

# COMPLEX GEOMETRY I: COMPLEX MANIFOLD AND ITS GEOMETRY

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## Part 1. Local theory

In this part, we mainly follows [Huy05] and [Dem12].

### 1. REVIEW OF COMPLEX ANALYSIS

**1.1. One variable case.** We first give a quick review about basic results in holomorphic functions of one variable. Fix an open subset  $U \subseteq \mathbb{C}$ . There are too many ways to define a holomorphic function, and all of them are equivalent.

**Definition 1.1.1** (holomorphic). A function  $f: U \rightarrow \mathbb{C}$  is called holomorphic at  $z_0 \in U$ , if there exists an open ball  $B_\varepsilon(z_0) \subseteq U$  with  $\varepsilon > 0$  such that  $f|_{B_\varepsilon(z_0)}$  can be written as convergent power series, that is

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \quad z \in B_\varepsilon(z_0)$$

$f$  is holomorphic on  $U$ , if  $f$  is holomorphic at any point of  $U$ .

*Remark 1.1.1* (Cauchy-Riemann equation). The second definition is given by Cauchy-Riemann equation. To be explicit, for a function  $f: U \rightarrow \mathbb{C}$ , we can regard it as a function defined on  $\mathbb{R}^2$ , and write it as  $f(x, y) = u(x, y) + \sqrt{-1}v(x, y)$ , where  $u, v$  are real-valued functions, then  $f$  is holomorphic if and only if  $u, v$  are continuously differentiable and satisfy the following Cauchy-Riemann equations:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

If we introduce the following two operators

$$\begin{aligned} \frac{\partial}{\partial z} &:= \frac{1}{2} \left( \frac{\partial}{\partial x} - \sqrt{-1} \frac{\partial}{\partial y} \right) \\ \frac{\partial}{\partial \bar{z}} &:= \frac{1}{2} \left( \frac{\partial}{\partial x} + \sqrt{-1} \frac{\partial}{\partial y} \right) \end{aligned}$$

Then Cauchy-Riemann equation is equivalent to  $\frac{\partial f}{\partial \bar{z}} = 0$ . Indeed,

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2} \left( \frac{\partial f}{\partial x} + \sqrt{-1} \frac{\partial f}{\partial y} \right) \\ &= \frac{1}{2} \left( \frac{\partial u}{\partial x} + \sqrt{-1} \frac{\partial v}{\partial x} + \sqrt{-1} \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \right) \\ &= 0 \end{aligned}$$

*Remark 1.1.2* (Cauchy integral formula). The third definition is given by Cauchy integral formula. To be explicit, a function  $f: U \rightarrow \mathbb{C}$  is holomorphic

if and only if  $f$  is continuously differentiable and for any  $B_\varepsilon(z_0) \subseteq U$ , the following formula holds

$$f(z_0) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial B_\varepsilon(z_0)} \frac{f(z)}{z - z_0} dz$$

Here are some standard facts in complex analysis, which can be found in any textbook.

**Theorem 1.1.1** (maximum principle). Let  $U \subseteq \mathbb{C}$  be open and connected. If  $f: U \rightarrow \mathbb{C}$  is holomorphic and non-constant, then  $|f|$  has no local maximum in  $U$ .

**Theorem 1.1.2** (identity theorem). If  $f, g: U \rightarrow \mathbb{C}$  are two holomorphic functions a connected open subset  $U \subseteq \mathbb{C}$  such that  $f(z) = g(z)$  for all  $z$  in a non-empty subset  $V$  of  $U$ , then  $f = g$ .

**Theorem 1.1.3** (Riemann extension theorem). Let  $f: B_\varepsilon(z_0) - \{z_0\} \rightarrow \mathbb{C}$  be a bounded holomorphic function, then  $f$  can be extended to a holomorphic function  $f: B_\varepsilon(z_0) \rightarrow \mathbb{C}$ .

**Theorem 1.1.4** (Riemann mapping theorem). Let  $U \subseteq \mathbb{C}$  be a simply connected proper open subset. Then  $U$  is biholomorphic to the unit ball.

**Theorem 1.1.5** (Liouville). Every bounded holomorphic function  $f: \mathbb{C} \rightarrow \mathbb{C}$  is constant.

*Remark 1.1.3.* Liouville theorem implies that  $\mathbb{C}$  is not biholomorphic to the unit ball. It's a striking difference to the real case since we know unit ball is homeomorphic to  $\mathbb{R}$ .

**1.2. Several variables case.** Now let  $U$  be an open subset of  $\mathbb{C}^n$ . For any  $w \in U$ , a polydisc  $B_\varepsilon(w) = \{z : |z^i - w_i| < \varepsilon_i\}$ , where  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ .

**Definition 1.2.1** (holomorphic). A function  $f: U \rightarrow \mathbb{C}$  is called holomorphic at point  $w \in U$ , if there exists a polydisc  $B_\varepsilon(w) \subseteq U$  such that the restriction of  $f|_{B_\varepsilon(w)}$  is given by power series

$$\sum_{i_1, \dots, i_n=0}^{\infty} a_{i_1 \dots i_n} (z_1 - w_1)^{i_1} \dots (z_n - w_n)^{i_n}$$

*Remark 1.2.1* (equivalent definitions of holomorphic function).

(1) A function  $f: U \rightarrow \mathbb{C}$  is holomorphic, if it satisfies Cauchy-Riemann equations for all coordinates  $z^i = x^i + \sqrt{-1}y^i$ , that is

$$\frac{\partial f}{\partial \bar{z}^i} = 0, \quad i = 1, 2, \dots, n$$

where  $\frac{\partial}{\partial \bar{z}^i} := \frac{1}{2}(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i})$

- (2) A function  $f: U \rightarrow \mathbb{C}$  is holomorphic if and only if  $f$  is continuously differentiable and for any  $z_0 \in U$ , the following formula holds

$$f(z) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\partial B_\varepsilon(w)} \frac{f(w)}{(z_1 - w_1) \dots (z_n - w_n)} dw_1 \dots dw_n$$

*Remark 1.2.2.* Other results such as maximum theorem, identity theorem and Liouville theorem generalize easily to the higher dimension. A version of Riemann extension still holds true. However, Riemann mapping theorem fails.

*Exercise 1.2.1.* Show that polydisc  $B_{(1,1)}(0) \subseteq \mathbb{C}^2$  is not biholomorphic to the unit disk  $D = \{z \in \mathbb{C}^2 : \|z\| < 1\}$ .

**Theorem 1.2.1** (Hartogs' theorem). Suppose  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$  and  $\varepsilon' = (\varepsilon'_1, \dots, \varepsilon'_n)$  are given such that for all  $i$  one has  $\varepsilon'_i < \varepsilon_i$ . If  $n > 1$ , then any holomorphic map  $f: B_\varepsilon(0) \setminus \overline{B_{\varepsilon'}(0)} \rightarrow \mathbb{C}$  can be uniquely extended to a holomorphic map  $f: B_\varepsilon(0) \rightarrow \mathbb{C}$ .

*Remark 1.2.3.* This is only valid in dimension at least two.

**Definition 1.2.2** (holomorphic). A function  $f: U \rightarrow \mathbb{C}^n$  is called holomorphic if all coordinate functions  $f_1, \dots, f_n$  are holomorphic functions  $U \rightarrow \mathbb{C}$ .

**Definition 1.2.3** (biholomorphic). A holomorphic map  $f: U \rightarrow V$  between two open subsets  $U, V \subseteq \mathbb{C}^n$  is biholomorphic if  $f$  is bijective and its inverse  $f^{-1}$  is also holomorphic.

**Definition 1.2.4** (complex Jacobian). Let  $f: U \rightarrow \mathbb{C}^n$  be a holomorphic map, the complex Jacobian of  $f$  at point  $z \in U$  is the matrix

$$J_{\mathbb{C}}(f)(z) := \left( \frac{\partial f_i}{\partial z^j}(z) \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$$

where  $f_i = z^i \circ f$ .

*Remark 1.2.4.* For each  $z \in U$ , the smooth map  $f: U \subseteq \mathbb{C}^m = \mathbb{R}^{2m} \rightarrow \mathbb{C}^n = \mathbb{R}^{2n}$  induces a  $\mathbb{R}$ -linear map, which is denoted by  $J_{\mathbb{R}}(f)(z): T_z \mathbb{R}^{2m} \rightarrow T_{f(z)} \mathbb{R}^{2n}$ . Suppose  $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}_{1 \leq i \leq m}$  and  $\{\frac{\partial}{\partial r^j}, \frac{\partial}{\partial s^j}\}_{1 \leq j \leq n}$  are local frames of  $T_z \mathbb{R}^{2m}$  and  $T_{f(z)} \mathbb{R}^{2n}$  respectively, then with respect to these basis one has

$$J_{\mathbb{R}}(f)(z) = \begin{pmatrix} \left( \frac{\partial u_i}{\partial x^j} \right) & \left( \frac{\partial u_i}{\partial y^j} \right) \\ \left( \frac{\partial v_i}{\partial x^j} \right) & \left( \frac{\partial v_i}{\partial y^j} \right) \end{pmatrix}$$

where  $u_i = r^i \circ f$  and  $v_i = s^i \circ f$ . If we consider its  $\mathbb{C}$ -linear extension, with respect to basis  $\{\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i}\}_{1 \leq i \leq m}$  and  $\{\frac{\partial}{\partial w^j}, \frac{\partial}{\partial \bar{w}^j}\}_{1 \leq j \leq n}$ , it can be written as

$$J_{\mathbb{R}}(f)(z) = \begin{pmatrix} \left( \frac{\partial f_i}{\partial z^j} \right) & \left( \frac{\partial f_i}{\partial \bar{z}^j} \right) \\ \left( \frac{\partial \bar{f}_i}{\partial z^j} \right) & \left( \frac{\partial \bar{f}_i}{\partial \bar{z}^j} \right) \end{pmatrix}$$

In particular, if  $f$  is holomorphic, then  $\det J_{\mathbb{R}}(f) = \det J_{\mathbb{C}}(f) \det \overline{J_{\mathbb{C}}(f)} = |\det J_{\mathbb{C}}(f)|^2 \geq 0$ .

**Definition 1.2.5** (regular value). Let  $U \subseteq \mathbb{C}^m$  be an open subset and let  $f: U \rightarrow \mathbb{C}^n$  be a holomorphic map,  $z \in U$  is called regular point, if  $J_{\mathbb{C}}(f)(z)$  is surjective. If every point  $z \in f^{-1}(w)$  is regular point, then  $w$  is called a regular value.

*Remark 1.2.5.* In particular, if  $f^{-1}(w) = \emptyset$ , then  $w$  is also called a regular value.

**Theorem 1.2.2** (inverse function theorem). Let  $f: U \rightarrow V$  be a holomorphic map between two open subsets  $U, V \subseteq \mathbb{C}^n$ . If  $z \in U$  is a regular point, then there exist open subsets  $z \in U' \subseteq U$  and  $f(z) \in V' \subseteq V$  such that  $f$  induces a biholomorphic map  $f: U' \rightarrow V'$ .

**Theorem 1.2.3** (implicit function theorem). Let  $U \subseteq \mathbb{C}^m$  be an open subset and let  $f: U \rightarrow \mathbb{C}^n$  be a holomorphic map, where  $m \geq n$ . Suppose  $z_0 \in U$  is a point such that

$$\det(J_{\mathbb{C}}(f)(z_0)) \neq 0$$

Then there exist open subsets  $U_1 \subseteq \mathbb{C}^{m-n}, U_2 \subseteq \mathbb{C}^n$  and a holomorphic map  $g: U_1 \rightarrow U_2$  such that  $U_1 \times U_2 \rightarrow U$  and  $f(z) = f(z_0)$  if and only if  $g(z_{n+1}, \dots, z_m) = (z_1, \dots, z_n)$ .

**Corollary 1.2.1.** Let  $U \subseteq \mathbb{C}^m$  be an open subset and  $f: U \rightarrow \mathbb{C}^n$  be a holomorphic map. Suppose that  $z_0 \in U$  such that  $J_{\mathbb{C}}(f)(z_0)$  has maximal rank. Then

- (1) If  $m \geq n$ , then there exists a biholomorphic map  $h: V \rightarrow U'$ , where  $U'$  is an open subset of  $U$  containing  $z_0$ , and  $V$  is an open subset of  $\mathbb{C}^n$  containing  $f(z_0)$ , such that  $f(h(z_1, \dots, z_n)) = (z_1, \dots, z_n)$ .
- (2) If  $m \leq n$ , then there exists a biholomorphic map  $g: V \rightarrow V'$ , where  $V, V'$  are open subsets of  $\mathbb{C}^n$  containing  $f(z_0)$ , such that  $g(f(z)) = (z_1, \dots, z_m, 0, \dots, 0)$ .

## 2. ALGEBRAIC AND ANALYTIC

## 2.1. Algebraic germ.

2.1.1. *Weierstrass' theorems.* Let  $f: B_\varepsilon(0) \rightarrow \mathbb{C}$  be a holomorphic function defined on polydisc  $B_\varepsilon(0)$ . For any  $w = (z_2, \dots, z_n)$  we denote  $f_w(z_1)$  the function  $f(z_1, \dots, z_n)$ . Now we're going to show that all zeros of  $f$  are caused by a factor of  $f$  which has the form of a Weierstrass polynomial.

**Definition 2.1.1** (Weierstrass polynomial). A Weierstrass polynomial is a polynomial in  $z_1$  of the form

$$z_1^d + \alpha_1(w)z_1^{d-1} + \dots + \alpha_d(w)$$

where coefficients  $\alpha_i(w)$  are holomorphic functions on some small disc in  $\mathbb{C}^{n-1}$  vanishing at the origin.

*Remark 2.1.1.* Recall the one variable case, any holomorphic function  $f(z)$  with a zero of order  $d$  at the origin can be written as  $z^d h(z)$ , where  $h(0) \neq 0$ . In fact,  $z^d$  is a Weierstrass polynomial since in this case,  $\alpha_i$  are constants which vanish at origin, that's exactly zero.

**Theorem 2.1.1** (Weierstrass preparation theorem). Let  $f: B_\varepsilon(0) \rightarrow \mathbb{C}$  be a holomorphic function on the polydisc  $B_\varepsilon(0)$ . Assume  $f(0) = 0$  and  $f_0(z_1) \neq 0$ . Then there exists a unique Weierstrass polynomial  $g_w(z_1)$  and a holomorphic function  $h$  on some smaller polydisc  $B_{\varepsilon'}(0) \subseteq B_\varepsilon(0)$  such that  $f = gh$  and  $h(0) \neq 0$ .

*Proof.* See Proposition 1.1.6 in Page8 of [Huy05].  $\square$

**Theorem 2.1.2** (Weierstrass division theorem). Let  $f \in \mathcal{O}_{\mathbb{C}^n,0}$  and let  $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$  be a Weierstrass polynomial of degree  $d$ . Then there exist  $r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$  of degree  $< d$  and  $h \in \mathcal{O}_{\mathbb{C}^n,0}$  such that  $f = gh + r$ . The functions  $h$  and  $r$  are uniquely determined.

*Proof.* See Proposition 1.1.17 in Page15 of [Huy05].  $\square$

2.1.2. *Stalk of sheaf of holomorphic functions.* Let's use  $\mathcal{O}_{\mathbb{C}^n}$  to denote the sheaf<sup>1</sup> of holomorphic functions on  $\mathbb{C}^n$ , and  $\mathcal{O}_{\mathbb{C}^n,0}$  to denote its stalk at origin. The elements in  $\mathcal{O}_{\mathbb{C}^n,0}$  are called germs. It's clear  $\mathcal{O}_{\mathbb{C}^n,0}$  is a local ring with maximal ideal  $\mathfrak{m}$  consisting of all functions that vanish at origin, which implies units in  $\mathcal{O}_{\mathbb{C}^n,0}$  are functions that don't vanish at origin.

By using Weierstrass preparation theorem, one can derive more about algebraic properties of  $\mathcal{O}_{\mathbb{C}^n,0}$ . For example, Weierstrass preparation theorem can be rephrased by saying that after an appropriate coordinate choice any function  $f \in \mathcal{O}_{\mathbb{C}^n,0}$  can be uniquely written as  $f = gh$ , where  $h \in \mathcal{O}_{\mathbb{C}^n,0}$  is a unit and  $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$  is a Weierstrass polynomial. Furthermore, it also shows the following important property.

<sup>1</sup>Sheaf and its cohomology are important tools we will use once and again, if you're not familiar with it, see Appendix A.



**Theorem 2.1.3.** The local ring  $\mathcal{O}_{\mathbb{C}^n,0}$  is a UFD.

*Proof.* We prove the assumption by induction on  $n$ . For  $n = 0$ , the ring  $\mathcal{O}_{\mathbb{C}^0,0} = \mathbb{C}$  is a field, and thus a UFD. Suppose that  $\mathcal{O}_{\mathbb{C}^{n-1},0}$  is a UFD, for  $f \in \mathcal{O}_{\mathbb{C}^n,0}$  we choose coordinates such that Weierstrass preparation theorem is applied, that is  $f = gh$ , where  $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$  is a Weierstrass polynomial and  $h$  is a unit in  $\mathcal{O}_{\mathbb{C}^n,0}$ . By induction we have  $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$  is UFD, then  $g$  can be written as a product of irreducible elements of  $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ . All that is left to show is that any irreducible element in  $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$  is also irreducible as an element in  $\mathcal{O}_{\mathbb{C}^n,0}$ .

Assume  $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$  is a Weierstrass polynomial which is written as the product of non-units  $g_i \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ . There are two cases:

- (1)  $g_i \in \mathcal{O}_{\mathbb{C}^{n-1},0}$ . By induction hypothesis,  $g_i$  can be written as the product of irreducible elements of  $\mathcal{O}_{\mathbb{C}^{n-1},0}$ , which are also irreducible in  $\mathcal{O}_{\mathbb{C}^n,0}$ .
- (2)  $g_i \notin \mathcal{O}_{\mathbb{C}^{n-1},0}$ . In this case,  $g_i$  satisfies the hypothesis of Weierstrass preparation theorem since  $g$  is a Weierstrass polynomial, then  $g_i$  is non-trivial on the  $z_1$ -line. So we can write  $g_i = \tilde{g}_i h_i$ , where  $\tilde{g}_i$  is also Weierstrass polynomial.

Note that degree of  $g$  as a polynomial in  $z_1$  is finite, then repeating above process leads to a decomposition, with factors are either irreducible Weierstrass polynomials or elements in  $\mathcal{O}_{\mathbb{C}^{n-1},0}$ .

Now it suffices to show any irreducible Weierstrass polynomial  $g$  is actually irreducible as an element of  $\mathcal{O}_{\mathbb{C}^n,0}$ . Suppose  $g = f_1 f_2$ , where  $f_1, f_2 \in \mathcal{O}_{\mathbb{C}^n,0}$  are non-units. We apply Weierstrass preparation theorem to obtain  $f_i = g_i h_i, i = 1, 2$ , and thus  $g = (g_1 g_2)(f_1 f_2)$ . By uniqueness one has  $g = g_1 g_2$ , which contradicts to the irreducibility of  $g$  as an element of  $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ .  $\square$

Another important fact is that  $\mathcal{O}_{\mathbb{C}^n,0}$  is noetherian, which follows from Weierstrass division theorem.

**Theorem 2.1.4.** The local UFD  $\mathcal{O}_{\mathbb{C}^n,0}$  is noetherian.

*Proof.* We prove the assumption by induction on  $n$ . For  $n = 0$ , it's clear since any field is noetherian. Suppose that  $\mathcal{O}_{\mathbb{C}^{n-1},0}$  is noetherian, then Hilbert's basis theorem implies  $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$  is also noetherian. Let  $I \subseteq \mathcal{O}_{\mathbb{C}^n,0}$  be a non-trivial idea and choose  $0 \neq f \in I$ . Changing coordinates if necessary, we may assume Weierstrass preparation theorem is applied, that is  $f = gh$ , where  $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$  is a Weierstrass polynomial and  $h$  is a unit in  $\mathcal{O}_{\mathbb{C}^n,0}$ , hence  $g \in I$ . Furthermore, we assume  $I \cap \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$  is generated by  $g_1, \dots, g_k$ .

For any other  $\tilde{f} \in I$ , the Weierstrass division theorem implies  $\tilde{f} = g\tilde{h} + r$  for some  $r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ . Since  $\tilde{f}, g\tilde{h} \in I$ , we have  $r \in I$  and therefore  $r \in I \cap \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ . Thus  $\tilde{f} = g\tilde{h} + \sum_{i=1}^k a_i g_i$ . This shows  $I$  is finitely generated by elements  $g, g_1, \dots, g_k$ .  $\square$

**Corollary 2.1.1.** Let  $g \in \mathcal{O}_{\mathbb{C}^n,0}$  be an irreducible element. If  $f \in \mathcal{O}_{\mathbb{C}^n,0}$  vanishes on  $Z(g) = \{z \mid g(z) = 0\}$ , then  $g$  divides  $f$ .

*Proof.* By Weierstrass preparation theorem we may assume  $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$  is a Weierstrass polynomial with degree  $d$ . By the Weierstrass division theorem one finds  $h \in \mathcal{O}_{\mathbb{C}^n,0}$  and  $r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$  of degree  $< d$  such that  $f = gh + r$ . For  $w \in \mathbb{C}^{n-1}$ , by assumption  $r_w$  vanishes on the zero set  $g_w$ . If all of zeros of  $g_w$  have multiplicity one, then  $r_w \equiv 0$  since  $r_w$  is of degree  $< d$ . Now it suffices to show the set  $w \in \mathbb{C}^{n-1}$  such that  $g_w$  has zeros with multiplicity  $> 1$  is quite “small”.

Since  $g$  is irreducible and  $\frac{\partial g}{\partial z_1}$  is of degree  $d-1$ , there exist elements  $h_1, h_2 \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$  and  $0 \neq \gamma \in \mathcal{O}_{\mathbb{C}^{n-1},0}$  such that  $h_1 g + h_2 \frac{\partial g}{\partial z_1} = \gamma$ . So if  $g_w$  has a zero  $\xi$  of multiplicity  $> 1$ , then  $\gamma(w) = h_1(\xi, w)g_w(\xi) + h_2(\xi, w)\frac{\partial g_w}{\partial z_1}(\xi) = 0$ . This shows such  $w$  is contained in the zero set of a non-trivial holomorphic function  $\gamma \in \mathcal{O}_{\mathbb{C}^{n-1},0}$ . Then the following exercise completes the proof.  $\square$

**Exercise 2.1.1.** Let  $U \subseteq \mathbb{C}^n$  be open and connected. Show that for any non-trivial holomorphic function  $f: U \rightarrow \mathbb{C}$  the complement  $U \setminus Z(f)$  of the zero set of  $f$  is connected and dense in  $U$ .

**2.2. Analytic germ.** For any  $f \in \mathcal{O}_{\mathbb{C}^n,0}$ ,  $Z(f)$  is not well-defined in fact since for another  $g \in \mathcal{O}_{\mathbb{C}^n,0}$ , which represents the same element with  $f$ ,  $Z(f)$  may not equal to  $Z(g)$ . However, there always exists an open neighborhood  $0 \in U \subseteq \mathbb{C}^n$  such that  $Z(f) \cap U = Z(g) \cap U$ .

**Definition 2.2.1** (germ of a set). The germ of a set in the origin  $0 \in \mathbb{C}^n$  is given by a subset  $X \subseteq \mathbb{C}^n$ . Two germs of a set in the origin  $X, Y \subseteq \mathbb{C}^n$  are same if there exists an open neighborhood  $0 \in U \subseteq \mathbb{C}^n$  such that  $X \cap U = Y \cap U$ .

Unless otherwise specified, in this section we only consider germ of a set in the origin, and for convenience we just call it a germ.

**Example 2.2.1.** For  $f \in \mathcal{O}_{\mathbb{C}^n,0}$ ,  $Z(f)$  is a germ.

**Definition 2.2.2** (analytic germ). A germ  $X \subseteq \mathbb{C}^n$  is called analytic if there exist elements  $f_1, \dots, f_k \in \mathcal{O}_{\mathbb{C}^n,0}$  such that  $X = Z(f_1, \dots, f_k) := \bigcap_{i=1}^k Z(f_i)$ .

**Example 2.2.2.** Let  $A$  be a subset of  $\mathcal{O}_{\mathbb{C}^n,0}$ . If we use  $(A)$  to denote the idea generated by  $A$ , then  $(A)$  is finitely generated since  $\mathcal{O}_{\mathbb{C}^n,0}$  is noetherian. Thus  $Z((A))$  is an analytic germ.

**Definition 2.2.3.** Let  $X \subseteq \mathbb{C}^n$  be a germ. Then  $I(X)$  denotes the set of all elements  $f \in \mathcal{O}_{\mathbb{C}^n,0}$  with  $X \subseteq Z(f)$ .

*Remark 2.2.1.* It's clear  $I(X)$  is an idea of  $\mathcal{O}_{\mathbb{C}^n,0}$ .

**Lemma 2.2.1.**

- (1) If  $X_1 \subseteq X_2$  are germs, then  $I(X_2) \subseteq I(X_1)$ .
- (2) If  $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$  are two ideas of  $\mathcal{O}_{\mathbb{C}^n,0}$ , then  $Z(\mathfrak{a}_2) \subseteq Z(\mathfrak{a}_1)$ .

- (3) For any analytic germ one has  $Z(I(X)) = X$ .  
 (4) For any idea  $\mathfrak{a}$  of  $\mathcal{O}_{\mathbb{C}^n,0}$  one has  $\mathfrak{a} \subseteq I(Z(\mathfrak{a}))$

*Proof.* (1), (2) and (4) are clear. For (3). It's clear  $X \subseteq Z(I(X))$ . On the other hand since  $X$  is analytic germ there exist elements  $f_1, \dots, f_k \in \mathcal{O}_{\mathbb{C}^n,0}$  such that  $X = Z(f_1, \dots, f_k)$  as germs, thus  $f_1, \dots, f_k \in I(X)$ , so by (2) we have  $Z(I(X)) \subseteq X = Z(f_1, \dots, f_k)$ . This completes the proof of (3).  $\square$

**Definition 2.2.4** (irreducible germ). An analytic germ is irreducible if the following condition is satisfied: If  $X = X_1 \cup X_2$ , where  $X_1, X_2$  are analytic germs, then  $X = X_1$  or  $X = X_2$ .

**Lemma 2.2.2.** An analytic germ  $X$  is irreducible if and only if  $I(X) \subseteq \mathcal{O}_{\mathbb{C}^n,0}$  is a prime ideal.

*Proof.* If  $X$  is irreducible and  $f_1 f_2 \in I(X)$ , then  $X \subseteq Z(f_1 f_2) = Z(f_1) \cup Z(f_2)$ , so we have  $X = (X \cap Z(f_1)) \cup (X \cap Z(f_2))$  is a union of analytic germs. Then by irreducibility one has  $X = X \cap Z(f_i)$  for some  $i = 1$  or  $i = 2$ , and thus at least one of functions  $f_1$  or  $f_2$  vanishes on  $X$ . This shows  $I(X)$  is prime.

Conversely, if  $I(X)$  is a prime ideal and let  $X = X_1 \cup X_2$  with  $X_1$  and  $X_2$  are analytic. If  $f_i \in I(X_i), i = 1, 2$ , then  $f_1 f_2 \in I(X)$  since

$$X = X_1 \cup X_2 \subseteq Z(f_1) \cup Z(f_2) = Z(f_1 f_2)$$

Hence  $f_1 \in I(X)$  or  $f_2 \in I(X)$ . Thus it suffices to shows that if  $X \neq X_1$  and  $X \neq X_2$ , there exist elements  $f_1 \in I(X_1) \setminus I(X)$  and  $f_2 \in I(X_2) \setminus I(X)$ . This follows immediately from (1) of Lemma 2.2.1.  $\square$

**Corollary 2.2.1.** For  $f \in \mathcal{O}_{\mathbb{C}^n,0}$ ,  $Z(f)$  is irreducible if and only if there exists an irreducible  $g \in \mathcal{O}_{\mathbb{C}^n,0}$  such that  $f = g^k$  for some  $k \in \mathbb{Z}_{>0}$ .

*Proof.* If  $f = g^k$  with  $g$  irreducible, then  $Z(f) = Z(g)$  and if  $h \in I(Z(g))$ , then  $g$  divides  $h$  by Corollary 2.1.1, this shows  $I(Z(g)) = (g)$  and thus  $Z(f)$  is irreducible since  $I(Z(f))$  also equals to  $(g)$ , which is prime. Conversely, if  $f = \prod g_i^{n_i}$ , then  $Z(f) = \bigcup Z(g_i)$ , which cannot be irreducible except for the case  $f = g^k$  for some irreducible  $g$ .  $\square$

**Lemma 2.2.3.** Every decreasing sequences of germs  $\{X_i\}$  is stationary.

*Proof.* Consider its corresponding sequence  $\{I(X_i)\}$ , it's an increasing sequence, thus it's stationary since  $\mathcal{O}_{\mathbb{C}^n,0}$  is noetherian, this completes the proof since for each  $i$ ,  $Z(I(X_i)) = X_i$ .  $\square$

**Theorem 2.2.1.** Every germ  $X$  admits a finite decomposition  $X = \bigcup_{i=1}^N X_i$ , where  $X_i$  is irreducible for each  $i$  and  $X_i \subsetneq X_j$  for  $i \neq j$ . The decomposition is unique apart from the ordering.

*Proof.* It suffices to show uniqueness since existence follows from above lemma. Assume  $X = \bigcup_{l=1}^{N'} X'_l$  is another decomposition, note that  $X_i = \bigcup_{l=1}^{N'} X_i \cap X'_l$ , we must have  $X_i = X_i \cap X'_{l(i)}$  since  $X_i$  is irreducible. Likewise

one has  $X'_{l(i)} \cap X_j$ , so we  $i = j$  since  $X_i \subsetneq X_j$  for  $i \neq j$ , and this shows  $X_i = X'_{l(j)}$   $\square$

### 2.3. Hilbert's Nullstellensatz.

**Theorem 2.3.1.** Let  $X \subseteq \mathbb{C}^n$  be an irreducible analytic germ defined by a prime ideal  $\mathfrak{p} \subseteq \mathcal{O}_{\mathbb{C}^n,0}$ . Then one can find a coordinate system

$$(z_1, \dots, z_{n-d}, z_{n-d+1}, \dots, z_n)$$

such that the projection  $(z_1, \dots, z_n) \rightarrow (z_{n-d+1}, \dots, z_n)$  induces a surjective map of germs  $X \rightarrow \mathbb{C}^d$  and such that the induced ring homomorphism  $\mathcal{O}_{\mathbb{C}^d,0} \rightarrow \mathcal{O}_{\mathbb{C}^n,0}/\mathfrak{p}$  is a finite integral ring extension.

*Proof.* See (4.19) of [Dem12].  $\square$

**Theorem 2.3.2** (Hilbert's Nullstellensatz). If  $I \subseteq \mathcal{O}_{\mathbb{C}^n,0}$  is any ideal, then  $\sqrt{I} = I(Z(I))$ .

*Proof.* It easy to see  $\sqrt{I} \subseteq I(Z(I))$ . Conversely, it suffices to show  $I(Z(I)) \subseteq \mathfrak{p}$  for all prime ideals containing  $I$  since  $\sqrt{I}$  is the intersection of all prime ideals  $\mathfrak{p}$  containing  $I$ . If one has

$$\mathfrak{p} = \sqrt{\mathfrak{p}} = I(Z(\mathfrak{p}))$$

then the results follows from  $Z(\mathfrak{p}) \subseteq Z(I)$ . Thus we reduce the problem to the case to  $I = \mathfrak{p}$  is a prime ideal. For  $f \in I(Z(\mathfrak{p}))$ , by Theorem 2.3.1 there exists an appropriate coordinate system  $(z_1, \dots, z_n)$  such that the induced element  $\bar{f} \in \mathcal{O}_{\mathbb{C}^n,0}/\mathfrak{p}$  satisfies an irreducible algebraic equation  $\bar{f}^k + a_1 \bar{f}^{k-1} + \dots + a_k = 0$  with  $a_i \in \mathcal{O}_{\mathbb{C}^d,0}$ . Since  $f$  vanishes along  $Z(\mathfrak{p})$ , the 0-th coefficient  $a_k$  does as well. As  $Z(\mathfrak{p}) \rightarrow \mathbb{C}^d$  is surjective, this shows  $a_k = 0$ . Hence above algebraic equation cannot be irreducible unless  $k = 1$ . Therefore  $\bar{f} = 0$  and thus  $f \in \mathfrak{p}$ .  $\square$

**Corollary 2.3.1.** There is a one to one correspondence between prime ideals of  $\mathcal{O}_{\mathbb{C}^n,0}$  and irreducible analytic germ given by  $X \mapsto I(X)$  and  $\mathfrak{p} \mapsto Z(\mathfrak{p})$ .

### 2.4. Dimension.

**Definition 2.4.1** (dimension). Let  $X$  be an irreducible analytic germ defined by a prime ideal  $\mathfrak{p} \subseteq \mathcal{O}_{\mathbb{C}^n,0}$ . Then the dimension of  $X$  is defined by  $n - \text{ht}\mathfrak{p}$ , where  $\text{ht}\mathfrak{p}$  is the height of  $\mathfrak{p}$ .

*Remark 2.4.1.* For arbitrary analytic germ is of dimension  $d$  if all its irreducible components are of the same dimension  $d$ .

*Remark 2.4.2.* If  $X \subseteq \mathbb{C}^n$  is an irreducible analytic germ of codimensional 1, then the prime ideal  $\mathfrak{p}$  defining  $X$  is of height 1. A basic result in commutative algebra says any prime ideal of height 1 in a UFD is principle. Therefore,  $\mathfrak{p} = (f)$  for some irreducible  $f \in \mathcal{O}_{\mathbb{C}^n,0}$ .

## 2.5. Meromorphic functions and relatively prime.

**Definition 2.5.1.** Let  $U \subseteq \mathbb{C}^n$  be an open subset. A meromorphic function  $f$  on  $U$  is a function on the complement of a nowhere dense subset  $S \subseteq U$  with the following property: There exist an open covering  $\{U_i\}$  of  $U$  and holomorphic functions  $g_i, h_i: U \rightarrow \mathbb{C}$  with  $h_i|_{U_i \setminus S} \cdot f|_{U_i \setminus S} = g_i|_{U_i \setminus S}$ .

*Remark 2.5.1.* For any  $z \in U$ , the meromorphic function  $f$  in a neighborhood of  $z$  is given by  $g/h$ , where  $g, h \in \mathcal{O}_{\mathbb{C}^n, z}$ . If we assume  $g, h$  are chosen to be relatively prime, then they're unique up to units.

**Proposition 2.5.1.** Let  $f \in \mathcal{O}_{\mathbb{C}^n, 0}$  be irreducible, then for sufficiently small  $\varepsilon$  and  $z \in B_\varepsilon(0)$  the induced element  $f \in \mathcal{O}_{\mathbb{C}^n, z}$  is irreducible.

*Proof.* Suppose  $f \in \mathcal{O}_{\mathbb{C}^n, z}$  is reducible, that is  $f = f_1 f_2$  where  $f_i \in \mathcal{O}_{\mathbb{C}^n, z}$  non-units, i.e.  $f_1(z) = f_2(z) = 0$ . Thus  $\frac{\partial f}{\partial z_1}(z) = \frac{\partial f_1}{\partial z_1}(z) f_2(z) + f_1(z) \frac{\partial f_2}{\partial z_1}(z) = 0$ .

Thus the set of points  $z \in B_\varepsilon(0)$  where  $f$  as an element of  $\mathcal{O}_{\mathbb{C}^n, z}$  is reducible is contained in the analytic set  $Z(f, \frac{\partial f}{\partial z_1})$ . Now it suffices to show it's a proper subset of  $Z(f)$  since  $f$  is irreducible, so is  $Z(f)$ . If not, then  $\frac{\partial f}{\partial z_1}$  would vanish on  $Z(f)$ . Since  $f$  is irreducible, we can apply Corollary 2.1.1 to obtain  $\frac{\partial f}{\partial z_1}$  divides  $f$ , a contradiction.  $\square$

**Proposition 2.5.2.** If  $f, g \in \mathcal{O}_{\mathbb{C}^n, 0}$  are relatively prime, then they're relatively prime in  $\mathcal{O}_{\mathbb{C}^n, z}$ , for  $z$  in a sufficiently small neighborhood of 0.

*Proof.* Without loss of generality, we may assume  $f, g \in \mathcal{O}_{\mathbb{C}^{n-1}, 0}[z_1]$  are Weierstrass polynomials, then  $f$  and  $g$  are relatively prime if and only if their resultant  $R \in \mathcal{O}_{\mathbb{C}^{n-1}}$  has non-zero germ at 0, therefore the germ of  $R$  is also non-zero in a sufficiently small neighborhood of 0.  $\square$

## Part 2. Complex manifold, vector bundle and their geometry

### 3. COMPLEX MANIFOLD

#### 3.1. Basic definitions and properties.

**Definition 3.1.1** (holomorphic atlas). A holomorphic atlas on a smooth manifold is an atlas  $\{(U_\alpha, \varphi_\alpha)\}$  of the form  $\varphi_\alpha : U_\alpha \cong \varphi_\alpha(U_\alpha) \subseteq \mathbb{C}^n$  such that transition functions  $\varphi_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_{\alpha\beta}) \rightarrow \varphi_\alpha(U_{\alpha\beta})$  are holomorphic functions. Furthermore,

- (1) the pair  $(U_\alpha, \varphi_\alpha)$  is called a holomorphic chart.
- (2) two holomorphic atlases are called equivalent, if the union of them is still a holomorphic atlas.

**Definition 3.1.2** (complex manifold). A complex  $n$ -manifold  $X$  is a smooth  $2n$ -manifold admitting an equivalence class of holomorphic atlases.

*Remark 3.1.1.* A complex manifold is called connected, compact, simply-connected and so on, if its underlying smooth manifold has this property.

**Definition 3.1.3** (submanifold). Let  $X$  be a complex  $n$ -manifold and  $Y \subseteq X$  be a smooth manifold of (real) dimension  $2k$ . Then  $Y$  is a complex submanifold if there exists a holomorphic atlas  $\{(U_i, \varphi_i)\}$  of  $X$  such that  $\varphi_i : U_i \cap Y \cong \varphi_i(U_i) \cap \mathbb{C}^k$ .

**Definition 3.1.4** (holomorphic map). Let  $X, Y$  be complex manifolds. A continuous map  $f : X \rightarrow Y$  is a holomorphic map if for any holomorphic charts  $(U, \varphi)$  and  $(U', \varphi')$  of  $X$  and  $Y$  respectively, the map  $\varphi' \circ f \circ \varphi^{-1} : \varphi(f^{-1}(U') \cap U) \rightarrow \varphi'(U')$  is holomorphic.

**Definition 3.1.5** (biholomorphic). Let  $X, Y$  be two complex manifolds.  $X, Y$  are called biholomorphic, if there exists a holomorphic homeomorphism  $f : X \rightarrow Y$ .

**Definition 3.1.6** (holomorphic function). A holomorphic function on complex manifold  $X$  is a holomorphic map  $f : X \rightarrow \mathbb{C}$ .

**Notation 3.1.1.** We always use  $\mathcal{O}_X$  to denote the sheaf of holomorphic functions on complex manifold  $X$ , and use  $\Gamma(U, \mathcal{O}_X)$  to denote sections over open subset  $U \subseteq X$ .

**Proposition 3.1.1.** Let  $X$  be a compact connected complex manifold. Then  $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$ .

*Proof.* It's clear from maximum principle. □

**Definition 3.1.7** (meromorphic function). A meromorphic function on a complex manifold  $X$  is a map  $f : X \rightarrow \coprod_{x \in X} Q(\mathcal{O}_{X,x})$  which associates to any  $x \in X$  an element  $f_x \in Q(\mathcal{O}_{X,x})$  such that for any  $x_0 \in X$  there exists a neighborhood  $U \subseteq X$  and two holomorphic functions  $g, h : U \rightarrow \mathbb{C}$  with  $f_x = g/h$  for all  $x \in U$ .

**Notation 3.1.2.** We always use  $\mathcal{K}_X$  to denote the sheaf of meromorphic functions on complex manifold  $X$ , and use  $K(X)$  to denote  $\Gamma(X, \mathcal{K}_X)$ .

**Definition 3.1.8** (algebraic dimension). The algebraic dimension of a compact connected complex manifold  $X$  is  $a(X) := \text{trdeg}_{\mathbb{C}} K(X)$ .

**Proposition 3.1.2** (Siegel). Let  $X$  be a compact connected complex  $n$ -manifold. Then

$$\text{trdeg}_{\mathbb{C}} K(X) \leq n$$

*Proof.* See Proposition 2.1.9 in Page 54 of [Huy05]. □

### 3.2. Analytic subvariety.

**Definition 3.2.1** (analytic subvariety). Let  $X$  be a complex manifold. An analytic subvariety of  $X$  is a closed subset  $Y \subseteq X$  such that for any  $x \in Y$  there exists an open neighborhood  $x \in U \subseteq X$  such that  $Y \cap U$  is a zero set of finitely many holomorphic functions  $f_1, \dots, f_k \in \mathcal{O}(U)$ .

*Remark 3.2.1.* Obviously, any analytic subvariety  $Y$  defines an analytic germ in any point  $z \in Y$ .

**Definition 3.2.2** (irreducible analytic subvariety). An analytic subvariety  $Y$  is called irreducible, if it cannot be written as the union  $Y = Y_1 \cup Y_2$  of two proper analytic subvarieties  $Y_i \subseteq Y, i = 1, 2$ .

Given an analytic subvariety  $Y$  of a complex manifold  $X$ .

**Definition 3.2.3** (regular). A point  $x \in Y$  is called regular point, if the functions  $f_1, \dots, f_k$  can be chosen such that  $\varphi(x) \in \varphi(U)$  is a regular point of holomorphic map  $f := (f_1 \circ \varphi^{-1}, \dots, f_k \circ \varphi^{-1}) : \varphi(U) \rightarrow \mathbb{C}^k$ , where  $(U, \varphi)$  is a local chart of  $x$ .

**Definition 3.2.4** (singular). A point  $x \in Y$  is singular, if it's not regular.

**Proposition 3.2.1.** The set of regular points  $Y_{\text{reg}} = Y \setminus Y_{\text{sing}}$  is a non-empty submanifold of  $X$ . Furthermore, if  $Y$  is irreducible, then  $Y_{\text{reg}}$  is connected.

**Definition 3.2.5** (dimension). The dimension of an irreducible analytic subvariety  $Y$  is defined by  $\dim Y = \dim Y_{\text{reg}}$ .

### 3.3. Basic examples.

**Example 3.3.1** (affine space). The  $n$ -dimensional complex plane  $\mathbb{C}^n$  is a complex manifold.

**Example 3.3.2** (complex tori). If  $V$  is a complex vector space of dimension  $n$  and  $\Gamma \subseteq V$  is a free abelian, discrete subgroup of order  $2n$ , then  $X = V/\Gamma$  is a complex manifold, which is called complex tori.

*Remark 3.3.1.* The underlying manifolds of complex tori with different  $\Gamma$  are not very interesting since they are all diffeomorphic to  $(S^1)^{2n}$ . However, if you pick two lattices  $\Gamma_1, \Gamma_2$  randomly, then  $\mathbb{C}^n/\Gamma_1$  and  $\mathbb{C}^n/\Gamma_2$  will not be biholomorphic to each other.

**Example 3.3.3** (projective space). The projective space  $\mathbb{CP}^n$  is a complex manifold. Indeed, atlas are given by  $U_i = \{[z] \in \mathbb{CP}^n \mid z^i \neq 0\}, 0 \leq i \leq n$ , and  $\varphi_i$  is defined as

$$\begin{aligned} \varphi_i: U_i &\rightarrow \mathbb{C}^n \\ [z] &\mapsto \left(\frac{z_0}{z^i}, \dots, \widehat{\frac{z^i}{z^i}}, \dots, \frac{z_n}{z^i}\right) \end{aligned}$$

The transition functions are calculated as follows: For  $i < j$

$$\varphi_i \circ \varphi_j^{-1}: (u_1, \dots, u_n) \mapsto \left(\frac{u_1}{u_i}, \dots, \widehat{\frac{u_i}{u_i}}, \dots, \frac{u_{j-1}}{u_i}, \frac{1}{u_i}, \frac{u_{j+1}}{u_i}, \dots, \frac{u_n}{u_i}\right)$$

It's holomorphic on  $U_i \cap U_j$ .

*Remark 3.3.2.*  $\mathbb{CP}^n$  is compact since  $\mathbb{CP}^n$  is diffeomorphic to  $S^{2n+1}/S^1$ , which is called Hopf fibration.

**Definition 3.3.1** (projective manifold). A complex manifold  $X$  is called projective if  $X$  is biholomorphic to a closed complex submanifold of some projective space  $\mathbb{CP}^N$ .

**Example 3.3.4** (Grassmannian manifold). The Grassmannian manifold

$$Gr(k, n+1) = \{k\text{-dimensional subspace of } \mathbb{C}^{n+1}\}$$

Now we're going to show  $Gr(k, n+1)$  is a manifold of dimension  $k(n+1-k)$ . Any  $W \in Gr(k, n+1)$  is generated by the rows of a  $k \times (n+1)$  matrix  $A$  of rank  $k$ . Let us denote the set of these matrices by  $M_{k,n+1}$ , which is an open subset of the set of all  $k \times (n+1)$  matrices. The latter space is a complex manifold which is canonically isomorphic to  $\mathbb{C}^{k(n+1)}$ . Thus we obtain a natural surjection  $\pi: M_{k,n+1} \rightarrow Gr(k, n+1)$ , which is the quotient by the natural action of  $GL(k, \mathbb{C})$  on  $M_{k,n+1}$ .

Let's fix an ordering  $\{B_1, \dots, B_m\}$  of all  $k \times k$ -minors of matrices  $A \in M_{k,n+1}$ . Define an open covering  $Gr(k, n+1) = \bigcup_{i=1}^m U_i$ , where  $U_i$  is the open subset  $\{\pi(A) \mid \det(B_i) \neq 0\}$ . Note that  $U_i$  is well-defined since if  $\pi(A) = \pi(A')$ , then  $A$  and  $A'$  differs an action of  $GL(k, \mathbb{C})$ , so  $\det(B_i) \neq 0$  if and only if  $\det(B'_i) \neq 0$ . So without lose of generality, we may assume  $A$  is of form  $(B_i, C_i)$ , where  $C_i$  is a  $k \times (n+1-k)$  matrix. Then the map  $\varphi_i: U_i \rightarrow \mathbb{C}^{k(n+1-k)}$ , given by  $\pi(A) \rightarrow B_i^{-1}C_i$  is well-defined, and  $\{(U_i, \varphi_i)\}$  will give atlas of  $Gr(k, n+1)$ , sicne all operations are matrix operation, thus they're holomorphic. This shows  $Gr(k, n+1)$  is a complex manifold with dimension  $k(n+1-k)$ .

