

CHERN INEQUALITIES

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ABSTRACT. It's a lecture note for studying the paper [\[Miy87\]](#).

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0. CONVENTIONS

- (1) An (algebraic) variety over a field k is an integral separated scheme of finite type over k .
- (2) A subvariety of a variety is a closed subscheme which is a variety.
- (3) A curve, surface or a threefold means a variety of dimension 1, 2 or 3.
- (4) A point on a scheme will always be a closed point.

1. PRELIMINARIES

In this section, unless otherwise specified, X always denotes a variety of dimension n over an algebraically closed field k .

1.1. Torsion-freeness and relexivity.

1.1.1. Torsion-freeness.

Definition 1.1.1. An \mathcal{O}_X -module \mathcal{F} is said to be **locally free sheaf** if there is an open covering $\{U_i\}$ of X such that $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r}$ holds for every U_i .

Definition 1.1.2. An \mathcal{O}_X -module \mathcal{F} is said to be **coherent sheaf** if

- (1) \mathcal{F} is of finite type.
- (2) For every open subset $U \subseteq X$ and every morphism $\alpha: \mathcal{O}_U^r \rightarrow \mathcal{F}|_U$, the kernel of α is of finite type.

Definition 1.1.3. A coherent sheaf \mathcal{F} on X is **torsion-free** if a stalk \mathcal{F}_x is a torsion-free $\mathcal{O}_{X,x}$ -module for every $x \in X$.

Definition 1.1.4. A coherent subsheaf \mathcal{F} of a torsion-free sheaf \mathcal{E} is said to be **saturated** if the quotient \mathcal{E}/\mathcal{F} is again torsion-free.

Proposition 1.1.1. Let X, Y be two varieties and $f: X \rightarrow Y$ be a dominant morphism. Then for any torsion-free \mathcal{O}_X -module \mathcal{F} , the direct image $f_*\mathcal{F}$ is a torsion-free \mathcal{O}_Y -module.

Proof. See Proposition 8.4.5 in [GD71]. □

Proposition 1.1.2. Let X be a normal variety. Then every torsion-free sheaf is locally free outside a set of codimension two.

Proof. See Proposition 5.1.7 in [Ish14]. □

Corollary 1.1.1. Every torsion-free sheaf on a smooth curve is locally free.

1.1.2. Reflexivity.

Definition 1.1.5. A coherent \mathcal{O}_X -module \mathcal{F} is said to be **reflexive** if the canonical homomorphism $\mathcal{F} \rightarrow \mathcal{F}^{**}$ is an isomorphism.

Proposition 1.1.3. Every locally free sheaf is reflexive, and every reflexive sheaf is torsion-free.

Proof. It follows from the definitions. □

Proposition 1.1.4. The dual sheaf of any coherent sheaf is reflexive.

Proof. See Proposition 5.5.18 in [Kob87]. □

1.2. Chow ring.

1.2.1. Cycles.

Definition 1.2.1. A k -cycle on X is a \mathbb{Z} -linear combination of irreducible subvarieties of dimension k .

Notation 1.2.1. The group of all k -cycles on X is denoted by $Z_k(X)$.

Definition 1.2.2. A **Weil divisor** on X is an $(n-1)$ -cycle.

Definition 1.2.3. A **Cartier divisor** on X is a global section of quotient sheaf $\mathcal{M}_X^*/\mathcal{O}_X^*$.

Definition 1.2.4. A k -cycle α on X is defined to be **rationally equivalent to zero** if there are finitely many $(k+1)$ -dimensional irreducible subvarieties $W_i \subseteq X$ and non-zero rational functions $f_i \in \mathbb{C}(W_i)$ such that

$$\alpha = \sum_i [\text{div}_{W_i}(f_i)],$$

where $\text{div}_{W_i}(f_i)$ is the divisor of the rational functions f_i on W_i .

Definition 1.2.5. The group of k -cycles modulo rational equivalences is defined to be $A_k(X)$, which is said to be the k -th **Chow group**.

Example 1.2.1. $A_{n-1}(X)$ is the group of Weil divisors modulo linear equivalence.

Notation 1.2.2. The group of Cartier divisors modulo linear equivalence is denoted by $\text{Pic}(X)$.

Remark 1.2.1. There is a group homomorphism from $\text{Pic}(X)$ to $A_{n-1}(X)$. In general it's neither injective nor surjective, but it's injective when X is normal and an isomorphism when X is smooth.

Definition 1.2.6. The group of **cycles of codimension k modulo rational equivalence** is defined to be $A^k(X) := A_{n-k}(X)$.

1.2.2. The intersection pairing.

Theorem 1.2.1. Let X be a smooth variety. There is a unique intersection product $A^r(X) \times A^s(X) \rightarrow A^{r+s}(X)$ for each r, s satisfying the axioms listed below

- (1) The intersection pairing makes $A^*(X)$ into a commutative associated graded ring with identity. It's called the **Chow ring** of X .
- (2) For any morphism $f: X \rightarrow Y$, $f^*: A^*(Y) \rightarrow A^*(X)$ is a ring homomorphism. If $g: Y \rightarrow Z$ is another morphism, then $f^* \circ g^* = (g \circ f)^*$.

- (3) If $f: X \rightarrow Y$ is a proper morphism, $f_*: A^*(X) \rightarrow A^*(Y)$ is a homomorphism of graded groups. If $g: Y \rightarrow Z$ is another proper morphism, then $g_* \circ f_* = (g \circ f)_*$.
- (4) If $f: X \rightarrow Y$ is a proper morphism, $x \in A^*(X)$ and $y \in A^*(Y)$, then

$$f_*(x \cdot f^*y) = f_*(x) \cdot y.$$

This is said to be the **projection formula**.

- (5) If Y, Z are cycles on X , and if $\Delta: X \rightarrow X \times X$ is the diagonal morphism, then

$$Y \cdot Z = \Delta^*(Y \times Z).$$

- (6) If Y and Z are subvarieties of X which intersect properly (meaning that every irreducible component of $Y \cap Z$ has codimension equal to $\text{codim } Y + \text{codim } Z$), then

$$Y \cdot Z = \sum i(Y, Z; W_j) W_j,$$

where the sum runs over the irreducible components W_j of $Y \cap Z$, and where the integer $i(Y, Z; W_j)$ depends only on a neighborhood of the generic point of W_j on X , which is said to be the **local intersection multiplicity** of Y and Z along W_j .

- (7) If Y is a subvariety of X , and Z is an effective Cartier divisor meeting Y properly, then $Y \cdot Z$ is just the cycle associated to the Cartier divisor $Y \cap Z$ on Y , which is defined by restricting the local equation of Z to Y .

Proof. See appendix A.1 in [Har77]. \square

Remark 1.2.2. If X is not smooth, the intersection pairing also makes sense in some settings, see [Ful98] for more details. For example, on any variety (or scheme), there is always an intersection pairing

$$\text{Pic}(X) \times A^k(X) \rightarrow A^{k+1}(X).$$

1.3. Chern classes.

1.3.1. Chern classes of locally free sheaf.

Definition 1.3.1. A locally free sheaf \mathcal{E} of rank r on X has **Chern classes** $c_i(\mathcal{E}) \in A^i(X)$ for all $0 \leq i \leq r$, which is defined by

$$\sum_{i=0}^r (-1)^i \pi^* c_i(\mathcal{E}) \xi^{r-i} = 0$$

in $A^r(\mathbb{P}(\mathcal{E}))$, where $\xi \in A^1(\mathbb{P}(\mathcal{E}))$ be the class of the divisor corresponding to $c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ and $\pi: \mathbb{P}(\mathcal{E}) \rightarrow X$ be the projection.

Definition 1.3.2. Let \mathcal{E} be a locally free sheaf of rank r on X . The **total Chern class** is

$$c(\mathcal{E}) = c_0(\mathcal{E}) + \cdots + c_r(\mathcal{E}) \in A^*(X).$$

Proposition 1.3.1.

