Curvatures of Left-invariant Metrics on Lie Groups

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In Riemannian geometry, it's natrual to ask the following questions:

- Given a smooth manifold M, does there exist a metric on M with certain curvature properties? For example, Hopf's conjecture.
- Conversely, given certain curvature properties, does there exist obstruction for manifolds? For example, Myers' theorem, Cartan-Hadamard theorem and so on.

- In this talk, we will consider above questions in the category of Lie groups with left-invariant metrics, and the main reference is [Mil76].

But why left-invariant metrics?

- It's easy to compute: The left-invariant metric on Lie group is the same thing as an inner product on its Lie algebra $\mathfrak{g}$, and it turns out the curvature information is encoded in the structure of Lie algebra.
- It contains lots of examples and unknown questions: Not every Lie group admits bi-invariant metrics, but every Lie group admits left-invariant metrics, and there are still many questions about left-invariant are unknown.

- Along the slides, we always assume G is an n-dimensional real Lie group, and $\mathfrak{g}$ is the associated Lie algebra, consisting of all left-invariant vector fields.
- Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathfrak{g}$. Then the structure constants α_{ijk} is defined by

$$[e_i, e_j] = \sum_{k=1}^{n} \alpha_{ijk} e_k.$$

In other words, $\alpha_{ijk} = \langle [e_i, e_j], e_k \rangle$. It's worth mentioning that the structure constants depends on the choice of basis, but the Lie algebra structure doesn't.

Lemma

Let G be a Lie group equipped with a left-invariant metric. Then with structure constants α_{ijk} as above, the sectional curvature $\kappa(e_1, e_2)$ is given by

$$\kappa(e_1, e_2) = \sum_k \Big\{ \frac{1}{2}\alpha_{12k}(-\alpha_{12k} + \alpha_{2k1} + \alpha_{k12}) - \frac{1}{4}(\alpha_{12k} - \alpha_{2k1} + \alpha_{k12})(\alpha_{12k} + \alpha_{2k1} - \alpha_{k12}) - \alpha_{k11}\alpha_{k22} \Big\}$$

Proof.

See Proof 52 in Appendix. □

- This explicit expression shows that the curvature can be computed completely from information about Lie algebra, together with its metric.
- Furthermore, the curvature depends continuously on the structure constants α_{ijk} and vanishes whenever they vanish.

Lemma

If the linear transformation $\mathrm{ad}(u)$ *is skew-symmetric, then*

$$\kappa(u, v) \geq 0$$

for all v*, where equality holds if and only if* u *is orthogonal to* $[v, \mathfrak{g}]$.

Proof.

Without lose of generality, we may assume u, v are orthonormal and $e_1, \dots, e_n$ is an orthonormal basis with $e_1 = u, e_2 = v$. Note that

$$\mathrm{ad}(u)e_i = [u, e_i] = \sum_k \alpha_{1ij} e_j.$$

Then the statement of $\mathrm{ad}(u)$ is skew-symmetric means that α_{1ij} is skew in the last two indices. This show $\kappa(u, v) = \sum_k (\alpha_{2k1})^2/4$, and the equality holds if and only if all $\alpha_{2k1} = 0$. □

- The following lemma shows the "skew-symmetric" is natural, since it comes from bi-invariant metric on connected Lie groups.

Lemma

A left-invariant metric on a connected Lie group is also right-invariant if and only if $\mathrm{ad}(u)$ *is skew-symmetric for all* $u \in \mathfrak{g}$.

Lemma

A connected Lie group admits a bi-invariant metric if and only if it's isomorphic to a product of a compact group and an abelian group.

Corollary

Every compact Lie group admits a left-invariant (and in fact a bi-invariant) metric with non-negative sectional curvature.

- Conversely, there is no satisfied description for Lie groups which possesse a left-invariant metric with non-negative sectional curvature. However, if we sharpen the inequality and require positive sectional curvature, then Wallach shown in [Wal72] examples are scarce indeed.

