HODGE THEORY AND COMPLEX ALGEBRAIC GEOMETRY

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ABSTRACT. It's a lecture note I typed for "Hodge theory and complex algebraic geometry" taught by Qizheng Yin, in spring 2022. This note mainly follows the blackboard-writing of Prof. Yin. I also add some details and my understandings in it.

Attention: there may be a considerable number of mistakes in this note, and that's all my fault, since I still have too many problems to work out.

CONTENTS

0. Overview

In this course, we will introduce two parts:

I Kähler manifold and Hodge decomposition;
II Hodge theory in algebra geometry.

For the first part, if X is a compact complex manifold, we can consider the following structures:

(1) Topology: $H^*_B(X, \mathbb{Z})$, singular cohomology, where B means "Betti".
(2) C^∞-structure: $H^*_{dR}(X, \mathbb{R}) = H^*(X, \Omega^\bullet_{X,\mathbb{R}})$, de Rham cohomology. In fact, de Rham theorem implies that
$$H^*_{dR}(X, \mathbb{R}) \cong H^*_B(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$$
(3) Complex structure: For $x \in X$, we use $T_{X,x}$ to denote its tangent space at x, its real dimension is $2n$. And there is a linear map $J_x : T_{X,x} \to T_{X,x}$ for any $x \in M$, such that $J_x^2 = -\mathrm{id}$. If we complexificate $T_{X,x}$, then we can decompose it into
$$T_{X,x} \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}_{X,x} \oplus T^{0,1}_{X,x}$$
where $T^{1,0}_{X,x}$ is the eigenspace belonging to eigenvalue $\sqrt{-1}$, and $T^{0,1}_{X,x}$ is the eigenspace belonging to eigenvalue $-\sqrt{-1}$.

If we consider its dual, we will get bundle/sheaf of differential forms, and we can also decompose them as follows
$$\Omega^1_{X,\mathbb{C}} = \Omega^1_{X,\mathbb{R}} \otimes \mathbb{C} = \Omega^{1,0}_X \oplus \Omega^{0,1}_X$$
and
$$\Omega^k_{X,\mathbb{C}} = \Omega^k_{X,\mathbb{R}} \otimes \mathbb{C} = \bigoplus_{p+q=k} \Omega^{p,q}_X$$
where $\Omega^{p,q}_X = \bigwedge^p \Omega^{1,0}_X \otimes \wedge^q \Omega^{0,1}_X$

Since we have such decomposition for differential forms, it's natural to ask if there is a similar decomposition for de Rham cohomology? In other words, do we have
$$H^k_{dR}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$
such that $\overline{H^{p,q}(X)} = H^{q,p}(X)$?

The Hodge decomposition says it's true for compact Kähler manifolds. It's a very beautiful result, connecting "Topology" and "Complex geometry", since de Rham cohomology reflects the topology information and
$$H^{p,q}(X) \cong H^q_{Dol}(X, \Omega^p_X)$$
where "Dol" means Dolbeault cohomology, reflects the holomorphic information of a complex manifold.

Here is some examples of Kähler manifolds. In fact, almost every interesting manifold is Kähler manifold:

Example 0.0.1. Riemann surfaces, complex torus, projective manifolds are Kähler manifolds.

We also need to know an example that is not Kähler manifold:

Example 0.0.2 (Hopf surface)**.** Consider $\mathbb{Z}$ acts on $\mathbb{C}^2\backslash\{0\}$ by $mz = \lambda^m z, m \in \mathbb{Z}$ for some $\lambda \in (0,1)$, then we define Hopf surface as follows

$$S = \mathbb{C}^2\backslash\{0\}/\mathbb{Z}$$

As we can see, S is diffeomorphic to $S^3 \times S^1$,then $\dim_{\mathbb{C}} H^1(S,\mathbb{C}) = 1$, so S can not be a Kähler manifold by Hodge's decomposition, since for a Kähler manifold, $\dim_{\mathbb{C}} H^1$ must be even.

Remark 0.0.3. By Chow's theorem/GAGA, a compact complex manifold X admitting an embedding into projective space can be defined by polynomial equations, i.e. X is a projective variety, so here comes the forth structure, and that's the second part of this course, we want to apply Hodge theory in algebraic geometry.

(4) Algebraic structure.

Part 1. Preliminaries

1. Complex manifold

1.1. Manifolds and vector bundles.

1.1.1. *Definitions and Examples.*

Definition 1.1.1 (complex manifold)**.** *A complex manifold consists of $(X, \{U_i, \phi_i\}_{i\in I})$, where X is a connected, Hausdorff topological space, $\{U_i\}_{i\in I}$ is an open cover of X such that the index set I is countable, and ϕ_i is a homeomorphism from U_i to an open subset V_i of $\mathbb{C}^n$, such that*

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is biholomorphic.

Remark 1.1.2. Such $\phi_i \circ \phi_j^{-1}$ is called transition function; n is called the dimension of X, denoted by $\dim_{\mathbb{C}} X$; $\{U_i, \phi_i\}_{i\in I}$ is called complex atlas. Two atlas are called equivalent, if the union of them is still an atlas.

Definition 1.1.3 (complex structure)**.** *A complex structure is an equivalence class of a complex atlas.*

Remark 1.1.4. Replace $\mathbb{C}^n$ by $\mathbb{R}^n$, and biholomorphism is replaced by homeomorphism or diffeomorphism, then we get topological manifold or smooth manifold. It's a philosophy, since these things are all trivial locally, so how does we glue them together really matter.

Remark 1.1.5. $V_i \subset \mathbb{C}^n$ usually can not be the whole $\mathbb{C}^n$. For example, there is no non-constant holomorphism from $\mathbb{C}$ to unit disk $\mathbb{D}$. More generally, X is called Brody hyperbodic if there is no non-constant holomorphism from $\mathbb{C}$ to X.

Example 1.1.6. Projective space $\mathbb{P}^n$ is a complex manifold. Atlas are $U_i = \{[z] \in \mathbb{P}^n \mid z_j \neq 0\}, 0 \leq i \leq n, \phi_i : U_i \to \mathbb{C}^n$ is defined as follows

$$[z] \mapsto (\frac{z_0}{z_i}, \dots, \frac{\widehat{z_i}}{z_i}, \dots, \frac{z_n}{z_i})$$

Transition functions are calculated as follows, for $i < j$

$$\phi_i \circ \phi_j^{-1} : (u_1, \dots, u_n) \mapsto (\frac{u_1}{u_i}, \dots, \frac{\widehat{u_i}}{u_i}, \dots, \frac{u_{j-1}}{u_i}, \frac{1}{u_i}, \frac{u_{j+1}}{u_i}, \dots, \frac{u_n}{u_i})$$

In fact, $\mathbb{P}^n$ is a compact complex manifold, since $\mathbb{P}^n$ is diffeomorphic to S^{2n+1}/S^1.

Example 1.1.7. Grassmannian manifold

$$Gr(k, n) = \{k\text{-dimensional subspace of } \mathbb{C}^n\}$$

Now we're going to show $Gr(k, n)$ is a manifold of dimension $k(n-k)$. An atlas for $Gr(k, n)$ is given as follows: The idea is to present linear space of dimension k as graphs of linear maps. To be precise, let's introduce some notations.

For any subset $I \subset \{1, \dots, n\}$ of the set of indices, let

$$I' = \{1, 2, \dots, n\} \backslash I$$

be its complement. And define $\mathbb{C}^I = \{x \in \mathbb{C}^n \mid x^i = 0, \forall i \in I'\}$. Clearly, if $|I| = k$, then $\mathbb{C}^I \in Gr(k, n)$. Note that $\mathbb{C}^{I'} = (\mathbb{C}^I)^\perp$. Let

$$U_I = \{E \in Gr(k, n) \mid E \cap \mathbb{C}^{I'} = \{0\}\}$$

Thus each $E \in U_I$ can be described as the graph of a unique linear map $A_I : \mathbb{C}^I \to \mathbb{C}^{I'}$, that is

$$E = \{y + A_I(y) \mid y \in \mathbb{C}^I\}$$

This gives a bijection

$$\varphi_I : U_I \to \mathrm{Hom}(\mathbb{C}^I, \mathbb{C}^{I'}) \cong \mathbb{C}^{k(n-k)}$$
$$E \mapsto \varphi_I(E) = A_I$$

Now it suffices to show this really is an atlas.

Example 1.1.8. Complex torus is $\mathbb{C}^n/\Lambda$ where Λ is a free abelian subgroup of $\mathbb{C}^n$ with rank $2n$, called a lattice.

Definition 1.1.9 (holomorphic map)**.** *Let X, Y be complex manifolds of dimension n, m, with atlas $(U_i, \phi_i : U_i \to V_i)$ and $(M_j, \psi_j : M_j \to N_j)$ respectively. A continous map $f : X \to Y$ is called holomorphic, if for any two charts, we have*

$$\psi_j \circ f \circ \phi_i^{-1} : V_i \to \psi_j(f(U_i) \cap M_j)$$

is holomorphic.

Definition 1.1.10 (holomorphic function)**.** *A holomorphic function on X is a holomorphic map $f : X \to \mathbb{C}$.*

Example 1.1.11. Let $S = (\mathbb{C}^2 \backslash \{0\})/\mathbb{Z}$ be Hopf surface, then

$$f : S \to \mathbb{P}^1 = (\mathbb{C}^2 \backslash \{0\})/\mathbb{C}^*$$

is a holomorphic map. The fibers of f are biholomorphic to 1-dimensional complex torus.

Proposition 1.1. *If X is a compact complex manifold, then every holomorphic function on X is constant.*

Proof. Standard conclusion in complex analysis. □

Definition 1.1.12 (immersion/submersion)**.** *A holomorphic map $f : X \to Y$ is called an immersion(resp submersion), if for all $x \in X$, there exists $(x \in U_i, \phi_i), (f(x) \in M_j, \psi_j)$, such that*

$$J_{\psi_j \circ f \circ \phi_i^{-1}}(\phi_i(x))$$

has the max rank $\dim X$(*resp* $\dim Y$)

Definition 1.1.13 (embedding)**.** *$f : X \to Y$ is an embedding, if it is immersion and $f : X \to f(X) \subset Y$ is homeomorphism.*

Definition 1.1.14 (submanifold)**.** *A closed connected subset Y of X is called a submanifold, if for all $x \in Y$, there exists $x \in U \subset X$ and a holomorphic submersion $f : U \to \mathbb{D}^k$ such that*

$$U \cap Y = f^{-1}(0)$$

where k is the codimension of Y in X.

Example 1.1.15 (regular value theorem)**.** Let X, Y be complex manifold with dimension n, m, If $y \in Y$ such that $\mathrm{rank} J_{f(x)}$ reaches maximum m for all $x \in f^{-1}(y)$, then $f^{-1}(y)$ is a submanifold of codimension m.

Definition 1.1.16 (projective manifold)**.** *A projective manifold*[1] *X is a submanifold of $\mathbb{P}^N$ of the form*

$$X = \{[z] \in \mathbb{P}^N \mid f_1(z) = \cdots = f_m(z) = 0\}$$

where f_i is a homogenous polynomial in $\mathbb{C}[z_0, \ldots, z_n]$

Remark 1.1.17. Here we always assume $(f_1, \ldots, f_m) \subset \mathbb{C}[z_0, \ldots, z_n]$ is a prime ideal, so the case that X is defined by polynomials like $f^2 = 0$ is not allowed, what's more, the condition that X is a manifold implies the following cases won't happen:

[1]Chow's theorem claims that every submanifold of $\mathbb{P}^n$ must be defined by a set of homogenous polynomials, so we can use this property to define a projective manifold, in convenient.

1. $f_1 f_2 = 0$;
2. X has a singular point.

Definition 1.1.18 (complete intersection)**.** *Let $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ be the projection, then X is a submanifold of codimension k if and only if*

$$J = (\frac{\partial f_i}{\partial z_j})_{\substack{1\leq i\leq m \\ 0\leq j\leq N}}$$

has rank k, for all $x \in \pi^{-1}(X)$. Then X is called a complete intersection, if $m = k$.

Example 1.1.19. Consider $C \subset \mathbb{P}^n$ defined by

$$xw - yz = y^2 - xz = z^2 - yw = 0$$

is not a complete intersection, called twisted cubic.

Example 1.1.20. Plücker embedding

$$\Phi : G(k, V) \hookrightarrow \mathbb{P}(\wedge^k V)$$

defined by $S \subset V$ with basis $s_1, \ldots, s_k$ is mapped to $[s_1 \wedge \cdots \wedge s_k]$. It's easy to check it's well-defined, this fact follows from the following lemma

Lemma 1.1.21. *Let W be a subspace of a finite dimensional vector space, and let $\mathcal{B}_1 = \{w_1, \ldots, w_k\}$ and $\mathcal{B}_2 = \{v_1, \ldots, v_k\}$ be two basis for W. Then $v_1 \wedge \cdots \wedge v_k = \lambda w_1 \wedge \cdots \wedge w_k$ for some λ lying in the base field.*

Proof. Write $w_j = a_{1j}v_1 + \cdots + a_{kj}v_k$. Then one can directly compute that

$$\begin{aligned} w_1 \wedge \cdots \wedge w_k &= (a_{11}v_1 + \cdots + a_{k1}v_k) \wedge \cdots \wedge (a_{1k}v_1 + \cdots + a_{kk}v_k) \\ &= \sum_{\sigma \in S_k} \epsilon(\sigma) a_{1\sigma(1)} \cdots a_{k\sigma(k)} v_1 \wedge \cdots \wedge v_k \\ &= \lambda v_1 \wedge \cdots \wedge v_k \end{aligned}$$

Note that λ we need is exactly the determinant of the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$. □

However, it's a little bit complicated to check it's injective. I will add the proof if I'm not tooo lazy in future.(smile)

Remark 1.1.22. Together above embedding with Chow's theorem, we have the fact that Grassmannian manifold is a variety, in fact.

1.1.2. *Vector bundle.*

Definition 1.1.23 (complex vector bundle)**.** *Let X be a differential manifold, E is a complex vector bundle of rank r on X*

1. (*Via total space*) *E is a differential manifold with surjective map $\pi : E \to X$, such that*
   (1) *For all $x \in X$, fiber E_x is a $\mathbb{C}$-vector space of dimension r;*
   (2) *For all $x \in X$, there exists $x \in U \subset X$ and $\pi^{-1}(U)$ is diffeomorphic to $U \times \mathbb{C}^r$ via h such that*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\pi"] \arrow[dr, "h"'] & \circlearrowleft & U \\ & U \times \mathbb{C}^r \arrow[ur, "p_1"'] \arrow[r, "p_2"] & \mathbb{C}^r \end{tikzcd}$$

and for all $y \in U$, $E_y \xrightarrow{p_2 \circ h} \mathbb{C}^r$ is a $\mathbb{C}$-vector space isomorphism.

Remark 1.1.24. Consider two local trivialization $(U_\alpha, h_\alpha), (U_\beta, h_\beta)$, then $h_\alpha \circ h_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$, this induces

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{\text{diff}} \mathrm{GL}(r, \mathbb{C})$$

such $g_{\alpha\beta}$ are called transition function[2], such that

$$\begin{aligned} g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} &= \mathrm{id} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma \\ g_{\alpha\alpha} &= \mathrm{id} \quad \text{on } U_\alpha \end{aligned}$$

2. (*Via transition function*) *E is the data of*
   (1) *open covering $\{U_\alpha\}$ of X;*
   (2) *transition functions $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{\textit{diff}} \mathrm{GL}(r, \mathbb{C})\}$, satisfies*

$$\begin{aligned} g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} &= \mathrm{id} \quad \textit{on } U_\alpha \cap U_\beta \cap U_\gamma \\ g_{\alpha\alpha} &= \mathrm{id} \quad \textit{on } U_\alpha \end{aligned}$$

Remark 1.1.25. The two definitions above are equivalent. The first definition implies the second clearly. The converse is a standard constructive method, which tell us how to glue things together using gluing data:

If we already have an open covering and a set of transition functions, the vector bundle E is defined to be the quotient of the disjoint union $\coprod_{U_\alpha}(U \times \mathbb{C}^r)$ by the equivalence relation that puts $(p', v') \in U_\beta \times \mathbb{C}^r$ equivalent to $(p, v) \in U_\alpha \times \mathbb{C}^r$ if and only if $p = p'$ and $v' = g_{\alpha\beta}(p)v$. To connect this definition with the previous one, define the map π to send the equivalence class of any given (p, v) to p.

Definition 1.1.26 (holomorphic vector bundle)**.** *X is a complex manifold, $\pi : E \to X$ is a complex vector bundle, given by $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{C})$. E is called a holomorphic vector bundle if $g_{\alpha\beta}$ is a holomorphic map.*

Exercise 1.1.27. Show that the total space of a holomorphic vector bundle E is a complex manifold.

Proof. Since we already have a complex structure on X, we need to pull it back to E using π and use the holomorphic transition functions to show it do gives a complex structure on E. □

Remark 1.1.28. Any complex manifold is in particular a differential manifold and any holomorphic vector bundle is in particular a complex vector bundle. However, a complex vector bundle does not admit any holomorphic

[2]Note that here "diff" means $g_{\alpha\beta}$ is a smooth map if we regard $\mathrm{GL}(r, \mathbb{C})$ as a differential manifold.

structure, this phenamena can be observed already in the case of line bundle, as we will see later.

Definition 1.1.29 (morphism between vector bundles)**.** *ϕ is a differential/holomorphic morphism of vector bundle on X of rank k, if $\phi : E \to F$ is differential/holomorphic map and fiberwise $\mathbb{C}$-linear of rank k.*

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi_1"'] & \circlearrowleft & F \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

Example 1.1.30. X is a differential/complex manifold, then $X \times \mathbb{C}^r$ is the trivial rank r complex/holomorphic vector bundle on X.

Example 1.1.31 (algebraic construction)**.** E, F are complex/holomorphic vector bundles on X, then $E \oplus F, E \otimes F, \mathrm{Hom}(E, F), E^* = \mathrm{Hom}(E, \mathbb{C}), \mathrm{Sym}^k E, \bigwedge^k E$ and $\det E$ are complex/holomorphic vector bundles.

Let's explain more explictly using transition functions. If E, F are given by transition functions $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$. WLOG, we may assume they share the same open covering $\{U_\alpha\}$, otherwise we can take their common refinement.

Then, for direct sum, we can define transition functions as

$$\begin{aligned} g''_{\alpha\beta} : U_\alpha \cap U_\beta &\to \mathrm{GL}(2r, \mathbb{C}) \\ x &\mapsto \mathrm{diag}(g_{\alpha\beta}(x), g'_{\alpha\beta}(x)) \end{aligned}$$

and similarly for tensor $E \otimes F$, we can define transition functions as $g_{\alpha\beta} \otimes g'_{\alpha\beta}$.

Now if we can define the dual vector bundle E^*, then in fact we can define $\mathrm{Hom}(E, F)$ as

$$\mathrm{Hom}(E, F) = E^* \otimes F$$

For dual vector bundle defined by $\{g_{\alpha\beta}\}$, in fact the transition functions are $\{(g_{\alpha\beta}^{-1})^T\}$, i.e. the transpose of the inverse. But it may be difficult to understand why? In fact, you will find it's just a fact in linear algebra.

Let's back to the definition via total space, it's natural to define the dual vector bundle of E, by defining all fibers to be the dual space of E_x. To elaborate, E^* is, first of all, the set of pairs $\{(p, l) \mid p \in X, \text{ and } l : E_x \to \mathbb{C} \text{ is a linear map.}\}$, and π maps (p, l) to $p \in X$. Furthermore, it's important to know what is the trivialization. If (U_α, h_α) is the trivialization of vector bundle E, defined by $E_p \ni (p, e) \mapsto (p, \lambda_\alpha(e)) \in U \times \mathbb{C}^r$, then we can define the trivialization of E^* as

$$\begin{aligned} h^*_\alpha : \pi^{-1}(U) &\to U \times \mathbb{C}^r \\ (p, l) &\mapsto (p, \lambda^*_\alpha(l)) \end{aligned}$$

where $\lambda^*_\alpha(l)$ can be seen as a functional on $\mathbb{C}^r$, such that $\lambda^*_\alpha(l)(\lambda_\alpha(e)) = l(e)$. It's quite natural to require that.

So in the language of linear algebra, if you have a matrix $A : \mathbb{C}^r \to \mathbb{C}^r$, then it induces a matrix of dual spaces $A' : (\mathbb{C}^r)^* \to (\mathbb{C}^r)^*$, then facts in linear algebra tells you $A' = (A^{-1})^T$, that's why here the relationship between $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ is transpose of inverse.

Remark 1.1.32. We should always hold such an ideal, all information of a vector bundle is encoded in its transition functions. So if the transition are trivial, i.e. identity matrix, then the vector bundle is just trivial one, or product bundle. So from the relationship between transition functions of vector bundle and its dual, if the vector bundle is a line bundle, i.e. $g_{\alpha\beta} \in \mathbb{C}\backslash\{0\}$, then

$$(g_{\alpha\beta}^{-1})^T g_{\alpha\beta} = g_{\alpha\beta}^{-1} g_{\alpha\beta} = \mathrm{id}$$

So the vector bundle $\mathrm{End}(L) = L^* \otimes L$ is the trivial bundle, but in general $\mathrm{End}(E)$ is not trivial. We will use this fact later.

Definition 1.1.33 (line bundle)**.** *A holomorphic line bundle L is a rank 1 vector bundle, i.e.*

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{holo} \mathbb{C}^*$$

Definition 1.1.34 (picard group)**.** *X is a complex manifold, then*

$$\mathrm{Pic}(X) = (\{\textit{holomorphic line bundles on } X\}/\textit{isomorphism}, \otimes)$$

called the picard group of X.

Remark 1.1.35. Clearly the identity of this group is trivial line bundle, and from Remark 1.1.32 we can see that the inverse element of line bundle L is its dual bundle L^*.

Example 1.1.36 (tautological line bundle)**.** Here is a special line bundle on $\mathbb{P}^n$

$$\begin{array}{c} L = \{([l], x) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid x \in l\} \subset \mathbb{P}^n \times \mathbb{C}^{n+1} \\ \Big\downarrow \pi \\ \mathbb{P}^n \end{array}$$

is called tautological line bundle. Consider open coverings

$$U_i = \{[l] = [l_1, \dots, l_n] \in \mathbb{P}^n \mid l_i \neq 0\}$$

there is a map $U_i \to \pi^{-1}(U_i)$, defined as

$$[l] \mapsto ([l], (\frac{l_0}{l_i}, \dots, 1, \dots, \frac{l_n}{l_i}))$$

and local trivialization $h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ defined as

$$([l], x) \mapsto ([l], \lambda)$$

where

$$x = \lambda(\frac{l_0}{l_i}, \dots, 1, \dots, \frac{l_n}{l_i})$$

so we can calculate transition function

$$h_i \circ h_j^{-1} : (U_i \cap U_j) \times \mathbb{C} \longrightarrow (U_i \cap U_j) \times \mathbb{C}$$
$$([l], \lambda_j) \mapsto ([l], \lambda_j(\frac{l_0}{l_j}, \dots, \frac{l_n}{l_j})) \mapsto ([l], \lambda_i)$$

such that

$$\lambda_j(\frac{l_0}{l_j}, \dots, \frac{l_n}{l_j}) = \lambda_i(\frac{l_0}{l_i}, \dots, \frac{l_n}{l_i})$$

which implies

$$\lambda_i = \lambda_j \frac{l_i}{l_j}$$

so we can see transition function $g_{ij} = l_i/l_j \in \mathbb{C}^*$. This line bundle L will be denoted by $\mathcal{O}_{\mathbb{P}^n}(-1)$.

Definition 1.1.37 (line bundles on $\mathbb{P}^n$)**.** *We can define*

$$\begin{aligned} \mathcal{O}_{\mathbb{P}^n}(-k) &= \mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes k}, \quad k \in \mathbb{N}^+ \\ \mathcal{O}_{\mathbb{P}^n}(k) &= (\mathcal{O}_{\mathbb{P}^n}(-k))^*, \quad k \in \mathbb{N}^+ \\ \mathcal{O}_{\mathbb{P}^n}(0) &= \mathbb{P}^n \times \mathbb{C}, \quad \textit{trivial line bundle.} \end{aligned}$$

Fact 1.1.38. In fact, line bundles listed above contain all possible line bundles over $\mathbb{P}^n$.

Example 1.1.39. More generally, consider

$$\begin{array}{c} E = \{([S], x) \in Gr(k, n) \times \mathbb{C}^n \mid x \in S\} \subset Gr(k, n) \times \mathbb{C}^n \\ \big\downarrow \pi \\ Gr(k, n) \end{array}$$

Definition 1.1.40 (section)**.** *$\pi : E \to X$ is a complex/holomorphic vector bundle, U an open subset of X. A section of E on U is a differential/holomorphic map $s : U \to E$, such that $\pi \circ s = \mathrm{id}_U$, denoted by $C^\infty(U, E)$ / $\Gamma(U, E)$.*

Remark 1.1.41. In particular, if $U = X$, section of E on X is called a global section.

Example 1.1.42. Global holomorphic sections of trivial holomorphic vector bundle are exactly holomorphic functions $f : X \to \mathbb{C}^r$.

Remark 1.1.43. In fact, global holomorphic sections are controlled heavily by holomorphic vector bundle itself. As we can seen from the above example: If X is a compact complex manifold, then all global holomorphic functions are only constant.

Conversely, global section can also control vector bundle to some extent. You can see the following Exercise as an example.

Exercise 1.1.44. Let $E \to X$ be a line bundle[3], then E is a trivial line bundle if and ony if there exists a non-vanishing global section s.

Proof. It's clear there exists a non-vanishing global section if E is trivial; Conversely, if there exists a non-vanishing global section s. Define the following map

$$\begin{aligned} \varphi : X \times \mathbb{C} &\to E \\ (x, \lambda) &\mapsto \lambda s(x) \end{aligned}$$

Now it suffices to show it's an isomorphism, i.e. the map $\varphi_x \to \{x\} \times \mathbb{C} \to E_x$ is an isomorphism of vector spaces. The map φ_x is given by $\lambda s(x)$, it's injective thus an isomorphism. Indeed, if $\lambda s(x) = 0$ then we have $\lambda = 0$ since $s(x) \neq 0$. □

Definition 1.1.45 (subbundle). *$\pi : E \to X$ is a complex/holomorphic vector bundle. $F \subset E$ is called a subbundle of rank s, if*

1. *For all $x \in X$, $F \cap E_x$ is a subvector space of dimension s.*
2. *$\pi|_F : F \to X$ induces a complex/holomorphic vector bundle.*

Remark 1.1.46. If F is a subbundle of E, then given a section of F, i.e. $\sigma : X \to F$ such that $\pi|_F \circ \sigma = \mathrm{id}_X$, then clearly we can extend it to a section of E.

Example 1.1.47. $\mathcal{O}_{\mathbb{P}^n}(-1) \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$, is a subbundle of rank 1.

Exercise 1.1.48.

$$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \begin{cases} 0, & k < 0 \\ \mathbb{C}, & k = 0 \\ \text{homogeneous polynomials in } n+1 \text{ variables of deg } k, & k > 0 \end{cases}$$

Proof. Let's see what happened for $k = -1$, the tautological line bundle. Since we have $\mathcal{O}_{\mathbb{P}^n}(-1)$ is a subbundle of trivial bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$. So we have a global section $\sigma \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-1))$ must be a global section of $\Gamma(\mathbb{P}^n, \mathbb{P}^n \times \mathbb{C}^{n+1})$. However, since $\mathbb{P}^n$ is a compact complex manifold, we have that global sections $\Gamma(\mathbb{P}^n, \mathbb{P}^n \times \mathbb{C}^{n+1})$ must be constant, i.e. for any $x \in \mathbb{P}^n$, $\sigma(x) = v$ is a constant. However, $v \in [l]$, for all $[l] \in \mathbb{P}^n$, which forces $v = 0$. Similarly we will get the result for case $k < 0$.

And for case $k = 0$, global sections are exactly constant so we have $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(0)) = \mathbb{C}$.

Now consider what will happen when $k > 0$. Take $k = 1$ for an example. Life is like a seesaw, so is mathmatics. If something is defined concisely, it must be quite difficult to compute. Since sections of a trivial bundle is easy to compute, so in practice, we always compute the sections of the trivialization of a vector bundle, and glue them together to get a global one, that's what we always do.

[3] Here we do not require it's holomorphic or not, as we can see in the proof, it doesn't matter.

For projective space $\mathbb{P}^n$, there exists a canonical affine cover $\{U_i\}$, where $U_i = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n \mid z_i \neq 0\} \cong \mathbb{C}^n$. And the transition functions g_{ij} of $\mathcal{O}_{\mathbb{P}^n}(1)$ is given by z_j/z_i, according to Example 1.1.31. So a global section is given by $s_i \in \Gamma(U_i, \mathcal{O}_{\mathbb{P}^n}(1))$ such that $s_i = s_j z_j/z_i$ on $U_i \cap U_j$. We claim that the only possible functions are of form $s_i = L/z_i$, where $L = a_0 z_0 + \cdots + a_n z_n$, that's a homogenous polynomial in $n+1$ variables of degree 1.

We compute explictly for $n = 1$ in order to really understand what's going on: A global section s of $\mathcal{O}_{\mathbb{P}^1}(1)$ is given by a function $s_0(z) = a + bz + cz^2 + \cdots \in \Gamma(U_0, \mathcal{O}_{\mathbb{P}^1}(1)) = k[z]$, where $z = z_1/z_0$, and a function $s_1(w) = \alpha + \beta w + \gamma w^2 + \cdots \in \Gamma(U_1, \mathcal{O}_{\mathbb{P}^1}(1)) = k[w]$ such that on $U_0 \cap U_1$

$$s_0(z) = a + bz + cz^2 + \ldots = z(\alpha + \beta w + \gamma w^2 + \ldots) = \alpha z + \beta + \gamma(1/z) + \ldots$$

This implies $a = \beta, b = \alpha$ and all other coefficients are zero. □

Example 1.1.49. For a morphism between vector bundles $\phi : E \to F$, $\ker \phi \subset E, \operatorname{im} \phi \subset F$ are subbundles.

Definition 1.1.50 (exact)**.** *A sequence of vector bundles*

$$S \xrightarrow{\phi} E \xrightarrow{\psi} Q$$

is called exact at E if $\ker \psi = \operatorname{im} \phi$;

Definition 1.1.51 (pullback)**.** *$f : X \to Y$ is a differential/holomorphic map, $\pi : E \to Y$ is a vector bundle, define*

$$f^*E = \{(x, e) \in X \times E \mid f(x) = \pi(e)\} \subset X \times E$$

is called the pullback of π.

Remark 1.1.52. To be somewhat more explicit, suppose $U \subset Y$ is a local trivialization, i.e. $\varphi_U : E|_U \to U \times \mathbb{C}^r$ with $\varphi_U(e) = (\pi(e), \lambda_U(e))$. Then we can define a local trivialization of f^*E on $f^{-1}(U) \subset X$, by

$$\begin{aligned} \varphi_U^* : f^*E|_{f^{-1}(U)} &\to f^{-1}(U) \times \mathbb{C}^r \\ (x, e) &\mapsto (x, \lambda_U(e)) \end{aligned}$$

and transition functions on $f^{-1}(U_\alpha) \cap f^{-1}(U_\beta)$ is given by $g_{\alpha\beta} \circ f$.

1.2. **Episode: sheaves.** Why we need sheaves here? As we have seen in the last section, the global sections of holomorphic vector bundle are very rare, but there are many local sections, we need to keep these information and learn the connection between global and local systemically. Sheaf is a power language for us to manage global and local at the same time. However, sheaf gives more information. Indeed, as we can see in Exercise 1.2.7, vector bundles are exactly locally free sheaves.

Definition 1.2.1 (sheaf)**.** *X is a topological space. A sheaf of abelian group $\mathscr{F}$ on X is the data of:*

1. *For any open subset U of X, $\mathscr{F}(U)$ is an abelian group.*

2. *If $U \subset V$ are two open subsets of X, then there is a group homomorphism $r_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$, such that*
   (1) $\mathscr{F}(\varnothing) = 0$
   (2) $r_{UU} = \mathrm{id}$
   (3) *If $W \subset U \subset V$, then $r_{UW} = r_{VW} \circ r_{UV}$*
   (4) *$\{V_i\}$ is an open covering of $U \subset X$, and $s \in \mathscr{F}(U)$. If $s|_{V_i} := r_{UV_i}(s) = 0, \forall i$, then $s = 0$.*
   (5) *$\{V_i\}$ is an open covering of $U \subset X$, and $s_i \in \mathscr{F}(V_i)$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there exists $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$.*

A sheaf which fails to meet $(4), (5)$ is called a presheaf. For example:

Example 1.2.2 (constant presheaf)**.** Let G be abelian group, the constant presheaf assign each open set U the group G itself. However, it's not a sheaf. Indeed, consider $U = U_1 \cup U_2$ with $U_1 \cap U_2 = \varnothing$. Consider $g_1 \in \mathscr{F}(U_1) = G, g_2 \in \mathrm{F}(U_2), g_1 \neq g_2$, then one can't find $g \in \mathscr{F}(U)$ such that $g|_{U_1} = g_1, g|_{U_2} = g_2$, since $g_1 \neq g_2$.

Example 1.2.3 (sheaf of sections of a holomorphic vector bundle)**.** If $\pi : E \to X$ is a holomorphic vector bundle over complex manifold X, then for any open $U \subseteq X$, define

$$\mathscr{F}(U) = \Gamma(U, E|_U)$$

This $\mathscr{F}$ will be denoted by $\mathcal{O}_X(E)$. In particular, if E is a trivial vector bundle, then $\mathcal{O}_X(E) = \mathcal{O}_X$, the sheaf of holomorphic function, also called the structure sheaf of X.

Definition 1.2.4 (morphism of sheaves on X)**.** *$\phi : \mathscr{F} \to \mathscr{G}$ is called a morphism of sheaves, if for any open subset U of X, there is a group homomorphism $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$, such that if $U \subset V$ are two open subsets of X, the the following diagram commutes*

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\phi(U)"] \arrow[d, "r_{UV}"] & \mathscr{G}(U) \arrow[d, "r_{UV}"] \\
\mathscr{F}(V) \arrow[r, "\phi(V)"] & \mathscr{G}(V)
\end{tikzcd}$$

Remark 1.2.5. For convenience, for $s \in \mathscr{F}(U)$, we often write $\varphi(s)$ instead of $\varphi(U)(s)$.

Example 1.2.6 (locally free sheaves)**.** A sheaf is called locally free, if there exists covering $\{U_\alpha\}$ such that $\mathscr{F}|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}^{\oplus r}$ of rank r. For $r = 1$, it is called invertible sheaf.

Exercise 1.2.7. There are correspondences:

$$\{\text{holomorphic vector bundles}\} \xleftrightarrow{1-1} \{\text{locally free sheaves}\}$$

$$\{\text{holomorphic line bundles}\} \xleftrightarrow{1-1} \{\text{invertible sheaves}\}$$

Proof. It suffices to prove the first correspondence. If we have a holomorphic vector bundle $\pi : E \to X$. Then consider the sheaf of sections $\mathcal{O}_X(E)$, We claim it's a locally free sheaf. Since we have local trivialization of holomorphic vector bundle $\{U_\alpha\}$. Then consider what's $\mathcal{O}_X(E)|_{U_\alpha}$. Since $E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^r$, then holomorphic sections of $U_\alpha \times \mathbb{C}^r \to U_\alpha$ are just holomorphic functions $f : U \to \mathbb{C}^r$, i.e. $\mathcal{O}_X(E|_{U_\alpha}) = \mathcal{O}_{U_\alpha}^{\oplus r}$. So sheaf $\mathcal{O}_X(E)$ is a locally free sheaf.

Conversely, if we have a locally free sheaf $\mathcal{E}$, how can we get a holomorphic vector bundle? Assume $\mathcal{E}$ is locally free over an open covering $\{U_\alpha\}$ of X, then we just need to glue $U_\alpha \times \mathbb{C}^r \to U_\alpha$ together to get a vector bundle. Therefore we need a family of gluing data $g_{\alpha\beta} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$. Since $\mathcal{E}$ is locally free, we have local isomorphism $f_\alpha : \mathcal{E}|_{U_\alpha} \to \mathcal{O}_{U_\alpha}^{\oplus r}$. Restricting to intersection $U_\alpha \cap U_\beta$, we get

$$f_{\alpha\beta} = f_\alpha|_{U_\alpha \cap U_\beta} \circ f_\beta^{-1}|_{U_\alpha \cap U_\beta} : \mathcal{O}_{U_\beta}^{\oplus r}|_{U_\alpha \cap U_\beta} \to \mathcal{O}_{U_\alpha}^{\oplus r}|_{U_\alpha \cap U_\beta}$$

Every such map is induced by a map

$$g_{\alpha\beta} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$$

that's gluing data we desire. □

For more about sheaves and its cohomology, see Appendix A.

1.3. **Tangent bundle.**

Definition 1.3.1 (tangent bundle)**.** *X is a differential manifold,* $\dim_{\mathbb{R}} X = n$*, and $\{U_\alpha, \phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n\}$ is a atlas of X. The* (*real*) *tangent bundle $T_{X,\mathbb{R}}$ is defined through transition functions*

$$\begin{aligned} g_{\alpha\beta} : U_\alpha \cap U_\beta &\overset{\textit{diff}}{\to} \mathrm{GL}(n, \mathbb{R}) \\ x &\mapsto J_{\phi_\alpha \circ \phi_\beta^{-1}}(\phi_\beta(x)) \end{aligned}$$

Then $T_{X,\mathbb{C}} = T_{X,\mathbb{R}} \otimes \mathbb{C}$ is a complex vector bundle, called the complexified tangent vector bundle.

Remark 1.3.2. The following statement may be a little bit boring, I write it down just to make myself more clear and to get familiar with two definition of vector bundle.

The tangent bundle $T_{X,\mathbb{R}}$ can be defined as the set

$$T_{X,\mathbb{R}} = \coprod_{x \in X} T_{X,x}$$

and note that there is a natural projection $\pi : T_{X,\mathbb{R}} \to X$, sending $v \in T_{X,x}$ to $x \in X$. Now we want to give a chart on $T_{X,\mathbb{R}}$ to make it into a differential manifold. Let $\{(U_i, \phi_i = (x_i^1, \ldots, x_i^n)\}$ be a chart of X, then we can define a chart on X by considering $\{(\pi^{-1}(U_i), \widetilde{\phi}_i)\}$, where $\widetilde{\phi}_i$ is defined through

$$\widetilde{\phi}_i(v) = (\phi_i(\pi(v)), (\mathrm{d}x_i^1)_{\pi(v)}(v), \ldots, (\mathrm{d}x_i^n)_{\pi(v)}(v)) \subset \mathbb{R}^n \times \mathbb{R}^n$$

note that such $\widetilde{\varphi}_i$ is bijective. And it's easy to equip $T_{X,\mathbb{R}}$ with a topology such that $\widetilde{\varphi}_i$ is diffeomorphism.

Now I need to calculate transition function to confirm myself as follows: For two charts $(U, \phi = (x_1, \ldots, x_n)), (V, \psi = (y_1, \ldots, y_n))$, then calculate

$$\widetilde{\psi} \circ \widetilde{\phi}^{-1} : \phi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$$

Note that

$$\widetilde{\phi}^{-1}(r_1, \cdots, r_n, u_1, \cdots, u_n) = \sum_i u_i \frac{\partial}{\partial x_i}|_{\phi^{-1}(r_1, \cdots, r_n)} \in T_{\phi^{-1}(r_1, \cdots, r_n)} M$$

But

$$\mathrm{d}y_j(\sum_i u_i \frac{\partial}{\partial x_i}) = \sum_i u_i(\frac{\partial}{\partial x_i}(y_j)) = \sum_i \frac{\partial y_i}{\partial x_j} u_i$$

Thus transition functions are

$$\begin{aligned}\widetilde{\psi} \circ \widetilde{\phi}^{-1}(r_1, \cdots, r_n, u_1, \cdots, u_n) &= (\psi \circ \phi^{-1}(r), (\sum_i \frac{\partial y_1}{\partial x_i}(r) u_i, \ldots, \sum_i \frac{\partial y_n}{\partial x_i}(r) u_i)) \\ &= (\psi \circ \phi^{-1}(r), (\frac{\partial y_j}{\partial x_i}(r))(\begin{matrix} u_1 \\ \vdots \\ u_n \end{matrix}))\end{aligned}$$

So transition function $g_{\alpha\beta}$ are exactly Jacobian of $\varphi_\alpha \circ \varphi_\beta^{-1}$.

