ČECH COHOMOLOGY

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Contents

1. Čech-de Rham Complex

1.1. **The Generalized Mayer-Vietoris Principle.** Mayer-Vietoris sequence allows us to compute the cohomology of the whole space M using an open covering $\mathfrak{U}$ consisting of two open subsets U, V. And in fact this argument can be generalized to countably many open subsets.

To make this generalization, we first reformulate the Mayer-Vietoris sequence in language of double complex $C^*(\mathfrak{U}, \Omega^*) = \bigoplus K^{p,q} = \bigoplus C^p(\mathfrak{U}, \Omega^q)$, where

$$\begin{aligned} K^{0,q} &= \Omega^q(U) \oplus \Omega^q(V) \\ K^{1,q} &= \Omega^q(U \cap V) \\ K^{2,q} &= 0, \quad p \geq 2 \end{aligned}$$

Remark 1.1.1. If you regard $\mathfrak{U}$ as a open covering consisting of countably many open subsets as the following way

$$\mathfrak{U} = \{U, V, \varnothing, \varnothing, \dots\}$$

And you can guess that we can generalize above double complex by considering the intersection of more open subsets, since here intersection of any three or more open subsets must be empty.

Note that there is two differential operators on double complex $C^*(\mathfrak{U}, \Omega^*)$, d in the vertical direction and difference operator δ in horizontal direction. Clearly d, δ commute with each other.

It's necessary for us to consider a more general double complex, since it's crucial ingredient of spectral sequences we will discuss later. For a double complex $K^{*,*}$ it's a complex with two differential operators d, δ in vertical and horizontal direction which commute with each other, and we can make it into a single graded complex[1] K^* in the following way:

Consider $K^n = \bigoplus_{p+q=n} K^{p,q}$ and operator is defined by

$$D = \delta + (-1)^p \mathrm{d}, \quad \text{on } K^{p,q}$$

D is a differential operator. Indeed, since K^n is a direct sum of $K^{p,q}$, so it suffice to take $\alpha^{p,q} \in K^{p,q}$ and check $D^2(\alpha^{p,q}) = 0$

$$\begin{aligned} D^2(\alpha^{p,q}) &= D(\delta\alpha^{p,q} + (-1)^p \mathrm{d}\alpha^{p,q}) \\ &= (-1)^{p+1}\mathrm{d}\delta\alpha^{p,q} + (-1)^p \delta\mathrm{d}\alpha^{p,q} \\ &= 0 \end{aligned}$$

Here we use the fact $\delta\alpha^{p,q} \in K^{p+1,q}$, $\mathrm{d}^2 = \delta^2 = 0$ and d commutes with δ.

Remark 1.1.2. $(-1)^p$ is crucial here, otherwise we won't get $D^2 = 0$.

Now Mayer-Vietoris sequence is reformulated as following

Theorem 1.1.1. The double complex $C^*(\mathfrak{U}, \Omega^*)$ computes the de Rham cohomology of M.

[1] Sometimes call it total complex.

Proof. The hallmark of proof is to show: A q-cochain α in double complex is D-cohomologous to a cochain with only top component, since such cochain is easy to deal with. Generally α may have two components

$$\alpha = \alpha_0 + \alpha_1, \quad \alpha_0 \in K^{0,q}, \alpha_1 \in K^{1,q-1}$$

But thanks to exactness of Mayer-Vietoris sequence:

$$0 \to \Omega^*(U \cup V) \xrightarrow{r} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{\delta} \Omega^*(U \cap V) \to 0$$

We can take $\beta \in \Omega^*(U) \oplus \Omega^*(V)$ such that $\delta\beta = \alpha_1$, then $\alpha - D\beta$ only contains top component. $\square$

Now we generalize our ideal as what we have mentioned in Remark 3.1.1. Fix an open covering $\mathfrak{U} = \{U_\alpha\}_{\alpha \in J}$ of M. And use $U_{\alpha\beta}$ to denote the intersection $U_\alpha \cap U_\beta$, similarly for $U_{\alpha\beta\gamma}$. Clearly there is a natural inclusion $\delta_\alpha \colon U_{\alpha\beta\gamma} \to U_{\beta\gamma}$. Thus we will have the following sequence of differential forms:

$$\Omega^*(M) \xrightarrow{r} \prod \Omega^*(U_{\alpha_0}) \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \prod_{\alpha_0 < \alpha_1} \Omega^*(U_{\alpha_0\alpha_1}) \overset{\delta_0}{\underset{\delta_2}{\overset{\delta_1}{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}}}} \prod_{\alpha_0<\alpha_1<\alpha_2} \Omega^*(U_{\alpha_0\alpha_1\alpha_2}) \to \dots$$

Recall difference operator δ we defined in Mayer-Vietoris is $\delta = \delta_0 - \delta_1$, here we also define generalized difference operator in a similar way: For $\omega \in \prod \Omega^q(U_{\alpha_0\dots\alpha_p})$, we define

$$(\delta\omega)_{\alpha_0\dots\alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0\dots\widehat{\alpha_i}\dots\alpha_{p+1}}$$

We also need this alternating sign in order to make $\delta^2 = 0$, that is making above sequence a complex.