- (1) $c_0(\mathcal{E}) = 1$ for any \mathcal{E} and $c_1(\mathcal{O}_X) = 1$ for any X .
- (2) If $f: X \rightarrow Y$ is a morphism and \mathcal{E} is locally free on Y , then $c_i(f^*\mathcal{E}) = f^*(c_i(\mathcal{E}))$.
- (3) If $0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$ is an exact sequence, then $c(\mathcal{F}) = c(\mathcal{E})c(\mathcal{G})$.
- (4) $c_i(\mathcal{E}^\vee) = (-1)^i c_i(\mathcal{E})$, where \mathcal{E}^\vee is the dual of \mathcal{E} .
- (5) $c_1(\wedge^r \mathcal{E}) = c_1(\mathcal{E})$ when \mathcal{E} has rank r .
- (6) If D is a Cartier divisor on X , then

$$c_1(\mathcal{O}_X(D)) = D.$$

Proof. See appendix A.3 in [Har77]. □

1.3.2. Chern classes of coherent sheaf. Let $F(X)$ be the free abelian group generated by the set of coherent sheaves (up to isomorphisms, otherwise it's not a set) on X , that is, an element of $F(X)$ is a formal linear combination $\sum_i n_i \mathcal{F}_i$, where $n_i \in \mathbb{Z}$ and \mathcal{F}_i is coherent. Let

$$(E) \quad 0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$$

be an exact sequence of sheaves, and we associate the element $Q(E) = \mathcal{F} - \mathcal{F}' - \mathcal{F}''$ of $F(X)$ to this exact sequence.

Definition 1.3.3. The **group of classes of sheaves** $K(X)$ on X is defined to be the quotient of $F(X)$ by the subgroup generated by the $Q(E)$, where E runs over all short exact sequences.

Definition 1.3.4. Let $F_1(X)$ be the free group generated by the set of locally free sheaves (up to isomorphisms), and $K_1(X)$ be the quotient of $F_1(X)$ by the subgroup generated by the $Q(E)$, where E runs over all short exact sequences of locally free sheaves.

Theorem 1.3.1 ([BS58]). Let X be a smooth quasi-projective variety. Then the homomorphism $\epsilon: K_1(X) \rightarrow K(X)$ is a bijection.

Corollary 1.3.1. The definition of Chern classes can be extended to arbitrary coherent sheaves.

1.4. Cones of divisors and curves.

1.4.1. *The cones of divisors.*

Definition 1.4.1. For two Cartier divisors D_1, D_2 on X , D_1 is **numerically equivalent** to D_2 if $D_1 \cdot C = D_2 \cdot C$ for all irreducible curves C .

Definition 1.4.2. The **Néron-Severi group** $N^1(X)_{\mathbb{Z}}$ is the quotient group of $\text{Pic}(X)$ by numerically equivalence, and

$$N^1(X)_{\mathbb{Q}} = N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}, \quad N^1(X)_{\mathbb{R}} = N^1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}.$$

Theorem 1.4.1. The Néron-Severi group $N^1(X)_{\mathbb{Z}}$ is a free abelian group of finite rank, and its rank is said to be the **Picard number**.

Definition 1.4.3. For two 1-cycles C, C' on X , C is **numerically equivalent** to C' if they have the same intersection number with every Cartier divisor.

Notation 1.4.1. The quotient group of $Z_1(X)$ by numerically equivalence is denoted by $N_1(X)_{\mathbb{Z}}$, and

$$N_1(X)_{\mathbb{Q}} = N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}, \quad N_1(X)_{\mathbb{R}} = N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}.$$

Remark 1.4.1. The intersection pairing

$$N^1(X)_{\mathbb{Z}} \times N_1(X)_{\mathbb{Z}} \rightarrow \mathbb{Z}$$

is by definition non-degenerate.

Definition 1.4.4. The **cone of effective curves** $\text{NE}(X)_{\mathbb{R}} \subseteq N_1(X)_{\mathbb{R}}$ is the cone spanned by non-negative linear combinations of curves, and $\overline{\text{NE}}(X)_{\mathbb{R}}$ is the **cone of pseudo-effective curves**, where $N_1(X)_{\mathbb{R}}$ is endowed with its usual topology as a \mathbb{R} -vector space.

1.4.2. *Nef cones and ample cones.*

Definition 1.4.5. A Cartier divisor on X is **nef (numerically effective)** if it has non-negative intersection with every irreducible curve on X .

Definition 1.4.6. The ample classes in $N^1(X)_{\mathbb{R}}$ forms an open cone $\text{NA}(X)_{\mathbb{R}}$, which is said to be **ample cone**.

Definition 1.4.7. The nef classes in $N^1(X)_{\mathbb{R}}$ forms a closed cone $\text{Nef}(X)_{\mathbb{R}}$, which is said to be **nef cone**.

Theorem 1.4.2. Let X be a projective variety.

- (1) The closure of the ample cone is the nef cone;
- (2) The interior of nef cone is the ample cone.

Proof. See Theorem 1.4.23 in [Laz04]. □

Theorem 1.4.3. Let X be a projective variety.

(1) The pseudo-effective cone is the closed cone dual to the nef cone, that is,

$$\overline{\text{NE}}(X)_{\mathbb{R}} = \{\gamma \in N_1(X)_{\mathbb{R}} \mid D \cdot \gamma \geq 0, \quad \forall D \in \overline{\text{NA}}(X)_{\mathbb{R}}\}.$$

(2)

$$\text{NA}(X)_{\mathbb{R}} = \{\gamma \in N^1(X)_{\mathbb{R}} \mid D \cdot \gamma > 0, \quad \forall D \in \overline{\text{NE}}(X)_{\mathbb{R}} - \{0\}\}.$$

Proof. See Theorem 1.4.28 and Theorem 1.4.29 in [Laz04]. \square

1.5. Asymptotic Riemann-Roch.

Theorem 1.5.1. Let X be a projective variety of dimension n and D be a Cartier divisor on X . Then

$$\chi(X, \mathcal{O}(mD)) = \frac{D^n}{n!} m^n + O(m^{n-1}).$$

More generally, for any coherent sheaf \mathcal{F} on X ,

$$\chi(X, \mathcal{F} \otimes \mathcal{O}_X(mD)) = \text{rank } \mathcal{F} \cdot \frac{D^n}{n!} m^n + O(m^{n-1}).$$

Proof. See Theorem 1.1.24 in [Laz04]. \square

2. TECHNIQUES

2.1. Semistable sheaves. Let X be a normal projective variety of dimension n over an algebraically closed field k of arbitrary characteristic.

Definition 2.1.1. The **average first Chern class** of a torsion-free sheaf \mathcal{E} is

$$\delta(\mathcal{E}) = \frac{c_1(\mathcal{E})}{\text{rank } \mathcal{E}} \in A^1(X)_{\mathbb{Q}}.$$

Definition 2.1.2. For a given $(n-1)$ -tuple $\mathfrak{A} = (H_1, \dots, H_{n-1}) \in \overline{\text{NA}}(X)_{\mathbb{Q}}^{n-1}$, the **average degree** with respect to \mathfrak{A} is the rational number $\delta_{\mathfrak{A}}(\mathcal{E}) = \delta(\mathcal{E})H_1 \dots H_{n-1}$.

Definition 2.1.3. A torsion-free sheaf \mathcal{E} is said to be **semistable** if

$$\delta_{\mathfrak{A}}(\mathcal{F}) \leq \delta_{\mathfrak{A}}(\mathcal{E})$$

for every non-zero subsheaf \mathcal{F} of \mathcal{E} .

Notation 2.1. If $\mathfrak{A} = ([H], \dots, [H])$, we use the terminology H -semistable instead of \mathfrak{A} -semistable.

Theorem 2.1.1 ([HN75]). Let \mathcal{E} be a torsion-free sheaf on X and $\mathfrak{A} \in \overline{\text{NA}}(X)_{\mathbb{Q}}^{n-1}$. Then there exists a unique filtration $\Sigma_{\mathfrak{A}}$,

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \dots \subsetneq \mathcal{E}_s = \mathcal{E},$$

which is called the **Harder-Narasimhan filtration**, such that

- (1) $\text{Gr}_i(\Sigma_{\mathfrak{A}}) = \mathcal{E}_i / \mathcal{E}_{i+1}$ is a torsion-free \mathfrak{A} -semistable sheaf;
- (2) $\delta_{\mathfrak{A}}(\text{Gr}_i(\Sigma_{\mathfrak{A}}))$ is a strictly decreasing function in i .

