

ALGEBRAIC CURVES

CHENGLONG YU, BOWEN LIU

CONTENTS

0. Preface	3
0.1. Notations	3
0.2. Motivations	4
1. Riemann surface and algebraic curves	8
1.1. Riemann surface	8
1.2. Algebraic curves	12
2. Ramification	19
2.1. Ramification covering	20
2.2. Riemann-Hurwitz formula	21
3. Bezout theorem	25
3.1. Statement and proof	25
3.2. Applications	26
4. Differential forms	31
4.1. Differential forms, differential operators and integrations	31
4.2. Holomorphic 1-form and meromorphic 1-form	33
4.3. Residue theorem	35
4.4. Poincaré-Hopf theorem	37
5. Normalization	39
5.1. Weierstrass preparation theorem	40
5.2. Resolution of singularities	42
5.3. Blow-up	45
5.4. Bezout formula for singular case	48
5.5. Plücker formula for singular case	48
6. Divisors	53
6.1. Motivations	53
6.2. Divisors	53
6.3. Global sections associated to divisors	57
6.4. Linear system and morphisms to projective space	60
7. Riemann-Roch theorem	64
7.1. Statement and applications	64
7.2. Proof of the theorem	67
8. Homework	68
8.1. Homework 1	68
8.2. Homework 2	72
8.3. Homework 3	76

8.4. Homework 4	79
8.5. Homework 5	81
8.6. Homework 6	86
References	91

0. PREFACE

0.1. **Notations.**

- (1) X, Y always denote Riemann surfaces.
- (2) C always denotes the algebraic plane curve.
- (3) $\Phi, \Psi: X \rightarrow Y$ always denote the holomorphic map between Riemann surfaces.
- (4) f, g sometimes denote functions (smooth, holomorphic, meromorphic), sometimes denote polynomials, and sometimes denote the convergent power series.
- (5) F, G always denote polynomials, and most of time they denote homogeneous polynomials given by polynomials f, g .
- (6) f_x always denote the partial derivative of f with respect to variable x .

0.2. Motivations.

0.2.1. *Meromorphic functions.* Let $U \subseteq \mathbb{C}$ be an open subset with coordinate $\{z\}$. In complex analysis we learnt that a meromorphic function f is a function that is holomorphic on all of U except for a set of isolated points, which are poles of the function. In other words, a meromorphic function can be regarded as a function $f: U \rightarrow \mathbb{C} \cup \{\infty\}$.

Topologically speaking, $\mathbb{C} \cup \{\infty\}$ is S^2 , and in fact there is a complex manifold structure on it. More precisely, we can glue two pieces of complex plane via $w = 1/z$ to obtain a complex manifold called Riemann sphere

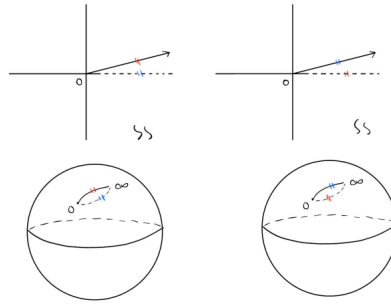
$$\mathbb{P}^1 = \mathbb{C} \cup_{\mathbb{C}^*} \mathbb{C},$$

and topologically \mathbb{P}^1 is exactly $\mathbb{C} \cup \{\infty\}$. By using this viewpoint, meromorphic function on U is exactly the same thing as holomorphic map from U to the Riemann sphere, and thus it gives us a lovely way to study meromorphic functions by using theories of holomorphic maps between Riemann surfaces, such as the number (counted with multiplicity) of zeros is equal to the number (counted with multiplicity) of poles.

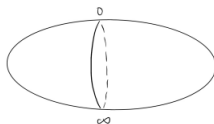
0.2.2. *Multivalueness of holomorphic functions.* For complex number $z = \rho e^{\sqrt{-1}\theta}$, where $\rho \in [0, \infty)$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, one has

$$(\sqrt{\rho} e^{\sqrt{-1}\theta/2})^2 = (\sqrt{\rho} e^{\sqrt{-1}(\theta/2+\pi)})^2 = z.$$

This shows there are two candidates for \sqrt{z} , and this phenomenon is called multivalueness of holomorphic function. If we define square root as $\sqrt{z} = \sqrt{\rho} e^{\sqrt{-1}\theta/2}$, then it's only well-defined on $\mathbb{C} \setminus [0, \infty)$, since it will “jump” when passing through the two sides of $[0, \infty)$, and $\mathbb{C} \setminus [0, \infty)$ is called a single value component of \sqrt{z} .



The ideal to solve this phenomenon is that, when passing the segment $[0, \infty)$, \sqrt{z} should come into another single value component. In other words, if we want to make square root \sqrt{z} defined on the whole complex plane, it should be no longer a function from \mathbb{C} to \mathbb{C} , but a function from \mathbb{C} to an object we obtained from gluing two single value components together. This construction also gives the Riemann sphere.

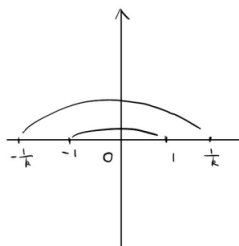


Similarly, $f(z) = \sqrt{1 - z^2}$ is well-defined on $\mathbb{C} \setminus [-1, 1]$, and it gives a well-defined function from \mathbb{C} to something obtained by gluing two copies of $\mathbb{C} \setminus [-1, 1]$, which is also the Riemann sphere.

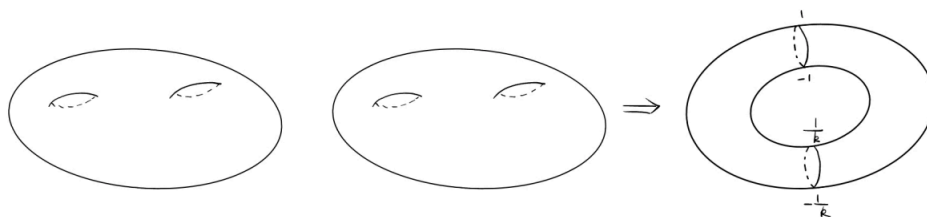
Now let's consider a more complicated example. For

$$f(z) = \sqrt{(1 - z^2)(1 - k^2 z^2)},$$

where $k \neq \pm 1$, it gives a well-defined function on \mathbb{C} minus two line segments connecting $-1, 1$ and $-1/k, 1/k$.



If we want to obtain a function defined on \mathbb{C} , we should glue two copies of above single value components. This gives a new Riemann surface called complex torus.



0.2.3. Abelian integrals.

Example 0.2.1 (arc-length of ellipse). For ellipse given by $(x/a)^2 + (y/b)^2 = 1$, by using parameterization

$$\begin{aligned} x &= a \cos \theta \\ y &= b \sin \theta, \end{aligned}$$

it's easy to see arc-length is given by

$$\begin{aligned} \int_{\theta_0}^{\theta_1} \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} d\theta &= a \int_{\theta_0}^{\theta_1} \sqrt{1 - k^2 \sin^2 \theta} d\theta \\ &\stackrel{z=\sin \theta}{=} \int_{z_0}^{z_1} \frac{\sqrt{1 - k^2 z^2}}{\sqrt{1 - z^2}} dz \\ &= \int_{z_0}^{z_1} \frac{1 - k^2 z^2}{\sqrt{(1 - k^2 z^2)(1 - z^2)}} dz, \end{aligned}$$

where $k = \sqrt{1 - b^2/a^2}$. For $k = 0$, since $\arcsin z$ is a primitive function of $1/\sqrt{1 - z^2}$, one has

$$\int_{z_0}^{z_1} \frac{1}{\sqrt{1 - z^2}} dz = \arcsin z_1 - \arcsin z_0.$$

The classical theory of “addition formula” gives

$$\sin(\alpha + \beta) = \sin \alpha \sqrt{1 - \sin^2 \beta} + \sqrt{1 - \sin^2 \alpha} \sin \beta.$$

In terms of integration

$$\int_0^{z_1} \frac{1}{\sqrt{1 - t^2}} dt + \int_0^{z_2} \frac{1}{\sqrt{1 - t^2}} dt = \int_0^{\sqrt{1 - z_2^2} + z_2 \sqrt{1 - z_1^2}} \frac{1}{\sqrt{1 - t^2}} dt.$$

For analogue of above case, if we define ellipse sine sn as

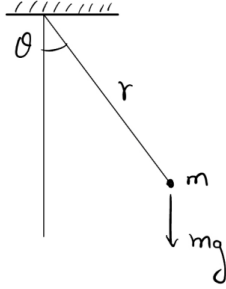
$$\int_0^{\arcsin z} \frac{1}{\sqrt{1 - k^2 \sin^2 t}} dt = \text{sn}^{-1}(z),$$

one can also show it satisfies some addition formula

$$\text{sn}(\alpha + \beta) = \frac{\text{sn} \alpha \sqrt{(1 - \text{sn}^2 \beta)(1 - k^2 \text{sn}^2 \beta)} + \text{sn} \beta \sqrt{(1 - \text{sn}^2 \alpha)(1 - k^2 \text{sn}^2 \alpha)}}{1 - k^2 \text{sn}^2 \alpha \text{sn}^2 \beta}.$$

However, the ellipse sine cannot be expressed as an elementary function, and this is closely related to the fact that $y^2 = (1 - z^2)(1 - k^2 z^2)$ is not a Riemann sphere.

Example 0.2.2 (simple pendulum). Suppose there is an object with mass m is released at $\theta = \alpha$ with zero initial velocity, and the length of pendulum is r .



The conservation of energy gives the following equation

$$\frac{1}{2}mr^2\left(\frac{d\theta}{dt}\right)^2 = mgr \cos \theta - mgr \cos \alpha.$$

In other words,

$$(0.1) \quad \left(\frac{d\theta}{dt}\right)^2 = 2\frac{g}{r}(\cos \theta - \cos \alpha) = 4\frac{g}{r}\left(\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\theta}{2}\right).$$

An approximation with θ sufficiently small, one has

$$\frac{d\theta}{dt} = \sqrt{\frac{g}{r}(\alpha^2 - \theta^2)}.$$

This shows

$$t = \int_0^\theta \sqrt{\frac{r}{g} \frac{1}{\alpha^2 - s^2}} ds.$$

Thus the period of the simple pendulum is given by

$$T = 4 \int_0^\alpha \sqrt{\frac{r}{g} \frac{1}{\alpha^2 - s^2}} ds = 2\pi \sqrt{\frac{r}{g}}.$$

However, if we don't use the approximation, and use substitution

$$\sin \varphi = \frac{\sin \frac{\theta}{2}}{\sin \frac{\alpha}{2}}$$

in (0.1), one has

$$\left(\frac{d\varphi}{dt}\right)^2 = \frac{g}{r}\left(1 - \sin^2 \frac{\alpha}{2} \sin^2 \varphi\right).$$

Then

$$t = \sqrt{\frac{r}{g}} \int_0^\varphi \frac{1}{\sqrt{1 - k^2 \sin^2 s}} ds,$$

where $k = \sin \frac{\alpha}{2}$, and thus explicit formula for the period of simple pendulum is

$$T = 4\sqrt{\frac{r}{g}} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2 s}} ds.$$

This is exactly ellipse integral.

Remark 0.2.1 (general case). Let f be a polynomial of two variables and $y = \Phi(X)$ be a solution of equation $f(x, y) = 0$. Then

$$\int R(x, f(x)) = 0$$

can be expressed as elementary function if and only if $\deg f = 0, 1, 2$, and in fact $\deg f$ is closely related to the topology of algebraic curves.

1. RIEMANN SURFACE AND ALGEBRAIC CURVES

1.1. Riemann surface.

1.1.1. Definitions.

Definition 1.1.1 (complex atlas). Let X be a topological space. A complex atlas on X consists of the following data:

- (1) $\{U_i\}_{i \in I}$ is an open covering of X .
- (2) For each $i \in I$, there exists a homeomorphism $\varphi_i: U_i \rightarrow \varphi_i(U_i) \subseteq \mathbb{C}$.
- (3) For $i, j \in I$, if $U_i \cap U_j \neq \emptyset$, then the transition function

$$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1}: \varphi_j(U_i \cap U_j) \rightarrow \varphi_i(U_i \cap U_j)$$

is holomorphic.

Remark 1.1.1. If $\{U_i, \varphi_i\}$ is a complex atlas on a topological space, then all transition functions φ_{ij} are not only holomorphic, but biholomorphic with inverse φ_{ji} .

Definition 1.1.2 (complex structure). Two complex atlas \mathcal{A}, \mathcal{B} are equivalent if $\mathcal{A} \cup \mathcal{B}$ is also a complex atlas, and a complex structure is an equivalent class of atlas on X .

Definition 1.1.3 (Riemann surface). A Riemann surface is a connected, second countable, Hausdorff topological space X together with a complex structure on X .

Remark 1.1.2. A Riemann surface X is a complex manifold with $\dim_{\mathbb{C}} X = 1$, and it's called a surface since $\dim_{\mathbb{R}} X = 2$.

1.1.2. Examples.

Example 1.1.1 (Riemann sphere). Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ be 2-sphere and $\{U_1 = S^2 \setminus (0, 0, 1), U_2 = S^2 \setminus (0, 0, -1)\}$ be an open covering of S^2 . Consider

$$\begin{aligned} \varphi_1: U_1 &\rightarrow \mathbb{C} \\ (x_1, x_2, x_3) &\mapsto \frac{x_1}{1 - x_3} + \sqrt{-1} \frac{x_2}{1 - x_3}, \end{aligned}$$

and

$$\begin{aligned} \varphi_2: U_2 &\rightarrow \mathbb{C} \\ (x_1, x_2, x_3) &\mapsto \frac{x_1}{1 + x_3} - \sqrt{-1} \frac{x_2}{1 + x_3}. \end{aligned}$$

A direct computation shows that

$$\left(\frac{x_1}{1 - x_3} + \sqrt{-1} \frac{x_2}{1 - x_3}\right) \left(\frac{x_1}{1 + x_3} - \sqrt{-1} \frac{x_2}{1 + x_3}\right) = \frac{x_1^2 + x_2^2}{1 - x_3^2} = 1,$$

and thus the transition function $\varphi_2 \circ \varphi_1^{-1}(z) = 1/z$. This shows $\{U_1, U_2\}$ is a complex atlas of S^2 . It's clear as a topological space S^2 is connected, second countable and Hausdorff, and thus S^2 is a Riemann surface, called Riemann sphere.

Remark 1.1.3. There is another construction of Riemann sphere, given by gluing two complex planes together on \mathbb{C}^* , and the gluing data on \mathbb{C}^* is given by $z \sim 1/w$. One thing to mention is that it's not clear object constructed in this way is Hausdorff. For example, if we glue two complex planes together on \mathbb{C}^* by using gluing data $z \sim w$, then the object obtained is not Hausdorff.

Example 1.1.2 (complex projective line). The complex projective line $\mathbb{P}^1 = \mathbb{C}^2 \setminus (0,0) / \sim$, where $(x, y) \sim (z, w)$ if and only if $(\lambda x, \lambda y) = (z, w)$ for some $\lambda \in \mathbb{C}^*$, and the equivalent class for (x, y) is denoted by $[x, y]$, called the homogenous coordinate. The quotient topology on \mathbb{P}^1 which makes it second countable, Hausdorff and compact. Consider

$$U_0 = \{[z, w] \mid z \neq 0\} \xrightarrow{\varphi_0} \mathbb{C}$$

where φ_0 is defined as $\varphi_0([z, w]) = z/w$. Similarly consider

$$U_1 = \{[z, w] \mid w \neq 0\} \xrightarrow{\varphi_1} \mathbb{C}$$

where φ_1 is defined as $\varphi_1([z, w]) = w/z$. For $z \in \varphi_1(U_0 \cap U_1)$, one has

$$z \xrightarrow{\varphi_1^{-1}} [z : 1] = [1 : \frac{1}{z}] \xrightarrow{\varphi_0} \frac{1}{z}.$$

This shows the transition function $\varphi_{01}(z) = 1/z$, which is holomorphic, and thus $\{(U_0, \varphi_0), (U_1, \varphi_1)\}$ gives a complex atlas on \mathbb{P}^1 .

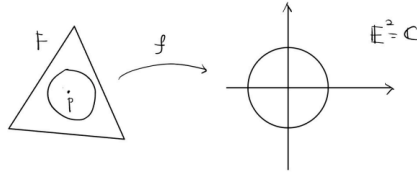
Remark 1.1.4 (complex projective space). The complex projective space \mathbb{P}^n is defined by $\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{0\} / \sim$, where $(x_0, x_1, \dots, x_n) \sim (y_0, y_1, \dots, y_n)$ if and only if there exists $\lambda \in \mathbb{C}^*$ such that $y_i = \lambda x_i$ holds for all $i = 0, 1, \dots, n$, and the equivalent class $[x_0 : x_1 : \dots : x_n]$ is call the homogenous coordinate of \mathbb{P}^n . There is a canonical affine open covering $\{(U_i, \varphi_i)\}$ of \mathbb{P}^n defined by

$$U_i = \{[x_0 : x_1 : \dots : x_n] \mid x_i \neq 0\} \xrightarrow{\varphi_i} \mathbb{C}^n,$$

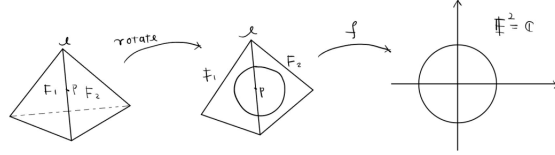
where $\varphi_i([x_0 : x_1 : \dots : x_n]) = (x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i)$, and it makes \mathbb{P}^n to be a complex n -manifold.

Example 1.1.3. Let P be a convex polyhedra in Euclidean 3-dimensional space \mathbb{E}^3 . Topologically P is S^2 , and let's construct a complex atlas on it.

