M. Peters and Joseph H. M. Steenbrink. *Mixed Hodge structures*, volume 52 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2008.