Definition 1.3.3 (holomorphic tangent bundle)**.** *X is a complex manifold, $\dim_{\mathbb{C}} X = n$, and $\{U_\alpha, \phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^n\}$ is an atlas of X. The holomorphic tangent bundle T_X is defined through transition functions*

$$\begin{aligned} g_{\alpha\beta} : U_\alpha \cap U_\beta &\to \mathrm{GL}(n, \mathbb{C}) \\ z &\mapsto J^{holo}_{\phi_\alpha \circ \phi_\beta^{-1}}(\phi_\beta(x)) \end{aligned}$$

where J^{holo} is holomorphic Jacobian.

Remark 1.3.4. For a holomorphic function $\phi_\alpha \circ \phi_\beta^{-1} : U_\beta \to U_\alpha$, its holomorphic Jacobian is the matrix

$$(\frac{\partial(\phi_\alpha \circ \phi_\beta^{-1})^j}{\partial z_i})_{1 \leq i,j \leq n}$$

Remark 1.3.5. Clearly, $T_X \neq T_{X,\mathbb{C}}$, even they don't have the same rank! For example, if X is a n-dimensional complex manifold, then

$$\dim T_X = n \neq 2n = \dim T_{X,\mathbb{C}}$$

Later we will see the relationship between them.

Remark 1.3.6 (sheaf viewpoint)**.** $\mathcal{O}_X$ is the sheaf of holomorphic function, then define the stalk at x is

$$\mathcal{O}_{X,x} := \varinjlim_{x \in U \subset X} \mathcal{O}_X(U)$$

The elements of $\mathcal{O}_{X,x}$ are called germs. For a tangent vector, we can take derivation in this direction, so

$$\text{tangent vector} \longrightarrow \text{derivation } D : \mathcal{O}_{X,x} \to \mathbb{C}$$

where a derivation is a map which satisfies
1. $\mathbb{C}$-linear
2. Leibniz rule $D(fg) = D(f)g + fD(g)$

In fact, the above correspondence is $1-1$, that is, every derivation arises from a tangent vector. So we have $T_{X,x} \cong$ space of derivation of $\mathcal{O}_{X,x}$, that's also a nice definition many authors prefer.

Definition 1.3.7 (cotangent bundle/anti-canonical bundle)**.** $\Omega_X = T^*_X$ *is called holomorphic cotangent bundle;* $K_X = \det \Omega_X$ *is called canonical bundle;* $K^*_X = \det T_X$ *is called the anti-canonical bundle.*

Example 1.3.8 (Euler sequence)**.** We calculate tangent bundle of $\mathbb{P}^n$ through the following exact sequence called Euler sequence[4].

$$0 \to \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \xrightarrow{\psi} T_{\mathbb{P}^n} \to 0$$

If we already have this exact sequence, then take the determinant we obtain

$$\begin{aligned} K^*_{\mathbb{P}^n} &= (\det \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1}) \otimes \det \mathcal{O}_{\mathbb{P}^n} \\ &= \mathcal{O}_{\mathbb{P}^n}(n+1) \end{aligned}$$

Let's clearify the Euler sequence in a geometry viewpoint: Let $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ denote the canonical projection from $\mathbb{C}^n\backslash\{0\} \to \mathbb{P}^n$. We know that basis of tangent vector at $z \in \mathbb{C}^{n+1}\backslash\{0\}$ is $\{\frac{\partial}{\partial z_0}, \dots, \frac{\partial}{\partial z_n}\}$, but these are not tangent vector for $\mathbb{P}^n$. Indeed, for a function f defined on $\mathbb{P}^n$, regard it as a function defined on $\mathbb{C}^n\backslash\{0\}$, we will have

$$f(\lambda z) = f(z), \quad \forall z \in \mathbb{C}^n\backslash\{0\}, \lambda \in \mathbb{C}^*$$

But if we consider the $g(z) = \frac{\partial f}{\partial z_i}(z)$, it won't give a function on $\mathbb{P}^n$, since

$$g(\lambda z) = \frac{\partial f}{\partial z_i}(\lambda z) = \frac{1}{\lambda}\frac{\partial f}{\partial z_i}(z) \neq \frac{\partial f}{\partial z_i}(z) = g(z), \quad \forall z \in \mathbb{C}^n\backslash\{0\}, \lambda \in \mathbb{C}^*$$

However, tangent vector $z_i\frac{\partial}{\partial z_i}$ will descend to tangent vector at $[z] \in \mathbb{P}^n$, since

$$g(\lambda z) = \lambda z_i \frac{\partial f}{\partial z_i}(\lambda z) = z_i \frac{\partial f}{\partial z_i}(z) = g(z), \quad \forall z \in \mathbb{C}^n\backslash\{0\}, \lambda \in \mathbb{C}^*$$

Now let's define ψ: Recall $z_0, \dots, z_n$ form basis of $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and span $(\mathcal{O}_{\mathbb{P}^n}(1))_{[z]}$, so can define

$$\begin{gathered} \psi : (\mathcal{O}_{\mathbb{P}^n}(1))^{\oplus n+1}_{[z]} \to T_{\mathbb{P}^n,[z]} \\ (0, \dots, \underbrace{z_i}_{j-th}, \dots, 0) \mapsto z_i \frac{\partial}{\partial z_j} \end{gathered}$$

[4]Refer to pages 408-409 of Griffiths-Harris for more details.

ϕ is defined as:

$$\begin{aligned}\phi : \mathcal{O}_{\mathbb{P}^n,[z]} &\to (\mathcal{O}_{\mathbb{P}^n}(1))^{\oplus n+1}_{[z]}\\ c &\to (cz_0, \dots, cz_n)\end{aligned}$$

It's clear ψ is surjective and ϕ is injective. Now let's show $\ker\psi = \operatorname{im}\phi$: If we apply $\psi \circ \phi$ to a constant c, then we obtain the a vector

$$V = cz_0 \frac{\partial}{\partial z_0} + \dots cz_n \frac{\partial}{\partial z_n}$$

Now let's show it's a zero vector: In fact, for a homogenous smooth function f defined on $\mathbb{R}^n \backslash \{0\}$ such that $f(\lambda x) = \lambda^d f(x)$, we have the famous relation

$$V(f) = df$$

which is discovered by Euler, and that's why this sequence is called Euler sequence. So clearly V is a zero vector in $\mathbb{P}^n$, since any function defined on $\mathbb{P}^n$ satisfies $f(\lambda x) = f(x)$, that is $d = 0$.

Exercise 1.3.9. For Grassmannian manifold $Gr(k, n)$, we have

$$0 \to E \to Gr(k, n) \otimes \mathbb{C}^n \to Q \to 0$$

Show that

$$T_{Gr(k,n)} \cong \operatorname{Hom}(E, Q)$$

Exercise 1.3.10 (adjunction formula)**.** Let $\pi : L \to X$ is a holomorphic line bundle, given $s \in \Gamma(X, L)$, suppose that $D = \{x \in X \mid s(x) = 0\}$ is a smooth submanifold of codimensional 1. Show that the following sequence is exact:

$$0 \to T_D \to T_X|_D \to L|_D \to 0$$

then we can get

$$K_D^* \cong K_D^* \otimes L|_D = (K_X^* \otimes L)|_D$$

Or dualizing gives

$$K_D \cong (K_X \otimes L)|_D$$

which is called adjunction formula.

Proof. First we have the following exact sequence

$$0 \to T_D \to T_X|_D \to T_X|_D/T_D \to 0$$

Sometimes we use N_D to denote $T_X|_D/T_D$, and call it norm bundle. So it suffices to show $L|_D$ is isomorphic to the norm bundle of D. Note that $s|_D = 0$ and s is not identically zero on X, so ds gives an isomorphism between $L|_D$ and N_D in fact.

Taking determinant we obtain adjunction formula, since for a exact sequence of vector bundle

$$0 \to A \to B \to C \to 0$$

we have

$$\det B = \det A \otimes \det C$$

And determinant of a line bundle is itself.

We often use adjunction formula when we understand X and K_X, and we want to compute K_D. In particular, let $X = \mathbb{P}^n$ and $L = \mathcal{O}_{\mathbb{P}^n}(-d)$, then Exercise 1.1.48 tells us that $D \subset \mathbb{P}^n$ is a smooth hypersurface defined by zero set of a homogenous polynomial with degree d. Then we have

$$\begin{aligned} K_D^* &\cong (K_X^* \otimes L)|_D \\ &= (\mathcal{O}_{\mathbb{P}^n}(n+1) \otimes \mathcal{O}_{\mathbb{P}^n}(-d))|_D \\ &\cong \mathcal{O}_{\mathbb{P}^n}(n+1-d)|_D \end{aligned}$$

and we call it

$$\begin{cases} \text{Fano}, & d < n+1 \\ \text{Calabi-Yau}, & d = n+1 \\ \text{General type}, & d > n+1 \end{cases}$$

□

Later we will define when a line bundle is positive. In other words, we say D is

$$\begin{cases} \text{Fano}, & \text{if } K_D^* \text{ is positive.} \\ \text{Calabi-Yau}, & \text{if } K_D^* \text{ is trivial.} \\ \text{General type}, & \text{if } K_D^* \text{ is negative.} \end{cases}$$

1.4. **Almost complex structure and integrable theorem.** Now let us talk about some linear algebra:

Consider a $2n$-dimensional real vector space V, a almost complex structure on V is a $\mathbb{R}$-linear transformation $J : V \to V$ such that $J^2 = -\operatorname{id}$. If V is the real vector space underlying a complex vector space, then $v \mapsto iv$ is an almost complex structure on V. Converse is still true, if there is an almost complex structure J on V, we can regard V as a complex vector space, by

$$(a + bi)v = av + bJ(v), \quad a, b \in \mathbb{R}$$

Indeed, the $\mathbb{R}$-linearity of J and the assumption $J^2 = -\operatorname{id}$ yield associative law, i.e

$$((a + ib)(c + id))v = (a + ib)((c + id)v), \quad v \in V$$

In particular, we have $i^2 v = -v$.

If we consider $V \otimes \mathbb{C}$, then J can extend to $V \otimes \mathbb{C}$ by $J(v \otimes \alpha) = J(v) \otimes \alpha$, then we can decompose $V \otimes \mathbb{C}$ into

$$\begin{aligned} V_{\mathbb{C}} = V \otimes \mathbb{C} &= V^{1,0} \oplus V^{0,1} \\ &= \{v \in V_{\mathbb{C}} \mid J(v) = iv\} \oplus \{v \in V_{\mathbb{C}} \mid J(v) = -iv\} \end{aligned}$$

such that $\overline{V^{1,0}} = V^{0,1}$, where conjugate means $\overline{v \otimes \alpha} = v \otimes \overline{\alpha}$. More explictly, for any $v \in V_{\mathbb{C}}$, we can decompose it into

$$v = \frac{1}{2}(v - iJ(v)) + \frac{1}{2}(v + iJ(v))$$

since the first part do lies in $V^{1,0}$, checked as follows

$$J(\frac{1}{2}(v - iJ(v))) = \frac{1}{2}(J(v) + iv)$$
$$i(\frac{1}{2}(v - iJ(v))) = \frac{1}{2}(iv + J(v))$$

and the latter part holds similarly.

Remark 1.4.1. One should be aware of the existence of two almost complex structure of $V_{\mathbb{C}}$. One is given by J and the other one by i. They coincide on the subspace $V^{1,0}$ but differ by a sign on $V^{0,1}$. We always regard $V_{\mathbb{C}}$ as the complex vector space with respect to i, and J is the additional structure that gives rise to the decomposition $V^{1,0} \oplus V^{0,1}$.

For a real vector space V of dimension d, there is a natural decomposition of its exterior algebra

$$\bigwedge^* V = \bigoplus_{k=0}^{d} \bigwedge^k V$$

We can do the same decomposition for the exterior algebra of $V_{\mathbb{C}}$ as

$$\bigwedge^* V_{\mathbb{C}} = \bigoplus_{k=0}^{d} \bigwedge^k V_{\mathbb{C}}$$

It's clear $\bigwedge^* V_{\mathbb{C}} = \bigwedge^* V \otimes \mathbb{C}$. Furthermore, we have the following decomposition

$$\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} V_{\mathbb{C}}^{p,q}$$

where $V_{\mathbb{C}}^{p,q} = \bigwedge^p V^{1,0} \otimes \bigwedge^q V^{0,1}$.

Now let's do what we have done on a manifold, firstly we need something playing the role of J.

Definition 1.4.2 (almost complex structure)**.** *X is a differential manifold of $\dim_{\mathbb{R}} X = 2n$. An almost complex structure on X is a complex structure on $T_{X,\mathbb{R}}$, i.e. an isomorphism of differential vector bundles $J : T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$ such that $J^2 = -\mathrm{id}$.*

It's natural to ask, if X is a complex manifold, and forget its complex structure and just regard it as a differential manifold, can we give a natural almost complex structure on it? That's the following example:

Example 1.4.3. X is a complex manifold, and $T_{X,\mathbb{R}}$ is its (real) tangent bundle if we just regard X as a differential manifold. Locally we have

$$T_{X,\mathbb{R}}|_U \cong U \times \mathbb{C}^n$$

for open subset U in X, where we regard $\mathbb{C}^n$ as a $2n$ dimension real vector space. There is a natural almost complex structure on it arising from multiplying i, so we get $J : T_{X,\mathbb{R}}|_U \to T_{X,\mathbb{R}}|_U$. More explictly, since there is a canonical basis of $T_{X,\mathbb{R}}|_U$ as follows:

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}$$

where $z_j = x_j + iy_j$ are standard coordinates on $\mathbb{C}^n$. We can write J explictly as follows

$$\begin{cases} J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i} \\ J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i} \end{cases}$$

If we want to get a global one, it suffices to glue them together. Consider any two trivializations $\widetilde{\varphi}_i : T_{X,\mathbb{R}}|_{U_i} \cong U_i \times \mathbb{C}^n, i = 1, 2$, where standard coordinates on U_1 is $\varphi_1(x) = (z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n)$ and the one on U_2 is $\varphi_2(x) = (g_1 = u_1 + iv_1, \ldots, g_n = u_n + iv_n)$. As we already know the transition functions between $T_{X,\mathbb{R}}|_{U_1}$ and $T_{X,\mathbb{R}}|_{U_2}$ is the Jacobian of $\varphi_1 \circ \varphi_2^{-1}$: For 2×2 part, we have

$$G_k \begin{pmatrix} \dfrac{\partial}{\partial x_k} \\ \dfrac{\partial}{\partial y_k} \end{pmatrix} := \begin{pmatrix} \dfrac{\partial u_j}{\partial x_k} & \dfrac{\partial u_j}{\partial y_k} \\ \dfrac{\partial v_j}{\partial x_k} & \dfrac{\partial v_j}{\partial y_k} \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial x_k} \\ \dfrac{\partial}{\partial y_k} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial}{\partial u_k} \\ \dfrac{\partial}{\partial v_k} \end{pmatrix}$$

For J, locally it looks like on $\{\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_k}\}$ or $\{\frac{\partial}{\partial u_k}, \frac{\partial}{\partial v_k}\}$:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

To check J can be glued together, it suffices to check G_k commutes with J on each 2×2 part. Cauchy-Riemann equation implies

$$G_k = \begin{pmatrix} \dfrac{\partial v_j}{\partial y_k} & \dfrac{\partial u_j}{\partial y_k} \\ -\dfrac{\partial u_j}{\partial y_k} & \dfrac{\partial v_j}{\partial y_k} \end{pmatrix} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

In this form it's easy to see G_k commutes with J, since

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -B & A \\ -A & -B \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

So complex structure gives an almost complex structure (naturally), but the question is: Does every complex structure on a complex manifold can be induced from a almost complex structure on an even-dimensional differential manifold? Unfortunately, it's false in general, but we have the following theorem.

Theorem 1.4.4 (Newlander-Nirenberg)**.** *Let (X, J) be a complex manifold, J is induced by a almost complex structure on X is equivalent to*

$$[T_X^{1,0}, T_X^{1,0}] \subset T_X^{1,0}$$

which is called an integrable condition.

1.5. **Operator ∂ and $\overline{\partial}$.** In this section, we will discuss the relationship between $T_X, T_{X,\mathbb{R}}, T_{X,\mathbb{C}}$ and so on, for a complex manifold X

First, we have $T_X \hookrightarrow T_{X,\mathbb{C}}$ as complex vector bundle[5], with image $T^{1,0}_{X,\mathbb{C}}$. In fact, if we consider locally, take $(z_1, \dots, z_n) \in U \subset \mathbb{C}^n, z_j = x_j + iy_j$ as a coordinate, then we can do the following identification

$$\Gamma(U, T_X) \ni \frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}) \in C^\infty(U, T^{1,0}_{X,\mathbb{C}})$$

moreover, we can define the conjugation as

$$\frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}) \in C^\infty(U, T^{0,1}_{X,\mathbb{C}})$$

that's non holomorphic part of $T_{U,\mathbb{C}}$. If we consider its dual space, we get the differential forms

$$\Omega^1_{X,\mathbb{C}} = \Omega^1_{X,\mathbb{R}} \otimes \mathbb{C} = \Omega^{1,0}_X \oplus \Omega^{0,1}_X$$

and take wedge product k times, then we get

$$\Omega^k_{X,\mathbb{C}} = \bigwedge^k \Omega^1_{X,\mathbb{C}} = \bigoplus_{p+q=k} \Omega^{p,q}_X, \quad \text{where } \Omega^{p,q}_X = \bigwedge^p \Omega^{1,0}_X \otimes (\bigwedge^q \Omega^{0,1}_X)$$

Remark 1.5.1. Let's see locally how to decompose $\Omega^1_{X,\mathbb{C}}$ into two parts with respect to the dual map of J. Choose a local coordinate $(z_1, \dots, z_n) \in U \subset \mathbb{C}^n, z_j = x_j + iy_j$. Then we have $C^\infty(U, \Omega^1_{X,\mathbb{C}}) = \mathrm{span}_{C^\infty(U)}\{\mathrm{d}x_1, \mathrm{d}y_1, \dots, \mathrm{d}x_n, \mathrm{d}y_n\}$. By definition

$$J^*(\mathrm{d}x_i)(\frac{\partial}{\partial x_i}) = \mathrm{d}x_i(J(\frac{\partial}{\partial x_i})) = \mathrm{d}x_i(\frac{\partial}{\partial y_i}) = 0$$
$$J^*(\mathrm{d}x_i)(\frac{\partial}{\partial y_i}) = \mathrm{d}x_i(J(\frac{\partial}{\partial y_i})) = \mathrm{d}x_i(-\frac{\partial}{\partial x_i}) = -1$$

so we have

$$J^*(\mathrm{d}x_i) = -\mathrm{d}y_i$$
$$J^*(\mathrm{d}y_i) = \mathrm{d}x_i$$

that is,

$$C^\infty(U, \Omega^{1,0}_{X,\mathbb{C}}) = \mathrm{span}_{C^\infty(U)}\{\mathrm{d}x_1 + i\mathrm{d}y_1, \dots, \mathrm{d}x_n + i\mathrm{d}y_n\}$$
$$C^\infty(U, \Omega^{0,1}_{X,\mathbb{C}}) = \mathrm{span}_{C^\infty(U)}\{\mathrm{d}x_1 - i\mathrm{d}y_1, \dots, \mathrm{d}x_n - i\mathrm{d}y_n\}$$

By our identification, the dual of $\dfrac{\partial}{\partial z_j}$ is

$$\mathrm{d}z_j = \mathrm{d}x_j + i\mathrm{d}y_j$$

and the dual of $\dfrac{\partial}{\partial \overline{z}_j}$ is

$$\mathrm{d}\overline{z}_j = \mathrm{d}x_j - i\mathrm{d}y_j$$

[5]However, not as a holomorphic vector bundle, since $T_{X,\mathbb{C}}$ contains part which is not holomorphic.

For any $\alpha \in C^\infty(X, \Omega^1_{X,\mathbb{C}})$, locally on U we have

$$\alpha = \sum \alpha_j \mathrm{d}x_j + \sum \beta_j \mathrm{d}y_j$$

then we can decompose it into

$$\alpha = \sum \frac{1}{2}(\alpha_j - i\beta_j)\mathrm{d}z_j + \sum \frac{1}{2}(\alpha_j + i\beta_j)\mathrm{d}\overline{z}_j$$

where the first part lies in $\Omega^{1,0}_U$ and the later part lies in $\Omega^{0,1}_U$.

Similarly we can take exterior product k times, get $\Omega^k_{X,\mathbb{C}}$ and decompose it into p, q parts.

Definition 1.5.2 (differential k-form)**.** *A k-form ω of type (p,q) is a differential section of $\Omega^{p,q}_X$, that is*

$$\omega \in C^\infty(X, \Omega^{p,q}_X) \subset C^\infty(X, \Omega^k_{X,\mathbb{C}})$$

Remark 1.5.3. It's quite neccessary for us to keep in mind how to distinguish a differential k-form what type it is, particularly for the case $k = 2$, since later we will study the first Chern class, a special $(1,1)$-form.

Let's firstly see it in a local viewpoint: Since $\Omega_{X,\mathbb{C}}$ is locally trivial, then for any section ω of $\Omega_{X,\mathbb{C}}$ we can write it as follows locally

$$\omega = \sum_i f_i \mathrm{d}z_i + g_i \mathrm{d}\overline{z}_i$$

Then after taking exterior product k times we get something which looks like

$$\sum_{\substack{|I|=p,|J|=q \\ p+q=k}} f_{IJ} \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J$$

So a k-form is a (p,q)-form if and only if locally it looks like

$$\sum_{|I|=p,|J|=q} f \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J$$

Or we can use a more global language: Let's elaborate in the case $k = 2$. By the definition of wedge of cotangent bundle, any section ω of $\Omega^2_{X,\mathbb{C}}$ is a skew-symmetric bilinear function defined on $C^\infty(X, T_{X,\mathbb{C}}) \times C^\infty(X, T_{X,\mathbb{C}})$. A 2-form ω is in type $(1,1)$ if and only if

$$\omega(C^\infty(X.T^{1,0}_X), C^\infty(X.T^{1,0}_X)) = \omega(C^\infty(X.T^{0,1}_X), C^\infty(X.T^{0,1}_X)) = 0$$

Similarly we can describe when a 2-form is in type $(2,0)$ or $(0,2)$.

Exercise 1.5.4. For $\mathbb{C}^n \cong \mathbb{R}^{2n}$, we have

$$\omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n = (\frac{i}{2})^n \mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\overline{z_n}$$

Proof. It suffices to show the case $n = 1$, and we can compute directly as follows

$$\begin{aligned}(\frac{i}{2})\mathrm{d}z \wedge \mathrm{d}\overline{z} &= (\frac{i}{2})(\mathrm{d}x + i\mathrm{d}y) \wedge (\mathrm{d}x - i\mathrm{d}y) \\ &= (\frac{i}{2})(-2i\mathrm{d}x \wedge \mathrm{d}y) \\ &= \mathrm{d}x \wedge \mathrm{d}y\end{aligned}$$

□

1.6. **Exterior differential.** Recall what we have done in the theory of de Rham cohomology: Let X be a differential manifold, with real dimension n, we have exterior differential

$$\mathrm{d} : C^\infty(X, \Omega^k_{X,\mathbb{R}}) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{R}})$$

such that

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \mathrm{d}\beta$$

Exterior differential has an important property:

$$\mathrm{d}^2 = 0$$

So we can consider such cochain complex

$$0 \to C^\infty(X, \Omega^0_{X,\mathbb{R}}) \to C^\infty(X, \Omega^1_{X,\mathbb{R}}) \to C^\infty(X, \Omega^2_{X,\mathbb{R}}) \to \cdots \to C^\infty(X, \Omega^n_{X,\mathbb{R}}) \to 0$$

with de Rham cohomology group

$$H^k(X, \mathbb{R}) := Z^k(X, \mathbb{R})/B^k(X, \mathbb{R})$$

The following theorem implies that the de Rham cohomology is just a topological data.

Theorem 1.6.1 (de Rham theorem)**.** $H^k(X, \mathbb{R})$ *computes the singular cohomology of X with real coefficient.*

Remark 1.6.2. We will prove de Rham theorem in Appendix A.

Theorem 1.6.3 (Poincaré lemma)**.** *Let $X = B(x_0, r_0) \subset \mathbb{R}^n$ is a open ball, then $H^k(X, \mathbb{R}) = 0, \forall k > 0$.*

Remark 1.6.4. Poincaré lemma implies that for small enough open set, the cohomology groups are trivial, so only for global differential forms, de Rham cohomology tells interesting information.

So let's see what is the complex version of above theory. Now let X be a complex manifold, with complex dimension n, then similar we also have an exterior derivative

$$\mathrm{d} : C^\infty(X, \Omega^k_{X,\mathbb{C}}) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{C}})$$

But there is also something interesting, we already know that we can decompose $\Omega^k_{X,\mathbb{C}}$, but for any $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{C}})$, we can also decompose $\mathrm{d}\alpha$.

Example 1.6.5. For $\alpha \in C^\infty(X, \Omega^0_{X,\mathbb{C}})$, then

$$\mathrm{d}\alpha \in C^\infty(X, \Omega^1_{X,\mathbb{C}}) = C^\infty(X, \Omega^{1,0}_X) \oplus C^\infty(X, \Omega^{0,1}_X)$$

Locally, we have

$$\begin{aligned}\mathrm{d}\alpha &= \sum \frac{\partial \alpha}{\partial x_j}\mathrm{d}x_j + \sum \frac{\partial \alpha}{\partial y_j}\mathrm{d}y_j \\ &= \sum \frac{1}{2}(\frac{\partial \alpha}{\partial x_j} - i\frac{\partial \alpha}{\partial y_j})\mathrm{d}z_j + \sum \frac{1}{2}(\frac{\partial \alpha}{\partial x_j} + i\frac{\partial \alpha}{\partial y_j})\mathrm{d}\overline{z}_j \\ &= \sum \frac{\partial \alpha}{\partial z_j}\mathrm{d}z_j + \sum \frac{\partial \alpha}{\partial \overline{z}_j}\mathrm{d}\overline{z}_j\end{aligned}$$

More generally, for $\alpha \in C^\infty(X, \Omega^{p,q}_X)$, then locally

$$\alpha = \sum_{|J|=p, |K|=q} \alpha_{J,K}\mathrm{d}z_J \wedge \mathrm{d}\overline{z}_K$$

then

$$\mathrm{d}\alpha = \sum_{|J|=p,|K|=q} \frac{\partial \alpha_{J,K}}{\partial z_l}\mathrm{d}z_l \wedge \mathrm{d}z_J \wedge \mathrm{d}\overline{z}_K + \sum_{|J|=p,|K|=q} \frac{\partial \alpha_{J,K}}{\partial \overline{z_l}}\mathrm{d}\overline{z_l} \wedge z_J \wedge \overline{z}_K$$

that is

$$\mathrm{d}\alpha \in C^\infty(X, \Omega^{p+1,q}_X) \oplus C^\infty(X, \Omega^{p,q+1}_X)$$

Definition 1.6.6 (partial operator)**.** *For $\alpha \in C^\infty(X, \Omega^{p,q}_X)$, we can define partial operator and its conjugation $\partial\alpha, \overline{\partial}\alpha$ as follows*

$$\mathrm{d}\alpha = \partial\alpha + \overline{\partial}\alpha$$

where $\partial\alpha \in C^\infty(X, \Omega^{p+1,q}_X), \overline{\partial}\alpha \in C^\infty(X, \Omega^{p,q+1}_X)$. More generally, if $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{C}})$, write $\alpha = \sum \alpha^{p,q}$, then we can define

$$\partial\alpha = \sum_{p,q} \partial\alpha^{p,q}, \quad \overline{\partial}\alpha = \sum_{p,q} \overline{\partial}\alpha^{p,q}$$

Remark 1.6.7. We have the following relations

1. Leibniz rule

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{\deg \alpha}\alpha \wedge \partial\beta$$

2. [6]

$$\partial^2 = \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0$$

So we can do the same thing for ∂ by consider the following cochain complex[7]

$$0 \to C^\infty(X, \Omega^{p,0}_X) \xrightarrow{\overline{\partial}} C^\infty(X, \Omega^{p,1}_X) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} C^\infty(X, \Omega^{p,n}_X) \to 0$$

[6] Hint: consider $\mathrm{d}^2 = (\partial + \overline{\partial})^2 = 0$

[7] You may wonder why don't we use ∂ to construct such cobchain complex. In fact, the two definitions are almost the same, since they conjugate to each other. However, the cohomology group of cochain complex defined by $\overline{\partial}$ is more meaningful, as we will see later.

Definition 1.6.8 (Dolbeault cohomology)**.**

$$H^{p,q}(X) := Z^{p,q}(X)/B^{p,q}(X) = H^q_{\overline{\partial}}(C^\infty(X, \Omega^{p,*}_X))$$

Here comes the key question we mentioned in the overview: Since we have $C^\infty(X, \Omega^k_{X,\mathbb{C}}) = \bigoplus_{p+q=k} C^\infty(X, \Omega^{p,q}_X)$, could we have the following decomposition?

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

Example 1.6.9. What is $H^{p,0}(X)$? Since $B^{p,0} = 0$, then

$$H^{p,0}(X) = Z^{p,0}(X) = \{\alpha \in C^\infty(X, \Omega^{p,0}_X) \mid \overline{\partial}\alpha = 0\}$$

Locally $\alpha = \sum_{|J|=p} \alpha_J \mathrm{d}z_J$, then

$$\overline{\partial}\alpha = \sum_{|J|=p} \frac{\partial \alpha_J}{\partial \overline{z}_K} \mathrm{d}\overline{z}_K \wedge \mathrm{d}z_J = 0 \implies \frac{\partial \alpha_J}{\partial \overline{z}_K} = 0$$

That is, α_J is holomorphic function. Since $\Omega^{p,0}_X \cong \Omega^p_X$ as complex vector bundle, we have $H^{p,0}(X) = \Gamma(X, \Omega^p_X)$[8].

Example 1.6.10. For a holomorphic map $f : X \to Y$ between complex manifold, then

$$f^* : C^\infty(Y, \Omega^k_{Y,\mathbb{C}}) \to C^\infty(X, \Omega^k_{X,\mathbb{C}})$$

Then[9]

$$f^* : C^\infty(Y, \Omega^{p,q}_{Y,\mathbb{C}}) \to C^\infty(X, \Omega^{p,q}_{X,\mathbb{C}})$$

and

$$f^* : H^{p,q}(Y) \to H^{p,q}(X)$$

so Dolbeault cohomology is a contravariant functor.

Example 1.6.11 (Dolbeault cohomology of a holomorphic vector bundle[10])**.** For a holomorphic vector bundle $E \to X$, we can also define

$$\overline{\partial}_E : C^\infty(X, \Omega^{0,q}_X \otimes E) \to C^\infty(X, \Omega^{0,q+1}_X \otimes E)$$

satisfies $\overline{\partial}^2_E = 0$. Let's elaborate this construction: Since any global section is glued together by local sections, we just need to define $\overline{\partial}_E$ for local sections and check is well-defined under the change of local chart. We can choose a local holomorphic frame $\{e_1, \dots, e_n\}$ for E on U, so any section $\sigma \in C^\infty(U, \Omega^{0,q}_X \otimes E)$ we can write $\sigma = \sum_i \varphi_i \otimes e_i$ for $\varphi_i \in C^\infty(U, \Omega^{0,q}_X)$. Then we can define

$$\overline{\partial}_E(\sigma) = \sum_i \overline{\partial}\varphi_i \otimes e_i$$

[8] This implies that Dolbeault cohomology do computes useful information.

[9] Check this, we need back to definition, a holomorphic map induces a tangent map $T_f : T_{X,\mathbb{C}} \to f^*T_{Y,\mathbb{C}}$,and consider its dual we get cotangent map $\Omega_f : f^*\Omega_{Y,\mathbb{C}} \to \Omega_{X,\mathbb{C}}$

[10] In previous, $E = \Omega^p_X$

It's clear that this definition is independent of the choice of local chart, since the transition functions are holomorphic and $\overline{\partial}$ kills them. Furthermore, $\overline{\partial}_E^2 = 0$ holds since $\overline{\partial}^2 = 0$.

So we can construct a cochain complex and define its cohomology, denoted by

$$H^q(X, E) = H^q_{\overline{\partial}_E}(C^\infty(X, \Omega_X^{0,*} \otimes E))$$

and we can show a result similar to Example 1.6.8.

$$H^0(X, E) = \Gamma(X, E)$$

Theorem 1.6.12 (Dolbeault lemma)**.** *Let $X = D(z_0, r_0) \subset \mathbb{C}^n$ be a polydisk, then*

$$H^{p,q}(X) = 0, \quad \forall p \geq 0, q > 0$$

1.7. **Čech cohomology.** Let X be a topological space, and $\mathcal{U} = (U_i)_{i\in I}$ be an open covering, such that I is countable and an ordered set. For all $i_0, \dots, i_p \in I$ write

$$U_{i_0 \dots i_p} = U_{i_0} \cap \dots \cap U_{i_p}$$

Let $\mathscr{F}$ be a sheaf of abelian group, define a chain complex $C^*(\mathcal{U}, \mathscr{F})$ as

$$0 \to C^0(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta} C^2(\mathcal{U}, \mathscr{F}) \to \dots$$

where

$$C^p = \prod_{i_0 < \dots < i_p} \mathscr{F}(U_{i_0 \dots i_p})$$

and δ is defined as

$$(\delta\alpha)_{i_0 \dots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \dots \widehat{i_k} \dots i_{p+1}}|_{U_{i_0 \dots i_{p+1}}}$$

Exercise 1.7.1 (Once and only once exercise in your whole life)**.** Check that $\delta \circ \delta = 0$

Thanks to Exercise 1.7.1 we can define Čech cohomology as the cohomology of above complex:

$$\check{H}^q(\mathcal{U}, \mathscr{F}) := H^q_\delta(C^*(\mathcal{U}, \mathscr{F}))$$

Example 1.7.2. We consider

$$\check{H}^0(\mathcal{U}, \mathscr{F}) = \{\alpha \in C^0(\mathcal{U}, \mathscr{F}) \mid \delta\alpha = 0\}$$

then if $\alpha = \prod_{i_0} \alpha_{i_0}$, then $\delta\alpha = 0$ implies

$$\alpha_i|_{U_i \cap U_j} = \alpha_j|_{U_i \cap U_j}$$

then we have

$$\check{H}^0(\mathcal{U}, \mathscr{F}) = \mathscr{F}(X)$$

In other words, the 0 dimensional Čech cohomology group of a sheaf is global section of itself.

However, we want out definition is independent of open cover, so

Definition 1.7.3 (Čech cohomology)**.** *We define Čech cohomology as*

$$\check{H}^q(X,\mathscr{F})=\varinjlim_{\mathcal{U}}\check{H}^q(\mathcal{U},\mathscr{F})$$

Remark 1.7.4. In other words, $\alpha=\alpha'\in\check{H}^q(X,\mathscr{F})$ is equivalent to there exists a common refinement $\mathcal{U}''$ such that

$$\alpha=\alpha'\in\check{H}^q(\mathcal{U}'',\mathscr{F})$$

Why we want to introduce Čech cohomology here? In fact, it provides a method to compute de Rham cohomology and Dolbeault cohomology we defined before. However, we will use sheaf cohomology to cover de Rham cohomology and Čech cohomology in Appendix A

Recall that if X is a complex manifold, $E\to X$ is a holomorphic vector bundle. Then we can define a sheaf of holomorphic sections, defined by

$$\mathcal{O}_X(E): U\mapsto\Gamma(U,E|_U)$$

then we get a Čech cohomology of this sheaf

$$\check{H}^q(X,\mathcal{O}_X(E))=\varinjlim_{\mathcal{U}}\check{H}^q(\mathcal{U},\mathcal{O}_X(E))$$

Theorem 1.7.5 (comparision)**.** *We have the following isomorphism*

$$\check{H}^q(X,\mathcal{O}_X(E))\cong H^q(X,E):=H^q_{\overline{\partial}_E}(C^\infty(X,\Omega_X^{0,*}\otimes E))$$

In particular, let $E=\Omega_X^p$, we have

$$\check{H}^q(X,\mathcal{O}_X(\Omega_X^p))\cong H^q(X,\Omega_X^p):=H^q_{\overline{\partial}}(C^\infty(X,\Omega_X^{p,*}))=H^{p,q}(X)$$

Remark 1.7.6. This remark tries to explain why de Rham cohomology reflects topological information, which Dolbeault cohomology reflects the holomorphic information: In fact, as you can see in Appendix A. The following sequence of sheaves

$$0\to E\to\Omega_X^{0,0}\otimes E\xrightarrow{\overline{\partial}_E}\Omega_E^{0,1}\otimes E\xrightarrow{\overline{\partial}_E}\cdots\xrightarrow{\overline{\partial}_E}\Omega_X^{0,n}\otimes E\to 0$$

is a fine resolution of E. Indeed, it's exact since there is Dolbeault lemma. It's fine since we can use partition of unity when we deal with smooth sections. Then the cohomology of complex $C^\infty(X,\Omega_X^{0,*}\otimes E)$ computes the cohomology of sheaf of holomorphic sections of E. If we take $E=\Omega^p$, that's Dolbeault cohomology, and that's why Dolbeault cohomology reflects holomorphic information.

We will see in Appendix B, the Čech cohomology equals to sheaf cohomology, and that's why it's called comparision theorem, since it compares Čech cohomology and sheaf cohomology.

Similarly, in de Rham cohomology, we have the same story. There is a sequence of sheaves

$$0\to\underline{\mathbb{C}}\xrightarrow{i}\Omega_{X,\mathbb{C}}^0\xrightarrow{\mathrm{d}}\Omega_{X,\mathbb{C}}^1\xrightarrow{\mathrm{d}}\cdots\xrightarrow{\mathrm{d}}\Omega_{X,\mathbb{C}}^n\to 0$$

where $\underline{\mathbb{C}}$ is the sheaf of locally constant functions, i.e.

$$\underline{\mathbb{C}} : U \mapsto \{\text{locally constant functions } f : U \to \mathbb{C}\}$$

It's a fine resolution. Indeed, Poincaré lemma implies the above sequence is also exact, and for differential k-forms we also have partition of unity. Sheaf cohomology tells us de Rham cohomology computes the cohomology of constant sheaf $\underline{\mathbb{C}}$. And this also explains why de Rham cohomology is just a topological information, since constant sheaf is just a pure topological information.

Theorem 1.7.7 (Leray)**.** *Let $\mathcal{U}$ be a covering such that for all $i_0 \dots i_k \in I$, and for all $q > 0$, we have*

$$H^q(U_{i_0 \dots i_k}, E|_{U_{i_0 \dots i_k}}) = 0$$

then $\mathcal{U}$ is called acyclic for E. Then

$$\check{H}^q(\mathcal{U}, \mathcal{O}_X(E)) \cong H^q(X, E)$$

Remark 1.7.8. This provides us a practical way to compute Čech cohomology.

