# REPRESENTATION THEORY

BOWEN LIU

Abstract. In this course, we will cover the following aspects:
1. Representation of finite groups.
2. Symmetric functions.
3. Lie groups and Lie algebra.
4. Representations of complex semisimple Lie algebra.
5. Representations of compact Lie groups.

## Contents

## 0. Introduction and overview

Group theory is the study of symmetrics of a mathmatics object. This is the point of view of geometry: given a geometry object $X$, what is its group of symmetries?

But representation theory reverse this question, given a group $G$, what object $X$ does it act on? Here we pay more attention on linear action, i.e., $X$ is a vector space.

We can compare with manifolds, since every abstract manifold can be embedded into $\mathbb{R}^n$, every abstract group can be embedded into $S_n$, according to Cayley's theorem as follows

**Theorem 0.0.1.** *Any finite group of order $n$ is isomorphic to a subgroup of the symmetric group $S_n$.*

In this course, we are interested in the following groups:

1. finite group, in particular symmetric group, Coxeters groups.
2. Lie groups over $\mathbb{R}$ and $\mathbb{C}$.

And representation theory is a very useful tool, one of the most important applications is the classification of finite simple groups, all kinds of finite simple groups are listed as follows

1. cyclic groups $C_p$ for prime $p$
2. alternating groups $A_n, n \geq 5$
3. 16 simple groups of Lie type
4. 26 sporadic groups

Among those sporadic groups, the largest one is the monster $M$, with order $|M| \sim 8 \cdot 10^{53}$, but the number of irrducible representations is only 194. As we will see, all irreducible representations of one group will reflect all imformation about it, so it's possible for us to learn the properties of monster group, by using its irreducible representations.

It's also worth mentioning that there is a crazy conjecture about monster group, called Monstrous Monnlight conjecture, proven by Borcherds in 1992, and he got his Fields medal in 1998.

# Part 1. Representation of finite group

## 1. Basic Definitions and Irreduciblity

### 1.1. Basic Definitions.

**Definition 1.1.1.** *Let $G$ be a finite group, $V$ is a finite-dimensional vector space over $k$. A* ***representation*** *of $G$ on $V$ is a group homomorphism $\rho: G \to \mathrm{GL}(V)$.*

**Notation 1.1.2.** We say $V$ is a representation of $G$ and often write $gv$ instead of $\rho(g)v$, we also say that $G$ acts on $V$.

**Remark 1.1.3.** We give following remarks:

1. $\rho$ equips $V$ with the $G$-module structure.
2. We will mostly work with $k = \mathbb{C}$, more generally, $V$ can be finite-dimensional $R$-module for a communicative ring with 1.
3. Let $B = (e_1, \dots, e_n)$ be a basis of $V$, for $\varphi \in \operatorname{End}_k V$, write $\varphi e_i = \sum a_{ji} e_j$, and let $A = (a_{ij}) \in M_n(k)$. If $\rho$ is a representation, the $\rho_B(g)$ is the matrix of $\rho(g)$ with respect to $B$. Then $g \to \rho_B(g)$ is a homomorphism from $G$ to $\mathrm{GL}(n,k)$, called the matrix representation.

**Definition 1.1.4.** *Let $V, W$ be two representations of finite group $G$. A linear map $\varphi : V \to W$ is a **map of representation** of $G$ if the following diagram commutes for all $g \in G$.*

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g"] & W \arrow[d, "g"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd}$$

**Definition 1.1.5.** *Let $\rho : G \to \mathrm{GL}(V)$ be a representation. A **subrepresentation** of $V$ is a vector subspace $W$ of $V$, such that $\rho(g)W \subseteq W, \forall g \in G$. For a subrepresentation $W$, the map $\rho(g)(v + W) := \rho(g)v + W$ defines a representation of $G$ on $V/W$, called the **quotient representation.***

**Lemma 1.1.6.** *For a map of representation $\varphi : V \to W$, the kernel of $\varphi$ is a subrepresentation of $V$, image and cokernel of $\varphi$ are subrepresentations of $W$.*

*Proof.* Trivial. □

By some standard linear algebra methods, we can construct new representations from old ones:

**Lemma 1.1.7.** *Let $\rho : G \to \mathrm{GL}(V), \sigma : G \to \mathrm{GL}(W)$ be a representation of $G$, then*

1. $\rho \oplus \sigma : G \to \mathrm{GL}(V \oplus W), g(v \oplus w) = gv \oplus gw$
2. $\rho \otimes \sigma : G \to \mathrm{GL}(V \otimes W), g(v \otimes w) = gv \otimes gw$
3. $\rho^{\otimes n} : G \to \mathrm{GL}(V^{\otimes n}), g(v^{\otimes n}) = (gv)^{\otimes n}$
4. $\wedge^n \rho : G \to \mathrm{GL}(\wedge V^n), g(v_1 \wedge \cdots \wedge v_n) = gv_1 \wedge \cdots \wedge gv_n$
5. $\operatorname{Sym}^n \rho : G \to \mathrm{GL}(\operatorname{Sym}^n V), g(v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n$
6. $\rho^\vee : G \to \mathrm{GL}(V^\vee), \rho^\vee(g) = (\rho(g)^t)^{-1}$
7. $\rho_{V,W} : G \to \operatorname{Hom}(V, W), (\rho(g)\varphi)(v) = \rho(g)\varphi(\rho(g^{-1}))$

*are representations of $G$.*

*Proof.* □

**Lemma 1.1.8.** *We have the following isomorphism*

$$\operatorname{Hom}_G(V, W) \cong \operatorname{Hom}(V, W)^G = G\textit{-invariants of } \operatorname{Hom}(V, W)$$

*Proof.* □

**Lemma 1.1.9.** *The following are isomorphisms of representations $U, V, W$ of $G$*

1. $\mathrm{Hom}(V, W) \cong V^\vee \otimes W$
2. $V \otimes (U \oplus W) \cong V \otimes U \oplus V \otimes W$
3. $\wedge^k(V \oplus W) \cong \bigoplus_{a+b=k} \wedge^a V \otimes \wedge^b W$
4. $\wedge^k(V^\vee) \cong (\wedge^k V)^\vee$
5. $\wedge^k(V^\vee) \cong \wedge^{n-k} V \otimes \det V^\vee$, *where* $n = \dim V, \det V = \wedge V^m$.

*Proof.* □

**Definition 1.1.10.** *Let $G$ be a group and $X$ be a set. A* ***group action*** *of $G$ on $X$ is a map $\sigma : G \to \mathrm{Aut}(X)$, such that*

1. $\sigma(g)x \in X, \forall x \in X$
2. $\sigma(gh)x = \sigma(g)\sigma(h)x, \forall x \in X$
3. $\sigma(e)x = x, \forall x \in X$

If we have such a group action, we can construct many useful representations

**Example 1.1.11.** Let $V$ be a finite-dimensional over $\mathbb{C}$ with basis $X$, and $G$ acts on $X$ via $\sigma$, we define $R_X : G \to \mathrm{GL}(V)$ as follows

$$R_X(g)(\sum_{x \in X} a_x e_x) = \sum_{x \in X} a_x e_{\sigma(g)x}$$

Here $R_X$ is called **permutation representation**.

And the following examples are based on above one.

**Example 1.1.12.** Choose $X$ to be $G$ considered as a set, and $G$ acts on $G$ by left multiply, then $R = R_G$ is called **regular representation**, in this case $V$ is denoted by $k[G]$, called group algebra.

**Example 1.1.13.** Let $V$ be the group algebra of $G$, and consider the map $\rho : G \to \mathrm{GL}(V)$ defined as follows

$$\rho(g)(\sum_{x \in X} a_x e_x) = \sum_{x \in X} \mathrm{sgn}(\sigma(g)) a_x e_{\sigma(g)x}$$

is called the **alternating representation**.

**Example 1.1.14.** Let $H$ be subgroup of $G$, and $X = \{g_1, \dots, g_n\}$ be a complete set of representatives of $G/H$, $G$ acts on $X$ by $g(g_i H) = gg_i H$. In this case, $R_X$ is called the **coset representation** of $G$ with respect to $H$.

Now we consider some concrete examples which we will use later.

**Example 1.1.15.** Consider $G = S_n$ and $X = \{1, 2, \dots, n\}$. Let $V = \mathbb{C}X$, and $W = \mathbb{C}(e_1 + \cdots + e_n) \subset V$. Consider the permutation representation $R_X$, then it's easy to see that $R_X|_W$ is trivial representation.

**Example 1.1.16.** Regular representation for $X = \{1,2,3\}$, we can write down explictly as follows

$$R(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R((12)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R((13)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$R((23)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad R((132)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad R((123)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

**Example 1.1.17.** A 2-dimension representation of $S_3$ : the symmetry of triangle, denoted by $V$

$$V(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V((12)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad V((13)) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$V((23)) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad V((132)) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad V((123)) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

## 1.2. **Irreduciblity.**

**Definition 1.2.1.** *A representation of $V$ is called **irreducible** if there is no proper invariant subspace $W$ of $V$; A representation of $V$ is called **indecomposable** if it can not be written as a direct sum of two nonzero subrepresentation.*

In fact, when we consider complex representation, the irreduciblility and indecomposablity coincides, stated as follows

**Theorem 1.2.2** (Maschke's theorem)**.** *Let $V$ be a representation of a finite group of $\mathbb{C}$, $W \subseteq V$ is a subrepresentation, then there is a complementary invariant subrepresentation $W'$ of $G$, such that $V = W \oplus W'$.*

**Remark 1.2.3.** Maschke theorem still holds when $\operatorname{char} k \nmid |G|$

**Remark 1.2.4.** Any continous representation of a compact group has this property, but group $(\mathbb{R}, +)$ does not, consider $a \mapsto \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$ fixes the x-axis, but there is no complementary subspace.

**Lemma 1.2.5** (schur lemma)**.** *Let $V, W$ be irreducible representations of finite group $G$, and $\varphi \in \operatorname{Hom}_G(V, W)$, then*

1. *either $\varphi$ is isomorphism, or $\varphi = 0$*
2. *If $V = W$, then $\varphi = \lambda I, \lambda \in \mathbb{C}$*

**Proposition 1.2.6.** *Let $\rho : G \to \mathrm{GL}(V)$ be representation of finite group, then there is a unique decomposition*

$$V = \bigoplus_{i=1}^{N} V_i^{a_i}$$

*where $V_i$ is distinct irrducible representations.*

## 1.3. **Representation of abelian groups and $S_3$.**

**Proposition 1.3.1.** *Let $G$ be a finite abelian group, then every irrducible representation of $G$ is 1-dimensional.*

**Remark 1.3.2.** Let $\rho : G \to \mathrm{GL}(V)$ be any representation, then map $\rho(g) : V \to V$ is in general not a map of representations, i.e., for $h \in G$,

$$\rho(g)(hv) \neq h(\rho(g)v)$$

In fact, we can prove $\rho(g) \in \mathrm{End}_G V$ if and only if $g \in \mathrm{Z}(G)$.

**Remark 1.3.3.** The converse statement also holds, see corollary3.20.

**Definition 1.3.4.** *Let $G$ be a finite group, then $G^\vee = \mathrm{Hom}_G(G, \mathbb{C}^*)$ is called the dual group.*

**Corollary 1.3.5.** *Let $G$ be a finite abelian group, then* $\mathrm{Irr}\, G \overset{1:1}{\Longleftrightarrow} G^\vee$

*Proof.* By the remark 2.25, if $G$ is abelian, then $G = \mathrm{Z}(G)$, then $\rho(g) \in \mathrm{End}_G V = \mathbb{C}^*, \forall g \in G$ and $V \in \mathrm{Irr}(G)$. □

For $S_3$, we have already seen the following representations:

1. trivial representation $U$, with dimension 1.
2. alternating representation $U'$, with dimension 1.
3. the regular representation $R$, with dimension 3.
4. the symmetric of the triangle $V$, with dimension 2.

And we also note that $R$ has a 1-dimensional subrepresentation $V' = \mathbb{C}(e_1 + e_2 + e_3)$, in fact, it's a trivial representation, hence it is isomorphic to $U$.

Consider the complementary subspace of $V'$ in $R$, denoted by $V'' = \{(v_1, v_2, v_2) \in V \mid v_1 + v_2 + v_2 = 0\}$, we can choose a basis $(\omega, 1, \omega^2), (1, \omega, \omega^2)$, where $\omega^3 = 1$.

Now, let $W$ be an arbitrary representation of $S_3$, consider $\mathbb{Z}/3\mathbb{Z} = \langle \sigma \rangle \subset S_3$, and decompose $W$ into

$$W = \bigoplus_{i=1}^{3} V_i^{\oplus a_i}, \quad V_i = \mathbb{C}v_i, \sigma v_i = \omega^i v_i$$

Let $\tau \in S_3$ be a transposition, such that

$$S_3 = \langle \sigma, \tau \rangle / (\tau\sigma\tau = \sigma^2)$$

then

$$\sigma(\tau v_i) = \tau(\sigma^2 v_i) = \tau(\omega^{2i} v_i) = \omega^{2i} \tau v_i$$

## 2. Character theory

In this section, $G$ denotes a finite group.

**Definition 2.0.1.** *Let $\rho : G \to \mathrm{GL}(V)$ be a representation, $\chi_V : G \to \mathbb{C}, g \mapsto \chi_V(g) = \mathrm{tr}(\rho(g))$ is a character of $\rho$.*

**Remark 2.0.2.** In fact, $\chi_V$ is a class function, i.e.,

$$\chi_V \in \mathscr{C}_G = \{f : G \to \mathbb{C} \mid f|_K = \text{constant}, \forall K \in \mathrm{Conj}(G)\}$$

The dimension of $\mathscr{C}_G = |\mathrm{Conj}(G)|$, and we have the following isomorphism

$$\mathscr{C}_G \cong \mathrm{Z}(\mathbb{C}[G])$$

defined by

$$f \mapsto \sum_{g \in G} f(g)g$$

**Proposition 2.0.3.** *Let $V, W$ be representations of $G$, then*

1. $\chi_{V \oplus W} = \chi_V + \chi_W$
2. $\chi_{V \otimes W} = \chi_V \chi_W$
3. $\chi_{V^\vee} = \overline{\chi_V}$
4. $\chi_{\mathrm{Sym}^2 V}(g) = \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2))$
5. $\chi_{\wedge^2 V}(g) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$

*Proof.* Note that $\{\lambda_i \lambda_j \mid i \leq j\}, \{\lambda_i \lambda_j \mid i < j\}$ are the eigenvalues of $g$ on $\mathrm{Sym}^2 V, \wedge^2 V$ respectively, then

$$\sum_{i \leq j} \lambda_i \lambda_j = \frac{1}{2}(\sum_{i,j} \lambda_i \lambda_j + \sum_i \lambda_i^2)$$

$$\sum_{i < j} \lambda_i \lambda_j = \frac{1}{2}(\sum_{i,j} \lambda_i \lambda_j - \sum_i \lambda_i^2)$$

$\square$

**Theorem 2.0.4** (The fixed point formula)**.** *Let $X$ be a finie set with an action by $G$. Let $V$ be the permutation representation. Let $X^g = \{x \in X \mid gx = x\}, g \in G$. Then $\chi_V(g) = |X^g|$*

*Proof.* Since $\mathrm{Aut}(X) \cong S_{|X|}$, the matrix $A$ representing $\rho(g)$ is a permutation matrix: if $ge_{x_i} = e_{x_j}$ for some $x_i, x_j \in X$, then

$$A_{ik} = \begin{cases} 1, & k = j \\ 0, & \text{otherwise} \end{cases}$$

Then, if $x_i \in X^g$, then $ge_{x_i} = e_{gx_i} = e_{x_i}$, that is $A_{ii} = 1$, so

$$\mathrm{tr}(\rho(g)) = \sum_{i : x_i \in X^g} A_{ii} = \sum_{i : x_i \in X^g} 1 = |X^g|$$

$\square$

**Definition 2.0.5.** *The character table of $G$ is a table with the conjugacy classes listed a cross, the irreducible representations listed on the left.*

**Example 2.0.6.** Character table for $S_3$

| | 1 | (12) | (123) |
|---|---|---|---|
| trivial $U$ | 1 | 1 | 1 |
| alternating $U'$ | 1 | $-1$ | 1 |
| standard $V$ | 2 | 0 | $-1$ |
| permutation $P$ | 3 | 1 | 0 |

Observe $W = U^{\oplus a} \oplus U'^{\oplus b} \oplus V^{\oplus c}$, then

$$\chi_W = a\chi_U + b\chi_{U'} + c\chi_V$$

Since $\chi_U, \chi_{U'}, \chi_V$ is independent, later we will see that $W$ is determined by $\chi_W$ up to isomorphism.