Theorem

SU(2) *is the only simply-connected Lie group admits a left-invariant metric with positive sectional curvature.*

- On the other hand, it's natural to ask on which Lie groups it admits a left-invariant metric with respect to which it's flat, that is all sectional curvatures vanish. A simple example is that if the Lie algebra $\mathfrak{g}$ is abelian. In fact, we have the following result.

Theorem

A Lie group with left-invariant metric is flat if and only if its Lie algebra splits as an orthogonal direct sum $\mathfrak{b} \oplus \mathfrak{u}$, where $\mathfrak{b}$ is an abelian subalgebra, $\mathfrak{u}$ is an abelian ideal, and $\mathrm{ad}(b)$ is skew-symmetric for every $b \in \mathfrak{b}$.

- Those with non-positive sectional curvature have been classified by Azencott and Wilson in [AW76]. Since the statements are complicated, we just give a qualitive result as follows.

Theorem

If a connected Lie group G has a left-invariant metric with non-positive sectional curvatures, then it's solvable.
If G is unimodular, then any left-invariant metric with non-positive sectional curvatures must actually be flat.

Example

Suppose the Lie algebra $\mathfrak{g}$ with $\dim \mathfrak{g} \geq 2$ has the property that the bracket $[x, y]$ is always equal to a linear combination of x and y. Then in fact one has

$$[x, y] = \ell(x)y - \ell(y)x,$$

where ℓ is a well-defined linear mapping from $\mathfrak{g}$ to the real number. Choosing any positive definite metric, the sectional curvatures are constant

$$K = -\|\ell\|^2.$$

Thus, in the non-abelian case $\ell \neq 0$, every possible metric on $\mathfrak{g}$ has constant negative sectional curvature.

Lemma

If the linear transformations $\mathrm{ad}(x)$ *is skew-symmetric, then* $\mathrm{Ric}(x) \geq 0$, *where the equality holds if and only if* x *is orthogonal to the commutator ideal* $[\mathfrak{g}, \mathfrak{g}]$.

Proof.

This follows immediately from Lemma 2. $\square$

Corollary

If the linear transformations $\mathrm{ad}(x)$ *is skew-symmetric and* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, *then Ricci curvature is positive.*

- The criterion for positive Ricci curvature is classical and elegant.

Theorem

A connected Lie group G admits a left-invariant metric with all positive Ricci curvature if and only if it's compact with finite fundamental group.

Proof.

In one direction this follows from the theorem of Myers which asserts that any complete Riemannian manifold with positive Ricci curvature is compact with finite fundamental group.

Continuation.

Conversely, if G is compact we can choose a bi-invariant metric, so that each $\mathrm{ad}(x)$ is skew-symmetric. If G has finite fundamental group, then its universal covering $\widetilde{G}$ is also compact.
Here we claim $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. For otherwise, $\mathfrak{g}$ admits a non-trivial abelianization which is an abelian Lie algebra, and thus there would exist a non-trivial Lie algebra holds from $\mathfrak{g}$ to the abelian Lie algebra $\mathbb{R}$.
Since $\widetilde{G}$ is simply-connected, this would induce a non-trivial homomorphism from $\widetilde{G}$ to the additive Lie group $\mathbb{R}$, but any non-trivial subgroup of $\mathbb{R}$ is non-compact, contradicting the hypothesis that $\widetilde{G}$ is compact. Now, using Lemma 10, it follows that G has positive Ricci curvature. □

Lemma

If u is orthogonal to the commutator ideal $[\mathfrak{g}, \mathfrak{g}]$, then $\mathrm{Ric}(u) \leq 0$, where the equality holds if and only if $\mathrm{ad}(u)$ is skew-symmetric.

Lemma

If a connected Lie group G has a left-invariant metric with non-negative Ricci curvature, then it's unimodular.

Proof.