Remark 1.1.3. It's a little weird in our definition about difference operator δ we require indices in $\omega_{\alpha_0\dots\alpha_p}$ are arranged in an increasing order, since by definition $U_{\alpha\beta} = U_{\beta\alpha}$. More generally we will allow indices in any order[2], by setting

$$\omega_{\dots\alpha\dots\beta\dots} = -\omega_{\dots\beta\dots\alpha\dots}$$

But we need to check it's consistent with definition of difference operator, that is Exercise 8.4 in Bott-Tu.

Proof. We need to check our new definition is compatible with δ, since for $(\delta\omega)_{\dots\beta\dots\alpha}$, there are two ways to define it:

(1) Interchange α with β first and take δ;
(2) Take δ first and interchange α with β.

[2]And we should do this, since sometimes it's difficult to ask these indices to be arranged in an increaing order, we will see this in later.

We need to show these two definitions are same. For terms $\omega_{\dots\beta\dots\widehat{\alpha_i}\dots\alpha\dots}$, it's clearly that it equals to $-\omega_{\dots\alpha\dots\widehat{\alpha_i}\dots\beta\dots}$ by definition. So it suffices to show

$$(-1)^i\omega_{\dots\widehat{\beta}\dots\alpha\dots}+(-1)^j\omega_{\dots\beta\dots\widehat{\alpha}\dots}=(-1)\{(-1)^i\omega_{\dots\widehat{\alpha}\dots\beta\dots}+(-1)^j\omega_{\dots\alpha\dots\widehat{\beta}\dots}\}$$

But by definition we can see

$$\omega_{\dots\widehat{\alpha}\dots\beta\dots}=(-1)^{j-i-1}\omega_{\dots\beta\dots\widehat{\alpha}\dots}$$
$$\omega_{\dots\alpha\dots\widehat{\beta}\dots}=(-1)^{j-i-1}\omega_{\dots\widehat{\beta}\dots\alpha\dots}$$

This completes the proof. □

As what we have seen in Mayer-Vietoris sequence, the exactness of Mayer-Vietoris plays an important role in proving isomorphism between cohomology of total complex and de Rham cohomology of the whole space. Here we still desire our generalized Mayer-Vietoris sequence is also exact.

Proposition 1.1.1 (The generalized Mayer-Vietoris sequence)**.** The following sequence is exact

$$\Omega^*(M)\xrightarrow{r}\prod\Omega^*(U_{\alpha_0})\xrightarrow{\delta}\prod\Omega^*(U_{\alpha_0\alpha_1})\xrightarrow{\delta}\prod\Omega^*(U_{\alpha_0\alpha_1\alpha_2})\to\dots$$

Proof. The proof here is quite similar to what we have done in Mayer-Vietoris sequence, that is using partition of unity to construct coboundaries we desired.

But if we regard the construction process as a homotopy operator, things become interesting. We construct an operator

$$K\colon\prod\Omega^*(U_{\alpha_0\dots\alpha_p})\to\prod\Omega^*(U_{\alpha_0\dots\alpha_{p-1}})$$

And we showed that

$$\delta K+K\delta=1$$

In other words, we showed that identity map is homotopic to zero map, but homotopic chain maps induce the same map between cohomology groups. So cohomology of this complex is isomorphic to trivial group, that is it's exact. □

By the same method, we can show

Proposition 1.1.2 (Generalized Mayer-Vietoris Principle)**.** The double complex $C^*(\mathfrak{U},\Omega^*)$ computes the cohomology of M. Furthermore, restriction map $r\colon\Omega^*(M)\to C^*(\mathfrak{U},\Omega^*)$ induces an isomorphism in cohomology.

Remark 1.1.4. The philosophy here is that if every row of an argumented double complex is exact, then the cohomology of total complex will compute the cohomology of the argumented column. We will revisit this ideal in spectral sequences.

Note that the rows and columns are symmetric with repest to the diagonal line, so we may desire if we make this double complex into a column argumented double complex, and all columns are exact, then the cohomology of total complex will reflect the cohomology of the argumented row.

It's natural to argument each column by the kernel of the bottom d, it consists of the locally constant functions defined on $U_{\alpha_0\dots\alpha_p}$, and we denote it by $C^*(\mathfrak{U},\mathbb{R})$. We can write this complex explicitly

$$C^0(\mathfrak{U},\mathbb{R}) \xrightarrow{\delta} C^1(\mathfrak{U},\mathbb{R}) \xrightarrow{\delta} C^2(\mathfrak{U},\mathbb{R}) \to \dots$$

We call the homology of this complex $H^*(\mathfrak{U},\mathbb{R})$, the Čech cohomology of the cover $\mathfrak{U}$. Note that it's a purely combinatorial object, since here we only care for the intersections of these open subsets, and it's computable.

But what's the condition for the columns are exact? The failure of p-th column to be exact is measured by

$$\prod H^q(U_{\alpha_0\dots\alpha_p})$$

So also by Poincaré lemma, if the open covering $\mathfrak{U}$ is a good cover, then all columns are exact, then we get

Theorem 1.1.2 (comparison theorem)**.** If $\mathfrak{U}$ is a good cover of the manifold M, then the Čech cohomology of the cover $\mathfrak{U}$ computes the de Rham cohomology of M, that is

$$H^*_{dR}(M) \cong H^*(\mathfrak{U},\mathbb{R})$$

Remark 1.1.5. If you're familiar with cohomology of sheaf, you will find that in fact de Rham cohomology computes the cohomology of constant sheaf $\mathbb{R}$, since the following is a fine resolution of $\mathbb{R}$

$$0 \to \mathbb{R} \to \Omega^1_M \to \Omega^2_M \to \dots$$

and de Rham cohomology is the cohomology of complex obtained by taking global section of above sheaf sequence. Later we will see if $\mathfrak{U}$ is a good cover of manifold M, then $H^*(\mathfrak{U},\mathbb{R})$ computes the Čech cohomology of constant sheaf $\mathbb{R}$. So this theorem in fact shows that Čech cohomology computes the cohomology of sheaf, and that's why it's called comparison theorem.