We can use this fact to get some interesting results. For example, since we can decompose

$$\chi_{V\otimes V} = (4,0,1) = (2,0,-1) + (1,1,1) + (1,-1,1)$$

So we can decompose

$$V \otimes V = U \oplus U' \oplus V$$

Similarly, we can decompose any representation of $S_3$ in the above way, if we know what does its character look like.

**Remark 2.0.7.** Note that different groups can have identical character tables, e.g., dihedral group

$$D_{4n} = \langle a, b \mid a^2 = b^{2n} = (ab)^2 = e\rangle$$

and quaternianic group

$$Q_{4n} = \langle a, b \mid a^2 = b^{2n}, (ab)^2 = e\rangle$$

have the same character table.

**Remark 2.0.8.** Nevertheless, characters can characterize the group $G$: order of $G$, order of all its normal subgroups, whether $G$ is simple or not.

**Proposition 2.0.9.** *Let $V$ be a representation of $G$. The map $\varphi = \frac{1}{|G|}\sum_{g\in G} g \in \operatorname{End} V$ as a projection from $V$ to $V^G = \{v \in V \mid gv = v, \forall g \in G\}$*

*Proof.* Let $w \in W$, $v = \varphi(w) = \frac{1}{|G|}\sum_{g\in G} gw$, then for any $h \in G$, we have

$$hv = \frac{1}{|G|}\sum_{g\in G} hgw = \frac{1}{|G|}\sum_{g\in G} gw = v$$

So $\operatorname{im}\varphi \subset V^G$.

Conversely, if $v \in V^G$, then $\varphi(v) = \frac{1}{|G|}\sum_{g\in G} gv = v$, this implies $V^G \subset \operatorname{im}\varphi$. Moreover, $\varphi \circ \varphi = \varphi$. $\square$

**Definition 2.0.10.** *We let* $(\alpha, \beta) = \sum_{g \in G} \overline{\alpha(g)}\beta(g)$ *denote a Hermitian inner product on* $\mathscr{C}_G$.

**Theorem 2.0.11** (First orthogonality relation)**.** *Let* $V, W \in \operatorname{Irr}(G)$*, then*

$$(\chi_V, \chi_W) = \begin{cases} 1, & V \cong W \\ 0, & \textit{otherwise} \end{cases}$$

*Proof.* If $V, W$ are irreducible representations, then schur's lemma implies

$$\dim \operatorname{Hom}(V, W)^G = \dim \operatorname{Hom}_G(V, W) = \begin{cases} 1, & V \cong W \\ 0, & \text{otherwise} \end{cases}$$

However, $\chi_{\operatorname{Hom}(V,W)} = \chi_{V^\vee \otimes W} = \chi_{V^\vee}\chi_W = \overline{\chi_V}\chi_W$.

Let $\varphi = \frac{1}{|G|}\sum_{g \in G} g \in \operatorname{End}(\operatorname{Hom}(V, W))$, then we have

$$\begin{aligned} \dim \operatorname{Hom}(V, W)^G =& \operatorname{tr}_{\operatorname{Hom}(V,W)^G} \varphi = \frac{1}{|G|}\sum_{g \in G} \operatorname{tr}_{\operatorname{Hom}(V,W)}(g) \\ =& \frac{1}{|G|}\sum_{g \in G} \chi_{\operatorname{Hom}(V,W)}(g) \\ =& \frac{1}{|G|}\sum_{g \in G} \overline{\chi_V(g)}\chi_W(g) \end{aligned}$$

□

**Corollary 2.0.12.** *Any representation of a finite group* $G$ *is determined by its character up to isomorphism, i.e.,* $V \cong W \iff \chi_V(g) = \chi_W(g), \forall g \in G$.

**Corollary 2.0.13.** *If* $V = \bigoplus_i V_i^{\oplus a_i}$*,* $V_i$ *are irreducible, distinct representings, then*

$$a_i = (\chi_{V_i}, \chi_V)$$

*In particular,* $V$ *is irreducible if and only if* $(\chi_V, \chi_V) = 1$.

**Corollary 2.0.14.** *The multiplicity of any irreducible representation* $V$ *of* $G$ *in the decomposition of the regular representation* $R = \mathbb{C}[G]$ *is equal to its dimension. In particular,* $|\operatorname{Irr}(G)| < \infty$.

*Proof.* Recall that $(e_g)_{g \in G}$ is a basis for $R$, and $ge_h = e_{gh}, \forall g, h \in G$. For the fixed point formula

$$\chi_R(g) = \begin{cases} 0, & g \neq e \\ |G|, & g = e \end{cases}$$

Then $R$ is not irreducible unless $G$ is trivial. Write $R = \bigoplus_i V_i^{\oplus a_i}$, then

$$a_i = (\chi_{V_i}, \chi_R) = \frac{1}{|G|}\chi_{V_i}(e)|G| = \dim V_i$$

□

**Remark 2.0.15.** If $R=\bigoplus_i V_i^{\oplus a_i}, a_i=\dim V_i$, then

$$|G|=\dim R=\sum_i(\dim V_i)^2$$

**Remark 2.0.16.** If $g\neq e$, then $0=\chi_R(g)=\sum_i \dim V_i\chi_{V_i}(g)$. If we know all but one row of character table, we can calculate the remaining one using this remark.

**Example 2.0.17.** Character table of $S_4$

We already have trivial representation, alternating representation and standard representation. Since $24=1+1+9+\sum_i(\dim V_i)^2$, so there exist two* other representation $\widetilde{V},W$, such that $\dim\widetilde{V}=3,\dim W=2$.

Consider $\widetilde{V}=U'\otimes V,\dim\widetilde{V}=3$, then

$$\chi_{\widetilde{V}}=\chi_{U'}\chi_V=(3,-1,0,1,-1)$$

Then

$$(\chi_{\widetilde{V}},\chi_{\widetilde{V}})=1$$

So it is irreducible. And the remaining one can be calculate from remark 3.16

| | 1 | (12) | (123) | (1234) | (12)(34) |
|---|---|---|---|---|---|
| trivial $U$ | 1 | 1 | 1 | 1 | 1 |
| alternating $U'$ | 1 | $-1$ | 1 | $-1$ | 1 |
| standard $V$ | 3 | 1 | 0 | $-1$ | $-1$ |
| $\widetilde{V}$ | 3 | $-1$ | 0 | 1 | $-1$ |
| $W$ | 2 | 0 | $-1$ | 0 | 2 |
| permutation $P$ | 4 | 2 | 1 | 0 | 0 |

**Proposition 2.0.18.** *Let $\alpha:G\to\mathbb{C}$ be any function. Set $\varphi_{\alpha,V}=\sum_{g\in G}\alpha(g)g:V\to V$ for any representation $V$. Then $\varphi_{\alpha,V}\in\operatorname{End}_G V$ for all $V$ if and only if $\alpha\in\mathscr{C}_G$.*

*Proof.* Condition for $\varphi_{\alpha,V}$ to be $G$-linear: For $h\in G$,

$$\begin{aligned}\varphi_{\alpha,V}(hv)&=\sum_g\alpha(g)g(hv)=\sum_g\alpha(h^{-1}gh)hgh^{-1}(hv)\\&=h(\sum_g\alpha(hgh^{-1})gv)\\&\overset{\alpha\text{ is class function}}{=}h(\sum_g\alpha(g)gv)=h\varphi_{\alpha,V}(v)\end{aligned}$$

*Why there is no other 1-dimensional representation? In fact, we will learn later that the number of irrducible representations is equal to the number of the conjuagate classes.

Conversely, Consider $\varphi_{\alpha,V}(hv) = h\varphi_{\alpha,V}(v)$ and take for $V$ the regular representation $R$. For $x \in G$,

$$\varphi_{\alpha,R}(he_x) = \varphi_{\alpha,R}(e_{hx}) = \sum_g \alpha(g)e_{hx} = \sum_g \alpha(g)e_{ghx}$$

But we also have

$$h(\varphi_{\alpha,R}(e_x)) = h(\sum_g \alpha(g)ge_x) = \sum_g \alpha(g)hge_x = \sum_g \alpha(g)e_{hgx} = \sum_g \alpha(h^{-1}gh)e_{ghx}$$

Thus $\alpha$ is a class function by comparing the coefficient of two side. $\square$

**Proposition 2.0.19.** *If $V = \bigoplus_i V_i^{\otimes a_i}$ is the isotypical decomposition, of a representation $V$. Then the projection $\pi_i : V \to V_i^{\otimes a_i}$ is given by*

$$\pi_i = \frac{\dim V_i}{|G|} \sum_{g\in G} \overline{\chi_{V_i}(g)}g$$

*Proof.* Let $W$ be fixed irrducible representation, $V$ be any representation. Since $\overline{\chi_W} \in \mathscr{C}_G$, then

$$\psi_{\overline{\chi_W},V} = \frac{1}{|G|} \sum_{g\in G} \overline{\chi_W(g)}g \in \mathrm{End}_G(V)$$

If $V$ is irreducible, then schur's lemma implies $\psi_{\overline{\chi_W},V} = \lambda\,\mathrm{id}$, where

$$\lambda = \frac{1}{\dim V} \mathrm{tr}_V\, \varphi_{\overline{\chi_W},V} = \frac{1}{\dim V \cdot |G|} \sum_{g\in G} \overline{\chi_W(g)}\chi_V(g) = \begin{cases} \frac{1}{\dim V}, & V \cong W \\ 0, & \text{otherwise} \end{cases}$$

If $V$ is arbitrary, then $\dim W \psi_{\overline{\chi_W},V}$ is a projection onto $W^a$ where $a$ is the number of times $W$ appears in $V$.

So, if $V = \bigoplus_i V_i^{\otimes a_i}$ is the isotypical decomposition, then

$$\pi_i = \frac{\dim V_i}{|G|} \sum_{g\in G} \overline{\chi_{V_i}(g)}g$$

is the projection onto $V_i^{\oplus a_i}$.

$\square$

**Proposition 2.0.20.**

$$|\operatorname{Irr}(G)| = |\operatorname{Conj}(G)|$$

*In other words, $\{\chi_{V_i} \mid V_i \in \operatorname{Irr}(G)\}$ forms an orthogonal basis for $\mathscr{C}_G$.*

*Proof.* Suppose $\alpha \in \mathscr{C}_G, (\alpha, \chi_V) = 0, \forall V \in \operatorname{Irr}(G)$, we must show $\alpha = 0$.

For any representation $V$, consider $\varphi_{\alpha,V}$, schur lemma implies $\varphi_{\alpha,V} = \lambda\,\mathrm{id}_V$, let $n = \dim V$, this implies

$$\lambda = \frac{1}{n}\mathrm{tr}(\varphi_{\alpha,V}) = \frac{1}{n}\sum_g \alpha(g)\chi_V(g) = \frac{|G|}{n}\overline{(\alpha, \chi_{V^\vee})} = 0$$

Thus $\varphi_{\alpha,V}=0$, that is,

$$\sum_g \alpha(g)g=0, \quad \text{for any representation } V \text{ of } G.$$

In particular, for $V=R$, the set $\{\rho(g)\in \operatorname{End} R \mid g\in G\}$ consists of linearly independent elements, thus $\alpha(g)=0, \forall g\in G$. □

**Corollary 2.0.21.** *If $G$ is a finite group, the following are equivalent*
1. *$G$ is abelian.*
2. *Every irrducible representation of $G$ has dimension 1.*

*Proof.* (2) → (1).

$$|G|=\sum_{i=1}^{|\operatorname{Conj}(G)|}(\dim V_i)^2=|\operatorname{Conj}(G)|$$

So $|K|=1, \forall K\in \operatorname{Conj}(G)$, that is, $G$ is abelian. □

**Proposition 2.0.22** (Second orthogonality relation)**.**

$$\sum_{i:V_i\in \operatorname{Irr}(G)}\overline{\chi_{V_i}(g)}\chi_{V_i}(h)=\begin{cases}\frac{|G|}{|K_g|}, & K_g=K_h\\ 0, & \textit{otherwise}\end{cases}$$

*where $K_g$ is the conjugacy class of $g$.*

*Proof.* Let $\chi_V,\chi_W$ be irreducible characters. First orthogonality relation implies

$$\delta_{V,W}=(\chi_V,\chi_W)=\frac{1}{|G|}=\sum_g\overline{\chi_V(g)}\chi_W(g)=\frac{1}{|G|}=\sum_{K\in \operatorname{Conj}(G)}\overline{\chi_V(K)}\chi_W(K)|K|$$

Then

$$U=(\sqrt{\frac{|K|}{|G|}}\chi_V(K))$$

is a unitary matrix. Orthogonality of the columns of $U$ yields the claim □

**Example 2.0.23.** [Monstrous Monnlight Conjecture] Let $G=\mathbb{M}$ be the monster group, i.e., the sporadic finite simple group with $|M|\sim 8\cdot 10^{53}$. One can show that $|\operatorname{Irr}(G)|=|\operatorname{Conj}(G)|=194$, a relatively small number.

To compare, $|\operatorname{Irr} S_{15}|=176, |\operatorname{Irr} S_{16}|=231$. Let $V_i\in \operatorname{Irr}(G)$ be ordered by their dimension.

| $V$ | $V_0$ | $V_1$ | $V_2$ | $V_3$ |
|---|---|---|---|---|
| $\dim V$ | 1 | 196883 | 21296876 | 842609256 |

Complex analysis tells Eisenstein series

$$G_k(\tau)=\sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}}\frac{1}{(m\tau+n)^k}$$

converges for $k \geq 3$ normally and defines a holomorphic function on $\mathbb{H}$. $G_k(\tau)$ admits a Fourier expansion

$$G_k(\tau) = \sum_{n=0}^{\infty} a_k(n) q^n, \quad q = e^{2\pi i \tau}$$

Consider

$$j(\tau) = \frac{172820 G_4(\tau)^3}{20 G_4(\tau)^3 + 49 G_6(\tau)^2}$$

Then $j(\tau) - 744 = q^{-1} + 196884q + 21493690q^2 + 864299970q^3 + \dots$

Mckay 1978 wrote a letter to Thompson

$$196884 = 196883 + 1$$

Thompson: the next term work similarly.

Suggestion: there exists $V = \bigoplus_{i=0}^{\infty} V_i$ infinitely-dimensional graded representation of $\mathbb{M}$ such that

$$\sum_{n=0}^{\infty} \chi_{V_n} q^{n-1} = j(q) - 744$$

Moreover,

$$T_q(\tau) = \sum_{n=0}^{\infty} \chi_{V_n}(g) q^{n-1} = \text{other well-known functions in complex analysis}$$

Corway-Norten verified this in 1979 on a computer.

Borcherds proved this conjecture in 1992 by $V$ the structure of a module over a vertex operator algebra.

**Definition 2.0.24.** *Let $G, H$ be finite groups, $V$ is a representation of $G$, $W$ is a representation of $H$, we define the external tensor product representation $V \boxtimes W$ of $G \times H$ by*

$$(g, h)(v, w) = gv \otimes hw, \quad \forall g \in G, h \in H, v \in V, w \in W.$$

*and extension by linearity to $V \otimes W$.*

*Similarly, we define a $G \times H$ action on $\operatorname{Hom}(V, W)$ by*

$$((g, h)\varphi)v = h\varphi(g^{-1}v), \quad g \in G, h \in H, v \in V, \varphi \in \operatorname{Hom}(V, W).$$

*and extension by linearity.*

**Remark 2.0.25.** We have

$$\operatorname{Hom}(V, W) \cong V^{\vee} \boxtimes W$$

as $G \times H$ representations.

