ANALYTIC COMPLEX GEOMETRY

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Contents

0. Preface

0.1. **To readers.** It's a lecture note about analytic complex geometry, and the main reference is [Dem97].

0.2. **Basic notations.**

1. M denotes a smooth real manifold, with tangent bundle TM and cotangent bundle T^*M.
2. ${}^s\mathcal{E}^p(M)$ denotes the space of C^s-global sections of $\bigwedge^p T^*M$, and $\mathcal{E}^p(M)$ denotes the space of smooth global sections of $\bigwedge^p T^*M$.
3. X denotes a smooth complex manifold, with tangent bundle TX and cotangent bundle T^*X.
4. ${}^s\mathcal{E}^{p,q}(X)$ denotes the space of C^s-global sections of $\bigwedge^{p,q} T^*X$, and $\mathcal{E}^{p,q}(X)$ denotes the space of smooth global sections of $\bigwedge^{p,q} T^*X$.

1. Currents

In this section, M is assumed to be an oriented smooth real manifold with dimension n.

1.1. **Currents on smooth manifold.**

Firstly we want to give a topology on the space of ${}^s\mathcal{E}^p(M)$ to make it to be a topological vector space. For $u \in {}^s\mathcal{E}^p(M)$, on coordinate open set $\Omega \subset M$ it can be written as

$$u = \sum_{|I|=p} u_I \mathrm{d}x^I$$

To each $L \Subset \Omega$ and every integer $s \in \mathbb{N}$, we associate a seminorm

$$P_{L,\Omega} = \sup_{x \in L} \max_{|\alpha| \leq s, |I|=p} |D^\alpha u_I(x)|$$

Since our manifolds are suppose to be Hausdorff, then M can be covered by countable coordinate set, that is $M = \bigcup_{k=1}^\infty \Omega_k$, and consider exhaustion for ech k

$$L_{k_1} \Subset L_{k_2} \Subset \cdots \Subset \Omega_k$$

then seminorms $\{P_{L_{k_m}, \Omega_k}\}$ gives a topology on ${}^s\mathcal{E}^p(M)$. Then according to Remark A.2.1, this topology is given by a translation invariant metric, and in this case it's complete, which makes ${}^s\mathcal{E}^p(M)$ a Fréchet space.

Let $K \Subset M$, ${}^s\mathcal{D}^p(K)$ is the subspace of elements $u \in {}^s\mathcal{E}^p(M)$ with compact support in K, together with induced topology. The ${}^s\mathcal{D}^p(M)$ denotes the set of all elements of ${}^s\mathcal{E}^p(M)$ with compact support, that is

$${}^s\mathcal{D}^p(M) = \bigcup_{K \Subset M} {}^s\mathcal{D}^p(K)$$

A sequence $u_l \to 0$ in ${}^s\mathcal{D}^p(M)$ if there exists $K \Subset M$ such that $\operatorname{supp} u_l \subset K$ for all $l \geq 1$ and $u_l \to 0$ in ${}^s\mathcal{E}^p(M)$.

Remark 1.1.1. Similarly one can define $\mathcal{D}^p(K), \mathcal{D}^p(M)$, in particular, if $p = 0$ and $M = \mathbb{R}^n$, then $\mathcal{D}^0(\mathbb{R}^n)$ is exactly the space of test functions.

Definition 1.1.1 (current)**.** The space of current of dimension p or degree $n-p$, denoted by $\mathcal{D}'_p(M) = \mathcal{D}'^{n-p}(M)$, is the space of linear functionals on $\mathcal{D}^p(M)$ such that the restriction on any $\mathcal{D}^p(K)$ is continous, where $K \Subset M$.

Notation 1.1.1. For a current $T \in \mathcal{D}'_p(M)$, $\langle T, u \rangle$ denotes the pairing between a current T and test form $u \in \mathcal{D}^p(M)$.

Remark 1.1.2. If a current T extends continuously to ${}^s\mathcal{D}^p(M)$, then T is called of order s.