Example 1.7.9. Consider $\mathcal{O}_X^\times \subset \mathcal{O}_X$, the sheaf of invertible holomorphic functions. Then we have

$$\check{H}^1(X, \mathcal{O}_X^\times) \cong \operatorname{Pic}(X)$$

2. Geometry of vector bundles

2.1. Connections.

Definition 2.1.1 (connection)**.** *X is a differential manifold, and $\pi : E \to X$ is a complex vector bundle. A connection on E is a $\mathbb{C}$-linear operator*

$$D : C^\infty(X, E) \to C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes E)$$

satisfying the Leibniz rule

$$D(f\sigma) = \mathrm{d}f \otimes \sigma + fD(\sigma)$$

for $f \in C^\infty(X)$ and $\sigma \in C^\infty(X, E)$.

Remark 2.1.2. In fact, if we ask D to satisfy the Leibniz rule, it induces

$$D : C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes E) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{C}} \otimes E)$$

for any k, by setting[11]

$$D(\varphi \otimes \sigma) = \mathrm{d}\varphi \otimes \sigma + (-1)^{\deg \varphi} \varphi \wedge (D\sigma)$$

for $\varphi \in C^\infty(X, \Omega^k_{X,\mathbb{C}})$ and $\sigma \in C^\infty(X, E)$.

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[11]Some authors may extend D by setting usual Leibniz rule, that is, without $(-1)^{\deg \varphi}$, it's not quit important. We will see that reason in local chart computation.

Remark 2.1.3. Let's see what's going on in local pointview. Locally around $x \in U \subset X$, then $\pi^{-1}(U) \cong U \times \mathbb{C}^r$, there is a basis $\{e_1, \dots, e_r\}$ for $\mathbb{C}^r$. For $\sigma \in C^\infty(U, E|_U)$, we have

$$\sigma = \sum_{j=1}^r s_j e_j, \quad s_j \in C^\infty(U)$$

By Leibniz rule, we have

$$D\sigma = \sum_{j=1}^r (\mathrm{d}s_j \otimes e_j + s_j De_j)$$

where $De_j \in C^\infty(U, \Omega^1_{X,\mathbb{C}} \otimes E)$. So we can write more explictly as

$$De_j = \sum_{i=1}^r a_{ij} \otimes e_i, \quad a_{ij} \in C^\infty(U, \Omega^1_{X,\mathbb{C}})$$

So we have

$$D\sigma = \sum_{j=1}^r (\mathrm{d}s_j \otimes e_j + \sum_{i=1}^r s_j a_{ij} \otimes e_i)$$

We can rewrite the above formula in frame of $\{e_1, \dots, e_r\}$ as

$$D\sigma = Ds = \mathrm{d}s + As$$

where

$$\sigma = s = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}, \quad A = (a_{ij}) \in C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes \mathrm{End}(E|_U))$$

Here we chose a local trivialization of the vector bundle E, so we may wonder what will happen if we change our choice. If $x \in U' \subset X$ is another trivialization, so $\pi^{-1}(U') \cong U' \times \mathbb{C}^r$, and $\{e'_1, \dots, e'_r\}$ is another local frame. Then

$$D\sigma = \begin{cases} Ds = \mathrm{d}s + As \\ Ds' = \mathrm{d}s' + A's' \end{cases}$$

so we wonder the relationship between A and A'. Transition functions between U and U' are

$$g : U \cap U' \to \mathrm{GL}(r, \mathbb{C})$$

so we have $s = gs'$ and $Ds = gDs'$. We compute as follows

$$\begin{aligned} \mathrm{d}s &= \mathrm{d}(gs') = (\mathrm{d}g)s' + g(\mathrm{d}s') = g(g^{-1}(\mathrm{d}g)s' + \mathrm{d}s') \\ \mathrm{d}s + As &= g(g^{-1}(\mathrm{d}g)s' + \mathrm{d}s' + g^{-1}Ags') \\ &= g(\mathrm{d}s' + (g^{-1}\mathrm{d}g + g^{-1}Ag)s') \end{aligned}$$

Since we have

$$\mathrm{d}s + As = gDs' = g(\mathrm{d}s' + A's')$$

So we have

$$A' = g^{-1}\mathrm{d}g + g^{-1}Ag$$

You may feel quite uncomfortable since A' does not conjugate to A under the change of the frame, but if we apply D twice, something interesting may happen.

Before that, we compute what does $D : C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes E) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{C}} \otimes E)$ look like locally:

Take $\sigma \in C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes E)$, if we choose a local frame $\{e_1, \dots, e_r\}$ on U, locally we can write as

$$\sigma = \sum_{j=1}^{r} s_j \otimes e_j, \quad s_j \in C^\infty(U, \Omega^k_{X,\mathbb{C}})$$

then

$$\begin{aligned}
D\sigma &= D(\sum_{j=1}^{r} s_j \otimes e_j) \\
&= \sum_{j=1}^{r} \mathrm{d}s_j \otimes e_j + (-1)^k s_j \wedge De_j \\
&= \sum_{j=1}^{r} (\mathrm{d}s_j \otimes e_j + (-1)^k s_j \wedge \sum_{i=1}^{r} a_{ij} \otimes e_i) \\
&= \sum_{j=1}^{r} (\mathrm{d}s_j \otimes e_j + \sum_{i=1}^{r} a_{ij} \wedge s_j \otimes e_i) \\
&= \mathrm{d}s + A \wedge s
\end{aligned}$$

So we know that $D : C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes E) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{C}} \otimes E)$ still looks like[12]

$$D\sigma = Ds = \mathrm{d}s + A \wedge s$$

Here we can see clearly what does $A \wedge s$ mean. Furthermore, we can see that $A \wedge A$ isn't trivial, unless in the case of A is a 1×1 matrix, in other words, E is a line bundle.

So we can compute as follows.

$$\begin{aligned}
D^2\sigma = D(\mathrm{d}s + As) &= \mathrm{d}(\mathrm{d}s + As) + A \wedge (\mathrm{d}s + As) \\
&= \mathrm{d}^2 s + \mathrm{d}(As) + A \wedge \mathrm{d}s + A \wedge As \\
&= \mathrm{d}^2 s + (\mathrm{d}A)s - A \wedge \mathrm{d}s + A \wedge \mathrm{d}s + A \wedge As \\
&= (\mathrm{d}A + A \wedge A)s
\end{aligned}$$

And we check what will happen if we choose another trivialization

$$\begin{aligned}
D^2\sigma = D^2 s = (\mathrm{d}A + A \wedge A)s &= (\mathrm{d}A + A \wedge A)gs' \\
&= gD^2 s' = g(\mathrm{d}A' + A' \wedge A')s'
\end{aligned}$$

[12] That's why we need $(-1)^{\deg \varphi}$ when we extend D.

so we have

$$\mathrm{d}A' + A' \wedge A' = g^{-1}(\mathrm{d}A + A \wedge A)g$$

that is, $\mathrm{d}A + A \wedge A$ behaves "well" under the change of frame, object with such property we always call it a "tensor"[13].

From discussion above, we can give the following definition

Definition 2.1.4 (curvature)**.** *There exists a global section $H_D \in C^\infty(X, \Omega^2_{X,\mathbb{C}} \otimes \mathrm{End}(E))$ such that*

$$D^2\sigma = H_D \wedge \sigma, \quad \forall \sigma \in C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes E)$$

such H_D is called the curvature tensor of connection D.

Definition 2.1.5 (Hermitian metric)**.** *X is a differential manifold, and $\pi : E \to X$ is a complex vector bundle. A Hermitian metric h on E is a Hermitian inner product on each fiber E_x, such that for all open subset $U \subset X$, and $\xi, \eta \in C^\infty(U, E|_U)$, we have*

$$\begin{aligned} \langle \xi, \eta \rangle : U &\to \mathbb{C} \\ x &\mapsto \langle \xi(x), \eta(x) \rangle \end{aligned}$$

is a smooth function.

Example 2.1.6. Locally, for $x \in U \subset X$, we have $\pi^{-1}(U) \cong U \times \mathbb{C}^r$, and $\{e_1, \dots, e_r\}$ is a local frame. Then our Hermitian metric is just a Hermitian matrix

$$H = (h_{\lambda\mu})$$

where $h_{\lambda\mu} \in C^\infty(U)$, defined by

$$h_{\lambda\mu}(x) = \langle e_\lambda(x), e_\mu(x) \rangle$$

Indeed, this Hermitian matrix can tell us how does the metric works. In our local frame, two sections $\xi = \sum_{i=1}^r \xi_i e_i, \eta = \sum_{i=1}^r \eta_i e_i \in C^\infty(U, E|_U)$ can be write as

$$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_r \end{pmatrix}, \eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_r \end{pmatrix}$$

[13]But what is a "tensor"? Here I quote a motto said by Leonard Susskind, a well-known physicist. I'm quite impressed when I first heard it in my childhood. "Tensor is something which behaves like a tensor."

Then

$$\begin{aligned}
h(\xi,\eta) &= h(\sum_{i=1}^r \xi_i e_i, \sum_{i=1}^r \eta_i e_i) \\
&= \sum_{i,j=1}^r \xi_i \overline{\eta_j} h(e_i, e_j) \\
&= \sum_{i,j=1}^r \xi_i \overline{\eta_j} h_{ij} \\
&= (\xi_1, \ldots, \xi_r) H \begin{pmatrix} \overline{\eta_1} \\ \vdots \\ \overline{\eta_r} \end{pmatrix} \\
&= \xi^t H \overline{\eta}
\end{aligned}$$

And take another $x \in U' \subset X$, $\pi^{-1}(U') \cong U' \times \mathbb{C}^r$, with $\{e_1', \ldots, e_r'\}$, and g is the transition function, we have

$$H' = g^t H \overline{g}$$

Proposition 2.1.7. *Every complex vector bundle admits a Hermitian metric*

Proof. Use partition of unity. □

Now for a complex vector bundle over a differential manifold, we have two structures on it, connection and Hermitian metric, so it's natural to require them to exist in a harmony.

For a Hermitian metric h, it induces a pairing $\{\cdot,\cdot\}$

$$C^\infty(X, \Omega^p_{X,\mathbb{C}} \otimes E) \times C^\infty(X, \Omega^q_{X,\mathbb{C}} \otimes E) \to C^\infty(X, \Omega^{p+q}_{X,\mathbb{C}})$$

We describe this pairing locally, consider $x \in U \subset X$, $\pi^{-1}(U) \cong U \times \mathbb{C}^r$, and $\{e_1, \ldots, e_r\}$ is a local frame. Then $\sigma \in C^\infty(X, \Omega^p_{X,\mathbb{C}} \otimes E)$ and $\eta \in C^\infty(X, \Omega^q_{X,\mathbb{C}} \otimes E)$ are in form

$$\sigma = \sum_{i=1}^r s_i \otimes e_i, \quad \tau = \sum_{j=1}^r t_j \otimes e_j$$

where s_i are p-forms and t_j are q-forms, then the pairing is locally look like

$$\begin{aligned}
\{\sigma, \tau\} &= \{\sum_{i=1}^r s_i \otimes e_i, \sum_{j=1}^r t_j \otimes e_j\} \\
&= \sum_{i,j=1}^r s_i \wedge t_j h(e_i, e_j) \\
&= \sum_{i,j=1}^r s_i \wedge t_j h_{ij} \\
&= s^t H \wedge \overline{t}
\end{aligned}$$

Using this pairing, we can define when a connection is called Hermitian.

Definition 2.1.8 (Hermitian connection)**.** *(E, h) is a Hermitian vector bundle on X. A connection D on E is Hermitian if for all $\sigma \in C^\infty(X, \Omega^p_{X,\mathbb{C}} \otimes E), \eta \in C^\infty(X, \Omega^q_{X,\mathbb{C}} \otimes E)$,*

$$\mathrm{d}\{\sigma, \tau\} = \{D\sigma, \tau\} + (-1)^{\deg \sigma}\{\sigma, D\tau\}$$

Since we know that a connection locally looks like $D = \mathrm{d} + A\wedge$. Then let's compute in a local frame to show what condition A needs to satisfy for a Hermitian connection.

Fixing a local frame, then $\sigma = s = (s_1, \dots, s_r)^t, \tau = t = (t_1, \dots, t_r)^t$. WLOG, we assume $\{e_1, \dots, e_r\}$ is a orthonormal basis, i.e. H is identity matrix, then

$$\{\sigma, \tau\} = s^t \wedge \bar{t}$$

Otherwise we need to consider the derivative of H, which will make things more complicated. Exercise 2.1.9 gives us an equivalent condition for A is a Hermitian connection.

If we already use orthonormal basis, we can directly compute as follows

$$\begin{aligned}
\mathrm{d}\{\sigma, \tau\} &= (\mathrm{d}s)^t \wedge \bar{t} + (-1)^{\deg \sigma} s^t \wedge \mathrm{d}\bar{t} \\
\{D\sigma, \tau\} &= (\mathrm{d}s + A \wedge s)^t \wedge \bar{t} = (\mathrm{d}s)^t \wedge \bar{t} + (-1)^{\deg \sigma} s^t \wedge A^t \wedge \bar{t} \\
\{\sigma, D\tau\} &= s^t \wedge (\overline{\mathrm{d}t + A \wedge t}) = s^t \wedge \mathrm{d}\bar{t} + s^t \wedge \overline{A} \wedge \bar{t}
\end{aligned}$$

then

$$\mathrm{d}\{\sigma, \tau\} - \{D\sigma, \tau\} - (-1)^{\deg \sigma}\{\sigma, D\tau\} = (-1)^{\deg \sigma + 1} s^t \wedge (A^t + \overline{A}) \wedge \bar{t}$$

So D is a Hermitian connection if and only if $A^t + \overline{A} = 0$.

Exercise 2.1.9. If connection D is locally given by A, and Hermitian metric is locally given by H. Show that A is Hermitian if and only if

$$\mathrm{d}H = A^t H + H\overline{A}$$

Proof. Let's compute $\mathrm{d}\{\sigma, \tau\}$ without choosing orthonormal basis:

$$\begin{aligned}
\mathrm{d}\{\sigma, \tau\} &= (\mathrm{d}s)^t H \wedge \bar{t} + (-1)^{\deg \sigma} s^t \wedge \mathrm{d}H \wedge \bar{t} + (-1)^{\deg \sigma} s^t H \wedge \mathrm{d}\bar{t} \\
\{D\sigma, \tau\} &= (\mathrm{d}s + A \wedge s)^t H \wedge \bar{t} = (\mathrm{d}s)^t H \wedge \bar{t} + (-1)^{\deg \sigma} s^t \wedge A^t H \wedge \bar{t} \\
\{\sigma, D\tau\} &= s^t H \wedge (\overline{\mathrm{d}t + A \wedge t}) = s^t H \wedge \mathrm{d}\bar{t} + s^t H\overline{A} \wedge \mathrm{d}\bar{t}
\end{aligned}$$

Thus

$$\mathrm{d}\{\sigma, \tau\} = \{D\sigma, \tau\} + (-1)^{\deg \sigma}\{\sigma, D\tau\} \Longleftrightarrow \mathrm{d}H = A^t H + H\overline{A}$$

□

Remark 2.1.10. Let's make it more beautiful. We define D^{adj}, adjoint connection, locally given by $-\overline{A}^t$ with respect to $H = I$, then we always have

$$\mathrm{d}\{\sigma, \tau\} = \{D\sigma, \tau\} + (-1)^{\deg \sigma}\{\sigma, D^{\mathrm{adj}}\tau\}$$

Take $\dfrac{1}{2}(D + D^{\mathrm{adj}})$, which is also a connection, locally looks like

$$\frac{1}{2}(A - \overline{A}^t)$$

is a Hermitian connection. So it's easy to get a Hermitian connection, just average A with its adjoint.

Proposition 2.1.11. *Every Hermitian vector bundle admits a Hermitian connection.*

Proof. Use partition of unity to show the existence of connection, and take the average of it and its adjoint connection. □

2.2. **Connections and metrics on holomorphic vector bundles.** In this section, let's see when the base space is a complex manifold, and the vector bundle is holomorphic, what will happen?

Recall that for a complex manifold X, we have

$$\Omega^1_{X,\mathbb{C}} = \Omega^{1,0}_X \oplus \Omega^{0,1}_X$$

Consider $E \to X$ is a complex vector bundle, and D is a connection, then we can decompose $D = D^{1,0} + D^{0,1}$ by composing the projection as follows

$$\begin{tikzcd}
 & & C^\infty(X, \Omega^{1,0}_X \otimes E) \\
C^\infty(X,E) \arrow[r, "D"] & C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes E) \arrow[ur] \arrow[dr] & \\
 & & C^\infty(X, \Omega^{0,1}_X \otimes E)
\end{tikzcd}$$

Locally, we have $D = \mathrm{d} + A$, then

$$D^{1,0} = \partial + A^{1,0}, \quad D^{0,1} = \overline{\partial} + A^{0,1}$$

both $D^{1,0}$ and $D^{0,1}$ satisfy Leibniz rule.

Now consider X is a complex manifold, and $E \to X$ is a holomorphic vector bundle. Recall that we already have

$$\overline{\partial}_E : C^\infty(X,E) \to C^\infty(X, \Omega^{0,1}_X \otimes E)$$

We want to compare $D^{0,1}_E$ and $\overline{\partial}_E$

Theorem 2.2.1 (Chern connection)**.** *X is a complex manifold, (E,h) is a Hermitian holomorphic vector bundle, then there exists a unique Hermitian connection D_E such that $D^{0,1}_E = \overline{\partial}_E$. D_E is called the Chern connection of (E,h).*

Proof. Uniqueness: If we already have $D^{0,1}_E = \overline{\partial}_E$. Locally $x \in U \subset X$, $\{e_1, \ldots, e_r\}$ is holomorphic local frame, and smooth section $\sigma = s = (s_1, \ldots, s_r)^t$. Then

$$D^{0,1}_E \sigma = \overline{\partial} s + A^{0,1} s = \overline{\partial}_E \sigma$$

If s is a holomorphic section, then $\overline{\partial}_E\sigma = \overline{\partial}s = 0$, which implies $A^{0,1} = 0$.

Since we have

$$\mathrm{d}H = \partial H + \overline{\partial}H = \underbrace{A^tH}_{\text{(1,0) part}} + \underbrace{H\overline{A}}_{\text{(0,1) part}}$$

then

$$\overline{\partial}H = H\overline{A}$$

So A is uniquely determined by

$$A = \overline{H}^{-1}\partial\overline{H}$$

Existence: It suffices to prove we can glue A together to get a global connection, i.e. compatible with holomorphic change of frames.

Consider another holomorphic local chart $x \in U' \subset X$, with frame $\{e'_1, \dots, e'_r\}$. And the metric with respect to this new frame is H', we have

$$H' = g^tH\overline{g}$$

Then

$$\begin{aligned} A' = \overline{H'^{-1}}\partial\overline{H'} &= g^{-1}\overline{H^{-1}(g^t)^{-1}}\partial(\overline{g^t}\overline{H}g) \\ &= g^{-1}\overline{H^{-1}(g^t)^{-1}}((\partial\overline{g^t}) + \overline{g^t}(\partial\overline{H})g + \overline{g^t}\overline{H}\partial g) \\ &= g^{-1}\overline{H^{-1}}(\partial\overline{H}g) + g^{-1}\mathrm{d}g \\ &= g^{-1}\mathrm{d}g + g^{-1}Ag \end{aligned}$$

As we desire. □

Corollary 2.2.2. *If X is a complex manifold, (E, h) is a Hermitian holomorphic vector bundle, D_E is Chern connection on it, and H_E is its curvature, called Chern curvature. If A is the matrix of D_E with respect to holomorphic local frame, then*

1. *A is of type $(1, 0)$, with $\partial A = -A \wedge A$*
2. *locally we have $H_E = \overline{\partial}A$, a form of type $(1, 1)$*
3. *$\overline{\partial}H_E = 0$*

Proof. For (1). Locally we have $A = \overline{H}^{-1}\partial\overline{H}$, so it's of type $(1, 0)$, and we compute

$$\begin{aligned} \partial A = \partial(\overline{H}^{-1}\partial\overline{H}) &= \partial\overline{H}^{-1} \wedge \partial\overline{H} \\ &= (-\overline{H}^{-1}(\partial\overline{H})\overline{H}^{-1}) \wedge \partial\overline{H} \\ &= -(\overline{H}^{-1}\partial\overline{H}) \wedge (\overline{H}^{-1}\partial\overline{H}) \\ &= -A \wedge A \end{aligned}$$

Note that in the second equality we use the identity

$$-\overline{H}\partial\overline{H}^{-1} = (\partial\overline{H})\overline{H}^{-1}$$

and this holds from $\partial(\overline{HH^{-1}}) = \partial\overline{\mathrm{I}_n} = 0$.

For (2). Chern curvature locally looks like

$$H_E = \mathrm{d}A + A \wedge A = \mathrm{d}A - \partial A = \overline{\partial} A$$

, which is of type $(1,1)$. And (3) is clear, since $\overline{\partial} H_E = \overline{\partial}^2 \overline{A} = 0$. □

The Chern connection of Hermitian holomorphic vector bundle behaves well with respect to bundle operations, as we see in the next two Exercises.

Exercise 2.2.3. (E, h) is a Hermitian holomorphic vector bundle, and $S \hookrightarrow E$ is a holomorphic subbundle. $S^{\perp}$ is defined by $(S^{\perp})_x = (S_x)^{\perp}$ with respect to h. We have $E = S \oplus S^{\perp}$ as complex vector bundle.[14] So we can define a natural projection

$$P_S : C^{\infty}(X, \Omega^k_{X,\mathbb{C}} \otimes E) \to C^{\infty}(X, \Omega^k_{X,\mathbb{C}} \otimes S)$$

Show that $D_S = P_S \circ D_E$.

Proof. Since Chern curvature is unique, so it suffices to check $P_s \circ D_E$ is Hermitian and satisfies $(P_s \circ D_E)^{0,1} = \overline{\partial}_S$. Take any section $\omega \in C^{\infty}(X, \Omega_{X,\mathbb{C}})$, then we have

$$P_s \circ D_E(\omega) = P_S(\overline{\partial}_E \omega) = \overline{\partial}_S \omega$$

And if σ, τ are two sections of S, then

$$\begin{aligned} \mathrm{d}\{\sigma, \tau\} &= \{D_E \sigma, \tau\} + (-1)^{\deg \sigma} \{\sigma, D_E \tau\} \\ &= \{P_S \circ D_E \sigma, \tau\} + (-1)^{\deg \sigma} \{\sigma, P_S \circ D_E \tau\} \end{aligned}$$

□

Exercise 2.2.4. If E and E' are two Hermitian holomorphic vector bundle, then

$$D_{E \otimes E'} = D_E \otimes \mathrm{id} + \mathrm{id} \otimes D_{E'}$$

Proof. The same as above. □

2.3. **Case of line bundle.** In this case we will consider a special case, i.e. line bundle, to find some interesting things.

Recall that if X is a differential manifold, and $\pi : X \to L$ is a complex line bundle, D is a connection on L. Since

$$H_D \in C^{\infty}(X, \Omega^2_{X,\mathbb{C}} \otimes \mathrm{End}(L))$$

But for line bundle, $\mathrm{End}(L) \cong L^* \otimes L \cong X \times \mathbb{C}$ is just trivial bundle. So in fact, $H_D \in C^{\infty}(X, \Omega^2_{X,\mathbb{C}})$, that is, curvature of connection is exactly a 2-form, without coefficient.

Furthermore, in a local pointview, D is represented by 1-form A, then $H_D = \mathrm{d}A + A \wedge A$, and for a line bundle, we clearly have $A \wedge A = 0$, since

[14] But in general, $S^{\perp}$ may not be a holomorphic subbundle of E. That is, if we have a short exact sequence of holomorphic vector bundle

$$0 \to S \to E \to Q \to 0$$

this exact sequence generally won's split. That's why we perfer short exact sequence rather than direct sum in algebraic geometry.

A is just a 1×1 matrix, and forms are skew symmetric. So $H_D = \mathrm{d}A$. A immediate consequence is that $\mathrm{d}H_D = 0$, i.e. H_D is a closed form, so we get

$$[H_D] \in H^2(X, \mathbb{C})$$

an element of de Rham cohomology group $H^2(X, \mathbb{C})$.

Now it's natural to ask what's the relationship between closed form from different connections. A surprising result is that they exactly lie in the same cohomology class.

If we consider another connection $\widetilde{D}$, and $\widetilde{A}$, let's compare H_D with $H_{\widetilde{D}}$. For all $\sigma \in C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes L)$, locally $\sigma = s$ with respect to $\{e_1, \dots, e_r\}$.

Then

$$\begin{aligned} D(\sigma) - \widetilde{D}(\sigma) &= (\mathrm{d}s + A \wedge s) - (\mathrm{d}s - \widetilde{A} \wedge s) \\ &= (A - \widetilde{A}) \wedge s \\ &= B \wedge \sigma \end{aligned}$$

where $B = (A - \widetilde{A}) \in C^\infty(X, \Omega^1_{X,\mathbb{C}})$.

Then

$$H_D - H_{\widetilde{D}} = \mathrm{d}B$$

So different connections give the same cohomology class in $H^2(X, \mathbb{C})$. What a beautiful result!

Definition 2.3.1 (First Chern class)**.** *Let $\pi : E \to X$ be a complex line bundle, D is any connection. Then define*

$$c_1(L) := [\frac{i}{2\pi} H_D] \in H^2(X, \mathbb{C})$$

, which is called the first Chern class of line bundle.

Remark 2.3.2. The first Chern class is a property of line bundle itself, and independent of connections on it. In other words, it's a topological information.

Let's explain why we need coefficient $\dfrac{i}{2\pi}$ here.

Lemma 2.3.3. *(E, h) is a Hermitian line bundle, and D is a Hermitian connection, then*

$$\frac{i}{2\pi} H_D \in C^\infty(X, \Omega^2_{X,\mathbb{R}})$$

hence $c_1(L) \in H^2(X, \mathbb{R})$.

Proof. Locally we have $\overline{A} = -A$

$$\overline{\frac{i}{2\pi} H_D} = -\frac{i}{2\pi} \overline{H_D} = -\frac{i}{2\pi} \overline{\mathrm{d}A} = -\frac{i}{2\pi} \mathrm{d}\overline{A} = \frac{i}{2\pi} \mathrm{d}A = \frac{i}{2\pi} H_D$$

□

Remark 2.3.4. However, why we need 2π here? Later we will see in fact $c_1(L) \in H^2(X, \mathbb{Z})$.

Exercise 2.3.5. Let $E \to X$ be a complex vector bundle of rank r, define

$$c_1(E) := c_1(\det E)$$

If $L \to X$ is a complex line bundle, show that

$$c_1(E \otimes L) = c_1(E) + rc_1(L)$$

Proof. First we need a formula about determinant and tensor product, we state it as follows: If E, F are two vector bundles, with rank r_1 and r_2, then we have

$$\det(E \otimes F) = (\det E)^{r_2} \otimes (\det F)^{r_1}$$

So we have

$$\det(E \otimes L) = \det E \otimes (\det L)^r$$

And use Exercise 2.2.4 to conclude. □

Now, Let's combine all we have together, to see what will happen. Let X be a complex manifold and L be a holomorphic line bundle, with a Hermitian metric. D_L is the Chern connection of L, and its Chern curvature is H_L.

By Corollary 2.2.2 and Lemma 2.3.3, we have

$$\frac{i}{2\pi} H_L \in C^\infty(X, \Omega^2_{X,\mathbb{R}}) \cap C^\infty(X, \Omega^{1,1}_X)$$

such that

$$\mathrm{d}(\frac{i}{2\pi} H_L) = \overline{\partial}(\frac{i}{2\pi} H_L) = 0$$

that is

$$[\frac{i}{2\pi} H_L] \in H^2(X, \mathbb{R}), \quad [\frac{i}{2\pi} H_L] \in H^{1,1}(X)$$

However, it's also neccessary to show cohomology class of Chern curvature is independent of metric h.

Example 2.3.6. Locally we have $x \in U \subset X$, with $\pi^{-1}(U) \cong U \times \mathbb{C}$, $\{e_1\}$ is the local frame. Then Hermitian metric is

$$H(z) = \langle e_1(z), e_1(z) \rangle = \|e_1(z)\|_h^2$$

Write $\varphi(z) = -\log H(z)$, a function $U \to \mathbb{R}$. Then

$$A = \overline{H}^{-1} \partial \overline{H} = e^{\varphi(z)} \partial e^{-\varphi(z)} = -\partial \varphi(z)$$

then

$$H_L = \overline{\partial} A = -\overline{\partial} \partial \varphi(z) = \partial \overline{\partial} \varphi(z)$$

then we have

$$\begin{aligned} \frac{i}{2\pi} H_L &= \frac{i}{2\pi} \partial \overline{\partial} \varphi(z) \\ &= \frac{i}{2\pi} \partial \overline{\partial} (-\log H(z)) \\ &= \frac{1}{2\pi i} \partial \overline{\partial} \log \|e_1(z)\|_h^2 \end{aligned}$$

Summarize as follows

Proposition 2.3.7. *X is a complex manifold, (L, h) is a Hermitian holomorphic line bundle. Then $c_1(L)$ is represented by a real $(1,1)$-form, given locally by*

$$\frac{i}{2\pi}H_L = \frac{1}{2\pi i}\partial\overline{\partial}\log\|e_1(z)\|_h^2$$

Exercise 2.3.8. Show that $[\frac{i}{2\pi}H_L] \in H^{1,1}(X)$ is independent of h.

Proof. Note that any two metric on a line bundle differences a function f which is positive everywhere, so we can write $\|e_1(z)\|_{h'} = e^f\|e_1(z)\|_h$ for some smooth function f. So by Proposition 2.3.7, we have the difference of Chern curvatures come from different metrics is $2\overline{\partial}\partial f$, as desired. □

2.4. **Positive line bundle.** Before we go into deeper, let's discuss some facts about linear algebra we will need. Let V be a n-dimensional complex vector space, and use $V_\mathbb{R}$ to denote the underlying real vector space, with real dimension $2n$. And J acts on $V_\mathbb{R}$ as $\times i$. Then

$$V_\mathbb{C} = V_\mathbb{R}\otimes\mathbb{C} = V^{1,0}\oplus V^{0,1}$$

Consider its dual space $W_\mathbb{R} = V_\mathbb{R}^*$, and $W_\mathbb{C} = W_\mathbb{R}\otimes\mathbb{C} = W^{1,0}\oplus W^{0,1}$. Then $W^{1,1} = W^{1,0}\otimes W^{0,1} = W^{1,1}\subset\bigwedge^2 W_\mathbb{C}$. For a Hermitian form $h: V\times V\to\mathbb{C}$, we have the following magic correspondence.

Lemma 2.4.1. *We have the following canonical correspondence*

$$\{\textit{Hermitian forms on } V\} \longleftrightarrow \{\textit{real } (1,1)\textit{-forms on } V_\mathbb{R}\}$$

Proof. Given a Hermitian form h, then $h\mapsto\omega = -\operatorname{Im}h$; Conversely, given such a form ω, we define

$$h = \omega(\cdot, J\cdot) - i\omega(\cdot,\cdot)$$

Now let's check it: In one direction, write $h = \operatorname{Re}h + i\operatorname{Im}h$, then

$$\operatorname{Re}h(u,v) + i\operatorname{Im}h(u,v) = h(u,v) = \overline{h(v,u)} = \operatorname{Re}h(v,u) - i\operatorname{Im}h(v,u)$$

So $\operatorname{Im}h$ is skew symmetric, that is, $\omega = -\operatorname{Im}h$ is an alternating real 2-form. Now we need to show ω is a $(1,1)$-form. Use Remark 1.5.3, we have $\omega\in W^{1,1}$ is equivalent to $\omega(V^{1,0}, V^{1,0}) = \omega(V^{0,1}, V^{0,1}) = 0$

Recall that $V^{1,0}$ is spanned by $u - iJ(u)$, then

$$\omega(u - iJ(u), v - iJ(v)) = \omega(u,v) - \omega(J(u), J(v)) - i(\omega(u, J(v)) + \omega(J(u), v))$$

Since

$$h(J(u), J(v)) = ih(u, J(v)) = i\times(-i)h(u,v) = h(u,v)$$

then

$$\omega(u,v) = \omega(J(u), J(v))$$

And note that

$$\operatorname{Re}h(u, J(v)) - i\omega(u, J(v)) = h(u, J(v)) = h(J(u), -v) = \operatorname{Re}h(J(u), -v) - i\omega(J(u), -v)$$

which implies

$$\omega(u, J(v)) + \omega(J(u), v) = 0$$

Similarly we can check $\omega(V^{0,1}, V^{0,1}) = 0$. So ω is a real 2-form of $(1,1)$ type.

Worth to be mentioned: From what we have done above, we see that a real 2-form ω is of type $(1,1)$ if and only if

$$\begin{cases} \omega(u,v) = \omega(J(u), J(v)) \\ \omega(u, J(v)) + \omega(J(u), v) = 0 \end{cases}$$

So if ω is a real $(1,1)$-form, then

$$\begin{aligned} \overline{h(u,v)} := \overline{\omega(u, J(v)) - i\omega(u,v)} &= \omega(u, J(v)) + i\omega(u,v) \\ &= -\omega(J(v), u) - i\omega(v,u) \\ &= -\omega(J^2(v), J(u)) - i\omega(v,u) \\ &= \omega(v, J(u)) - i\omega(v,u) \\ &= h(v,u) \end{aligned}$$

□

Remark 2.4.2. Though the correspondence above is canonical, we can choose a basis to see what's going on: If we choose a basis $z_1, \dots, z_n$ of V. Then Hermitian forms on V can be write as $h = \sum_{j,k} h_{jk} z_j^* \otimes \overline{z}_k^*$, where z_k^* is the dual basis of z_k and $h_{jk} = h(z_j, z_k)$ is a Hermitian matrix.

If we let $u = (u_1, \dots, u_n)^T, v = (v_1, \dots, v_n)^T$, then $h(u,v) = \sum_{jk} u_j h_{jk} \overline{v_k}$. By definition, we have

$$\begin{aligned} \omega(u,v) &= -\operatorname{Im} h(u,v) \\ &= \frac{i}{2}(\sum h_{jk} u_j \overline{v_k} - \sum \overline{h_{jk}}\, \overline{u_j} v_k) \\ &= \frac{i}{2}\sum (h_{jk} u_j \overline{v_k} - h_{jk} v_j \overline{u_k}) \\ &= \frac{i}{2}\sum h_{jk}(u_j \overline{v_k} - v_j \overline{u_k}) \end{aligned}$$

That is, the corresponding real $(1,1)$-form is

$$\omega = \frac{i}{2}\sum_{j,k} h_{jk} z_j^* \wedge \overline{z}_k^*$$

To be more explicit, if we choose $x_1, \dots, x_n, y_1, \dots, y_n$ to be basis of $V_{\mathbb{R}}$, then $z_i^* = x_i^* + i y_i^* \in (V_{\mathbb{C}})^*$.

Definition 2.4.3 (positive form)**.** *For a real* $(1,1)$*-form* ω*, it is called positive, if the corresponding Hermitian form* h *is positive definite.*

Definition 2.4.4 (positive line bundle)**.** *X is a complex manifold, L is a holomorphic line bundle. L is called positive if it admits a Hermitian metric h such that*

$$\frac{i}{2\pi} H_L$$

corresponds to a positive Hermitian metric on the holomorphic tangent bundle T_X.

Remark 2.4.5. For any $x \in X$, then
$$(\frac{i}{2\pi}H_L)_x \in (\Omega^2_{X,\mathbb{R}} \cap \Omega^{1,1}_X)_x$$
is a real $(1,1)$-form on $(T_{X,\mathbb{R}})_x$. Then by Lemma 2.4.1, we know that there is a one to one correspondence with Hermitian form on $T_{X,x}$. So globally we have that $\frac{i}{2\pi}H_L$ will correspond to a Hermitian metric on T_X.

Locally, we have
$$\frac{i}{2\pi}H_L = \frac{i}{2\pi}\partial\overline{\partial}\varphi(z) = \frac{i}{2\pi}\sum_{j,k}\frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k}\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k$$
Then L is positive is equivalent to the Hermitian metric
$$(\frac{\partial\varphi^2}{\partial z_j \partial \overline{z}_k})$$
is everywhere positive definite.

Exercise 2.4.6 (positive line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$)**.** First consider function
$$\begin{aligned} \varphi : \mathbb{C}^{n+1}\backslash\{0\} &\to \mathbb{R} \\ z &\mapsto \log(\sum_{j=0}^{n}|z_j|^2) \end{aligned}$$
then $(\frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k})$ is positive definite. Show that

1. $\frac{i}{2\pi}\partial\overline{\partial}\varphi(z)$ induces a real $(1,1)$-form on $\mathbb{P}^n$ such that $\mathrm{d}\omega = 0$.
2. φ comes from a Hermitian metric h on $\mathcal{O}_{\mathbb{P}^n}(1)$ called Fubini-Study metric.

Exercise 2.4.7. L is positive if and only if $L^{\otimes m}$ is positive for some $m \in \mathbb{N}_{\geq 0}$.

Proof. For a line bundle L locally we have the Hermitian metric corresponding to its curvature looking like
$$(\frac{\partial\varphi^2}{\partial z_j \partial \overline{z_k}})$$
and for $L^{\otimes m}, m \in \mathbb{N}_{\geq 0}$ we have
$$(m \cdot \frac{\partial\varphi^2}{\partial z_j \partial \overline{z_k}})$$
it's clear L is positive if and only if $L^{\otimes m}$ is. □

Exercise 2.4.8. Suppose X is a compact complex manifold, L is a positive line bundle, and M is any holomorphic line bundle, then there exists $N_0 \in \mathbb{N}$ such that $M \otimes L^{\otimes N}$ positive for $N \geq N_0$.

Proof. The proof is quite similar to Exercise 2.4.7, we need to check locally, but compactness is neccessary here. For an open subset U_1, locally we have the Hermitian metric corresponding to $M \otimes L^N$ looking like

$$(\frac{\partial \varphi_M^2}{\partial z_j \partial \overline{z_k}} + m \cdot \frac{\partial \varphi_L^2}{\partial z_j \partial \overline{z_k}})$$

So we can choose suffices large N_1 such that $M \otimes L^{\otimes N_1}$ is positive on U. Since X is compact, we can take a finite open covering $\{U_i\}$ of X and choose the largest N_i to be N we desired. $\square$

Remark 2.4.9. In fact, If we use language of algebraic geometry, then a positive line bundle is equivalent to an ample divisor.

2.5. **Lefschetz $(1,1)$-theorem.** Now we know that given a Hermitian holomorphic line bundle (L, h), then consider its Chern curvature we will get a real $(1,1)$-form. So we may wonder the converse of this statement. Is there any real $(1,1)$-form comes from such a Hermitian holomorphic line bundle? That's main theorem for this section.

Theorem 2.5.1 (Lefschetz $(1,1)$-theorem)**.** *X is a complex manifold, $\omega \in C^\infty(X, \Omega^2_{X,\mathbb{R}}) \cap C^\infty(X, \Omega^{1,1}_X)$, a real $(1,1)$-form, such that $\mathrm{d}\omega = 0$. And*

$$[\omega] \in \mathrm{im}(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R}))$$

Then there exists a Hermitian holomorphic line bundle (L, h) such that

$$\frac{i}{2\pi} H_L = \omega$$

Before proving this theorem, let's elaborate what does the following map mean

$$H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R})$$

since in de Rham cohomology, it's meaningless to say cohomology with $\mathbb{Z}$ coefficient. Here we use comparision $H^2(X,\mathbb{R}) \cong \check{H}^2(X,\underline{\mathbb{R}})$, where $\underline{\mathbb{R}}$ is the sheaf of local constant $\mathbb{R}$-functions and consider the map in terms of Čech cohomology

$$\check{H}^2(X,\underline{\mathbb{Z}}) \to \check{H}^2(X,\underline{\mathbb{R}})$$

We prove comparision theorem for $n = 2$ in an explict way, since later we will use it.

In sketch, the philosophy of this method is that we descend the degree of differential forms, but the price is we need to consider functions defined on intersections of many open subsets.