**Proposition 2.0.26.** *We have the following well-defined bijection:*

$$\begin{aligned} \operatorname{Irr}(G) \times \operatorname{Irr}(H) &\to \operatorname{Irr}(G \times H) \\ (V, W) &\to V \boxtimes W \end{aligned}$$

*Proof.* If suffices to look at characters. By property of trace we have

$$\chi_{V\boxtimes W}((g,h)) = \chi_V(g)\chi_W(h)$$

Recall that

$$\dim \mathrm{Hom}_G(V,W) = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_V(g)}\chi_W(g) = (\chi_V,\chi_W)_G$$

Then

$$\begin{aligned}(\chi_{V_1\boxtimes W_1},\chi_{V_2\boxtimes W_2}) &= \frac{1}{|G\times H|}\sum_{g,h\in G\times H}\overline{\chi_{V_1}(g)\chi_{W_1}(g)}\chi_{V_2}(g)\chi_{W_2}(g)\\ &= \frac{1}{|G|}\sum_{g}\overline{\chi_{V_1}(g)}\chi_{V_2}(g)\frac{1}{|G|}\sum_{h\in H}\overline{\chi_{W_1}(g)}\chi_{W_2}(g)\\ &= (\chi_{V_1},\chi_{V_2})_G(\chi_{W_1},\chi_{W_2})_H\end{aligned}$$

So $V\boxtimes W\in\mathrm{Irr}(G\times H)$, if $V\in\mathrm{Irr}(G), W\in\mathrm{Irr}(H)$.

By calculating the cardinality of both sides we get the desired result. □

## 3. Restriction and induced representation

**Definition 3.0.1** (restriction representation)**.** *Let $H<G$ be a subgroup, $V$ be a representation of $G$, we define $\mathrm{Res}\,V=\mathrm{Res}_H^G V: H\to\mathrm{GL}(V)$ to be the restriction of $V$ onto $H$, $\mathrm{Res}_H^G V$ is a representation of $H$.*

**Remark 3.0.2.** Restriction is transitive, i.e., for $K<H<G$, we have

$$\mathrm{Res}_K^H\mathrm{Res}_H^G=\mathrm{Res}_K^G$$

**Lemma 3.0.3.** *Let $H<G$, $W\in\mathrm{Irr}(H)$, then there exists $V\in\mathrm{Irr}(G)$ such that*

$$(\mathrm{Res}_H^G\chi_V,\chi_W)_H\neq 0$$

*Proof.* Consider the regular representation $R$, then

$$(\mathrm{Res}_H^G\chi_R,\chi_W)=\frac{|G|}{|H|}\chi_W(e)\neq 0$$

But the left term also equals to $\sum_i\dim V_i(\mathrm{Res}_H^G\chi_{V_i},\chi_W)_H$, so there must be at least one $V_i$, such that

$$(\mathrm{Res}_H^G\chi_{V_i},\chi_W)\neq 0$$

□

**Lemma 3.0.4.** *Let $H<G$, $V\in\mathrm{Irr}(G)$, $\mathrm{Res}_H^G V=\bigoplus W_i^{\otimes a_i}, W_i\in\mathrm{Irr}(W)$. Then $\sum a_i^2\leq[G:H]$ with equality if and only if $\chi_V(\sigma)=0,\forall\sigma\in G/H$.*

*Proof.* We have

$$\frac{1}{|G|}\sum_{h\in H}|\chi_V(h)|^2=(\mathrm{Res}_H^G V,\mathrm{Res}_H^G V)=\sum a_i^2$$

Since $V$ is irreducible, we have

$$\begin{aligned}1 = (\chi_V, \chi_V)_G &= \frac{1}{|G|}\sum_{g\in G}|\chi_V(g)|^2 \\ &= \frac{1}{|G|}(\sum_{h\in H}|\chi_V(h)|^2 + \sum_{\sigma\in G/H}|\chi_V(\sigma)|^2) \\ &= \frac{|H|}{|G|}\sum_i a_i^2 + \frac{1}{|G|}\sum_{\sigma\in G/H}|\chi_V(\sigma)|^2 \\ &\geq \frac{|H|}{|G|}\sum_i a_i^2\end{aligned}$$

□

**Proposition 3.0.5.** *Let $V, W$ be representation of $G$. Then $V \cong W \iff \operatorname{Res}_H^G V \cong \operatorname{Res}_H^G W$, for all cyclic subgroup $H$ of $G$.*

*Proof.* One direction is obvious, consider the other: Let $g \in G, H = \langle g\rangle$, then $\chi_V(g) = \chi_{\operatorname{Res}_H^G V}(g)$, the claim follows from $V \cong W \iff \chi_V(g) = \chi_W(g), \forall g \in G$. □

**Definition 3.0.6.** *Let $H < G$ be a subgroup, $\rho : G \to \mathrm{GL}(V)$ be a representation, $W \subset V$ be a $H$-invariant subspace, i.e., $\psi : H \to \mathrm{GL}(W)$ is a representation. Then the subspace $gW \subset V$ depends only on $gH$. Therefore, for $\sigma \in G/H$, we write $\sigma W = gW, g \in \sigma$. If $V$ has a unique decomposition $V = \bigoplus_{\sigma\in G/H} \sigma W$, we write $V = \operatorname{Ind} W = \operatorname{Ind}_H^G W$. In this case, $V$ is called a representation induced by $W$.*

**Remark 3.0.7.** Alternative formulations: for any $v \in V$, there exists a unique $v_\sigma \in \sigma W$, such that

$$v = \sum_{\sigma\in G/H} v_\sigma$$

or if $\{g_1, \dots, g_N\}$, $|N| = |G/H| = [G : H]$ is a complete system of representatives of $G/H$, then

$$V = \bigoplus_{i=1}^N g_i W$$

**Remark 3.0.8.**

$$\dim V = [G : H] \dim W$$

**Example 3.0.9.** Let $R$ be the regular representation of $G$, then

$$W = \bigoplus_{h\in H} \mathbb{C}e_h$$

is $H$-invariant. then $\psi : H \to \mathrm{GL}(W)$ is a representation, in fact, $W \cong R_H$ and clearly $R_G = \operatorname{Ind}_H^G R_H$.

**Example 3.0.10.** Let $H < G$ and $V$ the coset representation of $G$, i.e., $V$ has basis $(e_\sigma)_{\sigma\in G/H}$ and $ge_\sigma = e_{g\sigma}$. Then

$$W = \mathbb{C}e_{eH}$$

is $H$-invariant, and is the trivial representation of $H$, then

$$V = \operatorname{Ind}_H^G W$$

In particular, if $H = \{e\}$, then $V$ is the permutation representation $P$ of $G$, and $P = \operatorname{Ind}_{\{e\}}^G \mathbb{C}$.

**Example 3.0.11.** If $V_i = \operatorname{Ind}_H^G W_i, i = 1, 2$, then

$$V_1 \oplus V_2 = \operatorname{Ind}_H^G(W_1 \oplus W_2)$$

**Example 3.0.12.** If $V = \operatorname{Ind}_H^G W$, $W' \subset W$ is a $H$-invariant subspace, then

$$V' = \bigoplus_{\sigma\in G/H} \sigma W' \subset V$$

is $G$-invariant, and $V' = \operatorname{Ind}_H^G W'$.

**Proposition 3.0.13.** *Let $H < G$ be a subgroup, $\rho : G \to \mathrm{GL}(V)$ is induced by $\psi : H \to \mathrm{GL}(W)$, let $\rho' : G \to \mathrm{GL}(V')$ be any representation, $\phi \in \operatorname{Hom}_H(W, V')$, then there exists a unique $\Phi \in \operatorname{Hom}_G(V, V')$, such that*

$$\Phi|_W = \phi$$

*Proof.* For uniqueness: Let $\Phi \in \operatorname{Hom}_G(V, V')$ with $\Phi|_W = \phi$, and let $w \in \rho(g)W, g \in G$, then

$$\Phi(w) = \Phi(\rho(g)\rho(g^{-1})w) = \rho'(g)\Phi(\rho(g)^{-1}w) = \rho'(g)\phi(\rho(g)^{-1}w)$$

This determines $\Phi$ on $\rho(g)W$ for all $g \in G$, hence on $V$.

For existence: we define

$$\Phi(w) = \rho'(g)\phi(\rho(g)^{-1}w)$$

if $w \in \rho(g)W$, this is independent of the choice of $g$, since

$$\begin{aligned}\rho'(gh)\phi(\rho(gh)^{-1}w) &= \rho'(g)\rho'(h)\phi(\rho(h)^{-1}\rho(g)^{-1}w)\\ &= \rho'(g)\phi(\rho(h)\rho(h)^{-1}\rho(g)^{-1}w)\\ &= \rho'(g)\phi(\rho(g)^{-1}w)\end{aligned}$$

□

**Theorem 3.0.14.** *Let $H < G$ be a subgroup, and $\psi : H \to \mathrm{GL}(W)$ be a representation. Then there exists a representation $\rho : G \to \mathrm{GL}(V)$ induced by $W$, which is unique up to isomorphism.*

*Proof.* For existence: By example4.11 we may assume $W \in \operatorname{Irr}(H)$, $W'$ is isomorphic to a subrepresentation of $R_H$, since any $W' \in \operatorname{Irr}(H)$ appears in $R_H$. By example4.9 we have

$$R_G = \operatorname{Ind}_H^G R_H$$

and by example4.12 with $V = R_G, W = R_H$, we get

$$V' = \operatorname{Ind}_H^G W'$$

For uniqueness: Let $V = \operatorname{Ind}_H^G W, V' = \operatorname{Ind}_H^G W$, then proposition4.13 implies that there exists a unique $\Phi \in \operatorname{Hom}_G(V, V')$ such that $\Phi|_W = \mathrm{id}_W$, and $\Phi \circ \rho(g) = \rho'(g) \circ \Phi, \forall g \in G$. Then $\operatorname{Im}\Phi$ contains all $\rho'(g)W$, so $\operatorname{Im}\Phi = V'$.

By $\dim V = [G : H] \dim W = \dim V'$, we conclude $\Phi$ is an isomorphism. □

**Lemma 3.0.15.** *Let $V$ be a representation of $G$, and $H < G$ be a subgroup. Then*

$$V \otimes \operatorname{Ind}_H^G W = \operatorname{Ind}_H^G(\operatorname{Res}_H^G V \otimes W)$$

*Proof.* Note that

$$\begin{aligned} V \otimes \operatorname{Ind}_H^G W &= \bigoplus_{\sigma \in G/H} V \otimes \sigma W \\ &= \bigoplus_{\sigma \in G/H} \sigma(\operatorname{Res}_H^G V) \otimes \sigma W = \operatorname{Ind}_H^G(\operatorname{Res}_H^G V \otimes W) \end{aligned}$$

□

**Corollary 3.0.16.** *We have*

$$V \otimes P = \operatorname{Ind}_H^G(\operatorname{Res}_H^G V)$$

*where $P$ is permutation representation.*

*Proof.* Take $W$ as trivial representation, then this claim holds from lemma4.15. □

**Lemma 3.0.17.** Ind *is transitive.*

*Proof.*

$$\begin{aligned} \operatorname{Ind}_K^H \operatorname{Ind}_H^G &= \operatorname{Ind}_K^H \bigoplus_{\tau \in G/H} \tau V \\ &= \bigoplus_{\sigma \in H/K} \bigoplus_{\tau \in G/H} \sigma\tau V \\ &= \bigoplus_{\sigma' \in G/K} \sigma' V \\ &= \operatorname{Ind}_K^G V \end{aligned}$$

□

**Remark 3.0.18.** These results can also be obtained by looking at characters or using group algebra.

**Theorem 3.0.19.** *Let $H < G$ be a subgroup, and $\rho : G \to \mathrm{GL}(V), \psi : H \to \mathrm{GL}(W)$ be two representations, such that $V = \mathrm{Ind}_H^G W$. Then*

$$\chi_V(g) = \sum_{\sigma \in G/H} \chi_W(g_\sigma^{-1} g g_\sigma) = \frac{1}{|H|} \sum_{\substack{x \in G, \\ x^{-1}gx \in H}} \chi_W(x^{-1}gx)$$

*where $g_\sigma$ is any representative of $\sigma$.*

*Proof.* Let $V = \bigoplus_{\sigma \in G/H} \sigma W$, $\rho(g)$ permutes the $\sigma W$ among themselves, i.e., if $g_\sigma \in \sigma$ is a representative, we write $gg_\sigma = g_\tau h$ for some $\tau \in G/H, h \in H$.

$$g(g_\sigma W) = (g_\tau h)W = g_\tau(hW) = g_\tau W$$

Then we can calculate

$$\begin{aligned}
\chi_V(g) = \mathrm{tr}_V(\rho(g)) &= \sum_{\sigma \in G/H} \mathrm{tr}_{\sigma W}(\rho(g)) \\
&= \sum_{\sigma \in G/H} \chi_W(g_\sigma^{-1} g g_\sigma) = \sum_{\tau \in G/H} \chi_W(h^{-1} g_\tau^{-1} g g_\tau h) \\
&= \frac{1}{|H|} \sum_{\tau \in G/H} \sum_{h \in H} \chi_W(h^{-1} g_\tau^{-1} g g_\tau h) = \frac{1}{|H|} \sum_{\substack{x \in G, \\ x^{-1}gx \in H}} \chi_W(x^{-1}gx)
\end{aligned}$$

□

**Theorem 3.0.20** (Frobenius reciprocity)**.** *Let $H < G$ be a subgroup, $W$ be a representation of $H$, $U$ be a representation of $G$. Assume that $V = \mathrm{Ind}_H^G W$, then*

$$\mathrm{Hom}_H(W, \mathrm{Res}_H^G U) \cong \mathrm{Hom}_G(V, U)$$

*i.e., for $\varphi \in \mathrm{Hom}_H(W, \mathrm{Res}_H^G U)$ extends uniquely to $\tilde{\varphi} \in \mathrm{Hom}_G(V, U)$*

*Proof.* We write $V = \bigoplus_{\sigma \in G/H} \sigma W$, define $\tilde{\phi}$ on $\sigma W$ by the compostion

$$\sigma W \xrightarrow{g_\sigma^{-1}} W \xrightarrow{\varphi} U \xrightarrow{g_\sigma} U$$

This is independent of the choice of $g_\sigma$ since

$$g_\sigma h(\varphi(h^{-1} g_\sigma^{-1}(w))) = g_\sigma \varphi(hh^{-1} g_\sigma(w))$$

by $\varphi \in \mathrm{Hom}_H(W, \mathrm{Res}_H^G U)$ □

**Corollary 3.0.21.** *Let $H < G$ be a subgroup, $W$ be a representation of $H$, $U$ be a representation of $G$. Then*

$$(\chi_W, \mathrm{Res}_H^G \chi_U)_H = (\mathrm{Ind}_H^G \chi_W, \chi_U)_G$$

*Proof.* By linearity, we can assume $W, U$ are irreducible representations. This claim follows from the Frobenius reciprocity and schur's lemma

$$(\chi_V, \chi_U)_G = \dim \mathrm{Hom}_G(V, U)$$

□

**Example 3.0.22.** Let $G = S_3, H = S_2$. In $S_2$, the standard representation $V_2$ is isomorphic to the alternating representation $U_2'$. We have seen that $U_3, U_3', V_3$ are all irreducible representations of $S_3$.

And we can write down their character tables as follows

| | 1 | (12) |
|---|---|---|
| trivial $U_2$ | 1 | 1 |
| alternating $U_2'$ | 1 | $-1$ |

,

| | 1 | (12) | (123) |
|---|---|---|---|
| trivial $U_3$ | 1 | 1 | 1 |
| alternating $U_3'$ | 1 | $-1$ | 1 |
| standard $V_3$ | 2 | 0 | $-1$ |

Note that

$$\operatorname{Res} U_3 = U_2, \quad \operatorname{Res} U_3' = U_2', \quad \operatorname{Res} V_3 = U_2 \oplus U_2'$$

If we want to calculate Ind, firstly note that we have seen

$$P \otimes U = \operatorname{Ind}(\operatorname{Res} U), \quad U \text{ is any representation of } G$$

For $U = U_3$, we have $P = U_3 \oplus V_3 = \operatorname{Ind} U_2$.