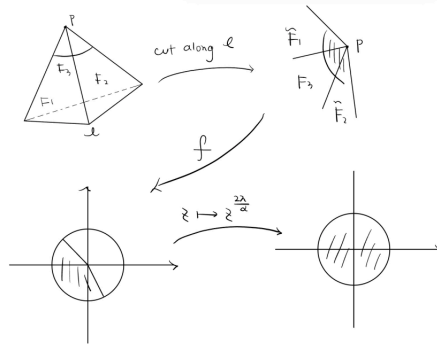
- (1) Suppose p is the interior point of some face F . Since F can be isometrically embedded into \mathbb{E}^2 , we choose an orientation-preserving, isometric embedding f which maps an open neighborhood U of p into $\mathbb{E}^2 = \mathbb{C}$.



- (2) Suppose p is the interior point of some edge $l = F_1 \cap F_2$. Firstly we rotate F_2 along l to the plane of F_1 , and then choose an orientation-preserving, isometric embedding f which maps an open neighborhood U of p into $\mathbb{E}^2 = \mathbb{C}$.



- (3) Suppose p is a vertex which is the intersection of three faces F_1, F_2 and F_3 . Firstly we cut it along some edge $l = F_1 \cap F_2$, and then rotate F_1, F_2 to the plane of F_3 . Then we use some orientation-preserving, isometric embedding f to map it into \mathbb{E}^2 , and finally composite it with $z \mapsto z^{2\pi/\alpha}$.



Exercise 1.1.1. Prove that above constructions give a complex atlas on convex polyhedra.

Remark 1.1.5. All of above three examples give complex structure on topological sphere S^2 , a non-trivial fact is that all of them are the “same” complex structure for S^2 (See Corollary 7.1.4).

Example 1.1.4 (complex torus). For non-zero $w_1, w_2 \in \mathbb{C}$ such that w_1, w_2 are \mathbb{R} -linearly independent, $L = \mathbb{Z}w_1 + \mathbb{Z}w_2$ is a discrete subgroup of $(\mathbb{C}, +)$. Let $\pi: \mathbb{C} \rightarrow T = \mathbb{C}/L$ be the natural projection. Then T equipped with the quotient topology is a connected, Hausdorff and second countable topological space. For $p \in T$, suppose z_0 is an inverse image of p . If we choose $\varepsilon \in \mathbb{R}_{>0}$ such that

$$B_{2\varepsilon}(0) \cap L = \{0\},$$

then $B_\varepsilon(z_0) \xrightarrow{\pi} \pi(B_\varepsilon(z_0)) \subseteq T$ is injective, and thus $\pi^{-1}: \pi(B_\varepsilon(z_0)) \rightarrow B_\varepsilon(z_0) \subseteq \mathbb{C}$ is a homeomorphism. Then $\{\pi(B_\varepsilon(\pi^{-1}(p)))\}_{p \in T}$ gives an open covering of T , and together with π^{-1} it gives a complex atlas of T .

Remark 1.1.6. It's clear complex structure constructed above depends on the choice of w_1, w_2 , but it's not obvious to see whether w_1, w_2 and w'_1, w'_2 give the same complex structure or not. In fact, they give the same complex structure if and only if they differ some elements in $\text{SL}(2, \mathbb{Z})$, and all complex structure on torus are arisen in this way in fact (See Proposition 7.1.2).

1.1.3. Morphisms.

Definition 1.1.4 (holomorphic map). Let X, Y be two Riemann surfaces and $\Phi: X \rightarrow Y$ be a continuous map. For $p \in X$, Φ is called holomorphic at x , if there exists a chart (U, φ) of x , and a chart (V, ψ) of $\Phi(x)$, such that

$$\psi \circ \Phi \circ \varphi^{-1}: \varphi(U \cap \Phi^{-1}(V)) \rightarrow \psi(V \cap \Phi(U))$$

is holomorphic at $\varphi(x)$. Moreover, Φ is called holomorphic in an open subset $W \subseteq X$, if Φ is holomorphic at any point in W .

Remark 1.1.7. It's clear the definition of holomorphic map is independent of the choice of charts, since change of coordinate is biholomorphic.

Definition 1.1.5 (isomorphism). Let $\Phi: X \rightarrow Y$ be a holomorphic map between Riemann surfaces. Φ is called an isomorphism if it's bijective and holomorphic.

Proposition 1.1.1. Let $\Phi: X \rightarrow Y$ be a holomorphic map between Riemann surfaces. Φ is an isomorphism if and only if Φ has an two-side inverse Ψ , and Ψ is holomorphic.

Proposition 1.1.2. The complex projective space is isomorphic to Riemann sphere.

Theorem 1.1.1 (open map theorem). Let $\Phi: X \rightarrow Y$ be a non-constant holomorphic map between Riemann surfaces. Then Φ is open.

Corollary 1.1.1. Let $\Phi: X \rightarrow Y$ be a non-constant holomorphic map between Riemann surfaces and X is compact. Then $\Phi(X) = Y$, and thus Y is compact.

Proof. By open map theorem, $\Phi(X)$ is an open subset of Y , and $\Phi(X)$ is compact in Y , since continuous function maps compact set to compact set. Then $\Phi(X)$ is both open and closed in Y , and thus $\Phi(X) = Y$ since Y is assumed to be connected. \square

Theorem 1.1.2. Let $\Phi: X \rightarrow Y$ be a non-constant holomorphic map between Riemann surfaces. Then for any $p \in Y$, $\Phi^{-1}(p)$ is a discrete set. In particular, if X is compact, then $\Phi^{-1}(p)$ is a non-empty finite set.

1.1.4. Meromorphic functions.

Definition 1.1.6 (singularity). Let X be a Riemann surface and f be a holomorphic function defined on $U \setminus \{x\}$ where $U \subseteq X$ is an open subset. The point x is called a removable singularity/pole/essential singularity, if there exists a chart (U, φ) of x , such that $f \circ \varphi^{-1}: \varphi(U) \rightarrow \mathbb{C}$ has $\varphi(x)$ as a removable singularity/pole/essential singularity.

Remark 1.1.8.

- (1) If $|f(x)|$ is bounded in a punctured neighborhood of x , then x is a removable singularity, and we can cancel the singularity by defining $f(x) = \lim_{y \rightarrow x} f(y)$.
- (2) If $\lim_{y \rightarrow x} |f(y)| = \infty$, then x is a pole.
- (3) If $\lim_{y \rightarrow x} |f(y)|$ doesn't exist, then p is a essential singularity.

Definition 1.1.7 (meromorphic function). Let X be a Riemann surface and f be a holomorphic function defined on $U \setminus \{x\}$ where $U \subseteq X$ is an open subset.

- (1) f is called a meromorphic function at x if x is either a removable singularity or a pole, or f is holomorphic at x .
- (2) f is called a meromorphic function on W , if it's meromorphic at any point $x \in W$.

Remark 1.1.9. If f, g are meromorphic on W , then $f \pm g, fg$ are meromorphic on W . If in addition, $g \not\equiv 0$, then f/g is also meromorphic on W . In other words, the set of meromorphic functions on W forms a field, which is called meromorphic function field.

Example 1.1.5. Consider f, g are two polynomials with $g \not\equiv 0$, then f/g is a meromorphic function on Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$. In fact, all meromorphic functions on S^2 are in this form.

Theorem 1.1.3 (discreteness of singularities and zeros). Let X be a Riemann surface and $W \subseteq X$ be an open subset. If f is a meromorphic function on W , then set of singularities and zeros of f is discrete, unless $f \equiv 0$.

Corollary 1.1.2. Let X be a compact Riemann surface.

- (1) If f is a non-zero meromorphic function, then f has finitely many poles and zeros on X .
- (2) If f, g are two meromorphic functions on an open subset $W \subseteq X$, and f agrees with g on a set with limit point in W , then $f \equiv g$.

1.2. Algebraic curves.

1.2.1. *Affine plane curves.* Let $V \subseteq \mathbb{C}$ be a connected open subset and g be a holomorphic function defined on U . The graph X of g , as a subset of \mathbb{C}^2 is defined by

$$\{(z, g(z)) \mid z \in U\}.$$

Given X the subspace topology, and let $\pi: X \rightarrow U$ be the projection to the first factor. Note that π is a homeomorphism, whose inverse sends the point $z \in U$ to the ordered pair $(z, g(z))$. This makes X a Riemann surface.

A generalization of the graph of holomorphic function is that we consider "Riemann surface" which is locally a graph, but perhaps not globally. The tools we use is implicit function theorem in fact.

Theorem 1.2.1 (The implicit function theorem). Let $f(z, w): \mathbb{C}^2 \rightarrow \mathbb{C}$ be holomorphic function of two variables and $X = \{(z, w) \in \mathbb{C}^2 \mid f(z, w) = 0\}$ be its zero locus. Let $p = (z_0, w_0)$ be a point of X and $\partial f / \partial z(p) \neq 0$. Then there exists a function $g(w)$ defined and holomorphic in a neighborhood of w_0 such that, near p , X is equal to the graph $z = g(w)$.

Method one. If we write $z = a + \sqrt{-1}b, w = c + \sqrt{-1}d$ and $f(z, w) = u + \sqrt{-1}v$, then u, v are smooth functions of a, b, c, d . Moreover, the Cauchy-Riemann equations give

$$\frac{\partial f}{\partial z} = \frac{\partial u}{\partial a} + \sqrt{-1} \frac{\partial v}{\partial a} = \frac{\partial v}{\partial b} - \sqrt{-1} \frac{\partial u}{\partial b} = A + \sqrt{-1}B.$$

Then

$$\frac{\partial(u, v)}{\partial(a, b)} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

and $\det \frac{\partial(u, v)}{\partial(a, b)} = A^2 + B^2 \neq 0$ if and only if $A + \sqrt{-1}B \neq 0$. Then the classical implicit function theorem implies the zero locus

$$\begin{cases} u = 0 \\ v = 0 \end{cases}$$

is locally given by

$$\begin{cases} a = a(c, d) \\ b = b(c, d). \end{cases}$$

In other words, $z = g(w)$. Now it suffices to compute $\partial g / \partial \bar{w}$ to show g is holomorphic. Again by Cauchy-Riemann equations

$$\frac{\partial f}{\partial w} = \frac{\partial u}{\partial c} + \sqrt{-1} \frac{\partial v}{\partial c} = \frac{\partial v}{\partial d} - \sqrt{-1} \frac{\partial u}{\partial d} = C + \sqrt{-1}D.$$

Then by chain rule one has

$$\begin{aligned} \frac{\partial(a, b)}{\partial(c, d)} &= \left(\frac{\partial(u, v)}{\partial(a, b)} \right)^{-1} \frac{\partial(u, v)}{\partial(c, d)} \\ &= \begin{pmatrix} A & B \\ -B & A \end{pmatrix}^{-1} \begin{pmatrix} C & D \\ -D & C \end{pmatrix} \\ &= \frac{1}{A^2 + B^2} \begin{pmatrix} AC + BD & AD - BC \\ BC - AD & BD + AC \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial g}{\partial \bar{w}} &= \frac{1}{2} \left(\frac{\partial}{\partial c} + \sqrt{-1} \frac{\partial}{\partial d} \right) (a + \sqrt{-1}b) \\ &= \frac{1}{2} \left(\frac{\partial a}{\partial c} + \sqrt{-1} \frac{\partial b}{\partial c} + \sqrt{-1} \frac{\partial a}{\partial d} - \frac{\partial b}{\partial d} \right) \\ &= 0 \end{aligned}$$

□

Method two. Firstly let's recall some basic facts in complex analysis: For a holomorphic function f defined on U , the integral

$$\frac{1}{2\pi\sqrt{-1}} \oint_{\partial U} \frac{f'(z)}{f(z)} dz$$

counts the number of zeros of $f(z)$ in U with multiplicity, and the integral

$$\frac{1}{2\pi\sqrt{-1}} \oint_{\partial U} z \frac{f'(z)}{f(z)} dz$$

is the summation of zeros of $f(z)$ in U . Now let's prove the implicit function theorem by using above observation. Fix $w = w_0$, the holomorphic function $f(z, w_0)$ has a zero at $z = z_0$, and we may choose an open neighborhood U of z_0 such that z_0 is the only zero of $f(z, w_0)$ in U since holomorphic function has discrete zeros. Consider the integral

$$\frac{1}{2\pi\sqrt{-1}} \oint_{\partial U} \frac{f_z(z, w)}{f(z, w)} dz = N(w),$$

which is well-defined on sufficiently small neighborhood D_{w_0} of w_0 . It gives a continuous, integer-valued function with $N(w_0) = 1$. This shows $N(w) = 1$ for all $w \in D_{w_0}$, and thus $f(z, w)$ has only one zero for every $w \in D_{w_0}$. Moreover, this zero point z is given by

$$\frac{1}{2\pi\sqrt{-1}} \oint_{\partial U} z \frac{f_z(z, w)}{f(z, w)} dz = g(w),$$

which is holomorphic with respect to w . □

Definition 1.2.1 (affine plane curve). An affine plane curve is the locus of zeros in \mathbb{C}^2 of a (non-trivial) polynomial $f(x, y)$.

Definition 1.2.2 (non-singular).

- (1) A polynomial $f(x, y)$ is non-singular at root x if either $\partial f / \partial x$ or $\partial f / \partial y$ is not zero at x , otherwise it's called singular.
- (2) The affine plane curve X defined by $f(x, y)$ is non-singular is non-singular at $p \in X$ if f is non-singular at x .
- (3) The curve X is non-singular if it's non-singular at each of its points.

Example 1.2.1. The affine plane curve $C \subseteq \mathbb{C}^2$ defined by $x^2 + y^2 - 1$ is non-singular.

Given a non-singular affine plane curve C , by the implicit function theorem, one has C is locally a graph, and thus it gives a complex structure of C . To be precise, suppose C is defined by the non-singular polynomial $f(x, w)$. Let $p = (x_0, y_0) \in C$ with $\partial f / \partial x(p) \neq 0$, then there exists a holomorphic function $g(x)$ such that in an open neighborhood U of p , C is the graph $w = g(x)$. Thus the projection $\pi: U \rightarrow \mathbb{C}$, which maps $(x, y) \rightarrow x$ is a homeomorphism from U to its image, which is an open subset in \mathbb{C} . This gives a complex chart of C .

A straightforward computation shows that complex charts given as above are compatible with each other, and thus it gives a complex structure on C . Moreover, C is second countable and Hausdorff, as a subspace of \mathbb{C}^2 . The only thing we need to check is C is connected. However, if f is an arbitrary non-singular polynomial, the affine plane curve defined by f may not be connected. For example, consider

$$f(x, y) = (x + y)(x + y - 1).$$

Then the affine plane curve defined by above non-singular polynomial is the union of two complex planes which do not meet. Later in Section 3.2.3 we will show that the plane curve defined by an irreducible polynomial must be connected. Thus we have the following theorem.

Theorem 1.2.2. A non-singular affine plane curve defined by an irreducible polynomial is a Riemann surface.

1.2.2. Projective plane curve.

Definition 1.2.3 (projective plane curve). Let F be a homogenous polynomial in $\mathbb{C}[x, y, z]$. A projective plane curve C defined by F is the zero locus of F , that is,

$$C = \{[x : y : z] \in \mathbb{P}^2 \mid F(x, y, z) = 0\}.$$

Remark 1.2.1 (relations between affine plane curve and projective plane curve). Given a projective plane curve C given by homogenous polynomial F . Consider

$$\begin{aligned} \varphi_0: U_0 = \mathbb{C}^2 &\rightarrow \mathbb{P}^2 \\ (y, z) &\mapsto [1 : y : z] \end{aligned}$$

Then $\varphi_0^{-1}(U_0 \cap C) = \{(y, z) \in \mathbb{C}^2 \mid F(1, y, z) = 0\}$ is an affine plane curve, and similarly there are other affine plane curves given by $\varphi_0^{-1}(U_1 \cap C)$ and $\varphi_0^{-1}(U_2 \cap C)$. Conversely, given an affine plane curve C defined by $f \in \mathbb{C}[y, z]$. Consider the homogenous polynomial $F(x, y, z)$ defined by

$$F(x, y, z) = x^d f\left(\frac{y}{x}, \frac{z}{x}\right)$$

where $d = \deg f$. Then F defines a projective plane curve such that the affine plane curve it gives on affine chart U_0 is exactly C .

Definition 1.2.4 (non-singular). A projective plane curve C is non-singular if the affine plane curves $\varphi_i^{-1}(U_i \cap C)$ are non-singular for $i = 0, 1, 2$, where $\varphi_i: U_i \rightarrow \mathbb{P}^2$ are standard affine covering of \mathbb{P}^2 .

Proposition 1.2.1. A projective plane curve $C = \{[x : y : z] : F(x, y, z) = 0\} \subseteq \mathbb{P}^2$ is non-singular if and only if

$$F = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$$

has no solution in \mathbb{P}^2 .

Proof. Since F is a homogenous polynomial, it satisfies the Euler's formula

$$dF = x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} + z \frac{\partial F}{\partial z},$$

where $d = \deg F$. Now let's start our proof as follows:

- (1) Suppose $F = \partial F/\partial x = \partial F/\partial y = \partial F/\partial z = 0$ has a solution (a, b, c) with $a \neq 0$. Then

$$\begin{aligned} \frac{\partial F}{\partial y}(1, \frac{b}{a}, \frac{c}{a}) &= \frac{1}{a^{d-1}} \frac{\partial F}{\partial y}(a, b, c) = 0 \\ \frac{\partial F}{\partial z}(1, \frac{b}{a}, \frac{c}{a}) &= \frac{1}{a^{d-1}} \frac{\partial F}{\partial z}(a, b, c) = 0 \\ F(1, \frac{b}{a}, \frac{c}{a}) &= \frac{1}{a^d} F(a, b, c) = 0. \end{aligned}$$

This shows the affine plane curve $\varphi_0^{-1}(U_0 \cap C)$ is singular, and thus C is singular.