More generally, proof for this theorem holds for comparison theorem for any sheaf, since if we choose a fine resolution of a given sheaf, then exactness of row is guaranteed by the existence of partition of unity, and exactness of column is guaranteed by the exactness of resolution.

1.2. **Explicit isomorphism between the Double Complex and de Rham and Čech.** Although we have already seen that $r\colon \Omega^*(M) \to C^*(\mathfrak{U},\Omega^*)$ induces an isomorphism of homology group, that is $H^*_{dR}(M) \cong H_D\{C^*(\mathfrak{U},\Omega^*)\}$, we still don't how does this isomorphism look like. In other words, given a Čech n-cocycle, what's the closed global form corresponding to it?

In fact, there is a chain map $f\colon C^*(\mathfrak{U},\Omega^*) \to \Omega^*(M)$ such that

(1) $f \circ r = 1$
(2) $r \circ f$ is chain homotopic to 1

As you can imagine, a Čech n-cochain is a collection of local information, and f collate these local things together to obtain the global form we want.

Proposition 1.2.1 (The Collating formula)**.** Let K be the homotopy operator we defined in Proposition 3.1.5. If $\alpha = \sum_{i=0}^{n} \alpha_i$ is a n-cochain and $D\alpha = \beta = \sum_{i=0}^{n+1} \beta_i$, then

$$f(\alpha) = \sum_{i=0}^{n} (-D''K)^i \alpha_i - \sum_{i=1}^{n+1} K(-D''K)^{i-1}\beta_i \in C^0(\mathfrak{U}, \Omega^*)$$

is a global form we desired. The homotopy operator

$$L\colon C^*(\mathfrak{U}, \Omega^*) \to \Omega^*(\mathfrak{U}, \Omega^*)$$

such that $1 - r \circ f = DL + LD$ is given by

$$L\alpha = \sum_{p=0}^{n-1} (L\alpha)_p$$

where

$$(L\alpha)_p = \sum_{i=p+1}^{n} K(-D''K)^{i-(p+1)}\alpha_i \in C^p(\mathfrak{U}, \Omega^{n-1-p})$$

Proof. Here we only check $f(\alpha)$ is a global form, that is $\delta f(\alpha) = 0$, since this is the only part missing in Bott-Tu. We divide $f(\alpha)$ into following two parts:

$$f(\alpha) = \underbrace{\sum_{i=0}^{n} (-D''K)^i \alpha_i}_{\text{part I}} - \underbrace{\sum_{i=1}^{n+1} K(-D''K)^{i-1}\beta_i}_{\text{part II}} \in C^0(\mathfrak{U}, \Omega^*)$$

For the first part, we have

$$\delta(\sum_{i=0}^{n} (-D''K)^i \alpha_i) = \delta\alpha_0 + \sum_{i=1}^{n} (-D''K)^i \delta\alpha_i + (-D''K)^{i-1} D''\alpha_i$$

Note that $\delta\alpha_i + D''\alpha_{i+1} = \beta_{i+1}$ for $0 \leq i \leq n-1$ and $\delta\alpha_n = \beta_{n+1}$, so we have

$$\delta(\sum_{i=0}^{n} (-D''K)^i \alpha_i) = \sum_{i=1}^{n+1} (-D''K)^{i-1}\beta_i$$

For the second part, since we have $K\delta + \delta K = 1$, so

$$\begin{aligned}\delta \sum_{i=1}^{n+1} K(-D''K)^{i-1}\beta_i &= (1 - K\delta) \sum_{i=1}^{n+1} (-D''K)^{i-1}\beta_i \\ &= \sum_{i=1}^{n+1} (-D''K)^{i-1}\beta_i - K\{\delta\beta_1 + \sum_{i=2}^{n+1} (-D''K)^{i-1}\delta\beta_i + (-D''K)^{i-2} D''\beta_i\}\end{aligned}$$

So it suffice to show

$$\delta\beta_1 + \sum_{i=2}^{n+1} (-D''K)^{i-1}\delta\beta_i + (-D''K)^{i-2} D''\beta_i = 0$$

Note that

$$\begin{cases}\delta\beta_i = \delta D''\alpha_i \\ D''\beta_i = D''\delta\alpha_{i-1} = -\delta D''\alpha_{i-1}\end{cases}$$

So adding these together we only have $(-D''K)^n\delta\beta_{n+1}$ left, and $\delta\beta_{n+1} = 0$ since $\beta_{n+1} = \delta\alpha_n$. This completes the proof. □

Corollary 1.2.1. If $\eta \in C^n(\mathfrak{U}, \mathbb{R})$ is a Čech cocycle, then the global closed form corresponding to it is given by

$$f(\eta) = (-1)^n(D''K)^n\eta$$

1.3. **The Tic-Tac-Toe Proof of the Künneth formula.** Now let's use theory we have developed to revisit Künneth formula, but we prove it in a weaker assumption: We replace M has a finite good cover by F has a finite dimensional cohomology.