If we want to calculate $\operatorname{Ind} V_2$, it's a little bit complicated.

By Frobenius reciprocity

$$\operatorname{Hom}_{S_3}(\operatorname{Ind} V_2, U_3) = \operatorname{Hom}_{S_2}(V_2, \operatorname{Res} U_3 = U_2) \overset{\text{schur}}{=} 0$$

$$\operatorname{Hom}_{S_3}(\operatorname{Ind} V_2, U_3') = \operatorname{Hom}_{S_2}(V_2, \operatorname{Res} U_3' = U_2') \overset{\text{schur}}{=} \mathbb{C}$$

$$\operatorname{Hom}_{S_3}(\operatorname{Ind} V_2, V_3) = \operatorname{Hom}_{S_2}(V_2, \operatorname{Res} V_3 = U_2 \oplus U_2') \overset{\text{schur}}{=} \mathbb{C}$$

So

$$\operatorname{Ind} V_2 = U_3' \oplus V_3$$

**Definition 3.0.23.** *Let $G$ be a finite group, and $R_k(G)$ be the free abelian group generated by all isomorphism classes of representations of $G$ over a field $k$, modulo the subsgroup generated by elements of the form $V + W - (V \oplus W)$. $R(G)$ is called the representation ring of $G$, or the Grothendieck group of $G$, denoted by $K_0(G)$, elements of $R(G)$ are called virtual representations.*

*The ring structure on $R(G)$ is the tensor product, defined on the generators of $R(G)$, and extended by linearity.*

**Remark 3.0.24.** We have the following remarks:

1. A character defines a ring homomorphism from $R(G)$ to $\mathscr{C}_G$
2. $\chi$ is injective is equivalent to a representation is determined by its character, the image of $\chi$ are called virtual characters.
3. $\chi_{\mathbb{C}} : R(G) \otimes_{\mathbb{Z}} \mathbb{C} \to \mathscr{C}_G$ is an isomorphism.
4. The virtual characters form a lattice $\Lambda \cong \mathbb{Z}^c \subset \mathscr{C}_G$. The actual characters form a cone $\Lambda_0 \cong \mathbb{N}^0 \subset \Lambda$.
5. By 3. we can define an inner product on $R(G)$ by
$$(V, W) = \dim \operatorname{Hom}_G(V, W)$$

**Example 3.0.25.** Let $G = C_n$, then $R(C_n) = \mathbb{Z}[x]/(x^n - 1)$, where $X$ correspond to the representation of a primitive $n$-th root of unity.

**Example 3.0.26.** $R(S_3) \cong \mathbb{Z}[x,y]/(xy-y, x^2-1, y^2-x-y-1)$. We can identify $x$ to the alternating representation $U'$, $y$ to the standard representation $V$ and 1 to the trivial representation.

Goal: Determine $R(S_n)$ for all $n$ and determine all irreducible representations of $S_n$ for all $n$.

# Part 2. Symmetric functions

## 4. Young tableau

**Definition 4.1** (Composition of $n$)**.** *A composition of $n$ is an odered sequence $(\alpha_1, \dots, \alpha_r)$ such that $\alpha_i \in \mathbb{Z}_{>0}$ and $\sum \alpha_i = n$; A weak composition of $n$ is a (finite or infinite) ordered sequence $(\alpha_1, \dots)$ such that $\alpha_i \in \mathbb{Z}_{>0}, \sum \alpha_i = n$ and $|\{i \in \mathbb{Z}_{>0} \mid \alpha_i \neq 0\}| < \infty$.*

**Definition 4.2** (Partition)**.** *A partition is any weak composition $\lambda = (\lambda_1, \dots)$ such that $\lambda_i \geq \lambda_{i+1}$ for all $i$. The nonzero $\lambda_i$ are called parts. The number of parts is the length of $\lambda$, denoted by $l(\lambda)$. $|\lambda| = \sum \lambda_i$ is the weight of $\lambda$. If $|\lambda| = n$, then we write $\lambda \vdash n$ and say $\lambda$ is a partition of $n$.*

**Notation 4.3.** The set consists of all partition of $n$ is denoted by $\mathcal{P}_n$.

**Notation 4.4** (Exponential notation)**.** If $j$ appears $m_j$ times in $\lambda$, we write $\lambda = (1^{m_1} 2^{m_2} \dots)$

**Lemma 4.5.** *We have the following correspondence*

$$\mathrm{Conj}(S_n) \longleftrightarrow \mathcal{P}_n$$

*Proof.* Recall that $w \in S_n$ factorizes uniquely as a product of disjoint cycles

$$w = (i_1 \dots i_{\alpha_1}) \dots (i_{n-\alpha_r+1} \dots i_n)$$

of order $\alpha_1, \dots, \alpha_r$. The order in which the cycles are listed is irrelevent.

If $\alpha_1 \geq \dots \geq \alpha_r$, then $\alpha = (\alpha_1, \dots, \alpha_r)$ is a partion of $n$, called the cycle type $\alpha(w)$ of $w$.

Let $v, w \in S_n$, if $v(i) = j$, then

$$w \circ v \circ w^{-1}(w(i)) = w(j)$$

so $v$ and $w \circ v \circ w^{-1}$ have the same cycle type, i.e. $\alpha(v) = \alpha(w \circ v \circ w^{-1})$. So $\alpha(w)$ determines $w \in S_n$ up to conjugacy. □

**Theorem 4.6** (Euler)**.** *$p(n) = |\mathcal{P}_n|$, where*

$$\sum_{n=0}^{\infty} p(n) x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

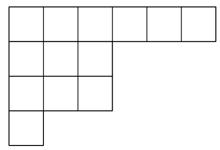
**Example 4.7.**

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p(n)$ | 1 | 1 | 2 | 3 | 5 | 7 | 11 | 15 | 22 | 30 | 42 |

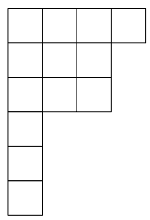
**Definition 4.8** (Young subgroup)**.** *For $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathcal{P}_n$. A Young subgroup is a subgroup of $S_n$ given as*

$$S_\lambda = S_{\{1,\dots,\lambda_1\}} \times S_{\{\lambda_1+1,\dots,\lambda_2\}} \times \cdots \times S_{\{n-\lambda_r+1,\dots,\lambda_n\}}$$

**Definition 4.9** (Young diagram)**.** *The Young diagram $D(\lambda)$ of $\lambda \in \mathcal{P}_n$ is $D(\lambda) = \{(i,j) \in \mathbb{Z}^2 \mid 1 \leq i \leq \lambda_j\}$. We draw a box for each point $(i,j)$.*

**Example 4.10.** $D((6,3,3,1)) =$ 

**Definition 4.11** (Conjugate of a partition)**.** *The conjugate of $\lambda \in \mathcal{P}_n$ is the partition $\lambda' \in \mathcal{P}_n$ whose Young diagram $D(\lambda')$ is the transpose of $D(\lambda)$.*

**Example 4.12.** $D((6,3,3,1))' =$ 

**Lemma 4.13.** *Let $\lambda$ be a partition, and $m \geq \lambda_1, n \geq \lambda'_1$. The $m+n$ numbers $\lambda_i + n - i (1 \leq i \leq n)$, $n - 1 + j - \lambda'_j (1 \leq j \leq m)$ are a permutation of $\{0, 1, 2, 3, \dots, m+n-1\}$*

*Proof.* Clearly $D(\lambda) \subset D(m^n)$. Take a path corresponding to $D(\lambda)$ from the lower left corner to the upper right corner, number the segment of the path by $0, 1, \dots, m+n-1$. The vertical segments are $\lambda_i + n - 1, 1 \leq i \leq n$. The horizontal segments (by transpotion) are $(m+n-1) - (\lambda'_j + m - j) = n - \lambda'_j + j - 1, 1 \leq j \leq m$. □

**Remark 4.14.** The lemma is equivalent to the identity

$$f_{\lambda,n}(t) + t^{m+n-1} f_{\lambda',m}(t^{-1}) = \frac{1 - t^{m+n}}{1-t}$$

**Definition 4.15** (Operations on partitions)**.** *Let $\lambda, \mu$ be partitions. We define $\lambda + \mu$ by $(\lambda + \mu)_i = \lambda_i + \mu_i$; $\lambda \cup \mu$ is partition in which $\lambda_i, \mu_j$ are arranged decreasing in order; $\lambda\mu$ is defined by $(\lambda\mu)_i = \lambda_i \mu_j$; $\lambda \times \mu$ is the partition in which* $\min\{\lambda_i, \mu_j\}$ *are arranged in decreasing order.*

**Example 4.16.** If we take $\lambda = (3,2,1)$ and $\mu = (2,2)$, compute as follows to see what's going on

$$\lambda + \mu = (5,4,1), \quad \lambda\mu = (6,4)$$
$$\lambda \cup \mu = (3,2,2,2,1), \quad \lambda \times \mu = (2,2,2,2,1,1)$$

**Lemma 4.17.** *We have the following relation between above operations*

$$(\lambda \cup \mu)' = \lambda' + \mu'$$
$$(\lambda \times \mu)' = \lambda'\mu'$$

*Proof.* $D(\lambda \cup \mu)$ is obtained from the rows of $D(\lambda)$ and $D(\mu)$ and arranging in order of decreasing length, so we have

$$(\lambda \cup \mu)'_k = \lambda'_k + \mu'_k$$

And

$$(\lambda \times \mu)'_k = \{(i, j) \in \mathbb{Z}^2 \mid \lambda_i \geq k, \mu_j \geq k\} = \lambda'_k \mu'_k$$

□

**Definition 4.18** (Orderings)**.** *Let $\lambda, \mu \in \mathcal{P}_n$, then*

1. *Containing order $C_n$: $(\lambda, \mu) \in C_n$ if and only if $\mu_i \leq \lambda_i, \forall i \geq 1$. We write $\mu \subseteq \lambda$ instead of $(\lambda, \mu) \in C_n$.*
2. *Reverse lexicographic ordering $L_n$: $(\lambda, \mu) \in L_n$ if and only if for $\lambda = \mu$ or the first non-vanishing difference $\lambda_i - \mu_i$ is positive.*
3. *reverse lexicographic ordering $L'_n$: $(\lambda, \mu) \in L'_n$ if and only if $\lambda = \mu$ or the first non-vanishing difference $\lambda^*_i - \mu^*_i$ is negative, where $\lambda^*_i = \lambda_{n+1-i}$.*
4. *Natural/Dominance ordering $N_n$: $(\lambda, \mu) \in N_n$ if and only if $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all $i \geq 1$. We write $\lambda \geq \mu$ instead of $(\lambda, \mu) \in N_n$.*

**Remark 4.19.** $C_n$ and $N_n$ are only partial orderings, but $L_n$ and $L'_n$ are total orderings.

**Definition 4.20** (Cover & Hasse diagram)**.** *If $(A, \leq)$ is a poset, $b, c \in A$, we say that $b$ is covered by $c$, written $b \prec c$ , if $b < c$ and there is no $d \in A$ such that $b < d < c$; The Hasse diagram of $A$ consists of vertices corresponding to element $a \in A$, and an arrow from the vertex $b$ to vertex $c$ if $b \prec c$.*

**Example 4.21.** If we consider dominance ordering on $\mathcal{P}_6$[†]

$$\begin{tikzcd}
& (6) & \\
& (5,1) \arrow[u] & \\
& (4,2) \arrow[u] & \\
(3^2) \arrow[ur] & & (4,1^2) \arrow[ul] \\
& (3,2,1) \arrow[ul] \arrow[ur] & \\
(3,1^3) \arrow[ur] & & (2^3) \arrow[ul] \\
& (2^2,1^2) \arrow[ul] \arrow[ur] & \\
& (2,1^4) \arrow[u] & \\
& (1^6) \arrow[u] &
\end{tikzcd}$$

[†] Here I really want to draw a Hasse diagram in the form of Young diagram, but there is no enough space for me to draw down all my ideas (smile).

**Lemma 4.22.** *Let $\lambda, \mu \in \mathcal{P}_n$. Then $\lambda \geq \mu$ implies $(\lambda, \mu) \in L_n \cap L'_n$*

*Proof.* Suppose that $\lambda \geq \mu$. Then either $\lambda_1 > \mu_1$, in which case $(\lambda, \mu) \in L_n$, or else $\lambda_1 = \mu_1$. In that case either $\lambda_2 > \mu_2$, in which case again $(\lambda, \mu) \in L_n$, or else $\lambda_2 = \mu_2$. Continuing in this way, we see that $(\lambda, \mu) \in L_n$.

Also, for each $i \geq 1$, we have

$$\begin{aligned} \lambda_{i+1} + \lambda_{i+2} + \ldots &= n - (\lambda_1 + \ldots + \lambda_i) \\ &\leqslant n - (\mu_1 + \ldots + \mu_i) \\ &= \mu_{i+1} + \mu_{i+2} + \ldots \end{aligned}$$

Hence the same reasoning as before shows that $(\lambda, \mu) \in L'_n$. □

**Lemma 4.23.** *Let $\lambda, \mu \in \mathcal{P}_n$, then $\lambda \geq \mu$ is equivalent to $\mu' \geq \lambda'$.*

*Proof.* It suffices to show one direction. Suppose $\lambda' \ngeq \mu'$, then for some $i \geq 1$, we have

$$(*) \qquad \begin{cases} \lambda'_1 + \cdots + \lambda'_j \leq \mu'_1 + \cdots + \mu'_j, & 1 \leq j \leq i-1 \\ \lambda'_1 + \cdots + \lambda'_i > \mu'_1 + \cdots + \mu'_i \end{cases}$$

which implies

$$\lambda'_i > \mu'_i$$

Let $l = \lambda'_i$ and $m = \mu'_i$. From $(*)$ it follows that

$$\lambda'_{i+1} + \lambda'_{i+2} + \cdots < \mu'_{i+1} + \mu'_{i+2} + \ldots$$

and denote this equation by $(**)$.

Now $\lambda'_{i+1} + \lambda'_{i+2} + \ldots$ is equal to the number of nodes in the diagram of $\lambda$ which lie to the right of the $i$-th column, and therefore

$$\lambda'_{i+1} + \lambda'_{i+2} + \cdots = \sum_{j=1}^{l} (\lambda_j - i)$$

Likewise

$$\mu'_{i+1} + \mu'_{i+2} + \ldots = \sum_{j=1}^{m} (\mu_j - i)$$

Hence from $(**)$ we have

$$\sum_{j=1}^{m} (\mu_j - i) > \sum_{j=1}^{l} (\lambda_j - i) \geqslant \sum_{j=1}^{m} (\lambda_j - i)$$

which implies

$$\mu_1 + \ldots + \mu_m > \lambda_1 + \ldots + \lambda_m$$

a contradiction. □

**Definition 4.24** (Young tableau)**.** *A Young tableau is a map $T(\lambda) : D(\lambda) \to \mathbb{N}$, defined by $(i, j) \mapsto T(\lambda)_{i,j} = k$. $\lambda$ is called the shape of $T(\lambda)$. If $T_{i,j} \leq T_{i,j+1}$ and $T_{i,j} < T_{i+1,j}$ for all $(i, j) \in D(\lambda)$, then $T(\lambda)$ is called semi-standard. Let $\alpha_k = |\{(i, j) \in D(\lambda) \mid T(\lambda)_{i,j} = k\}|$, then $\alpha = (\alpha_1, \ldots)$ is called the weight or type of $T(\lambda)$, If $\alpha = (1, 1, \ldots, 1)$, $T(\lambda)$ is called standard.*

**Example 4.25.** Consider the following two Young tableau

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&2&2&3&3&5\\ \hline 2&3&5&5\\ \cline{1-4} 4&4&7&7\\ \cline{1-4} 5&7\\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|c|c|}\hline 1&3&7&12&8&15\\ \hline 2&5&10&14\\ \cline{1-4} 4&8&11&16\\ \cline{1-4} 6&9\\ \cline{1-2}\end{array}$$

They are both Young tableau with shape $(6, 4, 4, 2)$, but the first one has type $(1, 3, 3, 2, 4, 0, 3)$, while the second one is standard.