Definition 1.1.2. For a current $T \in \mathcal{D}'_p(M)$, the support of T, denoted by $\operatorname{supp}(T)$, is the smallest closed set A such that $T|_{\mathcal{D}^p(M \setminus A)} = 0$.

The following two basic examples explains the terminology used for dimension and degree.

Example 1.1.1. Let $Z \subseteq M$ be an oriented closed submanifold with dimension p. The current of integration $[Z]$ is given by

$$\langle [Z], u\rangle := \int_Z u$$

where $u \in \mathcal{D}^p(M)$. It's clear that $[Z]$ is a current with $\operatorname{supp}[Z] = Z$, and its dimension is exactly the dimension of Z as a manifold.

Example 1.1.2. Let f be a p-form with L^1_{loc} coefficients, the T_f given by

$$\langle T_f, u\rangle = \int_M f \wedge u$$

where $u \in \mathcal{D}^{n-p}(M)$, is a current of degree p.

1.2. **Exterior derivative and wedge product on currents.**

1.2.1. *Exterior derivative.* As we have seen in Example 1.1.2, currents generalize the ideal of forms, and in this viewpoint, many of the operations for forms can also be extended to currents.

Let $T \in \mathcal{D}'^p(M)$, the exterior derivative $\mathrm{d}T$ is given by

$$\langle \mathrm{d}T, u\rangle := (-1)^{p+1}\langle T, \mathrm{d}u\rangle$$

where $u \in \mathcal{D}^{n-p-1}(M)$. The continuity of the linear functional $\mathrm{d}T$ follows from the exterior derivative d is continous, thus $\mathrm{d}T$ is a current of degree $p+1$.

Remark 1.2.1. If $T \in \mathcal{E}^p(M)$,

$$\langle \mathrm{d}T, u\rangle = \int_M \mathrm{d}T \wedge u = \int_M \mathrm{d}(T\wedge u) + (-1)^{p+1} T \wedge \mathrm{d}u = (-1)^{p+1}\int_M T \wedge \mathrm{d}u$$

That's why we define exterior derivative like this.

Example 1.2.1. Consider current T_f given by p-form with L^1_{loc} coefficients, then

$$\begin{aligned}
\langle T_{\mathrm{d}f}, u\rangle &= \int_M \mathrm{d}f \wedge u \\
&= \int_M \mathrm{d}(f\wedge u) + (-1)^{p+1} f \wedge \mathrm{d}u \\
&= (-1)^{p+1}\int_M f \wedge \mathrm{d}u \\
&= \langle \mathrm{d}T_f, u\rangle
\end{aligned}$$

This shows $T_{\mathrm{d}f} = \mathrm{d}T_f$, and that's why exterior derivative is defined like this.

Example 1.2.2. Consider current $T=[Z]$ given by a oriented closed submanifold of M with dimension p, then

$$\begin{aligned}\langle \mathrm{d}T, u\rangle &= (-1)^{n-p+1}\langle T, \mathrm{d}u\rangle \\ &= (-1)^{n-p+1}\int_Z \mathrm{d}u \\ &= (-1)^{n-p+1}\int_{\partial Z} u\end{aligned}$$

that is $\mathrm{d}T=(-1)^{n-p+1}[\partial Z]$.

1.2.2. *Wedge product.* Let $T\in \mathcal{D}'^{p}(M)$, $g\in\mathcal{E}^q(M)$, the wedge product $T\wedge g$ is a current of degree $p+q$, given by

$$\langle T\wedge g, u\rangle := \langle T, g\wedge u\rangle$$

where $u\in\mathcal{D}^{n-p-q}(M)$.

Proposition 1.2.1. Let $T\in \mathcal{D}'^{p}(M)$, $g\in\mathcal{E}^q(M)$, then

$$\mathrm{d}(T\wedge g)=\mathrm{d}T\wedge g+(-1)^p T\wedge \mathrm{d}g$$

1.3. **Currents on complex manifold.** Let X be a complex n-manifold, in particular, it's also a real manifold, then what we have talked can be used here, but there is a more explicit structure on $\mathcal{E}^k(X)$, that is

$$\mathcal{E}^k(X)=\bigoplus_{k=p+q}\mathcal{E}^{p,q}(X)$$

So $\mathcal{E}^{p,q}(X)$ is also endowed wtih a topology which is induced by seminorms, and $\mathcal{D}^{p,q}(X)$ is the space of smooth (p,q)-forms with compact supports.