Before proving the Künneth formula, let's make some general remarks: Let $\pi\colon E \to M$ be a map of manifolds, and $\mathfrak{U}$ is an open covering of M, then $\pi^{-1}(\mathfrak{U})$ is clearly an open covering of E. But in general $U_\alpha \cap U_\beta = \varnothing$ is not equivalent to $\pi^{-1}U_\alpha \cap \pi^{-1}U_\beta = \varnothing$. but one direction $U_\alpha \cap U_\beta = \varnothing \implies \pi^{-1}U_\alpha \cap \pi^{-1}U_\beta = \varnothing$ always holds. Indeed, if not, take $x \in \pi^{-1}U_\alpha \cap \pi^{-1}U_\alpha$, then $\pi(x) \in U_\alpha \cap U_\beta$, a contradiction. But the other direction may fail if π is not surjective, since we can't control parts which doesn't lie in the image of π by considering its inverse.

So if π is surjective, then the combinatorial property of $\mathfrak{U}$ and $\pi^{-1}(\mathfrak{U})$ are same. The double complex of $\pi^{-1}\mathfrak{U}$ computes the cohomology of E, which can be related to the cohomology of M. This is a powerful ideal of Čech cohomology.

However, things are not quite easy, since although $\pi^{-1}\mathfrak{U}$ is good cover, $\mathfrak{U}$ may not be a good cover. But for the case of vector bundle $\pi\colon E \to M$, the "goodness" of the cover is preserved. So we have

$$H^*_{dR}(E) \cong H^*_{dR}(M)$$

where $E \to M$ is a vector bundle.

Proposition 1.3.1 (Künneth formula)**.** If M and F are two manifolds and F has finite dimensional cohomology, then the de Rham cohomology of the product $M \times F$ is

$$H^*(M \times F) = H^*(M) \otimes H^*(F)$$

Proof. It suffices to show

$$\begin{aligned}\pi^*_{\mathfrak{U}}\colon H^*(F) \otimes C^*(\mathfrak{U}, \Omega^*) &\to C^*(\pi^{-1}\mathfrak{U}, \Omega^*) \\ [\omega_\alpha] \otimes \phi &\mapsto \rho^*\omega_\alpha \wedge \pi^*\phi\end{aligned}$$

induces an isomorphism in D-cohomology. It's clear $\pi^*_{\mathfrak{U}}$ induces an isomorphism of d-cohomology of these complexes. And the claim holds from following lemma:

Lemma 1.3.1 (Acyclic assembly lemma)**.** Whenever a homomorphism $f\colon K\to K'$ of double complex induces H_{d}-isomorphism, it also induces H_D-isomorphism.

Proof. It's an important lemma in homological algebra, but here we can assume only finitely many columns of K and K' are nonzero, since double complexes we concern possess this property. We prove this by induction on the number n of nonzero columns of double complex: If $n=1$, it's trivial since H_{d} is the same as H_D. If we have proven for $n<k$, then for $n=k$, consider the following exact sequence

$$\begin{tikzcd}
0 \arrow[r] & K_1 \arrow[r] \arrow[d, "f"] & K \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & K_1' \arrow[r] & K' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

where C, C' are first columns of K and K', and K_1, K_1' are subcomplexes of K, K' obtained by cutting out the first column. Then we get a long exact sequence of cohomology groups:

$$\begin{tikzcd}
\cdots \arrow[r] & H^{i-1}(C) \arrow[r] \arrow[d, "f^*"] & H^i(K_1) \arrow[r] \arrow[d, "f^*"] & H^i(K) \arrow[r] \arrow[d, "f^*"] & H^i(C) \arrow[r] \arrow[d, "f^*"] & \dots \\
\cdots \arrow[r] & H^{i-1}(C') \arrow[r] & H^i(K_1') \arrow[r] & H^i(K') \arrow[r] & H^i(C') \arrow[r] & \dots
\end{tikzcd}$$

Now use five lemma to complete the proof. □

□

1.4. **Čech cohomology of presheaf.** Now let's talk about the philosophy behind what we have done. When we define $C^*(\mathfrak{U},\Omega^*)$ and difference operator δ, the only two things we used are:

(1) For any open subset U, we have a group of differential forms on U;
(2) For two open subsets $V\subset U$, there is a natural restriction $\Omega^*(U)\to\Omega^*(V)$.

So in fact, we don't need the property of $\Omega^*(U)$ as a differential forms, it can be any group. Also we don't need our base space is a manifold, we can just consider on a topological space X.

So let's define Čech cohomology of presheaf $\mathscr{F}$ with repest to open covering $\mathfrak{U}$ on a topological space X: For any contravariant presheaf $\mathscr{F}$ as follows

$$0\to C^0(\mathfrak{U},\mathscr{F})\xrightarrow{\delta}C^1(\mathfrak{U},\mathscr{F})\xrightarrow{\delta}C^2(\mathfrak{U},\mathscr{F})\to\dots$$

and the cohomology of this complex is denoted by $H^*(\mathfrak{U},\mathscr{F})$. Similarly for a covariant presheaf $\mathscr{F}$, we can define

$$0\leftarrow C_0(\mathfrak{U},\mathscr{F})\xleftarrow{\delta}C_1(\mathfrak{U},\mathscr{F})\xleftarrow{\delta}C_2(\mathfrak{U},\mathscr{F})\leftarrow\dots$$

and consider its homology $H_*(\mathfrak{U}, \mathscr{F})$.