Sketch. Here we only give a sketch of proof of the existence. Put $\delta_{\mathfrak{A}}^{\max}(\mathcal{E}) := \sup\{\delta_{\mathfrak{A}}(\mathcal{F}) \mid 0 \neq \mathcal{F} \subseteq \mathcal{E} \text{ a coherent subsheaf}\}$. Then we need to prove that

- (1) $\delta_{\mathfrak{A}}^{\max}(\mathcal{E}) < \infty$;
- (2) There exists a saturated subsheaf $\mathcal{F}_1 \subseteq \mathcal{E}$ with maximal slope.

After that, suppose both \mathcal{F}_1 and \mathcal{F}_2 coherent subsheaves of rank r_1 and r_2 with maximal slope. By the following exact sequence

$$0 \rightarrow \mathcal{F}_1 \cap \mathcal{F}_2 \rightarrow \mathcal{F}_1 \oplus \mathcal{F}_2 \rightarrow \mathcal{F}_1 + \mathcal{F}_2 \rightarrow 0,$$

one has

$$\begin{aligned} c_1(\mathcal{F}_1 + \mathcal{F}_2) &= c_1(\mathcal{F}_1) + c_1(\mathcal{F}_2) - c_1(\mathcal{F}_1 \cap \mathcal{F}_2) \\ \text{rank}(\mathcal{F}_1 + \mathcal{F}_2) &= \text{rank}(\mathcal{F}_1) + \text{rank}(\mathcal{F}_2) - \text{rank}(\mathcal{F}_1 \cap \mathcal{F}_2). \end{aligned}$$

Then

$$\begin{aligned} \text{rank}(\mathcal{F}_1 + \mathcal{F}_2)\delta_{\mathfrak{A}}(\mathcal{F}_1 + \mathcal{F}_2) &= r_1\delta_{\mathfrak{A}}(\mathcal{F}_1) + r_2\delta_{\mathfrak{A}}(\mathcal{F}_2) - \text{rank}(\mathcal{F}_1 \cap \mathcal{F}_2)\delta_{\mathfrak{A}}(\mathcal{F}_1 \cap \mathcal{F}_2) \\ &\geq (r_1 + r_2)\delta_{\mathfrak{A}}^{\max}(\mathcal{E}) - \text{rank}(\mathcal{F}_1 \cap \mathcal{F}_2)\delta_{\mathfrak{A}}^{\max}(\mathcal{E}) \\ &= \text{rank}(\mathcal{F}_1 + \mathcal{F}_2)\delta_{\mathfrak{A}}^{\max}(\mathcal{E}). \end{aligned}$$

This shows $\mathcal{F}_1 + \mathcal{F}_2$ also has the maximal slope. By adding all these subsheaves together, this gives the **maximal \mathfrak{A} -destabilizing subsheaf** \mathcal{E}_1 . We repeat above process to obtain the maximal \mathfrak{A} -destabilizing subsheaf of $\mathcal{E}/\mathcal{E}_1$, and consider its preimage to obtain \mathcal{E}_2 , that is, $\mathcal{E}_2/\mathcal{E}_1 = (\mathcal{E}/\mathcal{E}_1)_1$. It remains to show $\delta_{\mathfrak{A}}(\mathcal{E}_1) > \delta_{\mathfrak{A}}(\mathcal{E}_2/\mathcal{E}_1)$. Indeed, otherwise we would have $\delta_{\mathfrak{A}}(\mathcal{E}_1) \leq \delta_{\mathfrak{A}}(\mathcal{E}_2)$, a contradiction. \square

Remark 2.1.1. The maximal \mathfrak{A} -destabilizing subsheaf of \mathcal{E} is characterized by the following properties:

- (1) $\delta_{\mathfrak{A}}(\mathcal{E}_1) \geq \delta_{\mathfrak{A}}(\mathcal{F})$ for every coherent subsheaf \mathcal{F} of \mathcal{E} ;
- (2) If $\delta_{\mathfrak{A}}(\mathcal{E}_1) = \delta_{\mathfrak{A}}(\mathcal{F})$ for $\mathcal{F} \subset \mathcal{E}$, then $\mathcal{F} \subset \mathcal{E}_1$.

Remark 2.1.2. The \mathfrak{A} -semistable filtration of the dual sheaf \mathcal{E}^* is essentially the same as that of \mathcal{E} , with each entry substituted by the duals of the quotient $\mathcal{E}/\mathcal{E}_{s-i}$.

Theorem 2.1.2. Let $\mathcal{E}_1^{\mathfrak{A}} \subset \mathcal{E}$ denote the maximal \mathfrak{A} -destabilizing subsheaf for $\mathfrak{A} \in \overline{\text{NA}}(X)_{\mathbb{Q}}^{n-1}$.

- (1) Let L be a closed affine segment joining $\mathfrak{A} \in \overline{\text{NA}}(X)_{\mathbb{Q}}^{n-1}$ and $\mathfrak{C} \in \overline{\text{NA}}(X)_{\mathbb{R}}^{n-1}$ and $\mathfrak{B} = (1-t)\mathfrak{A} + t\mathfrak{C}$ be a \mathbb{Q} -rational point on L . Then $\delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{B}}) = \delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{A}})$ whenever $0 < t < \epsilon$, where ϵ is a positive constant depends continuously on \mathfrak{C} provided \mathcal{E} and \mathfrak{A} is fixed.
- (2) Let $K \subset \overline{\text{NA}}(X)_{\mathbb{R}}^{n-1}$ be a compact subset and $\mathfrak{A} \in \overline{\text{NA}}(X)_{\mathbb{Q}}^{n-1}$ is away from K . Let $\mathfrak{A} \sharp K$ stands the union of the segments joining \mathfrak{A} and K . Then there exists an open neighborhood $U \subset N^1(X)_{\mathbb{R}}$ of \mathfrak{A} such that $\delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{B}}) = \delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{A}})$ for every $\mathfrak{B} \in U \cap (\mathfrak{A} \sharp K) \cap \overline{\text{NA}}(X)_{\mathbb{Q}}^{n-1}$.
- (3) If $\mathfrak{A} \in \text{NA}(X)_{\mathbb{Q}}^{n-1}$, then there exists an open neighborhood $U \subset \text{NA}(X)_{\mathbb{Q}}^{n-1}$ of \mathfrak{A} such that $\delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{B}}) = \delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{A}})$ for every $\mathfrak{B} \in U$.

Proof. For simplicity, we show the case $n = 2$ only, as the proof is quite similar for higher dimensions, and we denote $\mathfrak{C} = H \in \overline{\text{NA}}(X)_{\mathbb{Q}}$.

(1). At first we prove that $\delta_{\mathfrak{C}}(\mathcal{E}_1^{\mathfrak{B}}) \leq c$ for some constant c depending on \mathcal{E} , and on \mathfrak{C} continuously.

(a) If $\mathcal{E}^*(H)$ is globally generated, that is, there exists a surjective morphism $\mathcal{O}_X^{\oplus N} \rightarrow \mathcal{E}^*(H)$ for some integer N . By taking dual we have an

injective morphism $\mathcal{E} \rightarrow \mathcal{O}_X^{\oplus N}(H)$, and thus

$$\delta_{\mathfrak{C}}(\mathcal{E}_1^{\mathfrak{B}}) \leq c,$$

where c is a constant depending on \mathcal{E} , and on \mathfrak{C} continuously.

- (b) If H is ample, then there exists some integer m such that mH is globally generated, and thus in this case $\delta_{\mathfrak{C}}(\mathcal{E}_1^{\mathfrak{B}}) \leq c$ for some constant c depending on \mathcal{E} , and on \mathfrak{C} continuously.
- (c) If $H \in \overline{\text{NA}}(X)_{\mathbb{Q}}$, the same result still holds, as a nef divisor is a limit of ample divisors.

