SOLUTIONS TO ALGEBRA2-H

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ABSTRACT. This note contain solutions to homework of Algebra2-H (2024Spring), but we will omit proofs which are already shown in the textbook or quite trivial.

CONTENTS

1. Homework-1

1.1. **Solutions to 4.1.**

1. It suffices to note that $(u+1)^{-1} = (u^2-u+1)/3$.
2. Note that $u^8+1=0$, and by Eisenstein criterion it's easy to show that x^8+1 is irreducible.
4. It suffices to note that $[F(u):F(u^2)] \leq 2$.
5. Omit.
6. Omit.
7. Pick any $0 \neq v \in K \setminus F$, then by the explicit construction of $F(u)$, we may write

$$v = \frac{f(u)}{g(u)},$$

where $f, g \in F[x]$ with $g \neq 0$. In other words, one has $f(u) - vg(u) = 0$. On the other hand, $f(x) - vg(x) \not\equiv 0$, otherwise it leads to $v \in F$, since coefficients of f, g lie in F. This shows u satisfies a non-trivial polynomial with coefficients in K, and thus it's algebraic over K.
8. Omit.
9. If β is algebraic over F, then by exercise 7 one has $[F(\alpha):F(\beta)] < \infty$, and thus

$$[F(\alpha):F] = [F(\alpha):F(\beta)][F(\beta):F] < \infty,$$

a contradiction.
10 Since α is algebraic over $F(\beta)$, then there exists a non-trivial polynomial

$$P(x) = x^n + a_{n-1}(\beta)x^{n-1} + \cdots + a_0(\beta) \in F(\beta)[x]$$

such that $P(\alpha) = 0$. On the other hand, it's clear that β is transcendent over F, otherwise

$$[F(\alpha,\beta):F] = [F(\alpha,\beta):F(\beta)][F(\beta):F] < \infty,$$

a contradiction to α is transcendent over F. Thus by the explicit construction of $F(\beta)$, we may write

$$a_i(\beta) = \frac{f_i(\beta)}{g_i(\beta)},$$

where $f_i(x)$ and $g_i(x) \in F[x]$, while $g_i(x) \neq 0$. Now consider the polynomial

$$Q(x,y) = P(x)\prod_{i=1}^{n} g_i(y) \in F[x,y].$$

It's a polynomial satisfying $Q(\alpha,\beta) = 0$, which implies β is algebraic over $F(\alpha)$.

1.2. **Solutions to 4.2.**

2. It's clear $\mathbb{Q}(\sqrt{2}+\sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3})$. On the other hand, note that
$$\sqrt{3}-\sqrt{2} = (\sqrt{2}+\sqrt{3})^{-1} \in \mathbb{Q}(\sqrt{2}, \sqrt{3}).$$
This shows $\sqrt{2}, \sqrt{3} \in \mathbb{Q}(\sqrt{2}, \sqrt{3})$, and thus $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

Remark 1.2.1. In fact, any finite seperable extension is a simple extension, that is, a field extension generated by one element. This is called primitive element theorem.

3. Suppose there exists $a \in E$ such that $g(a) = 0$. Since g is irreducible over F, so it's the minimal polynomial of a over F. Thus
$$[F(a) : F] = \deg g = k.$$
On the other hand, $[E : F] = [E : F(a)][F(a) : F]$, a contradiction to $k \nmid [E : F]$.
5 Suppose K be a subring of E containing F. For any $0 \neq u \in K$, since E is algebraic over F, there exists a polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ such that $f(u) = 0$. Thus
$$u^{-1} = -\frac{1}{a_0}(u^{n-1} + a_{n-1}u^{n-2} + \cdots + a_1) \in K.$$
6. Omit.
7. It's clear $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$, since it's algebraic over $\mathbb{R}$, and it's algebraically closed.
(a) An algebraically closed field must contain infinitely many elements, otherwise if an algebraically closed E is a finite field with $|E| = q$, then $x^q - x + 1$ has no roots in E.
(b) An example is $[\mathbb{C} : \mathbb{R}] = 2$.
8. Firstly we prove that if $p_1, \ldots, p_n$ and p are distinct prime numbers, then $\sqrt{p} \notin \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_n})$ by induction. For $n = 1$, if $\sqrt{p} \in \mathbb{Q}(\sqrt{p_1})$, then there exists $a, b \in \mathbb{Q}$ such that
$$\sqrt{p} = a + \sqrt{p_1},$$
and thus $a^2 + b^2p_1 + 2ab\sqrt{p_1} = p$. Since $\sqrt{p_1} \notin \mathbb{Q}$, it leads to $ab = 0$. Both $a = 0$ and $b = 0$ will lead to contradictions. Now suppose the statement holds for $n = k-1$ and consider the case $n = k$. By induction hypothsis, one has
$$\sqrt{p}, \sqrt{p_k} \notin \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_{k-1}}).$$
If $\sqrt{p} \in \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_k})$, then
$$\sqrt{p} = c + d\sqrt{p_k},$$
where $c, d \in \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_{k-1}})$. By the same argument one has $cd = 0$, but $c \neq 0$, otherwise it contradicts to $\sqrt{p} \notin \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_{k-1}})$. This shows $\sqrt{p} = d\sqrt{p_k}$. Repeat above process for $d \in \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_{k-1}})$, one has
$$d = d_1\sqrt{p_{k-1}},$$

and thus

$$\sqrt{p} = d_{n-1}\sqrt{p_1 \dots p_k},$$

where $d_{n-1} \in \mathbb{Q}$, a contradiction. This shows $E = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \dots, \sqrt{p}, \dots)/\mathbb{Q}$ is an algebraic extension of infinite degree. Since $\overline{Q}$ is the algebraic closure of $\mathbb{Q}$, and E is algebraic over $\mathbb{Q}$, so $\overline{Q}$ is also the algebraic closure of E.

9. Omit.

10. Omit.

1.3. **Solutions to 4.3.**

1. Omit.

2. It suffices to show that $\sin 18^\circ$ is constructable. Suppose $\theta = 18^\circ$. Then $\sin 2\theta = \sin(\pi/2 - 3\theta) = \cos 3\theta$, and thus

$$2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta.$$

A simple computation yields

$$\cos\theta(4\sin^2\theta + 2\sin\theta - 1) = 0.$$

As a result, one has $\sin\theta = (\sqrt{5} - 1)/4$, which is constructable.

References

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