**Definition 4.26** (Kostka number)**.** *Let $\lambda \in \mathcal{P}_n$, $\alpha$ be a weak composition of $n$. Then Kostka number $K_{\lambda\alpha}$ is the number of semistandard tableau $T(\lambda)$ of weight $\alpha$.*

**Lemma 4.27.** *For $\lambda, \mu \in \mathcal{P}_n$, then $K_{\lambda\mu} = 0$ unless $\lambda \geq \mu$.*

*Proof.* Let $T(\lambda)$ be a Young tableau of weight $\mu$. For all $r \geq 1$, there are $\mu_1 + \cdots + \mu_r$ symbols $\leq r$ in $T(\lambda)$. Columns are strictly increasing, then these $\mu_1 + \cdots + \mu_r$ symbols must lie in the first $r$ rows. So

$$\mu_1 + \cdots + \mu_r \leq \lambda_1 + \cdots + \lambda_r, \quad \forall r \geq 1$$

That is, $\mu \leq \lambda$. $\square$

$S_n$ acts on $\mathbb{Z}^n$ by permuting coordinates, the fundamental domain for this action is

$$P_n = \{b \in \mathbb{Z}^n \mid b_n \geq \cdots \geq b_1\}$$

i.e. for $a \in \mathbb{Z}^n$, $S_n a \cap P_n = \{a^+\}$ for some $a^+ \in \mathbb{Z}^n$. In fact, $a^+$ is obtained from $a$ by rearranging $a_1, \ldots, a_n$ in decreasing order.

For $a, b \in \mathbb{Z}^n$, we define

$$a \geq b \iff a_1 + \cdots + a_i \geq b_1 + \cdots + b_i, \quad \forall i \geq 1$$

**Lemma 4.28.** *Let $a \in \mathbb{Z}^n$, then*

$$a \in P_n \iff a \geq wa, \forall w \in S_n$$

*Proof.* Suppose $a \in P_n$. If $wa = b$, then $(b_1, \ldots, b_n)$ is a permutation of $(a_1, \ldots, a_n)$, so $a_1 + \cdots + a_i \geq b_1 + \cdots + b_i, \forall i \geq 1$.

Conversely, if $a \geq wa$ for all $w \in S_n$. Then

$$(a_1, \ldots, a_n) \geq (a_1, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_n)$$

then we get

$$a_1 + \cdots + a_i \geq a_1 + \cdots + a_{i-1} + a_{i+1} \implies a_i \geq a_{i+1}$$

If we do this several times, we will see $a \in P_n$. $\square$

Let $\delta = (n-1, n-2, \ldots, 1, 0) \in P_n$, then we have

**Lemma 4.29.** *Let $a \in P_n$. Then for each $w \in S_n$, we have $(a+\delta-w\delta)^+ \geq a$.*

*Proof.* Since $\delta \in P_n$, then we have $\delta \geq w\delta$, hence

$$a + \delta - w\delta \geq a$$

Let $b = (a + \delta - w\delta)^+$. Then again by Lemma 4.28 we have

$$b \geq a + \delta - w\delta$$

Hence $b \geq a$. □

For each pair of integers $i, j$ such that $1 \leq i < j \leq n$ define $R_{ij} : \mathbb{Z}^n \to \mathbb{Z}^n$ by

$$R_{ij}(a) = (a_1, \ldots, a_i + 1, \ldots, a_j - 1, \ldots, a_n)$$

Any product $R = \prod_{i<j} R_{ij}^{r_{ij}}$ is called a raising operator. The order of the terms in the product is immaterial, since they commute with each other.

The following lemma explains why it is called raising:

**Lemma 4.30.** *Let $a \in \mathbb{Z}^n$ and let $R$ be a raising operator. Then*

$$Ra \geq a$$

*Proof.* For we may assume that $R = R_{ij}$, in which case the result is obvious. □

However, the converse of the lemma still holds

**Lemma 4.31.** *Let $a, b \in \mathbb{Z}^n$ be such that $a \leq b$ and $a_1 + \cdots + a_n = b_1 + \cdots + b_n$. Then there exists a raising operator $R$ such that $b = Ra$.*

*Proof.* We omit it here, since we won't use this result later. Readers may refer to [2] for more details. □

## 5. The ring of symmetric functions

The symmetric group $S_n$ acts on the ring $\mathbb{Z}[x_1, \ldots, x_n]$ of polynomials in $n$ variables $x_1, \ldots, x_n$ with integer coefficients by permuting the variables, that is

$$(wp)(x_1, \ldots, x_n) = p(x_{w(1)}, \ldots, x_{w(n)}), \quad w \in S_n, p \in \mathbb{Z}[x_1, \ldots, x_n]$$

**Definition 5.1** (Symmetric polynomial)**.** *$p \in \mathbb{Z}[x_1, \ldots, x_n]$ is called symmetric if it is invariant under the action of $S_n$.*

The symmetric polynomials form a subring

$$\Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^{S_n} \subset \mathbb{Z}[x_1, \ldots, x_n]$$

Note that $\Lambda_n$ is a graded ring, i.e. $\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k$, where $\Lambda_n^k = \{p \in \Lambda_n \mid \deg p = k\} \cup \{0\}$

**Definition 5.2.** *Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. We set $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. Let $\lambda$ be any partition of length $\leq n$. We define the polynomial*

$$m_\lambda(x_1, \ldots, x_n) = \sum_\alpha x^\alpha$$

*where $\alpha$ runs over all distinct permutation of $\lambda = (\lambda_1, \ldots, \lambda_n)$.*

**Example 5.3.** Let $n = 3$ and $\lambda = (2, 1, 0)$ to see what's going on

$$m_{(2,1)} = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + x_2^2 x_3$$

Since we have all permutations of $(2, 1, 0)$ are listed as follows

$$(2, 1, 0), (2, 0, 1), (1, 2, 0), (1, 0, 2), (0, 1, 2), (0, 2, 1)$$

**Remark 5.4.** The $(m_\lambda)_{l(\lambda)\leq n}$ form a $\mathbb{Z}$-basis of $\Lambda_n$. And $(m_\lambda)_{|\lambda|=k, l(\lambda)\leq n}$ form a a $\mathbb{Z}$-basis of $\Lambda_n^k$.

**Definition 5.5** (Inverse system)**.** *Let $(I, \leq)$ be a directed set. Let $(A_i)_{i\in I}$ be a family of groups, rings, modules, indexed by $I$, and $(f_{ij})_{i,j\in I}$ be a family of morphisms with $f_{ij} : A_i \to A_j$, such that*

1. $f_{ii} = \mathrm{id}_{A_i}$;
2. $f_{ij} = f_{ij} \circ f_{jk}$ *for all* $i, j, k \in I$

*The pair $(A_i, f_{ij})_{i,j\in I}$ is called an inverse system over $I$.*

**Definition 5.6** (Inverse limit)**.** *Let $(A_i, f_{ij})_{i,j\in I}$ be an inverse system. Let $x_i \in A_i, x_j \in A_j$. We define*

$$x_i \sim x_j \iff \textit{there exists } k \in I \textit{ with } i \leq k, j \leq k \textit{ and } f_{ki}(x_i) = f_{kj}(x_j)$$

*We define the inverse limit of this inverse system by*

$$\varprojlim_{i\in I} A_i = \prod A_i / \sim$$

We can use inverse limit to define our symmetric functions.

Let $k$ be fixed, let $m \geq n$, and consider

$$\mathbb{Z}[x_1, \ldots, x_m] \to \mathbb{Z}[x_1, \ldots, x_n]$$

Which sends each of $x_{n+1}, \ldots, x_m$ to zero and the other $x_i$ to themselves. On restriction to $\Lambda_m$ this gives a homomorphism as follows

$$\rho_{m,n} : \Lambda_m \to \Lambda_n$$

whose effect on the basis $(m_\lambda)$ is easily described as follows

$$m_\lambda(x_1, \ldots, x_m) \mapsto \begin{cases} m_\lambda(x_1, \ldots, x_n), & l(\lambda) \leq n \\ 0, & \text{otherwise} \end{cases}$$

$\rho_{m,n}$ is a surjective ring homomorphism.

On restriction to $\Lambda_m^k$ we have homomorphisms

$$\rho_{m,n}^k : \Lambda_m^k \to \Lambda_n^k$$

for all $k > 0$ and $m \geq n$, which are always surjective, and are bijective for $m \geq n \geq k$.

So we have $(\Lambda_n^k, \rho_{m,n}^k)$ is an inverse system over $\mathbb{N}$. We define

$$\Lambda^k = \varprojlim_n \Lambda_n^k$$

Let us clearify the elements in $\Lambda^k$, as what we defined, an element of $\Lambda^k$ is a sequence $f = (P_n)_{n\geq 0}$, where $P_n = P_n(x_1, \ldots, x_n)$ is a homogenous symmetric polynomial of degree $k$ in $x_1, \ldots, x_n$, and $f_m(x_1, \ldots, x_m, 0, \ldots, 0) = P_n(x_1, \ldots, x_n)$ whenever $m \geq n$. Since $\rho^k_{m,n}$ is an isomorphism for $m \geq n \geq k$, it follows that the projection

$$\rho^k_n : \Lambda^k \to \Lambda^k_n$$

which sends $f$ to $P_n$ is an isomorphism for all $n \geq k$, and hence that $\Lambda^k$ has a $\mathbb{Z}$-basis consisting of the monomial symmetric functions $m_\lambda$ (for all partitions $\lambda$ of $k$) defined by

$$\rho^k_n(m_\lambda) = m_\lambda(x_1, \ldots, x_n)$$

for all $n \geq k$. Hence $\Lambda^k$ is a free $\mathbb{Z}$-module of rank $p(k)$, the number of partitions of $k$.

**Example 5.7.** The above discussion may be a little abstract, let's compute a concrete example to show what's going on

If we let $m = 3, n = 2$, and let $\lambda = (1, 1)$, then

$$m_{(1,1)}(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_1 + x_2x_3 + x_3x_1 + x_3x_2$$

So

$$\rho_{3,2}(m_{(1,1)}(x_1, x_2, x_3)) = m_{(1,1)}(x_1, x_2) = x_1x_2 + x_2x_1$$

and in this case, $l(\lambda) = 2 = n$. If we let $\lambda = (1, 1, 1)$, then

$$\rho_{3,2}(m_{(1,1,1)}) = \rho_{3,2}(x_1x_2x_3) = 0$$

is quite natural.

Furthermore, if we let $k = n = 2, m = 3$, then obviously $\Lambda^2_3$ is spanned by

$$m_{(2,0)}(x_1, x_2, x_3) = x_1^2+x_2^2+x_3^3, \quad m_{(1,1)}(x_1, x_2, x_3) = x_1x_2+x_1x_3+x_2x_1+x_2x_3+x_3x_1+x_3x_2$$

and $\Lambda^2_2$ is spanned by

$$m_{(2,0)}(x_1, x_2) = x_1^2 + x_2^2, \quad m_{(1,1)}(x_1, x_2) = x_1x_2 + x_2x_1$$

So $\rho^2_{3,2}$ is clearly an isomorphism. Hope this example can help you to get a better understanding.

**Definition 5.8** (The ring of symmetric functions)**.** *We define*

$$\Lambda = \bigoplus_{k\geq 0} \Lambda^k$$

*$\Lambda$ is the free $\mathbb{Z}$-module generated by the $m_\lambda$ for all partitions $\lambda$, and is called the ring of symmetric functions. The $m_\lambda$ are called monomial symmetric functions.*

**Remark 5.9.** We have the following remarks

1. For any communicative ring $R$ in place of $\mathbb{Z}$, we can define a ring $\Lambda_R$ satisfying $\Lambda_R \cong \Lambda \otimes_{\mathbb{Z}} R$.
2. We have surjective ring homomorphisms $\rho_n = \bigoplus_{k\geq 0} \rho^k_n : \Lambda \to \Lambda_n, n \geq 0$. $\rho_n$ is an isomorphism in degrees $k \leq n$.

5.1. **Elementary symmetric function.** As we can see above, $m_\lambda$ for any $\lambda$ form a basis of the ring of symmetric functions. Now we will give several different basis of it, some of them are quite important to the representation theory of $S_n$.

First of them is elementary symmetric function

**Definition 5.10** (Elementary symmetric function)**.** *Let $e_0 = 1$ and $e_r = \sum_{i_1<\cdots<i_r} x_{i_1}\dots x_{i_r} = m_{(1^r)}$ for some $r \geq 1$.*

*For each partition $\lambda = (\lambda_1, \lambda_2, \dots)$ define $e_\lambda = e_{\lambda_1} e_{\lambda_2}\dots$. Then $e_\lambda$ is called elementary symmetric functions.*

**Remark 5.11.** The generating function for the $e_r$ is

$$E(t) = \sum_{r=0}^{\infty} e_r t^r = \prod_{i\geq 1}(1 + x_i t)$$

**Remark 5.12.** If the number of variables is finite, say $n$, then

$$\rho_n(e_r) = 0 \implies \sum_{r=0}^{n} e_r t^r = \prod_{i=1}^{n}(1 + x_i t) \in \Lambda_n[t]$$

**Lemma 5.13.** *Let $\lambda$ be a partition, $\lambda'$ its conjugate. Then*

$$e_{\lambda'} = m_\lambda + \sum_{\mu<\lambda} a_{\lambda\mu} m_\mu, \quad a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$$

*Proof.* When we multiply out the product $e_{\lambda'} = e_{\lambda'_1} e_{\lambda'_2}\dots$, we will obtain a sum of monomials, each of which is of the form

$$(x_{i_1} x_{i_2}\dots)(x_{j_1} x_{j_2}\dots)\dots = x^\alpha$$

where $i_1 < i_2 < \cdots < i_{\lambda'_1}, j_1 < j_2 < \cdots < j_{\lambda'_2}$, and so on.

Put the numbers $i_1, \dots, i_{\lambda'_1}$ into the first column of $D(\lambda)$ and similarly for the remaining numbers. The symbols $\leq r$ occur in the top $r$ rows of $D(\lambda)$. Hence we have

$$\alpha_1 + \cdots + \alpha_r \leq \lambda_1 + \cdots + \lambda_r$$

for each $r \geq 1$, i.e. we have $\alpha \leq \lambda$. If follows Lemma 4.28 that

$$e_{\lambda'} = \sum_{\mu\leq\lambda} a_{\lambda\mu} m_\mu$$

with $a_{\lambda\mu} \geq 0$ for each $\mu \leq \lambda$, and the argument above also shows that the monomial $x^\lambda$ occurs exactly once, so that $a_{\lambda\lambda} = 1$. □

**Proposition 5.14.** *We have*

$$\Lambda \cong \mathbb{Z}[e_1, e_2, \dots]$$

*and $e_r$ are algebraically independent over $\mathbb{Z}$.*

*Proof.* By above lemma, the $e_r$ form a $\mathbb{Z}$-basis since the $m_\lambda$ do so. Then every $f \in \Lambda$ uniquely expressible as a polynomial in $e_r, r \geq 0$. □

5.2. **Complete symmetric function.**

**Definition 5.15.** *Let* $h_0 = 1$*, and* $h_r = \sum_{\mu \vdash r} m_\mu, r \geq 1$*. For each partition* $\lambda = (\lambda_1, \lambda_2, \dots, )$*, we define* $h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots$*, called the complete symmetric functions.*

**Remark 5.16.** Note that $e_1 = h_1$. And it will be convenient to define $h_r, e_r = 0$ to be zero for $r < 0$.

**Lemma 5.17.** *The generating function of the* $h_r$ *is*

$$H(t) = \sum_{r \geq 0} h_r t^r = \prod_{i \geq 1} (1 - x_i t)^{-1}$$

*Furthermore, we have*

$$H(t)E(-t) = 1$$

*Proof.* To see the first, use the fact

$$\frac{1}{1 - x_i t} = \sum_k x_i^k t^k$$

and multiply these geometric series together.

Use the fact that the generating function of $e_r$ is

$$E(t) = \sum_{r \geq 0} e_r t^r = \prod_{i \geq 1} (1 + x_i t)$$

together with what we have proven to see the second. $\square$

**Remark 5.18.** $H(t)E(-t) = 1$ is equivalent to

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0$$

for all $n \geq 1$.

