COMPLEX GEOMETRY I: COMPLEX DIFFERENTIAL GEOMETRY

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Contents

Part 1. Basic Complex Geometry

In this part, we mainly follows [Huy05] and [Dem12].

1. Review of complex analysis

1.1. One variable case.

We first give a quick review about basic results in holomorphic functions of one variable. Fix an open subset $U \subseteq \mathbb{C}$. There are too many ways to define a holomorphic function, and all of them are equivalent.

Definition 1.1.1 (holomorphic)**.** A function $f\colon U \to \mathbb{C}$ is called holomorphic at $z_0 \in U$, if there exists an open ball $B_\varepsilon(z_0) \subseteq U$ with $\varepsilon > 0$ such that $f|_{B_\varepsilon(z_0)}$ can be written as convergent power series, that is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad z \in B_\varepsilon(z_0)$$

f is holomorphic om U, if f is holomorphic at any point of U.

Remark 1.1.1 (Cauchy-Riemann equation). The second definition is given by Cauchy-Riemann equation. To be explicit, for a function $f\colon U \to \mathbb{C}$, we can regard it as a function defined on $\mathbb{R}^2$, and write it as $f(x,y) = u(x,y) + \sqrt{-1}v(x,y)$, where u, v are real-valued functions, then f is holomorphic if and only if u, v are continuously differentiable and satisfy the following Cauchy-Riemann equations:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

If we introduce the following two operators

$$\begin{aligned} \frac{\partial}{\partial z} &:= \frac{1}{2}(\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}) \\ \frac{\partial}{\partial \overline{z}} &:= \frac{1}{2}(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}) \end{aligned}$$

Then Cauchy-Riemann equation is equivalent to $\frac{\partial f}{\partial \overline{z}} = 0$. Indeed,

$$\begin{aligned} \frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}(\frac{\partial f}{\partial x} + \sqrt{-1}\frac{\partial f}{\partial y}) \\ &= \frac{1}{2}(\frac{\partial u}{\partial x} + \sqrt{-1}\frac{\partial v}{\partial x} + \sqrt{-1}\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}) \\ &= 0 \end{aligned}$$

Remark 1.1.2 (Cauchy integral formula). The third definition is given by Cauchy integral formula. To be explicit, a function $f\colon U \to \mathbb{C}$ is holomorphic if and only if f is continuously differentiable and for any $B_\varepsilon(z_0) \subseteq U$, the following formula holds

$$f(z_0) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial B_\varepsilon(z_0)} \frac{f(z)}{z - z_0} \mathrm{d}z$$

Here are some standard facts in complex analysis, which can be found in any textbook.

Theorem 1.1.1 (maximum principle)**.** Let $U \subseteq \mathbb{C}$ be open and connected. If $f: U \to \mathbb{C}$ is holomorphic and non-constant, then $|f|$ has no local maximum in U.

Theorem 1.1.2 (identity theorem)**.** If $f, g: U \to \mathbb{C}$ are two holomorphic functions a connected open subset $U \subseteq \mathbb{C}$ such that $f(z) = g(z)$ for all z in a non-empty subset V of U, then $f = g$.

Theorem 1.1.3 (Riemann extension theorem)**.** Let $f: B_\varepsilon(z_0) - \{z_0\} \to \mathbb{C}$ be a bounded holomorphic function, then f can be extended to a holomorphic function $f: B_\varepsilon(z_0) \to \mathbb{C}$.

Theorem 1.1.4 (Riemann mapping theorem)**.** Let $U \subseteq \mathbb{C}$ be a simply-connected proper open subset. Then U is biholomorphic to the unit ball.

Theorem 1.1.5 (Liouville)**.** Every bounded holomorphic function $f: \mathbb{C} \to \mathbb{C}$ is constant.

Remark 1.1.3. Liouville theorem implies that $\mathbb{C}$ is not biholomorphic to the unit ball. It's a striking difference to the real case since we know unit ball is homeomorphic to $\mathbb{R}$.

1.2. **Several variables case.** Now let U be an open subset of $\mathbb{C}^n$. For any $w \in U$, a polydisc $B_\varepsilon(w) = \{z : |z^i - w_i| < \varepsilon_i\}$, where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$.

Definition 1.2.1 (holomorphic)**.** A function $f: U \to \mathbb{C}$ is called holomorphic at point $w \in U$, if there exists a polydisc $B_\varepsilon(w) \subseteq U$ such that the restriction of $f|_{B_\varepsilon(w)}$ is given by power series

$$\sum_{i_1,\dots,i_n=0}^{\infty} a_{i_1\dots i_n}(z_1 - w_1)^{i_1} \dots (z_n - w_n)^{i_n}$$

Remark 1.2.1 (equivalent definitions of holomorphic function)*.*

(1) A function $f: U \to \mathbb{C}$ is holomorphic, if it satisfies Cauchy-Riemann equations for all coordinates $z^i = x^i + \sqrt{-1}y^i$, that is

$$\frac{\partial f}{\partial \overline{z}_i} = 0, \quad i = 1, 2, \dots, n$$

where $\frac{\partial}{\partial z^i} := \frac{1}{2}(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i})$

(2) A function $f\colon U\to\mathbb{C}$ is holomorphic if and only if f is continuously differentiable and for any $z_0\in U$, the following formula holds

$$f(z)=\frac{1}{(2\pi\sqrt{-1})^n}\int_{\partial B_\varepsilon(w)}\frac{f(w)}{(z_1-w_1)\dots(z_n-w_n)}\mathrm{d}w_1\dots\mathrm{d}w_n$$

Remark 1.2.2. Other results such as maximum theorem, identity theorem and Liouville theorem generalize easily to the higher dimension. A version of Riemann extension still holds true. However, Riemann mapping theorem fails.

Exercise 1.2.1. Show that polydisc $B_{(1,1)}(0)\subseteq\mathbb{C}^2$ is not biholomorphic to the unit disk $D=\{z\in\mathbb{C}^2:\|z\|<1\}$.

Lemma 1.2.1 (local $\partial\overline{\partial}$-lemma)**.** Let ω be a real $(1,1)$-form defined on $\mathbb{C}^n$. Then ω is d-closed if and only if for any point $z\in\mathbb{C}^n$, there exists an open neighborhood U of z and a smooth function $\varphi\colon U\to\mathbb{R}$ such that

$$\omega=\sqrt{-1}\partial\overline{\partial}\varphi.$$

Lemma 1.2.2. Let $\varphi\colon\ \mathbb{C}^n\to\mathbb{R}$ be a smooth function such that $\partial\overline{\partial}f=0$. Then for any point $z\in\mathbb{C}^n$, there exists an open neighborhood U of z and a holomorphic functions $f\colon U\to\mathbb{C}$ such that $\varphi=\mathrm{Re}(f)$ over U.

Theorem 1.2.1 (Hartogs' theorem)**.** Suppose $\varepsilon=(\varepsilon_1,\dots,\varepsilon_n)$ and $\varepsilon'=(\varepsilon'_1,\dots,\varepsilon'_n)$ are given such that for all i one has $\varepsilon'_i<\varepsilon_i$. If $n>1$, then any holomorphic map $f\colon B_\varepsilon(0)\backslash\overline{B_{\varepsilon'}(0)}\to\mathbb{C}$ can be uniquely extended to a holomorphic map $f\colon B_\varepsilon(0)\to\mathbb{C}$.

Remark 1.2.3. This is only valid in dimension at least two.

Definition 1.2.2 (holomorphic)**.** A function $f\colon U\to\mathbb{C}^n$ is called holomorphic if all coordinate functions $f_1,\dots,f_n$ are holomorphic functions $U\to\mathbb{C}$.

Definition 1.2.3 (biholomorphic)**.** A holomorphic map $f\colon U\to V$ between two open subsets $U,V\subseteq\mathbb{C}^n$ is biholomorphic if f is bijective and its inverse f^{-1} is also holomorphic.

Definition 1.2.4 (complex Jacobian)**.** Let $f\colon U\to\mathbb{C}^n$ be a holomorphic map, the complex Jacobian of f at point $z\in U$ is the matrix

$$J_{\mathbb{C}}(f)(z):=\left(\frac{\partial f_i}{\partial z^j}(z)\right)_{\substack{1\le i\le n\\1\le j\le m}}$$

where $f_i=z^i\circ f$.

Remark 1.2.4. For each $z\in U$, the smooth map $f\colon U\subseteq\mathbb{C}^m=\mathbb{R}^{2m}\to\mathbb{C}^n=\mathbb{R}^{2n}$ induces a $\mathbb{R}$-linear map, which is denoted by $J_{\mathbb{R}}(f)(z)\colon T_z\mathbb{R}^{2m}\to T_{f(z)}\mathbb{R}^{2n}$. Suppose $\{\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^i}\}_{1\le i\le m}$ and $\{\frac{\partial}{\partial r^j},\frac{\partial}{\partial s^j}\}_{1\le j\le n}$ are local frames of $T_z\mathbb{R}^{2m}$ and $T_{f(z)}\mathbb{R}^{2n}$ respectively, then with respect to these basis one has

$$J_{\mathbb{R}}(f)(z)=\begin{pmatrix}(\frac{\partial u_i}{\partial x^j}) & (\frac{\partial u_i}{\partial y^j})\\(\frac{\partial v_i}{\partial x^j}) & (\frac{\partial v_i}{\partial y^j})\end{pmatrix}$$

where $u_i = r^i \circ f$ and $v_i = s^i \circ f$. If we consider its $\mathbb{C}$-linear extension, with respect to basis $\{\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z}^i}\}_{1\leq i\leq m}$ and $\{\frac{\partial}{\partial w^j}, \frac{\partial}{\partial \overline{w}^j}\}_{1\leq j\leq n}$, it can be written as

$$J_{\mathbb{R}}(f)(z) = \begin{pmatrix} (\frac{\partial f_i}{\partial z^j}) & (\frac{\partial f_i}{\partial \overline{z}^j}) \\ (\frac{\partial \overline{f}_i}{\partial z^j}) & (\frac{\partial \overline{f}_i}{\partial \overline{z}^j}) \end{pmatrix}$$

In particular, if f is holomorphic, then $\det J_{\mathbb{R}}(f) = \det J_{\mathbb{C}}(f) \det \overline{J_{\mathbb{C}}(f)} = |\det J_{\mathbb{C}}(f)|^2 \geq 0$.

Definition 1.2.5 (regular value)**.** Let $U \subseteq \mathbb{C}^m$ be an open subset and let $f\colon U \to \mathbb{C}^n$ be a holomorphic map, $z \in U$ is called regular point, if $J_{\mathbb{C}}(f)(z)$ is surjective. If every point $z \in f^{-1}(w)$ is regular point, then w is called a regular value.

Remark 1.2.5. In particular, if $f^{-1}(w) = \varnothing$, then w is also called a regular value.

Theorem 1.2.2 (inverse function theorem)**.** Let $f\colon U \to V$ be a holomorphic map between two open subsets $U, V \subseteq \mathbb{C}^n$. If $z \in U$ is a regular point, then there exist open subsets $z \in U' \subseteq U$ and $f(z) \in V' \subseteq V$ such that f induces a biholomorphic map $f\colon U' \to V'$.

Theorem 1.2.3 (implicit function theorem)**.** Let $U \subseteq \mathbb{C}^m$ be an open subset and let $f\colon U \to \mathbb{C}^n$ be a holomorphic map, where $m \geq n$. Suppose $z_0 \in U$ is a point such that

$$\det\left(J_{\mathbb{C}}(f)(z_0)\right) \neq 0$$

Then there exist open subsets $U_1 \subseteq \mathbb{C}^{m-n}, U_2 \subseteq \mathbb{C}^n$ and a holomorphic map $g\colon U_1 \to U_2$ such that $U_1 \times U_2 \to U$ and $f(z) = f(z_0)$ if and only if $g(z_{n+1}, \ldots, z_m) = (z_1, \ldots, z_n)$.

Corollary 1.2.1. Let $U \subseteq \mathbb{C}^m$ be an open subset and $f\colon U \to \mathbb{C}^n$ be a holomorphic map. Suppose that $z_0 \in U$ such that $J_{\mathbb{C}}(f)(z_0)$ has maximal rank. Then

(1) If $m \geq n$, then there exists a biholomorphic map $h\colon V \to U'$, where U' is an open subset of U containing z_0, and V is an open subset of $\mathbb{C}^n$ containing $f(z_0)$, such that $f(h(z_1, \ldots, z_n)) = (z_1, \ldots, z_n)$.
(2) If $m \leq n$, then there exists a biholomorphic map $g\colon V \to V'$, where V, V' are open subsets of $\mathbb{C}^n$ containing $f(z_0)$, such that $g(f(z)) = (z_1, \ldots, z_m, 0, \ldots, 0)$.

2. Local theory

2.1. Algebraic germ.

2.1.1. *Weierstrass' theorems.* Let $f\colon B_\varepsilon(0)\to\mathbb{C}$ be a holomorphic function defined on polydisc $B_\varepsilon(0)$. For any $w=(z_2,\dots,z_n)$ we denote $f_w(z_1)$ the function $f(z_1,\dots,z_n)$. Now we're going to show that all zeros of f are caused by a factor of f which has the form of a Weierstrass polynomial.

Definition 2.1.1 (Weierstrass polynomial)**.** A Weierstrass polynomial is a polynomial in z_1 of the form

$$z_1^d+\alpha_1(w)z_1^{d-1}+\dots+\alpha_d(w)$$

where coefficients $\alpha_i(w)$ are holomorphic functions on some small disc in $\mathbb{C}^{n-1}$ vanishing at the origin.

Remark 2.1.1. Recall the one variable case, any holomorphic function $f(z)$ with a zero of order d at the origin can be written as $z^d h(z)$, where $h(0)\neq 0$. In fact, z^d is a Weierstrass polynomial since in this case, α_i are constants which vanish at origin, that's exactly zero.

Theorem 2.1.1 (Weierstrass preparation theorem)**.** Let $f\colon B_\varepsilon(0)\to\mathbb{C}$ be a holomorphic function on the polydisc $B_\varepsilon(0)$. Assume $f(0)=0$ and $f_0(z_1)\neq 0$. Then there exists a unique Weierstrass polynomial $g_w(z_1)$ and a holomorphic function h on some smaller polydisc $B_{\varepsilon'}(0)\subseteq B_\varepsilon(0)$ such that $f=gh$ and $h(0)\neq 0$.

Proof. See Proposition 1.1.6 in Page8 of [Huy05]. □

Theorem 2.1.2 (Weierstrass division theorem)**.** Let $f\in\mathcal{O}_{\mathbb{C}^n,0}$ and let $g\in\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ be a Weierstrass polynomial of degree d. Then there exist $r\in\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ of degree $<d$ and $h\in\mathcal{O}_{\mathbb{C}^n,0}$ such that $f=gh+r$. The functions h and r are uniquely determined.

Proof. See Proposition 1.1.17 in Page15 of [Huy05]. □

2.1.2. *Stalk of sheaf of holomorphic functions.* Let's use $\mathcal{O}_{\mathbb{C}^n}$ to denote the sheaf[1] of holomorphic functions on $\mathbb{C}^n$, and use $\mathcal{O}_{\mathbb{C}^n,0}$ to denote its stalk at origin. The elements in $\mathcal{O}_{\mathbb{C}^n,0}$ are called germs. It's clear $\mathcal{O}_{\mathbb{C}^n,0}$ is a local ring with maximal ideal $\mathfrak{m}$ consisting of all functions that vanish at origin, which implies units in $\mathcal{O}_{\mathbb{C}^n,0}$ are functions that don't vanish at origin.

By using Weierstrass preparation theorem, one can derive more about algebraic properties of $\mathcal{O}_{\mathbb{C}^n,0}$. For example, Weierstrass preparation theorem can be rephrased by saying that after an appropriate coordinate choice any function $f\in\mathcal{O}_{\mathbb{C}^n,0}$ can be uniquely written as $f=gh$, where $h\in\mathcal{O}_{\mathbb{C}^n,0}$ is a unit and $g\in\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is a Weierstrass polynomial. Furthermore, it also shows the following important property.

[1] Sheaf and its cohomology are important tools we will use once and again, if you're not familiar with it, see Appendix 20.

Theorem 2.1.3. The local ring $\mathcal{O}_{\mathbb{C}^n,0}$ is a UFD.

Proof. We prove the assumption by induction on n. For $n = 0$, the ring $\mathcal{O}_{\mathbb{C}^n,0} = \mathbb{C}$ is a field, and thus a UFD. Suppose that $\mathcal{O}_{\mathbb{C}^{n-1},0}$ is a UFD, for $f \in \mathcal{O}_{\mathbb{C}^n,0}$ we choose coordinates such that Weierstrass preparation theorem is applied, that is $f = gh$, where $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is a Weierstrass polynomial and h is a unit in $\mathcal{O}_{\mathbb{C}^n,0}$. By induction we have $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is UFD, then g can be written as a product of irreducible elements of $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$. All that is left to show is that any irreducible element in $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is also irreducible as an element in $\mathcal{O}_{\mathbb{C}^n,0}$.

Assume $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is a Weierstrass polynomial which is written as the product of non-units $g_i \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$. There are two cases:

(1) $g_i \in \mathcal{O}_{\mathbb{C}^{n-1},0}$. By induction hypothesis, g_i can be written as the product of irreducible elements of $\mathcal{O}_{\mathbb{C}^{n-1},0}$, which are also irreducible in $\mathcal{O}_{\mathbb{C}^n,0}$.

(2) $g_i \notin \mathcal{O}_{\mathbb{C}^{n-1},0}$. In this case, g_i satisfies the hypothesis of Weierstrass preparation theorem since g is a Weierstrass polynomial, then g_i is non-trivial on the z_1-line. So we can write $g_i = \widetilde{g}_i h_i$, where $\widetilde{g}_i$ is also Weierstrass polynomial.

Note that degree of g as a polynomial in z_1 is finite, then repeating above process leads to a decomposition, with factors are either irreducible Weierstrass polynomials or elements in $\mathcal{O}_{\mathbb{C}^{n-1},0}$.

Now it suffices to show any irreducible Weierstrass polynomial g is actually irreducible as an element of $\mathcal{O}_{\mathbb{C}^n,0}$. Suppose $g = f_1 f_2$, where $f_1, f_2 \in \mathcal{O}_{\mathbb{C}^n,0}$ are non-units. We apply Weierstrass preparation theorem to obtain $f_i = g_i h_i, i = 1, 2$, and thus $g = (g_1 g_2)(f_1 f_2)$. By uniqueness one has $g = g_1 g_2$, which contradicts to the irreducibility of g as an element of $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$. □

Another important fact is that $\mathcal{O}_{\mathbb{C}^n,0}$ is noetherian, which follows from Weierstrass division theorem.

Theorem 2.1.4. The local UFD $\mathcal{O}_{\mathbb{C}^n,0}$ is noetherian.

Proof. We prove the assumption by induction on n. For $n = 0$, it's clear since any field is noetherian. Suppose that $\mathcal{O}_{\mathbb{C}^{n-1},0}$ is noetherian, then Hilbert's basis theorem implies $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is also noetherian. Let $I \subseteq \mathcal{O}_{\mathbb{C}^n,0}$ be a non-trivial idea and choose $0 \neq f \in I$. Changing coordinates if neccessary, we may assume Weierstrass preparation theorem is applied, that is $f = gh$, where $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is a Weierstrass polynomial and h is a unit in $\mathcal{O}_{\mathbb{C}^n,0}$, hence $g \in I$. Furthermore, we assume $I \cap \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is generated by $g_1, \ldots, g_k$.

For any other $\widetilde{f} \in I$, the Weierstrass division theorem implies $\widetilde{f} = g\widetilde{h} + r$ for some $r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$. Since $\widetilde{f}, g\widetilde{h} \in I$, we have $r \in I$ and therefore $r \in I \cap \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$. Thus $\widetilde{f} = g\widetilde{h} + \sum_{i=1}^k a_i g_i$. This shows I is finitely generated by elements $g, g_1, \ldots, g_k$. □

Corollary 2.1.1. Let $g \in \mathcal{O}_{\mathbb{C}^n,0}$ be an irreducible element. If $f \in \mathcal{O}_{\mathbb{C}^n,0}$ vanishes on $Z(g) = \{z \mid g(z) = 0\}$, then g divides f.

Proof. By Weierstrass preparation theorem we may assume $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is a Weierstrass polynomial with degree d. By the Weierstrass division theorem one finds $h \in \mathcal{O}_{\mathbb{C}^n,0}$ and $r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ of degree $< d$ such that $f = gh + r$. For $w \in \mathbb{C}^{n-1}$, by assumption r_w vanishes on the zero set g_w. If all of zeros of g_w have multiplicity one, then $r_w \equiv 0$ since r_w is of degree $< d$. Now it suffices to show the set $w \in \mathbb{C}^{n-1}$ such that g_w has zeros with multiplicity > 1 is quite "small".

Since g is irreducible and $\frac{\partial g}{\partial z_1}$ is of degree $d-1$, there exist elements $h_1, h_2 \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ and $0 \neq \gamma \in \mathcal{O}_{\mathbb{C}^{n-1},0}$ such that $h_1 g + h_2 \frac{\partial g}{\partial z_1} = \gamma$. So if g_w has a zero ξ of multiplicity > 1, then $\gamma(w) = h_1(\xi, w) g_w(\xi) + h_2(\xi, w) \frac{\partial g_w}{\partial z_1}(\xi) = 0$. This shows such w is contained in the zero set of a non-trivial holomorphic function $\gamma \in \mathcal{O}_{\mathbb{C}^{n-1},0}$. Then the following exercise completes the proof. $\square$

Exercise 2.1.1. Let $U \subseteq \mathbb{C}^n$ be open and connected. Show that for any non-trivial holomorphic function $f\colon U \to \mathbb{C}$ the complement $U \backslash Z(f)$ of the zero set of f is connected and dense in U.

2.2. **Analytic germ.** For any $f \in \mathcal{O}_{\mathbb{C}^n,0}$, $Z(f)$ is not well-defined in fact since for another $g \in \mathcal{O}_{\mathbb{C}^n,0}$, which represents the same element with f, $Z(f)$ may not equal to $Z(g)$. However, there always exists an open neighborhood $0 \in U \subseteq \mathbb{C}^n$ such that $Z(f) \cap U = Z(g) \cap U$.

Definition 2.2.1 (germ of a set)**.** The germ of a set in the origin $0 \in \mathbb{C}^n$ is given by a subset $X \subseteq \mathbb{C}^n$. Two germs of a set in the origin $X, Y \subseteq \mathbb{C}^n$ are same if there exists an open neighborhood $0 \in U \subseteq \mathbb{C}^n$ such that $X \cap U = Y \cap U$.

Unless otherwise specified, in this section we only consider germ of a set in the origin, and for convenience we just call it a germ.

Example 2.2.1. For $f \in \mathcal{O}_{\mathbb{C}^n,0}$, $Z(f)$ is a germ.

Definition 2.2.2 (analytic germ)**.** A germ $X \subseteq \mathbb{C}^n$ is called analytic if there exist elements $f_1, \dots, f_k \in \mathcal{O}_{\mathbb{C}^n,0}$ such that $X = Z(f_1, \dots, f_k) := \bigcap_{i=1}^k Z(f_i)$.

Example 2.2.2. Let A be a subset of $\mathcal{O}_{\mathbb{C}^n,0}$. If we use (A) to denote the idea generated by A, then (A) is finitely generated since $\mathcal{O}_{\mathbb{C}^n,0}$ is noetherian. Thus $Z((A))$ is an analytic germ.

Definition 2.2.3. Let $X \subseteq \mathbb{C}^n$ be a germ. Then $I(X)$ denotes the set of all elements $f \in \mathcal{O}_{\mathbb{C}^n,0}$ with $X \subseteq Z(f)$.

Remark 2.2.1. It's clear $I(X)$ is an idea of $\mathcal{O}_{\mathbb{C}^n,0}$.

Lemma 2.2.1.
(1) If $X_1 \subseteq X_2$ are germs, then $I(X_2) \subseteq I(X_1)$.
(2) If $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$ are two ideas of $\mathcal{O}_{\mathbb{C}^n,0}$, then $Z(\mathfrak{a}_2) \subseteq Z(\mathfrak{a}_1)$.

(3) For any analytic germ one has $Z(I(X)) = X$.
(4) For any idea $\mathfrak{a}$ of $\mathcal{O}_{\mathbb{C}^n,0}$ one has $\mathfrak{a} \subseteq I(Z(\mathfrak{a}))$

Proof. (1), (2) and (4) are clear. For (3). It's clear $X \subseteq Z(I(X))$. On the other hand since X is analytic germ there exist elements $f_1, \dots, f_k \in \mathcal{O}_{\mathbb{C}^n,0}$ such that $X = Z(f_1, \dots, f_k)$ as germs, thus $f_1, \dots, f_k \in I(X)$, so by (2) we have $Z(I(X)) \subseteq X = Z(f_1, \dots, f_k)$. This completes the proof of (3). □

Definition 2.2.4 (irreducible germ)**.** An analytic germ is irreducible if the following condition is satisfied: If $X = X_1 \cup X_2$, where X_1, X_2 are analytic germs, then $X = X_1$ or $X = X_2$.

Lemma 2.2.2. An analytic germ X is irreducible if and only if $I(X) \subseteq \mathcal{O}_{\mathbb{C}^n,0}$ is a prime ideal.

Proof. If X is irreducible and $f_1 f_2 \in I(X)$, then $X \subseteq Z(f_1 f_2) = Z(f_1) \cup Z(f_2)$, so we have $X = (X \cap Z(f_1)) \cup (X \cap Z(f_2))$ is a union of analytic germs. Then by irreducibility one has $X = X \cap Z(f_i)$ for some $i = 1$ or $i = 2$, and thus at least one of functions f_1 or f_2 vanishes on X. This shows $I(X)$ is prime.

Conversely, if $I(X)$ is a prime ideal and let $X = X_1 \cup X_2$ with X_1 and X_2 are analytic. If $f_i \in I(X_i), i = 1, 2$, then $f_1 f_2 \in I(X)$ since

$$X = X_1 \cup X_2 \subseteq Z(f_1) \cup Z(f_2) = Z(f_1 f_2)$$

Hence $f_1 \in I(X)$ or $f_2 \in I(X)$. Thus it suffices to shows that if $X \neq X_1$ and $X \neq X_2$, there exist elements $f_1 \in I(X_1) \backslash I(X)$ and $f_2 \in I(X_2) \backslash I(X)$. This follows immediately from (1) of Lemma 2.2.1. □

Corollary 2.2.1. For $f \in \mathcal{O}_{\mathbb{C}^n,0}$, $Z(f)$ is irreducible if and only if there exists an irreducible $g \in \mathcal{O}_{\mathbb{C}^n,0}$ such that $f = g^k$ for some $k \in \mathbb{Z}_{>0}$.

Proof. If $f = g^k$ with g irreducible, then $Z(f) = Z(g)$ and if $h \in I(Z(g))$, then g divides h by Corollary 2.1.1, this shows $I(Z(g)) = (g)$ and thus $Z(f)$ is irreducible since $I(Z(f))$ also equals to (g), which is prime. Conversely, if $f = \prod g_i^{n_i}$, then $Z(f) = \bigcup Z(g_i)$, which cannot be irreducible except for the case $f = g^k$ for some irreducible g. □

Lemma 2.2.3. Every decresing sequences of germs $\{X_i\}$ is stationary.

Proof. Consider its corresponding sequence $\{I(X_i)\}$, it's an increasing sequence, thus it's stationary since $\mathcal{O}_{\mathbb{C}^n,0}$ is noetherian, this completes the proof since for each i, $Z(I(X_i)) = X_i$. □

Theorem 2.2.1. Every germ X admits a finite decomposition $X = \bigcup_{i=1}^{N} X_i$, where X_i is irreducible for each i and $X_i \subsetneq X_j$ for $i \neq j$. The decomposition is unique apart from the ordering.

Proof. It suffices to show uniqueness since existence follows from above lemma. Assume $X = \bigcup_{l=1}^{N'} X'_l$ is another decomposition, note that $X_i = \bigcup_{l=1}^{N'} X_i \cap X'_l$, we must have $X_i = X_i \cap X'_{l(i)}$ since X_i is irreducible. Likewise one has $X'_{l(i)} \cap X_j$, so we $i = j$ since $X_i \subsetneq X_j$ for $i \neq j$, and this shows $X_i = X'_{l(j)}$ $\square$

2.3. Hilbert's Nullstellensatz.

Theorem 2.3.1. Let $X \subseteq \mathbb{C}^n$ be an irreducible analytic germ defined by a prime ideal $\mathfrak{p} \subseteq \mathcal{O}_{\mathbb{C}^n,0}$. Then one can find a coordinate system

$$(z_1, \dots, z_{n-d}, z_{n-d+1}, \dots, z_n)$$

such that the projection $(z_1, \dots, z_n) \to (z_{n-d+1}, \dots, z_n)$ induces a surjective map of germs $X \to \mathbb{C}^d$ and such that the induced ring homomorphism $\mathcal{O}_{\mathbb{C}^d,0} \to \mathcal{O}_{\mathbb{C}^n,0}/\mathfrak{p}$ is a finite integral ring extension.

Proof. See (4.19) of [Dem12]. $\square$

Theorem 2.3.2 (Hilbert's Nullstellensatz)**.** If $I \subseteq \mathcal{O}_{\mathbb{C}^n,0}$ is any ideal, then $\sqrt{I} = I(Z(I))$.

Proof. It easy to see $\sqrt{I} \subseteq I(Z(I))$. Conversely, it suffices to show $I(Z(I)) \subseteq \mathfrak{p}$ for all prime ideals containing I since $\sqrt{I}$ is the intersection of all prime ideals $\mathfrak{p}$ containing I. If one has

$$\mathfrak{p} = \sqrt{\mathfrak{p}} = I(Z(\mathfrak{p}))$$

then the results follows from $Z(\mathfrak{p}) \subseteq Z(I)$. Thus we reduce the problem to the case to $I = \mathfrak{p}$ is a prime ideal. For $f \in I(Z(\mathfrak{p}))$, by Theorem 2.3.1 there exists an appropriate coordinate system $(z_1, \dots, z_n)$ such that the induced element $\overline{f} \in \mathcal{O}_{\mathbb{C}^n,0}/\mathfrak{p}$ satisfies an irreducible algebraic equation $\overline{f}^k + a_1\overline{f}^{k-1} + \dots + a_k = 0$ with $a_i \in \mathcal{O}_{\mathbb{C}^d,0}$. Since f vanishes along $Z(\mathfrak{p})$, the 0-th coefficient a_k does as well. As $Z(\mathfrak{p}) \to \mathbb{C}^d$ is surjective, this shows $a_k = 0$. Hence above algebraic equation cannot be irreducible unless $k = 1$. Therefore $\overline{f} = 0$ and thus $f \in \mathfrak{p}$. $\square$

Corollary 2.3.1. There is a one to one correspondence between prime ideals of $\mathcal{O}_{\mathbb{C}^n,0}$ and irreducible analytic germ given by $X \mapsto I(X)$ and $\mathfrak{p} \to Z(\mathfrak{p})$.

2.4. Dimension.

Definition 2.4.1 (dimension)**.** Let X be an irreducible analytic germ defined by a prime ideal $\mathfrak{p} \subseteq \mathcal{O}_{\mathbb{C}^n,0}$. Then the dimension of X is defined by $n - \mathrm{ht}\mathfrak{p}$, where $\mathrm{ht}\mathfrak{p}$ is the height of $\mathfrak{p}$.

Remark 2.4.1. For arbitrary analytic germ is of dimension d if all its irreducible components are of the same dimension d.

Remark 2.4.2. If $X \subseteq \mathbb{C}^n$ is an irreducible analytic germ of codimensional 1, then the prime ideal $\mathfrak{p}$ defining X is of height 1. A basic result in communicative algebra says any prime ideal of height 1 in a UFD is principle. Therefore, $\mathfrak{p} = (f)$ for some irreducible $f \in \mathcal{O}_{\mathbb{C}^n,0}$.

2.5. **Meromorphic functions and relatively prime.**

Definition 2.5.1. Let $U \subseteq \mathbb{C}^n$ be an open subset. A meromorphic function f on U is a function on the complement of a nowhere dense subset $S \subseteq U$ with the following property: There exist an open covering $\{U_i\}$ of U and holomorphic functions $g_i, h_i \colon U \to \mathbb{C}$ with $h_i|_{U_i\setminus S} \cdot f|_{U_i\setminus S} = g_i|_{U_i\setminus S}$.

Remark 2.5.1. For any $z \in U$, the meromorphic function f in a neighborhood of z is given by g/h, where $g, h \in \mathcal{O}_{\mathbb{C}^n,z}$. If we assume g, h are chosen to be relatively prime, then they're unique up to units.

Proposition 2.5.1. Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ be irreducible, then for sufficiently small ε and $z \in B_\varepsilon(0)$ the induced element $f \in \mathcal{O}_{\mathbb{C}^n,z}$ is irreducible.

Proof. Suppose $f \in \mathcal{O}_{\mathbb{C}^n,z}$ is reducible, that is $f = f_1 f_2$ where $f_i \in \mathcal{O}_{\mathbb{C}^n,z}$ non-units, i.e. $f_1(z) = f_2(z) = 0$. Thus $\frac{\partial f}{\partial z_1}(z) = \frac{\partial f_1}{\partial z_1}(z) f_2(z) + f_1(z)\frac{\partial f_2}{\partial z_1}(z) = 0$.

Thus the set of points $z \in B_\varepsilon(0)$ where f as an element of $\mathcal{O}_{\mathbb{C}^n,z}$ is reducible is contained in the analytic set $Z(f, \frac{\partial f}{\partial z_1})$. Now it suffices to show it's a proper subset of $Z(f)$ since f is irreducible, so is $Z(f)$. If not, then $\frac{\partial}{\partial z_1}$ would vanish on $Z(f)$. Since f is irreducible, we can apply Corollary 2.1.1 to obtain $\frac{\partial f}{\partial z_1}$ divides f, a contradiction. □

Proposition 2.5.2. If $f, g \in \mathcal{O}_{\mathbb{C}^n,0}$ are relatively prime, then they're relatively prime in $\mathcal{O}_{\mathbb{C}^n,z}$, for z in a sufficiently small neighborhood of 0.

Proof. Without lose of generality, we may assume $f, g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ are Weierstrass polynomials, then f and g are relatively prime if and only if their resultant $R \in \mathcal{O}_{\mathbb{C}^{n-1}}$ has non-zero germ at 0, therefore the germ of R is also non-zero in a sufficiently small neighborhood of 0. □

3. Complex manifold

3.1. Basic definitions and properties.

Definition 3.1.1 (holomorphic atlas)**.** A holomorphic atlas on a smooth manifold is an atlas $\{(U_\alpha, \varphi_\alpha)\}$ of the form $\varphi_\alpha : U_\alpha \cong \varphi_\alpha(U_\alpha) \subseteq \mathbb{C}^n$ such that transition functions $\varphi_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_{\alpha\beta}) \to \varphi_\alpha(U_{\alpha\beta})$ are holomorphic functions. Furthermore,

(1) the pair $(U_\alpha, \varphi_\alpha)$ is called a holomorphic chart.
(2) two holomorphic atlases are called equivalent, if the union of them is still a holomorphic atlas.

Definition 3.1.2 (complex manifold)**.** A complex n-manifold X is a smooth $2n$-manifold admitting an equivalence class of holomorphic atlases.

Remark 3.1.1. A complex manifold is called connected, compact, simply-connected and so on, if its underlying real manifold has this property.

Definition 3.1.3 (submanifold)**.** Let X be a complex n-manifold and $Y \subseteq Y$ be a smooth manifold of (real) dimension $2k$. Then Y is a complex submanifold if there exists a holomorphic atlas $\{(U_i, \varphi_i)\}$ of X such that $\varphi_i : U_i \cap Y \cong \varphi_i(U_i) \cap \mathbb{C}^k$.

Definition 3.1.4 (holomorphic map)**.** Let X, Y be complex manifolds. A continuous map $f\colon X \to Y$ is a holomorphic map if for any holomorphic charts (U, φ) and (U', φ') of X and Y respectively, the map $\varphi' \circ f \circ \varphi^{-1} : \varphi(f^{-1}(U') \cap U) \to \varphi'(U')$ is holomorphic.

Definition 3.1.5 (biholomorphic)**.** Let X, Y be two complex manifolds. X, Y are called biholomorphic, if there exists a holomorphic homeomorphism $f\colon X \to Y$.

Definition 3.1.6 (holomorphic function)**.** A holomorphic function on complex manifold X is a holomorphic map $f\colon X \to \mathbb{C}$.

Notation 3.1.1. We always use $\mathcal{O}_X$ to denote the sheaf of holomorphic functions on complex manifold X, and use $\Gamma(U, \mathcal{O}_X)$ to denote sections over open subset $U \subseteq X$.

Proposition 3.1.1. Let X be a compact connected complex manifold. Then $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$.

Proof. It's clear from maximum principle. □

Definition 3.1.7 (meromorphic function)**.** A meromorphic function on a complex manifold X is a map $f\colon X \to \coprod_{x \in X} Q(\mathcal{O}_{X,x})$ which associates to any $x \in X$ an element $f_x \in Q(\mathcal{O}_{X,x})$ such that for any $x_0 \in X$ there exists a neighborhood $x_0 \in U \subseteq X$ and two holomorphic functions $g, h\colon U \to \mathbb{C}$ with $f_x = g/h$ for all $x \in U$.

Notation 3.1.2. We always use $\mathcal{K}_X$ to denote the sheaf of meromorphic functions on complex manifold X, and use $K(X)$ to denote $\Gamma(X, \mathcal{K}_X)$.

Definition 3.1.8 (algebraic dimension)**.** The algebraic dimension of a compact connected complex manifold X is $a(X) := \operatorname{trdeg}_{\mathbb{C}} K(X)$.

Proposition 3.1.2 (Siegel)**.** Let X be a compact connected complex n-manifold. Then

$$\operatorname{trdeg}_{\mathbb{C}} K(X) \leq n$$

Proof. See Proposition 2.1.9 in Page54 of [Huy05]. □

3.2. **Analytic subvariety.**

Definition 3.2.1 (analytic subvariety)**.** Let X be a complex manifold. An analytic subvariety of X is a closed subset $Y \subseteq X$ such that for any $x \in Y$ there exists an open neighborhood $x \in U \subseteq X$ such that $Y \cap U$ is a zero set of finitely many holomorphic functions $f_1, \dots, f_k \in \mathcal{O}(U)$.

Remark 3.2.1*.* Obviously, any analytic subvariety X defines an analytic germ in any point $z \in X$.

Definition 3.2.2 (irreducible analytic subvariety)**.** An analytic subvariety Y is called irreducible, if it cannot be written as the union $Y = Y_1 \cup Y_2$ of two proper analytic subvarieties $Y_i \subseteq Y, i = 1, 2$.

Given an analytic subvariety Y of a complex manifold X.

Definition 3.2.3 (regular)**.** A point $x \in Y$ is called regular point, if the functions $f_1, \dots, f_k$ can be chosen such that $\varphi(x) \in \varphi(U)$ is a regular point of holomorphic map $f := (f_1 \circ \varphi^{-1}, \dots, f_k \circ \varphi^{-1}) : \varphi(U) \to \mathbb{C}^k$, where (U, φ) is a local chart of x.

Definition 3.2.4 (singular)**.** A point $x \in Y$ is singular, if it's not regular.

Proposition 3.2.1. The set of regular points $Y_{\text{reg}} = Y \backslash Y_{\text{sing}}$ is a non-empty submanifold of X. Furthermore, if Y is irreducible, then Y_{reg} is connected.

Definition 3.2.5 (dimension)**.** The dimension of an irreducible analytic subvariety Y is defined by $\dim Y = \dim Y_{\text{reg}}$.

3.3. **Examples.**

Example 3.3.1 (affine space)**.** The n-dimensional complex plane $\mathbb{C}^n$ is a complex manifold.

Example 3.3.2 (complex tori)**.** If V is a complex vector space of dimension n and $\Gamma \subseteq V$ is a free abelian, discrete subgroup of order $2n$, then $X = V/\Gamma$ is a complex manifold, which is called complex tori.

Remark 3.3.1*.* The underlying manifolds of complex tori with different Γ are not very interesting since they are all diffeomorphic to $(S^1)^{2n}$. However, if you pick two lattices Γ_1, Γ_2 randomly, then $\mathbb{C}^n/\Gamma_1$ and $\mathbb{C}^n/\Gamma_2$ will not be biholomorphic to each other.

Example 3.3.3 (projective space)**.** The projective space $\mathbb{CP}^n$ is a complex manifold. Indeed, atlas are given by $U_i = \{[z] \in \mathbb{CP}^n \mid z^j \neq 0\}, 0 \leq i \leq n$, and φ_i is defined as

$$\begin{aligned}\varphi_i \colon U_i &\to \mathbb{C}^n \\ [z] &\mapsto (\frac{z_0}{z^i}, \dots, \widehat{\frac{z^i}{z^i}}, \dots, \frac{z_n}{z^i})\end{aligned}$$

The transition functions are calculated as follows: For $i < j$

$$\varphi_i \circ \varphi_j^{-1} \colon (u_1, \dots, u_n) \mapsto (\frac{u_1}{u_i}, \dots, \widehat{\frac{u_i}{u_i}}, \dots, \frac{u_{j-1}}{u_i}, \frac{1}{u_i}, \frac{u_{j+1}}{u_i}, \dots, \frac{u_n}{u_i})$$

It's holomorphic on $U_i \cap U_j$.

Remark 3.3.2. $\mathbb{CP}^n$ is compact since $\mathbb{CP}^n$ is diffeomorphic to S^{2n+1}/S^1, which is called Hopf fiberation.

Definition 3.3.1 (projective manifold)**.** A complex manifold X is called projective if X is biholomorphic to a closed complex submanifold of some projective space $\mathbb{CP}^N$.

Example 3.3.4 (Grassmannian manifold)**.** The Grassmannian manifold

$$Gr(k, n+1) = \{k\text{-dimensional subspace of } \mathbb{C}^{n+1}\}$$

Now we're going to show $Gr(k, n+1)$ is a manifold of dimension $k(n+1-k)$. Any $W \in Gr(k, n+1)$ is generated by the rows of a $k \times (n+1)$ matrix A of rank k. Let us denote the set of these matrices by $M_{k,n+1}$, which is an open subset of the set of all $k \times (n+1)$ matrices. The latter space is a complex manifold which is canonically isomorphic to $\mathbb{C}^{k(n+1)}$. Thus we obtain a natural surjection $\pi \colon M_{k,n+1} \to Gr(k, n+1)$, which is the quotient by the natural action of $\mathrm{GL}(k, \mathbb{C})$ on $M_{k,n+1}$.

Let's fix an ordering $\{B_1, \dots, B_m\}$ of all $k \times k$-minors of matrices $A \in M_{k,n+1}$. Define an open covering $Gr(k, n+1) = \bigcup_{i=1}^m U_i$, where U_i is the open subset $\{\pi(A) \mid \det(B_i) \neq 0\}$. Note that U_i is well-defined since if $\pi(A) = \pi(A')$, then A and A' differs an action of $\mathrm{GL}(k, \mathbb{C})$, so $\det(B_i) \neq 0$ if and only if $\det(B_i') \neq 0$. So without lose of generality, we may assume A is of form (B_i, C_i), where C_i is a $k \times (n+1-k)$ matrix. Then the map $\varphi_i : U_i \to \mathbb{C}^{k(n+1-k)}$, given by $\pi(A) \to B_i^{-1}C_i$ is well-defined, and $\{(U_i, \varphi_i)\}$ will give atlas of $Gr(k, n+1)$, sicne all operations are matrix operation, thus they're holomorphic. This shows $Gr(k, n+1)$ is a complex manifold with dimension $k(n+1-k)$.

Remark 3.3.3. If V is a complex vector space of dimension $n+1$, then $Gr(k, V)$ is defined as the set consisting of all k-dimensional subspaces of V, which is biholomorphic to $Gr(k, n+1)$.

Example 3.3.5 (Plücker embedding)**.** Let V be a complex vector space of dimension $n+1$, then

$$\Phi\colon Gr(k,V) \hookrightarrow \mathbb{CP}(\bigwedge^k V)$$

defined by $W \subseteq V$ with basis $w_1, \dots, w_k$ is mapped to $[w_1 \wedge \cdots \wedge w_k]$, is called Plücker embedding. It's well-defined, thanks to the following lemma.

Lemma 3.3.1. Let W be a complex vector space of dimension k, and $\mathcal{B}_1 = \{w_1, \dots, w_k\}$ and $\mathcal{B}_2 = \{v_1, \dots, v_k\}$ are two basis for W. Then $v_1 \wedge \cdots \wedge v_k = \lambda w_1 \wedge \cdots \wedge w_k$ for some $\lambda \in \mathbb{C}^*$.

Proof. If we express $w_j = a_{1j}v_1 + \cdots + a_{kj}v_k$, then direct computation shows that

$$\begin{aligned} w_1 \wedge \cdots \wedge w_k &= (a_{11}v_1 + \cdots + a_{k1}v_k) \wedge \cdots \wedge (a_{1k}v_1 + \cdots + a_{kk}v_k) \\ &= \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) a_{1\sigma(1)} \cdots a_{k\sigma(k)} v_1 \wedge \cdots \wedge v_k \\ &= \lambda v_1 \wedge \cdots \wedge v_k \end{aligned}$$

Note that λ is exactly the determinant of the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$. $\square$

Remark 3.3.4. It's a little bit complicated to check it's injective.

3.4. **Vector bundle.**

3.4.1. *In viewpoint of transition functions.*

Definition 3.4.1 (complex vector bundle)**.** Let X be a smooth manifold. A complex vector bundle E of rank r on X consists of the following data:

(1) E is a smooth manifold with surjective map $\pi\colon E \to X$, such that
   (1) For all $x \in X$, fibre E_x is a $\mathbb{C}$-vector space of dimension r.
   (2) For all $x \in X$, there exists $x \in U \subseteq X$ and there is a homeomorphism $\varphi\colon \pi^{-1}(U) \to U \times \mathbb{C}^r$ such that

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\pi"] \arrow[dr, "\varphi"'] & & U \\ & U \times \mathbb{C}^r \arrow[ur, "p_1"'] \arrow[r, "p_2"] & \mathbb{C}^r \end{tikzcd}$$

   and for all $y \in U$, $E_y \xrightarrow{p_2 \circ \varphi} \mathbb{C}^r$ is a $\mathbb{C}$-vector space isomorphism. (U, φ) is called a trivialization of E over U.

Remark 3.4.1 (transition functions)*.* Consider two local trivialization $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$. Then $\varphi_\alpha \circ \varphi_\beta^{-1}\colon (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$ induces

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{C})$$

where $g_{\alpha\beta}$ is called transition function. Furthermore, it satisfies

$$\begin{aligned} g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} &= \mathrm{id} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma \\ g_{\alpha\alpha} &= \mathrm{id} \quad \text{on } U_\alpha \end{aligned}$$

In fact, transition functions contain all information about this vector bundle since a vector bundle is locally trivial, so how are these trivial pieces glued together really matters.

Definition 3.4.2 (complex vector bundle)**.** Let X be a smooth manifold. A complex vector bundle E of rank r on X consists of the following data:

(1) open covering $\{U_\alpha\}$ of X.
(2) smooth functions $\{g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{C})\}$ satisfies

$$\begin{aligned} g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} &= \mathrm{id} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma \\ g_{\alpha\alpha} &= \mathrm{id} \quad \text{on } U_\alpha \end{aligned}$$

Remark 3.4.2. The two definitions above are equivalent. The first definition implies the second clearly. The converse is a standard constructive method: If we already have an open covering and a set of transition functions, the vector bundle E is defined to be the quotient of the disjoint union $\coprod_{U_\alpha}(U \times \mathbb{C}^r)$ by the equivalence relation that puts $(p', v') \in U_\beta \times \mathbb{C}^r$ equivalent to $(p, v) \in U_\alpha \times \mathbb{C}^r$ if and only if $p = p'$ and $v' = g_{\alpha\beta}(p)v$. To connect this definition with the previous one, define the map π to send the equivalence class of any given (p, v) to p.

Definition 3.4.3 (holomorphic vector bundle)**.** A holomorphic vector bundle $\pi\colon E \to X$ over a complex manifold X is a complex vector bundle with holomorphic transition functions.

Exercise 3.4.1. Show that the total space of a holomorphic vector bundle E is a complex manifold.

Proof. Since we already have a complex structure on X, we need to pull it back to E using π and use the holomorphic transition functions to show it do gives a complex structure on E. □

Example 3.4.1 (trivial bundle)**.** Let X be a smooth/complex manifold. Then $X \times \mathbb{C}^r$ is called trivial complex (holomorphic) vector bundle of rank r on X.

Definition 3.4.4 (subbundle)**.** Let $\pi\colon E \to X$ be a complex (holomorphic) vector bundle. $F \subseteq E$ is called a subbundle of rank s, if

(1) For all $x \in X$, $F \cap E_x$ is a subspace of E_x with dimension s.
(2) $\pi|_F\colon F \to X$ induces a complex (holomorphic) vector bundle.

3.4.2. *In viewpoint of sheaf.* One may refer to Appendix 20 for more details about sheaf.

Definition 3.4.5 (section)**.** Let X be a complex manifold and $\pi\colon E \to X$ be a complex (holomorphic) vector bundle. For any open subset $U \subseteq X$, a section of E over U is a smooth/holomorphic map $s\colon U \to E$ such that $\pi \circ s = \mathrm{id}_U$.