*Remark 3.3.3.* If  $V$  is a complex vector space of dimension  $n+1$ , then  $Gr(k, V)$  is defined as the set consisting of all  $k$ -dimensional subspaces of  $V$ , which is biholomorphic to  $Gr(k, n+1)$ .



**Example 3.3.5** (Plücker embedding). Let  $V$  be a complex vector space of dimension  $n + 1$ , then

$$\Phi: Gr(k, V) \hookrightarrow \mathbb{CP}(\bigwedge^k V)$$

defined by  $W \subseteq V$  with basis  $w_1, \dots, w_k$  is mapped to  $[w_1 \wedge \dots \wedge w_k]$ , is called Plücker embedding. It's well-defined, thanks to the following lemma.

**Lemma 3.3.1.** Let  $W$  be a complex vector space of dimension  $k$ , and  $\mathcal{B}_1 = \{w_1, \dots, w_k\}$  and  $\mathcal{B}_2 = \{v_1, \dots, v_k\}$  are two basis for  $W$ . Then  $v_1 \wedge \dots \wedge v_k = \lambda w_1 \wedge \dots \wedge w_k$  for some  $\lambda \in \mathbb{C}^*$ .

*Proof.* If we express  $w_j = a_{1j}v_1 + \dots + a_{kj}v_k$ , then direct computation shows that

$$\begin{aligned} w_1 \wedge \dots \wedge w_k &= (a_{11}v_1 + \dots + a_{k1}v_k) \wedge \dots \wedge (a_{1k}v_1 + \dots + a_{kk}v_k) \\ &= \sum_{\sigma \in S_k} \text{sign}(\sigma) a_{1\sigma(1)} \dots a_{k\sigma(k)} v_1 \wedge \dots \wedge v_k \\ &= \lambda v_1 \wedge \dots \wedge v_k \end{aligned}$$

Note that  $\lambda$  is exactly the determinant of the change of basis matrix from  $\mathcal{B}_1$  to  $\mathcal{B}_2$ .  $\square$

*Remark 3.3.4.* It's a little bit complicated to check it's injective.

### 3.4. Vector bundle.

#### 3.4.1. In viewpoint of transition functions.

**Definition 3.4.1** (complex vector bundle). Let  $X$  be a smooth manifold. A complex vector bundle  $E$  of rank  $r$  on  $X$  consists of the following data:

- (1)  $E$  is a smooth manifold with surjective map  $\pi: E \rightarrow X$ , such that
  - (1) For all  $x \in X$ , fibre  $E_x$  is a  $\mathbb{C}$ -vector space of dimension  $r$ .
  - (2) For all  $x \in X$ , there exists  $U \subseteq X$  and there is a homeomorphism  $\varphi: \pi^{-1}(U) \rightarrow U \times \mathbb{C}^r$  such that

$$\begin{array}{ccc}
 \pi^{-1}(U) & \xrightarrow{\pi} & U \\
 \searrow \varphi & \curvearrowright p_1 & \nearrow \\
 & U \times \mathbb{C}^r & \xrightarrow{p_2} \mathbb{C}^r
 \end{array}$$

and for all  $y \in U$ ,  $E_y \xrightarrow{p_2 \circ \varphi} \mathbb{C}^r$  is a  $\mathbb{C}$ -vector space isomorphism.  $(U, \varphi)$  is called a trivialization of  $E$  over  $U$ .

*Remark 3.4.1* (transition functions). Consider two local trivialization  $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ . Then  $\varphi_\alpha \circ \varphi_\beta^{-1}: (U_\alpha \cap U_\beta) \times \mathbb{C}^r \rightarrow (U_\alpha \cap U_\beta) \times \mathbb{C}^r$  induces

$$g_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow \text{GL}(r, \mathbb{C})$$

where  $g_{\alpha\beta}$  is called transition function. Furthermore, it satisfies

$$\begin{aligned}
 g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} &= \text{id} & \text{on } U_\alpha \cap U_\beta \cap U_\gamma \\
 g_{\alpha\alpha} &= \text{id} & \text{on } U_\alpha
 \end{aligned}$$

In fact, transition functions contain all information about this vector bundle since a vector bundle is locally trivial, so how are these trivial pieces glued together really matters.

**Definition 3.4.2** (complex vector bundle). Let  $X$  be a smooth manifold. A complex vector bundle  $E$  of rank  $r$  on  $X$  consists of the following data:

- (1) open covering  $\{U_\alpha\}$  of  $X$ .
- (2) smooth functions  $\{g_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow \text{GL}(r, \mathbb{C})\}$  satisfies

$$\begin{aligned}
 g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} &= \text{id} & \text{on } U_\alpha \cap U_\beta \cap U_\gamma \\
 g_{\alpha\alpha} &= \text{id} & \text{on } U_\alpha
 \end{aligned}$$

*Remark 3.4.2.* The two definitions above are equivalent. The first definition implies the second clearly. The converse is a standard constructive method: If we already have an open covering and a set of transition functions, the vector bundle  $E$  is defined to be the quotient of the disjoint union  $\coprod_{U_\alpha} (U_\alpha \times \mathbb{C}^r)$  by the equivalence relation that puts  $(p', v') \in U_\beta \times \mathbb{C}^r$  equivalent to  $(p, v) \in U_\alpha \times \mathbb{C}^r$  if and only if  $p = p'$  and  $v' = g_{\alpha\beta}(p)v$ . To connect this definition with the previous one, define the map  $\pi$  to send the equivalence class of any given  $(p, v)$  to  $p$ .

**Definition 3.4.3** (holomorphic vector bundle). A holomorphic vector bundle  $\pi: E \rightarrow X$  over a complex manifold  $X$  is a complex vector bundle with holomorphic transition functions.

**Exercise 3.4.1.** Show that the total space of a holomorphic vector bundle  $E$  is a complex manifold.

*Proof.* Since we already have a complex structure on  $X$ , we need to pull it back to  $E$  using  $\pi$  and use the holomorphic transition functions to show it does give a complex structure on  $E$ .  $\square$

**Example 3.4.1** (trivial bundle). Let  $X$  be a smooth/complex manifold. Then  $X \times \mathbb{C}^r$  is called trivial complex (holomorphic) vector bundle of rank  $r$  on  $X$ .

**Definition 3.4.4** (subbundle). Let  $\pi: E \rightarrow X$  be a complex (holomorphic) vector bundle.  $F \subseteq E$  is called a subbundle of rank  $s$ , if

- (1) For all  $x \in X$ ,  $F \cap E_x$  is a subspace of  $E_x$  with dimension  $s$ .
- (2)  $\pi|_F: F \rightarrow X$  induces a complex (holomorphic) vector bundle.

3.4.2. *In viewpoint of sheaf.* One may refer to Appendix A for more details about sheaf.

**Definition 3.4.5** (section). Let  $X$  be a complex manifold and  $\pi: E \rightarrow X$  be a complex (holomorphic) vector bundle. For any open subset  $U \subseteq X$ , a section of  $E$  over  $U$  is a smooth/holomorphic map  $s: U \rightarrow E$  such that  $\pi \circ s = \text{id}_U$ .

**Notation 3.4.1.** The set of all smooth or holomorphic sections over open subset  $U$  is denoted by  $C^\infty(U, E)$  or  $\Gamma(U, E)$ .

One reason why sheaf plays an important role of study of complex geometry is that you can regard a vector bundle as a special sheaf.

**Definition 3.4.6** (sheaf of sections). Let  $X$  be a complex manifold and  $\pi: E \rightarrow X$  a holomorphic vector bundle. Then its sheaf of sections is defined as

$$\mathcal{O}_X(E)(U) = \Gamma(U, E)$$

**Example 3.4.2.** Let  $E \rightarrow X$  be trivial holomorphic vector bundle. Then  $\mathcal{O}_X(E)$  is exactly sheaf of holomorphic functions.

**Example 3.4.3** (locally free sheaf). A sheaf  $\mathcal{F}$  over a topological space  $X$  is called locally free, if there exists an open covering  $\{U_\alpha\}$  of  $X$  such that  $\mathcal{F}|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}^{\oplus r}$  of rank  $r$ .

**Exercise 3.4.2.** Let  $X$  be a complex manifold. There is one to one correspondence over  $X$ :

$$\{\text{holomorphic vector bundles}\} \xleftrightarrow{1-1} \{\text{locally free sheaves}\}$$

*Proof.* If  $\pi: E \rightarrow X$  is a holomorphic vector bundle, then  $\mathcal{O}_X(E)$  is a locally free sheaf. Indeed since we have local trivialization of holomorphic vector bundle  $\{U_\alpha\}$ , that is  $E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^r$ , and it's known to all that sections of a trivial holomorphic function is exactly holomorphic functions, thus  $\mathcal{O}(E)|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}^{\oplus r}$ , that is  $\mathcal{O}_X(E)$  is a locally free sheaf.

Conversely, if  $\mathcal{E}$  is locally free over an open covering  $\{U_\alpha\}$  of  $X$ , then we just need to glue  $U_\alpha \times \mathbb{C}^r \rightarrow U_\alpha$  together to get a vector bundle. Therefore we need a family of gluing data  $g_{\alpha\beta}: (U_\alpha \cap U_\beta) \times \mathbb{C}^r \rightarrow (U_\alpha \cap U_\beta) \times \mathbb{C}^r$ . Since  $\mathcal{E}$  is locally free, we have local isomorphism  $f_\alpha: \mathcal{E}|_{U_\alpha} \rightarrow \mathcal{O}_{U_\alpha}^{\oplus r}$ . Restricting to intersection  $U_\alpha \cap U_\beta$ , we get

$$f_{\alpha\beta} = f_\alpha|_{U_\alpha \cap U_\beta} \circ f_\beta^{-1}|_{U_\alpha \cap U_\beta}: \mathcal{O}_{U_\beta}^{\oplus r}|_{U_\alpha \cap U_\beta} \rightarrow \mathcal{O}_{U_\alpha}^{\oplus r}|_{U_\alpha \cap U_\beta}$$

Every such map is induced by a map

$$g_{\alpha\beta}: (U_\alpha \cap U_\beta) \times \mathbb{C}^r \rightarrow (U_\alpha \cap U_\beta) \times \mathbb{C}^r$$

that's gluing data we desire.  $\square$

**Definition 3.4.7** (cohomology of vector bundle). Let  $E$  be a holomorphic vector bundle on a complex manifold  $X$ . Then its  $q$ -th cohomology  $H^q(X, E)$  is defined to be  $q$ -th sheaf cohomology of  $\mathcal{O}_X(E)$

**3.4.3. Algebraic construction.** Let  $E, F$  be complex (holomorphic) vector bundles on  $X$  with transition functions  $\{g_{\alpha\beta}\}, \{h_{\alpha\beta}\}$ . Then by algebraic construction we have

- (1)  $E \oplus F$ , given by transition functions  $\{\text{diag}(g_{\alpha\beta}, h_{\alpha\beta})\}$
- (2)  $E \otimes F$ , given by transition functions  $\{g_{\alpha\beta} \otimes h_{\alpha\beta}\}$ .
- (3)  $E^*$ , given by transition functions  $\{(g_{\alpha\beta}^{-1})^T\}$ .
- (4)  $\text{Hom}(E, F) := E^* \otimes F$ .
- (5)  $\bigwedge^k E$ , given by transition functions  $\{\bigwedge^k g_{\alpha\beta}\}$ .
- (6) Let  $f: X \rightarrow Y$  be a smooth/holomorphic map,  $\pi: E \rightarrow Y$  is a vector bundle with transition functions  $\{g_{\alpha\beta}\}$ , then transition functions of pullback bundle  $f^*E$  is given by  $\{g_{\alpha\beta} \circ f\}$ .

*Remark 3.4.3.* Here is an explicit construction of pullback bundle defined by

$$f^*E = \{(x, e) \in X \times E \mid f(x) = \pi(e)\} \subseteq X \times E$$

In fact, you can regard it as a push out as

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow{f} & Y \end{array}$$

In particular, pullback bundle has universal property.

3.4.4. *Hermitian structure.* Let  $X$  be a smooth manifold and  $\pi: E \rightarrow X$  be a complex vector bundle.

**Definition 3.4.8** (Hermitian metric). A Hermitian metric  $h$  on  $E$  is a global section of  $E^* \otimes \overline{E}$ .

*Remark 3.4.4* (local form). Let  $\{e_\alpha\}$  be a local frame of  $E$ . Then Hermitian metric  $h$  is given by

$$h = h_{\alpha\bar{\beta}} e^\alpha \otimes \bar{e}^\beta$$

where  $h_{\alpha\bar{\beta}}$  is a Hermitian matrix.

**Proposition 3.4.1.** Every complex vector bundle admits a Hermitian metric.

*Proof.* Use partition of unity. □

## 4. DIVISOR AND LINE BUNDLE

In this section, unless otherwise specified, we assume  $X$  is a complex manifold.

## 4.1. Line bundle.

**Definition 4.1.1** (line bundle). A complex (holomorphic) line bundle  $L$  is a complex (holomorphic) vector bundle of rank one.

**Proposition 4.1.1.** Let  $L$  be a complex line bundle over  $X$ . Then  $L \otimes L^*$  is the trivial bundle.

*Proof.* Suppose  $\{g_{\alpha\beta}\}$  is the transition functions of  $L$ , by Section 3.4.3 it's clear to see the transition functions of  $L^* \otimes L$  is

$$(g_{\alpha\beta}^{-1})^T g_{\alpha\beta} = g_{\alpha\beta}^{-1} g_{\alpha\beta} = \text{id}$$

This completes the proof.  $\square$

**Proposition 4.1.2.** Let  $L$  be a holomorphic over a compact complex manifold  $X$ . Then  $L$  is trivial if and only if both  $L$  and its dual  $L^*$  admit non-trivial global section.

**Proposition 4.1.3.** Let  $\pi: E \rightarrow X$  be a complex line bundle, then  $E$  is a trivial line bundle if and only if there exists a nowhere vanishing global section  $s$ .

*Proof.* It's clear there exists a nowhere vanishing global section if  $E$  is trivial. Conversely, if there exists a nowhere vanishing global section  $s$ . Consider the following map

$$\begin{aligned} \varphi: X \times \mathbb{C} &\rightarrow E \\ (x, \lambda) &\mapsto \lambda s(x) \end{aligned}$$

It's an isomorphism since fiberwisely one has  $\varphi_x(\lambda) = \lambda s(x)$ , and it's injective thus isomorphism since  $s(x) \neq 0$ .  $\square$

**Definition 4.1.2** (picard group). The picard group  $\text{Pic}(X)$  denotes set of all holomorphic line bundles on  $X$  up to isomorphism, whose group structure is given by tensor product.

**Proposition 4.1.4.** There is a natural isomorphism  $\text{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$ .

*Proof.* For a line bundle  $L$ , it's completely determined by its transition functions  $g_{\alpha\beta}: U_{\alpha\beta} \rightarrow \mathbb{C}^*$ , which is holomorphic functions. It gives rise to an element in  $\check{H}^1(X, \mathcal{O}_X^*)$  since  $g_{\alpha\beta}$  satisfies cocycle conditions. Furthermore, Čech cohomology<sup>2</sup> computes the sheaf cohomology for reasonable topological space, e.g. for manifolds.  $\square$

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<sup>2</sup>For more details, see Appendix A.

*Remark 4.1.1.* This proposition gives us a method to compute Picard group of a complex manifold since there is exponential sequence as follows

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X^* \rightarrow 0$$

which is a exact sequence of sheaves, then it gives a long exact sequence of cohomology groups as follows

$$\cdots \rightarrow H^1(X, \mathbb{Z}) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X^*) \rightarrow H^2(X, \mathbb{Z}) \rightarrow \cdots$$

Thus  $\text{Pic}(X)$  can in principle be computed by above exact sequence. Roughly speaking,  $\text{Pic}(X)$  has two parts:

- (1) A discrete part, measured by its image in  $H^2(X, \mathbb{Z})$ .
- (2) A continuous part coming from the  $H^1(X, \mathcal{O}_X)$ , which is possibly trivial.

#### 4.2. Divisor.

**Definition 4.2.1** (analytic hypersurface). An analytic hypersurface of  $X$  is an analytic subvariety  $Y \subseteq X$  of codimensional one.

*Remark 4.2.1.* By Remark 2.4.2 one has a hypersurface is locally given as the zero set of a non-trivial holomorphic function.

**Definition 4.2.2** (divisor). A divisor  $D$  on  $X$  is a locally finite<sup>3</sup> formal linear combination  $D = \sum a_i[Y_i]$  with  $Y_i \subseteq X$  are irreducible hypersurfaces and  $a_i \in \mathbb{Z}$ .

**Definition 4.2.3** (divisor group). The divisor group  $\text{Div}(X)$  is the set of all divisors endowed with the natural group structure.

**Definition 4.2.4** (effective). A divisor  $D = \sum a_i[Y_i]$  is called effective, if  $a_i \geq 0$  for all  $i$ . In this case, we write  $D \geq 0$ .

**Proposition 4.2.1.** Every hypersurfaces  $Y$  defines an effective divisor  $\sum[Y_i] \in \text{Div}(X)$ , where  $Y_i$  are irreducible components of  $Y$ .

*Proof.* It suffices to show the irreducible components of a hypersurface  $Y$  is locally finite.  $\square$

Let  $Y \subseteq X$  be a hypersurface and  $x \in Y$ . Suppose that  $Y$  defines an irreducible germ in  $x$ , that is this germ is the zero set of an irreducible  $g \in \mathcal{O}_{X,x}$ .

**Definition 4.2.5** (order). Let  $f$  be a meromorphic function in a neighborhood of  $x \in Y$ . Then the order  $\text{ord}_{Y,x}(f)$  of  $f$  in  $x$  with respect to  $Y$  is given by

$$f = g^{\text{ord}_{Y,x}(f)} h$$

where  $h \in \mathcal{O}_{X,x}^*$ .

*Remark 4.2.2.*

<sup>3</sup>The sum is called locally finite, if for any  $x \in X$ , there exists an open neighborhood  $U \subseteq X$  such that only finite many coefficients  $a_i \neq 0$  with  $Y_i \cap U \neq \emptyset$ .

- (1) The order of  $f$  in  $x$  with respect to  $Y$  is independent of the choice of  $g$  since any two irreducible  $g, g' \in \mathcal{O}_{X,x}$  with  $Z(g) = Z(g')$  only differs by an element in  $\mathcal{O}_{X,x}^*$ .
- (2) More globally, one can define order  $\text{ord}_Y(f)$  as  $\text{ord}_Y(f) = \text{ord}_{Y,x}(f)$  for  $x \in Y$  such that  $Y$  defines an irreducible germ in  $x$ . Such a point  $x \in Y$  always exists, for example, one can choose a regular point  $x \in Y_{\text{reg}}$ . Moreover, it's independent of the choice of  $x$  since

**Definition 4.2.6** (zeros and poles). Let  $f$  be a meromorphic function on  $X$ . Then

- (1)  $f$  has zeros of order  $d \geq 0$  along  $Y$  if  $\text{ord}_Y(f) = d$ .
- (2)  $f$  has poles of order  $d \geq 0$  along  $Y$  if  $\text{ord}_Y(f) = -d$ .

**Definition 4.2.7** (principal divisor). For  $f \in K(X)$ , the divisor associated to  $f$  is

$$(f) := \sum \text{ord}_Y(f)[Y]$$

where the sum is taken over all irreducible hypersurfaces  $Y \subseteq X$ . A divisor of this form is called principal.

*Remark 4.2.3.* The divisor  $(f)$  can be written as the difference of two effective divisors  $(f) = Z(f) - P(f)$ , where

$$Z(f) = \sum_{\text{ord}_Y(f) > 0} \text{ord}_Y(f)[Y], \quad P(f) = \sum_{\text{ord}_Y(f) < 0} \text{ord}_Y(f)[Y]$$

**Proposition 4.2.2.** There exists a natural isomorphism

$$H^0(X, \mathcal{K}_X^* / \mathcal{O}_X^*) \cong \text{Div}(X)$$

*Proof.* An element  $f \in H^0(X, \mathcal{K}_X^* / \mathcal{O}_X^*)$  is given by non-trivial meromorphic functions  $f_i \in K_X^*(U_i)$  such that  $f_i f_j^{-1}$  is a holomorphic function without zeros on  $U_i \cap U_j$ , where  $\{U_i\}$  is an open covering of  $X$ . Thus for any irreducible hypersurface  $Y \subseteq X$  with  $Y \cap U_i \cap U_j \neq \emptyset$ , one has  $\text{ord}_Y(f_i) = \text{ord}_Y(f_j)$ . Hence  $\text{ord}_Y(f)$  is well-defined for any irreducible hypersurface  $Y$ . Then one associates to  $f$  the divisor  $(f) = \sum \text{ord}_Y(f)[Y] \in \text{Div}(X)$ .

It's clear this map is a group homomorphism. To see it's bijective, we define the inverse as follows. If  $D = \sum a_i[Y_i] \in \text{Div}(X)$  is given, then there exists an open covering  $\{U_i\}$  of  $X$  such that  $Y_i \cap U_j$  is defined by  $g_{ij} \in \mathcal{O}(U_j)$  which is unique up to elements in  $\mathcal{O}^*(U_j)$ . Let  $f_j := \prod_i g_{ij}^{a_i} \in \mathcal{K}_X^*(U_j)$  since  $g_{ij}$  and  $g_{ik}$  defines the same irreducible hypersurface, they only differ by an element in  $\mathcal{O}^*(U_j \cap U_k)$ . Thus  $f$  glue to an element  $f \in H^0(X, \mathcal{K}_X^* / \mathcal{O}_X^*)$ . It's clear these two maps are inverse to each other.  $\square$

*Remark 4.2.4.* In algebraic geometry, elements in  $H^0(X, \mathcal{K}_X^* / \mathcal{O}_X^*)$  are called Cartier divisors and elements in  $\text{Div}(X)$  are called Weil divisors. Above isomorphism still holds in the algebraic setting under a weak smoothness assumption on  $X$ .



**Corollary 4.2.1.** There exists a natural group homomorphism

$$\begin{aligned} \mathrm{Div}(X) &\rightarrow \mathrm{Pic}(X) \\ D &\mapsto \mathcal{O}(D) \end{aligned}$$

where  $\mathcal{O}(D)$  is defined in the proof.

*Proof.* If  $D = \sum a_i[Y_i] \in \mathrm{Div}(X)$  corresponds to  $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ , which in turn is given by functions  $f_i \in \mathcal{K}_X^*(U_i)$  for an open covering  $\{U_i\}$ . Then we define  $\mathcal{O}(D) \in \mathrm{Div}(X)$  with transition functions  $\psi_{ij} := f_i f_j^{-1} \in \mathcal{O}_X^*(U_{ij})$ .

If  $D, D'$  are two divisors, without loss of generality we may assume they're given by  $\{f_i\}$  and  $\{f'_i\}$  respectively on the same open covering, then  $D + D'$ , then  $D + D'$  corresponds to  $\{f_i + f'_i\}$ . By definition  $\mathcal{O}(D + D')$  is described by  $\{\psi_{ij} \psi'_{ij}\}$ , hence  $\mathcal{O}(D + D') = \mathcal{O}(D) \otimes \mathcal{O}(D')$ .  $\square$

*Remark 4.2.5.* In fact, above corollary can be derived from the following exact sequence of sheaves

$$0 \rightarrow \mathcal{O}_X^* \rightarrow \mathcal{K}_X^* \rightarrow \mathcal{K}_X^*/\mathcal{O}_X^* \rightarrow 0$$

Then above group homomorphism is exactly the boundary map, the kernel of which coincides with the image of  $H^0(X, \mathcal{K}_X^*) \rightarrow H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ , and the latter by definition is the set of principal divisors.

**Definition 4.2.8** (linearly equivalent). Two divisors  $D, D'$  are called linearly equivalent, denoted by  $D \sim D'$ , if  $D - D'$  is a principal divisor.

**Corollary 4.2.2.** The group homomorphism  $\mathrm{Div}(X) \rightarrow \mathrm{Pic}(X)$  factorizes over an injection

$$\mathrm{Div}(X)/\sim \hookrightarrow \mathrm{Pic}(X)$$

**4.3. Relations between divisor and line bundle.** In general,  $\mathrm{Div}(X)/\sim \hookrightarrow \mathrm{Pic}(X)$  is a strict inclusion, but we will see if a line bundle admits a non-trivial global section, then it's contained in the image. In order to show this, we need to construct a canonical map

$$\begin{aligned} H^0(X, L) \setminus \{0\} &\rightarrow \mathrm{Div}(X) \\ s &\mapsto Z(s) \end{aligned}$$

The map is constructed as follows: Let  $L \in \mathrm{Pic}(X)$  on open covering  $\{U_i\}$  be trivialized by  $\psi_i: L|_{U_i} \rightarrow \mathcal{O}_{U_i}$ . Then divisor  $Z(s)$  is given by  $f := \{f_i := \psi_i(s|_{U_i})\} \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ .

**Proposition 4.3.1.** For  $0 \neq s \in H^0(X, L)$ , the line bundle  $\mathcal{O}(Z(s))$  is isomorphic to  $L$ .

**Proposition 4.3.2.** For any effective divisor  $D \in \mathrm{Div}(X)$ , there exists a section  $0 \neq s \in H^0(X, \mathcal{O}(D))$  with  $Z(s) = D$ .

**Corollary 4.3.1.** Non-trivial sections  $s_1 \in H^0(X, L_1)$  and  $s_2 \in H^0(X, L_2)$  define linearly equivalent divisors  $Z(s_1) \sim Z(s_2)$  if and only if  $L_1 \cong L_2$ .

*Proof.* If  $L_1 \cong L_2$ , then

If  $Z(s_1) \sim Z(s_2)$ , then by Corollary 4.2.2 one has  $\mathcal{O}(Z(s_1)) \cong \mathcal{O}(Z(s_2))$ , then this shows  $L_1 \cong L_2$  since  $\mathcal{O}(Z(s_i)) = L_i, i = 1, 2$ .  $\square$

**Corollary 4.3.2.** The image of the natural map  $\text{Div}(X) \rightarrow \text{Pic}(X)$  is generated by those line bundles  $L \in \text{Pic}(X)$  with  $H^0(X, L) \neq 0$ .

*Proof.* We have already seen if  $H^0(X, L) \neq 0$ , then  $L$  is contained in the image. Conversely, any divisor  $D = \sum a_i [Y_i]$  can be written as  $D = \sum a_i^+ [Y_i] - \sum a_j^- [Y_j]$  with  $a_k^\pm \geq 0$ , and thus  $\mathcal{O}(D) \cong \mathcal{O}(\sum a_i^+ [Y_i]) \otimes \mathcal{O}(\sum a_j^- [Y_j])^*$ . Both  $\mathcal{O}(\sum a_i^+ [Y_i])$  and  $\mathcal{O}(\sum a_j^- [Y_j])$  are associated to effective divisors, and therefore admit non-trivial global sections.  $\square$

*Remark 4.3.1.* For projective manifolds, the map  $\text{Div}(X) \rightarrow \text{Pic}(X)$  is surjective, but note that even for very easy manifolds, such as complex tori, this is no longer the case.

#### 4.4. Ample line bundle.

**Definition 4.4.1** (base point). Let  $L$  be a holomorphic line bundle on a complex manifold  $X$ . A point  $x \in X$  is a base point of  $L$  if  $s(x) = 0$  for all  $s \in H^0(X, L)$ .

**Notation 4.4.1.** The base locus  $\text{Bs}(L)$  is the set of all base points of  $L$ .

*Remark 4.4.1.* If  $\dim H^0(X, L) < \infty$ , we can choose a basis of global sections  $s_1, \dots, s_N$  of it, then  $\text{Bs}(L) = Z(s_1) \cap \dots \cap Z(s_N)$  is an analytic subvariety. Later we will see if  $X$  is compact, then  $\dim H^0(X, L) < \infty$ .

**Proposition 4.4.1.** Let  $L$  be a holomorphic line bundle on a complex manifold  $X$  and suppose  $s_1, \dots, s_N \in H^0(X, L)$  is a basis. Then

$$\begin{aligned} \varphi_L: X \setminus \text{Bs}(L) &\rightarrow \mathbb{CP}^N \\ x &\mapsto (s_0(x) : \dots : s_N(x)) \end{aligned}$$

defines a holomorphic map such that  $\varphi_L^* \mathcal{O}_{\mathbb{CP}^N}(-1) \cong L|_{X \setminus \text{Bs}(L)}$ .

**Definition 4.4.2** (ample line bundle). A holomorphic line bundle  $L$  on a complex manifold  $X$  is called ample if for some  $k > 0$  and some linear system in  $H^0(X, L^k)$  the associated map  $\varphi$  is an embedding.

*Remark 4.4.2.* By definition, a compact complex manifold is projective if and only if it admits an ample line bundle.

## 5. TANGENT AND COTANGENT BUNDLE

## 5.1. Complex and holomorphic tangent bundle.

**Definition 5.1.1** (complex tangent bundle). Let  $X$  be a smooth  $n$ -manifold with an atlas  $\{U_\alpha, \varphi_\alpha: U_\alpha \rightarrow V_\alpha \subseteq \mathbb{R}^n\}$ . Then (real) tangent bundle  $T_{\mathbb{R}}X$  is a vector bundle given by smooth transition functions

$$g_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow \mathrm{GL}(n, \mathbb{R})$$

$$x \mapsto J_{\mathbb{R}}(\varphi_\alpha \circ \varphi_\beta^{-1})(\varphi_\beta(x))$$

The complex tangent bundle  $T_{\mathbb{C}}X$  is defined as the complexification of  $T_{\mathbb{R}}X$ , that is,  $T_{\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C}$ .

**Definition 5.1.2** (holomorphic tangent bundle). Let  $X$  be a complex  $n$ -manifold, with an atlas  $\{U_\alpha, \varphi_\alpha: U_\alpha \rightarrow V_\alpha \subseteq \mathbb{C}^n\}$ . Then holomorphic tangent bundle  $TX$  is given by holomorphic transition functions

$$g_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow \mathrm{GL}(n, \mathbb{C})$$

$$z \mapsto J_{\mathbb{C}}(\varphi_\alpha \circ \varphi_\beta^{-1})(\varphi_\beta(z))$$

*Remark 5.1.1* (relations between complex tangent bundle and holomorphic tangent bundle). Let  $X$  be a complex  $n$ -manifold and  $\{z^i = x^i + \sqrt{-1}y^i\}_{1 \leq i \leq n}$  be a local coordinate of  $X$ . Then  $\{x^1, \dots, x^n, y^1, \dots, y^n\}$  gives a local coordinate of its underlying smooth  $2n$ -manifold, and there is an almost complex structure  $J$  on  $T_{\mathbb{R}}X$  given by

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}$$

$$J\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i}$$

Thus complex tangent bundle  $T_{\mathbb{C}}X$  can be decomposed as  $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$  with respect to  $J$ , with local frames as follows:

- (1)  $\{\frac{\partial}{\partial z^i} := \frac{1}{2}(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i})\}$  is a local frame of  $T^{1,0}X$ .
- (2)  $\{\frac{\partial}{\partial \bar{z}^i} := \frac{1}{2}(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i})\}$  is a local frame of  $T^{0,1}X$ .

Indeed, direct computation shows

$$J\left(\frac{\partial}{\partial z^i}\right) = \frac{1}{2}\left(\frac{\partial}{\partial y^i} + \sqrt{-1}\frac{\partial}{\partial x^i}\right) = \frac{\sqrt{-1}}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i}\right) = \sqrt{-1}\frac{\partial}{\partial z^i}$$

$$J\left(\frac{\partial}{\partial \bar{z}^i}\right) = \frac{1}{2}\left(\frac{\partial}{\partial y^i} - \sqrt{-1}\frac{\partial}{\partial x^i}\right) = -\frac{\sqrt{-1}}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i}\right) = -\sqrt{-1}\frac{\partial}{\partial \bar{z}^i}$$

and for any section  $s$  of  $T_{\mathbb{C}}X$ , one has the following decomposition

$$s = \frac{1}{2}(s - \sqrt{-1}J(s)) + \frac{1}{2}(s + \sqrt{-1}J(s))$$

Note that  $TX$  is isomorphic to  $(T_{\mathbb{R}}X, J)$  as complex vector bundle, and we claim  $(T_{\mathbb{R}}X, J)$  is isomorphic to  $T^{1,0}X$  as a complex vector bundle. Indeed, there is a natural inclusion  $T_{\mathbb{R}}X \hookrightarrow T_{\mathbb{C}}X$ , if we compose this inclusion with

projection  $T_{\mathbb{C}}X = T^{1,0}X \oplus T_X^{0,1} \rightarrow T^{1,0}X$  onto the first summand, we obtain an  $\mathbb{C}$ -isomorphism  $(T_{\mathbb{R}}X, J) \rightarrow T^{1,0}X$  with inverse map  $2\text{Re}(-)$ .

In particular,  $T^{1,0}X$  is isomorphic to  $TX$  as complex vector bundles, so we can endow  $T^{1,0}X$  with holomorphic structure such that  $T^{1,0}X$  is isomorphic to  $TX$  as holomorphic vector bundles, and thus we can use  $\{dz^i\}$  as local frame of holomorphic tangent bundle.

**5.2. Bidegree forms.** For complex manifold  $X$ , there is also an almost complex structure on  $\Omega_{X,\mathbb{R}}^1$ , that is dual bundle of  $T_{\mathbb{R}}X$ , and complexified dual space of  $T_{\mathbb{R}}X$  admits an analogous decomposition:

$$\Omega_{X,\mathbb{C}}^1 = \Omega_{X,\mathbb{R}}^1 \otimes \mathbb{C} = \Omega_X^{1,0} \oplus \Omega_X^{0,1}$$

There is also a decomposition on its  $k$ -th wedge product as follows:

$$\Omega_{X,\mathbb{C}}^k = \bigwedge^k \Omega_{X,\mathbb{C}}^1 = \bigoplus_{p+q=k} \Omega_X^{p,q}$$

where  $\Omega_X^{p,q} = \bigwedge^p \Omega_X^{1,0} \otimes \bigwedge^q \Omega_X^{0,1}$ .

**Definition 5.2.1** ( $(p, q)$ -form). A  $k$ -form  $\omega$  of type  $(p, q)$  is a smooth section of  $\Omega_X^{p,q}$ , that is

$$\omega \in C^\infty(X, \Omega_X^{p,q}) \subseteq C^\infty(X, \Omega_{X,\mathbb{C}}^k)$$

*Remark 5.2.1* (local form). Suppose  $\{z^1, \dots, z^n\}$  is a local coordinate of  $X$ , and denote  $z^i = x^i + \sqrt{-1}y^i$ , then  $\{dx^1, \dots, dx^n, dy^1, \dots, dy^n\}$  gives a local frame of  $\Omega_{X,\mathbb{R}}^1$ , and induced almost complex structure is given by

$$\begin{aligned} J^*(dx^i)\left(\frac{\partial}{\partial x^i}\right) &= dx^i\left(J\left(\frac{\partial}{\partial x^i}\right)\right) = dx^i\left(\frac{\partial}{\partial y^i}\right) = 0 \\ J^*(dx^i)\left(\frac{\partial}{\partial y^i}\right) &= dx^i\left(J\left(\frac{\partial}{\partial y^i}\right)\right) = dx^i\left(-\frac{\partial}{\partial x^i}\right) = -1 \end{aligned}$$

that is

$$\begin{aligned} J^*(dx^i) &= -dy^i \\ J^*(dy^i) &= dx^i \end{aligned}$$

and similarly we have

- (1)  $\{dz^i := dx^i + \sqrt{-1}dy^i\}$  is a local frame of  $\Omega_X^{1,0}$ .
- (2)  $\{d\bar{z}^i := dx^i - \sqrt{-1}dy^i\}$  is a local frame of  $\Omega_X^{0,1}$ .

For a  $k$ -form, it locally looks like

$$\sum_{\substack{|I|=p, |J|=q \\ p+q=k}} f_{IJ} dz^I \wedge d\bar{z}^J$$

where  $f_{IJ}$  are smooth functions, and a  $k$ -form is a  $(p, q)$ -form if and only if locally it looks like

$$\sum_{|I|=p, |J|=q} f_{IJ} dz^I \wedge d\bar{z}^J$$

**Exercise 5.2.1.** For  $\mathbb{C}^n \cong \mathbb{R}^{2n}$ , one has

$$dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n = \left(\frac{\sqrt{-1}}{2}\right)^n dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n$$

*Proof.* It suffices to show the case  $n = 1$ , and we can compute directly as follows

$$\begin{aligned} \left(\frac{\sqrt{-1}}{2}\right) dz \wedge d\bar{z} &= \left(\frac{\sqrt{-1}}{2}\right) (dx + \sqrt{-1}dy) \wedge (dx - \sqrt{-1}dy) \\ &= \left(\frac{\sqrt{-1}}{2}\right) (-2\sqrt{-1}dx \wedge dy) \\ &= dx \wedge dy \end{aligned}$$

□

**5.3. Dolbeault operators.** For complex manifold  $X$ , naturally there is a differential operator

$$d: C^\infty(X, \Omega_{X,\mathbb{C}}^k) \rightarrow C^\infty(X, \Omega_{X,\mathbb{C}}^{k+1})$$

Since we already know that we can decompose  $\alpha \in C^\infty(X, \Omega_{X,\mathbb{C}}^k)$ , so it's natural to ask how to decompose  $d\alpha \in C^\infty(X, \Omega_{X,\mathbb{C}}^{k+1})$ .

**Example 5.3.1.** For smooth function  $\alpha$  on  $X$ , locally direct computation shows

$$\begin{aligned} d\alpha &= \frac{\partial \alpha}{\partial x^i} dx^i + \frac{\partial \alpha}{\partial y^i} dy^i \\ &= \frac{1}{2} \left( \frac{\partial \alpha}{\partial x^i} - \sqrt{-1} \frac{\partial \alpha}{\partial y^i} \right) dz^i + \frac{1}{2} \left( \frac{\partial \alpha}{\partial x^i} + \sqrt{-1} \frac{\partial \alpha}{\partial y^i} \right) d\bar{z}^i \\ &= \frac{\partial \alpha}{\partial z^i} dz^i + \frac{\partial \alpha}{\partial \bar{z}^i} d\bar{z}^i \end{aligned}$$

If we denote

$$\begin{aligned} \partial \alpha &= \frac{\partial \alpha}{\partial z^i} dz^i \\ \bar{\partial} \alpha &= \frac{\partial \alpha}{\partial \bar{z}^i} d\bar{z}^i \end{aligned}$$

then  $d\alpha = \partial \alpha + \bar{\partial} \alpha$ , where  $\partial \alpha \in C^\infty(X, \Omega_X^{1,0})$  and  $\bar{\partial} \alpha \in C^\infty(X, \Omega_X^{0,1})$ . In general case, for  $\alpha \in C^\infty(X, \Omega_X^{p,q})$ , locally looks like

$$\alpha = \sum_{|I|=p, |J|=q} \alpha_{IJ} dz^J \wedge d\bar{z}^K$$

then

$$d\alpha = \sum_{|I|=p, |J|=q} \frac{\partial \alpha_{IJ}}{\partial z^l} dz^l \wedge dz^I \wedge d\bar{z}^J + \sum_{|I|=p, |J|=q} \frac{\partial \alpha_{IJ}}{\partial \bar{z}^l} d\bar{z}^l \wedge dz^I \wedge \bar{z}^J$$

Thus one can define

$$\begin{aligned} \partial: C^\infty(X, \Omega_X^{p,q}) &\rightarrow C^\infty(X, \Omega_X^{p+1,q}) \\ \bar{\partial}: C^\infty(X, \Omega_X^{p,q}) &\rightarrow C^\infty(X, \Omega_X^{p,q+1}) \end{aligned}$$

such that  $d = \partial + \bar{\partial}$ .

**Proposition 5.3.1.**

(1)

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \partial\beta$$

(2)

$$\partial^2 = \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0$$

According to (2) of Proposition 5.3.1 there is following cochain complex

$$(5.1) \quad 0 \rightarrow C^\infty(X, \Omega_X^{p,0}) \xrightarrow{\bar{\partial}} C^\infty(X, \Omega_X^{p,1}) \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} C^\infty(X, \Omega_X^{p,n}) \rightarrow 0$$

**Definition 5.3.1** (Dolbeault cohomology).

$$H^{p,q}(X) := H_{\bar{\partial}}^q(C^\infty(X, \Omega_X^{p,\bullet}))$$

*Remark 5.3.1.* Note that we have decomposition  $C^\infty(X, \Omega_{X,\mathbb{C}}^k) = \bigoplus_{p+q=k} C^\infty(X, \Omega_X^{p,q})$ , could we have the following decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

In fact, for compact Kähler manifold, such decomposition do holds, which is called hodge decomposition.

**Example 5.3.2.** Note that

$$H^{p,0}(X) = \{\alpha \in C^\infty(X, \Omega_X^{p,0}) \mid \bar{\partial}\alpha = 0\}$$

For  $\alpha \in C^\infty(X, \Omega_X^{p,0})$  locally written as  $\alpha = \sum_{|I|=p} \alpha_I dz^I$ , one has

$$\bar{\partial}\alpha = \sum_{|I|=p} \frac{\partial \alpha_I}{\partial \bar{z}^l} d\bar{z}^l \wedge dz^I = 0 \iff \frac{\partial \alpha_I}{\partial \bar{z}^l} = 0$$

which implies  $\alpha_I$  is a holomorphic function. This shows  $H^{p,0}(X) = \Gamma(X, \Omega_X^p)$ .

A natural question arises: what does this cohomology compute? In the context of smooth manifolds, de Rham cohomology calculates the cohomology of a constant sheaf, which heavily relies on the Poincaré lemma. In the complex setting, Dolbeault cohomology  $H^{p,q}(X)$  determines the  $q$ -th sheaf cohomology of  $\Omega_X^p$ , which is based on the following lemma.

**Proposition 5.3.2** ( $\bar{\partial}$ -Poincaré lemma). Let  $B$  be an sufficiently small open disc in  $\mathbb{C}^n$ . If  $\alpha \in C^\infty(B, \Omega_X^{p,q})$  is  $\bar{\partial}$ -closed and  $q > 0$ , then there exists  $\beta \in C^\infty(B, \Omega_X^{p,q-1})$  such that  $\alpha = \bar{\partial}\beta$ .

*Proof.* See Corollary 1.3.9 of Page47 of [Huy05].  $\square$

**Proposition 5.3.3** (functorial of Dolbeault cohomology). Let  $f: X \rightarrow Y$  be a holomorphic map between complex manifolds with pullback

$$f^*: C^\infty(Y, \Omega_{Y,\mathbb{C}}^k) \rightarrow C^\infty(X, \Omega_{X,\mathbb{C}}^k)$$

Then

$$f^*: C^\infty(Y, \Omega_{Y, \mathbb{C}}^{p,q}) \rightarrow C^\infty(X, \Omega_{X, \mathbb{C}}^{p,q})$$

and it induces

$$f^*: H^{p,q}(Y) \rightarrow H^{p,q}(X)$$

**Example 5.3.3** (Dolbeault cohomology of a holomorphic vector bundle<sup>4</sup>). For a holomorphic vector bundle  $E \rightarrow X$ , we can also define

$$\bar{\partial}_E: C^\infty(X, \Omega_X^{0,q} \otimes E) \rightarrow C^\infty(X, \Omega_X^{0,q+1} \otimes E)$$

satisfies  $\bar{\partial}_E^2 = 0$ . Let's elaborate this construction: Since any global section is glued together by local sections, we just need to define  $\bar{\partial}_E$  for local sections and check it is well-defined under the change of local chart. We can choose a local holomorphic frame  $\{e^1, \dots, e^n\}$  for  $E$  on  $U$ , so any section  $s \in C^\infty(U, \Omega_X^{0,q} \otimes E)$  locally can be written as  $s = s^i \otimes e_i$  with  $s^i \in C^\infty(U, \Omega_X^{0,q})$ . Then we can define

$$\bar{\partial}_E(s) = \bar{\partial}s^i \otimes e^i$$

It's clear that this definition is independent of the choice of local chart since the transition functions are holomorphic and  $\bar{\partial}$  kills them. Furthermore,  $\bar{\partial}_E^2 = 0$  holds since  $\bar{\partial}^2 = 0$ . Thus we can construct a cochain complex and define its cohomology, denoted by

$$H^q(X, E) = H_{\bar{\partial}_E}^q(C^\infty(X, \Omega_X^{0,\bullet} \otimes E))$$

and similarly  $H^q(X, E)$  computes the  $q$ -th sheaf cohomology of  $E$ .

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<sup>4</sup>In previous case,  $E = \Omega_X^{p,0}$

## 6. CONNECTIONS AND ITS CURVATURE

**6.1. Connections on complex vector bundle.** Let  $X$  be a complex manifold and  $\pi: E \rightarrow X$  be a complex vector bundle.

6.1.1. *Basic definitions.*

**Definition 6.1.1** (connection). A connection on  $E$  is a  $\mathbb{C}$ -linear operator

$$\nabla: C^\infty(X, E) \rightarrow C^\infty(X, \Omega_{X, \mathbb{C}}^1 \otimes E)$$

satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla s$$

for  $f \in C^\infty(X)$  and  $s \in C^\infty(X, E)$ .

*Remark 6.1.1* (connection form). Let  $\{e_\alpha\}$  be a local frame of  $E$ . Then any section  $s$  of  $E$  can be written as  $s = s^\alpha e_\alpha$ , and

$$\begin{aligned} \nabla(s^\alpha e_\alpha) &= ds^\alpha e_\alpha + s^\alpha \nabla s_\alpha \\ &= ds^\alpha e_\alpha + s^\alpha \omega_\alpha^\beta e_\beta \end{aligned}$$

where  $\omega_\alpha^\beta$  are 1-forms, which is called connection 1-form. In terms of Christoffel symbol, one has

$$\omega_\alpha^\beta = \Gamma_{i\alpha}^\beta dz^i + \Gamma_{\bar{i}\alpha}^\beta d\bar{z}^i$$

6.1.2. *Curvature form.* Now we're going to extend connection to something called exterior derivative defined on sections of vector bundle valued  $k$ -forms as follows

$$\begin{aligned} d^\nabla: C^\infty(X, \Omega_{X, \mathbb{C}}^k \otimes E) &\rightarrow C^\infty(X, \Omega_{X, \mathbb{C}}^{k+1} \otimes E) \\ \omega \otimes s &\mapsto d\omega \otimes s + (-1)^k \omega \wedge \nabla s \end{aligned}$$

**Definition 6.1.2** (curvature form). Let  $E$  be a complex vector bundle over a complex manifold  $X$  equipped with connection  $\nabla$ . There exists a section  $\Theta \in C^\infty(X, \Omega_{X, \mathbb{C}}^2 \otimes \text{End } E)$ , called curvature form, such that

$$(d^\nabla)^2 s = \Theta \wedge s$$

for all  $s \in C^\infty(X, \Omega_{X, \mathbb{C}}^k \otimes E)$ .