Definition 1.3.1. The space of currents of bidimension (p,q) or bidegree $(n-p,n-q)$, denoted by

$$\mathcal{D}'_{p,q}(X)=\mathcal{D}'^{(n-p,n-q)}(X)$$

is the topological dual of $\mathcal{D}^{p,q}(X)$.

Now let's try to explains a motivation for why we need currents. Let $\mathcal{E}^{p,q}$ be the locally free sheaf associated to $\bigwedge^{p,q}T^*X$, and Dolbeault-Grothendieck lemma says it gives a resolution of holomorphic p-forms, that is

$$0\to\Omega^p\to\mathcal{E}^{p,0}\xrightarrow{\overline{\partial}}\mathcal{E}^{p,1}\xrightarrow{\overline{\partial}}\dots$$

is an exact sequence of sheaves. Thus the Dolbeault cohomology in fact computes the sheaf cohomology of sheaf Ω^p.

The same story works for currents, in the setting of currents we also have $\overline{\partial}$-operator, and there is the following sequences

$$0\to\Omega^p\to\mathcal{D}'^{p,0}\xrightarrow{\overline{\partial}}\mathcal{D}'^{p,1}\xrightarrow{\overline{\partial}}\dots$$

A non-trivial fact says it's also an exact sequence, thus there is a new way to compute sheaf cohomology of sheaf of holomorphic p-forms.

The advantage of a smooth manifold is everything is smooth, but that's also its disadvantages, in the setting of currents, tools of functional analysis can be used to compute, and that's where advantages of currents lie.

2. Positive

2.1. **Positive** $(1,1)$ **current.** Let X be a complex n-manifold, in general one can consider the positiveness of a current over X with bidegree (p,q), but here we only consider the case of $(p,q)=(1,1)$.

Firstly in complex geometry we have seen for a real $(1,1)$-form u locally given by

$$u=\sqrt{-1}u_{ij}\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^j$$

It's called positive if matrix $(u_{ij(x)})_{i\times j}$ is semi-positive. Since $(1,1)$-form and $(n-1,n-1)$-form are dual to each other, then we can also define what is positive for $(n-1,n-1)$-form. More explicitly, for a $(n-1,n-1)$-form v locally given by

$$v=v_{ij}\mathrm{d}\widehat{z^i\wedge\mathrm{d}\overline{z}^j}$$

where $\mathrm{d}\widehat{z^i\wedge\mathrm{d}\overline{z}^j}$ is a $(n-1,n-1)$-form such that

$$\mathrm{d}\widehat{z^i\wedge\mathrm{d}\overline{z}^j}\wedge\mathrm{d}z^i\wedge\mathrm{d}z^j=(\sqrt{-1})^n\mathrm{d}z^1\wedge\cdots\wedge\mathrm{d}z^n\wedge\mathrm{d}\overline{z}^1\wedge\cdots\wedge\mathrm{d}\overline{z}^n$$

It's called positive if matrix $(v_{ij})_{i\times j}$ is positive.

Definition 2.1.1 (positive current)**.** Let T be a real $(1,1)$-current over X, it's called positive if

$$\langle T,v\rangle\geq 0$$

for any positive $(n-1,n-1)$-form $v\in\mathcal{D}^{n-1,n-1}(X)$.

2.2. **Pluri-subharmonic functions.** Consider $u=\log|z|$, then

$$\frac{\sqrt{-1}}{\pi}\partial\overline{\partial}u=\delta_0$$

Definition 2.2.1. $u\colon\Omega\to[-\infty,\infty]$ is called pluri-subhamonic function, if

1. u is upper semi-continous;
2. For any complex line $L\subseteq\mathbb{C}^n$, $u|_{\Omega\cap L}$ is subharmonic.