- (2) Conversely, if the projective plane curve defined by F is singular, without lose of generality we may assume $X_0 := \varphi_0^{-1}(U_0 \cap C)$ is singular. Then there exists a solution $(b, c) \in \mathbb{C}^2$ such that

$$F(1, b, c) = \frac{\partial F}{\partial y}(1, b, c) = \frac{\partial F}{\partial z}(1, b, c) = 0.$$

By Euler's formula one has

$$\frac{\partial F}{\partial x}(1, b, c) = dF(1, b, c) - b \frac{\partial F}{\partial y} - c \frac{\partial F}{\partial z} = 0.$$

As a consequence, $(1, a, b)$ is a solution of $F = \partial F/\partial x = \partial F/\partial y = \partial F/\partial z = 0$.

□

Theorem 1.2.3. Any non-singular projective plane curve C is a compact Riemann surface.

Proof. Later we will show that a non-singular homogenous polynomial must be irreducible (See Proposition 3.2.1). Then the three affine charts of C are non-singular affine plane curve defined by irreducible polynomials, and thus Riemann surfaces by Theorem 1.2.2. A straightforward computation shows that the complex structures on each affine charts are compatible, and thus C is a Riemann surface. Moreover, it's compact since \mathbb{P}^2 is compact and C is a closed subset of \mathbb{P}^2 . □

Remark 1.2.2. One way to understand projective plane curve is to regard it as a compactifications of affine plane curve.

Example 1.2.2 (Fermat curve). $x^d + y^d = z^d$ gives a non-singular projective plane curve.

Example 1.2.3. The polynomial $f(x, y) = y^2 - (1 - x^2)(1 - k^2x^2)$, $k \neq 0, \pm 1$ gives a non-singular affine plane curve C . Now we consider the compactification of C . Let $F(x, y, z)$ be the homogenous polynomial given by $f(x, y)$, that is,

$$F(x, y, z) = z^2y^2 - (z^2 - x^2)(z^2 - k^2x^2).$$

$F(x, y, z)$ gives a projective plane curve, and the affine plane curve it gives on the affine chart U_2 is exactly C , so it suffices to see the affine plane curves it gives on the other affine charts.

(1) The affine plane curve it gives on the affine chart U_1 is defined by

$$f(x, 1, z) = z^2 - (z^2 - x^2)(z^2 - k^2x^2).$$

In this case there is a new point $[0 : 1 : 0]$, which is singular.

(2) The affine plane curve it gives on the affine chart U_0 is defined by

$$f(1, y, z) = z^2y^2 - (z^2 - 1)(z^2 - k^2).$$

But in this case, there is no more new point since there is no solution satisfying $z = 0$.

In a summary, the compactification of the affine plane curve C adds a singular point, and later we will see how to handle with singularities by resolutions.

1.2.3. *Quadratic.* A homogenous polynomial F of degree 2 can be written as

$$F = (x, y, z)A \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where $A \in M_{3 \times 3}(\mathbb{C})$ is a symmetric matrix. In this section we will see the projective plane curve C defined by F is determined by the rank of A .

Proposition 1.2.2. If $\text{rk } A = 3$, then F is non-singular, and C is isomorphic to \mathbb{P}^1 .

Method one. If $\text{rk } A = 3$, then there exists $P \in \text{GL}(3, \mathbb{C})$ such that

$$P^T A P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

This shows that after a suitable change of coordinate, we may assume the projective plane curve C defined by F is $\{[x : y : z] \mid x^2 + y^2 - z^2 = 0\} \subseteq \mathbb{P}^2$. The following map gives an isomorphism between C and \mathbb{P}^1 .

$$\Phi: \mathbb{P}^1 \rightarrow C$$

$$[1 : t] \mapsto [1 - t^2 : 2t : 1 + t^2].$$

□

Method two. Consider the following holomorphic embedding

$$\begin{aligned}\Phi: \mathbb{P}^1 &\rightarrow \mathbb{P}^2 \\ [t_0 : t_1] &\mapsto [t_0^2 : t_0 t_1 : t_1^2].\end{aligned}$$

Note that the image of Φ is a projective plane curve defined by the equation $xz = y^2$. On the other hand, after a suitable change of coordinate we may also assume C is defined by this equation since there also exists $P \in \mathrm{GL}(3, \mathbb{C})$ such that

$$P^T A P = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

□

Proposition 1.2.3. If $\mathrm{rk} A = 2$, then C is isomorphic to the union of two \mathbb{P}^1 .

Proof. If $\mathrm{rk} A = 2$, then there exists $P \in \mathrm{GL}(3, \mathbb{C})$ such that

$$P^T A P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This shows the projective plane curve C is defined by $x^2 + y^2 = (x + \sqrt{-1}y)(x - \sqrt{-1}y)$, which is the union of two \mathbb{P}^1 which intersects at $[0 : 0 : 1]$. In particular, it's singular. □

Proposition 1.2.4. If $\mathrm{rk} A = 1$, then C is isomorphic to a double line.

Proof. If $\mathrm{rk} A = 1$, then there exists $P \in \mathrm{GL}(3, \mathbb{C})$ such that

$$P^T A P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This shows the projective plane curve C is defined by $x^2 = 0$, which is a singular projective plane curve called double line. □

2. RAMIFICATION

Topologically speaking a Riemann surface is an orientable 2-dimensional real manifold without boundary. In particular, the topology of a compact Riemann surface can be classified by its genus. So there is a natural question: Given a non-singular projective plane curve C defined by the homogenous polynomial $F(x, y, z) = y^2z - x(x - z)(x - \lambda z)$, $\lambda \neq 0, 1$, topologically C is a closed orientable surface, is there any way to compute its genus?

Consider the following map

$$\begin{aligned}\Phi: C \setminus [0 : 1 : 0] &\rightarrow \mathbb{P}^1 \\ [x : y : z] &\mapsto [x : z].\end{aligned}$$

It's clear that Φ is well-defined holomorphic map. If we desire to extend F to a holomorphic map $\tilde{\Phi}$ defined on C , we need to consider the behavior of C around $[0 : 1 : 0]$. On affine chart $U_1 = \{[x : 1 : z] \mid x, z \in \mathbb{C}\}$, it gives an affine plane curve defined by

$$f(x, z) = z - x(x - z)(x - \lambda z).$$

A direct computation shows that

$$\left. \frac{\partial f}{\partial z} \right|_{(0,0)} = 1, \quad \left. \frac{\partial f}{\partial x} \right|_{(0,0)} = 0.$$

Then by implicit function theorem, C is given by $[x : 1 : z(x)]$ locally around $[0 : 1 : 0]$, and

$$z'(0) = - \left. \frac{\partial p}{\partial x} \right|_{(0,0)} / \left. \frac{\partial p}{\partial z} \right|_{(0,0)} = 0/1 = 0.$$

Thus $x = 0$ is a removable singularity of $z(x)/x$, so it's reasonable to define $\tilde{\Phi}([0 : 1 : 0]) = [1 : 0]$ to give an extension of Φ since for $x \neq 0$,

$$\Phi([x : 1 : z(x)]) = [x : z(x)] = [1 : \frac{z(x)}{x}].$$

There are four special points for $\tilde{\Phi}: C \rightarrow \mathbb{P}^1$, listed as follows

$$\begin{aligned}[0 : 1 : 0] &\mapsto [1 : 0] \\ [0 : 0 : 1] &\mapsto [0 : 1] \\ [z : 0 : 1] &\mapsto [z : 1] \\ [\lambda z : 0 : 1] &\mapsto [\lambda z : 1].\end{aligned}$$

These points are called ramification points/values of $\tilde{\Phi}$, and besides these points, $\tilde{\Phi}$ is a double covering. Such a holomorphic map is called a ramification covering, and in this section we will show that all holomorphic maps between Riemann surfaces are ramification coverings. Moreover, we introduce the Riemann-Hurwitz formula, which gives a method to compute the genus of the ramification covering of a given space.

2.1. Ramification covering.

Theorem 2.1.1 (local normal form). Let $\Phi: X \rightarrow Y$ be a non-constant holomorphic map. Then there are local coordinates (U, φ) and (V, ψ) of p and $\Phi(p)$ respectively, such that

$$\psi \circ \Phi \circ \varphi^{-1}(z) = z^k$$

holds for all $z \in \varphi(U \cap \Phi^{-1}(V))$.

Proof. Firstly we fix a local coordinate (V, ψ) of $\Phi(p)$, and choose a local coordinate (U_1, φ_1) of p such that $\Phi(U) \subset V$. If we denote $\psi \circ \Phi \circ \varphi_1^{-1} = T$, then $T(0) = 0$. Suppose the Taylor expansion of T at $w = 0$ is

$$T(w) = \sum_{k=m}^{\infty} a_k w^k, \quad a_m \neq 0.$$

Then $T(w) = w^m S(w)$, where $S(w)$ is a holomorphic function with $S(0) \neq 0$, and thus there exists a holomorphic function $R(w)$ such that $R^m(w) = S(w)$.

Then $T(w) = (wR(w))^m = (\eta(w))^m$, where $\eta(0) = 0, \eta'(0) = R(0) \neq 0$. By inverse function theorem, there exists a sufficiently small neighborhood $U \subseteq U_1$ of p such that η is invertible in $\varphi_1(U)$, and thus this gives a new local coordinate of p as

$$U_1 \supseteq U \xrightarrow{\varphi_1} \varphi_1(U) \xrightarrow{\eta} \eta \circ \varphi_1(U) \subset \mathbb{C}.$$

If we define $\varphi = \eta \circ \varphi_1$, then with respect to (U, φ) and (V, ψ) , the local representation of Φ is given by

$$\psi \circ \Phi \circ \varphi^{-1}(z) = \psi \circ \Phi \circ \varphi_1^{-1} \circ \eta^{-1}(z) = T(\eta^{-1}(z)) = z^m.$$

□

Definition 2.1.1 (multiplicity). Let $\Phi: X \rightarrow Y$ be a holomorphic map between Riemann surfaces. If its local normal form at point $p \in X$ is given by $z \mapsto z^k$, then k is called the multiplicity¹ of Φ at p , denoted by $\text{mult}_p(\Phi)$.

Definition 2.1.2 (ramification point and ramification value). Let $\Phi: X \rightarrow Y$ be a holomorphic map between Riemann surfaces. A point $p \in X$ is called a ramification point if $\text{mult}_p(\Phi) > 1$, and the image of ramification point is called a ramification value.

Lemma 2.1.1. Let $\Phi: X \rightarrow Y$ be a non-constant holomorphic map between Riemann surfaces. A point $p \in X$ is a ramification point if there exists some local representation of Φ , denoted by T , such that $T'(0) = 0$.

Corollary 2.1.1. The set of ramification points of a holomorphic map is a discrete set.

¹Sometimes this number is also called ramification of F at p .

Theorem 2.1.2. Let $\Phi: X \rightarrow Y$ be a non-constant holomorphic map between compact Riemann surfaces and define

$$d_q(\Phi) = \sum_{p \in \Phi^{-1}(q)} \text{mult}_p(\Phi).$$

Then $d_q(\Phi)$ is independent of $q \in Y$, which is called the degree of Φ , and denoted by $\deg(\Phi)$.

Proof. Suppose $X = Y = \mathbb{D}$ are unit disks and $F: \mathbb{D} \rightarrow \mathbb{D}$ is a holomorphic map defined by $z \mapsto z^m$. Then it's easy to show $d_q(\Phi) = m$, for all $q \in \mathbb{D}$, since for $q = 0$, there is only one preimage of multiplicity m and for $q \neq 0$, there are m preimages of multiplicity 1.

Let's consider the general case. For $q \in Y$, since X is compact, $\Phi^{-1}(q)$ only consists of finitely many points, denoted by $\{p_1, \dots, p_k\}$. Fix a local coordinate (V, ψ) centered at $q \in Y$, for any $i = 1, \dots, k$, there is a local coordinate (U_i, φ_i) centered at $p_i \in X$ such that

$$\psi \circ \Phi \circ \varphi_i^{-1}(z) = z^{m_i}, \quad z \in \varphi_i(U_i),$$

where $m_i = \text{mult}_{p_i}(\Phi)$. If we choose another neighborhood $q \in W \subseteq V$ such that $\Phi^{-1}(W) \subseteq \bigcup_{i=1}^k U_i$, then for any $q \in W$, from the trivial case discussed before one has

$$d_q(\Phi) = \sum_{i=1}^k m_i.$$

This shows $d_q(\Phi)$ is a locally constant function, and thus $d_q(\Phi)$ is constant since Y is connected. \square

Corollary 2.1.2. A holomorphic map between compact Riemann surfaces is an isomorphism if and only if it has degree one.

Corollary 2.1.3. X is a compact Riemann surface, and f is a meromorphic function on X , then the number (counted with multiplicity) of zeros is equal to the number (counted with multiplicity) of poles.

Proof. Note that meromorphic function f on X is equivalent to the holomorphic map Φ from X to S^2 . Then the number of zeros is the multiplicity of Φ at zero and the number of poles is the multiplicity of Φ at ∞ . \square

2.2. Riemann-Hurwitz formula. In this section we talk about Riemann-Hurwitz formula, which computes the genus from a given ramification covering. Before that we recall some basic facts in topology. Let X be a compact oriented surface, the Euler number of X can be defined by the triangulation of X as follows: Suppose a triangulation of X is given with v vertices, e edges and t triangles. Then the Euler characteristic of X is defined by $v - e + t$. On the other hand, the Euler number can also be defined as

$$\chi(X) := \sum_i (-1)^i \dim H_i(X; \mathbb{R}),$$

where $H_i(X; \mathbb{R})$ is the i -th singular homology of X . The genus of X is defined by

$$\chi(X) = 2 - 2 \text{ genus}(X).$$

Theorem 2.2.1 (Riemann-Hurwitz formula). Let $\Phi: X \rightarrow Y$ be a non-constant holomorphic map between compact Riemann surfaces. Then

$$\chi(X) = \deg(\Phi)\chi(Y) - \sum_{p \in X} (\text{mult}_p(\Phi) - 1)$$

Proof. Choose a triangulation Δ of Y such that its vertex are exactly ramification values of F . Let v, e, t denote the number of vertices, edges and triangles of Δ respectively. Suppose Δ' is the triangulation of X obtained by pulling back Δ through F , and use v', e' and t' to denote the number of vertices, edges and triangles of Δ' respectively.

It's clear we have the following relations

$$t' = td, \quad e' = ed$$

where $d = \deg(\Phi)$. For $q \in Y$, note that

$$|\Phi^{-1}(q)| = \sum_{p \in \Phi^{-1}(q)} 1 = d + \sum_{p \in \Phi^{-1}(q)} (1 - \text{mult}_p(\Phi)).$$

Then the relation between v and v' is given by

$$\begin{aligned} v' &= \sum_{\text{vertex } q \text{ of } \Delta} |\Phi^{-1}(q)| \\ &= \sum_{\text{vertex } q \text{ of } \Delta} \left(d + \sum_{p \in \Phi^{-1}(q)} (1 - \text{mult}_p(\Phi)) \right) \\ &= vd + \sum_{\text{vertex } q \text{ of } \Delta} \left(\sum_{p \in \Phi^{-1}(q)} (1 - \text{mult}_p(\Phi)) \right) \\ &= vd + \sum_{p \in X} (1 - \text{mult}_p(\Phi)). \end{aligned}$$

Thus by the relation between Euler number and triangulation, we obtain the desired conclusion. \square

Remark 2.2.1. Since the set of ramification points is finite, then $\sum_{p \in X} (\text{mult}_p(\Phi) - 1)$ is a finite number, and for convenience we denote it by $B(\Phi)$. It describes how many ramification points of Φ are there on X .

Definition 2.2.1 (ramified holomorphic map). A holomorphic map Φ is called ramified if $B(\Phi) > 0$.

Definition 2.2.2 (unramified holomorphic map). A holomorphic map Φ is called unramified if $B(\Phi) = 0$.

Remark 2.2.2. A unramified holomorphic map is a covering map, and thus ramified holomorphic map is sometimes called ramified covering map.

Corollary 2.2.1. Let $\Phi: X \rightarrow Y$ be a non-constant holomorphic map between compact Riemann surfaces. Then

- (1) If Y is Riemann sphere and $\deg(\Phi) > 1$, then Φ must be ramified.
- (2) If $\text{genus}(X) = \text{genus}(Y) = 1$, then Φ must be unramified.
- (3) If $\text{genus}(X) = \text{genus}(Y) > 1$, then Φ must be an isomorphism.

Proof.

- (1) Since Riemann sphere has genus zero, one has

$$B(\Phi) = 2(\deg(\Phi) - 1) + 2\text{genus}(X) > 0.$$

- (2) By Riemann-Hurwitz formula we have

$$0 = 0 + B(\Phi).$$

- (3) By Riemann-Hurwitz formula we have

$$(1 - \deg(\Phi))(2\text{genus}(X) - 2) = B(\Phi).$$

Then $\deg(\Phi) = 1$, since $\deg(\Phi) \geq 1$, $2\text{genus}(X) - 2 > 0$ and $B(\Phi) \geq 0$.

□

2.2.1. Genus of projective plane curve. Now we're going to use Riemann-Hurwitz formula to compute the genus of projective plane curves. Firstly consider the example at the beginning of this section, that is, the non-singular projective plane curve C is defined by homogenous polynomial

$$F(x, y, z) = y^2z - x(x - z)(x - \lambda z),$$

where $\lambda \neq 0, 1$. The ramification covering $\tilde{\Phi}: C \rightarrow \mathbb{P}^1$ has degree 2, and the ramification values are $[1 : 0], [0 : 1], [z : 1], [\lambda z : 1]$. Then by Riemann-Hurwitz formula one has

$$\chi(C) = 2 \times 2 - 4$$

This shows the genus of C is 1.