*Remark 6.1.2* (local form). Let  $\{e_\alpha\}$  be a local frame of  $E$ . The curvature form  $\Theta$  can be written as

$$\Theta = \Theta_{i\bar{j}\alpha}^\beta dz^i \wedge d\bar{z}^j \otimes e^\alpha \otimes e_\beta$$

where  $\Theta_{i\bar{j}\alpha}^\beta$  can also be expressed in terms of Christoffel symbols just like what we have seen in [Liu23].



**6.2. Chern connection.** In this section, we will introduce additional structures to enhance the complexity of a vector bundle  $E$  over a complex manifold  $X$ . These structures include Hermitian metrics and complex structures. We will explore connections that align harmoniously with these structures, similar to our approach in Riemannian geometry. By doing so, we will obtain the Chern connection, which runs parallel to the Levi-Civita connection.

6.2.1. *Compatibility with Hermitian metric.*

**Definition 6.2.1** (Hermitian metric). Let  $E$  be a complex vector bundle. A Hermitian metric  $h$  on  $E$  is a smooth section of  $E^* \otimes \overline{E}^*$ .

*Remark 6.2.1* (local form). Let  $\{e_\alpha\}$  be a local frame of  $E$ . Then a Hermitian metric is determined by a positive definite Hermitian matrix  $(h_{\alpha\bar{\beta}})$ , that is

$$h = h_{\alpha\bar{\beta}} e^\alpha \otimes \bar{e}^\beta$$

where  $h_{\alpha\bar{\beta}} = h(e_\alpha, \bar{e}_\beta)$ .

**Definition 6.2.2** (Hermitian vector bundle). A complex vector bundle  $E$  together with a Hermitian metric  $h$  is called a Hermitian vector bundle  $(E, h)$ .

*Remark 6.2.2* (metric weight). Let  $L$  be a Hermitian line bundle. A Hermitian metric  $h$  is locally given by  $e^{-2\varphi}$ , where  $\varphi$  is a smooth function, which is called metric weight. Suppose  $\{g_{\alpha\beta}\}$  is transition function of  $L$  with respect to open covering  $\{U_\alpha\}$ . Then  $h$  is given by a collection  $\{h_\alpha \in C^\infty(U_\alpha)\}$  such that  $h_\alpha = |g_{\alpha\beta}|^{-2} h_\beta$ . In other words, a Hermitian metric is a collection of metric weights  $\{\varphi_\alpha \in C^\infty(U_\alpha)\}$  such that

$$\varphi_\alpha = \varphi_\beta + \log |g_{\alpha\beta}|$$

**Definition 6.2.3** (sesquilinear map). For a Hermitian vector bundle  $(E, h)$  over complex manifold  $X$ , there is a sesquilinear map

$$\begin{aligned} C^\infty(X, \Omega_{X, \mathbb{C}}^p \otimes E) \times C^\infty(X, \Omega_{X, \mathbb{C}}^q \otimes E) &\rightarrow C^\infty(X, \Omega_{X, \mathbb{C}}^{p+q}) \\ (s, t) &\mapsto \{s, t\} \end{aligned}$$

locally given by

$$\{s^\alpha e_\alpha, t^\beta e_\beta\} = h_{\alpha\bar{\beta}} s^\alpha \wedge \bar{t}^\beta$$

**Definition 6.2.4** (metric connection). A connection  $\nabla$  on a Hermitian vector bundle  $(E, h)$  is called a metric connection, if

$$d\langle s, t \rangle = \{\nabla s, t\} + \{s, \nabla t\}$$

where  $s, t$  are sections of  $E$ .

*Remark 6.2.3* (local form). If  $\{e_\alpha\}$  is a local frame of  $E$ , then

$$\begin{aligned} dh_{\alpha\bar{\beta}} &= d\langle e_\alpha, \bar{e}_\beta \rangle \\ &= \{\nabla e_\alpha, \bar{e}_\beta\} + \{e_\alpha, \nabla \bar{e}_\beta\} \\ &= \omega_\alpha^\gamma h_{\gamma\bar{\beta}} + \overline{\omega_\beta^\gamma} h_{\alpha\bar{\gamma}} \end{aligned}$$

So in matrix notation, we have

$$dh = \omega h + h \bar{\omega}^T$$

In particular, if we take  $\{e_\alpha\}$  to be orthogonal local frame of  $E$  with respect to  $h$ , we will find  $\omega + \bar{\omega}^T = 0$ , that is  $\omega$  is skew-Hermitian matrix.

**Proposition 6.2.1.** Let  $(E, h)$  be a Hermitian vector bundle over a complex manifold  $X$ . A connection  $\nabla$  is a metric connection if and only if

$$d\{s, t\} = \{\nabla s, t\} + (-1)^p \{s, \nabla t\}$$

where  $s \in C^\infty(X, \Omega_{X, \mathbb{C}}^p \otimes E)$  and  $t \in C^\infty(X, \Omega_{X, \mathbb{C}}^q \otimes E)$ .

**Proposition 6.2.2.** Let  $(E, h)$  be a Hermitian vector bundle equipped with connection  $\nabla^E$ . Then  $\nabla^E$  is a metric connection if and only if  $\nabla^{E^* \otimes \bar{E}^*} h = 0$ .

*Proof.* Direct computation shows

$$\begin{aligned} \nabla^{E^* \otimes \bar{E}^*} (h_{\alpha\bar{\beta}} e^\alpha \otimes \bar{e}^\beta) &= dh_{\alpha\bar{\beta}} \otimes e^\alpha \otimes \bar{e}^\beta + h_{\alpha\bar{\beta}} \nabla^{E^*} e^\alpha \otimes \bar{e}^\beta + h_{\alpha\bar{\beta}} e^\alpha \otimes \nabla^{\bar{E}^*} \bar{e}^\beta \\ &= dh_{\alpha\bar{\beta}} \otimes e^\alpha \otimes \bar{e}^\beta - h_{\alpha\bar{\beta}} \omega_\gamma^\alpha e^\gamma \otimes \bar{e}^\beta - h_{\alpha\bar{\beta}} \bar{\omega}_\gamma^\beta e^\alpha \otimes \bar{e}^\gamma \\ &= (dh_{\alpha\bar{\beta}} - \omega_\alpha^\gamma h_{\gamma\bar{\beta}} - \bar{\omega}_\beta^\gamma h_{\alpha\bar{\gamma}}) e^\alpha \otimes \bar{e}^\beta \end{aligned}$$

This shows desired result.  $\square$

**6.2.2. Compatibility with complex structure.** For a complex manifold  $X$ , we have decomposition

$$\Omega_{X, \mathbb{C}}^1 = \Omega_X^{1,0} \oplus \Omega_X^{0,1}$$

Let  $E \rightarrow X$  be a complex vector bundle with connection  $\nabla$ . Then we can decompose  $\nabla = \nabla^{1,0} + \nabla^{0,1}$  by composing the projection as follows

$$\begin{array}{ccc} & & C^\infty(X, \Omega_X^{1,0} \otimes E) \\ & \nearrow & \\ C^\infty(X, E) & \xrightarrow{\nabla} & C^\infty(X, \Omega_{X, \mathbb{C}}^1 \otimes E) \\ & \searrow & \\ & & C^\infty(X, \Omega_X^{0,1} \otimes E) \end{array}$$

If we write  $\nabla = d + \omega$  locally, then

$$\nabla^{1,0} = \partial + \omega^{1,0}$$

$$\nabla^{0,1} = \bar{\partial} + \omega^{0,1}$$

**Definition 6.2.5** (complex connection). A connection  $\nabla$  on a holomorphic vector bundle  $E$  over a complex manifold  $X$  is said to be compatible with complex structure if  $\nabla^{0,1} = \bar{\partial}_E$ .

*Remark 6.2.4* (local form). Let  $\{e_\alpha\}$  be a holomorphic local form of  $E$ , and denote

$$\nabla e_\alpha = (\Gamma_{i\alpha}^\beta dz^i + \Gamma_{\bar{i}\alpha}^\beta d\bar{z}^i) e_\beta$$

that is

$$\nabla^{0,1} e_\alpha = \Gamma_{\bar{i}\alpha}^\beta e_\beta d\bar{z}^i$$

But since  $\{e_\alpha\}$  is holomorphic, that is  $\bar{\partial}_E e_\alpha = 0$ , which implies  $\nabla$  is complex if and only if  $\Gamma_{\bar{i}\alpha}^\beta = 0$ .

**6.2.3. Chern connection.**

**Theorem 6.2.1** (Chern connection). Let  $X$  be a complex manifold,  $(E, h)$  a Hermitian holomorphic vector bundle. Then there exists a unique metric connection called Chern connection such that it's compatible with complex structure.

*Proof.* If metric connection  $\nabla$  is compatible with complex structure, then the following three equations are equivalent

$$\begin{aligned} dh &= \omega h + h \bar{\omega}^t \\ \partial h &= \omega h \\ \bar{\partial} h &= h \bar{\omega}^t \end{aligned}$$

since  $\omega$  is a  $(1, 0)$ -valued matrix. This shows Chern connection is uniquely determined by  $\omega = (\partial h)h^{-1}$ .  $\square$

*Remark 6.2.5* (local form). Chern connection is locally determined by

$$\frac{\partial h_{\alpha\bar{\beta}}}{\partial z^i} = \Gamma_{i\alpha}^\gamma h_{\gamma\bar{\beta}}$$

**Definition 6.2.6** (Chern curvature). Let  $X$  be a complex manifold and  $(E, h)$  be a Hermitian holomorphic vector bundle. The Chern curvature  $\Theta_h$  of  $(E, h)$  is defined as the curvature of Chern connection with respect to  $h$ .

**Corollary 6.2.1.** Let  $X$  be a complex manifold and  $(E, h)$  a Hermitian holomorphic vector bundle equipped with Chern connection  $\nabla$  locally given by  $\omega$ . Then

- (1)  $\partial\omega = \omega \wedge \omega$ .
- (2)  $\Theta_h = \bar{\partial}\omega$ .
- (3)  $\bar{\partial}\Theta_h = 0$ .

*Proof.* For (1). Since  $\omega = (\partial h)h^{-1}$ , then directly computation shows

$$\begin{aligned} \partial\omega &= -\partial h \wedge \partial(h^{-1}) \\ &= -\partial h \wedge (-h^{-1}\partial h h^{-1}) \\ &= (\partial h)h^{-1} \wedge (\partial h)h^{-1} \\ &= \omega \wedge \omega \end{aligned}$$

For (2).  $\Theta_h$  locally looks like

$$\Theta_h = d\omega - \omega \wedge \omega = d\omega - \partial\omega = \bar{\partial}\omega$$

For (3). It's clear from (2).  $\square$

*Remark 6.2.6* (local form). The Chern curvature can be expressed in terms of Christoffel symbol as follows

$$\Theta_h = \Theta_{i\bar{j}\alpha}^\gamma dz^i \wedge d\bar{z}^j \otimes e^\alpha \otimes e_\gamma$$

where  $\Theta_{i\bar{j}\alpha}^\gamma = -\frac{\partial \Gamma_{i\alpha}^\gamma}{\partial \bar{z}^j}$ . In other type one has

$$\begin{aligned} \Theta_{i\bar{j}\alpha\bar{\beta}} &= h_{\gamma\bar{\beta}} \Theta_{i\bar{j}\alpha}^\gamma \\ &= -h_{\gamma\bar{\beta}} \partial_{\bar{j}} (h^{\gamma\bar{\delta}} \frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i}) \\ &= -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} + h^{\gamma\bar{\delta}} \frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j} \end{aligned}$$

**6.2.4. Useful formulas of Chern connection.** Let  $X$  be a complex manifold and  $(E, h)$  be a Hermitian holomorphic vector bundle over  $X$ . Let  $\nabla$  be Chern connection determined by Christoffel symbol  $\Gamma_{i\alpha}^\beta$  on  $(E, h)$  with curvature  $\Theta_h$ . Suppose  $\{z^i\}$  is local coordinate of  $X$ ,  $\{e_\alpha\}$  is the local frame of  $E$ , and  $\{e^\alpha\}$  and  $\{\bar{e}_\alpha\}$  denote local frames of  $E^*$  and  $\bar{E}$  respectively.

**Proposition 6.2.3.**

$$\Gamma_{i\alpha}^\beta = h^{\beta\bar{\gamma}} \frac{\partial h_{\alpha\bar{\gamma}}}{\partial z^i}$$

*Proof.* See Remark 6.2.5.  $\square$

**Proposition 6.2.4.**

$$\begin{aligned} \Theta_{i\bar{j}\alpha}^\gamma &= -\frac{\partial \Gamma_{i\alpha}^\gamma}{\partial \bar{z}^j} \\ \Theta_{i\bar{j}\alpha\bar{\beta}} &= -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} + h^{\gamma\bar{\delta}} \frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j} \end{aligned}$$

*Proof.* See Remark 6.2.6.  $\square$

**Proposition 6.2.5.**

$$\begin{aligned} \nabla_{\frac{\partial}{\partial z^i}} e_\alpha &= \Gamma_{i\alpha}^\beta e_\beta, & \nabla_{\frac{\partial}{\partial \bar{z}^i}} e_\alpha &= 0, & \nabla_{\frac{\partial}{\partial z^i}} \bar{e}_\alpha &= 0, & \nabla_{\frac{\partial}{\partial \bar{z}^i}} \bar{e}_\alpha &= \Gamma_{i\alpha}^{\bar{\beta}} \bar{e}_\beta \\ \nabla_{\frac{\partial}{\partial z^i}} e^\alpha &= -\Gamma_{i\beta}^\alpha e^\beta, & \nabla_{\frac{\partial}{\partial \bar{z}^i}} e^\alpha &= 0, & \nabla_{\frac{\partial}{\partial z^i}} \bar{e}^\alpha &= 0, & \nabla_{\frac{\partial}{\partial \bar{z}^i}} \bar{e}^\alpha &= -\Gamma_{i\beta}^{\bar{\alpha}} \bar{e}^\beta \end{aligned}$$

*Proof.* It suffices to show the first two equalities, and others can be obtained from taking conjugates and dualities. The first one holds from definition of Christoffel symbol, and  $\Gamma_{i\alpha}^\beta = 0$  holds from the Remark 6.2.4.  $\square$

**Corollary 6.2.2.**

(1) For  $s \in C^\infty(X, E)$ , locally written as  $s = s^\alpha e_\alpha$ , one has

$$\begin{aligned}\nabla_{\frac{\partial}{\partial z^i}} s &= \left( \frac{\partial s^\beta}{\partial z^i} + s^\alpha \Gamma_{i\alpha}^\beta \right) e_\beta \\ \nabla_{\frac{\partial}{\partial \bar{z}^i}} s &= \frac{\partial s^\beta}{\partial \bar{z}^i} e_\beta\end{aligned}$$

(2) For  $s \in C^\infty(X, \bar{E})$ , locally written as  $s = s^\alpha \bar{e}_\alpha$ , one has

$$\begin{aligned}\nabla_{\frac{\partial}{\partial z^i}} s &= \frac{\partial s^\beta}{\partial z^i} \bar{e}_\beta \\ \nabla_{\frac{\partial}{\partial \bar{z}^i}} s &= \left( \frac{\partial s^\beta}{\partial \bar{z}^i} + s^\alpha \bar{\Gamma}_{i\alpha}^\beta \right) \bar{e}_\beta\end{aligned}$$

(3) For  $s \in C^\infty(X, E^*)$ , locally written as  $s = s_\alpha e^\alpha$ , one has

$$\begin{aligned}\nabla_{\frac{\partial}{\partial z^i}} s &= \left( \frac{\partial s_\beta}{\partial z^i} - s_\alpha \Gamma_{i\beta}^\alpha \right) e^\beta \\ \nabla_{\frac{\partial}{\partial \bar{z}^i}} s &= \frac{\partial s_\beta}{\partial \bar{z}^i} e^\beta\end{aligned}$$

(4) For  $s \in C^\infty(X, \bar{E}^*)$ , locally written as  $s = s_\alpha \bar{e}^\alpha$ , one has

$$\begin{aligned}\nabla_{\frac{\partial}{\partial z^i}} s &= \frac{\partial s_\beta}{\partial z^i} \bar{e}^\beta \\ \nabla_{\frac{\partial}{\partial \bar{z}^i}} s &= \left( \frac{\partial s_\beta}{\partial \bar{z}^i} - s_\alpha \bar{\Gamma}_{i\beta}^\alpha \right) \bar{e}^\beta\end{aligned}$$

**Proposition 6.2.6** (Ricci identity). For  $s \in C^\infty(X, E)$ , locally written as  $s = s^\alpha e_\alpha$ , one has

$$\nabla_{\frac{\partial}{\partial z^i}} \nabla_{\frac{\partial}{\partial \bar{z}^j}} s^\beta - \nabla_{\frac{\partial}{\partial \bar{z}^j}} \nabla_{\frac{\partial}{\partial z^i}} s^\beta = \Theta_{i\bar{j}\alpha}^\beta s^\alpha$$

*Proof.* Direct computation shows

$$\begin{aligned}\nabla_{\frac{\partial}{\partial z^i}} \nabla_{\frac{\partial}{\partial \bar{z}^j}} s &= \nabla_{\frac{\partial}{\partial z^i}} \left( \frac{\partial s^\alpha}{\partial \bar{z}^j} e_\alpha \right) \\ &= \frac{\partial^2 s^\beta}{\partial z^i \partial \bar{z}^j} e_\beta + \Gamma_{i\alpha}^\beta \frac{\partial s^\alpha}{\partial \bar{z}^j} e_\beta\end{aligned}$$

that is

$$\nabla_{\frac{\partial}{\partial z^i}} \nabla_{\frac{\partial}{\partial \bar{z}^j}} s^\beta = \frac{\partial^2 s^\beta}{\partial z^i \partial \bar{z}^j} + \Gamma_{i\alpha}^\beta \frac{\partial s^\alpha}{\partial \bar{z}^j}$$

Direct computation also shows

$$\nabla_{\frac{\partial}{\partial \bar{z}^j}} \nabla_{\frac{\partial}{\partial z^i}} s^\beta = \frac{\partial^2 s^\beta}{\partial \bar{z}^j \partial z^i} + \frac{\partial s^\alpha}{\partial \bar{z}^j} \Gamma_{i\alpha}^\beta + s^\alpha \frac{\partial \Gamma_{i\alpha}^\beta}{\partial \bar{z}^j}$$

Thus

$$\begin{aligned}\nabla_{\frac{\partial}{\partial z^i}} \nabla_{\frac{\partial}{\partial \bar{z}^j}} s^\beta - \nabla_{\frac{\partial}{\partial \bar{z}^j}} \nabla_{\frac{\partial}{\partial z^i}} s^\beta &= -s^\alpha \frac{\partial \Gamma_{i\alpha}^\beta}{\partial \bar{z}^j} \\ &= \Theta_{i\bar{j}\alpha}^\beta s^\alpha\end{aligned}$$

□

### 6.3. First Chern class.

6.3.1. *First Chern class of complex line bundle.* Let  $\pi: X \rightarrow L$  be a complex line bundle with connection  $\nabla$  over a complex manifold  $X$ . Then curvature  $\Theta$  is a global section of  $\Omega_{X,\mathbb{C}}^2$  since  $\text{End } L$  is trivial bundle. Furthermore,  $\Theta$  locally looks like  $d\omega$  since for line bundle  $\omega \wedge \omega = 0$ . An immediate consequence is  $d\Theta = 0$ , that is  $\Theta$  gives a cohomology class

$$[\Theta] \in H^2(X, \mathbb{C})$$

**Definition 6.3.1** (first Chern class of line bundle). Let  $L$  be a complex line bundle over complex manifold  $X$  equipped with connection  $\nabla$ . The first Chern class of  $L$  is defined as

$$c_1(L) := \left[ \frac{\sqrt{-1}}{2\pi} \Theta \right] \in H^2(X, \mathbb{C})$$

where  $\Theta$  is the curvature of  $\nabla$ .

**Proposition 6.3.1** (topological invariance).  $c_1(L) \in H^2(X, \mathbb{C})$  is independent of the choice of connection.

*Proof.* Let  $\tilde{\nabla}$  be another connection which is locally given by  $\tilde{\omega}$ . Then for section  $s$  of  $\Omega_{X,\mathbb{C}}^k \otimes L$ , one has

$$\begin{aligned} (\nabla - \tilde{\nabla})s &= (ds + \omega \wedge s) - (ds + \tilde{\omega} \wedge s) \\ &= (\omega - \tilde{\omega}) \wedge s \end{aligned}$$

Note that  $\omega - \tilde{\omega}$  is a global section of  $\Omega_{X,\mathbb{C}}^1$ , so  $\Theta - \tilde{\Theta}$  is exact. □

**Proposition 6.3.2.** Let  $\pi: L \rightarrow X$  be a complex line bundle over a complex manifold  $X$ . Then  $c_1(L) \in H^2(X, \mathbb{R})$ .

*Proof.* Equip  $L$  with a Hermitian metric  $h$ , then for a metric connection  $\nabla$ , locally we have

$$\bar{\omega} = -\omega$$

Thus

$$\overline{\frac{\sqrt{-1}}{2\pi} \Theta} = -\frac{\sqrt{-1}}{2\pi} \bar{\Theta} = -\frac{\sqrt{-1}}{2\pi} d\bar{\omega} = \frac{\sqrt{-1}}{2\pi} d\omega = \frac{\sqrt{-1}}{2\pi} \Theta$$

□

*Remark 6.3.1.* Here are two facts here we don't prove:

- (1)  $c_1(L) \in H^2(X, \mathbb{Z})$ .
- (2)  $L$  is determined by  $c_1(L)$ .

**Definition 6.3.2** (first Chern class of vector bundle). Let  $E$  be a complex vector bundle over complex manifold  $X$ . The first Chern class of  $E$  is defined to be the first Chern class of  $\det E$ .

**6.3.2. First Chern class of Hermitian holomorphic line bundle.** Let  $X$  be a complex manifold,  $(L, h)$  a Hermitian holomorphic line bundle, and  $\nabla$  is the Chern connection of  $(L, h)$  with Chern curvature  $\Theta_h$ . Then by Proposition 6.3.2 and Corollary 6.2.1, we have

$$\left[\frac{\sqrt{-1}}{2\pi}\Theta_h\right] \in H^2(X, \mathbb{R}) \cap H^{1,1}(X)$$

*Remark 6.3.2* (local form). Suppose Hermitian metric  $h$  is given by metric weight  $\{\varphi_\alpha\}$ , that is locally  $h = e^{-2\varphi_\alpha}$ . Then direct computation shows first Chern class is locally given by

$$\frac{\sqrt{-1}}{2\pi}\Theta_h = \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\varphi_\alpha$$

**Proposition 6.3.3.**  $\left[\frac{\sqrt{-1}}{2\pi}\Theta_h\right] \in H^{1,1}(X)$  is independent of  $h$ .

*Proof.* Note that any two metric on a line bundle differ a smooth function which is positive everywhere, so if  $h$  and  $h'$  are two different metrics, we can write  $\|e(z)\|_{h'} = e^f\|e(z)\|_h$  for some globally defined smooth function  $f$ . So by Remark 6.3.2, we have the difference of first Chern classes coming from different metrics is  $\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}f$ , and it's trivial in  $H^{1,1}(X)$  since  $f$  is globally defined.  $\square$

## 7. SECOND FUNDAMENTAL FORM

Let  $(E, h)$  be a Hermitian holomorphic vector bundle over complex manifold  $X$  with rank  $r$ . There is an exact sequence of holomorphic vector bundles

$$0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0$$

where  $S$  is a holomorphic subbundle of rank  $s$ , and  $Q$  is the quotient bundle, which is isomorphic to  $S^\perp$  as complex vector bundle.

**7.1. Second fundamental form of subbundle.** Suppose  $\nabla^E$  is the Chern connection on  $E$  and  $\pi_S: E \rightarrow S$  is the orthogonal projection. Then

$$\nabla^S := \pi_S \circ \nabla^E$$

is the Chern connection on  $S$ . Indeed,

- (1) It's clear  $\nabla^S$  is compatible with complex structure of  $S$  since  $\nabla^E$  is Chern connection of  $E$ , and  $S$  is holomorphic subbundle of  $E$ .
- (2) For sections  $s, t$  of  $S$ , one has

$$\begin{aligned} dh(s, t) &= h(\nabla^E s, t) + h(s, \nabla^E t) \\ &\stackrel{(a)}{=} h(\pi_S \circ \nabla^E s, t) + h(s, \pi_S \circ \nabla^E t) \\ &= h(\nabla^S s, t) + h(s, \nabla^S t) \end{aligned}$$

where (a) holds from  $\pi_S$  is orthogonal projection.

The second fundamental form of the subbundle  $S$  of  $E$  is defined as

$$B = \nabla^E - \nabla^S: C^\infty(X, S) \rightarrow C^\infty(X, \Omega_X^{1,0} \otimes Q)$$

Direct computation shows second fundamental form  $B$  is a section of  $\Omega_X^{1,0} \otimes \text{Hom}(S, Q)$ . For  $p \in X$ , suppose  $\{e_\alpha\}_{1 \leq \alpha \leq r}$  is a holomorphic local frame of  $E$  such that  $\{e_\alpha\}_{1 \leq \alpha \leq s}$  is a holomorphic local frame of  $S$  such that  $h_{\alpha\bar{\beta}}(p) = \delta_{\alpha\bar{\beta}}$ . By formula of Chern connection, for  $1 \leq \alpha \leq s$ , one has

$$\begin{aligned} \nabla^E e_\alpha(p) &= \sum_{\beta=1}^r \frac{h_{\alpha\bar{\beta}}}{\partial z^i}(p) dz^i \otimes e_\beta \\ \nabla^S e_\alpha(p) &= \sum_{\beta=1}^s \frac{h_{\alpha\bar{\beta}}}{\partial z^i}(p) dz^i \otimes e_\beta \end{aligned}$$

Then for  $1 \leq \alpha \leq s$ , one has

$$B e_\alpha(p) = \sum_{\beta=s+1}^r \frac{h_{\alpha\bar{\beta}}}{\partial z^i}(p) dz^i \otimes e_\beta$$

Thus with respect to local frame we choose, one has

$$B(p) = \sum_{\alpha=1}^s \sum_{\beta=s+1}^r \frac{h_{\alpha\bar{\beta}}}{\partial z^i}(p) dz^i \otimes e^\alpha \otimes e_\beta$$



One has its conjugate transpose<sup>5</sup> is

$$B^*(p) = \sum_{\beta=s+1}^r \sum_{\alpha=1}^s \frac{h_{\beta\bar{\alpha}}}{\partial \bar{z}^j}(p) d\bar{z}^j \otimes e^\beta \otimes e_\alpha$$

then  $B^* \wedge B$  gives a section of  $\Omega_X^{1,1} \otimes S^* \otimes S$ . To be explicit, for  $1 \leq \alpha \leq s$ , one has

$$\begin{aligned} B^* \wedge B e_\alpha(p) &= B^* \left( \sum_{\gamma=s+1}^r \frac{h_{\alpha\bar{\gamma}}}{\partial z^i}(p) dz^i \otimes e_\gamma \right) \\ &= - \sum_{\beta=1}^s \sum_{\gamma=s+1}^r \frac{h_{\alpha\bar{\gamma}}}{\partial z^i} \frac{h_{\gamma\bar{\beta}}}{\partial \bar{z}^j}(p) dz^i \wedge d\bar{z}^j e_\beta \end{aligned}$$

which implies

$$B^* \wedge B(p) = - \sum_{\alpha,\beta=1}^s \left\{ \sum_{\gamma=s+1}^r \frac{\partial h_{\alpha\bar{\gamma}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j}(p) \right\} dz^i \wedge d\bar{z}^j \otimes e^\alpha \otimes e_\beta$$

On the other hand, direct computation shows

$${}^E\Theta|_S(p) - {}^S\Theta(p) = \sum_{\alpha,\beta=1}^s \left\{ \sum_{\gamma=s+1}^r \frac{\partial h_{\alpha\bar{\gamma}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j}(p) \right\} dz^i \wedge d\bar{z}^j \otimes e^\alpha \otimes \bar{e}^\beta$$

Thus

$${}^S\Theta = {}^E\Theta|_S + B^* \wedge B$$

where  $B^*$  is conjugate transpose of  $B$  since  $e_\beta = \delta_{\beta\bar{\beta}} \bar{e}^\beta$ .

**Corollary 7.1.1.** If  $(E, h)$  is Griffiths or Nakano negative, so is the subbundle  $(S, h_S)$ .

*Proof.* It suffices to see if  $(E, h)$  is Nakano positive, so is the subbundle since Nakano positive implies Griffiths positive. Suppose  $(E, h)$  is Nakano positive. Note that for any non-zero matrix  $(u^{i\alpha})$ , one has

$$\sum_{\alpha,\beta=1}^s \sum_{\gamma=s+1}^r \frac{\partial h_{\alpha\bar{\gamma}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j} u^{i\alpha} \bar{u}^{j\beta} \geq 0$$

□

**7.2. Second fundamental form of quotient bundle.** Here we consider the following exact sequence

$$0 \rightarrow Q^* \rightarrow E^* \rightarrow S^* \rightarrow 0$$

The second fundamental form  $C$  of quotient bundle  $Q$  is defined as the second fundamental form of subbundle  $Q^*$ .

**Corollary 7.2.1.** If  $(E, h)$  is Griffiths or dual Nakano positive, so is the quotient bundle  $(Q, h_Q)$ .

<sup>5</sup>See section 2.1 in [Liu23] for more detail.

## 8. HERMITIAN GEOMETRY

**8.1. From Riemannian manifold to Hermitian manifold.** Let  $X$  be a complex  $n$ -manifold, and suppose  $\{z^i = x^i + \sqrt{-1}x^I\}$  is a local coordinate of  $X$ , where  $1 \leq i \leq n$  and  $n+1 \leq I = i+n \leq 2n$ . Then  $\{x^i, x^I\}$  gives a local coordinate of underlying smooth manifold of  $X$ , and there is a natural almost complex structure  $J$  on  $T_{\mathbb{R}}X$  which is given by

$$\begin{aligned} J\left(\frac{\partial}{\partial x^i}\right) &= \frac{\partial}{\partial x^I} \\ J\left(\frac{\partial}{\partial x^I}\right) &= -\frac{\partial}{\partial x^i} \end{aligned}$$

**Definition 8.1.1** (compatibility). A Riemannian metric  $g$  on  $T_{\mathbb{R}}X$  is called compatible with almost complex structure, if

$$g(V, W) = g(JV, JW)$$

for all  $V, W \in C^\infty(X, T_{\mathbb{R}}X)$ .

**Proposition 8.1.1.** Let  $g$  be a Riemannian metric on  $T_{\mathbb{R}}X$  which is compatible with  $J$  and locally given by

$$g = g_{ij}dx^i \otimes dx^j + g_{iJ}dx^i \otimes dx^J + g_{IJ}dx^I \otimes dx^J$$

Then

$$\begin{aligned} g_{ij} &= g_{IJ} \\ g_{iJ} &= g_{Ji} = -g_{jI} = -g_{Ij} \end{aligned}$$

*Proof.* Direct computation. □

**Notation 8.1.1.**

$$\begin{pmatrix} g^{il} & g^{iL} \\ g^{Il} & g^{IL} \end{pmatrix} \begin{pmatrix} g_{lj} & g_{lJ} \\ g_{Lj} & g_{LJ} \end{pmatrix} = I_{2n}$$

that is

$$\begin{aligned} g^{il}g_{lj} + g^{iL}g_{Lj} &= \delta_j^i \\ g^{il}g_{lJ} + g^{iL}g_{LJ} &= 0 \end{aligned}$$

Let  $g$  be a Riemannian metric which is compatible with  $J$ , and its  $\mathbb{C}$ -linear extension  $g_{\mathbb{C}}$  gives a matrix

$$G = \begin{pmatrix} (g_{\mathbb{C}}(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}))_{n \times n} & (g_{\mathbb{C}}(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}))_{n \times n} \\ (g_{\mathbb{C}}(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial z^j}))_{n \times n} & (g_{\mathbb{C}}(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}))_{n \times n} \end{pmatrix}$$

Direct computation shows

$$\begin{aligned} g_{\mathbb{C}}(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}) &= 0 \\ g_{\mathbb{C}}(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}) &= 0 \end{aligned}$$

and if we denote  $H = (h_{i\bar{j}})_{n \times n}$ , where

$$(8.1) \quad h_{i\bar{j}} := g_{\mathbb{C}}\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right) = \frac{1}{2}(g_{ij} + \sqrt{-1}g_{iJ})$$

Then

$$G = \begin{pmatrix} 0 & H \\ \overline{H} & 0 \end{pmatrix}$$

Note that  $G$  is a positive definite symmetric matrix, which implies  $H$  is a positive definite Hermitian matrix, that is  $h$  gives a Hermitian metric on  $T^{1,0}X$ .

**Proposition 8.1.2.**

$$\sqrt{\det g} = 2^n \det h$$

*Proof.* Direct computation shows

$$\begin{aligned} \det g &= \det \begin{pmatrix} (g_{ij})_{n \times n} & (g_{iJ})_{n \times n} \\ (g_{Ij})_{n \times n} & (g_{IJ})_{n \times n} \end{pmatrix} \\ &\stackrel{(1)}{=} \det \begin{pmatrix} (g_{ij})_{n \times n} & (g_{iJ})_{n \times n} \\ (-g_{iJ})_{n \times n} & (g_{ij})_{n \times n} \end{pmatrix} \\ &= \det \begin{pmatrix} (g_{ij} + \sqrt{-1}g_{iJ})_{n \times n} & (g_{iJ})_{n \times n} \\ (-g_{iJ} + \sqrt{-1}g_{ij})_{n \times n} & (g_{ij})_{n \times n} \end{pmatrix} \\ &= \det \begin{pmatrix} (g_{ij} + \sqrt{-1}g_{iJ})_{n \times n} & (g_{iJ})_{n \times n} \\ 0 & (g_{ij} + \sqrt{-1}g_{iJ})_{n \times n} \end{pmatrix} \\ &= (2^n \det h)^2 \end{aligned}$$

where (1) holds from Proposition 8.1.1.  $\square$

*Remark 8.1.1.* If  $h$  is a Hermitian metric on  $T^{1,0}X$ , there is also a Riemannian metric on  $T_{\mathbb{R}}X$  given by

$$\begin{aligned} g_{ij} &= 2\operatorname{Re}h_{i\bar{j}} \\ g_{iJ} &= 2\operatorname{Im}h_{i\bar{j}} \\ g_{Ij} &= -g_{iJ} \\ g_{IJ} &= g_{ij} \end{aligned}$$

**Notation 8.1.2.**  $h^{i\bar{j}}$  is defined by the  $(i, j)$ -entry of  $(H^{-1})^T$ , that is  $h^{i\bar{l}}h_{j\bar{l}} = \delta_j^i$ , and  $h^{\bar{i}j} := \overline{h^{i\bar{j}}}$ . Note that

$$G^{-1} = (G^{-1})^T = \begin{pmatrix} 0 & (h^{i\bar{j}})_{n \times n} \\ (h^{\bar{i}j})_{n \times n} & 0 \end{pmatrix}$$

**Proposition 8.1.3.**

$$h^{i\bar{j}} = 2(g^{ij} - \sqrt{-1}g^{iJ})$$

*Proof.* Direct computation shows

$$\begin{aligned}
h^{i\bar{l}}h_{j\bar{l}} &= (g^{il} - \sqrt{-1}g^{iL})(g_{jl} + \sqrt{-1}g_{jL}) \\
&= g^{il}g_{jl} + g^{iL}g_{jL} + \sqrt{-1}(g^{il}g_{jL} - g^{iL}g_{jl}) \\
&= g^{il}g_{lj} + g^{iL}g_{Lj} - \sqrt{-1}(g^{il}g_{lJ} + g^{iL}g_{Lj}) \\
&= \delta_j^i
\end{aligned}$$

□

**Definition 8.1.2** (fundamental form). The fundamental form  $\omega$  is defined as

$$\omega(V, W) := g_{\mathbb{C}}(JV, W)$$

where  $V, W \in C^\infty(X, T_{\mathbb{C}}X)$ .

**Proposition 8.1.4.**

$$\omega = \sqrt{-1}h_{i\bar{j}}dz^i \wedge d\bar{z}^j$$

In particular,  $\omega$  is a real  $(1, 1)$ -form.

*Proof.* Direct computation shows

$$\begin{aligned}
\omega\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) &= g_{\mathbb{C}}\left(J\left(\frac{\partial}{\partial z^i}\right), \frac{\partial}{\partial z^j}\right) = \sqrt{-1}g_{\mathbb{C}}\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = 0 \\
\omega\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right) &= g_{\mathbb{C}}\left(J\left(\frac{\partial}{\partial z^i}\right), \frac{\partial}{\partial \bar{z}^j}\right) = \sqrt{-1}g_{\mathbb{C}}\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right) = \sqrt{-1}h_{i\bar{j}} \\
\omega\left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial z^j}\right) &= g_{\mathbb{C}}\left(J\left(\frac{\partial}{\partial \bar{z}^i}\right), \frac{\partial}{\partial z^j}\right) = -\sqrt{-1}g_{\mathbb{C}}\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right) = -\sqrt{-1}h_{i\bar{j}} \\
\omega\left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}\right) &= g_{\mathbb{C}}\left(J\left(\frac{\partial}{\partial \bar{z}^i}\right), \frac{\partial}{\partial \bar{z}^j}\right) = -\sqrt{-1}g_{\mathbb{C}}\left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}\right) = 0
\end{aligned}$$

This shows

$$\begin{aligned}
\omega &= \sqrt{-1}h_{i\bar{j}}dz^i \otimes d\bar{z}^j - \sqrt{-1}h_{i\bar{j}}d\bar{z}^i \otimes dz^j \\
&= \sqrt{-1}h_{i\bar{j}}dz^i \otimes d\bar{z}^j - \sqrt{-1}h_{\bar{j}i}d\bar{z}^j \otimes dz^i \\
&= \sqrt{-1}h_{i\bar{j}}dz^i \wedge d\bar{z}^j
\end{aligned}$$

where the last step holds since  $h$  is Hermitian, that is  $h_{\bar{j}i} = h_{i\bar{j}}$ . □

**Proposition 8.1.5.**

$$2h + \sqrt{-1}\omega = g_{\mathbb{C}}$$

*Proof.* Direct computation shows

$$\begin{aligned}
h - \frac{1}{2}g_{\mathbb{C}} &= h_{i\bar{j}}dz^i \otimes d\bar{z}^j - \frac{1}{2}(h_{i\bar{j}}dz^i \otimes d\bar{z}^j + h_{i\bar{j}}d\bar{z}^i \otimes dz^j) \\
&= \frac{1}{2}h_{i\bar{j}}dz^i \otimes d\bar{z}^j - \frac{1}{2}h_{i\bar{j}}d\bar{z}^i \otimes dz^j \\
&= \frac{1}{2}h_{i\bar{j}}dz^i \wedge d\bar{z}^j \\
&= -\frac{\sqrt{-1}}{2}\omega
\end{aligned}$$

□

*Remark 8.1.2.* From above discussions, Hermitian metric  $h$ , Riemannian metric  $g$  on underlying smooth manifold and fundamental form  $\omega$  are the same things on a complex manifold  $X$ , and any of them gives a Hermitian structure.

**Theorem 8.1.1** (normal coordinate). Let  $(X, h)$  be a Hermitian manifold. For any  $p \in X$ , there exists a local holomorphic coordinate  $\{z^i\}$  centered at  $p$  such that

$$h_{i\bar{j}}(p) = \delta_{ij} \quad \text{and} \quad \frac{\partial h_{i\bar{j}}}{\partial \bar{z}^k} + \frac{\partial h_{i\bar{k}}}{\partial \bar{z}^j} = 0$$

*Proof.* Without loss of generality, we may assume

$$\omega = \sqrt{-1}(\delta_{i\bar{j}} + a_{i\bar{j}l}w^l + a_{i\bar{j}l}\bar{w}^l + O(|w|^2))dw^i \wedge d\bar{w}^j$$

□

**8.2. Curvatures of Hermitian manifold.** Sometimes we need to consider Hermitian holomorphic vector bundles over a Hermitian manifold, so there are two Hermitian metrics. In this case, in order to distinguish them, we always say a Hermitian holomorphic vector bundle  $(E, h)$  over a Hermitian manifold  $(X, g)$ .

**Definition 8.2.1** (curvatures of vector bundle). Let  $(E, h)$  be a Hermitian holomorphic vector bundle on Hermitian manifold  $(X, g)$ . Then

- (1) the first Chern-Ricci curvature of  $(E, h)$  is locally given by

$$\text{Ric}^{(1)}(h) = \sqrt{-1}h^{\alpha\bar{\beta}}\Theta_{i\bar{j}\alpha\bar{\beta}}dz^i \wedge d\bar{z}^j$$

- (2) the second Chern-Ricci curvature of  $(E, h)$  is locally given by

$$\text{Ric}^{(2)}(h) = \sqrt{-1}g^{i\bar{j}}\Theta_{i\bar{j}\alpha\bar{\beta}}e^\alpha \otimes \bar{e}^\beta$$

- (3) the Chern scalar curvature of  $(E, h)$  is locally given by

$$s = g^{i\bar{j}}h^{\alpha\bar{\beta}}\Theta_{i\bar{j}\alpha\bar{\beta}}$$

**Definition 8.2.2** (curvatures of Hermitian manifold). Let  $(X, h)$  be a Hermitian manifold. Then

- (1) the first (second) Chern-Ricci curvature of  $(X, h)$  is defined to be the first (second) Chern-Ricci curvature of its holomorphic tangent bundle.
- (2) the Chern scalar curvature of  $(X, h)$  is defined to be the Chern scalar curvature of its holomorphic tangent bundle.

**Proposition 8.2.1.** Let  $(E, h)$  be a Hermitian holomorphic vector bundle. The first Chern-Ricci curvature of  $(E, h)$  gives the first Chern class of  $(E, h)$  up to a scalar.

*Proof.* A direct computation shows

$$\frac{\sqrt{-1}}{2\pi} h^{\alpha\bar{\beta}} \left( -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} + h^{\gamma\bar{\delta}} \frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j} \right) = -\frac{\sqrt{-1}}{2\pi} \frac{\partial^2 \log \det(h_{\alpha\bar{\beta}})}{\partial z^i \partial \bar{z}^j}$$

Thus the first Chern-Ricci curvature of  $(E, h)$  gives the Chern curvature of  $(\det E, \det h)$  up a scalar  $1/2\pi$ , and by Definition 6.3.2 the first Chern class of  $(E, h)$  is defined to be first Chern class of  $\det E$ .  $\square$

**Definition 8.2.3** (holomorphic sectional curvature). Let  $(X, h)$  be a Hermitian manifold and  $v = v^i \frac{\partial}{\partial z^i} \in T_x X$  be a unit vector. The holomorphic sectional curvature in the direction  $v$  is defined as

$$\text{HSC}_x(v) := \Theta_{i\bar{j}k\bar{l}} v^i \bar{v}^j v^k \bar{v}^l$$

**Definition 8.2.4** (holomorphic bisectional curvature). Let  $(X, h)$  be a Hermitian manifold and  $v = v^i \frac{\partial}{\partial z^i}, w = w^i \frac{\partial}{\partial z^i} \in T_x X$  be unit vectors. The holomorphic sectional curvature in the direction  $v, w$  is defined as

$$\text{HBS}_x(v, w) := \Theta_{i\bar{j}k\bar{l}} v^i \bar{v}^j w^k \bar{w}^l$$

### 8.3. Useful formulas of Hermitian geometry.

#### Proposition 8.3.1.

$$\omega^n = (\sqrt{-1})^n n! \det(h_{i\bar{j}}) dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n$$

*Proof.* Direct computation shows

$$\begin{aligned} \omega^n &= (\sqrt{-1})^n (h_{i\bar{j}} dz^i \wedge d\bar{z}^j)^n \\ &= (\sqrt{-1})^n \sum_{\substack{\sigma \in S_n \\ \tau \in S_n}} h_{i_{\sigma(1)} \bar{j}_{\tau(1)}} \cdots h_{i_{\sigma(n)} \bar{j}_{\tau(n)}} dz^{i_{\sigma(1)}} \wedge d\bar{z}^{j_{\tau(1)}} \wedge \cdots \wedge dz^{i_{\sigma(n)}} \wedge d\bar{z}^{j_{\tau(n)}} \\ &= (\sqrt{-1})^n \sum_{\substack{\sigma \in S_n \\ \tau \in S_n}} (-1)^{|\sigma|} (-1)^{|\tau|} h_{i_{\sigma(1)} \bar{j}_{\tau(1)}} \cdots h_{i_{\sigma(n)} \bar{j}_{\tau(n)}} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \\ &= (\sqrt{-1})^n \sum_{\sigma \in S_n} \sum_{\rho \in S_n} (-1)^{|\rho|} h_{i_{\rho(1)} \bar{j}_1} \cdots h_{i_{\rho(n)} \bar{j}_n} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \\ &= (\sqrt{-1})^n n! \det(h_{i\bar{j}}) dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \end{aligned}$$

$\square$

#### Corollary 8.3.1.

$$\omega^{n-1} = (\sqrt{-1})^{n-1} (n-1)! \sum_{\substack{i_1 < \cdots < i_{n-1}, i_l \neq p \\ j_1 < \cdots < j_{n-1}, j_l \neq q}} h(p, q) dz^{i_1} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge dz^{i_{n-1}} \wedge d\bar{z}^{j_{n-1}}$$

where  $h(p, q)$  the cofactor of  $h$  without row  $p$  and column  $q$ .