Notation 3.4.1. The set of all smooth (or holomorphic) sections over open subset U is denoted by $C^\infty(U, E)$ (or $\Gamma(U, E)$).

One reason why sheaf plays an important role of study of complex geometry is that you can regard a vector bundle as a special sheaf.

Definition 3.4.6 (sheaf of sections)**.** Let X be a complex manifold and $\pi\colon E \to X$ a holomorphic vector bundle. Then its sheaf of sections is defined as

$$\mathcal{O}_X(E)(U) = \Gamma(U, E)$$

Example 3.4.2. Let $E \to X$ be trivial holomorphic vector bundle. Then $\mathcal{O}_X(E)$ is exactly sheaf of holomorphic functions.

Example 3.4.3 (locally free sheaf)**.** A sheaf $\mathscr{F}$ over a topological space X is called locally free, if there exists an open covering $\{U_\alpha\}$ of X such that $\mathscr{F}|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}^{\oplus r}$ of rank r.

Exercise 3.4.2. Let X be a complex manifold. There is one to one correspondence over X:

$$\{\text{holomorphic vector bundles}\} \xleftrightarrow{1-1} \{\text{locally free sheaves}\}$$

Proof. If $\pi\colon E \to X$ is a holomorphic vector bundle, then $\mathcal{O}_X(E)$ is a locally free sheaf. Indeed since we have local trivialization of holomorphic vector bundle $\{U_\alpha\}$, that is $E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^r$, and it's known to all that sections of a trivial holomorphic function is exactly holomorphic functions, thus $\mathcal{O}(E)|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}^{\oplus r}$, that is $\mathcal{O}_X(E)$ is a locally free sheaf.

Conversely, if $\mathcal{E}$ is locally free over an open covering $\{U_\alpha\}$ of X, then we just need to glue $U_\alpha \times \mathbb{C}^r \to U_\alpha$ together to get a vector bundle. Therefore we need a family of gluing data $g_{\alpha\beta}\colon (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$. Since $\mathcal{E}$ is locally free, we have local isomorphism $f_\alpha\colon \mathcal{E}|_{U_\alpha} \to \mathcal{O}_{U_\alpha}^{\oplus r}$. Restricting to intersection $U_\alpha \cap U_\beta$, we get

$$f_{\alpha\beta} = f_\alpha|_{U_\alpha\cap U_\beta} \circ f_\beta^{-1}|_{U_\alpha\cap U_\beta}\colon \mathcal{O}_{U_\beta}^{\oplus r}|_{U_\alpha\cap U_\beta} \to \mathcal{O}_{U_\alpha}^{\oplus r}|_{U_\alpha\cap U_\beta}$$

Every such map is induced by a map

$$g_{\alpha\beta}\colon (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$$

that's gluing data we desire. □

Definition 3.4.7 (cohomology of vector bundle)**.** Let E be a holomorphic vector bundle on a complex manifold X. Then its q-th cohomology $H^q(X, E)$ is defined to be q-th sheaf cohomology of $\mathcal{O}_X(E)$

3.4.3. *Algebraic construction.* Let E, F be complex (holomorphic) vector bundles on X with transition functions $\{g_{\alpha\beta}\}, \{h_{\alpha\beta}\}$ respectively. Then by algebraic construction we have

(1) $E \oplus F$, given by transition functions $\{\mathrm{diag}(g_{\alpha\beta}, h_{\alpha\beta})\}$
(2) $E \otimes F$, given by transition functions $\{g_{\alpha\beta} \otimes h_{\alpha\beta}\}$.
(3) E^*, given by transition functions $\{(g_{\alpha\beta}^{-1})^T\}$.
(4) $\mathrm{Hom}(E, F) := E^* \otimes F$.
(5) $\bigwedge^k E$, given by transition functions $\{\bigwedge^r g_{\alpha\beta}\}$.
(6) Let $f\colon X \to Y$ be a smooth/holomorphic map, $\pi\colon E \to Y$ is a vector bundle with transition functions $\{g_{\alpha\beta}\}$, then transition functions of pullback bundle f^*E is given by $\{g_{\alpha\beta} \circ f\}$.

Remark 3.4.3. Here is an explicit construction of pullback bundle defined by

$$f^*E = \{(x, e) \in X \times E \mid f(x) = \pi(e)\} \subseteq X \times E$$

In fact, you can regard it as a push out as

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

In particular, pullback bundle has universal property.

3.4.4. *Hermitian structure.* Let X be a smooth manifold and $\pi\colon E \to X$ be a complex vector bundle.

Definition 3.4.8 (Hermitian metric)**.** A Hermitian metric h on E is a global section of $E^* \otimes \overline{E}$.

Remark 3.4.4 (local form). Let $\{e_\alpha\}$ be a local frame of E. Then Hermitian metric h is given by

$$h = h_{\alpha\overline{\beta}} e^\alpha \otimes \overline{e}^\beta$$

where $h_{\alpha\overline{\beta}}$ is a Hermitian matrix.

Proposition 3.4.1. Every complex vector bundle admits a Hermitian metric.

Proof. Use partition of unity. □

3.4.5. *Line bundle.*

Definition 3.4.9 (line bundle)**.** A complex (or holomorphic) line bundle L is a complex (or holomorphic) vector bundle of rank one.

Proposition 3.4.2. Let L be a complex line bundle over X. Then $L \otimes L^*$ is the trivial bundle.

Proof. Suppose $\{g_{\alpha\beta}\}$ is the transition functions of L, by Section 3.4.3 it's clear to see the transition functions of $L^* \otimes L$ is

$$(g_{\alpha\beta}^{-1})^T g_{\alpha\beta} = g_{\alpha\beta}^{-1} g_{\alpha\beta} = \mathrm{id}$$

This completes the proof. □

Proposition 3.4.3. Let L be a holomorphic over a compact complex manifold X. Then L is trivial if and only if both L and its dual L^* admit non-trivial global section.

Proposition 3.4.4. Let $\pi\colon E \to X$ be a complex line bundle. Then E is a trivial line bundle if and ony if there exists a nowhere vanishing global section s.

Proof. It's clear there exists a nowhere vanishing global section if E is trivial. Conversely, if there exists a nowhere vanishing global section s. Consider the following map

$$\begin{aligned} \varphi\colon X \times \mathbb{C} &\to E \\ (x, \lambda) &\mapsto \lambda s(x) \end{aligned}$$

It's an isomorphism since fiberwisely one has $\varphi_x(\lambda) = \lambda s(x)$, and it's injective thus isomorphism since $s(x) \neq 0$. □

Definition 3.4.10 (picard group)**.** The picard group $\mathrm{Pic}(X)$ denotes set of all holomorphic line bundles on X up to isomorphism, whose group structure is given by tensor product.

Proposition 3.4.5. There is a natural isomorphism $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$.

Proof. For a line bundle L, it's completely determined by its transition functions $g_{\alpha\beta}\colon U_{\alpha\beta} \to \mathbb{C}^*$, which is holomorphic functions. It gives rise to an element in $\check{H}^1(X, \mathcal{O}_X^*)$ since $g_{\alpha\beta}$ satisfies cocycle conditions. Furthermore, Čech cohomology[2] computes the sheaf cohomology for reasonable topological space, e.g. for manifolds. $\square$

Remark 3.4.5. This proposition gives us a method to compute Picard group of a complex manifold since there is exponential sequence as follows

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

which is a exact sequence of sheaves, then it gives a long exact sequence of cohomology groups as follows

$$\cdots \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to \dots$$

Thus $\mathrm{Pic}(X)$ can in principle be computed by above exact sequence. Roughly speaking, $\mathrm{Pic}(X)$ has two parts:

(1) A discrete part, measured by its image in $H^2(X, \mathbb{Z})$.
(2) A continuous part coming from the $H^1(X, \mathcal{O}_X)$, which is possibly trivial.

Proposition 3.4.6. The set $\mathcal{O}_{\mathbb{CP}^n}(-1) \subseteq \mathbb{CP}^n \times \mathbb{C}^{n+1}$ that consists of all pairs $(l, z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1}$ with $z \in l$ forms a holomorphic line bundle, called tautological line bundle

Proof. Let $\pi\colon \mathcal{O}_{\mathbb{CP}^n}(-1) \to \mathbb{CP}^n$ be the projection to the first factor. Consider open covering $\{U_i\}_{i=0}^n$ of $\mathbb{CP}^n$, where

$$U_i = \{[l] = [l_0 : \dots : l_n] \in \mathbb{CP}^n \mid l_i \neq 0\}$$

A canonical trivialization of $\mathcal{O}_{\mathbb{CP}^n}(-1)$ over U_i is given by

$$\begin{aligned} \varphi_i\colon \pi^{-1}(U_i) &\to U_i \times \mathbb{C} \\ (l, z) &\mapsto (l, z_i) \end{aligned}$$

Its transition function is computed as follows

$$\begin{aligned} \varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times \mathbb{C} &\longrightarrow (U_i \cap U_j) \times \mathbb{C} \\ (l, w) &\mapsto (l, w\frac{l_i}{l_j}) \end{aligned}$$

where $l = (l_0 : \dots : l_n)$. This shows its transition function $g_{ij}(z) = z_i/z_j \in \mathbb{C}^*$ is holomorphic. $\square$

Definition 3.4.11 (line bundles on $\mathbb{CP}^n$)**.**

$$\begin{cases} \mathcal{O}_{\mathbb{CP}^n}(-k) = \mathcal{O}_{\mathbb{CP}^n}(-1)^{\otimes k} & k \in \mathbb{Z}_{>0} \\ \mathcal{O}_{\mathbb{CP}^n}(k) = (\mathcal{O}_{\mathbb{CP}^n}(-k))^* & k \in \mathbb{Z}_{>0} \\ \mathcal{O}_{\mathbb{CP}^n}(0) = \mathbb{CP}^n \times \mathbb{C} \end{cases}$$

[2]For more details, see Appendix 20.

Proposition 3.4.7. For $k \geq 0$. the space $\Gamma(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(k))$ is canonically isomorphic to the space $\mathbb{C}[z_0, \dots, z_n]_k$ of all homogenous polynomials of degree k.

Corollary 3.4.1. For $k < 0$ the line bundle $\mathcal{O}_{\mathbb{CP}^n}(k)$ admits no global holomorphic section.

Proof. It follows from above result and Proposition 3.4.3. □

3.5. **Euler sequence and adjunction formula.**

3.5.1. *Euler sequence.*

Proposition 3.5.1. On $\mathbb{CP}^n$ there exists a natural short exact sequence of holomorphic vector bundles

$$0 \to \mathcal{O}_{\mathbb{CP}^n} \xrightarrow{\phi} \mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus n+1} \xrightarrow{\psi} T\,\mathbb{CP}^n \to 0$$

Exercise 3.5.1. For Grassmannian manifold $Gr(k,n)$, we have

$$0 \to E \to Gr(k,n) \otimes \mathbb{C}^n \to Q \to 0$$

Show that

$$T_{Gr(k,n)} \cong \operatorname{Hom}(E, Q)$$

3.5.2. *Adjunction formula.*

Proposition 3.5.2 (adjunction formula)**.** Let $\pi \colon L \to X$ is a holomorphic line bundle and s be a holomorphic section of L. Suppose that $D = \{x \in X \mid s(x) = 0\}$ is a smooth submanifold of codimensional 1. Show that the following sequence is exact

$$0 \to TD \to TX|_D \to L|_D \to 0$$

As a consequence

$$K_D^* \cong K_D^* \otimes L|_D = (K_X^* \otimes L)|_D$$

Or equivalently

$$K_D \cong (K_X \otimes L)|_D$$

This is called adjunction formula.

Proof. Firstly we have the following exact sequence

$$0 \to TD \to TX|_D \to N_D \to 0$$

where N_D is the normal bundle. Now it suffices to show $L|_D$ is isomorphic to the normal bundle of D. Note that $s|_D = 0$ and s is not identically zero on X, so $\mathrm{d}s$ gives an isomorphism between $L|_D$ and N_D in fact. By taking determinant we obtain the adjunction formula since for a exact sequence of vector bundle

$$0 \to A \to B \to C \to 0$$

we have

$$\det B = \det A \otimes \det C$$

and determinant of a line bundle is itself. □

Example 3.5.1. Let $X = \mathbb{CP}^n$ and $L = \mathcal{O}_{\mathbb{CP}^n}(-d)$. Proposition 3.4.7 shows that $D \subseteq \mathbb{CP}^n$ is a smooth hypersurface defined by zero set of a homogenous polynomial with degree d. Then we have

$$\begin{aligned} K_D^* &\cong (K_X^* \otimes L)|_D \\ &= (\mathcal{O}_{\mathbb{CP}^n}(n+1) \otimes \mathcal{O}_{\mathbb{CP}^n}(-d))|_D \\ &\cong \mathcal{O}_{\mathbb{CP}^n}(n+1-d)|_D \end{aligned}$$

Remark 3.5.1. As a consequence, D is called

$$\begin{cases} \text{Fano} & d < n+1 \\ \text{Calabi-Yau} & d = n+1 \\ \text{general type} & d > n+1 \end{cases}$$

These concepts we will define later.

4. Divisor and line bundle

In this section, unless otherwise specified, we assume X is a complex manifold.

4.1. Divisor.

Definition 4.1.1 (analytic hypersurface)**.** An analytic hypersurface of X is an analytic subvariety $Y \subseteq X$ of codimensional one.

Remark 4.1.1. By Remark 2.4.2 one has a hypersurface is locally given as the zero set of a non-trivial holomorphic function.

Definition 4.1.2 (divisor)**.** A divisor D on X is a locally finite[3] formal linear combination $D = \sum a_i[Y_i]$ with $Y_i \subseteq X$ are irreducible hypersurfaces and $a_i \in \mathbb{Z}$.

Definition 4.1.3 (divisor group)**.** The divisor group $\mathrm{Div}(X)$ is the set of all divisors endowed with the natural group structure.

Definition 4.1.4 (effective)**.** A divisor $D = \sum a_i[Y_i]$ is called effective, if $a_i \geq 0$ for all i. In this case, we write $D \geq 0$.

Proposition 4.1.1. Every hypersurfaces Y defines an effective divisor $\sum[Y_i] \in \mathrm{Div}(X)$, where Y_i are irreducible components of Y.

Proof. It suffices to show the irreducible components of a hypersurface Y is locally finite. □

Let $Y \subseteq X$ be a hypersurface and $x \in Y$. Suppose that Y defines an irreducible germ in x, that is this germ is the zero set of an irreducible $g \in \mathcal{O}_{X,x}$.

Definition 4.1.5 (order)**.** Let f be a meromorphic function in a neighborhood of $x \in Y$. Then the order $\mathrm{ord}_{Y,x}(f)$ of f in x with respect to Y is given by

$$f = g^{\mathrm{ord}_{Y,x}(f)} h$$

where $h \in \mathcal{O}^*_{X,x}$.

Remark 4.1.2.

(1) The order of f in x with respect to Y is independent of the choice of g since any two irreducible $g, g' \in \mathcal{O}_{X,x}$ with $Z(g) = Z(g')$ only differs by an element in $\mathcal{O}^*_{X,x}$.
(2) More globally, one can define order $\mathrm{ord}_Y(f)$ as $\mathrm{ord}_Y(f) = \mathrm{ord}_{Y,x}(f)$ for $x \in Y$ such that Y defines an irreducible germ in x. Such a point $x \in Y$ always exists, for example, one can choose a regular point $x \in Y_{\mathrm{reg}}$. Moreover, it's independent of the choice of x since

[3] The sum is called locally finite, if for any $x \in X$, there exists an open neighborhood $x \in U \subseteq X$ such that only finite many coefficients $a_i \neq 0$ with $Y_i \cap U \neq \varnothing$.

Definition 4.1.6 (zeros and poles)**.** Let f be a meromorphic function on X. Then

(1) f has zeros of order $d \geq 0$ along Y if $\mathrm{ord}_Y(f) = d$.
(2) f has poles of order $d \geq 0$ along Y if $\mathrm{ord}_Y(f) = -d$.

Definition 4.1.7 (principal divisor)**.** For $f \in K(X)$, the divisor associated to f is

$$(f) := \sum \mathrm{ord}_Y(f)[Y]$$

where the sum is taken over all irreducible hypersurfaces $Y \subseteq X$. A divisor of this form is called principal.

Remark 4.1.3. The divisor (f) can be written as the difference of two effective divisors $(f) = Z(f) - P(f)$, where

$$Z(f) = \sum_{\mathrm{ord}_Y(f)>0} \mathrm{ord}_Y(f)[Y], \quad P(f) = \sum_{\mathrm{ord}_Y(f)<0} \mathrm{ord}_Y(f)[Y]$$

Proposition 4.1.2. There exists a natural isomorphism

$$H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \cong \mathrm{Div}(X)$$

Proof. An element $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ is given by non-trivial meromorphic functions $f_i \in K_X^*(U_i)$ such that $f_i f_j^{-1}$ is a holomorphic function without zeros on $U_i \cap U_j$, where $\{U_i\}$ is an open covering of X. Thus for any irreducible hypersurface $Y \subseteq X$ with $Y \cap U_i \cap U_j \neq \varnothing$, one has $\mathrm{ord}_Y(f_i) = \mathrm{ord}_Y(f_j)$. Hence $\mathrm{ord}_Y(f)$ is well-defined for any irreducible hypersurface Y. Then one associates to f the divisor $(f) = \sum \mathrm{ord}_Y(f)[Y] \in \mathrm{Div}(X)$.

It's clear this map is a group homomorphism. To see it's bijective, we define the inverse as follows. If $D = \sum a_i[Y_i] \in \mathrm{Div}(X)$ is given, then there exists an open covering $\{U_i\}$ of X such that $Y_i \cap U_j$ is defined by $g_{ij} \in \mathcal{O}(U_j)$ which is unique up to elements in $\mathcal{O}^*(U_j)$. Let $f_j := \prod_i g_{ij}^{a_i} \in \mathcal{K}_X^*(U_j)$ since g_{ij} and g_{ik} defines the same irreducible hypersurface, they only differ by an element in $\mathcal{O}^*(U_j \cap U_k)$. Thus f glue to an element $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. It's clear these two maps are inverse to each other. □

Remark 4.1.4. In algebraic geometry, elements in $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ are called Cartier divisors and elements in $\mathrm{Div}(X)$ are called Weil divisors. Above isomorphism still holds in the algebraic setting under a weak smoothness assumption on X.

Corollary 4.1.1. There exists a natural group homomorphism

$$\begin{aligned} \mathrm{Div}(X) &\to \mathrm{Pic}(X) \\ D &\mapsto \mathcal{O}(D) \end{aligned}$$

where $\mathcal{O}(D)$ is defined in the proof.

Proof. If $D = \sum a_i[Y_i] \in \mathrm{Div}(X)$ corresponds to $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$, which in turn is given by functions $f_i \in \mathcal{K}_X^*(U_i)$ for an open covering $\{U_i\}$. Then we define $\mathcal{O}(D) \in \mathrm{Div}(X)$ with transition functions $\psi_{ij} := f_i f_j^{-1} \in \mathcal{O}_X^*(U_{ij})$.

If D, D' are two divisors, without lose of generality we may assume they're given by $\{f_i\}$ and $\{f_i'\}$ respectively on the same open covering, then $D+D'$, then $D+D'$ corresponds to $\{f_i+f_i'\}$. By definition $\mathcal{O}(D+D')$ is described by $\{\psi_{ij}\psi_{ij}'\}$, hence $\mathcal{O}(D+D')=\mathcal{O}(D)\otimes\mathcal{O}(D')$. □

Remark 4.1.5. In fact, above corollary can be derived from the following exact sequence of sheaves

$$0\to\mathcal{O}_X^*\to\mathcal{K}_X^*\to\mathcal{K}_X^*/\mathcal{O}_X^*\to 0$$

Then above group homomorphism is exactly the boundary map, the kernel of which coincides with the image of $H^0(X,\mathcal{K}_X^*)\to H^0(X,\mathcal{K}_X^*/\mathcal{O}_X^*)$, and the latter by definition is the set of principal divisors.

Definition 4.1.8 (linearly equivalent)**.** Two divisors D, D' are called linearly equivalent, denoted by $D\sim D'$, if $D-D'$ is a principal divisor.

Corollary 4.1.2. The group homomorphism $\mathrm{Div}(X)\to\mathrm{Pic}(X)$ factorizes over an injection

$$\mathrm{Div}(X)/_\sim\hookrightarrow\mathrm{Pic}(X)$$

4.2. **Relations between divisor and line bundle.** In general, $\mathrm{Div}(X)/_\sim\hookrightarrow\mathrm{Pic}(X)$ is a strict inclusion, but we will see if a line bundle admits a non-trivial global section, then it's contained in the image. In order to show this, we need to construct a canonical map

$$\begin{aligned}H^0(X,L)\backslash\{0\}&\to\mathrm{Div}(X)\\ s&\mapsto Z(s)\end{aligned}$$

The map is constructed as follows: Let $L\in\mathrm{Pic}(X)$ on open covering $\{U_i\}$ be trivialized by $\psi_i\colon L|_{U_i}\to\mathcal{O}_{U_i}$. Then divisor $Z(s)$ is given by $f:=\{f_i:=\psi_i(s|_U\}\in H^0(X,\mathcal{K}_X^*/\mathcal{O}_X^*)$.

Proposition 4.2.1. For $0\neq s\in H^0(X,L)$, the line bundle $\mathcal{O}(Z(s))$ is isomorphic to L.

Proposition 4.2.2. For any effective divisor $D\in\mathrm{Div}(X)$, there exists a section $0\neq s\in H^0(X,\mathcal{O}(D))$ with $Z(s)=D$.

Corollary 4.2.1. Non-trivial sections $s_1\in H^0(X,L_1)$ and $s_2\in H^0(X,L_2)$ define linearly equivalent divisors $Z(s_1)\sim Z(s_2)$ if and only if $L_1\cong L_2$.

Proof. If $L_1\cong L_2$, then

If $Z(s_1)\sim Z(s_2)$, then by Corollary 4.1.2 one has $\mathcal{O}(Z(s_1))\cong\mathcal{O}(Z(s_2))$, then this shows $L_1\cong L_2$ since $\mathcal{O}(Z(s_i))=L_i, i=1,2$. □

Corollary 4.2.2. The image of the natural map $\mathrm{Div}(X)\to\mathrm{Pic}(X)$ is generated by those line bundles $L\in\mathrm{Pic}(X)$ with $H^0(X,L)\neq 0$.

Proof. We have already seen if $H^0(X,L)\neq 0$, then L is contained in the image. Conversely, any divisor $D=\sum a_i[Y_i]$ can be written as $D=\sum a_i^+[Y_i]-\sum a_j^-[Y_j]$ with $a_k^\pm\geq 0$, and thus $\mathcal{O}(D)\cong\mathcal{O}(\sum a_i^+[Y_i])\otimes\mathcal{O}(\sum a_j^-[Y_j])^*$. Both

$\mathcal{O}(\sum a_i^+[Y_i])$ and $\mathcal{O}(\sum a_j^-[Y_j])$ are associated to effective divisors, and therefore admit non-trivial global sections. $\square$

Remark 4.2.1. For projective manifolds, the map $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ is surjective, but note that even for very easy manifolds, such as complex tori, this is no longer the case.

4.3. **Ample line bundle.**

Definition 4.3.1 (base point)**.** Let L be a holomorphic line bundle on a complex manifold X. A point $x \in X$ is a base point of L if $s(x) = 0$ for all $s \in H^0(X, L)$.

Notation 4.3.1. The base locus $\mathrm{Bs}(L)$ is the set of all base points of L.

Remark 4.3.1. If $\dim H^0(X, L) < \infty$, we can choose a basis of global sections $s_1, \dots, s_N$ of it, then $\mathrm{Bs}(L) = Z(s_1) \cap \dots \cap Z(s_N)$ is an analytic subvariety. Later we will see if X is compact, then $\dim H^0(X, L) < \infty$.

Proposition 4.3.1. Let L be a holomorphic line bundle on a complex manifold X and suppose $s_1, \dots, s_N \in H^0(X, L)$ is a basis. Then

$$\begin{aligned} \varphi_L \colon X \backslash \mathrm{Bs}(L) &\to \mathbb{CP}^N \\ x &\mapsto (s_0(x) : \dots : s_N(x)) \end{aligned}$$

defines a holomorphic map such that $\varphi_L^* \mathcal{O}_{\mathbb{CP}^N}(-1) \cong L|_{X \backslash \mathrm{Bs}(L)}$.

Definition 4.3.2 (ample line bundle)**.** A holomorphic line bundle L on a complex manifold X is called ample if for some $k > 0$ and some linear system in $H^0(X, L^k)$ the associated map φ is an embedding.

Remark 4.3.2. By definition, a compact complex manifold is projective if and only if it admits an ample line bundle.

5. Tangent and cotangent bundle

5.1. Complex and holomorphic tangent bundle.

Definition 5.1.1 (complex tangent bundle)**.** Let X be a smooth n-manifold with an atlas $\{U_\alpha, \varphi_\alpha\colon U_\alpha\to V_\alpha\subseteq\mathbb{R}^n\}$. Then (real) tangent bundle $T_{\mathbb{R}}X$ is a vector bundle given by smooth transition functions

$$\begin{aligned}g_{\alpha\beta}\colon U_\alpha\cap U_\beta&\to \mathrm{GL}(n,\mathbb{R})\\ x&\mapsto J_{\mathbb{R}}(\varphi_\alpha\circ\varphi_\beta^{-1})(\varphi_\beta(x))\end{aligned}$$

The complex tangent bundle $T_{\mathbb{C}}X$ is defined as the complexification of $T_{\mathbb{R}}X$, that is, $T_{\mathbb{R}}X\otimes_{\mathbb{R}}\mathbb{C}$.

Definition 5.1.2 (holomorphic tangent bundle)**.** Let X be a complex n-manifold, with an atlas $\{U_\alpha, \varphi_\alpha\colon U_\alpha\to V_\alpha\subseteq\mathbb{C}^n\}$. Then holomorphic tangent bundle TX is given by holomorphic transition functions

$$\begin{aligned}g_{\alpha\beta}\colon U_\alpha\cap U_\beta&\to \mathrm{GL}(n,\mathbb{C})\\ z&\mapsto J_{\mathbb{C}}(\varphi_\alpha\circ\varphi_\beta^{-1})(\varphi_\beta(z))\end{aligned}$$

Remark 5.1.1 (relations between complex tangent bundle and holomorphic tangent bundle). Let X be a complex n-manifold and $\{z^i=x^i+\sqrt{-1}y^i\}_{1\leq i\leq n}$ be a local coordinate of X. Then $\{x^1,\dots,x^n,y^1,\dots,y^n\}$ gives a local coordinate of its underlying real $2n$-manifold, and there is an almost complex structure J on $T_{\mathbb{R}}X$ given by

$$\begin{aligned}J(\frac{\partial}{\partial x^i})&=\frac{\partial}{\partial y^i}\\ J(\frac{\partial}{\partial y^i})&=-\frac{\partial}{\partial x^i}\end{aligned}$$

Thus complex tangent bundle $T_{\mathbb{C}}X$ can be decomposed as $T_{\mathbb{C}}X=T^{1,0}X\oplus T^{0,1}X$ with respect to J, with local frames as follows:

(1) $\{\frac{\partial}{\partial z^i}:=\frac{1}{2}(\frac{\partial}{\partial x^i}-\sqrt{-1}\frac{\partial}{\partial y^i})\}$ is a local frame of $T^{1,0}X$.
(2) $\{\frac{\partial}{\partial \overline{z}^i}:=\frac{1}{2}(\frac{\partial}{\partial x^i}+\sqrt{-1}\frac{\partial}{\partial y^i})\}$ is a local frame of $T^{0,1}X$.

Indeed, direct computation shows

$$\begin{aligned}J(\frac{\partial}{\partial z^i})&=\frac{1}{2}(\frac{\partial}{\partial y^i}+\sqrt{-1}\frac{\partial}{\partial x^i})=\frac{\sqrt{-1}}{2}(\frac{\partial}{\partial x^i}-\sqrt{-1}\frac{\partial}{\partial y^i})=\sqrt{-1}\frac{\partial}{\partial z^i}\\ J(\frac{\partial}{\partial \overline{z}^i})&=\frac{1}{2}(\frac{\partial}{\partial y^i}-\sqrt{-1}\frac{\partial}{\partial x^i})=-\frac{\sqrt{-1}}{2}(\frac{\partial}{\partial x^i}+\sqrt{-1}\frac{\partial}{\partial y^i})=-\sqrt{-1}\frac{\partial}{\partial \overline{z}^i}\end{aligned}$$

and for any section s of $T_{\mathbb{C}}X$, one has the following decomposition

$$s=\frac{1}{2}(s-\sqrt{-1}J(s))+\frac{1}{2}(s+\sqrt{-1}J(s))$$

Note that TX is isomorphic to $(T_{\mathbb{R}}X, J)$ as complex vector bundle, and we claim $(T_{\mathbb{R}}X,J)$ is isomorphic to $T^{1,0}X$ as a complex vector bundle. Indeed, there is a natural inclusion $T_{\mathbb{R}}X\hookrightarrow T_{\mathbb{C}}X$, if we compose this inclusion with projection $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}_X \to T^{1,0}X$ onto the first summand, we obtain an $\mathbb{C}$-isomorphism $(T_{\mathbb{R}}X, J) \to T^{1,0}X$ with inverse map 2Re(-).

In particular, $T^{1,0}X$ is isomorphic to TX as complex vector bundles, so we can endow $T^{1,0}X$ with holomorphic structure such that $T^{1,0}X$ is isomorphic to TX as holomorphic vector bundles, and thus we can use $\{\mathrm{d}z^i\}$ as local frame of holomorphic tangent bundle.

5.2. **Bidegree forms.** For complex manifold X, there is also an almost complex structure on $\Omega^1_{X,\mathbb{R}}$, that is dual bundle of $T_{\mathbb{R}}X$, and complexified dual space of $T_{\mathbb{R}}X$ admits an analogous decomposition:

$$\Omega^1_{X,\mathbb{C}} = \Omega^1_{X,\mathbb{R}} \otimes \mathbb{C} = \Omega^{1,0}_X \oplus \Omega^{0,1}_X$$

There is also a decomposition on its k-th wedge product as follows:

$$\Omega^k_{X,\mathbb{C}} = \bigwedge^k \Omega^1_{X,\mathbb{C}} = \bigoplus_{p+q=k} \Omega^{p,q}_X$$

where $\Omega^{p,q}_X = \bigwedge^p \Omega^{1,0}_X \otimes \bigwedge^q \Omega^{0,1}_X$.

Definition 5.2.1 ((p,q)-form)**.** A k-form ω of type (p,q) is a smooth section of $\Omega^{p,q}_X$, that is

$$\omega \in C^\infty(X, \Omega^{p,q}_X) \subseteq C^\infty(X, \Omega^k_{X,\mathbb{C}})$$

Remark 5.2.1 (local form). Suppose $\{z^1, \dots, z^n\}$ is a local coordinate of X, and denote $z^i = x^i + \sqrt{-1}y^i$. Then $\{\mathrm{d}x^1, \dots, \mathrm{d}x^n, \mathrm{d}y^1, \dots, \mathrm{d}y^n\}$ gives a local frame of $\Omega^1_{X,\mathbb{R}}$, and induced almost complex structure is given by

$$\begin{aligned} J^*(\mathrm{d}x^i)(\frac{\partial}{\partial x^i}) &= \mathrm{d}x^i(J(\frac{\partial}{\partial x^i})) = \mathrm{d}x^i(\frac{\partial}{\partial y^i}) = 0 \\ J^*(\mathrm{d}x^i)(\frac{\partial}{\partial y^i}) &= \mathrm{d}x^i(J(\frac{\partial}{\partial y^i})) = \mathrm{d}x^i(-\frac{\partial}{\partial x^i}) = -1 \end{aligned}$$

that is

$$\begin{aligned} J^*(\mathrm{d}x^i) &= -\mathrm{d}y^i \\ J^*(\mathrm{d}y^i) &= \mathrm{d}x^i \end{aligned}$$

and similarly we have

(1) $\{\mathrm{d}z^i := \mathrm{d}x^i + \sqrt{-1}\mathrm{d}y^i\}$ is a local frame of $\Omega^{1,0}_X$.
(2) $\{\mathrm{d}\overline{z}^i := \mathrm{d}x^i - \sqrt{-1}\mathrm{d}y^i\}$ is a local frame of $\Omega^{0,1}_X$.

For a k-form, it locally looks like

$$\sum_{\substack{|I|=p,|J|=q \\ p+q=k}} f_{IJ}\mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J$$

where f_{IJ} are smooth functions, and a k-form is a (p,q)-form if and only if locally it looks like

$$\sum_{|I|=p,|J|=q} f_{IJ}\mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J$$

Exercise 5.2.1. For $\mathbb{C}^n \cong \mathbb{R}^{2n}$, one has

$$\mathrm{d}x^1 \wedge \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}x^n \wedge \mathrm{d}y^n = (\frac{\sqrt{-1}}{2})^n \mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^1 \wedge \cdots \wedge \mathrm{d}z^n \wedge \mathrm{d}\overline{z}^n$$

Proof. It suffices to show the case $n = 1$, and we can compute directly as follows

$$\begin{aligned}(\frac{\sqrt{-1}}{2})\mathrm{d}z \wedge \mathrm{d}\overline{z} &= (\frac{\sqrt{-1}}{2})(\mathrm{d}x + \sqrt{-1}\mathrm{d}y) \wedge (\mathrm{d}x - \sqrt{-1}\mathrm{d}y) \\ &= (\frac{\sqrt{-1}}{2})(-2\sqrt{-1}\mathrm{d}x \wedge \mathrm{d}y) \\ &= \mathrm{d}x \wedge \mathrm{d}y\end{aligned}$$

□

5.3. **Dolbeault operators.** For complex manifold X, naturally there is a differential operator

$$\mathrm{d}\colon C^\infty(X, \Omega^k_{X,\mathbb{C}}) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{C}})$$

Since there is a decomposition for $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{C}})$, it's natural to ask how to decompose $\mathrm{d}\alpha \in C^\infty(X, \Omega^{k+1}_{X,\mathbb{C}})$.

Example 5.3.1. For smooth function α on X, locally a direct computation shows

$$\begin{aligned}\mathrm{d}\alpha &= \frac{\partial\alpha}{\partial x^i}\mathrm{d}x^i + \frac{\partial}{\partial y^i}\mathrm{d}y^i \\ &= \frac{1}{2}(\frac{\partial\alpha}{\partial x^i} - \sqrt{-1}\frac{\partial\alpha}{\partial y^i})\mathrm{d}z^i + \frac{1}{2}(\frac{\partial\alpha}{\partial x^i} + \sqrt{-1}\frac{\partial\alpha}{\partial y^i})\mathrm{d}\overline{z}^i \\ &= \frac{\partial\alpha}{\partial z^i}\mathrm{d}z^i + \frac{\partial\alpha}{\partial \overline{z}^i}\mathrm{d}\overline{z}^i\end{aligned}$$

If we denote

$$\begin{aligned}\partial\alpha &= \frac{\partial\alpha}{\partial z^i}\mathrm{d}z^i \\ \overline{\partial}\alpha &= \frac{\partial\alpha}{\partial \overline{z}^i}\mathrm{d}\overline{z}^i\end{aligned}$$

then $\mathrm{d}\alpha = \partial\alpha + \overline{\partial}\alpha$, where $\partial\alpha \in C^\infty(X, \Omega^{1,0}_X)$ and $\overline{\partial}\alpha \in C^\infty(X, \Omega^{0,1}_X)$. In general case, for $\alpha \in C^\infty(X, \Omega^{p,q}_X)$, locally looks like

$$\alpha = \sum_{|I|=p,|J|=q} \alpha_{IJ}\mathrm{d}z^J \wedge \mathrm{d}\overline{z}^K$$

then

$$\mathrm{d}\alpha = \sum_{|I|=p,|J|=q} \frac{\partial\alpha_{IJ}}{\partial z^l}\mathrm{d}z^l \wedge \mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J + \sum_{|I|=p,|J|=q} \frac{\partial\alpha_{IJ}}{\partial \overline{z}^l}\mathrm{d}\overline{z}^l \wedge z^I \wedge \overline{z}^J$$

Thus one can define

$$\begin{aligned}\partial\colon C^\infty(X, \Omega^{p,q}_X) &\to C^\infty(X, \Omega^{p+1,q}_X) \\ \overline{\partial}\colon C^\infty(X, \Omega^{p,q}_X) &\to C^\infty(X, \Omega^{p,q+1}_X)\end{aligned}$$

such that $\mathrm{d} = \partial + \overline{\partial}$.

Proposition 5.3.1.
(1)
$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \partial\beta$$
(2)
$$\partial^2 = \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0$$

According to (2) of Proposition 5.3.1 there is following cochain complex

$$0 \to C^\infty(X, \Omega_X^{p,0}) \xrightarrow{\overline{\partial}} C^\infty(X, \Omega_X^{p,1}) \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} C^\infty(X, \Omega_X^{p,n}) \to 0 \tag{5.1}$$

Definition 5.3.1 (Dolbeault cohomology)**.**
$$H^{p,q}(X) := H^q_{\overline{\partial}}(C^\infty(X, \Omega_X^{p,\bullet}))$$

Remark 5.3.1. Note that we have decomposition $C^\infty(X, \Omega^k_{X,\mathbb{C}}) = \bigoplus_{p+q=k} C^\infty(X, \Omega_X^{p,q})$, could we have the following decomposition
$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$
In fact, for compact Kähler manifold, such decomposition do holds, which is called Hodge decomposition.

Example 5.3.2. Note that
$$H^{p,0}(X) = \{\alpha \in C^\infty(X, \Omega_X^{p,0}) \mid \overline{\partial}\alpha = 0\}$$
For $\alpha \in C^\infty(X, \Omega_X^{p,0})$ locally written as $\alpha = \sum_{|I|=p} \alpha_I \mathrm{d}z^I$, one has
$$\overline{\partial}\alpha = \sum_{|I|=p} \frac{\partial \alpha_I}{\partial \overline{z}^l} \mathrm{d}\overline{z}^l \wedge \mathrm{d}z^I = 0 \Longleftrightarrow \frac{\partial \alpha_I}{\partial \overline{z}^l} = 0$$
which implies α_I is a holomorphic function. This shows $H^{p,0}(X) = \Gamma(X, \Omega_X^p)$.

A natural question arises: what does this cohomology compute? In the context of smooth manifolds, de Rham cohomology calculates the cohomology of a constant sheaf, which heavily relies on the Poincaré lemma. In the complex setting, Dolbeault cohomology $H^{p,q}(X)$ determines the q-th sheaf cohomology of Ω_X^p, which is based on the following lemma.

Proposition 5.3.2 ($\overline{\partial}$-Poincaré lemma)**.** Let B be an sufficiently small open disc in $\mathbb{C}^n$. If $\alpha \in C^\infty(B, \Omega_X^{p,q})$ is $\overline{\partial}$-closed and $q > 0$, then there exists $\beta \in C^\infty(B, \Omega_X^{p,q-1})$ such that $\alpha = \overline{\partial}\beta$.

Proof. See Corollary 1.3.9 of Page47 of [Huy05]. □

Proposition 5.3.3 (functorial)**.** Let $f\colon X \to Y$ be a holomorphic map between complex manifolds with pullback
$$f^*\colon C^\infty(Y, \Omega^k_{Y,\mathbb{C}}) \to C^\infty(X, \Omega^k_{X,\mathbb{C}})$$

Then
$$f^*\colon C^\infty(Y,\Omega^{p,q}_{Y,\mathbb{C}})\to C^\infty(X,\Omega^{p,q}_{X,\mathbb{C}})$$
and it induces
$$f^*\colon H^{p,q}(Y)\to H^{p,q}(X)$$

Example 5.3.3 (Dolbeault cohomology of a holomorphic vector bundle[4])**.** For a holomorphic vector bundle $E\to X$, we can also define
$$\overline{\partial}_E\colon C^\infty(X,\Omega^{0,q}_X\otimes E)\to C^\infty(X,\Omega^{0,q+1}_X\otimes E)$$
satisfies $\overline{\partial}_E^2=0$. Let's elaborate this construction: Since any global section is glued together by local sections, we just need to define $\overline{\partial}_E$ for local sections and check is well-defined under the change of local chart. We can choose a local holomorphic frame $\{e^1,\dots,e^n\}$ for E on U, so any section $s\in C^\infty(U,\Omega^{0,q}_X\otimes E)$ locally can be written as $s=s^i\otimes e_i$ with $s^i\in C^\infty(U,\Omega^{0,q}_X)$. Then we can define
$$\overline{\partial}_E(s)=\overline{\partial}s^i\otimes e^i$$
It's clear that this definition is independent of the choice of local chart since the transition functions are holomorphic and $\overline{\partial}$ kills them. Furthermore, $\overline{\partial}_E^2=0$ holds since $\overline{\partial}^2=0$. Thus we can construct a cochain complex and define its cohomology, denoted by
$$H^q(X,E)=H^q_{\overline{\partial}_E}(C^\infty(X,\Omega^{0,\bullet}_X\otimes E))$$
and similarly $H^q(X,E)$ computes the q-th sheaf cohomology of E.

[4]In previous case, $E=\Omega^{p,0}_X$

Part 2. Complex Differential Geometry

6. Connections and its curvature

6.1. Connections on complex vector bundle. Let X be a complex manifold and $\pi\colon E\to X$ be a complex vector bundle.

6.1.1. *Basic definitions.*

Definition 6.1.1 (connection)**.** A connection on E is a $\mathbb{C}$-linear operator

$$\nabla\colon C^\infty(X,E)\to C^\infty(X,\Omega^1_{X,\mathbb{C}}\otimes E)$$

satisfying the Leibniz rule

$$\nabla(fs)=\mathrm{d}f\otimes s+f\nabla s$$

for $f\in C^\infty(X)$ and $s\in C^\infty(X,E)$.

Remark 6.1.1 (connection form)*.* Let $\{e_\alpha\}$ be a local frame of E. Then any section s of E can be written as $s=s^\alpha e_\alpha$, and

$$\begin{aligned}\nabla(s^\alpha e_\alpha)&=\mathrm{d}s^\alpha e_\alpha+s^\alpha\nabla s_\alpha\\&=\mathrm{d}s^\alpha e_\alpha+s^\alpha\omega^\beta_\alpha e_\beta\end{aligned}$$

where ω^β_α are 1-forms, which is called connection 1-form. In terms of Christoffel symbol, one has

$$\omega^\beta_\alpha=\Gamma^\beta_{i\alpha}\mathrm{d}z^i+\Gamma^\beta_{\bar{i}\alpha}\mathrm{d}\bar{z}^i$$

6.1.2. *Curvature form.* Now we're going to extend connection to something called exterior derivative defined on sections of vector bundle valued k-forms as follows

$$\begin{aligned}\mathrm{d}^\nabla\colon C^\infty(X,\Omega^k_{X,\mathbb{C}}\otimes E)&\to C^\infty(X,\Omega^{k+1}_{X,\mathbb{C}}\otimes E)\\\omega\otimes s&\mapsto \mathrm{d}\omega\otimes s+(-1)^k\omega\wedge\nabla s\end{aligned}$$

Definition 6.1.2 (curvature form)**.** Let E be a complex vector bundle over a complex manifold X equipped with connection ∇. There exists a section $\Theta\in C^\infty(X,\Omega^2_{X,\mathbb{C}}\otimes\operatorname{End}E)$, called curvature form, such that

$$(\mathrm{d}^\nabla)^2s=\Theta\wedge s$$

for all $s\in C^\infty(X,\Omega^k_{X,\mathbb{C}}\otimes E)$.

Remark 6.1.2 (local form)*.* Let $\{e_\alpha\}$ be a local frame of E. The curvature form Θ can be written as

$$\Theta=\Theta^\beta_{i\bar{j}\alpha}\mathrm{d}z^i\wedge\mathrm{d}\bar{z}^j\otimes e^\alpha\otimes e_\beta$$

where $\Theta^\beta_{i\bar{j}\alpha}$ can also be expressed in terms of Christoffel symbols just like what we have seen in [Liu23].

6.2. **Chern connection.** In this section, we will introduce additional structures to enhance the complexity of a vector bundle E over a complex manifold X. These structures include Hermitian metrics and complex structures. We will explore connections that align harmoniously with these structures, similar to our approach in Riemannian geometry. By doing so, we will obtain the Chern connection, which runs parallel to the Levi-Civita connection.

6.2.1. *Compatiblity with Hermitian metric.*

Definition 6.2.1 (Hermitian metric)**.** Let E be a complex vector bundle. A Hermitian metric h on E is a smooth section of $E^* \otimes \overline{E}^*$.

Remark 6.2.1 (local form)*.* Let $\{e_\alpha\}$ be a local frame of E. Then a Hermitian metric is determined by a positive definite Hermitian matrix $(h_{\alpha\overline{\beta}})$, that is

$$h = h_{\alpha\overline{\beta}} e^\alpha \otimes \overline{e}^\beta$$

where $h_{\alpha\overline{\beta}} = h(e_\alpha, \overline{e}_\beta)$.

Definition 6.2.2 (Hermitian vector bundle)**.** A complex vector bundle E together with a Hermitian metric h is called a Hermitian vector bundle (E, h).

Remark 6.2.2 (metric weight)*.* Let L be a Hermitian line bundle. A Hermitian metric h is locally given by $e^{-2\varphi}$, where φ is a smooth function, which is called metric weight. Suppose $\{g_{\alpha\beta}\}$ is transition function of L with respect to open covering $\{U_\alpha\}$. Then h is given by a collection $\{h_\alpha \in C^\infty(U_\alpha)\}$ such that $h_\alpha = |g_{\alpha\beta}|^{-2} h_\beta$. In other words, a Hermitian metric is a collection of metric weights $\{\varphi_\alpha \in C^\infty(U_\alpha)\}$ such that

$$\varphi_\alpha = \varphi_\beta + \log|g_{\alpha\beta}|$$

Definition 6.2.3 (sesquilinear map)**.** For a Hermitian vector bundle (E, h) over complex manifold X, there is a sesquilinear map

$$\begin{aligned} C^\infty(X, \Omega^p_{X,\mathbb{C}} \otimes E) \times C^\infty(X, \Omega^q_{X,\mathbb{C}} \otimes E) &\to C^\infty(X, \Omega^{p+q}_{X,\mathbb{C}}) \\ (s, t) &\mapsto \{s, t\} \end{aligned}$$

locally given by

$$\{s^\alpha e_\alpha, t^\beta e_\beta\} = h_{\alpha\overline{\beta}} s^\alpha \wedge \overline{t^\beta}$$

Definition 6.2.4 (metric connection)**.** A connection ∇ on a Hermitian vector bundle (E, h) is called a metric connection, if

$$\mathrm{d}\langle s, t\rangle = \{\nabla s, t\} + \{s, \nabla t\}$$

where s, t are sections of E.

Remark 6.2.3 (local form)*.* If $\{e_\alpha\}$ is a local frame of E, then

$$\begin{aligned} \mathrm{d}h_{\alpha\overline{\beta}} &= \mathrm{d}\langle e_\alpha, \overline{e}_\beta\rangle \\ &= \{\nabla e_\alpha, \overline{e}_\beta\} + \{e_\alpha, \nabla\overline{e}_\beta\} \\ &= \omega_\alpha^\gamma h_{\gamma\overline{\beta}} + \overline{\omega_\beta^\gamma} h_{\alpha\overline{\gamma}} \end{aligned}$$

So in matrix notation, we have

$$\mathrm{d}h = \omega h + h\overline{\omega}^T$$

In particular, if we take $\{e_\alpha\}$ to be orthogonal local frame of E with respect to h, we will find $\omega + \overline{\omega}^T = 0$, that is ω is skew-Hermitian matrix.

Proposition 6.2.1. Let (E, h) be a Hermitian vector bundle over a complex manifold X. A connection ∇ is a metric connection if and only if

$$\mathrm{d}\{s, t\} = \{\nabla s, t\} + (-1)^p\{s, \nabla t\}$$

where $s \in C^\infty(X, \Omega^p_{X,\mathbb{C}} \otimes E)$ and $t \in C^\infty(X, \Omega^q_{X,\mathbb{C}} \otimes E)$.

Proposition 6.2.2. Let (E, h) be a Hermitian vector bundle equipped with connection ∇^E. Then ∇^E is a metric connection if and only if $\nabla^{E^*\otimes\overline{E}^*}h = 0$.