Suppose on contrary that G were not unimodular. Then the unimodular kernel $\mathfrak{u} = \{x \in \mathfrak{g} \mid \mathrm{tr}\,\mathrm{ad}(x) = 0\}$, which is an ideal containing $[\mathfrak{g}, \mathfrak{g}]$, doesn't equal to $\mathfrak{g}$. Choosing a unit vector b orthogonal to $\mathfrak{u}$, we would have $\mathrm{tr}\,\mathrm{ad}(b) \neq 0$. Hence $\mathrm{ad}(b)$ could not be skew-symmetric, and it would follow by Lemma 13 that $\mathrm{Ric}(b) < 0$, a contradiction. □

- In general, there is no obstruction for the existence of negative Ricci curvature for manifolds with dimension ≥ 3. See [GY86] and [Loh94].
- But for left-invariant metrics on Lie group, we will see later that simple group $\mathrm{SL}(2, \mathbb{R})$ and the unimodular solvable group $E(1, 1)$ both admit non-flat left-invariant metrics with non-positive Ricci curvature. It seems unlikely that any higher dimensional simple group admits such a metric.

Theorem

If the Lie algebra of G contains linearly independent vectors x, y, z such that

$$[x, y] = z,$$

then there exists a left-invariant metric such that $\mathrm{Ric}(x) < 0$ *and* $\mathrm{Ric}(z) > 0$.

Proof.

Choose a fixed basis $b_1, \dots, b_n$ with $b_1 = x, b_2 = y, b_3 = z$. For any real number $\varepsilon > 0$, consider an auxiliary basis $e_1, \dots, e_n$ defined by $e_1 = \varepsilon b_1, e_2 = \varepsilon b_2, e_i = \varepsilon^2 b_i$ for $i \geq 3$. Define a left-invariant metric by requiring that $e_1, \dots, e_n$ should be orthonormal. Let $\mathfrak{g}_\varepsilon$ denote the Lie algebra $\mathfrak{g}$ equipped with this particular metric and particular orthonormal basis.

Continuation.

Setting $[e_i, e_j] = \sum \alpha_{ijk} e_k$, the structure constants α_{ijk} are clearly functions of ε. Now consider the limit $\varepsilon \to 0$. Then each α_{ijk} tends to a well-defined limit, and thus we obtain a limit Lie algebra $\mathfrak{g}_0$ with prescribed metric and prescribed orthonormal basis. Furthermore, the bracket operator in $\mathfrak{g}_0$ is given by

$$[e_1, e_2] = -[e_2, e_1] = e_3,$$

with $[e_i, e_j] = 0$ otherwise. Applying Lemma 10 and Lemma 13 it follows that

$$\mathrm{Ric}(e_1) < 0 < \mathrm{Ric}(e_3)$$

are satisfied in $\mathfrak{g}_0$. But these Ricci curvatures must vary continuously as we vary the structure constants, so it follows that $\mathrm{Ric}(e_1) < 0 < \mathrm{Ric}(e_3)$ whenever ε is sufficiently close to zero. $\square$

- According to Eliasson in [Eli71], any smooth manifold of dimension ≥ 3 admits a Riemannian metric of negative scalar curvature.
- However, metrics of non-negative scalar curvature do not always exist. From [Lic63], one cannot have a metric with non-negative scalar curvature, except possibly identically zero, on a compact spin manifold whose $\widehat{A}$-genus is not zero.
- Moreover, Kazdan and Warner showed in [KW75] there are no topological obstructions to scalar curvature which are negative somewhere.

- For left-invariant metrics, the situation of negative scalar curvature can be described as follows.

Theorem

If the Lie group is solvable, then every left-invariant metric on G is either flat, or else has negative scalar curvature.

Theorem

If the Lie algebra of G is not abelian, then G possesses a left-invariant metric of negative scalar curvature.

- There remains the question as to which Lie groups admit left-invariant metrics of positive scalar curvature.

Theorem (Wallach)

Let G be a connected Lie group. If the universal covering of G is not homomorphic to Euclidean space, then G admits a left-invariant metric of positive scalar curvature.

- To prove this, we need the following basic result of Lie groups.