*Proof.* Direct computation shows

$$\begin{aligned}
 \omega^{n-1} &= (\sqrt{-1})^{n-1} (h_{i\bar{j}} dz^i \wedge d\bar{z}^j)^{n-1} \\
 &= (\sqrt{-1})^{n-1} \sum_{\substack{\sigma \in S_{n-1} \\ \tau \in S_{n-1}}} h_{i_{\sigma(1)}\bar{j}_{\tau(1)}} \cdots h_{i_{\sigma(n-1)}\bar{j}_{\tau(n-1)}} dz^{i_{\sigma(1)}} \wedge d\bar{z}^{j_{\tau(1)}} \wedge \cdots \wedge dz^{i_{\sigma(n-1)}} \wedge d\bar{z}^{j_{\tau(n-1)}} \\
 &= (\sqrt{-1})^{n-1} \sum_{\substack{i_1 < \cdots < i_{n-1}, i_l \neq p \\ j_1 < \cdots < j_{n-1}, j_l \neq q}} \sum_{\substack{\sigma \in S_{n-1} \\ \rho \in S_{n-1}}} (-1)^{|\rho|} h_{i_{\rho(1)}\bar{j}_1} \cdots h_{i_{\rho(n-1)}\bar{j}_{n-1}} dz^{i_1} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge dz^{i_{n-1}} \wedge d\bar{z}^{j_{n-1}} \\
 &= (\sqrt{-1})^{n-1} (n-1)! \sum_{\substack{i_1 < \cdots < i_{n-1}, i_l \neq p \\ j_1 < \cdots < j_{n-1}, j_l \neq q}} h(p, q) dz^{i_1} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge dz^{i_{n-1}} \wedge d\bar{z}^{j_{n-1}}
 \end{aligned}$$

□

**Corollary 8.3.2.**

$$\omega^{n-2} = (\sqrt{-1})^{n-2} (n-2)! \sum_{\substack{i_1 < \cdots < i_{n-2}, i_l \neq p, s \\ j_1 < \cdots < j_{n-2}, j_l \neq q, t}} h \begin{pmatrix} p, q \\ s, t \end{pmatrix} dz^{i_1} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge dz^{i_{n-2}} \wedge d\bar{z}^{j_{n-2}}$$

where  $h \begin{pmatrix} p, q \\ s, t \end{pmatrix}$  the cofactor of  $h$  without rows  $p, s$  and columns  $q, t$ .

**Proposition 8.3.2.** Let  $(X, \omega)$  be Hermitian  $n$ -manifold. Then  $\omega^n/n!$  is the volume form of the underlying smooth manifold with respect to  $g$ .

*Proof.* Direct computation shows

$$\begin{aligned}
 \omega^n &\stackrel{(1)}{=} (\sqrt{-1})^n n! \det(h_{i\bar{j}}) dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \\
 &\stackrel{(2)}{=} n! 2^n \det(h_{i\bar{j}}) dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n \\
 &\stackrel{(3)}{=} n! \text{vol}
 \end{aligned}$$

where

- (1) holds from Proposition 8.3.1.
- (2) holds from Exercise 5.2.1.
- (3) holds from Proposition 8.1.2.

□

**Proposition 8.3.3.** Let  $(X, \omega)$  be a Hermitian  $n$ -manifold.

(1) If  $\varphi$  is a real  $(1, 1)$ -form, then

$$\varphi \wedge \omega^{n-1} = \frac{1}{n} \text{tr}_\omega \varphi \cdot \omega^n$$

(2) If  $\varphi$  is a  $(1, 0)$ -form, then

$$\sqrt{-1} \varphi \wedge \bar{\varphi} \wedge \omega^{n-1} = \frac{1}{n} |\varphi|^2 \cdot \omega^n$$

(3) If  $\varphi$  is a  $(0, 1)$ -form, then

$$\sqrt{-1}\varphi \wedge \bar{\varphi} \wedge \omega^{n-1} = -\frac{1}{n}|\varphi|^2 \cdot \omega^n$$

*Proof.* For (1). Suppose  $\varphi = \sqrt{-1}\varphi_{i\bar{j}}dz^i \wedge d\bar{z}^j$ . Then by Proposition 8.3.1 one has

$$\frac{1}{n} \operatorname{tr}_\omega \varphi \cdot \omega^n = (\sqrt{-1})^n (n-1)! h^{i\bar{j}} \varphi_{i\bar{j}} \det h dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n$$

and by Corollary 8.3.1 one has

$$\begin{aligned} \varphi \wedge \omega^{n-1} &= (\sqrt{-1})^n (n-1)! \varphi_{i\bar{j}} dz^i \wedge d\bar{z}^j \wedge \sum_{\substack{i_1 < \cdots < i_{n-1}, i_l \neq p \\ j_1 < \cdots < j_{n-1}, j_l \neq q}} h(p, q) dz^{i_1} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge dz^{i_{n-1}} \wedge d\bar{z}^{j_{n-1}} \\ &= (\sqrt{-1})^n (n-1)! \sum_{1 \leq p, q \leq n} \varphi_{p\bar{q}} (-1)^{p+q} h(p, q) dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \end{aligned}$$

Note that by expansion of determinant one has

$$h^{i\bar{j}} \varphi_{i\bar{j}} \det h = h^{i\bar{j}} \varphi_{i\bar{j}} \sum_{k=1}^n h_{k\bar{j}} (-1)^{k+j} h(k, j) = \sum_{1 \leq i, j \leq n} \varphi_{i\bar{j}} (-1)^{i+j} h(i, j)$$

For (2). Suppose  $\varphi = \varphi_i dz^i$ . Then

$$\sqrt{-1}\varphi \wedge \bar{\varphi} = \sqrt{-1}\varphi_i \bar{\varphi}_{\bar{j}} dz^i \wedge d\bar{z}^j$$

is a real  $(1, 1)$ -form, and it's clear

$$\operatorname{tr}_\omega \sqrt{-1}\varphi \wedge \bar{\varphi} = |\varphi|^2$$

then by (1) we obtain desired result, and (3) follows from (2) directly.  $\square$

**Proposition 8.3.4.** Let  $(E, h)$  be a Hermitian vector bundle over a Hermitian manifold  $(X, g)$  and  $\varphi$  be a real  $(1, 1)$ -form. Then

$$\{\varphi, \varphi\} \frac{\omega^{n-2}}{(n-2)!} = (|\operatorname{tr}_\omega \varphi|^2 - |\varphi|^2) \frac{\omega^n}{n!}$$

*Proof.* Suppose  $\varphi = \sqrt{-1}\varphi_{i\bar{j}}^\alpha dz^i \wedge d\bar{z}^j \otimes e_\alpha$ . Then by Corollary 8.3.2 one has

$$\text{LHS} = (\sqrt{-1})^n \varphi_{i\bar{j}}^\alpha \overline{\varphi_{k\bar{l}}^\beta} h_{\alpha\bar{\beta}} dz^i \wedge \bar{z}^j \wedge dz^l \wedge \bar{z}^k \wedge \sum_{\substack{i_1 < \cdots < i_{n-2}, i_l \neq p, s \\ j_1 < \cdots < j_{n-2}, j_l \neq q, t}} g \binom{p, q}{s, t} dz^{i_1} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge dz^{i_{n-2}} \wedge d\bar{z}^{j_{n-2}}$$

If we want to insert  $dz^i \wedge d\bar{z}^j \wedge dz^l \wedge d\bar{z}^k$  into  $\sum_{\substack{i_1 < \cdots < i_{n-2}, i_l \neq p, s \\ j_1 < \cdots < j_{n-2}, j_l \neq q, t}} dz^{i_1} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge dz^{i_{n-2}} \wedge d\bar{z}^{j_{n-2}}$ , there are the following four cases

$$\begin{cases} i = p, l = s, j = q, k = t \\ i = p, l = s, j = t, k = q \\ i = s, l = p, j = q, k = t \\ i = s, l = p, j = t, k = q \end{cases}$$



So case by case one has

$$\begin{aligned} & \varphi_{i\bar{j}}^\alpha \overline{\varphi_{k\bar{l}}^\beta} dz^i \wedge d\bar{z}^j \wedge dz^l \wedge d\bar{z}^k \wedge \sum_{\substack{i_1 < \dots < i_{n-2}, i_l \neq p, s \\ j_1 < \dots < j_{n-2}, j_l \neq q, t}} dz^{i_1} \wedge d\bar{z}^{j_1} \wedge \dots \wedge dz^{i_{n-2}} \wedge d\bar{z}^{j_{n-2}} \\ &= \sum_{\substack{1 \leq p, q \leq n \\ 1 \leq s, t \leq n}} (\varphi_{p\bar{q}}^\alpha \overline{\varphi_{t\bar{s}}^\beta} - \varphi_{p\bar{t}}^\alpha \overline{\varphi_{q\bar{s}}^\beta} - \varphi_{s\bar{q}}^\alpha \overline{\varphi_{t\bar{p}}^\beta} + \varphi_{s\bar{t}}^\alpha \overline{\varphi_{q\bar{p}}^\beta}) (-1)^{p+q+t+s} dz^1 \wedge d\bar{z}^1 \wedge \dots \wedge dz^n \wedge d\bar{z}^n \end{aligned}$$

On the other hand, direct computation shows

$$\begin{aligned} |\mathrm{tr}_\omega \varphi|^2 &= \varphi_{i\bar{j}}^\alpha \overline{\varphi_{k\bar{l}}^\beta} g^{i\bar{j}} g^{l\bar{k}} h_{\alpha\bar{\beta}} \\ |\varphi^2| &= \varphi_{i\bar{j}}^\alpha \overline{\varphi_{k\bar{l}}^\beta} g^{i\bar{k}} g^{l\bar{j}} h_{\alpha\bar{\beta}} \end{aligned}$$

and thus

$$\mathrm{RHS} = (\sqrt{-1})^n \varphi_{i\bar{j}}^\alpha \overline{\varphi_{k\bar{l}}^\beta} h_{\alpha\bar{\beta}} (g^{i\bar{j}} g^{l\bar{k}} - g^{i\bar{k}} g^{l\bar{j}}) \det g$$

Note that Laplacian's theorem implies

$$\begin{aligned} (g^{i\bar{j}} g^{l\bar{k}} - g^{i\bar{k}} g^{l\bar{j}}) \det g &= (g^{i\bar{j}} g^{l\bar{k}} - g^{i\bar{k}} g^{l\bar{j}}) \sum_{\substack{1 \leq p, s \leq n \\ 1 \leq j, k \leq n}} (g_{p\bar{j}} g_{s\bar{k}} - g_{s\bar{j}} g_{p\bar{k}}) (-1)^{p+j+s+k} g \binom{p, j}{s, k} \\ &= (\delta_p^i \delta_s^l - \delta_s^i \delta_p^l - \delta_s^i \delta_p^l + \delta_p^i \delta_s^l) (-1)^{p+j+s+k} g \binom{p, j}{s, k} \end{aligned}$$

This shows the RHS equals to the LHS.  $\square$

**Proposition 8.3.5.** Let  $(X, h)$  be a Hermitian manifold. Then

$$\langle dz^i \wedge \alpha, \beta \rangle = \langle \alpha, h^{p\bar{i}} t_p \beta \rangle$$

holds for forms  $\alpha, \beta$  with appropriate bidegrees.

#### 8.4. Gauduchon metric.

**Theorem 8.4.1** (Gauduchon metric). If  $X$  be a complex manifold, then there exists a Hermitian metric  $\omega$  such that

$$\partial \bar{\partial} \omega^{n-1} = 0$$

which is called Gauduchon metric.

*Proof.* See [Gau77].  $\square$

**Corollary 8.4.1.** Let  $f: X \rightarrow \mathbb{R}$  be a smooth function on a compact complex manifold  $X$ . If

$$\sqrt{-1} \partial \bar{\partial} f \geq 0$$

then  $f$  is a constant.

*Proof.* Let  $\omega$  be the Gauduchon metric on  $X$ . Then

$$0 \leq \int_X \sqrt{-1} \partial \bar{\partial} f \wedge \omega^{n-1} = \int_X \sqrt{-1} f \wedge \partial \bar{\partial} \omega^{n-1} = 0$$

which implies  $\sqrt{-1}\partial\bar{\partial}f = 0$ . Note that

$$(8.2) \quad \partial\bar{\partial}f^2 = \partial(2f\bar{\partial}f) = 2\partial f \wedge \bar{\partial}f + 2f\partial\bar{\partial}f = 2\partial f \wedge \bar{\partial}f$$

Then

$$\begin{aligned} 0 &\stackrel{(1)}{=} \int_X \sqrt{-1}\partial\bar{\partial}f^2 \wedge \omega^{n-1} \\ &\stackrel{(2)}{=} 2 \int_X \sqrt{-1}\partial f \wedge \bar{\partial}f \wedge \omega^{n-1} \\ &\stackrel{(3)}{=} \frac{2}{n} \int_X |\partial f|^2 \omega^n \end{aligned}$$

where

(1) holds from  $\omega$  is a Gauduchon metric.

(2) holds from equation (8.2).

(3) holds from (2) of Proposition 8.3.3.

This shows  $\partial f = 0$ , and this also shows  $df = 0$  since  $f$  is real-valued, that is  $f$  is a constant.  $\square$

**Corollary 8.4.2.** Let  $f: X \rightarrow \mathbb{R}$  be a smooth function on a compact complex manifold  $X$ . If there exists a Hermitian metric  $\omega$  such that

$$\sqrt{-1} \operatorname{tr}_\omega \partial\bar{\partial}f \geq 0$$

then  $f$  is a constant.

*Proof.* By (1) of Proposition 8.3.3 one has

$$\int_X \sqrt{-1} \operatorname{tr}_\omega \partial\bar{\partial}f \wedge \omega^n = \int_X n\sqrt{-1}\partial\bar{\partial}f \wedge \omega^{n-1}$$

The argument in above corollary still works.  $\square$

## 9. KÄHLER GEOMETRY

### 9.1. Kähler manifold.

**Definition 9.1.1** (Kähler manifold). A Hermitian manifold  $(X, h)$  is called a Kähler manifold, if its fundamental form  $\omega$  is d-closed<sup>6</sup>.

*Remark 9.1.1.* Note that  $d\omega = 0$  is equivalent to  $\partial\omega = 0$ , and is also equivalent to  $\bar{\partial}\omega = 0$  since  $\omega$  is a real  $(1, 1)$ -form.

*Remark 9.1.2* (local form). By Proposition 8.1.4 one has

$$\omega = \sqrt{-1}h_{i\bar{j}}dz^i \wedge d\bar{z}^j$$

So Kähler condition  $d\omega = 0$  can be computed explicitly as follows

$$\begin{aligned} d\omega &= \sqrt{-1}d(h_{i\bar{j}}dz^i \wedge d\bar{z}^j) \\ &= \sqrt{-1}\left(\frac{\partial h_{i\bar{j}}}{\partial z^k}dz^k \wedge dz^i \wedge d\bar{z}^j - \frac{\partial h_{i\bar{j}}}{\partial \bar{z}^k}dz^i \wedge d\bar{z}^k \wedge d\bar{z}^j\right) \\ &= 0 \end{aligned}$$

So locally Kähler condition can be written as follows

$$\begin{aligned} \partial_k h_{i\bar{j}} &= \partial_i h_{k\bar{j}} \\ \partial_{\bar{k}} h_{i\bar{j}} &= \partial_{\bar{j}} h_{i\bar{k}} \end{aligned}$$

holds for all  $i, j, k$ .

**Proposition 9.1.1.** Let  $(X, h)$  be a Kähler manifold. Then the first Chern-Ricci curvature coincides with the second Chern-Ricci curvature.

*Proof.* Note that

$$\Theta_{i\bar{j}k\bar{l}} = -\frac{\partial^2 h_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + h^{p\bar{q}} \frac{\partial h_{k\bar{q}}}{\partial z^i} \frac{\partial h_{p\bar{l}}}{\partial \bar{z}^j}$$

Thus if  $(X, h)$  is Kähler, then

$$\Theta_{i\bar{j}k\bar{l}} = \Theta_{k\bar{j}i\bar{l}} = \Theta_{i\bar{l}k\bar{j}}$$

As a consequence, one has

$$\text{Ric}^{(1)}(h) = \sqrt{-1}h^{k\bar{l}}\Theta_{i\bar{j}k\bar{l}}dz^i \wedge d\bar{z}^j = \sqrt{-1}h^{k\bar{l}}\Theta_{k\bar{l}i\bar{j}}dz^i \wedge d\bar{z}^j = \text{Ric}^{(2)}(h)$$

This completes the proof.  $\square$

**Definition 9.1.2** (Kähler-Einstein metric). A Kähler metric  $\omega$  is called a Kähler-Einstein metric, if there exists  $\lambda \in \mathbb{R}$  such that

$$\text{Ric}(\omega) = \lambda\omega$$

**Example 9.1.1.** Any complex curve<sup>7</sup>  $X$  is Kähler since  $d\omega = 0$  automatically holds.

<sup>6</sup> $\omega$  is called Kähler form and  $h$  is called Kähler metric.

<sup>7</sup>In other words, a Riemann surface.

**Proposition 9.1.2.** A submanifold of a Kähler manifold is still Kähler.

*Proof.* If  $(X, \omega)$  is a Kähler manifold and  $Y$  is a submanifold, the restriction of  $\omega$  to  $Y$  gives Kähler form of  $Y$ .  $\square$

**Proposition 9.1.3.** Let  $(X, \omega)$  be a compact Kähler  $n$ -manifold. Then  $H^{2k}(X, \mathbb{R}) \neq 0$  for  $0 \leq k \leq n$ .

*Proof.* Note that  $d(\omega^k) = 0$  holds for  $0 \leq k \leq n$  since  $d\omega = 0$ , that is  $[\omega^k] \in H^{2k}(X, \mathbb{R})$ . By Proposition 8.3.2 one has  $\omega^n = n! \text{vol}$ , so the integral pairing

$$\int_X \omega^k \wedge \omega^{n-k} = n! \int_X \text{vol} \neq 0$$

implies  $[\omega^k] \neq 0$  for  $0 \leq k \leq n$ .  $\square$

**Theorem 9.1.1.** Let  $(X, h)$  be a Kähler manifold. Then locally we can choose a holomorphic coordinate  $(z^1, \dots, z^n)$  such that

$$h_{i\bar{j}}(z) = \delta_{i\bar{j}} - \Theta_{i\bar{j}k\bar{l}}(p) z^k \bar{z}^l + O(|z|^2)$$

## 9.2. When Levi-Civita connection encounters Chern connection.

Let  $X$  be a complex  $n$ -manifold and  $\{z^i = x^i + \sqrt{-1}x^I\}$  be a local coordinate of  $X$ , where  $1 \leq i \leq n$  and  $n+1 \leq I = i+n \leq 2n$ . Then  $\{x^i, x^I\}$  gives a local coordinate of underlying smooth manifold of  $X$ . Let  $g$  be a Riemannian metric on  $T_{\mathbb{R}}X$  which is compatible with natural almost complex structure  $J$  on  $T_{\mathbb{R}}X$  with Levi-Civita connection  $\nabla$ . Now consider  $\mathbb{C}$ -linear extension of  $\nabla$

$$\tilde{\nabla}: C^\infty(X, T_{\mathbb{C}}X) \rightarrow C^\infty(X, \Omega_{X, \mathbb{C}}^1 \otimes T_{\mathbb{C}}X)$$

It's clear  $\tilde{\nabla}$  gives a connection on  $T_{\mathbb{C}}X$ .

**Notation 9.2.1.** For  $A \in \{1, \dots, n, \bar{1}, \dots, \bar{n}\}$ , we denote

$$z^A = \begin{cases} z^i & A = i \\ \bar{z}^i & A = \bar{i} \end{cases}$$

**Proposition 9.2.1.** Let  $\{\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i}\}$  be a local frame of  $T_{\mathbb{C}}X$ . Then

$$\tilde{\nabla}_{\frac{\partial}{\partial z^A}} \frac{\partial}{\partial z^B} = \Gamma_{AB}^C \frac{\partial}{\partial z^C} = \frac{1}{2} h^{CE} \left( \frac{\partial h_{EB}}{\partial z^A} + \frac{\partial h_{AE}}{\partial z^B} - \frac{\partial h_{AB}}{\partial z^E} \right) \frac{\partial}{\partial z^C}$$

where  $A, B, C, E \in \{1, \dots, n, \bar{1}, \dots, \bar{n}\}$ .

*Proof.* By proof of Koszul formula, it suffices to check

- (1)  $\tilde{\nabla}$  is compatible with  $g_{\mathbb{C}}$ .
- (2)  $\tilde{\nabla}$  is torsion-free, that is for  $X, Y \in C^\infty(X, T_{\mathbb{C}}X)$ , one has

$$\tilde{\nabla}_X Y - \tilde{\nabla}_Y X = [X, Y]$$

Above two claims can be checked easily by using the fact  $\nabla$  is Levi-Civita connection and the  $\mathbb{C}$ -linearity of  $\tilde{\nabla}$ .  $\square$

**Corollary 9.2.1.**

$$\begin{aligned}\overline{\Gamma_{ij}^k} &= \Gamma_{ij}^k = \frac{1}{2} h^{k\bar{l}} \left( \frac{\partial h_{j\bar{l}}}{\partial z^i} + \frac{\partial h_{i\bar{l}}}{\partial z^j} \right) \\ \overline{\Gamma_{ij}^k} &= \Gamma_{ij}^k = \frac{1}{2} h^{k\bar{l}} \left( \frac{\partial h_{j\bar{l}}}{\partial \bar{z}^i} - \frac{\partial h_{j\bar{i}}}{\partial \bar{z}^l} \right) \\ \overline{\Gamma_{ij}^k} &= \Gamma_{ij}^k = 0\end{aligned}$$

*Proof.* Note that

$$h_{ij} = h_{\bar{i}\bar{j}} = h^{ij} = h^{\bar{i}\bar{j}} = 0$$

□

**Theorem 9.2.1.** Let  $(X, h)$  be a Kähler manifold with induced Riemannian metric  $g$  on underlying smooth manifold, and suppose  $\nabla$  is Levi-Civita connection with respect to  $g$ . Then Chern connection with respect to  $h$  can be obtained from the restriction of  $\mathbb{C}$ -linear extension of  $\nabla$  to  $T^{1,0}X$ .

*Proof.* Let  $\tilde{\nabla}$  be the  $\mathbb{C}$ -linear extension of  $\nabla$  and  $\{\frac{\partial}{\partial z^i}\}$  be a local frame of  $T^{1,0}X$ . Then by definition one has

$$\tilde{\nabla}_{\frac{\partial}{\partial z^j}} = \Gamma_{ij}^k dz^i \otimes \frac{\partial}{\partial z^k} + \overline{\Gamma_{ij}^k} dz^i \otimes \frac{\partial}{\partial \bar{z}^k} + \Gamma_{ij}^k d\bar{z}^i \otimes \frac{\partial}{\partial z^k} + \overline{\Gamma_{ij}^k} d\bar{z}^i \otimes \frac{\partial}{\partial \bar{z}^k}$$

By Corollary 9.2.1 one has  $\overline{\Gamma_{ij}^k} = 0$  automatically, and if Kähler condition holds, then

$$\begin{aligned}\Gamma_{ij}^k &= h^{k\bar{l}} \frac{\partial h_{j\bar{l}}}{\partial z^i} \\ \Gamma_{ij}^k &= \overline{\Gamma_{ij}^k} = 0\end{aligned}$$

Thus  $\tilde{\nabla}|_{T^{1,0}X}$  gives a connection on  $T^{1,0}X$ , and by formula of Chern connection, it's exactly the Chern connection with respect to  $h$ . □

**9.3. Curvatures of Kähler metric and underlying Riemannian metric.** In this section, let  $(X, h)$  be a Kähler manifold with induced Riemannian metric  $g$  on underlying smooth manifold, and suppose  $\{z^i = x^i + \sqrt{-1}x^I\}$  is a local coordinate of  $X$ , where  $1 \leq i \leq n$  and  $n+1 \leq I = i+n \leq 2n$ .

**Notation 9.3.1.**

(1)

$$\begin{aligned}R_{ijkl} &= R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) \\ R_{ijKL} &= R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^K}, \frac{\partial}{\partial x^L}\right)\end{aligned}$$

where  $R$  is curvature tensor of Levi-Civita connection  $\nabla$  with respect to  $g$ .

(2)

$$\Theta_{i\bar{j}k\bar{l}} = \Theta\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^l}\right)$$

where  $\Theta$  is Chern curvature with respect to  $h$ .

**Lemma 9.3.1.**

$$\begin{aligned} R_{ijkl} &= R_{ijKL} \\ R_{ijKl} &= -R_{ijkL} \end{aligned}$$

*Proof.* It follows from Kähler condition.  $\square$

**Corollary 9.3.1.**

$$\begin{aligned} R_{ij} &= R_{IJ} \\ R_{iJ} &= R_{Ji} = -R_{jI} = -R_{Ij} \end{aligned}$$

*Proof.* It follows from Proposition 8.1.1 and Lemma 9.3.1.  $\square$

**Theorem 9.3.1.**

$$\Theta_{i\bar{j}} = \frac{1}{2}(R_{ij} + \sqrt{-1}R_{iJ})$$

where  $\Theta_{i\bar{j}}$  is given by  $\text{Ric}(h) = \sqrt{-1}\Theta_{i\bar{j}}dz^i \wedge d\bar{z}^j$ , while  $R_{ij}$  and  $R_{iJ}$  are Riemannian Ricci curvatures.

**Corollary 9.3.2.** Let  $(X, h)$  be a Kähler manifold with induced Riemannian metric  $g$  on underlying smooth manifold. Then

- (1)  $h$  has positive Chern-Ricci curvature if and only if  $g$  has positive Ricci curvature.
- (2)  $h$  is Kähler-Einstein with Einstein constant  $\lambda$  if and only if  $g$  is an Einstein metric with Einstein constant  $\lambda$ .

**Corollary 9.3.3.** Let  $(X, h)$  be a Kähler manifold with induced Riemannian metric  $g$  on underlying smooth manifold. Let  $s_R$  be the Riemannian scalar curvature and  $s$  is the Chern scalar curvature. Then  $s_R = 2s$ .

*Proof.* Direct computation shows

$$\begin{aligned} s &= h^{i\bar{j}}\Theta_{i\bar{j}} \\ &= 2(g^{ij} - \sqrt{-1}g^{iJ}) \cdot \frac{1}{2}(R_{ij} + \sqrt{-1}R_{iJ}) \\ &= g^{ij}R_{ij} + g^{iJ}R_{iJ} \\ s_R &= g^{ij}R_{ij} + g^{iJ}R_{iJ} + g^{Ij}R_{Ij} + g^{IJ}R_{IJ} \\ &\stackrel{(1)}{=} 2g^{ij}R_{ij} + 2g^{iJ}R_{iJ} \\ &= 2s \end{aligned}$$

where (1) holds from Proposition 8.1.1 and Corollary 9.3.1.  $\square$

## 10. COMPLEX PROJECTIVE SPACE

## 10.1. Tautological line bundle.

**Proposition 10.1.1.** The set  $\mathcal{O}_{\mathbb{CP}^n}(-1) \subseteq \mathbb{CP}^n \times \mathbb{C}^{n+1}$  that consists of all pairs  $(l, z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1}$  with  $z \in l$  forms a holomorphic line bundle, called tautological line bundle

*Proof.* Let  $\pi: \mathcal{O}_{\mathbb{CP}^n}(-1) \rightarrow \mathbb{CP}^n$  be the projection to the first factor. Consider open covering  $\{U_i\}_{i=0}^n$  of  $\mathbb{CP}^n$ , where

$$U_i = \{[l] = [l_0 : \dots : l_n] \in \mathbb{CP}^n \mid l_i \neq 0\}$$

A canonical trivialization of  $\mathcal{O}_{\mathbb{CP}^n}(-1)$  over  $U_i$  is given by

$$\begin{aligned} \varphi_i: \pi^{-1}(U_i) &\rightarrow U_i \times \mathbb{C} \\ (l, z) &\mapsto (l, z_i) \end{aligned}$$

Its transition function is computed as follows

$$\begin{aligned} \varphi_i \circ \varphi_j^{-1}: (U_i \cap U_j) \times \mathbb{C} &\longrightarrow (U_i \cap U_j) \times \mathbb{C} \\ (l, w) &\mapsto (l, w \frac{l_i}{l_j}) \end{aligned}$$

where  $l = (l_0 : \dots : l_n)$ . This shows its transition function  $g_{ij}(z) = z_i/z_j \in \mathbb{C}^*$  is holomorphic.  $\square$

**Definition 10.1.1** (line bundles on  $\mathbb{CP}^n$ ).

$$\begin{cases} \mathcal{O}_{\mathbb{CP}^n}(-k) = \mathcal{O}_{\mathbb{CP}^n}(-1)^{\otimes k} & k \in \mathbb{Z}_{>0} \\ \mathcal{O}_{\mathbb{CP}^n}(k) = (\mathcal{O}_{\mathbb{CP}^n}(-k))^* & k \in \mathbb{Z}_{>0} \\ \mathcal{O}_{\mathbb{CP}^n}(0) = \mathbb{CP}^n \times \mathbb{C} \end{cases}$$

**Proposition 10.1.2.** For  $k \geq 0$ , the space  $\Gamma(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(k))$  is canonically isomorphic to the space  $\mathbb{C}[z_0, \dots, z_n]_k$  of all homogenous polynomials of degree  $k$ .

**Corollary 10.1.1.** For  $k < 0$  the line bundle  $\mathcal{O}_{\mathbb{CP}^n}(k)$  admits no global holomorphic section.

*Proof.* It follows from above result and Proposition 4.1.2.  $\square$

## 10.2. Euler sequence and adjunction formula.

## 10.2.1. Euler sequence.

**Proposition 10.2.1.** On  $\mathbb{CP}^n$  there exists a natural short exact sequence of holomorphic vector bundles

$$0 \rightarrow \mathcal{O}_{\mathbb{CP}^n} \xrightarrow{\phi} \mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus n+1} \xrightarrow{\psi} T\mathbb{CP}^n \rightarrow 0$$

**Exercise 10.2.1.** For Grassmannian manifold  $Gr(k, n)$ , we have

$$0 \rightarrow E \rightarrow Gr(k, n) \otimes \mathbb{C}^n \rightarrow Q \rightarrow 0$$

Show that

$$T_{Gr(k, n)} \cong \text{Hom}(E, Q)$$

### 10.2.2. Adjunction formula.

**Proposition 10.2.2** (adjunction formula). Let  $\pi: L \rightarrow X$  is a holomorphic line bundle and  $s$  be a holomorphic section of  $L$ . Suppose that  $D = \{x \in X \mid s(x) = 0\}$  is a smooth submanifold of codimensional 1. Show that the following sequence is exact

$$0 \rightarrow TD \rightarrow TX|_D \rightarrow L|_D \rightarrow 0$$

As a consequence

$$K_D^* \cong K_D^* \otimes L|_D = (K_X^* \otimes L)|_D$$

Or equivalently

$$K_D \cong (K_X \otimes L)|_D$$

This is called adjunction formula.

*Proof.* Firstly we have the following exact sequence

$$0 \rightarrow TD \rightarrow TX|_D \rightarrow N_D \rightarrow 0$$

where  $N_D$  is the normal bundle. Now it suffices to show  $L|_D$  is isomorphic to the normal bundle of  $D$ . Note that  $s|_D = 0$  and  $s$  is not identically zero on  $X$ , so  $ds$  gives an isomorphism between  $L|_D$  and  $N_D$  in fact. By taking determinant we obtain the adjunction formula since for a exact sequence of vector bundle

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

we have

$$\det B = \det A \otimes \det C$$

and determinant of a line bundle is itself.  $\square$

**Example 10.2.1.** Let  $X = \mathbb{CP}^n$  and  $L = \mathcal{O}_{\mathbb{CP}^n}(-d)$ . Proposition 10.1.2 shows that  $D \subseteq \mathbb{CP}^n$  is a smooth hypersurface defined by zero set of a homogenous polynomial with degree  $d$ . Then we have

$$\begin{aligned} K_D^* &\cong (K_X^* \otimes L)|_D \\ &= (\mathcal{O}_{\mathbb{CP}^n}(n+1) \otimes \mathcal{O}_{\mathbb{CP}^n}(-d))|_D \\ &\cong \mathcal{O}_{\mathbb{CP}^n}(n+1-d)|_D \end{aligned}$$

*Remark 10.2.1.* As a consequence,  $D$  is called

$$\begin{cases} \text{Fano} & d < n+1 \\ \text{Calabi-Yau} & d = n+1 \\ \text{general type} & d > n+1 \end{cases}$$

These concepts we will define later.



### 10.3. Fubini-Study metric.

**Proposition 10.3.1** (Fubini-Study metric). Let  $\mathbb{CP}^n = \bigcup_{i=0}^n U_i$  be the canonical open covering, that is  $U_i = \{(z^0 : \dots : z^n) \mid z^i \neq 0\}$ . Then there is a Kähler metric  $\omega_{FS}$  on  $\mathbb{CP}^n$  such that

$$\omega_{FS}|_{U_i} = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left( \frac{\sum_{l=0}^n |z^l|^2}{|z^i|^2} \right)$$

*Proof.* Note that  $\omega_{FS}|_{U_i}$  can be written as

$$\omega_{FS}|_{U_i} = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left( \sum_{l=1}^n |w^l|^2 + 1 \right)$$

where  $w^l = z^l/z^i$ . The following steps show that  $\{\omega_{FS}|_{U_i}\}_{i=0}^n$  gives a real  $(1,1)$ -form which is closed.

(1) It's globally defined since direct computation shows

$$\begin{aligned} \log \left( \frac{\sum_{k=0}^n |z^k|^2}{|z^i|^2} \right) &= \log \left( \frac{|z^j|^2}{|z^i|^2} \frac{\sum_{k=0}^n |z^k|^2}{|z^j|^2} \right) \\ &= \log \left( \frac{|z^j|^2}{|z^i|^2} \right) + \log \left( \frac{\sum_{k=0}^n |z^k|^2}{|z^j|^2} \right) \\ &= \log \left( \frac{\sum_{k=0}^n |z^k|^2}{|z^j|^2} \right) \end{aligned}$$

where the last equality holds since  $|z^j|^2/|z^i|^2$  is a nowhere vanishing holomorphic function.

(2) It's real since  $\overline{\partial \bar{\partial}} = \bar{\partial} \partial = -\partial \bar{\partial}$ .

(3) It's  $\partial$ -closed since each  $\omega_{FS}|_{U_i}$  is  $\partial$ -closed.

It remains to show  $\omega$  is positive. A direct computation yields

$$\partial \bar{\partial} \log \left( 1 + \sum_{l=1}^n |w^l|^2 \right) = \frac{1}{(1 + \sum_{l=1}^n |w^l|^2)^2} h_{i\bar{j}} dw^i \wedge d\bar{w}^j$$

where  $h_{i\bar{j}} = (1 + \sum_{l=1}^n |w^l|^2) \delta_{ij} - w^i \bar{w}^j$ . Now it suffices to show  $h_{i\bar{j}}$  is positive definite, for  $u \neq 0$ , one has

$$\begin{aligned} u^T (h_{i\bar{j}}) \bar{u} &= (u, u) + (w, w)(u, u) - u^T \bar{w} w^T \bar{u} \\ &= (u, u) + (w, w)(u, u) - (u, w)(w, u) \\ &= (u, u) + (w, w)(u, u) - \overline{(w, u)}(w, u) \\ &= (u, u) + (w, w)(u, u) - |(w, u)|^2 > 0 \end{aligned}$$

□

**Corollary 10.3.1.** Any projective manifold is Kähler.

*Proof.* By Proposition 9.1.2, the submanifold of Kähler manifold is still Kähler. □

**Proposition 10.3.2.**  $\mathcal{O}_{\mathbb{CP}^n}(1)$  is a positive holomorphic line bundle.

*Proof.* Note that line bundle  $\mathcal{O}_{\mathbb{CP}^n}(1)$  can be given transition functions  $\{U_i, g_{ij}\}$ , where  $\{U_i\}$  is canonical open covering and  $g_{ij} = z^j/z^i$ . Consider  $h_i: U_i \rightarrow \mathbb{R}_{>0}$  given by

$$h_i = \frac{|z^i|^2}{\sum_{l=0}^n |z^l|^2}$$

Then  $h_i$  can be glued together to obtain a Hermitian metric on  $\mathcal{O}_{\mathbb{CP}^n}(1)$  since  $h_i = h_j |g_{ij}|^2$ . And it's clear to see Hermitian metric corresponding to curvature of Chern connection with respect to this metric is Fubini-Study metric.  $\square$

*Remark 10.3.1.* Note that  $\mathcal{O}_{\mathbb{CP}^n}(-1)$  is a subbundle of  $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ , so we can obtain a natural Hermitian metric of  $\mathcal{O}_{\mathbb{CP}^n}(-1)$  by restricting standard Hermitian metric of  $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ , and Hermitian metric on  $\mathcal{O}_{\mathbb{CP}^n}(1)$  we defined before is exactly the dual metric of this natural metric.

**10.4. Curvatures of projective space.** Fubini-Study metric  $\omega_{FS}$  on  $\mathbb{CP}^n$  is locally given by

$$\omega_{FS} = \sqrt{-1} h_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

where

$$h_{i\bar{j}} = \frac{(1 + \sum_{l=1}^n |z^l|^2) \delta_{ij} - z^i \bar{z}^j}{(1 + \sum_{l=1}^n |z^l|^2)^2}$$

Direct computation shows

$$\Theta_{i\bar{j}k\bar{l}} = h_{i\bar{j}} h_{k\bar{l}} + h_{k\bar{j}} h_{i\bar{l}}$$

and consequently

$$\text{Ric}(\omega_{FS}) = (n+1)\omega_{FS}$$

This shows Fubini-Study metric is a Kähler-Einstein metric with Einstein constant  $n+1$ .

### Part 3. Hodge decomposition

#### 11. HODGE THEORY

**11.1. Hodge star and adjoint operators.** Let  $(X, \omega)$  be a compact Hermitian manifold. A  $(p, q)$ -form  $\alpha$  can be locally written as

$$\alpha = \frac{1}{p! \times q!} \alpha_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q}$$

Then for  $\alpha, \beta \in C^\infty(X, \Omega_X^{p,q})$ , the local inner product is defined as

$$\langle \alpha, \beta \rangle = \frac{1}{p! \times q!} h^{i_1 \bar{k}_1} \dots h^{i_p \bar{k}_p} h^{l_1 \bar{j}_1} \dots h^{l_q \bar{j}_q} \alpha_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} \overline{\beta_{k_1 \dots k_p \bar{l}_1 \dots \bar{l}_q}}$$

which is a smooth function on  $X$ .

**Definition 11.1.1** (inner product on  $(p, q)$ -form). An inner product on the space of  $(p, q)$ -form is defined as

$$(\alpha, \beta) := \int_X \langle \alpha, \beta \rangle \frac{\omega^n}{n!}$$

where  $\alpha, \beta \in C^\infty(X, \Omega_X^{p,q})$ . This also gives an inner product on  $\Omega_{X,\mathbb{C}}^k = \bigoplus_{p+q=k} \Omega_X^{p,q}$ .

Holding the inner product  $(-, -)$ , the formal adjoint operator of  $d$  is defined as an operator

$$d^* : C^\infty(X, \Omega_{X,\mathbb{C}}^{p+q}) \rightarrow C^\infty(X, \Omega_{X,\mathbb{C}}^{p+q-1})$$

satisfying  $(\alpha, d\beta) = (d^*\alpha, \beta)$  for  $\alpha, \beta$  with appropriate degrees, similarly one can define  $\partial^*$  and  $\bar{\partial}^*$ . In order to construct these adjoint operators, we need to introduce the well-known Hodge star operator.

**Definition 11.1.2** (Hodge star operator). There exists an operator

$$\star : C^\infty(X, \Omega_X^{p,q}) \rightarrow C^\infty(X, \Omega_X^{n-q, n-p})$$

such that

$$(\alpha, \beta) = \int_X \alpha \wedge \star \bar{\beta}$$

*Remark 11.1.1.* It's well-defined since  $\bar{\beta}$  is a  $(q, p)$ -form, and thus  $\star \bar{\beta}$  is a  $(n-p, n-q)$ -form.

**Lemma 11.1.1.**

- (1)  $\star 1 = \omega^n / n!$
- (2)  $\star \omega = \omega^{n-1} / (n-1)!$
- (3)  $\overline{\star \psi} = \star \bar{\psi}$
- (4)  $\star \star = (-1)^{p+q}$  on  $C^\infty(X, \Omega_X^{p,q})$
- (5)  $(\star \varphi, \star \psi) = (\varphi, \psi)$

**Proposition 11.1.1.**  $d^* = -\star d \star$

*Proof.* For arbitrary  $\alpha \in C^\infty(X, \Omega_{X, \mathbb{C}}^{p+q})$  and  $\beta \in C^\infty(X, \Omega_{X, \mathbb{C}}^{p+q+1})$ , then

$$\begin{aligned}
(d\alpha, \beta) &= \int_X d\alpha \wedge \star \beta \\
&= \int_X d(\alpha \wedge \star \beta) - (-1)^{p+q} \alpha \wedge d \star \beta \\
&= (-1)^{p+q+1} \int_X \alpha \wedge d \star \beta \\
&\stackrel{(1)}{=} (-1)^{p+q+1} (-1)^{2n-(p+q+1)+1} \int_X \alpha \wedge \star \star d \star \beta \\
&= -(\alpha, \star d \star \beta)
\end{aligned}$$

where (1) holds from (4) of Lemma 11.1.1.  $\square$

**Proposition 11.1.2.**

$$\begin{aligned}
\partial^* &= -\star \bar{\partial} \star \\
\bar{\partial}^* &= -\star \partial \star
\end{aligned}$$

*Proof.* Direct computation.  $\square$

**Definition 11.1.3** (Lefschetz operator). Let  $(X, \omega)$  be a compact Kähler manifold. The Lefschetz operator is defined as

$$\begin{aligned}
L: C^\infty(X, \Omega_X^{p,q}) &\rightarrow C^\infty(X, \Omega_X^{p+1,q+1}) \\
\alpha &\mapsto \omega \wedge \alpha
\end{aligned}$$

**Lemma 11.1.2.**  $\Lambda := L^* = (-1)^{p+q} \star L \star$  on  $(p, q)$ -forms.

*Proof.* For  $\alpha \in C^\infty(X, \Omega_X^{p,q})$ ,  $\beta \in C^\infty(X, \Omega_X^{p+1,q+1})$ , direct computation shows

$$\begin{aligned}
(L\alpha, \beta) &= \int_X L\alpha \wedge \star \beta \\
&= \int_X \omega \wedge \alpha \wedge \star \beta \\
&\stackrel{(1)}{=} \int_X \alpha \wedge \omega \wedge \star \beta \\
&\stackrel{(2)}{=} \int_X \alpha \wedge (-1)^{p+q} \star \star \omega \wedge \star \beta \\
&= (\alpha, (-1)^{p+q} \star L \star \beta)
\end{aligned}$$

where

(1) holds from  $\omega$  is a 2-form.

(2) holds from (4) of Lemma 11.1.1.  $\square$

### 11.2. Hodge theorem.

**Definition 11.2.1** (Laplacian). Laplacian  $\Delta_\bullet$  is an operator defined by  $\Delta_\bullet := \bullet\bullet^* + \bullet^*\bullet$ , where  $\bullet$  can be  $d, \partial$  and  $\bar{\partial}$ .

**Definition 11.2.2** (harmonic). A form  $\alpha$  is called  $\Delta_\bullet$ -harmonic if  $\Delta_\bullet\alpha = 0$ , where  $\bullet$  can be  $d, \partial$  and  $\bar{\partial}$ .

**Notation 11.2.1.**  $\mathcal{H}^k$  denotes the space of  $\Delta_d$ -harmonic  $k$ -forms, and  $\mathcal{H}^{p,q}$  denotes the space of  $\Delta_{\bar{\partial}}$ -harmonic forms of type  $(p, q)$ .

**Lemma 11.2.1.**  $\alpha$  is  $\Delta_\bullet$ -harmonic if and only if  $\bullet\alpha = 0, \bullet^*\alpha = 0$ , where  $\bullet$  can be  $d, \partial$  and  $\bar{\partial}$ .

*Proof.* Direct computation shows

$$\begin{aligned} (\alpha, \Delta_d\alpha) &= (\alpha, dd^*\alpha) + (\alpha, d^*d\alpha) \\ &= \|d^*\alpha\|^2 + \|d\alpha\|^2 \end{aligned}$$

This shows  $\alpha$  is  $\Delta_d$ -harmonic if and only if  $d\alpha = d^*\alpha = 0$ , the other cases are same.  $\square$

**Theorem 11.2.1** (Hodge theorem). Let  $(X, h)$  be a compact Hermitian  $n$ -manifold. Then

- (1)  $\mathcal{H}^{p,q}$  is finite dimensional.
- (2) There is a decomposition  $C^\infty(X, \Omega_X^{p,q}) = \mathcal{H}^{p,q} \oplus \Delta_{\bar{\partial}}(C^\infty(X, \Omega_X^{p,q}))$ , which is orthogonal with respect to inner products in Definition 11.1.1.

**Corollary 11.2.1.** There is the following orthonormal decomposition

$$C^\infty(X, \Omega_X^{p,q}) = \mathcal{H}^{p,q} \oplus \bar{\partial}(C^\infty(X, \Omega_X^{p,q-1})) \oplus \bar{\partial}^*(C^\infty(X, \Omega_X^{p,q+1}))$$

**Corollary 11.2.2.**

$$\begin{aligned} \ker \bar{\partial} &= \mathcal{H}^{p,q} \oplus \bar{\partial}^*(C^\infty(X, \Omega_X^{p,q-1})) \\ \ker \bar{\partial}^* &= \mathcal{H}^{p,q} \oplus \bar{\partial}(C^\infty(X, \Omega_X^{p,q+1})) \end{aligned}$$

**Corollary 11.2.3.** The natural map  $\mathcal{H}^{p,q} \rightarrow H^{p,q}(X)$  is an isomorphism. In particular,  $H^{p,q}(X)$  is finite dimensional.

In order to give the following isomorphism

$$\star: \mathcal{H}^{p,q} \rightarrow \mathcal{H}^{n-q, n-p}$$

Parallel to the real case<sup>8</sup>, it suffices to have

$$\star \circ \Delta_{\bar{\partial}} = \Delta_{\bar{\partial}} \circ \star$$

But something bad happens since we only have  $\bar{\partial}^* = -\star\partial\star$ , and direct computation only yields  $\Delta_{\bar{\partial}}\circ\star = \star\circ\Delta_{\partial}$ . So it fails generally since  $\Delta_{\bar{\partial}} \neq \Delta_{\partial}$ . There are two ways to deal with this gap. The first way is that we will see later if  $X$  is compact Kähler manifold, then  $\Delta_{\partial} = \Delta_{\bar{\partial}}$ , that is Theorem 12.1.1. Then

<sup>8</sup>See Hodge theory in [Liu23].

**Corollary 11.2.4.** If  $(X, \omega)$  is a compact Kähler  $n$ -manifold, then  $\star: \mathcal{H}^{p,q} \rightarrow \mathcal{H}^{n-q, n-p}$  is an isomorphism.