Proof. Direct computation shows

$$\begin{aligned}
\nabla^{E^*\otimes\overline{E}^*}(h_{\alpha\overline{\beta}}e^\alpha \otimes \overline{e}^\beta) &= \mathrm{d}h_{\alpha\overline{\beta}} \otimes e^\alpha \otimes \overline{e}^\beta + h_{\alpha\overline{\beta}}\nabla^{E^*}e^\alpha \otimes \overline{e}^\beta + h_{\alpha\overline{\beta}}e^\alpha \otimes \nabla^{\overline{E}^*}\overline{e}^\beta \\
&= \mathrm{d}h_{\alpha\overline{\beta}} \otimes e^\alpha \otimes \overline{e}^\beta - h_{\alpha\overline{\beta}}\omega^\alpha_\gamma e^\gamma \otimes \overline{e}^\beta - h_{\alpha\overline{\beta}}\overline{\omega^\beta_\gamma}e^\alpha \otimes \overline{e}^\gamma \\
&= (\mathrm{d}h_{\alpha\overline{\beta}} - \omega^\gamma_\alpha h_{\gamma\overline{\beta}} - \overline{\omega^\gamma_\beta}h_{\alpha\overline{\gamma}})e^\alpha \otimes \overline{e}^\beta
\end{aligned}$$

This shows desired result. $\square$

6.2.2. *Compatiblity with complex structure.* For a complex manifold X, we have decomposition

$$\Omega^1_{X,\mathbb{C}} = \Omega^{1,0}_X \oplus \Omega^{0,1}_X$$

Let $E \to X$ be a complex vector bundle with connection ∇. Then we can decompose $\nabla = \nabla^{1,0} + \nabla^{0,1}$ by composing the projection as follows

$$\begin{tikzcd}
 & & C^\infty(X, \Omega^{1,0}_X \otimes E) \\
C^\infty(X, E) \arrow[r, "\nabla"] & C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes E) \arrow[ur] \arrow[dr] & \\
 & & C^\infty(X, \Omega^{0,1}_X \otimes E)
\end{tikzcd}$$

If we write $\nabla = \mathrm{d} + \omega$ locally, then

$$\begin{aligned}
\nabla^{1,0} &= \partial + \omega^{1,0} \\
\nabla^{0,1} &= \overline{\partial} + \omega^{0,1}
\end{aligned}$$

Definition 6.2.5 (complex connection)**.** A connection ∇ on a holomorphic vector bundle E over a complex manifold X is said to be compatible with complex structure if $\nabla^{0,1} = \overline{\partial}_E$.

Remark 6.2.4 (local form). Let $\{e_\alpha\}$ be a holomorphic local form of E, and denote

$$\nabla e_\alpha = (\Gamma_{i\alpha}^\beta \mathrm{d}z^i + \Gamma_{\bar{i}\alpha}^\beta \mathrm{d}\bar{z}^i)e_\beta$$

that is

$$\nabla^{0,1} e_\alpha = \Gamma_{\bar{i}\alpha}^\beta e_\beta \mathrm{d}\bar{z}^i$$

But since $\{e_\alpha\}$ is holomorphic, that is $\overline{\partial}_E e_\alpha = 0$, which implies ∇ is complex if and only if $\Gamma_{\bar{i}\alpha}^\beta = 0$.

6.2.3. *Chern connection.*

Theorem 6.2.1 (Chern connection)**.** Let X be a complex manifold, (E, h) a Hermitian holomorphic vector bundle. Then there exists a unique metric connection called Chern connection such that it's compatible with complex structure.

Proof. If metric connection ∇ is compatible with complex structure, then the following three equations are equivalent

$$\begin{aligned}
\mathrm{d}h &= \omega h + h\overline{\omega}^t \\
\partial h &= \omega h \\
\overline{\partial} h &= h\overline{\omega}^t
\end{aligned}$$

since ω is a $(1,0)$-valued matrix. This shows Chern connection is uniquely determined by $\omega = (\partial h)h^{-1}$. □

Remark 6.2.5 (local form). Chern connection is locally determined by

$$\frac{\partial h_{\alpha\overline{\beta}}}{\partial z^i} = \Gamma_{i\alpha}^\gamma h_{\gamma\overline{\beta}}$$

Definition 6.2.6 (Chern curvature)**.** Let X be a complex manifold and (E, h) be a Hermitian holomorphic vector bundle. The Chern curvature Θ_h of (E, h) is defined as the curvature of Chern connection with respect to h.

Corollary 6.2.1. Let X be a complex manifold and (E, h) a Hermitian holomorphic vector bundle equipped with Chern connection ∇ locally given by ω. Then

(1) $\partial\omega = \omega \wedge \omega$.
(2) $\Theta_h = \overline{\partial}\omega$.
(3) $\overline{\partial}\Theta_h = 0$.

Proof. For (1). Since $\omega = (\partial h)h^{-1}$, then directly computation shows

$$\begin{aligned}
\partial\omega &= -\partial h \wedge \partial(h^{-1}) \\
&= -\partial h \wedge (-h^{-1}\partial h h^{-1}) \\
&= (\partial h)h^{-1} \wedge (\partial h)h^{-1} \\
&= \omega \wedge \omega
\end{aligned}$$

For (2). Θ_h locally looks like

$$\Theta_h = \mathrm{d}\omega - \omega \wedge \omega = \mathrm{d}\omega - \partial\omega = \overline{\partial}\omega$$

For (3). It's clear from (2). □

Remark 6.2.6 (local form). The Chern curvature can be expressed in terms of Christoffel symbol as follows

$$\Theta_h = \Theta^{\gamma}_{i\bar{j}\alpha} \mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j \otimes e^{\alpha} \otimes e_{\gamma}$$

where $\Theta^{\gamma}_{i\bar{j}\alpha} = -\frac{\partial \Gamma^{\gamma}_{i\alpha}}{\partial \overline{z}^j}$. In other type one has

$$\begin{aligned}\Theta_{i\bar{j}\alpha\overline{\beta}} &= h_{\gamma\overline{\beta}}\Theta^{\gamma}_{i\bar{j}\alpha} \\ &= -h_{\gamma\overline{\beta}}\partial_{\bar{j}}(h^{\gamma\overline{\delta}}\frac{\partial h_{\alpha\overline{\delta}}}{\partial z^i}) \\ &= -\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j} + h^{\gamma\overline{\delta}}\frac{\partial h_{\alpha\overline{\delta}}}{\partial z^i}\frac{\partial h_{\gamma\overline{\beta}}}{\partial \overline{z}^j}\end{aligned}$$

6.2.4. *Useful formulas of Chern connection.* Let X be a complex manifold and (E, h) be a Hermitian holomorphic vector bundle over X. Let ∇ be Chern connection determined by Christoffel symbol $\Gamma^{\beta}_{i\alpha}$ on (E, h) with curvature Θ_h. Suppose $\{z^i\}$ is local coordinate of X, $\{e_{\alpha}\}$ is the local frame of E, and $\{e^{\alpha}\}$ and $\{\overline{e}_{\alpha}\}$ denote local frames of E^* and $\overline{E}$ respectively.

Proposition 6.2.3.

$$\Gamma^{\beta}_{i\alpha} = h^{\beta\overline{\gamma}}\frac{\partial h_{\alpha\overline{\gamma}}}{\partial z^i}$$

Proof. See Remark 6.2.5. □

Proposition 6.2.4.

$$\begin{aligned}\Theta^{\gamma}_{i\bar{j}\alpha} &= -\frac{\partial \Gamma^{\gamma}_{i\alpha}}{\partial \overline{z}^j} \\ \Theta_{i\bar{j}\alpha\overline{\beta}} &= -\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j} + h^{\gamma\overline{\delta}}\frac{\partial h_{\alpha\overline{\delta}}}{\partial z^i}\frac{\partial h_{\gamma\overline{\beta}}}{\partial \overline{z}^j}\end{aligned}$$

Proof. See Remark 6.2.6. □

Proposition 6.2.5.

$$\begin{aligned}&\nabla_{\frac{\partial}{\partial z^i}} e_{\alpha} = \Gamma^{\beta}_{i\alpha}e_{\beta}, \quad \nabla_{\frac{\partial}{\partial \overline{z}^i}} e_{\alpha} = 0, \quad \nabla_{\frac{\partial}{\partial z^i}} \overline{e}_{\alpha} = 0, \quad \nabla_{\frac{\partial}{\partial \overline{z}^i}} \overline{e}_{\alpha} = \Gamma^{\overline{\beta}}_{\overline{i}\overline{\alpha}}\overline{e}_{\beta} \\ &\nabla_{\frac{\partial}{\partial z^i}} e^{\alpha} = -\Gamma^{\alpha}_{i\beta}e^{\beta}, \quad \nabla_{\frac{\partial}{\partial \overline{z}^i}} e^{\alpha} = 0, \quad \nabla_{\frac{\partial}{\partial z^i}} \overline{e}^{\alpha} = 0, \quad \nabla_{\frac{\partial}{\partial \overline{z}^i}} \overline{e}^{\alpha} = -\Gamma^{\overline{\alpha}}_{\overline{i}\overline{\beta}}\overline{e}^{\beta}\end{aligned}$$

Proof. It suffices to show the first two equalities, and others can be obtained from taking conjugates and dualities. The first one holds from definition of Christoffel symbol, and $\Gamma^{\beta}_{\bar{i}\alpha} = 0$ holds from the Remark 6.2.4. □

Corollary 6.2.2.

(1) For $s \in C^\infty(X, E)$, locally written as $s = s^\alpha e_\alpha$, one has

$$\nabla_{\frac{\partial}{\partial z^i}} s = (\frac{\partial s^\beta}{\partial z^i} + s^\alpha \Gamma^\beta_{i\alpha}) e_\beta$$

$$\nabla_{\frac{\partial}{\partial \overline{z}^i}} s = \frac{\partial s^\beta}{\partial \overline{z}^i} e_\beta$$

(2) For $s \in C^\infty(X, \overline{E})$, locally written as $s = s^{\overline{\alpha}} \overline{e}_\alpha$, one has

$$\nabla_{\frac{\partial}{\partial z^i}} s = \frac{\partial s^{\overline{\beta}}}{\partial z^i} \overline{e}_\beta$$

$$\nabla_{\frac{\partial}{\partial \overline{z}^i}} s = (\frac{\partial s^{\overline{\beta}}}{\partial \overline{z}^i} + s^{\overline{\alpha}} \Gamma^{\overline{\beta}}_{\overline{i}\overline{\alpha}}) \overline{e}_\beta$$

(3) For $s \in C^\infty(X, E^*)$, locally written as $s = s_\alpha e^\alpha$, one has

$$\nabla_{\frac{\partial}{\partial z^i}} s = (\frac{\partial s_\beta}{\partial z^i} - s_\alpha \Gamma^\alpha_{i\beta}) e^\beta$$

$$\nabla_{\frac{\partial}{\partial \overline{z}^i}} s = \frac{\partial s_\beta}{\partial \overline{z}^i} e^\beta$$

(4) For $s \in C^\infty(X, \overline{E}^*)$, locally written as $s = s_{\overline{\alpha}} \overline{e}^\alpha$, one has

$$\nabla_{\frac{\partial}{\partial z^i}} s = \frac{\partial s_{\overline{\beta}}}{\partial z^i} \overline{e}^\beta$$

$$\nabla_{\frac{\partial}{\partial \overline{z}^i}} s = (\frac{\partial s_{\overline{\beta}}}{\partial \overline{z}^i} - s_{\overline{\alpha}} \Gamma^{\overline{\gamma}}_{\overline{i}\overline{\beta}}) \overline{e}^\beta$$

Proposition 6.2.6 (Ricci identity)**.** For $s \in C^\infty(X, E)$, locally written as $s = s^\alpha e_\alpha$, one has

$$\nabla_{\frac{\partial}{\partial z^i}} \nabla_{\frac{\partial}{\partial \overline{z}^j}} s^\beta - \nabla_{\frac{\partial}{\partial \overline{z}^j}} \nabla_{\frac{\partial}{\partial z^i}} s^\beta = \Theta^\beta_{i\overline{j}\alpha} s^\alpha$$

Proof. Direct computation shows

$$\begin{aligned}\nabla_{\frac{\partial}{\partial z^i}} \nabla_{\frac{\partial}{\partial \overline{z}^j}} s &= \nabla_{\frac{\partial}{\partial z^i}} (\frac{\partial s^\alpha}{\partial \overline{z}^j} e_\alpha) \\ &= \frac{\partial^2 s^\beta}{\partial z^i \partial \overline{z}^j} e_\beta + \Gamma^\beta_{i\alpha} \frac{\partial s^\alpha}{\partial \overline{z}^j} e_\beta\end{aligned}$$

that is

$$\nabla_{\frac{\partial}{\partial z^i}} \nabla_{\frac{\partial}{\partial \overline{z}^j}} s^\beta = \frac{\partial^2 s^\beta}{\partial z^i \partial \overline{z}^j} + \Gamma^\beta_{i\alpha} \frac{\partial s^\alpha}{\partial \overline{z}^j}$$

Direct computation also shows

$$\nabla_{\frac{\partial}{\partial \overline{z}^j}} \nabla_{\frac{\partial}{\partial z^i}} s^\beta = \frac{\partial^2 s^\beta}{\partial \overline{z}^j \partial z^i} + \frac{\partial s^\alpha}{\partial \overline{z}^j} \Gamma^\beta_{i\alpha} + s^\alpha \frac{\partial \Gamma^\beta_{i\alpha}}{\partial \overline{z}^j}$$

Thus

$$\begin{aligned}\nabla_{\frac{\partial}{\partial z^i}} \nabla_{\frac{\partial}{\partial \overline{z}^j}} s^\beta - \nabla_{\frac{\partial}{\partial \overline{z}^j}} \nabla_{\frac{\partial}{\partial z^i}} s^\beta &= -s^\alpha \frac{\partial \Gamma^\beta_{i\alpha}}{\partial \overline{z}^j} \\ &= \Theta^\beta_{i\overline{j}\alpha} s^\alpha\end{aligned}$$

□

6.3. **First Chern class.**

6.3.1. *First Chern class of complex line bundle.* Let $\pi\colon X\to L$ be a complex line bundle with connection ∇ over a complex manifold X. Then curvature Θ is a global section of $\Omega^2_{X,\mathbb{C}}$ since $\operatorname{End} L$ is trivial bundle. Furthermore, Θ locally looks like $\mathrm{d}\omega$ since for line bundle $\omega\wedge\omega=0$. An immediate consequence is $\mathrm{d}\Theta=0$, that is Θ gives a cohomology class

$$[\Theta]\in H^2(X,\mathbb{C})$$

Definition 6.3.1 (first Chern class of line bundle)**.** Let L be a complex line bundle over complex manifold X equipped with connection ∇. The first Chern class of L is defined as

$$c_1(L):=[\frac{\sqrt{-1}}{2\pi}\Theta]\in H^2(X,\mathbb{C})$$

where Θ is the curvature of ∇.

Proposition 6.3.1 (topological invariance)**.** $c_1(L)\in H^2(X,\mathbb{C})$ is independent of the choice of connection.

Proof. Let $\widetilde{\nabla}$ be another connection which is locally given by $\widetilde{\omega}$. Then for section s of $\Omega^k_{X,\mathbb{C}}\otimes L$, one has

$$\begin{aligned}(\nabla-\widetilde{\nabla})s&=(\mathrm{d}s+\omega\wedge s)-(\mathrm{d}s+\widetilde{\omega}\wedge s)\\&=(\omega-\widetilde{\omega})\wedge s\end{aligned}$$

Note that $\omega-\widetilde{\omega}$ is a global section of $\Omega^1_{X,\mathbb{C}}$, so $\Theta-\widetilde{\Theta}$ is exact. □

Proposition 6.3.2. Let $\pi\colon L\to X$ be a complex line bundle over a complex manifold X. Then $c_1(L)\in H^2(X,\mathbb{R})$.

Proof. Equip L with a Hermitian metric h, then for a metric connection ∇, locally we have

$$\overline{\omega}=-\omega$$

Thus

$$\overline{\frac{\sqrt{-1}}{2\pi}\Theta}=-\frac{\sqrt{-1}}{2\pi}\overline{\Theta}=-\frac{\sqrt{-1}}{2\pi}\mathrm{d}\overline{\omega}=\frac{\sqrt{-1}}{2\pi}\mathrm{d}\omega=\frac{\sqrt{-1}}{2\pi}\Theta$$

□

Remark 6.3.1. Here are two facts here we don't prove:

(1) $c_1(L)\in H^2(X,\mathbb{Z})$.
(2) L is determined by $c_1(L)$.

Definition 6.3.2 (first Chern class of vector bundle)**.** Let E be a complex vector bundle over complex manifold X. The first Chern class of E is defined to be the first Chern class of $\det E$.

6.3.2. *First Chern class of Hermitian holomorphic line bundle.* Let X be a complex manifold, (L, h) a Hermitian holomorphic line bundle, and ∇ is the Chern connection of (L, h) with Chern curvature Θ_h. Then by Proposition 6.3.2 and Corollary 6.2.1, we have

$$[\frac{\sqrt{-1}}{2\pi}\Theta_h] \in H^2(X, \mathbb{R}) \cap H^{1,1}(X)$$

Remark 6.3.2 (local form). Suppose Hermitian metric h is given by metric weight $\{\varphi_\alpha\}$, that is locally $h = e^{-2\varphi_\alpha}$. Then direct computation shows first Chern class is locally given by

$$\frac{\sqrt{-1}}{2\pi}\Theta_h = \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}\varphi_\alpha$$

Proposition 6.3.3. $[\frac{\sqrt{-1}}{2\pi}\Theta_h] \in H^{1,1}(X)$ is independent of h.

Proof. Note that any two metric on a line bundle differ a smooth function which is positive everywhere, so if h and h' are two different metrics, we can write $\|e(z)\|_{h'} = e^f\|e(z)\|_h$ for some globally defined smooth function f. So by Remark 6.3.2, we have the difference of first Chern classes coming from different metrics is $\frac{\sqrt{-1}}{\pi}\overline{\partial}\partial f$, and it's trivial in $H^{1,1}(X)$ since f is globally defined. □

6.4. **Lefschetz $(1,1)$-theorem.** Now we know that given a Hermitian holomorphic line bundle (L, h), its Chern curvature we will get a real $(1,1)$-form. So we may wonder the converse of this statement. Is there any real $(1,1)$-form comes from such a Hermitian holomorphic line bundle? That's main theorem for this section.

Theorem 6.4.1 (Lefschetz $(1,1)$-theorem)**.** Let X be a complex manifold and $[\omega] \in H^2(X, \mathbb{R}) \cap H^{1,1}(X)$. If

$$[\omega] \in \operatorname{im}\{H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})\},$$

then there exists a Hermitian holomorphic line bundle (L, h) such that

$$\frac{\sqrt{-1}}{2\pi}\Theta_h = \omega$$

Before proving this theorem, let's explain these notations. Here $H^2(X, \mathbb{Z})$ and $H^2(X, \mathbb{R})$ are sheaf cohomology of constant sheaves $\underline{\mathbb{Z}}$ and $\underline{\mathbb{R}}$. By de Rham theorem[5], there is no difference between sheaf cohomology of $\underline{\mathbb{R}}$ and de Rham cohomology, but it's meaningless to consider de Rham cohomology with $\mathbb{Z}$-coefficient.

Here we use the isomorphisms

$$H^2(X, \mathbb{Z}) \cong \check{H}^2(X, \underline{\mathbb{Z}})$$
$$H^2(X, \mathbb{R}) \cong \check{H}^2(X, \underline{\mathbb{R}})$$

[5]See appendix 20.

and consider the map in terms of Čech cohomology

$$\check{H}^2(X, \underline{\mathbb{Z}}) \to \check{H}^2(X, \underline{\mathbb{R}}).$$

Above isomorphism is called comparision theorem, and can be proved by technique of spectral sequences in general. Here we give an explicit construction in dimension two, that is, construct a Čech 2-cocycle from a closed 2-form.

In sketch, the philosophy of this construction is that we can descend the degree of differential forms, but the price we pay is to consider functions defined on intersections of many open subsets.

Proof of comparision theorem in dimension two. Let X be a smooth manifold and $Z^1(X) \subset \Omega^1_{X,\mathbb{R}}$ be the sheaf of closed 1-form. Then we have the following sequence of sheaves

$$0 \to \underline{\mathbb{R}} \to C^\infty(X) \xrightarrow{\mathrm{d}} Z^1 \to 0.$$

By Poincaré lemma it's an exact sequence. Similarly, there is also an exact sequence

$$0 \to Z^1 \to \Omega^1_{X,\mathbb{R}} \xrightarrow{\mathrm{d}} Z^2 \to 0,$$

where Z^2 is the sheaf of closed 2-forms. By the definition of de Rham cohomology, we have

$$H^2(X, \mathbb{R}) = \frac{C^\infty(X, Z^2)}{\mathrm{d}C^\infty(X, \Omega^1_{X,\mathbb{R}})}$$

In order to avoid taking limit in Čech cohomology, we choose a good enough open covering[6] $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ such that

(1)
$$\mathrm{d}\colon C^\infty(U_\alpha, \Omega^1_{U_\alpha,\mathbb{R}}) \to C^\infty(U_\alpha, Z^2)$$
is surjective for any $\alpha \in I$.

(2)
$$\mathrm{d}\colon C^\infty(U_\alpha \cap U_\beta) \to C^\infty(U_\alpha \cap U_\beta, Z^1)$$
is surjective for any $\alpha, \beta \in I$.

Let ω be a closed 2-form. For any $\alpha \in I$, we choose $A_\alpha \in C^\infty(U_\alpha, \Omega^1_{U_\alpha,\mathbb{R}})$ such that

$$\omega|_{U_\alpha} = \mathrm{d}A_\alpha.$$

Then

$$\prod_{\alpha,\beta}(A_\alpha - A_\beta)$$

is a Čech 1-cocycle in $C^1(\mathfrak{U}, Z^1)$ since $\mathrm{d}(A_\alpha - A_\beta)|_{U_\alpha\cap U_\beta} = \omega - \omega = 0$. For any $\alpha, \beta \in I$, we choose $f_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta)$ such that

$$(A_\alpha - A_\beta)_{\alpha\beta} = \mathrm{d}f_{\alpha\beta}.$$

[6] In fact, it's called Leray covering, and Leray's theorem about Čech cohomology says that the Čech cohomology with respect to Leray covering is exactly the Čech cohomology.

Note that
$$f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}|_{U_\alpha\cap U_\beta\cap U_\gamma}$$
is d-closed by the same reason, and thus it's locally constant. Then
$$\check{\omega} = \prod_{\alpha,\beta,\gamma}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta})$$
is a Čech 2-cocycle in $C^2(\mathfrak{U}, \underline{\mathbb{R}})$. Then by Leray's theorem, we obtain a Čech 2-cocycle $\check{\omega} \in \check{H}^2(X, \underline{\mathbb{R}})$ from a closed 2-form $\omega \in H^2(X, \mathbb{R})$. $\square$

Now let's prove Lefschetz $(1,1)$-theorem.

Proof of theorem 6.4.1. Let $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ be an open covering consisting of open polydisk such that for all $\alpha, \beta \in I$, the intersection $U_\alpha \cap U_\beta$ is simply-connected.

For a d-closed real $(1,1)$-form ω, after a refinement if neccessary, Lemma 1.2.1 implies that there exist smooth functions $\varphi_\alpha \colon U_\alpha \to \mathbb{R}$ such that
$$\omega|_{U_\alpha} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi_\alpha.$$
Then on any two intersection $U_\alpha \cap U_\beta$, one has $\partial\overline{\partial}(\varphi_\alpha - \varphi_\beta) = 0$. Again after a refinement if neccessary, Lemma 1.2.2 implies that there exist holomorphic functions $f_{\alpha\beta}$ such that
$$(\varphi_\alpha - \varphi_\beta)|_{U_\alpha\cap U_\beta} = 2\mathrm{Re}(f_{\alpha\beta}) = f_{\alpha\beta} + \overline{f_{\alpha\beta}}.$$
For $\prod f_{\alpha\beta} \in C^1(\mathfrak{U}, \mathcal{O}_X)$, one has
$$(\delta f)_{\alpha\beta\gamma} = (f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta})|_{U_\alpha\cap U_\beta\cap U_\gamma}$$
Since $2\mathrm{Re}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta})_{\alpha\beta\gamma} = 0$, it must be a locally constant pure imaginary number, that is, it lies in $2\pi\sqrt{-1}\underline{\mathbb{R}}(U_\alpha \cap U_\beta \cap U_\gamma)$.

For real 1-form
$$A_\alpha = \frac{\sqrt{-1}}{4\pi}(\overline{\partial}\varphi_\alpha - \partial\varphi_\alpha),$$
a direct computation shows that $\omega|_{U_\alpha} = \mathrm{d}A_\alpha$, and that's why we define A_α in this form.

Recall what we have done in the proof of comparision theorem: If we want to find Čech cocycle which corresponding to ω, we need to consider $A_\alpha - A_\beta$ on the intersection $U_\alpha \cap U_\beta$. A direct computation shows that
$$\begin{aligned}\partial(\varphi_\beta - \varphi_\alpha) &= \partial(f_{\alpha\beta} + \overline{f_{\alpha\beta}})\\ &= \partial f_{\alpha\beta}\\ &= \mathrm{d}f_{\alpha\beta}\\ \overline{\partial}(\varphi_\beta - \varphi_\alpha) &= \mathrm{d}\overline{f_{\alpha\beta}}.\end{aligned}$$
Thus
$$(A_\beta - A_\alpha)_{\alpha\beta} = \frac{\sqrt{-1}}{4\pi}\mathrm{d}(\overline{f_{\alpha\beta}} - f_{\alpha\beta}) = \frac{1}{2\pi}\mathrm{d}(\mathrm{Im}(f_{\alpha\beta})).$$

Then the Čech 2-cocycle $\check{\omega}$ corresponding to ω is

$$\begin{aligned}\check{\omega} &= \prod(\frac{1}{2\pi}\mathrm{Im}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}))_{\alpha\beta\gamma} \\ &= \prod(\frac{1}{2\pi\sqrt{-1}}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}))_{\alpha\beta\gamma}.\end{aligned}$$

By hypothesis one has $[\check{\omega}]$ is an image of $[\prod n_{\alpha\beta\gamma}] \in \check{H}^2(X, \underline{\mathbb{Z}})$. However, it doesn't mean that $f_{\alpha\beta}$ are exactly integers, but not too bad, we just need some correction terms, that is

$$\prod(\frac{1}{2\pi\sqrt{-1}}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}))_{\alpha\beta\gamma} = \prod n_{\alpha\beta\gamma} + \delta(\prod c_{\alpha\beta})$$

where $\prod(c_{\alpha\beta}) \in C^1(\mathfrak{U}, \underline{R})$ is 1-cochain. If we define $f'_{\alpha\beta} = f_{\alpha\beta} - 2\pi\sqrt{-1}c_{\alpha\beta}$, then

$$(f'_{\beta\gamma} - f'_{\alpha\gamma} + f'_{\alpha\beta})_{\alpha\beta\gamma} = 2\pi\sqrt{-1}n_{\alpha\beta\gamma} \in 2\pi\sqrt{-1}\underline{\mathbb{Z}}(U_\alpha \cap U_\beta \cap U_\gamma).$$

Now consider the holomorphic function from $U_\alpha \cap U_\beta$ to $\mathbb{C}^*$ defined by $g_{\alpha\beta} = \exp(-f'_{\alpha\beta})$. A direct computation shows that it satisfies the cocycle condition

$$g_{\beta\gamma}g_{\alpha\gamma}^{-1}g_{\alpha\beta} = 1,$$

since $e^{2\pi\sqrt{-1}} = 1$. Then $\{g_{\alpha\beta}\}$ is a collection of transition functions, and gives a holomorphic line bundle L.

Now it suffices to construct a Hermitian metric on L, and calculate its curvature to complete the proof. Note that

$$(\varphi_\alpha - \varphi_\beta)_{U_\alpha \cap U_\beta} = 2\mathrm{Re}(f_{\alpha\beta}) = 2\mathrm{Re}(f_{\alpha\beta})' = -\log|g_{\alpha\beta}|^2.$$

Consider the Hermitian metric h, which is locally given by

$$h_\alpha = \exp(-\varphi_\alpha)$$

on U_α. It's well-defined since $h_\beta = |g_{\alpha\beta}|^2 h_\alpha = g^T_{\alpha\beta}h_\alpha\overline{g_{\alpha\beta}}$. Moreover,

$$\frac{\sqrt{-1}}{2\pi}\Theta_h = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_\alpha = \omega.$$

This completes the proof. □

Remark 6.4.1. Consider the exponential sequence

$$0 \to \mathbb{Z} \xrightarrow{2\pi\sqrt{-1}} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

and the induced long exact sequence

$$\cdots \to H^1(X, \mathcal{O}_X^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to \dots.$$

Let L be a holomorphic line bundle determined by its transition functions $\{g_{\alpha\beta}\}$. The proof for Lefschetz $(1,1)$-theorem shows that δ maps $\{g_{\alpha\beta}\}$ to $-c_1(L)$.

7. Hermitian geometry

7.1. Hermitian manifold and Riemannian manifold.

A Hermitian manifold is a complex manifold X together with a Hermitian metric h on holomorphic tangent bundle TX. One way to construct a Hermitian metric on TX is to consider special Riemannian metric on the underlying real manifold.

Suppose $\{z^i = x^i + \sqrt{-1}x^I\}$ is a local coordinate of X, where $1 \leq i \leq n$ and $n+1 \leq I = i+n \leq 2n$. Then $\{x^i, x^I\}$ gives a local coordinate of underlying real manifold of X, and there is a natural almost complex structure J on $T_{\mathbb{R}}X$ which is given by

$$\begin{aligned} J(\frac{\partial}{\partial x^i}) &= \frac{\partial}{\partial x^I} \\ J(\frac{\partial}{\partial x^I}) &= -\frac{\partial}{\partial x^i} \end{aligned}$$

Definition 7.1.1 (compatiblity)**.** A Riemannian metric g on $T_{\mathbb{R}}X$ is called compatible with almost complex structure, if

$$g(V, W) = g(JV, JW)$$

for all $V, W \in C^\infty(X, T_{\mathbb{R}}X)$.

Proposition 7.1.1. Let g be a Riemannian metric on $T_{\mathbb{R}}X$ which is compatible with J and locally given by

$$g = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j + g_{iJ}\mathrm{d}x^j \otimes \mathrm{d}x^J + g_{Ij}\mathrm{d}x^I \otimes \mathrm{d}x^j + g_{IJ}\mathrm{d}x^I \otimes \mathrm{d}x^J$$

Then

$$\begin{aligned} g_{ij} &= g_{IJ} \\ g_{iJ} &= g_{Ji} = -g_{jI} = -g_{Ij} \end{aligned}$$

Proof. Direct computation. □

Notation 7.1.1.

$$\begin{pmatrix} g^{il} & g^{iL} \\ g^{Il} & g^{IL} \end{pmatrix} \begin{pmatrix} g_{lj} & g_{lJ} \\ g_{Lj} & g_{LJ} \end{pmatrix} = \mathrm{I}_{2n}$$

In other words,

$$\begin{aligned} g^{il}g_{lj} + g^{iL}g_{Lj} &= \delta^i_j \\ g^{il}g_{lJ} + g^{iL}g_{LJ} &= 0 \end{aligned}$$

Let g be a Riemannian metric on underlying real manifold which is compatible with J. Then its $\mathbb{C}$-linear extension $g_{\mathbb{C}}$ gives a matrix

$$G = \begin{pmatrix} \left(g_{\mathbb{C}}(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j})\right)_{n\times n} & \left(g_{\mathbb{C}}(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j})\right)_{n\times n} \\ \left(g_{\mathbb{C}}(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial z^j})\right)_{n\times n} & \left(g_{\mathbb{C}}(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j})\right)_{n\times n} \end{pmatrix}$$

Direct computation shows that

$$g_{\mathbb{C}}(\frac{\partial}{\partial z^i},\frac{\partial}{\partial z^j})=0$$
$$g_{\mathbb{C}}(\frac{\partial}{\partial \overline{z}^i},\frac{\partial}{\partial \overline{z}^j})=0$$

and if we denote $H=(h_{i\overline{j}})_{n\times n}$, where

$$h_{i\overline{j}}:=g_{\mathbb{C}}(\frac{\partial}{\partial z^i},\frac{\partial}{\partial \overline{z}^j})=\frac{1}{2}(g_{ij}+\sqrt{-1}g_{iJ}) \tag{7.1}$$

Then

$$G=\begin{pmatrix}0 & H\\ \overline{H} & 0\end{pmatrix}$$

Moreover, H is a positive definite Hermitian matrix since G is a positive definite symmetric matrix. Thus h gives a Hermitian metric on $T^{1,0}X$, which makes X a Hermitian manifold. Conversely, if h is a Hermitian metric on $T^{1,0}X$, there is also a Riemannian metric on $T_{\mathbb{R}}X$ given by

$$\begin{aligned} g_{ij}&=2\mathrm{Re}h_{i\overline{j}}\\ g_{iJ}&=2\,\mathrm{im}\,h_{i\overline{j}}\\ g_{Ij}&=-g_{iJ}\\ g_{IJ}&=g_{ij}\end{aligned}$$

From now on, g always denotes a Riemannian metric on the underlying real manifold which is compatible with J, and h be the Hermitian metric corresponding to g.

Proposition 7.1.2.

$$\sqrt{\det g}=2^n\det h$$

Proof. Direct computation shows

$$\begin{aligned}\det g&=\det\begin{pmatrix}(g_{ij})_{n\times n} & (g_{iJ})_{n\times n}\\ (g_{Ij})_{n\times n} & (g_{IJ})_{n\times n}\end{pmatrix}\\ &\overset{(1)}{=}\det\begin{pmatrix}(g_{ij})_{n\times n} & (g_{iJ})_{n\times n}\\ (-g_{iJ})_{n\times n} & (g_{ij})_{n\times n}\end{pmatrix}\\ &=\det\begin{pmatrix}(g_{ij}+\sqrt{-1}g_{iJ})_{n\times n} & (g_{iJ})_{n\times n}\\ (-g_{iJ}+\sqrt{-1}g_{ij})_{n\times n} & (g_{ij})_{n\times n}\end{pmatrix}\\ &=\det\begin{pmatrix}(g_{ij}+\sqrt{-1}g_{iJ})_{n\times n} & (g_{iJ})_{n\times n}\\ \mathrm{O} & (g_{ij}+\sqrt{-1}g_{iJ})_{n\times n}\end{pmatrix}\\ &=(2^n\det h)^2\end{aligned}$$

where (1) holds from Proposition 7.1.1. □

Notation 7.1.2. $h^{i\bar{j}}$ is defined by the (i,j)-entry of $(H^{-1})^T$, that is $h^{i\bar{l}}h_{j\bar{l}} = \delta^i_j$, and $h^{\bar{i}j} := \overline{h^{i\bar{j}}}$. Note that

$$G^{-1} = (G^{-1})^T = \begin{pmatrix} 0 & (h^{i\bar{j}})_{n\times n} \\ (h^{\bar{i}j})_{n\times n} & 0 \end{pmatrix}$$

Proposition 7.1.3.

$$h^{i\bar{j}} = 2(g^{ij} - \sqrt{-1}g^{iJ})$$

Proof. Direct computation shows

$$\begin{aligned} h^{i\bar{l}}h_{j\bar{l}} &= (g^{il} - \sqrt{-1}g^{iL})(g_{jl} + \sqrt{-1}g_{jL}) \\ &= g^{il}g_{jl} + g^{iL}g_{jL} + \sqrt{-1}(g^{il}g_{jL} - g^{iL}g_{jL}) \\ &= g^{il}g_{lj} + g^{iL}g_{Lj} - \sqrt{-1}(g^{il}g_{lJ} + g^{iL}g_{Lj}) \\ &= \delta^i_j \end{aligned}$$

□

Definition 7.1.2 (fundamental form)**.** The fundamental form ω of g is defined as

$$\omega(V,W) := g_{\mathbb{C}}(JV, W)$$

where $V, W \in C^\infty(X, T_{\mathbb{C}}X)$.

Proposition 7.1.4.

$$\omega = \sqrt{-1}h_{i\bar{j}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j$$

In particular, ω is a real $(1,1)$-form.

Proof. Direct computation shows

$$\begin{aligned} \omega(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}) &= g_{\mathbb{C}}(J(\frac{\partial}{\partial z^i}), \frac{\partial}{\partial z^j}) = \sqrt{-1}g_{\mathbb{C}}(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}) = 0 \\ \omega(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}) &= g_{\mathbb{C}}(J(\frac{\partial}{\partial z^i}), \frac{\partial}{\partial \bar{z}^j}) = \sqrt{-1}g_{\mathbb{C}}(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}) = \sqrt{-1}h_{i\bar{j}} \\ \omega(\frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial z^i}) &= g_{\mathbb{C}}(J(\frac{\partial}{\partial \bar{z}^i}), \frac{\partial}{\partial z^j}) = -\sqrt{-1}g_{\mathbb{C}}(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}) = -\sqrt{-1}h_{\bar{i}j} \\ \omega(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}) &= g_{\mathbb{C}}(J(\frac{\partial}{\partial \bar{z}^i}), \frac{\partial}{\partial \bar{z}^j}) = -\sqrt{-1}g_{\mathbb{C}}(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}) = 0 \end{aligned}$$

This shows

$$\begin{aligned} \omega &= \sqrt{-1}h_{i\bar{j}}\mathrm{d}z^i \otimes \mathrm{d}\bar{z}^j - \sqrt{-1}h_{\bar{i}j}\mathrm{d}\bar{z}^i \otimes \mathrm{d}z^j \\ &= \sqrt{-1}h_{i\bar{j}}\mathrm{d}z^i \otimes \mathrm{d}\bar{z}^j - \sqrt{-1}h_{\bar{j}i}\mathrm{d}\bar{z}^j \otimes \mathrm{d}z^i \\ &= \sqrt{-1}h_{i\bar{j}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j \end{aligned}$$

where the last step holds since h is Hermitian, that is $h_{\bar{j}i} = h_{i\bar{j}}$. □

Proposition 7.1.5.

$$2h + \sqrt{-1}\omega = g_{\mathbb{C}}$$

Proof. Direct computation shows

$$\begin{aligned} h-\frac{1}{2}g_{\mathbb{C}} &= h_{i\bar{j}}\mathrm{d}z^i\otimes\mathrm{d}\bar{z}^j-\frac{1}{2}(h_{i\bar{j}}\mathrm{d}z^i\otimes\mathrm{d}\bar{z}^j+h_{\bar{i}j}\mathrm{d}\bar{z}^i\otimes\mathrm{d}z^j) \\ &= \frac{1}{2}h_{i\bar{j}}\mathrm{d}z^i\otimes\mathrm{d}\bar{z}^j-\frac{1}{2}h_{\bar{i}j}\mathrm{d}\bar{z}^i\otimes\mathrm{d}z^j \\ &= \frac{1}{2}h_{i\bar{j}}\mathrm{d}z^i\wedge\mathrm{d}\bar{z}^j \\ &= -\frac{\sqrt{-1}}{2}\omega \end{aligned}$$

□

Remark 7.1.1. In fact, Hermitian metric h, Riemannian metric g on underlying real manifold and fundamental form ω are the same things on a complex manifold X, and any of them gives a Hermitian structure on X.

$$\begin{tikzcd} & h \arrow[dl, bend right, "1"'] \arrow[dr, bend left, "3"] & \\ \omega \arrow[ur, bend left, "2"] \arrow[rr, "5"] & & g_{\mathbb{C}} \arrow[ul, bend right, "4"'] \arrow[ll, bend left, "6"] \end{tikzcd}$$

The explict correspondences are listed as follows:

1	$2\omega(\text{-},\text{-})=-\mathrm{Im}h(\text{-},\text{-})$
2	$2h(\text{-},\text{-})=\omega(\text{-},J\text{-})-\sqrt{-1}\omega(\text{-},\text{-})$
3	$2g_{\mathbb{C}}(\text{-},\text{-})=\mathrm{Re}h(\text{-},\text{-})$
4	$2h(\text{-},\text{-})=g_{\mathbb{C}}(\text{-},\text{-})-\sqrt{-1}g_{\mathbb{C}}(J\text{-},\text{-})$
5	$g_{\mathbb{C}}(\text{-},\text{-})=\omega(\text{-},J\text{-})$
6	$\omega(\text{-},\text{-})=g_{\mathbb{C}}(J\text{-},\text{-})$

In later discussion, we may say a Hermitian manifold (X,h) or (X,ω) when we're emphasizing its Hermitian metric or fundamental form.

Theorem 7.1.1 (normal coordinate)**.** Let (X,h) be a Hermitian manifold. For any $p\in X$, there exists a local holomorphic coordinate $\{z^i\}$ centered at p such that

$$h_{i\bar{j}}(p)=\delta_{i\bar{j}}\quad\text{and}\quad\frac{\partial h_{i\bar{j}}}{\partial\bar{z}^k}+\frac{\partial h_{i\bar{k}}}{\partial\bar{z}^j}=0$$

Proof. Without lose of generality, we may assume

$$\omega=\sqrt{-1}(\delta_{i\bar{j}}+a_{i\bar{j}l}w^l+a_{i\bar{j}\bar{l}}\overline{w}^l+O(|w|^2))\mathrm{d}w^i\wedge\mathrm{d}\overline{w}^j$$

□

7.2. **Curvatures of Hermitian manifold.** Sometimes we need to consider Hermitian holomorphic vector bundles over a Hermitian manifold, so there are two Hermitian metrics. In this case, in order to distinguish them, we always say a Hermitian holomorphic vector bundle (E,h) over a Hermitian manifold (X,g).

Definition 7.2.1 (curvatures of vector bundle)**.** Let (E,h) be a Hermitian holomorphic vector bundle on Hermitian manifold (X,g). Then

(1) the first Chern-Ricci curvature of (E,h) is locally given by

$$\operatorname{Ric}^{(1)}(h)=\sqrt{-1}h^{\alpha\overline{\beta}}\Theta_{i\overline{j}\alpha\overline{\beta}}\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^j$$

(2) the second Chern-Ricci curvature of (E,h) is locally given by

$$\operatorname{Ric}^{(2)}(h)=\sqrt{-1}g^{i\overline{j}}\Theta_{i\overline{j}\alpha\overline{\beta}}e^\alpha\otimes\overline{e}^\beta$$

(3) the Chern scalar curvature of (E,h) is locally given by

$$s=g^{i\overline{j}}h^{\alpha\overline{\beta}}\Theta_{i\overline{j}\alpha\overline{\beta}}$$

Remark 7.2.1. For convenience, we always use the following notations.

(1) $\operatorname{Ric}^{(1)}(h)=\sqrt{-1}\operatorname{tr}_h\Theta_h$ and $\operatorname{Ric}^{(2)}(h)=\sqrt{-1}\operatorname{tr}_g\Theta_h$, where Θ_h is the Chern curvature of (E,h).
(2) For a real $(1,1)$-form φ locally written as $\sqrt{-1}\varphi_{i\overline{j}}\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^j$, $\operatorname{tr}_g\varphi$ denotes the function $g^{i\overline{j}}\varphi_{i\overline{j}}$. In particular, the Chern scalar curvature is denoted by $\operatorname{tr}_g\operatorname{tr}_h\Theta_h$, where Θ_h is the Chern curvature of (E,h).

Definition 7.2.2 (curvatures of Hermitian manifold)**.** Let (X,h) be a Hermitian manifold. Then

(1) the first (or second) Chern-Ricci curvature of (X,h) is defined to be the first (or second) Chern-Ricci curvature of its holomorphic tangent bundle.
(2) the Chern scalar curvature of (X,h) is defined to be the Chern scalar curvature of its holomorphic tangent bundle.

Proposition 7.2.1. Let (E,h) be a Hermitian holomorphic vector bundle. The first Chern-Ricci curvature of (E,h) gives the first Chern class of (E,h) up to a scalar.

Proof. A direct computation shows

$$\frac{\sqrt{-1}}{2\pi}h^{\alpha\overline{\beta}}\left(-\frac{\partial^2h_{\alpha\overline{\beta}}}{\partial z^i\partial\overline{z}^j}+h^{\gamma\overline{\delta}}\frac{\partial h_{\alpha\overline{\delta}}}{\partial z^i}\frac{\partial h_{\gamma\overline{\beta}}}{\partial\overline{z}^j}\right)=-\frac{\sqrt{-1}}{2\pi}\frac{\partial^2\log\det(h_{\alpha\overline{\beta}})}{\partial z^i\partial\overline{z}^j}$$

Thus the first Chern-Ricci curvature of (E,h) gives the Chern curvature of $(\det E,\det h)$ up a scalar $1/2\pi$, and by Definition 6.3.2 the first Chern class of (E,h) is defined to be first Chern class of $\det E$. □

Definition 7.2.3 (holomorphic sectional curvature)**.** Let (X,h) be a Hermitian manifold and $v=v^i\frac{\partial}{\partial z^i}\in T_pX$ be a unit vector. The holomorphic sectional curvature in the direction v is defined as

$$\mathrm{HSC}_p(v):=\Theta_{i\bar{j}k\bar{l}}v^i\overline{v}^jv^k\overline{v}^l$$

Definition 7.2.4 (holomorphic bisectional curvature)**.** Let (X,h) be a Hermitian manifold and $v=v^i\frac{\partial}{\partial z^i}, w=w^i\frac{\partial}{\partial z^i}\in T_pX$ be unit vectors. The holomorphic sectional curvature in the direction v,w is defined as

$$\mathrm{HBSC}_p(v,w):=\Theta_{i\bar{j}k\bar{l}}v^i\overline{v}^jw^k\overline{w}^l$$

7.3. **Useful formulas of Hermitian geometry.** In this section we collect some useful formulas in Hermitian geometry. Unless otherwise specified, we assume (X,h) is a Hermitian n-manifold with fundamental form ω, and g is the Riemannian metric on the underlying real manifold.

Proposition 7.3.1.

$$\omega^n=(\sqrt{-1})^n n!\det(h_{i\bar{j}})\mathrm{d}z^1\wedge\mathrm{d}\overline{z}^1\wedge\cdots\wedge\mathrm{d}z^n\wedge\mathrm{d}\overline{z}^n$$

Proof. Direct computation shows

$$\begin{aligned}
\omega^n&=(\sqrt{-1})^n(h_{i\bar{j}}\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^j)^n\\
&=(\sqrt{-1})^n\sum_{\substack{\sigma\in S_n\\ \tau\in S_n}}h_{i_{\sigma(1)}\overline{j_{\tau(1)}}}\dots h_{i_{\sigma(n)}\overline{j_{\tau(n)}}}\mathrm{d}z^{i_{\sigma(1)}}\wedge\mathrm{d}\overline{z}^{j_{\tau(1)}}\wedge\cdots\wedge\mathrm{d}z^{i_{\sigma(n)}}\wedge\mathrm{d}\overline{z}^{j_{\tau(n)}}\\
&=(\sqrt{-1})^n\sum_{\substack{\sigma\in S_n\\ \tau\in S_n}}(-1)^{|\sigma|}(-1)^{|\tau|}h_{i_{\sigma(1)}\overline{j_{\tau(1)}}}\dots h_{i_{\sigma(n)}\overline{j_{\tau(n)}}}\mathrm{d}z^1\wedge\mathrm{d}\overline{z}^1\wedge\cdots\wedge\mathrm{d}z^n\wedge\mathrm{d}\overline{z}^n\\
&=(\sqrt{-1})^n\sum_{\sigma\in S_n}\sum_{\rho\in S_n}(-1)^{|\rho|}h_{i_{\rho(1)}\overline{j_1}}\dots h_{i_{\rho(n)}\overline{j_n}}\mathrm{d}z^1\wedge\mathrm{d}\overline{z}^1\wedge\cdots\wedge\mathrm{d}z^n\wedge\mathrm{d}\overline{z}^n\\
&=(\sqrt{-1})^n n!\det(h_{i\bar{j}})\mathrm{d}z^1\wedge\mathrm{d}\overline{z}^1\wedge\cdots\wedge\mathrm{d}z^n\wedge\mathrm{d}\overline{z}^n
\end{aligned}$$

∎

Corollary 7.3.1.

$$\omega^{n-1}=(\sqrt{-1})^{n-1}(n-1)!\sum_{\substack{i_1<\cdots<i_{n-1},i_l\neq p\\ j_1<\cdots<j_{n-1},j_l\neq q}}h(p,q)\mathrm{d}z^{i_1}\wedge\mathrm{d}\overline{z}^{j_1}\wedge\cdots\wedge\mathrm{d}z^{i_{n-1}}\wedge\mathrm{d}\overline{z}^{j_{n-1}}$$

where $h(p,q)$ the cofactor of h without row p and column q.

Proof. Direct computation shows

$$\begin{aligned}
\omega^{n-1} &= (\sqrt{-1})^{n-1}(h_{i\bar{j}}\mathrm{d}z^i\wedge \mathrm{d}\bar{z}^j)^{n-1}\\
&= (\sqrt{-1})^{n-1}\sum_{\substack{\sigma\in S_{n-1}\\ \tau\in S_{n-1}}} h_{i_{\sigma(1)}\overline{j_{\tau(1)}}}\dots h_{i_{\sigma(n-1)}\overline{j_{\tau(n-1)}}}\mathrm{d}z^{i_{\sigma(1)}}\wedge \mathrm{d}\bar{z}^{j_{\tau(1)}}\wedge\dots\wedge \mathrm{d}z^{i_{\sigma(n-1)}}\wedge \mathrm{d}\bar{z}^{j_{\tau(n-1)}}\\
&= (\sqrt{-1})^{n-1}\sum_{\substack{i_1<\dots<i_{n-1},i_l\neq p\\ j_1<\dots<j_{n-1},j_l\neq q}}\sum_{\substack{\sigma\in S_{n-1}\\ \rho\in S_{n-1}}}(-1)^{|\rho|}h_{i_{\rho(1)}\overline{j_1}}\dots h_{i_{\rho(n-1)}\overline{j_{n-1}}}\mathrm{d}z^{i_1}\wedge \mathrm{d}\bar{z}^{j_1}\wedge\dots\wedge \mathrm{d}z^{i_{n-1}}\wedge \mathrm{d}\bar{z}^{j_{n-1}}\\
&= (\sqrt{-1})^{n-1}(n-1)!\sum_{\substack{i_1<\dots<i_{n-1},i_l\neq p\\ j_1<\dots<j_{n-1},j_l\neq q}} h(p,q)\mathrm{d}z^{i_1}\wedge \mathrm{d}\bar{z}^{j_1}\wedge\dots\wedge \mathrm{d}z^{i_{n-1}}\wedge \mathrm{d}\bar{z}^{j_{n-1}}
\end{aligned}$$

□

Corollary 7.3.2.

$$\omega^{n-2} = (\sqrt{-1})^{n-2}(n-2)!\sum_{\substack{i_1<\dots<i_{n-2},i_l\neq p,s\\ j_1<\dots<j_{n-2},j_l\neq q,t}} h\begin{pmatrix}p,q\\ s,t\end{pmatrix}\mathrm{d}z^{i_1}\wedge \mathrm{d}\bar{z}^{j_1}\wedge\dots\wedge \mathrm{d}z^{i_{n-2}}\wedge \mathrm{d}\bar{z}^{j_{n-2}}$$

where $h\begin{pmatrix}p,q\\ s,t\end{pmatrix}$ the cofactor of h without rows p,s and columns q,t.