However, it's not a beautiful definition, since our definition may depend on the choice of open covering $\mathfrak{U}$. So it's necessary for us to ask what will happen when we change the choice of open covering.

Lemma 1.4.1. Given $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ an open cover and $\mathfrak{V} = \{V_\beta\}_{\beta \in J}$ a refinement, if ϕ, ψ are two refinement maps $J \to I$, then there is a homotopy operator between $\phi^\#$ and $\psi^\#$.

Proof. Define $K \colon C^q(\mathfrak{U}, \mathscr{F}) \to C^{q-1}(\mathfrak{V}, \mathscr{F})$ by

$$(K\omega)(V_{\beta_0 \dots \beta_{q-1}}) = \sum (-1)^i \omega(U_{\phi(\beta_0)\dots\phi(\beta_i)\psi(\beta_i)\dots\psi(\beta_{q-1})})$$

And we can check[3]

$$\psi^\# - \phi^\# = \delta K + K\delta$$

as follows: Take a cochain $\omega \in C^q(\mathfrak{U}, \mathscr{F})$, and an intersection of open covers $V_{\beta_0 \dots \beta_q}$, then it's easy to see

$$\psi^\# - \phi^\#(\omega)(V_{\beta_0\dots\beta_q}) = \omega(U_{\psi(\beta_0)\dots\psi(\beta_q)}) - \omega(U_{\phi(\beta_0)\dots\phi(\beta_q)})$$

Now let's compute $\delta K\omega$ as follows:

$$\begin{aligned}
\delta K(\omega)(V_{\beta_0\dots\beta_q}) &= \sum_i (-1)^i K\omega(V_{\beta_0\dots\widehat{\beta_i}\dots\beta_q}) \\
&= \underbrace{\sum_{i\le j} (-1)^{i+j}\omega(U_{\phi(\beta_0)\dots\widehat{\phi(\beta_i)}\dots\phi(\beta_{j+1})\psi(\beta_{j+1})\dots\psi(\beta_q)})}_{\text{part I}} \\
&\quad + \underbrace{\sum_{i>j} (-1)^{i+j}\omega(U_{\phi(\beta_0)\dots\phi(\beta_j)\psi(\beta_j)\dots\widehat{\psi(\beta_j)}\dots\psi(\beta_q)})}_{\text{part II}}
\end{aligned}$$

Similarly we have $K\delta\omega$ as follows

$$\begin{aligned}
K\delta\omega(V_{\beta_0\dots\beta_q}) &= \sum_j (-1)^j \delta\omega(U_{\phi(\beta_0)\dots\phi(\beta_j)\psi(\beta_j)\dots\psi(\beta_q)}) \\
&= \underbrace{\sum_{i<j} (-1)^{i+j}\omega(U_{\phi(\beta_0)\dots\widehat{\phi(\beta_i)}\dots\phi(\beta_j)\psi(\beta_j)\dots\psi(\beta_q)})}_{\text{part III}} \\
&\quad + \underbrace{\sum_{i>j} (-1)^{i+j}\omega(U_{\phi(\beta_0)\dots\phi(\beta_j)\psi(\beta_j)\dots\widehat{\psi(\beta_{i-1})}\dots\psi(\beta_q)})}_{\text{part IV}} \\
&\quad + \underbrace{\sum_j \omega(U_{\phi(\beta_0)\dots\widehat{\phi(\beta_j)}\psi(\beta_j)\dots\psi(\beta_q)})}_{\text{part V}}
\end{aligned}$$

[3] An exercise you only check once in your whole life.

Note that part I cancels with part III, since if you fix i, you will find j-th terms of part I and part III are equal but differ a sign. Similarly you can find part II and part IV almost cancel each other, but

$$\text{part II} + \text{part IV} = \underbrace{\sum_j -\omega(U_{\phi(\beta_0)\ldots\phi(\beta_j)\widehat{\psi(\beta_j)}\psi(\beta_{j+1})\ldots\psi(\beta_q)})}_{\text{part VI}}$$

and it's clear to see that

$$\text{part V} + \text{part VI} = \omega(U_{\psi(\beta_0)\ldots\psi(\beta_q)}) - \omega(U_{\phi(\beta_0)\ldots\phi(\beta_q)})$$

as desired. This completes the proof. $\square$

So for two different open covering $\mathfrak{U}, \mathfrak{B}$ such that $\mathfrak{B}$ is a refinement of $\mathfrak{U}$, then there is a natural homomorphism

$$f_{\mathfrak{U}\mathfrak{B}} \colon H^*(\mathfrak{U}, \mathscr{F}) \to H^*(\mathfrak{B}, \mathscr{F})$$

Furthermore, if there are three open covering such that $\mathfrak{C}$ is a refinement of $\mathfrak{B}$, and $\mathfrak{B}$ is a refinement of $\mathfrak{U}$. then we have

$$f_{\mathfrak{U}\mathfrak{C}} = f_{\mathfrak{U}\mathfrak{B}} f_{\mathfrak{B}\mathfrak{C}}$$

So if we give a partial order on set of all open coverings, that is $\mathfrak{U} < \mathfrak{B}$, if $\mathfrak{B}$ is a refinement of $\mathfrak{U}$, we obtain a direct system $\{H^*(\mathfrak{U}, \mathscr{F}), f_{\mathfrak{U}\mathfrak{B}}\}$. The direct limit of this direct system

$$\check{H}^*(X, \mathscr{F}) := \varinjlim_{\mathfrak{U}} H^*(\mathfrak{U}, \mathscr{F})$$

is called Čech cohomology of X with values in the presheaf $\mathscr{F}$.