Remark 2.2.1. subharmonic i.e. For all $a\in\Omega,\xi\in\mathbb{C}^n$ with $|\xi|\ll 1$, one has

$$u(a)\leq\int_0^{2\pi}u(a+e^{\sqrt{-1}\theta}\xi)\mathrm{d}\theta$$

Notation 2.2.1. The space of pluri-subhamonic functions on Ω is denoted by $\mathrm{Psh}(\Omega)$.

Proposition 2.2.1. Here are some basic properties of pluri-subhamonic functions

1. pluri-subhamonic function is subharmonic.
2. $u\in\mathrm{Psh}(\Omega)$, if Ω is connected, then $u\equiv-\infty$ or $u\in L^1_{\mathrm{loc}}(\Omega)$.
3. If $\{u_k\}$ is a sequence of pluri-subhamonic functions, u_k descends to u, then u is pluri-subhamonic.
4. Let $u\in\mathrm{Psh}(\Omega)\cap L^1_{\mathrm{loc}}(\Omega)$ and $(\rho_\varepsilon)_{\epsilon>0}$ be a family of modifiers, then $u_\varepsilon:=u*\rho_\varepsilon\in C^\infty(U_\varepsilon)\cap\mathrm{Psh}(U_\varepsilon)$, and u_ε descends to u as $\varepsilon\to 0$.

5. If $u \in C^2(\Omega)$, then $u \in \mathrm{Psh}(\Omega)$ if and only if $(\frac{\partial^2 u}{\partial z^i \partial \bar{z}^j})$ is semi-positive, that is $\sqrt{-1}\partial\overline{\partial}u \geq 0$.
6. (*a*) Let $u \in \mathrm{Psh}(\Omega) \cap L^1_{\mathrm{loc}}$, then $\sqrt{-1}\partial\overline{\partial}u$ is a positive $(1,1)$-current.
   (*b*) Given a distribution φ on Ω, then $\sqrt{-1}\partial\overline{\partial}\varphi \geq 0$ in the sense of current, then $\varphi = u$ for some $u \in \mathrm{Psh}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$.

Appendix A. Topological vector spaces

In this appendix we mainly follows [Rud74].

A.1. **Basic definitions and first properties.**

All vector spaces are assumed to be over $\mathbb{R}$ or $\mathbb{C}$.

Definition A.1.1 (balance)**.** Let X be a vector space, a set $B \subset X$ is said to be balanced if $\alpha B \subset B$ for all scalars α with $|\alpha| < 1$.

Definition A.1.2 (invariant metric)**.** A metric d on a vector space X is called invariant, if

$$d(x+z, y+z) = d(x, y)$$

for all $x, y, z \in X$.

Definition A.1.3. A topological vector space is a vector space X with topology τ such that

1. every point of X is closed set;
2. the vector space operations are continuous with respect to τ.

Remark A.1.1. In the vector space context, the term local base always means a local base at 0, that is a collection $\mathscr{B}$ of neighborhoods of 0 such that every neighborhoods of 0 contains a member of $\mathscr{B}$.

Definition A.1.4 (types of topological vector space)**.** Let X be a topological vector space with topology τ.

1. X is locally convex if there is a local base $\mathscr{B}$ whose members are convex.
2. X is locally bounded if 0 has a bounded neighborhood.
3. X is locally compact if 0 has a neighborhood whose closure is compact.
4. X is metrizable if τ is compatible with some metric d.
5. X is a F-space if its topology is induced by a complete invariant metric d.
6. X is a Fréchet space if X is a locally convex F-space.
7. X is normable if there is a norm on X such that the metric induced by the norm is compatible with τ.
8. X has Heine-Borel property if every closed and bounded subset of X is compact.

Remark A.1.2. Here is a list of some relations between these properties of a topological vector space X.

1. If X is locally bounded, then X has a countable local base.
2. X is metrizable if and only if X has a countable local base.
3. X is normable if and only if X is locally convex and locally bounded.
4. X has finite dimension if and only if X is locally compact.
5. If a locally bounded space X has the Heine-Borel property, then X has finite dimension.