Since $e_r$ are algebraically independent, we may define a homomorphism of graded rings as follows

**Definition 5.19.**

$$\begin{aligned} \omega : \Lambda &\to \Lambda \\ e_r &\mapsto h_r \end{aligned}$$

**Lemma 5.20.** $\omega$ *is a involution.*

*Proof.* The relations

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0, \quad \forall n \geq 1$$

are symmetric with respect to interchanging $e_r$ and $h_r$. $\square$

**Proposition 5.21.** *We have*

$$\Lambda \cong \mathbb{Z}[h_1, h_2, \dots]$$

*and* $h_r$ *are algebraically independent over* $\mathbb{Z}$*.*

*Proof.* Follows from that $\omega^2 = \mathrm{Id}$, that is $\omega$ is an automorphism of $\Lambda$. $\square$

**Remark 5.22.** If the number of variables is finite, say $n$, then $\omega|_\Lambda = \mathrm{id}\,|_{\Lambda_n}$, and $\Lambda_n \cong \mathbb{Z}[h_1, \dots, h_n]$ with $h_r$ are algebraically independent over $\mathbb{Z}$, but $h_{r+1}, \dots$ are nonzero polynomials in $h_1, \dots, h_n$.

**Remark 5.23.** We could define $f_\lambda = \omega(m_\lambda)$ and would obtain another basis of $\Lambda$, but these play no role later on.

Remark 5.18 lead to a determinant identity which we shall make use of later. Let $N$ be a positive integer and consider the matrices of $N+1$ rows and columns

$$H = (h_{i-j})_{0\leq i,j\leq N}, \quad E = ((-1)^{i-j} e_{i-j})_{0\leq i,j\leq N}$$

Then $E, H$ are lower unitriangular, so we have $\det E = \det H = 1$. Moreover, Remark 5.18 shows that

$$\sum_{r=0}^{N} (-1)^r e_r h_{n-r} = 0$$

which implies that

$$EH = \mathrm{Id}$$

It follows that each minor of $H$ is equal to the complementary cofactor of $E^T$, the transpose of $E$.

Now let $\lambda, \mu$ be partitions of length $\leq p$ such that $\lambda', \mu'$ have length $\leq p$. $p + q = N + 1$. And consider the minor of $H$ with row indices $\lambda_i + p - i (1 \leq i \leq p)$ and columns indices $\mu_i + p - i (1 \leq i \leq p$. By Lemma 4.13 the complementary cofactor of $E^T$ has row indices $p-1+j-\lambda'_j(1 \leq j \leq q$ and column indices $p - 1 + j - \mu'_j(1 \leq j \leq p)$. Hence we have

$$\det(h_{\lambda_1-\mu_j-i+j})_{1<i,j<p} = (-1)^{|\lambda|+|\mu|} \det((-1)^{\lambda'_i-\mu'_j-i+j} e_{\lambda'_i-\mu'_j-i+j})_{1<i,j<q}$$

The minus signs cancel out, and we have proven the following results:

**Lemma 5.24.** *Let $\lambda, \mu$ be partitions of length $\leq p$ such that $\lambda', \mu'$ have length $\leq p$. $p + q = N + 1$. Then*

$$\det(h_{\lambda_i-\mu_j-i+j})_{0\leq i,j\leq p} = \det(e_{\lambda'_i-\mu'_i-i+j})_{0\leq i,j\leq q}$$

*In particular, if $\mu = \varnothing$, then $\det(h_{\lambda_i-i+j}) = \det(e_{\lambda'_j-i+j})$.*

### 5.3. **Power sums.**

**Definition 5.25.** *Let $p_r = \sum_i x_i^r = m_{(r)}, r \geq 1$, $p_r$ is call the $r$-th power sum. For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, we define $p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots$*

**Lemma 5.26.** *The generating function of $p_r$ is*

$$P(t) = \sum_{r\geq 1} p_r t^{r-1} = \frac{H(t)}{H'(t)}$$

*Furthermore, we have the following properties*

1. $P(-t) = \frac{E'(t)}{E(t)}$
2. $nh_n = \sum_{r=1}^{n} p_r h_{n-r}$
3. $ne_n = \sum_{r=1}^{n} (-1)^{r-1} p_r e_{n-r}$

*Proof.* We compute as follows

$$\begin{aligned} P(t) &= \sum_{i\geq 1}\sum_{r\geq 1} x_i^r t^{r-1} \\ &= \sum_{i\geq 1} \frac{x_i}{1-x_i t} \\ &= \sum_{i\geq 1} \frac{\mathrm{d}}{\mathrm{d}t} \log\left(\frac{1}{1-x_i t}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \log \prod_{i\geq 1} (1-x_i t)^{-1} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \log H(t) \\ &= \frac{H'(t)}{H(t)} \end{aligned}$$

Similarly we have $P(-t) = \frac{\mathrm{d}}{\mathrm{d}t} \log E(t)$.

From above we have

$$\begin{aligned} nh_n &= \sum_{r=1}^{n} p_r h_{n-r} \\ ne_n &= \sum_{r=1}^{n} (-1)^{r-1} p_r e_{n-r} \end{aligned}$$

for $n \geq 1$. □

**Remark 5.27.** The second and third equations enable us to express the $h's$ and the $e's$ in terms of the $p's$, and vice versa. In fact, the third equations are due to Isaac Newton, and are known as Newton's formulas. And from the second formula, it is clear that $h_n \in \mathbb{Q}[p_1, \ldots, p_n]$ and $p_n \in \mathbb{Z}[h_1, \ldots, h_n]$, and hence

$$\mathbb{Q}[p_1, \ldots, p_n] = \mathbb{Q}[h_1, \ldots, h_n]$$

Since the $h_r$ are algebraically independent over $\mathbb{Z}$, and hence also over $\mathbb{Q}$, it follows that:

**Proposition 5.28.** $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}[p_1, p_2, \ldots]$ *and the* $p_r$ *are algebraically independent over* $\mathbb{Q}$*. The* $p_r$ *form a* $\mathbb{Q}$*-basis for* $\Lambda_{\mathbb{Q}}$*.*

**Definition 5.29.** *Let* $\lambda = (1^{m_1} 2^{m_2} \ldots)$ *be a partition in exponential notation. We define*

$$\begin{aligned} \varepsilon_\lambda &= (-1)^{m_2+m_4+\ldots} = (-1)^{|\lambda|-l(\lambda)} \\ z_\lambda &= \prod_{j\geq 1} j^{m_j} m_j! \end{aligned}$$

**Remark 5.30.** Let $w \in S_n$ with cycle type $\alpha(w) = (1^{m_1}2^{m_2}\dots)$, then

$$\varepsilon_{\alpha(w)} = \begin{cases} 1, & w \text{ is even} \\ -1, & w \text{ is odd} \end{cases}$$

so we have $S_n \to \{\pm 1\}$ defined by $w \mapsto \varepsilon_{\alpha(w)}$ is the usual sign homomorphism.

**Lemma 5.31.** $\omega(p_\lambda) = \varepsilon_\lambda p_\lambda$

*Proof.* Since we have

$$\omega(E(t)) = H(t), \omega(H(t)) = E(t)$$

then we have

$$\omega(P(t)) = \omega(\frac{H'(t)}{H(t)}) = \frac{E'(t)}{E(t)} = P(-t)$$

then

$$\omega(p_n) = (-1)^{n-1}p_n, \quad \forall n \geq 1$$

then

$$\omega(p_\lambda) = (-1)^{\sum \lambda_i - \sum 1} p_\lambda = \varepsilon^\lambda p_\lambda$$

□

**Lemma 5.32.** *We have*

$$\begin{aligned} H(t) &= \sum_\lambda \frac{1}{z_\lambda} p_\lambda t^{|\lambda|}, \quad h_n = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} p_\lambda \\ E(t) &= \sum_\lambda \frac{\varepsilon_\lambda}{z_\lambda} p_\lambda t^{|\lambda|}, \quad e_n = \sum_{\lambda \vdash n} \frac{\varepsilon_\lambda}{z_\lambda} p_\lambda \end{aligned}$$

*Proof.* It suffices to prove the identity in the first row, since the one in the second row then follows by applying the involution $\omega$ and using the fact that $p_k$ is an eigenvector of $\omega$ with respect to $\varepsilon_\lambda$.

We compute as follows,

$$\begin{aligned} H(z) &= \exp \sum_{r \geq 1} p_r t^r / r \\ &= \prod_{r \geq 1} \exp(p_r t^r / r) \\ &= \prod_{r \geq 1} \sum_{m_r = 0}^{\infty} (p_r t^r)^{m_r} / r^{m_r} m_r! \\ &= \sum_\lambda z_\lambda^{-1} p_\lambda t^{|\lambda|} \end{aligned}$$

The first step follows from Lemma 5.26. □

## 6. SCHUR FUNCTIONS

**Lemma 6.1.** *Let $A_n = \{f \in \mathbb{Z}[x_1, \dots, x_n] \mid w(f) = \operatorname{sgn}(w)f, \forall w \in S_n\}$, then $A_n$ is a free module of rank 1 over $\Lambda_n$.*

*Proof.* Let $f \in A_n$, then $x_i - x_j, i \neq j$ divides $f$, since $f|_{x_i = x_j} = 0$, so we have $\prod_{i<j}(x_i - x_j)$ divides $f$. Then

$$f = \prod_{i<j}(x_i - x_j)g, \quad g \in \Lambda_n$$

So $A_n$ is generated by $\prod_{i<j}(x_i - x_j)$ over $\Lambda_n$, i.e. $A_n = \prod_{i<j}(x_i - x_j)\Lambda_n$ □

Let $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ be a monomial, and consider the polynomial $a_\alpha$ obtained by antisymmetrizing $x^\alpha$, that is

$$a_\alpha = \sum_{w \in S_n} \operatorname{sgn}(w) w(x^\alpha)$$

Clearly $a_\alpha$ is skew-symmetric, i.e. $a_\alpha \in A_n$. In particular, therefore $a_\alpha$ vanishes unless $\alpha_1, \dots, \alpha_n$ are all distinct. Hence we may as well assume that $\alpha_1 > \cdots > \alpha_n \geq 0$. And we may write $\alpha = \lambda + \delta$, where $\lambda$ is a partition[‡] with length $\leq n$ and $\delta = (n-1, n-2, \dots, 1, 0)$. Then

$$a_\alpha = a_{\lambda+\delta} = \sum_{w \in S_n} \operatorname{sgn}(w) w(x^{\lambda+\delta})$$

which can be written as a determinant.

**Lemma 6.2.** *Let $\lambda$ be a partition $l(\lambda) \leq n$, then*

1. *$a_{\lambda+\delta} = \det(x_i^{\lambda_j+n-j})_{1 \leq i,j \leq n}$. In particular, $a_\delta = \det(x_i^{n-j})_{1 \leq i,j \leq n} = \prod(x_i - x_j)$ is the Vandermonde determinant.*
2. *$a_{\lambda+\delta}$ is divisible by $a_\delta$.*

*Proof.* 1. follows from the Leibniz formula for the determinant $\det A = \sum_{w \in S_n} \operatorname{sgn}(w) \prod_{i=1}^r a_{i,w(i)}$.

2. follows from Lemma 6.1. □

**Definition 6.3.** *Let $\lambda$ be a partition, $l(\lambda) \leq n$, and $\delta = (n-1, n-2, \dots, 0) \in \mathbb{Z}_{\geq 0}^n$. We define the schur polynomial*

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta} \in \Lambda_n$$

Notice that the definition of $s_\lambda$ makes sense for any integer vector $\lambda \in \mathbb{Z}^n$ such that $\lambda + \delta$ has no negative parts. If $\lambda_i + n - i$ are not all distinct, then $s_\lambda = 0$. If they are all distinct, then we have $\lambda + \delta = w(\mu + \delta)$ for some $w \in S_n$ and some partition $\mu$, and $s_\lambda = \operatorname{sgn}(w) s_\mu$.

---

[‡] $\lambda$ is indeed a partition. Take an example, $\alpha_1 + 1 - n \geq \alpha_2 + 2 - n$ holds, since $\alpha_1 > \alpha_2$ is equivalent to $\alpha_1 \geq \alpha_2 + 1$

The polynomial $a_{\lambda+\delta}$ where $\lambda$ runs through all partitions of length $\leq n$, form a basis of $A_n$. Multiplication by $a_\delta$ is an isomorphism of $\Lambda_n$ onto $A_n$, since $A_n$ is the free $\Lambda_n$-module generated by $a_\delta$.

So we have proven

**Lemma 6.4.** *The schur polynomial $s_\lambda$, where $\lambda$ is a partition with $l(\lambda) \leq n$, form a $\mathbb{Z}$-basis of $\Lambda_n$.*

**Proposition 6.5.** *The $s_\lambda$ for all partitions $\lambda$ form a $\mathbb{Z}$-basis of $\Lambda$, called schur functions. The $s_\lambda$ for all partitions $\lambda$ with $|\lambda| = k$ form a $\mathbb{Z}$-basis of $\Lambda^k$.*

*Proof.* From the definition it follows that

$$a_{\lambda+\delta+(k^n)} = \prod_{i=1}^{n} x_i^k a_{\lambda+\delta}, \quad s_{\lambda+(k^n)} = s_\lambda$$

□

**Proposition 6.6.**

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \leq i,j \leq n}, \quad n \leq l(\lambda)$$
$$s_\lambda = \det(e_{\lambda'_i - i + j})_{1 \leq i,j \leq m}, \quad m \leq l(\lambda')$$

*Proof.* □

**Corollary 6.7.** *We have the following properties*

1. $\omega(s_\lambda) = s_{\lambda'}$
2. $s_{(n)} = h_n, s_{(1^n)} = e_n$

## 7. Orthogonality

Let $x = (x_1, x_2, x_3, \dots), y = (y_1, y_2, y_3, \dots)$ be finite or infinite sequences of variables. We denote the symmetric functions of the $x's$ by $s_\lambda(x), p_\lambda(x)$, etc. and the symmetric functions of the $y's$ by $s_\lambda(y), p_\lambda(y)$, etc.

**Proposition 7.1.** *We give three series expansions for the product*

$$\begin{aligned}\prod_{i,j}(1 - x_i y_j)^{-1} &= \sum_\lambda \frac{1}{z_\lambda} p_\lambda(x) p_\lambda(y) \\ &= \sum_\lambda h_\lambda(x) m_\lambda(y) \\ &= \sum_\lambda s_\lambda(x) s_\lambda(y)\end{aligned}$$

*Proof.* For the first one, Since we have

$$H(t) = \prod_i (1 - x_i t)^{-1} = \sum_\lambda z_k^{-1} p_\lambda t^{|\lambda|}$$

Choose as variables $x_iy_j$, then

$$\begin{aligned}\prod_{i,j}(1-x_iy_jt)^{-1} = H(t) &= \sum_\lambda \frac{1}{z_\lambda} p_\lambda(x_1y_1,\dots,x_iy_j,\dots,x_ny_n)t^{|\lambda|} \\ &= \sum_\lambda \frac{1}{z_\lambda} p_\lambda(x)p_\lambda(y)t^{|\lambda|}\end{aligned}$$

and set $t=1$ to get desired result.

For the second one,

$$\begin{aligned}\prod_{i,j}(1-x_iy_jt)^{-1} &= \prod_j H(y_j) \\ &= \prod_j \sum_{r=0}^{\infty} h_r(x)y_j^r \\ &= \sum_\alpha h_\alpha(x)y^\alpha \\ &= \sum_\lambda h_\lambda(x)m_\lambda(y)\end{aligned}$$

where $\alpha$ runs through all sequences $(\alpha_1,\alpha_2,\dots)$ of non-negative integers such that $\sum \alpha_i < \infty$, and $\lambda$ runs through all partitions.

