TOPCIS IN COMPLEX ALGEBRAIC GEOMETRY

BOWEN LIU

Contents

0. Preface

0.1. **Introduction.** In this lecture, the object we're most interested in is the complex variety.

Definition 0.1 (complex variety)**.** A complex algebraic variety or simply a complex variety is a quasi-projective[1] variety over $\mathbb{C}$.

Definition 0.2 (non-singular)**.** A complex variety X is non-singular if the sheaf of Kähler differentials $\Omega_{X/\mathbb{C}}$ is locally free.

Given any non-singular projective complex variety X, one can show that $X \subseteq \mathbb{CP}^n$ is a submanifold by using inverse function theorem. Conversely, Chow showed that

Theorem 0.1 (Chow)**.** Any compact complex submanifold[2] of complex projective space must be a complex variety.

Chow's theorem implies that there is a deep connection between complex manifolds and complex varieties, and thus techniques from complex differential geometry may be used to solve some questions in algebraic geometry, such as corollaries of Calabi-Yau theorem. On the other hand, motivated by Chow's theorem, it's natural to ask whether a compact complex manifold can be (holomorphically) embedded into complex projective space or not.

Theorem 0.2 (Riemann)**.** Any compact Riemann surface can be embedded into $\mathbb{CP}^N$.

Theorem 0.3 (Kodaira)**.** A compact complex manifold with a positive holomorphic line can be embedded into $\mathbb{CP}^N$.

Remark 0.1*.* In fact, Riemann's result can be obtained from Kodaira's embedding. Given a Hermitian holomorphic line bundle (L, h), its Chern curvature $\sqrt{-1}\Theta_h$ represents the first Chern class $c_1(L)$, and $\partial\overline{\partial}$-lemma shows that any real $(1,1)$-form which represents $c_1(L)$ can be realized as the Chern curvature of some Hermitian metric h. Thus if we want to see whether a holomorphic line bundle is positive or not, it suffice to compute its first Chern class, and there always exists holomorphic line with positive first Chern class[3].

The Kähler manifold is an important object in the complex differential geometry, which lies in the intersection of Riemannian manifold, complex manifold and symplectic manifold, and has many elegant properties. One of the most profound results is the Hodge decomposition.

[1] $X \subseteq \mathbb{CP}^n$ is a projective variety if it's the zero-locus of some (finite) family of homogeneous polynomials, that generate a prime ideal, and it's called quasi-projective if it's an open subset of a projective variety.

[2] In fact, "submanifold" can be replaced by analytic subvariety, that is, we allow some singularities.

[3] For holomorphic line bundle L over Riemann surface, the "positivity" of first Chern class is determined by its degree, that is, holomorphic line bundle with positive degree has positive first Chern class.

Theorem 0.4 (Hodge)**.** Let (X, ω) be a compact Kähler manifold. Then there is a decomposition

$$H^n(X) \cong \bigoplus_{p+q=n} H^{p,q}(X),$$

where $H^{p,q}(X)$ is the Dolbeault cohomology of X.

Remark 0.2. The Hodge decomposition is independent of the choice of Kähler form ω, but for the proof, we need to use theory of harmonic forms and Kähler identities.

The Hodge decomposition has lots of consequences in algebraic geometry. Let X be a non-singular projective complex variety. The algebraic de Rham complex is defined by

$$\Omega^\bullet_{X/\mathbb{C}} \colon \mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega_{X/\mathbb{C}} \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Omega^n_{X/\mathbb{C}},$$

where $n = \dim X$, and the algebraic de Rham cohomology is defined by the hypercohomology of above complex as follows

$$H^k_{alg}(X) = \mathbb{H}^k(\Omega^\bullet_{X/\mathbb{C}}),$$

where $k \in \mathbb{Z}_{\geq 0}$. Note that there is a natural filteration on algebraic de Rham complex

$$\Omega^\bullet_{X/\mathbb{C}} = F^0\Omega^\bullet_{X/\mathbb{C}} \supseteq F^1\Omega^\bullet_{X/\mathbb{C}} \supseteq \cdots \supseteq F^n\Omega^\bullet_{X/\mathbb{C}} = \{0\},$$

where

$$F^p\Omega^\bullet_{X/\mathbb{C}} \colon 0 \to \cdots \to 0 \to \Omega^p_{X/\mathbb{C}} \to \cdots \to \Omega^n_{X/\mathbb{C}}.$$

This filteration gives the Hodge to de Rham spectral sequence.