Proposition 7.3.2. $\omega^n/n!$ is the volume form of the underlying real manifold with respect to g.

Proof. Direct computation shows

$$\begin{aligned}
\omega^n &\overset{(1)}{=} (\sqrt{-1})^n n!\det(h_{i\bar{j}})\mathrm{d}z^1\wedge \mathrm{d}\bar{z}^1\wedge\dots\wedge \mathrm{d}z^n\wedge \mathrm{d}\bar{z}^n\\
&\overset{(2)}{=} n!2^n\det(h_{i\bar{j}})\mathrm{d}x^1\wedge \mathrm{d}y^1\wedge\dots\wedge \mathrm{d}x^n\wedge \mathrm{d}y^n\\
&\overset{(3)}{=} n!\,\mathrm{vol}
\end{aligned}$$

where

(1) holds from Proposition 7.3.1.
(2) holds from Exercise 5.2.1.
(3) holds from Proposition 7.1.2.

□

Proposition 7.3.3.

(1) If φ is a real $(1,1)$-form, then

$$\varphi\wedge\omega^{n-1} = \frac{1}{n}\operatorname{tr}_\omega\varphi\cdot\omega^n$$

(2) If φ is a $(1,0)$-form, then

$$\sqrt{-1}\varphi\wedge\bar{\varphi}\wedge\omega^{n-1} = \frac{1}{n}|\varphi|^2\cdot\omega^n$$

(3) If φ is a $(0,1)$-form, then

$$\sqrt{-1}\varphi\wedge\overline{\varphi}\wedge\omega^{n-1}=-\frac{1}{n}|\varphi|^2\cdot\omega^n$$

Proof. For (1). Suppose $\varphi=\sqrt{-1}\varphi_{i\overline{j}}\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^j$. Then by Proposition 7.3.1 one has

$$\frac{1}{n}\operatorname{tr}_\omega\varphi\cdot\omega^n=(\sqrt{-1})^n(n-1)!h^{i\overline{j}}\varphi_{i\overline{j}}\det h\mathrm{d}z^1\wedge\mathrm{d}\overline{z}^1\wedge\cdots\wedge\mathrm{d}z^n\wedge\mathrm{d}\overline{z}^n$$

and by Corollary 7.3.1 one has

$$\begin{aligned}\varphi\wedge\omega^{n-1}&=(\sqrt{-1})^n(n-1)!\varphi_{i\overline{j}}\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^j\wedge\sum_{\substack{i_1<\cdots<i_{n-1},i_l\neq p\\ j_1<\cdots<j_{n-1},j_l\neq q}}h(p,q)\mathrm{d}z^{i_1}\wedge\mathrm{d}\overline{z}^{j_1}\wedge\cdots\wedge\mathrm{d}z^{i_{n-1}}\wedge\mathrm{d}\overline{z}^{j_{n-1}}\\&=(\sqrt{-1})^n(n-1)!\sum_{1\leq p,q\leq n}\varphi_{p\overline{q}}(-1)^{p+q}h(p,q)\mathrm{d}z^1\wedge\mathrm{d}\overline{z}^1\wedge\cdots\wedge\mathrm{d}z^n\wedge\mathrm{d}\overline{z}^n\end{aligned}$$

Note that by expansion of determinant one has

$$h^{i\overline{j}}\varphi_{i\overline{j}}\det h=h^{i\overline{j}}\varphi_{i\overline{j}}\sum_{k=1}^n h_{k\overline{j}}(-1)^{k+j}h(k,j)=\sum_{1\leq i,j\leq n}\varphi_{i\overline{j}}(-1)^{i+j}h(i,j)$$

For (2). Suppose $\varphi=\varphi_i\mathrm{d}z^i$. Then

$$\sqrt{-1}\varphi\wedge\overline{\varphi}=\sqrt{-1}\varphi_i\overline{\varphi_j}\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^j$$

is a real $(1,1)$-form, and it's clear

$$\operatorname{tr}_\omega\sqrt{-1}\varphi\wedge\overline{\varphi}=|\varphi|^2$$

then by (1) we obtain desired result, and (3) follows from (2) directly. □

Proposition 7.3.4.

$$\langle\mathrm{d}z^i\wedge\alpha,\beta\rangle=\langle\alpha,h^{p\overline{i}}\iota_p\beta\rangle$$

holds for α,β with appropriate bidegrees.

Proposition 7.3.5. Let (E,h) be a Hermitian holomorphic vector bundle over a Hermitian manifold (X,g) and φ be a real $(1,1)$-form. Then

$$\{\varphi,\varphi\}\frac{\omega^{n-2}}{(n-2)!}=(|\operatorname{tr}_\omega\varphi|^2-|\varphi|^2)\frac{\omega^n}{n!}$$

Proof. Suppose $\varphi=\sqrt{-1}\varphi^\alpha_{i\overline{j}}\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^j\otimes e_\alpha$. Then by Corollary 7.3.2 one has

$$\mathrm{LHS}=(\sqrt{-1})^n\varphi^\alpha_{i\overline{j}}\overline{\varphi^\beta_{k\overline{l}}}h_{\alpha\overline{\beta}}\mathrm{d}z^i\wedge\overline{z}^j\wedge\mathrm{d}z^l\wedge\overline{z}^k\wedge\sum_{\substack{i_1<\cdots<i_{n-2},i_l\neq p,s\\ j_1<\cdots<j_{n-2},j_l\neq q,t}}g\begin{pmatrix}p,q\\ s,t\end{pmatrix}\mathrm{d}z^{i_1}\wedge\mathrm{d}\overline{z}^{j_1}\wedge\cdots\wedge\mathrm{d}z^{i_{n-2}}\wedge\mathrm{d}\overline{z}^{j_{n-2}}$$

If we want to insert $\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^j\wedge\mathrm{d}z^l\wedge\mathrm{d}\overline{z}^k$ into $\sum_{\substack{i_1<\cdots<i_{n-2},i_l\neq p,s\\ j_1<\cdots<j_{n-2},j_l\neq q,t}}\mathrm{d}z^{i_1}\wedge\mathrm{d}\overline{z}^{j_1}\wedge\cdots\wedge\mathrm{d}z^{i_{n-2}}\wedge\mathrm{d}\overline{z}^{j_{n-2}}$, there are the following four cases

$$\begin{cases} i=p,l=s,j=q,k=t\\ i=p,l=s,j=t,k=q\\ i=s,l=p,j=q,k=t\\ i=s,l=p,j=t,k=q\end{cases}$$

So case by case one has

$$\begin{aligned}&\varphi^\alpha_{i\bar{j}}\overline{\varphi^\beta_{k\bar{l}}}\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^j\wedge\mathrm{d}z^l\wedge\mathrm{d}\overline{z}^k\wedge\sum_{\substack{i_1<\cdots<i_{n-2},i_l\neq p,s\\ j_1<\cdots<j_{n-2},j_l\neq q,t}}\mathrm{d}z^{i_1}\wedge\mathrm{d}\overline{z}^{j_1}\wedge\cdots\wedge\mathrm{d}z^{i_{n-2}}\wedge\mathrm{d}\overline{z}^{j_{n-2}}\\ =&\sum_{\substack{1\leq p,q\leq n\\ 1\leq s,t\leq n}}(\varphi^\alpha_{p\bar{q}}\overline{\varphi^\beta_{t\bar{s}}}-\varphi^\alpha_{p\bar{t}}\overline{\varphi^\beta_{q\bar{s}}}-\varphi^\alpha_{s\bar{q}}\overline{\varphi^\beta_{t\bar{p}}}+\varphi^\alpha_{s\bar{t}}\overline{\varphi^\beta_{q\bar{p}}})(-1)^{p+q+t+s}\mathrm{d}z^1\wedge\mathrm{d}\overline{z}^1\wedge\cdots\wedge\mathrm{d}z^n\wedge\mathrm{d}\overline{z}^n\end{aligned}$$

On the other hand, direct computation shows

$$\begin{aligned}|\operatorname{tr}_\omega\varphi|^2&=\varphi^\alpha_{i\bar{j}}\overline{\varphi^\beta_{k\bar{l}}}g^{i\bar{j}}g^{l\bar{k}}h_{\alpha\overline{\beta}}\\ |\varphi^2|&=\varphi^\alpha_{i\bar{j}}\overline{\varphi^\beta_{k\bar{l}}}g^{i\overline{k}}g^{l\bar{j}}h_{\alpha\overline{\beta}}\end{aligned}$$

and thus

$$\mathrm{RHS}=(\sqrt{-1})^n\varphi^\alpha_{i\bar{j}}\overline{\varphi^\beta_{k\bar{l}}}h_{\alpha\overline{\beta}}(g^{i\bar{j}}g^{l\overline{k}}-g^{i\overline{k}}g^{l\bar{j}})\det g$$

Note that Laplacian's theorem implies

$$\begin{aligned}(g^{i\bar{j}}g^{l\overline{k}}-g^{i\overline{k}}g^{l\bar{j}})\det g&=(g^{i\bar{j}}g^{l\overline{k}}-g^{i\overline{k}}g^{l\bar{j}})\sum_{\substack{1\leq p,s\leq n\\ 1\leq j,k\leq n}}(g_{p\bar{j}}g_{s\overline{k}}-g_{s\bar{j}}g_{p\overline{k}})(-1)^{p+j+s+k}g\begin{pmatrix}p,j\\ s,k\end{pmatrix}\\ &=(\delta^i_p\delta^l_s-\delta^i_s\delta^l_p-\delta^i_s\delta^l_p+\delta^i_p\delta^l_s)(-1)^{p+j+s+k}g\begin{pmatrix}p,j\\ s,k\end{pmatrix}\end{aligned}$$

This shows the RHS equals to the LHS. □

7.4. **Gauduchon metric.**

Theorem 7.4.1 (Gauduchon metric)**.** If X be a complex manifold, then there exists a Hermitian metric ω such that

$$\partial\overline{\partial}\omega^{n-1}=0,$$

which is called Gauduchon metric.

Proof. See [Gau77]. □

Corollary 7.4.1. Let $f\colon X\to\mathbb{R}$ be a smooth function on a compact complex manifold X. If

$$\sqrt{-1}\partial\overline{\partial}f\geq 0,$$

then f is a constant.

Proof. Let ω be the Gauduchon metric on X. Then

$$0 \leq \int_X \sqrt{-1}\partial\overline{\partial} f \wedge \omega^{n-1} = \int_X \sqrt{-1} f \wedge \partial\overline{\partial}\omega^{n-1} = 0$$

which implies $\sqrt{-1}\partial\overline{\partial} f = 0$. Note that

$$\partial\overline{\partial} f^2 = \partial(2f\overline{\partial} f) = 2\partial f \wedge \overline{\partial} f + 2f\partial\overline{\partial} f = 2\partial f \wedge \overline{\partial} f \tag{7.2}$$

Then

$$\begin{aligned} 0 &\overset{(1)}{=} \int_X \sqrt{-1}\partial\overline{\partial} f^2 \wedge \omega^{n-1} \\ &\overset{(2)}{=} 2\int_X \sqrt{-1}\partial f \wedge \overline{\partial f} \wedge \omega^{n-1} \\ &\overset{(3)}{=} \frac{2}{n}\int_X |\partial f|^2 \omega^n \end{aligned}$$

where

(1) holds from ω is a Gauduchon metric.
(2) holds from equation (7.2).
(3) holds from (2) of Proposition 7.3.3.

This shows $\partial f = 0$, and this also shows $\mathrm{d}f = 0$ since f is real-valued, that is f is a constant. □

Corollary 7.4.2. Let $f\colon X \to \mathbb{R}$ be a smooth function on a compact complex manifold X. If there exists a Hermitian metric ω such that

$$\sqrt{-1}\operatorname{tr}_\omega \partial\overline{\partial} f \geq 0,$$

then f is a constant.

Proof. By (1) of Proposition 7.3.3 one has

$$\int_X \sqrt{-1}\operatorname{tr}_\omega \partial\overline{\partial} f \wedge \omega^n = \int_X n\sqrt{-1}\partial\overline{\partial} f \wedge \omega^{n-1}$$

The argument in above corollary still works. □

8. Kähler geometry

8.1. Kähler manifold.

Definition 8.1.1 (Kähler manifold)**.** A Hermitian manifold (X, h) is called a Kähler manifold, if its fundamental form ω is d-closed[7].

Remark 8.1.1. Note that $\mathrm{d}\omega = 0$ is equivalent to $\partial\omega = 0$, and is also equivalent to $\overline{\partial}\omega = 0$ since ω is a real $(1,1)$-form.

Remark 8.1.2 (local form). By Proposition 7.1.4 one has

$$\omega = \sqrt{-1}h_{i\bar{j}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j$$

So Kähler condition $\mathrm{d}\omega$ can be computed explicitly as follows

$$\begin{aligned}
\mathrm{d}\omega &= \sqrt{-1}\mathrm{d}(h_{i\bar{j}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j) \\
&= \sqrt{-1}(\frac{\partial h_{i\bar{j}}}{\partial z^k}\mathrm{d}z^k \wedge \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j - \frac{\partial h_{i\bar{j}}}{\partial \bar{z}^k}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^k \wedge \mathrm{d}\bar{z}^j) \\
&= 0
\end{aligned}$$

So locally Kähler condition can be written as follows

$$\begin{aligned}
\partial_k h_{i\bar{j}} &= \partial_i h_{k\bar{j}} \\
\partial_{\bar{k}} h_{i\bar{j}} &= \partial_{\bar{j}} h_{i\bar{k}}
\end{aligned}$$

holds for all i, j, k.

Proposition 8.1.1. Let (X, h) be a Kähler manifold. Then the first Chern-Ricci curvature coincides with the second Chern-Ricci curvature.

Proof. Note that

$$\Theta_{i\bar{j}k\bar{l}} = -\frac{\partial^2 h_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + h^{p\bar{q}}\frac{\partial h_{k\bar{q}}}{\partial z^i}\frac{\partial h_{p\bar{l}}}{\partial \bar{z}^j}$$

Thus if (X, h) is Kähler, then

$$\Theta_{i\bar{j}k\bar{l}} = \Theta_{k\bar{j}i\bar{l}} = \Theta_{i\bar{l}k\bar{j}}$$

As a consequence, one has

$$\mathrm{Ric}^{(1)}(h) = \sqrt{-1}h^{k\bar{l}}\Theta_{i\bar{j}k\bar{l}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j = \sqrt{-1}h^{k\bar{l}}\Theta_{k\bar{l}i\bar{j}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j = \mathrm{Ric}^{(2)}(h)$$

This completes the proof. □

Definition 8.1.2 (Kähler-Einstein metric)**.** A Kähler metric ω is called a Kähler-Einstein metric, if there exists $\lambda \in \mathbb{R}$ such that

$$\mathrm{Ric}(\omega) = \lambda\omega$$

Example 8.1.1. Any complex curve[8] X is Kähler since $\mathrm{d}\omega = 0$ automatically holds.

---

[7] ω is called Kähler form and h is called Kähler metric.

[8] In other words, a Riemann surface.

Proposition 8.1.2. A submanifold of a Kähler manifold is still Kähler.

Proof. If (X,ω) is a Kähler manifold and Y is a submanifold, the restriction of ω to Y gives Kähler form of Y. $\square$

Proposition 8.1.3. Let (X,ω) be a compact Kähler n-manifold. Then $H^{2k}(X,\mathbb{R})\neq 0$ for $0\leq k\leq n$.

Proof. Note that $\mathrm{d}(\omega^k)=0$ holds for $0\leq k\leq n$ since $\mathrm{d}\omega=0$, that is $[\omega^k]\in H^{2k}(X,\mathbb{R})$. By Proposition 7.3.2 one has $\omega^n=n!\,\mathrm{vol}$, so the integral pairing

$$\int_X\omega^k\wedge\omega^{n-k}=n!\int_X\mathrm{vol}\neq 0$$

implies $[\omega^k]\neq 0$ for $0\leq k\leq n$. $\square$

Theorem 8.1.1. Let (X,h) be a Kähler manifold. Then locally we can choose a holomorphic coordinate $(z^1,\dots,z^n)$ such that

$$h_{i\bar{j}}(z)=\delta_{i\bar{j}}-\Theta_{i\bar{j}k\bar{l}}(p)z^k\bar{z}^l+O(|z|^2)$$

8.2. **Levi-Civita connection encounters Chern connection.** Let X be a complex n-manifold and $\{z^i=x^i+\sqrt{-1}x^I\}$ be a local coordinate of X, where $1\leq i\leq n$ and $n+1\leq I=i+n\leq 2n$. Then $\{x^i,x^I\}$ gives a local coordinate of underlying real manifold of X. Let g be a Riemannian metric on $T_{\mathbb{R}}X$ which is compatible with natural almost complex structure J on $T_{\mathbb{R}}X$ with Levi-Civita connection ∇. Now consider $\mathbb{C}$-linear extension of ∇

$$\widetilde{\nabla}\colon C^\infty(X,T_{\mathbb{C}}X)\to C^\infty(X,\Omega^1_{X,\mathbb{C}}\otimes T_{\mathbb{C}}X)$$

It's clear $\widetilde{\nabla}$ gives a connection on $T_{\mathbb{C}}X$.

Notation 8.2.1. For $A\in\{1,\dots,n,\bar{1},\dots,\bar{n}\}$, we denote

$$z^A=\begin{cases}z^i & A=i\\ \bar{z}^i & A=\bar{i}\end{cases}$$

Proposition 8.2.1. Let $\{\frac{\partial}{\partial z^i},\frac{\partial}{\partial\bar{z}^i}\}$ be a local frame of $T_{\mathbb{C}}X$. Then

$$\widetilde{\nabla}_{\frac{\partial}{\partial z^A}}\frac{\partial}{\partial z^B}=\Gamma^C_{AB}\frac{\partial}{\partial z^C}=\frac{1}{2}h^{CE}(\frac{\partial h_{EB}}{\partial z^A}+\frac{\partial h_{AE}}{\partial z^B}-\frac{\partial h_{AB}}{\partial z^E})\frac{\partial}{\partial z^C}$$

where $A,B,C,E\in\{1,\dots,n,\bar{1},\dots,\bar{n}\}$.

Proof. By proof of Koszul formula, it suffices to check

(1) $\widetilde{\nabla}$ is compatible with $g_{\mathbb{C}}$.
(2) $\widetilde{\nabla}$ is torsion-free, that is for $X,Y\in C^\infty(X,T_{\mathbb{C}}X)$, one has

$$\widetilde{\nabla}_XY-\widetilde{\nabla}_YX=[X,Y]$$

Above two claims can be checked easily by using the fact ∇ is Levi-Civita connection and the $\mathbb{C}$-linearity of $\widetilde{\nabla}$. $\square$

Corollary 8.2.1.

$$\overline{\Gamma_{\bar{i}\bar{j}}^{\bar{k}}} = \Gamma_{ij}^{k} = \frac{1}{2}h^{k\bar{l}}(\frac{\partial h_{j\bar{l}}}{\partial z^i} + \frac{\partial h_{i\bar{l}}}{\partial z^j})$$
$$\overline{\Gamma_{i\bar{j}}^{\bar{k}}} = \Gamma_{\bar{i}j}^{k} = \frac{1}{2}h^{k\bar{l}}(\frac{\partial h_{j\bar{l}}}{\partial \bar{z}^i} - \frac{\partial h_{j\bar{i}}}{\partial \bar{z}^l})$$
$$\overline{\Gamma_{ij}^{\bar{k}}} = \Gamma_{\bar{i}\bar{j}}^{k} = 0$$

Proof. Note that

$$h_{ij} = h_{\bar{i}\bar{j}} = h^{ij} = h^{\bar{i}\bar{j}} = 0$$

□

Theorem 8.2.1. Let (X, h) be a Kähler manifold with induced Riemannian metric g on underlying real manifold, and suppose ∇ is Levi-Civita connection with respect to g. Then Chern connection with respect to h can be obtained from the restriction of $\mathbb{C}$-linear extension of ∇ to $T^{1,0}X$.

Proof. Let $\widetilde{\nabla}$ be the $\mathbb{C}$-linear extension of ∇ and $\{\frac{\partial}{\partial z^i}\}$ be a local frame of $T^{1,0}X$. Then by definition one has

$$\widetilde{\nabla}\frac{\partial}{\partial z^j} = \Gamma_{ij}^{k}\mathrm{d}z^i \otimes \frac{\partial}{\partial z^k} + \Gamma_{ij}^{\bar{k}}\mathrm{d}z^i \otimes \frac{\partial}{\partial \bar{z}^k} + \Gamma_{\bar{i}j}^{k}\mathrm{d}\bar{z}^i \otimes \frac{\partial}{\partial z^k} + \Gamma_{\bar{i}j}^{\bar{k}}\mathrm{d}\bar{z}^i \otimes \frac{\partial}{\partial \bar{z}^k}$$

By Corollary 8.2.1 one has $\Gamma_{ij}^{\bar{k}} = 0$ automatically, and if Kähler condition holds, then

$$\Gamma_{ij}^{k} = h^{k\bar{l}}\frac{\partial h_{j\bar{l}}}{\partial z^i}$$
$$\Gamma_{\bar{i}j}^{k} = \Gamma_{\bar{i}j}^{\bar{k}} = 0$$

Thus $\widetilde{\nabla}|_{T^{1,0}X}$ gives a connection on $T^{1,0}X$, and by formula of Chern connection, it's exactly the Chern connection with respect to h. □

8.3. **Curvatures of Kähler metric.** In this section, let (X, h) be a Kähler manifold with induced Riemannian metric g on underlying real manifold, and suppose $\{z^i = x^i + \sqrt{-1}x^I\}$ is a local coordinate of X, where $1 \leq i \leq n$ and $n + 1 \leq I = i + n \leq 2n$.

Notation 8.3.1.

(1)

$$R_{ijkl} = R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l})$$
$$R_{ijKL} = R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^K}, \frac{\partial}{\partial x^L})$$

where R is curvature tensor of Levi-Civita connection ∇ with respect to g.

(2)

$$\Theta_{i\bar{j}k\bar{l}} = \Theta(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^l})$$

where Θ is Chern curvature with respect to h.

8.3.1. *Ricci curvature and scalar curvature.*

Lemma 8.3.1.

$$R_{ijkl} = R_{ijKL}$$
$$R_{ijKl} = -R_{ijkL}$$

Proof. It follows from Kähler condition. □

Corollary 8.3.1.

$$R_{ij} = R_{IJ}$$
$$R_{iJ} = R_{Ji} = -R_{jI} = -R_{Ij}$$

Proof. It follows from Proposition 7.1.1 and Lemma 8.3.1. □

Theorem 8.3.1.

$$\Theta_{i\bar{j}} = \frac{1}{2}(R_{ij} + \sqrt{-1}R_{iJ})$$

where $\Theta_{i\bar{j}}$ is given by $\mathrm{Ric}(h) = \sqrt{-1}\Theta_{i\bar{j}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j$, while R_{ij} and R_{iJ} are Riemannian Ricci curvatures.

Proof. Suppose $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^I}\}$ is an orthonormal frame of real tangent bundle with respect to g, and thus $\{u_i := \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial x^I})\}$ gives an orthonormal frame of holomorphic tangent bundle with respect to h. Then

$$\begin{aligned}
\Theta_{i\bar{j}} &= \sum_k \left(R(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}, u_k, \bar{u}_k)\right) \\
&= \sum_k \frac{1}{2}\left(\underbrace{R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, u_k, \bar{u}_k)}_{\text{part I}} + \underbrace{\sqrt{-1}R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^J}, u_k, \bar{u}_k)}_{\text{part II}}\right)
\end{aligned}$$

For part I, one has

$$\begin{aligned}
\sum_k R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, u_k, \bar{u}_k) &= \sum_k \frac{1}{2}R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} - \sqrt{-1}\frac{\partial}{\partial x^K}, \frac{\partial}{\partial x^k} + \sqrt{-1}\frac{\partial}{\partial x^K}) \\
&= \sum_k \frac{\sqrt{-1}}{2}(R_{ijkK} - R_{ijKk}) \\
&= \sum_k \sqrt{-1}R_{ijkK} \\
&= \sum_k \sqrt{-1}(-R_{kijK} - R_{jkiK}) \\
&= \sum_k \sqrt{-1}(R_{kiJk} + R_{KiJK}) \\
&= \sqrt{-1}R_{iJ}
\end{aligned}$$

Here we used Lemma 8.3.1 and first Bianchi identity. Similarly for part II one has

$$\begin{aligned}\sum_k \sqrt{-1}R(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^J},u_k,\overline{u}_k) &= \sum_k \frac{\sqrt{-1}}{2}R(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^J},\frac{\partial}{\partial x^k}-\sqrt{-1}\frac{\partial}{\partial x^K},\frac{\partial}{\partial x^k}+\sqrt{-1}\frac{\partial}{\partial x^K})\\ &= \sum_k -R_{iJkK}\\ &= \sum_k (R_{kiJK}+R_{JkiK})\\ &= \sum_k (R_{kijk}+R_{KijK})\\ &= R_{ij}\end{aligned}$$

This shows the desired result. □

Corollary 8.3.2. Let (X,h) be a Kähler manifold with induced Riemannian metric g on underlying real manifold. Then

(1) h has positive Chern-Ricci curvature if and only if g has positive Ricci curvature.
(2) h is Kähler-Einstein with Einstein constant λ if and only if g is an Einstein metric with Einstein constant λ.

Corollary 8.3.3. Let (X,h) be a Kähler manifold with induced Riemannian metric g on underlying real manifold. Let s_R be the Riemannian scalar curvature and s is the Chern scalar curvature. Then $s_R=2s$.

Proof. On one hand, direct computation shows

$$\begin{aligned}s &= h^{i\bar{j}}\Theta_{i\bar{j}}\\ &= 2(g^{ij}-\sqrt{-1}g^{iJ})\cdot\frac{1}{2}(R_{ij}+\sqrt{-1}R_{iJ})\\ &= g^{ij}R_{ij}+g^{iJ}R_{iJ}\end{aligned}$$

On the other hand,

$$\begin{aligned}s_R &= g^{ij}R_{ij}+g^{iJ}R_{iJ}+g^{Ij}R_{Ij}+g^{IJ}R_{ij}\\ &\overset{(1)}{=} 2g^{ij}R_{ij}+2g^{iJ}R_{iJ}\\ &= 2s\end{aligned}$$

where (1) holds from Proposition 7.1.1 and Corollary 8.3.1. □

8.3.2. *Holomorphic sectional curvature and holomorphic bisectional curvature.* Recall that for a Hermitian manifold (X,h) and unit vectors $v=v^i\frac{\partial}{\partial z^i}, w=w^i\frac{\partial}{\partial z^i}\in T_pX$, the holomorphic sectional curvature is defined by

$$\mathrm{HSC}_p(v)=\Theta_{i\bar{j}k\bar{l}}v^i\overline{v}^jv^k\overline{v}^l$$

and the holomorphic bisectional curvature is defined by

$$\mathrm{HBSC}_p(v, w) = \Theta_{i\bar{j}k\bar{l}} v^i \overline{v}^j w^k \overline{w}^l$$

Now suppose (X, h) is a Kähler manifold, we will see the holomorphic sectional curvature and holomorphic bisectional curvature are closely related to the Riemannian sectional curvature.

Suppose unit vectors $v, w \in T_pX$ are given by

$$\begin{aligned} v &= \frac{1}{\sqrt{2}}(x - \sqrt{-1}Jx) \\ w &= \frac{1}{\sqrt{2}}(y - \sqrt{-1}Jy) \end{aligned}$$

where x, y are real vectors with $g(x, x) = g(y, y) = 1$. Then the holomorphic bisectional curvature is computed by

$$\begin{aligned} R(v, \overline{v}, w, \overline{w}) &= \frac{1}{2} R(x - \sqrt{-1}Jx, x + \sqrt{-1}Jx, w, \overline{w}) \\ &= \sqrt{-1} R(x, Jx, w, \overline{w}) \\ &= -R(x, Jx, y, Jy) \\ &= R(x, y, y, x) + R(x, Jy, Jy, x) \end{aligned}$$

In particular, the holomorphic sectional curvature of v is exactly the sectional curvature of the plane spanned by x and Jx.

Note that the holomorphic bisectional curvature is a sum of two sectional curvatures. Hence the holomorphic bisectional curvature carries less information than the sectional curvature. On the other hand, the holomorphic bisectional curvature carries more information that the Ricci curvature, since for real vector x, one has

$$\mathrm{Ric}_p(x) = \sum_{i=1}^{n} \left(R(\frac{\partial}{\partial x^i}, x, x, \frac{\partial}{\partial x^i}) + R(J(\frac{\partial}{\partial x^i}), x, x, J(\frac{\partial}{\partial x^i})) \right)$$

As a consequence, positive sectional curvature implies positive holomorphic bisectional curvature, and positive holomorphic bisectional curvature implies positive Ricci curvature.

Definition 8.3.1 (constant holomorphic sectional curvature)**.** Let (X, h) be a Kähler manifold. (X, h) has constant holomorphic sectional curvature c if

$$\mathrm{HSC}_p(v) = c$$

for all unit vector $v \in T_pX$.

Remark 8.3.1. In other words, the sectional curvature of all J-invariant planes equal to c, that it, $R(x, Jx, Jx, x) = c$ for all unit real vector x.

The definition given above is the the most natural way to define constant holomorphic sectional curvature, and now let's try to give another descriptions which is easy to use.

Proposition 8.3.1. Let (X,h) be a Kähler manifold with constant holomorphic sectional curvature c. Then $R = cR_0$, where

$$\begin{aligned}R_0(X,Y,Z,W) = \frac{1}{4}\{&g(X,W)g(Y,Z) - g(Y,W)g(X,Z) - g(X,JZ)g(Y,JW)\\ &+g(X,JW)g(Y,JZ) - 2g(X,JY)g(Z,JW)\}\end{aligned}$$

Proof. See Proposition 7.3 of Chapter IX in [KN69]. □

Proposition 8.3.2. Let (X,h) be a Kähler manifold. It has constant holomorphic sectional curvature c if and only if

$$\Theta_{i\bar{j}k\bar{l}} = \frac{c}{2}(h_{i\bar{j}}h_{k\bar{l}} + h_{i\bar{l}}h_{k\bar{j}})$$

Proof. A direct computation shows

$$\begin{aligned}\Theta_{i\bar{j}k\bar{l}} =& R(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^l})\\ =& cR_0(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^l})\\ =& \frac{c}{4}\left\{g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^l})g(\frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial z^k}) - g(\frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial \bar{z}^l})g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^k}) - g(\frac{\partial}{\partial z^i}, J\frac{\partial}{\partial z^k})g(\frac{\partial}{\partial \bar{z}^j}, J\frac{\partial}{\partial \bar{z}^l})\right.\\ &\left.+g(\frac{\partial}{\partial z^i}, J\frac{\partial}{\partial \bar{z}^l})g(\frac{\partial}{\partial \bar{z}^j}, J\frac{\partial}{\partial z^k}) - 2g(\frac{\partial}{\partial z^i}, J\frac{\partial}{\partial \bar{z}^j})g(\frac{\partial}{\partial z^k}, J\frac{\partial}{\partial \bar{z}^l})\right\}\\ =& \frac{c}{4}(h_{i\bar{l}}h_{k\bar{j}} + h_{i\bar{l}}h_{k\bar{j}} + 2h_{i\bar{j}}h_{k\bar{l}})\\ =& \frac{c}{2}(h_{i\bar{l}}h_{k\bar{j}} + h_{i\bar{j}}h_{k\bar{l}})\end{aligned}$$

On the other hand, it's clear (X,h) has constant holomorphic sectional curvature c if $\Theta_{i\bar{j}k\bar{l}} = \frac{c}{2}(h_{i\bar{j}}h_{k\bar{l}} + h_{i\bar{l}}h_{k\bar{j}})$. □

Remark 8.3.2.

(1) Above formula differs a sign from the one given in Proposition 7.6 of Chapter IX in [KN69], since the curvature notation we defined here differs a sign from the one defined by Kobayashi.
(2) In [Tia00], he called this by constant holomorphic bisectional curvature, and it maybe a bit confusing for beginners. If (X,h) has constant holomorphic sectional curvature, in this case you have that the bisectional curvature of X,Y equals

$$R(X,JX,JY,Y) = \frac{c}{2}\left(1 + g(X,Y)^2 + g(X,JY)\right)$$

As you can see, this expression is not constant (as you vary X,Y among unit vectors), but it varies between $c/2$ and c according to the relative position of the 2-planes spanned by (X,JX) and (Y,JY). In particular you see that there is no such thing as a non-flat "Kähler manifold with constant holomorphic bisectional curvature", because if there was such a thing, in particular the holomorphic sectional curvature would be constant, but then the curvature tensor would be given by the above formula, and the bisectional curvature would actually be NOT constant unless $c = 0$, that is, all the curvatures are constant because the metric is flat.

8.4. **Fubini-Study metric.**

Proposition 8.4.1 (Fubini-Study metric)**.** Let $\mathbb{CP}^n = \bigcup_{i=0}^n U_i$ be the canonical open covering, that is $U_i = \{(z^0 : \cdots : z^n) \mid z^i \neq 0\}$. Then there is a Kähler metric ω_{FS} on $\mathbb{CP}^n$, called Fubini-Study metric, such that

$$\omega_{FS}|_{U_i} = \frac{\sqrt{-1}}{2}\partial\overline{\partial}\log\left(\frac{\sum_{l=0}^n |z^l|^2}{|z^i|^2}\right)$$

Proof. Note that $\omega_{FS}|_{U_i}$ can be written as

$$\omega_{FS}|_{U_i} = \frac{\sqrt{-1}}{2}\partial\overline{\partial}\log(\sum_{l=1}^n |w^l|^2 + 1)$$

where $w^l = z^l/z^i$. The following steps show that $\{\omega_{FS}|_{U_i}\}_{i=0}^n$ gives a real $(1,1)$-form which is closed.

(1) It's globally defined since direct computation shows

$$\begin{aligned}
\log(\frac{\sum_{k=0}^n |z^l|^2}{|z^i|^2}) &= \log(\frac{|z^j|^2}{|z^i|^2}\frac{\sum_{k=0}^n |z^l|^2}{|z^j|^2})) \\
&= \log(\frac{|z^j|^2}{|z^i|^2}) + \log(\frac{\sum_{k=0}^n |z^l|^2}{|z^j|^2}) \\
&= \log(\frac{\sum_{k=0}^n |z^l|^2}{|z^j|^2})
\end{aligned}$$

where the last equality holds since $|z^j|^2/|z^i|^2$ is a nowhere vanishing holomorphic function.

(2) It's real since $\overline{\partial\overline{\partial}} = \overline{\partial}\partial = -\partial\overline{\partial}$.

(3) It's ∂-closed since each $\omega_{FS}|_{U_i}$ is ∂-closed.

It remains to show ω is positive. A direct computation yields

$$\partial\overline{\partial}\log(1 + \sum_{l=1}^n |w^l|^2) = \frac{1}{(1 + \sum_{l=1}^n |w^l|^2)^2} h_{i\overline{j}}\mathrm{d}w^i \wedge \mathrm{d}\overline{w}^j$$

where $h_{i\overline{j}} = (1+\sum_{l=1}^n |w^l|^2)\delta_{ij} - w^i\overline{w}^j$. Now it suffices to show $h_{i\overline{j}}$ is positive definite, for $u \neq 0$, one has

$$\begin{aligned}
u^T(h_{i\overline{j}})\overline{u} &= (u,u) + (w,w)(u,u) - u^T\overline{w}w^T\overline{u} \\
&= (u,u) + (w,w)(u,u) - (u,w)(w,u) \\
&= (u,u) + (w,w)(u,u) - \overline{(w,u)}(w,u) \\
&= (u,u) + (w,w)(u,u) - |(w,u)|^2 > 0
\end{aligned}$$

□

Corollary 8.4.1. Any projective manifold is Kähler.

Proof. By Proposition 8.1.2, the submanifold of Kähler manifold is still Kähler. ☐

Proposition 8.4.2. $\mathcal{O}_{\mathbb{CP}^n}(1)$ is a positive holomorphic line bundle.

Proof. Note that line bundle $\mathcal{O}_{\mathbb{CP}^n}(1)$ can be given transition functions $\{U_i, g_{ij}\}$, where $\{U_i\}$ is canonical open covering and $g_{ij} = z^j/z^i$. Consider $h_i\colon U_i \to \mathbb{R}_{>0}$ given by

$$h_i = \frac{|z^i|^2}{\sum_{l=0}^n |z^l|^2}$$

Then h_i can be glued together to obtain a Hermitian metric on $\mathcal{O}_{\mathbb{CP}^n}(1)$ since $h_i = h_j|g_{ij}|^2$. And it's clear to see Hermitian metric corresponding to curvature of Chern connection with respect to this metric is Fubini-Study metric. ☐

Remark 8.4.1. Note that $\mathcal{O}_{\mathbb{CP}^n}(-1)$ is a subbundle of $\mathbb{CP}^n \times \mathbb{C}^{n+1}$, so we can obtain a natural Hermitian metric of $\mathcal{O}_{\mathbb{CP}^n}(-1)$ by restricting standard Hermitian metric of $\mathbb{CP}^n \times \mathbb{C}^{n+1}$, and Hermitian metric on $\mathcal{O}_{\mathbb{CP}^n}(1)$ we defined before is exactly the dual metric of this natural metric.

Theorem 8.4.1. The Fubini-Study metric ω_{FS} on $\mathbb{CP}^n$ is a Kähler-Einstein metric with Einstein constant $n+1$ and has constant holomorphic sectional curvature.

Remark 8.4.2.

(1) In our definition (or the one given by Kobayashi in [KN69]), Fubini-Study metric has constant holomorphic sectional curvature 2, and in the definition given by Gang Tian in [Tia00], Fubini-Study metric has constant holomorphic sectional curvature 1, but it doesn't matter since up to a rescaling they're all the same.

(2) There are other models that have constant holomorphic sectional curvature, such as

Example 8.4.1. Let $X = \mathbb{C}^n$ equipped with $\omega = \sqrt{-1}/2\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^i$. Then (X, ω) is flat, and thus has constant holomorphic sectional curvature 0.

Example 8.4.2. Let $X = \mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$ equipped with

$$\omega = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log(1 - |z|^2)$$

Then (X, ω) has constant holomorphic sectional curvature -2.

Also, there is a theorem parallel to the Hopf's theorem in Riemannian geometry.

Theorem 8.4.2 (uniformizaiton theorem). If (X, h) is a complete Kähler manifold of constant holomorphic sectional curvature, then its universal covering is one of above examples. Moverover, up to rescaling, h pulls back to one of the metrics in the above examples.

Part 3. Hodge theory

9. Hodge theorem

9.1. Hodge star and adjoint operators. Let (X,ω) be a compact Hermitian manifold. A (p,q)-form α can be locally written as

$$\alpha=\frac{1}{p!\times q!}\alpha_{i_1\dots i_p\overline{j}_1\dots\overline{j}_q}\mathrm{d}z^{i_1}\wedge\dots\wedge\mathrm{d}z^{i_p}\wedge\mathrm{d}\overline{z}^{j_1}\wedge\dots\wedge\mathrm{d}\overline{z}^{j_q}$$

Then for $\alpha,\beta\in C^\infty(X,\Omega_X^{p,q})$, the local inner product is defined as

$$\langle\alpha,\beta\rangle=\frac{1}{p!\times q!}h^{i_1\overline{k}_1}\dots h^{i_p\overline{k}_p}h^{l_1\overline{j}_1}\dots h^{l_q\overline{j}_q}\alpha_{i_1\dots i_p\overline{j}_1\dots\overline{j}_q}\overline{\beta_{k_1\dots k_p\overline{l}_1\dots\overline{l}_q}}$$

which is a smooth function on X.

Definition 9.1.1 (inner product on (p,q)-form)**.** An inner product on the space of (p,q)-form is defined as

$$(\alpha,\beta):=\int_X\langle\alpha,\beta\rangle\frac{\omega^n}{n!}$$

where $\alpha,\beta\in C^\infty(X,\Omega_X^{p,q})$. This also gives an inner product on $\Omega^k_{X,\mathbb{C}}=\bigoplus_{p+q=k}\Omega_X^{p,q}$.

Holding the inner product (-, -), the formal adjoint operator of d is defined as an operator

$$\mathrm{d}^*\colon C^\infty(X,\Omega^k_{X,\mathbb{C}})\to C^\infty(X,\Omega^{k-1}_{X,\mathbb{C}})$$

satisfying $(\alpha,\mathrm{d}\beta)=(\mathrm{d}^*\alpha,\beta)$ for α,β with appropriate degrees, similarly one can define ∂^* and $\overline{\partial}^*$. In order to construct these adjoint operators, we need to introduce the well-known Hodge star operator.

Definition 9.1.2 (Hodge star operator)**.** There exists an operator

$$\star: C^\infty(X,\Omega_X^{p,q})\to C^\infty(X,\Omega_X^{n-q,n-p})$$

such that

$$(\alpha,\beta)=\int_X\alpha\wedge\star\overline{\beta}$$

Remark 9.1.1. It's well-defined since $\overline{\beta}$ is a (q,p)-form, and thus $\star\overline{\beta}$ is a $(n-p,n-q)$-form.

Lemma 9.1.1.

(1) $\star 1=\omega^n/n!$
(2) $\star\omega=\omega^{n-1}/(n-1)!$
(3) $\overline{\star\psi}=\star\overline{\psi}$
(4) $\star\star=(-1)^{p+q}$ on $C^\infty(X,\Omega_X^{p,q})$
(5) $(\star\varphi,\star\psi)=(\varphi,\psi)$

Proposition 9.1.1. $\mathrm{d}^*=-\star\mathrm{d}\star$

Proof. For arbitrary $\alpha \in C^\infty(X, \Omega^{p+q}_{X,\mathbb{C}})$ and $\beta \in C^\infty(X, \Omega^{p+q+1}_{X,\mathbb{C}})$, then

$$\begin{aligned}(\mathrm{d}\alpha, \beta) &= \int_X \mathrm{d}\alpha \wedge \star\beta \\ &= \int_X \mathrm{d}(\alpha \wedge \star\beta) - (-1)^{p+q}\alpha \wedge \mathrm{d} \star \beta \\ &= (-1)^{p+q+1} \int_X \alpha \wedge \mathrm{d} \star \beta \\ &\overset{(1)}{=} (-1)^{p+q+1}(-1)^{2n-(p+q+1)+1} \int_X \alpha \wedge \star\star \mathrm{d} \star \beta \\ &= -(\alpha, \star\mathrm{d} \star \beta)\end{aligned}$$

where (1) holds from (4) of Lemma 9.1.1. □

Proposition 9.1.2.

$$\begin{aligned}\partial^* &= - \star \overline{\partial} \star \\ \overline{\partial}^* &= - \star \partial \star\end{aligned}$$

Proof. Direct computation. □

Definition 9.1.3 (Lefschetz operator)**.** Let (X, ω) be a compact Kähler manifold. The Lefschetz operator is defined as

$$\begin{aligned}L\colon C^\infty(X, \Omega^{p,q}_X) &\to C^\infty(X, \Omega^{p+1,q+1}_X) \\ \alpha &\mapsto \omega \wedge \alpha\end{aligned}$$

Lemma 9.1.2. $\Lambda := L^* = (-1)^{p+q} \star L \star$ on (p, q)-forms.

Proof. For $\alpha \in C^\infty(X, \Omega^{p,q}_{X,}), \beta \in C^\infty(X, \Omega^{p+1,q+1}_X)$, direct computation shows

$$\begin{aligned}(L\alpha, \beta) &= \int_X L\alpha \wedge \star\beta \\ &= \int_X \omega \wedge \alpha \wedge \star\beta \\ &\overset{(1)}{=} \int_X \alpha \wedge \omega \wedge \star\beta \\ &\overset{(2)}{=} \int_X \alpha \wedge (-1)^{p+q} \star \star\omega \wedge \star\beta \\ &= (\alpha, (-1)^{p+q} \star L \star \beta)\end{aligned}$$

where

(1) holds from ω is a 2-form.
(2) holds from (4) of Lemma 9.1.1.

□

9.2. **Hodge theorem.**

Definition 9.2.1 (Laplacian)**.** Laplacian $\Delta_\bullet$ is an operator defined by $\Delta_\bullet := \bullet\bullet^* + \bullet^*\bullet$, where $\bullet$ can be d, ∂ and $\overline{\partial}$.

Definition 9.2.2 (harmonic)**.** A form α is called $\Delta_\bullet$-harmonic if $\Delta_\bullet\alpha = 0$, where $\bullet$ can be d, ∂ and $\overline{\partial}$.

Notation 9.2.1. $\mathcal{H}^k$ denotes the space of Δ_{d}-harmonic k-forms, and $\mathcal{H}^{p,q}$ denotes the space of $\Delta_{\overline{\partial}}$-harmonic forms of type (p,q).

Lemma 9.2.1. *α is $\Delta_\bullet$-harmonic if and only if $\bullet\alpha = 0, \bullet^*\alpha = 0$, where $\bullet$ can be d, ∂ and $\overline{\partial}$.*

Proof. Direct computation shows

$$\begin{aligned}(\alpha, \Delta_{\mathrm{d}}\alpha) &= (\alpha, \mathrm{d}\mathrm{d}^*\alpha) + (\alpha, \mathrm{d}^*\mathrm{d}\alpha)\\ &= \|\mathrm{d}^*\alpha\|^2 + \|\mathrm{d}\alpha\|^2\end{aligned}$$

This shows α is Δ_{d}-harmonic if and only if $\mathrm{d}\alpha = \mathrm{d}^*\alpha = 0$, the other cases are same. □

Theorem 9.2.1 (Hodge theorem)**.** *Let (X, h) be a compact Hermitian n-manifold. Then*

1. *$\mathcal{H}^{p,q}$ is finite dimensional.*
2. *There is a decomposition $C^\infty(X, \Omega_X^{p,q}) = \mathcal{H}^{p,q} \oplus \Delta_{\overline{\partial}}(C^\infty(X, \Omega_X^{p,q}))$, which is orthogonal with respect to inner products in Definition 9.1.1.*

Corollary 9.2.1. *There is the following orthonormal decomposition*

$$C^\infty(X, \Omega_X^{p,q}) = \mathcal{H}^{p,q} \oplus \overline{\partial}(C^\infty(X, \Omega_X^{p,q-1}) \oplus \overline{\partial}^*(C^\infty(X, \Omega_X^{p,q+1}))$$

Corollary 9.2.2.

$$\begin{aligned}\ker\overline{\partial} &= \mathcal{H}^{p,q} \oplus \overline{\partial}^*(C^\infty(X, \Omega_X^{p,q-1}))\\ \ker\overline{\partial}^* &= \mathcal{H}^{p,q} \oplus \overline{\partial}(C^\infty(X, \Omega_X^{p,q+1}))\end{aligned}$$

Corollary 9.2.3. *The natural map $\mathcal{H}^{p,q} \to H^{p,q}(X)$ is an isomorphism. In particular, $H^{p,q}(X)$ is finite dimensional.*

In order to give the following isomorphism

$$\star\colon \mathcal{H}^{p,q} \to \mathcal{H}^{n-q,n-p}$$

Parallel to the real case[9], it suffices to have

$$\star \circ \Delta_{\overline{\partial}} = \Delta_{\overline{\partial}} \circ \star$$

But something bad happens since we only have $\overline{\partial}^* = -\star\partial\star$, and direct computation only yields $\Delta_{\overline{\partial}} \circ \star = \star \circ \Delta_\partial$. So it fails generally since $\Delta_{\overline{\partial}} \neq \Delta_\partial$. There are two ways to deal with this gap. The first way is that we will see later if X is compact Kähler manifold, then $\Delta_\partial = \Delta_{\overline{\partial}}$, that is Theorem 10.1.1. Then

[9]See Hodge theory in [Liu23].

Corollary 9.2.4. If (X,ω) is a compact Kähler n-manifold, then $\star\colon \mathcal{H}^{p,q}\to\mathcal{H}^{n-q,n-p}$ is an isomorphism.

Another way is to consider

$$\begin{aligned}\overline{\star}\colon C^\infty(X,\Omega_X^{p,q}) &\to C^\infty(X,\Omega_X^{n-p,n-q})\\ \alpha &\mapsto \star\overline{\alpha}\end{aligned}$$

then direct computation shows

$$\overline{\star}\circ\Delta_{\overline{\partial}}=\Delta_{\overline{\partial}}\circ\overline{\star}$$

Corollary 9.2.5. If (X,h) is a compact Hermitian manifold, then $\overline{\star}\colon\mathcal{H}^{p,q}\to\mathcal{H}^{n-p,n-q}$ is an isomorphism.

Corollary 9.2.6. $H^{p,q}(X)\cong H^{n-p,n-q}(X)$.

Remark 9.2.1. This is a special case of Serre duality.