Theorem (Iwasawa)

Let G be a connected Lie group. Then

1. *Every compact subgroup is contained in a maximal compact subgroup H, which is necessary a connected Lie group.*
2. *This maximal compact subgroup is unique up to conjugation.*
3. *As a topological space, G is homeomorphic with the product of H and some Euclidean space $\mathbb{R}^n$.*

Proof.

See [Iwa49]. □

Corollary

The universal covering of a connected Lie group G is homeomorphic to a Euclidean space if and only if every compact subgroup of G is abelian.

Proof.

If every compact subgroup of G is abelian, then by Theorem 19, G is homeomorphic to the product of an abelian Lie group and some Euclidean space $\mathbb{R}^n$. Note that any abelian (real) Lie group must be $(S^1)^k \times \mathbb{R}^m$. This shows the universal covering of G is homeomorphic to a Euclidean space.
Conversely, if there exists a non-abelian compact subgroup, then the maximal compact subgroup H is also non-abelian. Note that the universal covering of any connected compact non-abelian Lie group is not homeomorphic to the Euclidean space, and thus G cannot be homeomorphic to some Euclidean. $\square$

Proof of Theorem 18.

Since the universal covering of G is not Euclidean, there exists a compact non-abelian subgroup H, and by Iwasawa's theorem we may assume H is connected. Since H is compact, we can construct an inner product on $\mathfrak{g}$ which is invariant under $\mathrm{Ad}(H)$. Let $e_1, \dots, e_m$ be an orthonormal basis for the Lie algebra of H, and extend to an orthonormal basis $e_1, \dots, e_n$ for $\mathfrak{g}$. Since inner product on $\mathfrak{g}$ is $\mathrm{Ad}(H)$-invariant, we see that $\mathrm{ad}(e_1), \dots, \mathrm{ad}(e_m)$ must be skew-symmetric.

Fixing any $\varepsilon > 0$, consider a new basis $e'_1, \dots, e'_n$ defined by

$$e'_1 = e_1, \dots, e'_m = e_m, \quad e'_{m+1} = \varepsilon e_{m+1}, \dots, e'_n = \varepsilon e_n.$$

Choose a new inner product so that basis $e'_1, \dots, e'_n$ is orthogonal. The symbol $\mathfrak{g}_\varepsilon$ will denote the Lie algebra provided with this new inner product, and with this specified orthonormal basis.

Continuation.

It's clear the structure constants of $\mathfrak{g}_\varepsilon$ are continuous functions of ε, so there is a well-defined limit algebra $\mathfrak{g}_0$ with prescribed inner product and prescribed orthonormal basis. Evidently $\mathfrak{g}_0$ splits as an orthogonal direct sum $\mathfrak{h} \oplus \mathfrak{u}$, where $\mathfrak{h}$ is the subalgebra spanned by $e'_1, \dots, e'_m$ and $\mathfrak{u}$ is the abelian ideal spanned by $e'_{m+1}, \dots, e'_n$. Applying Lemma 26 we see that $\nabla_u = 0$ for all $u \in \mathfrak{u}$, so

$$R(x, u) = \nabla_x \nabla_u - \nabla_u \nabla_x - \nabla_{[x,u]} = 0,$$

and thus $\kappa(x, u) = 0$ for all $x \in \mathfrak{g}$. In particular, the Ricci curvature $\mathrm{Ric}(u) = 0$ for all $u \in \mathfrak{u}$. On the other hand, by Lemma 10 we have $\mathrm{Ric}(b) \geq 0$ for $b \in \mathfrak{h}$, and the equality does not always hold since $\mathfrak{h}$ is not abelian. Therefore the scalar curvature $\rho = \sum_i \mathrm{Ric}(e'_i)$ of the limit algebra $\mathfrak{g}_0$ is positive. It follows by continuity that $\rho(\mathfrak{g}_\varepsilon) > 0$ whenever ε is sufficiently small. $\square$

- In this section we study 3-dimensional Lie algebra, and a useful tool is the "cross product operation".
- If u, v are elements of a 3-dimensional vector space which is provided with an inner product and a preferred orientation, then the cross product $u \times v$ is defined. This product is bilinear and skew-symmetric as a function of u and v. The vector $u \times v$ is orthogonal to both u and v and has length $\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2$. Its direction is determined by the requirement that the triple $u, v, u \times v$ is positively oriented.

