A SURVEY ON RIGIDITY OF COMPLEX PROJECTIVE SPACE

BOWEN LIU

Abstract. It's an interesting, also important to consider the rigidity of $\mathbb{CP}^n$, that is to determine whether a complex manifold with certain geometrical and topological information is biholomorphic to $\mathbb{CP}^n$ or not. In this note we introduce some classical results and key techniques to prove them, and introduce some refinements given recently.

Contents

1. Classical results

In 1957, Hirzebruch and Kodaira showed in [HK57] that if a Kähler manifold M is diffeomorphic to $\mathbb{CP}^n$, then

(1) M is biholomorphic to $\mathbb{CP}^n$ if n is odd.
(2) M is biholomorphic to $\mathbb{CP}^n$ if n is even and $c_1(M) \neq -(n+1)\omega$, where $\omega \in H^2(M, \mathbb{Z})$ is a positive generator.

The key technique of their proof is that if M is diffeomorphic to $\mathbb{CP}^n$, then the total Pontrjagin class of M is the same as the one of $\mathbb{CP}^n$. Hirzebruch also observed that their results can be improved if the Pontrjagin calsses are topological invariants, which is the first problem he proposed in [Hir54], and he also observed exotic complex structure of $\mathbb{CP}^3$ is closely related to the complex structure of S^6 in [Hir54], and a detailed proof of this observation can be found in the closing remarks in [Tos17].

Novikov proved that rational Pontrjagin classes of a closed smooth manifold are indeeed homeomorphism invariants in [Nov65], and since M has torsion-free integral cohomology if M is homeomorphic to $\mathbb{CP}^n$, we obtain the invariance of integral Pontrjagin classes, which improves the results of Hirzebruch from diffeomorphism to homeomorphism.

Later in [Yau77] Yau proved the case $c_1(M) = -(n+1)\omega$ can not happen by using his Chern number inequality, and he also proved that in the case of surface, the hypothesis "homeomorphism" can be relaxed to "homotopy equivalence" and the Kähler condition is unnecessary. All in all, the stories till now can be summarized as the following two theorems.

Theorem 1.1 (Hirzebruch-Kodaira [HK57], Yau [Yau77])**.** If a Kähler manifold is homeomorphic to $\mathbb{CP}^n$, then it must be biholomorphic to $\mathbb{CP}^n$.

Theorem 1.2 (Yau [Yau77])**.** If a compact surface is homotopy equivalent to $\mathbb{CP}^n$, then it must be biholomorphic to $\mathbb{CP}^n$.

Having above two beautiful theorems, one can still ask the following questions about the rigidity of $\mathbb{CP}^n$.

(1) Is "Kähler" in Theorem 1.1 necessary? In other words, whether a complex manifold which is homotopy equivalent/homeomorphic/diffeomorphic to $\mathbb{CP}^n$ must be $\mathbb{CP}^n$ or not.
(2) Can we generalize Theorem 1.2 to higher dimensions?

For the first tough question, there are only some negative results for the first tough question. It's shown in [MY68] (for $n = 3$) and [Hsi66] (for $n \geq 4$) that for every $n \geq 3$ the homotopy type of $\mathbb{CP}^n$ supports infinitely many inequivalent smooth structure distinguished by their Pontrjagin classes. But for the second one, there are some good news.

Firstly Fujita showed in [Fuj80] that if M is a Fano n-manifold such that the cohomology ring $H^*(M, \mathbb{Z})$ is isomorphic to $H^*(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$, then M is biholomorphic to $\mathbb{CP}^n$ when $n \leq 5$. Not long after that, Lanteri and Struppa extended the results of Fujita in dimension 3 in [LS86]. To be explicit, they showed that any compact Kähler 3-manifold with the same cohomology ring as $\mathbb{CP}^3$ is biholomorphic to $\mathbb{CP}^3$.

Up to now, the best results for the second question are given by Libgober and Wood in [LW90]. They showed that a compact Kähler manifold which is homotopy equivalent to $\mathbb{CP}^n$ must be biholomorphic to $\mathbb{CP}^n$ when $n \leq 6$. In section 3, we will introduce some remarks given by Ping Li in [Li16], which refined the classical results under some symmetry condition and finiteness of fundamental group.