Another way is to consider

$$\begin{aligned} \bar{\star}: C^\infty(X, \Omega_X^{p,q}) &\rightarrow C^\infty(X, \Omega_X^{n-p, n-q}) \\ \alpha &\mapsto \star \bar{\alpha} \end{aligned}$$

then direct computation shows

$$\bar{\star} \circ \Delta_{\bar{\partial}} = \Delta_{\bar{\partial}} \circ \bar{\star}$$

**Corollary 11.2.5.** If  $(X, h)$  is a compact Hermitian manifold, then  $\bar{\star}: \mathcal{H}^{p,q} \rightarrow \mathcal{H}^{n-p, n-q}$  is an isomorphism.

**Corollary 11.2.6.**  $H^{p,q}(X) \cong H^{n-p, n-q}(X)$ .

*Remark 11.2.1.* This is a special case of Serre duality.

### 11.3. Useful formulas of adjoint operators.

**Proposition 11.3.1.** Let  $(X, h)$  be a compact Kähler manifold. Then locally

$$\begin{cases} \partial = dz^i \wedge \nabla_i \\ \partial^* = -h^{i\bar{j}} \iota_{\bar{i}} \circ \nabla_j = -h^{i\bar{j}} \nabla_j \circ \iota_{\bar{i}} \end{cases} \quad \begin{cases} \bar{\partial} = d\bar{z}^i \wedge \nabla_{\bar{i}} \\ \bar{\partial}^* = -h^{i\bar{j}} \iota_{\bar{j}} \circ \nabla_i = -h^{i\bar{j}} \nabla_i \circ \iota_{\bar{j}} \end{cases}$$

*Proof.* Here we only give the proof of the case  $\partial$  and  $\partial^*$ , the proof for the other two cases are same. It suffices to check pointwisely, and at each point we may also choose normal coordinate in Theorem 9.1.1. For  $(p, q)$ -form  $\alpha$ , locally written as  $\alpha = \alpha_{J\bar{K}} dz^J \wedge d\bar{z}^K$ . Then

$$\partial\alpha = \frac{\partial\alpha_{J\bar{K}}}{\partial z^i} dz^i \wedge dz^J \wedge d\bar{z}^K$$

and

$$\begin{aligned} dz^i \wedge \nabla_i \alpha &= dz^i \wedge \nabla_i (\alpha_{J\bar{K}} dz^J \wedge d\bar{z}^K) \\ &= dz^i \wedge \frac{\partial\alpha_{J\bar{K}}}{\partial z^i} dz^J \wedge d\bar{z}^K + \alpha_{J\bar{K}} \nabla_i (dz^J \wedge d\bar{z}^K) \\ &\stackrel{(1)}{=} \frac{\partial\alpha_{J\bar{K}}}{\partial z^i} dz^i \wedge dz^J \wedge d\bar{z}^K \end{aligned}$$

where (1) holds from our choice of normal coordinate. To see formula of  $\partial^*$ , take arbitrary forms  $\alpha, \beta$  with appropriate bidegrees, then

$$\begin{aligned} (\partial\alpha, \beta) &= (dz^i \wedge \nabla_i \alpha, \beta) \\ &\stackrel{(2)}{=} (\nabla_i \alpha, h^{p\bar{i}} \iota_{\bar{i}} \beta) \\ &\stackrel{(3)}{=} -(\alpha, h^{p\bar{i}} \nabla_i \circ \iota_{\bar{i}} \beta) \end{aligned}$$

where

(2) holds from Proposition 8.3.5.

(3) holds from Stokes' theorem and the fact Chern connection is compatible with metric.

This shows

$$\partial^* = -h^{i\bar{j}} \nabla_j \circ \iota_i \stackrel{(4)}{=} -h^{i\bar{j}} \iota_i \circ \nabla_j$$

where (4) holds from  $\iota_i \circ \nabla_j = \nabla_j \circ \iota_i$ .  $\square$

**Proposition 11.3.2.** Let  $(X, \omega)$  be a compact Kähler manifold. Then locally

$$\Lambda = \sqrt{-1} h^{i\bar{j}} \iota_i \circ \iota_{\bar{j}} = -\sqrt{-1} h^{i\bar{j}} \iota_{\bar{j}} \circ \iota_i$$

*Proof.* For arbitrary forms  $\alpha, \beta$  with appropriate bidegrees, direct computation shows

$$\begin{aligned} (\omega \wedge \alpha, \beta) &= (\sqrt{-1} h_{i\bar{j}} dz^i \wedge d\bar{z}^j \wedge \alpha, \beta) \\ &\stackrel{(1)}{=} (\sqrt{-1} h_{i\bar{j}} d\bar{z}^j \wedge \alpha, h^{p\bar{i}} \iota_p \beta) \\ &\stackrel{(2)}{=} (\sqrt{-1} h_{i\bar{j}} \alpha, h^{p\bar{i}} h^{j\bar{q}} \iota_{\bar{q}} \circ \iota_p \beta) \\ &\stackrel{(3)}{=} (\alpha, -\sqrt{-1} h_{j\bar{i}} h^{p\bar{i}} h^{j\bar{q}} \iota_{\bar{q}} \circ \iota_p \beta) \\ &= (\alpha, -\sqrt{-1} h^{p\bar{i}} \iota_{\bar{i}} \circ \iota_p \beta) \end{aligned}$$

where

(1) and (2) hold from Proposition 8.3.5.

(3) holds from  $h_{i\bar{j}}$  is Hermitian, that is  $\overline{h_{i\bar{j}}} = h_{j\bar{i}}$ .

This shows

$$\Lambda = -\sqrt{-1} h^{i\bar{j}} \iota_{\bar{j}} \circ \iota_i \stackrel{(4)}{=} \sqrt{-1} h^{i\bar{j}} \iota_i \circ \iota_{\bar{j}}$$

where (4) holds from  $\iota_i \circ \iota_{\bar{j}} = -\iota_{\bar{j}} \circ \iota_i$ .  $\square$

## 12. HODGE DECOMPOSITION

## 12.1. Kähler identities.

**Definition 12.1.1** (commutator of differential operators). Let  $A, B$  be two differential operators. The commutator of  $A, B$  is defined as

$$[A, B] := AB - (-1)^{\deg A \deg B} BA$$

**Lemma 12.1.1** (Jacobi identity). Let  $A, B, C$  be differential operators. Then

$$(-1)^{\deg A \deg C} [A, [B, C]] + (-1)^{\deg B \deg A} [B, [C, A]] + (-1)^{\deg C \deg B} [C, [A, B]] = 0$$

*Remark 12.1.1.* In our case, the degree of  $d, d^*, \partial, \partial^*, \bar{\partial}, \bar{\partial}^*$  is one, and the degree of  $L$  and  $\Lambda$  is zero<sup>9</sup>.

**Proposition 12.1.1** (Kähler identities). If  $(X, \omega)$  is a compact Kähler manifold, then

$$\begin{aligned} [\bar{\partial}^*, L] &= \sqrt{-1} \partial \\ [\partial^*, L] &= -\sqrt{-1} \cdot \bar{\partial} \\ [\Lambda, \bar{\partial}] &= -\sqrt{-1} \partial^* \\ [\Lambda, \partial] &= \sqrt{-1} \cdot \bar{\partial}^* \end{aligned}$$

*Proof.* By taking conjugates and adjoints, it suffices to prove the first identity, which is a first order identity of differential equation. But by Theorem 9.1.1, locally we have  $h_{i\bar{j}} = \delta_{i\bar{j}} + O(|\xi^2|)$ . Thus it suffices to check Kähler identity for the case  $U \subseteq \mathbb{C}^n$  equipped with standard Hermitian metric.

Suppose  $(p, q)$ -form  $\alpha$  is locally given by  $\alpha = \alpha_{JK} dz^J \wedge d\bar{z}^K$ , then by Proposition 11.3.1 one has  $\bar{\partial}^* \alpha = -\sum_l \iota_{\frac{\partial}{\partial \bar{z}^l}} \frac{\partial \alpha}{\partial z^l}$ . Thus

$$\begin{aligned} [\bar{\partial}^*, L]\alpha &= \bar{\partial}^*(\omega \wedge \alpha) - \omega \wedge \bar{\partial}^* \alpha \\ &= -\sum_l \iota_{\frac{\partial}{\partial \bar{z}^l}} \frac{\partial}{\partial z^l} (\omega \wedge \alpha) + \omega \wedge \sum_l \iota_{\frac{\partial}{\partial \bar{z}^l}} \frac{\partial \alpha}{\partial z^l} \\ &\stackrel{(1)}{=} -\sum_l \iota_{\frac{\partial}{\partial \bar{z}^l}} (\omega \wedge \frac{\partial \alpha}{\partial z^l}) + \omega \wedge \sum_l \iota_{\frac{\partial}{\partial \bar{z}^l}} \frac{\partial \alpha}{\partial z^l} \\ &= -\left\{ \sum_l \left( \iota_{\frac{\partial}{\partial \bar{z}^l}} \omega \right) \wedge \frac{\partial \alpha}{\partial z^l} + \omega \wedge \sum_l \iota_{\frac{\partial}{\partial \bar{z}^l}} \frac{\partial \alpha}{\partial z^l} \right\} + \sum_l \omega \wedge \iota_{\frac{\partial}{\partial \bar{z}^l}} \frac{\partial \alpha}{\partial z^l} \\ &= -\sum_l \left( \iota_{\frac{\partial}{\partial \bar{z}^l}} \omega \right) \wedge \frac{\partial \alpha}{\partial z^l} \\ &\stackrel{(2)}{=} \sqrt{-1} \sum_l dz^l \wedge \frac{\partial \alpha}{\partial z^l} \\ &= \sqrt{-1} \partial \alpha \end{aligned}$$

<sup>9</sup>You can try to understand this thing in a following way: operators  $d, d^*, \partial, \partial^*, \bar{\partial}, \bar{\partial}^*$  take derivatives, but  $L$  and  $\Lambda$  are linear operators.



where

(1) holds from  $\omega$  is a closed  $(1, 1)$ -form.

(2) holds from Proposition 8.1.4, that is  $\omega = \sqrt{-1} \sum_{i=1}^n dz^i \wedge d\bar{z}^i$ .

□

**Theorem 12.1.1.** Let  $(X, \omega)$  be a compact Kähler manifold. Then

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$$

*Proof.* Since

$$\Delta_d = (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial})$$

By the fourth Kähler identity, one has

(1) The first term can be computed as

$$\begin{aligned} (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) &= (\partial + \bar{\partial})(\partial^* - \sqrt{-1}\Lambda\partial + \sqrt{-1}\partial\Lambda) \\ &= \partial\partial^* - \sqrt{-1}\partial\Lambda\partial + \bar{\partial}\partial^* - \sqrt{-1}\cdot\bar{\partial}\Lambda\partial + \sqrt{-1}\cdot\bar{\partial}\partial\Lambda \end{aligned}$$

(2) The second term can be computed as

$$\begin{aligned} (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial}) &= (\partial^* - \sqrt{-1}\Lambda\partial + \sqrt{-1}\partial\Lambda)(\partial + \bar{\partial}) \\ &= \partial^*\partial + \sqrt{-1}\partial\Lambda\partial + \partial^*\bar{\partial} - \sqrt{-1}\Lambda\partial\bar{\partial} + \sqrt{-1}\partial\Lambda\bar{\partial} \end{aligned}$$

By the third Kähler identity, one has

$$\partial^* = \sqrt{-1}[\Lambda, \bar{\partial}] = \sqrt{-1}\Lambda\bar{\partial} - \sqrt{-1}\bar{\partial}\Lambda$$

then

$$\begin{aligned} \bar{\partial}\partial^* &= \bar{\partial}(\sqrt{-1}\Lambda\bar{\partial} - \sqrt{-1}\cdot\bar{\partial}\Lambda) = \sqrt{-1}\cdot\bar{\partial}\Lambda\bar{\partial} \\ \partial^*\bar{\partial} &= (\sqrt{-1}\Lambda\bar{\partial} - \sqrt{-1}\cdot\bar{\partial}\Lambda)\bar{\partial} = -\sqrt{-1}\cdot\bar{\partial}\Lambda\bar{\partial} = -\bar{\partial}\partial^* \end{aligned}$$

Now we have

$$\begin{aligned} \Delta_d &= \Delta_\partial - \sqrt{-1}\cdot\bar{\partial}\Lambda\partial - \sqrt{-1}\Lambda\partial\bar{\partial} + \sqrt{-1}\bar{\partial}\partial\Lambda + \sqrt{-1}\partial\Lambda\bar{\partial} \\ &= \Delta_\partial + \sqrt{-1}(\Lambda\bar{\partial}\partial - \bar{\partial}\Lambda\partial) + \sqrt{-1}(\partial\Lambda\bar{\partial} - \bar{\partial}\partial\Lambda) \\ &= \Delta_\partial + \sqrt{-1}[\Lambda, \bar{\partial}]\partial + \sqrt{-1}\partial[\Lambda, \bar{\partial}] \\ &= \Delta_\partial + \partial^*\partial + \partial\partial^* \\ &= 2\Delta_\partial \end{aligned}$$

□

**Corollary 12.1.1.** On a compact Kähler manifold,  $\Delta_d$ -harmonic is equivalent to  $\Delta_\partial$ -harmonic, and is equivalent to  $\Delta_{\bar{\partial}}$ -harmonic.

**Corollary 12.1.2.** Let  $(X, \omega)$  be a Kähler manifold and  $\alpha$  be a  $(p, q)$ -form. Then  $\Delta_d\alpha$  is still a  $(p, q)$ -form.

*Proof.* It's clear to see  $\Delta_\partial\alpha$  is still a  $(p, q)$ -form.

□

**Exercise 12.1.1.** Show that for compact Kähler manifold we have

$$[\Delta_d, L] = 0$$

$$[L, \Lambda] = (k - n) \text{id} \quad \text{on } C^\infty(X, \Omega_{X, \mathbb{C}}^k)$$

*Proof.* For the first equation, we have  $\Delta_d = 2\Delta_\partial = 2(\partial\partial^* + \partial^*\partial)$ . Thus

$$[\Delta_d, L] = 2([\partial\partial^*, L] + [\partial^*\partial, L]) = 2(\partial[\partial^*, L] + [\partial^*, L]\partial)$$

The last equality holds by the fact that  $L$  commutes with  $\partial$  since  $\omega$  is  $\partial$ -closed. Now we use the identity  $[\partial^*, L] = -\sqrt{-1} \cdot \bar{\partial}$ , which anticommutes with  $\partial$  to conclude.

For the second equation, without lose of generality it suffices to check on  $U \subseteq \mathbb{C}^n$  equipped with standard Hermitian metric since we are considering operators of order zero. Suppose  $\varphi = \varphi_{IJ} dz^I d\bar{z}^J$  is a  $k$ -form with type  $(p, q)$ . A direct computation shows

$$\begin{aligned} L\Lambda\varphi &= L \left( \sqrt{-1} \sum_{i=1}^n \varphi_{IJ} \iota_i \circ \iota_{\bar{i}} (dz^I \wedge d\bar{z}^J) \right) \\ &= L \left( \sqrt{-1} \sum_{i=1}^n (-1)^p \varphi_{IJ} \iota_i dz^I \wedge \iota_{\bar{i}} d\bar{z}^J \right) \\ &= (\sqrt{-1})^2 \sum_{i,j=1}^n (-1)^{2p-1} \varphi_{IJ} dz^j \wedge \iota_i dz^I \wedge d\bar{z}^j \iota_{\bar{i}} d\bar{z}^J \\ \Lambda L\varphi &= \Lambda \left( \sqrt{-1} \sum_{j=1}^n (-1)^p \varphi_{IJ} dz^j \wedge dz^I \wedge d\bar{z}^j \wedge d\bar{z}^J \right) \\ &= (\sqrt{-1})^2 \sum_{i,j=1}^n (-1)^p \varphi_{IJ} \iota_i \circ \iota_{\bar{i}} (dz^j \wedge dz^I \wedge d\bar{z}^j \wedge d\bar{z}^J) \\ &= (\sqrt{-1})^2 \sum_{i,j=1}^n (-1)^{2p+1} \varphi_{IJ} \iota_i (dz^j \wedge dz^I \wedge \iota_{\bar{i}} (d\bar{z}^j \wedge d\bar{z}^J)) \\ &= (\sqrt{-1})^2 \sum_{i,j=1}^n (-1)^{2p+1} \varphi_{IJ} \iota_i \left( dz^j \wedge dz^I \wedge \underbrace{(\delta_{\bar{i}}^j d\bar{z}^J - d\bar{z}^j \wedge \iota_{\bar{i}} d\bar{z}^J)}_A \right) \\ &= (\sqrt{-1})^2 \sum_{i,j=1}^n (-1)^{2p+1} \varphi_{IJ} (\delta_{\bar{i}}^j dz^I \wedge A - dz^j \wedge \iota_i dz^I \wedge A) \end{aligned}$$

Then

$$\begin{aligned} L\Lambda\varphi - \Lambda L\varphi &= (\sqrt{-1})^2 \sum_{j=1}^n \varphi_{IJ} \left( dz^I \wedge d\bar{z}^J - dz^I \wedge d\bar{z}^j \wedge \iota_{\bar{j}} dz^J - dz^j \wedge \iota_j dz^I \wedge d\bar{z}^J \right) \\ &= (k - n)\varphi \end{aligned}$$

□

### 12.2. Hodge decomposition.

**Theorem 12.2.1.** Let  $(X, h)$  be a compact Kähler manifold,  $\alpha = \sum_{p+q=k} \alpha^{p,q}$ . Then  $\alpha$  is harmonic if and only if  $\alpha^{p,q}$  is harmonic, that is

$$\mathcal{H}^k \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}$$

with  $\overline{\mathcal{H}^{p,q}} = \mathcal{H}^{q,p}$ .

*Proof.* It follows from  $\Delta_d$  preserves bidegree.  $\square$

**Theorem 12.2.2** (Hodge decomposition). Let  $(X, h)$  be a compact Kähler manifold. Then

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X)$$

with  $\overline{H^{p,q}(X)} = H^{q,p}(X)$ .

*Proof.* It follows from there are natural isomorphisms  $H^k(X, \mathbb{C}) \cong \mathcal{H}^k \otimes \mathbb{C}$  and  $H^{p,q}(X) \cong \mathcal{H}^{p,q}$ .  $\square$

**Corollary 12.2.1.** Let  $(X, h)$  be a compact Kähler manifold. Then

$$b_k = \sum_{p+q=k} h^{p,q}$$

with  $h^{p,q} = h^{q,p}$ , where  $b_k = \dim H^k(X, \mathbb{C})$  and  $h^{p,q} = \dim H^{p,q}(X)$ .

**Corollary 12.2.2.**  $b_k$  is even when  $k$  is odd.

**Corollary 12.2.3.**  $b_k \neq 0$  when  $k$  is even.

*Proof.*  $h^{k,k} \neq 0$  since  $0 \neq \omega^k \in H^{k,k}(X)$ .  $\square$

There are many relations between  $h^{p,q}$ , and we can draw a picture as follows, called Hodge diamond since it has the same symmetry as a diamond.

$$\begin{array}{ccccccc}
 & & & h^{0,0} & & & b_0 \\
 & & & & h^{1,0} & & h^{0,1} & & b_1 \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & b_2 \\
 & & \ddots & & \vdots & & \ddots & & \vdots \\
 \text{Hodge} \updownarrow & h^{n,0} & \dots & \text{Serre} & \dots & h^{0,n} & & b_n \\
 & & \ddots & & \vdots & & \ddots & & \vdots \\
 & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & b_{2n-2} \\
 & & & h^{n,n-1} & & h^{n-1,n} & & & b_{2n-1} \\
 & & & & h^{n,n} & & & & b_{2n} \\
 & & & & \longleftrightarrow & & & & \\
 & & & & \text{conjugation} & & & & 
 \end{array}$$

**Example 12.2.1.**

$$H^{p,q}(\mathbb{CP}^n) = \begin{cases} \mathbb{C} & 0 \leq p = q \leq n \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* It's known to all that the singular cohomology of  $\mathbb{CP}^n$  with complex coefficient is

$$H^k(\mathbb{CP}^n, \mathbb{C}) = \begin{cases} \mathbb{C} & k \text{ is even} \\ 0 & k \text{ is odd} \end{cases}$$

Thus it's clear to compute Dolbeault cohomology of  $\mathbb{CP}^n$  using the symmetry of Hodge diamond.  $\square$

**12.3. Bott-Chern cohomology.** Review what we have done: We have already proven one of the main theorems in this course, that is, Hodge decomposition. But along the way we used the Kähler metric on a Kähler manifold, a question is that: (In)dependence of the Kähler metric? The answer is that our decomposition is independent of the choice of Kähler metric, shown by Bott-Chern cohomology.

**Definition 12.3.1** (Bott-Chern cohomology). Let  $X$  be a complex manifold. The Bott-Chern cohomology is defined as

$$H_{\text{BC}}^{p,q}(X) := \frac{Z_{\text{BC}}^{p,q} := \{\alpha \in C^\infty(X, \Omega_X^{p,q}) \mid d\alpha = 0\}}{\partial\bar{\partial}C^\infty(X, \Omega_X^{p-1,q-1})}$$

*Remark 12.3.1.* There is a natural map

$$Z_{\text{BC}}^{p,q}(X) \rightarrow H^{p+q}(X, \mathbb{C})$$

which descends to

$$H_{\text{BC}}^{p,q}(X) \rightarrow H^{p+q}(X, \mathbb{C})$$

since  $\partial\bar{\partial}\beta = d\bar{\partial}\beta$ . On the other hand, there is also a natural map

$$Z_{\text{BC}}^{p,q}(X) \rightarrow H^{p,q}(X)$$

which descends to

$$H_{\text{BC}}^{p,q}(X) \rightarrow H^{p,q}(X)$$

since  $\partial\bar{\partial}\beta = -\bar{\partial}\partial\beta$ . So if we can prove there are isomorphisms between

$$H_{\text{BC}}^{p,q}(X) \cong H^{p,q}(X)$$

$$\bigoplus_{p+q=k} H_{\text{BC}}^{p,q}(X) \cong H^k(X, \mathbb{C})$$

then Hodge decomposition is canonical, that is independent of choice of Kähler metric since Bott-Chern cohomology is independent of the choice of Kähler metric.

**Lemma 12.3.1** ( $\partial\bar{\partial}$ -lemma). Let  $(X, \omega)$  be a compact Kähler manifold and  $\alpha$  be a d-closed  $(p, q)$ -form. If  $\alpha$  is  $\bar{\partial}$ -exact or  $\partial$ -exact, then there exists a  $(p-1, q-1)$ -form such that

$$\alpha = \partial\bar{\partial}\beta$$

*Proof.* Suppose  $\alpha$  is  $\bar{\partial}$ -exact. Then  $\alpha = \bar{\partial}\gamma$  for some  $(p, q-1)$ -form  $\gamma$ , and Hodge's theorem implies  $\gamma$  has decomposition

$$\gamma = a + \partial b + \partial^* c$$

where  $a$  is  $\Delta_{\partial}$ -harmonic, and  $b, c$  are forms with appropriate degrees. Direct computation shows

$$\begin{aligned} \alpha = \bar{\partial}\gamma &= \bar{\partial}a + \bar{\partial}\partial b + \bar{\partial}\partial^* c \\ &= -\partial\bar{\partial}b + \bar{\partial}\partial^* c \\ &= -\partial\bar{\partial}b - \partial^*\bar{\partial}c \end{aligned}$$

Now it suffices to show  $-\partial^*\bar{\partial}c = 0$ . A trick here is to note that

$$0 = \partial\alpha = -\partial\bar{\partial}\partial^*c \implies \partial^*\bar{\partial}c \in \ker \partial \cap \text{im } \partial^* = 0 \implies \partial^*\bar{\partial}c = 0$$

So we have

$$\alpha = \partial\bar{\partial}(-b)$$

as desired.  $\square$

**Corollary 12.3.1.** Let  $(X, \omega)$  be a compact Kähler manifold. Then

- (1)  $H_{\text{BC}}^{p,q}(X) \rightarrow H^{p,q}(X)$  is an isomorphism.
- (2)  $\bigoplus_{p+q=k} H_{\text{BC}}^{p,q}(X) \rightarrow H^k(X, \mathbb{C})$  is an isomorphism.

*Proof.* Here we only prove the first isomorphism. From Remark 12.3.1, there is a canonical map  $H_{\text{BC}}^{p,q}(X) \rightarrow H^{p,q}(X)$ , and if we choose a Kähler metric, we have  $H^{p,q}(X) \cong \mathcal{H}^{p,q}$ , we will show our canonical map is both surjective and injective via this chosen metric.

- (1) To see surjectivity: For element in  $H^{p,q}(X)$  we choose a  $\Delta_{\bar{\partial}}$ -harmonic representative. Since  $\Delta_{\bar{\partial}}$ -harmonic is equivalent to  $\Delta_d$ -harmonic, so this representative is also  $d$ -closed.
- (2) To see injectivity: Suppose we have  $[\alpha] \in H_{\text{BC}}^{p,q}(X)$  such that  $\alpha$  is trivial in  $H^{p,q}(X)$ , that is  $\bar{\partial}$ -exact. Then Lemma 12.3.1, that is  $\partial\bar{\partial}$ -lemma implies it's trivial in Bott-Chern cohomology.  $\square$

**Corollary 12.3.2.** A Hermitian metric  $\omega$  is Kähler if and only if it can be written locally as

$$\omega = \sqrt{-1}\partial\bar{\partial}f$$

where  $f$  is a real-valued smooth function.

*Proof.* It's clear if  $\omega$  is locally written as  $\sqrt{-1}\partial\bar{\partial}f$ , then it gives a Kähler metric. Conversely, a Kähler metric  $\omega$  is an element in  $H^{1,1}(X)$ , and we have already shown that  $H^{1,1}(X) = H_{\text{BC}}^{1,1}(X)$ , and Dolbeault lemma implies Dolbeault cohomology vanishes on open subset which is sufficiently small, this completes the proof.  $\square$

## 13. APPLICATIONS OF HODGE THEORY

**13.1. Adjoint operators on bundle valued forms.** Let  $(E, h)$  be a Hermitian holomorphic vector bundle on Hermitian  $n$ -manifold  $X$ , what we have done can be generalized to bundle valued forms. More explicitly, for  $\varphi, \psi \in C^\infty(X, \Omega_X^{p,q} \otimes E)$ , locally written as  $\varphi = \varphi^\alpha e_\alpha, \psi = \psi^\beta e_\beta$ , then local inner product is given by

$$\langle \varphi, \psi \rangle := h_{\alpha\bar{\beta}} \langle \varphi^\alpha, \psi^\beta \rangle$$

and  $\varphi \wedge \bar{\psi}$  is defined as

$$\varphi \wedge \bar{\psi} := \varphi^\alpha \wedge \bar{\psi}^\beta h_{\alpha\bar{\beta}}$$

The inner product on  $C^\infty(X, \Omega_X^{p,q} \otimes E)$  is given by

$$(\varphi, \psi) := \int_X \langle \varphi, \psi \rangle \frac{\omega^n}{n!}$$

where  $\varphi, \psi \in C^\infty(X, \Omega_X^{p,q} \otimes E)$ . The hodge star operator is defined as an operator

$$\star_E: C^\infty(X, \Omega_X^{p,q} \otimes E) \rightarrow C^\infty(X, \Omega_X^{n-q, n-p} \otimes E)$$

such that

$$(\varphi, \psi) = \int_X \varphi \wedge \star_E \bar{\psi}$$

Let  $\nabla$  be the Chern connection of  $(E, h)$ . Then  $\nabla^{0,1} = \bar{\partial}_E$ , and if we set  $\nabla^{1,0} = \partial_E$ , then

$$\begin{aligned} \Theta_h &= \nabla^2 \\ &= \partial_E^2 + \partial_E \bar{\partial}_E + \bar{\partial}_E \partial_E + \bar{\partial}_E^2 \\ &= [\partial_E, \bar{\partial}_E] \end{aligned}$$

**Exercise 13.1.1.** Give formulas of  $\partial_E^*$  and  $\bar{\partial}_E^*$  in terms of Hodge star  $\star_E$ .

Laplacians  $\Delta_{\partial_E}, \Delta_{\bar{\partial}_E}$  can be defined in a same way, and there is also a Hodge theorem, which gives the following decomposition

$$C^\infty(X, \Omega_X^{p,q} \otimes E) = \mathcal{H}^{p,q}(X, E) \oplus \text{im } \bar{\partial}_E \oplus \text{im } \bar{\partial}_E^*$$

and by the same argument we have

$$H^{p,q}(X, E) \cong \mathcal{H}^{p,q}$$

Lefschetz operator is defined as follows

$$\begin{aligned} L: C^\infty(X, \Omega_X^{p,q} \otimes E) &\rightarrow C^\infty(X, \Omega_X^{p+1, q+1} \otimes E) \\ \alpha &\mapsto \omega \wedge \alpha \end{aligned}$$

and  $\Lambda$  is the formal adjoint of  $L$ . If  $(X, \omega)$  is also a Kähler  $n$ -manifold, then there are also Kähler identities

$$\begin{aligned} [\bar{\partial}_E^*, L] &= \sqrt{-1} \partial \\ [\partial_E^*, L] &= -\sqrt{-1} \cdot \bar{\partial} \\ [\Lambda, \partial] &= -\sqrt{-1} \partial^* \\ [\Lambda, \bar{\partial}] &= \sqrt{-1} \cdot \bar{\partial}^* \end{aligned}$$

and

$$[L, \Lambda] = (p + q - n) \text{id}$$

holds on  $E$ -valued  $(p, q)$ -forms.

### 13.2. Serre duality.

**Theorem 13.2.1** (Serre duality). Let  $X$  be a compact complex  $n$ -manifold and  $E$  be a holomorphic vector bundle. Then there exists a non-degenerate  $\mathbb{C}$ -linear pairing

$$\begin{aligned} H^{p,q}(X, E) \times H^{n-p, n-q}(X, E^*) &\rightarrow \mathbb{C} \\ ([\alpha], [\beta]) &\mapsto \int_X \alpha \wedge \beta \end{aligned}$$

In particular, we have

$$H^{p,q}(X, E) = H^{n-p, n-q}(X, E^*)^*$$

*Sketch of the proof.* Firstly endow  $E$  with a Hermitian metric  $h$ , and show

$$\Delta_{\bar{\partial}_E^*} \circ \bar{\kappa}_E = \bar{\kappa}_E \circ \Delta_{\bar{\partial}_E}$$

Then

$$\bar{\kappa}_E: \mathcal{H}^{p,q}(X, E) \xrightarrow{\cong} \mathcal{H}^{n-p, n-q}(X, E^*)$$

and Hodge theorem implies that

$$\mathcal{H}^{p,q}(X, E) \cong H^{p,q}(X, E)$$

For all  $\alpha \in \mathcal{H}^{p,q}(X, E)$ ,  $\beta \in \mathcal{H}^{n-p, n-q}(X, E^*)$ , we have  $\beta = \bar{\kappa}_E \gamma$  for some  $\gamma \in \mathcal{H}^{p,q}(X, E)$ , and thus

$$\int_X \alpha \wedge \beta = \int_X \alpha \wedge \bar{\kappa}_E \gamma = \langle \alpha, \gamma \rangle$$

is non-degenerate.  $\square$

**Corollary 13.2.1.** Let  $X$  be a compact complex  $n$ -manifold and  $E$  be a holomorphic vector bundle. Then

$$H^{p,q}(X) = H^{n-p, n-q}(X)^*$$

*Proof.* Consider  $E = \mathcal{O}_X$  in Serre duality, and then desired result holds from the fact  $\mathcal{O}_X^* = \mathcal{O}_X$ .  $\square$

*Remark 13.2.1.* This recovers Corollary 11.2.6.

**Corollary 13.2.2.** Let  $X$  be a compact complex  $n$ -manifold and  $E$  be a holomorphic vector bundle. Then

$$H^q(X, E) = H^{n-q}(X, K_X \otimes E^*)^*$$

*Proof.* Set  $p = 0$  in Serre duality one has

$$H^{0,q}(X, E) \cong H^{n,n-q}(X, E^*)^*$$

which gives desired result. □



## Part 4. Positivity and vanishing theorems

### 14. POSITIVITY

**14.1. Positivity of line bundle.** Let  $(L, h)$  be a Hermitian holomorphic line bundle over a complex manifold  $X$ . Then  $\frac{\sqrt{-1}}{2\pi}\Theta_h$  gives a real  $(1, 1)$ -form, so it corresponds to a Hermitian form on  $TX$ .

**Definition 14.1.1** (positive line bundle). Let  $L$  be a holomorphic line bundle over  $X$ .  $L$  is called positive if it admits a Hermitian metric  $h$  such that the Hermitian form corresponding to  $\frac{\sqrt{-1}}{2\pi}\Theta_h$  is positive definite.

*Remark 14.1.1.* The Kodaira embedding theorem implies positive line bundle is exactly ample divisor in algebraic geometry.

*Remark 14.1.2* (local form). Locally, one has

$$\frac{\sqrt{-1}}{2\pi}\Theta_h = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h = \frac{\sqrt{-1}}{2\pi}\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j}dz^i \wedge d\bar{z}^j$$

where  $\varphi = -\log h$ . Thus  $L$  is positive if and only if the Hermitian matrix  $(\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j})$  is positive definite everywhere.

**Proposition 14.1.1.** If  $X$  admits a positive holomorphic line bundle, then  $X$  is Kähler.

*Proof.* The first Chern class of  $(L, h)$  gives its Kähler form. □

**Proposition 14.1.2.**  $L$  is positive if and only if  $L^{\otimes m}$  is positive for some  $m \in \mathbb{N}_{\geq 0}$ .

*Proof.* For a line bundle  $L$  locally we have the Hermitian metric corresponding to its curvature looking like

$$\left(\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j}\right)$$

and for  $L^{\otimes m}, m \in \mathbb{N}_{\geq 0}$  we have

$$\left(m \cdot \frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j}\right)$$

It's clear  $L$  is positive if and only if  $L^{\otimes m}$  is. □

**Exercise 14.1.1.** Let  $X$  be a compact complex manifold and  $L$  be a positive line bundle. For any holomorphic line bundle  $L'$ , there exists  $N_0 \in \mathbb{N}$  such that  $L' \otimes L^{\otimes N}$  positive for  $N \geq N_0$ .

*Proof.* The proof is quite similar to above exercise, we need to check locally, but compactness is necessary here. Over an open subset  $U_1$ , locally we have the Hermitian metric corresponding to  $L' \otimes L^m$  looking like

$$\left(\frac{\partial^2\varphi_{L'}}{\partial z^i\partial\bar{z}^j} + m \cdot \frac{\partial^2\varphi_L}{\partial z^i\partial\bar{z}^j}\right)$$

So we can choose suffices large  $N_1$  such that  $M \otimes L^{\otimes N_1}$  is positive on  $U$ . Since  $X$  is compact, we can take a finite open covering  $\{U_i\}$  of  $X$  and choose the largest  $N_i$  to be  $N$  we desired.  $\square$

**14.2. Positivity of vector bundles.** Let  $(E, h)$  be a Hermitian holomorphic vector bundle of rank  $r$  over a complex  $n$ -manifold  $X$  with Chern connection  $\nabla$ . In local frame, its Chern curvature is given by

$$\Theta_h = \Theta_{i\bar{j}\alpha\bar{\beta}}^\beta dz^i \wedge d\bar{z}^j \otimes e^\alpha \otimes e_\beta$$

**Definition 14.2.1** (positivity).

- (1)  $(E, h)$  is said to be Griffiths positive, if for any non-zero  $(u^i) \in \mathbb{C}^n$  and  $(v^\alpha) \in \mathbb{C}^r$

$$\Theta_{i\bar{j}\alpha\bar{\beta}} u^i \bar{u}^j v^\alpha \bar{v}^\beta > 0$$

- (2)  $(E, h)$  is said to be Nakano positive, if for any non-zero matrix  $(u^{i\alpha})$

$$\Theta_{i\bar{j}\alpha\bar{\beta}} u^{i\alpha} \bar{u}^{j\beta} > 0$$

- (3)  $(E, h)$  is said to be dual Nakano positive, if for any non-zero matrix  $(u^{i\alpha})$

$$\Theta_{i\bar{j}\alpha\bar{\beta}} u^{i\beta} \bar{u}^{j\alpha} > 0$$

*Remark 14.2.1.* The semi-positivity and negativity can be defined in the same way.

**Proposition 14.2.1.**

- (1) If  $(E, h)$  is Nakano positive or dual Nakano positive, then  $(E, h)$  is Griffiths positive.  
(2)  $(E, h)$  is Nakano positive if and only if  $(E^*, h^*)$  is dual Nakano negative.

*Proof.* For (1). If  $(E, h)$  is Nakano positive, then for non-zero  $(u^i) \in \mathbb{C}^n$  and  $(v^\alpha) \in \mathbb{C}^r$ , consider matrix  $(u^{i\alpha})$  defined by  $u^{i\alpha} := u^i v^\alpha$ , then

$$\Theta_{i\bar{j}\alpha\bar{\beta}} u^{i\alpha} \bar{u}^{j\beta} v^\alpha \bar{v}^\beta = \Theta_{i\bar{j}\alpha\bar{\beta}} u^{i\alpha} \bar{u}^{j\beta} > 0$$

The same argument holds for the case  $(E, h)$  is dual Nakano positive. (2) follows from the relation between curvature form of  $(E, h)$  and  $(E^*, h^*)$ , see Section 4.3 of [Liu23].  $\square$

**Proposition 14.2.2.** Let

$$0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0$$

be an exact sequence of holomorphic vector bundles.

- (1) If  $(E, h)$  is Griffiths or dual Nakano positive, then so is  $(Q, h^Q)$ .  
(2) If  $(E, h)$  is Griffiths or Nakano negative, then so is  $(S, h^S)$ .

**Proposition 14.2.3.** If Hermitian holomorphic vector bundle  $(E, h)$  is Griffiths positive, then  $(E \otimes \det E, h \otimes \det h)$  is Nakano positive and dual Nakano positive.

## 15. VANISHING THEOREMS

## 15.1. Kodaira vanishing theorem.

**Theorem 15.1.1** (Bochner-Kodaira-Nakano identity). Let  $(X, \omega)$  be a compact Kähler manifold and  $(E, h)$  a Hermitian holomorphic vector bundle. Then

$$\Delta_{\bar{\partial}_E} = [\sqrt{-1}\Theta_h, \Lambda] + \Delta_{\partial_E}$$

*Proof.* Direct computation shows

$$\begin{aligned} \Delta_{\bar{\partial}_E} &= [\bar{\partial}_E, \bar{\partial}_E^*] \\ &= -\sqrt{-1}[\bar{\partial}_E, [\Lambda, \partial_E]] \\ &= -\sqrt{-1}[\Lambda, [\partial_E, \bar{\partial}_E]] - \sqrt{-1}[\partial_E, [\bar{\partial}_E, \Lambda]] \\ &= -\sqrt{-1}[\Lambda, \Theta_h] - \sqrt{-1}[\partial_E, \sqrt{-1}\partial_E^*] \\ &= [\sqrt{-1}\Theta_h, \Lambda] + \Delta_{\partial_E} \end{aligned}$$

□

**Corollary 15.1.1** (Bochner-Kodaira-Nakano inequality). Let  $(X, \omega)$  be a compact Kähler manifold and  $(E, h)$  a Hermitian holomorphic vector bundle. Then for  $\alpha \in C^\infty(X, \Omega_X^{p,q} \otimes E)$ , one has

$$([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) \leq (\Delta_{\bar{\partial}_E}\alpha, \alpha)$$

In particular, if  $\alpha$  is  $\Delta_{\bar{\partial}_E}$ -harmonic, then  $([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) \leq 0$ .

*Proof.* Direct computation shows

$$\begin{aligned} (\Delta_{\bar{\partial}_E}\alpha, \alpha) - ([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) &= (\Delta_{\partial_E}\alpha, \alpha) \\ &= \|\partial_E\alpha\|^2 + \|\partial_E^*\alpha\|^2 \geq 0 \end{aligned}$$

□

**Corollary 15.1.2.** Let  $X$  be a complex manifold. If Hermitian holomorphic vector bundle  $(E, h)$  is Griffiths positive, then  $(\det E, \det h)$  is a positive holomorphic line bundle. In particular,  $X$  is Kähler.

**Theorem 15.1.2** (Kodaira-Akizuki-Nakano vanishing). Let  $X$  be a compact  $n$ -manifold,  $(L, h)$  a positive Hermitian holomorphic line bundle. Then

$$H^{p,q}(X, L) = 0$$

for  $p + q > n$ .

*Proof.* Let  $X$  be endowed with Kähler metric  $\omega$  given by Chern curvature of  $L$ , then there is an isomorphism  $H^{p,q}(X, L) \cong \mathcal{H}^{p,q}(X, L)$ . For  $\alpha \in \mathcal{H}^{p,q}(X, L)$ , Corollary 15.1.1 implies

$$[\sqrt{-1}\Theta_h, \Lambda]\alpha \leq 0$$

On the other hand,

$$([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) = 2\pi(p + q - n)\|\alpha\|^2 \geq 0$$

Thus if  $p + q > n$ , one has  $\alpha = 0$ , this completes the proof.  $\square$

**Corollary 15.1.3** (Kodaira vanishing). Let  $X$  be a compact  $n$ -manifold and  $(L, h)$  be a positive holomorphic line bundle over  $X$ . Then

$$H^q(X, K_X \otimes L) = 0$$

for  $q > 0$ .

*Proof.* Just note that

$$H^q(X, K_X \otimes L) = H^{n,q}(X, L)$$

$\square$

**Corollary 15.1.4.** Let  $(X, \omega)$  be a compact Kähler  $n$ -manifold. If  $(L, h)$  is a semi-positive line bundle and  $\text{rk } \Theta_h \geq k$ , then

$$H^{p,q}(X, L) = 0$$

for  $p + q \geq 2n - k + 1$ .

**Exercise 15.1.1.** Compute all  $H^q(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(k))$  for all  $k, q$ .

**Definition 15.1.1** (Fano). A Fano manifold is a compact Kähler manifold with positive anti-canonical bundle  $K_X^* = \det TX$ .

**Proposition 15.1.1.** Let  $X$  be a Fano manifold, then

$$H^q(X, \mathcal{O}_X) = 0$$

for all  $q > 0$ .

*Proof.* Note that  $\mathcal{O}_X = K_X \otimes K_X^*$ .  $\square$

**Theorem 15.1.3** (Serre vanishing). Let  $X$  be a compact complex  $n$ -manifold and  $(L, h)$  be a positive holomorphic line bundle over  $X$ . For any holomorphic vector bundle  $E$  on  $X$ , there exists a constant  $m_0$  such that for all  $m \geq m_0$

$$H^q(X, E \otimes L^{\otimes m}) = 0$$

for  $q > 0$ .

*Proof.* If  $X$  is endowed with Kähler metric  $\omega$  given by Chern curvature of  $L$  and  $E$  is endowed with a Hermitian metric  $h$ , then  $H^{p,q}(X, E \otimes L^{\otimes m}) \cong \mathcal{H}^{p,q}(X, E \otimes L^{\otimes m})$ . For  $\alpha \in \mathcal{H}^{p,q}(X, E \otimes L^{\otimes m})$ , one has

$$\begin{aligned} ([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) + 2\pi m(p + q - n)\|\alpha\|^2 &\stackrel{(1)}{=} ([\sqrt{-1}\Theta_{E \otimes L^{\otimes m}}, \Lambda]\alpha, \alpha) \\ &\stackrel{(2)}{\leq} 0 \end{aligned}$$

where

(1) holds from  $\Theta_{E \otimes L^{\otimes m}} = \Theta^E \otimes \text{id} + m(\text{id} \otimes \Theta^L)$ .

(2) holds from Corollary 15.1.1, that is Bochner-Kodaira-Nakano inequality.

On the other hand, Cauchy inequality implies that

$$([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) \geq -C\|\alpha\|^2$$

where constant  $C$  is the norm of  $[\sqrt{-1}\Theta_h, \Lambda]$ . So if we have  $2\pi m(p + q - n) - C > 0$ , the argument in proof of Kodaira vanishing theorem implies  $\alpha = 0$ . Consider  $p = n, q > 0, m_0 \geq \frac{C}{2\pi}$ , then for all  $m \geq m_0$  and  $q > 0$ , one has

$$H^{n,q}(X, E \otimes L^{\otimes m}) = 0$$

that is to say  $H^q(X, K_X \otimes E \otimes L^{\otimes m}) = 0$ . So in order to show  $H^q(X, E \otimes L^{\otimes m}) = 0$ , it suffices to consider  $K_X^* \otimes E$  at beginning, and then we will obtain

$$H^q(X, K_X \otimes K_X^* \otimes E \otimes L^{\otimes m}) = H^q(X, E \otimes L^{\otimes m}) = 0$$

This completes the proof.  $\square$

## 15.2. Nakano vanishing.

**Theorem 15.2.1** (Nakano vanishing). Let  $X$  be a compact complex manifold and  $(E, h)$  a Hermitian holomorphic vector bundle over  $X$ .

(1) If  $(E, h)$  is Nakano positive, then

$$H^{n,q}(X, E) = 0$$

for  $q \geq 1$ .

(2) If  $(E, h)$  is dual Nakano positive, then

$$H^{p,n}(X, E) = 0$$

for  $p \geq 1$ .

*Proof.* Here we only give the proof of the first one, the second can be derived from the same argument. Suppose  $\{\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}\}$  is a holomorphic local frame which is orthonormal at point  $p \in X$ , and  $\varphi \in H^{n,q}(X, E)$  which is locally written as  $\varphi = \varphi_I^\alpha dz^1 \wedge \dots \wedge dz^n \wedge d\bar{z}^I \otimes e_\alpha$  where  $|I| = q$ . Direct computation shows

$$\begin{aligned} \langle \sqrt{-1}[\Theta_h, \Lambda]\varphi, \varphi \rangle &= \langle \sqrt{-1}\Theta_h \Lambda \varphi, \varphi \rangle \\ &\stackrel{(1)}{=} -\langle \Theta_{k\bar{l}\alpha}^\gamma dz^k \wedge d\bar{z}^l \sum_m \iota_m \iota_{\bar{m}} \varphi_I^\alpha dz^1 \wedge \dots \wedge dz^n \wedge d\bar{z}^I \otimes e_\gamma, \varphi_J^\beta dz^1 \wedge \dots \wedge dz^n \wedge d\bar{z}^J \otimes e_\beta \rangle \\ &= -\Theta_{k\bar{l}\alpha\bar{\beta}} \varphi_I^\alpha \overline{\varphi_J^\beta} \langle dz^k \wedge d\bar{z}^l \wedge \sum_m \iota_m \iota_{\bar{m}} dz^1 \wedge \dots \wedge dz^n \wedge d\bar{z}^I, dz^1 \wedge \dots \wedge dz^n \wedge d\bar{z}^J \rangle \\ &= \sum_k \Theta_{k\bar{l}\alpha\bar{\beta}} \varphi_I^\alpha \overline{\varphi_J^\beta} \langle d\bar{z}^l \wedge \iota_{\bar{k}} d\bar{z}^I, d\bar{z}^J \rangle \\ &= \sum_{I,J} \left\{ \Theta_{k\bar{l}\alpha\bar{\beta}} \left( \sum_{k \in I} \varphi_I^\alpha \right) \overline{\left( \sum_{l \in J} \varphi_J^\beta \right)} \right\} \\ &\stackrel{(2)}{\geq} 0 \end{aligned}$$

where

(1) holds from Proposition 11.3.2 since by Corollary 15.1.2 one has  $X$  is Kähler.