But note that on manifold M good cover is cofinal in this partial order, so we can only use good cover to compute direct limit. Furthermore, if we consider constant presheaf $\underline{R}$, which is defined as

$$\mathbb{R}(U) := \{f \colon U \to \mathbb{R} \mid f \text{ is locally constant}\}$$

then generalized Mayer-Vietoris implies

$$H^*(\mathfrak{U}, \mathbb{R}) \cong H^*_{dR}(M)$$

when $\mathfrak{U}$ is a good cover. Thus

$$\begin{aligned}\check{H}^*(M, \mathbb{R}) : &= \varinjlim_{\mathfrak{U}} H^*(\mathfrak{U}, \mathbb{R}) \\ &= \varinjlim H^*_{dR}(M) \\ &= H^*_{dR}(M)\end{aligned}$$

Remark 1.4.1. In fact, constant presheaf we defined here is a sheaf. And some authors define constant presheaf $\mathscr{F}$ valued abelian group G as follows:

$$\mathscr{F}(U) := G$$

for any open set U. This presheaf won't be a sheaf in general. So please be caution about notations in Bott-Tu here.

Remark 1.4.2. Here we only computes the Čech cohomology of presheaf $\mathscr{F}$, but we're always concern about Čech cohomology of sheaf. So it's natural to ask the relations between Čech cohomology of a presheaf $\mathscr{F}$ and Čech cohomology of its sheafication $\mathscr{F}^+$. In fact, we will prove on a Hausdorff paracompact topological space X we have

$$\check{H}^*(X, \mathscr{F}) \cong \check{H}^*(X, \mathscr{F}^+)$$

Firstly let's establish a lemma

Lemma 1.4.2. Suppose that for any presheaf $\mathscr{F}$ of abelian group such that its sheafication $\mathscr{F}^+$ is zero, the group $\check{H}^q(X, \mathscr{F})$ are all zero. Then for arbitary presheaf $\mathscr{F}$ of abelian group, we have

$$\check{H}^*(X, \mathscr{F}) \cong \check{H}^*(X, \mathscr{F}^+)$$

Proof. Consider the exact sequence of presheaves

$$0 \to \ker(\mathscr{F} \to \mathscr{F}^+) \to \mathscr{F} \to \operatorname{im}(\mathscr{F} \to \mathscr{F}^+) \to 0$$
$$0 \to \operatorname{im}(\mathscr{F} \to \mathscr{F}^+) \to \mathscr{F}^+ \to \operatorname{coker}(\mathscr{F} \to \mathscr{F}^+) \to 0$$

Let $\mathscr{K}$ denote the presheaf $\ker(\mathscr{F} \to \mathscr{F}^+)$ and $\mathscr{C}$ denote the presheaf $\operatorname{coker}(\mathscr{F} \to \mathscr{F}^+)$, it's easy to see $\mathscr{K}^+ = \mathscr{C}^+ = 0$, since sheafication commutes with limits and colimits.

Then consider the long exact sequence of Čech cohomology group induced by above two short sequences, we obtain

$$\cdots \to \check{H}^q(X, \mathscr{K}) \to \check{H}^q(X, \mathscr{F}) \to \check{H}^q(X, \operatorname{im}(\mathscr{F} \to \mathscr{F}^+)) \to \check{H}^{q+1}(X, \mathscr{K}) \to \dots$$
$$\cdots \to \check{H}^q(X, \operatorname{im}(\mathscr{F} \to \mathscr{F}^+)) \to \check{H}^q(X, \mathscr{F}^+) \to \check{H}^q(X, \mathscr{C}) \to \check{H}^{q+1}(X, \operatorname{im}(\mathscr{F} \to \mathscr{F}^+)) \to \dots$$

which implies the following two morphisms are isomorphisms

$$\check{H}^q(X, \mathscr{F}) \to \check{H}^q(X, \operatorname{im}(\mathscr{F} \to \mathscr{F}^+))$$
$$\check{H}^q(X, \operatorname{im}(\mathscr{F} \to \mathscr{F}^+)) \to \check{H}^q(X, \mathscr{F}^+)$$

This completes the proof. □

Use this lemma, given a Čech cocycle of a presheaf $\mathscr{F}$ with $\mathscr{F}^+$ which is defined on some cover, to construct a refinement of the cover on which every component of the cocycle vanishes.