X is a differential manifold, and $Z^1 \subset \Omega^1_{X,\mathbb{R}}$, sheaf of closed 1-form. Then we have the following exact sequence of sheaves

$$0 \to \underline{\mathbb{R}} \to C^\infty(X) \xrightarrow{\mathrm{d}} Z^1 \to 0$$

Locally constant functions are clearly smooth functions, such that d acts on them is zero, so the exactness for the first two is trivial. But for the last one, it is equivalent to that a closed form locally must be an exact form, that's Poincaré lemma.

Similarly, define $Z^2 \subset \Omega^2_{X,\mathbb{R}}$, sheaf of closed 2-forms. then

$$0 \to Z^1 \to \Omega^1_{X,\mathbb{R}} \xrightarrow{\mathrm{d}} Z^2 \to 0$$

This sequence is exact for the same reason.

By the definition of de Rham cohomology, we have

$$H^2(X,\mathbb{R}) = \frac{C^\infty(X,Z^2)}{\mathrm{d}C^\infty(X,\Omega^1_{X,\mathbb{R}})}$$

In order to avoid the limit in the definition of Čech cohomology, we take open covering $\mathcal{U} = \{U_\alpha\}$ good enough, such that

$$\mathrm{d} : C^\infty(U_\alpha, \Omega^1_{U_\alpha,\mathbb{R}}) \to C^\infty(U_\alpha, Z^2)$$

is surjective for any α. And

$$\mathrm{d} : C^\infty(U_\alpha \cap U_\beta) \to C^\infty(U_\alpha \cap U_\beta, Z^1)$$

is surjective for any α, β.

If ω is a closed real 2-form, i.e. $[\omega] \in H^2(X,\mathbb{R})$. For any α, choose $A_\alpha \in C^\infty(U_\alpha, \Omega^1_{U_\alpha,\mathbb{R}})$ such that

$$\omega|_{U_\alpha} = \mathrm{d}A_\alpha$$

then

$$\prod_{\alpha,\beta}(A_\alpha - A_\beta)$$

is a Čech 1-cocchain in $C^1(\mathcal{U}, Z^1)$, it's d closed since $\mathrm{d}(A_\alpha - A_\beta)|_{U_\alpha \cap U_\beta} = \omega - \omega = 0$.

For any α, β, choose $f_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta)$, such that

$$(A_\alpha - A_\beta)_{\alpha\beta} = \mathrm{d}f_{\alpha\beta}$$

then

$$f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}|_{U_\alpha \cap U_\beta \cap U_\gamma}$$

is closed, hence locally constant,

$$\check{\omega} = \prod_{\alpha,\beta,\gamma}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta})$$

is a Čech 2-cocycle, in $C^2(\mathcal{U}, \underline{\mathbb{R}})$. We have $\delta\check{\omega} = 0$, and $[\omega]$ corresponds to $[\check{\omega}]$, that's the explict construction for comparision theorem in dimension 2. In fact, the general case is proved in the same method.

And we also the some lemmas in multiply complex analysis.

Lemma 2.5.2. *Locally on a polydisk $D \subset \mathbb{C}^n$, and $\omega \in C^\infty(D, \Omega^2_{D,\mathbb{R}}) \cap C^\infty(D, \Omega^{1,1}_D)$ is a* d *closed real $(1,1)$-form. Then there exists a smooth function $\varphi : D \to \mathbb{R}$ such that*

$$\omega = i\partial\overline{\partial}\varphi$$

Proof. Poincaré lemma implies that $\omega = \mathrm{d}A = \mathrm{d}(A^{1,0}+A^{0,1}) = (\partial+\overline{\partial})(A^{1,0}+A^{0,1})$, and since A is real, then $\overline{A^{1,0}} = A^{0,1}$.

Since ω is a $(1,1)$-form, then

$$\begin{cases} \partial A^{1,0} = 0 \\ \overline{\partial} A^{0,1} = 0 \\ \omega = \overline{\partial} A^{1,0} + \partial A^{0,1} \end{cases}$$

Dolbeault lemma implies that $A^{0,1} = \overline{\partial} f$, so $A^{1,0} = \partial \overline{f}$, so we have

$$\begin{aligned} \omega &= \overline{\partial}\partial\overline{f} + \partial\overline{\partial} f \\ &= \partial\overline{\partial}(f - \overline{f}) \\ &= i\partial\overline{\partial}\varphi \end{aligned}$$

□

Lemma 2.5.3. *Locally on $U \subset \mathbb{C}^n$, a simply connected open subset, and a smooth function $\varphi : U \to \mathbb{R}$, such that $\partial\overline{\partial}\varphi = 0$[15]. Then there exists a holomorphic functions $f : U \to \mathbb{C}$, such that $\varphi = \mathrm{Re}(f)$.*

Now let's prove Lefschetz $(1,1)$-theorem

Proof. Let's first see how does the above two lemmas play a role in our proof. We will choose a good enough open cover $\mathcal{U} = \{U_\alpha\}$ of open polydisk such that for all α, β, we have $U_\alpha \cap U_\beta$ is simply connected.

Since ω is a d closed real $(1,1)$-form, Lemma 2.5.2 implies that there exists smooth function $\varphi_\alpha : U_\alpha \to \mathbb{R}$ such that

$$\omega|_{U_\alpha} = \frac{i}{2\pi}\partial\overline{\partial}\varphi_\alpha$$

On any two intersection $U_\alpha \cap U_\beta$, we have $\partial\overline{\partial}(\varphi_\alpha - \varphi_\beta) = 0$, then Lemma 2.5.3 implies that there exists a holomorphic function $f_{\alpha\beta}$, such that

$$(\varphi_\alpha - \varphi_\beta)|_{U_\alpha \cap U_\beta} = 2\,\mathrm{Re}(f_{\alpha\beta}) = f_{\alpha\beta} + \overline{f_{\alpha\beta}}$$

Consider $\prod f_{\alpha\beta} \in C^1(\mathcal{U}, \mathcal{O}_X)$, then

$$(\delta f)_{\alpha\beta\gamma} = (f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta})|_{U_\alpha \cap U_\beta \cap U_\gamma}$$

Note that $2\,\mathrm{Re}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta})_{\alpha\beta\gamma} = 0$, so it must be a locally constant imaginary number, i.e. it lies in $2\pi i \underline{\mathbb{R}}(U_\alpha \cap U_\beta \cap U_\gamma)$.

Consider real form[16]

$$A_\alpha = \frac{i}{4\pi}(\overline{\partial}\varphi_\alpha - \partial\varphi_\alpha)$$

and by directly computing, we can note that $\omega|_{U_\alpha} = \mathrm{d}A_\alpha$, and that's why we define A_α in this method.

[15] Such φ is called pluriharmonic

[16] Here we need to consider some queer coefficients, in order to get a beautiful result. In fact, we need to use $e^{2\pi i} = 1$, a god given formula.

Similar to what we have done in the proof of comparision theorem, we want to consider $A_\alpha - A_\beta$ on the intersection $U_\alpha \cap U_\beta$. So we compute the difference of each term of A_α and A_β as follows

$$\begin{aligned}\partial(\varphi_\beta - \varphi_\alpha) &= \partial(f_{\alpha\beta} + \overline{f_{\alpha\beta}}) \\ &= \partial f_{\alpha\beta} \\ &= \mathrm{d} f_{\alpha\beta}\end{aligned}$$

Similarly we have

$$\overline{\partial}(\varphi_\beta - \varphi_\alpha) = \mathrm{d}\overline{f_{\alpha\beta}}$$

then

$$(A_\beta - A_\alpha)_{\alpha\beta} = \frac{i}{4\pi}\mathrm{d}(\overline{f_{\alpha\beta}} - f_{\alpha\beta}) = \frac{1}{2\pi}\mathrm{d}(\mathrm{im}(f_{\alpha\beta}))$$

Via $H^2(X,\mathbb{R}) \cong \check{H}^2(X,\mathbb{R})$, $[\omega]$ corresponds to $[\check{\omega}]$, above process is just what we have done in the proof of comparision theorem, so we have

$$\begin{aligned}\check{\omega} &= \prod(\frac{1}{2\pi}\,\mathrm{im}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}))_{\alpha\beta\gamma} \\ &= \prod(\frac{1}{2\pi i}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}))_{\alpha\beta\gamma}\end{aligned}$$

Hypothesis tells that $[\check{\omega}]$ is an image of $[\prod n_{\alpha\beta\gamma}] \in \check{H}^2(X,\underline{\mathbb{Z}})$. However, it doesn't mean that $f_{\alpha\beta}$ are exactly integers, but not too bad, we just need some correction terms, that is

$$\prod(\frac{1}{2\pi i}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}))_{\alpha\beta\gamma} = \prod n_{\alpha\beta\gamma} + \delta(\prod c_{\alpha\beta})$$

where $\prod(c_{\alpha\beta})$ is real 1-cochain.

So we set $f'_{\alpha\beta} = f_{\alpha\beta} - 2\pi i c_{\alpha\beta}$. Then

$$\frac{1}{2\pi i}(f'_{\beta\gamma} - f'_{\alpha\gamma} + f'_{\alpha\beta})_{\alpha\beta\gamma} = 2\pi i n_{\alpha\beta\gamma} \in 2\pi i\underline{\mathbb{Z}}(U_\alpha \cap U_\beta \cap U_\gamma)$$

Note that $e^{2\pi i} = 1$, Then consider $g_{\alpha\beta} = \exp(-f'_{\alpha\beta})$, a holomorphic from $U_\alpha \cap U_\beta$ to $\mathbb{C}^*$, it satisfies the cocycle condition

$$g_{\beta\gamma}g_{\alpha\gamma}^{-1}g_{\alpha\beta} = 1$$

so we get a holomorphic line bundle L.

Remark 2.5.4. It's important to keep in mind vector bundles are encoded in their gluing data, and you can regard it as an element in $\check{H}^1$. So if we want to get a holomorphic line bundle, we need to determine its transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ satisfying the cocycle conditions, that is, $\prod g_{\alpha\beta} \in C^1(\mathcal{U}, \mathcal{O}_X^*)$ such that $\delta(\prod g_{\alpha\beta}) = 1$, i.e. $[\prod g_{\alpha\beta}] \in \check{H}^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X)$. That's what Example 1.7.9 already tells us.

Now we need to give a Hermitian metric on this holomorphic line bundle H, and calculate its curvature to complete the proof.

Note that

$$(\varphi_\alpha - \varphi_\beta)_{U_\alpha \cap U_\beta} = 2\,\mathrm{Re}(f_{\alpha\beta}) = 2\,\mathrm{Re}(f_{\alpha\beta})' \quad = -\log|g_{\alpha\beta}|^2$$

then we get a Hermitian metric

$$H_\alpha = -\exp(-\varphi_\alpha), \quad \text{on } U_\alpha$$

Indeed, since $H_\beta = |g_{\alpha\beta}|^2 H_\alpha = g_{\alpha\beta}^T H_\alpha \overline{g_{\alpha\beta}}$.

Finally,

$$\frac{i}{2\pi} H_L = \frac{i}{2\pi} \partial\overline{\partial}\varphi_\alpha = \omega$$

This completes the proof. □

Remark 2.5.5. Now if we already have a holomorphic line bundle L, determined by its transition functions $g_{\alpha\beta}$. We can try to reverse what we have done above. That is, take its logarithm, consider its alternating sum and divide it by $2\pi i$, then we get an element in $\check{H}^2(X, \mathbb{R})$, and that's exact $-c_1$, where c_1 is the first Chern class.

However, we can rephrase it as a basical operation in homological algebra, consider the exponential sequence

$$0 \to \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1$$

Taking cohomology we will get a boundary map ∂

$$\mathrm{Pic}(X) \cong \check{H}^1(X, \mathcal{O}_X^*) \xrightarrow{\partial} \check{H}^2(X, \underline{\mathbb{Z}}) \to \check{H}^2(X, \underline{\mathbb{R}})$$

that's just what we have done, so this boundary map sometimes is denoted by $-c_1$.

2.6. **Hypersurface and Divisors.** This section we will briefly introduce some definitions and theorems about hypersurfaces and divisors wihout proofs.

Definition 2.6.1 (hypersurface)**.** *X is a complex manifold, a hypersurface of X is a closed subset $D \subset X$ such that for all $x \in D$, there exists an open subset $U \subset X$ containing x and a nonzero holomorphic function $f : U \to \mathbb{C}$ such that*

$$D \cap U = \{x \in U \mid f(x) = 0\}$$

Remark 2.6.2. $x \in D$ is called smooth, if we can choose holomorphic function $f : D \to \mathbb{C}$ as a submersion. The set of all smooth point is denoted by D_{sm}; If $D_{sm} = D$, then $D_{sm} = D$. And note that we do not assume D is connected in the definition.

Exercise 2.6.3. Let $D \subset X$ be a smooth hypersurface, then there exists an open covering $\{U_\alpha\}$ of X and assign each U_α a holomorphic submersion f_α, such that

$$D \cap U_\alpha = \{x \in U_\alpha \mid f_\alpha(x) = 0\}$$

Then $g_{\alpha\beta} = f_\alpha / f_\beta : U_\alpha \cap U_\beta \to \mathbb{C}^*$ is a holomorphic function. Then we get a holomorphic line bundle, denoted by $\mathcal{O}_X(D)$ on X. In particular, if we take $X = \mathbb{P}^n, D = \mathbb{P}^{n-1}$, then $\mathcal{O}_X(D) = \mathcal{O}_{\mathbb{P}^n}(1)$.

Proof. In the case $X = \mathbb{P}^n, D = \mathbb{P}^{n-1} = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n \mid x_0 = 0\}$, we can choose a covering of D as $\{U_i\}_{i=1}^n$, where $U_i = \{(x_0 : x_1 : \cdots : x_n) \in \mathbb{P}^n \mid x_i \neq 0\}$ and take $f_i = x_0$. $\square$

Remark 2.6.4. In fact, we can drop the assumption of "smoothness" by some technical method.

Lemma 2.6.5. *If $D \subset X$ is a hypersurface, then $D_{sm} \subset D$ is open and dense.*

Proposition 2.1. *If $D \subset X$ is a hypersurface, then there exists a holomorphic line bundle $\mathcal{O}_X(D)$ on X with global section σ, such that $D = \{x \in X \mid \sigma(x) = 0\}$*

Proof. (Sketch.) Set $Z = D \backslash D_{sm}$. Then $D_{sm} \subset X \backslash Z$ is a smooth hypersurface. Then we get transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \backslash Z \to \mathbb{C}^*$, so we get a holomorphic line bundle over $X \backslash Z$. Then Lemma 2.6.5 tells us Z contains no hypersurface of X, and Hartgos theorem tells us we can extend this holomorphic function $g_{\alpha\beta}$ to $U_\alpha \cap U_\beta$. So we define a holomorphic line bundle. $\square$

Definition 2.6.6 (irreducible hypersurface)**.** *A hypersurface $D \subset X$ is called irreducible, if it can not be written as union of two hypersurface.*

Definition 2.6.7 (divisor)**.** *X is a complex manifold, a divisor on X is a finite formal sum*

$$D = \sum_i a_i D_i$$

where $a_i \in \mathbb{Z}$ and D_i is a irreducible hypersurface. And formally we can define

$$\mathcal{O}_X(D) := \bigotimes_i \mathcal{O}_X(D_i)^{\otimes a_i}$$

Remark 2.6.8. We have seen that from divisors we can get a holomorphic line bundle. So it's natural to guess all holomorphic line bundle are arised in this form. However, it's false.

Example 2.6.9. There exists a complex torus with no hypersurface, but there is non trivial holomorphic line bundle on it.

But

Theorem 2.6.10. *If X is a projective manifold, then for any holomorphic line bundle L, there exists a divisor D ($D = D_1 - D_2$, and D_1, D_2 are hypersurface in X), such that*

$$L \cong \mathcal{O}_X(D)$$

Let X be a compact complex manifold of dimension n, and $D \subset X$ is a smooth hypersurface, we can define

$$D : C^\infty(X, \Omega^{2n-2}_{X,\mathbb{R}}) \to \mathbb{R}$$
$$\omega \mapsto \int_D \omega|_D$$

and if ω is a exact form, then $\int_D \omega|_D = 0$ by Stokes.

So we get a $[D] : H^{2n-2}(X, \mathbb{R}) \to \mathbb{R}$, then Poincaré duality tells us we get $[D] \in H^2(X, \mathbb{R})$.

Surprisingly,

Theorem 2.6.11 (Lelong-Poincaré)**.** *X is a compact complex manifold, $D \subset X$ is a smooth hypersurface, then*

$$[D] = c_1(\mathcal{O}_X(D)) \in H^2(X, \mathbb{R})$$

Remark 2.6.12. We can also drop smoothness condition. We need to make sense of $[D] \in H^2(X, \mathbb{R})$, i.e. we need to check the following integral make sense:

$$\int_{D_{sm}} \omega|_{D_{sm}}$$

It's not trivial, since D_{sm} is just an open subset, and integral over an open subset may be quite bad.

Part 2. **Hodge theory**

3. Kähler manifold

3.1. **Definitions and Examples.**

Definition 3.1.1 (Kähler manifold)**.** *Let X be a complex manifold, h is a positive Hermitian metric on TX, and ω is the real $(1,1)$-form corresponding to h. X is called a Kähler manifold, if $\mathrm{d}\omega = 0$.*

Remark 3.1.2. Note that the condition $\mathrm{d}\omega = 0$ is equivalent to $\partial\omega = 0$, and is also equivalent to $\overline{\partial}\omega = 0$.

Remark 3.1.3. Our definition of Kähler manifold is the complex Hermitian viewpoint. But Kähler manifold in fact is an intersection of three interesting objects: Complex manifold, Symplectic manifold and Riemannian manifold.

Let's see from the symplectic viewpoint: If (X, ω) is a symplectic manifold, where X is a differential manifold and ω is a d closed real non-degenerate symplectic form. (X, ω) is called a Kähler manifold, if there exists an integrable almost complex structure J on $T_{X,\mathbb{R}}$ such that $g(u, v) := \omega(w, Ju)$ is positive definite, that is g is a Riemannian metric.

Also, we can define Kähler manifold from the differential geometry viewpoint. Let (X, g) be a Riemannian manifold, where g is a Riemannian metric. (X, g) is called Kähler if there exists an integrable almost complex structure J on $T_{X,\mathbb{R}}$ satisfying $g(Ju, Jv) = g(u, v)$ and preserved by parallel transport with respect to Levi-Civita connection.

Anyway, the hallmarks of a Kähler manifold are “complex structure”, “positive” and “closed”.

Remark 3.1.4. For a Kähler manifold X, we have J, h, ω, g on it. Now we want to elaborate to how do these things intersect with each other. One thing we need to keep in mind all the way is the identification $T_{X,x} \longleftrightarrow T_{X,\mathbb{R},x}$. We draw a diagram as follows

$$\begin{tikzcd}
& h \arrow[dl, bend right, "1"'] \arrow[dr, bend left, "3"] & \\
\omega \arrow[ur, bend right, "2"'] \arrow[rr, "5"] & & g \arrow[ul, bend left, "4"'] \arrow[ll, bend left, "6"]
\end{tikzcd}$$

And the explict correspondence is listed as follows

$$\begin{array}{cc}
1 & \omega(u, v) = -\operatorname{Im} h(u, v) \\
2 & h(u, v) = \omega(u, Jv) - i\omega(u, v) \\
3 & g(u, v) = \operatorname{Re} h(u, v) \\
4 & h(u, v) = g(u, v) - ig(Ju, v) \\
5 & g(u, v) = \omega(u, Jv) \\
6 & \omega(u, v) = g(Ju, v)
\end{array}$$

Example 3.1.5. Any complex curve[17] X is Kähler. Since $\mathrm{d}\omega = 0$ automatically holds.

Example 3.1.6. If X admits a positive holomorphic line bundle, then X is Kähler, since we can take ω to be its first Chern class. In particular, $\mathbb{P}^n$ is Kähler, since $\mathcal{O}_{\mathbb{P}^n}(1)$ is a positive holomorphic line bundle of it, with respect to Fubini-Study metric.

Exercise 3.1.7. Show that a submanifold of a Kähler manifold is still Kähler. In particular, any projective manifold is Kähler.

Proof. If X is a Kähler manifold and Y is a submanifold, h is a positive Hermitian metric on TX such that its corresponding real $(1,1)$-form is d closed, then consider its restriction on Y to conclude. □

Exercise 3.1.8. If (X, ω) is a Kähler manifold with $\dim_{\mathbb{C}} X = n$. Show that $\dfrac{\omega^n}{n!}$ is the volume form of X as a Riemannian manifold with respect to g. Furthermore, since $\mathrm{d}\omega = 0$, then $\mathrm{d}(\omega^k) = 0, 0 \leq k \leq n$, then

$$[\omega^k] \in H^{2k}(X, \mathbb{R})$$

[17] In other words, a Riemann surface

Deduce that if X is a compact Kähler manifold, then $[\omega^k] \neq 0$ for all $0 \leq k \leq n$. So $H^{2k}(X, \mathbb{R}) \neq 0$.

Proof. In fact, $\dfrac{\omega^n}{n!} = \mathrm{vol}$ holds for any complex manifold with a positive Hermitian metric h, the Kähler condition does not really come into play, since it's just a problem of linear algebra.

Let's begin by fixing a point $p \in X$ and a frame for the T_pX, orthonormal with respect to the Riemannian metric g defined by h on X, call it $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}$, and use $\mathrm{d}x_1, \mathrm{d}y_1, \ldots, \mathrm{d}x_n, \mathrm{d}y_n$ to denote its dual, and $\frac{\partial}{\partial z_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}), \mathrm{d}z_i = \mathrm{d}x_i + i\mathrm{d}y_i$ By Remark 2.4.2, we have

$$\omega_p = \frac{i}{2}\sum_{j=1}^{n} \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$$

Since the volume form is just

$$\mathrm{vol}_p = \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n$$

and compare it with ω^n, directly compute

$$\omega_p^n = n!\frac{i^n}{2^n}\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\overline{z_n}$$

we can compute $n = 2$ to feel what's going on:

$$\begin{aligned}
\omega_p^2 &= (\frac{i}{2})^2(\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} + \mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2}) \wedge (\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} + \mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2}) \\
&= (\frac{i}{2})^2(\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} \wedge \mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2} + \mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2} \wedge \mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1}) \\
&= 2(\frac{i}{2})^2\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} \wedge \mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2}
\end{aligned}$$

By Exercise 1.5.4 we have

$$\mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}y_2 = (\frac{i}{2})^2\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} \wedge \mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2}$$

Thus we get desired result.

Now suppose X is a compact Kähler manifold, then consider the integral pairing

$$\int_X \omega^k \wedge \omega^{n-k} = \int_X \mathrm{vol} = \mathrm{vol}(X) \neq 0 \implies [\omega^k] \neq 0$$

□

As we can see from the definition of Kähler manifold, all of the requirements are local, but from the above exercise, we can see a surprising thing, that is the cohomology groups with even dimension must be non-trivial, it's a global result.

To some extent, this reflects the philosophy of Hodge theory, that is how does locally good property control global cohomology. Kähler is a locally good property, and the following theorem may cultivate you such an intuition.

Theorem 3.1.9. *X is a Kähler manifold, then locally around $x \in U \subset X$, we can choose a holomorphic coordinate $(\xi_1, \dots, \xi_n)$ such that $h_{jk} = \delta_{jk} + O(|\xi|^2)$*

Proof. With linear change of coordinate, we can assume

$$\omega = \frac{i}{2}\sum_{jk} h_{jk} \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_K$$

where $h_{jk} = \delta_{jk} + O(|\xi|)$, that is

$$h_{jk} = \delta_{jk} + \sum_l (a_{jkl}z_l + a'_{jkl}\overline{z_l}) + O(|\xi|^2)$$

What we need to do is to kill the first order term. Since h_{jk} is Hermitian, that is $h_{jk} = \overline{h_{kj}}$. So we have

$$\overline{a_{kjl}} = a'_{jkl} \tag{3.1}$$

To get above result we only use the fact that h_{jk} is Hermitian, but we also have that ω is also ∂ closed, we compute directly

$$\partial\omega = \frac{i}{2}\sum_{jkl} \frac{\partial h_{jk}}{\partial z_l} \mathrm{d}z_l \wedge \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k$$

If we want $\partial\omega = 0$, we need the coefficients of $\mathrm{d}z_l \wedge \mathrm{d}z_j$ and $\mathrm{d}z_j \wedge \mathrm{d}z_l$ are equal so that they can cancel with each other, that is

$$\frac{\partial h_{jk}}{\partial z_l} = \frac{\partial h_{lk}}{\partial z_j} \implies a_{jkl} = a_{lkj}$$

Set $\xi_k = z_k + \frac{1}{2}\sum_{jl} a_{jkl}z_jz_l$, this is a holomorphic change of coordinate, so $\xi_1, \dots, \xi_n$ is also a holomorphic coordinate. Then

$$\begin{aligned}
\mathrm{d}\xi_k &= \mathrm{d}z_k + \frac{1}{2}\sum_{jl} a_{jkl}(z_l\mathrm{d}z_j + z_j\mathrm{d}z_l) \\
&= \mathrm{d}z_k + \frac{1}{2}\sum_{jl}(a_{jkl} + a_{lkj})z_l\mathrm{d}z_j \\
&= \mathrm{d}z_k + \sum_{jl} a_{jkl}z_l\mathrm{d}z_j
\end{aligned}$$

So we have

$$\frac{i}{2}\sum_k \mathrm{d}\xi_k \wedge \mathrm{d}\overline{\xi_k} = \frac{i}{2}\sum_k \mathrm{d}z_k \wedge \mathrm{d}\overline{z}_k + \frac{i}{2}\sum_{jkl}(\overline{a_{jkl}z_l}\mathrm{d}z_k \wedge \mathrm{d}\overline{z}_j + a_{jkl}z_l\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j) + O(|\xi|^2)$$

By (3.1), we have

$$\sum_{jkl}\overline{a_{jkl}z_l}\mathrm{d}z_k \wedge \mathrm{d}\overline{z}_j = \sum_{jkl}\overline{a_{kjl}z_l}\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k = \sum_{jkl} a'_{jkl}\overline{z_l}\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k$$

So we have

$$\begin{aligned}\frac{i}{2}\sum_k \mathrm{d}\xi_k \wedge \mathrm{d}\overline{\xi_k} &= \frac{i}{2}\sum_{jk}(\delta_{jk} + \sum_l a_{jkl} z_l + a'_{jkl}\overline{z_l})\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k + O(|\xi|^2)\\ &= \omega + O(|\xi|^2)\end{aligned}$$

This completes the proof. □

3.2. **Differential operators.** Now let's introduce some functionals, since we can do this when we have metric and we do have it on vector bundles.

If X is an oriented differential manifold, with $\dim_{\mathbb{R}} X = n$. (E, g_E) is an Euclidean real vector bundle, that is[18]

$$\begin{aligned} g_E : C^\infty(X,E) \times C^\infty(X,E) &\to C^\infty(X)\\ (\alpha,\beta) &\mapsto \{\alpha,\beta\}\end{aligned}$$

a bilinear mapping.

Suppose there exists a Riemannian metric g on X, then we have a volume form vol with respect to g. Use such volume form, we can define L^2-inner product on $C^\infty_c(X,E)$ as

$$(\alpha,\beta)_{L^2} = \int_X \{\alpha,\beta\}\,\mathrm{vol}$$

For any two Euclidean real vector bundles $(E,g_E),(F,g_F)$, and two linear operators

$$\begin{aligned} P &: C^\infty_c(X,E) \to C^\infty_c(X,F)\\ P^* &: C^\infty_c(X,F) \to C^\infty_c(X,E)\end{aligned}$$

We say that P and P^* are formally adjoints[19] if

$$(P\alpha,\beta)_{L^2} = (\alpha, P^*\beta)_{L^2}, \quad \forall \alpha \in C^\infty_c(X,E), \beta \in C^\infty_c(X,F)$$

We are interested in the case $E = \Omega^k_{X,\mathbb{R}}$, and $P = \mathrm{d}$, that is

$$\mathrm{d} : C^\infty_c(X, \Omega^k_{X,\mathbb{R}}) \to C^\infty_c(X, \Omega^{k+1}_{X,\mathbb{R}})$$

Claim that in this case, its adjoint do exists

$$\mathrm{d}^* : C^\infty_c(X, \Omega^{k+1}_{X,\mathbb{R}}) \to C^\infty_c(X, \Omega^{k}_{X,\mathbb{R}})$$

To define this, we need the Hodge star operator. Let's first make it clear in the case of vector space.

Example 3.2.1. Let V be an oriented n-dimensional Euclidean vector space, with inner product $\langle\ ,\ \rangle$, and let $W = V^*$. There exists a canonical volume form $\mathrm{vol} \in \bigwedge^n W \cong \mathbb{R}$. More explictly, if $\{e_1,\dots,e_n\}$ is a orthonormal basis of V, and $\{e^1,\dots,e^n\}$ is the dual basis in W, then canonical volume form is $e^1\wedge\cdots\wedge e^n$.

[18] Don't confuse g_E here with pairing $\{\ ,\ \}$ we used before.

[19] In order to avoid quite hard functional analysis, we use such formal definition, but we will see later, in our interested case, such adjoints really exist.

From linear algebra we already know the wedge product

$$\bigwedge^k W \times \bigwedge^{n-k} W \xrightarrow{\wedge} \bigwedge^n W$$

is non-degenerate. So for any $\beta \in \bigwedge^k W$, we can define $*\beta \in \bigwedge^{n-k} W$ such that

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \operatorname{vol}, \quad \forall \alpha \in \bigwedge^k W$$

where $\langle\ ,\ \rangle$ is the inner product induced from W. To be more explicit, for $u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \in \bigwedge^k W$, we define their inner product as

$$\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \rangle = \det(\langle u_i, v_j \rangle)_{k \times k}$$

Clearly we have $*1 = \operatorname{vol}$ and $*\operatorname{vol} = 1$. Indeed, by definition, take $1, \alpha \in \bigwedge^0 W \cong \mathbb{R}$, then

$$\alpha * 1 = \alpha \wedge *1 = \alpha \operatorname{vol} \implies *1 = \operatorname{vol}$$

Similar for $*\operatorname{vol}$. Furthermore,

$$*e^i = (-1)^{i-1} e^1 \wedge \dots \widehat{e^i} \wedge \cdots \wedge e^n$$

To show this, we also need to back to definition. For any $\alpha \in W$, write it as $\alpha = \sum_{i=1}^n a_i e^i$, then

$$\begin{aligned}
\alpha \wedge *e^i &= \langle \alpha, e^i \rangle \operatorname{vol} \\
&= \langle \sum_{i=1}^n a_i e^i, e^i \rangle \operatorname{vol} \\
&= a_i \operatorname{vol} \\
&= a_i e^1 \wedge \cdots \wedge e^n
\end{aligned}$$

From this equation, it's easy to see what $*e^i$ is exactly.

Last but not least,

$$** = *^2 = (-1)^{k(n-k)} \operatorname{id}, \quad \text{on } \bigwedge^k W$$

the proof of it is also a routine, we omit it here.

We can carry what we have done to bundles of differential forms, since differential forms are just covector spaces living on a manifold smoothly.

Definition 3.2.2 (Hodge star operator)**.** *There exists*

$$* : C^\infty(X, \Omega^k_{X,\mathbb{R}}) \to C^\infty(X, \Omega^{n-k}_{X,\mathbb{R}})$$

such that

$$\alpha \wedge *\beta = \{\alpha, \beta\} \operatorname{vol}, \quad \forall \alpha \in C^\infty(X, \Omega^k_{X,\mathbb{C}})$$

Remark 3.2.3. In particular, if $\alpha, \beta \in C^\infty_c(X, \Omega^k_{X,\mathbb{R}})$, then

$$(\alpha, \beta)_{L^2} = \int_X \alpha \wedge *\beta$$

Lemma 3.2.4. $\mathrm{d}^* = (-1)^{nk+1} * \mathrm{d} * : C_c^\infty(X, \Omega^{k+1}_{X,\mathbb{C}}) \to C_c^\infty(X, \Omega^k_{X,\mathbb{C}})$

Proof. For $\alpha \in C_c^\infty(X, \Omega^k_{X,\mathbb{R}})$ and $\beta \in C_c^\infty(X, \Omega^{k+1}_{X,\mathbb{R}})$, then

$$\begin{aligned}
(\mathrm{d}\alpha, \beta)_{L^2} &= \int_X \mathrm{d}\alpha \wedge *\beta \\
&= \int_X \mathrm{d}(\alpha \wedge *\beta) - (-1)^k \alpha \wedge \mathrm{d} * \beta \\
&= (-1)^{k+1} \int_X \alpha \wedge \mathrm{d} * \beta \\
&= (-1)^{k+1}(-1)^{k(n-k)} \int_X \alpha \wedge * * \mathrm{d} * \beta \\
&= (-1)^{nk+1}(\alpha, *\mathrm{d} * \beta)_{L^2}
\end{aligned}$$

□

Although we have a formula for d^*, we can have a more computable formula for d^* in trivial case.

Recall contraction or interior product: Given a $\theta \in C^\infty(X, T_{X,\mathbb{R}})$ and $u \in C^\infty(X, \Omega^k_{X,\mathbb{R}})$, we have

$$\iota_\theta u \in C^\infty(X, \Omega^{k-1}_{X,\mathbb{R}})$$

defined by $\iota_\theta(u)(v_1, \ldots, v_{k-1}) = u(\theta, v_1, \ldots, v_{k-1})$. Locally, if $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ is a local frame of $T_{X,\mathbb{R}}$, then

$$\iota_{\frac{\partial}{\partial x_m}}(\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_k}) = \begin{cases} 0, & m \notin \{j_1, \ldots, j_k\} \\ (-1)^{l-1}\mathrm{d}x_{j_1} \wedge \ldots \widehat{\mathrm{d}x_{j_l}} \wedge \cdots \wedge \mathrm{d}x_{j_k}, & m = j_l \end{cases}$$

Furthermore, ι_θ satisfies the Leibniz rule.

Let $U \subset \mathbb{R}^n$ with standard Euclidean metric on $T_{U,\mathbb{R}}$, write $u = \sum_{|J|=k} u_J \mathrm{d}x_J$. Then

$$\mathrm{d}^* u = -\sum_{l=1}^n \sum_{|J|=k} \frac{\partial u_J}{\partial x_l} \iota_{\frac{\partial}{\partial x_l}} \mathrm{d}x_J$$

Let's check case $n = 1$ for an example. Take a 1-form $u = f\mathrm{d}x$, by definition $\mathrm{d}^* = (-1) * \mathrm{d} * = - * \mathrm{d} *$. And take any other 1-form $g\mathrm{d}x$

$$g\mathrm{d}x \wedge *(f\mathrm{d}x) = \langle g\mathrm{d}x, f\mathrm{d}x \rangle \mathrm{d}x = gf\mathrm{d}x \implies *(f\mathrm{d}x) = f$$

So we compute as follows

$$\begin{aligned}
- * \mathrm{d} * u &= - * \mathrm{d} * (f\mathrm{d}x) \\
&= - * \mathrm{d}f \\
&= - * (\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x) \\
&= -\frac{\mathrm{d}f}{\mathrm{d}x}
\end{aligned}$$

as desired.

Now, let's consider about n dimensional complex manifold X, endowed with a Hermitian metric h, and ω is the real $(1,1)$-form corresponds to h. As we have seen in Exercise 3.1.8, we have

$$\mathrm{vol} = \frac{\omega^n}{n!}$$

with respect to g. Since we have the following decomposition

$$\Omega^k_{X,\mathbb{C}} = \bigoplus_{p+q=k} \Omega^{p,q}_X$$

You can guess that we can descend our Hodge star operator to these $\Omega^{p,q}_X$.

Definition 3.2.5 (Hodge star operator)**.** *Hodge star operator is a* $\mathbb{C}$*-linear operator*

$$* : C^\infty(X, \Omega^k_{X,\mathbb{C}}) \to C^\infty(X, \Omega^{2n-k}_{X,\mathbb{C}})$$

such that

$$\alpha \wedge *\overline{\beta} = \{\alpha, \beta\}\,\mathrm{vol}, \quad \forall \alpha, \beta \in C^\infty(X, \Omega^k_{X,\mathbb{C}})$$

Remark 3.2.6. We should be more careful when dealing with inner product on $\Omega^k_{X,\mathbb{C}}$, since it's not quite similar to what we have discussed in Example 3.2.1. Given a Hermitian metric h on X and $\{\mathrm{d}z_1, \dots, \mathrm{d}z_n\}$ is a unitary coframe with respect to this metric. Then inner product on $\Omega^{p,q}_{X,\mathbb{C}}, p+q=k$ is given by taking the basis $\{\mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J\}_{|I|=p,|J|=q}$ to be orthonormal and of length $\|\mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J\|^2 = 2^k$. Recall that $\|\mathrm{d}z_i\|^2 = 2$ on $\mathbb{C}^n$.

Let's see how $*$ acts on $\Omega^{p,q}_X$. Consider $\alpha, \beta \in \Omega^{p,q}_X$, since we already know vol is a (n,n)-form. In order to get a (n,n)-form from $\alpha \wedge *\overline{\beta}$, we need $*\overline{\beta}$ is a $(n-p, n-q)$ form, that is, $*\beta$ is a $(n-q, n-p)$-form. So we will have

$$* : \Omega^{p,q}_X \cong \Omega^{n-q,n-p}_X$$

In other words, we have

$$* : C^\infty(X, \Omega^{p,q}_X) \to C^\infty(X, \Omega^{n-q,n-p}_X)$$

Use Hodge star operator, we can also define the adjoints of d, ∂ and $\overline{\partial}$.

$$\begin{aligned}
\mathrm{d}^* &= - * \mathrm{d} * \\
\partial^* &= - * \overline{\partial} * \\
\overline{\partial}^* &= - * \partial *
\end{aligned}$$

Similarly we can calculate them in a standard Hermitian metric.

Example 3.2.7. Take $U \subset \mathbb{C}^n$ with standard Hermitian metric. For any (p,q)-form u, we have

$$u = \sum_{|J|=p,|K|=q} = u_{JK} \mathrm{d}z_J \wedge \mathrm{d}\overline{z}_K$$

then we have

$$\partial^* u = -2\sum_{l=1}^{n}\sum_{|J|=p,|K|=q}\frac{\partial u_{JK}}{\partial \overline{z_l}}\iota_{\frac{\partial}{\partial z_l}}\mathrm{d}z_J\wedge \mathrm{d}\overline{z}_K$$
$$\overline{\partial}^* u = -2\sum_{l=1}^{n}\sum_{|J|=p,|K|=q}\frac{\partial u_{JK}}{\partial z_l}\iota_{\frac{\partial}{\partial \overline{z_l}}}\mathrm{d}z_J\wedge \mathrm{d}\overline{z}_K$$

Definition 3.2.8 (Laplacians)**.**

$$\begin{aligned}\Delta_{\mathrm{d}} &= \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}\\ \Delta_{\partial} &= \partial\partial^* + \partial^*\partial\\ \Delta_{\overline{\partial}} &= \overline{\partial\partial}^* + \overline{\partial}^*\overline{\partial}\end{aligned}$$

Example 3.2.9. Take $U \subset \mathbb{R}^n$ with standard Euclidean metric. For $f \in C^\infty(U)$, we have

$$\begin{aligned}\Delta_{\mathrm{d}} f &= \mathrm{d}\mathrm{d}^* f + \mathrm{d}^*\mathrm{d}f\\ &= \mathrm{d}^*\mathrm{d}f\\ &= \mathrm{d}^*(\sum_j \frac{\partial f}{\partial x_j}\mathrm{d}x_j)\\ &= -\sum_j \frac{\partial^2 f}{\partial x_j^2}\\ &= -\Delta f\end{aligned}$$

That's why we call it Laplacian.