Theorem 0.5 (E_1-degeneration)**.** Let X be a non-singular projective complex variety. The Hodge to de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_{X/\mathbb{C}}) \Longrightarrow H^{p+q}_{alg}(X)$$

degenerates at E_1-page, and

$$\dim_{\mathbb{C}} H^p(X, \Omega^q_{X/\mathbb{C}}) = \dim_{\mathbb{C}} H^q(X, \Omega^p_{X/\mathbb{C}}).$$

Remark 0.3. The inequality

$$\dim H^k_{alg}(X) \leq \sum_{p+q=k} H^q(X, \Omega^p_{X/\mathbb{C}})$$

always holds, and the equality holds if and only if the Hodge to de Rham spectral sequence degenerates at E_1-page.

There are several important developments in Kähler geometry after Hodge and Kodaira, such as the solution to Calabi conjecture given by Shing-Tung Yau, and the connection between stable vector bundles and Hermitian-Yang-Mills metrics proved by Uhlenbeck-Yau.

Theorem 0.6 (Calabi-Yau)**.** Let (X,ω) be a compact Kähler manifold and χ be a real $(1,1)$-form that represents the first Chern class. Then there exists a unique $\omega_h \in [\omega]$ such that $\mathrm{Ric}(\omega_h) = \chi$.

Corollary 0.1. There exists a unique Ricci-flat Kähler metric on compact Kähler manifold with vanishing first Chern class.

Now let's introduce some algebraic consequence of Calabi-Yau theorem.

Theorem 0.7. Let X be a non-singular projective complex variety with ample canonical bunle K_X. Then

$$(-1)^n \left(c_1^n(X) - \frac{2(n+1)}{n} c_1^{n-2}(X) c_2(X) \right) \leq 0.$$

Moreover, the equality holds if and only if X is a locally symmetric variety of ball type.

Corollary 0.2. If X is a locally symmetric variety of ball type, then X^σ is again a locally symmetric variety of ball type for any $\sigma \in \mathrm{Aut}(\mathbb{C})$.

Theorem 0.8. Let X be a non-singular projective complex variety with $c_1(X) = 0$. Then for any ample line bundle L on X,

$$c_2(X) \cdot L^{n-2} \geq 0.$$

Moreover, the equality holds if and only if X is an abelian variety.

To state Uhlenbeck-Yau's theorem, we need the following preparations.

Definition 0.3 (slope)**.** Let (X,ω) be a compact Kähler and E be a holomorphic vector bundle. The slope of E with respect to ω is defined by

$$\mu_\omega(E) = \frac{\deg_\omega(E)}{\operatorname{rk} E},$$

where

$$\deg_\omega(E) = \int_X c_1(E) \cdot \omega^{n-1}.$$

Definition 0.4 (stability)**.** Let (X,ω) be a compact Kähler and E be a holomorphic vector bundle.

(1) E is μ_ω-stable if for all subbundle $F \subseteq E$, one has

$$\mu_\omega(F) < \mu_\omega(E).$$

(2) E is μ_ω-semistable if for all subbundle $F \subseteq E$, one has

$$\mu_\omega(F) \leq \mu_\omega(E).$$

Definition 0.5 (Hermitian-Yang-Mills metric)**.** Let (X,ω) be a compact Kähler and E be a holomorphic vector bundle. A Hermitian metric h on E is called Hermitian-Yang-Mills if

$$\wedge_\omega \Theta_h = \lambda \,\mathrm{id}_E,$$

where $\lambda \in \mathbb{R}$.

Theorem 0.9 (Uhlenbeck-Yau)**.** Let (X, ω) be a compact Kähler and E be a μ_ω-stable holomorphic bundle. Then there exists a unique Hermitian-Yang-Mills metric on E.

It also has lots of algebraic consequences.

Theorem 0.10. Let X be a non-singular projective complex variety and H be a line bundle. Let E be a μ_H-semistable vector bundle. Then

$$\left(c_1^2(E) - \frac{\mathrm{rk}(E)+1}{\mathrm{rk}(E)} c_2(E)\right) \cdot H^{n-2} \geq 0$$

Theorem 0.11. Let X be a non-singular projective complex variety and H be a line bundle. Let E be a μ_H-semistable vector bundle. Then

$$H^p(X, \Omega_X^q \otimes E \otimes H) = 0$$

for all $p + q > \dim X$.

In particular, above vanishing theorem generalizes the classical Kodaira vanishing theorem, which is a consequence of Hodge theory.

0.2. **Outlines.**

0.2.1. *Part I: Hodge theory.*

(1) Existence of harmonic forms.
(2) Kähler condition and Hodge package.
(3) Kodaira's vanishing theorem.
(4) Cartier descent theorem.
(5) de Rham decomposition theorem of Deligne and Illusie's theorem.
(6) Hodge symmetry.

0.2.2. *Part II: Non-abelian Hodge theory.*

(1) Existence of Hermitian-Yang-Mills metrics.
(2) Higgs bundle and the variant.
(3) Non-abelian Hodge theory.
(4) Ogus-Vologodsky theorem.
(5) Higgs-de Rham flow.

References

Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, P.R. China,

Email address: liubw22@mails.tsinghua.edu.cn