Furthermore, we put $c' = \delta_{\mathfrak{C}}(\mathcal{E}_1^{\mathfrak{A}})$. By the definition of the maximal destabilizing sheaves, we get

$$\delta_{\mathfrak{B}}(\mathcal{E}_1^{\mathfrak{A}}) \leq \delta_{\mathfrak{B}}(\mathcal{E}_1^{\mathfrak{B}}).$$

As $\delta_{\mathfrak{B}}$ is a linear function in $\mathfrak{B} = (1-t)\mathfrak{A} + t\mathfrak{C}$, this inequality is rewritten as

$$(1-t)\delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{A}}) + t\delta_{\mathfrak{C}}(\mathcal{E}_1^{\mathfrak{A}}) \leq (1-t)\delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{B}}) + t\delta_{\mathfrak{C}}(\mathcal{E}_1^{\mathfrak{B}}).$$

Hence

$$\begin{aligned} \delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{B}}) &\leq \delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{A}}) \leq \delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{B}}) + \frac{t}{1-t}(\delta_{\mathfrak{C}}(\mathcal{E}_1^{\mathfrak{B}}) - \delta_{\mathfrak{C}}(\mathcal{E}_1^{\mathfrak{A}})) \\ &\leq \delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{B}}) + \frac{t}{1-t}(c - c'). \end{aligned}$$

Note that $\delta(\mathcal{E}_1^{\mathfrak{A}}), \delta(\mathcal{E}_1^{\mathfrak{B}}) \in (1/r!)A^1(X)_{\mathbb{Z}}$ and $\mathfrak{A} \in (1/m)N^1(X)_{\mathbb{Z}}$ for some positive integer m . Therefore, if

$$\frac{t}{1-t}(c - c') < \frac{1}{r!m},$$

then $\delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{A}}) = \delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{B}})$.

(2). Let U be the open ball centered at \mathfrak{A} with radius $r = \inf_{\mathfrak{C} \in K} \epsilon(\mathcal{E}, \mathfrak{A}, \mathfrak{C})d(\mathfrak{A}, \mathfrak{C})$, where d stands for Euclidean metric.

(3). Let $K \subset \text{NA}(X)_{\mathbb{R}}^{n-1}$ be a sphere centered at \mathfrak{A} and apply (2). \square

Corollary 2.1.1. Given a compact subset $K \subset \overline{\text{NA}}(X)_{\mathbb{R}}^{n-1}$ and $\mathfrak{A} \in \overline{\text{NA}}(X)_{\mathbb{Q}}^{n-1}$ is away from K , the \mathfrak{B} -semistable filtration is a refinement of \mathfrak{A} -semistable filtration for all $\mathfrak{B} \in (\mathfrak{A} \sharp K)_{\mathbb{Q}}$ sufficiently near \mathfrak{A} .

Proof. By (2) of above theorem, we have $\mathcal{E}_1^{\mathfrak{B}} \subseteq \mathcal{E}_1^{\mathfrak{A}}$ for all $\mathfrak{B} \in (\mathfrak{A} \sharp K)_{\mathbb{Q}}$ sufficiently near \mathfrak{A} . If \mathcal{E} is semistable, it's clear that the \mathfrak{B} -semistable filtration of \mathcal{E} is a refinement of \mathfrak{A} -semistable filtration of \mathcal{E} , and the general case is obtained by repeating above process for each semistable grade $\mathcal{E}_i/\mathcal{E}_{i+1}$. \square

Corollary 2.1.2. Let \mathcal{E} be a torsion-free sheaf on X . Then the function $\delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{A}})$ is a continuous, piecewise multilinear function on $\text{NA}(X)_{\mathbb{Q}}^{n-1}$, and continuous on any \mathbb{Q} -rational segment of $\overline{\text{NA}}(X)_{\mathbb{Q}}^{n-1}$.

Proof. Note that if both \mathfrak{A} and \mathfrak{B} in $\mathrm{NA}(X)_{\mathbb{Q}}^{n-1}$, then there exists some open neighborhood of \mathfrak{A} containing \mathfrak{B} , and there also exists some open neighborhood of \mathfrak{B} containing \mathfrak{A} . By the symmetry we have $\mathcal{E}_1^{\mathfrak{B}} = \mathcal{E}_1^{\mathfrak{A}}$, and thus $\delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{A}})$ is continuous on $\mathrm{NA}(X)_{\mathbb{Q}}^{n-1}$. The same argument shows $\delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{A}})$ is also continuous in any rational segment of $\overline{\mathrm{NA}}(X)_{\mathbb{Q}}^{n-1}$. \square

2.2. A numerical criterion for semistability on curves. Through this section, the ground field k is always an algebraically closed field with characteristic 0 except Lemma 2.2.1, and C is a smooth complete curve.

2.2.1. Projective bundle on curves. Let \mathcal{E} be a locally free sheaf of rank r on C and $\pi: \mathbb{P}(\mathcal{E}) \rightarrow C$ the associated projective bundle with tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

Definition 2.2.1. The **normalized hyperplane class** $\lambda_{\mathcal{E}}$ is the numerical class of $c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) - \pi^* \delta(\mathcal{E}) \in N^1(\mathbb{P}(\mathcal{E}))_{\mathbb{Q}}$.

Remark 2.2.1. The normalized hyperplane class $\lambda_{\mathcal{E}}$ is uniquely determined by two properties:

- (1) $\lambda_{\mathcal{E}}^r = 0$.
- (2) $\lambda_{\mathcal{E}}$ on each fiber is numerically equivalent to the hyperplane.

Proposition 2.2.1. The class of relative anti-canonical divisor $-K_{\mathbb{P}(\mathcal{E})} + \pi^* K_C$ equals $r\lambda_{\mathcal{E}}$.

Proof. It follows from the relative Euler sequence, that is,

$$0 \rightarrow \Omega_{\mathbb{P}(\mathcal{E})/C}^1 \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \otimes \pi^* \mathcal{E}^* \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})} \rightarrow 0.$$

\square

Proposition 2.2.2.

- (1) $\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\ell)) = \mathcal{S}^{\ell} \mathcal{E}$ for $\ell \geq 0$ and $\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\ell)) = 0$ for $\ell < 0$.
- (2) $R^i \pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\ell)) = 0$ for $0 < i < n$.
- (3) $R^n \pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\ell)) = 0$ for $\ell > -n - 1$.

Proof. See Exercise III 8.4 in [Har77]. \square

Proposition 2.2.3. The Néron-Severi group of $\mathbb{P}(\mathcal{E})$ is

$$N^1(\mathbb{P}(\mathcal{E})) = \mathbb{R} \lambda_{\mathcal{E}} \oplus \pi^* N^1(X),$$

and the group of numerically equivalent 1-cycles is

$$N_1(\mathbb{P}(\mathcal{E})) = \lambda_{\mathcal{E}}^{r-2} N^1(\mathbb{P}(\mathcal{E})).$$

Proof. See Proposition V 2.3 in [Har77]. \square

2.2.2. *Criterion.*

Lemma 2.2.1. Let f be a seperable surjective k -morphism of a smooth complete curve C' onto C . Then a locally free sheaf \mathcal{E} is semistable if and only if $f^*\mathcal{E}$ is semistable.

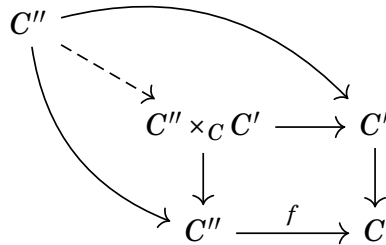
Proof. Firstly let's prove "if" part. Let $\mathcal{G} \subseteq \mathcal{E}$ be a non-zero subsheaf. Then $\delta(f^*\mathcal{G}) \leq \delta(f^*\mathcal{E})$ as $f^*\mathcal{E}$ is semistable, and thus $\delta(\mathcal{G}) \leq \delta(\mathcal{E})$.