A.2. **Seminorms and local convexity.**

Definition A.2.1 (seminorm)**.** A seminorm on a vector space X is a real-valued function p on X such that

1. $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$;
2. $p(\alpha x) = |\alpha| p(x)$ for all $x \in X$ and scalars α;
3. $p(x) \neq 0$ if $x \neq 0$.

Definition A.2.2 (separating)**.** A family $\mathscr{P}$ of seminorms on X is said to be separating if to each $x \neq 0$ corresponds at least one $p \in \mathscr{P}$ with $p(x) \neq 0$.

Seminorms are closely to local convexity in two ways: In every locally convex space there exists a separating family of continous seminorms. Conversely, if $\mathscr{P}$ is a separating family of seminorms on a vector space X, then $\mathscr{P}$ can be used to define a locally convex topology on X with the property that every $p \in \mathscr{P}$ is continous.

Theorem A.2.1. Suppose $\mathscr{P}$ is a separating family of seminorms on a vector space X, associate to each $p \in \mathscr{P}$ and to each positive integer n the set

$$V(p,n) = \{x \colon p(x) < \frac{1}{n}\}$$

Let $\mathscr{B}$ be the collection of all finite intersections of the sets $V(p,n)$, then $\mathscr{B}$ is a convex balanced local base for a topology τ on X, which turns X into a locally convex space such that

1. every $p \in \mathscr{P}$ is continous;
2. a set $E \subset X$ is bounded if and only if every $p \in \mathscr{P}$ is bounded on E.

Remark A.2.1*.* If $\mathscr{P} = \{p_i \mid i = 1, 2, 3 \dots\}$ is a countable separating family of seminorms on X, then $\mathscr{P}$ induces a topology τ with a countable local base, thus it's metrizable. However, in this case, a compatible translation invariant metric can be defined directly in terms of $\{p_i\}$, that is

$$d(x,y) = \sum_{i=1}^{\infty} \frac{2^{-i} p_i(x-y)}{1 + p_i(x-y)}$$

A.3. **Examples of Fréchet space.** In this section we introduce some function spaces which will be used in later work with distributions. The notation $C^\infty(\Omega)$ denotes the vector space consisting of smooth functions over Ω. If $K \Subset \Omega$, then $\mathcal{D}(K)$ denotes subspace of $C^\infty(\Omega)$ consisting of those smooth functions with support in K. Now we're going to define topology on it to make it into a Fréchet space such that $\mathcal{D}(K)$ is a closed subspace of $C^\infty(\Omega)$ whenever $K \Subset \Omega$, and thus $\mathcal{D}(K)$ is also a Fréchet space.

To do this, choose a exhaustion K_i of Ω, and define seminorms $p_N, N = 1, 2, \dots$ by setting

$$p_N(f) = \max\{|D^\alpha f(x)| : x \in K_N, |\alpha| \leq N\}$$

They define a metrizable locally convex topology on $C^\infty(\Omega)$ according Remark A.2.1. For each $x \in \Omega$, the functional $f \to f(x)$ is continous with respect to this topology, and note that $\mathcal{D}(K)$ is the intersection of the null spaces of these functionals, as x ranges over the complement of K, thus $\mathcal{D}(K)$ is closed in $C^\infty(M)$.

To show $C^\infty(\Omega)$ is a Fréchet space, now it suffices to show it's complete. Note that a local base is given by the sets

$$V_N = \{f \in C^\infty(\Omega) : p_N(f) < \frac{1}{N}\}$$

If $\{f_i\}$ is a Cauchy sequence in $C^\infty(\Omega)$ and if N is fixed, then $f_i - f_j \in V_N$ if i and j are sufficiently large. Thus $|D^\alpha f_i - D^\alpha f_j| < \frac{1}{N}$ on K_N if $|\alpha| \leq N$. It follows that each $D^\alpha f_i$ converges uniformly on compact subsets of Ω to a function g_α. In particular, f_i converges to g_0. It's clear $g_0 \in C^\infty(\Omega)$ with $g_\alpha = D^\alpha g_0$, and $f_i \to g$ with respect to the topology of $C^\infty(\Omega)$.