For the third one is sometimes called Cauchy formula, we compute as

$$\begin{aligned}a_\delta(x)a_\delta(y)\prod_{i,j=1}^{n}(1-x_iy_j)^{-1} &= a_\delta(x)\sum_{w\in S_n}\operatorname{sgn}(w)w(y^\delta)\sum_\lambda h_\lambda(x)m_\lambda(y) \\ &= a_\delta(x)\sum_{w\in S_n}\sum_\lambda \operatorname{sgn}(w)y^{w\delta}h_\lambda(x)\sum_{\substack{\alpha \text{ is the} \\ \text{permutation of } \lambda}} y^\alpha \\ &= a_\delta(x)\sum_{w\in S_n,\alpha\in\mathbb{N}^n}\operatorname{sgn}(w)h_\alpha(x)y^{\alpha+w\delta} \\ &= \sum_{w\in S_n,\beta\in\mathbb{N}^n}(a_\delta(x)\operatorname{sgn}(w)h_{\beta-w\delta}(x))y^\beta \\ &= \sum_{\beta\in\mathbb{N}^n}a_\beta(x)y^\beta \qquad (\alpha_\beta = 0 \text{ if } \beta\neq w(\lambda+\delta), w\in S_n) \\ &= \sum_{w\in S_n}\sum_\lambda w(a_{\lambda+\delta}(x))y^{w(\lambda+\delta)} \\ &= \sum_\lambda a_{\lambda+\delta}(x)\sum_{w\in S_n}\operatorname{sgn}(w)w(y^{\lambda+\delta}) \\ &= \sum_\lambda a_{\lambda+\delta}(x)a_{\lambda+\delta}(y)\end{aligned}$$

This proves in the case of $n$ variables $x_i$ and $n$ variables $y_i$, now let $n\to\infty$ as usual to complete the proof. $\square$

**Definition 7.2.** *We define a $\mathbb{Z}$-valued bilinear form $\langle\cdot,\cdot\rangle : \Lambda\times\Lambda\to\mathbb{Z}$ by requiring*

$$\langle h_\lambda, m_\mu\rangle = \delta_{\lambda\mu}$$

*for all partitions $\lambda,\mu$, where $\delta_{\lambda\mu}$ is the Kronecker delta.*

**Lemma 7.3.** *For each $n\geq 0$, let $(u_\lambda),(v_\lambda)$ be $\mathbb{Q}$-bases of $\Lambda^n_{\mathbb{Q}}$, indexed by the partition $\lambda$ of $n$. Then the following condition are equivalent:*

1. $\langle \mu_\lambda, v_\mu\rangle = \delta_{\lambda\mu}$ *for all* $\lambda,\mu$.
2. $\sum_\lambda u_\lambda(x)v_\lambda(y) = \prod_{i,j}(1-x_iy_j)^{-1}$.

*Proof.* Let

$$u_\lambda = \sum_\rho a_{\lambda\rho}h_\rho,\quad v_\mu = \sum_\sigma b_{\mu\sigma}m_\sigma$$

then

$$\langle u_\lambda, v_\mu\rangle = \sum_\rho a_{\lambda\rho}b_{\mu\rho}$$

so the first statement is equivalent to

$$\sum_\rho a_{\lambda\rho}b_{\mu\rho} = \delta_{\lambda\mu}$$

And note that the second statement is equivalent to

$$\sum_\lambda u_\lambda(x)v_\lambda(y) = \sum_\rho h_\rho(x)m_\rho(y)$$

so it is also equivalent to

$$\sum_\lambda a_{\lambda\rho}b_{\lambda\sigma} = \delta_{\rho\sigma}$$

This completes the proof. □

So together with Proposition 7.1 with Lemma 7.3, it follows that

$$\langle p_\lambda, p_\mu\rangle = \delta_{\lambda\mu}z_\lambda$$

so that the $p_\lambda$ form an orthogonal basis of $\Lambda_{\mathbb{Q}}$. Likewise we have

$$\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}$$

so that $s_\lambda$ form an orthonormal basis of $\Lambda$, and the $s_\lambda$ such that $|\lambda| = n$ form an orthonormal basis of $\Lambda^n$.

Any other orthonormal basis of $\Lambda^n$ must therefore be obtained from the basis $(s_\lambda)$ by transformation by an orthonormal integer matrix. The only such matrices are signed permutation matrices, therefore the orthonormal relation $s_\lambda$ satisfied characterizes the $s_\lambda$ up to order and sign.

**Lemma 7.4.** $\omega : \Lambda\to\Lambda$ *is an isometry for* $\langle\cdot,\cdot\rangle$.

*Proof.* Since we have $\omega(p_\lambda) = \varepsilon_\lambda p_\lambda$, hence we

$$\langle \omega(p_\lambda, \omega(p_\mu)\rangle = \varepsilon_\lambda\varepsilon_\mu\langle p_\lambda, p_\mu\rangle = \varepsilon_\lambda\varepsilon_\mu z_\lambda\delta_{\lambda\mu} = \langle p_\lambda, p_\mu\rangle$$

since $(\varepsilon_\lambda)^2 = 1$. This completes the proof. □

7.1. **Transition matrices.** Let $\lambda, \mu$ be partitions, we define

$$\{\lambda\}^j = \{\mu \subset \lambda \mid |\mu| = |\lambda| - j, 0 \leq \lambda_i' - \mu_i' \leq 1, \forall i\}$$
$$\{\lambda\}_j = \{\mu \subset \lambda \mid |\mu| = |\lambda| + j, \lambda_i' \leq \mu_i' \leq \lambda_i' + 1, \forall i\}$$

**Definition 7.5.** *A* ***flag*** $\mu_\bullet$ *is a sequence of partitions*

$$\mu_n \subset \mu_{n-1} \subset \cdots \subset \mu_0 = \lambda$$

*such that* $\mu_i \in \{\mu_{i-1}\}^{a_i}$ *for some* $a_i \geq 0$*, and all* $1 \leq i \leq n$*. The sequence* $a = (a_1, \dots, a_n)$ *is called the weight of* $\mu_0$*.*

**Definition 7.6.** *A flag is called* ***complete*** *if* $n = |\lambda|$*.*

**Example 7.7.** Consider $\lambda = (6,4,4,2)$, we can get a flag as follows by removing boxes.

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&2&2&3&3&5\\\hline 2&3&5&5\\\cline{1-4} 4&4&7&7\\\cline{1-4} 5&7\\\cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|c|c|c|}\hline 1&2&2&3&3&5\\\hline 2&3&5&5\\\cline{1-4} 4&4\\\cline{1-2} 5\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|c|c|c|}\hline 1&2&2&3&3&5\\\hline 2&3&5&5\\\cline{1-4} 4&4\\\cline{1-2} 5\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|c|c|}\hline 1&2&2&3&3\\\hline 2&3\\\cline{1-2} 4&4\\\cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|c|c|}\hline 1&2&2&3&3\\\hline 2&3\\\cline{1-2}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&2\\\hline 2\\\cline{1-1}\end{array}\qquad \begin{array}{|c|}\hline 1\\\hline\end{array}\qquad \varnothing$$

where we have

$$\mu_0 = (6,4,4,2) \supset \mu_1 = (6,4,2,1) \supset \mu_2 = (6,4,2,1) \supset \mu_3 = (5,2,2) \supset$$
$$\mu_4 = (5,2) \supset \mu_5 = (3,1) \supset \mu_6 = (1) \supset \mu_7 = \varnothing$$

and

$$a_1 = 3, a_2 = 0, a_3 = 4, a_4 = 2, a_5 = 3, a_6 = 3, a_7 = 1$$

that is $a = (3,0,4,2,3,3,1)$

**Lemma 7.8.**

$$\{\textit{semistandard Young tableau } T(\lambda\} \longleftrightarrow \{\textit{flag } \mu_\bullet \textit{ such that } \mu_0 = \lambda\}$$

*Proof.* Let $n = |\lambda|$. Given $\mu_\bullet$ with $\mu_0 = \lambda$, define $T(\lambda)$ by filling all the $a_i$ boxes of $\mu_i - \mu_{i+1}$ with $n - i$, $1 \leq i \leq n$. Then $u_i \in \{\mu_{i-1}\}^{a_i}$ implies all columns are strictly increasing and $a_i \geq 0$ implies all rows are increasing.

Given a semistandard Young tableau $T(\lambda)$ of weight $a = (a_1, \dots, a_n)$, remove $a_i$ boxes whoses entry is $n - i + 1$ to obtain $\mu_i$ and set $\mu_0 = \lambda$. Rows of $T(\lambda)$ are increasing implies $|\mu_i| - |\mu_{i-1}| = a_{i-1} \geq 0$ and columns of $T(\lambda)$ are strictly increasing implies at most one box in each column is removed, that is $0 \leq \mu_{i-1}' - \mu_i' \leq 1$. $\square$

Recall that we have

$$s_{(n)} = h_n, \quad s_{(1^n)} = e_n$$

**Proposition 7.9** (Pier's formula)**.** *We have*

1. $s_\lambda e_j = \sum_{\mu \in \{\lambda\}_j} s_\mu$
2. $s_\lambda h_j = \sum_{\mu' \in \{\lambda'\}_j} s_\mu$

*Proof.* Let $\lambda = (\lambda_1, \dots, \lambda_n)$ with $n$ sufficiently large by allowing some $\lambda_i$ to be zero.

$$s_\lambda e_i a_\delta = a_{\lambda+\delta} e_i \in A_r$$

implies

$$a_{\lambda+\delta} = \sum_\mu B_{\lambda\mu} a_{\mu+\delta}$$

Let $l_i = \lambda_i + n - i$, then the only way to obtain a monomial $x_1^{m_1} \dots x_n^{m_n}$ with $m_1 > m_2 > \dots > m_n$ in $a_{\lambda+\delta} e_i$ is possibly by $x_1^{l_1} \dots x_n^{l_n} x_{j_1} \dots x_{j_n}$. This monomial has strictly decreasing exponents if and only if the following is satisfied: Set

$$\mu_k = \begin{cases} \lambda_k, & k \notin \{j_1, \dots, j_i\} \\ \lambda_k + 1, & k \in \{j_1, \dots, j_i\} \end{cases}$$

Then $\mu_1 \geq \dots \geq \mu_n$, i.e. $\mu \in \{\lambda\}_i$. The coefficient of such a monomial is $B_{\lambda\mu} = 1$, so we have

$$a_{\lambda+\delta} e_i = \sum_{\mu \in \{\lambda\}_i} a_{\mu+\delta}$$

And the second equation follows from the first since $\omega(e_n) = h_n, \omega(s_\lambda) = s_{\lambda'}$. $\square$

Use the following, we can express $s_\lambda$ with $x_n = 1$ in terms of $s_\mu$ in $n-1$ variables.

**Lemma 7.10.** $s_\lambda(x_1, \dots, x_{n-1}, 1) = \sum_{j=0}^{|\lambda|} \sum_{\mu \in \{\lambda\}_j} s_\mu(x_1, \dots, x_{n-1})$

*Proof.* By Cauchy formula

$$\begin{aligned} \sum_\lambda s_\lambda(x_1, \dots, x_{n-1}, 1) s_\lambda(y_1, \dots, y_n) &= \prod_{i=1}^{n-1} \prod_{j=1}^{n} (1 - x_i y_j)^{-1} \prod_{j=1}^{n} (1 - y_j)^{-1} \\ &= \sum_\mu s_\mu(x_1, \dots, x_{n-1}) s_\mu(y_1, \dots, y_n) \sum_{j=0}^{\infty} h_j(y_1, \dots, y_n) \\ &= \sum_\mu s_\mu(x_1, \dots, x_{n-1}) \sum_{j=0}^{\infty} \sum_{\lambda' \in \{\mu'\}_j} s_\lambda(y_1, \dots, y_n) \end{aligned}$$

Comparing the coefficients of $s_\lambda(y_1, \dots, y_n)$, we have

$$\begin{aligned} s_\lambda(x_1, \dots, x_{n-1}, 1) &= \sum_{j=0}^{\infty} \sum_{\mu, \lambda' \in \{\mu'\}_j} s_\mu(x_1, \dots, x_{n-1}) \\ &= \sum_{j=0}^{|\lambda|} \sum_{\mu' \in \{\lambda\}_j} s_\mu(x_1, \dots, x_{n-1}) \end{aligned}$$

since $\lambda' \in \{\mu'\}_j$ implies $j \leq |\lambda| = n$. $\square$

**Lemma 7.11.** *We can write*

$$s_\lambda(x_1,\dots,x_n) = \sum_{\substack{\mu_\bullet=(\varnothing\subset\mu\subset\lambda)\\ a=|\lambda|-|\mu|}} x_n^a s_\mu(x_1,\dots,x_{n-1})$$

*Proof.* $s_\lambda(x_1,\dots,x_n)$ is homogenous of degree $|\lambda|$, then

$$\begin{aligned} s_\lambda(x_1,\dots,x_n) &= x_n^{|\lambda|} s_\lambda(\frac{x_1}{x_n},\dots,\frac{x_{n-1}}{x_n},1) \\ &= x_n^{|\lambda|}\sum_{j=0}^{|\lambda|}\sum_{\mu\in\{\lambda\}^j} s_\mu(\frac{x_1}{x_n},\dots,\frac{x_{n-1}}{x_n}) \\ &= \sum_{j=0}^{|\lambda|}\sum_{\mu\in\{\lambda\}^j} x_n^{|\lambda|-|\mu|} s_\mu(x_1,\dots,x_{n-1}) \end{aligned}$$

□

**Theorem 7.12.** *We have*

$$s_\lambda(x_1,\dots,x_n) = \sum_{\substack{T \text{ is semistandard}\\ \text{Young tableau of sharp } \lambda}} x^T$$

*where*

$$x^T = \prod_{i=1}^n x_i^{a_{n-i+1}}$$

*and $a$ is the weight of $T(\lambda)$.*

*Proof.*

$$s_\lambda(x_1,\dots,x_n) = \sum x_n^{a_1}x_{n-1}^{a_2}\dots x_{n-i+1}^{a_i} s_\mu(x_1,\dots,x_{n-i})$$

where the sumation runs over $\mu_\bullet = (\mu_i \subset \mu_{i-1} \subset \dots \subset \mu_0 = \lambda)$ such that $|\mu_i| - |\mu_{i-1}| = a_i$ and $0 \le \mu_i' - \mu_{i-1}' \le 1$. Then we have

$$\begin{aligned} s_\lambda(x_1,\dots,x_n) &= \sum_{\mu_\cdot \text{ is a flag of } \lambda}\prod_{i=1}^n x_i^{a_{n-1+i}} \\ &= \sum x^T \end{aligned}$$

where $T$ runs over all semistandard Young tableau as desired. □

**Remark 7.13.** In combinatorics this statement is taken as a definition, and all the properties of $s_\lambda$ are derived from this. In particular, $s_\lambda \in \Lambda_n^k$ where $k = |\lambda|$.

**Corollary 7.14.** $s_\lambda = \sum_{\mu\le\lambda} K_{\lambda\mu} m_\lambda$, *where $K_{\lambda\mu}$ is Kostka number.*

**Example 7.15.** Let $n = 3$ and $\lambda = (3,3,1)$ to compute $s_\lambda(x_1,x_2,x_3)$ use above property. All we need to do is to find out all semistandard Young tableaus, and compute the weight of flags which correspond to them.