9.3. **Useful formulas of adjoint operators.**

Proposition 9.3.1. Let (X,h) be a compact Kähler manifold. Then locally

$$\begin{cases}\partial=\mathrm{d}z^i\wedge\nabla_i\\ \partial^*=-h^{i\overline{j}}\iota_i\circ\nabla_{j}=-h^{i\overline{j}}\nabla_{j}\circ\iota_i\end{cases}\qquad\begin{cases}\overline{\partial}=\mathrm{d}\overline{z}^i\wedge\nabla_{\overline{i}}\\ \overline{\partial}^*=-h^{i\overline{j}}\iota_{\overline{j}}\circ\nabla_i=-h^{i\overline{j}}\nabla_i\circ\iota_{\overline{j}}\end{cases}$$

Proof. Here we only give the proof of the case ∂ and ∂^*, the proof for the other two cases are same. It suffices to check pointwisely, and at each point we may also choose normal coordinate in Theorem 8.1.1. For (p,q)-form α, locally written as $\alpha=\alpha_{J\overline{K}}\mathrm{d}z^J\wedge\mathrm{d}\overline{z}^K$. Then

$$\partial\alpha=\frac{\partial\alpha_{J\overline{K}}}{\partial z^i}\mathrm{d}z^i\wedge\mathrm{d}z^J\wedge\mathrm{d}\overline{z}^K$$

and

$$\begin{aligned}\mathrm{d}z^i\wedge\nabla_i\alpha&=\mathrm{d}z^i\wedge\nabla_i(\alpha_{J\overline{K}}\mathrm{d}z^J\wedge\mathrm{d}\overline{z}^K)\\ &=\mathrm{d}z^i\wedge\frac{\partial\alpha_{J\overline{K}}}{\partial z^i}\mathrm{d}z^J\wedge\mathrm{d}\overline{z}^K+\alpha_{J\overline{K}}\nabla_i(\mathrm{d}z^J\wedge\mathrm{d}\overline{z}^K)\\ &\overset{(1)}{=}\frac{\partial\alpha_{J\overline{K}}}{\partial z^i}\mathrm{d}z^i\wedge\mathrm{d}z^J\wedge\mathrm{d}\overline{z}^K\end{aligned}$$

where (1) holds from our choice of normal coordinate. To see formula of ∂^*, take arbitrary forms α,β with appropriate bidegrees, then

$$\begin{aligned}(\partial\alpha,\beta)&=(\mathrm{d}z^i\wedge\nabla_i\alpha,\beta)\\ &\overset{(2)}{=}(\nabla_i\alpha,h^{p\overline{i}}\iota_p\beta)\\ &\overset{(3)}{=}-(\alpha,h^{p\overline{i}}\nabla_i\circ\iota_p\beta)\end{aligned}$$

where

(2) holds from Proposition 7.3.4.

(3) holds from Stokes' theorem and the fact Chern connection is compatible with metric.

This shows

$$\partial^* = -h^{i\bar{j}}\nabla_j \circ \iota_i \overset{(4)}{=} -h^{i\bar{j}}\iota_i \circ \nabla_j$$

where (4) holds from $\iota_i \circ \nabla_j = \nabla_j \circ \iota_i$. $\square$

Proposition 9.3.2. Let (X, ω) be a compact Kähler manifold. Then locally

$$\Lambda = \sqrt{-1}h^{i\bar{j}}\iota_i \circ \iota_{\bar{j}} = -\sqrt{-1}h^{i\bar{j}}\iota_{\bar{j}} \circ \iota_i$$

Proof. For arbitrary forms α, β with appropriate bidegrees, direct computation shows

$$\begin{aligned}
(\omega \wedge \alpha, \beta) &= (\sqrt{-1}h_{i\bar{j}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j \wedge \alpha, \beta) \\
&\overset{(1)}{=} (\sqrt{-1}h_{i\bar{j}}\mathrm{d}\bar{z}^j \wedge \alpha, h^{p\bar{i}}\iota_p\beta) \\
&\overset{(2)}{=} (\sqrt{-1}h_{i\bar{j}}\alpha, h^{p\bar{i}}h^{j\bar{q}}\iota_{\bar{q}} \circ \iota_p\beta) \\
&\overset{(3)}{=} (\alpha, -\sqrt{-1}h_{j\bar{i}}h^{p\bar{i}}h^{j\bar{q}}\iota_{\bar{q}} \circ \iota_p\beta) \\
&= (\alpha, -\sqrt{-1}h^{p\bar{i}}\iota_{\bar{i}} \circ \iota_p\beta)
\end{aligned}$$

where

(1) and (2) hold from Proposition 7.3.4.
(3) holds from $h_{i\bar{j}}$ is Hermitian, that is $\overline{h_{i\bar{j}}} = h_{j\bar{i}}$.

This shows

$$\Lambda = -\sqrt{-1}h^{i\bar{j}}\iota_{\bar{j}} \circ \iota_i \overset{(4)}{=} \sqrt{-1}h^{i\bar{j}}\iota_i \circ \iota_{\bar{j}}$$

where (4) holds from $\iota_i \circ \iota_{\bar{j}} = -\iota_{\bar{j}} \circ \iota_i$. $\square$

10. Hodge decomposition

10.1. Kähler identities.

Definition 10.1.1 (commutor of differential operators)**.** Let A, B be two differential operators. The commutor of A, B is defined as

$$[A, B] := AB - (-1)^{\deg A \deg B} BA$$

Lemma 10.1.1 (Jacobi identity)**.** Let A, B, C be differential operators. Then

$$(-1)^{\deg A \deg C}[A, [B, C]] + (-1)^{\deg B \deg A}[B, [C, A]] + (-1)^{\deg C \deg B}[C, [A, B]] = 0$$

Remark 10.1.1. In our case, the degree of $\mathrm{d}, \mathrm{d}^*, \partial, \partial^*, \overline{\partial}, \overline{\partial}^*$ is one, and the degree of L and Λ is zero[10].

Proposition 10.1.1 (Kähler identities)**.** If (X, ω) is a compact Kähler manifold, then

$$[\overline{\partial}^*, L] = \sqrt{-1}\partial$$
$$[\partial^*, L] = -\sqrt{-1} \cdot \overline{\partial}$$
$$[\Lambda, \overline{\partial}] = -\sqrt{-1}\partial^*$$
$$[\Lambda, \partial] = \sqrt{-1} \cdot \overline{\partial}^*$$

Proof. By taking conjugates and adjoints, it suffices to prove the first identity, which is a first order identity of differential equation. But by Theorem 8.1.1, locally we have $h_{i\bar{j}} = \delta_{i\bar{j}} + O(|\xi^2|)$. Thus it suffices to check Kähler identity for the case $U \subseteq \mathbb{C}^n$ equipped with standard Hermitian metric.

Suppose (p, q)-form α is locally given by $\alpha = \alpha_{JK} \mathrm{d}z^J \wedge \mathrm{d}\overline{z}^K$, then by Proposition 9.3.1 one has $\overline{\partial}^* \alpha = -\sum_l \iota_{\frac{\partial}{\partial \overline{z}^l}} \frac{\partial \alpha}{\partial z^l}$. Thus

$$\begin{aligned}
[\overline{\partial}^*, L]\alpha &= \overline{\partial}^*(\omega \wedge \alpha) - \omega \wedge \overline{\partial}^* \alpha \\
&= -\sum_l \iota_{\frac{\partial}{\partial \overline{z}^l}} \frac{\partial}{\partial z^l}(\omega \wedge \alpha) + \omega \wedge \sum_l \iota_{\frac{\partial}{\partial \overline{z}^l}} \frac{\partial \alpha}{\partial z^l} \\
&\overset{(1)}{=} -\sum_l \iota_{\frac{\partial}{\partial \overline{z}^l}} (\omega \wedge \frac{\partial \alpha}{\partial z^l}) + \omega \wedge \sum_l \iota_{\frac{\partial}{\partial \overline{z}^l}} \frac{\partial \alpha}{\partial z^l} \\
&= -\{\sum_l (\iota_{\frac{\partial}{\partial \overline{z}^l}} \omega) \wedge \frac{\partial \alpha}{\partial z^l} + \omega \wedge \iota_{\frac{\partial}{\partial \overline{z}^l}} \frac{\partial \alpha}{\partial z^l}\} + \sum_l \omega \wedge \iota_{\frac{\partial}{\partial \overline{z}^l}} \frac{\partial \alpha}{\partial z^l} \\
&= -\sum_l (\iota_{\frac{\partial}{\partial \overline{z}^l}} \omega) \wedge \frac{\partial \alpha}{\partial z^l} \\
&\overset{(2)}{=} \sqrt{-1} \sum_l \mathrm{d}z^l \wedge \frac{\partial \alpha}{\partial z^l} \\
&= \sqrt{-1}\partial\alpha
\end{aligned}$$

[10]You can try to understand this thing in a following way: operators $\mathrm{d}, \mathrm{d}^*, \partial, \partial^*, \overline{\partial}, \overline{\partial}^*$ take derivatives, but L and Λ are linear operators.

where

(1) holds from ω is a closed $(1,1)$-form.
(2) holds from Proposition 7.1.4, that is $\omega = \sqrt{-1}\sum_{i=1}^n \mathrm{d}z^i \wedge \mathrm{d}\overline{z}^i$.

□

Theorem 10.1.1. Let (X,ω) be a compact Kähler manifold. Then

$$\Delta_{\mathrm{d}} = 2\Delta_{\partial} = 2\Delta_{\overline{\partial}}$$

Proof. Since

$$\Delta_{\mathrm{d}} = (\partial+\overline{\partial})(\partial^*+\overline{\partial}^*) + (\partial^*+\overline{\partial}^*)(\partial+\overline{\partial})$$

By the fourth Kähler identity, one has

(1) The first term can be computed as

$$\begin{aligned}(\partial+\overline{\partial})(\partial^*+\overline{\partial}^*) &= (\partial+\overline{\partial})(\partial^* - \sqrt{-1}\Lambda\partial + \sqrt{-1}\partial\Lambda)\\ &= \partial\partial^* - \sqrt{-1}\partial\Lambda\partial + \overline{\partial}\partial^* - \sqrt{-1}\cdot\overline{\partial}\Lambda\partial + \sqrt{-1}\cdot\overline{\partial}\partial\Lambda\end{aligned}$$

(2) The second term can be computed as

$$\begin{aligned}(\partial^*+\overline{\partial}^*)(\partial+\overline{\partial}) &= (\partial^* - \sqrt{-1}\Lambda\partial + \sqrt{-1}\partial\Lambda)(\partial+\overline{\partial})\\ &= \partial^*\partial + \sqrt{-1}\partial\Lambda\partial + \partial^*\overline{\partial} - \sqrt{-1}\Lambda\partial\overline{\partial} + \sqrt{-1}\partial\Lambda\overline{\partial}\end{aligned}$$

By the third Kähler identity, one has

$$\partial^* = \sqrt{-1}[\Lambda,\overline{\partial}] = \sqrt{-1}\Lambda\overline{\partial} - \sqrt{-1}\overline{\partial}\Lambda$$

then

$$\begin{aligned}\overline{\partial}\partial^* &= \overline{\partial}(\sqrt{-1}\Lambda\overline{\partial} - \sqrt{-1}\cdot\overline{\partial}\Lambda) = \sqrt{-1}\cdot\overline{\partial}\Lambda\overline{\partial}\\ \partial^*\overline{\partial} &= (\sqrt{-1}\Lambda\overline{\partial} - \sqrt{-1}\cdot\overline{\partial}\Lambda)\overline{\partial} = -\sqrt{-1}\cdot\overline{\partial}\Lambda\overline{\partial} = -\overline{\partial}\partial^*\end{aligned}$$

Now we have

$$\begin{aligned}\Delta_{\mathrm{d}} &= \Delta_{\partial} - \sqrt{-1}\cdot\overline{\partial}\Lambda\partial - \sqrt{-1}\Lambda\partial\overline{\partial} + \sqrt{-1}\overline{\partial}\partial\Lambda + \sqrt{-1}\partial\Lambda\overline{\partial}\\ &= \Delta_{\partial} + \sqrt{-1}(\Lambda\overline{\partial}\partial - \overline{\partial}\Lambda\partial) + \sqrt{-1}(\partial\Lambda\overline{\partial} - \partial\overline{\partial}\Lambda)\\ &= \Delta_{\partial} + \sqrt{-1}[\Lambda,\overline{\partial}]\partial + \sqrt{-1}\partial[\Lambda,\overline{\partial}]\\ &= \Delta_{\partial} + \partial^*\partial + \partial\partial^*\\ &= 2\Delta_{\partial}\end{aligned}$$

□

Corollary 10.1.1. On a compact Kähler manifold, Δ_{d}-harmonic is equivalent to Δ_{∂}-harmonic, and is equivalent to $\Delta_{\overline{\partial}}$-harmonic.

Corollary 10.1.2. Let (X,ω) be a Kähler manifold and α be a (p,q)-form. Then $\Delta_{\mathrm{d}}\alpha$ is still a (p,q)-form.

Proof. It's clear to see $\Delta_{\partial}\alpha$ is still a (p,q)-form. □

Exercise 10.1.1. Show that for compact Kähler manifold we have

$$\begin{aligned}
&[\Delta_{\mathrm{d}}, L] = 0\\
&[L, \Lambda] = (k-n)\,\mathrm{id} \quad \text{on } C^\infty(X, \Omega^k_{X,\mathbb{C}})
\end{aligned}$$

Proof. For the first equation, we have $\Delta_{\mathrm{d}} = 2\Delta_\partial = 2(\partial\partial^* + \partial^*\partial)$. Thus

$$[\Delta_{\mathrm{d}}, L] = 2([\partial\partial^*, L] + [\partial^*\partial, L]) = 2(\partial[\partial^*, L] + [\partial^*, L]\partial)$$

The last equality holds by the fact that L commutes with ∂ since ω is ∂-closed. Now we use the identity $[\partial^*, L] = -\sqrt{-1}\cdot\overline{\partial}$, which anticommutes with ∂ to conclude.

For the second equation, without lose of generality it suffices to check on $U \subseteq \mathbb{C}^n$ equipped with standard Hermitian metric since we are considering operators of order zero. Suppose $\varphi = \varphi_{IJ}\mathrm{d}z^I\mathrm{d}\overline{z}^J$ is a k-form with type (p,q). A direct computation shows

$$\begin{aligned}
L\Lambda\varphi =& L\left(\sqrt{-1}\sum_{i=1}^n \varphi_{IJ}\iota_i\circ\iota_{\overline{i}}(\mathrm{d}z^I\wedge\mathrm{d}\overline{z}^J)\right)\\
=& L\left(\sqrt{-1}\sum_{i=1}^n (-1)^p\varphi_{IJ}\iota_i\mathrm{d}z^I\wedge\iota_{\overline{i}}\mathrm{d}\overline{z}^J\right)\\
=& (\sqrt{-1})^2\sum_{i,j=1}^n (-1)^{2p-1}\varphi_{IJ}\mathrm{d}z^j\wedge\iota_i\mathrm{d}z^I\wedge\mathrm{d}\overline{z}^j\iota_{\overline{i}}\mathrm{d}\overline{z}^J\\
\Lambda L\varphi =& \Lambda\left(\sqrt{-1}\sum_{j=1}^n (-1)^p\varphi_{IJ}\mathrm{d}z^j\wedge\mathrm{d}z^I\wedge\mathrm{d}\overline{z}^j\wedge\mathrm{d}\overline{z}^J\right)\\
=& (\sqrt{-1})^2\sum_{i,j=1}^n (-1)^p\varphi_{IJ}\iota_i\circ\iota_{\overline{i}}(\mathrm{d}z^j\wedge\mathrm{d}z^I\wedge\mathrm{d}\overline{z}^j\wedge\mathrm{d}\overline{z}^J)\\
=& (\sqrt{-1})^2\sum_{i,j=1}^n (-1)^{2p+1}\varphi_{IJ}\iota_i\left(\mathrm{d}z^j\wedge\mathrm{d}z^I\wedge\iota_{\overline{i}}(\mathrm{d}\overline{z}^j\wedge\mathrm{d}\overline{z}^J)\right)\\
=& (\sqrt{-1})^2\sum_{i,j=1}^n (-1)^{2p+1}\varphi_{IJ}\iota_i\left(\mathrm{d}z^j\wedge\mathrm{d}z^I\wedge\underbrace{(\delta_{\overline{i}}^{\overline{j}}\mathrm{d}\overline{z}^J - \mathrm{d}\overline{z}^j\wedge\iota_{\overline{i}}\mathrm{d}\overline{z}^J)}_{A}\right)\\
=& (\sqrt{-1})^2\sum_{i,j=1}^n (-1)^{2p+1}\varphi_{IJ}(\delta_i^j\mathrm{d}z^I\wedge A - \mathrm{d}z^j\wedge\iota_i\mathrm{d}z^I\wedge A)
\end{aligned}$$

Then

$$\begin{aligned}
L\Lambda\varphi - \Lambda L\varphi &= (\sqrt{-1})^2\sum_{j=1}^n\varphi_{IJ}\left(\mathrm{d}z^I\wedge\mathrm{d}\overline{z}^J - \mathrm{d}z^I\wedge\mathrm{d}\overline{z}^j\wedge\iota_{\overline{j}}\mathrm{d}z^J - \mathrm{d}z^j\wedge\iota_j\mathrm{d}z^I\wedge\mathrm{d}\overline{z}^J\right)\\
&= (k-n)\varphi
\end{aligned}$$

□

10.2. **Hodge decomposition.**

Theorem 10.2.1. Let (X, h) be a compact Kähler manifold, $\alpha = \sum_{p+q=k} \alpha^{p,q}$. Then α is harmonic if and only if $\alpha^{p,q}$ is harmonic, that is

$$\mathcal{H}^k \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}$$

with $\overline{\mathcal{H}^{p,q}} = \mathcal{H}^{q,p}$.

Proof. It follows from Δ_{d} preserves bidegree. □

Theorem 10.2.2 (Hodge decomposition)**.** Let (X, h) be a compact Kähler manifold. Then

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X)$$

with $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

Proof. It follows from there are natural isomorphisms $H^k(X, \mathbb{C}) \cong \mathcal{H}^k \otimes \mathbb{C}$ and $H^{p,q}(X) \cong \mathcal{H}^{p,q}$. □

Corollary 10.2.1. Let (X, h) be a compact Kähler manifold. Then

$$b_k = \sum_{p+q=k} h^{p,q}$$

with $h^{p,q} = h^{q,p}$, where $b_k = \dim H^k(X, \mathbb{C})$ and $h^{p,q} = \dim H^{p,q}(X)$.

Corollary 10.2.2. b_k is even when k is odd.

Corollary 10.2.3. $b_k \neq 0$ when k is even.

Proof. $h^{k,k} \neq 0$ since $0 \neq \omega^k \in H^{k,k}(X)$. □

There are many relations between $h^{p,q}$, and we can draw a picture as follows, called Hodge diamond since it has the same symmetry as a diamond.

$$\begin{array}{cccccccccc}
 & & & & h^{0,0} & & & & & b_0 \\
 & & & h^{1,0} & & h^{0,1} & & & & b_1 \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & & b_2 \\
 & \iddots & & & \vdots & & & \ddots & & \vdots \\
\text{Hodge} \updownarrow & h^{n,0} & \cdots & & \overset{\curvearrowleft}{\text{Serre}} & & \cdots & & h^{0,n} & b_n \\
 & \ddots & & & \vdots & & & \iddots & & \vdots \\
 & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & & b_{2n-2} \\
 & & & h^{n,n-1} & & h^{n-1,n} & & & & b_{2n-1} \\
 & & & & h^{n,n} & & & & & b_{2n} \\
 & & & & \overset{\longleftrightarrow}{\text{conjugation}} & & & & &
\end{array}$$

Example 10.2.1.

$$H^{p,q}(\mathbb{CP}^n) = \begin{cases} \mathbb{C} & 0 \leq p = q \leq n \\ 0 & \text{otherwise} \end{cases}$$

Proof. It's known to all that the singular cohomology of $\mathbb{CP}^n$ with complex coefficient is

$$H^k(\mathbb{CP}^n, \mathbb{C}) = \begin{cases} \mathbb{C} & k \text{ is even} \\ 0 & k \text{ is odd} \end{cases}$$

Thus it's clear to compute Dolbeault cohomology of $\mathbb{CP}^n$ using the symmetry of Hodge diamond. □

10.3. **Bott-Chern cohomology.** Review what we have done: We have already proven one of the main theorems in this course, that is, Hodge decomposition. But along the way we used the Kähler metric on a Kähler manifold, a question is that: (In)denpendence of the Kähler metric? The anwser is that our decomposition is independent of the choice of Kähler metric, shown by Bott-Chern cohomology.

Definition 10.3.1 (Bott-Chern cohomology)**.** Let X be a complex manifold. The Bott-Chern cohomology is defined as

$$H^{p,q}_{\mathrm{BC}}(X) := \frac{Z^{p,q}_{\mathrm{BC}} := \{\alpha \in C^\infty(X, \Omega^{p,q}_X) \mid \mathrm{d}\alpha = 0\}}{\partial\overline{\partial} C^\infty(X, \Omega^{p-1,q-1}_X)}$$

Remark 10.3.1. There is a natural map

$$Z^{p,q}_{\mathrm{BC}}(X) \to H^{p+q}(X, \mathbb{C})$$

which descends to

$$H^{p,q}_{\mathrm{BC}}(X) \to H^{p+q}(X, \mathbb{C})$$

since $\partial\overline{\partial}\beta = \mathrm{d}\overline{\partial}\beta$. On the other hand, there is also a natural map

$$Z^{p,q}_{\mathrm{BC}}(X) \to H^{p,q}(X)$$

which descends to

$$H^{p,q}_{\mathrm{BC}}(X) \to H^{p,q}(X)$$

since $\partial\overline{\partial}\beta = -\overline{\partial}\partial\beta$. So if we can prove there are isomorphisms between

$$H^{p,q}_{\mathrm{BC}}(X) \cong H^{p,q}(X)$$

$$\bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X) \cong H^k(X, \mathbb{C})$$

then Hodge decomposition is canonical, that is independent of choice of Kähler metric since Bott-Chern cohomology is independent of the choice of Kähler metric.

Lemma 10.3.1 ($\partial\overline{\partial}$-lemma)**.** Let (X, ω) be a compact Kähler manifold and α be a d-closed (p, q)-form. If α is $\overline{\partial}$-exact or ∂-exact, then there exists a $(p-1, q-1)$-form such that

$$\alpha = \partial\overline{\partial}\beta$$

Proof. Suppose α is $\overline{\partial}$-exact. Then $\alpha = \overline{\partial}\gamma$ for some $(p, q-1)$-form γ, and Hodge's theorem implies γ has decomposition

$$\gamma = a + \partial b + \partial^* c$$

where a is Δ_∂-harmonic, and b, c are forms with appropriate degrees. Direct computation shows

$$\begin{aligned}\alpha = \overline{\partial}\gamma &= \overline{\partial}a + \overline{\partial}\partial b + \overline{\partial}\partial^* c \\ &= -\partial\overline{\partial}b + \overline{\partial}\partial^* c \\ &= -\partial\overline{\partial}b - \partial^*\overline{\partial}c\end{aligned}$$

Now it suffices to show $-\partial^*\overline{\partial}c = 0$. A trick here is to note that

$$0 = \partial\alpha = -\partial\partial^*\overline{\partial}c \implies \partial^*\overline{\partial}c \in \ker\partial \cap \operatorname{im}\partial^* = 0 \implies \partial^*\overline{\partial}c = 0$$

So we have

$$\alpha = \partial\overline{\partial}(-b)$$

as desired. □

Corollary 10.3.1. Let (X, ω) be a compact Kähler manifold. Then

(1) $H^{p,q}_{\mathrm{BC}}(X) \to H^{p,q}(X)$ is an isomorphism.
(2) $\bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X) \to H^k(X, \mathbb{C})$ is an isomorphism.

Proof. Here we only prove the first isomorphism. From Remark 10.3.1, there is a canonical map $H^{p,q}_{\mathrm{BC}}(X) \to H^{p,q}(X)$, and if we choose a Kähler metric, we have $H^{p,q}(X) \cong \mathcal{H}^{p,q}$, we will show our canonical map is both surjective and injective via this chosen metric.

(1) To see surjectivity: For element in $H^{p,q}(X)$ we choose a $\Delta_{\overline{\partial}}$-harmonic representative. Since Δ_∂-harmonic is equivalent to Δ_{d}-harmonic, so this representative is also d-closed.
(2) To see injectivity: Suppose we have $[\alpha] \in H^{p,q}_{\mathrm{BC}}(X)$ such that α is trivial in $H^{p,q}(X)$, that is $\overline{\partial}$-exact. Then Lemma 10.3.1, that is $\partial\overline{\partial}$-lemma implies it's trivial in Bott-Chern cohomology.

□

Corollary 10.3.2. A Hermitian metric ω is Kähler if and only if it can be written locally as

$$\omega = \sqrt{-1}\partial\overline{\partial}f$$

where f is a real-valued smooth function.

Proof. It's clear if ω is locally written as $\sqrt{-1}\partial\overline{\partial}f$, then it gives a Kähler metric. Conversely, a Kähler metric ω is an element in $H^{1,1}(X)$, and we have already shown that $H^{1,1}(X) = H^{1,1}_{\mathrm{BC}}(X)$, and Dolbeault lemma implies Dolbeault cohomology vanishes on open subset which is sufficiently small, this completes the proof. □

11. Applications of Hodge theory

11.1. **Adjoint operators on bundle valued forms.** Let(E,h) be a Hermitian holomorphic vector bundle on Hermitian n-manifold X, what we have done can be generalized to bundle valued forms. More explicitly, for $\varphi,\psi\in C^\infty(X,\Omega_X^{p,q}\otimes E)$, locally written as $\varphi=\varphi^\alpha e_\alpha,\psi=\psi^\beta e_\beta$, then local inner product is given by

$$\langle\varphi,\psi\rangle:=h_{\alpha\overline{\beta}}\langle\varphi^\alpha,\psi^\beta\rangle$$

and $\varphi\wedge\psi$ is defined as

$$\varphi\wedge\overline{\psi}:=\varphi^\alpha\wedge\overline{\psi^\beta}h_{\alpha\overline{\beta}}$$

The inner product on $C^\infty(X,\Omega_X^{p,q}\otimes E)$ is given by

$$(\varphi,\psi):=\int_X\langle\varphi,\psi\rangle\frac{\omega^n}{n!}$$

where $\varphi,\psi\in C^\infty(X,\Omega_X^{p,q}\otimes E)$. The Hodge star operator is defined as an operator

$$\star_E\colon C^\infty(X,\Omega_X^{p,q}\otimes E)\to C^\infty(X,\Omega_X^{n-q,n-p}\otimes E)$$

such that

$$(\varphi,\psi)=\int_X\varphi\wedge\star_E\overline{\psi}$$

Let ∇ be the Chern connection of (E,h). Then $\nabla^{0,1}=\overline{\partial}_E$, and if we set $\nabla^{1,0}=\partial_E$, then

$$\begin{aligned}\Theta_h&=\nabla^2\\&=\partial_E^2+\partial_E\overline{\partial}_E+\overline{\partial}_E\partial_E+\overline{\partial}_E^2\\&=[\partial_E,\overline{\partial}_E]\end{aligned}$$

Exercise 11.1.1. Give formulas of ∂_E^* and $\overline{\partial}_E^*$ in terms of Hodge star $\star_E$.

Laplacians $\Delta_{\partial_E},\Delta_{\overline{\partial}_E}$ can be defined in a same way, and there is also a Hodge theorem, which gives the following decomposition

$$C^\infty(X,\Omega_X^{p,q}\otimes E)=\mathcal{H}^{p,q}(X,E)\oplus\operatorname{im}\overline{\partial}_E\oplus\operatorname{im}\overline{\partial}_E^*$$

and by the same argument we have

$$H^{p,q}(X,E)\cong\mathcal{H}^{p,q}$$

Lefschetz operator is defined as follows

$$\begin{aligned}L\colon C^\infty(X,\Omega^{p,q}\otimes E)&\to C^\infty(X,\Omega^{p+1,q+1}\otimes E)\\\alpha&\mapsto\omega\wedge\alpha\end{aligned}$$

and Λ is the formal adjoint of L. If (X,ω) is also a Kähler n-manifold, then there are also Kähler identities

$$
\begin{aligned}
[\overline{\partial}_E^*, L] &= \sqrt{-1}\partial \\
[\partial_E^*, L] &= -\sqrt{-1}\cdot\overline{\partial} \\
[\Lambda, \partial] &= -\sqrt{-1}\partial^* \\
[\Lambda, \partial] &= \sqrt{-1}\cdot\overline{\partial}^*
\end{aligned}
$$

and

$$[L,\Lambda] = (p+q-n)\,\mathrm{id}$$

holds on E-valued (p,q)-forms.

11.2. **Serre duality.**

Theorem 11.2.1 (Serre duality)**.** Let X be a compact complex n-manifold and E be a holomorphic vector bundle. Then there exists a non-degenerate $\mathbb{C}$-linear pairing

$$
\begin{aligned}
H^{p,q}(X,E)\times H^{n-p,n-q}(X,E^*) &\to \mathbb{C} \\
([\alpha],[\beta]) &\mapsto \int_X \alpha\wedge\beta
\end{aligned}
$$

In particular, we have

$$H^{p,q}(X,E) = H^{n-p,n-q}(X,E^*)^*$$

Sketch of the proof. Firstly endow E with a Hermitian metric h, and show

$$\Delta_{\overline{\partial}_E^*}\circ\overline{\star}_E = \overline{\star}_E\circ\Delta_{\overline{\partial}_E}$$

Then

$$\overline{\star}_E\colon \mathcal{H}^{p,q}(X,E)\xrightarrow{\cong}\mathcal{H}^{n-p,n-q}(X,E^*)$$

and Hodge theorem implies that

$$\mathcal{H}^{p,q}(X,E)\cong H^{p,q}(X,E)$$

For all $\alpha\in\mathcal{H}^{p,q}(X,E),\beta\in\mathcal{H}^{n-p,n-q}(X,E^*)$, we have $\beta=\overline{\star}_E\gamma$ for some $\gamma\in\mathcal{H}^{p,q}(X,E)$, and thus

$$\int_X\alpha\wedge\beta = \int_X\alpha\wedge\overline{\star}_E\gamma = \langle\alpha,\gamma\rangle$$

is non-degenerate. □

Corollary 11.2.1. Let X be a compact complex n-manifold and E be a holomorphic vector bundle. Then

$$H^{p,q}(X) = H^{n-p,n-q}(X)^*$$

Proof. Consider $E=\mathcal{O}_X$ in Serre duality, and then desired result holds from the fact $\mathcal{O}_X^*=\mathcal{O}_X$. □

Remark 11.2.1. This recovers Corollary 9.2.6.

Corollary 11.2.2. Let X be a compact complex n-manifold and E be a holomorphic vector bundle. Then

$$H^q(X,E) = H^{n-q}(X, K_X \otimes E^*)^*$$

Proof. Set $p=0$ in Serre duality one has

$$H^{0,q}(X,E) \cong H^{n,n-q}(X,E^*)^*$$

which gives desired result. □

12. Lefschetz decomposition

12.1. **Lefschetz decomposition.**

Proposition 12.1.1. Let (X, ω) be a Kähler n-manifold. Then $L^{n-k}\colon C^\infty(X, \Omega^k_{X,\mathbb{R}}) \to C^\infty(X, \Omega^{2n-k}_{X,\mathbb{R}})$ is an isomorphism for $k \leq n$.

Proof. In fact, we will prove that L^r are injective for all $1 \leq r \leq n-k$. As a consequence, L^{n-k} is an isomorphism since $\Omega^k_{X,\mathbb{R}}$ has the same rank as $\Omega^{2n-k}_{X,\mathbb{R}}$. In Exercise 10.1.1 we have shown that

$$[L, \Lambda]\alpha = (k-n)\alpha, \quad \forall \alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}}).$$

Then

$$\begin{aligned}[L^r, \Lambda] &= L^r\Lambda - \Lambda L^r \\ &= L(L^{r-1}\Lambda - \Lambda L^{r-1}) + L\Lambda L^{r-1} - \Lambda L L^{r-1} \\ &= L[L^{r-1}, \Lambda] + [L, \Lambda]L^{r-1}.\end{aligned}$$

By induction it's easy to show the following identity

$$[L^r, \Lambda]\alpha = (r(k-n) + r(r-1))L^{r-1}\alpha, \quad \forall \alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}}).$$

For $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}})$, if $L^r\alpha = 0, r \leq n-k$, then

$$\begin{aligned}L^r\Lambda\alpha &= [L^r, \Lambda]\alpha \\ &= (r(k-n) + r(r-1))L^{r-1}\alpha.\end{aligned}$$

In other words, we have

$$L^{r-1}(L\Lambda\alpha - (r(k-n) + r(r-1))\alpha) = 0. \tag{12.1}$$

Now let's prove L^r is injective by induction on r: It's clear L is injective, and suppose L^{r-1} is injective. Then by (12.1) one has

$$L\Lambda\alpha = (r(k-n) + r(r-1))\alpha.$$

If we denote $\beta = \Lambda\alpha$, and apply L^r to both side of above equation, then we have

$$L^{r+1}\beta = (r(k-n) + r(r-1))L^r\alpha = 0,$$

where $\beta \in C^\infty(X, \Omega^{k-2}_{X,\mathbb{R}})$. It's clear $\beta = 0$ if β is a smooth function. Then by induction on k, we have $\beta = 0$, and thus $\alpha = 0$. □

Definition 12.1.1 (primitive form)**.** Let (X, ω) be a Kähler n-manifold. A k-form α is called primitive if $L^{n-k+1}\alpha = 0$.

Exercise 12.1.1. A k-form α is primitive if and only if $\Lambda\alpha = 0$.

Proof. For an n-form α, α is primitive if and only if $L\alpha = 0$. On the other hand, the Exercise 10.1.1 implies that

$$[L, \Lambda] = (k-n)\,\mathrm{id}, \quad \text{on } C^\infty(X, \Omega^k_{X,\mathbb{R}}).$$

This shows if $n=k$, then L and Λ commutes. Thus we have α is primitive if and only if $\Lambda\alpha=0$, since

$$\Lambda\alpha=0\iff L\Lambda\alpha=0\iff\Lambda L\alpha=0\iff L\alpha=0$$

and the first and last equality we use the fact that L is injective on $\Omega^k_{X,\mathbb{R}}, k\leq n$ and Λ is injective on $\Omega^{n+2}_{X,\mathbb{R}}$. In general case, we have

$$[L^r,\Lambda]\alpha=(r(k-n)+r(r-1))L^{r-1}\alpha$$

and in particular for $r=n-k+1$ where k is the degree of α, we have

$$[L^r,\Lambda]\alpha=0$$

The argument can be repeated to conclude. □

Proposition 12.1.2. For any k-form α, there exists a unique decomposition

$$\alpha=\sum_r L^r\alpha_r,$$

where α_r is primitive $(k-2r)$-form.

Proof. Firstly let's prove the uniqueness: If $\sum_r L^r\alpha_r=0$ with primitive α_r, we need to show $\alpha_r=0$. If not, then take the largest r_m such that $\alpha_{r_m}\neq 0$. By the choice of α_{r_m}, L^{n-k+r_m} kills everything in $\sum_r L^r\alpha_r$ but $L^{r_m}a_{r_m}$. Then

$$0=L^{n-k+r_m}(\sum_r L^r\alpha_r)=L^{n-k+r_m}(L^{r_m}\alpha_{r_m})\neq 0,$$

which is a contradiction.

Now let's prove the existence: Since $L^{n-k+2}\colon C^\infty(X,\Omega^{k-2}_{X,\mathbb{R}})\to C^\infty(X,\Omega^{2n-k+2}_{X,\mathbb{R}}$ is an isomorphism, then there exists $\beta\in C^\infty(X,\Omega^{k-2}_{X,\mathbb{R}})$ such that

$$L^{n-k+1}\alpha=L^{n-k+2}\beta.$$

Then $\alpha-L\beta$ is primitive a primitive k-form, that is

$$\alpha=(\alpha-L\beta)+L\beta.$$

By induction on k, we have primitive decomposition for $\beta\in C^\infty(X,\Omega^{k-2}_{X,\mathbb{R}})$. and this completes the proof. □

Remark 12.1.1. If we define $H=[L,\Lambda]$, then (L,H,Λ) generates an $\mathfrak{sl}_2$-action on $\bigoplus_k C^\infty(X,\Omega^k_{X,\mathbb{R}})$.

In fact, the Lefschetz operator also defines a map between cohomology groups

$$\begin{aligned} L\colon H^k(X,\mathbb{R})&\to H^{k+2}(X,\mathbb{R})\\ [\alpha]&\mapsto[\omega\wedge\alpha]. \end{aligned}$$

Now let's see it's well-defined:

(1) If α is closed, then

$$\mathrm{d}(\omega\wedge\alpha)=\mathrm{d}\omega\wedge\alpha+\omega\wedge\mathrm{d}\alpha=0.$$

(2) If $\alpha = \mathrm{d}\beta$, then

$$\omega \wedge \mathrm{d}\beta = \mathrm{d}\omega \wedge \beta + \omega \wedge \mathrm{d}\beta = \mathrm{d}(\omega \wedge \beta).$$

Theorem 12.1.1 (hard Lefschetz theorem[11])**.** Let (X, ω) be a compact Kähler n-manifold. Then

$$L^{n-k}\colon H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R})$$

is an isomorphism for $1 \leq k \leq n$.

Proof. In Exercise 10.1.1 we have shown $[\Delta_{\mathrm{d}}, L] = 0$, so the Lefschetz operator induces a map between harmonic forms as follows

$$L^{n-k}\colon \mathcal{H}^k \to \mathcal{H}^{2n-k}.$$

By Proposition 12.1.1 L^{n-k} is injective and $\mathcal{H}^k, \mathcal{H}^{2n-k}$ have the same dimension, we obtain the desired result. □

Definition 12.1.2 (primitive form)**.** Let (X, ω) be a compact Kähler n-manifold. For $[\alpha] \in H^k(X, \mathbb{R})$, it's called primitive, if $L^{n-k+1}[\alpha] = 0$.

Notation 12.1.1. $H^k(X, \mathbb{R})_{\mathrm{prim}}$ denotes the set of all primitive forms.

Corollary 12.1.1 (Lefschetz decomposition)**.** There is the following decomposition

$$H^k(X, \mathbb{R}) = \bigoplus_r L^r H^{k-2r}(X, \mathbb{R})_{\mathrm{prim}}.$$

Remark 12.1.2. If $[\omega] \in H^2(X, \mathbb{Z})$, such as ω comes from a positive holomorphic line bundle, then we can state theorem and corollary for $H^k(X, \mathbb{Q})$.

Moreover, we have the following isomorphism

$$L^{n-k}\colon H^{p,q}(X) \to H^{n-q,n-p}(X)$$

for $k = p + q \leq n$.

Corollary 12.1.2. Let (X, ω) be a compact Kähler n-manifold. Then for $2 \leq k \leq n$, one has $b_{k-2} \leq b_k$ and $h^{p-1,q-1} \leq h^{p,q}$ with $k = p + q$.

12.2. **Hodge index.**

12.2.1. *Surface case.*

Example 12.2.1. For open subset $U \subseteq \mathbb{C}^2$ equipped with canonical Kähler form

$$\omega = \sqrt{-1}\left(\mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^1 + \mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^2\right).$$

The volume form is given by

$$\mathrm{vol} = \frac{\omega^2}{2!} = -\left(\mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^2\right).$$

[11]Though proof of this theorem is quite easy using tools we have, but it's quite hard for Lefschetz, since during his time, there is no Hodge theorem. Here we use L to denote Lefschetz operator, in order to honor Lefschetz.

Suppose α is a $(2,0)$ form written as

$$\alpha = a\mathrm{d}z^1 \wedge \mathrm{d}z^2.$$

Now we're going to compute $\star\overline{\alpha}$, which is also a $(2,0)$-form. Suppose $\star\overline{\alpha} = b\mathrm{d}z^1 \wedge \mathrm{d}z^2$. Then by definition, for an arbitrary $(0,2)$-form β one has

$$\langle \beta, \alpha\rangle \operatorname{vol} = \beta \wedge \star\overline{\alpha}.$$

In particular, if we choose $\beta = \mathrm{d}\overline{z}^1 \wedge \mathrm{d}\overline{z}^2$, then

$$\begin{aligned}
\beta \wedge \star\overline{\alpha} &= -b\mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^2 \\
\{\beta, \alpha\} \operatorname{vol} &= \{a\mathrm{d}z^1 \wedge \mathrm{d}z^2, \mathrm{d}\overline{z}^1 \wedge \mathrm{d}\overline{z}^2\} \times -\mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^2 \\
&= -a\mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^2.
\end{aligned}$$

This shows $\alpha = \star\overline{\alpha}$. By the same computation one has $\alpha = \star\overline{\alpha}$ holds for a $(0,2)$-form α. On the other hand, it's clear $(2,0)$-form and $(0,2)$-form are automatically primitive.

Now we're going to see if a $(1,1)$-form α is primitive, what's the relation between α and $\star\overline{\alpha}$. For $(1,1)$-form α, written as

$$\alpha = a_{11}\mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^1 + a_{22}\mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^2 + a_{12}\mathrm{d}z_1 \wedge \mathrm{d}\overline{z}^2 + a_{21}\mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^1.$$

A direct computation shows that

$$\star\overline{\alpha} = a_{22}\mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^1 + a_{11}\mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^2 - a_{12}\mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^2 - a_{21}\mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^1.$$

On the other hand,

$$L\alpha = \omega \wedge \alpha = \sqrt{-1}(a_{11} + a_{22})\mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^2.$$

Then

$$L\alpha = 0 \iff a_{11} + a_{22} = 0 \iff \star\overline{\alpha} = -\alpha.$$

Lemma 12.2.1. Let (X, ω) be a Kähler surface. If (p,q)-form α is primitive 2-form, then

$$\star\overline{\alpha} = (-1)^p\alpha.$$

Proof. By taking normal coordinate, it suffices to consider $U \subseteq \mathbb{C}^2$, and that's exactly what we have done in Example 12.2.1. $\square$

Let X be a compact Kähler surface. The Poincaré duality and Stokes theorem imply that we have the following well-defined non-degenerate pairing

$$\begin{aligned}
Q\colon H^2(X,\mathbb{R}) \times H^2(X,\mathbb{R}) &\to \mathbb{R} \\
([\alpha],[\beta]) &\mapsto \int_X \alpha \wedge \beta
\end{aligned}$$

Then we obtain a Hermitian form by considering

$$H([\alpha],[\beta]) = Q([\alpha],\overline{[\beta]}).$$

Lemma 12.2.2. The Lefschetz decomposition $H^2(X,\mathbb{R}) = H^2(X,\mathbb{R})_{\mathrm{prim}} \oplus \mathbb{R}\cdot[\omega]$ is orthonormal with respect to Q.

Proof.

$$Q([\omega],[\alpha]) = \int_X \omega\wedge\alpha = \int_X L\alpha = 0$$

for α is primitive and harmonic. □

Theorem 12.2.1. $H^2(X,\mathbb{C})_{\mathrm{prim}} = \bigoplus_{p+q=2} H^{p,q}(X)_{\mathrm{prim}}$ is orthonormal with respect to H, and $(-1)^p H$ is positive definite on $H^{p,q}(X)_{\mathrm{prim}}$.

Proof. It's clear above decomposition is orthonormal, since

$$\int_X \alpha = 0$$

if α is not a $(2,2)$-form. To see $(-1)^p H$ is positive definite on $H^{p,q}(X)_{\mathrm{prim}}$, we take a harmonic representative α for any non-zero primitive cohomology class in $H^{p,q}(X)_{\mathrm{prim}}$. Then

$$\begin{aligned}(-1)^p H([\alpha],[\alpha]) &= (-1)^p \int_X \alpha\wedge\overline{\alpha}\\ &= (-1)^{p+q}\int_X \alpha\wedge\star\overline{\alpha}\\ &= \|\alpha\|^2 > 0.\end{aligned}$$

This shows $(-1)^p H$ is positive definite on $H^{p,q}(X)_{\mathrm{prim}}$. □

Corollary 12.2.1 (Hodge index)**.** The index of H defined on $H^2(X,\mathbb{C})\cap H^{1,1}(X)$ is $(1,h^{1,1}-1)$.

Proof. Note that there is the following decomposition

$$H^2(X,\mathbb{C})\cap H^{1,1}(X) = H^{1,1}(X)_{\mathrm{prim}}\oplus\mathbb{C}[\omega],$$

and we have already shown that H is negative definite on $H^{1,1}(X)_{\mathrm{prim}}$. Then the index for H on $H^2(X,\mathbb{C})\cap H^{1,1}(X)$ is $(1,h^{1,1}-1)$. □

12.2.2. *General case.* In this section we will introduce a more general case: Let (X,ω) be a compact Kähler n-manifold. Then by Lefschetz decomposition we have

$$H^k(X,\mathbb{R}) = \bigoplus_r L^r H^{k-2r}(X,\mathbb{R})_{\mathrm{prim}},\quad k\le n,$$

and by Hodge decomposition we have a more explicit decomposition

$$H^k(X,\mathbb{C})_{\mathrm{prim}} = \bigoplus_{p+q=k} H^{p,q}(X)_{\mathrm{prim}}.$$

As we have seen in the case of surface, H will be positive definite or negative definite in these (p,q) components. Now we introduce some symbols, in order to get a neater result.

Consider

$$Q\colon H^k(X,\mathbb{R})\times H^k(X,\mathbb{R})\to\mathbb{R}$$
$$([\alpha],[\beta])\mapsto(-1)^{\frac{k(k-1)}{2}}\int_X\omega^{n-k}\wedge\alpha\wedge\beta.$$

Then Q is a bilinear form, and it is symmetric when k is even and anti-symmetric when k is odd.

Definition 12.2.1 (Weil operator)**.** The Weil operator $\mathsf{C}\colon H^k(X,\mathbb{C})\to H^k(X,\mathbb{C})$ is defined by $\mathsf{C}|_{H^{p,q}(X)}\mapsto\sqrt{-1}^{p-q}\,\mathrm{id}$.

Remark 12.2.1. The Weil operator C maps $H^k(X,\mathbb{R})$ to $H^k(X,\mathbb{R})$ in fact:

$$\mathsf{C}|_{\overline{H^{p,q}(X)}}=\mathsf{C}_{H^{q,p}(X)}=\sqrt{-1}^{q-p}\,\mathrm{id}=\overline{\sqrt{-1}^{p-q}}\,\mathrm{id}=\overline{\mathsf{C}|_{H^{p,q}(X)}}.$$

Now we define

$$H\colon H^k(X,\mathbb{C})\times H^k(X,\mathbb{C})\to\mathbb{C}$$
$$([\alpha],[\beta])\mapsto Q(\mathsf{C}[\alpha],\overline{[\beta]}).$$

In other words, we have

$$H([\alpha],[\beta])=(-1)^{\frac{k(k-1)}{2}}\sqrt{-1}^{p-q}\int_X\omega^{n-k}\wedge\alpha\wedge\overline{\beta},\quad \alpha,\beta\in H^{p,q}(X).$$

Exercise 12.2.1. H is a Hermitian form on $H^{p,q}(X)$.

Proof. For $[\alpha],[\beta]\in H^{p,q}(X)$, one has

$$\begin{aligned}\overline{H([\alpha],[\beta])}&=(-1)^{\frac{k(k-1)}{2}}(-1)^{p-q}\sqrt{-1}^{p-q}\int_X\omega^{n-k}\wedge\overline{\alpha}\wedge\beta\\&=(-1)^{\frac{k(k-1)}{2}}(-1)^{p-q}\sqrt{-1}^{p-q}(-1)^{(p+q)^2}\int_X\omega^{n-k}\wedge\beta\wedge\overline{\alpha}.\end{aligned}$$

Note that

$$(p+q)^2-p-q=2pq+p(p-1)+q(q-1)$$

is always even, this completes the proof. □

Lemma 12.2.3. Let α be a primitive (p,q)-form with $p+q=k$. Then

$$\star\alpha=(-1)^{\frac{k(k+1)}{2}}\sqrt{-1}^{p-q}\frac{L^{n-k}\alpha}{(n-k)!}.$$

Theorem 12.2.2 (Hodge-Riemann bilinear relations)**.**
(1) $H^k(X,\mathbb{R})=\bigoplus_r L^rH^{k-2r}(X,\mathbb{R})_{\mathrm{prim}}$ is orthonormal with respect to Q.
(2) $H^k(X,\mathbb{C})_{\mathrm{prim}}=\bigoplus_{p+q=k}H^{p,q}(X)_{\mathrm{prim}}$ is orthonormal with respect to H.
(3) H is positive definite on $H^{p,q}(X)_{\mathrm{prim}}$.

Proof. For (1). For $r<s$, note that

$$\omega^{n-k}\wedge L^r\gamma\wedge L^s\delta=(L^{n-k+r+s}\gamma)\wedge\delta=0$$

since $L^{n-k+2r+1}\gamma=0$ and $r<s$.

For (2). If α is a (p,q)-form, and β is (p',q')-form, and $(p,q)\neq(p',q')$, then $\omega^{n-k}\wedge\alpha\wedge\overline{\beta}$ is not a (n,n)-form.