1.5. **Meaning of lower dimension Čech cohomology.** Note that a presheaf $\mathscr{F}$ on a manifold M is a sheaf if and only if for each open covering $\{U_\alpha\}$ we have the following sequence is exact

$$0 \to \mathscr{F}(M) \to \prod_{\alpha} \mathscr{F}(U_\alpha) \to \prod_{\alpha,\beta} \mathscr{F}(U_{\alpha\beta})$$

Thus as you can see $\check{H}^0(\mathfrak{U}, \mathscr{F})$ computes the global section of $\mathscr{F}$ for any $\mathfrak{U}$, which implies $\check{H}^0(M, \mathscr{F}) = \mathscr{F}(M)$.

It's natural to ask is there any meaning for $\check{H}^1$ or $\check{H}^2$ and so on? Let's see some cases:

1.5.1. *Classification of real line bundle.* Recall that for any (real) vector bundle $E \to M$, there exists a open covering $\mathfrak{U} = \{U_\alpha\}$ and transition functions, that is smooth functions

$$g_{\alpha\beta}\colon U_{\alpha\beta} \to \mathrm{GL}(n, \mathbb{R})$$

Or more explicitly,

$$g_{\alpha\beta}\colon U_{\alpha\beta} \to O(n)$$

since there always exists Riemannian metrics.

So you may define a "presheaf" $\mathscr{F}$, that is

$$\mathscr{F}(U) := \{f\colon U \to O(n) \mid f \text{ is smooth}\}$$

And you can realize these transition functions $g_{\alpha\beta}$ as an element in $g \in \check{H}^1(\mathfrak{U}, \mathscr{F})$, since you can realize cocycle condition as a $\delta g = 0$, and that's why it's called cocycle condition. And you can rewrite condition for up to isomorphism as a condition for coboundary.

However, we always define a presheaf valued in an abelian groups and if it's not valued in abelian groups, something complicated will happen (you can search for your own interest: non-abelian cohomology).

So we're interested in the case $n = 1$, that's the case real line bundle, since in this case $O(1) = \mathbb{Z}/2\mathbb{Z}$ is abelian. In this case, we can identify the set of real line bundle up to isomorphism with Čech cohomology group $\check{H}^1(M, \mathbb{Z}/2\mathbb{Z})$, and that's Stiefel–Whitney class.

1.5.2. *Classification of complex line bundle.* Now let's consider complex line bundle. Note that in this case transition functions are smooth functions $g_{\alpha\beta}\colon U_{\alpha\beta} \to \mathbb{C}^*$, so similarly we can realize the set of complex line bundle up to isomorphism as $\check{H}^1(M, C^\infty(\mathbb{C}^*))$. Let's explain more explicitly: Consider the following exact sequence of sheaves

$$0 \to C^\infty(\mathbb{Z}) \to C^\infty(\mathbb{C}) \to C^\infty(\mathbb{C}^*)$$

where $C^\infty(\bullet)(U) = \{f\colon U \to \bullet \mid f \text{ is smooth}\}$, $\bullet$ can be $\mathbb{Z}, \mathbb{C}, \mathbb{C}^*$. Note that $C^\infty(\mathbb{Z})$ is exactly constant sheaf $\mathbb{Z}$.

Consider the long exact sequence of Čech cohomology induced by this short exact sequence

$$\cdots \to \check{H}^1(M, C^\infty(\mathbb{C})) \to \check{H}^1(M, C^\infty(\mathbb{C}^*)) \to \check{H}^2(M, \mathbb{Z}) \to \check{H}^2(M, C^\infty(\mathbb{C})) \to \dots$$

Note that $C^\infty(\mathbb{C})$ is a "fine" sheaf, thus $\check{H}^*(M, C^\infty(\mathbb{C})) = 0$, thus $\check{H}^1(M, C^\infty(\mathbb{C}^*)) \cong \check{H}^2(M, \mathbb{Z})$, that is every complex line bundle is determined by its (first) Chern class.

Remark 1.5.1. If you're more familiar with complex geometry, you will find that the set of all holomorphic line bundle up to isomorphism over a complex manifold X is a group, called Picard group, sometimes denoted by $\mathrm{Pic}(X)$, and that's isomorphic to

$$\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$$

this is sometimes called Lefschetz (1,1) theorem.

1.6. **Locally constant sheaf.** Let $\mathfrak{U}$ be an open covering, the nerve of $\mathfrak{U}$ is a simplicial complex constructed as follows: To every open subset U_α, we associate with a vertice α. If $U_\alpha \cap U_\beta \neq \varnothing$, we connect the vertices α and β with an edge. If $U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$, then we fill in the face of the triangle $\alpha\beta\gamma$ and so on.

For example, the nerve of open covering in Example 9.1 in Bott-Tu can be visualized as a hollow triangle, or in a more abstract way[4], we can describe it as the following family of sets:

$$\varnothing, \{\alpha\}, \{\beta\}, \{\gamma\}, \{\alpha,\beta\}, \{\alpha,\gamma\}, \{\beta,\gamma\}$$

So as you can see, the nerve of an open covering contains all data about open covering itself, or in other words, nerves are equivalent to open coverings.

Note that the ingredients you need to compute Čech cohomology are data about open covering and what does restriction map of sheaf look like. But if we compute the Čech cohomology of constant sheaf, restriction maps are just natural map, so there is nothing to care about, that is, the Čech cohomology of constant sheaf is determined by the nerve of open covering, and it's just a simplicial homology of nerve, which is easy to be computed by a computer mechanically.

However, if we consider some other sheaves, things become interesting, for example, Čech cohomology of locally constant sheaf.