(2) holds from  $(E, h)$  is Nakano positive.

But by Corollary 15.1.1, that is Bochner-Kodaira-Nakano inequality, one has  $\langle \sqrt{-1}[\Theta_h, \Lambda]\varphi, \varphi \rangle \leq 0$ . This shows  $\varphi = 0$  as before.  $\square$

**Corollary 15.2.1.** Let  $X$  be a compact complex  $n$ -manifold with  $n \geq 2$ . Then there doesn't exist metric  $h$  such that  $(TX, h)$  is Nakano positive.

*Proof.* Suppose  $(TX, h)$  is Nakano positive, then

$$\begin{aligned} H^{1,1}(X) &= H^{0,1}(X, T^*X) \\ &\stackrel{(1)}{=} H^{n,n-1}(X, TX) \\ &\stackrel{(2)}{=} 0 \end{aligned}$$

where

(1) holds from Serre duality.

(2) holds from Nakano vanishing theorem.

However, by Proposition 14.2.1 and Corollary 15.1.2 one has  $X$  is a Kähler manifold, which leads to a contradiction since  $H^{1,1}(X) \neq 0$  for a Kähler manifold.  $\square$

**Corollary 15.2.2.** Let  $X$  be a compact complex  $n$ -manifold with  $n \geq 2$ . Then there doesn't exist metric  $h$  such that  $(T^*X, h)$  is dual Nakano positive.

*Proof.* Note that

$$H^{n-1,n}(X, T^*X) = H^{1,0}(X, TX) = H^0(X, \text{End}(TX)) \neq 0$$

$\square$

**Example 15.2.1.** If  $n \geq 2$ ,  $(\mathbb{CP}^n, \omega_{\text{FS}})$

(1) is dual Nakano positive.

(2) is Nakano semi-positive, but not Nakano positive.

*Proof.* By Section 10.4 one has the curvature of Fubini-Study metric is

$$\Theta_{i\bar{j}k\bar{l}} = h_{i\bar{j}}h_{k\bar{l}} + h_{i\bar{l}}h_{k\bar{j}}$$

For  $p \in X$ , if we consider normal coordinate at  $p$ , then for any non-zero matrix  $(u^{ij})$ , a direct computation shows

$$\begin{aligned} \Theta_{i\bar{j}k\bar{l}}u^{il}\bar{u}^{jk} &= h_{i\bar{j}}h_{k\bar{l}}u^{il}\bar{u}^{jk} + h_{i\bar{l}}h_{k\bar{j}}u^{il}\bar{u}^{jk} \\ &= \sum_{i,j} |u^{ij}|^2 + u^{ii}\bar{u}^{jj} \\ &= \sum_{i \neq j} |u^{ij}|^2 + \frac{1}{2} \sum_{i,j} |u^{ii} + u^{jj}|^2 \\ &> 0 \end{aligned}$$

which implies  $(\mathbb{CP}^n, \omega_{FS})$  is dual Nakano positive.

$$\begin{aligned} \Theta_{i\bar{j}k\bar{l}} u^{ik} \bar{u}^{jl} &= h_{i\bar{j}} h_{k\bar{l}} u^{ik} \bar{u}^{jl} + h_{i\bar{l}} h_{k\bar{j}} u^{ik} \bar{u}^{jl} \\ &= \sum_{i,j} |u^{ij}|^2 + u^{ij} \bar{u}^{ji} \\ &= \frac{1}{2} \sum_{i,j} |u^{ij} + u^{ji}|^2 \\ &\geq 0 \end{aligned}$$

which implies  $(\mathbb{CP}^n, \omega_{FS})$  is Nakano semi-positive, and it's not Nakano positive by Corollary 15.2.1.  $\square$

### 15.3. Griffiths vanishing.

**Theorem 15.3.1** (Griffiths vanishing). If  $(E, h)$  is Griffiths positive, then

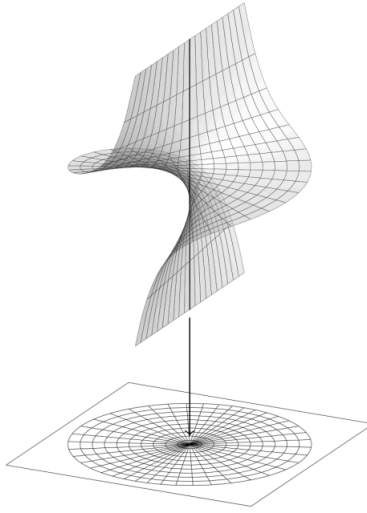
- (1)  $H^{n,n}(X, E) = 0$ .
- (2)  $H^{p,q}(X, E) = 0$  for  $p + q \geq n + \text{rk } E$ .

## 16. KODAIRA EMBEDDING AND CHOW THEOREM

**16.1. Blow-up.** In this section we will introduce a technical tool we need in the proof of Kodaira embedding, which is blow-up at a point.

**Definition 16.1.1** (blow-up).  $U \times \mathbb{CP}^{n-1} \supset \tilde{U} := \{((x_1, \dots, x_n), (y_1 : \dots : y_n)) \mid x_i y_j = x_j y_i\}$

*Remark 16.1.1.* The most vivid way to understand blow-up is to consider the fibres of projection  $\tilde{U} \rightarrow U$ : If  $x \neq 0$ , then the fibre of  $x$  is just a point since the ratio of  $y_i$  is uniquely determined. But for  $x = 0$ , there is no restriction for  $y_i$ , so you get the whole projective space  $\mathbb{CP}^{n-1}$ . Thus as you can imagine, there is nothing happening except the origin, sounds like a boom. For example, the following figure shows the case of  $n = 2$



**Lemma 16.1.1.**  $\tilde{U} \subseteq U \times \mathbb{CP}^{n-1}$  is a submanifold of dimension  $n$ .

Since blow-up is a local operation, so it can be done on a complex manifold. If  $X$  is a complex manifold with dimension  $n$  with  $x \in X$ , and  $\{U_i\}$  is an open covering such that  $x \in U_1$  and  $x \notin U_i, i \neq 1$ , then we can show that  $\tilde{U}_1 \cup (\bigcup_{i \neq 1} U_i)$  glue together a new complex manifold with dimension  $n$ . This is called blow-up of  $X$  at point  $x$ , and it's denoted by  $\tilde{X}$ . Similarly there is a natural projection  $\pi: \tilde{X} \rightarrow X$  and  $\pi^{-1}(x)$  is biholomorphic to  $\mathbb{CP}^{n-1}$ , which is called exceptional divisor and denote it by  $E$ .

**Exercise 16.1.1.** If  $X$  is compact Kähler manifold, then  $\tilde{X}$  is also a compact Kähler manifold.

**Exercise 16.1.2.** The idea sheaf of exceptional divisor  $\mathcal{I}_E \cong \mathcal{O}_{\tilde{X}}(E)^*$

**Proposition 16.1.1.** The canonical bundle  $K_{\tilde{X}}$  of the blow-up  $\tilde{X}$  is isomorphic to  $\pi^* K_X \otimes \mathcal{O}_{\tilde{X}}(E)^{\otimes n-1}$ .



**Corollary 16.1.1.** For the exceptional divisor  $E = \mathbb{CP}^n \subseteq \tilde{X} \rightarrow X$  one has  $\mathcal{O}(E)|_E \cong \mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$ .

*Proof.* The adjunction formula implies

$$K_E \cong (K_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(E))|_E$$

Then by Proposition 16.1.1 and  $K_E = \mathcal{O}_{\mathbb{CP}^{n-1}}(-n)$  one has

$$\mathcal{O}_{\mathbb{CP}^{n-1}}(-n) \cong (\pi^* K_X \otimes \mathcal{O}_{\tilde{X}}(E)^{\otimes n})|_E$$

Note that  $E$  is a fibre of  $\pi$ , so  $\pi^* K_X|_E$  is trivial. Thus

$$\mathcal{O}_{\mathbb{CP}^{n-1}}(-n) \cong \mathcal{O}_{\tilde{X}}(E)^{\otimes n}|_E$$

On the other hand, the only possible line bundles on  $\mathbb{CP}^{n-1}$  take the form  $\mathcal{O}_{\mathbb{CP}^{n-1}}(k)$ ,  $k \in \mathbb{Z}$ . Thus  $\mathcal{O}_{\tilde{X}}(E)|_E \cong \mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$ .  $\square$

The main reason we need blow-up here is that the sections after blow-up is the “same” as the one before.

**Proposition 16.1.2.** For a holomorphic line bundle  $L$ , one has

$$H^0(X, L^{\otimes m}) = H^0(\tilde{X}, \pi^* L^{\otimes m})$$

holds for arbitrary  $m \in \mathbb{Z}_{\geq 0}$ .

*Proof.* If  $X$  is one-dimensional, then  $\pi: \tilde{X} \rightarrow X$  is an isomorphism and thus  $H^0(X, L^{\otimes m}) = H^0(\tilde{X}, \pi^* L^{\otimes m})$ .

Now let's consider the case  $\dim_{\mathbb{C}} X \geq 2$ . Given an element  $s \in H^0(X, L^{\otimes m})$ , we can get an element in  $H^0(\tilde{X}, \pi^* L^{\otimes m})$  by composing projection  $\pi$ . Conversely, for  $\tilde{s} \in H^0(\tilde{X}, \pi^* L^{\otimes m})$ , it can be restricted to  $\tilde{X} \setminus E = X \setminus \{x\}$ , and then extended to a global section of  $L^{\otimes m}$  by Hartogs theorem.  $\square$

## 16.2. Kodaira embedding.

**Theorem 16.2.1** (Kodaira embedding). Let  $X$  be a compact complex manifold. The following statements are equivalent:

- (1) There exists a holomorphic embedding  $\varphi: X \hookrightarrow \mathbb{CP}^N$ .
- (2) There exists a integral Kähler form  $\omega$  on  $X$ , that is,  $[\omega] \in \text{im}\{H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{R})\}$
- (3) There exists a positive holomorphic line bundle on  $X$ .

*Remark 16.2.1.* (1) clearly implies (2), and (2) implies (3) is Lefschetz (1, 1)-theorem, so the heart of the proof is (3) to (1).

*Sketch.* Use holomorphic global sections  $H^0(X, L^{\otimes m})$  for sufficiently large  $m$  to construct  $\varphi: X \hookrightarrow \mathbb{CP}^N$ . We need to show the following three things:

- (1) For sufficiently large  $m$ ,  $L^{\otimes m}$  is globally generated, which means for all  $x \in X$ , there exists a global section  $s \in H^0(X, L^{\otimes m})$  such that  $s(x) \neq 0$ . Then for all  $x \in X$ ,  $H_x = \{s \in H^0(X, L^{\otimes m}) \mid s(x) = 0\}$  is a hyper-surface. Thus we get a holomorphic map  $\varphi: X \rightarrow \mathbb{CP}(H^0(X, L^{\otimes m})^*)$ , defined by  $x \mapsto H_x$ . Indeed since any hypersurface in  $H^0(X, L^{\otimes m})$  is a

line in  $H^0(X, L^{\otimes m})^*$ , that is an element in  $\mathbb{CP}(H^0(X, L^{\otimes m})^*)$ . And you will find  $\varphi$  is holomorphic since we're using holomorphic sections.

- (2) For more sufficiently large<sup>10</sup>  $m$ ,  $L^{\otimes m}$  separates points, that is for all  $x, y \in X$ , there exists  $s \in H^0(X, L^{\otimes m})$  such that  $s(x) \neq 0, s(y) = 0$ . Thus in this case our  $\varphi$  is injective.
- (3) For more more sufficiently large  $m$ ,  $L^{\otimes m}$  separates tangent vectors, that is, for all  $x \in X, u \in T_{X,x}$  there exists  $s \in H^0(X, L^{\otimes m})$  such that  $s(x) = 0$  and  $ds(u) \neq 0$ . Thus in this case our  $\varphi$  is an immersion, together with  $X$  is compact we have  $\varphi$  is an embedding.

*Remark 16.2.2.* We can also describe  $\varphi$  more explicitly. Locally around  $x_0$ , choose a basis  $s_0, \dots, s_N$  of  $H^0(X, L^{\otimes m})$  such that  $s_0(x_0) \neq 0$ . Then there exists a neighborhood  $U$  of  $x_0$  such that  $s_0(x) \neq 0$  for all  $x \in U$ . Then

$$\frac{s_1}{s_0}, \dots, \frac{s_N}{s_0} \in H^0(U, \mathcal{O}_U)$$

So we can define

$$\begin{aligned} \varphi|_U: U &\rightarrow \mathbb{CP}^N \\ x &\mapsto (1, \frac{s_1}{s_0}(x), \dots, \frac{s_N}{s_0}(x)) \end{aligned}$$

And you can check it's same as what we have defined without choosing a basis.

Here we only give a sketch proof of the first statement, the proofs for second and third are similar, but more complicated.

We want to detect the value of a section at a point is zero or not. Sheaves give us a good way to describe such thing. For  $x \in X$ , consider ideal sheaf of  $x$

$$\mathcal{I}_x = \{s \in \mathcal{O}_X \mid s(x) = 0\} \subseteq \mathcal{O}_X$$

Then sections we are searching for is  $H^0(X, \mathcal{I}_x \otimes L^{\otimes m})$ . For computation, we have exact sequence of sheaf

$$0 \rightarrow \mathcal{I}_x \otimes L^{\otimes m} \rightarrow L^{\otimes m} \rightarrow L^{\otimes m}|_x \rightarrow 0$$

And using Čech cohomology we can derive a long exact sequence

$$0 \rightarrow H^0(X, \mathcal{I}_x \otimes L^{\otimes m}) \rightarrow H^0(X, L^{\otimes m}) \rightarrow \mathbb{C} \rightarrow \check{H}^1(X, \mathcal{I}_x \otimes L^{\otimes m}) \rightarrow \dots$$

Our goal is to show  $H^0(X, \mathcal{I}_x \otimes L^{\otimes m}) \neq 0$ . If  $\check{H}^1(X, \mathcal{I}_x \otimes L^{\otimes m}) = 0$  for sufficiently large  $m$ , then we can get desired result.

For Čech cohomology we know a little, but we know quite a lot for Dolbeault cohomology. So an ideal is to turn idea sheaf into a line bundle and use Dolbeault cohomology to compute.

Similarly, we have

$$H^0(X, \mathcal{I}_x \otimes L^{\otimes m}) = H^0(\tilde{X}, \mathcal{I}_E \otimes \pi^* L^{\otimes m}) = H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(E)^* \otimes \pi^* L^{\otimes m})$$

And that's why blow-up works since  $\mathcal{I}_x \otimes L^{\otimes m}$  is just a sheaf, and it's a little difficult for us to compute the cohomology of sheaf, but after blow-up,

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<sup>10</sup>Larger than  $m$  is step one.

we make it to a line bundle  $\mathcal{O}_{\tilde{X}}(E)^* \otimes \pi^* L^{\otimes m}$ , and Dolbeault cohomology comes into its place.

Consider the following short exact sequence of sheaves on  $\tilde{X}$ :

$$0 \rightarrow \mathcal{I}_E \otimes \pi^* L^{\otimes m} \rightarrow \pi^* L^{\otimes m} \rightarrow \pi^* L^{\otimes m}|_E \rightarrow 0$$

So we get a long exact sequence

$$0 \rightarrow H^0(\tilde{X}, \mathcal{I}_E \otimes \pi^* L^{\otimes m}) \rightarrow H^0(\tilde{X}, \pi^* L^{\otimes m}) \rightarrow \mathbb{C} \rightarrow \check{H}^1(\tilde{X}, \mathcal{I}_E \otimes \pi^* L^{\otimes m}) \rightarrow \dots$$

But

$$\check{H}^1(\tilde{X}, \mathcal{I}_E \otimes \pi^* L^{\otimes m}) = H_{Dol}^1(\tilde{H}, \mathcal{I}_E \otimes \pi^* L^{\otimes m})$$

Claim  $H_{Dol}^1(\tilde{H}, \mathcal{I}_E \otimes \pi^* L^{\otimes m}) = 0$ , when  $m$  is sufficiently large. Indeed, note that

$$\mathcal{I}_E \otimes \pi^* L^{\otimes m} \cong K_{\tilde{X}} \otimes \{\mathcal{O}_{\tilde{X}}(E)^{* \otimes n} \otimes \pi^*(K_X^* \otimes L^{\otimes m})\}$$

So by Kodaira vanishing, it suffices to show the following line bundle is positive when  $m$  is sufficiently large:

$$\mathcal{O}_{\tilde{X}}(E)^{* \otimes n} \otimes \pi^*(K_X^* \otimes L^{\otimes m})$$

In fact,  $K_X^* \otimes L^{\otimes m}$  will be positive on  $X$  when  $m$  is sufficiently large. But when we pull it back something bad may happen since  $\pi^*(K_X^* \otimes L^{\otimes m})$  is positive except along  $E$ . However,  $\mathcal{O}_{\tilde{X}}(E)^{* \otimes n}|_E = \mathcal{O}_{\mathbb{CP}^{n-1}}(n)$ , so two parts work together to give a positive line bundle. To be more explicit, take any Hermitian metric on  $\mathcal{O}_{\tilde{X}}(E)^{* \otimes n}$  extending (Fubini-Study) $^{\otimes n}$ , then its positive on  $E$ , but may not be positive otherwise. However we can choose  $m$  sufficiently large to offset its negative impact.

This completes the proof of first part of Kodaira embedding, for second and third, arguments are similar, but we need more blow-ups and things become complicated.  $\square$

**Corollary 16.2.1.** Let  $(X, \omega)$  be a compact Kähler manifold such that  $H^{2,0}(X) = H^{0,2}(X) = 0$ . Then  $X$  is a projective manifold.

*Proof.* Hodge decomposition implies that  $H^{2,0}(X) = H^{0,2}(X) = 0$ , and thus  $H^2(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = H^2(X, \mathbb{C}) = H^{1,1}(X)$ . Let  $[\alpha_1], \dots, [\alpha_n] \in H^2(X, \mathbb{Q})$  be a basis such that  $\alpha_i$  is harmonic and of type  $(1, 1)$ . Since the Kähler form  $\omega$  is real, harmonic<sup>11</sup> and of type  $(1, 1)$ . Then

$$\omega = \sum_i \lambda_i \alpha_i, \quad \lambda_i \in \mathbb{R}$$

For  $\mu_i \in \mathbb{Q}$  sufficiently close to  $\lambda_i$ , one still has  $\sum_i \mu_i \alpha_i$  is positive. Thus  $\sum_i \mu_i \alpha_i$  gives a Kähler form. Take  $N$  sufficiently large such that  $[N \sum_i \mu_i \alpha_i] \in \text{im}(H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{R}))$ . Applying Kodaira embedding to complete the proof.  $\square$

**Corollary 16.2.2.** Fano manifold is projective.

*Proof.* Since for Fano manifold, all  $H^{0,p}(X) = 0, q > 0$ .  $\square$

<sup>11</sup>It's harmonic since  $[\Delta_d, L] = 0$ .

**16.3. Chow's theorem.** Since we already embed a compact complex manifold into projective space, there is no reason for us to avoid Chow's theorem. A wonderful theorem lies in the intersection of algebraic and analytic.

**Theorem 16.3.1** (Chow). Every closed complex submanifold  $X \subseteq \mathbb{CP}^n$  is algebraic, that is, defined by polynomial equations.

*Remark 16.3.1.* Finally, for holomorphic line on compact complex manifolds: positive is equivalent to ample.

Although Chow's theorem can be derived from GAGA proved by Serre, in an elegant way using sheaf theory, here we give a sketch of a classical proof of Chow's theorem.

We need to show every closed complex submanifold  $X$  of  $\mathbb{CP}^n$  is algebraic, our ideal is to construct an analytic hypersurface in a Grassmannian manifold  $Gr(r, n)$  with Plücker embedding  $\varphi: Gr(r, n) \hookrightarrow \mathbb{CP}(\bigwedge^r \mathbb{C}^n)$ , and use some facts about it:

- (1)  $\text{Pic}(Gr(r, n)) = \mathbb{Z} \cdot \varphi^* \mathcal{O}_{\mathbb{CP}}(1)$
- (2) Every closed analytic hypersurface of  $Gr(r, n)$  is algebraic.

If the analytic hypersurface  $W$  we construct in Grassmannian manifold can determine  $X$  algebraically, that is  $W$  is algebraic implies  $X$  is, then we complete the proof.

The philosophy here is to convert a submanifold with arbitrary codimension in  $\mathbb{CP}^n$  to a hypersurface, the cost we pay is that we need to consider Grassmannian manifold rather than  $\mathbb{CP}^n$ . But it do works!

Let's be more explicit:

**Definition 16.3.1** (analytic subset). A closed analytic subset in  $\mathbb{CP}^n$  is a closed subset, locally defined by some holomorphic equations.

*Remark 16.3.2.* We can replace closed complex submanifold by closed analytic subset in Chow's theorem since we can not avoid singularity, and it doesn't matter in fact.

However, although we allow singularities, singularities won't be too much: Let  $X \subseteq \mathbb{CP}^n$  be an irreducible closed analytic subset of dimension  $r$ . Then there exists closed analytic subset  $X_{\text{sing}} \subseteq X$  such that  $X \setminus X_{\text{sing}}$  is smooth and dense. Furthermore,  $X \setminus X_{\text{sing}}$  is a submanifold of  $\mathbb{CP}^n$  of dimension  $r$ .

Now fix  $X$ , an irreducible closed analytic subset of dimension  $r$  in  $\mathbb{CP}^n$ . Let  $V \in Gr(n-r, n+1)$ , then  $\mathbb{CP}(V) \subseteq \mathbb{CP}^n$  with dimension  $n-r-1$ . So as you can imagine, an object with dimension  $r$  and an object with dimension  $n-r-1$  may fail to intersect with each other.

Let

$$W = \{V \in Gr(n-r, n+1) \mid \mathbb{CP}(V) \cap X \neq \emptyset\}$$

Claim:

- (1)  $W$  is a closed analytic hypersurface of  $Gr(n-r, n+1)$ .
- (2)  $W$  determines  $X$  algebraically.

Here we give a sketch of proof of Claims:

For (1). Consider the following diagram

$$\begin{array}{ccc} \mathbb{CP}(E) & \xrightarrow{q} & \mathbb{CP}^n \supset X \\ \downarrow p & & \\ Gr(n-r, n+1) & & \end{array}$$

where  $E \rightarrow Gr(n-r, n+1)$  is tautological bundle of Grassmannian manifold. Then we can write<sup>12</sup>

$$W = p(q^{-1}(X))$$

Since  $q$  is holomorphic and  $X$  is closed and analytic, then  $q^{-1}(X)$  is also closed and analytic. But the difficulty is  $p(q^{-1}(X))$  is also closed and analytic, and this holds from the following fact.

**Theorem 16.3.2.**  $p$  is holomorphic and proper<sup>13</sup>.

Now we show that  $W$  is a hypersurface, and that's just a computation for dimension: We already know the dimension of  $Gr(n-r, n+1)$  is  $(n-r)(n+1-(n-r)) = (n-r)(r+1)$ , so we need to show the dimension of  $W$  is  $(n-r)(r+1)-1$ .

First, let's consider the fibre of  $q$ : it consists of subspaces  $V \subseteq \mathbb{C}^{n+1}$  of dimension  $n-r$  containing a given line  $l$ , and that's another Grassmannian manifold  $Gr(n-r-1, n)$ , if we consider  $V \mapsto V/l$ , and its dimension is  $(n-r-1)(r+1)$ . So the dimension of  $q^{-1}(X)$  is  $r + (n-r-1)(r+1) = (n-r)(r+1) - 1 = \dim Gr(n-r, n+1) - 1$ .

So we may desire the property of  $p$  is not too bad so that we will obtain  $\dim p(q^{-1}(X)) = \dim q^{-1}(X)$  as we desired. It suffices to show that there exists a dense open subset  $U \subseteq q^{-1}(X)$ , such that  $p|_U$  has finite fibres. In fact, we will show it's one to one correspondence.

Consider fibre of  $p_X : q^{-1}(X) \rightarrow Gr(n-r, n+1)$  over given  $V \in Gr(n-r, n+1)$ , and that's  $\mathbb{CP}(V) \cap X$ . So we may desire almost every  $V$  such that this intersection is just a point. It suffices to show that the complement of

$$\{(V, x) \in q^{-1}(X) \mid \mathbb{CP}(V) \cap X \text{ has only one smooth point } x\}$$

is closed analytic of dimension less than  $\dim q^{-1}(X)$ . There are three cases:

- (1)  $\mathbb{CP}(V) \cap X$  contains  $x \in X_{\text{sing}}$ , singular locus of  $X$ . But  $\dim_{\mathbb{C}} q^{-1}(X_{\text{sing}}) < \dim_{\mathbb{C}} q^{-1}(X)$ .
- (2)  $\mathbb{CP}(V) \cap X$  has at least two points.
- (3)  $\mathbb{CP}(V) \cap X$  not transverse intersection at  $x$ .

Then  $W = p(q^{-1}(X)) \subseteq Gr(n-r, n+1)$  is a closed analytic hypersurface.

For (2). Consider  $q_W : p^{-1}(W) \rightarrow \mathbb{CP}^n$ . Claim

$$X = \{x \in \mathbb{CP}^n \mid q_W^{-1}(x) = q^{-1}(x)\}$$

<sup>12</sup>Why?

<sup>13</sup>Proper means: For any  $Y \subseteq \mathbb{CP}(E)$  closed and analytic, then  $p(Y)$  is closed and analytic.

And there are some equivalent descriptions:

$$\begin{aligned} q_W^{-1}(x) = q^{-1}(x) &\iff p^{-1}(W) \cap q^{-1}(x) = q^{-1}(x) \\ &\iff q^{-1}(x) \subseteq p^{-1}(W) \end{aligned}$$

Clearly, if  $x \in X$ , then  $q^{-1}(x) \in p^{-1}(W)$  since  $W = p(q^{-1}(X))$ . For the other direction: we can translate it as: If  $y \notin X$ , then we need to find  $V \in Gr(n-r, n+1)$  containing  $l = \langle y \rangle$ , but  $\mathbb{CP}(V) \cap X = \emptyset$ .

To see this: Use projection from  $y$ , that is  $\mathbb{CP}^n \xrightarrow{\pi_y} \mathbb{CP}^{n-1}$ . Since  $y \notin X$ , then  $\pi_y|_X$  is well-defined and has finite fibres. Note that  $\mathbb{CP}(V) \cap X \neq \emptyset$  if and only if  $\mathbb{CP}(V/\langle y \rangle) \cap \pi_y(X) = \emptyset$ . From computation before, we know it's a condition for hypersurface. So it's easy to choose  $V$  we desire.

Conclusion (from analytic to algebraic):  $p$  and  $q$  is algebraic, and  $W$  is algebraic, so we obtain  $p^{-1}(W)$  is also algebraic. So  $q_W$  is also algebraic. Thus  $X$  is algebraic.

## 17. MORE BOCHNER TECHNIQUES

**17.1. Obstruction to holomorphic vector fields.** In Riemannian geometry, there is the following theorem which shows there are obstructions of Killing vector fields or harmonic 1-forms.

**Theorem 17.1.1.** Let  $(M, g)$  be a compact Riemannian manifold.

- (1) If  $\text{Ric}(g) < 0$ , then there is no non-trivial Killing vector field.
- (2) If  $\text{Ric}(g) > 0$ , then  $b_1(M) = 0$ .

In the realm of complex geometry, analogous obstructions exist, drawing a parallel between Killing fields and holomorphic vector fields. However, it is important to note that not every Killing vector field is a holomorphic vector field.

**Theorem 17.1.2.** On a compact Kähler manifold, a Killing vector field  $X$  is holomorphic if and only if

- (1)  $\text{div} X = 0$ .
- (2)  $\nabla^* X = 0$ .

**Lemma 17.1.1.** Let  $(M, g)$  be a compact Riemannian manifold,  $X$  is Killing if and only if  $\text{div} X = 0$  and  $\nabla^* \nabla X = (\text{Ric}(X))^\sharp$ .

**Lemma 17.1.2.** Let  $(M, \omega)$  be a compact Kähler manifold. A vector field  $X$  is holomorphic if and only if  $\nabla^* \nabla X = (\text{Ric}(X))^\sharp$ .

**Theorem 17.1.3.** Let  $(M, \omega)$  be a compact Kähler manifold with  $\text{Ric}(\omega) < 0$ . Then there is no non-trivial holomorphic vector field.

*Proof.* Let  $X$  be a non-trivial holomorphic vector field locally given by  $X = X^i \frac{\partial}{\partial z^i}$  and  $\omega = \sqrt{-1} h_{i\bar{j}} dz^i \wedge d\bar{z}^j$ . If we define  $u = |X|^2$ , a direct computation shows

$$\begin{aligned} \sqrt{-1} \partial \bar{\partial} u &= \sqrt{-1} (\partial \{ \nabla^{0,1} X, X \} + \{ X, \nabla^{1,0} X \}) \\ &\stackrel{(1)}{=} \sqrt{-1} \partial \{ X, \nabla^{1,0} X \} \\ &= \sqrt{-1} (\{ \nabla^{1,0} X, \nabla^{1,0} X \} + \{ X, \nabla^{0,1} \nabla^{1,0} X \}) \\ &\stackrel{(2)}{=} \sqrt{-1} (\{ \nabla X, \nabla X \} + \{ X, \Theta_h X \}) \end{aligned}$$

where (1) and (2) holds from  $X$  is a holomorphic vector field. Note that

$$\begin{aligned} \text{tr}_\omega \sqrt{-1} \{ \nabla X, \nabla X \} &= \text{tr}_\omega (\sqrt{-1} \nabla_i X^j \overline{\nabla_k X^l} h_{j\bar{l}} dz^i \wedge d\bar{z}^k) \\ &= \nabla_i X^j \overline{\nabla_k X^l} h_{j\bar{l}} h^{i\bar{k}} \\ &= |\nabla X|^2 \\ \text{tr}_\omega \sqrt{-1} \{ X, \Theta_h X \} &= -\text{tr}_\omega \{ \sqrt{-1} \Theta_h X, X \} \\ &= -\text{tr}_\omega (\sqrt{-1} \Theta_{i\bar{j}\alpha}^\beta X^\alpha \overline{X^\gamma} h_{\beta\bar{\gamma}} dz^i \wedge d\bar{z}^j) \\ &= -\Theta_{i\bar{j}\alpha\bar{\gamma}} X^\alpha \overline{X^\gamma} \\ &= -\text{Ric}(X, X) \end{aligned}$$

This shows

$$\mathrm{tr}_\omega \sqrt{-1} \partial \bar{\partial} u = |\nabla X|^2 - \mathrm{Ric}(X, X) > 0$$

If  $u$  obtains its maximum at some point  $p \in M$ , then  $\mathrm{tr}_\omega \sqrt{-1} \partial \bar{\partial} u(p) \leq 0$ , a contradiction.  $\square$

**Corollary 17.1.1.** Let  $(M, \omega)$  be a compact Kähler manifold. If  $\mathrm{Ric}(\omega) \leq 0$ , then the following statements are equivalent:

- (1)  $X$  is parallel.
- (2)  $X$  is Killing.
- (3)  $X$  is holomorphic.

*Proof.* As shown in Theorem 17.1.3, if  $\mathrm{Ric}(\omega) \leq 0$ , for a non-trivial holomorphic vector field  $X$  one has

$$\mathrm{tr}_\omega \sqrt{-1} \partial \bar{\partial} u = |\nabla X|^2 - \mathrm{Ric}(X, X) \geq 0$$

Then by the same argument one has  $\nabla X = 0$ , that is every holomorphic vector field is parallel, and it's clear every parallel vector field is Killing.  $\square$

**Corollary 17.1.2.** Let  $(M, \omega)$  be a compact complex manifold with negative holomorphic sectional curvature. Then there is no non-trivial holomorphic vector field.

*Proof.* Let  $X$  be a non-trivial holomorphic vector field locally given by  $X = X^i \frac{\partial}{\partial z^i}$  and  $u = |X|^2$ . The same argument shows

$$\sqrt{-1} \partial \bar{\partial} u(X, X) = |\nabla_X X|^2 - \Theta_{i\bar{j}k\bar{l}} X^i \bar{X}^j X^k \bar{X}^l$$

Then maximum principle completes the proof.  $\square$

## 17.2. Obstruction to holomorphic 1-form.

**Theorem 17.2.1.** Let  $(M, \omega)$  be a compact Kähler manifold. If  $\mathrm{Ric}(\omega) > 0$ . Then there is no non-trivial holomorphic 1-form.

**Corollary 17.2.1.** Let  $(M, \omega)$  be compact Kähler manifold and  $\mathrm{Ric}(\omega) \geq 0$ . Then any holomorphic 1-form is parallel.

**17.3. Rigidity of complex projective space.** The rigidity of complex projective space is a fascinating subject in complex geometry. It involves determining whether a geometric object satisfies certain conditions to be  $\mathbb{CP}^n$  or not, and it has produced numerous interesting results. Yau demonstrated that any Kähler manifold, which is homeomorphic to  $\mathbb{CP}^n$ , is also biholomorphic to  $\mathbb{CP}^n$ . He also resolved a conjecture put forth by Frankel in 1961, which stated that a compact Kähler manifold with positive bisectional curvature must be biholomorphic to  $\mathbb{CP}^n$ . Additionally, there exists an even stronger result.

**Theorem 17.3.1** ([FLW17]). Let  $(X, \omega)$  be a compact Kähler manifold with positive orthogonal bisectional curvature. Then  $X$  is biholomorphic to  $\mathbb{CP}^n$ .



In this section we try to use Bochner technique to show the following easier result.

**Proposition 17.3.1.** Let  $(X, \omega)$  be a compact Kähler manifold with positive orthogonal bisectional curvature. Then  $H^{1,1}(X) = \mathbb{C}$ .

## 18. SCHWARZ LEMMAS

In this section, we will present the powerful formulas called Schwarz calculation, which extend the Bochner techniques discussed earlier.

## 18.1. Schwarz lemmas for holomorphic bisectional curvature.

**Lemma 18.1.1.** Let  $f: (M, g) \rightarrow (N, h)$  be a holomorphic map between Hermitian manifolds and  $\nabla$  be the induced connection on  $T^*X \otimes f^*(TN)$  by Chern connections. Then in the local holomorphic coordinates  $\{z^i\}$  and  $\{w^\alpha\}$  on  $M$  and  $N$  respectively, one has

$$\partial\bar{\partial}u = \{\nabla df, \nabla df\} + \left( (\Theta_g)_{i\bar{j}k\bar{l}} g^{k\bar{q}} g^{p\bar{l}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} - (\Theta_h)_{\alpha\bar{\beta}\gamma\bar{\delta}} (f_i^\alpha \overline{f_j^\beta}) (g^{p\bar{q}} f_p^\gamma \overline{f_q^\delta}) \right) dz^i \wedge d\bar{z}^j$$

and

$$\mathrm{tr}_\omega \sqrt{-1} \partial\bar{\partial}u = |\nabla df|^2 + (g^{i\bar{j}} (\Theta_g)_{i\bar{j}k\bar{l}}) g^{k\bar{q}} g^{p\bar{l}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} - (\Theta_h)_{\alpha\bar{\beta}\gamma\bar{\delta}} (g^{i\bar{j}} f_i^\alpha \overline{f_j^\beta}) (g^{p\bar{q}} f_p^\gamma \overline{f_q^\delta})$$

where  $u = \mathrm{tr}_g(f^*\omega_h) = |\nabla f|^2$ ,  $f = f_i^\alpha dz^i \otimes e_\alpha$  and  $e_\alpha = f^*(\frac{\partial}{\partial w^\alpha})$ . Furthermore, one has

$$\mathrm{tr}_\omega \sqrt{-1} \partial\bar{\partial} \log u \geq \frac{1}{u} \left( (g^{i\bar{j}} (\Theta_g)_{i\bar{j}k\bar{l}}) g^{k\bar{q}} g^{p\bar{l}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} - (\Theta_h)_{\alpha\bar{\beta}\gamma\bar{\delta}} (g^{i\bar{j}} f_i^\alpha \overline{f_j^\beta}) (g^{p\bar{q}} f_p^\gamma \overline{f_q^\delta}) \right)$$

holds outside the set of critical points of  $f$ .

**Corollary 18.1.1.** Let  $g, h$  be two Hermitian metrics on a complex manifold  $M$ . Then

$$\mathrm{tr}_g \sqrt{-1} \partial\bar{\partial} \log \mathrm{tr}_g \omega_h \geq \frac{1}{\mathrm{tr}_g \omega_h} \left( (g^{i\bar{j}} (\Theta_g)_{i\bar{j}k\bar{l}}) g^{k\bar{q}} g^{p\bar{l}} h_{p\bar{q}} - (\Theta_h)_{i\bar{j}p\bar{q}} g^{i\bar{j}} g^{p\bar{q}} \right)$$

and

$$\mathrm{tr}_h \sqrt{-1} \partial\bar{\partial} \log \mathrm{tr}_g \omega_h \geq \frac{1}{\mathrm{tr}_g \omega_h} \left( (h^{i\bar{j}} (\Theta_g)_{i\bar{j}k\bar{l}}) g^{k\bar{q}} g^{p\bar{l}} h_{p\bar{q}} - (h^{i\bar{j}} (\Theta_h)_{i\bar{j}p\bar{q}}) g^{p\bar{q}} \right)$$

**Corollary 18.1.2.** Let  $f: (M, g) \rightarrow (N, h)$  be a holomorphic map between Hermitian manifolds. Suppose

- (1) the second Chern-Ricci curvature  $\mathrm{Ric}^{(2)}(\omega_g) \geq a\omega_g$  for some  $a$ ;
- (2) the holomorphic bisectional curvature  $(N, h)$  is bounded from above by  $b$ , that is

$$\Theta_h(w, \bar{w}, v, \bar{v}) \leq b|w|^2|v|^2$$

Then

$$\mathrm{tr}_g \sqrt{-1} \partial\bar{\partial}u \geq au - bu^2$$

where  $u = \mathrm{tr}_g(f^*\omega_h) = |\nabla f|^2$ . Furthermore, outside the set of critical points of  $f$ , one has

$$\mathrm{tr}_g \sqrt{-1} \partial\bar{\partial} \log u \geq a - bu$$

*Proof.* Note that  $\text{Ric}^{(2)}(\omega_g) \geq a\omega_g$  implies  $(\Theta_g)_{i\bar{j}k\bar{l}}g^{k\bar{l}} \geq ag_{i\bar{j}}$ . Thus

$$\begin{aligned} (g^{i\bar{j}}(\Theta_g)_{i\bar{j}k\bar{l}})g^{k\bar{q}}g^{p\bar{l}}h_{\alpha\bar{\beta}}f_p^\alpha\overline{f_q^\beta} &= g^{i\bar{j}}(\Theta_g)_{i\bar{j}k\bar{l}}\frac{g^{k\bar{l}}g_{k\bar{l}}}{n}g^{k\bar{q}}g^{p\bar{l}}h_{\alpha\bar{\beta}}f_p^\alpha\overline{f_q^\beta} \\ &\geq \frac{ag^{i\bar{j}}g_{i\bar{j}}}{n}g_{k\bar{l}}g^{k\bar{q}}g^{p\bar{l}}h_{\alpha\bar{\beta}}f_p^\alpha\overline{f_q^\beta} \\ &= ag^{p\bar{l}}f_p^\alpha\overline{f_l^\beta} \\ &= au \end{aligned}$$

On the other hand, by using normal coordinate on  $M$  it's easy to see

$$(\Theta_h)_{\alpha\bar{\beta}\gamma\bar{\delta}}(g^{i\bar{j}}f_i^\alpha\overline{f_j^\beta})(g^{p\bar{q}}f_p^\gamma\overline{f_q^\delta}) \leq b(f_i^\alpha\overline{f_j^\beta}g^{i\bar{j}}h_{\alpha\bar{\beta}})(f_p^\gamma\overline{f_q^\delta}g^{p\bar{q}}h_{\gamma\bar{\delta}}) = bu^2$$

Then by Lemma 18.1.1 this completes the proof.  $\square$

**Lemma 18.1.2** ([Yau75]). Let  $(M, g)$  be a complete Riemannian manifold with Ricci curvature bounded from below and  $f \in C^2(M, \mathbb{R})$  be bounded from above. Then there exists  $\{p_k\} \subseteq M$  such that

- (1)  $\lim_k |\nabla f(p_k)| = 0$ .
- (2)  $\limsup_k \Delta f(p_k) \leq 0$ .
- (3)  $\lim_k f(p_k) = \sup f$ .

**Theorem 18.1.1** ([Yau78a]). Let  $(M, \omega_g)$  be a complete Kähler manifold with  $\text{Ric}(\omega_g) \geq a\omega_g$  and  $(N, \omega_h)$  be a Hermitian manifold with holomorphic bisectional curvature  $\leq b < 0$ . If  $f: M \rightarrow N$  is a non-constant holomorphic map, then  $a < 0$  and

$$f^*\omega_h \leq \frac{a}{b}\omega_g$$

*Proof.* Let  $\Delta_g$  denotes the operator  $\text{tr}_g \sqrt{-1}\partial\bar{\partial}$ . By Corollary 18.1.2 one has

$$\Delta_g u \geq au - bu^2$$

where  $u = \text{tr}_g(f^*\omega_h)$ . It suffices to show  $\sup_M u \leq a/b$ . Now let's consider the following two cases:

- (1) If  $\sup_M u < \infty$ , then by Lemma 18.1.2 there exists a sequence  $\{p_k\}$  such that  $\limsup \Delta u(p_k) \leq 0$ , and  $\lim u(p_k) = \sup_M u$  and so

$$\Delta_g u(p_k) \geq au(p_k) - bu^2(p_k)$$

By taking  $\limsup$ , we deduce that

$$0 \geq a \sup_M u - b(\sup_M u)^2$$

Since  $\sup_M u > 0$  and  $b < 0$ , one has  $a < 0$  and  $\sup_M u \leq a/b$ .

- (2) If  $\sup_M u = \infty$ , consider

$$v = \frac{1}{\sqrt{u+c}}$$

for some constant  $c > 0$ . Direct computation shows

$$\begin{aligned}\Delta_g v &= -\frac{1}{2}v^3\Delta_g u + \frac{3}{v}|\nabla v|^2 \\ \Delta_g u &= \frac{2}{v^3}\left(\frac{3}{v}|\nabla v|^2 - \Delta_g v\right) \geq au - bu^2\end{aligned}$$

that is

$$6|\nabla v|^2 - 2v\Delta_g v \geq auv^4 - bu^2v^4$$

By Lemma 18.1.2 again there exists a sequence  $\{p_k\}$  such that  $\lim_k(-v)(p_k) = \sup_M(-v) = 0$  and  $\nabla v(p_k) = 0$  and

$$\limsup_k \Delta_g v(p_k) \leq 0$$

Then

$$0 \geq \limsup(6|\nabla v|^2 - 2v\Delta_g v)(p_k) \geq \limsup(auv^4 - bu^2v^4)(p_k) = -b$$

which is a contradiction to  $b > 0$ .

In a word, only the first case will happen, and this completes the proof.  $\square$

**Corollary 18.1.3.** Let  $f: (\mathbb{D}, \omega) \rightarrow (\mathbb{D}, \omega)$  be a holomorphic map between unit disk with Poincaré metric. Then

$$f^*(\omega) \leq \omega$$

*Proof.* It's clear since unit disk with Poincaré metric has constant holomorphic bisectional curvature  $-1$  and  $\text{Ric}(\omega) = -\omega$ .  $\square$

**Corollary 18.1.4.** Let  $f: (M, g) \rightarrow (N, h)$  be a holomorphic map between two Hermitian manifolds. If

- (1)  $(M, g)$  is complete Kähler with  $\text{Ric}(g) \geq 0$ .
- (2)  $(N, h)$  has negative holomorphic bisectional curvature.

Then  $f$  is constant.

*Proof.* Suppose the holomorphic bisectional curvature of  $N$  is bounded above by  $b < 0$ . If there exists a non-constant holomorphic map from  $(M, g)$  to  $(N, h)$ , then it contradicts to  $\text{Ric}(\omega_g) \geq 0$  since by Theorem 18.1.1 one has if  $\text{Ric}(\omega_g) \geq a\omega_g$ , then  $a < 0$ .  $\square$

**Corollary 18.1.5.** Let  $f: (M, g) \rightarrow (N, h)$  be a holomorphic map between two Hermitian manifolds such that  $M$  is compact with  $\text{Ric}^{(2)}(\omega_g) \geq 0$  and  $N$  has non-positive holomorphic bisectional curvature. If one of the following statements holds, then  $f$  is a constant.

- (1)  $\text{Ric}^{(2)}(\omega_g) > 0$  at some point.
- (2)  $N$  has negative holomorphic bisectional curvature at some point.

*Proof.* Let  $\omega_G$  be a Gauduchon metric of  $M$  and  $u = \text{tr}_g(f^*\omega_h)$ . By Lemma 18.1.1 the integration

$$\int_M \text{tr}_\omega \sqrt{-1} \partial \bar{\partial} u \cdot \omega_G^n = 0$$

implies  $\nabla df = 0$  and

$$(g^{i\bar{j}}(\Theta_g)_{i\bar{j}k\bar{l}})g^{k\bar{q}}g^{p\bar{l}}h_{\alpha\bar{\beta}}f_p^\alpha\overline{f_q^\beta} = (\Theta_h)_{\alpha\bar{\beta}\gamma\bar{\delta}}(g^{i\bar{j}}f_i^\alpha\overline{f_j^\beta})(g^{p\bar{q}}f_p^\gamma\overline{f_q^\delta}) = 0$$

Hence  $|df|$  is a constant, and it's clear from the second equation that (1) or (2) implies  $df = 0$ , that is  $f$  is a constant.  $\square$

## 18.2. Schwarz lemmas for holomorphic sectional curvature.

**Lemma 18.2.1.** There is the following identity

$$\int_{\mathbb{CP}^{n-1}} \frac{\xi^i \bar{\xi}^j \xi^k \bar{\xi}^l}{|\xi|^4} \omega_{FS}^{n-1} = \frac{\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj}}{n(n+1)}$$

where  $[\xi^1 : \dots : \xi^n]$  are homogenous coordinates on  $\mathbb{CP}^{n-1}$  and  $\omega_{FS}$  is the Fubini-Study metric.