For (3). To see H is positive definite on $H^{p,q}(X)_{\mathrm{prim}}$, we take a harmonic representative α for any non-zero primitive cohomology class in $H^{p,q}(X)_{\mathrm{prim}}$. Then

$$H([\alpha],[\alpha])=(-1)^{\frac{k(k-1)}{2}}\sqrt{-1}^{p-q}\int_X\omega^{n-k}\wedge\alpha\wedge\overline{\alpha}$$

By Lemma 12.2.3 one has

$$\begin{aligned}\star\overline{\alpha}&=(-1)^{\frac{k(k+1)}{2}}\sqrt{-1}^{q-p}\frac{L^{n-k}\overline{\alpha}}{(n-k)!}\\&=(-1)^{\frac{k(k-1)}{2}}\sqrt{-1}^{p-q}\frac{L^{n-k}\overline{\alpha}}{(n-k)!}.\end{aligned}$$

Then

$$H([\alpha],[\alpha])=(n-k)!\int_X\alpha\wedge\star\overline{\alpha}=(n-k)!\|\alpha\|^2>0$$

□

Corollary 12.2.2 (Hodge index theorem)**.** Let X be a compact Kähler n-manifold with n is even[12]. Then $\int_X\alpha\wedge\beta$ on $H^n(X,\mathbb{R})$ is of signature

$$\sum_{p,q}(-1)^p h^{p,q}$$

where summation runs over all p,q.

Proof. Note that the signature of $\int_X\alpha\wedge\beta$ on $H^n(X,\mathbb{R})$ is the same as the signature of $\int_X\alpha\wedge\overline{\beta}$ on $H^n(X,\mathbb{C})$. We write

$$H^n(X,\mathbb{C})=\bigoplus_{\substack{p+q+2r=n\\ r,p,q\in\mathbb{Z}_{\geq 0}}}L^rH^{p,q}(X)_{\mathrm{prim}}$$

Then Hodge-Riemann bilinear theorem implies that $\int_X\alpha\wedge\overline{\beta}$ is $(-1)^p$-definite on $L^rH^{p,q}(X)_{\mathrm{prim}}$, where we used the fact n is even. Then we have the signature is

$$\sum_{p+q+2r=n}(-1)^p h^{p,q}_{\mathrm{prim}}.$$

But $h^{p,q}_{\mathrm{prim}}=h^{p,q}-h^{p-1,q-1}$, so

$$\sum_{p+q+2r=n}(-1)^p(h^{p,q}-h^{p-1,q-1}).$$

Note that $p+q=n$ counted once and $p+q<n$ counted twice, so rewrite it as

$$\sum_{p+q\text{ even}}(-1)^p h^{p,q},$$

[12]In this case $\int_X\alpha\wedge\beta$ is symmetric on $H^n(X,\mathbb{R})$.

since $h^{p,q} = h^{n-p,n-q}$. And this is also equivalent to sum all p, q, since

$$\sum_{p+q \text{ odd}} (-1)^p h^{p,q} = 0$$

This completes the proof. □

Example 12.2.2. For surface, we have

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X)_{\text{prim}} \oplus \mathbb{C}[\omega] \oplus H^{0,2}(X)$$

Then this corollary implies

$$h^{0,0} + h^{2,0} - h^{1,1} + h^{0,2} + h^{2,2} = h^{2,0} + h^{0,2} + (1 - (h^{1,1} - 1)),$$

which recovers what we have done in the case of surface.

Part 4. Positivity and vanishing theorems

13. Positivity

13.1. **Positivity of line bundle.** Let (L, h) be a Hermitian holomorphic line bundle over a complex manifold X. Then $\frac{\sqrt{-1}}{2\pi}\Theta_h$ gives a real $(1,1)$-form, so it corresponds to a Hermitian form on TX.

Definition 13.1.1 (positive line bundle)**.** Let L be a holomorphic line bundle over X. L is called positive if it admits a Hermitian metric h such that the Hermitian form corresponding to $\frac{\sqrt{-1}}{2\pi}\Theta_h$ is positive definite.

Remark 13.1.1. The Kodaira embedding theorem implies positive line bundle is exactly ample divisor in algebraic geometry.

Remark 13.1.2 (local form). Locally, one has

$$\frac{\sqrt{-1}}{2\pi}\Theta_h = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h = \frac{\sqrt{-1}}{2\pi}\frac{\partial^2\varphi}{\partial z^i\partial\overline{z}^j}\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^j$$

where $\varphi = -\log h$. Thus L is positive if and only if the Hermitian matrix $(\frac{\partial^2\varphi}{\partial z^i\partial\overline{z}^j})$ is positive definite everywhere.

Proposition 13.1.1. If X admits a positive holomorphic line bundle, then X is Kähler.

Proof. The first Chern class of (L, h) gives its Kähler form. □

Proposition 13.1.2. L is positive if and only if $L^{\otimes m}$ is positive for some $m \in \mathbb{N}_{\geq 0}$.

Proof. For a line bundle L locally we have the Hermitian metric corresponding to its curvature looking like

$$(\frac{\partial^2\varphi}{\partial z^i\partial\overline{z}^j})$$

and for $L^{\otimes m}, m \in \mathbb{N}_{\geq 0}$ we have

$$(m\cdot\frac{\partial^2\varphi}{\partial z^i\partial\overline{z}^j})$$

It's clear L is positive if and only if $L^{\otimes m}$ is. □

Exercise 13.1.1. Let X be a compact complex manifold and L be a positive line bundle. For any holomorphic line bundle L', there exists $N_0 \in \mathbb{N}$ such that $L'\otimes L^{\otimes N}$ positive for $N \geq N_0$.

Proof. The proof is quite similar to above exercise, we need to check locally, but compactness is neccessary here. Over an open subset U_1, locally we have the Hermitian metric corresponding to $L'\otimes L^m$ looking like

$$(\frac{\partial^2\varphi_{L'}}{\partial z^i\partial\overline{z}^j} + m\cdot\frac{\partial^2\varphi_L}{\partial z^i\partial\overline{z}^j})$$

So we can choose suffices large N_1 such that $M \otimes L^{\otimes N_1}$ is positive on U. Since X is compact, we can take a finite open covering $\{U_i\}$ of X and choose the largest N_i to be N we desired. □

13.2. **Positivity of vector bundles.** Let (E, h) be a Hermitian holomorphic vector bundle of rank r over a complex n-manifold X with Chern connection ∇. In local frame, its Chern curvature is given by

$$\Theta_h = \Theta_{i\bar{j}\alpha}^{\beta} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j \otimes e^\alpha \otimes e_\beta$$

Definition 13.2.1 (positivity)**.**

(1) (E, h) is said to be Griffiths positive, if for any non-zero $(u^i) \in \mathbb{C}^n$ and $(v^\alpha) \in \mathbb{C}^r$
$$\Theta_{i\bar{j}\alpha\bar{\beta}} u^i \overline{u}^j v^\alpha \overline{v}^\beta > 0$$
(2) (E, h) is said to be Nakano positive, if for any non-zero matrix $(u^{i\alpha})$
$$\Theta_{i\bar{j}\alpha\bar{\beta}} u^{i\alpha} \overline{u}^{j\beta} > 0$$
(3) (E, h) is said to be dual Nakano positive, if for any non-zero matrix $(u^{i\alpha})$
$$\Theta_{i\bar{j}\alpha\bar{\beta}} u^{i\beta} \overline{u}^{j\alpha} > 0$$

Remark 13.2.1. The semi-positivity and negativity can be defined in the same way.

Proposition 13.2.1.

(1) If (E, h) is Nakano positive or dual Nakano positive, then (E, h) is Griffiths positive.
(2) (E, h) is Nakano positive if and only if (E^*, h^*) is dual Nakano negative.

Proof. For (1). If (E, h) is Nakano positive, then for non-zero $(u^i) \in \mathbb{C}^n$ and $(v^\alpha) \in \mathbb{C}^r$, consider matrix $(u^{i\alpha})$ defined by $u^{i\alpha} := u^i v^\alpha$, then

$$\Theta_{i\bar{j}\alpha\bar{\beta}} u^i \overline{u}^j v^\alpha \overline{v}^\beta = \Theta_{i\bar{j}\alpha\bar{\beta}} u^{i\alpha} \overline{u}^{j\beta} > 0$$

The same argument holds for the case (E, h) is dual Nakano positive. (2) follows from the relation between curvature form of (E, h) and (E^*, h^*), see Section 4.3 of [Liu23]. □

Proposition 13.2.2. Let

$$0 \to S \to E \to Q \to 0$$

be an exact sequence of holomorphic vector bundles.

(1) If (E, h) is Griffiths or dual Nakano positive, then so is (Q, h^Q).
(2) If (E, h) is Griffiths or Nakano negative, then so is (S, h^S).

Proposition 13.2.3. If Hermitian holomorphic vector bundle (E, h) is Griffiths positive, then $(E \otimes \det E, h \otimes \det h)$ is Nakano positive and dual Nakano positive.

14. Vanishing theorems

14.1. **Kodaira vanishing theorem.**

Theorem 14.1.1 (Bochner-Kodaira-Nakano identity)**.** Let (X,ω) be a compact Kähler manifold and (E,h) a Hermitian holomorphic vector bundle. Then

$$\Delta_{\overline{\partial}_E} = [\sqrt{-1}\Theta_h, \Lambda] + \Delta_{\partial_E}$$

Proof. Direct computation shows

$$\begin{aligned}
\Delta_{\overline{\partial}_E} &= [\overline{\partial}_E, \overline{\partial}_E^*] \\
&= -\sqrt{-1}[\overline{\partial}_E, [\Lambda, \partial_E]] \\
&= -\sqrt{-1}[\Lambda, [\partial_E, \overline{\partial}_E]] - \sqrt{-1}[\partial_E, [\overline{\partial}_E, \Lambda]] \\
&= -\sqrt{-1}[\Lambda, \Theta_h] - \sqrt{-1}[\partial_E, \sqrt{-1}\partial_E^*] \\
&= [\sqrt{-1}\Theta_h, \Lambda] + \Delta_{\partial_E}
\end{aligned}$$

□

Corollary 14.1.1 (Bochner-Kodaira-Nakano inequality)**.** Let (X,ω) be a compact Kähler manifold and (E,h) a Hermitian holomorphic vector bundle. Then for $\alpha \in C^\infty(X, \Omega_X^{p,q} \otimes E)$, one has

$$([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) \leq (\Delta_{\overline{\partial}_E}\alpha, \alpha)$$

In particular, if α is $\Delta_{\overline{\partial}_E}$-harmonic, then $([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) \leq 0$.

Proof. Direct computation shows

$$\begin{aligned}
(\Delta_{\overline{\partial}_E}\alpha, \alpha) - ([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) &= (\Delta_{\partial_E}\alpha, \alpha) \\
&= \|\partial_E\alpha\|^2 + \|\partial_E^*\alpha\|^2 \geq 0
\end{aligned}$$

□

Corollary 14.1.2. Let X be a complex manifold. If Hermitian holomorphic vector bundle(E,h) is Griffiths positive, then $(\det E, \det h)$ is a positive holomorphic line bundle. In particular, X is Kähler.

Theorem 14.1.2 (Kodaira-Akizuki-Nakano vanishing)**.** Let X be a compact n-manifold, (L,h) a positive Hermitian holomorphic line bundle. Then

$$H^{p,q}(X,L) = 0$$

for $p+q > n$.

Proof. Let X be endowed with Kähler metric ω given by Chern curvature of L, then there is an isomorphism $H^{p,q}(X,L) \cong \mathcal{H}^{p,q}(X,L)$. For $\alpha \in \mathcal{H}^{p,q}(X,L)$, Corollary 14.1.1 implies

$$[\sqrt{-1}\Theta_h, \Lambda]\alpha \leq 0$$

On the other hand,

$$([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) = 2\pi(p+q-n)\|\alpha\|^2 \geq 0$$

Thus if $p+q>n$, one has $\alpha=0$, this completes the proof. □

Corollary 14.1.3 (Kodaira vanishing)**.** Let X be a compact n-manifold and (L,h) be a positive holomorphic line bundle over X. Then

$$H^q(X,K_X\otimes L)=0$$

for $q>0$.

Proof. Just note that

$$H^q(X,K_X\otimes L)=H^{n,q}(X,L)$$

□

Corollary 14.1.4. Let (X,ω) be a compact Kähler n-manifold. If (L,h) is a semi-positive line bundle and $\operatorname{rk}\Theta_h\geq k$, then

$$H^{p,q}(X,L)=0$$

for $p+q\geq 2n-k+1$.

Exercise 14.1.1. Compute all $H^q(\mathbb{CP}^n,\mathcal{O}_{\mathbb{CP}^n}(k))$ for all k,q.

Definition 14.1.1 (Fano)**.** A Fano manifold is a compact Kähler manifold with positive anti-canonical bundle $K_X^*=\det TX$.

Proposition 14.1.1. Let X be a Fano manifold, then

$$H^q(X,\mathcal{O}_X)=0$$

for all $q>0$.

Proof. Note that $\mathcal{O}_X=K_X\otimes K_X^*$. □

Theorem 14.1.3 (Serre vanishing)**.** Let X be a compact complex n-manifold and (L,h) be a positive holomorphic line bundle over X. For any holomorphic vector bundle E on X, there exists a constant m_0 such that for all $m\geq m_0$

$$H^q(X,E\otimes L^{\otimes m})=0$$

for $q>0$.

Proof. If X is endowed with Kähler metric ω given by Chern curvature of L and E is endowed with a Hermitian metric h, then $H^{p,q}(X,E\otimes L^{\otimes m})\cong\mathcal{H}^{p,q}(X,E\otimes L^{\otimes m})$. For $\alpha\in\mathcal{H}^{p,q}(X,E\otimes L^{\otimes m})$, one has

$$\begin{aligned}([\sqrt{-1}\Theta_h,\Lambda]\alpha,\alpha)+2\pi m(p+q-n)\|\alpha\|^2&\overset{(1)}{=}([\sqrt{-1}\Theta_{E\otimes L^{\otimes m}},\Lambda]\alpha,\alpha)\\&\overset{(2)}{\leq}0\end{aligned}$$

where

(1) holds from $\Theta_{E\otimes L^{\otimes m}}=\Theta^E\otimes\operatorname{id}+m(\operatorname{id}\otimes\Theta^L)$.
(2) holds from Corollary 14.1.1, that is Bochner-Kodaira-Nakano inequality.

On the other hand, Cauchy inequality implies that

$$([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) \geq -C\|\alpha\|^2$$

where constant C is the norm of $[\sqrt{-1}\Theta_h, \Lambda]$. So if we have $2\pi m(p+q-n) - C > 0$, the argument in proof of Kodaira vanishing theorem implies $\alpha = 0$. Consider $p = n, q > 0, m_0 \geq \frac{C}{2\pi}$, then for all $m \geq m_0$ and $q > 0$, one has

$$H^{n,q}(X, E \otimes L^{\otimes m}) = 0$$

that is to say $H^q(X, K_X \otimes E \otimes L^{\otimes m}) = 0$. So in order to show $H^q(X, E \otimes L^{\otimes m}) = 0$, it suffices to consider $K_X^* \otimes E$ at beginning, and then we will obtain

$$H^q(X, K_X \otimes K_X^* \otimes E \otimes L^{\otimes m}) = H^q(X, E \otimes L^{\otimes m}) = 0$$

This completes the proof. □

14.2. **Nakano vanishing.**

Theorem 14.2.1 (Nakano vanishing)**.** Let X be a compact complex manifold and (E, h) a Hermitian holomorphic vector bundle over X.

(1) If (E, h) is Nakano positive, then
$$H^{n,q}(X, E) = 0$$
for $q \geq 1$.

(2) If (E, h) is dual Nakano positive, then
$$H^{p,n}(X, E) = 0$$
for $p \geq 1$.

Proof. Here we only give the proof of the first one, the second can be derived from the same argument. Suppose $\{\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}\}$ is a holomorphic local frame which is orthonormal at point $p \in X$, and $\varphi \in H^{n,q}(X, E)$ which is locally written as $\varphi = \varphi^\alpha_{\overline{I}} \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n \wedge \mathrm{d}\overline{z}^I \otimes e_\alpha$ where $|I| = q$. Direct computation shows

$$\begin{aligned}
\langle \sqrt{-1}[\Theta_h, \Lambda]\varphi, \varphi\rangle &= \langle \sqrt{-1}\Theta_h \Lambda\varphi, \varphi\rangle \\
&\overset{(1)}{=} -\langle \Theta^\gamma_{k\bar{l}\alpha} \mathrm{d}z^k \wedge \mathrm{d}\overline{z}^l \sum_m \iota_m \iota_{\overline{m}} \varphi^\alpha_{\overline{I}} \mathrm{d}z^1 \wedge \mathrm{d}z^n \wedge \mathrm{d}\overline{z}^I \otimes e_\gamma, \varphi^\beta_{\overline{J}} \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n \wedge \mathrm{d}\overline{z}^J \otimes e_\beta\rangle \\
&= -\Theta_{k\bar{l}\alpha\overline{\beta}} \varphi^\alpha_{\overline{I}} \overline{\varphi^\beta_{\overline{J}}} \langle \mathrm{d}z^k \wedge \mathrm{d}\overline{z}^l \wedge \sum_m \iota_m \iota_{\overline{m}} \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n \wedge \mathrm{d}\overline{z}^I, \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n \wedge \mathrm{d}\overline{z}^J\rangle \\
&= \sum_k \Theta_{k\bar{l}\alpha\overline{\beta}} \varphi^\alpha_{\overline{I}} \overline{\varphi^\beta_{\overline{J}}} \langle \mathrm{d}\overline{z}^l \wedge \iota_{\overline{k}} \mathrm{d}\overline{z}^I, \mathrm{d}\overline{z}^J\rangle \\
&= \sum_{I,J} \left\{ \Theta_{k\bar{l}\alpha\overline{\beta}} (\sum_{k\in I} \varphi^\alpha_{\overline{I}}) \overline{(\sum_{l \in J} \varphi^\beta_{\overline{J}})} \right\} \\
&\overset{(2)}{\geq} 0
\end{aligned}$$

where

(1) holds from Proposition 9.3.2 since by Corollary 14.1.2 one has X is Kähler.
(2) holds from (E, h) is Nakano positive.

But by Corollary 14.1.1, that is Bochner-Kodaira-Nakano inequality, one has $\langle\sqrt{-1}[\Theta_h, \Lambda]\varphi, \varphi\rangle \leq 0$. This shows $\varphi = 0$ as before. $\square$

Corollary 14.2.1. Let X be a compact complex n-manifold with $n \geq 2$. Then there doesn't exist metric h such that (TX, h) is Nakano positive.

Proof. Suppose (TX, h) is Nakano positive, then

$$\begin{aligned} H^{1,1}(X) &= H^{0,1}(X, T^*X) \\ &\overset{(1)}{=} H^{n,n-1}(X, TX) \\ &\overset{(2)}{=} 0 \end{aligned}$$

where

(1) holds from Serre duality.
(2) holds from Nakano vanishing theorem.

However, by Proposition 13.2.1 and Corollary 14.1.2 one has X is a Käher manifold, which leads to a contradiction since $H^{1,1}(X) \neq 0$ for a Kähler manifold. $\square$

Corollary 14.2.2. Let X be a compact complex n-manifold with $n \geq 2$. Then there doesn't exist metric h such that (T^*X, h) is dual Nakano positive.

Proof. Note that

$$H^{n-1,n}(X, T^*X) = H^{1,0}(X, TX) = H^0(X, \mathrm{End}(TX)) \neq 0$$

$\square$

Example 14.2.1. If $n \geq 2$, $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$ is dual Nakano positive and Nakano semi-positive, but not Nakano positive.

Proof. By Theorem 8.4.1 one has the curvature of Fubini-Study metric is

$$\Theta_{i\bar{j}k\bar{l}} = h_{i\bar{j}}h_{k\bar{l}} + h_{i\bar{l}}h_{k\bar{j}}$$

For $p \in X$, if we consider normal coordinate at p, then for any non-zero matrix (u^{ij}), a direct computation shows

$$\begin{aligned} \Theta_{i\bar{j}k\bar{l}}u^{il}\overline{u}^{jk} &= h_{i\bar{j}}h_{k\bar{l}}u^{il}\overline{u}^{jk} + h_{i\bar{l}}h_{k\bar{j}}u^{il}\overline{u}^{jk} \\ &= \sum_{i,j}|u^{ij}|^2 + u^{ii}\overline{u}^{jj} \\ &= \sum_{i\neq j}|u^{ij}|^2 + \frac{1}{2}\sum_{i,j}|u^{ii} + u^{jj}|^2 \\ &> 0 \end{aligned}$$

which implies $(\mathbb{CP}^n, \omega_{FS})$ is dual Nakano positive.

$$\begin{aligned}
\Theta_{i\bar{j}k\bar{l}} u^{ik}\overline{u}^{jl} &= h_{i\bar{j}}h_{k\bar{l}}u^{ik}\overline{u}^{jl} + h_{i\bar{l}}h_{k\bar{j}}u^{ik}\overline{u}^{jl} \\
&= \sum_{i,j} |u^{ij}|^2 + u^{ij}\overline{u}^{ji} \\
&= \frac{1}{2}\sum_{i,j} |u^{ij} + u^{ji}|^2 \\
&\geq 0
\end{aligned}$$

which implies $(\mathbb{CP}^n, \omega_{FS})$ is Nakano semi-positive, and it's not Nakano positive by Corollary 14.2.1. □

14.3. **Griffiths vanishing.**

Theorem 14.3.1 (Griffiths vanishing)**.** If (E, h) is Griffiths positive, then

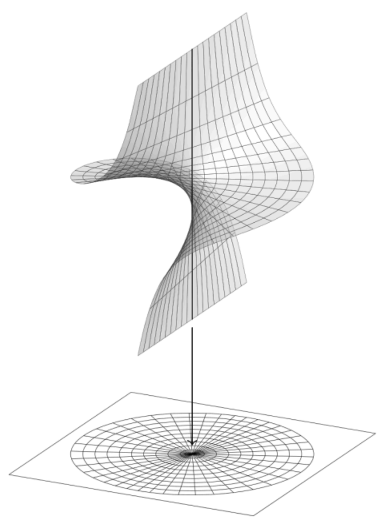
(1) $H^{n,n}(X, E) = 0$.
(2) $H^{p,q}(X, E) = 0$ for $p + q \geq n + \operatorname{rk} E$.

15. Kodaira embedding and Chow theorem

15.1. **Blow-up.** In this section we will introduce a technical tool we need in the proof of Kodaira embedding, which is blow-up at a point.

Definition 15.1.1 (blow-up)**.** $U \times \mathbb{CP}^{n-1} \supset \widetilde{U} := \{((x_1, \dots, x_n), (y_1 : \cdots : y_n)) \mid x_i y_j = x_j y_i\}$

Remark 15.1.1. The most vivid way to understand blow-up is to consider the fibres of projection $\widetilde{U} \to U$: If $x \neq 0$, then the fibre of x is just a point since the ratio of y_i is uniquely determined. But for $x = 0$, there is no restriction for y_i, so you get the whole projective space $\mathbb{CP}^{n-1}$. Thus as you can imagine, there is nothing happening except the origin, sounds like a boom. For example, the following figure shows the case of $n = 2$



Lemma 15.1.1. *$\widetilde{U} \subseteq U \times \mathbb{CP}^{n-1}$ is a submanifold of dimension n.*

Since blow-up is a local operation, so it can be done on a complex manifold. If X is a complex manifold with dimension n with $x \in X$, and $\{U_i\}$ is an open covering such that $x \in U_1$ and $x \notin U_i, i \neq 1$, then we can show that $\widetilde{U_1} \cup (\bigcup_{i \neq 1} U_i)$ glue together a new complex manifold with dimension n. This is called blow-up of X at point x, and it's denoted by $\widetilde{X}$. Similarly there is a natural projection $\pi \colon \widetilde{X} \to X$ and $\pi^{-1}(x)$ is biholomorphic to $\mathbb{CP}^{n-1}$, which is called exceptional divisor and denote it by E.

Exercise 15.1.1. If X is compact Kähler manifold, then $\widetilde{X}$ is also a compact Kähler manifold.

Exercise 15.1.2. The idea sheaf of exceptional divisor $\mathcal{I}_E \cong \mathcal{O}_{\widetilde{X}}(E)^*$

Proposition 15.1.1. *The canonical bundle $K_{\widetilde{X}}$ of the blow-up $\widetilde{X}$ is isomorphic to $\pi^* K_X \otimes \mathcal{O}_{\widetilde{X}}(E)^{\otimes n-1}$.*

Corollary 15.1.1. For the exceptional divisor $E = \mathbb{CP}^n \subseteq \widetilde{X} \to X$ one has $\mathcal{O}(E)|_E \cong \mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$.

Proof. The adjunction formula implies

$$K_E \cong (K_{\widetilde{X}} \otimes \mathcal{O}_{\widetilde{X}}(E))|_E$$

Then by Proposition 15.1.1 and $K_E = \mathcal{O}_{\mathbb{CP}^{n-1}}(-n)$ one has

$$\mathcal{O}_{\mathbb{CP}^{n-1}}(-n) \cong (\pi^* K_X \otimes \mathcal{O}_{\widetilde{X}}(E)^{\otimes n})|_E$$

Note that E is a fibre of π, so $\pi^* K_X|_E$ is trivial. Thus

$$\mathcal{O}_{\mathbb{CP}^{n-1}}(-n) \cong \mathcal{O}_{\widetilde{X}}(E)^{\otimes n}|_E$$

On the other hand, the only possible line bundles on $\mathbb{CP}^{n-1}$ take the form $\mathcal{O}_{\mathbb{CP}^{n-1}}(k), k \in \mathbb{Z}$. Thus $\mathcal{O}_{\widetilde{X}}(E)|_E \cong \mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$. □

The main reason we need blow-up here is that the sections after blow-up is the “same” as the one before.

Proposition 15.1.2. For a holomorphic line bundle L, one has

$$H^0(X, L^{\otimes m}) = H^0(\widetilde{X}, \pi^* L^{\otimes m})$$

holds for arbitrary $m \in \mathbb{Z}_{\geq 0}$.

Proof. If X is one-dimensional, then $\pi\colon \widetilde{X} \to X$ is an isomorphism and thus $H^0(X, L^{\otimes m}) = H^0(\widetilde{X}, \pi^* L^{\otimes m})$.

Now let's consider the case $\dim_{\mathbb{C}} X \geq 2$. Given an element $s \in H^0(X, L^{\otimes m})$, we can get an element in $H^0(\widetilde{X}, \pi^* L^{\otimes m})$ by composing projection π. Conversely, for $\widetilde{s} \in H^0(\widetilde{X}, \pi^* L^{\otimes m})$, it can be restricted to $\widetilde{X} \backslash E = X \backslash \{x\}$, and then extended to a global section of $L^{\otimes m}$ by Hartogs theorem. □

15.2. Kodaira embedding.

Theorem 15.2.1 (Kodaira embedding)**.** Let X be a compact complex manifold. The following statements are equivalent:

(1) There exists a holomorphic embedding $\varphi\colon X \hookrightarrow \mathbb{CP}^N$.
(2) There exists a integral Kähler form ω on X, that is, $[\omega] \in \operatorname{im}\{H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})\}$
(3) There exists a positive holomorphic line bundle on X.

Remark 15.2.1. (1) clealy implies (2), and (2) implies (3) is Lefschetz $(1,1)$-theorem, so the heart of the proof is (3) to (1).

Sketch. Use holomorphic global sections $H^0(X, L^{\otimes m})$ for sufficiently large m to construct $\varphi\colon X \hookrightarrow \mathbb{CP}^N$. We need to show the following three things:

(1) For sufficiently large m, $L^{\otimes m}$ is globally generated, which means for all $x \in X$, there exists a global section $s \in H^0(X, L^{\otimes m})$ such that $s(x) \neq 0$. Then for all $x \in X$, $H_x = \{s \in H^0(X, L^{\otimes m}) \mid s(x) = 0\}$ is a hypersurface. Thus we get a holomorphic map $\varphi\colon X \to \mathbb{CP}(H^0(X, L^{\otimes m})^*)$, defined by $x \mapsto H_x$. Indeed since any hypersurface in $H^0(X, L^{\otimes m})$ is a

line in $H^0(X, L^{\otimes m})^*$, that is an element in $\mathbb{CP}(H^0(X, L^{\otimes m})^*)$. And you will find φ is holomorphic since we're using holomorphic sections.

(2) For more sufficiently large[13] m, $L^{\otimes m}$ separetes points, that is for all $x, y \in X$, there exists $s \in H^0(X, L^{\otimes m})$ such that $s(x) \neq 0, s(y) = 0$. Thus in this case our φ is injective.

(3) For more more sufficiently large m, $L^{\otimes m}$ separetes tangent vectors, that is, for all $x \in X, u \in T_{X,x}$ there exists $s \in H^0(X, L^{\otimes m})$ such that $s(x) = 0$ and $\mathrm{d}s(u) \neq 0$. Thus in this case our φ is an immersion, together with X is compact we have φ is an embedding.

Remark 15.2.2. We can also describe φ more explicitly. Locally around x_0, choose a basis $s_0, \ldots, s_N$ of $H^0(X, L^{\otimes m})$ such that $s_0(x_0) \neq 0$. Then there exists a neighborhood U of x_0 such that $s_0(x) \neq$ for all $x \in U$. Then

$$\frac{s_1}{s_0}, \ldots, \frac{s_N}{s_0} \in H^0(U, \mathcal{O}_U)$$

So we can define

$$\begin{aligned} \varphi|_U \colon U &\to \mathbb{CP}^N \\ x &\mapsto (1, \frac{x_1}{x_0}(x), \ldots, \frac{s_N}{s_0}(x)) \end{aligned}$$

And you can check it's same as what we have defined without choosing a basis.

Here we only give a sketch proof of the first statement, the proofs for second and third are similar, but more complicated.

We want to detect the value of a section at a point is zero or not. Sheaves give us a good way to describe such thing. For $x \in X$, consider ideal sheaf of x

$$\mathcal{I}_x = \{s \in \mathcal{O}_X \mid s(x) = 0\} \subseteq \mathcal{O}_X$$

Then sections we are searching for is $H^0(X, \mathcal{I}_x \otimes L^{\otimes m})$. For computation, we have exact sequence of sheaf

$$0 \to \mathcal{I}_x \otimes L^{\otimes m} \to L^{\otimes m} \to L^{\otimes m}|_x \to 0$$

And using Čech cohomology we can derive a long exact sequence

$$0 \to H^0(X, \mathcal{I}_x \otimes L^{\otimes m}) \to H^0(X, L^{\otimes}) \to \mathbb{C} \to \check{H}^1(X, \mathcal{I}_x \otimes L^{\otimes m}) \to \ldots$$

Our goal is to show $H^0(X, \mathcal{I}_x \otimes L^{\otimes m}) \neq 0$. If $\check{H}^1(X, \mathcal{I}_x \otimes L^{\otimes m}) = 0$ for sufficiently large m, then we can get desired result.

For Čech cohomology we know a little, but we know quite a lot for Dolbeault cohomology. So an ideal is to turn idea sheaf into a line bundle and use Dolbeault cohomology to compute.

Similarly, we have

$$H^0(X, \mathcal{I}_x \otimes L^{\otimes m}) = H^0(\widetilde{X}, \mathcal{I}_E \otimes \pi^* L^{\otimes m}) = H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(E)^* \otimes \pi^* L^{\otimes m})$$

And that's why blow-up works since $\mathcal{I}_x \otimes L^{\otimes m}$ is just a sheaf, and it's a little difficult for us to compute the cohomology of sheaf, but after blow-up,

[13]Larger than m is step one.

we make it to a line bundle $\mathcal{O}_{\widetilde{X}}(E)^* \otimes \pi^* L^{\otimes m}$, and Dolbeault cohomology comes into its place.

Consider the following short exact sequence of sheaves on $\widetilde{X}$:

$$0 \to \mathcal{I}_E \otimes \pi^* L^{\otimes m} \to \pi^* L^{\otimes m} \to \pi^* L^{\otimes m}|_E \to 0$$

So we get a long exact sequence

$$0 \to H^0(\widetilde{X}, \mathcal{I}_E \otimes \pi^* L^{\otimes m}) \to H^0(\widetilde{X}, \pi^* L^{\otimes m}) \to \mathbb{C} \to \check{H}^1(\widetilde{X}, \mathcal{I}_E \otimes \pi^* L^{\otimes m}) \to \dots$$

But

$$\check{H}^1(\widetilde{X}, \mathcal{I}_E \otimes \pi^* L^{\otimes m}) = H^1_{Dol}(\widetilde{H}, \mathcal{I}_E \otimes \pi^* L^{\otimes m})$$

Claim $H^1_{Dol}(\widetilde{H}, \mathcal{I}_E \otimes \pi^* L^{\otimes m}) = 0$, when m is sufficiently large. Indeed, note that

$$\mathcal{I}_E \otimes \pi^* L^{\otimes m} \cong K_{\widetilde{X}} \otimes \{\mathcal{O}_{\widetilde{X}}(E)^{*\otimes n} \otimes \pi^*(K_X^* \otimes L^{\otimes m})\}$$

So by Kodaira vanishing, it suffices to show the following line bundle is positive when m is sufficiently large:

$$\mathcal{O}_{\widetilde{X}}(E)^{*\otimes n} \otimes \pi^*(K_X^* \otimes L^{\otimes m})$$

In fact, $K_X^* \otimes L^{\otimes m}$ will be positive on X when m is sufficiently large. But when we pull it back something bad may happen since $\pi^*(K_X^* \otimes L^{\otimes m})$ is positive except along E. However, $\mathcal{O}_{\widetilde{X}}(E)^{*\otimes n}|_E = \mathcal{O}_{\mathbb{CP}^{n-1}}(n)$, so two parts work together to give a positive line bundle. To be more explicit, take any Hermitian metric on $\mathcal{O}_{\widetilde{X}}(E)^{*\otimes n}$ extending (Fubini-Study)$^{\otimes n}$, then its positive on E, but may not be positive otherwise. However we can choose m sufficiently large to offset its negative impact.

This completes the proof of first part of Kodaira embedding, for second and third, arguments are similar, but we need more blow-ups and things become complicated. □

Corollary 15.2.1. Let (X, ω) be a compact Kähler manifold such that $H^{2,0}(X) = H^{0,2}(X) = 0$. Then X is a projective manifold.

Proof. Hodge decomposition implies that $H^{2,0}(X) = H^{0,2}(X) = 0$, and thus $H^2(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = H^2(X, \mathbb{C}) = H^{1,1}(X)$. Let $[\alpha_1], \dots, [\alpha_n] \in H^2(X, \mathbb{Q})$ be a basis such that α_i is harmonic and of type $(1,1)$. Since the Kähler form ω is real, harmonic[14] and of type $(1,1)$. Then

$$\omega = \sum_i \lambda_i \alpha_i, \quad \lambda_i \in \mathbb{R}$$

For $\mu_i \in \mathbb{Q}$ sufficiently close to λ_i, one still has $\sum_i \mu_i \alpha_i$ is positive. Thus $\sum_i \mu_i \alpha_i$ gives a Kähler form. Take N sufficiently large such that $[N \sum_i \mu_i \alpha_i] \in \operatorname{im}(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R}))$. Applying Kodaira embedding to complete the proof. □

Corollary 15.2.2. Fano manifold is projective.

Proof. Since for Fano manifold, all $H^{0,p}(X) = 0, q > 0$. □

[14] It's harmonic since $[\Delta_{\mathrm{d}}, L] = 0$.

15.3. **Chow's theorem.** Since we already embed a compact complex manifold into projective space, there is no reason for us to avoid Chow's theorem. A wonderful theorem lies in the intersection of algebraic and analytic.

Theorem 15.3.1 (Chow)**.** Every closed complex submanifold $X \subseteq \mathbb{CP}^n$ is algebraic, that is, defined by polynomial equations.

Remark 15.3.1. Finally, for holomorphic line on compact complex manifolds: positive is equivalent to ample.

Although Chow's theorem can be derived from GAGA proved by Serre, in an elegant way using sheaf theory, here we give a sketch of a classical proof of Chow's theorem.

We need to show every closed complex submanifold X of $\mathbb{CP}^n$ is algebraic, our ideal is to construct an analytic hypersurface in a Grassmannian manifold $Gr(r,n)$ with Plücker embedding $\varphi\colon Gr(r,n) \hookrightarrow \mathbb{CP}(\bigwedge^r \mathbb{C}^n)$, and use some facts about it:

(1) $\mathrm{Pic}(Gr(r,n)) = \mathbb{Z} \cdot \varphi^* \mathcal{O}_{\mathbb{CP}}(1)$
(2) Every closed analytic hypersurface of $Gr(r,n)$ is algebraic.

If the analytic hypersurface W we construct in Grassmannian manifold can determine X algebraically, that is W is algebraic implies X is, then we complete the proof.

The philosophy here is to convert a submanifold with arbitrary codimension in $\mathbb{CP}^n$ to a an hypersurface, the cost we pay is that we need to consider Grassmannian manifold rather than $\mathbb{CP}^n$. But it do works!

Let's be more explicit:

Definition 15.3.1 (analytic subset)**.** A closed analytic subset in $\mathbb{CP}^n$ is a closed subset, locally defined by some holomorphic equations.

Remark 15.3.2. We can replace closed complex submanifold by closed analytic subset in Chow's theorem since we can not avoid singularity, and it doesn't matter in fact.

However, although we allow singularities, singularities won't be too much: Let $X \subseteq \mathbb{CP}^n$ be an irreducible closed analytic subset of dimension r. Then there exists closed analytic subset $X_{\mathrm{sing}} \subseteq X$ such that $X \backslash X_{\mathrm{sing}}$ is smooth and dense. Furthermore, $X \backslash X_{\mathrm{sing}}$ is a submanifold of $\mathbb{CP}^n$ of dimension r.

Now fix X, an irreducible closed analytic subset of dimension r in $\mathbb{CP}^n$. Let $V \in Gr(n-r, n+1)$, then $\mathbb{CP}(V) \subseteq \mathbb{CP}^n$ with dimension $n-r-1$. So as you can imagine, an object with dimension r and an object with dimension $n-r-1$ may fail to intersect with each other.

Let
$$W = \{V \in Gr(n-r, n+1) \mid \mathbb{CP}(V) \cap X \neq \varnothing\}$$
Claim:

(1) W is a closed analytic hypersurface of $Gr(n-r, n+1)$.
(2) W determines X algebraically.

Here we give a sketch of proof of Claims:

For (1). Consider the following diagram

$$\begin{tikzcd} \mathbb{CP}(E) \arrow[r, "q"] \arrow[d, "p"] & \mathbb{CP}^n \supset X \\ Gr(n-r, n+1) & \end{tikzcd}$$

where $E \to Gr(n-r, n+1)$ is tautological bundle of Grassmannian manifold. Then we can write[15]

$$W = p(q^{-1}(X))$$

Since q is holomorphic and X is closed and analytic, then $q^{-1}(X)$ is also closed and analytic. But the difficulty is $p(q^{-1}(X))$ is also closed and analytic, and this holds from the following fact.

Theorem 15.3.2. p is holomorphic and proper[16].

Now we show that W is a hypersurface, and that's just a computation for dimension: We already know the dimension of $Gr(n-r, n+1)$ is $(n-r)(n+1-(n-r)) = (n-r)(r+1)$, so we need to show the dimension of W is $(n-r)(r+1)-1$.

First, let's consider the fibre of q: it consists of subspaces $V \subseteq \mathbb{C}^{n+1}$ of dimension $n-r$ containing a given line l, and that's another Grassmannian manifold $Gr(n-r-1, n)$, if we consider $V \mapsto V/l$, and its dimension is $(n-r-1)(r+1)$. So the dimension of $q^{-1}(X)$ is $r + (n-r-1)(r+1) = (n-r)(r+1) - 1 = \dim Gr(n-r, n+1) - 1$.

So we may desire the property of p is not too bad so that we will obtain $\dim p(q^{-1}(X)) = \dim q^{-1}(X)$ as we desired. It suffices to show that there exists a dense open subset $U \subseteq q^{-1}(X)$, such that $p|_U$ has finite fibres. In fact, we will show it's one to one correspondence.

Consider fibre of $p_X : q^{-1}(X) \to Gr(n-r, n+1)$ over given $V \in Gr(n-r, n+1)$, and that's $\mathbb{CP}(V) \cap X$. So we may desire almost every V such that this intersection is just a point. It suffices to show that the complement of

$$\{(V, x) \in q^{-1}(X) \mid \mathbb{CP}(V) \cap X \text{ has only one smooth point } x\}$$

is closed analytic of dimension less than $\dim q^{-1}(X)$. There are three cases:

(1) $\mathbb{CP}(V) \cap X$ contains $x \in X_{\mathrm{sing}}$, singular locus of X. But $\dim_{\mathbb{C}} q^{-1}(X_{\mathrm{sing}}) < \dim_{\mathbb{C}} q^{-1}(X)$.
(2) $\mathbb{CP}(V) \cap X$ has at least two points.
(3) $\mathbb{CP}(V) \cap X$ not transverse intersection at x.

Then $W = p(q^{-1}(X)) \subseteq Gr(n-r, n+1)$ is a closed analytic hypersurface.

For (2). Consider $q_W : p^{-1}(W) \to \mathbb{CP}^n$. Claim

$$X = \{x \in \mathbb{CP}^n \mid q_W^{-1}(x) = q^{-1}(x)\}$$

[15] Why?

[16] Proper means: For any $Y \subseteq \mathbb{CP}(E)$ closed and analytic, then $p(Y)$ is closed and analytic.

And there are some equivalent descriptions:

$$\begin{aligned} q_W^{-1}(x) = q^{-1}(x) &\Longleftrightarrow p^{-1}(W) \cap q^{-1}(x) = q^{-1}(x) \\ &\Longleftrightarrow q^{-1}(x) \subseteq p^{-1}(W) \end{aligned}$$

Clearly, if $x \in X$, then $q^{-1}(x) \in p^{-1}(W)$ since $W = p(q^{-1}(X))$. For the other direction: we can translate it as: If $y \notin X$, then we need to find $V \in Gr(n-r, n+1)$ containing $l = \langle y \rangle$, but $\mathbb{CP}(V) \cap X = \varnothing$.

To see this: Use projection from y, that is $\mathbb{CP}^n \xrightarrow{\pi_y} \mathbb{CP}^{n-1}$. Since $y \notin X$, then $\pi_y|_X$ is well-defined and has finite fibres. Note that $\mathbb{CP}(V) \cap X \neq \varnothing$ if and only if $\mathbb{CP}(V/\langle y \rangle) \cap \pi_y(X) = \varnothing$. From computation before, we know it's a condition for hypersurface. So it's easy to choose V we desire.

Conclusion (from analytic to algebraic): p and q is algebraic, and W is algebraic, so we obtain $p^{-1}(W)$ is also algebraic. So q_W is also algebraic. Thus X is algebraic.

16. More Bochner techniques

16.1. **Obstruction to holomorphic vector fields.**

In Riemannian geometry, there is the following theorem which shows there are obstructions of Killing vector fields or harmonic 1-forms.

Theorem 16.1.1. Let (M, g) be a compact Riemannian manifold.

(1) If $\mathrm{Ric}(g) < 0$, then there is no non-trivial Killing vector field.
(2) If $\mathrm{Ric}(g) > 0$, then $b_1(M) = 0$.

In the realm of complex geometry, analogous obstructions exist, drawing a parallel between Killing fields and holomorphic vector fields. However, it is important to note that not every Killing vector field is a holomorphic vector field.

Theorem 16.1.2. On a compact Kähler manifold, a Killing vector field X is holomorphic if and only if

(1) $\mathrm{div} X = 0$.
(2) $\nabla^* X = 0$.

Lemma 16.1.1. Let (M, g) be a compact Riemannian manifold. A vector field X is Killing if and only if $\mathrm{div} X = 0$ and $\nabla^*\nabla X = (\mathrm{Ric}(X))^\sharp$.

Lemma 16.1.2. Let (M, ω) be a compact Kähler manifold. A vector field X is holomorphic if and only if $\nabla^*\nabla X = (\mathrm{Ric}(X))^\sharp$.

Theorem 16.1.3. Let (M, ω) be a compact Kähler manifold with $\mathrm{Ric}(\omega) < 0$. Then there is no non-trivial holomorphic vector field.

Proof. Let X be a non-trivial holomorphic vector field locally given by $X = X^i \frac{\partial}{\partial z^i}$ and $\omega = \sqrt{-1} h_{i\bar{j}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j$. If we define $u = |X|^2$, a direct computation shows

$$\begin{aligned}
\sqrt{-1}\partial\bar{\partial} u &= \sqrt{-1}\left(\partial\{\nabla^{0,1} X, X\} + \{X, \nabla^{1,0} X\}\right) \\
&\overset{(1)}{=} \sqrt{-1}\partial\{X, \nabla^{1,0} X\} \\
&= \sqrt{-1}\left(\{\nabla^{1,0} X, \nabla^{1,0} X\} + \{X, \nabla^{0,1}\nabla^{1,0} X\right) \\
&\overset{(2)}{=} \sqrt{-1}\left(\{\nabla X, \nabla X\} + \{X, \Theta_h X\}\right)
\end{aligned}$$

where (1) and (2) holds from X is a holomorphic vector field. Note that

$$\begin{aligned}
\mathrm{tr}_\omega \sqrt{-1}\{\nabla X, \nabla X\} &= \mathrm{tr}_\omega(\sqrt{-1}\nabla_i X^j \overline{\nabla_k X^l} h_{j\bar{l}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^k) \\
&= \nabla_i X^j \overline{\nabla_k X^l} h_{j\bar{l}} h^{i\bar{k}} \\
&= |\nabla X|^2 \\
\mathrm{tr}_\omega \sqrt{-1}\{X, \Theta_h X\} &= -\mathrm{tr}_\omega\{\sqrt{-1}\Theta_h X, X\} \\
&= -\mathrm{tr}_\omega(\sqrt{-1}\Theta_{i\bar{j}\alpha}^{\beta} X^\alpha \overline{X}^\gamma h_{\beta\bar{\gamma}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j) \\
&= -h^{i\bar{j}} \Theta_{i\bar{j}\alpha\bar{\gamma}} X^\alpha \overline{X}^\gamma \\
&= -\mathrm{Ric}(X, X)
\end{aligned}$$

This shows

$$\mathrm{tr}_\omega \sqrt{-1}\partial\overline{\partial}u = |\nabla X|^2 - \mathrm{Ric}(X, X) > 0$$

If u obtains its maximum at some point $p \in M$, then $\mathrm{tr}_\omega \sqrt{-1}\partial\overline{\partial}u(p) \leq 0$, a contradiction. □

Corollary 16.1.1. Let (M, ω) be a compact Kähler manifold. If $\mathrm{Ric}(\omega) \leq 0$, then the following statements are equivalent:

(1) X is parallel.
(2) X is Killing.
(3) X is holomorphic.

Proof. As shown in Theorem 16.1.3, if $\mathrm{Ric}(\omega) \leq 0$, for a non-trivial holomorphic vector field X one has

$$\mathrm{tr}_\omega \sqrt{-1}\partial\overline{\partial}u = |\nabla X|^2 - \mathrm{Ric}(X, X) \geq 0$$

Then by the same argument one has $\nabla X = 0$, that is every holomorphic vector field is parallel, and it's clear every parallel vector field is Killing. □

Corollary 16.1.2. Let (M, ω) be a compact complex manifold with negative holomorphic sectional curvature. Then there is no non-trivial holomorphic vector field.

Proof. Let X be a non-trivial holomorphic vector field locally given by $X = X^i\frac{\partial}{\partial z^i}$ and $u = |X|^2$. The same argument shows

$$\sqrt{-1}\partial\overline{\partial}u(X, X) = |\nabla_X X|^2 - \Theta_{i\bar{j}k\bar{l}}X^i\overline{X}^j X^k\overline{X}^l$$

Then maximum principle completes the proof. □

16.2. **Obstruction to holomorphic 1-form.**

Theorem 16.2.1. Let (M, ω) be a compact Kähler manifold. If $\mathrm{Ric}(\omega) > 0$. Then there is no non-trivial holomorphic 1-form.

Corollary 16.2.1. Let (M, ω) be compact Kähler manifold and $\mathrm{Ric}(\omega) \geq 0$. Then any holomorphic 1-form is parallel.

16.3. **Rigidity of complex projective space.** The rigidity of complex projective space is a fascinating subject in complex geometry. It involves determining whether a geometric object satisfies certain conditions to be $\mathbb{CP}^n$ or not, and it has produced numerous interesting results. Yau demonstrated that any Kähler manifold, which is homeomorphic to $\mathbb{CP}^n$, is also biholomorphic to $\mathbb{CP}^n$. He also resolved a conjecture put forth by Frankel in 1961, which stated that a compact Kähler manifold with positive bisectional curvature must be biholomorphic to $\mathbb{CP}^n$. Additionally, there exists an even stronger result.

Theorem 16.3.1 ([FLW17])**.** Let (X, ω) be a compact Kähler manifold with positive orthogonal bisectional curvature. Then X is biholomorphic to $\mathbb{CP}^n$.

In this section we try to use Bochner technique to show the following easier result.

Proposition 16.3.1. Let (X, ω) be a compact Kähler manifold with positive orthogonal bisectional curvature. Then $H^{1,1}(X) = \mathbb{C}$.

17. Schwarz lemmas

In this section, we will present the powerful formulas called Schwarz calculation, which extend the Bochner techniques discussed earlier.