Definition 1.6.1 (locally constant sheaf)**.** A sheaf $\mathscr{F}$ is called a locally constant sheaf valued in G, if there exists an open covering $\{U_\alpha\}$ such that $\mathscr{F}|_{U_\alpha}$ is constant sheaf G.

Remark 1.6.1. So you can obtain a locally constant sheaf by gluing constant sheaf G defined on a open covering $\{U_\alpha\}$. More explicitly, if on each open subsets U_α we have a constant sheaf G, to glue them together, we need to define something like a "transition function": On each $U_{\alpha\beta}$, we require restriction map $\rho^\alpha_{\alpha\beta}\colon G(U_\alpha) \to G(U_{\alpha\beta})$ is an isomorphism, thus we can define

$$g_{\alpha\beta} = \rho^\alpha_{\alpha\beta} \circ (\rho^\beta_{\beta\alpha})^{-1}\colon G(U_\alpha \cap U_\beta) \to G(U_\beta \cap U_\alpha) \in \operatorname{Aut} G$$

Furthermore, we require $g_{\alpha\beta}$ satisfies cocycle condition. Note that $g_{\alpha\beta}$ is locally constant function valued in G.

For example, sheaf in Exercise 10.7 of Bott-Tu is a locally constant sheaf. It's easy to see that cohomology of this locally constant sheaf is zero in each dimension.

1.7. **Monodromy.** As we have seen in the last section, locally constant sheaf may not be a constant sheaf. So it's natural to ask can we find all locally constant sheaf up to isomorphism?

In fact, if we consider locally constant sheaf valued vector space, there is the following category equivalence:

[4]Recall that an abstract simplicial complex is a family of sets, which is closed in taking subsets.

(1) Locally constant sheaf valued $\mathbb{R}^n$ up to isomorphism;
(2) $\pi_1(M)$ representations, that is homomorphism from $\pi_1(M) \to \mathrm{GL}(n, \mathbb{R})$ up to conjugacy;
(3) Flat vector bundles with fiber $\mathbb{R}^n$ over M up to isomorphism.

Remark 1.7.1. Here we can use any vector space to replace $\mathbb{R}^n$.

Corollary 1.7.1. Every locally constant sheaf valued in $\mathbb{Z}_2$ is constant.

Proof. Note that $\mathrm{GL}(\mathbb{Z}_2)$ is trivial. □

Locally constant sheaf, which is also called local system is a quite interesting object, although it's a little difficult to prove above equivalences formally, we can give a short explainations which is quite informal.

1.7.1. *Equivalence between flat bundle and local system.*

Definition 1.7.1 (flat vector bundle)**.** A vector bundle E is called flat vector bundle, if it admits a trivialization such that transition functions are locally constant.

Remark 1.7.2. It's not hard to see that a flat vector bundle admits a connection such that its curvature form vanishes. In fact, the following two definitions for flatness are same:

(1) E admits a trivialization such that transition functions are locally constant;
(2) E admits a connection such that its curvature form vanishes.

Now let' see why locally constant sheaf valued vector space is the same as flat vector bundles. If we already have a flat vector bundle, that is an open covering $\{U_\alpha\}$ and a set of locally constant functions $\{g_{\alpha\beta}\}$. Note that $g_{\alpha\beta}$ are isomorphisms and satisfy cocycle condition, thus we can use these $g_{\alpha\beta}$ to glue constant sheaf to obtain a locally constant sheaf. Convesely, for a locally constant sheaf $\mathscr{F}$, then there exists an open covering $\{U_\alpha\}$ such that $\mathscr{F}|_{U_\alpha}$ is constant sheaf and transition functions $g_{\alpha\beta}$ between these constant sheaves are exactly locally constant functions, thus we can use these $g_{\alpha\beta}$ to define a flat vector bundle. All information is encoded in these $g_{\alpha\beta}$!

1.7.2. *Equivalence between π_1-representation and local system.* Recall that we can define $H^1(X, \mathbb{R})$ as the dual space of $H_1(X, \mathbb{R})$, that is

$$H^1(X, \mathbb{R}) := \{f \colon H_1(X, \mathbb{R}) \to \mathbb{R} \mid f \text{ is linear}\}$$

Note that Hurewicz theorem implies that $H_1(X, \mathbb{R})$ is the abelianization of $\pi_1(X)$ and $\mathbb{R}$ is abelian, thus in fact we can regard $H^1(X, \mathbb{R})$ as a one dimensional π_1-representation $\pi_1(X) \to \mathbb{R}$.

So in general, you can realize a local system as an element in $\check{H}^1(\mathfrak{U}, \mathrm{GL}(n, \mathbb{R}))$, where

$$\mathrm{GL}(n, \mathbb{R})(U) = \{f \colon U \to \mathrm{GL}(n, \mathbb{R}) \mid f \text{ is locally constant}\}$$

From what we have discussed in $n = 1$, you can get a feeling that this is the set of π_1-representation $\pi_1(X) \to \mathrm{GL}(n, \mathbb{R})$ up to conjugacy.

Remark 1.7.3. These explainations are quite informal and just want to give you an intuition, please refer to this page[5] for a detailed proof, or find a good reference to get yourself understand totally if you're interested in this topic.

[5]http://mcs.unife.it/alex.massarenti/files/Local%20Systems.pdf

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