Example 2.2.1 (Fermat curve). Let C be the projective plane curve defined by the homogenous polynomial $F(x, y, z) = x^d + y^d - z^d$. A direct computation shows C is non-singular, and thus it gives a Riemann surface. Consider the holomorphic map

$$\begin{aligned} \Phi: C &\rightarrow \mathbb{P}^1 \\ [x : y : z] &\mapsto [x : z]. \end{aligned}$$

Note that

$$y^d = z^d - x^d = (x - \alpha_1 z) \dots (x - \alpha_d z),$$

where $\alpha_1, \dots, \alpha_d \in \mathbb{C}$ are different d -th unit roots. Then Φ is a ramification covering of degree d , and has d ramification values. Then by Riemann-Hurwitz formula,

$$\chi(C) = 2 \times d - d(d - 1).$$

This shows the genus of C is $(d-1)(d-2)/2$.

Remark 2.2.3. In general, for a non-singular projective plane curve C defined by a homogenous polynomial of degree d , the genus of C is $(d-1)(d-2)/2$, and this is called Plücker's formula or genus-degree formula. Moreover, if C is singular, then the genus of the normalization of C is

$$\frac{(d-1)(d-2)}{2} - \delta,$$

where $\delta > 0$ is related to the type of singularities of C .

3. BEZOUT THEOREM

3.1. Statement and proof. Let C, C' be a non-singular projective plane curves defined by a homogenous polynomials F, G such that F, G has no common divisors². In this section we will show how to count the number of the intersections of C and C' .

Definition 3.1.1 (intersection number). The intersection number at point $p \in C \cap C'$ is the order of zero of G at p on some affine chart on C . The intersection number of C and C' is the summation of intersection numbers of all intersections $p \in C \cap C'$.

Remark 3.1.1. Note that the change of affine charts does not change the vanishing order of a polynomial. This shows the intersection number of an intersection is well-defined. For convenience, the intersection number at point p is denoted by $(C, C')_p$, and the intersection number of C and C' is denoted by (C, C') , that is,

$$(C, C') = \sum_{p \in C \cap C'} (C, C')_p.$$

A standard exercise to show $(C, C')_p = (C', C)_p$.

Theorem 3.1.1 (Bezout theorem). Let C, C' be two non-singular projective plane curves defined by homogenous polynomials F, G such that F, G has no common divisors. Then the intersection number

$$(C, C') = ed,$$

where $\deg F = e, \deg G = d$.

Proof. Let L be a linear homogenous polynomial such that $L \nmid F$ and H be the projective line defined by L . Consider the holomorphic map

$$\Phi: C \rightarrow \mathbb{P}^1$$

$$[x : y : z] \mapsto [L^d : G].$$

Since C is compact, by Corollary 1.1.1 one has Φ is surjective.

(1) Suppose Φ is a non-constant holomorphic map. Note that the order of zeros of Φ equals (C, H^d) , and the order of poles of Φ equals to (C, C') . Then

$$(C, H^d) = (C, C').$$

since both order of zeros and order of poles are degree of Φ . By definition one has

$$(C, H^d) = d(C, H).$$

Now it suffices to show a projective plane curve defined by a homogenous polynomial with degree e intersects a projective line e times, which is straightforward.

²Since both F and G are irreducible, this assumption exclude the trivial cases $F \mid G$ and $G \mid F$.

- (2) If Φ is a constant holomorphic map, then there exists a constant $\lambda \in \mathbb{C}^*$ such that $G = \lambda L^d$. Again one has

$$(C, L^d) = (C, \lambda H^d) = (C, C'),$$

since $\lambda \neq 0$.

□

3.2. Applications.

3.2.1. *Plücker formula.* In this section we will prove Plücker formula as a consequence of Bezout theorem, but before that we prove a technique lemma.

Lemma 3.2.1. Let C be a projective plane curve of degree d . Then there exists an affine coordinate $[x : y : 1] \subseteq \mathbb{P}^2$ such that C is given by the following equation

$$f(x, y) = y^d + a_1(x)y^{d-1} + \cdots + a_d(x) = 0,$$

where $a_i(x) \in \mathbb{C}[x]$ with $\deg a_i(x) \leq i$, or $a_j(x) = 0$.

Proof. Let $[z : w : 1]$ be an arbitrary affine coordinate of \mathbb{P}^2 and C is defined by

$$p'(z, w) = 0, \quad \deg p' = d.$$

If p' is not of the necessary form, then consider the following coordinate transformation

$$\begin{aligned} z &= x + \lambda y \\ w &= y. \end{aligned}$$

Consider the coefficient $b(\lambda)$ of the term involving y^n in $f'(x + \lambda y, y)$. It's clear $b(\lambda)$ is a non-zero polynomial in λ , and hence can equal 0 for only a finite number of values of λ . Then we choose λ such that $b(\lambda) \neq 0$, and for such a chosen λ , we consider

$$f(x, y) = \frac{1}{b(\lambda)} f'(x + \lambda y, y).$$

Then in affine coordinate $[x : y : 1]$, the equation of C is

$$f(x, y) = 0,$$

which satisfies our desire. □

Corollary 3.2.1 (Plücker formula). Let $C \subseteq \mathbb{P}^2$ be a non-singular projective plane curve of degree d . Then the genus of C is $(d-1)(d-2)/2$.

Proof. By Lemma 3.2.1, without lose of generality we may assume C is defined by the non-singular homogenous polynomial F with

$$F(x, y, z) = y^d - a_1(x, z)y^{d-1} - \cdots - a_d(x, z).$$

Then consider the following holomorphic map

$$\begin{aligned} \Phi: C &\rightarrow \mathbb{P}^1 \\ [x : y : z] &\mapsto [x : z], \end{aligned}$$

which is a ramification covering in fact. Now by Riemann-Hurwitz formula it suffices to compute the ramification data of Φ . On affine charts $U_2 = \{[x : y : 1]\} \subseteq \mathbb{P}^2$, C is defined by

$$f(x, y) = y^d - a_1(x, 1)y^{d-1} - \cdots - a_d(x, 1) = 0.$$

- (1) If $f_y(x_0, y_0) \neq 0$, then by implicit function theorem, around the point $[x_0 : y_0 : 1]$, the affine plane curve $C \cap U_2$ is given by $[x : y(x) : 1]$, and thus Φ is a local diffeomorphism at this point.
- (2) If $f_y(x_0, y_0) = 0$, then $f_x(x_0, y_0) \neq 0$ since f is non-singular. By implicit function theorem again, around $[x_0 : y_0 : 1]$, there exists a local coordinate $y \mapsto [x(y) : y : 1]$, and Φ is given by $y \mapsto x(y)$. By chain rule one has

$$x'(y) = -(f_x(x(y), y))^{-1} f_y(x(y), y).$$

This shows the order of zero of $x'(y)$ equals to the order of zero of $f_y(x(y), y)$.

This shows $B(\Phi) = \sum_{p \in C} (\text{mult}_p(\Phi) - 1)$ is exactly the intersection number of F and F_y , and since both F and F_y are non-singular homogenous polynomial, by Bezout theorem one has

$$B(\Phi) = d(d-1).$$

By Rmann-Hurwitz formula, the genus of C is $(d-1)(d-2)/2$. \square

3.2.2. Non-singular homogenous polynomial is irreducible. Another application of Bezout theorem is that any non-singular homogenous polynomial is irreducible.

Proposition 3.2.1. Let F be a non-singular homogenous polynomial. Then F is irreducible.

Proof. On contrary we suppose $F = F_1 F_2$. By chain rule of derivative it's easy to see both F_1 and F_2 are non-singular. Then by Bezout theorem, F_1 and F_2 have at least a common zero, which contradicts to P is non-singular, since P is singular at the common zero of F_1 and F_2 , which can be shown by chain rule of derivatives again. \square

3.2.3. Connectness of irreducible plane curve. In this section, we will prove the connectness of plane curves as we mentioned before. In fact, we will prove the following stronger theorem.

Theorem 3.2.1. Let F be an irreducible homogenous polynomial and C be the projective plane curve defined by F . Then the set of singularities S is finite, and $C \setminus S$ is connected.

Before starting the proof, we prepare some basic facts we will use.

Lemma 3.2.2. If R is a UFD and

$$\begin{aligned} f &= a_0 x^m + a_1 x^{m-1} + \cdots + a_m, \\ g &= b_0 x^n + b_1 x^{n-1} + \cdots + b_n \end{aligned}$$

are polynomials in $R[x]$ with $a_0 \neq 0, b_0 \neq 0$. Then f, g has a non-trivial common divisor if and only if there exists $F, G \in R[x]$ with $\deg F < m, \deg G < n$ such that

$$f \cdot G = F \cdot g.$$

Proof. On one hand, if f, g has a non-trivial common divisor h , then

$$f = h \cdot F$$

$$g = h \cdot G.$$

This shows $f \cdot G = F \cdot g$, where $\deg F < \deg f \leq m$ and $\deg G < \deg g \leq n$.

On the other hand, if $f \cdot G = F \cdot g$ with $\deg F < m$ and $\deg G < n$, then all factors of f cannot be all factors of F since $\deg f > \deg F$. Hence there exists a non-trivial divisor of f which is also a divisor of g since $R[x]$ is UFD by Gauss lemma. \square

Suppose

$$F(x) = A_0x^{m-1} + \cdots + A_{m-1}$$

$$G(x) = B_0x^{n-1} + \cdots + B_{n-1}.$$

Then $f \cdot G = F \cdot g$ if and only if

$$(3.1) \quad \begin{cases} a_0B_0 = b_0A_0 \\ a_1B_0 + a_0B_1 = b_1A_0 + b_0A_1 \\ \vdots \\ a_mB_{m-1} = b_nA_{m-1}. \end{cases}$$

Thus $f \cdot G = F \cdot g$ has non-zero solutions F, G if and only if (3.1) has a non-zero solution $(A_0, \dots, A_{m-1}, B_0, \dots, B_{n-1})$. Then by basic theory of systems of linear equations, (3.1) has a non-zero solution if and only if the following determinant equals to zero.

$$(3.2) \quad \det \begin{pmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 & b_1 & b_0 & \cdots & 0 \\ a_2 & a_1 & \cdots & 0 & b_2 & b_1 & \cdots & 0 \\ \vdots & \vdots & \cdots & a_0 & \vdots & \vdots & \cdots & b_0 \\ a_m & a_{m-1} & \cdots & \vdots & b_n & b_{n-1} & \cdots & \vdots \\ 0 & a_m & \cdots & \vdots & 0 & b_n & \cdots & \vdots \\ \vdots & \vdots & \cdots & a_{m-1} & \vdots & \vdots & \cdots & b_{n-1} \\ 0 & 0 & \cdots & a_m & 0 & 0 & \cdots & b_n \end{pmatrix}$$

Definition 3.2.1 (resultant). If R is a ring and

$$f = a_0x^m + a_1x^{m-1} + \cdots + a_m,$$

$$g = b_0x^n + b_1x^{n-1} + \cdots + b_n$$

are polynomials in $R[x]$. The resultant of f, g is defined as the determinant in (3.2), and denoted by $\mathcal{R}(f, g)$.

Theorem 3.2.2. If R is a UFD and

$$f = a_0x^m + a_1x^{m-1} + \cdots + a_m,$$

$$g = b_0x^n + b_1x^{n-1} + \cdots + b_n$$

are polynomials in $R[x]$ with $a_0 \neq 0$, then

- (1) f, g have a non-trivial common divisor if and only if $\mathcal{R}(f, g) = 0$;
- (2) there exists polynomial $\alpha, \beta \in R[x]$, with $\deg \alpha < n, \deg \beta < m$ such that

$$\alpha(x)f(x) + \beta(x)g(x) = \mathcal{R}(f, g).$$

Definition 3.2.2 (discriminant). Let R be a ring and $f \in R[x]$. The discriminant of p is defined by $\mathcal{D}(f) := \mathcal{R}(f, f')$, where f' is the formal derivative of f .

Corollary 3.2.2. Let R be a UFD and $f \in R[x]$. Then f has a multiple root if and only if $\mathcal{D}(f) = 0$.

Now let's start the proof of Theorem 3.2.1.

Proof. Firstly let's show F has only finitely many singularities. By Lemma 3.2.1, without loss of generality we may assume C is defined by

$$f(x, y) = y^d + a_1(x)y^{d-1} + \cdots + a_d(x)$$

on some affine chart. If we regard $f(x, y)$ and $f_y(x, y)$ as elements in $\mathbb{C}[x][y]$, then $\mathcal{R}(f, f_y) \in \mathbb{C}[x]$, which is a non-zero polynomial since $f(x, y)$ is irreducible. By Theorem 3.2.2 there exists $\alpha, \beta \in \mathbb{C}[x, y]$ such that

$$\alpha(x, y)f(x, y) + \beta(x, y)f_y(x, y) = \mathcal{R}(f, f_y)(x).$$

If point (x_0, y_0) such that $f(x_0, y_0) = f_y(x_0, y_0) = 0$, then

$$\mathcal{R}(f, f_y)(x_0) = 0.$$

This shows $f(x, y) = f_y(x, y) = 0$ has finitely many solutions, and thus C only has finitely many singularities on this affine chart. On the other hand, the infinity line $z = 0$ only intersects with C finitely many times, and thus there are at most finitely many singularities on $z = 0$. As a consequence, C has only finitely many singularities.

To prove $C^* = C \setminus S$ is connected, it suffices to show C^* is connected on the affine chart $U_2 = \{[x : y : 1]\}$ since

$$C^* \cap U_2 \subseteq C^* \subseteq C = \overline{C^* \cap U_2},$$

and a basic fact in point set topology says that if a set is connected, so is its closure. For convenience, in the following proof we still use C to denote the affine plane curve $C \cap U_2$. Now consider the ramification covering

$$\Phi: C \rightarrow \mathbb{C}$$

$$(x, y) \mapsto x.$$

If we define $B = \{x_0 \in \mathbb{C} \mid \mathcal{R}(f, f_y)(x_0) = 0\} \subseteq \mathbb{C}^1$, then the argument in the proof of Corollary 3.2.1 can be used here to show $\Phi: C \setminus \Phi^{-1}(B) \rightarrow \mathbb{C} \setminus B$

is a local diffeomorphism. Thus $\Phi: C \setminus \Phi^{-1}(B) \rightarrow \mathbb{C} \setminus B$ is a covering map on each component of $C \setminus \Phi^{-1}(B)$ since Φ is a proper.

For each point $x_0 \notin B$, the fiber $\Phi^{-1}(x_0)$ are exactly the d distinct solutions of $y^d + a_1(x_0)y^{d-1} + \dots + a_d(x_0) = 0$ has d distinct solutions, denoted by $\{y_1(x_0), \dots, y_d(x_0)\}$. By the basic theory of covering space, there is an action of the fundamental group $\pi_1(\mathbb{C} \setminus B, x_0)$ on the fiber $\Phi^{-1}(x_0)$. To be precise, given $[\gamma] \in \pi_1(\mathbb{C} \setminus B, x_0)$, we choose arbitrary representative $\gamma \in [\gamma]$ and consider its lift $\tilde{\gamma}$, which is independent of the choice of γ . If $y_i(x_0)$ and $y_j(x_0)$ are endpoints of $\tilde{\gamma}$, then $[\gamma] \cdot y_i(x_0) = y_j(x_0)$. Thus it's clear to see the number of connected components of $C \setminus \Phi^{-1}(B)$ equals to the number of orbits of $\Phi^{-1}(x_0)$ under the $\pi_1(\mathbb{C} \setminus B, x_0)$ -action.

Suppose $\{y_1(x_0), \dots, y_l(x_0)\}$ is an orbit of $\pi_1(\mathbb{C} \setminus B, x_0)$ -action. Then for any $x \notin B$, we choose a path $\gamma: [0, 1] \rightarrow \mathbb{C} \setminus B$ connecting x_0 and x . Then γ has l different liftings ending at points $y_1(x), \dots, y_l(x)$, which can be extended as holomorphic functions defined on an open neighborhood of x . If we define

$$\begin{aligned} \sigma_1(x) &= \sum_i y_i(x) \\ \sigma_2(x) &= \sum_{i < j} y_i(x) y_j(x) \\ &\vdots \\ \sigma_l(x) &= y_1(x) \dots y_d(x), \end{aligned}$$

then $\sigma_i(x)$ does not depend on the choice of paths connecting x_0 and x , and thus $\sigma_i(x)$ are holomorphic functions defined over $\mathbb{C} \setminus B$. By Rouché's theorem, one can see these $\sigma_i(x)$ has polynomial growth, that is, there exists constants C and N such that

$$|\sigma_i(x)| < C|x|^N$$

holds for all $i = 1, \dots, l$. Then by Riemann extension theorem one has $\sigma_i(x)$ are defined on \mathbb{C} , and they are polynomials of x in fact. Note that

$$(y - y_1(x)) \dots (y - y_l(x)) \mid f(x, y).$$

Then

$$g(x, y) = y^d - \sigma_1(x)y^{d-1} + \sigma_2(x)y^{d-2} + \dots + (-1)^l \sigma_l(x) \in \mathbb{C}[x, y]$$

also divides $f(x, y)$. But since $f(x, y)$ is irreducible, one has $f = g$, and thus the $\pi_1(\mathbb{C} \setminus B, x_0)$ -action is transitive as desired. \square

4. DIFFERENTIAL FORMS

4.1. Differential forms, differential operators and integrations.

4.1.1. *Differential forms.* Firstly let's consider the differential forms defined on an open subset $U \subseteq \mathbb{C}$. Suppose $\{z\}$ is the coordinate on \mathbb{C} . Then

- (1) A smooth 1-form is of the form $f dz + g d\bar{z}$, where f, g are smooth functions, and the set of all smooth 1-forms defined on U is denoted by $\mathcal{A}^1(U)$.
- (2) A smooth 1-form is a $(1, 0)$ -form, if it's of the form $f dz$, where f is a smooth function, and the set of all $(1, 0)$ -form defined on U is denoted by $\mathcal{A}^{1,0}(U)$.
- (3) A smooth 1-form is a $(0, 1)$ -form, if it's of the form $f d\bar{z}$, where f is a smooth function, and the set of all $(0, 1)$ -form defined on U is denoted by $\mathcal{A}^{0,1}(U)$.
- (4) A smooth 1-form is a holomorphic 1-form, if it's of the form $f dz$, where f is a holomorphic function, and the set of all holomorphic 1-form defined on U is denoted by $\Omega^1(U)$.
- (5) A smooth 2-form is of the form $f dz \wedge d\bar{z}$, where f is a smooth function, and the set of all smooth 2-forms defined on U is denoted by $\mathcal{A}^2(U)$.
- (6) A holomorphic 2-form is of the form $f dz \wedge d\bar{z}$, where f is a holomorphic function, and the set of all holomorphic 2-forms defined on U is denoted by $\Omega^2(U)$.