Conversely, suppose \mathcal{E} is semistable. Without lose of generality we may assume f is a Galois morphism with Galois group G , which acts on $f^*\mathcal{E}$. If $f^*\mathcal{E}$ is not semistable and \mathcal{F}_1 be the maximal destabilizing subbundle of $f^*\mathcal{E}$. For any $g \in G$, we have $g^*\mathcal{F}_1 = \mathcal{F}_1$ as the maximal destabilizing subsheaf is unique. Hence there exists a subbundle \mathcal{E}_1 of \mathcal{E} such that $f^*\mathcal{E}_1 = \mathcal{F}_1$, and by "if" part \mathcal{E}_1 is semistable. On the other hand, by semistability we have $\mathcal{E}_1 = \mathcal{E}$, and thus $\mathcal{F}_1 = f^*\mathcal{E}$. This completes the proof. \square

Theorem 2.2.1. The following conditions are equivalent:

- (1) \mathcal{E} is semistable;
- (2) $\lambda_{\mathcal{E}}$ is nef;
- (3) $\overline{\text{NA}}(\mathbb{P}(\mathcal{E})) = \mathbb{R}_+ \lambda_{\mathcal{E}} \oplus \mathbb{R}_+ \pi^* d$, where d is a positive generator of $N^1(C)_{\mathbb{Z}} \cong \mathbb{Z}$;
- (4) $\overline{\text{NE}}(\mathbb{P}(\mathcal{E})) = \mathbb{R}_+ \lambda_{\mathcal{E}}^{r-1} \oplus \mathbb{R}_+ \lambda_{\mathcal{E}}^{r-2} \pi^* d$;
- (5) Every effective divisor on $\mathbb{P}(\mathcal{E})$ is nef.

Proof. (1) to (2). If $\lambda_{\mathcal{E}}$ is not nef, then there exists an irreducible curve $C' \subset \mathbb{P}(\mathcal{E})$ with $C' \lambda_{\mathcal{E}} < 0$. It's clear¹ that C' is mapped surjectively onto C . Let C'' be the normalization of C' and $f: C'' \rightarrow C$ be the composition of $C'' \rightarrow C' \rightarrow C$. Then by the base change $f: C'' \rightarrow C$, the multi-section C' becomes a union of cross sections C''_i on the projective bundle $\mathbb{P}(f^*\mathcal{E})$ over C'' , and $C''_i \lambda_{\mathbb{P}(f^*\mathcal{E})}$ is evidently negative since $C' \lambda_{\mathcal{E}} < 0$. For a section $s: C'' \rightarrow C''_i \subset \mathbb{P}(f^*\mathcal{E})$, it gives a line bundle $\mathcal{L} = s^* \mathcal{O}_{\mathbb{P}(f^*\mathcal{E})}(1)$ on C'' , which has degree $C''_i c_1(\mathcal{O}_{\mathbb{P}(f^*\mathcal{E})}(1)) = C''_i \lambda_{f^*\mathcal{E}} + \delta(f^*\mathcal{E}) < \delta(f^*\mathcal{E})$, so that $f^*\mathcal{E}$ is unstable, and thus \mathcal{E} is unstable by Lemma 2.2.1.



¹Otherwise we have $C' \lambda_{\mathcal{E}} > 0$.

(2) to (4). If $\lambda_{\mathcal{E}}^{r-2}(a\lambda_{\mathcal{E}} + b\pi^*d)$ is pseudo-effective and $\lambda_{\mathcal{E}}$ is nef, then

$$b = \lambda_{\mathcal{E}}^{r-1}(a\lambda_{\mathcal{E}} + b\pi^*d) \geq 0.$$

On the other hand, $\lambda_{\mathcal{E}}^{r-1}$ is pseudo-effective since $\lambda_{\mathcal{E}}$ is nef, and thus $a \geq 0$.

The equivalent between (3) and (4) is straightforward since the nef cone is the closed cone dual to the pseudo-effective cone (Theorem 1.4.3).

(3) and (4) to (5). Since $\lambda_{\mathcal{E}}$ is nef, $\lambda_{\mathcal{E}} + \epsilon\pi^*d$ is ample for any positive real number ϵ . Assume $a\lambda_{\mathcal{E}} + b\pi^*d$ is an effective divisor. Then the 1-cycles $(a\lambda_{\mathcal{E}} + b\pi^*d)(\lambda_{\mathcal{E}} + \epsilon\pi^*d)^{r-2}$ is effective, and thus their limit $(a\lambda_{\mathcal{E}} + b\pi^*d)\lambda_{\mathcal{E}}^{r-2}$ is pseudo-effective. Then by (4) one has $a, b \geq 0$, and thus $a\lambda_{\mathcal{E}} + b\pi^*d$ is nef by (3).

(5) to (1). Suppose that \mathcal{E} is unstable and let \mathcal{E}_1 be the maximal destabilizing subbundle. Let α be a rational number with $\delta(\mathcal{E}_1) > \alpha > \delta(\mathcal{E})$. Then by the Riemann-Roch theorem,

$$\begin{aligned} H^0(C, \mathcal{S}^N \mathcal{E}_1(-N\alpha d)) &\subseteq H^0(C, \mathcal{S}^N \mathcal{E}(-N\alpha d)) \\ &\cong H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(N) \otimes \pi^* \mathcal{O}_C(-N\alpha d)) \end{aligned}$$

is non-trivial for sufficiently large N . Then $N\{\lambda_{\mathcal{E}} + (\delta(\mathcal{E}) - \alpha)\pi^*d\}$ is effective but clearly not nef. \square

2.2.3. Semipositive and semistability.

Definition 2.2.2. Let D be a \mathbb{Q} -Cartier divisor on C . A \mathbb{Q} -torsion-free sheaf $\mathcal{F} = \mathcal{E}(D)$ is said to be **ample** or **semipositive** if $\xi + \pi^*D$ is ample or nef, where $\xi = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$.

Definition 2.2.3. A \mathbb{Q} -torsion-free sheaf \mathcal{F} is said to be **negative** or **seminegative** if \mathcal{F}^* is ample or semipositive.

Proposition 2.2.4. The direct sums, tensor products, symmetric products and exterior products of ample (or semipositive) \mathbb{Q} -torsion-free sheaves are all ample (or semipositive).

Theorem 2.2.2. Let \mathcal{E} be a vector bundle on C . Then \mathcal{E} is semistable if and only if $\mathcal{E}(-\delta(\mathcal{E}))$ is semipositive.

Proof. It follows from Theorem 2.2.1. \square

Corollary 2.2.1. Let \mathcal{E} be a vector bundle on C . Then \mathcal{E} is semistable if and only if $\mathcal{E}(-\delta(\mathcal{E}))$ is seminegative.

Proof. It suffices to note that \mathcal{E} is semistable if and only if \mathcal{E}^* is semistable. \square

Corollary 2.2.2.

- (1) The \mathbb{Q} -vector bundle $\mathcal{E}(-D)$ is seminegative if and only if $\deg D \geq \deg \delta(\mathcal{E}_1)$, where \mathcal{E}_1 is the maximal destabilizing subsheaf of \mathcal{E} .
- (2) The \mathbb{Q} -vector bundle $\mathcal{E}(-D)$ is negative if and only if $\deg D > \deg \delta(\mathcal{E}_1)$, where \mathcal{E}_1 is the maximal destabilizing subsheaf of \mathcal{E} .
- (3) The \mathbb{Q} -vector bundle $\mathcal{E}(D)$ is semipositive if and only if $\deg D \geq \deg \delta((\mathcal{E}^*)_1)$.
- (4) The \mathbb{Q} -vector bundle $\mathcal{E}(D)$ is positive if and only if $\deg D > \deg \delta((\mathcal{E}^*)_1)$.

Proof. For simplicity we only prove the first statement, and the proof is quite similar for others.

Let $\mathcal{E}_1 \subset \dots \subset \mathcal{E}_s = \mathcal{E}$ be the semistable filtration of \mathcal{E} . Since $\mathcal{G}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable and $\deg \delta(\mathcal{G}_i)$ is decreasing in i , one has $\mathcal{G}_i(-\delta(\mathcal{E}_1))$ is seminegative for all i , and thus $\mathcal{E}(-\delta(\mathcal{E}_1))$ is seminegative. If $\deg D \geq \deg \delta(\mathcal{E}_1)$, then $\mathcal{E}(-D)$ is also seminegative.

Conversely, if $\deg D$ is smaller than $\deg \delta(\mathcal{E}_1)$ for a \mathbb{Q} -divisor D , then $\mathcal{E}(-D)$, containing an ample \mathbb{Q} -vector bundle $\mathcal{E}_1(-D)$, is never seminegative. \square

Corollary 2.2.3. A semistable vector bundle \mathcal{E} on C is ample (resp. semipositive, seminegative, negative) if and only if its degree is positive (resp. semipositive, seminegative, negative).