Definition 3.2.10 (harmonic)**.** *Let X be an oriented Riemannian manifold or a complex Hermitian manifold. A (compactly supported) form α is called $\Delta_\bullet$-harmonic if $\Delta_\bullet \alpha = 0$. Here $\bullet$ can be* d, ∂ *and* $\overline{\partial}$*.*

Lemma 3.2.11. *α is Δ_{d}-harmonic if and only if* $\mathrm{d}\alpha = 0, \mathrm{d}^*\alpha = 0$*. Same for ∂ and $\overline{\partial}$.*

Proof. Note that

$$\begin{aligned}(\alpha, \Delta_{\mathrm{d}}\alpha)_{L^2} &= (\alpha, \mathrm{d}\mathrm{d}^*\alpha)_{L^2} + (\alpha, \mathrm{d}^*\mathrm{d}\alpha)_{L^2}\\ &= \|\mathrm{d}^*\alpha\|^2 + \|\mathrm{d}\alpha\|^2\end{aligned}$$

Cases for ∂ and $\overline{\partial}$ can be proved using the same argument. □

We're not going to study so much functionals. The reason we introduce above things is that they're neccessary for us to know what Hodge theorem is talking about.

The statement of the Hodge theorem is as follows, there are two versions, that is, real version and complex version. But for Hodge, they're the same things, just elliptic differential equations. Let's see real version first.

Theorem 3.2.12 (Hodge theorem). *Let (X, g) be a compact oriented Riemannian manifold, with $\dim_{\mathbb{R}} X = n$. Let $\mathcal{H}^k$ denote the space of Δ_{d}-harmonic k-forms, a subset of $C^\infty(X, \Omega^k_{X,\mathbb{R}})$. Then*

1. *$\mathcal{H}^k$ is finite dimensional.*
2. *decomposition $C^\infty(X, \Omega^k_{X,\mathbb{R}}) = \mathcal{H}^k \oplus \Delta_{\mathrm{d}}(C^\infty(X, \Omega^k_{X,\mathbb{R}}))$. Furthermore, it is orthonormal with respect to $(\ ,\ )_{L_2}$.*

Remark 3.2.13. Although we omit the proof of Hodge theorem here[20], we can see why $\mathcal{H}^k$ is orthonormal to $\Delta_{\mathrm{d}}(C^\infty(X, \Omega^k_{X,\mathbb{R}}))$.

Take a harmonic k-form α and $\Delta_{\mathrm{d}}\beta$ where β is also a k-form. Then

$$\begin{aligned}
(\alpha, \Delta_{\mathrm{d}}\beta)_{L^2} &= (\alpha, \mathrm{d}\mathrm{d}^*\beta)_{L^2} + (\alpha, \mathrm{d}^*\mathrm{d}\beta)_{L^2} \\
&= (\mathrm{d}^*\alpha, \mathrm{d}^*\beta)_{L^2} + (\mathrm{d}\alpha, \mathrm{d}\beta)_{L^2} \\
&= (\mathrm{d}\mathrm{d}^*\alpha, \beta)_{L^2} + (\mathrm{d}^*\mathrm{d}\alpha, \beta)_{L^2} \\
&= (\Delta_{\mathrm{d}}\alpha, \beta)_{L^2} \\
&= 0
\end{aligned}$$

From above computation, we see that for any two k-forms α, β, we have

$$(\alpha, \Delta_{\mathrm{d}}\beta)_{L^2} = (\Delta_{\mathrm{d}}\alpha, \beta)_{L^2}$$

In other words, Laplacian is a self-adjoint operator.

Corollary 3.2.14. *More explictly, we have the following orthonormal decomposition*

$$C^\infty(X, \Omega^k_{X,\mathbb{R}}) = \mathcal{H}^k \oplus \mathrm{d}(C^\infty(X, \Omega^{k-1}_{X,\mathbb{R}})) \oplus \mathrm{d}^*(C^\infty(X, \Omega^{k+1}_{X,\mathbb{R}}))$$

Proof. It suffices to show $\mathrm{d}(C^\infty(X, \Omega^{k-1}_{X,\mathbb{R}}))$ is orthonormal to $\mathrm{d}^*(C^\infty(X, \Omega^{k+1}_{X,\mathbb{R}}))$, and the proof is quite easy. Take $\mathrm{d}\alpha$ and $\mathrm{d}^*\beta$, where α is a $k-1$-form and β is a $k+1$-form. Then

$$(\mathrm{d}\alpha, \mathrm{d}^*\beta)_{L_2} = (\mathrm{d}^2\alpha, \beta)_{L_2} = 0$$

□

Corollary 3.2.15.

$$\begin{aligned}
\ker \mathrm{d} &= \mathcal{H}^k \oplus \mathrm{d}^*(C^\infty(X, \Omega^{k-1}_{X,\mathbb{R}})) \\
\ker \mathrm{d}^* &= \mathcal{H}^k \oplus \mathrm{d}(C^\infty(X, \Omega^{k-1}_{X,\mathbb{R}}))
\end{aligned}$$

Proof. Clear from Corollary 3.2.14. □

Corollary 3.2.16. *The natural map $\mathcal{H}^k \to H^k(X, \mathbb{R})$ is an isomorphism![21] In other words, every element in $H^k(X, \mathbb{R})$ is represented by a unique Δ_{d}-harmonic form.*

Proof. Clear from Corollary 3.2.15. □

[20] In fact, almost every textbook omits it.

[21] Here we use exclamatory mark to show it's a surprising result.

Corollary 3.2.17. *$*: \mathcal{H}^k \to \mathcal{H}^{n-k}$ is an isomorphism.*

Proof. It suffices to show $*$ maps harmonic forms to harmonic forms, since we already have $*$ maps k-forms to k-forms By Lemma 3.2.10, we just need to show $\mathrm{d} * \alpha = \mathrm{d}^* * \alpha = 0$ for a harmonic form α. Directly compute as follows

$$\begin{aligned}\mathrm{d} * \alpha &= (-1)^{\bullet_1} * * \mathrm{d} * \alpha = (-1)^{\bullet_2} * \mathrm{d}^* \alpha = 0\\ \mathrm{d}^* * \alpha &= (-1)^{\bullet_3} * \mathrm{d} * * \alpha = (-1)^{\bullet_4} * \mathrm{d}\alpha = 0\end{aligned}$$

Here we use $\bullet, \bullet'$ to denote the power of (-1), since it's not neccessary for us to know what exactly it is. □

Remark 3.2.18. In fact, above corollary follows from the following identity

$$\Delta_{\mathrm{d}} \circ * = * \circ \Delta_{\mathrm{d}}$$

which can be directly[22] checked. In other words, Hodge star commutes with Laplacian Δ_{d}. Here gives a method of computation: From what we have done in the proof, we will see

$$\begin{aligned}*\mathrm{d}^*\mathrm{d} &= (-1)^{\bullet_2}\mathrm{d} * \mathrm{d} = (-1)^{\bullet_2 + \bullet_4}\mathrm{d}\mathrm{d}^* *\\ *\mathrm{d}\mathrm{d}^* &= (-1)^{\bullet_4}\mathrm{d}^* * \mathrm{d}^* = (-1)^{\bullet_2 + \bullet_4}\mathrm{d}^*\mathrm{d} *\end{aligned}$$

So all we need to do is to figure out the precise number of $\bullet_2, \bullet_4$ and show that $\bullet_2 + \bullet_4$ is even.

Corollary 3.2.19 (Poincaré duality)**.** *$H^k(X, \mathbb{R}) \cong H^{n-k}(X, \mathbb{R})$.*

Proof. Clear from Corollary 3.2.16 and Corollary 3.2.17. □

From above corollaries of Hodge theorem, I think you've convinced yourself that Hodge theorem is really a quite meaningful theorem. Now let's elaborate complex version of it.

Theorem 3.2.20 (Hodge theorem)**.** *Let (X, h) be a compact complex Hermitian manifold with $\dim_{\mathbb{C}} X = n$. Let $\mathcal{H}^{p,q}$ be the space of $\Delta_{\overline{\partial}}$ harmonic forms of type (p, q), a subset of $C^\infty(X, \Omega_X^{p,q})$. Then*

1. *$\mathcal{H}^{p,q}$ is finite dimensional.*
2. *decomposition $C^\infty(X, \Omega_X^{p,q}) = \mathcal{H}^{p,q} \oplus \Delta_{\overline{\partial}}(C^\infty(X, \Omega_X^{p,q}))$. Furthermore, it is orthonormal with respect to $(\ ,\ )_{L_2}$.*

The story here is the same as the real version.

Corollary 3.2.21. *More explictly, we have the following orthonormal decomposition*

$$C^\infty(X, \Omega_X^{p,q}) = \mathcal{H}^k \oplus \overline{\partial}(C^\infty(X, \Omega_X^{p,q-1}) \oplus \overline{\partial}^*(C^\infty(X, \Omega_X^{p,q+1}))$$

Corollary 3.2.22.

$$\begin{aligned}\ker \overline{\partial} &= \mathcal{H}^{p,q} \oplus \overline{\partial}^*(C^\infty(X, \Omega_X^{p,q-1}))\\ \ker \overline{\partial}^* &= \mathcal{H}^{p,q} \oplus \overline{\partial}(C^\infty(X, \Omega_X^{p,q+1}))\end{aligned}$$

[22]But not easily

Corollary 3.2.23. *The natural map $\mathcal{H}^{p,q} \to H^{p,q}_{Dol}(X)$ is an isomorphism! In particular, $H^{p,q}_{Dol}(X)$ is finite dimensional.*

Parallel to what have happened in the real version, we may desire that

$$* : \mathcal{H}^{p,q} \to \mathcal{H}^{n-q,n-p}$$

is an isomorphism. In order to have such an isomorphism, we may desire the following identity holds:

$$* \circ \Delta_{\overline{\partial}} = \Delta_{\overline{\partial}} \circ *$$

But something bad happens, since we only have $\overline{\partial}^* = - * \partial *$, and if we use the same method we will get $\Delta_{\overline{\partial}} \circ * = * \circ \Delta_{\partial}$. So it fails generally since $\Delta_{\overline{\partial}} \neq \Delta_{\partial}$ in general.

There are two ways to fix it. The first way is that we will see later if X is compact Kähler manifold, then $\Delta_{\partial} = \Delta_{\overline{\partial}}$. Then

Corollary 3.2.24. *(X, ω) is a compact Kähler manifold with dimension n, then $* : \mathcal{H}^{p,q} \to \mathcal{H}^{n-q,n-p}$ is an isomorphism*

Another way is to fix the operator $*$, we define

$$\begin{aligned} \overline{*} : \Omega^{p,q}_X &\to \Omega^{n-p,n-q}_X \\ \beta &\mapsto * \overline{\beta} \end{aligned}$$

then

$$\overline{*} \Delta_{\overline{\partial}} = \Delta_{\overline{\partial}} \overline{*}$$

Corollary 3.2.25. *(X, h) is a compact Hermitian manifold, then $\overline{*} : \mathcal{H}^{p,q} \to \mathcal{H}^{n-p,n-q}$ is an isomorphism.*

Corollary 3.2.26. *$H^{p,q}_{Dol}(X) \cong H^{n-p,n-q}_{Dol}(X)$.*

Remark 3.2.27. This is a special case of Serre duality.

Untill now, the same things happen simultaneously in the real world and complex world, but they do not intersect with each other, just like parallel universes. But, as we will see soon, the Kähler condition plays a role of a "wormhole", connecting these two parallel universes.[23]

3.3. **Differential operators on Kähler manifolds.** (X, ω) is a Kähler manifold

Definition 3.3.1.

$$\begin{aligned} L : C^\infty(X, \Omega^k_{X,\mathbb{R}}) &\to C^\infty(X, \Omega^{k+2}_{X,\mathbb{R}}) \\ \alpha &\mapsto \omega \wedge \alpha \end{aligned}$$

Lemma 3.3.2. $\Lambda := L^* = (-1)^k * L *$.

[23]So romantic.

Proof. For $\alpha \in C_c^\infty(X, \Omega^k_{X,\mathbb{R}}), \beta \in C_c^\infty(X, \Omega^{k+2}_{X,\mathbb{R}})$. Compute

$$\begin{aligned}\{L\alpha, \beta\} \operatorname{vol} &= L\alpha \wedge *\beta \\ &= \omega \wedge \alpha \wedge *\beta \\ &= \alpha \wedge \omega \wedge *\beta \\ &= \alpha \wedge (-1)^{k(2n-k)} * *\omega \wedge *\beta \\ &= \alpha \wedge *((-1)^{k(2n-k)} * L * \beta) \\ &= \{\alpha, (-1)^k * L * \beta\} \operatorname{vol}\end{aligned}$$

□

Remark 3.3.3. If A, B are two differential operators, we define the commutor of A, B as

$$[A, B] := AB - (-1)^{\deg A \deg B} BA$$

As what we have learnt in Lie algebra, commutor should satisfy Jacobi identity. Here is a similar one:

$$(-1)^{\deg A \deg C}[A,[B,C]]+(-1)^{\deg B \deg A}[B,[C,A]]+(-1)^{\deg C \deg B}[C,[A,B]] = 0$$

In our case, the degree of $\mathrm{d}, \mathrm{d}^*, \partial, \partial^*, \overline{\partial}, \overline{\partial}^*$ is one, and the degree of L and Λ is zero[24].

Now we have eight differential operators, and Kähler condition implies that there are some relations between them.

Proposition 3.3.4 (Kähler identities)**.** *If (X, ω) is a Kähler manifold, then we have*

$$\begin{aligned}\left[\overline{\partial}^*, L\right] &= i\partial \\ [\partial^*, L] &= -i\overline{\partial} \\ [\Lambda, \partial] &= -i\partial^* \\ [\Lambda, \partial] &= i\bar{\partial}^*\end{aligned}$$

Example 3.3.5. Let $U \subset \mathbb{C}^n$ be an open subset with standard Hermitian metric, then $\omega = \frac{i}{2}\sum \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$. For any compactly supported (p, q)-form $u = \sum_{J,K} u_{JK} \mathrm{d}z_J \wedge \mathrm{d}\overline{z}_K$.

By Example 3.2.7, we have

$$\begin{aligned}\overline{\partial}^* u &= -2\sum_l \sum_{J,K} \frac{\partial u_{JK}}{\partial z_l} \iota_{\frac{\partial}{\partial \overline{z_l}}} \mathrm{d}z_J \wedge \mathrm{d}\overline{z}_K \\ &= -2\sum_l \iota_{\frac{\partial}{\partial \overline{z_l}}} \frac{\partial u}{\partial z_l}\end{aligned}$$

[24]You can try to understand this thing in a following way: operators $\mathrm{d}, \mathrm{d}^*, \partial, \partial^*, \overline{\partial}, \overline{\partial}^*$ do take differentials, but L and Λ not.

So we have

$$\begin{aligned}\left[\overline{\partial}^*, L\right]u &= \overline{\partial}^*(\omega\wedge u) - \omega\wedge\overline{\partial}^* u\\ &= -2\sum_l \iota_{\frac{\partial}{\partial\overline{z}_l}}\frac{\partial}{\partial z_l}(\omega\wedge u) + \omega\wedge 2\sum_l \iota_{\frac{\partial}{\partial\overline{z}_l}}\frac{\partial u}{\partial z_l}\end{aligned}$$

Since ω is a closed $(1,1)$-form, then

$$\frac{\partial}{\partial z_l}(\omega\wedge u) = \omega\wedge\frac{\partial u}{\partial z_l}$$

So we have

$$\begin{aligned}\iota_{\frac{\partial}{\partial\overline{z}_l}}\frac{\partial}{\partial z_l}(\omega\wedge u) &= \iota_{\frac{\partial}{\partial\overline{z}_l}}(\omega\wedge\frac{\partial u}{\partial z_l})\\ &= (\iota_{\frac{\partial}{\partial\overline{z}_l}}\omega)\wedge\frac{\partial u}{\partial z_l} + \omega\wedge\iota_{\frac{\partial}{\partial\overline{z}_l}}\frac{\partial u}{\partial z_l}\end{aligned}$$

Then

$$\begin{aligned}\left[\overline{\partial}^*, L\right]u &= -2\sum_l(\iota_{\frac{\partial}{\partial\overline{z}_l}}\omega)\wedge\frac{\partial u}{\partial z_l}\\ &= i\sum_l \mathrm{d}z_l\wedge\frac{\partial u}{\partial z_l}\\ &= i\sum_l\sum_{J,K}\frac{\partial u_{JK}}{\partial z_l}\mathrm{d}z_l\wedge\mathrm{d}z_J\wedge\mathrm{d}\overline{z}_K\\ &= i\partial u\end{aligned}$$

Proof. By conjugating and taking adjoints, it suffices to prove the first identity, that is a first order identity of differential equation

$$\left[\overline{\partial}^*, L\right] = i\partial$$

But by Theorem 3.1.9, locally we have $h_{jk} = \delta_{jk} + O(|\xi^2|)$. Thus Kähler identity holds from the $U\subset\mathbb{C}^n$ case. □

Theorem 3.3.6. (X,ω) *is a Kähler manifold. Then*

$$\Delta_{\mathrm{d}} = 2\Delta_{\partial} = 2\Delta_{\overline{\partial}}$$

In particular, Δ_{d}*-harmonic is equivalent to* Δ_{∂}*-harmonic and is equivalent to* $\Delta_{\overline{\partial}}$*-harmonic.*

Proof. Directly compute

$$\Delta_{\mathrm{d}} = (\partial+\overline{\partial})(\partial^*+\overline{\partial}^*) + (\partial^*+\overline{\partial}^*)(\partial+\overline{\partial})$$

Use the forth Kähler identity, we firt compute the first term

$$\begin{aligned}(\partial+\overline{\partial})(\partial^*+\overline{\partial}^*) &= (\partial+\overline{\partial})(\partial^* - i\Lambda\partial + i\partial\Lambda)\\ &= \partial\partial^* - i\partial\Lambda\partial + \overline{\partial}\partial^* - i\overline{\partial}\Lambda\partial + i\overline{\partial}\partial\Lambda\end{aligned}$$

And the second term

$$
\begin{aligned}
(\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) &= (\partial^* - i\Lambda\partial + i\partial\Lambda)(\partial + \overline{\partial}) \\
&= \partial^*\partial + i\partial\Lambda\partial + \partial^*\overline{\partial} - i\Lambda\partial\overline{\partial} + i\partial\Lambda\overline{\partial}
\end{aligned}
$$

Use the third Kähler identity, we have

$$\partial^* = i[\Lambda, \overline{\partial}] = i\Lambda\overline{\partial} - i\overline{\partial}\Lambda$$

then

$$
\begin{aligned}
\overline{\partial}\partial^* &= \overline{\partial}(i\Lambda\overline{\partial} - i\overline{\partial}\Lambda) = i\overline{\partial}\Lambda\overline{\partial} \\
\partial^*\overline{\partial} &= (i\Lambda\overline{\partial} - i\overline{\partial}\Lambda)\overline{\partial} = -i\overline{\partial}\Lambda\overline{\partial} \\
&= -\overline{\partial}\partial^*
\end{aligned}
$$

Now we have

$$
\begin{aligned}
\Delta_{\mathrm{d}} &= \Delta_\partial - i\overline{\partial}\Lambda\partial - i\Lambda\partial\overline{\partial} + i\overline{\partial}\partial\Lambda + i\partial\Lambda\overline{\partial} \\
&= \Delta_\partial + i(\Lambda\overline{\partial}\partial - \overline{\partial}\Lambda\partial) + i(\partial\Lambda\overline{\partial} - \partial\overline{\partial}\Lambda) \\
&= \Delta_\partial + i[\Lambda, \overline{\partial}]\partial + i\partial[\Lambda, \overline{\partial}] \\
&= \Delta_\partial + \partial^*\partial + \partial\partial^* \\
&= 2\Delta_\partial
\end{aligned}
$$

□

Exercise 3.3.7. Show that for Kähler manifold we have

$$
\begin{aligned}
&[\Delta_{\mathrm{d}}, L] = 0 \\
&[L, \Lambda] = (k - n)\,\mathrm{id} \quad \text{on } C^\infty(X, \Omega^k_{X,\mathbb{C}})
\end{aligned}
$$

Proof. For the first one, we have $\Delta_{\mathrm{d}} = 2\Delta_\partial = 2(\partial\partial^* + \partial^*\partial)$. Thus

$$[\Delta_{\mathrm{d}}, L] = 2([\partial\partial^*, L] + [\partial^*\partial, L]) = 2(\partial[\partial^*, L] + [\partial^*, L]\partial)$$

The last equality holds by the fact that L commutes with ∂, since ω is ∂ closed. Now we use the identity $[\partial^*, L] = -i\overline{\partial}$, which anticommutes with ∂. Thus we have the desired result.

For the second one, since we are considering operators of order zero, thus WLOG we will assume that the metric is the standard flat metric. Recall that L is the exterior product with $\omega = \frac{i}{2}\sum_i \mathrm{d}z_i \wedge \mathrm{d}\overline{z_i}$. Let A_j be the operator given by the exterior product with $\frac{i}{2}\mathrm{d}z_j \wedge \mathrm{d}z_j$

$$[L, \Lambda]\alpha =$$

□

Corollary 3.3.8. *(X, ω) is a Kähler manifold, and α is a (p, q)-form, then $\Delta_{\mathrm{d}}\alpha$ is still a (p, q)-form.*

Proof. $\Delta_\partial\alpha$ is still a (p, q)-form is a clear fact. □

Theorem 3.3.9. *(X,ω) is a Kähler manifold, $\alpha = \sum_{p+q=k} \alpha^{p,q}$. Then α is harmonic if and only if $\alpha^{p,q}$ is harmonic. That is*

$$\mathcal{H}^k \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}$$

with $\overline{\mathcal{H}^{p,q}} = \mathcal{H}^{q,p}$.

Theorem 3.3.10 (Hodge decomposition)**.** *(X,ω) is a compact Kähler manifold. Then*

$$H^k(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X)$$

with $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

Corollary 3.3.11. *(X,ω) is a compact Kähler manifold. Set $b_k = \dim H^k(X,\mathbb{C})$ and $h^{p,q} = \dim H^{p,q}(X)$. Then*

$$b_k = \sum_{p+q=k} h^{p,q}$$

with $h^{p,q} = h^{q,p}$.

Corollary 3.3.12. *b_k is even when k is odd.*

Corollary 3.3.13. *$b_k \neq 0$ when k is even.*

Proof. $h^{k,k} \neq 0$, since $0 \neq \omega^k \in H^{k,k}(X)$. $\square$

There are many relations between $h^{p,q}$, and we can draw a picture as follows, called Hodge diamond, since it has the same symmetry as a diamond.

$$\begin{array}{ccccccccccc}
 & & & & & h^{0,0} & & & & & b_0 \\
 & & & & h^{1,0} & & h^{0,1} & & & & b_1 \\
 & & & h^{2,0} & & h^{1,1} & & h^{0,2} & & & b_2 \\
 & & \iddots & & & \vdots & & & \ddots & & \vdots \\
\text{Hodge} \updownarrow & h^{n,0} & & \cdots & & \underset{\text{Serre}}{\curvearrowleft} & & \cdots & & h^{0,n} & b_n \\
 & & \ddots & & & \vdots & & & \iddots & & \vdots \\
 & & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & & b_{2n-2} \\
 & & & & h^{n,n-1} & & h^{n-1,n} & & & & b_{2n-1} \\
 & & & & & h^{n,n} & & & & & b_{2n} \\
 & & & & & \underset{\text{conjugation}}{\longleftrightarrow} & & & & &
\end{array}$$

Example 3.3.14.

$$H^{p,q}(\mathbb{P}^n) = \begin{cases} \mathbb{C}, & 0 \leq p = q \leq n \\ 0, & \text{otherwise} \end{cases}$$

Proof. It's known to all that the singular cohomology of $\mathbb{R}^n$ with complex coefficient are

$$H^k(\mathbb{P}^n, \mathbb{C}) = \begin{cases} \mathbb{C}, & k \text{ is even} \\ 0, & k \text{ is odd} \end{cases}$$

And it's clear to compute Dolbeault cohomology of $\mathbb{P}^n$ using the symmetry of Hodge diamond. □

3.4. **Bott-Chern cohomology.** Review what we have done: We have already proven one of the main theorems in this course, that is, Hodge decomposition. But along the way we used the Kähler metric on a Kähler manifold, so our decomposition may depend on it. A question is that: (In)denpendence of the Kähler metric? The anwser is yes, shown by Bott-Chern cohomology.

Definition 3.4.1 (Bott-Chern cohomology)**.** *X is a complex manifold, we define*

$$H^{p,q}_{BC}(X) := \frac{Z^{p,q}_{BC} := \{\alpha \in C^\infty(X, \Omega^{p,q}) \mid \mathrm{d}\alpha = 0\}}{\partial\overline{\partial} C^\infty(X, \Omega^{p-1,q-1}_X)}$$

Remark 3.4.2. There is a natural map

$$Z^{p,q}_{BC}(X) \to H^{p+q}(X, \mathbb{C})$$

and since $\partial\overline{\partial}\beta = \mathrm{d}\overline{\partial}\beta$, then it descends to

$$H^{p,q}_{BC}(X) \to H^{p+q}(X, \mathbb{C})$$

On the other hand, we also have a natural map

$$Z^{p,q}_{BC}(X) \to H^{p,q}(X)$$

and since $\partial\overline{\partial}\beta = -\overline{\partial}\partial\beta$, then it also descends to

$$H^{p,q}_{BC}(X) \to H^{p,q}(X)$$

In our definition about Bott-Chern cohomology, there's nothing relative to the choice of metrics. So if we can prove there is an isomorphism between

$$H^{p,q}_{BC}(X) \cong H^{p,q}(X), \qquad \bigoplus_{p+q=k} H^{p,q}_{BC}(X) \cong H^k(X, \mathbb{C})$$

we can say our Hodge decomposition is canonical.

Lemma 3.4.3 ($\partial\overline{\partial}$-lemma)**.** *$(X, \omega)$ is a compact Kähler manifold, α is a* d*-closed (p, q)-form, i.e. $\alpha \in Z^{p,q}_{BC}(X)$. If α is $\overline{\partial}$-exact or ∂-exact, then there exists a $(p-1, q-1)$-form such that*

$$\alpha = \partial\overline{\partial}\beta$$

Proof. We have $\mathrm{d}\alpha = \partial\alpha = \overline{\partial}\alpha = 0$, and $\alpha = \overline{\partial}\gamma$. Hodge's theorem implies that we can write γ as

$$\gamma = a + \partial b + \partial^* c$$

where a is Δ_∂-harmonic. Directly compute

$$\begin{aligned}\alpha = \overline{\partial}\gamma &= \overline{\partial}a + \overline{\partial}\partial b + \overline{\partial}\partial^* c\\ &= -\partial\overline{\partial}b + \overline{\partial}\partial^* c\\ &= -\partial\overline{\partial}b - \partial^*\overline{\partial}c\end{aligned}$$

What we need to do is to show $-\partial^*\overline{\partial}c = 0$. A trick here is to note that

$$0 = \partial\alpha = -\partial\partial^*\overline{\partial}c \implies \partial^*\overline{\partial}c \in \ker(\partial)\cap\mathrm{im}(\partial^*) \implies \partial^*\overline{\partial}c = 0$$

So we have

$$\alpha = \partial\overline{\partial}(-b)$$

as desired. □

Corollary 3.4.4. *$H^{p,q}_{BC}(X) \to H^{p,q}(X)$ is an isomorphism, and $\bigoplus_{p+q=k} H^{p,q}_{BC}(X) \to H^k(X,\mathbb{C})$ is an isomorphism.*

Proof. Here we only prove the first isomorphism. From Remark 3.4.2 we have a canonical map $H^{p,q}_{BC}(X) \to H^{p,q}(X)$, and if we choose a metric, we have $H^{p,q}(X) \cong \mathcal{H}^{p,q}$, we will show our canonical map is both surjective and injective using this chosen metric.

Surjectivity. For $H^{p,q}(X)$ we choose a $\Delta_{\overline{\partial}}$-harmonic representative. Since Δ_∂-harmonic is equivalent to Δ_{d}-harmonic, so this representative is also d-closed.

Injectivity. Suppose we have $\gamma \in H^{p,q}_{BC}(X)$, represented by $[\alpha]$, where $\alpha \in Z^{p,q}_{BC}(X)$ such that $\overline{\partial}\alpha = 0$, that is $\overline{\partial}$-exact. By $\partial\overline{\partial}$-lemma we have that $\alpha \in \partial\overline{\partial}C^\infty(X, \Omega_X^{p-1,q-1})$, that is $0 = \gamma \in H^{p,q}_{BC}(X)$. □

3.5. **Lefschetz decomposition.**

Let (X,ω) be a n dimensional Kähler manifold, then

$$\begin{aligned}L : \Omega^k_{X,\mathbb{R}} &\to \Omega^{k+2}_{X,\mathbb{R}}\\ \alpha &\mapsto \omega\wedge\alpha\end{aligned}$$

Proposition 3.5.1. *$L^{n-k} : \Omega^k_{X,\mathbb{R}} \to \Omega^{2n-k}_{X,\mathbb{R}}$ is an isomorphism for $k \leq n$.*

Proof. Since $\Omega^k_{X,\mathbb{R}}$ has the same rank with $\Omega^{2n-k}_{X,\mathbb{R}}$, it suffices to show L^{n-k} is injective, and it suffices to show that it's injective in each fiber. Recall that $L^* = \Lambda = (-1)^k * L* : \Omega^k_{X,\mathbb{R}} \to \Omega^{k-2}_{X,\mathbb{R}}$, and

$$[L,\Lambda]\alpha = (n-k)\alpha, \quad \forall \alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}})$$

then

$$\begin{aligned}[L^r,\Lambda] &= L^r\Lambda - \Lambda L^r\\ &= L(L^{r-1}\Lambda - \Lambda L^{r-1}) + L\Lambda L^{r-1} - \Lambda L L^{r-1}\\ &= L[L^{r-1},\Lambda] + [L,\Lambda]L^{r-1}\end{aligned}$$

So we can prove the following identity by induction on r:

$$[L^r,\Lambda]\alpha = (r(k-n) + r(r-1))L^{r-1}\alpha, \quad \forall\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}})$$

Suppose $L^r\alpha = 0, r \leq n-k$, then

$$\begin{aligned} L^r\Lambda\alpha &= [L^r, \Lambda]\alpha \\ &= (r(k-n) + r(r-1))L^{r-1}\alpha \end{aligned}$$

So we have

$$L^{r-1}(L\Lambda\alpha - (r(k-n) + r(r-1))\alpha) = 0$$

In other words, from $\alpha \in \ker L^r$, we get something in $\ker L^{r-1}$. Clearly L is injective, so we have the following identity by induction on r:

$$L\Lambda\alpha = (r(k-n) + r(r-1))\alpha$$

Let $\beta = \Lambda\alpha$, and apply L^r to both side we have

$$L^{r+1}\beta = (r(k-n) + r(r-1))L^r\alpha = 0$$

where $\beta \in C^\infty(X, \Omega^{k-2}_{X,\mathbb{R}})$. By induction on k, we have $\beta = 0$, so we have $\alpha = 0$. □

Remark 3.5.2. From above proof, we can see all $L^r, r \leq n-k$ are injective.

Definition 3.5.3 (primitive form)**.** (X, ω) *is a Kähler manifold, with dimension* $n \geq k$*.* $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}})$ *is called primitive if* $L^{n-k+1}\alpha = 0$*.*

Exercise 3.5.4. $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}}), k \leq n$ is primitive if and only if $\Lambda\alpha = 0$.

Proof. Let's first see what will happen for a primitive n-form α. α is primitive if and only if $L\alpha = 0$. Recall that Exercise 3.3.7 implies that

$$[L, \Lambda] = (k-n)\,\mathrm{id}, \quad \text{on } C^\infty(X, \Omega^k_{X,\mathbb{C}})$$

So if $k = n$, then L and Λ commutes, so we have α is primitive if and only if $\Lambda\alpha = 0$, since

$$\Lambda\alpha = 0 \iff L\Lambda\alpha = 0 \iff \Lambda L\alpha = 0 \iff L\alpha = 0$$

and the first and last equality we use the fact that L is injective on $\Omega^k_{X,\mathbb{R}}, k \leq n$ and Λ is injective on $\Omega^{n+2}_{X,\mathbb{R}}$.

More generally, we have

$$[L^r, \Lambda]\alpha = (r(k-n) + r(r-1))L^{r-1}\alpha$$

and in particular for $r = n-k+1$ where k is the degree of α, we have

$$[L^r, \Lambda]\alpha = 0$$

The argument can be repeated as above to conclude. □

Proposition 3.5.5. *For all* $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}}), k \leq n$*. Then there exists a unique decomposition*

$$\alpha = \sum_r L^r\alpha_r$$

with $\alpha_r \in C^\infty(X, \Omega^{k-2r}_{X,\mathbb{R}})$ *is primitive.*

Proof. Uniqueness. Suppose $\sum_r L^r\alpha_r = 0$ with primitive α_r. We want to show $\alpha_r = 0$. If not, then take the largest r_m such that $\alpha_{r_m} \neq 0$. By our choice, L^{n-k+r_m} kills everything in $\sum_r L^r\alpha_r$ but $L^{r_m}a_{r_m}$, so

$$0 = L^{n-k+r_m}(\sum_r L^r\alpha_r) = L^{n-k+r_m}(L^{r_m}\alpha_{r_m}) \neq 0$$

A contradiction.

Existence. For $L^{n-k+1}\alpha \in C^\infty(X, \Omega^{2n-k+2}_{X,\mathbb{R}})$, then there exists $\beta \in C^\infty(X, \Omega^{k-2}_{X,\mathbb{R}})$ such that

$$L^{n-k+1}\alpha = L^{n-k+2}\beta \implies L^{n-k+1}(\alpha - L\beta) = 0$$

So $\alpha - L\beta$ is primitive, that is

$$\alpha = (\alpha - L\beta) + L\beta$$

By induction on k, we have primitive decomposition for $\beta \in C^\infty(X, \Omega^{k-2}_{X,\mathbb{R}})$. This completes the proof. $\square$

Remark 3.5.6. Set $H = [L, \Lambda]$, we have proven that (L, H, Λ) generates an $\mathfrak{sl}_2$-action on $\bigoplus_k \Omega^k_{X,\mathbb{R}}$, hence on $\bigoplus_k C^\infty(X, \Omega^k_{X,\mathbb{R}})$.

In cohomology, we can define the following map

$$\begin{aligned} L : H^k(X, \mathbb{R}) &\to H^{k+2}(X, \mathbb{R}) \\ [\alpha] &\mapsto [\omega \wedge \alpha] \end{aligned}$$

Clearly, it's well-defined, it suffices to check that L maps closed forms to closed forms and exact forms to exact forms. If α is closed, then

$$\mathrm{d}(\omega \wedge \alpha) = \mathrm{d}\omega \wedge \alpha + \omega \wedge \mathrm{d}\alpha = 0$$

since ω is closed. And if $\alpha = \mathrm{d}\beta$, then

$$\omega \wedge \mathrm{d}\beta = \mathrm{d}\omega \wedge \beta + \omega \wedge \mathrm{d}\beta = \mathrm{d}(\omega \wedge \beta)$$

So, as you can see, L is well-defined mainly relys on the fact that ω is closed.

Theorem 3.5.7 (Hard Lefschetz theorem[25])**.** *(X, ω) is a compact Kähler manifold with dimension n, then*

$$L^{n-k} : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R})$$

is an isomorphism for $k \leq n$.

Proof. Use $[\Delta_{\mathrm{d}}, L] = 0$, we have

$$L^{n-k} : \mathcal{H}^k \to \mathcal{H}^{2n-k}$$

By the fact that L^{n-k} is injective and $\mathcal{H}^k, \mathcal{H}^{2n-k}$ have the same dimension, we obtain the desired result. $\square$

[25]Though proof of this theorem is quite easy using tools we have, but it's quite hard for Lefschetz, since during his time, there is no Hodge! And we use L to denote this operator, in order to honor Lefschetz.

Definition 3.5.8 (primitive form)**.** *(X,ω) is a compact Kähler manifold, with dimension n. For $[\alpha]\in H^k(X,\mathbb{R})$, we call it primitive, if $L^{n-k+1}[\alpha]=0$. We use $H^k(X,\mathbb{R})_{\text{prim}}$ to denote the set of all primitive forms.*

Corollary 3.5.9 (Lefschetz decomposition)**.** *For all $[\alpha]\in H^k(X,\mathbb{R}), k\le n$, we have following unique decomposition*

$$[\alpha]=\sum_r L^r[\alpha_r]$$

where $[\alpha_r]\in H^{k-2r}(X,\mathbb{R})_{\text{prim}}$. In other words

$$H^k(X,\mathbb{R})=\bigoplus_r L^rH^{k-2r}(X,\mathbb{R})_{\text{prim}}$$

Remark 3.5.10. If $[\omega]\in H^2(X,\mathbb{Z})$, such as ω comes from a positive holomorphic line bundle. Then we can state theorem and corollary for $H^k(X,\mathbb{Q})$.

Moreover, we have the following isomorphism

$$L^{n-k}:H^{p,q}(X)\to H^{n-q,n-p}(X)$$

for $k=p+q\le n$.

Corollary 3.5.11. *For a compact Kähler manifold (X,ω) with dimension n Then $b_{k-2}\le b_k$ and $h^{p-1,q-1}\le h^{p,q}$ for $k=p+q\le n$.*

3.6. **Hodge index.** Firstly we show the baby case: surfaces

Example 3.6.1. For open subset $U\subset\mathbb{C}^2$, we have

$$\omega=\frac{i}{2}(\mathrm{d}z_1\wedge\mathrm{d}\overline{z_1}+\mathrm{d}z_2\wedge\mathrm{d}\overline{z_2})$$

Then we have volume form

$$\mathrm{vol}=\frac{\omega^2}{2!}=-\frac{1}{4}(\mathrm{d}z_1\wedge\mathrm{d}\overline{z_1}\wedge\mathrm{d}z_2\wedge\mathrm{d}\overline{z_2})$$

If we take a 2-form α of type $(2,0)$, then

$$\alpha=a\mathrm{d}z_1\wedge\mathrm{d}z_2$$

then its Hodge star $*\overline{\alpha}$ is also a $(2,0)$-form, we assume $*\overline{\alpha}=b\mathrm{d}z_1\wedge\mathrm{d}z_2$. If we want to determine b, just take an arbitrary $(0,2)$-form and compute. For convenience let's take $\beta=\mathrm{d}\overline{z_1}\wedge\mathrm{d}\overline{z_2}$, then

$$\beta\wedge*\overline{\alpha}=-b\mathrm{d}z_1\wedge\mathrm{d}\overline{z_1}\wedge\mathrm{d}z_2\wedge\mathrm{d}\overline{z_2}$$

$$\{\beta,\alpha\}\,\mathrm{vol}=\{a\mathrm{d}z_1\wedge\mathrm{d}z_2,\mathrm{d}\overline{z_1}\wedge\mathrm{d}\overline{z_2}\}\times\frac{-1}{4}\mathrm{d}z_1\wedge\mathrm{d}\overline{z_1}\wedge\mathrm{d}z_2\wedge\mathrm{d}\overline{z_2}$$

and by definition of inner product on $\Omega^{2,0}_{X,\mathbb{C}}$, we have

$$\{a\mathrm{d}z_1\wedge\mathrm{d}z_2,\mathrm{d}\overline{z_1}\wedge\mathrm{d}\overline{z_2}\}=2^2a=4a$$

So we have $a=b$, that is, $\alpha=*\overline{\alpha}$. Directly compute we have

$$L\alpha=\omega\wedge\alpha=0$$

which means a 2-form α of type $(2,0)$ is always primitive. And computations for $(0,2)$-form is similar, we also have $*\overline{\alpha}=\alpha$ for a $(0,2)$-form.