Remark 4.1.1. It's clear $\mathcal{A}^1(U) = \mathcal{A}^{1,0}(U) \oplus \mathcal{A}^{0,1}(U)$.

If we want to define differential forms on Riemann surfaces, a natural idea is to define them on each coordinate chart, and glue them together in a suitable way, so we need to know what will happen under the holomorphic change of coordinate.

Suppose $\Phi: U \rightarrow V$ is a holomorphic function between open subsets $U, V \subseteq \mathbb{C}$ and $\theta = f dw + g d\bar{w}$ is a smooth 1-form on V . Then the pullback of θ is defined by

$$\Phi^*(\theta) = f(\Phi(z))\Phi'(z)dz + g(\Phi(z))\overline{\Phi'(z)}d\bar{z}.$$

Similarly, if $\theta = f dw \wedge d\bar{w}$ is a smooth 2-form, then the pullback is defined by

$$\Phi^*(\theta) = f(\Phi(z))|\Phi'(z)|^2 dz \wedge d\bar{z}.$$

In fact, pullback is a contravariant functor.

Definition 4.1.1 (differential k -form). A differential k -form θ on a Riemann surface X assigns to any local coordinate $\varphi: U \rightarrow V$ a k -form α , and assignments are compatible³ with the charts.

³This means if $U' \xrightarrow{\varphi'} V'$ is another local coordinate assigned with k -form β , then

$$\Phi^*(\beta) = \alpha,$$

where $\Phi = \varphi' \circ \varphi^{-1}(z)$.

Definition 4.1.2 ((1, 0)-form and (0, 1)-form). A differential 1-form θ on a Riemann surface X is called

- (1) a (1, 0)-form, if it can be represented as $f dz$ locally, where f is a smooth function;
- (2) a (0, 1)-form, if it can be represented as $f d\bar{z}$ locally, where f is a smooth function.

Definition 4.1.3 (holomorphic 1-form). A holomorphic 1-form θ on a Riemann surface X is a differential (1, 0)-form which can be locally represented as $f(z)dz$, where f is a holomorphic function.

4.1.2. *Differential operators.* Given a smooth function f defined on an open subset $U \subseteq \mathbb{C}$, one has

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}.$$

Then we can define the operators ∂ and $\bar{\partial}$ on smooth functions as follows

$$\begin{aligned}\partial f &:= \frac{\partial f}{\partial z} dz \\ \bar{\partial} f &:= \frac{\partial f}{\partial \bar{z}} d\bar{z}.\end{aligned}$$

Similarly for a smooth 1-form $\theta = f dz + g d\bar{z}$. Then

$$d\theta = \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz + \frac{\partial g}{\partial z} dz \wedge d\bar{z} = \left(\frac{\partial g}{\partial z} - \frac{\partial f}{\partial \bar{z}} \right) dz \wedge d\bar{z}.$$

Then we can define the operators ∂ and $\bar{\partial}$ on smooth 1-form $\theta = f dz + g d\bar{z}$ as follows

$$\begin{aligned}\partial\theta &:= \partial g \wedge d\bar{z} \\ \bar{\partial}\theta &:= \bar{\partial} f \wedge dz.\end{aligned}$$

Thus we have constructed differential operators d, ∂ and $\bar{\partial}$ on open subset $U \subseteq \mathbb{C}$, and above constructions can also be paralld to the Riemann surface X .

Theorem 4.1.1.

- (1) $d = \partial + \bar{\partial}$.
- (2) $d^2 = \partial^2 = \bar{\partial}^2 = 0$.
- (3) $\partial\bar{\partial} = -\bar{\partial}\partial$.
- (4) A (1, 0)-form θ is holomorphic if and only if $d\theta = \bar{\partial}\theta = 0$.
- (5) d, ∂ and $\bar{\partial}$ satisfy the Leibniz rule, and commute with pullback.

4.1.3. *Integrations of differential forms.* Let θ be a smooth 1-form on a Riemann surface X and γ be a piecewise smooth curve on X . Suppose the curve γ is divided into $\gamma = \gamma_1 \cup \cdots \cup \gamma_n$, such that $\gamma_i: [a_i, b_i] \rightarrow U_i$, where (U_i, φ_i) is a local coordinate. If θ is given by $f_i dz_i + g_i d\bar{z}_i$ in the local chart (U_i, φ_i) , then the integration of θ along γ is defined by

$$\int_{\gamma} \theta = \sum_{i=1}^n \int_{\gamma_i} \theta := \sum_{i=1}^n \int_{a_i}^{b_i} \{f \cdot z'_i(t) + g \cdot \overline{z'_i(t)}\} dt.$$

Similarly, if η is a 2-form and D is a region on X , we also divide D into $D = D_1 \cup \cdots \cup D_n$ such that each D_i lies in some local chart (U_i, φ_i) . If we write $z_i = x_i + \sqrt{-1}y_i$, then

$$dz_i \wedge d\bar{z}_i = (dx_i + \sqrt{-1}dy_i) \wedge (dx_i - \sqrt{-1}dy_i) = -2\sqrt{-1}dx_i \wedge dy_i.$$

Thus if η is given locally by

$$f dz_i \wedge d\bar{z}_i,$$

then the integration is defined by

$$\int_D \eta = \sum_{i=1}^n \int_{D_i} \eta := \sum_{i=1}^n \int_{\varphi_i(D_i)} -2\sqrt{-1}f dx_i \wedge dy_i.$$

Theorem 4.1.2 (Stokes). Let X be a Riemann surface and θ be a smooth 1-form. If D is a compact region with piecewise smooth boundary ∂D , then

$$\int_D d\theta = \int_{\partial D} \theta.$$

4.2. Holomorphic 1-form and meromorphic 1-form.

4.2.1. Holomorphic 1-form.

Example 4.2.1. Consider the non-singular affine plane curve C defined by $f(x, y) = y^2 - x(x-1)(x-\lambda) = 0$. Then dx/y is a holomorphic 1-form on C .

- (1) For point (x, y) with $y \neq 0$, dx/y is a well-defined holomorphic 1-form.
- (2) For point (x, y) with $y = 0$, since C is non-singular, at this point one has $f_x \neq 0$. Note that $f(x, y) = 0$ holds on C , and thus one has $f_x dx + f_y dy = 0$ holds on C , which implies

$$\frac{dx}{2y} = -\frac{dy}{f_x}.$$

This shows dx/y is always a well-defined holomorphic 1-form on C .

More generally, arguments shown in above example can be used to prove the following proposition.

Proposition 4.2.1. Let C be a non-singular affine plane curve defined by $f(x, y) = 0$. Then

$$\omega = \frac{dx}{f_y} = \frac{dy}{f_x}$$

is a holomorphic 1-form on C .

Proposition 4.2.2. Let C be a non-singular projective plane curve defined by $F(x, y, z) = 0$ with $\deg F \geq 3$. Then the holomorphic 1-form

$$\omega = \frac{dx}{F_y(x, y, 1)}$$

on the affine piece $\{z = 1\}$ extends to a holomorphic 1-form on C .

Proof. Firstly we extend the holomorphic 1-form as follows

$$\omega = \frac{d(x/z)}{F_y(x/z, y/z, z/z)}.$$

Then on the affine piece defined by $x = 1$, one has

$$\omega = -\frac{z^{d-3}dz}{F_y(1, y, z)} = \frac{z^{d-3}dz}{F_z(1, y, z)}.$$

Thus if $d \geq 3$, the extension of ω is a holomorphic 1-form defined on C . \square

Remark 4.2.1. More generally, if $g(x, y) \in \mathbb{C}[x, y]$ is a polynomial, then by the same argument one can show that the holomorphic 1-form

$$\omega = \frac{g(x, y)dx}{F_y(x, y, 1)}$$

defined on affine piece also extends to a holomorphic 1-form on C if $\deg g \leq d - 3$. Note that the dimension of vector space consisting of homogenous polynomial with degree $d - 3$ in three variables are $\binom{d-1}{2} = (d-1)(d-2)/2$. In fact, any holomorphic 1-form on C is of this form, and dimension of vector space consisting of all holomorphic 1-forms is genus. This gives an another viewpoint to genus formula.

4.2.2. Meromorphic 1-forms.

Definition 4.2.1 (meromorphic 1-form). A meromorphic 1-form θ on a Riemann surface X is a smooth $(1, 0)$ -form which can be locally represented as $f(z)dz$, where f is a meromorphic function.

Recall that given a meromorphic function f on a Riemann surface X , for $p \in X$, we can choose a local coordinate z centered at p , and consider the Laurent series of $f \circ \varphi^{-1}(z)$ as

$$f(z) = \sum_{n=m}^{\infty} c_n z^n, \quad c_m \neq 0.$$

The order of f at p is defined by m and denoted by $\text{ord}_p(f)$.

Lemma 4.2.1. $\text{ord}_p(f)$ is independent of the choice of local coordinate.

Proof. A meromorphic function f on a Riemann surface X corresponds to a holomorphic map $\Phi: X \rightarrow S^2$. If p is a zero point of f , then $\text{ord}_p(f) = \text{mult}_p(\Phi)$, and if p is a pole of f , then $\text{ord}_p(f) = -\text{mult}_p(\Phi)$. \square

Let θ be a meromorphic 1-form on Riemann surface X , in local coordinate z centered at p , we can write

$$\theta = f(z)dz$$

so we can define $\text{ord}_p(\theta) = \text{ord}_p(f)$, and clearly it's independent of the choice of local coordinate.

Example 4.2.2. Let $X = S^2 = \mathbb{C} \cup \{\infty\}$ and $\{(\mathbb{C}, z), (\mathbb{C}, w)\}$ be an atlas of S^2 , where the transition is given by $w = 1/z$. Consider 1-form θ which locally looks like dz on (\mathbb{C}, z) . Using holomorphic change of coordinate, we have θ looks like

$$\theta = \frac{-1}{z^2} dz$$

on (\mathbb{C}, w) . Clearly θ is a meromorphic 1-form, and

$$\text{ord}_p(\theta) = \begin{cases} 0, & p \in S^2 \setminus \{\infty\} \\ -2, & p = \infty \end{cases}$$

Then

$$\sum_{p \in S^2} \text{ord}_p(\theta) = -2.$$

Example 4.2.3. Let $\{(\mathbb{C}, z), (\mathbb{C}, w)\}$ be an atlas of \mathbb{C}_∞ with transition function $z = 1/w$. Suppose meromorphic 1-form θ is locally given by a rational function $r(z)$ on (\mathbb{C}, z) , where

$$r(z) = c \prod_{j=1}^n (z - \lambda_j)^{a_j},$$

where $c \neq 0, a_i \in \mathbb{Z}, \lambda_j \neq \lambda_i \in \mathbb{C}$. Thus in the coordinate chart (\mathbb{C}, w) , one has

$$\theta = c \prod_{j=1}^n \left(\frac{1}{w} - \lambda_j\right)^{a_j} \left(-\frac{1}{w^2}\right) dw$$

This shows

$$\text{ord}_p(\theta) = \begin{cases} a_j, & p = \lambda_j \\ -2 - \sum_{j=1}^n a_j, & p = \infty. \end{cases}$$

Then

$$\sum_{p \in S^2} \text{ord}_p(\theta) = -2.$$

Remark 4.2.2. This shows for a meromorphic 1-form θ on Riemann sphere S^2 , one always has

$$\sum_{p \in S^2} \text{ord}_p(\theta) = 2 \text{genus}(S^2) - 2.$$

Later we will see it's not a coincidence.

4.3. Residue theorem. Let θ be a meromorphic 1-form on a Riemann surface X . Suppose θ is locally given by $f dz$, where f is a meromorphic function. The order of f lose too many information given by the coefficient of its Laurent series and we want to keep track coefficients which are invariant under the holomorphic change of local coordinate. Luckily, there exists such an invariant, that is -1 -th coefficient of Laurent series c_{-1} .

Definition 4.3.1 (residue). The residue of a meromorphic 1-form θ is defined by $\text{Res}_p(\theta) = c_{-1}$.

The following lemma shows that the residue is independent of the choice of local coordinate, and gives a formula to compute it.

Lemma 4.3.1. Let D be any compact region in Riemann surface X such that $p \in D \setminus \partial D$, ∂D is piecewise smooth, and θ cannot have poles in $D \setminus \{p\}$. Then

$$\text{Res}_p(\theta) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial D} \theta.$$

Proof. Choose $D' \subseteq D$ such that $p \in D' \setminus \partial D'$, $\partial D'$ is smooth, and D' is contained in a local chart with local coordinate z centered at p . In this local chart, we can write θ as

$$\theta = \left(\sum_{n=m}^{\infty} c_n z^n \right) dz.$$

Then

$$\int_{\partial D} \theta - \int_{\partial D'} \theta = \int_{D \setminus D'} d\theta = 0,$$

where the last equality holds since θ is holomorphic in $D \setminus D'$. As a consequence,

$$\int_{\partial D} \theta = \int_{\partial D'} \theta = \int_{\varphi(\partial D')} \left(\sum_{n=m}^{\infty} c_n z^n \right) dz = 2\pi\sqrt{-1}c_{-1} = 2\pi\sqrt{-1} \text{Res}_p(\theta).$$

□

Theorem 4.3.1 (residue theorem). Let X be a compact Riemann surface and θ be a meromorphic 1-form on X . Then

$$\sum_{p \in X} \text{Res}_p(\theta) = 0$$

Proof. Since X is compact, there are only finitely many poles of θ , denoted by $\{p_1, \dots, p_k\}$. For each $1 \leq j \leq k$, we can choose a neighborhood D_j of p_j which plays the role of D' in Lemma 4.3.1. Then

$$\sum_{p \in X} \text{Res}(\theta) = \sum_{j=1}^k \text{Res}_{p_j}(\theta) = \frac{1}{2\pi\sqrt{-1}} \sum_{j=1}^k \int_{\partial D_j} \theta = \frac{1}{2\pi\sqrt{-1}} \int_{D \setminus \bigcup_{j=1}^k D_j} d\theta = 0.$$

□

Corollary 4.3.1. Let X be a compact Riemann surface and f be a meromorphic function on X . Then

$$\sum_{p \in X} \text{ord}_p(f) = 0$$

Proof. It suffices to note that

$$\text{ord}_p(f) = \text{Res}_p\left(\frac{df}{f}\right).$$

□

4.4. Poincaré-Hopf theorem. Let M be a real closed 2-manifold and σ be a smooth 1-form with isolated zeros. Suppose σ is locally given by $\sigma = udx + vdy$ on an open neighborhood U of zero p such that $U \setminus \{p\}$ contains no zero of σ . Then the index of σ at p , denoted by $\text{Ind}_p(\sigma)$, is defined by the degree of the following map

$$\begin{aligned} \Phi: S^1(\epsilon) &\rightarrow S^1 \\ (x, y) &\mapsto \frac{(u, v)}{\sqrt{u^2 + v^2}}, \end{aligned}$$

where $S^1(\epsilon)$ is the sphere of radius ϵ contained in U . The Poincaré-Hopf theorem⁴ says that

$$\sum_{i=1}^k \text{Ind}_{p_i}(\sigma) = \chi(M),$$

where $\{p_1, \dots, p_k\}$ are all zeros of σ . Moreover, Poincaré-Hopf theorem still holds if σ is smooth except finitely many singularities, by adding the index of these singularities. In this section we will use Poincaré-Hopf theorem to show that the phenomenon we have seen in Example 4.2.2 and Example 4.2.3 are not coincidences.

Theorem 4.4.1. Let X be a compact Riemann surface and θ be a meromorphic 1-form on X . Then

$$\sum_{p \in X} \text{ord}_p(\theta) = -\chi(X) = 2g - 2.$$

Proof. Consider the 1-form $\sigma = \text{Re}(\theta)$, which is a smooth 1-form besides the poles of θ , and the zeros of σ are exactly the one of θ . For any zero or pole $p \in X$ of θ , without loss of generality we may assume θ is of the form $z^m dz$ locally. Then

$$\sigma = r^m (\cos(m\theta)dx - \sin(m\theta)dy)$$

where $r = |z|$. Thus the index at point p is

$$\begin{aligned} \text{Ind}_p(\sigma) &= \frac{1}{2\pi} \int_0^{2\pi} \cos(m\theta) d \sin(-m\theta) + \sin(-m\theta) d \cos(m\theta) \\ &= \frac{1}{2\pi} \int_0^{2\pi} -m (\sin(-m\theta)^2 + \cos(-m\theta)^2) d\theta \\ &= -\text{ord}_p(\theta). \end{aligned}$$

⁴See page 35 of [Mil65].

Thus by Poincaré-Hopf theorem one has

$$\sum_{p \in X} \text{ord}_p(\theta) = -\chi(X).$$

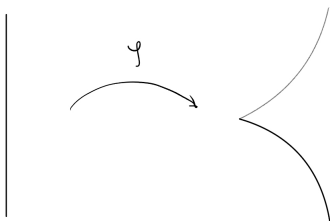
□

Remark 4.4.1. In fact, above theorem is equivalent to the Riemann-Hurwitz formula, which is left an exercise in homework.