2. Key techniques

In this section we collect some key techniques shown in [Tos17] and [Li16], which are used to determine whether a compact Kähler manifold is biholomorphic to $\mathbb{CP}^n$ or not under certain geometrical of topological conditions.

2.1. Rigidity criterion.

In order to determine whether a compact Kähler manifold is biholomorphic to $\mathbb{CP}^n$ or not, we mainly use the following criterion which is given by Kobayashi and Ochiai in [KO73].

Theorem 2.1. If M is a compact Kähler manifold and L is a positive line bundle on M with $\int_M c_1^n(L) = 1$ and $\dim H^0(M, L) = n+1$, then M is biholomorphic to $\mathbb{CP}^n$.

Proof. Let $\{\varphi_1, \dots, \varphi_n\}$ be a basis of $H^0(M, L)$ and let $D_j = \{\varphi_j = 0\}$ be the corresponding divisors. Define $V_n = M$ and

$$V_{n-k} = D_1 \cap \cdots \cap D_k$$

for $1 \leq k \leq n$.

Lemma 2.1. For each $0 \leq r \leq n$ we have that

(1) V_{n-r} is irreducible, of dimension $n-r$ and Poincaré dual to $c_1^r(L)$.
(2) The sequence
$$0 \to \operatorname{span}\{\varphi_1, \dots, \varphi_r\} \to H^0(M, L) \to H^0(V_{n-r}, L)$$
is exact, where the last map is given by restriction.

Proof. The proof of the lemma is by induction on r, the case $r = 0$ being obvious. Supppose (1) and (2) holds for $r-1$, we see that V_{n-r+1} is irreducible and that φ_r is not identically zero on it. Hence $V_{n-r} = \{x \in V_{n-r+1} \mid \varphi_r(x) = 0\}$ is an effective divisor on V_{n-r+1} and so it can be expressed as a sum of irreducible subvarieties of dimension $n-r$. Since $c_1^{r-1}(L)$ is dual to V_{n-r+1} and $c_1(L)$ is dual to D_r we see that $c_1^r(L)$ is dual to V_{n-r}. If V_{n-r} were reducible, then $V_{n-r} = V' + V''$ and so

$$\begin{aligned}
1 &= \int_M c_1^n(L) \\
&= \int_M c_1^r(L) c_1^{n-r}(L) \\
&= \int_{V_{n-r}} c_1^{n-r}(L) \\
&= \int_{V'} c_1^{n-r}(L) + \int_{V''} c_1^{n-r}(L)
\end{aligned}$$

But since L is positive, the last two terms are both positive integers, and this is a contradiction. Thus (1) is proved. As for (2). the restriction sequence

$$0 \to \mathcal{O}_{V_{n-r+1}} \to \mathcal{O}_{V_{n-r+1}}(L) \to \mathcal{O}_{V_{n-r}}(L) \to 0$$

gives

$$0 \to H^0(V_{n-r+1}, \mathcal{O}) \to H^0(V_{n-r+1}, L) \to H^0(V_{n-r}, L) \to \dots$$

where the first map is given by multiplication by φ_r. This means the kernel of the restriction map $H^0(V_{n-r+1}, L) \to H^0(V_{n-r}, L)$ is spanned by φ_r. This together with the statement in (2) for $r-1$ proves (2) for (2). ☐

Now we apply Lemma 2.1 with $r = n$ and see that V_0 is a single point and that φ_{n+1} does not vanish there. So given any point of M there is a section of L that does not vanish there, that is L is base-point free. Then L induces a holomorphic map by

$$\begin{aligned} f\colon M &\to \mathbb{CP}^n \\ x &\mapsto \{\varphi \in H^0(M, L) \mid \varphi(x) = 0\} \end{aligned}$$