List as follows

$$\begin{ytableau} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 \end{ytableau} \qquad (3,3,1) \supset_1 (3,3) \supset_3 (3) \supset_3 \varnothing$$

$$\begin{ytableau} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 \end{ytableau} \qquad (3,3,1) \supset_2 (3,2) \supset_2 (3) \supset_3 \varnothing$$

$$\begin{ytableau} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 \end{ytableau} \qquad (3,3,1) \supset_3 (3,1) \supset_1 (3) \supset_3 \varnothing$$

$$\begin{ytableau} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 \end{ytableau} \qquad (3,3,1) \supset_2 (3,2) \supset_3 (2) \supset_2 \varnothing$$

$$\begin{ytableau} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 3 \end{ytableau} \qquad (3,3,1) \supset_3 (3,1) \supset_2 (2) \supset_2 \varnothing$$

$$\begin{ytableau} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 3 \end{ytableau} \qquad (3,3,1) \supset_3 (3,1) \supset_3 (1) \supset_1 \varnothing$$

so we have

$$s_{(3,3,1)} = x_1x_2^3x_3^3 + x_1^2x_2^2x_3^3 + x_1^3x_2x_3^3 + x_1^2x_2^3x_3 + x_1^3x_2^2x_3^2 + x_1^3x_2^3x_3$$

Now we have already know the relations between bases $(s_\lambda)$ and $(m_\lambda)$, We also want to know

$$s_\lambda = \sum F_{\lambda\mu} p_\mu$$

**Definition 7.16.** *We arrange partition with respect to the reverse lexicographic order $L_n$, i.e. $(n)$ is first and $(1^n)$ is last. A matrix $(M_{\lambda\mu})$ indexed by $\lambda, \mu \in \mathcal{P}_n$ is said to be* ***strictly upper triangle****, if $M_{\lambda\mu} = 0$ unless $\lambda \geq \mu$; And* ***strictly upper unitriangular*** *if also $M_{\lambda\lambda} = 1$ for all $\lambda \in \mathcal{P}_n$; Similarly for strictly lower unitriangular.*

We set $U_n$ be the set of all strictly upper unitriangular matrices and $U'_n$ be the set of all strictly lower unitriangular matrices.

**Lemma 7.17.** *$U_n, U'_n$ are groups with respect to matrix multiplication.*

*Proof.* Let $M, N \in U_n$, then we have

$$(MN)_{\lambda\mu} = \sum_\nu M_{\lambda\nu} N_{\nu\mu} = 0$$

unless there exists $\nu$ such that $\lambda \geq \nu \geq \mu$, i.e. unless $\lambda \geq \mu$. For the same reason we have

$$(MN)_{\lambda\lambda} = M_{\lambda\lambda} N_{\lambda\lambda} = 1$$

i.e. $MN \in U_n$.

Consider $\sum_\mu M_{\lambda\nu} x_\mu = y_\lambda$, If $\nu \leq \lambda$, these equations involve $x_\mu$ for $\mu \leq \nu$, hence $\mu \leq \lambda$. The same is true for the equivalent set of equations

$$\sum_\mu (M^{-1})_{\lambda\mu} y_\mu = x_\mu$$

implies $(M^{-1})_{\lambda\mu} = 0$ unless $\mu \leq \lambda$. □

**Lemma 7.18.** *Let*

$$J = \begin{cases} 1, & \mu = \lambda' \\ 0, & otherwise \end{cases}$$

*Then $M \in U_n$ is equivalent to $JMJ \in U_n'$*

*Proof.* If let $N = JMJ$, then we have $N_{\lambda\mu} = M_{\mu'\lambda'}$. Then by Lemma 4.23, we have $\lambda \geq \mu$ is equivalent to $\mu' \geq \lambda'$. This completes the proof. □

**Definition 7.19.** *Let $(u_\lambda), (v_\lambda)$ be $\mathbb{Q}$ bases for $\Lambda$. We denote by $M(u, v)$ the matrix $(M_{\lambda\mu})$ of coefficients in the equations*

$$u_\lambda = \sum_\mu M_{\lambda\mu} v_\mu$$

*and $M(u, v)$ is called the transition matrix from $(v_\lambda)$ to $(u_\lambda)$.*

**Lemma 7.20.** *Let $(u_\lambda), (v_\lambda), (w_\lambda)$ be $\mathbb{Q}$ bases of $\Lambda$, and let $(u'_\lambda), (v'_\lambda)$ be the dual bases of $(u_\lambda), (v_\lambda)$ with respect to $\langle \cdot, \cdot \rangle$. Then*

$$\begin{aligned} M(u, v)M(v, w) &= M(v, w) \\ M(v, u) &= M(u, v)^{-1} \\ M(v', u') &= M(v, u)^T = M(u, v)^* \\ M(wv, wu) &= M(u, v) \end{aligned}$$

*where $T$ means transpose and $*$ means transpose of inverse.*

**Proposition 7.21.** *The matrix $(K_{\lambda\mu})$ is in $U_n$.*

*Proof.* By Lemma 4.27, we have $K_{\lambda\mu} = 0$ unless $\lambda \geq \mu$. In particular, we have $K_{\lambda\lambda} = 1$. □

**Remark 7.22.** In fact, all transition matrices between bases $e_\lambda, h_\lambda, m_\lambda, s_\lambda$ can be expressed in terms of $J$ and $K$

**Definition 7.23.** *Let $L$ denote the transition matrix $M(p, m)$, i.e.*

$$p_\lambda = \sum_\mu L_{\lambda\mu} m_\mu$$

**Definition 7.24.** *Let $\lambda$ be partition, $l(\lambda) = r$. Let $f : [1, r] \subset \mathbb{Z} \to \mathbb{Z}_{\geq 0}$. We define $f(\lambda)$ to be the vector whose $i$-th component is*

$$f(\lambda)_i = \sum_{f(j)=i} \lambda_j, \quad i \geq 1$$

**Proposition 7.25.** $L_{\lambda\mu} = |\{f : \mathbb{Z} \to \mathbb{Z}_{\geq 0} \mid f(\lambda) = \mu\}|$

*Proof.* Note that

$$\begin{aligned} p_\lambda &= p_{\lambda_1} p_{\lambda_2} \cdots \\ &= \sum_{f:[1,l(\lambda)]\to\mathbb{Z}_{\geq 0}} x_{f(1)}^{\lambda_1} x_{f(2)}^{\lambda_2} \cdots \\ &= \sum_f x^{f(\lambda)} \\ &= \sum_\mu \sum_{f(\lambda)=\mu} \sum_{w\in S_n} x^{w(\mu)} \end{aligned}$$

and $\sum_{w\in S_n} x^{w(\mu)}$ is just $m_\mu$. $\square$

**Definition 7.26.** *Let $\lambda, \mu$ be partitions, $\lambda$ is a refinement of $\mu$ if $\lambda = \bigcup_{i\geq 1} \lambda^{(i)}$ such that $\lambda^{(i)}$ is a partition of $\mu_j$. We write $\lambda \leq_R \mu$.*

**Lemma 7.27.** *We have*

1. *$\lambda \leq_R \mu$ is equivalent to $\mu = f(\lambda)$ for some $f : [1, l(\lambda)] \to \mathbb{N}$.*
2. *$\leq_R$ is a partial order on $\mathcal{P}_n$.*
3. *$\lambda \leq_R \mu$ implies $\lambda \leq \mu$.*

*Proof.* See problem set. $\square$

**Corollary 7.28.** *We have*

1. *$L = (L_{\lambda\mu}) \in U_n'$*
2. *$M(p, s) = M(p, m)M(s, m)^{-1} = LK^{-1}$*

## 8. Representation of $S_n$

Now finally we come back to our topic, representation theory, and use what we have learnt about symmetric functions to see what's the irreducible representation ring of $S_n$ is.

Recall we have a bilinear form on $C(G, \mathbb{C})$, defined by

$$(f, g)_G = \frac{1}{|G|} \sum_{x\in G} f(x)g(x^{-1})$$

We extend it to functions $f : G \to A$, and $A$ is any communicative $\mathbb{C}$-algebra. We also extend restriction $\operatorname{Res}_H^G$ and induction $\operatorname{Ind}_H^G$ from functions $f : G \to \mathbb{C}$ to $f : G \to A$. Then Frobenius reciprocity still holds, i.e. For $H \leq G$, and $\chi : G \to A, \psi : H \to A$ are functions. If $\chi$ is a class function, then

$$(\operatorname{Ind}_H^G \psi, \chi)_G = (\psi, \operatorname{Res}_H^G \chi)_H$$

**Lemma 8.1.** *Let $m, n \in \mathbb{N}$. We embed $S_m \times S_n$ into $S_{m+n}$ by making $S_m$ and $S_n$ act on complementary subsets of $\{1, \ldots, m + n\}$. Then:*

1. *All such subgroups are conjugate to each other*

2. *If $v \in S_n$ has cycle type $\alpha(v)$, $w \in S_n$ has cycle type $\alpha(w)$, then $v \times w \in S_{n+m}$ is well-defined up to conjugate in $S_{m+n}$ with cycle type $\alpha(v \times w) = \alpha(v) \cup \alpha(w)$.*
3. *Let $\psi : S_n \to \Lambda, w \mapsto p_{\alpha(w)}$. Then in the setting of 2., $\psi(v \times w) = \psi(v)\psi(w)$.*

*Proof.* Clear. □

**Definition 8.2.** *Let $R^n$ denote the $\mathbb{Z}$-module generated by $V \in \mathrm{Irr}(S_n)$ modulo the relations $V + W - V \oplus W$. Set $R = \bigoplus_{n \geq 0} R^n$, where $S_0 = \{e\}$ and $R^0 = \mathbb{Z}$.*

For $V \in R^m, W \in R^n$, let $V \boxtimes W$ be the corresponding representation of $S_m \times S_n$. Set

$$V \bullet W = \mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} (V \boxtimes W)$$

For $V = \bigoplus_{n \geq 0} V_n, W = \bigoplus_{n \geq 0} W_n$, where $V_n, W_n \in R^n$, we set

$$(V, W) = \sum_{n \geq 0} (V_n, W_n)_{S_n}$$

with

$$(V_n, W_n)_{S_n} = \dim \mathrm{Hom}_{S_n}(V_n, W_n)$$

**Proposition 8.3.** *For $R$, we have*
1. *$(R, \bullet)$ is a communicative graded ring.*
2. *$(\cdot, \cdot) : R \times R \to \mathbb{Z}$ is a well-defined scalar product on $R$.*

*Proof.* Omit. □

**Definition 8.4.** *The **Frobenius characteristic** is the map*

$$\mathrm{ch} : R \to \Lambda_{\mathbb{C}} = \Lambda \otimes \mathbb{C}$$
$$V \mapsto \mathrm{ch}(V)$$

*where $\mathrm{ch}^n(V) = (\chi_V, \psi)_{S_n} = \dfrac{1}{n!} \sum_{w \in S_n} \chi_V(w)\psi(w^{-1})$ for $V \in R^n$.*

**Lemma 8.5.** *Let $V \in R^n$. Then*

$$\mathrm{ch}^n(V) = \sum_{|\lambda| = n} z_\lambda^{-1} \chi_V(K_\lambda) p_\lambda$$

*where $\chi_V(K_\lambda) = \chi_V(w)$ for $w \in K_\lambda \in \mathrm{Conj}(S_n)$.*

*Proof.* Firstly, we have

$$\mathrm{ch}^n(V) = \frac{1}{n!} \sum_{w \in S_n} \chi_V(w) p_{\alpha(w)}$$

since $\psi(w^{-1}) = p_{\alpha(w^{-1})} = p_{\alpha(w)}$. Note that $\chi_V(w) = \chi_V(w')$ if $\alpha(w) = \alpha(w') \in \mathrm{Conj}(S_n)$ and $|K_\lambda| = n! z_\lambda^{-1}$, then

$$\mathrm{ch}^n(V) = \frac{1}{n!} \sum_{\lambda \in \mathrm{Conj}(S_n)} |K_\lambda| \chi_V(K_\lambda) p_\lambda \quad = \sum_{|\lambda| = n} z_\lambda^{-1} \chi_V(K_\lambda) p_\lambda$$

as desired. ◻

**Proposition 8.6.** ch *is an isometry, i.e. for* $V, W \in R^n$*, we have*

$$\langle \mathrm{ch}^n(V), \mathrm{ch}^n(W) \rangle = (V, W)$$

*Proof.* Note that

$$\begin{aligned}
\langle \mathrm{ch}^n(V), \mathrm{ch}^n(W) \rangle &= \sum_{\lambda,\mu} z_\lambda^{-1} z_\mu^{-1} \chi_V(K_\lambda)\chi_W(K_\mu)\langle p_\lambda, p_\mu \rangle \\
&= \sum_{\lambda} z_\lambda^{-1} \chi_V(K_\lambda)\chi_W(K_\lambda) \\
&= \frac{1}{n!}\sum_{\lambda} |K_\lambda| \chi_V(K_\lambda)\chi_W(K_\lambda) \\
&= (\chi_V, \chi_W)_{S_n} \\
&= (V, W)_{R^n}
\end{aligned}$$

◻

**Proposition 8.7.** ch *is an isometric ring isomorphism* $R \cong \Lambda_{\mathbb{C}}$.

*Proof.* It suffices to show ring isomorphism:

For $V \in R^m, W \in R^n$, we have

$$\begin{aligned}
\mathrm{ch}(V \bullet W) &= \mathrm{ch}(\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}}(V \boxtimes W)) \\
&= (\chi_{\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}}(V \boxtimes W)}, \psi)_{S_{m+n}} \\
&= (\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}}(\chi_{V \boxtimes W}), \psi)_{S_{m+n}} \\
&= (\chi_{V \boxtimes W}, \mathrm{Res}_{S_m \times S_n}^{S_{m+n}} \psi)_{S_m \times S_n} \\
&= (\chi_V, \psi)_{S_n}(\chi_W, \psi)_{S_m} \\
&= \mathrm{ch}(V)\,\mathrm{ch}(W)
\end{aligned}$$

i.e. ch is a homomorphism.

Let $\eta = \chi_{U_n}$, where $U_n$ is trivial representation of $S_n$. Then

$$\mathrm{ch}(U_n) = \sum_{\lambda} z_\lambda^{-1} p_\lambda = h_\lambda$$

If $\lambda \vdash n$, let $\eta_\lambda = \eta_{\lambda_1}\eta_{\lambda_2}$, which implies $\eta_\lambda$ is a character of $S_n$, and

$$H_\lambda = \mathrm{Ind}_{S_{\lambda_1} \times \cdots \times S_{\lambda_n}}^{S_n}(U_{\lambda_1} \boxtimes \cdots \boxtimes U_{\lambda_n})$$

so we have $\mathrm{ch}(H_\lambda) = h_\lambda$.

Recall that

$$s_\lambda = \det(h_{\lambda_i - i + j})_{i,j}$$

For each $\lambda \vdash n$. Let $V^\lambda \in R^n$ be the isomorphism class of a representation such that

$$\chi^\lambda = \chi_{V^\lambda} = \det(\eta_{\lambda_i - i + j})_{i,j}$$

Then $\mathrm{ch}(V^\lambda) = s_\lambda$.

By the following computation

$$(\chi^\lambda, \chi^\mu) = \langle \mathrm{ch}(V^\lambda), \mathrm{ch}(V^\mu)\rangle = \langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}$$

So $\pm\chi^\lambda$ is an irreducible character of $S_n$. Since we have $|\operatorname{Conj}(S_n)| = p_n = |\operatorname{Irr}(S_n)|$, then $\chi^\lambda$ are all characters of $S_n$, so $(V^\lambda)_{\lambda\vdash n}$ forms a basis of $R^n$, so we have $\mathrm{ch}|_{R_n}$ is an isomorphism. This completes the proof. □

**Theorem 8.8** (Frobenius). *The irreducible characters of $S_n$ are $\chi^\lambda, \lambda \vdash n$. Moreover, the dimension of $V^\lambda$ is $K_{\lambda(1^n)}$, the number of standard Young tableau of shape $\lambda$.*

*Proof.* It remains to show that $\chi^\lambda$ and not $-\chi^\lambda$ is an irreducible character. Need to show $\chi_\lambda(e) > 0$, where $e \in K_{(1^n)} \in \operatorname{Conj}(S_n)$.

$$s_\lambda = \mathrm{ch}(V^\lambda) = \sum_\nu z_\nu^{-1}\chi_\nu(K_\lambda)p_\nu$$

then

$$\langle s_\lambda, p_\mu\rangle = \sum_\nu z_\nu^{-1}\chi_\nu(K_\lambda)\langle p_\nu, p_\mu\rangle = \chi_\mu(K_\lambda)$$

since $\langle p_\nu, p_\mu\rangle = z_\mu\delta_{\mu\nu}$.

Then

$$\dim(V^\lambda) = \chi^\lambda(e) = \chi_\lambda(K_{(1^n)}) = \langle s_\lambda, p_1^n\rangle = \langle s_\lambda, p_{(1^n)}\rangle = K_{\lambda(1^n)}$$

□

School of Mathematics, Shandong University, Jinan, 250100, P.R. China,
*Email address*: bowenl@mail.sdu.edu.cn