5. NORMALIZATION

In this section we will deal with singularities of algebraic curves. Roughly speaking, after resolving all singularities of a curve C , we should obtain a Riemann surface, which is “isomorphic” to C besides these singularities. Before the formal definitions, let’s see some examples of singularities we have already seen.

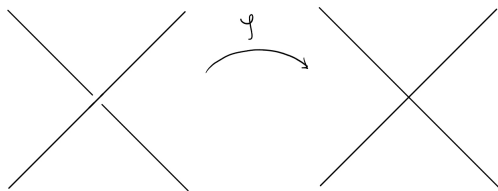
Example 5.1. The affine plane curve C defined by $x^2 - y^2 = 0$ has a singular point $(0, 0)$. Geometrically speaking, there are two projective line intersect at the point $(0, 0)$, which cause the singularity. Thus one way to solve the singularity is to “split” these two lines.



Formally speaking, we should consider the disjoint union of two copy of \mathbb{C} , which is mapped to C as follows

$$\begin{aligned} \Phi: \mathbb{C} \amalg \mathbb{C} &\rightarrow C \\ \{t_1\}, \{t_2\} &\mapsto (t_1, t_1), (t_2, -t_2). \end{aligned}$$

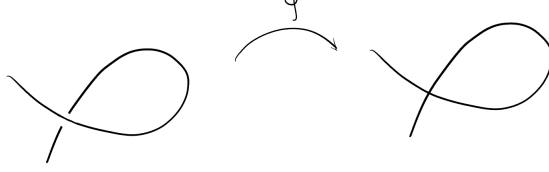
Example 5.2. The affine plane curve C defined by $x^2 - y^3 = 0$ has a singular point $(0, 0)$. To solve this singularity, geometrically thinking we should pull this curve “straightly”, which can be seen as



Formally speaking, we should consider the parameterization

$$\begin{aligned} \Phi: \mathbb{C} &\rightarrow C \\ t &\mapsto (t^3, t^2). \end{aligned}$$

Example 5.3. The affine plane curve defined by $y^2 - x^2(x - 1) = 0$ has a singular point $(0, 0)$. From the following picture we can see that if we want to solve the singularity, we should also “split” the two part which intersect at $(0, 0)$, as what we have done in the Example 5.1.



5.1. Weierstrass preparation theorem. Denote

$$\mathbb{C}\{x\} = \left\{ \sum_{k=1}^{\infty} a_k x^k \mid \text{convergent series with positive convergence radius.} \right\}$$

$$\mathbb{C}\{x, y\} = \left\{ \sum_{k=1}^{\infty} a_{kl} x^k y^l \mid \text{convergent series with positive convergence radius.} \right\}$$

They are called germs of holomorphic functions.

Definition 5.1.1 (Weierstrass polynomial). An element $f(x, y) \in \mathbb{C}\{x, y\}$ is called a Weierstrass polynomial if $f(x, y) = y^d + a_1(x)y^{d-1} + \cdots + a_d(x) \in \mathbb{C}\{x, y\}$, where $a_i(x) \in \mathbb{C}\{x\}$ and $a_i(0) = 0$.

Theorem 5.1.1 (Weierstrass preparation theorem). If $f \in \mathbb{C}\{x, y\}$ such that $f(0, y)$ is not identically zero, then there exist a unique $u \in \mathbb{C}\{x, y\}^*$ and a unique Weierstrass polynomial w such that $f = uw$.

Proof. Firstly we may assume $f(0, 0) = 0$, otherwise $f \in \mathbb{C}\{x, y\}^*$ and there is nothing to prove. If so, then $f(0, y)$ has an isolated zero at $y = 0$, that is, there exists $\epsilon > 0$ such that

$$\{f(0, y) = 0\} \cap \{|y| \leq \epsilon\} = \{y = 0\}.$$

By continuity we may choose $\rho > 0$ sufficiently small such that $f(x, y) \neq 0$ on $\{|x| < \rho, |y| = \epsilon\}$. Then the number of zeros of $f(x, y)$ in $|y| \leq \epsilon$ for a fixed x is computed by

$$n(x) = \frac{1}{2\pi\sqrt{-1}} \int_{|y|=\epsilon} \frac{f_y(x, y)}{f(x, y)} dy,$$

which is an integer-valued holomorphic function, and thus $n(x) \equiv m$ is a constant.

For all $|x| < \rho$, suppose $y_1(x), \dots, y_m(x)$ are zeros of $f(x, y)$ contained in $\{|y| \leq \epsilon\}$. Then we claim that $w(x, y) = (y - y_1(x)) \cdots (y - y_m(x))$ is a Weierstrass polynomial. Indeed, note that

$$\sigma_k(x) := \sum_{i=1}^m y_i^k(x) = \frac{1}{2\pi\sqrt{-1}} \int_{|y|=\epsilon} y^k \frac{f_y(x, y)}{f(x, y)} dy$$

are holomorphic, and thus if we write

$$w(x, y) = y^m + a_1(x)y^{m-1} + \cdots + a_m(x),$$

then $a_i(x)$ are polynomials of $\sigma_1(x), \dots, \sigma_m(x)$. This shows $a_i(x) \in \mathbb{C}\{x\}$, and $a_i(0) = 0$ for all i since $y_i(0) = 0$ for all i .

For convenience we denote $D = \{|x| < \rho, |y| \leq \epsilon\}$. By definition $u(x, y) = f(x, y)/w(x, y)$ is well-defined in $D \setminus \{w = 0\}$. For fixed $|x| < \rho$, by construction $w(x, y)$ and $f(x, y)$ have the same zeros in y . Therefore $u(x, y) \neq 0$ on D and $u(x, y)$ is holomorphic in variable y for each x . Now for given y_0 with $|y_0| < \epsilon$, one has

$$u(x, y_0) = \frac{1}{2\pi\sqrt{-1}} \int_{|y|=\epsilon} \frac{u(x, y)}{y - y_0} dy.$$

This shows $u(x, y)$ is holomorphic in variables x and y , and thus one has $u(x, y)$ is holomorphic. Moreover, since u has no zeros, it has a non-zero constant term $u(0, 0)$, and thus $u \in \mathbb{C}\{x, y\}^*$.

Finally let's see the uniqueness. If $f = u'w'$ in D , then

$$w' = y^d + c_1(x)y^{d-1} + \dots + c_d(x) = (y - y_1(x)) \dots (y - y_m(x)) = w.$$

This shows $w = w'$ and thus $u = u'$. \square

Corollary 5.1.1. $\mathbb{C}\{x, y\}$ is UFD.

Proof. Firstly note that $\mathbb{C}\{x\}$ is UFD, since for $f \in \mathbb{C}\{x\}$, one has

$$f = x^\mu g,$$

where $g \in \mathbb{C}\{x\}$ is a unit. Then by Gauss lemma one has $\mathbb{C}\{x\}[y]$ is UFD. Now for $f \in \mathbb{C}\{x, y\}$, suppose $f = x^\mu g$ with $g(0, y) \neq 0$. Since μ is unique, it suffices to show the unique factorization for g . By Weierstrass preparation theorem there is a decomposition

$$g(x, y) = uw.$$

Since Weierstrass polynomial w belongs to $\mathbb{C}\{x\}[y]$ which is UFD, there is a unique decomposition

$$w = w_1^{p_1} \dots w_k^{p_k},$$

where $w_i \in \mathbb{C}\{x\}[y]$ is monic irreducible. Now we need to show each w_i is irreducible in $\mathbb{C}\{x, y\}$. If not, suppose $w_i = a_i b_i$ in $\mathbb{C}\{x, y\}$. Then $w_i(0, y) \neq 0$ implies both $a_i(0, y) \neq 0$ and $b_i(0, y) \neq 0$, and again by Weierstrass preparation theorem one has

$$\begin{aligned} a_i &= u'_i w'_i \\ b_i &= u''_i w''_i, \end{aligned}$$

where $w'_i, w''_i \in \mathbb{C}\{x\}[y]$. Since the decomposition in Weierstrass preparation theorem is unique, one has $u'_i u''_i = 1$ and $w_i = w'_i w''_i$ in $\mathbb{C}\{x\}[y]$, a contradiction. Thus we obtain a decomposition of g into

$$g = uw_1^{p_1} \dots w_k^{p_k},$$

where u is a unit in $\mathbb{C}\{x, y\}$ and w_i are irreducible in $\mathbb{C}\{x, y\}$.

Now let's prove the uniqueness of decomposition of g . Suppose g is decomposed as

$$g = uw_1^{p_1} \dots w_k^{p_k} = v\tilde{w}_1^{q_1} \dots \tilde{w}_l^{q_l}.$$

Since $g(0, y) \not\equiv 0$, then again by Weierstrass preparation theorem we may decompose these w_i and \tilde{w}_j into

$$\begin{aligned} w_i &= u_i w'_i \\ \tilde{w}_j &= v_j \tilde{w}'_j. \end{aligned}$$

By the uniqueness of the decomposition in Weierstrass preparation and the factorization in $\mathbb{C}\{x\}[y]$, one has $\{w_i\}$ and $\{\tilde{w}_j\}$ are the same up to ordering, and $l = k$. This completes the proof. \square

Remark 5.1.1. Although $y^2 - x^2(x-1)$ is irreducible in $\mathbb{C}[x, y]$, it's reducible in $\mathbb{C}\{x\}[y]$, that is,

$$(y - x\sqrt{x-1})(y + x\sqrt{x-1}).$$

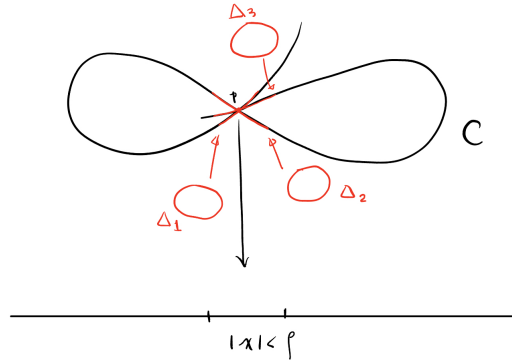
In the following section we will see such local decomposition gives the local resolution of singularities.

5.2. Resolution of singularities. Let C be an irreducible projective plane curve with singularities $\text{Sing}(C)$. A normalization of C is a compact Riemann surface \tilde{C} together with a continuous map $\Phi: \tilde{C} \rightarrow C$ such that Φ is surjective and

$$\Phi: \tilde{C} \setminus \Phi^{-1}(\text{Sing}(C)) \rightarrow C \setminus \text{Sing}(C)$$

is an isomorphism. In this section we will use unique factorization of $\mathbb{C}\{x, y\}$ to construct the normalization of C . Firstly let's give a rough ideal about what we're going to do.

The idea is to find sufficiently small $D = \{|x| < \rho, |y| < \epsilon\}$ such that $C \cap D$ decomposed into several pieces⁵ $C_1 \cup \dots \cup C_l$, where each C_i is homeomorphic to a disk and the union attaches them only at their centers. If we have constructed homeomorphisms φ_i from disk Δ_i to C_i for each i and repeat this procedure for all singularities, then we may construct the normalization \tilde{C} by adding these C_i to $C \setminus \text{Sing}(C)$ in a suitable way.



⁵In fact, suppose $f = uw_1 \dots w_l$ in $\mathbb{C}\{x, y\}$, where w_1, \dots, w_l are distinct irreducible Weierstrass polynomials and $u \in \mathbb{C}\{x, y\}^*$. Then each C_i is the zero locus $\{w_i = 0\}$.

In the following sections we will explain above procedures in detail. The construction of homeomorphism for each singularity is called local resolution and adding these C_i to $C \setminus \text{Sing}(C)$ is called the global resolution.

5.2.1. Local resolution of singularities. Let C be an irreducible projective plane curve with singularities $\text{Sing}(C)$. Suppose p is a singularity of C , and without loss of generality we may assume $p = [1 : 0 : 0]$ by after a suitable $\text{PGL}(3, \mathbb{C})$ transformation. Moreover, we may put the affine equation of C in the following form

$$f(x, y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x) = 0,$$

where $a_i(x) \in \mathbb{C}[x]$. Since $f(x, y)$ is irreducible, it has no multiple divisors in $\mathbb{C}[x][y]$, so $\mathcal{R}(f, f_y) \neq 0$ in $\mathbb{C}[x]$. Regardless of whether we are in $\mathbb{C}[x][y]$ or in $\mathbb{C}\{x\}[y]$, the resultant $\mathcal{R}(f, f_y)$ is the same, which implies $f(x, y)$ has no multiple divisors in $\mathbb{C}\{x\}[y]$. Then $f(x, y)$ is decomposed into the product of distinct irreducible factors in $\mathbb{C}\{x\}[y]$ as follows

$$f = f_1 \cdots f_l.$$

Moreover, from $f(0, y) = y^n$, one can see every f_i must satisfy $f_i(0, y) \neq 0$. Thus by Weierstrass preparation theorem one has in $\mathbb{C}\{x, y\}$ one has the following decomposition

$$f_i = u_i w_i,$$

where $u_i \in \mathbb{C}\{x, y\}^*$ and w_i is a Weierstrass polynomial. Thus $f(x, y)$ is decomposed into the product of irreducible Weierstrass polynomials in $\mathbb{C}\{x, y\}$ as follows

$$f = u w_1 \cdots w_l.$$

In order to avoid messy notations, in the following discussion we use w to denote one of the irreducible Weierstrass polynomials appeared in above decomposition.

Note that $\mathcal{R}(w, w_y)(x) \neq 0$ implies $\mathcal{R}(w, w_y)(x)$ can only have isolated zeros. And since $w(0, y) = y^k$ has multiple roots, then $\mathcal{R}(w, w_y)(0) = 0$. Then there exists sufficiently small $\rho > 0$ such that for each $x \neq 0$ in

$$D = \{x \in \mathbb{C} \mid |x| < \rho\},$$

one has $\mathcal{R}(w, w_y)(x) \neq 0$. Then

$$w(x, y) = \prod_{\nu=1}^k (y - y_\nu(x)),$$

where $y_\nu(x)$'s are roots of $w(x, y)$. Moreover, for $0 \neq x \in D$, $\mathcal{R}(w, w_y)(x) \neq 0$, so that

$$w_y(x, y) \neq 0.$$

Then by the implicit function theorem, every $y_\nu(x)$ is locally a holomorphic function, and it can be uniquely analytically extended to a holomorphic function defined on $D \setminus \{x \in \mathbb{R}_{\geq 0}\}$, still denoted by $y_\nu(x)$. Now analytically

extend $y_\nu(x)$ across the cut line, the $y_\nu^*(x)$ obtained after this continuation must still satisfy

$$f(x, y_\nu^*(x)) = 0,$$

and thus $y_\nu^*(x)$ is one of the $y_1(x), \dots, y_k(x)$. In other words, the monodromy is given by a permutation τ of $\{1, \dots, k\}$. By the same argument used in the proof of connectness of irreducible projective plane curve, one can show that the permutation τ is a k -cycle.

Theorem 5.2.1. Notations as above, and denote $\Delta = \{t \in \mathbb{C} \mid |t| < \rho^{1/k}\}$. Then the map

$$\begin{aligned} \varphi: \Delta &\rightarrow \mathbb{C}^2 \\ t &\mapsto (t^k, y_\nu(t^k)), \end{aligned}$$

is a well-defined holomorphic map and φ is injective from Δ onto

$$C^\Delta = \{(x, y) \in \mathbb{C}^2 \mid |x| < \rho, |y| < \epsilon, w(x, y) = 0\}.$$

Furthermore, φ is a biholomorphic from $\Delta \setminus \{0\}$ onto $C^\Delta \setminus \{(0, 0)\}$.

Proof. As t wraps one around the origin of Δ , t^k wraps around the origin k times. This shows when t wraps one around the origin once, $y_\nu(t^k)$ remains unchanged since the monodromy is given by a k -cycle. In this way, $y_\nu(t^k)$ defines a single-valued holomorphic function for $0 < |t| < \rho^{1/k}$, which can be extended to a holomorphic function defined on Δ by Riemann extension theorem. From this one can see $\phi: \Delta \rightarrow \mathbb{C}^2$ is a well-defined holomorphic map.

To see φ is injective: If $(t_1^k, y_\nu(t_1^k)) = (t_2^k, y_\nu(t_2^k))$, then $t_2 = (\xi_k)^\ell t_1$ for some $\ell \in \mathbb{Z}$, where ξ_k is the k -th unit root, and thus

$$y_\nu((\xi_k)^\ell t_1^k) = y_\nu(t_1^k).$$

Note that only when the variable x wraps around the origin km times does the value $y_\nu(x)$ remains unchanged. Therefore one has $k \mid \ell$, which implies ϕ is injective. Moreover, as t varies in Δ , one can see $y_\nu(t^k)$ passes all possible values of $y_1(x), \dots, y_k(x)$ for $|x| \leq \rho$, which shows ϕ maps Δ onto C^Δ .

By implicit function theorem, there is a Riemann surface structure on $C^\Delta \setminus \{(0, 0)\}$, and thus φ is biholomorphic since it's holomorphic and both injective and surjective. \square

5.2.2. Global resolution of singularities. Let C be an irreducible projective plane curve with singularities $\text{Sing}(C)$ and $C^* = C \setminus \text{Sing}(C)$. For the singularity p , according to the method of the proceeding section, there exists m open discs Δ_i together with l holomorphic maps φ_i such that

$$\varphi_i: \Delta_i \setminus \{0\} \rightarrow C^*$$

is a biholomorphic map onto the image set. Now we use these φ_i to glue discs Δ_i to C^* to get the following Riemann surface

$$\tilde{C} = C^* \bigcup_{\varphi_1} \Delta_1 \bigcup_{\varphi_2} \Delta_2 \cdots \bigcup_{\varphi_l} \Delta_l.$$

Repeat this procedure for each singularity and then we obtain the normalization \tilde{C} of C .