**Theorem 18.2.1.** Let  $(M, g)$  be a Hermitian  $n$ -manifold,  $(N, h)$  be a Kähler  $n$ -manifold and  $f: (M, g) \rightarrow (N, h)$  be a non-constant holomorphic map. Suppose that

- (1)  $\text{Ric}^{(2)}(g) \geq -\lambda\omega_g + \mu f^*\omega_h$  for continuous functions  $\lambda, \mu$  with  $\mu \geq 0$ .
- (2) holomorphic sectional curvature of  $h$  is bounded from above by a continuous functions  $-\kappa \leq 0$ .

Then

$$\text{tr}_g \sqrt{-1} \partial \bar{\partial} u \geq -\lambda u + \left( \frac{(r+1)f^*\kappa}{2r} + \frac{\mu}{n} \right) u^2$$

where  $r$  is maximal rank of  $df$  and  $u = \text{tr}_g(f^*\omega_h)$ . Furthermore, outside the critical points of  $f$  one has

$$\text{tr}_g \sqrt{-1} \partial \bar{\partial} \log u \geq -\lambda + \left( \frac{(r+1)f^*\kappa}{2r} + \frac{\mu}{n} \right) u$$

*Proof.* Let  $\Delta_g$  denote the operator  $= \text{tr}_g \sqrt{-1} \partial \bar{\partial}$ . Then as computation in Lemma 18.1.1 one has

$$\Delta_g u = |\nabla df|^2 + \underbrace{(g^{i\bar{j}}(\Theta_g)_{i\bar{j}k\bar{l}})g^{k\bar{q}}g^{p\bar{l}}h_{\alpha\bar{\beta}}f_p^\alpha\overline{f_q^\beta}}_I - \underbrace{(\Theta_h)_{\alpha\bar{\beta}\gamma\bar{\delta}}(g^{i\bar{j}}f_i^\alpha\overline{f_j^\beta})(g^{p\bar{q}}f_p^\gamma\overline{f_q^\delta})}_{II}$$

For part  $I$ , one has

$$\begin{aligned} I &\geq g^{i\bar{j}}(\Theta_h)_{i\bar{j}k\bar{l}} \frac{g^{k\bar{l}}g_{k\bar{l}}}{n} g^{k\bar{q}}g^{p\bar{l}}h_{\alpha\bar{\beta}}f_p^\alpha\overline{f_q^\beta} \\ &= (-\lambda g_{k\bar{l}} + \mu g^{i\bar{j}}g_{k\bar{l}}h_{\gamma\bar{\delta}}f_i^\gamma\overline{f_j^\delta})g^{k\bar{q}}g^{p\bar{l}}h_{\alpha\bar{\beta}}f_p^\alpha\overline{f_q^\beta} \\ &= (-\lambda g_{k\bar{l}} + \mu h_{\gamma\bar{\delta}}f_k^\gamma\overline{f_l^\delta})g^{k\bar{q}}g^{p\bar{l}}h_{\alpha\bar{\beta}}f_p^\alpha\overline{f_q^\beta} \\ &= -\lambda\mu + \frac{\mu}{n}\mu^2 \end{aligned}$$

For part *II*, by taking normal coordinate at  $p \in M$  and  $f(p) \in N$  we may assume  $f_i^\alpha = \lambda_i \delta_i^\alpha$  with  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{r(p)} > \lambda_{r(p)+1} = 0$ , where  $r(p)$  is rank of  $df(p)$ . Then  $\text{tr}_g(f^* \omega_h) = \sum_{i=1}^n \lambda_i^2$ . Hence

$$\begin{aligned}
II &= (\Theta_h)_{\alpha\bar{\beta}\gamma\bar{\delta}}(g^{i\bar{j}} f_i^\alpha \bar{f}_j^\beta)(g^{p\bar{q}} f_p^\gamma \bar{f}_q^\delta) \\
&= \sum_{i,k=1}^n (\Theta_h)_{\alpha\bar{\beta}\gamma\bar{\delta}} f_i^\alpha \bar{f}_i^\beta f_k^\gamma \bar{f}_k^\delta \\
&\stackrel{(a)}{=} \sum_{i,j,k,l=1}^n (\Theta_h)_{\alpha\bar{\beta}\gamma\bar{\delta}} f_i^\alpha \bar{f}_j^\beta f_k^\gamma \bar{f}_l^\delta \left( \frac{\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj}}{2} \right) \\
&\stackrel{(b)}{=} \frac{n(n+1)}{2} (\Theta_h)_{\alpha\bar{\beta}\gamma\bar{\delta}} f_i^\alpha \bar{f}_j^\beta f_k^\gamma \bar{f}_l^\delta \int_{\mathbb{CP}^{n-1}} \frac{\xi^i \bar{\xi}^j \xi^k \bar{\xi}^l}{|\xi|^4} \omega_{FS}^{n-1} \\
&= \frac{n(n+1)}{2} \int_{\mathbb{CP}^{n-1}} (\Theta_h)_{\alpha\bar{\beta}\gamma\bar{\delta}} \frac{(f_i^\alpha \xi^i)(\bar{f}_j^\beta \bar{\xi}^j)(f_k^\gamma \xi^k)(\bar{f}_l^\delta \bar{\xi}^l)}{|\xi|^4} \omega_{FS}^{n-1} \\
&\stackrel{(c)}{\leq} -\frac{\kappa(f(p))n(n+1)}{2} \int_{\mathbb{CP}^{n-1}} \frac{(|f_i^\alpha \xi^i|^2)^2}{|\xi|^4} \omega_{FS}^{n-1}
\end{aligned}$$

where

(a) holds from  $(N, h)$  is Kähler.

(b) holds from Lemma 18.2.1.

(c) holds from holomorphic sectional curvature of  $h$  is bounded from above by  $-\kappa$ .

Since  $f_i^\alpha = \lambda_i \delta_i^\alpha$ , one has

$$\begin{aligned}
\int_{\mathbb{CP}^{n-1}} \frac{(|f_i^\alpha \xi^i|^2)^2}{|\xi|^4} \omega_{FS}^{n-1} &= \frac{1}{n(n+1)} \sum_{i,j,\alpha,\beta} \lambda_i \lambda_j \lambda_i \lambda_j \delta_i^\alpha \delta_j^\alpha \delta_k^\beta \delta_l^\beta (\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj}) \\
&= \frac{1}{n(n+1)} \left( \left( \sum_{\alpha} \lambda_{\alpha}^2 \right)^2 + \sum_{\alpha} \lambda_{\alpha}^4 \right) \\
&\stackrel{(d)}{\geq} \frac{1}{n(n+1)} \cdot \frac{r+1}{r} \left( \sum_{\alpha} \lambda_{\alpha}^2 \right)^2
\end{aligned}$$

where (d) holds since  $r$  is the maximal numbers of non-zero elements of  $\lambda_{\alpha}$ . Hence one has

$$II \leq -\frac{(r+1)f^* \kappa}{2r} \left( \sum_{\alpha} \lambda_{\alpha}^2 \right)^2 = -\frac{(r+1)f^* \kappa}{2r} u^2$$

This completes the proof.  $\square$

**Corollary 18.2.1.** Let  $(M, \omega)$  be a complete Kähler manifold with Ricci curvature bounded from below by a positive constant. Then  $M$  is compact, and there is no non-trivial holomorphic map from  $M$  into a Kähler manifold with non-positive holomorphic sectional curvature.

**Corollary 18.2.2.** Let  $(N, h)$  be a Hermitian manifold with non-positive holomorphic sectional curvature. Then any holomorphic map from  $\mathbb{CP}^n$  to  $N$  is constant. In particular,  $N$  contains no rational curves.

**Corollary 18.2.3.** Let  $(N, h)$  be a Hermitian manifold with negative holomorphic sectional curvature. Then any holomorphic map from torus  $\mathbb{T}^2$  to  $N$  is constant.

## Part 5. Existence of Kähler-Einstein metric

### 19. CALABI-YAU THEOREM

**19.1. Introduction.** The investigation into the existence of Kähler-Einstein metrics is a compelling and extensive topic that traces back to 1954. In which year Calabi proposed the famous Calabi conjecture, which was finally solved by Yau in 1976. Firstly, Calabi proved the uniqueness of the solution and laid out the program of proving the existence by the method of continuity and also pointed out the openness and the need of a priori estimates in [Cal57]. The very easy a priori zeroth order estimate for the case of negative first Chern class was firstly given by Aubin in [Aub78], but he did not apply his a priori estimate to the continuity method. Instead he used the method of variation which is rather difficult to comprehend.

Yau made the important contribution of using Moser's method of integration by parts and iteration by Sobolev inequality to get a priori zeroth order estimate.

**Theorem 19.1** ([MR054],[Yau78b]). Let  $(X, \omega_g)$  be a compact Kähler manifold. If  $\Omega$  is a real  $(1, 1)$ -form which represents  $2\pi c_1(X)$ , then there exists a unique metric  $\omega \in [\omega_g]$  such that  $\text{Ric}(\omega) = \Omega$ .

This remarkable result establishes several related results which are of fundamental importance in the study of complex manifolds.

**Corollary 19.1.1.** Let  $(X, \omega_g)$  be a compact Kähler manifold with  $c_1(X) = 0$ . Then there exists a unique Ricci flat metric.

**Corollary 19.1.2.** Let  $(X, \omega_g)$  be a compact Kähler manifold with  $c_1(X) > 0$ . Then  $M$  is simply-connected.

**Corollary 19.1.3** ([Yau77]). Every complex surface which is homotopic equivalent to  $\mathbb{CP}^2$  is biholomorphic to  $\mathbb{CP}^2$ .

### 19.2. The Monge-Ampère equation and priori estimates.

**19.2.1. The reformulation of Calabi conjecture in Monge-Ampère equation.** The Calabi conjecture can be reduced to a problem of fully non-linear partial differential equations. By Lemma 12.3.1, that is  $\partial\bar{\partial}$ -lemma, one has

$$\omega = \omega_g + \sqrt{-1}\partial\bar{\partial}\varphi$$

where  $\varphi$  is the smooth function we desire. Again by  $\partial\bar{\partial}$ -lemma one also has

$$\text{Ric}(\omega_g) = \Omega + \sqrt{-1}\partial\bar{\partial}F$$

where  $F$  is a smooth function which is unique up to a constant. If we consider the normalization

$$\int_X e^F \omega_g^n = \int_X \omega_g^n$$



then  $F$  is unique. Suppose  $\text{Ric}(\omega) = \Omega$ . Then

$$\text{Ric}(\omega_g) - \text{Ric}(\omega) = \sqrt{-1}\partial\bar{\partial}F$$

that is

$$\sqrt{-1}\partial\bar{\partial}\log\frac{\omega^n}{\omega_g^n} = \sqrt{-1}\partial\bar{\partial}F$$

which is equivalent to the following equation

$$\omega^n = e^{F+C}\omega_g^n$$

and by normalization of  $F$  one has  $C = 0$ . Thus in order to solve Calabi conjecture, it suffices to solve the following complex Monge-Ampère equation:

$$(19.1) \quad \begin{cases} (\omega_g + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^F\omega_g^n \\ \int_X e^F\omega_g^n = 1 \end{cases}$$

**Theorem 19.2.1** ([Cal57]). The solution of (19.1) is unique.

*Proof.* Suppose  $\varphi_1, \varphi_2$  are two solutions. Then

$$\begin{aligned} 0 &= \omega_1^n - \omega_2^n = (\omega_1 - \omega_2)(\omega_1^{n-1} + \cdots + \omega_2^{n-1}) \\ &= \sqrt{-1}\partial\bar{\partial}(\varphi_1 - \varphi_2)(\omega_1^{n-1} + \cdots + \omega_2^{n-1}) \end{aligned}$$

If we define  $\psi = \varphi_1 - \varphi_2$ , then

$$\begin{aligned} -\int_X \psi \sqrt{-1}\partial\bar{\partial}\psi \wedge (\omega_1^{n-1} + \cdots + \omega_2^{n-1}) &\stackrel{(1)}{=} \int_X \partial\psi \wedge \bar{\partial}\psi \wedge (\omega_1^{n-1} + \cdots + \omega_2^{n-1}) \\ &\stackrel{(2)}{\geq} \frac{1}{n} |\partial\psi|^2 \omega_1^n \end{aligned}$$

where

(1) holds from integration by parts and  $\omega_1, \omega_2$  are Kähler forms.

(2) holds from (2) of Proposition 8.3.3 and positivity of  $\omega_1, \omega_2$ .

This shows  $\partial\psi = 0$ , and thus  $\psi$  is a constant, which completes the proof.  $\square$

To solve the existence of solution of (19.1), Yau used the continuity method. Consider a sequence of equations as follows

$$(19.2) \quad \det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \frac{e^{tF}}{\int_X e^{tF}\omega_g^n} \det(g_{i\bar{j}})$$

It's clear (19.2) is solvable at  $t = 0$ , and thus consider the following set is non-empty

$$I = \{t \in [0, 1] \mid \text{equation (19.2) is solvable}\}$$

The openness of  $I$  can be relatively easily demonstrated through the application of the inverse function theorem. However, the true challenge lies in establishing its closedness. To accomplish this, a series of rigorous a priori estimates are essential.

19.2.2.  $C^0$ -estimate.

**Theorem 19.2.2** ( $C^0$ -estimate). Suppose  $\varphi$  is a solution of (19.1). Then there exists constant  $C$  depending on  $X, \omega_g, F$  such that

$$\sup \varphi - \inf \varphi \leq C$$

*Proof.* Without lose of generality we may assume  $\sup \varphi = -1$ . Then Green formula says

$$\varphi(x) = \frac{1}{V} \int \varphi(y) \omega_g^n - \frac{1}{V} \int G(x, y) \Delta \varphi(y) \omega_g^n$$

and  $\omega - \omega_g = \sqrt{-1} \partial \bar{\partial} \varphi$  implies

$$\mathrm{tr}_g \sqrt{-1} \partial \bar{\partial} \varphi = \mathrm{tr}_g \omega - n \geq n$$

Assume  $G \geq 0$ , if  $\varphi(p) = -1$ , then

$$\begin{aligned} -1 &= \varphi(p) \\ &= \frac{1}{V} \int \varphi(y) \omega_g^n - \frac{1}{V} \int G(p, y) \Delta \varphi(y) \omega_g^n \\ &\leq \frac{1}{V} \int \varphi(y) \omega_g^n + \frac{1}{V} \int n G(p, y) \omega_g^n \end{aligned}$$

this shows

$$\|\varphi\|_{L^1} = - \int \varphi \omega_g^n \leq C_1$$

For  $L_2$ -estimate, note that

$$\begin{aligned} \int \varphi (e^F - 1) \omega_g^n &= \int \varphi (\omega^n - \omega) \\ &= \int \varphi (\sqrt{-1} \partial \bar{\partial} \varphi) (\omega^{n-1} + \dots + \omega_g^{n-1}) \\ &\leq -\frac{1}{n} \int |\partial \varphi|^2 \omega_g^n \end{aligned}$$

that is

$$\int |\partial \varphi|^2 \omega_g^n \leq n \sup |e^F - 1| \int |\varphi| \omega_g^n \leq C_2$$

that is  $\|\nabla \varphi\|_{L^2}^2 \leq C_3$ , and by Poincaré inequality one has

$$\|\varphi - \bar{\varphi}\|_{L^2} \leq C_4 \|\nabla \varphi\|_{L^2}$$

that is  $\|\varphi\|_{L^2} \leq C_5$ .

For  $p \geq 2$ , set  $\psi = -\varphi$ , one has

$$\begin{aligned}
 \int \psi^{p-1}(e^F - 1)\omega_g^n &= - \int \psi^{p-1}\sqrt{-1}\partial\bar{\partial}\psi \wedge (\omega^{n-1} + \dots + \omega_g^{n-1}) \\
 &= \int (p-1)\psi^{p-2}\sqrt{-1}\partial\psi \wedge \bar{\partial}\psi \wedge (\omega^{n-1} + \dots + \omega_g^{n-1}) \\
 &\geq \int (p-1)\psi^{p-2}\sqrt{-1}\partial\psi \wedge \bar{\partial}\psi \wedge \omega_g^{n-1} \\
 &= \frac{4(p-1)}{p^2} \int \sqrt{-1}\partial\psi^{\frac{p}{2}} \wedge \bar{\partial}\psi^{\frac{p}{2}} \wedge \omega_g^{n-1} \\
 &= \frac{4(p-1)}{np^2} \int |\partial\psi^{\frac{p}{2}}|^2 \omega_g^n
 \end{aligned}$$

This gives

$$\int |\partial\psi^{\frac{p}{2}}|^2 \omega_g^n \leq C_6 p \int |\varphi|^{p-1} \omega_g^n$$

that is

$$\|\nabla\psi\|_{L^2}^2 \leq C_6 p \|\psi\|_{L^{p-1}}^{p-1}$$

The Sobolev inequality shows for all  $f \in W^{1,q}(X, \omega_g)$ , one has

$$\|f\|_{L^{\frac{2nq}{2n-q}}} \leq C_7 \|f\|_{W^{1,q}}$$

Set  $f = \psi^{\frac{p}{2}}$  and  $q = 2$ , one has

$$\begin{aligned}
 \left( \int \psi^{\frac{np}{n-1}} \omega_g^n \right)^{\frac{n-1}{n}} &= \|\psi\|_{L^{\frac{np}{n-1}}}^p = \|\psi\|_{L^{\frac{2n}{n-1}}}^2 \\
 &\leq C_8 \left( \int |\nabla\psi^{\frac{p}{2}}|^2 \omega_g^n + \int \psi \omega_g^n \right) \\
 &\leq C_9 (p \|\psi\|_{L^{p-1}}^{p-1} + \|\psi\|_{L^p}^p) \\
 &\leq C_{10} p \|\psi\|_{L^p}^p
 \end{aligned}$$

This gives

$$\|\psi\|_{L^{\frac{np}{n-1}}} \leq C_{10}^{\frac{1}{p}} p^{\frac{1}{p}} \|\psi\|_{L^p}$$

Set  $p_k = (\frac{n}{n-1})^k p$ , then

$$\|\psi\|_{L^{p_{k+1}}} \leq (C_{10} p_k)^{\frac{1}{p_k}} \|\psi\|_{L^{p_k}} \leq \prod_{j=0}^k (C_{10} p_j)^{\frac{1}{p_j}} \|\psi\|_{L^p}$$

This gives

$$\|\psi\|_{L^\infty} \leq C \|\psi\|_{L^p}$$

□

19.2.3.  $C^2$ -estimate.

**Theorem 19.2.3.** Suppose  $\omega_h$  is a solution of (19.1). Then there exists a constant  $c$  depending on  $X, \omega_g, F$  such that

$$c^{-1}\omega_g \leq \omega_h \leq c\omega_g$$

*Proof.* It suffices to prove  $\text{tr}_g \omega_h \leq c$  since linear algebra yields the following inequality

$$\text{tr}_h \omega_g \leq \frac{1}{n-1} (\text{tr}_g \omega_h)^{n-1} \frac{\omega_g^n}{\omega_h^n}$$

By Corollary 18.1.1, that is Schwarz computation, one has

$$\Delta_h \log \text{tr}_g \omega_h \geq \frac{1}{\text{tr}_g \omega_h} \left( h^{i\bar{j}}(\Theta_g)_{i\bar{j}k\bar{l}} g^{k\bar{q}} g^{p\bar{l}} h_{p\bar{q}} - h^{i\bar{j}}(\Theta_h)_{i\bar{j}p\bar{q}} g^{p\bar{q}} \right)$$

where  $\Delta_h = \text{tr}_h \sqrt{-1} \partial \bar{\partial}$ . Since  $(X, g)$  is a given Kähler manifold, one has its curvature is bounded from below by  $-B$  as follows

$$(\Theta_g)_{i\bar{j}k\bar{l}} \geq -B(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}})$$

Then

$$\begin{aligned} h^{i\bar{j}}(\Theta_g)_{i\bar{j}k\bar{l}} g^{k\bar{q}} g^{p\bar{l}} h_{p\bar{q}} &\geq -B(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}) g^{k\bar{q}} g^{p\bar{l}} h^{i\bar{j}} h_{p\bar{q}} \\ &= -B(1 + \frac{1}{n}) \delta_i^p \delta_j^q h^{i\bar{j}} h_{p\bar{q}} \\ &= -(n+1)B \end{aligned}$$

On the other hand, one has

$$h^{i\bar{j}}(\Theta_h)_{i\bar{j}p\bar{q}} g^{p\bar{q}} = \left( g^{i\bar{j}}(\Theta_g)_{i\bar{j}p\bar{q}} - F_{p\bar{q}} \right) g^{p\bar{q}} = s - F_{p\bar{q}} g^{p\bar{q}}$$

where  $\text{Ric}(\omega_g) = \Omega + \sqrt{-1} \partial \bar{\partial} F$  and  $s$  is the scalar curvature of  $(X, g)$ . Then by  $\text{tr}_h \omega_g \text{tr}_g \omega_h \geq n^2$  one has

$$\begin{aligned} \Delta_h \log \text{tr}_g \omega_h &\geq -\frac{1}{n^2} ((n+1)B + c_0) \text{tr}_h \omega_g \\ &\geq -2B \text{tr}_h \omega_g - \frac{c_0}{n^2} \text{tr}_g \omega_h \end{aligned}$$

where  $c_0 = (s - F_{p\bar{q}} g^{p\bar{q}})$ . Note that

$$\Delta_h \varphi = h^{j\bar{k}} \varphi_{j\bar{k}} = h^{j\bar{k}} (h_{j\bar{k}} - g_{j\bar{k}}) = n - \text{tr}_h \omega_g$$

Then there exists an appropriate  $\lambda$  such that

$$\Delta_h (\log \text{tr}_g \omega_h + \lambda \varphi) \geq \text{tr}_h \omega_g - C_1$$

If  $\log \text{tr}_g \omega_h + \lambda \varphi$  obtain its maximum at  $p \in X$ , then  $\text{tr}_h \omega_g(p) \leq C_1$ , and again by trick of linear algebra one has

$$\text{tr}_g \omega_h(p) \leq C_2$$

for some constant  $C_2$ . Since  $p$  is the point such that  $\log \text{tr}_g \omega_h + \lambda \varphi$  obtains its maximum, then by  $C^0$ -estimate there exists some constant  $C$  such that

$$\text{tr}_g \omega_h \leq C$$

This completes the proof.  $\square$

19.2.4.  *$C^3$ -estimate.* Let  $(\Gamma_g)_{ij}^k$  and  $(\Gamma_h)_{ij}^k$  denote the Christoffel symbols of  $g$  and  $h$  respectively, and set

$$S_{ij}^k = (\Gamma_h)_{ij}^k - (\Gamma_g)_{ij}^k$$

Suppose we have the following  $C^2$ -estimate

$$c^{-1}\omega_g \leq \omega_h \leq c\omega_g$$

**Lemma 19.2.1.** There exists a constant  $C$  depending on  $X, \omega_g, F, c$  such that

$$\Delta_h |S|_h^2 \geq -C |S|_h^2 - C$$

**Lemma 19.2.2.** There exists a constant  $C$  depending on  $X, \omega_g, F, c$  such that

$$|S| < C$$

### 19.3. Proof of Calabi-Yau theorem.

**Theorem 19.3.1.** Let  $(X, \omega_g)$  be a compact Kähler manifold. Then for any  $k \geq 3$  and  $F \in C^k(X, \mathbb{R})$ , the complex Monge-Ampère equation

$$\begin{cases} \omega_h^n = e^F \omega_g^n \\ \omega_h = \omega_g + \sqrt{-1} \partial \bar{\partial} \varphi \\ \int_X e^F \omega_g^n = \int_X \omega_g^n \end{cases}$$

has a solution  $\varphi \in C^{k+1, \alpha}(X)$ .

### 19.4. Aubin-Yau theorem.

19.4.1. *Uniqueness of Kähler-Einstein metric when  $c_1(X) < 0$ .*

**Lemma 19.4.1.** Let  $X$  be a compact complex manifold. Then there exists at most one Kähler metric  $\omega$  such that  $\text{Ric}(\omega) = -\omega$ .

*Proof.* Suppose that there are two Kähler metrics  $\omega_1$  and  $\omega_2$  such that

$$\text{Ric}(\omega_1) = -\omega_1$$

$$\text{Ric}(\omega_2) = -\omega_2$$

By  $\partial\bar{\partial}$ -lemma, there exists  $\varphi \in C^\infty(X, \mathbb{R})$  such that  $\omega_1 = \omega_2 + \sqrt{-1} \partial \bar{\partial} \varphi$ , so one has

$$\sqrt{-1} \partial \bar{\partial} \varphi = \sqrt{-1} \partial \bar{\partial} \log \frac{\omega_1^n}{\omega_2^n}$$

In other words, there exists a constant  $c$  such that

$$\omega_1^n = e^{\varphi+c} \omega_2^n$$

Suppose  $\varphi$  attains its maximum at point  $p$ , that is,  $\sqrt{-1} \partial \bar{\partial} \varphi(p) \leq 0$ . Then

$$\omega_1 = \omega_2 + \sqrt{-1} \partial \bar{\partial} \varphi \leq \omega_2$$

which implies  $e^{\varphi(p)+c} \leq 1$ . This shows  $\varphi + c \leq \varphi(p) + c \leq 0$ . Similarly, if  $\varphi$  attains its minimum at  $q$ , then  $e^{\varphi(q)+c} \geq 1$  and  $\varphi + c \geq \varphi(q) + c \geq 0$ . Hence  $\varphi + c \equiv 0$ , and thus  $\omega_1 = \omega_2$ .  $\square$

#### 19.4.2. The reformulation of Aubin-Yau theorem in Monge-Ampère equation.

**Theorem 19.4.1** (Aubin-Yau). Let  $X$  be a complex manifold with  $c_1(X) < 0$ . Then there exists a unique Kähler metric  $\omega \in 2\pi c_1(X)$  such that  $\text{Ric}(\omega) = -\omega$ .

*Proof.* Let  $\omega_g$  be any Kähler metric in  $[-2\pi c_1(X)]$ . Then by  $\partial\bar{\partial}$ -lemma there exists a smooth function  $F$  such that

$$\text{Ric}(\omega_g) = -\omega_g + \sqrt{-1}\partial\bar{\partial}F$$

since  $[\text{Ric}(\omega_g)] = 2\pi c_1(X)$ . Let  $\omega$  be another Kähler metric such that  $[\omega] = [\omega_g]$ . Again by  $\partial\bar{\partial}$ -lemma there exists a smooth function  $\varphi$  such that

$$\omega = \omega_g + \sqrt{-1}\partial\bar{\partial}\varphi$$

Then  $\text{Ric}(\omega) = -\omega$  is equivalent to

$$\sqrt{-1}\partial\bar{\partial} \log \frac{\omega^n}{\omega_g^n} = \sqrt{-1}\partial\bar{\partial}(F + \varphi)$$

or in other words, there exists a constant  $C$  such that

$$\omega^n = e^{F+\varphi+C}\omega_g^n$$

By rescaling  $F$  we may assume  $C = 0$ , so Aubin-Yau is equivalent to solve the following complex Monge-Ampère equation:

$$(19.3) \quad \begin{cases} (\omega_g + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{F+\varphi}\omega_g^n \\ \int_X e^{F+\varphi}\omega_g^n = 1 \end{cases}$$

$\square$

#### 19.4.3. $C^0$ -estimate.

**Theorem 19.4.2** ( $C^0$ -estimate). Suppose  $\varphi$  is a solution of (19.3). Then

$$\sup_X |\varphi| \leq \sup_X |F|$$

*Proof.* As argument in Lemma 19.4.1, at maximum point  $p$  of  $\varphi$ , one has  $\varphi + F(p) \leq \varphi(p) + F(p) \leq 0$ , so

$$\sup_X \varphi \leq -F(p) \leq \sup_X |F|$$

Similarly, at minimum point  $q$  of  $\varphi$ ,  $\varphi + F(q) \geq \varphi(q) + F(q) \geq 0$  and

$$\inf_X \varphi \geq -F(q) \geq -\sup_X |F|$$

Hence  $\sup_X |\varphi| \leq \sup_X |F|$ .  $\square$

19.4.4.  $C^2$ -estimate.

**Theorem 19.4.3** ( $C^2$ -estimate). There exists a uniform constant  $C$  depending on  $X, \omega_g, F$  such that

$$C^{-1}\omega_g \leq \omega \leq C\omega_g$$

*Proof.* The same as Theorem 19.2.3.  $\square$

19.4.5.  $C^3$ -estimate. Given the  $C^0$ -estimate and  $C^2$ -estimate, the  $C^3$ -estimate is very similar to that in the proof of Calabi-Yau theorem. Here we formulate a general setup. Let  $(\Gamma_g)_{ij}^k$  and  $(\Gamma_h)_{ij}^k$  denote the Christoffel symbols of  $g$  and  $h$  respectively, and set

$$S_{ij}^k = (\Gamma_h)_{ij}^k - (\Gamma_g)_{ij}^k$$

Suppose we have the following  $C^2$ -estimate

$$c^{-1}\omega_g \leq \omega_h \leq c\omega_g$$

**Theorem 19.4.4** ( $C^3$ -estimate).

19.4.6. *Proof of Aubin-Yau theorem.*

## Part 6. Deformations of complex structure

### 20. DEFORMATIONS OF COMPLEX STRUCTURE

**20.1. The Maurer-Cartan equation.** Recall that a complex structure on a smooth manifold  $M$  is encoded by an integrable almost complex structure  $J$ , and two complex manifolds  $(M, J)$  and  $(M', J')$  are isomorphic if there exists a diffeomorphism  $F: M \rightarrow M'$  such that  $dF \circ J = J' \circ dF$ . Thus the set of all complex structures on a fixed smooth manifold  $M$  is the quotient of the set

$$\mathcal{A}_c(M) := \{J \mid J \text{ is an integrable almost complex structure on } M\}$$

of all complex structures by the action of the diffeomorphism group. Firstly let's consider the set

$$\mathcal{A}_{ac}(M) := \{J \mid J \text{ is an almost complex structure on } M\}$$

For arbitrary almost complex structure  $J \in \mathcal{A}_{ac}(M)$ , it's uniquely determined by a decomposition of the complexified tangent bundle  $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$  with  $J$  is  $\sqrt{-1} \text{ id}$  on  $T^{1,0}M$  and  $-\sqrt{-1} \text{ id}$  on  $T^{0,1}M$ . In fact, giving  $T^{0,1}M \subseteq T_{\mathbb{C}}M$  is enough, since we can set  $T^{1,0}M = \overline{T^{0,1}M}$ . If  $J(t)$  is a continuous family of almost complex structure with  $J(0) = J$ , there is a continuous family of such decompositions  $T_{\mathbb{C}}M = T_t^{1,0}M \oplus T_t^{0,1}M$ , or equivalently, of subspaces  $T_t^{0,1}M \subseteq T_{\mathbb{C}}M$ .

Thus for small  $t$  the deformations  $J(t)$  of  $J$  gives a map

$$\phi(t): T^{0,1}M \rightarrow T^{1,0}M$$

with  $v + \phi(t)v \in T_t^{1,0}M$ . Conversely, if  $\phi(t): T^{0,1}M \rightarrow T^{1,0}M$  is given, then one defines for small  $t$

$$T_t^{1,0}M := (\text{id} + \phi(t))(T^{0,1}M)$$

Here the condition  $t$  to be “small” has to be imposed in order to ensure that with this definition  $T_t^{1,0}M \subseteq T_{\mathbb{C}}M \rightarrow T^{1,0}M$  is an isomorphism. Thus deformations of almost complex structure is encoded by such a map  $\phi(t)$ . Now for convenience we assume  $J$  is an integrable almost complex, and thus we denote  $(M, J)$  by  $X$ . In particular, there is  $\bar{\partial}$  operator on the holomorphic tangent bundle  $TX$ , which can be applied to  $\phi(t) \in C^\infty(X, \Omega_X^{0,1} \otimes T^{1,0}X)$ .

The following proposition shows that the integrability condition for deformation of the almost complex structure can be rephrased in terms of  $\phi(t)$ .

**Proposition 20.1.1.** The integrability condition is equivalent to the Maurer-Cartan equation

$$\bar{\partial}\phi(t) + [\phi(t), \phi(t)] = 0$$



*Proof.* It suffices to compute locally so we write  $\phi(t)$  with respect to local coordinate  $\{z^1, \dots, z^n\}$  as follows

$$\phi(t) = \phi_i^j(t) d\bar{z}^i \otimes \frac{\partial}{\partial z_j}$$

The integrability condition implies  $[T_t^{0,1}X, T_t^{0,1}X] \subseteq T_t^{0,1}X$ , so in particular one has

$$\left[ \frac{\partial}{\partial \bar{z}_i} + \phi(t) \left( \frac{\partial}{\partial \bar{z}_i} \right), \frac{\partial}{\partial \bar{z}_j} + \phi(t) \left( \frac{\partial}{\partial \bar{z}_j} \right) \right] \in T_t^{0,1}X$$

A direct computation yields

$$\left( \left[ \frac{\partial}{\partial \bar{z}^i}, \phi_k^l(t) \frac{\partial}{\partial z^l} \right] + \left[ \phi_i^j(t) \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k} \right] + \left[ \phi_i^j(t) \frac{\partial}{\partial z^j}, \phi_k^l(t) \frac{\partial}{\partial z^l} \right] \right) \in T_t^{0,1}X$$

Note that

$$\begin{aligned} \left[ \frac{\partial}{\partial \bar{z}^i}, \phi_k^l(t) \frac{\partial}{\partial z^l} \right] &= \frac{\partial \phi_k^l(t)}{\partial \bar{z}^i} \frac{\partial}{\partial z^l} \\ \left[ \phi_i^j(t) \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k} \right] &= -\frac{\partial \phi_i^j(t)}{\partial \bar{z}^k} \frac{\partial}{\partial z^j} \end{aligned}$$

This shows

$$\left[ \frac{\partial}{\partial \bar{z}^i}, \phi_k^l(t) \frac{\partial}{\partial z^l} \right] + \left[ \phi_i^j(t) \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k} \right] = \left( \frac{\partial \phi_k^j(t)}{\partial \bar{z}^i} - \frac{\partial \phi_i^j(t)}{\partial \bar{z}^k} \right) \frac{\partial}{\partial z^j} = \bar{\partial} \phi(t) \left( \frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^k} \right)$$

On the other hand, note that

$$[\phi(t), \phi(t)] = d\bar{z}^i \wedge d\bar{z}^k \left[ \phi_i^j(t) \frac{\partial}{\partial z^j}, \phi_k^l(t) \frac{\partial}{\partial z^l} \right]$$

Thus

$$\left[ \phi_i^j(t) \frac{\partial}{\partial z^j}, \phi_k^l(t) \frac{\partial}{\partial z^l} \right] = [\phi(t), \phi(t)] \left( \frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^k} \right)$$

So integrability condition implies

$$\bar{\partial}(t) + [\phi(t), \phi(t)] \left( \frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^k} \right) \in T_t^{0,1}X$$

This shows  $\bar{\partial}(t) + [\phi(t), \phi(t)]$  is a section of  $\Omega_X^{0,2} \otimes (T^{1,0}X \cap T_t^{0,1}X)$ , but for sufficiently small  $t$  one has  $T^{1,0}X \cap T_t^{0,1}X = 0$ . This shows

$$\bar{\partial}(t) + [\phi(t), \phi(t)] = 0$$

Conversely, if the Maurer-Cartan equation holds, then the integrability condition holds for a local frame of  $T_t^{0,1}X$ , and thus for all sections of  $T_t^{0,1}X$ .  $\square$

Now let's consider the power series expansion of a given deformation  $\phi(t)$  as follows

$$\phi(t) = \phi_0 + \phi_1 t + \phi_2 t^2 + \dots$$

The Maurer-Cartan equation gives

$$\bar{\partial} \left( \sum_{i=1}^{\infty} \phi_i t^i \right) + \sum_{i,j=1}^{\infty} [\phi_i, \phi_j] t^{i+j} = 0$$

This yields a recursive system of equations

$$\begin{aligned} 0 &= \bar{\partial} \phi_1 \\ 0 &= \bar{\partial} \phi_2 + [\phi_1, \phi_1] \\ &\vdots \\ 0 &= \bar{\partial} \phi_k + \sum_{0 < i < k} [\phi_i, \phi_{k-i}] \end{aligned}$$

In particular, the first-order deformation of the complex structure is determined by a  $\bar{\partial}$ -closed  $(0, 1)$ -form  $\phi_1$  with valued in  $TX$ . Thus it determines an element  $[\phi_1] \in H^1(X, TX)$ .

**Definition 20.1.1** (Kodaira-Spencer class). The Kodaira-Spencer class of a one-parameter deformation  $J(t)$  of a complex structure  $J$  is the induced cohomology class  $[\phi_1] \in H^1(X, TX)$ .

**Proposition 20.1.2.** Let  $X$  be a complex manifold. There is a natural bijection between all first-order deformations of  $X$  and elements of  $H^1(X, TX)$ .

**Corollary 20.1.1.** A first-order deformations  $v \in H^1(X, TX)$  cannot be integrated if  $[v, v] \in H^2(X, TX)$  does not vanish.

## Part 7. Appendix

### APPENDIX A. SHEAF AND ITS COHOMOLOGY

#### A.1. Basic definitions and properties.

**Definition A.1.1** (sheaf). Let  $X$  be a topological space. A sheaf of abelian group  $\mathcal{F}$  on  $X$  is the data of:

- (1) For any open subset  $U$  of  $X$ ,  $\mathcal{F}(U)$  is an abelian group.
- (2) If  $U \subseteq V$  are two open subsets of  $X$ , then there is a group homomorphism  $r_{UV}: \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ , such that
  - (a)  $\mathcal{F}(\emptyset) = 0$ .
  - (b)  $r_{UU} = \text{id}$ .
  - (c) If  $W \subseteq U \subseteq V$ , then  $r_{UW} = r_{VW} \circ r_{UV}$ .
  - (d)  $\{V_i\}$  is an open covering of  $U \subseteq X$ , and  $s \in \mathcal{F}(U)$ . If  $s|_{V_i} := r_{UV_i}(s) = 0, \forall i$ , then  $s = 0$ .
  - (e) If  $\{V_i\}$  is an open covering of  $U \subseteq X$ , and  $s_i \in \mathcal{F}(V_i)$  such that  $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ , then there exists  $s \in \mathcal{F}(U)$  such that  $s|_{V_i} = s_i$ .

**Definition A.1.2** (presheaf). A sheaf without (d) and (e) is called a presheaf.

**Example A.1.1** (constant presheaf). For an abelian group  $G$ , the constant presheaf assign each open set  $U$  the group  $G$  itself. However, it's not a sheaf. Indeed, consider  $U = U_1 \cup U_2$  with  $U_1 \cap U_2 = \emptyset$ . Consider  $g_1 \in \mathcal{F}(U_1) = G, g_2 \in \mathcal{F}(U_2), g_1 \neq g_2$ , one cannot find  $g \in \mathcal{F}(U)$  such that  $g|_{U_1} = g_1, g|_{U_2} = g_2$  since  $g_1 \neq g_2$ .

**Definition A.1.3** (morphism of sheaves).  $\phi: \mathcal{F} \rightarrow \mathcal{G}$  is called a morphism of sheaves, if for any open subset  $U$  of  $X$ , there is a group homomorphism  $\phi(U): \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  such that if  $U \subseteq V$  are two open subsets of  $X$ , then the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \downarrow r_{UV} & & \downarrow r_{UV} \\ \mathcal{F}(V) & \xrightarrow{\phi(V)} & \mathcal{G}(V) \end{array}$$

**Notation A.1.1.** For convenience, for  $s \in \mathcal{F}(U)$ , we often write  $\varphi(s)$  instead of  $\varphi(U)(s)$ .

**A.2. Sheafification.** In Example A.1.1, we come across a presheaf that is not a sheaf. To obtain a sheaf from a presheaf, we require a process known as sheafification. One approach to defining sheafification is through its universal property.

**Definition A.2.1** (sheafification). Given a presheaf  $\mathcal{F}$  there is a sheaf  $\mathcal{F}^+$  and a morphism  $\theta: \mathcal{F} \rightarrow \mathcal{F}^+$  with the property that for any sheaf  $\mathcal{G}$  and any morphism  $\varphi: \mathcal{F} \rightarrow \mathcal{G}$  there is a unique morphism  $\bar{\varphi}: \mathcal{F}^+ \rightarrow \mathcal{G}$  such that  $\varphi = \bar{\varphi} \circ \theta$ . In other words, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} \\
\downarrow \theta & \nearrow \bar{\varphi} & \\
\mathcal{F}^+ & & 
\end{array}$$

Although the universal property shows that if the sheafification exists, it's determined uniquely up to unique isomorphism, how can we show that there does exist a sheafification? To give an explicit construction, we need to consider stalks of a presheaf.

**Definition A.2.2** (stalks). For a presheaf  $\mathcal{F}$  and  $p \in X$ , the stalk at  $p$  is defined as

$$\mathcal{F}_p = \varinjlim_{p \in U} \mathcal{F}(U)$$

*Remark A.2.1* (alternative definition). In order to avoid language of direct limit, we give a more useful and equivalent description of stalk: For  $p \in U \cap V$ ,  $s_U \in \mathcal{F}(U)$  and  $s_V \in \mathcal{F}(V)$  are considered equivalent if there exists  $x \in W \subseteq U \cap V$  such that the restrictions of the two sections coincide on some smaller neighborhood of  $x$ . An element  $s_p \in \mathcal{F}_p$ , which is called a germ, is an equivalence class  $[s_U]$ , and for  $s \in \mathcal{F}(U)$ , the germ given by  $s$  is denoted by  $s|_p$ .

*Remark A.2.2* (morphisms on stalks). Given a morphism of sheaves  $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ , it induces a morphism of abelian groups  $\varphi_p: \mathcal{F}_p \rightarrow \mathcal{G}_p$  as follows:

$$\begin{aligned}
\varphi_p: \mathcal{F}_p &\rightarrow \mathcal{G}_p \\
s|_p &\mapsto \varphi(s)|_p
\end{aligned}$$

As you can imagine, stalks are quite local information, and the difference between sheaf and presheaf is that whether a local information can glue together uniquely or not. So stalks of presheaf and its sheafification should be the same. And one way to construct sheafification is to glue stalks together in a suitable way. Construct  $\mathcal{F}^+(U)$  as a set of functions

$$f: U \rightarrow \coprod_{p \in U} \mathcal{F}_p$$

such that  $f(p) \in \mathcal{F}_p$  and for every  $p \in U$  there is an open set  $V_p \subseteq U$  and  $t \in \mathcal{F}(V_p)$  such that for all  $q \in V_p$  we have the germ  $t|_q = f(q)$ .

**Proposition A.2.1.**  $\mathcal{F}^+$  is a sheaf.

*Proof.*

- (1) Let  $U$  be an open set,  $\{V_i\}$  an open covering of  $U$ , and  $s \in \mathcal{F}^+(U)$  such that  $s|_{V_i} = 0$  for all  $i$ , then  $s$  must be zero: It suffices to show  $s(p) = 0$  for all  $p \in U$ . Take any  $p \in U$ , then there exists an open set  $V_i$  contains  $p$ , hence  $s(p) = s|_{V_i}(p) = 0$ .

(2) Suppose for each  $i$ , we have  $s_i \in \mathcal{F}^+(V_i)$  such that

$$s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$$

We can construct  $s \in \mathcal{F}^+(U)$  such that  $s|_{V_i} = s_i$  directly: take any  $p \in U$  and  $V_i$  containing  $p$ , define  $s(p) = s_i(p)$ . This is well-defined since  $s_i$  agree on the intersections. All is left to check is that  $s$  satisfies the requirements of the sheafification. The first condition is trivial. For the second one, just consider that you can apply the condition to  $s_i$ , and this will give you an open neighborhood  $W_i$  contained in  $V_i$  and containing  $p$ , with  $t_i \in \mathcal{F}(W_i)$  as above. Since  $W_i$  is open in  $V_i$ , which is open in  $U$ , so  $W_i$  is suitable also for the function  $s$  we have just defined.  $\square$

*Remark A.2.3.* From this construction, you can see the stalk of  $\mathcal{F}^+$  at  $p$  is exactly  $\mathcal{F}_p$ . Check it by definition.

Now let's define the canonical morphism  $\theta: \mathcal{F} \rightarrow \mathcal{F}^+$  as follows: For open  $U \subseteq X$ , and  $s \in \mathcal{F}(U)$ , define

$$\begin{aligned} \theta(s) : U &\rightarrow \coprod_{p \in U} \mathcal{F}_p \\ p &\mapsto s|_p \end{aligned}$$

Note that if  $\mathcal{F}$  is already a sheaf, we desire canonical morphism  $\theta$  is an isomorphism. Indeed, if  $s_p = 0$  for all  $p \in U$ , so there exists an open covering  $\{V_i\}$  of  $U$  such that  $s|_{V_i} = 0$ , by axioms of sheaf we obtain  $s = 0$ , this is injectivity. For surjectivity: take  $f \in \mathcal{F}^+(U)$ . Since for each  $p \in U$  there exists  $p \in V_p \subseteq U$  and  $t \in \mathcal{F}(V_p)$  such that  $f(p) = t|_p$ , then glue these  $t$  together to get our desired  $s$  such that  $\theta(s) = f$ .