Take a 2-form α, say of type $(1,1)$, that is

$$\alpha = a_{11}\mathrm{d}z_1\wedge\mathrm{d}\overline{z_1}+a_{22}\mathrm{d}z_2\wedge\mathrm{d}\overline{z_2}+a_{12}\mathrm{d}z_1\wedge\mathrm{d}\overline{z_2}+a_{21}\mathrm{d}z_2\wedge\mathrm{d}\overline{z_1}$$

We can also compute its Hodge star, but it's a little bit complicated, so we just list the result as follows

$$*\overline{\alpha} = a_{22}\mathrm{d}z_1\wedge\mathrm{d}\overline{z_1}+a_{11}\mathrm{d}z_2\wedge\mathrm{d}\overline{z_2}-a_{12}\mathrm{d}z_1\wedge\mathrm{d}\overline{z_2}-a_{21}\mathrm{d}z_2\wedge\mathrm{d}\overline{z_1}$$

Directly compute we have

$$L\alpha=\omega\wedge\alpha=\frac{i}{2}(a_{11}+a_{22})\mathrm{d}z_1\wedge\mathrm{d}\overline{z_1}\wedge\mathrm{d}z_2\wedge\mathrm{d}\overline{z_2}$$

So

$$L\alpha=0\iff a_{11}+a_{22}=0\iff *\overline{\alpha}=-\alpha$$

Lemma 3.6.2. *(X,ω) is a Kähler surface, $\alpha\in C^\infty(X,\Omega_X^{p,q})$ is primitive 2-form, then*

$$*\overline{\alpha}=(-1)^p\alpha$$

Proof. It suffices to do $U\subset\mathbb{C}^2$, and that's what we have done in Example 3.6.1. □

Now let's see the case that X is a compact Kähler surface. Poincaré duality implies that we have the following non-degenerate pairing

$$\begin{aligned}Q: H^2(X,\mathbb{R})\times H^2(X,\mathbb{R})&\to\mathbb{R}\\ ([\alpha],[\beta])&\mapsto\int_X\alpha\wedge\beta\end{aligned}$$

Then we get a Hermitian form by defining

$$H([\alpha],[\beta])=Q([\alpha],\overline{[\beta]})$$

Lemma 3.6.3. *Lefschetz decomposition for $H^2(X,\mathbb{R})=H^2(X,\mathbb{R})_{\mathrm{prim}}\oplus\mathbb{R}\cdot[\omega]$ is orthonormal with respect to Q. If we use complex coefficient, then with respect to H.*

Proof.

$$Q([\omega],[\alpha])=\int_X\omega\wedge\alpha=\int_X L\alpha=0$$

for α is primitive and harmonic. □

Theorem 3.6.4. *$H^2(X,\mathbb{C})_{\mathrm{prim}}=\bigoplus_{p+q=2}H^{p,q}(X)_{\mathrm{prim}}$ is orthonormal with respect to H. Furthermore, $(-1)^pH$ is positive definite on $H^{p,q}(X)_{\mathrm{prim}}$.*

Proof. Orthonormality is almost trivial, since only integrate $(2,2)$-form we can get a non-zero result. Take a harmonic form α such that $[\alpha] \in H^{p,q}(X)_{\text{prim}} \backslash \{0\}, p+q=2$, then

$$\begin{aligned}(-1)^p H([\alpha],[\alpha]) &= (-1)^p \int_X \alpha \wedge \overline{\alpha} \\ &= (-1)^{p+q} \int_X \alpha \wedge *\overline{\alpha} \\ &= \|\alpha\|^2 > 0\end{aligned}$$

So $(-1)^p H$ is positive definite on $H^{p,q}(X)_{\text{prim}}$. $\square$

Corollary 3.6.5 (Hodge index theorem: surface case)**.** *H on $H^2(X,\mathbb{C}) \cap H^{1,1}(X)$ is of index $(1, h^{1,1}-1)$.*

Proof. Note that for $H^2(X,\mathbb{C}) \cap H^{1,1}(X)$ we have the following decomposition

$$H^2(X,\mathbb{C}) \cap H^{1,1}(X) = H^{1,1}(X)_{\text{prim}} \oplus \mathbb{C}[\omega]$$

And H is negative definite on $H^{1,1}(X)_{\text{prim}}$, so index for H on $H^2(X,\mathbb{C}) \cap H^{1,1}(X)$ is $(1, h^{1,1}-1)$. $\square$

Now let's see a more general case: (X,ω) is a compact Kähler manifold with dimension n. Then by Lefschetz decomposition we have

$$H^k(X,\mathbb{R}) = \bigoplus_r L^r H^{k-2r}(X,\mathbb{R})_{\text{prim}}, \quad k \leq n$$

And by Hodge decomposition we have each piece can be decomposed into smaller one

$$H^k(X,\mathbb{C})_{\text{prim}} = \bigoplus_{p+q=k} H^{p,q}(X)_{\text{prim}}$$

As we have seen in the case of surface, H will be positive definite or negative definite in such small piece. The same thing happens in higher dimension. Now we introduce some symbols, in order to get a neater result.

Let $\varepsilon(k) = (-1)^{\frac{k(k-1)}{2}}$, and we define

$$\begin{aligned}Q : H^k(X,\mathbb{R}) \times H^k(X,\mathbb{R}) &\to \mathbb{R} \\ ([\alpha],[\beta]) &\mapsto \varepsilon(k) \int_X \omega^{n-k} \wedge \alpha \wedge \beta\end{aligned}$$

This Q is just a bilinear form, and it is symmetric when k is even and anti-symmetric when k is odd.

Definition 3.6.6 (Weil operator)**.** *Weil operator is defined as follows*

$$\begin{aligned}\mathsf{C} : H^k(X,\mathbb{R}) &\to H^k(X,\mathbb{R}) \\ \mathsf{C}|_{H^{p,q}(X)} &\mapsto i^{p-q}\,\mathrm{id}\end{aligned}$$

Remark 3.6.7. Weil operator C is a real operator. Indeed, we can check directly as follows

$$\mathsf{C}|_{\overline{H^{p,q}(X)}} = \mathsf{C}_{H^{q,p}(X)} = i^{q-p}\operatorname{id} = \overline{i^{p-q}}\operatorname{id} = \overline{\mathsf{C}|_{H^{p,q}(X)}}$$

Now we define

$$\begin{aligned} H : H^k(X,\mathbb{C}) \times H^k(X,\mathbb{C}) &\to \mathbb{C} \\ ([\alpha],[\beta]) &\mapsto Q(\mathsf{C}[\alpha],\overline{[\beta]}) \end{aligned}$$

In other words, we have

$$H([\alpha],[\beta]) = (-1)^{\frac{k(k-1)}{2}} i^{p-q}\int_X \omega^{n-k}\wedge\alpha\wedge\overline{\beta}, \quad \alpha,\beta\in H^{p,q}(X)$$

Exercise 3.6.8. H is a Hermitian form on $H^{p,q}(X)$.

Proof. Take $[\alpha],[\beta]\in H^{p,q}(X)$, then

$$\begin{aligned} \overline{H([\alpha],[\beta])} &= (-1)^{\frac{k(k-1)}{2}}(-1)^{p-q} i^{p-q}\int_X \omega^{n-k}\wedge\overline{\alpha}\wedge\beta \\ &= (-1)^{\frac{k(k-1)}{2}}(-1)^{p-q} i^{p-q}(-1)^{(p+q)^2}\int_X \omega^{n-k}\wedge\beta\wedge\overline{\alpha} \end{aligned}$$

Note that

$$(p+q)^2 - p - q = 2pq + p(p-1) + q(q-1)$$

is always even, this completes the proof. □

Theorem 3.6.9 (Hodge-Riemann bilinear relations)**.** *We have following results:*

1. $H^k(X,\mathbb{R}) = \bigoplus_r L^r H^{k-2r}(X,\mathbb{R})_{\mathrm{prim}}$ *is orthonormal with respect to* Q*.*
2. $H^k(X,\mathbb{C})_{\mathrm{prim}} = \bigoplus_{p+q=k} H^{p,q}(X)_{\mathrm{prim}}$ *is orthonormal with respect to* H*.*
3. H *is positive definite on* $H^{p,q}(X)_{\mathrm{prim}}$*.*

Proof. 1. Take $r<s$, note that

$$\omega^{n-k}\wedge L^r\gamma\wedge L^s\delta = (L^{n-k+r+s}\gamma)\wedge\delta = 0$$

since $L^{n-k+2r+1}\gamma = 0$ and $r<s$.

2. If α is a (p,q)-form, and β is (p',q')-form, and $(p,q)\neq(p',q')$, then $\omega^{n-k}\wedge\alpha\wedge\overline{\beta}$ is not a (n,n)-form.

3. We need to establish the following lemma

Lemma 3.6.10. $\alpha\in C^\infty(X,\Omega_X^{p,q})$ *is a primitive* k*-form, then*

$$*\alpha = (-1)^{\frac{k(k+1)}{2}} i^{p-q}\frac{L^{n-k}\alpha}{(n-k)!}$$

Take α which is harmonic such that it represents $[\alpha]\in H^{p,q}(X)_{\mathrm{prim}}\backslash\{0\}$. Then

$$H([\alpha],[\alpha]) = (-1)^{\frac{k(k-1)}{2}} i^{p-q}\int_X \omega^{n-k}\wedge\alpha\wedge\overline{\alpha}$$

Note that

$$\begin{aligned}*\overline{\alpha} &= (-1)^{\frac{k(k+1)}{2}} i^{q-p} \frac{L^{n-k}\overline{\alpha}}{(n-k)!} \\ &= (-1)^{\frac{k(k-1)}{2}} i^{p-q} \frac{L^{n-k}\overline{\alpha}}{(n-k)!}\end{aligned}$$

so we have

$$H([\alpha],[\alpha]) = (n-k)!\int_X \alpha\wedge *\overline{\alpha} = (n-k)!\|\alpha\|^2 > 0$$

□

Corollary 3.6.11 (Hodge index theorem)**.** *Let $k = n = \dim_{\mathbb{C}} X$ and k is even*[26]*. Then $\int_X \alpha\wedge\beta$ on $H^n(X,\mathbb{R})$ is of signature*

$$\sum_{p,q}(-1)^p h^{p,q}$$

where summation runs over all p,q.

Proof. Note that the signature of $\int_X \alpha\wedge\beta$ on $H^n(X,\mathbb{R})$ is the same as the signature of $int_X\alpha\wedge\overline{\beta}$ on $H^n(X,\mathbb{C})$.

We write

$$H^n(X,\mathbb{C}) = \bigoplus_{r,p,q} L^r H^{p,q}(X)_{\mathrm{prim}}, \quad p+q+2r = n$$

Then Hodge-Riemann bilinear theorem implies that $\int_X \alpha\wedge\overline{\beta}$ is $(-1)^p$-definite on $L^r H^{p,q}(X)_{\mathrm{prim}}$, here we need the requirement n is even.

Then we have the signature is

$$\sum_{p+q+2r=n}(-1)^p h^{p,q}_{\mathrm{prim}}$$

We also have $h^{p,q}_{\mathrm{prim}} = h^{p,q} - h^{p-1,q-1}$. Then signature is

$$\sum_{p+q+2r=n}(-1)^p(h^{p,q} - h^{p-1,q-1})$$

Note that $p+q=n$ counted once and $p+q<n$ counted twice, so rewrite it as

$$\sum_{p+q \text{ even}}(-1)^p h^{p,q}$$

since $h^{p,q} = h^{n-p,n-q}$. And this is also equivalent to sum all p,q, since

$$\sum_{p+q \text{ odd}}(-1)^p h^{p,q} = 0$$

This completes the proof. □

[26]So that $\int_X \alpha\wedge\beta$ is symmetric on $H^n(X,\mathbb{R})$.

Example 3.6.12. For surface, we have

$$H^2(X,\mathbb{C}) = H^{2,0}(X)\oplus H^{1,1}(X)_{\mathrm{prim}}\oplus\mathbb{C}[\omega]\oplus H^{0,2}(X)$$

Then this corollary implies

$$h^{0,0}+h^{2,0}-h^{1,1}+h^{0,2}+h^{2,2}=h^{2,0}+h^{0,2}+(1-(h^{1,1}-1))$$

recover what we have done in the case of surface.

4. Applications of Hodge theory

4.1. **Serre duality.** Let X be a compact complex manifold, and $E\to X$ be a holomorphic vector bundle. Recall

$$H^q(X,E)=H^q_{\overline{\partial}_E}(C^\infty(X,\Omega_X^{0,\bullet}\otimes E))$$

One can also define

$$H^{p,q}(X,E)=H^q_{\overline{\partial}_E}(C^\infty(X,\Omega_X^{p,\bullet}\otimes E))$$

But it does not give anything new, since

$$H^{p,q}(X,E)=H^q(X,\Omega_X^p\otimes E)$$

We already defined Hodge star operator $\overline{*}: C^\infty(X,\Omega_X^{p,q})\to C^\infty(X,\Omega_X^{n-p,n-q})$, we also want to do it for the forms with coefficients in E, that is, if (E,h) is a Hermitian holomorphic vector bundle, we want to define

$$\overline{*}_E: C^\infty(X,\Omega_X^{p,q}\otimes E)\to C^\infty(X,\Omega_X^{n-p,n-q}\otimes E^*)$$

We define it locally. Take $x\in U\subset X$, then we define a $\mathbb{C}$-antilinear map on each fiber

$$\begin{aligned}\tau_x: E_x&\to E_x^*\\ e_x&\mapsto\langle\bullet,e_x\rangle\end{aligned}$$

then we get a $\mathbb{C}$-antilinear map

$$\tau: E\to E^*$$

For $\varphi\otimes e\in C^\infty(U,\Omega_U^{p,q}\otimes E)$, we define

$$\overline{*}_E(\varphi\otimes e):=*\overline{\varphi}\otimes\tau(e)$$

In other words, we want to have

$$\alpha\wedge\overline{*}_E\beta=\{\alpha,\beta\}\,\mathrm{vol}$$

If D_E is the Chern connection, then $D_E^{0,1}=\overline{\partial}_E$, we set $D_E^{1,0}=\partial_E$, then its Chern curvature is that

$$H_E=D_E^2=\partial_E^2+\partial_E\overline{\partial}_E+\overline{\partial}_E\partial_E+\overline{\partial}_E^2$$

Recall that H_E is a $(1,1)$-form, with coefficients in $\mathrm{End}(X)$, then $\partial_E^2=0$
So,

$$H_E=\partial_E\overline{\partial}_E+\overline{\partial}_E\partial_E=[\partial_E,\overline{\partial}_E]$$

We want to define formal adjoints $\partial_E^*,\overline{\partial}_E^*$. In fact, we can write them down directly.

Exercise 4.1.1. Give formulas in terms of $\overline{*}_E$

So we can define Laplacians $\Delta_{\partial_E}, \Delta_{\overline{\partial}_E}$, and Hodge appears again! We will have

$$C^\infty(X, \Omega_X^{p,q} \otimes E) = \mathcal{H}^{p,q}(X, E) \oplus \operatorname{im} \overline{\partial}_E \oplus \operatorname{im} \overline{\partial}_E^*$$

and by the same argument we have

$$H^{p,q}(X, E) \cong \mathcal{H}^{p,q}$$

Theorem 4.1.2 (Serre duality)**.** *Let X be a compact complex manifold, E is a holomorphic vector bundle, then there exists a non-degenerate $\mathbb{C}$-linear pairing*

$$H^{p,q}(X, E) \times H^{n-p,n-q}(X, E^*) \to \mathbb{C}$$
$$([\alpha], [\beta]) \mapsto \int_X \alpha \wedge \beta$$

In particular, we have

$$H^{p,q}(X, E) = H^{n-p,n-q}(X, E^*)^*$$

Before the proof of Serre duality, let's recall how do we prove Poincaré duality using Hodge theory. Poincaré duality states that if X is a compact oriented Riemannian manifold with $\dim_{\mathbb{R}} X = n$, then the pairing

$$H^{p,q}(X, \mathbb{R}) \times H^{n-p,n-q}(X, \mathbb{R}) \to \mathbb{R}$$
$$([\alpha], [\beta]) \mapsto \int_X \alpha \wedge \beta$$

is non-degenerate. So we have $H^k(X, \mathbb{R}) = H^{n-k}(X, \mathbb{R})^*$. Our proof via Hodge theory is that

$$* : \mathcal{H}^k \xrightarrow{\cong} \mathcal{H}^{n-k}$$

and Hodge theorem imply that

$$\mathcal{H}^k \cong H^k(X, \mathbb{R})$$

For $\alpha \in \mathcal{H}^k, \beta \in \mathcal{H}^{n-k}$, we have $\beta = *\gamma$ for some $\gamma \in \mathcal{H}^k$, so

$$\int_X \alpha \wedge \beta = \int_X \alpha \wedge *\gamma = \langle \alpha, \gamma \rangle_{L^2}$$

is non-degenerate.

Our proof of Serre duality is quite similar to the one of Poincaré duality.

Proof. Sketch. Endow E with a Hermitian metric h. Firstly show

$$\Delta_{\overline{\partial}_E^*} \overline{*}_E = \overline{*}_E \Delta_{\overline{\partial}_E}$$

then

$$\overline{*}_E : \mathcal{H}^{p,q}(X, E) \xrightarrow{\cong} \mathcal{H}^{n-p,n-q}(X, E^*)$$

and Hodge theorem implies that

$$\mathcal{H}^{p,q}(X, E) \cong H^{p,q}(X, E)$$

For all $\alpha \in \mathcal{H}^{p,q}(X,E), \beta \in \mathcal{H}^{n-p,n-q}(X,E^*)$, we have $\beta = \overline{*}_E\gamma$ for some $\gamma \in \mathcal{H}^{p,q}(X,E)$, then

$$\int_X \alpha \wedge \beta = \int_X \alpha \wedge \overline{*}_E\gamma = \langle \alpha, \gamma \rangle_{L^2}$$

is non-degenerate. □

Example 4.1.3. Let $E = \mathcal{O}_X$, then $H^{p,q}(X) = H^{n-p,n-q}(X)^*$. This recovers Corollary 3.2.26.

Example 4.1.4. Let $p = 0$, we have

$$H^q(X,E) = H^{n-q}(X, K_X \otimes E^*)^*$$

where n is the dimension of X. This form of Serre duality may be the one you learnt in a more serious algebraic geometry course.

4.2. **Kodaira vanishing.** Now if we let X be a compact Kähler manifold with dimension n. (E,h) as above. Similarly we can define

$$\begin{aligned} L : C^\infty(X, \Omega^{p,q} \otimes E) &\to C^\infty(X, \Omega^{p+1,q+1} \otimes E) \\ \alpha &\mapsto \omega \wedge \alpha \end{aligned}$$

and $\Lambda = L^*$, formal adjoint. We also have Kähler identities

$$\begin{aligned} \left[\overline{\partial}^*_E, L\right] &= i\partial \\ [\partial^*_E, L] &= -i\overline{\partial} \\ [\Lambda, \partial] &= -i\partial^* \\ [\Lambda, \partial] &= i\bar{\partial}^* \end{aligned}$$

and

$$[L, \Lambda] = (p+q-n)\,\mathrm{id}, \quad \text{on } C^\infty(X, \Omega^{p,q}_X \otimes E)$$

Untill now, all things are same as the things without coefficient E. But

Theorem 4.2.1 (Bochner-Kodaira-Nakano identity)**.**

$$\Delta_{\overline{\partial}_E} = [iH_E, \Lambda] + \Delta_{\partial_E}$$

Proof. By definition, we have

$$\begin{aligned} \Delta_{\overline{\partial}_E} &= [\overline{\partial}_E, \overline{\partial}^*_E] \\ &= -i[\overline{\partial}_E, [\Lambda, \partial_E]] \\ &= -i[\Lambda, [\partial_E, \overline{\partial}_E]] - i[\partial_E, [\overline{\partial}_E, \Lambda]] \\ &= -i[\Lambda, H_E] - i[\partial_E, i\partial^*_E] \\ &= [iH_E, \Lambda] + \Delta_{\partial_E} \end{aligned}$$

□

Corollary 4.2.2 (Bochner-Kodaira-Nakano inequality). *For $\alpha \in C^\infty(X, \Omega_X^{p,q} \otimes E)$. Then*

$$([iH_E, \Lambda]\alpha, \alpha)_{L^2} \leq (\Delta_{\overline{\partial}_E}\alpha, \alpha)_{L^2}$$

In particular, if α is $\Delta_{\overline{\partial}_E}$-harmonic, then $([iH_E, \Lambda]\alpha, \alpha)_{L^2} \leq 0$.

Proof.

$$\begin{aligned}(\Delta_{\overline{\partial}_E}\alpha, \alpha)_{L^2} - ([iH_E, \Lambda]\alpha, \alpha)_{L^2} &= (\Delta_{\partial_E}\alpha, \alpha) \\ &= \|\partial_E \alpha\|^2 + \|\partial_E^* \alpha\|^2 \geq 0\end{aligned}$$

□

Theorem 4.2.3 (Kodaira-Akizuki-Nakano vanishing). *(X, ω) is a compact Kähler manifold with dimension n. (L, h) is a positive holomorphic line bundle. Then*

$$H^{p,q}(X, L) = 0, \quad p + q > n$$

In particular, if we let $p = n$, then

$$H^q(X, K_X \otimes L) = 0, \quad q > 0$$

Proof. Use $\Delta_{\overline{\partial}_L}$-harmonic forms, that is $H^{p,q}(X, L) \cong \mathcal{H}^{p,q}(X, L)$. For $\alpha \in \mathcal{H}^{p,q}(X, L)$, then BKN inequality says

$$([iH_E, \Lambda]\alpha, \alpha)_{L^2} \leq 0$$

On the other hand, we can choose $\omega = \dfrac{i}{2\pi} H_L$, since L is positive. So we have

$$[iH_L, \Lambda]\alpha = 2\pi(p + q - n)\alpha$$

Thus

$$([iH_E, \Lambda]\alpha, \alpha)_{L^2} = 2\pi(p + q - n)\|\alpha\|^2 \geq 0$$

Thus

$$([iH_E, \Lambda]\alpha, \alpha)_{L^2} = 0$$

That is $\alpha = 0$. □

Corollary 4.2.4 (Kodaira vanishing). *(X, ω) is a compact Kähler manifold with dimension n. (L, h) is a positive holomorphic line bundle. Then*

$$H^q(X, K_X \otimes L) = 0, \quad q > 0$$

Proof. Let $p = n$. □

Exercise 4.2.5. Compute all $H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$ for all k, q.

Definition 4.2.6 (Fano). *Fano manifold is a compact Kähler manifold with $K_X^* = \det T_X$ is positive.*

Exercise 4.2.7. Let X be a Fano manifold. Show that

$$H^q(X, \mathcal{O}_X) = 0, \quad \forall q > 0$$

Proof. Note that $\mathcal{O}_X = K_X \otimes K_X^*$. □

Theorem 4.2.8 (Serre vanishing). *(X,ω) is a compact Kähler manifold with dimension n. (L,h) is a holomorphic line bundle. For any holomorphic vector bundle E on X. Then there exists a constant m_0 such that for all $m\geq m_0$*

$$H^q(X,E\otimes L^{\otimes m})=0,\quad q>0$$

Proof. Endow E with a Hermitian metric and consider $H^{p,q}(X,E\otimes L^{\otimes m})\cong\mathcal{H}^{p,q}(X,E\otimes L^{\otimes m})$. For $\alpha\in\mathcal{H}^{p,q}(X,E\otimes L^{\otimes m})$. BKN inequality implies that

$$([iH_{E\otimes L^{\otimes m}},\Lambda]\alpha,\alpha)_{L^2}\leq 0$$

Recall that $D_{E\otimes L^{\otimes m}}=D_E\otimes 1+1\otimes D_{L^{\otimes m}}$. So

$$H_{E\otimes L^{\otimes m}}=H_E\otimes 1+m(1\otimes H_L)\in C^\infty(X,\Omega^{1,1}_X\otimes\mathrm{End}(E\otimes L^{\otimes m}))$$

Choose $\omega=\dfrac{i}{2\pi}H_L$, since L is positive. Then

$$([iH_{E\otimes L^{\otimes m}},\Lambda]\alpha,\alpha)_{L^2}=([iH_E,\Lambda]\alpha,\alpha)_{L^2}+2\pi m(p+q-n)\|\alpha\|^2\leq 0$$

Cauchy inequality implies that

$$([iH_E,\Lambda]\alpha,\alpha)_{L^2}\geq -C\|\alpha\|^2$$

where C is the norm of $[iH_E,\Lambda]$.

So if we have $2\pi m(p+q-n)-c>0$, then

$$([iH_{E\otimes L^{\otimes m}},\Lambda]\alpha,\alpha)_{L^2}\geq 0$$

Thus $\alpha=0$ as desired.

So we take $p=n,q>0,m_0\geq\dfrac{c}{2\pi}$, we have

$$H^{n,q}(X,E\otimes L^{\otimes m})=H^q(X,K_X\otimes E\otimes L^{\otimes m})=0,\quad\forall m\geq m_0,q>0$$

Since E is arbitrary, use $K_X^*\otimes E$ instead[27]. □

4.3. **Kodaira embedding.**

Theorem 4.3.1 (Kodaira embedding). *X is a compact complex manifold, the following statements are equivalent*

1. *There exists a holomorphic embedding $\varphi:X\hookrightarrow\mathbb{P}^N$.*
2. *There exists a integral Kähler form ω on X, that is, $[\omega]\in\mathrm{im}(H^2(X,\mathbb{Z})\to H^2(X,\mathbb{R}))$*
3. *There exists a positive holomorphic line bundle on X.*

Remark 4.3.2. 1 clealy implies 2, and 2 implies 3 is Lefschetz $(1,1)$-theorem. So the heart of the proof is 3 to 1.

Remark 4.3.3. We will shall see 1 together with Chow's theorem, $X\subset\mathbb{P}^N$ can be written as a zero set of homogenous polynomials. Thus X is a projective manifold in our definition.

Before the proof of Kodaira embedding, let's see some corollaries.

[27]Note that m_0 might change in the process.

Corollary 4.3.4. *X is a compact Kähler manifold such that $H^2(X,\mathcal{O}_X)=H^{0,2}(X)=0$. Then X is a projective manifold.*

Proof. Hodge decomposition implies that $H^{2,0}(X)=H^{0,2}(X)=0$, so $H^2(X,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}=H^2(X,\mathbb{C})=H^{1,1}(X)$. Choose basis $[\alpha_1],\dots,[\alpha_n]\in H^2(X,\mathbb{Q})$ such that α_i is harmonic and of type $(1,1)$. Since the Kähler form ω is real, harmonic[28] and of type $(1,1)$. Then

$$\omega=\sum_i \lambda_i\alpha_i,\quad \lambda_i\in\mathbb{R}$$

For $\mu_i\in\mathbb{Q}$ sufficiently close to λ_i, then $\sum_i\mu_i\alpha_i$ is still positive. Thus $\sum_i\mu_i\alpha_i$ gives a Kähler form. Take N sufficiently large such that $[N\sum_i\mu_i\alpha_i]\in\operatorname{im}(H^2(X,\mathbb{Z})\to H^2(X,\mathbb{R}))$. Applying Kodaira embedding together with Chow's theorem to complete the proof. □

Corollary 4.3.5. *Fano manifold is projective.*

Proof. Since for Fano manifold, all $H^{0,p}(X)=0,q>0$. □

Now we give the proof of Kodaira embedding.

Proof. (Sketch.) Use holomorphic global sections $\Gamma(X,L^{\otimes m})$ for sufficiently large m to construct $\varphi:X\hookrightarrow\mathbb{P}^N$. We need to show the following three things:

1. For sufficiently large m, $L^{\otimes m}$ is globally generated, which means for all $x\in X$, there exists a global section $s\in\Gamma(X,L^{\otimes m})$ such that $s(x)\neq 0$. Then for all $x\in X$, $H_x=\{s\in\Gamma(X,L^{\otimes m})\mid s(x)=0\}$ is a hypersurface. Thus we get a holomorphic map $\varphi:X\to\mathbb{P}(\Gamma(X,L^{\otimes m})^*)$, defined by $x\mapsto H_x$. Indeed, since any hypersurface in $\Gamma(X,L^{\otimes m})$ is a line in $\Gamma(X,L^{\otimes m})^*$, that is an element in $\mathbb{P}(\Gamma(X,L^{\otimes m})^*)$. And you will find φ is holomorphic since we're using holomorphic sections.
2. For more sufficiently large[29] m, $L^{\otimes m}$ separetes points, that is for all $x,y\in X$, there exists $s\in\Gamma(X,L^{\otimes m})$ such that $s(x)\neq 0, s(y)=0$. Thus in this case our φ is injective.
3. For more more sufficiently large m, $L^{\otimes m}$ separetes tangent vectors, that is, for all $x\in X, u\in T_{X,x}$ there exists $s\in\Gamma(X,L^{\otimes m})$ such that $s(x)=0$ and $\mathrm{d}s(u)\neq 0$. Thus in this case our φ is an immersion, together with X is compact we have φ is an embedding.

Remark 4.3.6. We can also describe φ more explicitly. Locally around x_0, choose a basis $s_0,\dots,s_N$ of $\Gamma(X,L^{\otimes m})$ such that $s_0(x_0)\neq 0$. Then there exists a neighborhood U of x_0 such that $s_0(x)\neq$ for all $x\in U$. Then

$$\frac{s_1}{s_0},\dots,\frac{s_N}{s_0}\in\Gamma(U,\mathcal{O}_U)$$

[28] It's harmonic since $[\Delta_{\mathrm{d}},L]=0$.

[29] Larger than m is step one.

So we can define

$$\begin{aligned}\varphi|_U : U &\to \mathbb{P}^N\\ x &\mapsto (1, \frac{x_1}{x_0}(x), \dots, \frac{s_N}{s_0}(x))\end{aligned}$$

And you can check it's same as what we have defined without choosing a basis.

Here we only give a sketch proof of the first statement, the proofs for second and third are similar, but more complicated.

We want to detect the value of a section at a point is zero or not. Sheaves give us a good way to describe such thing. For $x \in X$, consider ideal sheaf of x

$$\mathcal{I}_x = \{s \in \mathcal{O}_X \mid s(x) = 0\} \subset \mathcal{O}_X$$

Then sections we are searching for is $\Gamma(X, \mathcal{I}_x \otimes L^{\otimes m})$. For computation, we have exact sequence of sheaf

$$0 \to \mathcal{I}_x \otimes L^{\otimes m} \to L^{\otimes m} \to L^{\otimes m}|_x \to 0$$

And using Čech cohomology we can derive a long exact sequence

$$0 \to \Gamma(X, \mathcal{I}_x \otimes L^{\otimes m}) \to \Gamma(X, L^{\otimes}) \to \mathbb{C} \to \check{H}^1(X, \mathcal{I}_x \otimes L^{\otimes m}) \to \dots$$

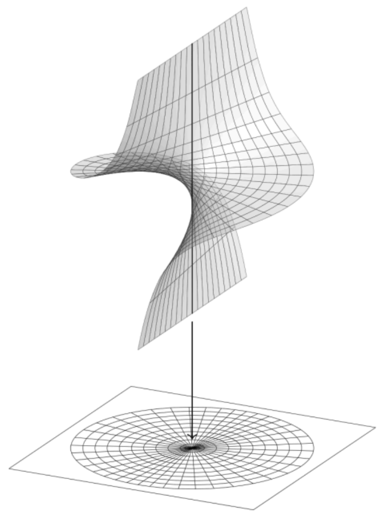
Our goal is to show $\Gamma(X, \mathcal{I}_x \otimes L^{\otimes m}) \neq 0$. If $\check{H}^1(X, \mathcal{I}_x \otimes L^{\otimes m}) = 0$ for sufficiently large m, then we can get desired result.

For Čech cohomology we know a little, but we know quite a lot for Dolbeault cohomology. So an ideal is to turn idea sheaf into a line bundle and use Dolbeault cohomology to compute.

A technical tool we need is blowing up a point. For example: If we want to blow up the origin near a neighborhood $0 \in U \subset \mathbb{C}^n$. Here is the definition

Definition 4.3.7 (blowing up)**.** $U \times \mathbb{P}^{n-1} \supset \widetilde{U} := \{((x_1, \dots, x_n), (y_1 : \dots : y_n)) \mid x_i y_j = x_j y_i\}$

Remark 4.3.8. How to understand blowing up? The most vivid way is to consider the fibers of projection $\widetilde{U} \to U$: If $x \neq 0$, then the fiber of x is just a point, since the ratio of y_i is uniquely determined; But for $x = 0$, then there is no restriction for y_i, that is you get the whole projective space $\mathbb{P}^{n-1}$. So as you can imagine, there is nothing happening except the origin, sounds like a boom. For example, the following figure shows the case of $n = 2$



In Definition 4.3.7, we only gives a definition in sense of set theory, but after blowing up we will get a complex manifold.

Exercise 4.1. $\widetilde{U} \subset U \times \mathbb{P}^{n-1}$ is a submanifold of dimension n.

As we have seen, blowing up is a local operation, so we can do it on a manifold. If X is a complex manifold with dimension n, and $x \in X$. $\{U_i\}$ is an open covering such that $x \in U_1$ and $x \notin U_i, i \neq 1$. Then we can show that $\widetilde{U_1} \cup (\bigcup_{i\neq 1} U_i)$ glue together a new complex manifold with dimension n, and denote it by $\widetilde{X}$, and we also have a natural projection $\pi : \widetilde{X} \to X$.

Similarly we have $\pi^{-1}(x)$ is biholomorphic to $\mathbb{P}^{n-1}$, and call it exceptional divisor and denote it by E. Projection $\pi|_{\widetilde{X}\setminus E}$ is a biholomorphic map.

Exercise 4.3.9. X is compact Kähler manifold, then $\widetilde{X}$ is also a compact Kähler manifold.

Exercise 4.3.10. The idea sheaf of exceptional divisor $\mathcal{I}_E \cong \mathcal{O}_{\widetilde{X}}(E)^*$

Exercise 4.3.11. Show that

$$K_{\widetilde{X}} \cong \pi^* K_X \otimes \mathcal{O}_{\widetilde{X}}(E)^{\otimes n-1}$$

Hints:

1. Reduce to $X = U \subset \mathbb{C}^n, x = 0$.
2. Show that $K_{\widetilde{\mathbb{C}^n}} \cong \mathcal{O}_{\widetilde{\mathbb{C}^n}}(E)^{\otimes n-1}$.

The adjunction formula implies

$$K_E \cong (K_{\widetilde{X}} \otimes \mathcal{O}_{\widetilde{X}}(E))|_E$$

So together Exercise 4.3.11 and $K_E = \mathcal{O}_{\mathbb{P}^{n-1}}(-n)$ we have

$$\mathcal{O}_{\mathbb{P}^{n-1}}(-n) \cong (\pi^* K_X \otimes \mathcal{O}_{\widetilde{X}}(E)^{\otimes n})|_E$$

But E is a fiber of π, thus $\pi^* K_X|_E$ is trivial, so

$$\mathcal{O}_{\mathbb{P}^{n-1}}(-n) \cong \mathcal{O}_{\widetilde{X}}(E)^{\otimes n}|_E$$

However we already know the only possible line bundles on $\mathbb{P}^{n-1}$ take the form $\mathcal{O}_{\mathbb{P}^{n-1}}(k), k \in \mathbb{Z}$. Thus $\mathcal{O}_{\widetilde{X}}(E)|_E \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$.

The main reason we need blowing up here is that the sections after blowing up is the "same" as the one before. To be explicit, we have $\Gamma(X, L^{\otimes m}) = \Gamma(\widetilde{X}, \pi^* L^{\otimes m})$. Indeed, take a section in $\Gamma(X, L^{\otimes m})$, we can get a section in $\Gamma(\widetilde{X}, \pi^* L^{\otimes m})$ by composing projection π; Conversely, briefly speaking, a section defined on a dense subset of $\widetilde{X}$ determines a global one: Note that $\widetilde{X} \backslash E$ is biholomorphic to $X \backslash \{x\}$, then by pulling sections defined on X back we get a section defined on $\widetilde{X} \backslash E$, and Hartogs extension will tell the anwser.

Similarly, we have

$$\Gamma(X, \mathcal{I}_x \otimes L^{\otimes m}) = \Gamma(\widetilde{X}, \mathcal{I}_E \otimes \pi^* L^{\otimes m}) = \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(E)^* \otimes \pi^* L^{\otimes m})$$

And that's why blowing up works, since $\mathcal{I}_x \otimes L^{\otimes m}$ is just a sheaf, and it's a little difficult for us to compute the cohomology of sheaf, but after blowing up, we make it to a line bundle $\mathcal{O}_{\widetilde{X}}(E)^* \otimes \pi^* L^{\otimes m}$, and Dolbeault cohomology comes into its place.

Consider the following short exact sequence of sheaves on $\widetilde{X}$:

$$0 \to \mathcal{I}_E \otimes \pi^* L^{\otimes m} \to \pi^* L^{\otimes m} \to \pi^* L^{\otimes m}|_E \to 0$$

So we get a long exact sequence

$$0 \to \Gamma(\widetilde{X}, \mathcal{I}_E \otimes \pi^* L^{\otimes m}) \to \Gamma(\widetilde{X}, \pi^* L^{\otimes m}) \to \mathbb{C} \to \check{H}^1(\widetilde{X}, \mathcal{I}_E \otimes \pi^* L^{\otimes m}) \to \dots$$

But

$$\check{H}^1(\widetilde{X}, \mathcal{I}_E \otimes \pi^* L^{\otimes m}) = H^1_{Dol}(\widetilde{H}, \mathcal{I}_E \otimes \pi^* L^{\otimes m})$$

Claim $H^1_{Dol}(\widetilde{H}, \mathcal{I}_E \otimes \pi^* L^{\otimes m}) = 0$, when m is sufficiently large. Indeed, note that

$$\mathcal{I}_E \otimes \pi^* L^{\otimes m} \cong K_{\widetilde{X}} \otimes \{\mathcal{O}_{\widetilde{X}}(E)^{*\otimes n} \otimes \pi^*(K_X^* \otimes L^{\otimes m})\}$$

So by Kodaira vanishing, it suffices to show the following line bundle is positive when m is sufficiently large:

$$\mathcal{O}_{\widetilde{X}}(E)^{*\otimes n} \otimes \pi^*(K_X^* \otimes L^{\otimes m})$$

In fact, $K_X^* \otimes L^{\otimes m}$ will be positive on X when m is sufficiently large. But when we pull it back something bad may happen, since $\pi^*(K_X^* \otimes L^{\otimes m})$ is positive except along E. However, $\mathcal{O}_{\widetilde{X}}(E)^{*\otimes n}|_E = \mathcal{O}_{\mathbb{P}^{n-1}}(n)$, so two parts work together to give a positive line bundle. To be more explicit, take any Hermitian metric on $\mathcal{O}_{\widetilde{X}}(E)^{*\otimes n}$ extending (Fubini-Study)$^{\otimes n}$, then its positive on E, but may not be positive otherwise. However we can choose m sufficiently large to offset its negative impact.

This completes the proof of first part of Kodaira embedding, for second and third, arguments are similar, but we need more blowing ups and things become complicated. □

4.4. **Chow's theorem.** Since we already embed a compact complex manifold into projective space, there is no reason for us to avoid Chow's theorem. A wonderful theorem lies in the intersection of algebraic and analytic.

Theorem 4.4.1 (Chow)**.** *Every closed complex submanifold $X \subset \mathbb{P}^n$ is algebraic, i.e. defined by polynomial equations.*

Remark 4.4.2. Finally, for holomorphic line on compact complex manifolds: positive is equivalent to ample.

Although Chow's theorem can be derived from GAGA proved by Serre, in an elegant way using sheaf theory, here we give a sketch of a classical proof of Chow's theorem.

We need to show every closed complex submanifold X of $\mathbb{P}^n$ is algebraic, our ideal is to construct an analytic hypersurface in a Grassmannian manifold $Gr(r,n)$ with Plücker embedding $\varphi : Gr(r,n) \hookrightarrow \mathbb{P}(\bigwedge^r \mathbb{C}^n)$, and use some facts about it:

1. $\mathrm{Pic}(Gr(r,n)) = \mathbb{Z} \cdot \varphi^* \mathcal{O}_{\mathbb{P}}(1)$
2. Every closed analytic hypersurface of $Gr(r,n)$ is algebraic.