17.1. Schwarz lemmas for holomorphic bisectional curvature.

Lemma 17.1.1. Let $f\colon (M,g) \to (N,h)$ be a holomorphic map between Hermitian manifolds and ∇ be the induced connection on $T^*X \otimes f^*(TN)$ by Chern connections. Then in the local holomorphic coordinates $\{z^i\}$ and $\{w^\alpha\}$ on M and N respectively, one has

$$\partial\overline{\partial}u = \{\nabla \mathrm{d}f, \nabla \mathrm{d}f\} + \Big((\Theta_g)_{i\bar{j}k\bar{l}} g^{k\bar{q}} g^{p\bar{l}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} - (\Theta_h)_{\alpha\bar{\beta}\gamma\bar{\delta}} (f_i^\alpha \overline{f_j^\beta})(g^{p\bar{q}} f_p^\gamma \overline{f_q^\delta})\Big)\, \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j$$

and

$$\operatorname{tr}_\omega \sqrt{-1}\partial\overline{\partial}u = |\nabla \mathrm{d}f|^2 + (g^{i\bar{j}} (\Theta_g)_{i\bar{j}k\bar{l}}) g^{k\bar{q}} g^{p\bar{l}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} - (\Theta_h)_{\alpha\bar{\beta}\gamma\bar{\delta}} (g^{i\bar{j}} f_i^\alpha \overline{f_j^\beta})(g^{p\bar{q}} f_p^\gamma \overline{f_q^\delta})$$

where $u = \operatorname{tr}_g(f^*\omega_h) = |\nabla f|^2, f = f_i^\alpha \mathrm{d}z^i \otimes e_\alpha$ and $e_\alpha = f^*(\frac{\partial}{\partial w^\alpha})$. Furthermore, one has

$$\operatorname{tr}_\omega \sqrt{-1}\partial\overline{\partial}\log u \geq \frac{1}{u}\Big((g^{i\bar{j}} (\Theta_g)_{i\bar{j}k\bar{l}}) g^{k\bar{q}} g^{p\bar{l}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} - (\Theta_h)_{\alpha\bar{\beta}\gamma\bar{\delta}} (g^{i\bar{j}} f_i^\alpha \overline{f_j^\beta})(g^{p\bar{q}} f_p^\gamma \overline{f_q^\delta})\Big)$$

holds outside the set of critical points of f.

Corollary 17.1.1. Let g, h be two Hermitian metrics on a complex manifold M. Then

$$\operatorname{tr}_g \sqrt{-1}\partial\overline{\partial}\log \operatorname{tr}_g \omega_h \geq \frac{1}{\operatorname{tr}_g \omega_h}\Big((g^{i\bar{j}} (\Theta_g)_{i\bar{j}k\bar{l}}) g^{k\bar{q}} g^{p\bar{l}} h_{p\bar{q}} - (\Theta_h)_{i\bar{j}p\bar{q}} g^{i\bar{j}} g^{p\bar{q}}\Big)$$

and

$$\operatorname{tr}_h \sqrt{-1}\partial\overline{\partial}\log \operatorname{tr}_g \omega_h \geq \frac{1}{\operatorname{tr}_g \omega_h}\Big((h^{i\bar{j}} (\Theta_g)_{i\bar{j}k\bar{l}}) g^{k\bar{q}} g^{p\bar{l}} h_{p\bar{q}} - (h^{i\bar{j}} (\Theta_h)_{i\bar{j}p\bar{q}}) g^{p\bar{q}}\Big)$$

Corollary 17.1.2. Let $f\colon (M,g) \to (N,h)$ be a holomorphic map between Hermitian manifolds. Suppose

(1) the second Chern-Ricci curvature $\mathrm{Ric}^{(2)}(\omega_g) \geq a\omega_g$ for some a;
(2) the holomorphic bisectional curvature (N,h) is bounded from above by b, that is

$$\Theta_h(w, \overline{w}, v, \overline{v}) \leq b|w|^2|v|^2$$

Then

$$\operatorname{tr}_g \sqrt{-1}\partial\overline{\partial}u \geq au - bu^2$$

where $u = \operatorname{tr}_g(f^*\omega_h) = |\nabla f|^2$. Furthermore, outside the set of critical points of f, one has

$$\operatorname{tr}_g \sqrt{-1}\partial\overline{\partial}\log u \geq a - bu$$

Proof. Note that $\mathrm{Ric}^{(2)}(\omega_g) \geq a\omega_g$ implies $(\Theta_g)_{i\bar{j}k\bar{l}}g^{k\bar{l}} \geq ag_{i\bar{j}}$. Thus

$$\begin{aligned}(g^{i\bar{j}}(\Theta_g)_{i\bar{j}k\bar{l}})g^{k\bar{q}}g^{p\bar{l}}h_{\alpha\bar{\beta}}f_p^\alpha\overline{f_q^\beta} &= g^{i\bar{j}}(\Theta_g)_{i\bar{j}k\bar{l}})\frac{g^{k\bar{l}}g_{k\bar{l}}}{n}g^{k\bar{q}}g^{p\bar{l}}h_{\alpha\bar{\beta}}f_p^\alpha\overline{f_q^\beta} \\ &\geq \frac{ag^{i\bar{j}}g_{i\bar{j}}}{n}g_{k\bar{l}}g^{k\bar{q}}g^{p\bar{l}}h_{\alpha\bar{\beta}}f_p^\alpha\overline{f_q^\beta} \\ &= ag^{p\bar{l}}f_p^\alpha\overline{f_l^\beta} \\ &= au\end{aligned}$$

On the other hand, by using normal coordinate on M it's easy to see

$$(\Theta_h)_{\alpha\bar{\beta}\gamma\bar{\delta}}(g^{i\bar{j}}f_i^\alpha\overline{f_j^\beta})(g^{p\bar{q}}f_p^\gamma\overline{f_q^\delta}) \leq b(f_i^\alpha\overline{f_j^\beta}g^{i\bar{j}}h_{\alpha\bar{\beta}})(f_p^\gamma\overline{f_q^\delta}g^{p\bar{q}}h_{\gamma\bar{\delta}}) = bu^2$$

Then by Lemma 17.1.1 this completes the proof. □

Lemma 17.1.2 ([Yau75])**.** Let (M, g) be a complete Riemannian manifold with Ricci curvature bounded from below and $f \in C^2(M, \mathbb{R})$ be bounded from above. Then there exists $\{p_k\} \subseteq M$ such that

(1) $\lim_k |\nabla f(p_k)| = 0$.
(2) $\limsup_k \Delta f(p_k) \leq 0$.
(3) $\lim_k f(p_k) = \sup f$.

Theorem 17.1.1 ([Yau78a])**.** Let (M, ω_g) be a complete Kähler manifold with $\mathrm{Ric}(\omega_g) \geq a\omega_g$ and (N, ω_h) be a Hermitian manifold with holomorphic bisectional curvature $\leq b < 0$. If $f\colon M \to N$ is a non-constant holomorphic map, then $a < 0$ and

$$f^*\omega_h \leq \frac{a}{b}\omega_g$$

Proof. Let Δ_g denotes the operator $\mathrm{tr}_g \sqrt{-1}\partial\bar{\partial}$. By Corollary 17.1.2 one has

$$\Delta_g u \geq au - bu^2$$

where $u = \mathrm{tr}_g(f^*\omega_h)$. It suffices to show $\sup_M u \leq a/b$. Now let's consider the following two cases:

(1) If $\sup_M u < \infty$, then by Lemma 17.1.2 there exists a sequence $\{p_k\}$ such that $\limsup \Delta u(p_k) \leq 0$, and $\lim u(p_k) = \sup_M u$ ans so

$$\Delta_g u(p_k) \geq au(p_k) - bu^2(p_k)$$

By taking limsup, we deduce that

$$0 \geq a\sup_M u - b(\sup_M u)^2$$

Since $\sup_M u > 0$ and $b < 0$, one has $a < 0$ and $\sup_M u \leq a/b$.

(2) If $\sup_M u = \infty$, consider

$$v = \frac{1}{\sqrt{u + c}}$$

for some constant $c > 0$. Direct computation shows

$$\Delta_g v = -\frac{1}{2}v^3\Delta_g u + \frac{3}{v}|\nabla v|^2$$
$$\Delta_g u = \frac{2}{v^3}(\frac{3}{v}|\nabla v|^2 - \Delta_g v) \geq au - bu^2$$

that is

$$6|\nabla v|^2 - 2v\Delta_g v \geq auv^4 - bu^2v^4$$

By Lemma 17.1.2 again there exists a sequence $\{p_k\}$ such that $\lim_k(-v)(p_k) = \sup_M(-v) = 0$ and $\nabla v(p_k) = 0$ and

$$\limsup_k \Delta_g v(p_k) \leq 0$$

Then

$$0 \geq \limsup(6|\nabla v|^2 - 2v\Delta_g v)(p_k) \geq \limsup(auv^4 - buv^4)(p_k) = -b$$

which is a contradiction to $b > 0$.

In a word, only the first case will happen, and this completes the proof. □

Corollary 17.1.3. Let $f\colon (\mathbb{D}, \omega) \to (\mathbb{D}, \omega)$ be a holomorphic map between unit disk with Poincaré metric. Then

$$f^*(\omega) \leq \omega$$

Proof. It's clear since unit disk with Poincaré metric has constant holomorphic bisectional curvature -1 and $\mathrm{Ric}(\omega) = -\omega$. □

Corollary 17.1.4. Let $f\colon (M, g) \to (N, h)$ be a holomorphic map between two Hermitian manifolds. If

(1) (M, g) is complete Kähler with $\mathrm{Ric}(g) \geq 0$.
(2) (N, h) has negative holomorphic bisectional curvature.

Then f is constant.

Proof. Suppose the holomorphic bisectional curvature of N is bounded above by $b < 0$. If there exists a non-constant holomorphic map from (M, g) to (N, h), then it contradicts to $\mathrm{Ric}(\omega_g) \geq 0$ since by Theorem 17.1.1 one has if $\mathrm{Ric}(\omega_g) \geq a\omega_g$, then $a < 0$. □

Corollary 17.1.5. Let $f\colon (M, g) \to (N, h)$ be a holomorphic map between two Hermitian manifolds such that M is compact with $\mathrm{Ric}^{(2)}(\omega_g) \geq 0$ and N has non-positive holomorphic bisectional curvature. If one of the following statements holds, then f is a constant.

(1) $\mathrm{Ric}^{(2)}(\omega_g) > 0$ at some point.
(2) N has negative holomorphic bisectional curvature at some point.

Proof. Let ω_G be a Gauduchon metric of M and $u = \mathrm{tr}_g(f^*\omega_h)$. By Lemma 17.1.1 the integration

$$\int_M \mathrm{tr}_\omega \sqrt{-1}\partial\bar{\partial}u \cdot \omega_G^n = 0$$

implies $\nabla \mathrm{d} f=0$ and

$$(g^{i\bar{j}}(\Theta_g)_{i\bar{j}k\bar{l}})g^{k\bar{q}}g^{p\bar{l}}h_{\alpha\bar{\beta}}f_p^\alpha\overline{f_q^\beta}=(\Theta_h)_{\alpha\bar{\beta}\gamma\bar{\delta}}(g^{i\bar{j}}f_i^\alpha\overline{f_j^\beta})(g^{p\bar{q}}f_p^\gamma\overline{f_q^\delta})=0$$

Hence $|\mathrm{d}f|$ is a constant, and it's clear from the second equation that (1) or (2) implies $\mathrm{d}f=0$, that is f is a constant. □

17.2. **Schwarz lemmas for holomorphic sectional curvature.**

Lemma 17.2.1. There is the following identity

$$\int_{\mathbb{CP}^{n-1}}\frac{\xi^i\overline{\xi}^j\xi^k\overline{\xi}^l}{|\xi|^4}\omega_{FS}^{n-1}=\frac{\delta_{ij}\delta_{kl}+\delta_{il}\delta_{kj}}{n(n+1)}$$

where $[\xi^1:\cdots:\xi^n]$ are homogenous coordinates on $\mathbb{CP}^{n-1}$ and ω_{FS} is the Fubini-Study metric.

Theorem 17.2.1. Let (M,g) be a Hermitian n-manifold, (N,h) be a Kähler n-manifold and $f\colon(M,g)\to(N,h)$ be a non-constant holomorphic map. Suppose that

(1) $\mathrm{Ric}^{(2)}(g)\geq-\lambda\omega_g+\mu f^*\omega_h$ for continuous functions λ,μ with $\mu\geq 0$.
(2) holomorphic sectional curvature of h is bounded from above by a continuous functions $-\kappa\leq 0$.

Then

$$\mathrm{tr}_g\sqrt{-1}\partial\overline{\partial}u\geq-\lambda u+\left(\frac{(r+1)f^*\kappa}{2r}+\frac{\mu}{n}\right)u^2$$

where r is maximal rank of $\mathrm{d}f$ and $u=\mathrm{tr}_g(f^*\omega_h)$. Furthermore, outside the critical points of f one has

$$\mathrm{tr}_g\sqrt{-1}\partial\overline{\partial}\log u\geq-\lambda+\left(\frac{(r+1)f^*\kappa}{2r}+\frac{\mu}{n}\right)u$$

Proof. Let Δ_g denote the operator $=\mathrm{tr}_g\sqrt{-1}\partial\overline{\partial}$. Then as computation in Lemma 17.1.1 one has

$$\Delta_g u=|\nabla\mathrm{d}f|^2+\underbrace{(g^{i\bar{j}}(\Theta_g)_{i\bar{j}k\bar{l}})g^{k\bar{q}}g^{p\bar{l}}h_{\alpha\bar{\beta}}f_p^\alpha\overline{f_q^\beta}}_{\mathrm{I}}-\underbrace{(\Theta_h)_{\alpha\bar{\beta}\gamma\bar{\delta}}(g^{i\bar{j}}f_i^\alpha\overline{f_j^\beta})(g^{p\bar{q}}f_p^\gamma\overline{f_q^\delta})}_{\mathrm{II}}$$

For part I, one has

$$\begin{aligned}I&\geq g^{i\bar{j}}(\Theta_h)_{i\bar{j}k\bar{l}}\frac{g^{k\bar{l}}g_{k\bar{l}}}{n}g^{k\bar{q}}g^{p\bar{l}}h_{\alpha\bar{\beta}}f_p^\alpha\overline{f_q^\beta}\\&=(-\lambda g_{k\bar{l}}+\mu g^{i\bar{j}}g_{k\bar{l}}h_{\gamma\bar{\delta}}f_i^\gamma\overline{f_j^\delta})g^{k\bar{q}}g^{p\bar{l}}h_{\alpha\bar{\beta}}f_p^\alpha\overline{f_q^\beta}\\&=(-\lambda g_{k\bar{l}}+\mu h_{\gamma\bar{\delta}}f_k^\gamma\overline{f_l^\delta})g^{k\bar{q}}g^{p\bar{l}}h_{\alpha\bar{\beta}}f_p^\alpha\overline{f_q^\beta}\\&=-\lambda\mu+\frac{\mu}{n}\mu^2\end{aligned}$$

For part II, by taking normal coordinate at $p \in M$ and $f(p) \in N$ we may assume $f_i^\alpha = \lambda_i \delta_i^\alpha$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r(p)} > \lambda_{r(p)+1} = 0$, where $r(p)$ is rank of $\mathrm{d}f(p)$. Then $\mathrm{tr}_g(f^*\omega_h) = \sum_{i=1}^n \lambda_i^2$. Hence

$$
\begin{aligned}
II &= (\Theta_h)_{\alpha\overline{\beta}\gamma\overline{\delta}}(g^{i\overline{j}} f_i^\alpha \overline{f_j^\beta})(g^{p\overline{q}} f_p^\gamma \overline{f_q^\delta}) \\
&= \sum_{i,k=1}^n (\Theta_h)_{\alpha\overline{\beta}\gamma\overline{\delta}} f_i^\alpha \overline{f_i^\beta} f_k^\gamma \overline{f_k^\delta} \\
&\overset{(a)}{=} \sum_{i,j,k,l=1}^n (\Theta_h)_{\alpha\overline{\beta}\gamma\overline{\delta}} f_i^\alpha \overline{f_j^\beta} f_k^\gamma \overline{f_l^\delta} \left(\frac{\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj}}{2} \right) \\
&\overset{(b)}{=} \frac{n(n+1)}{2} (\Theta_h)_{\alpha\overline{\beta}\gamma\overline{\delta}} f_i^\alpha \overline{f_j^\beta} f_k^\gamma \overline{f_l^\delta} \int_{\mathbb{CP}^{n-1}} \frac{\xi^i \overline{\xi}^j \xi^k \overline{\xi}^l}{|\xi|^4} \omega_{FS}^{n-1} \\
&= \frac{n(n+1)}{2} \int_{\mathbb{CP}^{n-1}} (\Theta_h)_{\alpha\overline{\beta}\gamma\overline{\delta}} \frac{(f_i^\alpha \xi^i)(\overline{f_j^\beta \xi^j})(f_k^\gamma \xi^\gamma)(\overline{f_l^\delta \xi^l})}{|\xi|^4} \omega_{FS}^{n-1} \\
&\overset{(c)}{\leq} -\frac{\kappa(f(p)) n(n+1)}{2} \int_{\mathbb{CP}^{n-1}} \frac{(|f_i^\alpha \xi^i|^2)^2}{|\xi|^4} \omega_{FS}^{n-1}
\end{aligned}
$$

where

(a) holds from (N, h) is Kähler.
(b) holds from Lemma 17.2.1.
(c) holds from holomorphic sectional curvature of h is bounded from above by $-\kappa$.

Since $f_i^\alpha = \lambda_i \delta_i^\alpha$, one has

$$
\begin{aligned}
\int_{\mathbb{CP}^{n-1}} \frac{(|f_i^\alpha \xi^i|^2)^2}{|\xi|^4} \omega_{FS}^{n-1} &= \frac{1}{n(n+1)} \sum_{i,j,\alpha,\beta} \lambda_i \lambda_j \lambda_k \lambda_l \delta_i^\alpha \delta_j^\alpha \delta_k^\beta \delta_l^\beta (\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj}) \\
&= \frac{1}{n(n+1)} \left((\sum_\alpha \lambda_\alpha^2)^2 + \sum_\alpha \lambda_\alpha^4 \right) \\
&\overset{(d)}{\geq} \frac{1}{n(n+1)} \cdot \frac{r+1}{r} (\sum_\alpha \lambda_\alpha^2)^2
\end{aligned}
$$

where (d) holds since r is the maximal numbers of non-zero elements of λ_α. Hence one has

$$
II \leq -\frac{(r+1) f^*\kappa}{2r} (\sum_\alpha \lambda_\alpha^2)^2 = -\frac{(r+1) f^*\kappa}{2r} u^2
$$

This completes the proof. □

Corollary 17.2.1. Let (M, ω) be a complete Kähler manifold with Ricci curvature bounded from below by a positive constant. Then M is compact, and there is no non-trivial holomorphic map from M into a Kähler manifold with non-positive holomorphic sectional curvature.

Corollary 17.2.2. Let (N, h) be a Hermitian manifold with non-positive holomorphic sectional curvature. Then any holomorphic map from $\mathbb{CP}^n$ to N is constant. In particular, N contains no rational curves.

Corollary 17.2.3. Let (N, h) be a Hermitian manifold with negative holomorphic sectional curvature. Then any holomorphic map from torus $\mathbb{T}^2$ to N is constant.

Part 5. Existence of Kähler-Einstein metric

18. Calabi-Yau theorem

18.1. **Introduction.** The investigation into the existence of Kähler-Einstein metrics is a compelling and extensive topic that traces back to 1954. In which year Calabi proposed the famous Calabi conjecture, which was finally solved by Yau in 1976. Firstly, Calabi proved the uniqueness of the solution and laid out the program of proving the existence by the method of continuity and also pointed out the openness and the need of a priori estimates in [Cal57]. The very easy a priori zeroth oreder estimate for the case of negative first Chern class was firstly given by Aubin in [Aub78], but he did not apply his a priori estimate to the continuity method. Instead he used the method of variation which is rather difficult to comprehend.

Yau made the important contribution of using Morser's method of integration by parts and iteration by Sobolev inequality to get a priori zeroth oreder estimate.

Theorem 18.1 ([MR054],[Yau78b])**.** Let (X, ω_g) be a compact Kähler manifold. If Ω is a real $(1,1)$-form which represents $2\pi c_1(X)$, then there exists a unique metric $\omega \in [\omega_g]$ such that $\mathrm{Ric}(\omega) = \Omega$.

This remarkable result establishes several related results which are of fundamental importance in the study of complex manifolds.

Corollary 18.1.1. Let (X, ω_g) be a compact Kähler manifold with $c_1(X) = 0$. Then there exists a unique Ricci flat metric.

Corollary 18.1.2. Let (X, ω_g) be a compact Kähler manifold with $c_1(X) > 0$. Then M is simply-connected.

Corollary 18.1.3 ([Yau77])**.** Every complex surface which is homotopic equivalent to $\mathbb{CP}^2$ is biholomorphic to $\mathbb{CP}^2$

18.2. The Monge-Amperé equation and priori estimates.

18.2.1. *The reformulation of Calabi conjecture in Monge-Amperé equation.* The Calabi conjecture can be reduced to a problem of fully non-linear partial differential equations. By Lemma 10.3.1, that is $\partial\overline{\partial}$-lemma, one has

$$\omega = \omega_g + \sqrt{-1}\partial\overline{\partial}\varphi$$

where φ is the smooth function we desire. Again by $\partial\overline{\partial}$-lemma one also has

$$\mathrm{Ric}(\omega_g) = \Omega + \sqrt{-1}\partial\overline{\partial}F$$

where F is a smooth function which is unique up to a constant. If we consider the normalization

$$\int_X e^F \omega_g^n = \int_X \omega_g^n$$

then F is unique. Suppose $\mathrm{Ric}(\omega)=\Omega$. Then

$$\mathrm{Ric}(\omega_g)-\mathrm{Ric}(\omega)=\sqrt{-1}\partial\overline{\partial}F$$

that is

$$\sqrt{-1}\partial\overline{\partial}\log\frac{\omega^n}{\omega_g^n}=\sqrt{-1}\partial\overline{\partial}F$$

which is equivalent to the following equation

$$\omega^n=e^{F+C}\omega_g^n$$

and by nomalization of F one has $C=0$. Thus in order to solve Calabi conjecture, it suffices to solve the following complex Monge-Amperé equation:

$$\begin{cases}(\omega_g+\sqrt{-1}\partial\overline{\partial}\varphi)^n=e^F\omega_g^n\\ \int_X e^F\omega_g^n=1\end{cases}\tag{18.1}$$

Theorem 18.2.1 ([Cal57])**.** The solution of (18.1) is unique.

Proof. Suppose φ_1,φ_2 are two solutions. Then

$$\begin{aligned}0=\omega_1^n-\omega_2^n&=(\omega_1-\omega_2)(\omega_1^{n-1}+\cdots+\omega_2^{n-1})\\&=\sqrt{-1}\partial\overline{\partial}(\varphi_1-\varphi_2)(\omega_1^{n-1}+\cdots+\omega_2^{n-1})\end{aligned}$$

If we define $\psi=\varphi_1-\varphi_2$, then

$$\begin{aligned}-\int_X\psi\sqrt{-1}\partial\overline{\partial}\psi\wedge(\omega_1^{n-1}+\cdots+\omega_2^{n-1})&\overset{(1)}{=}\int_X\partial\psi\wedge\overline{\partial}\psi\wedge(\omega_1^{n+1}+\cdots+\omega_2^{n-1})\\&\overset{(2)}{\geq}\frac{1}{n}|\partial\psi|^2\omega_1^n\end{aligned}$$

where

(1) holds from integration by parts and ω_1,ω_2 are Kähler forms.
(2) holds from (2) of Proposition 7.3.3 and positivity of ω_1,ω_2.

This shows $\partial\psi=0$, and thus ψ is a constant, which completes the proof. □

To solve the existence of solution of (18.1), Yau used the continuity method. Consider a sequence of equations as follows

$$\det(g_{i\bar{j}}+\varphi_{i\bar{j}})=\frac{e^{tF}}{\int_X e^{tF}\omega_g^n}\det(g_{i\bar{j}})\tag{18.2}$$

It's clear (18.2) is solvable at $t=0$, and thus consider the following set is non-empty

$$I=\{t\in[0,1]\mid \text{equation (18.2) is solvable}\}$$

The openness of I can be relatively easily demonstrated through the application of the inverse function theorem. However, the true challenge lies in establishing its closedness. To accomplish this, a series of rigorous a priori estimates are essential.

18.2.2. C^0-*estimate.*

Theorem 18.2.2 (C^0-estimate)**.** Suppose φ is a solution of (18.1). Then there exists constant C depending on X, ω_g, F such that

$$\sup\varphi - \inf\varphi \leq C$$

Proof. Without lose of generality we may assume $\sup\varphi = -1$. Then Green formula says

$$\varphi(x) = \frac{1}{V}\int \varphi(y)\omega_g^n - \frac{1}{V}\int G(x,y)\Delta\varphi(y)\omega_g^n$$

and $\omega - \omega_g = \sqrt{-1}\partial\overline{\partial}\varphi$ implies

$$\operatorname{tr}_g \sqrt{-1}\partial\overline{\partial}\varphi = \operatorname{tr}_g \omega - n \geq n$$

Assume $G \geq 0$, if $\varphi(p) = -1$, then

$$\begin{aligned}
-1 &= \varphi(p) \\
&= \frac{1}{V}\int \varphi(y)\omega_g^n - \frac{1}{V}\int G(p,y)\Delta\varphi(y)\omega_g^n \\
&\leq \frac{1}{V}\int \varphi(y)\omega_g^n + \frac{1}{V}\int nG(p,y)\omega_g^n
\end{aligned}$$

this shows

$$\|\varphi\|_{L^1} = -\int \varphi\omega_g^n \leq C_1$$

For L_2-estimate, note that

$$\begin{aligned}
\int \varphi(e^F - 1)\omega_g^n &= \int \varphi(\omega^n - \omega) \\
&= \int \varphi(\sqrt{-1}\partial\overline{\partial}\varphi)(\omega^{n-1} + \cdots + \omega_g^{n-1}) \\
&\leq -\frac{1}{n}\int |\partial\varphi|^2\omega_g^n
\end{aligned}$$

that is

$$\int |\partial\varphi|^2\omega_g^n \leq n\sup|e^F - 1|\int |\varphi|\omega_g^n \leq C_2$$

that is $\|\nabla\varphi\|_{L^2}^2 \leq C_3$, and by Poincaré inequality one has

$$\|\varphi - \overline{\varphi}\|_{L^2} \leq C_4\|\nabla\varphi\|_{L^2}$$

that is $\|\varphi\|_{L^2} \leq C_5$.

For $p \geq 2$, set $\psi = -\varphi$, one has

$$\begin{aligned}\int \psi^{p-1}(e^F - 1)\omega_g^n &= -\int \psi^{p-1}\sqrt{-1}\partial\overline{\partial}\psi \wedge (\omega^{n-1} + \cdots + \omega_g^{n-1}) \\ &= \int (p-1)\psi^{p-2}\sqrt{-1}\partial\psi \wedge \overline{\partial}\psi \wedge (\omega^{n-1} + \cdots + \omega_g^{n-1}) \\ &\geq \int (p-1)\psi^{p-2}\sqrt{-1}\partial\psi \wedge \overline{\partial}\psi \wedge \omega_g^{n-1} \\ &= \frac{4(p-1)}{p^2}\int \sqrt{-1}\partial\psi^{\frac{p}{2}} \wedge \overline{\partial}\psi^{\frac{p}{2}} \wedge \omega_g^{n-1} \\ &= \frac{4(p-1)}{np^2}\int |\partial\psi^{\frac{p}{2}}|^2\omega_g^n\end{aligned}$$

This gives

$$\int |\partial\psi^{\frac{p}{2}}|\omega_g^n \leq C_6 p\int |\varphi|^{p-1}\omega_g^n$$

that is

$$\|\nabla\psi\|_{L^2}^2 \leq C_6 p\|\psi\|_{L^{p-1}}^{p-1}$$

The Sobolev inequality shows for all $f \in W^{1,q}(X, \omega_g)$, one has

$$\|f\|_{L^{\frac{2nq}{2n-q}}} \leq C_7\|f\|_{W^{1,q}}$$

Set $f = \psi^{\frac{p}{2}}$ and $q = 2$, one has

$$\begin{aligned}(\int \psi^{\frac{np}{n-1}}\omega_g^n)^{\frac{n-1}{n}} &= \|\psi\|^p_{L^{\frac{np}{n-1}}} = \|\psi\|^2_{L^{\frac{2n}{n-1}}} \\ &\leq C_8(\int |\nabla\psi^{\frac{p}{2}}|^2\omega_g^n + \int \psi\omega_g^n) \\ &\leq C_9(p\|\psi\|_{L^{p-1}}^{p-1} + \|\psi\|_{L^p}^p) \\ &\leq C_{10}p\|\psi\|_{L^p}^p\end{aligned}$$

This gives

$$\|\psi\|_{L^{\frac{np}{n-1}}} \leq C_{10}^{\frac{1}{p}} p^{\frac{1}{p}}\|\psi\|_{L^p}$$

Set $p_k = (\frac{n}{n-1})^k p$, then

$$\|\psi\|_{L^{p_{k+1}}} \leq (C_{10}P_k)^{\frac{1}{p_k}}\|\psi\|_{L^{p_k}} \leq \prod_{j=0}^{k}(C_{10}p_j)^{\frac{1}{p_j}}\|\psi\|_{L^p}$$

This gives

$$\|\psi\|_{L^\infty} \leq C\|\psi\|_{L^p}$$

□

18.2.3. *C^2-estimate.*

Theorem 18.2.3. Suppose ω_h is a solution of (18.1). Then there exists a constant c depending on X, ω_g, F such that

$$c^{-1}\omega_g \leq \omega_h \leq c\omega_g$$

Proof. It suffices to prove $\operatorname{tr}_g \omega_h \leq c$ since linear algebra yields the following inequality

$$\operatorname{tr}_h \omega_g \leq \frac{1}{n-1}(\operatorname{tr}_g \omega_h)^{n-1}\frac{\omega_g^n}{\omega_h^n}$$

By Corollary 17.1.1, that is Schwarz computation, one has

$$\Delta_h \log \operatorname{tr}_g \omega_h \geq \frac{1}{\operatorname{tr}_g \omega_h}\left(h^{i\bar{j}}(\Theta_g)_{i\bar{j}k\bar{l}}g^{k\bar{q}}g^{p\bar{l}}h_{p\bar{q}} - h^{i\bar{j}}(\Theta_h)_{i\bar{j}p\bar{q}}g^{p\bar{q}}\right)$$

where $\Delta_h = \operatorname{tr}_h \sqrt{-1}\partial\bar{\partial}$. Since (X, g) is a given Käher manifold, one has its curvature is bounded from below by $-B$ as follows

$$(\Theta_g)_{i\bar{j}k\bar{l}} \geq -B(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}})$$

Then

$$\begin{aligned} h^{i\bar{j}}(\Theta_g)_{i\bar{j}k\bar{l}}g^{k\bar{q}}g^{p\bar{l}}h_{p\bar{q}} &\geq -B(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}})g^{k\bar{q}}g^{p\bar{l}}h^{i\bar{j}}h_{p\bar{q}} \\ &= -B(1+\frac{1}{n})\delta_i^p\delta_j^q h^{i\bar{j}}h_{p\bar{q}} \\ &= -(n+1)B \end{aligned}$$

On the other hand, one has

$$h^{i\bar{j}}(\Theta_h)_{i\bar{j}p\bar{q}}g^{p\bar{q}} = \left(g^{i\bar{j}}(\Theta_g)_{i\bar{j}p\bar{q}} - F_{p\bar{q}}\right)g^{p\bar{q}} = s - F_{p\bar{q}}g^{p\bar{q}}$$

where $\operatorname{Ric}(\omega_g) = \Omega + \sqrt{-1}\partial\bar{\partial}F$ and s is the scalar curvature of (X, g). Then by $\operatorname{tr}_h \omega_g \operatorname{tr}_g \omega_h \geq n^2$ one has

$$\begin{aligned} \Delta_h \log \operatorname{tr}_g \omega_h &\geq -\frac{1}{n^2}\left((n+1)B + c_0\right)\operatorname{tr}_h \omega_g \\ &\geq -2B\operatorname{tr}_h \omega_g - \frac{c_0}{n^2}\operatorname{tr}_g \omega_h \end{aligned}$$

where $c_0 = (s - F_{p\bar{q}}g^{p\bar{q}})$. Note that

$$\Delta_h \varphi = h^{j\bar{k}}\varphi_{j\bar{k}} = h^{j\bar{k}}(h_{j\bar{k}} - g_{j\bar{k}}) = n - \operatorname{tr}_h \omega_g$$

Then there exists an appropriate λ such that

$$\Delta_h(\log \operatorname{tr}_g \omega_h + \lambda\varphi) \geq \operatorname{tr}_h \omega_g - C_1$$

If $\log \operatorname{tr}_g \omega_h + \lambda\varphi$ obtain its maximum at $p \in X$, then $\operatorname{tr}_h \omega_g(p) \leq C_1$, and again by trick of linear algebra one has

$$\operatorname{tr}_g \omega_h(p) \leq C_2$$

for some constant C_2. Since p is the point such that $\log \operatorname{tr}_g \omega_h + \lambda\varphi$ obtains its maximum, then by C^0-estimate there exists some constant C such that

$$\operatorname{tr}_g \omega_h \leq C$$

This completes the proof. □

18.2.4. *C^3-estimate.* Let $(\Gamma_g)_{ij}^k$ and $(\Gamma_h)_{ij}^k$ denote the Christoffel symbols of g and h respectively, and set

$$S_{ij}^k = (\Gamma_h)_{ij}^k - (\Gamma_g)_{ij}^k$$

Suppose we have the following C^2-estimate

$$c^{-1}\omega_g \leq \omega_h \leq c\omega_g$$

Lemma 18.2.1. There exists a constant C depending on X, ω_g, F, c such that

$$\Delta_h |S|_h^2 \geq -C|S|_h^2 - C$$

Lemma 18.2.2. There exists a constant C depending on X, ω_g, F, c such that

$$|S| < C$$

18.3. **Proof of Calabi-Yau theorem.**

Theorem 18.3.1. Let (X, ω_g) be a compact Kähler manifold. Then for any $k \geq 3$ and $F \in C^k(X, \mathbb{R})$, the complex Monge-Amperé equation

$$\begin{cases} \omega_h^n = e^F \omega^n g \\ \omega_h = \omega_g + \sqrt{-1}\partial\overline{\partial}\varphi \\ \int_X e^F \omega_g^n = \int_X \omega_g^n \end{cases}$$

has a solution $\varphi \in C^{k+1,\alpha}(X)$.

18.4. **Aubin-Yau theorem.**

18.4.1. *Uniqueness of Kähler-Einstein metric when $c_1(X) < 0$.*

Lemma 18.4.1. Let X be a compact complex manifold. Then there exists at most one Kähler metric ω such that $\operatorname{Ric}(\omega) = -\omega$.

Proof. Suppose that there are two Kähler metrics ω_1 and ω_2 such that

$$\operatorname{Ric}(\omega_1) = -\omega_1$$
$$\operatorname{Ric}(\omega_2) = -\omega_2$$

By $\partial\overline{\partial}$-lemma, there exists $\varphi \in C^\infty(X, \mathbb{R})$ such that $\omega_1 = \omega_2 + \sqrt{-1}\partial\overline{\partial}\varphi$, so one has

$$\sqrt{-1}\partial\overline{\partial}\varphi = \sqrt{-1}\partial\overline{\partial}\log\frac{\omega_1^n}{\omega_2^n}$$

In other words, there exists a constant c such that

$$\omega_1^n = e^{\varphi + c}\omega_2^n$$

Suppose φ attains its maximum at point p, that is, $\sqrt{-1}\partial\overline{\partial}\varphi(p) \leq 0$. Then

$$\omega_1 = \omega_2 + \sqrt{-1}\partial\overline{\partial}\varphi \leq \omega_2$$

which implies $e^{\varphi(p)+c} \leq 1$. This shows $\varphi + c \leq \varphi(p) + c \leq 0$. Similarly, if φ attains its minimum at q, then $e^{\varphi(q)+c} \geq 1$ and $\varphi + c \geq \varphi(q) + c \geq 0$. Hence $\varphi + c \equiv 0$, and thus $\omega_1 = \omega_2$. □

18.4.2. *The reformulation of Aubin-Yau theorem in Monge-Amperé equation.*

Theorem 18.4.1 (Aubin-Yau)**.** Let X be a complex manifold with $c_1(X) < 0$. Then there exists a unique Kähler metric $\omega \in 2\pi c_1(X)$ such that $\mathrm{Ric}(\omega) = -\omega$.

Proof. Let ω_g be any Kähler metric in $[-2\pi c_1(X)]$. Then by $\partial\overline{\partial}$-lemma there exists a smooth function F such that

$$\mathrm{Ric}(\omega_g) = -\omega_g + \sqrt{-1}\partial\overline{\partial}F$$

since $[\mathrm{Ric}(\omega_g)] = 2\pi c_1(X)$. Let ω be another Kähler metric such that $[\omega] = [\omega_g]$. Again by $\partial\overline{\partial}$-lemma there exists a smooth function φ such that

$$\omega = \omega_g + \sqrt{-1}\partial\overline{\partial}\varphi$$

Then $\mathrm{Ric}(\omega) = -\omega$ is equivalent to

$$\sqrt{-1}\partial\overline{\partial}\log\frac{\omega^n}{\omega_g^n} = \sqrt{-1}\partial\overline{\partial}(F + \varphi)$$

or in other words, there exists a constant C such that

$$\omega^n = e^{F+\varphi+C}\omega_g^n$$

By rescaling F we may assume $C = 0$, so Aubin-Yau is equivalent to solve the following complex Monge-Amperé equation:

$$\begin{cases} (\omega_g + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{F+\varphi}\omega_g^n \\ \int_X e^{F+\varphi}\omega_g^n = 1 \end{cases} \tag{18.3}$$

□

18.4.3. *C^0-estimate.*

Theorem 18.4.2 (C^0-estimate)**.** Suppose φ is a solution of (18.3). Then

$$\sup_X |\varphi| \leq \sup_X |F|$$

Proof. As argument in Lemma 18.4.1, at maximum point p of φ, one has $\varphi + F(p) \leq \varphi(p) + F(p) \leq 0$, so

$$\sup_X \varphi \leq -F(p) \leq \sup_X |F|$$

Similarly, at minimum point q of φ, $\varphi + F(q) \geq \varphi(q) + F(q) \geq 0$ and

$$\inf_X \varphi \geq -F(q) \geq -\sup_X |F|$$

Hence $\sup_X |\varphi| \leq \sup_X |F|$. □

18.4.4. *C^2-estimate.*

Theorem 18.4.3 (C^2-estimate)**.** There exists a uniform constant C depending on X, ω_g, F such that

$$C^{-1}\omega_g \leq \omega \leq C\omega_g$$

Proof. The same as Theorem 18.2.3. □

18.4.5. *C^3-estimate.* Given the C^0-estimate and C^2-estimate, the C^3-estimate is very similar to that in the proof of Calabi-Yau theorem. Here we formulate a general setup. Let $(\Gamma_g)_{ij}^k$ and $(\Gamma_h)_{ij}^k$ denote the Christoffel symbols of g and h respectively, and set

$$S_{ij}^k = (\Gamma_h)_{ij}^k - (\Gamma_g)_{ij}^k$$

Suppose we have the following C^2-estimate

$$c^{-1}\omega_g \leq \omega_h \leq c\omega_g$$

Theorem 18.4.4 (C^3-estimate)**.**

18.4.6. *Proof of Aubin-Yau theorem.*

Part 6. Deformations of complex structure

19. Deformations of complex structure

19.1. The Maurer-Cartan equation.

Recall that a complex structure on a smooth manifold M is encoded by an integrable almost complex structure J, and two complex manifolds (M, J) and (M', J') are isomorphic if there exists a diffeomorphism $F\colon M \to M'$ such that $\mathrm{d}F \circ J = J' \circ \mathrm{d}F$. Thus the set of all complex structures on a fixed smooth manifold M is the quotient of the set

$$\mathcal{A}_c(M) := \{J \mid J \text{ is an integrable almost complex structure on } M\}$$

of all complex structures by the action of the diffeomorphism group. Firstly let's consider the set

$$\mathcal{A}_{ac}(M) := \{J \mid J \text{ is an almost complex structure on } M\}$$

For arbitrary almost complex structure $J \in \mathcal{A}_{ac}(M)$, it's uniquely determined by a decomposition of the complexified tangent bundle $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ with J is $\sqrt{-1}\,\mathrm{id}$ on $T^{1,0}M$ and $-\sqrt{-1}\,\mathrm{id}$ on $T^{0,1}M$. In fact, giving $T^{0,1}M \subseteq T_{\mathbb{C}}M$ is enough, since we can set $T^{1,0}M = \overline{T^{0,1}M}$. If $J(t)$ is a continuous family of almost complex structure with $J(0) = J$, there is a continuous family of such decompositions $T_{\mathbb{C}}M = T^{1,0}_t M \oplus T^{0,1}_t M$, or equivalently, of subspaces $T^{0,1}_t M \subseteq T_{\mathbb{C}}M$.

Thus for small t the deformations $J(t)$ of J gives a map

$$\phi(t)\colon T^{0,1}M \to T^{1,0}M$$

with $v + \phi(t)v \in T^{1,0}_t M$. Conversely, if $\phi(t)\colon T^{0,1}M \to T^{1,0}M$ is given, then one defines for small t

$$T^{1,0}_t M := (\mathrm{id} + \phi(t))(T^{0,1}M)$$

Here the condition t to be "small" has to be imposed in order to ensure that with this definition $T^{1,0}_t M \subseteq T_{\mathbb{C}}M \to T^{1,0}M$ is an isomorphism. Thus deformations of almost complex structure is encoded by such a map $\phi(t)$. Now for convenience we assume J is an integrable almost complex, and thus we denote (M, J) by X. In particular, there is $\overline{\partial}$ operator on the holomorphic tangent bundle TX, which can be applied to $\phi(t) \in C^\infty(X, \Omega^{0,1}_X \otimes T^{1,0}X)$.

The following proposition shows that the integrability condition for deformation of the almost complex structure can be rephrased in terms of $\phi(t)$.

Proposition 19.1.1. The integrability condition is equivalent to the Maurer-Cartan equation

$$\overline{\partial}\phi(t) + [\phi(t), \phi(t)] = 0$$

Proof. It suffices to compute locally so we write $\phi(t)$ with respect to local coordinate $\{z^1, \dots, z^n\}$ as follows

$$\phi(t) = \phi_i^j(t) \mathrm{d}\overline{z}^i \otimes \frac{\partial}{\partial z_j}$$

The integrability condition implies $[T_t^{0,1}X, T_t^{0,1}X] \subseteq T_t^{0,1}X$, so in particular one has

$$\left[\frac{\partial}{\partial \overline{z}_i} + \phi(t)(\frac{\partial}{\partial \overline{z}_i}), \frac{\partial}{\partial \overline{z}_j} + \phi(t)(\frac{\partial}{\partial \overline{z}_j})\right] \in T_t^{0,1}X$$

A direct computation yields

$$\left(\left[\frac{\partial}{\partial \overline{z}^i}, \phi_k^l(t)\frac{\partial}{\partial z^l}\right] + \left[\phi_i^j(t)\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^k}\right] + \left[\phi_i^j(t)\frac{\partial}{\partial z^j}, \phi_k^l(t)\frac{\partial}{\partial z^l}\right]\right) \in T_t^{0,1}X$$

Note that

$$\left[\frac{\partial}{\partial \overline{z}^i}, \phi_k^l(t)\frac{\partial}{\partial z^l}\right] = \frac{\partial \phi_k^l(t)}{\partial \overline{z}^i}\frac{\partial}{\partial z^l}$$

$$\left[\phi_i^j(t)\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^k}\right] = -\frac{\partial \phi_i^j(t)}{\partial \overline{z}^k}\frac{\partial}{\partial z^j}$$

This shows

$$\left[\frac{\partial}{\partial \overline{z}^i}, \phi_k^l(t)\frac{\partial}{\partial z^l}\right] + \left[\phi_i^j(t)\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^k}\right] = \left(\frac{\partial \phi_k^j(t)}{\partial \overline{z}^i} - \frac{\partial \phi_i^j(t)}{\partial \overline{z}^k}\right)\frac{\partial}{\partial z^j} = \overline{\partial}\phi(t)(\frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial \overline{z}^k})$$

On the other hand, note that

$$[\phi(t), \phi(t)] = \mathrm{d}\overline{z}^i \wedge \mathrm{d}\overline{z}^k \left[\phi_i^j(t)\frac{\partial}{\partial z^j}, \phi_k^l(t)\frac{\partial}{\partial z^l}\right]$$

Thus

$$\left[\phi_i^j(t)\frac{\partial}{\partial z^j}, \phi_k^l(t)\frac{\partial}{\partial z^l}\right] = [\phi(t), \phi(t)](\frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial \overline{z}^k})$$

So integrability condition implies

$$\overline{\partial}(t) + [\phi(t), \phi(t)](\frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial \overline{z}^k}) \in T_t^{0,1}X$$

This shows $\overline{\partial}(t) + [\phi(t), \phi(t)]$ is a section of $\Omega_X^{0,2} \otimes (T^{1,0}X \cap T_t^{0,1}X)$, but for sufficiently small t one has $T^{1,0}X \cap T_t^{0,1}X = 0$. This shows

$$\overline{\partial}(t) + [\phi(t), \phi(t)] = 0$$

Conversely, if the Maurer-Cartan equation holds, then the integrability condition holds for a local frame of $T_t^{0,1}X$, and thus for all sections of $T_t^{0,1}X$. □

Now let's consider the power series expansion of a given deformation $\phi(t)$ as follows

$$\phi(t) = \phi_0 + \phi_1 t + \phi_2 t^2 + \dots$$

The Maurer-Cartan equation gives

$$\overline{\partial}\left(\sum_{i=1}^{\infty}\phi_i t^i\right)+\sum_{i,j=1}^{\infty}[\phi_i,\phi_j]t^{i+j}=0$$

This yields a recursive system of equations

$$\begin{aligned}
0 &= \overline{\partial}\phi_1\\
0 &= \overline{\partial}\phi_2+[\phi_1,\phi_1]\\
&\vdots\\
0 &= \overline{\partial}\phi_k+\sum_{0<i<k}[\phi_i,\phi_{k-i}]
\end{aligned}$$

In particular, the first-order deformation of the complex structure is determined by a $\overline{\partial}$-closed $(0,1)$-form ϕ_1 with valued in TX. Thus it determines an element $[\phi_1]\in H^1(X,TX)$.

Definition 19.1.1 (Kodaira-Spencer class)**.** The Kodaira-Spencer class of a one-parameter deformation $J(t)$ of a complex structure J is the induced cohomology class $[\phi_1]\in H^1(X,TX)$.

Proposition 19.1.2. Let X be a complex manifold. There is a natural bijection between all first-order deformations of X and elements of $H^1(X,TX)$.

Corollary 19.1.1. A first-order deformations $v\in H^1(X,TX)$ cannot be integrated if $[v,v]\in H^2(X,TX)$ does not vanish.

Part 7. Appendix

20. Sheaf and Cohomology

20.1. **Sheaves.** Along this section, X denotes a topological space.

20.1.1. *Definitions and Examples.*

Definition 20.1.1 (sheaf)**.** A presheaf of abelian group $\mathscr{F}$ on X consisting of the following data:

(1) For any open subset U of X, $\mathscr{F}(U)$ is an abelian group.
(2) If $U \subseteq V$ are two open subsets of X, then there is a group homomorphism $r_{UV} \colon \mathscr{F}(U) \to \mathscr{F}(V)$. Moreover, above data satisfy
  I $\mathscr{F}(\varnothing) = 0$.
  II $r_{UU} = \mathrm{id}$.
  III If $W \subseteq U \subseteq V$ are open subsets of X, then $r_{UW} = r_{VW} \circ r_{UV}$.

Moreover, $\mathscr{F}$ is called a sheaf if it satisfies the following extra conditions

  IV Let $\{V_i\}_{i \in I}$ be an open covering of open subset $U \subseteq X$ and $s \in \mathscr{F}(U)$. If $s|_{V_i} := r_{UV_i}(s) = 0$ for all $i \in I$, then $s = 0$.
  V Let $\{V_i\}_{i \in I}$ be an open covering of open subset $U \subseteq X$ and $s_i \in \mathscr{F}(V_i)$. If $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for all $i, j \in I$, then there exists $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$ for all $i \in I$.

Example 20.1.1 (constant presheaf)**.** For an abelian group G, the constant presheaf assign each open subset U the group G itself, but in general it's not a sheaf.

Definition 20.1.2 (morphism of presheaves)**.** A morphism $\varphi \colon \mathscr{F} \to \mathscr{G}$ between presheaves consisting of the following data:

(1) For any open subset U of X, there is a group homomorphism $\varphi(U) \colon \mathscr{F}(U) \to \mathscr{G}(U)$.
(2) If $U \subseteq V$ are two open subsets of X, then the following diagram commutes

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{G}(U) \\ \downarrow{\scriptstyle r_{UV}} & & \downarrow{\scriptstyle r_{UV}} \\ \mathscr{F}(V) & \xrightarrow{\varphi(V)} & \mathscr{G}(V) \end{array}$$

Notation 20.1.1. For convenience, for $s \in \mathscr{F}(U)$, we often write $\varphi(s)$ instead of $\varphi(U)(s)$.

Remark 20.1.1. The morphisms between sheaves are defined as morphisms of presheaves.

Definition 20.1.3 (isomorphism)**.** A morphism of presheaves $\varphi \colon \mathscr{F} \to \mathscr{G}$ is called an isomorphism if it has two-sided inverse, that is, there exists a morphism of presheaves $\psi \colon \mathscr{G} \to \mathscr{F}$ such that $\psi\varphi = \mathrm{id}_{\mathscr{F}}$ and $\varphi\psi = \mathrm{id}_{\mathscr{G}}$.