5.3. Blow-up. In this section we will introduce a method to determine the irreducible decomposition of a polynomial $f(x, y)$ in $\mathbb{C}\{x, y\}$. Without loss of generality we may assume $f(0, 0) = 0$ and write

$$f(x, y) = f_m(x, y) + f_{m+1}(x, y) + \dots,$$

where f_i are homogenous polynomials of degree i , and m is the smallest integer such that $f_d \neq 0$.

- (1) If $m = 1$, then $(0, 0)$ is a non-singular point. In this case, $\{f_1(x, y) = 0\}$ is the (unique) tangent line of $f(x, y) = 0$ at $(0, 0)$.
- (2) If $m \geq 2$, then $\{f_m(x, y) = 0\}$ is the union of lines, and this set is called the tangent cone of $f(x, y) = 0$.

After a suitable linear transformation, we may assume the tangent cone of $f(x, y) = 0$ at $(0, 0)$ does not contain $\{x = 0\}$. Thus

$$(5.1) \quad f(x, y) = (y - \alpha_1 x) \dots (y - \alpha_m x) + f_{m+1}(x, y) + \dots,$$

where $\alpha_i \in \mathbb{C}$.

Definition 5.3.1 (ordinary singularity). If $\{f_m = 0\}$ is the union of m distinct lines, then $(0, 0)$ is called an ordinary m -tuple singularity.

Example 5.3.1. If $f(x, y) = y^2 + x^2(x - 1)$, then the tangent cone of $f(x, y) = 0$ at $(0, 0)$ is the union of lines $y = \pm x$, and thus $(0, 0)$ is an ordinary 2-tuple singularity.

Example 5.3.2. If $f(x, y) = y^2 - x^3$, then the tangent cone of $f(x, y) = 0$ at $(0, 0)$ is $y^2 = 0$, that is, the union of double copies of $y = 0$. In this case $(0, 0)$ is not an ordinary singularity. The singularity of this type is called a cusp.

Proposition 5.3.1. If $f(x, y)$ has ordinary m -tuple singularity at $(0, 0)$, then $f(x, y)$ decomposes as a product of m irreducible factors in $\mathbb{C}\{x, y\}$ as follows

$$f(x, y) = u \prod_{i=1}^m (y - x h_i(x)),$$

where $u \in \mathbb{C}\{x, y\}^*$.

Proof. It's left as an exercise in homework, and here we list the key steps of the proof as follows.

- (1) Denote by $w = y/x$,

$$g(x, w) = \frac{f(x, xw)}{x^m} \in \mathbb{C}\{x, w\}.$$

Prove that g converges in a product of discs

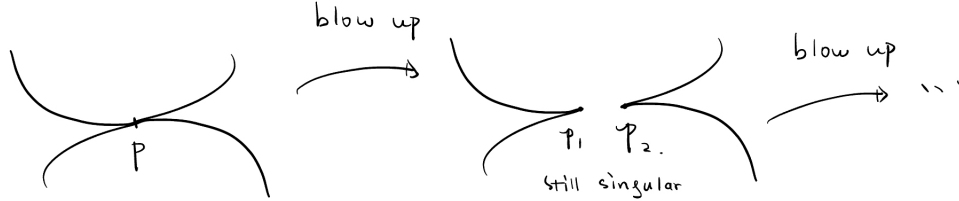
$$D_{\rho_1} \times D_{\rho_2} = \{(x, w) \mid |x| < \rho_1, |w| < \rho_2\}$$

that contains $(0, \alpha_i)$.

- (2) Prove that $g(0, \alpha_i) = 0$ and $\partial g / \partial w(0, \alpha_i) \neq 0$ and hence $g(x, w) = 0$ has a solution $w = h_i(x)$ near $(0, \alpha_i)$ with $h_i(x) \in \mathbb{C}\{x\}$ and $h_i(0) = \alpha_i$.
- (3) Prove that $\prod (y - xh_i(x)) \mid f(x, y)$ and $f(x, y)$ equals to the product of m irreducible factors up to units in $\mathbb{C}\{x, y\}$.

□

The procedure by introducing a new variable w such that $y = xw$ to consider the polynomial $g(x, w)$ in the proof of above proposition is called “blow up”. Note that locally around $x = 0$, there is no difference between $f(x, y) = 0$ and $g(x, w) = 0$ except $x = 0$, but $g(0, w)$ may has lots of solutions, denoted by $\alpha_1, \dots, \alpha_k$. (You can think that one point is blown up to several points geometrically.)



If $g(x, w)$ is non-singular at all these points, then $g(x, w) = 0$ can be viewed as a local normalization of $f(x, y) = 0$ at $(0, 0)$. Otherwise we may need to blow up again along those singularities. In particular, Proposition 5.3.1 shows that if $(0, 0)$ is an ordinary m -tuple singularity, then after blowing up once, you can get the desired local normalization.

Example 5.3.3. For $f(x, y) = x^2 - y^2$, we know that $f(x, y) = 0$ is the union of two line at origin and thus $(0, 0)$ is a singularity. By considering

$$g(x, w) = \frac{f(x, xw)}{x^2} = \frac{x^2 - x^2w^2}{x^2} = 1 - w^2.$$

One can see that for $x = 0$, $g(0, w)$ has solutions $w = \pm 1$, and that's exactly $y = \pm x$.

Example 5.3.4. For $f(x, y) = y^2 - x^2 + x^3 - y^3 = 0$, one has

$$g(x, w) = \frac{f(x, xw)}{x^2} = \frac{x^2w^2 - x^2 + x^3 - x^3w^3}{x^2} = w^2 - 1 + x - xw^3.$$

Then $x = 0$ has solutions $w = \pm 1$. A direct computation shows $\partial g / \partial w = 2w - 3xw^2$, and thus

$$\left. \frac{\partial g}{\partial w} \right|_{x=0, w=\pm 1} = \pm 1 \neq 0.$$

By implicit function theorem there exists $w_1(x), w_2(x) \in \mathbb{C}\{x\}$ such that $w_1(0) = 1$ and $w_2(0) = -1$. Then $f(x, y) = 0$ is parameterized by

$$\begin{aligned} y &= xw_1(x) \\ y &= xw_2(x) \end{aligned}$$

locally around $(0, 0)$.

Example 5.3.5. For $f(x, y) = y^2 - y^3 + x^3 = 0$, one has

$$g(x, w) = \frac{f(x, xw)}{x^2} = \frac{x^2w^2 - x^3w^3 + x^3}{x^2} = w^2 - xw^3 + x = 0.$$

Then $g(0, w)$ has solution $w = 0$. Note that

$$\begin{aligned} \partial g / \partial x &= w^3 + 1 \\ \partial g / \partial w &= 2w - 3xw^2. \end{aligned}$$

Then

$$\left. \frac{\partial g}{\partial x} \right|_{x=0, w=0} = 1 \neq 0,$$

By implicit function theorem there exists $x = x(w)$ with $x(0) = 0$. Note that

$$x'(0) = - \left. \frac{\partial g / \partial w}{\partial g / \partial x} \right|_{w=0} = \frac{0}{1} = 0,$$

and $x''(0) \neq 0$. Then $f(x, y) = 0$ is parameterized by

$$\begin{cases} x(w) = w^2(c + \dots) \\ y(w) = w^3(c + \dots) \end{cases}$$

locally around $(0, 0)$.

Example 5.3.6. For $y^2 - y^3 - x^4 = 0$, $y = xw$, one has

$$g_1(x, w) = \frac{f(x, xw)}{x^2} = \frac{x^2w^2 - x^3y^3 - x^4}{x^2} = w^2 - xw^3 - x^2 = 0.$$

Then $g_1(0, w) = 0$ has solution $w = 0$. Since

$$\left. \frac{\partial g_1}{\partial x} \right|_{(0,0)} = 0, \quad \left. \frac{\partial g_1}{\partial w} \right|_{(0,0)} = 0,$$

one has $(0, 0)$ is still a singular point of $g_1(x, w)$, so we may blow it up again by setting $w = xt$. By doing this, one has

$$g_2(x, t) = \frac{g_1(x, xt)}{x^2} = \frac{x^2t^2 - x^4t^3 - x^2}{x^2} = t^2 - x^2t^3 - 1.$$

Then $g_2(0, t)$ has solutions $t = \pm 1$. Note that

$$\left. \frac{\partial g_2}{\partial t} \right|_{(0, \pm 1)} = \pm 2 \neq 0.$$

Then by implicit function theorem there exists $t_1(x), t_2(x) \in \mathbb{C}\{x\}$ such that $t_1(0) = 1$ and $t_2(0) = -1$, and thus $f(x, y) = 0$ is parameterized by

$$\begin{aligned} y &= x^2 t_1(x) \\ y &= x^2 t_2(x) \end{aligned}$$

locally around $(0, 0)$.

5.4. Bezout formula for singular case. In Theorem 3.1.1 we have shown the Bezout theorem under the assumption of C, C' are non-singular projective plane curve. In general case Bezout theorem still holds, if we use the right definition for the intersection number.

Definition 5.4.1 (intersection number). Let C, C' be irreducible projective plane curves defined by homogenous polynomials F, G such that F, G has no common divisors, and $p \in C \cap C'$. If U is an open neighborhood of p such that the local decomposition of C into irreducibles as

$$C \cap U = C_1 \cup \cdots \cup C_k,$$

with local normalizations $\varphi_i: \Delta_i \rightarrow C_i$, then the intersection number at p is defined by

$$(C, C')_p = \sum_{i=1}^k \text{ord}_{t=0}(G(\varphi_i(t))).$$

The intersection number of C and C' , denoted by (C, C') , is the summation

$$(C, C') = \sum_{p \in C \cap C'} (C, C')_p.$$

Remark 5.4.1. The definition given above is compatible with the intersection number in the non-singular case, since if C is non-singular, then locally there is only one piece and the local normalization is a biholomorphism.

Theorem 5.4.1 (Bezout). Let C, C' be irreducible projective plane curves defined by homogenous polynomials F, G such that F, G has no common divisors. Then the intersection number

$$(C, C') = ed,$$

where $\deg F = e$ and $\deg G = d$.

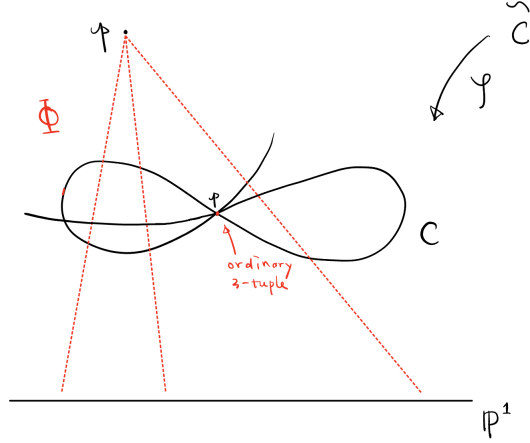
5.5. Plücker formula for singular case.

5.5.1. Riemann-Hurwitz approach.

Theorem 5.5.1. Let C be an irreducible projective plane curve defined by the homogenous polynomial F of degree d with $\text{Sing}(C) = \{p_1, \dots, p_k\}$, and $\varphi: \tilde{C} \rightarrow C$ be its normalization. Suppose each p_i is an ordinary m_i -tuple singularity. Then

$$\text{genus}(\tilde{C}) = \binom{d-1}{2} - \sum_{i=1}^k \binom{m_i}{2} \geq 0$$

Proof. Choose a point $p \notin C$ and not lies in the tangent cone of C for any singularity. The point p defines a projection Φ from C to the projective line \mathbb{P}^1 , and the composition $\tilde{\Phi} = \Phi \circ \varphi$ is a holomorphic map from \tilde{C} to \mathbb{P}^1 .



Now it suffices to figure out the ramification data of $\tilde{\Phi}$ and use Riemann-Hurwitz formula to compute the genus of \tilde{C} .

(1) If $q \in C \setminus \text{Sing}(C)$, then

$$\text{mult}_{\varphi^{-1}(q)} \tilde{\Phi} - 1 = \text{mult}_q \Phi - 1 = (F, F_y)_q$$

(2) If $q \in \text{Sing}(C)$, then for each $q_i \in \pi^{-1}(q)$, one has $\text{mult}_{q_i} \tilde{\Phi} = 1$ since \tilde{C} is locally defined by irreducible linear function.

This shows the ramification data

$$B(\tilde{\Phi}) = \sum_{q \in C \setminus \text{Sing}(C)} (F, F_y)_q.$$

Now let's figure out the intersection number of singular points. Suppose $p \in \text{Sing}(C)$ is a m -tuple singularity. For convenience we may assume $p = [0 : 0 : 1]$ and denote $f(x, y) = F(x, y, 1)$. Since p is an ordinary m -tuple singularity, by Proposition 5.3.1 one has

$$f(x, y) = (y - y_1(x)) \cdots (y - y_m(x)),$$

where $y_i(x) = a_i x + o(x^2)$. Then there are m local normalizations at point p , which are given by $\varphi_i(t) = (t, y_i(t))$. This shows the intersection number

at singularity p is given by

$$\begin{aligned}
(f, f_y)_{(0,0)} &= \sum_{i=1}^m \text{ord}_{t=0} f_y(\varphi_i(t)) \\
&= \sum_{i=1}^m \text{ord}_{t=0} \prod_{j \neq i} (y_i(t) - y_j(t)) \\
&= \sum_{i=1}^m m - 1 \\
&= m(m-1).
\end{aligned}$$

Then by Bezout theorem (Theorem 5.4.1), one has

$$B(\tilde{\Phi}) = d(d-1) - \sum_{i=1}^k m_i(m_i-1).$$

By Riemann-Hurwitz formula one has

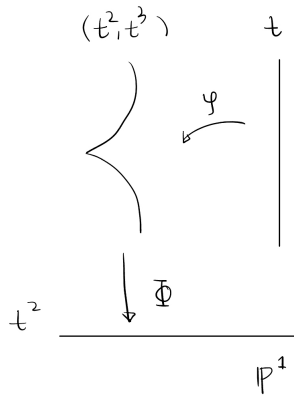
$$2 \text{genus}(\tilde{C}) - 2 = -2d + d(d-1) - \sum_{i=1}^k m_i(m_i-1),$$

and thus

$$\text{genus}(\tilde{C}) = \binom{d-1}{2} - \sum_{i=1}^k \binom{m_i}{2}.$$

□

Remark 5.5.1. Above argument shows that the genus of the normalization \tilde{C} only depends on the degree of C and the type of singularities. For example, suppose $p \in C$ is a cusp, that is, locally it's given by $y^2 = x^3$, and its local normalization is given by $\varphi: t \mapsto (t^2, t^3)$.



The preimage $\varphi^{-1}(0)$ is a ramification point of $\tilde{\Phi}$, with $\text{mult}_{\varphi^{-1}(0)} \tilde{\Phi} = 2$, and thus

$$B(\tilde{\Phi}) = \sum_{q \in C \setminus \text{Sing}(C)} (f, f_y)_q + \sum_{q \text{ is a cusp}} 1.$$

On the other hand, the intersection number at the cusp is

$$(f, f_y)_p = \text{ord}_{t=0} 2t^3 = 3.$$

Then by the same argument one can see that one more cusp will decrease the genus of \tilde{C} by 1.

In general, for each type singularity we may define a δ -invariant, such that one more such type singularity will decrease the genus of normalization by δ . As we have seen, the δ -invariant for m -tuple singularity is $\binom{m}{2}$ and the δ -invariant for cusp is 1.

5.5.2. Poincaré-Hopf approach. In this section we introduce another approach to compute the genus of the normalization by Poincaré-Hopf theorem, from which it's relatively easy to compute the δ -invariance of singularity.

Suppose C is a projective plane curve defined by homogenous polynomial F . Consider

$$\eta = \frac{dx}{F_y(x, y, 1)} = -\frac{dy}{F_x(x, y, 1)}.$$

If F is non-singular, then η has no zeros or poles on the affine piece $C \cap \{[x : y : 1]\}$. On $\{z = 0\}$, a direct computation shows

$$\eta = -\frac{z^{d-3}dz}{F_y(1, y, z)} = \frac{z^{d-3}dz}{F_x(x, 1, z)}.$$

Thus η gives a meromorphic 1-form on C , and by using Bezout theorem one has

$$\sum_{p \in C} \text{ord}_p(\eta) = (d-3)d.$$

Then Poincaré-Hopf theorem implies that $\text{genus}(C) = (d-1)(d-2)/2$.

Now let's generalize above arguments to the case C is singular. Suppose $f(x, y)$ has singularity at $(0, 0)$ with multiplicity m , and $x = 0$ is not in the tangent cone of f at $(0, 0)$. Now consider the blow up at the singularity $(0, 0)$, that is,

$$g(x, w) = \frac{f(x, xw)}{x^m}.$$

A direct computation by chain rule shows that

$$x^{m-1}g_w(x, w) = f_y(x, xw).$$

Thus

$$\eta = \frac{dx}{f_y(x, y)} = \frac{dx}{x^{m-1}g_w(x, xw)} = x^{-(m-1)} \frac{dx}{g_w(x, w)}.$$

If $g(0, w) = 0$ has solutions $\alpha_1, \dots, \alpha_k$, and $g(x, w)$ is non-singular at these points, then $g(x, w) = 0$ gives the normalization of C , denoted by \tilde{C} .

Then by Poincaré-Hopf theorem one has

$$\begin{aligned} 2 \operatorname{genus}(\tilde{C}) - 2 &= \sum_{p \in \tilde{C}} (x^{-(m-1)}, x)_p + \sum_{p \in \tilde{C}} \operatorname{ord}_p \left(\frac{dx}{g_w(x, w)} \right) \\ &= -m(m-1) + d(d-3). \end{aligned}$$

Otherwise we repeat above procedures again to each singularity $(0, w_i)$, and by induction we have the δ -invariance for singularity $(0, 0)$ is

$$\delta = \binom{m}{2} + \binom{m_1}{2} + \cdots + \binom{m_k}{2} + \cdots$$

Example 5.5.1. The δ -invariance of ordinary m -tuple singularity is $\binom{m}{2}$, since after blowing up once, it's already non-singular, which coincides with previous result.