- Let G be a connected 3-dimensional Lie group with left-invariant metric. Choose an orientation for the Lie algebra of G, so that the cross product is defined.

Lemma

The bracket product operation in Lie algebra $\mathfrak{g}$ is related to the cross product operation by the formula

$$[u, v] = L(u \times v),$$

where L is a uniquely defined linear mapping from $\mathfrak{g}$ to itself. The Lie group G is unimodular if and only if L is symmetric.

Proof.

Let $\mathfrak{g}$ be a 3-dimensional Lie algebra with an inner product and preferred orientation. Choose an oriented orthonormal basis e_1, e_2, e_3, define the linear transformation $L\colon \mathfrak{g} \to \mathfrak{g}$ by $L(e_1) = [e_2, e_3], L(e_2) = [e_3, e_1], L(e_3) = [e_1, e_2]$. Then the identity $L(e_i \times e_j) = [e_i, e_j]$ is true for all basis elements, hence $L(x \times y) = [x, y]$ for all x and y. Setting

$$L(e_i) = \sum \alpha_{ij} e_j.$$

Note that

$$\begin{aligned} \operatorname{tr} \operatorname{ad}(e_1) &= -\alpha_{23} + \alpha_{32} \\ \operatorname{tr} \operatorname{ad}(e_2) &= -\alpha_{31} + \alpha_{13} \\ \operatorname{tr} \operatorname{ad}(e_3) &= -\alpha_{12} + \alpha_{21}. \end{aligned}$$

Thus $\mathfrak{g}$ is unimodular if and only if (α_{ij}) is symmetric, or in other words if and only if L is symmetric. □

- Now let's specialize to the unimodular case. If L is symmetric, then there exists an orthonormal basis e_1, e_2, e_3 consisting of eigenvectors, $Le_i = \lambda_i e_i$. Replacing e_1 by $-e_1$ if necessary, we may assume e_1, e_2, e_3 is positively oriented. The bracket product operation is then given by $[e_1, e_2] = L(e_3) = \lambda_3 e_3$, with similar expressions for the other $[e_i, e_j]$. Thus we obtain the following normal form

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3,$$

for the bracket product operation in a 3-dimensional unimodular Lie algebra.

- For convenience, we denote $\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i$. Then the curvature properties are described as follows.

Theorem

The orthonormal basis e_1, e_2, e_3, chosen as before, diagonalizes the Ricci quadratic form, the principal Ricci curvatures being given by

$$\mathrm{Ric}(e_1) = 2\mu_2\mu_3, \quad \mathrm{Ric}(e_2) = 2\mu_1\mu_3, \quad \mathrm{Ric}(e_3) = 2\mu_1\mu_2.$$

As a consequence, the scalar curvature is given by $s = 2(\mu_2\mu_3 + \mu_1\mu_3 + \mu_1\mu_2)$.

Corollary

In the 3-dimensional unimodular case, the determinant of the Ricci quadratic form is always non-negative. If this determinant is zero, then at least two of the principal Ricci curvatures must be zero.

- There are now just six distinct cases, which we tabulate as follows. By changing signs if necessary, we assume that at most one of the structure constants $\lambda_1, \lambda_2, \lambda_3$ is negative.

Signs of $\lambda_1, \lambda_2, \lambda_3$	Associated Lie group	Description
$+, +, +$	SU(2)	compact, simple
$+, +, -$	SL$(2, \mathbb{R})$	noncompact, simple
$+, +, 0$	$E(2)$	solvable
$+, -, 0$	$E(1, 1)$	solvable
$+, 0, 0$	Heisenberg group	nilpotent
$0, 0, 0$	$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$	abelian

Corollary

For any left-invariant metric on the Heisenberg group, the Ricci quadratic form has signature $(+, -, -)$ and the scalar curvature s is negative. Furthermore, the principal Ricci curvatures satisfy

$$|\operatorname{Ric}(e_1)| = |\operatorname{Ric}(e_2)| = |\operatorname{Ric}(e_1)| = \rho.$$

Proof.