Remark 20.1.2. A morphism of presheaves $\varphi \colon \mathscr{F} \to \mathscr{G}$ is an isomorphism if and only if for every open subset $U \subseteq X$, $\varphi(U) \to \mathscr{G}(U)$ is an isomorphism of abelian groups.

20.1.2. *Stalks.*

Definition 20.1.4 (stalks)**.** For a presheaf $\mathscr{F}$ and $p \in X$, the stalk at p is defined as

$$\mathscr{F}_p = \varinjlim_{p \in U} \mathscr{F}(U)$$

Remark 20.1.3 (alternative definition). In order to avoid language of direct limit, we give a more useful but equivalent description of stalk: For $p \in U \cap V$, $s_U \in \mathscr{F}(U)$ and $s_V \in \mathscr{F}(V)$ are equivalent if there exists $x \in W \subseteq U \cap V$ such that $s_U|_W = s_V|_W$. An element $s_p \in \mathscr{F}_p$, which is called a germ, is an equivalence class $[s_U]$, and for $s \in \mathscr{F}(U)$, the germ given by s is denoted by $s|_p$.

Notation 20.1.2.

(1) For $s \in \mathscr{F}(U)$ and $p \in U$, $s|_p$ denotes the equivalent class it gives.
(2) For $s_p \in \mathscr{F}_p$, $s \in \mathscr{F}(U)$ denotes the section such that $s|_p = s_p$.

Definition 20.1.5 (morphisms on stalks)**.** Given a morphism of sheaves $\varphi\colon \mathscr{F} \to \mathscr{G}$, it induces a morphism of abelian groups $\varphi_p\colon \mathscr{F}_p \to \mathscr{G}_p$ as follows:

$$\begin{aligned} \varphi_p\colon \mathscr{F}_p &\to \mathscr{G}_p \\ s_p &\mapsto \varphi(s)|_p. \end{aligned}$$

Remark 20.1.4. It's neccessary to check the φ_p is well-defined since there are different choices s such that $s|_p = s_p$.

Proposition 20.1.1. Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism between sheaves. Then φ is an isomorphism if and only if the induced map $\varphi_p\colon \mathscr{F}_p \to \mathscr{G}_p$ is an isomorphism for every $p \in X$.

Proof. It's clear if φ is an isomorphism between sheaves, then it induces an isomorphism between stalks. Conversely, it suffices to show $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$ is an isomorphism for every open subset $U \subseteq X$.

(1) Injectivity: For $s, s' \in \mathscr{F}(U)$ such that $\varphi(s) = \varphi(s')$, by passing to stalks one has $\varphi_p(s|_p) = \varphi_p(s'|_p)$ for every $p \in U$, and thus $s|_p = s'|_p$ since φ_p is an isomorphism. By definition of stalks there exists an open subset $V_p \subseteq U$ containing p such that s agrees with s' on V_p. Then it gives an open covering $\{V_p\}$ of U, and by axiom (IV) one has $s = s'$ on U.
(2) Surjectivity: For $t \in \mathscr{G}(U)$, by passing to stalks there exists $s_p \in \mathscr{F}_p$ such that $\varphi_p(s_p) = t|_p$ for every $p \in U$ since φ_p is surjective. By definition of stalks there exists an open subset $V_p \subseteq U$ containing p and $s \in \mathscr{F}(V_p)$ such that $\varphi(s) = t$ on V_p. This gives a collection of sections defined on an open covering $\{V_p\}$ of U, and by injectivity we proved above one has these sections agree with each other on the intersections. Then by axiom (V) there exists a section $s \in \mathscr{F}(U)$ such that $\varphi(s) = t$.

$\square$

20.1.3. *Sheafification.* In Example 20.1.1, we come across a presheaf that is not a sheaf. To obtain a sheaf from a presheaf, we require a process known as sheafification. One approach to defining sheafification is through its universal property.

Definition 20.1.6 (sheafification)**.** Given a presheaf $\mathscr{F}$ there is a sheaf $\mathscr{F}^+$ and a morphism $\theta\colon \mathscr{F} \to \mathscr{F}^+$ with the property that for any sheaf $\mathscr{G}$ and any morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$ there is a unique morphism $\overline{\varphi}\colon \mathscr{F}^+ \to \mathscr{G}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\varphi"] \arrow[d, "\theta"] & \mathscr{G} \\ \mathscr{F}^+ \arrow[ur, "\overline{\varphi}"] & \end{tikzcd}$$

The universal property shows that if the sheafification exists, then it's unique up to a unique isomorphism. One way to give an explicit construction of sheafification is to glue stalks together in a suitable way. Let $\mathscr{F}^+(U)$ be a set of functions

$$f\colon U \to \coprod_{p\in U} \mathscr{F}_p$$

such that $f(p) \in \mathscr{F}_p$ and for every $p \in U$ there is an open subset $V_p \subseteq U$ containing p and $t \in \mathscr{F}(V_p)$ such that $t|_q = f(q)$ for all $q \in V_p$.

Proposition 20.1.2. $\mathscr{F}^+$ is the sheafication of $\mathscr{F}$.

Proof. Firstly let's show $\mathscr{F}^+$ is a sheaf: It's clear $\mathscr{F}^+$ is a presheaf, so it suffices to check conditions (IV) and (V) in the definition. Let $U \subseteq X$ be an open subset and $\{V_i\}$ be an open covering of U.

(1) If $s \in \mathscr{F}^+(U)$ such that $s|_{V_i} = 0$ for all i, then s must be zero: It suffices to show $s(p) = 0$ for all $p \in U$. For any $p \in U$, then there exists an open subset V_i contains p, hence $s(p) = s|_{V_i}(p) = 0$.

(2) Suppose there exists a collection of sections $\{s_i \in \mathscr{F}^+(V_i)\}_{i\in I}$ such that

$$s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j}$$

holds for all $i, j \in I$. Now we construct $s \in \mathscr{F}^+(U)$ as follows: For $p \in U$ and V_i containing p, we define $s(p) = s_i(p)$. This is well-defined since s_i agree on the intersections, so it remains to show $s \in \mathscr{F}^+(U)$. It's clear $s(p) \in \mathscr{F}_p$. For $p \in U$, there exists V_i containing p, and thus there exists $W_i \subseteq V_i$ containing p and $t \in \mathscr{F}(W_i)$ such that $t|_q = s_i(q) = s(q)$ for all $q \in V_p$.

There is a canonical morphism $\theta\colon \mathscr{F} \to \mathscr{F}^+$ as follows: For open subset $U \subseteq X$, and $s \in \mathscr{F}(U)$, $\theta(s)$ is defined by

$$\begin{aligned} \theta(s)\colon U &\to \coprod_{p\in U} \mathscr{F}_p \\ p &\mapsto s|_p. \end{aligned}$$

Note that if $\mathscr{F}$ is a sheaf, the canonical morphism $\theta\colon \mathscr{F} \to \mathscr{F}^+$ is an isomorphism.

(1) Injectivity: If $s \in \mathscr{F}(U)$ such that $s|_p = 0$ for all $p \in U$, then there exists an open covering $\{V_i\}_{i\in I}$ of U such that $s|_{V_i} = 0$, by axiom (IV) of sheaf one has $s = 0$.
(2) Surjectivity: For $f \in \mathscr{F}^+(U)$ and $p \in U$, there exists $p \in V_p \subseteq U$ and $t \in \mathscr{F}(V_p)$ such that $f(p) = t|_p$ by construction of $\mathscr{F}^+$. Then glue these sections together to get our desired s such that $\theta(s) = f$.

Finally let's show $\mathscr{F}^+$ statisfies the universal property of sheafification. A morphism of presheaves $\varphi\colon \mathscr{F} \to \mathscr{G}$ induces a map on stalks

$$\varphi_p\colon \mathscr{F}_p \to \mathscr{G}_p.$$

For $f \in \mathscr{F}^+(U)$, the composite of f with the map

$$\coprod_{p\in U} \varphi_p\colon \coprod_{p\in U} \mathscr{F}_p \to \coprod_{p\in U} \mathscr{G}_p$$

gives a map $\widetilde{\varphi}(f)\colon U \to \coprod_{p\in U} \mathscr{G}_p$, and in fact $\widetilde{\varphi}(f) \in \mathscr{G}^+(U)$: For $p \in U$, $\widetilde{\varphi}(f)(p) \in \mathscr{G}_p$ since $f(p) \in \mathscr{F}_p$ and $\varphi_p\colon \mathscr{F}_p \to \mathscr{G}_p$. If for all $q \in V_p$ we have $t|_q = f(q)$, then

$$\widetilde{\varphi}(f)(q) = \varphi_q(f(q)) = \varphi_q(t|_q) = \varphi(t)|_q.$$

Since $\mathscr{G}$ is a sheaf, the canonical morphism $\theta'\colon \mathscr{G} \to \mathscr{G}^+$ is an isomorphism, so we can define $\overline{\varphi}\colon = \theta'^{-1} \circ \widetilde{\varphi}$. Now let's show $\varphi = \overline{\varphi} \circ \theta = \theta'^{-1} \circ \widetilde{\varphi} \circ \theta$. It's easy to show they coincide on each stalk since $\varphi_p = \theta_p'^{-1} \circ \widetilde{\varphi}_p \circ \theta_p$, and thus $\varphi = \overline{\varphi} \circ \theta$ by Proposition 20.1.1. Furthermore, uniqueness follows from the fact that $\overline{\varphi}_p$ is uniquely determined by φ_p. □

Remark 20.1.5. From the construction, one can see the stalk of $\mathscr{F}^+$ at p is exactly $\mathscr{F}_p$.

Remark 20.1.6. The sheafification can be described in a more fancy language: Since we have sheaf of abelian groups on X as a category, denote it by $\underline{Ab}_X$, and presheaf is a full subcategory of $\underline{Ab}_X$, there is a natural inclusion functor ι from category of sheaf to category of presheaf. The sheafification is the adjoint functor of ι.

Example 20.1.2 (constant sheaf)**.** For an abelian group G, the associated constant sheaf $\underline{G}$ is the sheafication of the constant presheaf. By the construction of sheafification, $\underline{G}$ can be explicitly expressed as

$$\underline{G}(U) = \{\text{locally constant function } f\colon U \to G\}$$

20.1.4. *Exact sequence of sheaf.* Given a morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$ between sheaves of abelian groups, there are the following presheaves

$$U \mapsto \ker \varphi(U)$$
$$U \mapsto \operatorname{im} \varphi(U)$$
$$U \mapsto \operatorname{coker} \varphi(U),$$

since $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$ is a group homomorphism.

Proposition 20.1.3. Kernel of a morphism between sheaves is a sheaf.

Proof. Let $\{V_i\}_{i\in I}$ be an open covering of U.

(1) For $s \in \ker\varphi(U)$, if $s|_{V_i} = 0$, then $s = 0$ since s is also in $\mathscr{F}(U)$.
(2) If there exists $s_i \in \ker\varphi(V_i)$ such that $s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j}$, then they glue together to get $s \in \mathscr{F}(U)$. Note that
$$\varphi(U)(s)|_{V_i} = \varphi(V_i)(s|_{V_i}) = \varphi(V_i)(s_i) = 0$$
Then $s \in \ker\varphi(U)$.

□

But the image of morphism may not be a sheaf. Although we can prove the first requirement in the same way, the proof for the second requirement fails: If there exists $s_i \in \operatorname{im}\varphi(V_i)$, and we can glue them together to get a $s \in \mathscr{G}(U)$, but s may not be the image of some $t \in \mathscr{F}(U)$. The cokernel fails to be a sheaf for the same reason.

Definition 20.1.7 (image and cokernel)**.** Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism between sheaves of abelian groups. Then the image and cokernel of φ is defined to be the sheafification of the following presheaves
$$U \mapsto \operatorname{im}\varphi(U)$$
$$U \mapsto \operatorname{coker}\varphi(U)$$
respectively.

Definition 20.1.8 (exact)**.** For a sequence of sheaves:
$$\cdots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^{i} \xrightarrow{\varphi^{i}} \mathscr{F}^{i+1} \to \dots$$
It's called exact at $\mathscr{F}^i$, if $\ker\varphi^i = \operatorname{im}\varphi^{i-1}$. If a sequence is exact at everywhere, then it's an exact sequence of sheaves.

Definition 20.1.9 (short exact sequence)**.** An exact sequence of sheaves is called a short exact sequence if it looks like
$$0 \to \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{H} \to 0$$

Proposition 20.1.4. Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism between sheaves of abelian groups. Then for any $p \in X$, one has
$$(\ker\varphi)_p = \ker\varphi_p$$
$$(\operatorname{im}\varphi)_p = \operatorname{im}\varphi_p.$$

Proof. For (1). It's clear $(\ker\varphi)_p \subseteq \ker\varphi_p$. Conversely, if $s_p \in \ker\varphi_p$, then $\varphi_p(s_p) = 0 \in \mathscr{G}_p$. In other words, there exists an open subset U containing p and $s \in \mathscr{F}(U)$ such that $s|_p = s_p$ and $\varphi(s)|_p = 0$, which implies there is another open subset V containing p such that $\varphi(s)|_V = 0$. Hence $\varphi(s|_V) = 0$, that is, $s|_V \in \ker\varphi(V)$. Thus $s_p = (s|_V)|_p \in (\ker\varphi)_p$.

For (2). It's clear $(\operatorname{im}\varphi)_p \subseteq \operatorname{im}\varphi_p$ since the sheafication doesn't change stalk. Conversely, if $s_p \in \operatorname{im}\varphi_p$, then there exists $t_p \in \mathscr{F}_p$ such that $\varphi_p(t_p) = s_p$. Suppose $t \in \mathscr{F}(U)$ is a section of some open subset U containing p such that $t|_p = t_p$. Then $\varphi(t)|_p = \varphi_p(t_p) = s_p$. In other words, s_p is in the stalk of the image presheaf at p, but the sheafication doesn't change stalk, so we have $s_p \in (\operatorname{im}\varphi)_p$. □

Corollary 20.1.1. The sequence of sheaves

$$\cdots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^{i} \xrightarrow{\varphi^{i}} \mathscr{F}^{i+1} \to \cdots$$

is exact if and only if the sequence of abelian groups are exact

$$\cdots \to \mathscr{F}^{i-1}_p \xrightarrow{\varphi^{i-1}_p} \mathscr{F}^{i}_p \xrightarrow{\varphi^{i}_p} \mathscr{F}^{i+1}_p \to \cdots$$

for all $p \in X$.

Corollary 20.1.2. The the sequence of sheaves

$$0 \to \mathscr{F} \to \mathscr{G}$$

is exact if and only if for any open subset U, the following sequence of abelian groups is exact

$$0 \to \mathscr{F}(U) \to \mathscr{G}(U).$$

Method one. For any open subset $U \subseteq X$, one has

$$\varphi(U) \colon \mathscr{F}(U) \to \mathscr{G}(U)$$

is injective, since by definition we have for any open subset $U \subseteq X$, $\ker\varphi(U) = 0$, that is injectivity. □

Method two. Or from another point of view, for each $p \in U$, we have

$$\varphi_p \colon \mathscr{F}_p \to \mathscr{G}_p$$

is injective. That is $\ker\varphi_p = 0$. So we obtain $(\ker\varphi(U))_p = 0$ for all $p \in U$. But for a section $s \in \mathscr{F}(U)$ if we have $s|_p = 0$, then we must have $s = 0$. So we obtain $\ker\varphi(U) = 0$. □

Example 20.1.3 (exponential sequence)**.** Let X be a complex manifold and $\mathcal{O}_X$ be its holomorphic function sheaf. Then

$$0 \to 2\pi\sqrt{-1}\underline{\mathbb{Z}} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

is an exact sequence of sheaves, called exponential sequence.

Proof. The difficulty is to show exp is surjective on stalks at $p \in X$. That is we need to construct logarithms of functions $g \in \mathcal{O}_X^*(U)$ for U, a neighborhood of p. We may choose U is simply-connected, then define

$$\log g(q) = \log g(p) + \int_{\gamma_q} \frac{\mathrm{d}g}{g}$$

for $q \in U$, where γ_q is a path from p to q in U, and the definition is independent of the choice of γ_q since U is simply-connected. □

Remark 20.1.7. In fact, U is simply-connected is crucial for constructing logarithm. If we consider $X = \mathbb{C}$ and $U = \mathbb{C}\setminus\{0\}$, then

$$\exp\colon \mathcal{O}_X(U) \to \mathcal{O}_X^*(U)$$

cannot be surjective.

20.2. **Derived functor formulation of Sheaf Cohomology.** The category $\underline{Ab}_X$: sheaves of abelian groups on X. In this section we will introduce sheaf cohomology by considering it as a derived functor.

Given an exact sequence of sheaf as follows

$$0 \to \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}''.$$

By taking its section over open subset U, we obtain a sequence of abelian groups

$$0 \to \mathscr{F}'(U) \xrightarrow{\phi(U)} \mathscr{F}(U) \xrightarrow{\psi(U)} \mathscr{F}''(U).$$

Above sequence is not only exact at $\mathscr{F}'(U)$, but also is exact at $\mathscr{F}(U)$. In other words, the functor given by taking section over open subset is a left exact functor.

(1) Firstly let's show $\ker\psi(U) \supseteq \operatorname{im}\phi(U)$. For $s \in \mathscr{F}'(U)$, if we want to show $\psi\circ\phi(s) = 0$, it suffices to show $(\psi\circ\phi(s))|_p = 0$ for all $p \in U$ since $\mathscr{F}''$ is a sheaf. For any $p \in U$, by considering stalk at p we obtain an exact sequence of abelian groups

$$0 \to \mathscr{F}'_p \xrightarrow{\phi_p} \mathscr{F}_p \xrightarrow{\psi_p} \mathscr{F}''_p.$$

Then we obtain $\psi_p\circ\phi_p(s|_p) = 0$, which implies $(\psi\circ\phi(s))|_p = 0$.

(2) Conversely, Given $s \in \ker\psi(U)$, we have $s|_p \in \ker\psi_p$ for any $p \in U$. By exactness of stalks, there exists $t_p \in \mathscr{F}'_p$ such that $\phi_p(t_p) = s|_p$. Thus there exists an open subset V_i containing p and $t_i \in \mathscr{F}'(V_i)$ such that $\phi(t_i) = s|_{V_i}$. Now it suffices to show these t_i can be glued together to obtain $t \in \mathscr{F}(U)$, and since $\mathscr{F}$ is a sheaf, it suffices to check these t_i agree on intersections $V_i\cap V_j$. Note that $\phi(t_i - t_j|_{V_i\cap V_i}) = s|_{V_i\cap V_j} - s|_{V_i\cap V_j} = 0$, then these t_i agree on intersections since ϕ is injective.

Remark 20.2.1. From above argument, we can see that

$$0 \to \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}''$$

is exact if and only if for any open subset $U \subseteq X$

$$0 \to \mathscr{F}'(U) \xrightarrow{\phi(U)} \mathscr{F}(U) \xrightarrow{\psi(U)} \mathscr{F}''(U)$$

is exact.

In homological algebra, we always consider the derived functor of a left or right-exact functor. In particular, the functor of taking global section is a left exact functor, and its right derived functor defines the cohomology of a sheaf. Before we come into the definition of derived functor, firstly let's define the injective resolution of a sheaf.

Definition 20.2.1 (injective)**.** A sheaf $\mathcal{I}$ is injective if $\mathrm{Hom}(-, \mathcal{I})$ is an exact functor.

Definition 20.2.2 (injective resolution)**.** Let $\mathscr{F}$ be a sheaf. An injective resolution of $\mathscr{F}$ is an exact sequence

$$0 \to \mathscr{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \dots$$

where $\mathcal{I}^i$ are injective for all i.

Theorem 20.2.1. Every sheaf admits an injective resolution.

Theorem 20.2.2. Let $\mathscr{F} \to \mathcal{I}^\bullet$ and $\mathscr{G} \to \mathcal{G}^\bullet$ are two resolutions and $\phi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Then there exists a morphism $\widetilde{\phi}\colon \mathcal{I}^\bullet \to \mathcal{G}^\bullet$ which lifts ϕ, which is unique up to homotopy.

Definition 20.2.3 (sheaf cohomology)**.** Let $\mathscr{F}$ be a sheaf of abelian groups. Then

$$H^p(X, \mathscr{F}) := H^p(\mathcal{I}^\bullet(X)).$$

Remark 20.2.2. The Theorem 20.2.2 shows that the definition of sheaf cohomology is independent of the choice of injective resolution.

Example 20.2.1. By definition, the 0-th cohomology is exact the global section

$$H^0(X, \mathscr{F}) := \ker\left\{\mathcal{I}^0(X) \to \mathcal{I}^1(X)\right\}.$$

Thus $H^0(X, \mathscr{F}) = \mathscr{F}(X)$, the global sections of sheaf.

Example 20.2.2. If $\mathscr{F}$ is a injective sheaf, then $H^i(X, \mathscr{F}) = 0$ for all $i > 0$, since the sheaf cohomology of injective sheaf can be computed by using the following special injective resolution

$$0 \to \mathscr{F} \xrightarrow{\mathrm{id}} \mathscr{F} \to 0 \to 0 \to \dots$$

Theorem 20.2.3 (zig-zag)**.** If

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$

is a short sequence of sheaves, then there is an induced long exact sequence of abelian groups

$$0 \to H^0(X, \mathscr{F}) \to H^0(X, \mathscr{G}) \to H^0(X, \mathscr{H}) \to H^1(X, \mathscr{F}) \to H^1(X, \mathscr{G}) \to \dots$$

Definition 20.2.4 (direct image)**.** Let $f\colon X \to Y$ be continuous map between topological spaces and $\mathscr{F}$ be a sheaf of abelian groups on X. The direct image of $\mathscr{F}$, denoted by $f_*\mathscr{F}$, is a sheaf on Y defined by

$$f_*\mathscr{F}(U) := \mathscr{F}(f^{-1}(U)).$$

Proposition 20.2.1. $f_*\colon \underline{Ab}_X \to \underline{Ab}_Y$ is a left exact functor.

Proof. Given an exact sequence of sheaves on X

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''.$$

Then we need to show

$$0 \to f_*\mathscr{F}' \to f_*\mathscr{F} \to f_*\mathscr{F}''$$

is also an exact sequence of sheaves on Y. By Remark 20.2.1 it suffices to show that for any open subset $V \subseteq Y$, we have the following exact sequence

$$0 \to f_*\mathscr{F}'(V) \to f_*\mathscr{F}(V) \to f_*\mathscr{F}''(V),$$

and that's exactly

$$0 \to \mathscr{F}'(f^{-1}(V)) \to \mathscr{F}(f^{-1}(V)) \to \mathscr{F}''(f^{-1}(V)).$$

Since f is continuous, then $f^{-1}(V)$ is an open subset in X, and thus above sequence of abelian is exact since $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ is exact. □

20.3. **Acyclic resolution.** In practice it may be difficult for us to choose an injective resolution, so we usual other resolutions to compute sheaf cohomology.

Definition 20.3.1 (acyclic sheaf)**.** A sheaf $\mathscr{F}$ is acyclic if $H^i(X, \mathscr{F}) = 0$ for all $i > 0$.

Example 20.3.1. Injective sheaf is acyclic.

Definition 20.3.2 (acyclic resolution)**.** Let $\mathscr{F}$ be a sheaf. An acyclic resolution of $\mathscr{F}$ is an exact sequence

$$0 \to \mathscr{F} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \mathcal{A}^2 \to \dots$$

where $\mathcal{A}^i$ is acyclic for all i.

Proposition 20.3.1. The cohomology of sheaf $\mathscr{F}$ can be computed using acyclic resolution.

In fact, it's a quite homological techniques, called dimension shifting, so we will state this technique in language of homological algebra. Let's see a baby version of it.

Example 20.3.2. Let $\mathcal{F}$ be a left exact functor and $0 \to A \to M_1 \to B \to 0$ be an exact sequence with M_1 is $\mathcal{F}$-acyclic. Then $R^{i+1}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$ for $i > 0$, and $R^1\mathcal{F}(A)$ is the cokernel of $\mathcal{F}(M_1) \to \mathcal{F}(B)$.

Proof. By considering the long exact sequence induced by $0 \to A \to M^1 \to B \to 0$, one has

$$R^i\mathcal{F}(M^1) \to R^i\mathcal{F}(B) \to R^{i+1}\mathcal{F}(A) \to R^{i+1}\mathcal{F}(M^1)$$

(1) If $i > 0$, then $R^i\mathcal{F}(M^1) = R^{i+1}\mathcal{F}(M^1) = 0$ since M^1 is $\mathcal{F}$-acyclic, and thus $R^{i+1}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$ for $i > 0$.

(2) If $i = 0$, then
$$0 \to \mathcal{F}(M^1) \to \mathcal{F}(B) \to R^1\mathcal{F}(A) \to 0$$
implies $R^1\mathcal{F}(A) = \operatorname{coker}\{\mathcal{F}(M^1) \to \mathcal{F}(B)\}$.

□

Now let's prove dimension shifting in a general setting.

Lemma 20.3.1 (dimension shifting)**.** If
$$0 \to A \to M^1 \to M^2 \to \cdots \to M^m \to B \to 0$$
is exact with M^i is $\mathcal{F}$-acyclic, then $R^{i+m}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$ for $i > 0$, and $R^m\mathcal{F}(A)$ is the cokernel of $\mathcal{F}(M^m) \to \mathcal{F}(B)$.

Proof. Prove it by induction on m. For $m = 1$, we already see it in Example 20.3.2. Assume it holds for $m < k$, then for $m = k$, let's split $0 \to A \to M^1 \to M^2 \to \cdots \to M^k \xrightarrow{\mathrm{d}_k} B \to 0$ into two exact sequences
$$0 \to A \to M^1 \to M^2 \to \cdots \to M^{k-1} \to \ker \mathrm{d}_k \to 0$$
$$0 \to \ker \mathrm{d}_k \to M^k \xrightarrow{\mathrm{d}_k} B \to 0.$$
Then by induction hypothesis, for $i > 0$ we have
$$R^{i+k-1}\mathcal{F}(A) \cong R^i\mathcal{F}(\ker \mathrm{d}_k)$$
$$R^{i+1}\mathcal{F}(\ker \mathrm{d}_k) \cong R^i\mathcal{F}(B).$$
Combine these two isomorphisms together we obtain $R^{i+k}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$ for $i > 0$, as desired. For $i = 0$, it suffices to let $i = 1$ in $R^{i+k-1}\mathcal{F}(A) \cong R^i\mathcal{F}(\ker \mathrm{d}_k)$, then we obtain
$$R^k\mathcal{F}(A) = R^1\mathcal{F}(\ker \mathrm{d}_k) = \operatorname{coker}\{\mathcal{F}(M^k) \to \mathcal{F}(B)\}.$$
This completes the proof. □

Corollary 20.3.1. If $0 \to A \to M^\bullet$ is a $\mathcal{F}$-acyclic resolution, then $R^i\mathcal{F}(A) = H^i(\mathcal{F}(M^\bullet))$.

Proof. Truncate the resolution as
$$0 \to A \to M^0 \to M^1 \to \ldots M^{i-1} \to B \to 0$$
$$0 \to B \to M^i \to M^{i+1} \to \ldots$$
Since we already have $R^i\mathcal{F}(A) = \operatorname{coker}\{\mathcal{F}(M^{i-1}) \to \mathcal{F}(B)\}$, and $\mathcal{F}$ is left exact, one has
$$\mathcal{F}(B) = \ker\{\mathcal{F}(M^i) \to \mathcal{F}(M^{i+1})\}.$$
Thus we obtain
$$R^i\mathcal{F}(A) = \operatorname{coker}\{\mathcal{F}(M^{i-1}) \to \ker\{\mathcal{F}(M^i) \to \mathcal{F}(M^{i+1})\}\} = H^i(\mathcal{F}(M^\bullet)).$$

□

20.4. **Examples about acyclic sheaf.**

20.4.1. *Flabby sheaf.* First kind of acyclic sheaf is flabby[17] sheaf.

Definition 20.4.1 (flabby)**.** A sheaf $\mathscr{F}$ is flabby if for all open $U \subseteq V$, the restriction map $\mathscr{F}(V) \to \mathscr{F}(U)$ is surjective.

Now let's see some examples about flabby sheaves.

Example 20.4.1. A constant sheaf on an irreducible topological space is flabby.

Proof. Note that the constant presheaf on a irreducible topological space is a sheaf in fact, and it's easy to see this constant presheaf is flabby. □

In particular, we have

Example 20.4.2. Let X be an algebraic variety. Then constant sheaf $\mathbb{Z}_X$ is flabby.

Example 20.4.3. If $\mathscr{F}$ is a flabby sheaf on X, and $f\colon X \to Y$ is a continuous map, then $f_*\mathscr{F}$ is a flabby sheaf on Y.

Proof. For $V \subseteq W$ in Y, it suffices to show $f_*\mathscr{F}(W) \to f_*\mathscr{F}(V)$ is surjective, and that's

$$\mathscr{F}(f^{-1}W) \to \mathscr{F}(f^{-1}V)$$

it's surjective since $\mathscr{F}$ is flabby. □

Example 20.4.4. An injective sheaf is flabby.

Proof. Let $\mathcal{I}$ be an injective sheaf and $V \subseteq U$ be open subsets. Now we define sheaf $\underline{\mathbb{Z}}_U$ on X by

$$\underline{\mathbb{Z}}_U := \begin{cases} \underline{\mathbb{Z}}(W) & W \subseteq U \\ 0 & \text{otherwise} \end{cases}$$

where $\underline{\mathbb{Z}}$ is constant sheaf valued in $\mathbb{Z}$, and similarly we define sheaf $\underline{\mathbb{Z}}_V$. By construction one has $\underline{\mathbb{Z}}_U(W) = \underline{\mathbb{Z}}_V(W)$ unless $W \subseteq U$ and $W \not\subseteq V$. Thus we obtain an exact sequence

$$0 \to \underline{\mathbb{Z}}_V \to \underline{\mathbb{Z}}_U.$$

Applying the functor $\mathrm{Hom}(-, \mathcal{I})$, which is exact, we obtain an exact sequence

$$\mathrm{Hom}(\underline{\mathbb{Z}}_U, \mathcal{I}) \to \mathrm{Hom}(\underline{\mathbb{Z}}_V, \mathcal{I}) \to 0.$$

Now let's explain why we need such a weird sheaf $\underline{\mathbb{Z}}_U$. In fact, we will prove $\mathrm{Hom}(\underline{\mathbb{Z}}_U, \mathcal{I}) = \mathcal{I}(U)$. Indeed since $\varphi\colon \underline{\mathbb{Z}}_U \to \mathcal{I}$ is a sheaf morphism. Then if $W \not\subseteq U$, then $\varphi(U)$ must be zero. If $W = U$, then the group of sections of $\underline{\mathbb{Z}}_U(U)$ over any connected component is simply $\mathbb{Z}$ and hence $\varphi(U)$ on this connected component is determined by the image of $1 \in \mathbb{Z}$. Thus $\varphi(U)$ can be thought of an element of $\mathcal{I}(U)$. Now on any proper open subset of U, φ is determined by restriction maps. Hence $\mathrm{Hom}(\underline{\mathbb{Z}}_U, \mathcal{I}) = \mathcal{I}(U)$, as desired.

[17] Some authors also call this flasque.

The same argument shows $\mathrm{Hom}(\mathbb{Z}_U, \mathcal{I}) = \mathcal{I}(V)$, and thus we obtain an exact sequence

$$\mathcal{I}(U) \to \mathcal{I}(V) \to 0,$$

which shows $\mathcal{I}$ is flabby. $\square$

Our goal is to prove a flabby sheaf is acyclic, but we still need some property of flabby sheaves.

Proposition 20.4.1. If $0 \to \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and $\mathscr{F}'$ is flabby, then for any open subset U, the sequence

$$0 \to \mathscr{F}'(U) \xrightarrow{\phi(U)} \mathscr{F}(U) \xrightarrow{\psi(U)} \mathscr{F}''(U) \to 0$$

is exact.

Proof. It suffices to show $\mathscr{F}(U) \to \mathscr{F}''(U) \to 0$ is exact. Here we only gives a sketch of the proof. Since we have exact sequence on stalks for each $p \in U$ as follows

$$0 \to \mathscr{F}'_p \xrightarrow{\phi_p} \mathscr{F}_p \xrightarrow{\psi_p} \mathscr{F}''_p \to 0$$

Then for each $s \in \mathscr{F}''(U)$, there exists $t_p \in \mathscr{F}_p$ such that $\psi_p(t_p) = s|_p$, so there exists open subset $V_p \subseteq U$ containing p and $t \in \mathscr{F}(V_p)$ such that $\psi(t) = s|_{V_p}$. If we can glue these t together then we get a section in $\mathscr{F}(U)$ and is mapped to s, which completes the proof. However, they may not equal on the intersection. But things are not too bad, consider another point q and $t' \in \mathscr{F}(V_q)$ such that $\psi(t') = s|_{V_q}$, $(t'-t)|_{V_p \cap V_q} \in \ker \psi(V_p \cap V_q) = \mathrm{im}\, \phi(V_p \cap V_q)$. So there exists $t'' \in \mathscr{F}'(V_p \cap V_q)$ such that

$$\phi(t'') = (t'-t)|_{V_p \cap V_q}$$

Now since $\mathscr{F}'$ is flabby, then there exists $t''' \in \mathscr{F}(V_p)$ such that $t'''|_{V_p \cap V_q} = t''$. And consider $t + \phi(t''') \in \mathscr{F}(V_p)$, which will coincide with t' on $V_p \cap V_q$. After above corrections, we can glue t after correction together. $\square$

Proposition 20.4.2. If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ and $\mathscr{F}$ are flabby, then $\mathscr{F}''$ is flabby.

Proof. Take $V \subseteq U$ and consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathscr{F}'(U) \arrow[r] \arrow[d] & \mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{F}''(U) \arrow[r] \arrow[d, dashed] & 0 \arrow[d] \\
0 \arrow[r] & \mathscr{F}'(V) \arrow[r] & \mathscr{F}(V) \arrow[r] & \mathscr{F}''(V) \arrow[r] & 0
\end{tikzcd}$$

Then the desired result follows from five lemma. $\square$

Proposition 20.4.3. A flabby sheaf is acyclic.

Proof. Let $\mathscr{F}$ be a flabby sheaf. Since there are enough injective objects in the category of sheaf of abelian groups, there is an exact sequence

$$0 \to \mathscr{F} \to \mathcal{I} \to \mathscr{Q} \to 0$$

with $\mathcal{I}$ is injective. By Example 20.4.4 we have $\mathcal{I}$ is flabby, and thus by Proposition 20.4.2 we have $\mathscr{Q}$ is flabby. Consider the long exact sequence induced from above short exact sequence

$$\mathscr{F}(X) \to \mathcal{I}(X) \to \mathscr{Q}(X) \to H^1(X, \mathscr{F}) \to H^1(X, \mathcal{I}) \to \dots$$

Note that $H^1(X, \mathcal{I}) = 0$ since $\mathcal{I}$ is injective, and thus acyclic. Then $H^1(X, \mathscr{F}) = \operatorname{coker}\{\mathcal{I}(X) \to \mathscr{Q}(X)\}$. But Proposition 20.4.1 shows that $\mathcal{I}(X) \to \mathscr{Q}(X)$ is surjective since $\mathscr{F}$ is flabby, so $H^1(X, \mathscr{F}) = 0$.

Now let's prove $H^k(X, \mathscr{F}) = 0$ for $k > 0$ by induction on k, and above argument shows it's true for $k = 1$. Assume this holds for $k < n$, and consider

$$\cdots \to H^{n-1}(X, \mathscr{Q}) \to H^n(X, \mathscr{F}) \to H^n(X, \mathcal{I}) \to H^n(X, \mathscr{Q}) \to \dots$$

By induction hypothesis, we can reduce above sequence to

$$\cdots \to 0 \to H^n(X, \mathscr{F}) \to 0 \to H^n(X, \mathscr{Q}) \to \dots$$

which implies $H^n(X, \mathscr{F}) = 0$. This completes the proof. $\square$

20.4.2. *Soft sheaf.* The second kind of acyclic sheaves is called soft sheaves, which is quit similar to flabby.

Definition 20.4.2 (soft)**.** A sheaf $\mathscr{F}$ over X is soft if for any closed subset $S \subseteq X$ the restriction map $\mathscr{F}(X) \to \mathscr{F}(S)$ is surjective.

Remark 20.4.1. For closed subset S, the section over it is defined by

$$\mathscr{F}(S) := \varinjlim_{S \subseteq U} \mathscr{F}(U)$$

Parallel to Proposition 20.4.1 and Proposition 20.4.2, soft sheaf has the following properties:

Proposition 20.4.4. If $0 \to \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and $\mathscr{F}'$ is soft, then the following sequence

$$0 \to \mathscr{F}'(X) \xrightarrow{\phi(X)} \mathscr{F}(X) \xrightarrow{\psi(X)} \mathscr{F}''(X) \to 0$$

is exact.

Proposition 20.4.5. If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ and $\mathscr{F}$ are soft, then $\mathscr{F}''$ is soft.

Proposition 20.4.6. A soft sheaf is acyclic.

So you may wonder, what's the difference between flabby and soft since the definitions are quite similar, and both of them are acyclic. Clearly by definition of sections over a closed subset, we know that every flabby sheaf is soft, but converse fails

Example 20.4.5. The sheaf of smooth functions on a smooth manifold is soft but not flabby.

Lemma 20.4.1. If $\mathscr{M}$ is a sheaf of modules over a soft sheaf of rings $\mathscr{R}$, then $\mathscr{M}$ is a soft sheaf.

Proof. Let $s \in \mathscr{M}(K)$ for some closed subset $K \subseteq X$. Then s extends to some open neighborhood U of K. Let $\rho \in \mathscr{R}(K \cup (X \backslash U))$ be defined by

$$\rho = \begin{cases} 1, & \text{on } K \\ 0, & \text{on } X \backslash U \end{cases}$$

Since $\mathscr{R}$ is soft, then ρ extends to a section over X, then $\rho \circ s$ is the desired extension of s. □

20.4.3. *Fine sheaf.* Another important kind of acyclic sheaves, which behaves like sheaf of differential forms Ω_X^k is called fine sheaf. Recall what is a partition of unity: Let $U = \{U_i\}_{i\in I}$ be a locally finite open covering of topological space X. A partition of unity subordinate to U is a collection of continuous functions $f_i \colon U_i \to [0,1]$ for each $i \in I$ such that its support lies in U_i, and for any $x \in X$

$$\sum_{i\in I} f_i(x) = 1.$$

Definition 20.4.3 (fine sheaf)**.** A fine sheaf $\mathscr{F}$ on X is a sheaf of $\mathscr{A}$-modules, where $\mathscr{A}$ is a sheaf of rings such that for every locally finite open covering $\{U_i\}_{i\in I}$ of X, there is a partition of unity

$$\sum_{i\in I} \rho_i = 1$$

where $\rho_i \in \mathscr{A}(X)$ and $\operatorname{supp}(\rho_i) \subseteq U_i$.

Remark 20.4.2. For a sheaf $\mathscr{F}$ on X and a section $s \in \mathscr{F}(X)$, its support is defined as

$$\operatorname{supp}(s) := \overline{\{x \in X : s|_x \neq 0\}}.$$

Proposition 20.4.7. A fine sheaf is acyclic.

Proof. Let $\mathscr{F}$ be a sheaf of $\mathscr{A}$-modules and a fine sheaf. And choose a injective resolution

$$0 \to \mathscr{F} \xrightarrow{\mathrm{d}} \mathcal{I}^0 \xrightarrow{\mathrm{d}} \mathcal{I}^0 \xrightarrow{\mathrm{d}} \mathcal{I}^1 \xrightarrow{\mathrm{d}} \dots$$

such that $\mathcal{I}^i$ are injective sheaves of $\mathscr{A}$-modules. Let $s \in \mathcal{I}^p(X)$ such that $\mathrm{d}s = 0$. Then by exactness of injective resolution we have X is covered by open subsets U_i such that for each i there is an element $t_i \in \mathcal{I}^{p-1}(U_i)$ such that $\mathrm{d}t_i = s|_{U_i}$. By passing to a refinement we may assume that the cover $\{U_i\}$ is locally finite. Let $\{\rho_i\}$ be a partition of unity subordinate to $\{U_i\}$. Then we have $t = \sum \rho_i t_i \in \mathscr{I}^{p-1}(X)$ such that $\mathrm{d}t = s$. This completes the proof. □

Example 20.4.6. Let M be a smooth manifold and $\pi\colon E \to M$ be a vector bundle. Then the sheaf of smooth sections of E is a $C^\infty(M)$-module sheaf, which is a fine sheaf. In particular, the sheaf of tangent bundle, sheaf of differential forms Ω_M and k-forms Ω_M^k are fine sheaves.

Remark 20.4.3. As a consequence, it's meaningless to compute cohomology of sheaf of differential k-forms, or any other vector bundle over a smooth manifold. But in complex version, something interesting happens: Let $(X, \mathcal{O}_X)$ be a complex manifold and $\pi\colon E \to X$ be a holomorphic vector bundle. Then the sheaf of holomorphic sections of E is not a fine sheaf since there is no partition of unity may not be holomorphic, so the cohomology of holomorphic vector bundle is not trivial, and that's what Dolbeault cohomology computes.

For fine sheaf and soft sheaf, we have

Lemma 20.4.2. Fine sheaf is soft.

Proof. Let $\mathscr{F}$ be a fine sheaf, $S \subseteq X$ closed and $s \in \mathscr{F}(S)$. Let $\{U_i\}$ be an open covering of S and $s_i \in \mathscr{F}(U_i)$ such that

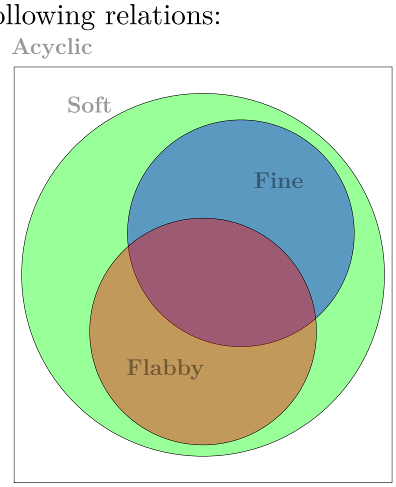
$$s_i|_{S\cap U_i} = s|_{S\cap U_i}.$$

Let $U_0 = X - S$, and $s_0 = 0$. Then $\{U_i\}\coprod\{U_0\}$ is an open covering of X. Without lose of generality, we assume this open covering is locally finite and choose a partition of unity $\{\rho_i\}$ subordinate to it. Then

$$\overline{s} := \sum_i \rho_i(s_i)$$

is a section in $\mathscr{F}(X)$ which extends s. $\square$

Remark 20.4.4. Until now, we have shown that soft, fine and flabby sheaves are acyclic. Lemma 20.4.2 shows fine sheaf is soft, and by definition a flabby sheaf is soft. The Example 20.4.5 shows that soft sheaf may not be flabby, and constant sheaf on an irreducible space is flabby but not fine. In a summary, we have the following relations:



20.5. **Proof of de Rham theorem using sheaf cohomology.** As we already know, for constant sheaf $\underline{\mathbb{R}}$ over a smooth manifold M, we have the following fine resolution

$$0 \to \underline{\mathbb{R}} \xrightarrow{i} \Omega^0_M \xrightarrow{\mathrm{d}} \Omega^1_M \xrightarrow{\mathrm{d}} \Omega^2_M \xrightarrow{\mathrm{d}} \dots$$

And de Rham cohomology computes the sheaf cohomology of $\underline{\mathbb{R}}$. de Rham theorem implies that de Rham cohomology equals to the singular cohomology with real coefficient. So if we can give constant sheaf another resolution using singular cochains, we may derive the de Rham cohomology.

We state this in a general setting: Let X be a topological manifold, and a constant sheaf $\underline{G}$ over X, where G is an abelian group. Let $S^p(U, G)$ be the group of singular cochains in U with coefficients in G, and let δ denote the coboundary operator.

Let $\mathscr{S}^p(G)$ be the sheaf over X generated by the presheaf $U \mapsto S^p(U, G)$, with induced differential mapping $\mathscr{S}^p(G) \xrightarrow{\delta} \mathscr{S}^{p+1}(G)$.

Similar to Poincaré lemma, we have for a unit ball U in Euclidean space, we have the following sequence

$$\dots \to S^{p-1}(U, G) \xrightarrow{\delta} S^p(U, G) \xrightarrow{\delta} S^{p+1}(U, G) \to \dots$$

is exact. So we have the following resolution of the constant sheaf $\underline{G}$

$$0 \to \underline{G} \to \mathscr{S}^0(G) \xrightarrow{\delta} \mathscr{S}^1(G) \xrightarrow{\delta} \mathscr{S}^2(G) \to \dots$$

Remark 20.5.1. If M is a smooth manifold, then we can consider smooth chains, that is $f\colon \Delta^p \to U$, where f is a smooth function. The corresponding results above still hold, and we have a resolution by smooth cochains with coefficients in G:

$$0 \to \underline{G} \to \mathscr{S}^\bullet_\infty(G)$$

So if we choose $G = \mathbb{R}$, then it suffices to show $0 \to \underline{\mathbb{R}} \to \mathscr{S}^\bullet_\infty(\mathbb{R})$ is an acyclic resolution, then we obtain de Rham theorem.

First, note that $\mathscr{S}^p_\infty$ is a $\mathscr{S}^0_\infty$-module, given by cup product on open subsets. Then by Lemma 20.4.1 and the fact $\mathscr{S}^0_\infty$ is soft we know that it's a soft resolution. This completes the proof.

20.6. **Hypercohomology.** In homological algebra, the hypercohomology is a generalization of cohomology functor which takes as input not objects in abelian category but instead chain complexes of objects.

One of the motivations for hypercohomology is to generalize the zig-zag lemma, that is, the short exact sequence of sheaves induces a long exact sequence of cohomology groups. It turns out hypercohomology gives techniques for constructing a similar cohomological associated long exact sequence from an arbitrary long exact sequence

$$0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \dots \to \mathscr{F}_k \to 0$$

Now let's give the definition of hypercohomology: Let $\mathscr{F}^\bullet\colon \dots \to \mathscr{F}^{i-1} \to \mathscr{F}^i \to \mathscr{F}^{i+1} \to \cdots$ be a complex of sheaves of abelian groups, which is bounded from below, that is, $\mathscr{F}^n = 0$ for $n \ll 0$. Then $\mathscr{F}^\bullet$ admits an injective resolution $\mathscr{F}^\bullet \to \mathcal{I}^\bullet$. In other words,

$$\begin{tikzcd} \dots \arrow[r] & \mathscr{F}^{i-1} \arrow[r] \arrow[d] & \mathscr{F}^i \arrow[r] \arrow[d] & \mathscr{F}^{i+1} \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & \mathcal{I}^{i-1} \arrow[r] & \mathcal{I}^i \arrow[r] & \mathcal{I}^{i+1} \arrow[r] & \dots \end{tikzcd}$$

such that

(1) All $\mathcal{I}^i$ are injective sheaves.
(2) The induced homomorphism $H^i(\mathscr{F}^\bullet) \to H^i(\mathcal{I}^\bullet)$ is an isomorphism.

The hypercohomology of $\mathscr{F}^\bullet$ is defined by

$$\mathbb{H}^i(X, \mathscr{F}^\bullet) := H^i(\Gamma(X, \mathcal{I}^\bullet))$$

Definition 20.6.1. For a sheaf $\mathscr{F}$, $\mathscr{F}^\bullet[n]$ is a sheaf of complex defined by

$$(\mathscr{F}^\bullet[n])^i = \begin{cases} \mathscr{F} & i = n \\ 0 & \text{otherwise.} \end{cases}$$

Example 20.6.1. Let $\mathscr{F}$ be a sheaf and $0 \to \mathscr{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \dots$ be an injective resolution of $\mathscr{F}$. Then

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\ 0 \arrow[r] & \mathcal{I}^0 \arrow[r] & \mathcal{I}^1 \arrow[r] & \mathcal{I}^2 \arrow[r] & \dots \end{tikzcd}$$

is an injective resolution of $\mathscr{F}^\bullet[0]$. Indeed, $\mathcal{I}^i$ are injective for all $i \geq 0$, and

$$H^i(\mathcal{I}^\bullet) = \begin{cases} \mathscr{F}, & n = 0 \\ 0, & \text{otherwise} \end{cases} = H^i(\mathscr{F}^\bullet[0])$$

So by definition of hypercohomology, we have $\mathbb{H}^i(X, \mathscr{F}^\bullet[0]) = H^i(\Gamma(X, \mathcal{I}^\bullet)) = H^i(X, \mathscr{F}^\bullet)$. In general, one has

$$\mathbb{H}^i(X, \mathscr{F}^\bullet[n]) \cong H^{i+n}(X, \mathscr{F}).$$

Theorem 20.6.1 (zig-zag)**.** Let $0 \to \mathscr{F}^\bullet \to \mathscr{G}^\bullet \to \mathscr{H}^\bullet \to 0$ be a short exact sequence of complexes of sheaves which are bounded from below. Then there is an induced long exact sequence

$$\dots \to \mathbb{H}^{i-1}(X, \mathscr{H}^\bullet) \to \mathbb{H}^i(X, \mathscr{F}^\bullet) \to \mathbb{H}^i(X, \mathscr{G}^\bullet) \to \mathbb{H}^i(X, \mathscr{H}^\bullet) \to \mathbb{H}^{i+1}(X, \mathscr{F}^\bullet) \to \dots$$

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