Proof. Take $D = 0$ in Corollary 2.2.2. \square

Corollary 2.2.4. Let \mathcal{E} and \mathcal{F} be semistable bundles on C . Then $\mathcal{E} \otimes \mathcal{F}$ and $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ are also semistable.

Proof. It follows from the semipositive bundle tensor with semipositive bundle is still semipositive. \square

Corollary 2.2.5. Let \mathcal{E} and \mathcal{F} be two vector bundles. Then $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ is negative if and only if $\deg \delta(\mathcal{F}_1) + \deg \delta((\mathcal{E}^*)_1) < 0$. As a consequence, $\mathcal{H}om(\mathcal{E}_1, \mathcal{E}/\mathcal{E}_1)$ is negative.

Proof. For the first part, note that $\mathcal{H}om(\mathcal{E}, \mathcal{F}) = \mathcal{E}^* \otimes \mathcal{F}$ and take $D = 0$ in Corollary 2.2.2. For the half part, it suffices to note $(\mathcal{E}/\mathcal{E}_1)_1 = \mathcal{E}_2/\mathcal{E}_1$. \square

Corollary 2.2.6. A vector bundle is semistable if and only if $\mathcal{S}^n \mathcal{E}$ is semistable, where $n \geq 2$.

Proposition 2.2.5. Let \mathcal{E} be a vector bundle on C . The following conditions are equivalent:

- (1) \mathcal{E} is semistable;
- (2) $\mathcal{E}(-D)$ is negative with D is a \mathbb{Q} -divisor of degree $\delta(\mathcal{E}) + (1/2r!)$.

Proof. The implication (1) to (2) follows from Corollary 2.2.1.

Conversely, assume (2) and let \mathcal{E}_1 be the maximal destabilizing subsheaf. Then by Corollary 2.2.2 we have $\mathcal{E}(-D)$ is negative if and only if $\deg D > \deg \delta(\mathcal{E}_1)$ so that

$$\delta(\mathcal{E}) \leq \delta(\mathcal{E}_1) < \delta(\mathcal{E}) + \frac{1}{2r!}.$$

On the other hand, both $\deg \delta(\mathcal{E}_1)$ and $\deg \delta(\mathcal{E})$ sit in $(1/r!)\mathbb{Z}$. Hence we have $\deg \delta(\mathcal{E}_1) = \deg \delta(\mathcal{E})$, and thus $\mathcal{E}_1 \cong \mathcal{E}$. \square

Corollary 2.2.7. Let $\mathcal{C} \rightarrow T$ be a proper smooth family of irreducible curves, where \mathcal{C} and T are k -varieties. Let \mathcal{E} be a vector bundle on \mathcal{C} . Then the set

$$S(T) = \{t \in T \mid \mathcal{E} \text{ is semistable on } C_t\}$$

is a Zariski open subset of T .

2.3. Mumford-Mehta-Ramanathan's theorem.

Theorem 2.3.1 ([MR82]). Let X be a complex normal projective variety of dimension n and \mathcal{E} be a torsion-free sheaf. Let H_1, \dots, H_{n-1} be ample Cartier divisors. Then for sufficiently large integers m_1, \dots, m_{n-1} , the maximal destabilizing subsheaf \mathcal{F} of $\mathcal{E}|_C$ extends to a saturated subsheaf² of \mathcal{E} on X if C is a general complete intersection curve of $|m_i H_i|$'s.

2.4. The Bogomolov-Gieseker inequality for semistable sheaves. In this section, the ground field k is always algebraically closed of characteristic zero.

Lemma 2.4.1. Let X be a normal projective variety of dimension n and $\mathfrak{A} \in \text{NA}(X)_{\mathbb{R}}^{n-1}$. Let \mathcal{E} be an \mathfrak{A} -semistable torsion-free sheaf on X , with its first Chern class being a \mathbb{Q} -Cartier divisor. Let D be a non-zero effective Cartier divisor on X . Then

$$H^0(X, \mathcal{S}^{rt} \mathcal{E}(-tc_1(\mathcal{E}) - D)) = 0$$

for every positive integer t such that $tc_1(\mathcal{E})$ is an integral Cartier divisor.

Proof. For a generic curve C in X , by Theorem 2.3.1 one has $\mathcal{S}^{rt} \mathcal{E}(-tc_1(\mathcal{E}))|_C$ is semistable since $\mathcal{S}^{rt} \mathcal{E}$ is semistable. If

$$H^0(X, \mathcal{S}^{rt} \mathcal{E}(-tc_1(\mathcal{E}) - D)) \neq 0,$$

then there is an inclusion $\mathcal{O}_C(D) \rightarrow \mathcal{S}^{rt} \mathcal{E}(-tc_1(\mathcal{E}))|_C$. But $\deg \delta(\mathcal{O}_C(D)) > 0$ since D is effective and $\mathcal{S}^{rt} \mathcal{E}(-tc_1(\mathcal{E})) = \mathcal{S}^{rt} \{\mathcal{E}(-\delta(\mathcal{E}))\}$ has degree zero on every curve. This contradicts to $\mathcal{S}^{rt} \mathcal{E}(-tc_1(\mathcal{E}))|_C$ is semistable. \square

²Such an extension of \mathcal{F} is necessarily the maximal (H_1, \dots, H_{n-1}) -destabilizing subsheaf of \mathcal{E} and hence unique.

Corollary 2.4.1. Let things be as Lemma 2.4.1 and L be a fixed Cartier divisor. Then $h^0(X, \mathcal{S}^{rt} \mathcal{E}(-tc_1(\mathcal{E}) + L))$ is bounded by a polynomial of degree $r - 1$ in t .

Proof. For simplicity of the notation, put $\mathcal{F}^t = \mathcal{S}^{rt} \mathcal{E}(-tc_1(\mathcal{E}))$. The proof is by induction on the dimension n of X . If $n = 1$, let D be a divisor of degree d such that $D - L$ is effective. Then there is a natural exact sequence

$$H^0(X, \mathcal{F}^t(L - D)) \rightarrow H^0(X, \mathcal{F}^t(L)) \rightarrow H^0(D, \mathcal{F}^t(L))$$

of which the first term vanishes by Lemma 2.4.1, where the last term is a k -vector space of dimension $d \binom{rt+r-1}{rt} = d \binom{rt+r-1}{r-1}$. This completes the proof of $n = 1$.

For $n \geq 2$, let $\mathfrak{A} \equiv (H_1, \dots, H_{n-1}) \pmod{(\mathbb{Q}_+^\times)^{n-1}}$, where H_i is integral and ample. Let Y be a general hyperplane section in $|mH_i|$ for sufficiently large m such that $\mathcal{E}|_Y$ is (H_1, \dots, H_{n-2}) -semistable on Y and $Y - L$ is ample. Note that such a number m , though possibly very large, is independent of t . Consider the exact sequence

$$H^0(X, \mathcal{F}^t(L - Y)) \rightarrow H^0(X, \mathcal{F}^t(L)) \rightarrow H^0(Y, \mathcal{F}^t(L)).$$

The first term vanishes by Lemma 2.4.1 and the dimension of the last term is bounded by a polynomial of degree $r - 1$ by the induction hypothesis. This completes the proof. \square

Theorem 2.4.1 (The Bogomolov-Gieseker inequality). Let S be a smooth projective surface over k . If \mathcal{E} is an H -semistable torsion-free sheaf of rank r on S , where H is an ample divisor, then

$$(r - 1)c_1^2(\mathcal{E}) \leq 2rc_2(\mathcal{E}).$$

Proof. From Corollary 2.4.1, it follows that neither $h^0(S, \mathcal{S}^{rt} \mathcal{E}(-tc_1(\mathcal{E})))$ nor $h^2(S, \mathcal{S}^{rt} \mathcal{E}(-tc_1(\mathcal{E}))) = h^0(S, \mathcal{S}^{rt} \mathcal{E}^*(-tc_1(\mathcal{E}^*)) + K_S)$ grows like t^{r+1} . Hence we obtain the inequality

$$\chi(S, \mathcal{S}^{rt} \mathcal{E}(-tc_1(\mathcal{E}))) \leq \text{polynomial of degree } r \text{ in } t.$$