If the analytic hypersurface W we construct in Grassmannian manifold can determine X algebraically, that is W is algebraic implies X is, then we complete the proof.

The philosophy here is to convert a submanifold with arbitrary codimension in $\mathbb{P}^n$ to a an hypersurface, the cost we pay is that we need to consider Grassmannian manifold rather than $\mathbb{P}^n$. But it do works!

Let's be more explicit:

Definition 4.4.3 (analytic subset)**.** *A closed analytic subset in $\mathbb{P}^n$ is a closed subset, locally defined by some holomorphic equations.*

Remark 4.4.4. We can replace closed complex submanifold by closed analytic subset in Chow's theorem, since we can not avoid singularity, and it doesn't matter in fact.

However, although we allow singularities, singularities won't be too much: Let $X \subset \mathbb{P}^n$ be an irreducible closed analytic subset of dimension r. Then there exists closed analytic subset $X_{\text{sing}} \subset X$ such that $X \backslash X_{\text{sing}}$ is smooth and dense. Furthermore, $X \backslash X_{\text{sing}}$ is a submanifold of $\mathbb{P}^n$ of dimension r.

Now fix X, an irreducible closed analytic subset of dimension r in $\mathbb{P}^n$. Let $V \in Gr(n-r, n+1)$, then $\mathbb{P}(V) \subset \mathbb{P}^n$ with dimension $n-r-1$. So as you can imagine, an object with dimension r and an object with dimension $n-r-1$ may fail to intersect with each other.

Let

$$W = \{V \in Gr(n-r, n+1) \mid \mathbb{P}(V) \cap X \neq \varnothing\}$$

Claim:

1. W is a closed analytic hypersurface of $Gr(n-r, n+1)$.
2. W determines X algebraically.

Here we give a sketch of proof of Claims:

For (1). Consider the following diagram

$$\begin{tikzcd} \mathbb{P}(E) \arrow[r, "q"] \arrow[d, "p"] & \mathbb{P}^n \supset X \\ Gr(n-r, n+1) & \end{tikzcd}$$

where $E \to Gr(n-r, n+1)$ is tautological bundle of Grassmannian manifold. Then we can write[30]

$$W = p(q^{-1}(X))$$

Since q is holomorphic and X is closed and analytic, then $q^{-1}(X)$ is also closed and analytic. But the difficulty is $p(q^{-1}(X))$ is also closed and analytic, and this holds from the following fact.

Fact 4.4.5. p is holomorphic and proper[31].

Now we show that W is a hypersurface, and that's just a computation for dimension: We already know the dimension of $Gr(n-r, n+1)$ is $(n-r)(n+1-(n-r)) = (n-r)(r+1)$, so we need to show the dimension of W is $(n-r)(r+1)-1$.

First, let's consider the fiber of q: it consists of subspaces $V \subset \mathbb{C}^{n+1}$ of dimension $n-r$ containing a given line l, and that's another Grassmannian manifold $Gr(n-r-1, n)$, if we consider $V \mapsto V/l$, and its dimension is $(n-r-1)(r+1)$. So the dimension of $q^{-1}(X)$ is $r + (n-r-1)(r+1) = (n-r)(r+1) - 1 = \dim Gr(n-r, n+1) - 1$.

So we may desire the property of p is not too bad so that we will obtain $\dim p(q^{-1}(X)) = \dim q^{-1}(X)$ as we desired. It suffices to show that there exists a dense open subset $U \subset q^{-1}(X)$, such that $p|_U$ has finite fibers. In fact, we will show it's one to one correspondence.

Consider fiber of $p_X : q^{-1}(X) \to Gr(n-r, n+1)$ over given $V \in Gr(n-r, n+1)$, and that's $\mathbb{P}(V) \cap X$. So we may desire almost every V such that this intersection is just a point. It suffices to show that the complement of

$$\{(V, x) \in q^{-1}(X) \mid \mathbb{P}(V) \cap X \text{ has only one smooth point } x\}$$

is closed analytic of dimension less than $\dim q^{-1}(X)$. There are three cases:

1. $\mathbb{P}(V) \cap X$ contains $x \in X_{\text{sing}}$, singular locus of X. But $\dim_{\mathbb{C}} q^{-1}(X_{\text{sing}}) < \dim_{\mathbb{C}} q^{-1}(X)$.
2. $\mathbb{P}(V) \cap X$ has at least two points.
3. $\mathbb{P}(V) \cap X$ not transverse intersection at x.

Then $W = p(q^{-1}(X)) \subseteq Gr(n-r, n+1)$ is a closed analytic hypersurface.

For (2). Consider $q_W : p^{-1}(W) \to \mathbb{P}^n$. Claim

$$X = \{x \in \mathbb{P}^n \mid q_W^{-1}(x) = q^{-1}(x)\}$$

[30] Why?

[31] Proper means: For any $Y \subset \mathbb{P}(E)$ closed and analytic, then $p(Y)$ is closed and analytic.

And there are some equivalent descriptions:

$$\begin{aligned} q_W^{-1}(x) = q^{-1}(x) &\Longleftrightarrow p^{-1}(W) \cap q^{-1}(x) = q^{-1}(x) \\ &\Longleftrightarrow q^{-1}(x) \subset p^{-1}(W) \end{aligned}$$

Clearly, if $x \in X$, then $q^{-1}(x) \in p^{-1}(W)$, since $W = p(q^{-1}(X))$. For the other direction: we can translate it as: If $y \notin X$, then we need to find $V \in Gr(n-r, n+1)$ containing $l = \langle y \rangle$, but $\mathbb{P}(V) \cap X = \varnothing$.

To see this: Use projection from y, that is $\mathbb{P}^n \xrightarrow{\pi_y} \mathbb{P}^{n-1}$. Since $y \notin X$, then $\pi_y|_X$ is well-defined and has finite fibers. Note that $\mathbb{P}(V) \cap X \neq \varnothing$ if and only if $\mathbb{P}(V/\langle y \rangle) \cap \pi_y(X) = \varnothing$. From computation before, we know it's a condition for hypersurface. So it's easy to choose V we desire.

Conclusion (from analytic to algebraic): p and q is algebraic, and W is algebraic, so we obtain $p^{-1}(W)$ is also algebraic. So q_W is also algebraic. Thus X is algebraic.

5. Hodge structure

5.1. Basic definitions and examples.

Definition 5.1.1 (Hodge-structure)**.** *A $\mathbb{Z}$-Hodge structure of weight k is*

1. *a finitely generated $\mathbb{Z}$-module $V_{\mathbb{Z}}$;*
2. *a decomposition*

$$V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$.

Remark 5.1.2. We can also define $\mathbb{Q}, \mathbb{R}$-Hodge structure.

Definition 5.1.3 (effective)**.** *A Hodge structure V is called effective if $V^{p,q} = 0$ unless $p, q \geq 0$.*

Definition 5.1.4 (Hodge filtration)**.** *Given a Hodge structure $(V_{\mathbb{Z}}, V^{p,q})$, Hodge filtration $F^pV_{\mathbb{C}}$ is defined by*

$$F^pV_{\mathbb{C}} = \bigoplus_{p' \geq p} V^{p',q}$$

Thus

$$\dots F^{p-1}V_{\mathbb{C}} \supset F^pV_{\mathbb{C}} \supset F^{p+1}V_{\mathbb{C}} \supset \dots$$

Remark 5.1.5. From a Hodge filtration, one can recover Hodge structure as follows:

$$\begin{aligned} V_{\mathbb{C}} &= F^pV_{\mathbb{C}} \oplus \overline{F^{k-p+1}V_{\mathbb{C}}} \qquad (*) \\ V^{p,q} &= F^pV_{\mathbb{C}} \cap \overline{F^pV_{\mathbb{C}}} \end{aligned}$$

So $(V_{\mathbb{Z}}, V^{p,q})$ is equivalent to $(V_{\mathbb{Z}}, F^pV_{\mathbb{C}})$ satisfying $(*)$.

Example 5.1.6. X is a compact Kähler manifold, then $V_{\mathbb{Z}} = H^k(X, \mathbb{Z}), V^{p,q} = H^{p,q}(X)$ is a Hodge structure.

Example 5.1.7. If V, W are two Hodge structure of weight k and l, then

$$V \otimes W, \operatorname{Hom}(V, W), V^* = \operatorname{Hom}(V, \mathbb{Z}), V^{\otimes n}, \operatorname{Sym}^n V, \bigwedge^n V$$

are Hodge structures. And you can check weight of first three Hodge structures are $k+l, l-k, -k$, and the weight of last three Hodge structures are nk.

Example 5.1.8 (Tate)**.** $\mathbb{Z}_{(1)} = 2\pi i\mathbb{Z} \subset \mathbb{C}$ of type $(-1,-1)$ is called Tate Hodge structure. And we can define

$$\begin{cases} \mathbb{Z}_{(n)} = \mathbb{Z}_{(1)}^{\otimes n} \\ \mathbb{Z}_{(0)} = \mathbb{Z} \\ \mathbb{Z}_{(-1)} = \mathbb{Z}_{(1)}^{*} \\ \mathbb{Z}_{(-n)} = \mathbb{Z}_{(-1)}^{\otimes n} \end{cases}$$

Example 5.1.9 (Tate twist)**.** Given a Hodge structure V, its Tate twist is defined by

$$V(n) := V \otimes \mathbb{Z}_{(n)}$$

Now let's define morphism of Hodge structure. However, we can only define morphism between Hodge structures such that the difference of their weight is even.

Let $(V_{\mathbb{Z}}, V^{p,q})$ and $(W_{\mathbb{Z}}, W^{p,q})$ be two Hodge structure of weight k and $k+2r$, where $r \in \mathbb{Z}$.

Definition 5.1.10 (morphism of Hodge structure)**.** $\phi : V_{\mathbb{Z}} \to W_{\mathbb{Z}}$ *is a morphism of abelian groups. It's called a morphism of Hodge structure of type* (r,r)*, if* $\phi_{\mathbb{C}}$ *satisfies*

$$\phi_{\mathbb{C}}(V^{p,q}) \subset W^{p+r,q+r} \iff \phi_{\mathbb{C}}(F^pV_{\mathbb{C}}) \subset F^{p+r}W_{\mathbb{C}}$$

Lemma 5.1.11 (strictness)**.** *If* $\phi : V \to W$ *is a morphism of Hodge structure of type* (r,r)*. Then* ϕ *is strict with the Hodge filtration, i.e.*

$$\operatorname{im}\phi \cap F^{p+r}W_{\mathbb{C}} = \phi(F^pV_{\mathbb{C}})$$

Proof. If $\alpha = \phi(\beta) \in F^{p+r}W_{\mathbb{C}}$. We write $\beta = \sum_{p'+q=k} \beta^{p',q}$. Then

$$\alpha = \sum_{p'+q=k} \phi(\beta^{p',q}), \quad \phi(\beta^{p',q}) \in F^{p'+r}W_{\mathbb{C}}$$

Since $\alpha \in F^{p+r}W_{\mathbb{C}}$, then $\phi(\beta^{p',q}) = 0$ for $p' < p$. Thus

$$\alpha = \sum_{p' \geq p} \phi(\beta^{p',q}) = \phi(\sum_{p' \geq p} \beta^{p',q}) \in \phi(F^pV_{\mathbb{C}})$$

□

Remark 5.1.12. Strictness implies $\operatorname{im}\phi$ has a canonical Hodge structure of weight $k+2r$.

Example 5.1.13. Given a morphism ϕ of Hodge structure, then $\ker\phi, \operatorname{im}\phi, \operatorname{coker}\phi$ are Hodge structure.

Remark 5.1.14. Thus we obtain an abelian category: Objects are Hodge structure of given weight k, and morphisms are morphisms of Hodge structure of type $(0,0)$.

Example 5.1.15. $f : X \to Y$ is a holomorphic map between compact Kähler manifold, then $f^* : H^k(Y,\mathbb{Z}) \to H^k(X,\mathbb{Z})$ is a morphism of Hodge structure of type $(0,0)$.

Example 5.1.16 (Gysin pushforward)**.** If X, Y are two compact Kähler manifold such that $\dim Y = \dim X + r$, then

$$\begin{tikzcd} H^k(X,\mathbb{Z}) \arrow[r] \arrow[d] & H^{k+2r}(Y,\mathbb{Z}) \\ H_{2\dim X-k}(X,\mathbb{Z}) \arrow[r, "f_*"] & H_{2\dim X-k}(Y,\mathbb{Z}) \arrow[u] \end{tikzcd}$$

We obtain Gysin pushforward f_* as above, a morphism of Hodge structure of type (r,r).

5.2. **Polarization.**

Definition 5.2.1 (polarized)**.** *A polarized $\mathbb{Z}$-Hodge structure of weight k is $\mathbb{Z}$-Hodge structure $(V_{\mathbb{Z}}, F^pV_{\mathbb{C}})$ of weight k together with a morphism of Hodge structure $Q : V \otimes V \to \mathbb{Z}$ of type $(-k,-k)$ such that*

$$\begin{aligned} H : V_{\mathbb{C}} \otimes V_{\mathbb{C}} &\to \mathbb{C} \\ (\alpha,\beta) &\mapsto Q(C\alpha, \overline{\beta}) \end{aligned}$$

is a positive Hermitian form.

Remark 5.2.2. *H is Hermitian implies Q is*

$$\begin{cases} \textit{symmetric}, & k \textit{ is even} \\ \textit{anti-symmetric}, & k \textit{ is odd} \end{cases}$$

Example 5.2.3. X is a compact complex manifold of dimension n, with a positive holomorphic line bundle $L \to X$. Then we have $L : H^k(X.\mathbb{Z}) \to H^{k+2}(X,\mathbb{Z})$, and

$$H^k(X,\mathbb{Z})_{\text{prim}} = \ker L^{n-k+1}$$

has a Hodge of weight k. Then

$$\begin{aligned} Q : H^k(X,\mathbb{Z})_{\text{prim}} \otimes H^k(X,\mathbb{Z})_{\text{prim}} &\to \mathbb{Z} \\ (\alpha,\beta) &\mapsto \varepsilon(k)\int_X L^{n-k}\alpha\wedge\beta \end{aligned}$$

Remark 5.2.4. About the definition of a polarization

1. $Q : V \otimes V \to \mathbb{Z}$ is a morphism of Hodge structure is equivalent to (1) of Hodge-Riemann bilinear relations.

2. $H : V_{\mathbb{C}} \otimes V_{\mathbb{C}} \to \mathbb{C}$ is a positive Hermitian form is equivalent to (2) of Hodge-Riemann bilinear relations.

Lemma 5.2.5 (semisimplity)**.** *Let W be a $\mathbb{Q}$-Hodge structure, $V \subset W$ a $\mathbb{Q}$-sub Hodge structure, if W is polarized, then so is V. Furthermore, there is a decomposition of polarized Hodge structure*

$$W = V \oplus V'$$

Proof. Set $V'_{\mathbb{Q}}$ to be the orthonormal complement of $V_{\mathbb{Q}}$ with respect to Q, where Q is the polarization form of W. Check

$$V'^{p,q} = (V^{p,q})^{\perp H}$$

$\square$

Remark 5.2.6. The category of polarized $\mathbb{Q}$-Hodge structure of a given weight k is abelian and semisimple.

5.3. **Weight one Hodge structure.** Now let's consider some examples which are not so trivial.

5.3.1. *Picard variety.* Let X be a compact Kähler manifold, recall the exponential sequence

$$0 \to \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1$$

from this we can obtain a long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(X,\mathbb{Z}) \longrightarrow H^0(X,\mathcal{O}_X) \longrightarrow H^0(X,\mathcal{O}_X^*) \\ \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^*) \\ \xrightarrow{c_1} H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O}_X) \longrightarrow H^2(X,\mathcal{O}_X^*) \longrightarrow \end{aligned}$$

For the first row, in fact it's the following sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{\exp} \mathbb{C}^*$$

So $H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X)$ is injective. So we obtain such a short exact sequence

$$0 \to H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} \ker\{H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)\} \to 0$$

Furthermore, by definition we have $H^q(X,\mathcal{O}_X) = H^{0,q}(X)$. So from this short exact sequence we can see

1. $\mathrm{Pic}(X) \xrightarrow{c_1} \mathrm{NS}$[32]$(X) = H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ is surjective. This is a version of the Lefschetz $(1,1)$-theorem.

[32]Néron-Severi group of X

2. $\rho := \operatorname{rank} \mathrm{NS}(X)$, called Picard rank, then for a compact Kähler manifold, we have $0 \leq \rho \leq h^{1,1}(X)$. ρ can be zero since there exists a compact Kähler manifold such that is no holomorphic line bundle with non-trivial first Chern class. In particular, if X is projective, then $\rho \geq 1$, since first Chern class of its positive line bundle is not trivial.
3. $\mathrm{Pic}^0(X) := \ker c_1 = H^{0,1}(X)/H^1(X,\mathbb{Z})$, called Picard variety of X. It's a complex torus. Indeed, note that there is a natural inclusion $H^1(X,\mathbb{Z}) \to H^1(X,\mathbb{C})$, since by universal coefficient theorem we have $H^1(X,\mathbb{Z})$ is torsion-free. And the map $H^1(X,\mathbb{Z}) \to H^{0,1}(X)$ is the composition of following maps
$$H^1(X,\mathbb{Z}) \hookrightarrow H^1(X,\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X) \to H^{0,1}(X)$$
Since $\overline{H^1(X,\mathbb{Z})} = H^1(X,\mathbb{Z})$, then $H^1(X,\mathbb{Z}) \cap H^{1,0}(X) = 0$. Thus we obtain $H^1(X,\mathbb{Z}) \hookrightarrow H^{0,1}(X)$, and clealy it's a lattice.

So in a summary, we have the following exact sequence
$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \xrightarrow{c_1} \mathrm{NS}(X) \to 0$$

Example 5.3.1. Let X be a compact complex curve, then $\mathrm{NS}(X) = \mathbb{Z}$, and its Picard variety is sometimes denoted by $J(X)$, called Jacobian of X.

5.3.2. *Albanese variety.* Now let's introduce Albanese variety: Let X be a compact Kähler manifold of dimension n. Consider
$$H^{2n-1}(X,\mathbb{Z}) \to H^{2n-1}(X,\mathbb{C}) = H^{n,n-1}(X) \oplus H^{n-1,n}(X) \to H^{n-1,n}(X)$$
Then Albanese varieties is defined by
$$\mathrm{Alb}(X) := H^{n-1,n}(X)/\operatorname{im} H^{2n-1}(X,\mathbb{Z})$$
Note that $H^{2n-1}(X,\mathbb{Z})$ may not be torsion-free, so we need to consider the image of the map, and it's also a complex torus.

By Poincaré duality and Serre duality, we have
$$\begin{aligned} H^{2n-1}(X,\mathbb{Z}) &\cong H_1(X,\mathbb{Z}) \\ H^{n-1,n}(X) &\cong H^{1,0}(X)^* = H^0(X,\Omega_X^1)^* \end{aligned}$$
Then
$$\mathrm{Alb}(X) = H^0(X,\Omega^1)^*/\operatorname{im} H_1(X,\mathbb{Z})$$
The advantage of this expression is we can give a quite geometrical description
$$\begin{aligned} H_1(X,\mathbb{Z}) &\to H^0(X,\Omega_X^1)^* \\ [\gamma] &\mapsto (\omega \mapsto \int_\gamma \omega) \end{aligned}$$

Exercise 5.3.2. Show that for X compact Kähler, then $\omega \mapsto \int_\gamma \omega$ only depends on the homology class $[\gamma]$.

Remark 5.3.3. There is a duality between Picard variety and Albanese variety.

$$\mathrm{Pic}^0(X) \overset{\text{dual}}{\longleftrightarrow} \mathrm{Alb}(X)$$
$$H^1(X,\mathbb{Z}) \longleftrightarrow H_1(X,\mathbb{Z})/\text{torsion}$$
$$H^{0,1}(X) \subset H^1(X,\mathbb{C}) \longleftrightarrow H^{1,0}(X)^* \subset H^1(X,\mathbb{C})^*$$

Now you may wonder why we need Albanese variety, since it's just a duality of Picard variety. In fact, we will have more geometrical tool to deal with Albanese variety.

Choose a base point $x_0 \in X$, then

Definition 5.3.4 (Albanese map)**.** *Albanese map is defined as follows*

$$a : X \to \mathrm{Alb}(X)$$
$$x \mapsto (\omega \mapsto \int_{x_0}^{x} \omega)$$

Exercise 5.3.5. Show that Albanese map is holomorphic.

Proposition 5.3.6 (universal property)**.** *For all $\varphi : X \to A$, a holomorphic map from X to a complex torus A such that $\varphi(x_0) = 0_A$. Then there exists a unique holomorphic map of complex tori g such that the following diagram commutes*

$$\begin{tikzcd} X \arrow[rr, "a"] \arrow[dr, "\varphi"'] & & \mathrm{Alb}(X) \arrow[dl, dashed, "g"] \\ & A & \end{tikzcd}$$

Before proof of this universal property, let's recall some property of a complex torus: For a complex torus $A = V/\Lambda$, you can identify $V = T_{A,0_A}$, the tangent space at 0_A, and it's $(\Omega^1_{A,0_A})^*$ by definition. Furthermore, every global 1-form is obtained from $\Omega^1_{A,0_A}$ by moving parallel, so $(\Omega^1_{A,0_A})^* = H^0(A,\Omega^1_A)^*$, and $\Lambda = H_1(A,\mathbb{Z})$. In particular, then $\mathrm{Alb}(A) = A$.

Proof. Existence of g. If we write $\mathrm{Alb}(X)$ as $H^0(X,\Omega^1_X)^*/\operatorname{im} H_1(X,\mathbb{Z})$ and $A = H^0(A,\Omega^1_A)^*/H_1(A,\mathbb{Z})$, then the existence of a map g from $\mathrm{Alb}(X)$ to A is equivalent to the existence $g_* : H_1(X,\mathbb{Z}) \to H_1(A,\mathbb{Z})$ such that

$$g_* \otimes_{\mathbb{Z}} \mathbb{R} : H^0(X,\Omega^1_A)^* \to H^0(A,\Omega^1_A)^*$$

is $\mathbb{C}$-linear. Take $g_* = \varphi_* : H_1(X,\mathbb{Z}) \to H_1(A,\mathbb{Z})$. To see that $\varphi = g \circ a$, we have $\varphi_* = g_*$, by dualizing we will obtain

$$\varphi^* = (g \circ a)^* : H^0(A,\Omega^1_A) \to H^0(X,\Omega^1_X)$$

Thus φ and $g \circ a$ have the same differential for any $x \in X$, and $\varphi(x_0) = g \circ a(x_0) = 0_A$. Thus $\varphi = g \circ a$ in a neighborhood U of x_0. Consider the following closed analytic

$$\{x \in X \mid \varphi(x) = g \circ a(x)\}$$

which contains an open subset U. Thus it must be the whole space, that is $\varphi = g \circ a$.

Uniqueness. Consider

$$a^k : X^k \to \mathrm{Alb}(X)$$
$$(x_1, \ldots, x_k) \mapsto \sum a(x_i)$$

Let's prove a^k is surjective, when k is sufficiently large. It suffices to show that a^k is submersion at some point $(x_1, \ldots, x_k) \in X^k$, since a^k is proper, then $\operatorname{im} a^k$ is a closed analytic and contains an open subset. Equivalently,

$$(a^k)^* : H^0(\mathrm{Alb}(X), \Omega^1_{\mathrm{Alb}(X)}) \to H^0(X^k, \Omega^1_{X^k}) \to \Omega^1_{X^k,(x_1,\ldots,x_k)}$$

is injective. Note that $H^0(\mathrm{Alb}(X), \Omega^1_{\mathrm{Alb}(X)}) = H^0(X, \Omega^1_X)$ and $\Omega^1_{X^k,(x_1,\ldots,x_k)} = \Omega^1_{X,x_1} \oplus \cdots \oplus \Omega^1_{X,x_k}$. Since $H^0(X, \Omega^1_X)$ is finite dimensional, so it's possible to distinguish different differential forms by considering its value in sufficiently large points. □

Remark 5.3.7. In fact, we have proven the following equivalent category

$$\{\text{torsion-free effective } \mathbb{Z}\text{-Hodge structure of weight } 1\}\} \Longleftrightarrow \{\text{complex tori}\}$$

More explictly, For a torsion-free effective $\mathbb{Z}$-Hodge structure $(V_{\mathbb{Z}}, V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1})$, then we obtain a complex tori $V^{0,1}/V_{\mathbb{Z}}$; Conversely, given a complex tori A, then we obtain

$$V_{\mathbb{Z}} = H^1(\mathrm{Pic}^0(A), \mathbb{Z})$$

Let $L \to X$ be a positive holomorphic line bundle. Consider $H^1(X, \mathbb{Z})$ is primitive. Then there exists

$$Q : H^1(X, \mathbb{Z}) \times H^1(X, \mathbb{Z}) \to \mathbb{Z}$$
$$(\alpha, \beta) \mapsto \int L^{n-1} \alpha \wedge \beta$$

Consider the following identification

$$H^1(X, \mathbb{Z}) \cong H_1(\mathrm{Pic}^0(X), \mathbb{Z})$$

Then

$$Q \in \bigwedge^2 (H^1(X, \mathbb{Z})^*) \cong \bigwedge^2 (H_1(\mathrm{Pic}^0(X), \mathbb{Z})^*) = \bigwedge^2 H^1(\mathrm{Pic}^0(X), \mathbb{Z}) \cong H^2(\mathrm{Pic}^0(X), \mathbb{Z})$$

After tensoring $\mathbb{R}$, we have

$$H^1(X, \mathbb{R}) \cong H_1(\mathrm{Pic}^0(X), \mathbb{R}) \cong T_{\mathrm{Pic}^0(X),\mathbb{R},0} \quad \text{real tangent space at } 0$$

Thus we obtain a constant real (in fact, it's integral) 2-form ω on $\mathrm{Pic}^0(X)$.

Proposition 5.3.8. *ω is a Kähler form.*

Proof. Hint:

1. ω is $(1,1)$-form. Indeed, $\omega = 0$ on $T^{1,0}_{\mathrm{Pic}^0(X),0}$ since $Q = 0$ on $H^{1,0}(X)$.

2. ω is positive is equivalent to positivity of H on $H^{1,0}(X)$.

□

Corollary 5.3.9. *X is projective, then* $\mathrm{Pic}^0(X)$ *is projective.*

Proof. A positive holomorphic line bundle on X will induce a positive holomorphic line bundle on $\mathrm{Pic}^0(X)$. □

Definition 5.3.10 (abelian variety)**.** *A projective complex torus is called a complex abelian variety.*

Remark 5.3.11. In fact, we have proven the following equivalent category

$$\{\text{polarized torsion-free effective } \mathbb{Z}\text{-Hodge structure of weight } 1\}\} \Longleftrightarrow \{\text{complex abelian variety}\}$$

Part 3. Appendix

A. Appendix for Sheaf and its Cohomology

A.1. **Sheafification.** In this section we will consider sheafification.

Recall Example 1.2.2, we encounter a presheaf which is not a sheaf. So we may wonder how can we get a sheaf from this presheaf? And that's sheafification.

There are too many ways to define sheafification. One way is to define by its universal property:

Definition A.1.1 (sheafification)**.** *Given a presheaf $\mathscr{F}$ there is a sheaf $\mathscr{F}^+$ and a morphism $\theta : \mathscr{F} \to \mathscr{F}^+$ with the property that for any sheaf $\mathscr{G}$ and any morphism $\varphi : \mathscr{F} \to \mathscr{G}$ there is a unique morphism $\overline{\varphi} : \mathscr{F}^+ \to \mathscr{G}$ such that $\varphi = \overline{\varphi} \circ \theta$. In other words, the following diagram commutes:*

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\varphi"] \arrow[d, "\theta"] & \mathscr{G} \\
\mathscr{F}^+ \arrow[ur, "\overline{\varphi}"] &
\end{tikzcd}$$

Although the universal property shows that if the sheafification exists, it's determined uniquely up to unique isomorphism, how can we show that there do exists a sheafification?

To give an explict construction, we need to consider stalks of a presheaf.

Definition A.1.2 (stalks)**.** *For a presheaf $\mathscr{F}$ and $x \in X$, stalk at x is defined as*

$$\mathscr{F}_x = \varinjlim_{x \in U} \mathscr{F}(U)$$

Remark A.1.3 (alternative definition)**.** In order to avoid language of inverse limit, we give a more useful and equivalent description of stalk: an element $s_x \in \mathscr{F}_x$, which is called a germ, is an equivalence class $[s_U]$, where $s_U \in \mathscr{F}(U)$ and $x \in U$. Two such sections s_U and s_V are considered equivalent if the restrictions of the two sections coincide on some neighborhood of x. For $s \in \mathscr{F}(U), x \in U$, we use $s|_x$ to denote its equivalence class.

Remark A.1.4 (morphisms on stalks)**.** Given a morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$, it induces a morphism of abelian groups $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ as follows:

$$\begin{aligned} \varphi_p : \mathscr{F}_p &\to \mathscr{G}_p \\ s_p &\mapsto \varphi(s)|_p \end{aligned}$$

and it's easy to check φ_p is well-defined.

As you can imagine, stalks are quite local information, and the difference between sheaf and presheaf is that whether a local information can glue together uniquely or not. So stalks of presheaf and its sheafification should be the same. And one way to construct sheafification is to glue stalks together in a suitable way.

Construct $\mathscr{F}^+(U)$ as a set of functions

$$f : U \to \coprod_{p\in U} \mathscr{F}_p$$

such that $f(p) \in \mathscr{F}_p$ and for every $p \in U$ there is an open set $p \in V_p \subseteq U$ and $t \in \mathscr{F}(V_p)$ such that for all $q \in V_p$ we have the germ $t|_q = f(q)$.

$\mathscr{F}^+$ is a sheaf. Indeed:

1. Let U be an open set, $\{V_i\}$ an open covering of U, and $s \in \mathscr{F}^+(U)$ such that $s|_{V_i} = 0$ for all i, then s must be zero: It suffices to show $s(p) = 0$ for all $p \in U$. Take any $p \in U$, then there exists an open set V_i contains p, hence $s(p) = s|_{V_i}(p) = 0$;
2. Suppose for each i, we have $s_i \in \mathscr{F}^+(V_i)$ such that
$$s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j}$$
We can construct $s \in \mathscr{F}^+(U)$ such that $s|_{V_i} = s_i$ directly: take any $p \in U$ and V_i containing p, define $s(p) = s_i(p)$. This is well-defined since s_i agree on the intersections. All is left to check is that s satisfies the requirements of the sheafification. The first condition is trivial. For the second one, just consider that you can apply the condition to s_i, and this will give you an open neighborhood W_i contained in V_i and containing p, with $t_i \in \mathscr{F}(W_i)$ as above. Since W_i is open in V_i, which is open in U, so W_i is suitable also for the function s we have just defined.

Remark A.1.5. From this construction, you can see the stalk of $\mathscr{F}^+$ at p is exactly $\mathscr{F}_p$. Check it by definition.

Now let's define the canonical morphism $\theta : \mathscr{F} \to \mathscr{F}^+$ as follows: For open $U \subseteq X$, and $s \in \mathscr{F}(U)$, define

$$\begin{aligned}\theta(s) : U &\to \coprod_{p\in U} \mathscr{F}_p \\ p &\mapsto s|_p\end{aligned}$$

Note that if $\mathscr{F}$ is already a sheaf, we desire canonical morphism θ is an isomorphism. Indeed, if $s_p = 0$ for all $p \in U$, so there exists an open covering $\{V_i\}$ of U such that $s|_{V_i} = 0$, by axioms of sheaf we obtain $s = 0$, this is injectivity; For surjectivity: take $f \in \mathscr{F}^+(U)$. Since for each $p \in U$ there exists $p \in V_p \subseteq U$ and $t \in \mathscr{F}(V_p)$ such that $f(p) = t|_p$, then glue these t together to get our desired s such that $\theta(s) = f$.

Finally let's construct $\overline{\varphi}$: A map of presheaves $\varphi : \mathscr{F} \to \mathscr{G}$ induces a map on stalks

$$\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$$

Thus for $f \in \mathscr{F}^+(U)$, we can compose f with the map

$$\coprod_{p\in U} \varphi_p : \coprod_{p\in U} \mathscr{F}_p \to \coprod_{p\in U} \mathscr{G}_p$$

to get a map $U \to \coprod_{p\in U} \mathscr{G}_p$. Thus we get a morphism $\widetilde{\varphi} : \mathscr{F}^+ \to \mathscr{G}^+$. Indeed, $\widetilde{\varphi}(f)(p) \in \mathscr{G}_p$, since $f(p) \in \mathscr{F}_p$ and $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$; If for all $q \in V_p$ we have $t|_q = f(q)$, then

$$\widetilde{\varphi}(f)(q) = \varphi_q(f(q)) = \varphi_q(t|_q) = \varphi(t)|_q$$

So $\widetilde{\varphi}(f) \in \mathscr{G}^+$. Since $\mathscr{G}$ is assumed to be sheaf, then canonical morphism $\theta' : \mathscr{G} \to \mathscr{G}^+$ is isomorphic, so we obtain $\overline{\varphi} := \theta'^{-1} \circ \widetilde{\varphi}$. Now let's show $\varphi = \overline{\varphi} \circ \theta = \theta'^{-1} \circ \widetilde{\varphi} \circ \theta$. It's suffice to show they coincide on each stalk since both $\mathscr{F}^+$ and $\mathscr{G}$ are sheaves, and it's quite easy to see this, since $\varphi_p = \theta_p'^{-1} \circ \widetilde{\varphi}_p \circ \theta_p$. Furthermore, uniqueness follows from the fact that $\overline{\varphi}_p$ is uniquely determined by φ_p.

Remark A.1.6. We can describe sheafification in a more fancy language: Since we have sheaf of abelian groups on X as a category, denote it by $\underline{Ab}_X$, and presheaf is a full subcategory of $\underline{Ab}_X$, there is a natural inclusion functor ι from category of sheaf to category of presheaf. Then sheafification is the adjoint functor of ι.

A.2. **More examples on sheaves.**

Example A.2.1 (constant sheaf)**.** Let G be abelian group, the associated constant sheaf $\underline{G}$ is the sheafication of the presheaf

$$U \mapsto G$$

Use the construction of sheafification, we can write $\underline{G}$ more explicitly as

$$\underline{G}(U) = \{\text{locally constant function } f : U \to G\}$$

Example A.2.2 (ringed space)**.** A ringed space is the data of space + functions. For different spaces, we can define different functions:

1. Let X be a topological space, then $\mathscr{C}_X$ is defined by: For any open subset U, we define
$$\mathscr{C}_X(U) := \{\text{continous functions } f : U \to \mathbb{R}\}$$
2. Let M be a smooth manifold, then C^∞_M is defined by: For any open subset U, we define
$$C^\infty_M(U) := \{\text{smooth functions } f : U \to \mathbb{R}\}$$
3. Let X be a complex manifold, then $\mathcal{O}_X$ is defined by: For any open subset U, we define
$$\mathcal{O}_X(U) := \{\text{holomorphic functions} f : U \to \mathbb{C}\}$$
4. Let X be an algebraic variety, then $\mathcal{O}_X$ is defined by: For any open subset U, we define
$$\mathcal{O}_X(U) := \{\text{regular functions on } U\}$$

Example A.2.3 (Sheaf of modules on a ringed space)**.** Let $(X, \mathcal{O}_X)$ be a a ringed space. A sheaf of $\mathcal{O}_X$-module is a sheaf $\mathscr{M}$ such that for any open $U \subseteq X$, $\mathscr{M}(U)$ is an $\mathcal{O}_X(U)$-module and the module structure is compatible with the restriction.

Example A.2.4. Let $(X, \mathcal{O}_X)$ be an algebraic variety, then we have (quasi) coherent sheaves of $\mathcal{O}_X$-modules.

Example A.2.5. Let $(X, \mathcal{O}_X)$ be a complex manifold and let $\pi : E \to M$ be a holomorphic vector bundle. Then by Example 1.2.3 we know that E defines a sheaf. In fact E is a sheaf of $\mathcal{O}_X$-modules.

A.3. **Exact sequence of sheaf.** We can consider the kernel and cokernel if we already have a morphism of objects. So it's natural to define similar conceptions for morphisms of sheaves.

Given a morphism $\varphi : \mathscr{F} \to \mathscr{G}$ between sheaves of abelian groups, we define its kernel $\ker\varphi$ as a presheaf by assigning each open subset U the abelian group $\ker\varphi(U)$, since $\varphi(U)$ is a morphism of abelian groups.

Similarly, we can define its image or cokernel as a presheaf by assigning each open subset U the abelian group $\operatorname{im}\varphi(U)$ or $\operatorname{coker}\varphi(U)$.

It's natural to ask whether the kernel, image or cokernel of morphism φ between sheaves are still sheaves or not? Unfortunately, only kernel of φ is still a sheaf, its image or cokernel may fail to be sheaf in our definition.

Why kernel of a morphism $\varphi : \mathscr{F} \to \mathscr{G}$ is still a sheaf? Let's check by definition: Take $s \in \ker\varphi(U)$, and take an open covering $\{V_i\}_{i\in I}$ of U. Then $s|_{V_i} = 0$ must imply $s = 0$ since s is also in $\mathscr{F}(U)$; If there exists $s_i \in \ker\varphi(V_i)$ such that $s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j}$, so they glue together to get $s \in \mathscr{F}(U)$, and we need to check $\varphi(U)(s) = 0$. But we have

$$\varphi(U)(s)|_{V_i} = \varphi(V_i)(s|_{V_i}) = \varphi(V_i)(s_i) = 0$$

Then we obtain $\varphi(U)(s) = 0$.

But image of morphism may not be a sheaf: Although we can prove the first requirement in a same way, for the second something bad happens. If there exists $s_i \in \operatorname{im}\varphi(V_i)$, and we can glue them together to get a $s \in \mathscr{G}(U)$, but s may not be the image of some $t \in \mathscr{F}(U)$. The cokernel fails to be a sheaf for the same reason.

So we may change our definition about image and cokernel: To define the image and cokernel of a morphism to be the sheafification of our previous definition.

For a sequence of sheaves:

$$\cdots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^{i} \xrightarrow{\varphi^{i}} \mathscr{F}^{i+1} \to \cdots$$

It's called exact at $\mathscr{F}^i$, if $\ker\varphi^i = \operatorname{im}\varphi^{i-1}$. If a sequence is exact at everywhere, then it's an exact sequence of sheaves.

However, there is a better description of exactness of sequence of sheaves, that is looking its stalks:

Proposition A.3.1. *The sequence of sheaves*

$$\cdots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^{i} \xrightarrow{\varphi^{i}} \mathscr{F}^{i+1} \to \cdots$$

is exact if and only if the sequence of abelian groups are exact

$$\cdots \to \mathscr{F}^{i-1}_x \xrightarrow{\varphi^{i-1}_x} \mathscr{F}^{i}_x \xrightarrow{\varphi^{i}_x} \mathscr{F}^{i+1}_x \to \cdots$$

for all $x \in X$.

Proof. It suffices to show for any morphism $\varphi : \mathscr{F} \to \mathscr{G}$, we have $(\ker\varphi)_p = \ker\varphi_p, (\operatorname{im}\varphi)_p = \operatorname{im}\varphi_p$. Let's fix $p \in X$ and check by definition.

For (1). It's clear $(\ker\varphi)_p \subseteq \ker\varphi_p$; Conversely, take $s_p \in \ker\varphi_p$, then $\varphi_p(s_p) = 0 \in \mathscr{G}_p$. In other words, there exists an open set U containing p and $s \in \mathscr{F}(U)$ such that $s|_p = s_p$ and $\varphi(s)|_p = 0$, which implies there is an open set V containing p such that $\varphi(s)|_V = 0$. Hence $\varphi(s|_V) = 0$, that is $s|_V \in \ker\varphi(V)$. Thus $s_p = (s|_V)|_p \in (\ker\varphi)_p$.