If $y \in \mathbb{CP}^n$ corresponds to a hyperplane, which is spanned by some sections $\varphi_1, \dots, \varphi_n$, then $f(x) = y$ if and only if $\varphi_1(x) = \cdots = \varphi_n(x) = 0$. Again by Lemma 2.1 with $r = n$ says that $x = V_0$ exists and is unique. This shows f is a bijection, and since any bijective holomorphic map is biholomorphic, this completes the proof. ☐

If M is a compact Kähler manifold which has the same cohomology group with $\mathbb{CP}^n$, then by a simple argument of long exact sequence one has

$$c_1\colon \operatorname{Pic}(M) \to H^2(M, \mathbb{Z})$$

is bijective, which allows us to construct holomorphic line bundle L with a given 2-form as its first Chern class. In order to satisfy the conditions in Theorem 2.1, one way is to give a computable formula of $\chi(M, L)$ and use vanishing theorems to conclude

$$\dim H^0(M, L) = \chi(M, L)$$

In order to derive such a formula, the invariance of Pontrjagin classes plays an important role.

2.2. **Invariance of Pontrjagin classes.**

Theorem 2.2. *Let M be a compact complex n-manifold admitting Pontrjagin classes*

$$p_i(M) = \binom{n+1}{i} \omega^{2i}$$

for $0 \le i \le \lfloor n/2 \rfloor$, where ω is the generator of $H^2(M, \mathbb{Z})$. If L is a holomorphic line bundle on M, then

$$\chi(M, L) = \int_M e^{c_1(L) + \frac{c_1(M)}{2}} \left(\frac{\omega/2}{\sinh(\omega/2)} \right)^{n+1}$$

Proof. By Hirzebruch-Riemann-Roch one has

$$\chi(M,L) := \sum_{p\geq 0}(-1)^p \dim H^p(M,L) = \int_M e^{c_1(L)}\,\mathrm{Td}(M)$$

where $\mathrm{Td}(M)$ is the Todd genus of M. The Todd genus is defined in terms of the Chern classes of M, but since in our case we only know the Pontrjagin classes of M, we need to express $\mathrm{Td}(M)$ in terms of these. To do this, we use the identity

$$\mathrm{Td}(M) = e^{\frac{c_1(M)}{2}}\hat{A}(M)$$

Recall that $\hat{A}(M)$ is the $\hat{A}$ genus of M defined by

$$\hat{A}(M) = \prod_{j\geq 0}\frac{\sqrt{\gamma_j}/2}{\sinh(\sqrt{\gamma_j}/2)}$$

where γ_j is given by the following formal summation

$$\sum_{j\geq 0} p_j(M)x^j = \prod_{j\geq 1}(1+\gamma_j x)$$

Now thanks to the expression of Pontrjagin of M, one has

$$\sum_{j\geq 0} p_j(M)x^j = (1+\omega^2 x)^{n+1}$$

which gives $\gamma_1 = \cdots = \gamma_{n+1} = \omega^2$ and $\gamma_j = 0$ for $j > n+1$. Thus we obtain

$$\begin{aligned}\chi(M,L) &= \int_M e^{c_1(L)}\,\mathrm{Td}(M)\\ &= \int_M e^{c_1(L)+\frac{c_1(M)}{2}}\hat{A}(M)\\ &= \int_M e^{c_1(L)+\frac{c_1(M)}{2}}\left(\frac{\omega/2}{\sinh(\omega/2)}\right)^{n+1}\end{aligned}$$

□

Corollary 2.1. Let M be a compact Kähler manifold which is homeomorphic to $\mathbb{CP}^n$. If ω is a positive generator of $H^2(M,\mathbb{Z})$, then $c_1(M) = (n+1)\omega$ or $-(n+1)\omega$, with the latter only possibly occurring when n is even.