Appendix B. Distribution theory

B.1. **Introduction.** The theory of distributions frees differential calculus from certain problems arisen from non-smooth functions exist. This is done by extending it to a class of objects called distribution which is much larger than the class of smooth functions.

To motivate the definitions to come, let's firstly consider $n = 1$ and the integrals that follow are taken with respect to Lebesgue measure. Let $\mathcal{D} = \mathcal{D}(\mathbb{R})$ denotes the vector space consisting of all $\phi \in C^\infty(\mathbb{R})$ with compact support. Then $\int f\phi$ exists for every locally integerable f and every $\phi \in \mathcal{D}$. If f happens to be continously differentiable, then

$$\int f'\phi = -\int f\phi'$$

and if f is smooth, then

$$\int f^{(k)}\phi = (-1)^k \int f\phi^{(k)}$$

for arbitrary $k \in \mathbb{Z}_{\geq 1}$. However, the right hands of above equalities make sense whether f is smooth or not, then we can therefore assign a "k-th derivative" to every f that is locally integerable as a functional on $\mathcal{D}$ that sends ϕ to $(-1)^k \int f\phi^{(k)}$, and distributions will be defined to be those linear functionals on $\mathcal{D}$ that are continuous with respect to a certain topology we will define later.

B.2. **Test function spaces.** Let $\Omega \subset \mathbb{R}^n$ be a non-empty open set, for each compact $K \subset \Omega$, we have already seen $\mathcal{D}(K)$ is a Fréchet space in section A.3. The union of the spaces $\mathcal{D}(K)$ as K ranges over all compact subsets of Ω, is called test function space $\mathcal{D}(\Omega)$. It's clear $\mathcal{D}(\Omega)$ is a vector space, now let's introduce the norms

$$\|\phi\|_N = \max\{|D^\alpha \phi(x)| : x \in \Omega, |\alpha| \leq N\}$$

where $\phi \in \mathcal{D}(\Omega)$ and $N \geq 0$. According to Theorem A.2.1, it gives a locally convex topology on $\mathcal{D}(\Omega)$.

The restriction of these norms to any fix $\mathcal{D}(K) \subset \mathcal{D}(\Omega)$ induce the same topology on $\mathcal{D}(K)$ as topology we have given before, but this topology has the disadvantages of not being complete. For example, take $n = 1, \Omega = \mathbb{R}$ and pick $\phi \in \mathcal{D}(\Omega)$ with support in $[0, 1]$ and $\phi > 0$ in $(0, 1)$. Consider

$$\psi_m(x) = \phi(x-1) + \frac{1}{2}\phi(x-2) + \cdots + \frac{1}{m}\phi(x-m)$$

Then $\{\psi_m\}$ is a Cauchy sequence in the suggested topology of $\mathcal{D}(\Omega)$, but the limit of $\{\psi_m\}$ does not have compact support, hence not in $\mathcal{D}(\Omega)$. In particular, $\mathcal{D}(\Omega)$ is not a Fréchet space with respect to above topology.

Now we define another locally convex topology τ on $\mathcal{D}(\Omega)$ which is complete, the fact that this τ is not metrizable is only minor inconvenience, as we will see.

Definition B.2.1. Let Ω be a non-empty open set in $\mathbb{R}^n$.

1. For every compact $K \subset \Omega$, τ_K denotes the Fréchet space topology of $\mathcal{D}(K)$;
2. β is the collection of all convex balanced sets $W \subset \mathcal{D}(\Omega)$ such that $\mathcal{D}(K) \cap W \in \tau_K$ for every compact $K \subset \Omega$;
3. τ is the collection of all unions of sets of the form $\phi + W$, with $\phi \in \mathcal{D}(\Omega)$ and $W \in \beta$.

Definition B.2.2 (distribution)**.** A linear functional on $\mathcal{D}(\Omega)$ which is contінous with respect to the topology τ described in Definition B.2.1 is called a distribution in Ω.