For (2). It's clear $(\operatorname{im}\varphi)_p \subseteq \operatorname{im}\varphi_p$, since $(\operatorname{im}\varphi)_p$ is the same stalk of the presheaf of image before sheafification; Conversely, if $s_p \in \operatorname{im}\varphi_p$, we have some $t_p \in \mathscr{F}_p$ such that $\varphi_p(t_p) = s_p$. Suppose $t \in \mathscr{F}(U)$ is a section of some open set U containing p such that $t|_p = t_p$. Then $\varphi(t)|_p = \varphi_p(t_p) = s_p$, so s_p is in the stalk of the image presheaf at p. But the stalk at a point remains the same after sheafification, we have $s_p \in (\operatorname{im}\varphi)_p$. □

Remark A.3.2. The proof for the first part is a routine, but the proof for the half part shows the hallmark of sheafification: Stalks are not changed!.

Now let's consider a special exact sequence: short exact sequence:

$$0 \to \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{H} \to 0$$

In this case, φ is called injective and ψ is called surjective. Attention: For any open subset $U \subseteq X$, we will have

$$\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$$

is injective. Indeed, by definition we have for any open subset $U \subseteq X$, $\ker\varphi(U) = 0$, that is injectivity. Or from another point of view, for each $p \in U$, we have

$$\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$$

is injective. That is $\ker\varphi_p = 0$. So we obtain $(\ker\varphi(U))_p = 0$ for all $p \in U$. But for a section $s \in \mathscr{F}(U)$ if we have $s|_p = 0$, then we must have $s = 0$. So we obtain $\ker\varphi(U) = 0$.

The second pointview may be a little talk nonsense, but we will see it can explain why $\psi(U) : \mathscr{G}(U) \to \mathscr{H}(U)$ may not be surjective in general. Since from

$$\psi_p : \mathscr{G}_p \to \mathscr{H}_p$$

is surjective we can only get "locally surjective", and from locally surjectivity you may not get a global one. The reason for why does image fail to be a sheaf appears again.

Example A.3.3 (exponential sequence)**.** Let X be a complex manifold and $\mathcal{O}_X$ is its holomorphic function sheaf. Then

$$0 \to 2\pi i\underline{\mathbb{Z}} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

is an exact sequence of sheaves, called exponential sequence.

Proof. The difficulty is to show exp is surjective on stalks at $p \in X$. That is we need to construct logarithms of functions $g \in \mathcal{O}_X^*(U)$ for U, a neighborhood of p. We may choose U is simply-connected, then define

$$\log g(q) = \log g(p) + \int_{\gamma_q} \frac{\mathrm{d}g}{g}$$

for $q \in U$, where γ_q is a path from p to q in U, and our definition is independent of the choice of γ_q since U is simply-connected.

In fact, U is simply-connected is crucial for constructing logarithm. If we consider $X = \mathbb{C}$ and $U = \mathbb{C}\backslash\{0\}$, then

$$\exp : \mathcal{O}_X(U) \to \mathcal{O}_X^*(U)$$

won't be surjective. □

A.4. **Derived functor formulation of sheaf cohomology.** The category $\underline{Ab}_X$: sheaves of abelian groups on X. In this section we will introduce sheaf cohomology by considering it as a derived functor.

For an exact sequence of sheaf:

$$0 \to \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}''$$

If we take its section on U, we get a sequence of abelian groups

$$0 \to \mathscr{F}'(U) \xrightarrow{\phi(U)} \mathscr{F}(U) \xrightarrow{\psi(U)} \mathscr{F}''(U)$$

We already know this sequence is still exact at $\mathscr{F}'(U)$, now let's it's still exact at $\mathscr{F}(U)$, that is

$$\ker\psi(U) = \operatorname{im}\phi(U)$$

Let's first show $\ker\psi(U) \supseteq \operatorname{im}\phi(U)$. Take $s \in \mathscr{F}'(U)$, and we want to show $\psi\phi(s) = 0$. It suffices to show $\psi\phi(s)|_p = 0$ for all $p \in U$, since $\mathscr{F}''$ is a sheaf. For any $p \in U$, consider its stalk we obtain an exact sequence of abelian groups

$$0 \to \mathscr{F}'_p \xrightarrow{\phi_p} \mathscr{F}_p \xrightarrow{\psi_p} \mathscr{F}''_p$$

then we obtain $\psi_p\phi_p(s|_p) = 0$, that is $\psi\phi(s)|_p$.

On the other hand. Take $s \in \ker\psi(U)$, then for any $p \in U$ we have $s|_p \in \ker\psi_p$. By exactness of stalks, there exists $t_p \in \mathscr{F}'_p$ such that $\phi_p(t_p) = s|_p$. So there exists an open subset V_i containing p and $t_i \in \mathscr{F}'(V_i)$ such that $\phi(t_i) = s|_{V_i}$. We claim that these t_i can be glued together to obtain $t \in \mathscr{F}(U)$. Since $\mathscr{F}$ is a sheaf, it suffices to check these t_i agree on intersections $V_i \cap V_j$. This follows from the injectivity of ϕ, since $\phi(t_i - t_j|_{V_i \cap V_i}) = s|_{V_i \cap V_j} - s|_{V_i \cap V_j} = 0$.

Remark A.4.1. From above argument, we can see that

$$0 \to \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}''$$

is exact if and only if for any open subset $U \subseteq X$

$$0 \to \mathscr{F}'(U) \xrightarrow{\phi(U)} \mathscr{F}(U) \xrightarrow{\psi(U)} \mathscr{F}''(U)$$

is exact.

In homological algebra, we can consider the derived functor of a left or right-exact functor. Here as we can see, take sections (in particular, global sections) of a sheaf is a left exact functor.

So, as what we did in homological, we need choose a injective resolution and consider the cohomology of the sequence of its global sections to define the sheaf cohomology.

Definition A.4.2 (injective)**.** *A sheaf $\mathcal{I}$ is injective if* $\mathrm{Hom}(-, \mathcal{I})$ *is an exact functor.*

Fact A.4.3. $\underline{Ab}_X$ is an abelian category with enough injectives. Namely, every sheaf $\mathscr{F}$ can be realized as a subsheaf of some injective sheaf.

Definition A.4.4 (injective resolution)**.** *Let $\mathscr{F}$ be a sheaf, an injective resolution of $\mathscr{F}$ is an exact sequence*

$$0 \to \mathscr{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \dots$$

where $\mathcal{I}^i, i = 0, 1, 2, \dots$ are injective.

Fact A.4.5. Every sheaf admits an injective resolution.

Fact A.4.6. Let $\mathscr{F} \to \mathcal{I}^\bullet$ and $\mathscr{G} \to \mathcal{G}^\bullet$ are two resolutions, and $\phi : \mathscr{F} \to \mathscr{G}$ is a homeomorphism of sheaves. Then there exists $\widetilde{\phi} : \mathcal{I}^\bullet \to \mathcal{G}^\bullet$. Although the lifting of ϕ may not be unique, but they are homotopic.

Definition A.4.7 (sheaf cohomology)**.** *Let $\mathscr{F}$ be a sheaf of abelian groups, then*

$$H^p(X, \mathscr{F}) := H^p(\mathcal{I}^\bullet(X))$$

Remark A.4.8. This definition is independent of the choice of injective resolution thanks to $A.4.6$.

Example A.4.9. By definition,

$$H^0(X, \mathscr{F}) := \ker\{\mathcal{I}^0(X) \to \mathcal{I}^1(X)\}$$

Thus $H^0(X, \mathscr{F}) = \mathscr{F}(X)$, the global sections of sheaf.

Example A.4.10. If $\mathscr{F}$ is a injective sheaf, then $H^i(X, \mathscr{F}) = 0$ for all $i > 0$. It's clear if we choose the following injective resolution

$$0 \to \mathscr{F} \xrightarrow{\mathrm{id}} \mathscr{F} \to 0 \to 0 \to \dots$$

Proposition A.4.11 (zig-zag)**.** *If*

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$

is a short sequence of sheaves, then there is an induced long exact sequence of abelian groups

$$0 \to H^0(X, \mathscr{F}) \to H^0(X, \mathscr{G}) \to H^0(X, \mathscr{H}) \to H^1(X, \mathscr{F}) \to H^1(X, \mathscr{G}) \to \dots$$

Definition A.4.12 (direct image)**.** *Let $f : X \to Y$ be continous map between topological spaces. Let $\mathscr{F}$ be a sheaf of abelian groups on X. A sheaf $f_*\mathscr{F}$ on Y is defined by*

$$f_*\mathscr{F}(U) := \mathscr{F}(f^{-1}(U))$$

$f_*\mathscr{F}$ is called direct image sheaf of $\mathscr{F}$.

Proposition A.4.13. *$f_* : \underline{Ab}_X \to \underline{Ab}_Y$ is a left exact functor.*

Proof. Given an exact sequence of sheaves on X

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$$

Then we need to show

$$0 \to f_*\mathscr{F}' \to f_*\mathscr{F} \to f_*\mathscr{F}''$$

is also an exact sequence on Y. But from $A.4.1$ it suffices to check for any $V \in Y$, we have the following exact sequence

$$0 \to f_*\mathscr{F}'(V) \to f_*\mathscr{F}(V) \to f_*\mathscr{F}''(V)$$

and that's

$$0 \to \mathscr{F}'(f^{-1}(V)) \to \mathscr{F}(f^{-1}(V)) \to \mathscr{F}''(f^{-1}(V))$$

and since f is continous, then $f^{-1}(V)$ is an open subset in X. This completes the proof. □

Since we obtain another left exact functor f_*, we can consider its derived functor.

Definition A.4.14 (higher direct image sheaves)**.** *Let $0 \to \mathscr{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathscr{F}$. The higher direct image sheaf is defined by: For open U,*

$$R^i f_*\mathscr{F}(U) := H^i(f_*\mathcal{I}^\bullet(U))$$

For higher direct image sheaves, it has similar properties parallel to $A.4.9$, $A.4.10$ and $A.4.11$, since these are properties shared by derived functors.

A.5. **Computation for cohomology.** Since it may be difficult for us to choose an injective resolution, we usual other resolutions to compute sheaf cohomology.

Definition A.5.1 (acyclic sheaf)**.** *A sheaf $\mathscr{F}$ is acyclic if $H^i(X, \mathscr{F}) = 0, \forall i > 0$.*

Example A.5.2. Injective sheaf is acyclic.

Definition A.5.3 (acyclic resolution)**.** *Let $\mathscr{F}$ be a sheaf, an acyclic resolution of $\mathscr{F}$ is an exact sequence*

$$0 \to \mathscr{F} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \mathcal{A}^2 \to \dots$$

where $\mathcal{A}^i, i = 0, 1, 2, \dots$ are acyclic.

Proposition A.5.4. *The cohomology of sheaf $\mathscr{F}$ can be computed using acyclic resolution.*

In fact, it's a quite homological techniques, called dimension shifting. So we will state this technique in language of homological algebra. Let's see a baby version of it.

Example A.5.5. Let $\mathcal{F}$ be a left exact functor, and $0 \to A \to M_1 \to B \to 0$ is exact with M_1 is $\mathcal{F}$-acyclic. Then $R^{i+1}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$ for $i > 0$, and $R^1\mathcal{F}(A)$ is the cokernel of $\mathcal{F}(M_1) \to \mathcal{F}(B)$.

Proof. Consider the long exact sequence induced by $0 \to A \to M_1 \to B \to 0$, then we obtain

$$R^i\mathcal{F}(M_1) \to R^i\mathcal{F}(B) \to R^{i+1}\mathcal{F}(A) \to R^{i+1}\mathcal{F}(M_1)$$

If $i > 0$, then $R^i\mathcal{F}(M_1) = R^{i+1}\mathcal{F}(M_1) = 0$ since M_1 is $\mathcal{F}$-acyclic. So we obtain $R^{i+1}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$. If $i = 0$, then

$$0 \to \mathcal{F}(M_1) \to \mathcal{F}(B) \to R^1\mathcal{F}(A) \to 0$$

implies $R^1\mathcal{F}(A) = \operatorname{coker}\{\mathcal{F}(M_1) \to \mathcal{F}(B)\}$ □

Now let's prove dimension shifting in a general setting.

Lemma A.5.6 (dimension shifting)**.** *If*

$$0 \to A \to M_1 \to M_2 \to \cdots \to M_m \to B \to 0$$

is exact with M_i is $\mathcal{F}$-acyclic. Then $R^{i+m}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$ for $i > 0$, and $R^m\mathcal{F}(A)$ is the cokernel of $\mathcal{F}(M_m) \to \mathcal{F}(B)$.

Proof. Prove it by induction on m. For $m = 1$, we already see it in Example $A.5.5$. Assume this is holds for $m < k$, then for $m = k$, let's split $0 \to A \to M_1 \to M_2 \to \cdots \to M_k \xrightarrow{\mathrm{d}_k} B \to 0$ into two exact sequences

$$0 \to A \to M_1 \to M_2 \to \cdots \to M_{k-1} \to \ker \mathrm{d}_k \to 0$$

$$0 \to \ker \mathrm{d}_k \to M_k \xrightarrow{\mathrm{d}_k} B \to 0$$

Then by induction, for $i > 0$ we have

$$R^{i+k-1}\mathcal{F}(A) \cong R^i\mathcal{F}(\ker \mathrm{d}_k)$$
$$R^{i+1}\mathcal{F}(\ker \mathrm{d}_k) \cong R^i\mathcal{F}(B)$$

Combine these two isomorphisms together we obtain $R^{i+k}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$ for $i > 0$, as desired. For $i = 0$, it suffices to let $i = 1$ in $R^{i+k-1}\mathcal{F}(A) \cong R^i\mathcal{F}(\ker \mathrm{d}_k)$, then we obtain

$$R^k\mathcal{F}(A) = R^1\mathcal{F}(\ker \mathrm{d}_k) = \mathrm{coker}\{\mathcal{F}(M_k) \to \mathcal{F}(B)\}$$

This completes the proof. □

Corollary A.5.7. $0 \to A \to M_\bullet$ *is a* $\mathcal{F}$*-acyclic resolution, then* $R^i\mathcal{F}(A) = H^i(\mathcal{F}(M))$.

Proof. Truncate the resolution as

$$0 \to A \to M_0 \to M_1 \to \dots M_{i-1} \to B \to 0$$
$$0 \to B \to M_i \to M_{i+1} \to \dots$$

Since we already have $R^i\mathcal{F}(A) = \mathrm{coker}\{\mathcal{F}(M_{i-1}) \to \mathcal{F}(B)\}$. Note $\mathcal{F}$ is left exact, then

$$\mathcal{F}(B) = \ker\{\mathcal{F}(M_i) \to \mathcal{F}(M_{i+1})\}$$

Thus we obtain

$$R^i\mathcal{F}(A) = \mathrm{coker}\{\mathcal{F}(M_{i-1}) \to \ker\{\mathcal{F}(M_i) \to \mathcal{F}(M_{i+1})\}\} = H^i(\mathcal{F}(M))$$

□

A.6. Examples about acyclic sheaf.

A.6.1. *Flabby sheaf.* First kind of acyclic sheaf is flabby[33] sheaf.

Definition A.6.1 (flabby)**.** *A sheaf* $\mathscr{F}$ *is flabby if all open* $U \subseteq V$*, the restriction map* $\mathscr{F}(V) \to \mathscr{F}(U)$ *is surjective.*

Now let's see some examples about flabby sheaves.

Example A.6.2. A constant sheaf on an irreducible topological space is flabby.

Proof. Note that the constant presheaf on a irreducible topological space is a sheaf in fact. And it's easy to see this constant presheaf is flabby. □

In particular, we have

Example A.6.3. Let X be an algebraic variety, so any two non-empty open sets intersect non-trivially. Then constant sheaf $\mathbb{Z}_X$ is flabby.

Example A.6.4. If $\mathscr{F}$ is a flabby sheaf on X, and $f : X \to Y$ is a continous map, then $f_*\mathscr{F}$ is a flabby sheaf on Y.

[33] Some authors also call this flasque.

Proof. For $V \subset W$ in Y, it suffices to show $f_*\mathscr{F}(W) \to f_*\mathscr{F}(V)$ is surjective, and that's

$$\mathscr{F}(f^{-1}W) \to \mathscr{F}(f^{-1}V)$$

it's surjective since $\mathscr{F}$ is flabby. □

Example A.6.5. An injective sheaf is flabby.

Proof. Let $\mathcal{I}$ be an injective sheaf and let $V \subseteq U$ be an inclusion of open sets. We define a sheaf $\underline{\mathbb{Z}}_U$ on X by

$$\underline{\mathbb{Z}}_U := \begin{cases} \underline{\mathbb{Z}}(W), & W \subseteq U \\ 0, & \text{otherwise} \end{cases}$$

where $\underline{\mathbb{Z}}$ is constant sheaf valued in $\mathbb{Z}$. Similarly we can define $\underline{\mathbb{Z}}_V$, and we have $\underline{\mathbb{Z}}_U(W) = \underline{\mathbb{Z}}_V(W)$ unless $W \subseteq U$ and $W \not\subseteq V$. Thus we obtain an exact sequence

$$0 \to \underline{\mathbb{Z}}_V \to \underline{\mathbb{Z}}_U$$

Applying the functor $\mathrm{Hom}(-, \mathcal{I})$, which is exact, we obtain an exact sequence

$$\mathrm{Hom}(\underline{\mathbb{Z}}_U, \mathcal{I}) \to \mathrm{Hom}(\underline{\mathbb{Z}}_V, \mathcal{I}) \to 0$$

is exact. Now let's see why we need such a weird sheaf $\underline{\mathbb{Z}}_U$. In fact, we will prove $\mathrm{Hom}(\underline{\mathbb{Z}}_U, \mathcal{I}) = \mathcal{I}(U)$. Indeed, since $\varphi : \underline{\mathbb{Z}}_U \to \mathcal{I}$ is a sheaf morphism. Then if $W \not\subseteq U$, then $\varphi(U)$ must be zero. If $W = U$, then the group of sections of $\underline{\mathbb{Z}}_U(U)$ over any connected component is simply $\mathbb{Z}$ and hence $\varphi(U)$ on this connected component is determined by the image of $1 \in \mathbb{Z}$. Thus $\varphi(U)$ can be thought of an element of $\mathcal{I}(U)$. Now on any proper open subset of U, φ is determined by restriction maps. Hence $\mathrm{Hom}(\underline{\mathbb{Z}}_U, \mathcal{I}) = \mathcal{I}(U)$, as desired. We can do similar things for V, and we obtain an exact sequence

$$\mathcal{I}(U) \to \mathcal{I}(V) \to 0$$

Thus $\mathcal{I}$ is flabby. □

Our goal is to prove a flabby sheaf is acyclic, but we still need some property of flabby sheaves.

Proposition A.6.6. *If* $0 \to \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \to 0$ *is an exact sequence of sheaves, and* $\mathscr{F}'$ *is flabby, then for any open set* U*, the sequence*

$$0 \to \mathscr{F}'(U) \xrightarrow{\phi(U)} \mathscr{F}(U) \xrightarrow{\psi(U)} \mathscr{F}''(U) \to 0$$

is exact.

Proof. It suffices to show $\mathscr{F}(U) \to \mathscr{F}''(U) \to 0$ is exact. And it may be quit hard than it looks and here we give a sketch of proof.

Since we have exact sequence on stalks for each $p \in U$ as follows

$$0 \to \mathscr{F}'_p \xrightarrow{\phi_p} \mathscr{F}_p \xrightarrow{\psi_p} \mathscr{F}''_p \to 0$$

Then for each $s \in \mathscr{F}''(U)$, then there exists $t_p \in \mathscr{F}_p$ such that $\psi_p(t_p) = s|_p$. So there exists $V_p \subseteq U$ containing p and $t \in \mathscr{F}(V_p)$ such that $\psi(t) =$

$s|_{V_p}$. If we can glue these t together then we get a section in $\mathcal{F}(U)$ and is mapped to s, which completes the proof. However, they may not equal on the intersection. But things are not too bad, consider another point q and $t' \in \mathscr{F}(V_q)$ such that $\psi(t') = s|_{V_q}$, $(t'-t)|_{V_p \cap V_q} \in \ker \psi(V_p \cap V_q) = \operatorname{im} \phi(V_p \cap V_q)$. So there exists $t'' \in \mathscr{F}'(V_p \cap V_q)$ such that

$$\phi(t'') = (t'-t)|_{V_p \cap V_q}$$

Now since $\mathscr{F}'$ is flabby, then there exists $t''' \in \mathscr{F}(V_p)$ such that $t'''|_{V_p \cap V_q} = t''$. And consider $t + \phi(t''') \in \mathscr{F}(V_p)$, which will coincide with t' on $V_p \cap V_q$. After above corrections, we can glue t after correction together. $\square$

Proposition A.6.7. *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ and $\mathscr{F}$ are flabby, then $\mathscr{F}''$ is flabby.*

Proof. Take $V \subseteq U$ and consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathscr{F}'(U) \arrow[r] \arrow[d] & \mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{F}''(U) \arrow[r] \arrow[d, dashed] & 0 \arrow[d] \\
0 \arrow[r] & \mathscr{F}'(V) \arrow[r] & \mathscr{F}(V) \arrow[r] & \mathscr{F}''(V) \arrow[r] & 0
\end{tikzcd}$$

Five lemma implies the anwser. $\square$

Proposition A.6.8. *A flabby sheaf is acyclic.*

Proof. Let $\mathscr{F}$ be a flabby sheaf. Since there are enough injectives, there is an exact sequence

$$0 \to \mathscr{F} \to \mathcal{I} \to \mathscr{Q} \to 0$$

with $\mathcal{I}$ is injective. By Example $A.6.5$ we have $\mathcal{I}$ is flabby. And by Proposition $A.6.7$ we have $\mathscr{Q}$ is flabby. Consider the long exact sequence induced from above short exact sequence

$$\mathscr{F}(X) \to \mathcal{I}(X) \to \mathscr{Q}(X) \to H^1(X, \mathscr{F}) \to H^1(X, \mathcal{I}) \to \dots$$

Since injective sheaf is acyclic, then $H^1(X, \mathcal{I}) = 0$. Then $H^1(X, \mathscr{F}) = \operatorname{coker}\{\mathcal{I}(X) \to \mathscr{Q}(X)\}$. But from Proposition $A.6.6$ we know that it's surjective. So $H^1(X, \mathscr{F}) = 0$.

Now we prove $H^k(X, \mathscr{F}) = 0$ for $k > 0$ by induction on k. We already show $k = 1$. Assume this holds for $k < n$. Then consider

$$\dots \to H^{n-1}(X, \mathscr{F}) \to H^{n-1}(X, \mathcal{I}) \to H^{n-1}(X, \mathscr{Q}) \to H^n(X, \mathscr{F}) \to H^n(X, \mathcal{I}) \to H^n(X, \mathscr{Q}) \to \dots$$

From we already know, we can reduce above sequence to

$$\dots \to 0 \to 0 \to 0 \to H^n(X, \mathscr{F}) \to 0 \to H^n(X, \mathscr{Q}) \to \dots$$

which implies $H^n(X, \mathscr{F}) = 0$. This completes the proof. $\square$

A.6.2. *Soft sheaf.* The second kind of acyclic sheaves is called soft sheaves, which is quit similar to flabby.

Definition A.6.9 (soft)**.** *A sheaf $\mathscr{F}$ over X is soft if for any closed subset $S \subseteq X$ the restriction map $\mathscr{F}(X) \to \mathscr{F}(S)$ is surjective.*

Remark A.6.10. Here we need to know how to define sections over a closed subset: For closed subset S,

$$\mathscr{F}(S) := \varinjlim_{S \subset U} \mathscr{F}(U)$$

Quite similar to Proposition A.6.6 and A.6.7, soft sheaf has the following properties:

Proposition A.6.11. *If $0 \to \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and $\mathscr{F}'$ is soft, then the following sequence*

$$0 \to \mathscr{F}'(X) \xrightarrow{\phi(X)} \mathscr{F}(X) \xrightarrow{\psi(X)} \mathscr{F}''(X) \to 0$$

is exact.

Proposition A.6.12. *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ and $\mathscr{F}$ are soft, then $\mathscr{F}''$ is soft.*

Proposition A.6.13. *A soft sheaf is acyclic.*

So you may wonder, what's the difference between flabby and soft, since the definitions are quite similar, and both of them are acyclic. Clearly by definition of sections over a closed subset, we know that every flabby sheaf is soft, but converse fails

Example A.6.14. The sheaf of smooth functions on a smooth manifold is soft but not flabby.

Lemma A.6.15. *If $\mathscr{M}$ is a sheaf of modules over a soft sheaf of rings $\mathscr{R}$, then $\mathscr{M}$ is a soft sheaf.*

Proof. Let $s \in \mathscr{M}(K)$ for K a closed subset of X. Then s extends to some open neighborhood U of K. Let $\rho \in \mathscr{R}(K \cup (X - U))$ be defined by

$$\rho = \begin{cases} 1, & \text{on } K \\ 0, & \text{on } X - U \end{cases}$$

Since $\mathscr{R}$ is soft, then ρ extends to a section over X, then ρs is the desired extension of s. $\square$

A.6.3. *Fine sheaf.* Another important kind of acyclic sheaves, which behaves like sheaf of differential forms Ω^k_X is called fine sheaf. Recall what is a partition of unity: Let $U = \{U_i\}_{i \in I}$ be a locally finite open cover of X. A partition of unity subordinate to U is a collection of continous or smooth functions $f_i : U_i \to [0,1]$ for each $i \in I$ such that its support lies in U_i, and for any $x \in X$

$$\sum_{i\in I} f_i(x) = 1$$

Note that for a smooth manifold M, then sheaf of differential k-forms is a C^∞_M-module sheaf.

Definition A.6.16 (fine sheaf)**.** *A fine sheaf $\mathscr{F}$ on X is a sheaf of $\mathscr{A}$-modules, where $\mathscr{A}$ is a sheaf of rings such that for every locally finite open cover $\{U_i\}_{i\in I}$ of X, there is a partition of unity*

$$\sum_{i\in I} \rho_i = 1$$

where $\rho_i \in \mathscr{A}(X)$ and $\operatorname{supp}(\rho_i) \subseteq U_i$.

Remark A.6.17. It' neccessary to give an explict definition of support of a section: For a sheaf $\mathscr{F}$ on X and a section $s \in \mathscr{F}(X)$, its support is defined as

$$\operatorname{supp}(s) := \overline{\{x \in X : s|_x \neq 0\}}$$

Proposition A.6.18. *A fine sheaf is acyclic.*

Proof. Let $\mathscr{F}$ be a sheaf of $\mathscr{A}$-modules and a fine sheaf. And choose a injective resolution

$$0 \to \mathscr{F} \xrightarrow{\mathrm{d}} \mathcal{I}^0 \xrightarrow{\mathrm{d}} \mathcal{I}^0 \xrightarrow{\mathrm{d}} \mathcal{I}^1 \xrightarrow{\mathrm{d}} \dots$$

such that $\mathcal{I}^i$ are injective sheaves of $\mathscr{A}$-modules. Let $s \in \mathcal{I}^p(X)$ such that $\mathrm{d}s = 0$. Then by exactness of injective resolution we have X is covered by open sets U_i such that for each i there is an element $t_i \in \mathcal{I}^{p-1}(U_i)$ such that $\mathrm{d}t_i = s|_{U_i}$. By passing to a refinement we may assume that the cover $\{U_i\}$ is locally finite. Let $\{\rho_i\}$ be a partition of unity subordinate to $\{U_i\}$. Then we have $t = \sum \rho_i t_i \in \mathscr{I}^{p-1}(X)$ such that $\mathrm{d}t = s$. This completes the proof. □

Example A.6.19. Let M be a manifold and let $\pi : E \to M$ be a vector bundle, then sheaf of smooth sections of E is also a fine sheaf.

Example A.6.20. Tangent sheaf T_M, and its tensor $T_M^{\otimes k}$, sheaf of differential forms Ω_M and k-forms Ω^k_M are fine sheaves.

Remark A.6.21. So from Example A.6.19, we know that it's meaningless to compute cohomology of sheaf of differential k-forms, or any other vector bundle over a smooth manifold. But in complex version, something interesting happens:

Let $(X, \mathcal{O}_X)$ be a complex manifold, and let $\pi : E \to X$ a holomorphic vector bundle. Then the sheaf of holomorphic sections of E is not a fine sheaf, since there is no partition of unity we use in the sense of smooth. So we may compute cohomology of some holomorphic vector bundle, and that's what does Dolbeault cohomology compute.

For fine sheaf and soft sheaf, we have

Lemma A.6.22. *Fine sheaf is soft*

Proof. Let $\mathscr{F}$ be a fine sheaf, $S \subseteq X$ closed and $s \in \mathscr{F}(S)$. Let $\{U_i\}$ be an open covering of S and $s_i \in \mathscr{F}(U_i)$ such that

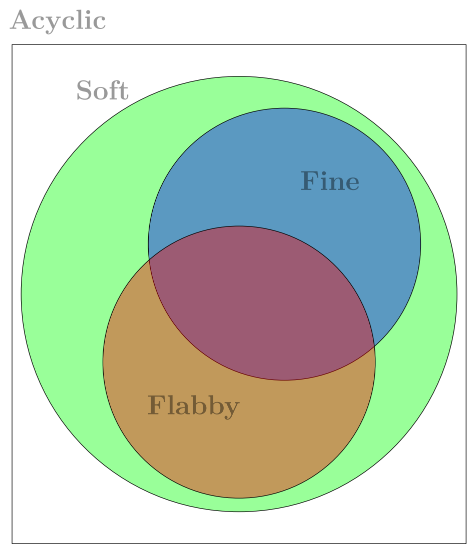
$$s_i|_{S \cap U_i} = s|_{S \cap U_i}$$

Let $U_0 = X - S$, and $s_0 = 0$. Then $\{U_i\} \coprod \{U_0\}$ is an open covering of X, WLOG we assume this open covering is locally finite and choose a partition of unity $\{\rho_i\}$ subordinate to it. Set

$$\overline{s} := \sum_i \rho_i(s_i) \in \mathscr{F}(X)$$

which extends s. □

Remark A.6.23. For flabby, soft, fine and acyclic sheaves. In fact we have the following relations:



Indeed, constant sheaf on an irreducible space is flabby but not fine, sheaf of smooth functions on a smooth manifold is fine but not flabby.

A.7. **Proof of de Rham theorem using sheaf cohomology.** As we already know, for constant sheaf $\underline{\mathbb{R}}$ over a smooth manifold M, we have the following fine resolution

$$0 \to \underline{\mathbb{R}} \xrightarrow{i} \Omega^0_M \xrightarrow{\mathrm{d}} \Omega^1_M \xrightarrow{\mathrm{d}} \Omega^2_M \xrightarrow{\mathrm{d}} \dots$$

And de Rham cohomology computes the sheaf cohomology of $\underline{\mathbb{R}}$. de Rham theorem implies that de Rham cohomology equals to the singular cohomology with real coefficient. So if we can give constant sheaf another resolution using singular cochains, we may derive the de Rham cohomology.

We state this in a general setting: Let X be a topological manifold, and a constant sheaf $\underline{G}$ over X, where G is an abelian group. Let $S^p(U, G)$ be the group of singular cochains in U with coefficients in G, and let δ denote the coboundary operator.

Let $\mathscr{S}^p(G)$ be the sheaf over X generated by the presheaf $U \mapsto S^p(U,G)$, with induced differential mapping $\mathscr{S}^p(G) \xrightarrow{\delta} \mathscr{S}^{p+1}(G)$.

Similar to Poincaré lemma, we have for a unit ball U in Euclidean space, we have the following sequence

$$\dots \to S^{p-1}(U,G) \xrightarrow{\delta} S^p(U,G) \xrightarrow{\delta} S^{p+1}(U,G) \to \dots$$

is exact. So we have the following resolution of the constant sheaf $\underline{G}$

$$0 \to \underline{G} \to \mathscr{S}^0(G) \xrightarrow{\delta} \mathscr{S}^1(G) \xrightarrow{\delta} \mathscr{S}^2(G) \to \dots$$

Remark A.7.1. If M is a smooth manifold, then we can consider smooth chains, that is $f : \Delta^p \to U$, where f is a smooth function. The corresponding results above still hold, and we have a resolution by smooth cochains with coefficients in G:

$$0 \to \underline{G} \to \mathscr{S}^\bullet_\infty(G)$$

So if we choose $G = \mathbb{R}$, then it suffices to show $0 \to \underline{\mathbb{R}} \to \mathscr{S}^\bullet_\infty(\mathbb{R})$ is an acyclic resolution, then we obtain de Rham theorem.

First, note that $\mathscr{S}^p_\infty$ is a $\mathscr{S}^0_\infty$-module, given by cup product on open sets. Then by Lemma A.6.15 and the fact $\mathscr{S}^0_\infty$ is soft we know that it's a soft resolution. This completes the proof.

A.8. **Hypercohomology.** In homological algebra, the hypercohomology is a generalization of cohomology functor which takes as input not objects in abelian category but instead chain complexes of objects.

One of the motivations for hypercohomology comes from the fact that there isn't an obvious generalization of cohomological long exact sequences associated to short exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

It turns out hypercohomology gives techniques for constructing a similar cohomological associated long exact sequence from an arbitrary long exact sequence

$$0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \dots \to \mathscr{F}_k \to 0$$

Now let's clearify the definition of hypercohomology: Let $\mathscr{F}^\bullet : \dots \to \mathscr{F}^{i-1} \to \mathscr{F}^i \to \mathscr{F}^{i+1} \to \dots$ be a complex of sheaves of abelian groups. Assume $\mathscr{F}^\bullet$ is bounded below, i.e. $\mathscr{F}^n = 0$ for $n \ll 0$. Then $\mathscr{F}^\bullet$ admits an injective resolution $\mathscr{F}^\bullet \to \mathcal{I}^\bullet$. In other words

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \mathscr{F}^{i-1} & \longrightarrow & \mathscr{F}^i & \longrightarrow & \mathscr{F}^{i+1} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & \mathcal{I}^{i-1} & \longrightarrow & \mathcal{I}^i & \longrightarrow & \mathcal{I}^{i+1} & \longrightarrow & \dots \end{array}$$

such that

1. All $\mathcal{I}^i$ are injective sheaves;

2. The induced homeomorphism $H^i(\mathscr{F}^\bullet) \to H^i(\mathcal{I}^\bullet)$ is an isomorphism.

The hypercohomology of $\mathscr{F}^\bullet$ is defined by

$$H^i(X, \mathscr{F}^\bullet) := H^i(\Gamma(X, \mathcal{I}^\bullet))$$

Notation A.8.1. Let $\mathscr{F}^\bullet[n]$ be the complex obtained from $\mathscr{F}$ by

$$(\mathscr{F}^\bullet[n])^i = \begin{cases} \mathscr{F}, & i = n \\ 0, & \text{otherwise} \end{cases}$$

Example A.8.2. Let $\mathscr{F}$ be a sheaf and consider $\mathscr{F}^\bullet[0]$, that is,

$$0 \to \underbrace{\mathscr{F}}_{\text{deg zero}} \to 0 \to 0 \to \dots$$

If $0 \to \mathscr{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \dots$ is an injective resolution of $\mathscr{F}$, then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{F} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}^0 & \longrightarrow & \mathcal{I}^1 & \longrightarrow & \mathcal{I}^2 & \longrightarrow & \dots \end{array}$$

is an injective resolution of $\mathscr{F}^\bullet$. Indeed, $\mathcal{I}^i$ are injective for all $i \geq 0$. Furthermore,

$$H^i(\mathcal{I}^\bullet) = \begin{cases} \mathscr{F}, & n = 0 \\ 0, & \text{otherwise} \end{cases} = H^i(\mathscr{F}^\bullet[0])$$

So by definition of hypercohomology, we have $H^i(X, \mathscr{F}^\bullet[0]) = H^i(\Gamma(X, \mathcal{I}^\bullet)) = H^i(X, \mathscr{F}^\bullet)$. For general case, we have

$$H^i(X, \mathscr{F}^\bullet[n]) \cong H^{i+n}(X, \mathscr{F})$$

Proposition A.8.3 (zig-zag in hypercohomology)**.** *Let $0 \to \mathscr{F}^\bullet \to \mathscr{G}^\bullet \to \mathscr{H}^\bullet \to 0$ be a short exact sequence of bounded below complex. Then there is an induced long exact sequence*

$$\dots \to H^{i-1}(X, \mathscr{H}^\bullet) \to H^i(X, \mathscr{F}^\bullet) \to H^i(X, \mathscr{G}^\bullet) \to H^i(X, \mathscr{H}^\bullet) \to H^{i+1}(X, \mathscr{F}^\bullet) \to \dots$$

B. Spectral sequence

Spectral sequence provide a useful tool to compute cohomology. We will focus on a special form of spectral sequence as follows:

1. For each $r > r_0$, an r-th page

$$E_r = \bigoplus_{p,q \in \mathbb{Z}} E_r^{p,q}$$

2. $\mathrm{d}_r : E_r \to E_r$ with $\mathrm{d}_r \circ \mathrm{d}_r = 0$, where $\mathrm{d}_r = \sum_{p,q} \mathrm{d}_r^{p,q}$.

$$\mathrm{d}_r^{p,q} : E_r^{p,q} \to E_r^{p+r, q-r+1}$$

3. $E_{r+1} = H^\bullet(E_r, \mathrm{d}_r)$. More precisely, we have

$$E_{r+1}^{p,q} = \frac{\ker\{E_r^{p,q} \to E_r^{p+r,q-r+1}\}}{\mathrm{im}\{E_r^{p-r,q+r-1} \to E_r^{p,q}\}}$$

4. $E_r^{p,q} = 0$ for all $p < 0, q < 0$.

So what's infinite page $E_\infty^{p,q}$? For each (p,q). there exists $r(p,q)$ such that for all $r' \geq r(p,q)$, we have $E_{r'}^{p,q} = E_{r(p,q)}^{p,q}$. Then we have

$$E_\infty^{p,q} = E_{r(p,q)}^{p,q}$$

We say that the spectral sequence $E_r^{p,q}$ converges to H^n if

1. H^n admits a filtration

$$0 = F^{n+1}H^n \subset F^n H^n \subset \cdots \subset F^1 H^n \subset F^0 H^n = H^n$$

2. $E_\infty^{p,q} = F^p H^{p+q} / F^{p+1} H^{p+q}$.

And we use notation $E_r^{p,q} \implies H^{p+q}$.

We say that the spectral sequence degenerates at r-page if $\mathrm{d}_{r'} = 0$ for all $r' \geq r$. As a consequence, $E_\infty = E_r$.

Example B.0.1 (Leray spectral sequence)**.** Let $f : X \to Y$ be a continous map of topological spaces. Let $\mathscr{F}$ be a sheaf of abelian groups on X. Then there is a spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_* \mathscr{F}) \implies H^{p+q}(X, \mathscr{F})$$

Example B.0.2 (Čech-to-derived functor spectral sequence)**.** There is a spectral sequence

$$E_2^{p,q} = \check{H}^p(\mathfrak{U}, H^q(X, \mathscr{F})) \implies H^{p+q}(X, \mathscr{F})$$

where $\mathscr{F}$ is a sheaf on X and $H^q(X, \mathscr{F})$ is the presheaf

$$U \mapsto H^q(U, \mathscr{F})$$

As a consequence, if $H^q(U_{i_0 \dots i_p}, \mathscr{F}) = 0$ for all $q > 0$ and $p \geq 0$, then

$$\check{H}^q(\mathfrak{U}, \mathscr{F}) = H^p(X, \mathscr{F})$$

for all p.

Example B.0.3. Let $\mathscr{F}^\bullet : 0 \to \mathscr{F}^0 \to \mathscr{F}^1 \to \dots$ be a complex of sheaves. Then there are two spectral sequences both converges to the hypercohomology

$$E_1^{p,q} = H^q(X, \mathscr{F}^p) \implies H^{p+q}(X, \mathscr{F})$$
$$E_2^{p,q} = H^p(X, H^q(\mathscr{F}^\bullet)) \implies H^{p+q}(X, \mathscr{F})$$

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