Proof. Since ω is a generator of $H^2(M,\mathbb{Z})$, we may write $c_1(M) = \lambda\omega$. The reduction mod 2 of $c_1(M)$ is the second Stiefel-Whitney class $w_2(M) \in H^2(M,\mathbb{Z})$, which is a topological invariant. Hence it is equals to $w_2(\mathbb{CP}^n)$ which equals $c_1(\mathbb{CP}^n) \equiv n+1 \pmod 2$. This shows $c_1(M) = (n+1+2s)\omega$ for some $s \in \mathbb{Z}$. Since rational Pontrjagin classes are topological invariants, and M has torsion-free integral cohomology, one has the Pontrjagin classes are the same as $\mathbb{CP}^n$, that is

$$p_i(M) = \binom{n+1}{i}\omega^{2i}$$

By theorem 2.2, one has

$$\chi(M,\mathcal{O}) = \int_M e^{\frac{n+1+2s}{2}\omega}\left(\frac{\omega/2}{\sinh(\omega/2)}\right)^{n+1} = \int_M e^{s\omega}\left(\frac{\omega}{1-e^{-\omega}}\right)^{n+1}.$$

where the last equality holds by the identity

$$\frac{x}{1-e^{-x}} = e^{\frac{x}{2}}\frac{x/2}{\sinh(x/2)}$$

In order to compute above integral, note that $\int_M \omega^n = 1$ since ω is a positive generator of $H^2(M,\mathbb{Z})$, and the integrals over M of all other powers of ω are zero by definition, so it suffices to compute the coefficient of x^n in the power series expansion of

$$e^{sx}\left(\frac{x}{1-e^{-x}}\right)^{n+1}$$

Let $F(z) = e^{sz}(\frac{z}{1-e^{-z}})^{n+1}$. Then Cauchy's integral formula shows that the coefficient that we are interested in equals the contour integral

$$\frac{1}{2\pi\sqrt{-1}}\oint\frac{F(z)}{z^{n+1}}\mathrm{d}z = \frac{1}{2\pi\sqrt{-1}}\oint\frac{e^{sz}}{(1-e^{-z})^{n+1}}$$

A standard computation yields

$$\chi(M,\mathcal{O}) = \binom{n+s}{n}.$$

On the other hand, since M is Kähler and has the same cohomology group as $\mathbb{CP}^n$, one has $\chi(M,\mathcal{O}) = 1$. This shows $\binom{n+s}{n} = 1$, which can be rewritten as

$$n! = (s+n)\dots(s+1).$$

So if n is ood this implies $s = 0$, while if n is even, s is either 0 or $-n-1$. $\square$

3. Refinements of Ping Li

3.1. **Topological conditions.** As shown in [Tos17], Corollary 2.1 is the key step in proof of Theorem 1.1. However, one can see that "homeomorphism" is a quite strong assumption, and one can only assume necessary topological conditions to give the same results.

Proposition 3.1 ([Li16])**.** Let M be a compact Kähler manifold. If its integral cohomology ring and Pontrjagin classes are the same as those of $\mathbb{CP}^n$, then $c_1(M) = (n+1)\omega$ or $-(n+1)\omega$, with the latter only possibly occurring when n is even, where ω is a positive generator of $H^2(M, \mathbb{Z})$.

By the same proof, Ping Li obtained the same results as Hirzebruch and Kodaira in 1957, and in order to exclude the case $c_1(M) < 0$, he added a condition about finiteness of fundamental group, since M is compact, then its universal covering must also be compact if $\pi_1(M)$ is finite. As a conclusion, he showed the following result.

Theorem 3.1 ([Li16])**.** Let M be a compact Kähler manifold having the same integral cohomology ring and Pontrjagin classes as $\mathbb{CP}^n$. Then

(1) M is biholomorphic to $\mathbb{CP}^n$ if n is odd.
(2) M is biholomorphic to $\mathbb{CP}^n$ if n is even and the fundamental group $\pi_1(M)$ is finite.

3.2. **Symmetry condition.** Recall a classical conjecture which was posed by Petrie in [Pet72], asserts that if M is an n-dimensional homotopy complex projective space[1] admitting an (effective and smooth) S^1-action, then its total Pontrjagin class is the same as the one of $\mathbb{CP}^n$.

Petrie himself verified this conjecture [Pet73] under the stronger hypothesis that an n-dimensional torus acts (effectively and smoothly) on M. Dessai and Wilking improved on Petrie's result by showing that the conjecture holds if a torus whose dimension is larger than $(n+1)/4$ acts on M in [DW04]. Then combining Theorem 3.1 with Dessai–Wilking's result, Ping Li gives his second observation.