Notation B.2.1. The space of all distributions in Ω is denoted by $\mathcal{D}'(\Omega)$.

Theorem B.2.1. If Λ is a linear functional on $\mathcal{D}(\Omega)$, the following two conditions are equivalent:

1. $\Lambda \in \mathcal{D}'(\Omega)$;
2. To every compact $K \subset \Omega$ corresponds a non-negative integer N and a constant $C < \infty$ such that the inequality
$$|\Lambda\phi| \leq C\|\phi\|_N$$
holds for every $\phi \in \mathcal{D}(K)$.

Example B.2.1. Suppose f is a locally integerable function in Ω, consider
$$\Lambda_f(\phi) = \int_\Omega f\phi \mathrm{d}x$$
where $\phi \in \mathcal{D}(\Omega)$. Note that $|\Lambda_f(\phi)| \leq (\int_K |f|)\|\phi\|_0$, then according to Theorem B.2.1 one has $\Lambda_f \in \mathcal{D}'(\Omega)$. It's customary to identify the distribution Λ_f with the function f and to say that such distributions are functions.

Example B.2.2. If μ is a positive measure on Ω with $\mu(K) < \infty$ for every compact $K \subset \Omega$, then
$$\Lambda_\mu(\phi) = \int_\Omega \phi \mathrm{d}\mu$$
defines a distribution Λ_μ in Ω, which is usually identified with μ.

Appendix C. Subharmonic functions

C.1. **Definition.** Let u is a Borel function on $\overline{B}(a,r)$ which is bounded above or below, consider the mean values of u over the ball or sphere

$$\mu_B(u;a,r) = \frac{1}{\alpha_m r^m}\int_{B(a,r)} u(x)\mathrm{d}\lambda(x)$$
$$\mu_S(u;a,r) = \frac{1}{\sigma_{m-1} r^{m-1}}\int_{S(a,r)} u(x)\mathrm{d}\sigma(x)$$

As $\mathrm{d}\lambda = \mathrm{d}r\mathrm{d}\sigma$, these mean values are related by

$$\begin{aligned}\mu_B(u;a,r) &= \frac{1}{\alpha_m r^m}\int_0^r \sigma_{m-1}t^{m-1}\mu_S(u;a,t)\mathrm{d}t \\ &= m\int_0^1 t^{m-1}\mu_S(u;a,rt)\mathrm{d}t\end{aligned}$$

Theorem C.1.1 (subharmonic)**.** Let $u\colon \Omega \to [-\infty,\infty]$ be an upper semi-continous function, the following various forms of mean value inequalities are equivalent:

1. $u(a) \le \mu_S(u;a,r)$ holds for all $\overline{B}(a,r) \subset \Omega$;
2. $u(a) \le \mu_B(u;a,r)$ holds for all $\overline{B}(a,r) \subset \Omega$;
3. For $a \in \Omega$, there exists a sequence (r_n) descends to 0 such that
$$u(a) \le \mu_B(u;a,r_n)$$
holds for arbitrary n.
4. For $a \in \Omega$, there exists a sequence (r_n) descends to 0 such that
$$u(a) \le \mu_S(u;a,r_n)$$
holds for arbitrary n.

A function u satisfying one of the above properties is said to be a subharmonic on Ω.

Notation C.1.1. The set of subharmonic functions over Ω is denoted by $\mathrm{Sh}(\Omega)$

C.2. **First properties.**

Theorem C.2.1 (maximum principle)**.** If u is subharmonic in Ω, then

$$\sup_\Omega u = \limsup_{\Omega\ni z\to\partial\Omega\cup\{\infty\}} u(z)$$

and $\sup_K u = \sup_{\partial K} u(z)$ for every compact subset $K \subset \Omega$.

Theorem C.2.2. For any decreasing sequence (u_k) of subharmonic functions, the limit $u = \lim u_k$ is subharmonic.

Theorem C.2.3. If Ω is connected and $u \in \mathrm{Sh}(\Omega)$, then either $u \equiv -\infty$ or $u \in L^1_{\mathrm{loc}}(\Omega)$.

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