On the other hand, we have

$$\begin{aligned} \chi(S, \mathcal{S}^{rt} \mathcal{E}(-tc_1(\mathcal{E}))) &\stackrel{(1)}{=} \chi(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(rt) \otimes \pi^* \mathcal{O}_S(-tc_1(\mathcal{E}))) \\ &\stackrel{(2)}{=} \frac{t^{r+1}}{(r+1)!} \{rc_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) - \pi^* c_1(\mathcal{E})\}^{r+1} + O(t^r) \\ &\stackrel{(3)}{=} \frac{(rt)^{r+1}}{(r+1)!} \left\{ -c_2(\mathcal{E}) + \frac{r-1}{2r} c_1^2(\mathcal{E}) \right\} + O(t^r), \end{aligned}$$

where

- (1) holds from the projection formula;
 (2) holds from by the asymptotic Riemann-Roch theorem (Theorem 1.5.1);
 (3) holds from the following standard computation

$$\begin{aligned}
 \left\{ \xi - \frac{\pi^* c_1(\mathcal{E})}{r} \right\}^{r+1} &= \left\{ \xi^r - \pi^* c_1(\mathcal{E}) \xi^{r-1} + \frac{r-1}{2r} \pi^* c_1^2(\mathcal{E}) \xi^{r-2} \right\} \left\{ \xi - \frac{\pi^* c_1(\mathcal{E})}{r} \right\} \\
 &= \left\{ \pi^* c_2(\mathcal{E}) \xi^{r-2} + \frac{r-1}{2r} \pi^* c_1^2(\mathcal{E}) \xi^{r-2} \right\} \left\{ \xi - \frac{\pi^* c_1(\mathcal{E})}{r} \right\} \\
 &= \left\{ -c_2(\mathcal{E}) + \frac{r-1}{2r} c_1^2(\mathcal{E}) \right\},
 \end{aligned}$$

where $\xi = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$.

This completes the proof. \square

Corollary 2.4.2. Let \mathcal{E} be a locally free sheaf of rank r on a smooth projective surface S . Let L be an ample integral divisor on S such that $\mathcal{E}(-\delta(\mathcal{E}) + L)$ is ample and $\mathcal{E}(-\delta(\mathcal{E}) - L)$ is negative (as \mathbb{Q} -vector bundles). Assume the inequality $2rc_2(\mathcal{E}) < (r-1)c_1^2(\mathcal{E})$ and put

$$\alpha = \frac{(r-1)c_1^2(\mathcal{E}) - 2rc_2(\mathcal{E})}{6r^2(r+1)L^2} \in \mathbb{Q}.$$

Then

- (1) Either $\mathcal{S}^t \mathcal{E}(-t\delta(\mathcal{E}))$ or $\mathcal{S}^t \mathcal{E}^*(-t\delta(\mathcal{E}^*))$ contains the ample line bundle $\mathcal{O}_S(t\alpha L)$, where t is any very large integer such that $t\delta(\mathcal{E})$ and $t\alpha$ are integral.
 (2) For any nef divisor D , the maximal D -destabilizing subsheaf \mathcal{E}_1^D has normalized degree not less than

$$\delta_D(\mathcal{E}) + \frac{\alpha LD}{r}$$

with respect to D .

Proof. (1). For simplicity, we put $\mathcal{F} = \mathcal{E}(-\delta(\mathcal{E}))$. By the standard computation we have

$$\chi(S, \mathcal{S}^t \mathcal{F}) = \frac{t^{r+1}}{(r+1)!} \left\{ -c_2(\mathcal{E}) + \frac{r-1}{2r} c_1^2(\mathcal{E}) \right\} + O(t^r).$$

Hence, by the Serre duality, we infer that $h^0(S, \mathcal{S}^t \mathcal{F})$ or $h^0(S, \mathcal{S}^t \mathcal{F}^*)$ is

$$\geq \frac{t^{r+1}}{4r(r+1)!} \left\{ (r-1)c_1^2(\mathcal{E}) - 2rc_2(\mathcal{E}) \right\} + O(t^r).$$

Assume the first case and consider the following natural exact sequences

$$\begin{aligned} 0 \rightarrow H^0(S, \mathcal{S}^t \mathcal{F}(-t\alpha L)) &\rightarrow H^0(S, \mathcal{S}^t \mathcal{F}) \rightarrow H^0(C, \mathcal{S}^t \mathcal{F}), \\ 0 \rightarrow H^0(C, \mathcal{S}^t \mathcal{F}(-tL)) &\rightarrow H^0(C, \mathcal{S}^t \mathcal{F}) \rightarrow H^0(D, \mathcal{S}^t \mathcal{F}), \end{aligned}$$

where C is a general curve linearly equivalent to $t\alpha L$ and D is a 0-cycle of degree $t^2\alpha L^2$. The first term of the second sequence vanishes as $\mathcal{S}^t \mathcal{F}(-tL) = \mathcal{S}^t\{\mathcal{F}(-L)\}$ is negative. Hence $h^0(C, \mathcal{S}^t \mathcal{F})$ is bounded by

$$\begin{aligned} t^2\alpha(\text{rank } \mathcal{S}^t \mathcal{F})L^2 &\equiv \frac{\alpha t^{r+1}}{(r-1)!}L^2 \\ &\equiv \frac{t^{r+1}}{6(r+1)!r} \{(r-1)c_1^2(\mathcal{E}) - 2rc_2(\mathcal{E})\} \pmod{O(t^r)}. \end{aligned}$$

This shows $H^0(S, \mathcal{S}^t \mathcal{F}(-t\alpha L))$ is non-zero whenever t is very large in view of the first exact sequence, and thus such a non-zero global section gives the inclusion $\mathcal{O}_S(t\alpha L) \hookrightarrow \mathcal{S}^t \mathcal{F}$. Similarly, the second case will yield $H^0(S, \mathcal{S}^t \mathcal{F}^*(-t\alpha L)) \neq 0$.

(2). It suffices to consider the following cases:

(a) If $\mathcal{S}^t \mathcal{F}$ contains $\mathcal{O}_S(t\alpha L)$, then

$$\delta_D(\mathcal{E}_1^D) - \delta_D(\mathcal{E}) \geq \alpha LD.$$

(b) If $\mathcal{S}^t \mathcal{F}^*$ contains $\mathcal{O}_S(t\alpha L)$, then

$$\delta_D(\mathcal{E}_1^D) - \delta_D(\mathcal{E}) \geq \frac{1}{r} \left\{ \delta_D((\mathcal{E}^*)^D) - \delta_D(\mathcal{E}^*) \right\} \geq \frac{\alpha LD}{r}.$$

This completes the proof. \square

Corollary 2.4.3. Let \mathcal{E} be a torsion-free sheaf of rank r on a normal projective variety X of dimension n and H_1, \dots, H_{n-2} be ample Cartier divisors. Let D be a nef Cartier divisor on X . Assume that $H_1 \dots H_{n-2} D$ is not numerically trivial. If \mathcal{E} is (H_1, \dots, H_{n-2}, D) -semistable, then

$$(r-1)c_1^2(\mathcal{E})H_1 \dots H_{n-2} \leq 2rc_2(\mathcal{E})H_1 \dots H_{n-2}.$$

Proof. Suppose the contrary. We may assume that \mathcal{E} is a vector bundle in codimension 2 by taking double dual. Fix an ample divisor H_0 such that $\mathcal{E}(-\delta(\mathcal{E})+H_0)$ and $\mathcal{E}^*(\delta(\mathcal{E})+H_0)$ are both ample. Let H be an arbitrary ample divisor. Then by Theorem 2.3.1 there exist positive integer m_1, \dots, m_{n-2} depending on H such that $H|_S$ -semistable filtration of $\mathcal{E}|_S$ coincides with the restriction of (H_1, \dots, H_{n-2}, H) -semistable filtration of \mathcal{E} to a generic complete intersection surface $S = (m_1 H_1) \dots (m_{n-2} H_{n-2})$.