Theorem 3.2 ([Li16])**.** If a compact Kähler manifold is homotopy equivalent to $\mathbb{CP}^n$ and acted on effectively and smoothly by a torus whose dimension is larger than $(n+1)/4$, then it must be biholomorphic to $\mathbb{CP}^n$.

3.3. **Refinement of results in $n = 4$.** By using some concrete computations in $n = 4$. Ping Li showed the following result.

Theorem 3.3 ([Li16])**.** A compact Kähler manifold with finite fundamental group and having the same integral cohomology ring as $\mathbb{CP}^4$ is biholomorphic to $\mathbb{CP}^4$.

[1]A smooth closed $2n$-dimensional manifold is called an n-dimensional homotopy complex projective space if it's homotopy equivalent to $\mathbb{CP}^n$.

As is now well known that a Fano manifold is simply connected, which is a corollary of the celebrated Calabi–Yau theorem, the conditions of the fundamental group being finite and having the same integral cohomology ring are strictly weaker than the assumptions in Fujita's result.

References

[DW04] Anand Dessai and Burkhard Wilking. Torus actions on homotopy complex projective spaces. *Math. Z.*, 247(3):505–511, 2004.

[Fuj80] Takao Fujita. On topological characterizations of complex projective spaces and affine linear spaces. *Proc. Japan Acad. Ser. A Math. Sci.*, 56(5):231–234, 1980.

[Hir54] Friedrich Hirzebruch. Some problems on differentiable and complex manifolds. *Ann. of Math. (2)*, 60:213–236, 1954.

[HK57] F. Hirzebruch and K. Kodaira. On the complex projective spaces. *J. Math. Pures Appl. (9)*, 36:201–216, 1957.

[Hsi66] Wu-chung Hsiang. A note on free differentiable actions of S^1 and S^3 on homotopy spheres. *Ann. of Math. (2)*, 83:266–272, 1966.

[KO73] Shoshichi Kobayashi and Takushiro Ochiai. Characterizations of complex projective spaces and hyperquadrics. *J. Math. Kyoto Univ.*, 13:31–47, 1973.

[Li16] Ping Li. Some remarks on the uniqueness of the complex projective spaces. *Bull. Lond. Math. Soc.*, 48(2):379–385, 2016.

[LS86] A. Lanteri and D. Struppa. Projective manifolds with the same homology as $\mathbf{P}^k$. *Monatsh. Math.*, 101(1):53–58, 1986.

[LW90] Anatoly S. Libgober and John W. Wood. Uniqueness of the complex structure on Kähler manifolds of certain homotopy types. *J. Differential Geom.*, 32(1):139–154, 1990.

[MY68] Deane Montgomery and C. T. Yang. Differentiable actions on homotopy seven spheres. II. In *Proc. Conf. on Transformation Groups (New Orleans, La., 1967)*, pages 125–134. Springer, New York, 1968.

[Nov65] S. P. Novikov. Rational Pontrjagin classes. Homeomorphism and homotopy type of closed manifolds. I. *Izv. Akad. Nauk SSSR Ser. Mat.*, 29:1373–1388, 1965.

[Pet72] Ted Petrie. Smooth S^1 actions on homotopy complex projective spaces and related topics. *Bull. Amer. Math. Soc.*, 78:105–153, 1972.

[Pet73] Ted Petrie. Torus actions on homotopy complex projective spaces. *Invent. Math.*, 20:139–146, 1973.

[Tos17] Valentino Tosatti. Uniqueness of $\mathbb{CP}^n$. *Expo. Math.*, 35(1):1–12, 2017.

[Yau77] Shing Tung Yau. Calabi's conjecture and some new results in algebraic geometry. *Proc. Nat. Acad. Sci. U.S.A.*, 74(5):1798–1799, 1977.

Mathematics Department, Tsinghua University, Beijing, 100084, P.R. China,
Email address: `liubw22@mails.tsinghua.edu.cn`