Finally let's construct  $\bar{\varphi}$ : A map of presheaves  $\varphi: \mathcal{F} \rightarrow \mathcal{G}$  induces a map on stalks

$$\varphi_p: \mathcal{F}_p \rightarrow \mathcal{G}_p$$

Thus for  $f \in \mathcal{F}^+(U)$ , we can compose  $f$  with the map

$$\coprod_{p \in U} \varphi_p: \coprod_{p \in U} \mathcal{F}_p \rightarrow \coprod_{p \in U} \mathcal{G}_p$$

to get a map  $U \rightarrow \coprod_{p \in U} \mathcal{G}_p$ . Thus we get a morphism  $\tilde{\varphi}: \mathcal{F}^+ \rightarrow \mathcal{G}^+$ . Indeed,  $\tilde{\varphi}(f)(p) \in \mathcal{G}_p$  since  $f(p) \in \mathcal{F}_p$  and  $\varphi_p: \mathcal{F}_p \rightarrow \mathcal{G}_p$ . If for all  $q \in V_p$  we have  $t|_q = f(q)$ , then

$$\tilde{\varphi}(f)(q) = \varphi_q(f(q)) = \varphi_q(t|_q) = \varphi(t)|_q$$

So  $\tilde{\varphi}(f) \in \mathcal{G}^+$ . Since  $\mathcal{G}$  is assumed to be sheaf, then canonical morphism  $\theta': \mathcal{G} \rightarrow \mathcal{G}^+$  is isomorphic, so we obtain  $\bar{\varphi} = \theta'^{-1} \circ \tilde{\varphi}$ . Now let's show  $\varphi = \bar{\varphi} \circ \theta = \theta'^{-1} \circ \tilde{\varphi} \circ \theta$ . It's suffice to show they coincide on each stalk since both  $\mathcal{F}^+$  and  $\mathcal{G}$  are sheaves, and it's quite easy to see this since  $\varphi_p = \theta_p'^{-1} \circ \tilde{\varphi}_p \circ \theta_p$ . Furthermore, uniqueness follows from the fact that  $\bar{\varphi}_p$  is uniquely determined by  $\varphi_p$ .

*Remark A.2.4.* We can describe sheafification in a more fancy language: Since we have sheaf of abelian groups on  $X$  as a category, denote it by  $\underline{Ab}_X$ , and presheaf is a full subcategory of  $\underline{Ab}_X$ , there is a natural inclusion functor  $\iota$  from category of sheaf to category of presheaf. Then sheafification is the adjoint functor of  $\iota$ .

### A.3. More examples on sheaves.

**Example A.3.1** (constant sheaf). Let  $G$  be abelian group. The associated constant sheaf  $\underline{G}$  is the sheafification of the constant presheaf. By the construction of sheafification, we can write  $\underline{G}$  more explicitly as

$$\underline{G}(U) = \{\text{locally constant function } f: U \rightarrow G\}$$

**Example A.3.2** (ringed space). A ringed space are data consisting of space and functions. For different spaces, we can define different functions:

- (1) For topological space  $X$ ,  $\mathcal{C}_X$  is defined by: For any open subset  $U$ , we define

$$\mathcal{C}_X(U) := \{\text{continuous functions } f: U \rightarrow \mathbb{R}\}$$

- (2) For smooth manifold  $M$ ,  $C_M^\infty$  is defined by: For any open subset  $U$ , we define

$$C_M^\infty(U) := \{\text{smooth functions } f: U \rightarrow \mathbb{R}\}$$

- (3) For complex manifold  $X$ ,  $\mathcal{O}_X$  is defined by: For any open subset  $U$ , we define

$$\mathcal{O}_X(U) := \{\text{holomorphic functions } f: U \rightarrow \mathbb{C}\}$$

- (4) For algebraic variety  $X$ ,  $\mathcal{O}_X$  is defined by: For any open subset  $U$ , we define

$$\mathcal{O}_X(U) := \{\text{regular functions on } U\}$$

**Example A.3.3** (sheaf of modules on a ringed space). Let  $(X, \mathcal{O}_X)$  be a ringed space. A sheaf of  $\mathcal{O}_X$ -module is a sheaf  $\mathcal{M}$  such that for any open  $U \subseteq X$ ,  $\mathcal{M}(U)$  is an  $\mathcal{O}_X(U)$ -module and the module structure is compatible with the restriction.

**Example A.3.4.** Let  $(X, \mathcal{O}_X)$  be a complex manifold and  $\pi: E \rightarrow M$  be a holomorphic vector bundle. Then global section of  $E$  gives a sheaf of  $\mathcal{O}_X$ -modules.

**A.4. Exact sequence of sheaf.** We can consider the kernel and cokernel if we already have a morphism of objects. So it's natural to define similar conceptions for morphisms of sheaves.

Given a morphism  $\varphi: \mathcal{F} \rightarrow \mathcal{G}$  between sheaves of abelian groups, we define its kernel  $\ker \varphi$  as a presheaf by assigning each open subset  $U$  the abelian group  $\ker \varphi(U)$  since  $\varphi(U)$  is a morphism of abelian groups.

Similarly, we can define its image or cokernel as a presheaf by assigning each open subset  $U$  the abelian group  $\text{im } \varphi(U)$  or  $\text{coker } \varphi(U)$ .

It's natural to ask whether the kernel, image or cokernel of morphism  $\varphi$  between sheaves are still sheaves or not? Unfortunately, only kernel of  $\varphi$  is still a sheaf, its image or cokernel may fail to be sheaf in our definition.

Why kernel of a morphism  $\varphi: \mathcal{F} \rightarrow \mathcal{G}$  is still a sheaf? Let's check by definition: Take  $s \in \ker \varphi(U)$ , and take an open covering  $\{V_i\}_{i \in I}$  of  $U$ . Then  $s|_{V_i} = 0$  must imply  $s = 0$  since  $s$  is also in  $\mathcal{F}(U)$ . If there exists  $s_i \in \ker \varphi(V_i)$  such that  $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ , so they glue together to get  $s \in \mathcal{F}(U)$ , and we need to check  $\varphi(U)(s) = 0$ . But we have

$$\varphi(U)(s)|_{V_i} = \varphi(V_i)(s|_{V_i}) = \varphi(V_i)(s_i) = 0$$

Then we obtain  $\varphi(U)(s) = 0$ .

But image of morphism may not be a sheaf: Although we can prove the first requirement in a same way, for the second something bad happens. If there exists  $s_i \in \operatorname{im} \varphi(V_i)$ , and we can glue them together to get a  $s \in \mathcal{G}(U)$ , but  $s$  may not be the image of some  $t \in \mathcal{F}(U)$ . The cokernel fails to be a sheaf for the same reason.

So we may change our definition about image and cokernel: To define the image and cokernel of a morphism to be the sheafification of our previous definition.

For a sequence of sheaves:

$$\dots \rightarrow \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^i \xrightarrow{\varphi^i} \mathcal{F}^{i+1} \rightarrow \dots$$

It's called exact at  $\mathcal{F}^i$ , if  $\ker \varphi^i = \operatorname{im} \varphi^{i-1}$ . If a sequence is exact at everywhere, then it's an exact sequence of sheaves.

However, there is a better description of exactness of sequence of sheaves, that is looking its stalks:

**Proposition A.4.1.** The sequence of sheaves

$$\dots \rightarrow \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^i \xrightarrow{\varphi^i} \mathcal{F}^{i+1} \rightarrow \dots$$

is exact if and only if the sequence of abelian groups are exact

$$\dots \rightarrow \mathcal{F}_x^{i-1} \xrightarrow{\varphi_x^{i-1}} \mathcal{F}_x^i \xrightarrow{\varphi_x^i} \mathcal{F}_x^{i+1} \rightarrow \dots$$

for all  $x \in X$ .

*Proof.* It suffices to show for any morphism  $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ , we have  $(\ker \varphi)_p = \ker \varphi_p$ ,  $(\operatorname{im} \varphi)_p = \operatorname{im} \varphi_p$ . Let's fix  $p \in X$  and check by definition.

For (1). It's clear  $(\ker \varphi)_p \subseteq \ker \varphi_p$ . Conversely, take  $s_p \in \ker \varphi_p$ , then  $\varphi_p(s_p) = 0 \in \mathcal{G}_p$ . In other words, there exists an open set  $U$  containing  $p$  and  $s \in \mathcal{F}(U)$  such that  $s|_p = s_p$  and  $\varphi(s)|_p = 0$ , which implies there is an open set  $V$  containing  $p$  such that  $\varphi(s)|_V = 0$ . Hence  $\varphi(s|_V) = 0$ , that is  $s|_V \in \ker \varphi(V)$ . Thus  $s_p = (s|_V)|_p \in (\ker \varphi)_p$ .

For (2). It's clear  $(\operatorname{im} \varphi)_p \subseteq \operatorname{im} \varphi_p$  since  $(\operatorname{im} \varphi)_p$  is the same stalk of the presheaf of image before sheafification. Conversely, if  $s_p \in \operatorname{im} \varphi_p$ , we have some  $t_p \in \mathcal{F}_p$  such that  $\varphi_p(t_p) = s_p$ . Suppose  $t \in \mathcal{F}(U)$  is a section of some open set  $U$  containing  $p$  such that  $t|_p = t_p$ . Then  $\varphi(t)|_p = \varphi_p(t_p) = s_p$ , so  $s_p$

is in the stalk of the image presheaf at  $p$ . But the stalk at a point remains the same after sheafification, we have  $s_p \in (\text{im } \varphi)_p$ .  $\square$

*Remark A.4.1.* The proof for the first part is a routine, but the proof for the half part shows the hallmark of sheafification: Stalks are not changed!.

Now let's consider a special exact sequence: short exact sequence:

$$0 \rightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \rightarrow 0$$

In this case,  $\varphi$  is called injective and  $\psi$  is called surjective. Attention: For any open subset  $U \subseteq X$ , we will have

$$\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$$

is injective. Indeed, by definition we have for any open subset  $U \subseteq X$ ,  $\ker \varphi(U) = 0$ , that is injectivity. Or from another point of view, for each  $p \in U$ , we have

$$\varphi_p : \mathcal{F}_p \rightarrow \mathcal{G}_p$$

is injective. That is  $\ker \varphi_p = 0$ . So we obtain  $(\ker \varphi(U))_p = 0$  for all  $p \in U$ . But for a section  $s \in \mathcal{F}(U)$  if we have  $s|_p = 0$ , then we must have  $s = 0$ . So we obtain  $\ker \varphi(U) = 0$ .

The second pointview may be a little talk nonsense, but we will see it can explain why  $\psi(U) : \mathcal{G}(U) \rightarrow \mathcal{H}(U)$  may not be surjective in general. Since from

$$\psi_p : \mathcal{G}_p \rightarrow \mathcal{H}_p$$

is surjective we can only get “locally surjective”, and from locally surjectivity you may not get a global one. The reason for why does image fail to be a sheaf appears again.

**Example A.4.1** (exponential sequence). Let  $X$  be a complex manifold and  $\mathcal{O}_X$  be its holomorphic function sheaf. Then

$$0 \rightarrow 2\pi\sqrt{-1}\mathbb{Z} \rightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \rightarrow 0$$

is an exact sequence of sheaves, called exponential sequence.

*Proof.* The difficulty is to show  $\exp$  is surjective on stalks at  $p \in X$ . That is we need to construct logarithms of functions  $g \in \mathcal{O}_X^*(U)$  for  $U$ , a neighborhood of  $p$ . We may choose  $U$  is simply-connected, then define

$$\log g(q) = \log g(p) + \int_{\gamma_q} \frac{dg}{g}$$

for  $q \in U$ , where  $\gamma_q$  is a path from  $p$  to  $q$  in  $U$ , and our definition is independent of the choice of  $\gamma_q$  since  $U$  is simply-connected.

In fact,  $U$  is simply-connected is crucial for constructing logarithm. If we consider  $X = \mathbb{C}$  and  $U = \mathbb{C} \setminus \{0\}$ , then

$$\exp : \mathcal{O}_X(U) \rightarrow \mathcal{O}_X^*(U)$$

won't be surjective.  $\square$



**A.5. Derived functor formulation of sheaf cohomology.** The category  $\underline{Ab}_X$ : sheaves of abelian groups on  $X$ . In this section we will introduce sheaf cohomology by considering it as a derived functor.

For an exact sequence of sheaf:

$$0 \rightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$$

If we take its section on  $U$ , we get a sequence of abelian groups

$$0 \rightarrow \mathcal{F}'(U) \xrightarrow{\phi(U)} \mathcal{F}(U) \xrightarrow{\psi(U)} \mathcal{F}''(U)$$

We already know this sequence is still exact at  $\mathcal{F}'(U)$ , now let's it's still exact at  $\mathcal{F}(U)$ , that is

$$\ker \psi(U) = \operatorname{im} \phi(U)$$

Let's first show  $\ker \psi(U) \supseteq \operatorname{im} \phi(U)$ . Take  $s \in \mathcal{F}'(U)$ , and we want to show  $\psi\phi(s) = 0$ . It suffices to show  $\psi\phi(s)|_p = 0$  for all  $p \in U$  since  $\mathcal{F}''$  is a sheaf. For any  $p \in U$ , consider its stalk we obtain an exact sequence of abelian groups

$$0 \rightarrow \mathcal{F}'_p \xrightarrow{\phi_p} \mathcal{F}_p \xrightarrow{\psi_p} \mathcal{F}''_p$$

then we obtain  $\psi_p\phi_p(s|_p) = 0$ , that is  $\psi\phi(s)|_p$ .

On the other hand. Take  $s \in \ker \psi(U)$ , then for any  $p \in U$  we have  $s|_p \in \ker \psi_p$ . By exactness of stalks, there exists  $t_p \in \mathcal{F}'_p$  such that  $\phi_p(t_p) = s|_p$ . So there exists an open subset  $V_i$  containing  $p$  and  $t_i \in \mathcal{F}'(V_i)$  such that  $\phi(t_i) = s|_{V_i}$ . We claim that these  $t_i$  can be glued together to obtain  $t \in \mathcal{F}'(U)$ . Since  $\mathcal{F}$  is a sheaf, it suffices to check these  $t_i$  agree on intersections  $V_i \cap V_j$ . This follows from the injectivity of  $\phi$  since  $\phi(t_i - t_j|_{V_i \cap V_j}) = s|_{V_i \cap V_j} - s|_{V_i \cap V_j} = 0$ .

*Remark A.5.1.* From above argument, we can see that

$$0 \rightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$$

is exact if and only if for any open subset  $U \subseteq X$

$$0 \rightarrow \mathcal{F}'(U) \xrightarrow{\phi(U)} \mathcal{F}(U) \xrightarrow{\psi(U)} \mathcal{F}''(U)$$

is exact.

In homological algebra, we can consider the derived functor of a left or right-exact functor. Here as we can see, take sections (in particular, global sections) of a sheaf is a left exact functor.

So, as what we did in homological, we need choose a injective resolution and consider the cohomology of the sequence of its global sections to define the sheaf cohomology.

**Definition A.5.1** (injective). A sheaf  $\mathcal{I}$  is injective if  $\operatorname{Hom}(-, \mathcal{I})$  is an exact functor.

**Theorem A.5.1.**  $\underline{Ab}_X$  is an abelian category with enough injectives. Namely, every sheaf  $\mathcal{F}$  can be realized as a subsheaf of some injective sheaf.

**Definition A.5.2** (injective resolution). Let  $\mathcal{F}$  be a sheaf, an injective resolution of  $\mathcal{F}$  is an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^1 \rightarrow \mathcal{I}^2 \rightarrow \dots$$

where  $\mathcal{I}^i, i = 0, 1, 2, \dots$  are injective.

**Theorem A.5.2.** Every sheaf admits an injective resolution.

**Theorem A.5.3.** Let  $\mathcal{F} \rightarrow \mathcal{I}^\bullet$  and  $\mathcal{G} \rightarrow \mathcal{G}^\bullet$  are two resolutions, and  $\phi : \mathcal{F} \rightarrow \mathcal{G}$  is a homomorphism of sheaves. Then there exists  $\tilde{\phi} : \mathcal{I}^\bullet \rightarrow \mathcal{G}^\bullet$ . Although the lifting of  $\phi$  may not be unique, but they are homotopic.

**Definition A.5.3** (sheaf cohomology). Let  $\mathcal{F}$  be a sheaf of abelian groups, then

$$H^p(X, \mathcal{F}) := H^p(\mathcal{I}^\bullet(X))$$

*Remark A.5.2.* This definition is independent of the choice of injective resolution thanks to [A.5.3](#).

**Example A.5.1.** By definition,

$$H^0(X, \mathcal{F}) := \ker\{\mathcal{I}^0(X) \rightarrow \mathcal{I}^1(X)\}$$

Thus  $H^0(X, \mathcal{F}) = \mathcal{F}(X)$ , the global sections of sheaf.

**Example A.5.2.** If  $\mathcal{F}$  is a injective sheaf, then  $H^i(X, \mathcal{F}) = 0$  for all  $i > 0$ . It's clear if we choose the following injective resolution

$$0 \rightarrow \mathcal{F} \xrightarrow{\text{id}} \mathcal{F} \rightarrow 0 \rightarrow 0 \rightarrow \dots$$

**Proposition A.5.1** (zig-zag). If

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$$

is a short sequence of sheaves, then there is an induced long exact sequence of abelian groups

$$0 \rightarrow H^0(X, \mathcal{F}) \rightarrow H^0(X, \mathcal{G}) \rightarrow H^0(X, \mathcal{H}) \rightarrow H^1(X, \mathcal{F}) \rightarrow H^1(X, \mathcal{G}) \rightarrow \dots$$

**Definition A.5.4** (direct image). Let  $f : X \rightarrow Y$  be continuous map between topological spaces. Let  $\mathcal{F}$  be a sheaf of abelian groups on  $X$ . A sheaf  $f_*\mathcal{F}$  on  $Y$  is defined by

$$f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U))$$

$f_*\mathcal{F}$  is called direct image sheaf of  $\mathcal{F}$ .

**Proposition A.5.2.**  $f_* : \underline{Ab}_X \rightarrow \underline{Ab}_Y$  is a left exact functor.

*Proof.* Given an exact sequence of sheaves on  $X$

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}''$$

Then we need to show

$$0 \rightarrow f_*\mathcal{F}' \rightarrow f_*\mathcal{F} \rightarrow f_*\mathcal{F}''$$

is also an exact sequence on  $Y$ . But by Remark A.5.1 it suffices to check for any  $V \in Y$ , we have the following exact sequence

$$0 \rightarrow f_*\mathcal{F}'(V) \rightarrow f_*\mathcal{F}(V) \rightarrow f_*\mathcal{F}''(V)$$

and that's

$$0 \rightarrow \mathcal{F}'(f^{-1}(V)) \rightarrow \mathcal{F}(f^{-1}(V)) \rightarrow \mathcal{F}''(f^{-1}(V))$$

and since  $f$  is continuous, then  $f^{-1}(V)$  is an open subset in  $X$ . This completes the proof.  $\square$

Since we obtain another left exact functor  $f_*$ , we can consider its derived functor.

**Definition A.5.5** (higher direct image sheaves). Let  $0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^\bullet$  be an injective resolution of  $\mathcal{F}$ . The higher direct image sheaf is defined by: For open  $U$ ,

$$R^i f_*\mathcal{F}(U) := H^i(f_*\mathcal{I}^\bullet(U))$$

For higher direct image sheaves, it has similar properties parallel to A.5.1, A.5.2 and A.5.1 since these are properties shared by derived functors.

**A.6. Computation for cohomology.** Since it may be difficult for us to choose an injective resolution, we use other resolutions to compute sheaf cohomology.

**Definition A.6.1** (acyclic sheaf). A sheaf  $\mathcal{F}$  is acyclic if  $H^i(X, \mathcal{F}) = 0, \forall i > 0$ .

**Example A.6.1.** Injective sheaf is acyclic.

**Definition A.6.2** (acyclic resolution). Let  $\mathcal{F}$  be a sheaf, an acyclic resolution of  $\mathcal{F}$  is an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{A}^0 \rightarrow \mathcal{A}^1 \rightarrow \mathcal{A}^2 \rightarrow \dots$$

where  $\mathcal{A}^i, i = 0, 1, 2, \dots$  are acyclic.

**Proposition A.6.1.** The cohomology of sheaf  $\mathcal{F}$  can be computed using acyclic resolution.

In fact, it's a quite homological techniques, called dimension shifting. So we will state this technique in language of homological algebra. Let's see a baby version of it.

**Example A.6.2.** Let  $\mathcal{F}$  be a left exact functor, and  $0 \rightarrow A \rightarrow M_1 \rightarrow B \rightarrow 0$  is exact with  $M_1$  is  $\mathcal{F}$ -acyclic. Then  $R^{i+1}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$  for  $i > 0$ , and  $R^1\mathcal{F}(A)$  is the cokernel of  $\mathcal{F}(M_1) \rightarrow \mathcal{F}(B)$ .

*Proof.* Consider the long exact sequence induced by  $0 \rightarrow A \rightarrow M_1 \rightarrow B \rightarrow 0$ , then we obtain

$$R^i\mathcal{F}(M_1) \rightarrow R^i\mathcal{F}(B) \rightarrow R^{i+1}\mathcal{F}(A) \rightarrow R^{i+1}\mathcal{F}(M_1)$$

If  $i > 0$ , then  $R^i\mathcal{F}(M_1) = R^{i+1}\mathcal{F}(M_1) = 0$  since  $M_1$  is  $\mathcal{F}$ -acyclic. So we obtain  $R^{i+1}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$ . If  $i = 0$ , then

$$0 \rightarrow \mathcal{F}(M_1) \rightarrow \mathcal{F}(B) \rightarrow R^1\mathcal{F}(A) \rightarrow 0$$

implies  $R^1\mathcal{F}(A) = \text{coker}\{\mathcal{F}(M_1) \rightarrow \mathcal{F}(B)\}$   $\square$

Now let's prove dimension shifting in a general setting.

**Lemma A.6.1** (dimension shifting). If

$$0 \rightarrow A \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_m \rightarrow B \rightarrow 0$$

is exact with  $M_i$  is  $\mathcal{F}$ -acyclic. Then  $R^{i+m}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$  for  $i > 0$ , and  $R^m\mathcal{F}(A)$  is the cokernel of  $\mathcal{F}(M_m) \rightarrow \mathcal{F}(B)$ .

*Proof.* Prove it by induction on  $m$ . For  $m = 1$ , we already see it in Example A.6.2. Assume this holds for  $m < k$ , then for  $m = k$ , let's split  $0 \rightarrow A \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_k \xrightarrow{d_k} B \rightarrow 0$  into two exact sequences

$$0 \rightarrow A \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_{k-1} \rightarrow \ker d_k \rightarrow 0$$

$$0 \rightarrow \ker d_k \rightarrow M_k \xrightarrow{d_k} B \rightarrow 0$$

Then by induction, for  $i > 0$  we have

$$R^{i+k-1}\mathcal{F}(A) \cong R^i\mathcal{F}(\ker d_k)$$

$$R^{i+1}\mathcal{F}(\ker d_k) \cong R^i\mathcal{F}(B)$$

Combine these two isomorphisms together we obtain  $R^{i+k}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$  for  $i > 0$ , as desired. For  $i = 0$ , it suffices to let  $i = 1$  in  $R^{i+k-1}\mathcal{F}(A) \cong R^i\mathcal{F}(\ker d_k)$ , then we obtain

$$R^k\mathcal{F}(A) = R^1\mathcal{F}(\ker d_k) = \text{coker}\{\mathcal{F}(M_k) \rightarrow \mathcal{F}(B)\}$$

This completes the proof.  $\square$

**Corollary A.6.1.**  $0 \rightarrow A \rightarrow M_\bullet$  is a  $\mathcal{F}$ -acyclic resolution, then  $R^i\mathcal{F}(A) = H^i(\mathcal{F}(M))$ .

*Proof.* Truncate the resolution as

$$0 \rightarrow A \rightarrow M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_{i-1} \rightarrow B \rightarrow 0$$

$$0 \rightarrow B \rightarrow M_i \rightarrow M_{i+1} \rightarrow \cdots$$

Since we already have  $R^i\mathcal{F}(A) = \text{coker}\{\mathcal{F}(M_{i-1}) \rightarrow \mathcal{F}(B)\}$ . Note  $\mathcal{F}$  is left exact, then

$$\mathcal{F}(B) = \ker\{\mathcal{F}(M_i) \rightarrow \mathcal{F}(M_{i+1})\}$$

Thus we obtain

$$R^i\mathcal{F}(A) = \text{coker}\{\mathcal{F}(M_{i-1}) \rightarrow \ker\{\mathcal{F}(M_i) \rightarrow \mathcal{F}(M_{i+1})\}\} = H^i(\mathcal{F}(M))$$

$\square$

## A.7. Examples about acyclic sheaf.

A.7.1. *Flabby sheaf*. First kind of acyclic sheaf is flabby<sup>14</sup> sheaf.

**Definition A.7.1** (flabby). A sheaf  $\mathcal{F}$  is flabby if all open  $U \subseteq V$ , the restriction map  $\mathcal{F}(V) \rightarrow \mathcal{F}(U)$  is surjective.

Now let's see some examples about flabby sheaves.

**Example A.7.1.** A constant sheaf on an irreducible topological space is flabby.

*Proof.* Note that the constant presheaf on a irreducible topological space is a sheaf in fact. And it's easy to see this constant presheaf is flabby.  $\square$

In particular, we have

**Example A.7.2.** Let  $X$  be an algebraic variety, so any two non-empty open sets intersect non-trivially. Then constant sheaf  $\mathbb{Z}_X$  is flabby.

**Example A.7.3.** If  $\mathcal{F}$  is a flabby sheaf on  $X$ , and  $f: X \rightarrow Y$  is a continuous map, then  $f_*\mathcal{F}$  is a flabby sheaf on  $Y$ .

*Proof.* For  $V \subseteq W$  in  $Y$ , it suffices to show  $f_*\mathcal{F}(W) \rightarrow f_*\mathcal{F}(V)$  is surjective, and that's

$$\mathcal{F}(f^{-1}W) \rightarrow \mathcal{F}(f^{-1}V)$$

it's surjective since  $\mathcal{F}$  is flabby.  $\square$

**Example A.7.4.** An injective sheaf is flabby.

*Proof.* Let  $\mathcal{I}$  be an injective sheaf and let  $V \subseteq U$  be an inclusion of open sets. We define a sheaf  $\mathbb{Z}_U$  on  $X$  by

$$\mathbb{Z}_U := \begin{cases} \mathbb{Z}(W) & W \subseteq U \\ 0 & \text{otherwise} \end{cases}$$

where  $\mathbb{Z}$  is constant sheaf valued in  $\mathbb{Z}$ . Similarly we can define  $\mathbb{Z}_V$ , and we have  $\mathbb{Z}_U(W) = \mathbb{Z}_V(W)$  unless  $W \subseteq U$  and  $W \not\subseteq V$ . Thus we obtain an exact sequence

$$0 \rightarrow \mathbb{Z}_V \rightarrow \mathbb{Z}_U$$

Applying the functor  $\text{Hom}(-, \mathcal{I})$ , which is exact, we obtain an exact sequence

$$\text{Hom}(\mathbb{Z}_U, \mathcal{I}) \rightarrow \text{Hom}(\mathbb{Z}_V, \mathcal{I}) \rightarrow 0$$

is exact. Now let's see why we need such a weird sheaf  $\mathbb{Z}_U$ . In fact, we will prove  $\text{Hom}(\mathbb{Z}_U, \mathcal{I}) = \mathcal{I}(U)$ . Indeed since  $\varphi: \mathbb{Z}_U \rightarrow \mathcal{I}$  is a sheaf morphism. Then if  $W \not\subseteq U$ , then  $\varphi(U)$  must be zero. If  $W = U$ , then the group of sections of  $\mathbb{Z}_U(U)$  over any connected component is simply  $\mathbb{Z}$  and hence  $\varphi(U)$  on this connected component is determined by the image of  $1 \in \mathbb{Z}$ . Thus  $\varphi(U)$  can be thought of an element of  $\mathcal{I}(U)$ . Now on any proper open subset

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<sup>14</sup>Some authors also call this flasque.

of  $U$ ,  $\varphi$  is determined by restriction maps. Hence  $\text{Hom}(\underline{\mathbb{Z}}_U, \mathcal{I}) = \mathcal{I}(U)$ , as desired. We can do similar things for  $V$ , and we obtain an exact sequence

$$\mathcal{I}(U) \rightarrow \mathcal{I}(V) \rightarrow 0$$

Thus  $\mathcal{I}$  is flabby.  $\square$

Our goal is to prove a flabby sheaf is acyclic, but we still need some property of flabby sheaves.

**Proposition A.7.1.** If  $0 \rightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \rightarrow 0$  is an exact sequence of sheaves, and  $\mathcal{F}'$  is flabby, then for any open set  $U$ , the sequence

$$0 \rightarrow \mathcal{F}'(U) \xrightarrow{\phi(U)} \mathcal{F}(U) \xrightarrow{\psi(U)} \mathcal{F}''(U) \rightarrow 0$$

is exact.

*Proof.* It suffices to show  $\mathcal{F}(U) \rightarrow \mathcal{F}''(U) \rightarrow 0$  is exact. And it may be quit hard than it looks and here we give a sketch of proof.

Since we have exact sequence on stalks for each  $p \in U$  as follows

$$0 \rightarrow \mathcal{F}'_p \xrightarrow{\phi_p} \mathcal{F}_p \xrightarrow{\psi_p} \mathcal{F}''_p \rightarrow 0$$

Then for each  $s \in \mathcal{F}''(U)$ , then there exists  $t_p \in \mathcal{F}_p$  such that  $\psi_p(t_p) = s|_p$ . So there exists  $V_p \subseteq U$  containing  $p$  and  $t \in \mathcal{F}(V_p)$  such that  $\psi(t) = s|_{V_p}$ . If we can glue these  $t$  together then we get a section in  $\mathcal{F}(U)$  and is mapped to  $s$ , which completes the proof. However, they may not equal on the intersection. But things are not too bad, consider another point  $q$  and  $t' \in \mathcal{F}(V_q)$  such that  $\psi(t') = s|_{V_q}$ ,  $(t' - t)|_{V_p \cap V_q} \in \ker \psi(V_p \cap V_q) = \text{im } \phi(V_p \cap V_q)$ . So there exists  $t'' \in \mathcal{F}'(V_p \cap V_q)$  such that

$$\phi(t'') = (t' - t)|_{V_p \cap V_q}$$

Now since  $\mathcal{F}'$  is flabby, then there exists  $t''' \in \mathcal{F}'(V_p)$  such that  $t'''|_{V_p \cap V_q} = t''$ . And consider  $t + \phi(t''') \in \mathcal{F}(V_p)$ , which will coincide with  $t'$  on  $V_p \cap V_q$ . After above corrections, we can glue  $t$  after correction together.  $\square$

**Proposition A.7.2.** If  $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$  is an exact sequence of sheaves, and if  $\mathcal{F}'$  and  $\mathcal{F}$  are flabby, then  $\mathcal{F}''$  is flabby.

*Proof.* Take  $V \subseteq U$  and consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}'(U) & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{F}''(U) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}'(V) & \longrightarrow & \mathcal{F}(V) & \longrightarrow & \mathcal{F}''(V) & \longrightarrow & 0 \end{array}$$

Then five lemma completes the proof.  $\square$

**Proposition A.7.3.** A flabby sheaf is acyclic.

*Proof.* Let  $\mathcal{F}$  be a flabby sheaf. Since there are enough injectives, there is an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{I} \rightarrow \mathcal{Q} \rightarrow 0$$

with  $\mathcal{I}$  is injective. By Example A.7.4 we have  $\mathcal{I}$  is flabby. And by Proposition A.7.2 we have  $\mathcal{Q}$  is flabby. Consider the long exact sequence induced from above short exact sequence

$$\mathcal{F}(X) \rightarrow \mathcal{I}(X) \rightarrow \mathcal{Q}(X) \rightarrow H^1(X, \mathcal{F}) \rightarrow H^1(X, \mathcal{I}) \rightarrow \dots$$

Since injective sheaf is acyclic, then  $H^1(X, \mathcal{I}) = 0$ . Then  $H^1(X, \mathcal{F}) = \text{coker}\{\mathcal{I}(X) \rightarrow \mathcal{Q}(X)\}$ . But from Proposition A.7.1 we know that it's surjective. So  $H^1(X, \mathcal{F}) = 0$ .

Now we prove  $H^k(X, \mathcal{F}) = 0$  for  $k > 0$  by induction on  $k$ . We already show  $k = 1$ . Assume this holds for  $k < n$ . Then consider

$$\dots \rightarrow H^{n-1}(X, \mathcal{F}) \rightarrow H^{n-1}(X, \mathcal{I}) \rightarrow H^{n-1}(X, \mathcal{Q}) \rightarrow H^n(X, \mathcal{F}) \rightarrow H^n(X, \mathcal{I}) \rightarrow H^n(X, \mathcal{Q}) \rightarrow \dots$$

From we already know, we can reduce above sequence to

$$\dots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow H^n(X, \mathcal{F}) \rightarrow 0 \rightarrow H^n(X, \mathcal{Q}) \rightarrow \dots$$

which implies  $H^n(X, \mathcal{F}) = 0$ . This completes the proof.  $\square$

**A.7.2. Soft sheaf.** The second kind of acyclic sheaves is called soft sheaves, which is quit similar to flabby.

**Definition A.7.2** (soft). A sheaf  $\mathcal{F}$  over  $X$  is soft if for any closed subset  $S \subseteq X$  the restriction map  $\mathcal{F}(X) \rightarrow \mathcal{F}(S)$  is surjective.

*Remark A.7.1.* Here we need to know how to define sections over a closed subset: For closed subset  $S$ ,

$$\mathcal{F}(S) := \varinjlim_{S \subseteq U} \mathcal{F}(U)$$

Quite similar to Proposition A.7.1 and A.7.2, soft sheaf has the following properties:

**Proposition A.7.4.** If  $0 \rightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \rightarrow 0$  is an exact sequence of sheaves, and  $\mathcal{F}'$  is soft, then the following sequence

$$0 \rightarrow \mathcal{F}'(X) \xrightarrow{\phi(X)} \mathcal{F}(X) \xrightarrow{\psi(X)} \mathcal{F}''(X) \rightarrow 0$$

is exact.

**Proposition A.7.5.** If  $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$  is an exact sequence of sheaves, and if  $\mathcal{F}'$  and  $\mathcal{F}$  are soft, then  $\mathcal{F}''$  is soft.

**Proposition A.7.6.** A soft sheaf is acyclic.

So you may wonder, what's the difference between flabby and soft since the definitions are quite similar, and both of them are acyclic. Clearly by definition of sections over a closed subset, we know that every flabby sheaf is soft, but converse fails

**Example A.7.5.** The sheaf of smooth functions on a smooth manifold is soft but not flabby.

**Lemma A.7.1.** If  $\mathcal{M}$  is a sheaf of modules over a soft sheaf of rings  $\mathcal{R}$ , then  $\mathcal{M}$  is a soft sheaf.

*Proof.* Let  $s \in \mathcal{M}(K)$  for  $K$  a closed subset of  $X$ . Then  $s$  extends to some open neighborhood  $U$  of  $K$ . Let  $\rho \in \mathcal{R}(K \cup (X - U))$  be defined by

$$\rho = \begin{cases} 1, & \text{on } K \\ 0, & \text{on } X - U \end{cases}$$

Since  $\mathcal{R}$  is soft, then  $\rho$  extends to a section over  $X$ , then  $\rho s$  is the desired extension of  $s$ .  $\square$

**A.7.3. Fine sheaf.** Another important kind of acyclic sheaves, which behaves like sheaf of differential forms  $\Omega_X^k$  is called fine sheaf. Recall what is a partition of unity: Let  $U = \{U_i\}_{i \in I}$  be a locally finite open cover of  $X$ . A partition of unity subordinate to  $U$  is a collection of continuous or smooth functions  $f_i : U_i \rightarrow [0, 1]$  for each  $i \in I$  such that its support lies in  $U_i$ , and for any  $x \in X$

$$\sum_{i \in I} f_i(x) = 1$$

Note that for a smooth manifold  $M$ , then sheaf of differential  $k$ -forms is a  $C_M^\infty$ -module sheaf.

**Definition A.7.3** (fine sheaf). A fine sheaf  $\mathcal{F}$  on  $X$  is a sheaf of  $\mathcal{A}$ -modules, where  $\mathcal{A}$  is a sheaf of rings such that for every locally finite open cover  $\{U_i\}_{i \in I}$  of  $X$ , there is a partition of unity

$$\sum_{i \in I} \rho_i = 1$$

where  $\rho_i \in \mathcal{A}(X)$  and  $\text{supp}(\rho_i) \subseteq U_i$ .

*Remark A.7.2.* It's necessary to give an explicit definition of support of a section: For a sheaf  $\mathcal{F}$  on  $X$  and a section  $s \in \mathcal{F}(X)$ , its support is defined as

$$\text{supp}(s) := \overline{\{x \in X : s|_x \neq 0\}}$$

**Proposition A.7.7.** A fine sheaf is acyclic.

*Proof.* Let  $\mathcal{F}$  be a sheaf of  $\mathcal{A}$ -modules and a fine sheaf. And choose a injective resolution

$$0 \rightarrow \mathcal{F} \xrightarrow{d} \mathcal{I}^0 \xrightarrow{d} \mathcal{I}^1 \xrightarrow{d} \mathcal{I}^2 \xrightarrow{d} \dots$$

such that  $\mathcal{I}^i$  are injective sheaves of  $\mathcal{A}$ -modules. Let  $s \in \mathcal{I}^p(X)$  such that  $ds = 0$ . Then by exactness of injective resolution we have  $X$  is covered by open sets  $U_i$  such that for each  $i$  there is an element  $t_i \in \mathcal{I}^{p-1}(U_i)$  such that  $dt_i = s|_{U_i}$ . By passing to a refinement we may assume that the cover  $\{U_i\}$  is locally finite. Let  $\{\rho_i\}$  be a partition of unity subordinate to  $\{U_i\}$ . Then we have  $t = \sum \rho_i t_i \in \mathcal{I}^{p-1}(X)$  such that  $dt = s$ . This completes the proof.  $\square$



**Example A.7.6.** Let  $M$  be a manifold and let  $\pi: E \rightarrow M$  be a vector bundle, then sheaf of smooth sections of  $E$  is also a fine sheaf.

**Example A.7.7.** Tangent sheaf  $T_M$ , and its tensor  $T_M^{\otimes k}$ , sheaf of differential forms  $\Omega_M$  and  $k$ -forms  $\Omega_M^k$  are fine sheaves.

*Remark A.7.3.* So from Example A.7.6, we know that it's meaningless to compute cohomology of sheaf of differential  $k$ -forms, or any other vector bundle over a smooth manifold. But in complex version, something interesting happens:

Let  $(X, \mathcal{O}_X)$  be a complex manifold, and let  $\pi: E \rightarrow X$  a holomorphic vector bundle. Then the sheaf of holomorphic sections of  $E$  is not a fine sheaf since there is no partition of unity we use in the sense of smooth. So we may compute cohomology of some holomorphic vector bundle, and that's what does Dolbeault cohomology compute.

For fine sheaf and soft sheaf, we have

**Lemma A.7.2.** Fine sheaf is soft

*Proof.* Let  $\mathcal{F}$  be a fine sheaf,  $S \subseteq X$  closed and  $s \in \mathcal{F}(S)$ . Let  $\{U_i\}$  be an open covering of  $S$  and  $s_i \in \mathcal{F}(U_i)$  such that

$$s_i|_{S \cap U_i} = s|_{S \cap U_i}$$

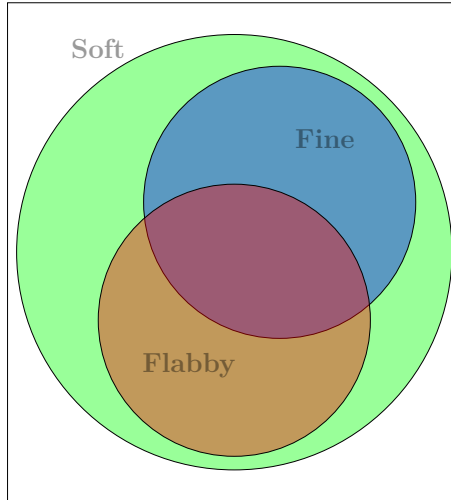
Let  $U_0 = X - S$ , and  $s_0 = 0$ . Then  $\{U_i\} \amalg \{U_0\}$  is an open covering of  $X$ , WLOG we assume this open covering is locally finite and choose a partition of unity  $\{\rho_i\}$  subordinate to it. Set

$$\bar{s} := \sum_i \rho_i(s_i) \in \mathcal{F}(X)$$

which extends  $s$ . □

*Remark A.7.4.* For flabby, soft, fine and acyclic sheaves. In fact we have the following relations:

Acyclic



Indeed, constant sheaf on an irreducible space is flabby but not fine, sheaf of smooth functions on a smooth manifold is fine but not flabby.

**A.8. Proof of de Rham theorem using sheaf cohomology.** As we already know, for constant sheaf  $\underline{\mathbb{R}}$  over a smooth manifold  $M$ , we have the following fine resolution

$$0 \rightarrow \underline{\mathbb{R}} \xrightarrow{i} \Omega_M^0 \xrightarrow{d} \Omega_M^1 \xrightarrow{d} \Omega_M^2 \xrightarrow{d} \dots$$

And de Rham cohomology computes the sheaf cohomology of  $\underline{\mathbb{R}}$ . de Rham theorem implies that de Rham cohomology equals to the singular cohomology with real coefficient. So if we can give constant sheaf another resolution using singular cochains, we may derive the de Rham cohomology.

We state this in a general setting: Let  $X$  be a topological manifold, and a constant sheaf  $\underline{G}$  over  $X$ , where  $G$  is an abelian group. Let  $S^p(U, G)$  be the group of singular cochains in  $U$  with coefficients in  $G$ , and let  $\delta$  denote the coboundary operator.

Let  $S^p(G)$  be the sheaf over  $X$  generated by the presheaf  $U \mapsto S^p(U, G)$ , with induced differential mapping  $S^p(G) \xrightarrow{\delta} S^{p+1}(G)$ .

Similar to Poincaré lemma, we have for a unit ball  $U$  in Euclidean space, we have the following sequence

$$\dots \rightarrow S^{p-1}(U, G) \xrightarrow{\delta} S^p(U, G) \xrightarrow{\delta} S^{p+1}(U, G) \rightarrow \dots$$

is exact. So we have the following resolution of the constant sheaf  $\underline{G}$

$$0 \rightarrow \underline{G} \rightarrow S^0(G) \xrightarrow{\delta} S^1(G) \xrightarrow{\delta} S^2(G) \rightarrow \dots$$

*Remark A.8.1.* If  $M$  is a smooth manifold, then we can consider smooth chains, that is  $f: \Delta^p \rightarrow U$ , where  $f$  is a smooth function. The corresponding results above still hold, and we have a resolution by smooth cochains with coefficients in  $G$ :

$$0 \rightarrow \underline{G} \rightarrow S_\infty^\bullet(G)$$

So if we choose  $G = \mathbb{R}$ , then it suffices to show  $0 \rightarrow \underline{\mathbb{R}} \rightarrow S_\infty^\bullet(\mathbb{R})$  is an acyclic resolution, then we obtain de Rham theorem.

First, note that  $S_\infty^p$  is a  $S_\infty^0$ -module, given by cup product on open sets. Then by Lemma A.7.1 and the fact  $S_\infty^0$  is soft we know that it's a soft resolution. This completes the proof.

**A.9. Hypercohomology.** In homological algebra, the hypercohomology is a generalization of cohomology functor which takes as input not objects in abelian category but instead chain complexes of objects.

One of the motivations for hypercohomology comes from the fact that there isn't an obvious generalization of cohomological long exact sequences associated to short exact sequence

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$$

It turns out hypercohomology gives techniques for constructing a similar cohomological associated long exact sequence from an arbitrary long exact sequence

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \cdots \rightarrow \mathcal{F}_k \rightarrow 0$$

Now let's clarify the definition of hypercohomology: Let  $\mathcal{F}^\bullet : \cdots \rightarrow \mathcal{F}^{i-1} \rightarrow \mathcal{F}^i \rightarrow \mathcal{F}^{i+1} \rightarrow \cdots$  be a complex of sheaves of abelian groups. Assume  $\mathcal{F}^\bullet$  is bounded below, i.e.  $\mathcal{F}^n = 0$  for  $n \ll 0$ . Then  $\mathcal{F}^\bullet$  admits an injective resolution  $\mathcal{F}^\bullet \rightarrow \mathcal{I}^\bullet$ . In other words

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathcal{F}^{i-1} & \longrightarrow & \mathcal{F}^i & \longrightarrow & \mathcal{F}^{i+1} \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \mathcal{I}^{i-1} & \longrightarrow & \mathcal{I}^i & \longrightarrow & \mathcal{I}^{i+1} \longrightarrow \cdots \end{array}$$

such that

- (1) All  $\mathcal{I}^i$  are injective sheaves.
- (2) The induced homomorphism  $H^i(\mathcal{F}^\bullet) \rightarrow H^i(\mathcal{I}^\bullet)$  is an isomorphism.

The hypercohomology of  $\mathcal{F}^\bullet$  is defined by

$$H^i(X, \mathcal{F}^\bullet) := H^i(\Gamma(X, \mathcal{I}^\bullet))$$

**Notation A.9.1.** Let  $\mathcal{F}^\bullet[n]$  be the complex obtained from  $\mathcal{F}$  by

$$(\mathcal{F}^\bullet[n])^i = \begin{cases} \mathcal{F}, & i = n \\ 0, & \text{otherwise} \end{cases}$$

**Example A.9.1.** Let  $\mathcal{F}$  be a sheaf and consider  $\mathcal{F}^\bullet[0]$ , that is,

$$0 \rightarrow \underbrace{\mathcal{F}}_{\text{deg zero}} \rightarrow 0 \rightarrow 0 \rightarrow \cdots$$

If  $0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^1 \rightarrow \cdots$  is an injective resolution of  $\mathcal{F}$ , then

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{I}^0 & \longrightarrow & \mathcal{I}^1 & \longrightarrow & \mathcal{I}^2 \longrightarrow \cdots \end{array}$$

is an injective resolution of  $\mathcal{F}^\bullet$ . Indeed,  $\mathcal{I}^i$  are injective for all  $i \geq 0$ . Furthermore,

$$H^i(\mathcal{I}^\bullet) = \begin{cases} \mathcal{F}, & n = 0 \\ 0, & \text{otherwise} \end{cases} = H^i(\mathcal{F}^\bullet[0])$$

So by definition of hypercohomology, we have  $H^i(X, \mathcal{F}^\bullet[0]) = H^i(\Gamma(X, \mathcal{I}^\bullet)) = H^i(X, \mathcal{F}^\bullet)$ . For general case, we have

$$H^i(X, \mathcal{F}^\bullet[n]) \cong H^{i+n}(X, \mathcal{F})$$

**Proposition A.9.1** (zig-zag in hypercohomology). Let  $0 \rightarrow \mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet \rightarrow \mathcal{H}^\bullet \rightarrow 0$  be a short exact sequence of bounded below complex. Then there is an induced long exact sequence

$$\cdots \rightarrow H^{i-1}(X, \mathcal{H}^\bullet) \rightarrow H^i(X, \mathcal{F}^\bullet) \rightarrow H^i(X, \mathcal{G}^\bullet) \rightarrow H^i(X, \mathcal{H}^\bullet) \rightarrow H^{i+1}(X, \mathcal{F}^\bullet) \rightarrow \cdots$$

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