By Corollary 2.4.3, we have

$$\begin{aligned} \delta(\mathcal{E}_1^{(\mathfrak{B}, H)})SH - \delta(\mathcal{E}^{(\mathfrak{B}, H)})SH &= \delta_H((\mathcal{E}|_S)_1^H) - \delta_H(\mathcal{E}|_S) \\ &\geq c \{(r-1)c_1^2(\mathcal{E}|_S) - 2rc_2(\mathcal{E}|_S)\} (H, H_0)_S / (H_0^2)_S \\ &= c \{[(r-1)c_1^2(\mathcal{E}) - 2rc_2(\mathcal{E})]S\} HH_0S / H_0^2S, \end{aligned}$$

where $\mathfrak{B} = (H_1, \dots, H_{n-2})$ and c is a positive constant. Therefore, by dividing out both sides by $m_1 \dots m_{n-2}$, we obtain the inequality

$$\delta_{(\mathfrak{B}, H)}(\mathcal{E}_1^{(\mathfrak{B}, H)}) \geq \delta_{(\mathfrak{B}, H)}(\mathcal{E}) + cHH_0H_1 \dots H_{n-2}.$$

By the continuity of the function $\delta_{\mathfrak{A}}(\mathcal{E}_1^{\mathfrak{A}})$ on a segment joining (\mathfrak{B}, D) and (\mathfrak{B}, H) , we have

$$\delta_{(\mathfrak{B}, D)}(\mathcal{E}_1^{(\mathfrak{B}, D)}) \geq \delta_{(\mathfrak{B}, H)}(\mathcal{E}) + cDH_0H_1 \dots H_{n-2} > \delta_{(\mathfrak{B}, D)}(\mathcal{E}),$$

a contradiction. \square

Corollary 2.4.4. Let \mathcal{E} be a torsion-free sheaf of rank r on a normal projective variety X of dimension n and H_1, \dots, H_{n-2} be ample Cartier divisors. If

$$\{(r-1)c_1^2(\mathcal{E}) - 2rc_2(\mathcal{E})\}H_1 \dots H_{n-2} > 0,$$

then \mathcal{E} is (H_1, \dots, H_{n-2}, D) -unstable for any non-zero nef divisor D .

2.5. Semistability in positive and mixed characteristic.

2.5.1. Semistability in positive characteristic. Let C be a smooth complete curve over an algebraically closed field k of characteristic $p > 0$.

Definition 2.5.1. A vector bundle \mathcal{E} on C is said to be **strongly semistable** if, for every positive integer s , $(F^s)^*\mathcal{E}$ is semistable.

Remark 2.5.1. If C is an elliptic curve, it's known that every semistable bundle is strongly semistable, but that is not the case when $g(C) \geq 2$.

Proposition 2.5.1. If \mathcal{E} is strongly semistable on C , then $f^*\mathcal{E}$ is semistable for any surjective k -morphism $f: C' \rightarrow C$.

Proof. Let C'' be a smooth model of the separable closure of C . The natural projection $C' \rightarrow C''$ is pure inseparable and hence $C' = F^{-s}C''$ for some non-negative integer s (Proposition IV 2.5 of [Har77]). Thus we get the commutative diagram

$$\begin{array}{ccc} C' & \xrightarrow{F^s} & C'' \\ g \downarrow & & \downarrow h \\ F^{-s}C & \xrightarrow{F^s} & C \end{array}$$

Since \mathcal{E} is strongly semistable, we have $(F^s)^*$ is semistable on $F^{-s}C$, and thus $f^*\mathcal{E} = g^*(F^s)^*\mathcal{E}$ is also semistable by Lemma 2.2.1 as g is seperable. \square

Remark 2.5.2. The Theorem 2.2.1 and its corollaries still hold in positive characteristic if the “semistability” is subsituted by “strong semistability”.

2.5.2. Semistability in mixed characteristic. Let X be a smooth projective variety over a noetherian integral domain R of characteristic zero and \mathcal{E} be a torsion-free sheaf on X . Fix $\mathfrak{A} \in \overline{\text{NA}}(X/R)_{\mathbb{Q}}^{n-1}$, where n is the relative dimension of X . Then the set of geometric points $t \in \text{Spec} R$ such that \mathcal{E}_t/X_t is \mathfrak{A} -semistable forms an open subset.

On the contrary, we know very little about the strong semistability of the reductions of a semistable sheaves.

Question 2.5.1. Let C be an irreducible smooth projective curve over a noetherian integral domain R of characteristic zero. Assume that a locally free sheaf \mathcal{E} on C is \mathfrak{A} -semistable on the generic fibre C_* . Let S be the set of primes of positive characteristic on $\text{Spec} R$ such that \mathcal{E} is strongly semistable. Is S a dense subset of $\text{Spec} R$?

2.6. Generic semipositive theorem for cotangent bundle. From now on, all varieties are defined over an algebraically closed field k of characteristic 0. Let X be a normal projective variety of dimension n .

Definition 2.6.1. Let $\mathfrak{B} \in \overline{\text{NA}}(X)_{\mathbb{Q}}^{n-2}$.

- (1) A torsion-free sheaf \mathcal{E} on X is said to be **generically \mathfrak{B} -seminegative** if, for every numerically effective \mathbb{Q} -Cartier divisor D on X , its maximal (\mathfrak{B}, D) -destabilizing subsheaf \mathcal{E}_1 satisfies $\delta_{(\mathfrak{B}, D)}(\mathcal{E}_1) < 0$.
- (2) A torsion-free sheaf \mathcal{E} on X is said to be **generically \mathfrak{B} -semipositive** if \mathcal{E}^* is generically \mathfrak{B} -seminegative.

Lemma 2.6.1. Let \mathcal{E} be a torsion-free sheaf on X and

$$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_s = \mathcal{E}$$

be the (\mathfrak{B}, D) -semistable filtration of \mathcal{E} and put $\alpha_i = \delta_{(\mathfrak{B}, D)}(\mathcal{E}_i/\mathcal{E}_{i-1})$. Then $\alpha_1 > \cdots > \alpha_s \geq 0$ for every $D \in \overline{\text{NA}}(X)_{\mathbb{Q}}$ if \mathcal{E} is generically \mathfrak{B} -semipositive.

Proof. It follows from the definition. \square

Theorem 2.6.1. Let $\mathfrak{B} = (H_1, \dots, H_{n-2}) \in \text{NA}(X)_{\mathbb{Q}}^{n-2}$ and \mathcal{E} be a generically \mathfrak{B} -semipositive torsion-free sheaf on X . Then

$$c_2(\mathcal{E})H_1 \dots H_{n-2} \geq 0$$

holds.

Theorem 2.6.2. Let $\mathfrak{B} = (H_1, \dots, H_{n-2}) \in \mathrm{NA}(X)_{\mathbb{Q}}^{n-2}$. Then the torsion-free sheaf $\rho_* \Omega_{X'}^1$ is generically \mathfrak{B} -semipositive unless X is uniruled, where $\rho: X' \rightarrow X$ denotes an arbitrary resolution.

3. RESULTS

3.1. Semipositivity of $3c_2 - c_1^2$.

Proposition 3.1.1. Let X be a non-uniruled, normal projective variety of dimension n with \mathbb{Q} -Cartier canonical divisor K_X which is nef. Let $\mathfrak{B} \in \text{NA}(X)_{\mathbb{Q}}^{n-2}$ such that $K_X^2|\mathfrak{B}|$ is positive. Then

$$\{3c_2(\mathcal{E}) - c_1(\mathcal{E})^2\}|\mathfrak{B}| \geq 0,$$

where $\mathcal{E} = \rho_*\Omega_{X'}^1$, and $\rho: X' \rightarrow X$ is an arbitrary resolution.

3.2. Non-negativity of the Kodaira dimension of minimal threefolds.

3.2.1. The Gorenstein case.

Theorem 3.2.1. Let X be a normal projective Gorenstein threefold with only canonical singularities (X is Gorenstein if and only if K_X is a Cartier divisor). Assume K_X is nef. Then the Euler characteristic $\chi(X, \mathcal{O}_X)$ is non-negative. In particular, either $h^0(X, \mathcal{O}_X(K_X))$ or $h^1(X, \mathcal{O}_X)$ is non-zero, and thus $\kappa(X) \geq 0$.

3.2.2. The K_X^2 is numerically non-trivial case.

Theorem 3.2.2. Let X be a normal projective Gorenstein threefold with only isolated singularities. Assume the \mathbb{Q} -Cartier divisor K_X is nef and K_X^2 is numerically non-trivial. Then $\kappa(X) \geq 0$.

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