Taking $\lambda_2 = \lambda_3 = 0$ one has $\mu_1 = -\mu_2 = -\mu_3 = -\lambda_1/2$. Thus

$$\rho = 2(-\frac{1}{4}\lambda_1^2 - \frac{1}{4}\lambda_1^2 + \frac{1}{4}\lambda_1^2) = -\frac{1}{2}\lambda_1^2,$$

and $\operatorname{Ric}(e_1) = -\operatorname{Ric}(e_2) = -\operatorname{Ric}(e_3) = -\rho$. □

Corollary

Let G be either $\mathrm{SL}(2,\mathbb{R})$ or $E(1,1)$. Then depending the choice of left-invariant metric the signature of the Ricci quadratic form can be either $(+,-,-)$ or $(0,0,-)$. However, the scalar curvature ρ must always be strictly negative.

Proof.

If $\lambda_1 = 0$ while λ_2 and λ_3 have oppsite sign, then

$$u_1 = \frac{1}{2}(\lambda_2 + \lambda_3), \quad u_2 = \frac{1}{2}(\lambda_3 - \lambda_2), \quad u_3 = \frac{1}{2}(\lambda_2 - \lambda_3).$$

Thus

$$\rho = -\frac{1}{2}(\lambda_2 - \lambda_3)^2 < 0.$$

Since λ_2 and λ_3 has oppsite sign, then $\lambda_2 \neq \lambda_3$, and thus $\mathrm{Ric}(e_1) = 2u_2u_3 < 0$.

Continuation.

If $\lambda_2 = -\lambda_3$, then $u_1 = 0$, and thus $\mathrm{Ric}(e_2) = \mathrm{Ric}(e_3) = 0$. In this case, the signature of Ricci quadratic form is $(-, 0, 0)$. If $\lambda_2 > 0 > \lambda_3$, then $u_1 > 0, u_2 < 0, u_3 > 0$. This shows the signature of Ricci quadratic form is $(+, -, -)$. Similarly for the case $\lambda_3 > 0 > \lambda_2$.

If the λ_i are all non-zero with say $\lambda_1 < 0 < \lambda_2, \lambda_3$, then the computation $\partial\rho/\partial\lambda_1 = -\lambda_1 + \lambda_2 + \lambda_3$ shows that ρ is monotone as a function of λ_1 for $\lambda_1 \leq 0$. Therefore

$$\rho(\lambda_1, \lambda_2, \lambda_3) < \rho(0, \lambda_2, \lambda_3) = -\frac{1}{2}(\lambda_2 - \lambda_3)^2 \leq 0.$$

By similar argument as above, more information about Ricci quadratic form can be derived. $\square$

- Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathfrak{g}$ and α_{ijk} be the structure constants of $\mathfrak{g}$.

Lemma

Let ∇ be the Levi-Civita connection corresponding to the left-invariant metric on G. Then

$$\nabla_{e_i} e_j = \sum_k \frac{1}{2}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij})e_k.$$

Proof.

It follows from the following formula

$$\langle \nabla_x y, z \rangle = \frac{1}{2}\left(\langle [x, y], z\rangle - \langle [y, z], x\rangle + \langle [z, x], y\rangle\right).$$

Proof of Lemma 1.

Recall that the sectional curvature is given by

$$\kappa(u, v) = \langle R(u, v)v, u\rangle,$$

where $R(u, v)v = \nabla_u \nabla_v v - \nabla_v \nabla_u v - \nabla_{[u,v]} v$. Then inserting formula in Lemma 26 into the definition, we easily obtain the desired formula. □

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Thanks!