Example 5.5.2. For $f(x, y) = y^n - x^m$, with $\gcd(m, n) = 1$. If $n < m$, then

$$g(x, w) = \frac{f(x, xw)}{x^n} = \frac{x^n w^n - x^m}{x^n} = w^n - x^{m-n}.$$

This shows

$$\delta(n, m) = \delta(n, m-n) + \binom{n}{2}.$$

Then by induction one has the δ -invariance of singularity $(0, 0)$ is $(m-1)(n-1)/2$.

6. DIVISORS

Along this section, we always assume X is a compact Riemann surface.

6.1. Motivations. If $X \rightarrow \mathbb{P}^2$ is biholomorphic to a non-singular projective plane curve of degree d , then the genus of X is given by $(d-1)(d-2)/2$. In other words, the genus of X cannot be arbitrary integers. In general, one may also ask whether there exists some \mathbb{P}^N such that X can be embedded into \mathbb{P}^N holomorphically.

6.2. Divisors.

Definition 6.2.1 (divisors). A divisor on X is a formal sum $D = \sum_{p \in X} D(p) \cdot p$, where $D(p) \in \mathbb{Z}$ such that $D(p) \neq 0$ for only finitely many p .

Notation 6.2.1. $\text{Div}(X)$ denotes the free abelian group generated by divisors on X .

Definition 6.2.2 (degree). For $D \in \text{Div}(X)$, the degree of D is defined by

$$\deg(D) = \sum_{p \in X} D(p).$$

Remark 6.2.1. The degree gives a group homomorphism $\deg: \text{Div}(X) \rightarrow \mathbb{Z}$. The kernel of \deg is denoted by $\text{Div}^0(X)$, that is,

$$\text{Div}^0(X) := \{D \in \text{Div}(X) \mid \deg(D) = 0\}.$$

6.2.1. Principal divisor.

Definition 6.2.3 (principal divisor). If $f \neq 0$ is a meromorphic function on X , the principal divisor corresponding to f is

$$\text{div}(f) := \sum_{p \in X} \text{ord}_p(f) \cdot p.$$

Notation 6.2.2. $\text{PDiv}(X)$ denotes the set of all principal divisors on X .

Lemma 6.2.1. Suppose f, g are meromorphic functions on X .

1. $\text{div}(fg) = \text{div}(f) + \text{div}(g)$.
2. For $g \neq 0$, $\text{div}(f/g) = \text{div}(f) - \text{div}(g)$.
3. For $f \neq 0$, $\text{div}(1/f) = -\text{div}(f)$.

Corollary 6.2.1. $\text{PDiv}(X)$ is a subgroup of $\text{Div}(X)$.

Example 6.2.1 (divisor of zeros or poles). Let $f \neq 0$ be a meromorphic function on X , the zero divisor is defined by

$$\text{div}_0(f) := \sum_{\substack{p \in X \\ \text{ord}_p(f) > 0}} \text{ord}_p(f) \cdot p,$$

and the pole divisor is defined by

$$\text{div}_\infty(f) := - \sum_{\substack{p \in X \\ \text{ord}_p(f) < 0}} \text{ord}_p(f) \cdot p.$$

It's clear

$$\operatorname{div}(f) = \operatorname{div}_0(f) - \operatorname{div}_\infty(f).$$

Definition 6.2.4 (linearly equivalent). For $D_1, D_2 \in \operatorname{Div}(X)$, if $D_1 - D_2$ is a principal divisor, then D_1, D_2 are called linearly equivalent, and denoted by $D_1 \sim D_2$.

Example 6.2.2. $\operatorname{div}_0(f)$ is linearly equivalent to $\operatorname{div}_\infty(f)$.

Lemma 6.2.2. If $f \not\equiv 0$ is a meromorphic function on X , then

$$\deg(\operatorname{div}(f)) = 0$$

Proof. It follows from Corollary 2.1.3. □

Corollary 6.2.2.

$$\operatorname{PDiv}(X) \subseteq \operatorname{Div}^0(X).$$

A natural question is that when $\operatorname{PDiv}(X) = \operatorname{Div}^0(X)$. The following theorem shows that the statement holds for $X = \mathbb{P}^1$, and for higher genus case, this statement fails even for torus.

Theorem 6.2.1. $\operatorname{PDiv}(\mathbb{P}^1) = \operatorname{Div}^0(\mathbb{P}^1)$.

Proof. For $D \in \operatorname{Div}^0(\mathbb{P}^1)$, we may write is as

$$D = \sum_{i=1}^n e_i \cdot \lambda_i + e_\infty \cdot \infty, \quad \lambda_i \in \mathbb{C},$$

where $e_\infty = -\sum_{i=1}^n e_i$. Then for the meromorphic function given by $f = \prod_{i=1}^n (z - \lambda_i)^{e_i}$, one has $\operatorname{div}(f) = D$. □

Corollary 6.2.3. For $D_1, D_2 \in \operatorname{Div}(\mathbb{P}^1)$, $D_1 \sim D_2$ if and only if $\deg(D_1) = \deg(D_2)$.

Theorem 6.2.2. Let $X = \mathbb{C}/L$ be the complex torus, where $L = \mathbb{Z}w_1 + \mathbb{Z}w_2$ is a lattice. Then

$$\operatorname{Div}^0(X)/\operatorname{PDiv}(X) \cong X.$$

Proof. Consider the following group homomorphism

$$A: \operatorname{Div}(X) \rightarrow X$$

$$\sum_{p \in X} n_p \cdot p \mapsto \sum_{p \in X} n_p p,$$

where $\sum_{p \in X} n_p p$ is the addition structure of X . Note that $A|_{\operatorname{Div}^0(X)}$ is surjective, since for any $p \in X$, one has

$$A(p - 0) = p,$$

and $p - 0$ is a divisor with degree zero. Now it suffices to show that $\ker A = \operatorname{PDiv}(X)$. □

Remark 6.2.2. It's a special case of the Abel theorem.

6.2.2. Canonical divisor.

Definition 6.2.5 (canonical divisor). Let θ be a meromorphic 1-form on X . The canonical divisor K given by θ is defined by

$$K := \sum_{p \in X} \text{ord}_p(\theta) \cdot p.$$

Lemma 6.2.3. If f is a meromorphic function, and θ is a meromorphic 1-form, then $f\theta$ is also a meromorphic 1-form, and

$$\text{div}(f\theta) = \text{div}(f) + \text{div}(\theta).$$

Conversely, we have

Lemma 6.2.4. If θ_1, θ_2 are meromorphic 1-form, then there exists a meromorphic function f such that

$$\theta_1 = f\theta_2$$

Proof. Suppose locally meromorphic 1-forms θ_1, θ_2 are given by

$$\theta_1 = f_1 dz, \quad \theta_2 = f_2 dz.$$

Then we can define a meromorphic function f locally by f_1/f_2 . The construction is independent of the choice of local charts, since factors coming from the change of charts with cancel with each other, as one of them is on the denominator and the other one is on the numerator. \square

Corollary 6.2.4. The difference of any two canonical divisors is a principal divisor.

Corollary 6.2.5. The canonical divisors have the same degree. Moreover, the degree of canonical divisor is $2 \text{genus}(X) - 2$.

Proof. It follows from Poincaré-Hopf theorem. \square

6.2.3. Pullback of divisors. Let $\Phi: X \rightarrow Y$ be a non-constant holomorphic map. For point $q \in Y$, we regard it as a divisor and define the pullback of it as follows

$$\Phi^*(q) := \sum_{p \in \Phi^{-1}(q)} \text{mult}_p(\Phi) \cdot p.$$

Then for any divisor $D \in \text{Div}(Y)$, its pullback is defined by

$$\Phi^*(D) = \sum_{q \in Y} D(q) \cdot \Phi^*(q).$$

Moreover, for any $q \in Y$, one has

$$\deg(\Phi^*(q)) = \sum_{p \in \Phi^{-1}(q)} \text{mult}_p(\Phi) = \deg(\Phi).$$

Then since taking degree is a group homomorphism, one has

$$\deg(\Phi^*(D)) = \sum_{q \in Y} D(q) \deg(\Phi^*(q)) = \deg(\Phi) \deg(D).$$

Lemma 6.2.5. Let $\Phi: X \rightarrow Y$ be a non-constant holomorphic map.

- (1) $\Phi^*: \text{Div}(Y) \rightarrow \text{Div}(X)$ is a group homomorphism.
- (2) $\Phi^*(\text{PDiv}(Y)) \subseteq \text{PDiv}(X)$.

Proof. It's clear that Φ^* is a group homomorphism. For (2). Let $f \not\equiv 0$ be a meromorphic function on Y . Then we claim that

$$\Phi^*(\text{div}(f)) = \text{div}(f \circ \Phi).$$

To see this, for any $p \in X$, we have

$$\begin{aligned} \Phi^*(\text{div}(f))(p) &= \text{mult}_p(\Phi) \text{div}(f)(\Phi(p)) \\ &= \text{mult}_p(\Phi) \text{ord}_{\Phi(p)}(f) \\ &= \text{ord}_p(f \circ \Phi) \\ &= \text{div}(f \circ \Phi)(p). \end{aligned}$$

□

Corollary 6.2.6. Let $\Phi: X \rightarrow Y$ be a non-constant holomorphic map. If $D_1 \sim D_2$ on Y , then $\Phi^*(D_1) \sim \Phi^*(D_2)$ on X .

Definition 6.2.6 (ramification divisor). Let $\Phi: X \rightarrow Y$ be a holomorphic map between compact Riemann surfaces. The ramification divisor is defined by

$$R_\Phi := \sum_{p \in X} (\text{mult}_p(\Phi) - 1) \cdot p.$$

Remark 6.2.3. Recall that Riemann-Hurwitz formula says that

$$2 \text{ genus}(X) - 2 = \deg(\Phi)(2 \text{ genus}(Y) - 2) + \sum_{p \in X} (\text{mult}_p(\Phi) - 1).$$

If θ is a non-zero meromorphic 1-form on Y , then $\Phi^*(\theta)$ is also a meromorphic 1-form, and thus

$$\begin{aligned} \deg(\text{div}(\theta)) &= 2 \text{ genus}(Y) - 2 \\ \deg(\text{div}(\Phi^*\theta)) &= 2 \text{ genus}(X) - 2. \end{aligned}$$

Then the Riemann-Hurwitz formula can be written as

$$\deg(\text{div}(\Phi^*(\theta))) = \deg(\Phi) \deg(\text{div}(\theta)) + \deg(R_\Phi) = \deg(\Phi^*(\text{div}(\theta))) + \deg(R_\Phi).$$

As a consequence, the pullback of a canonical divisor $\text{div}(\theta)$ by a holomorphic map Φ is still a canonical divisor if and only if Φ is not ramified.

6.2.4. Partial order of divisors.

Definition 6.2.7 (effective divisors). A divisor D on X is called effective divisor if $D(p) \geq 0$ for all $p \in X$, and denote it by $D \geq 0$. For the same reason, $D > 0$ if $D(p) > 0$ for all $p \in X$.

Remark 6.2.4. For any divisor D , it can be written as a difference of two effective divisors as follows

$$D = \sum_{\substack{p \in X \\ D(p) \geq 0}} D(p) \cdot p - \sum_{\substack{p \in X \\ D(p) < 0}} -D(p) \cdot p.$$

Definition 6.2.8 (partial order). For two divisors $D_1, D_2 \in \text{Div}(X)$, we say $D_1 \geq D_2$ if $D_1 - D_2 \geq 0$.

6.3. Global sections associated to divisors.

6.3.1. *Global sections of $\mathcal{O}(D)$.* Given $D \in \text{Div}(X)$, consider the following set⁶

$$\Gamma(X, \mathcal{O}(D)) := \{f \in \mathcal{M}(X) \mid \text{div}(f) + D \geq 0\}.$$

Moreover, if $f \equiv 0$, we define $\text{ord}_p(f) = \infty$, and thus $0 \in \Gamma(X, \mathcal{O}(D))$.

Remark 6.3.1. $\Gamma(X, \mathcal{O}(D))$ consists of meromorphic functions with poles not too bad.

- (1) If $D(p) = -n < 0$, then p must be a zero of f with order $\geq n$;
- (2) If $D(p) = n > 0$, then p may be a pole, but its order at least won't be larger than n .

It's clear that there is a \mathbb{C} -vector space structure on $\Gamma(X, \mathcal{O}(D))$. Later (in Corollary 6.3.1) we will show that $\Gamma(X, \mathcal{O}(X))$ is a finite-dimensional \mathbb{C} -vector space, and we use $\ell(D)$ to denote $\dim_{\mathbb{C}} \Gamma(X, \mathcal{O}(D))$ for convenience. Before that, let's see some basic properties and examples.

Lemma 6.3.1. For $D_1, D_2 \in \text{Div}(X)$. If $D_1 \leq D_2$, then $\Gamma(X, \mathcal{O}(D_1)) \subseteq \Gamma(X, \mathcal{O}(D_2))$.

Lemma 6.3.2. If $\deg(D) < 0$, then $\Gamma(X, \mathcal{O}(D)) = \{0\}$.

Proof. If $f \in \Gamma(X, \mathcal{O}(D))$ and $f \neq 0$, then one has $\text{div}(f) + D \geq 0$. By taking degree one has

$$0 = \deg(\text{div}(f)) \geq -\deg(D) > 0.$$

A contradiction. □

Lemma 6.3.3. If $D_1 \sim D_2$ are two divisors, then $\Gamma(X, \mathcal{O}(D_1)) \cong \Gamma(X, \mathcal{O}(D_2))$ as \mathbb{C} -vector spaces.

Proof. Since $D_1 \sim D_2$, there exists a meromorphic function h such that $D_1 = D_2 + \text{div}(h)$. For any $f \in \Gamma(X, \mathcal{O}(D_1))$, then

$$\text{div}(fh) = \text{div}(f) + \text{div}(h) \geq -D_1 + D_1 - D_2 = -D_2.$$

Thus one can define a linear map

$$\begin{aligned} \mu_h: \Gamma(X, \mathcal{O}(D_1)) &\rightarrow \Gamma(X, \mathcal{O}(D_2)) \\ f &\mapsto fh. \end{aligned}$$

⁶In algebraic geometry, a Cartier divisor corresponds to a line bundle $\mathcal{O}(D)$, and here $\Gamma(X, \mathcal{O}(D))$ is exactly the global section of line bundle $\mathcal{O}(D)$.

with inverse $\mu_{h^{-1}}: \Gamma(X, \mathcal{O}(D_2)) \rightarrow \Gamma(X, \mathcal{O}(D_1))$. This shows $\Gamma(X, \mathcal{O}(D_1)) \cong \Gamma(X, \mathcal{O}(D_2))$. \square

Example 6.3.1. If $D = 0$, then $\Gamma(X, \mathcal{O}(0))$ consists of holomorphic function, and since X is compact, one has

$$\Gamma(X, \mathcal{O}(0)) \cong \mathbb{C}.$$

In particular, if $D \in \text{PDiv}(X)$, then $\Gamma(X, \mathcal{O}(D)) \cong \Gamma(X, \mathcal{O}(0)) \cong \mathbb{C}$.

Example 6.3.2. Suppose D is a divisor on \mathbb{P}^1 with $\deg(D) \geq 0$, which is written by

$$D = \sum_{i=1}^n e_i \cdot \lambda_i + e_\infty \cdot \infty.$$

Consider the function

$$f_D(z) = \prod_{i=1}^n (z - \lambda_i)^{-e_i}.$$

Then we claim that

$$\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(D)) = \{g(z)f_D(z) \mid g(z) \text{ is a polynomial of degree at most } \deg(D)\}.$$

For a polynomial $g(z)$ of degree d , one has $\text{div}(g) \geq -d \cdot$, and note that

$$\text{div}(f_D) = \sum_i e_i \cdot \lambda_i + \left(\sum_i e_i\right) \cdot \infty.$$

Then

$$\begin{aligned} \text{div}(g(z)f_D(z)) + D &= \text{div}(g) + \text{div}(f_D) + D \\ &\geq \left(\sum_i e_i + e_\infty - d\right) \cdot \infty \\ &\geq 0. \end{aligned}$$

Conversely, for any function $h \in \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(D))$, consider $g = h/f_D$. A direct computation shows

$$\begin{aligned} \text{div}(g) &= \text{div}(h) - \text{div}(f_D) \\ &\geq -D - \text{div}(f_D) \\ &= \left(-\sum_i e_i - e_\infty\right) \cdot \infty \\ &= -\deg(D) \cdot \infty. \end{aligned}$$

This shows that g can admit no poles in the finite part \mathbb{C} , and can have a pole of order at most $\deg(D)$. This forces g to be a polynomial of degree at most $\deg(D)$. In particular, one has

$$\ell(\mathcal{O}_{\mathbb{P}^1}(D)) = \begin{cases} 0, & \deg(D) < 0 \\ 1 + \deg(D), & \deg(D) \geq 0 \end{cases}$$

Example 6.3.3. Suppose X is a compact Riemann surface with genus ≥ 1 and D is a divisor given by a point p . Then we claim that

$$\Gamma(X, \mathcal{O}(p)) = \mathbb{C}.$$

Note that if $f \in \Gamma(X, \mathcal{O}(p))$, then f is only allowed to have a simple pole at p . If f is non-constant, then it gives a holomorphic map $\Phi: X \rightarrow \mathbb{P}^1$ with $\Phi^{-1}(\infty) = \{p\}$. This shows $\deg(\Phi) = 1$, and thus it's an isomorphism, a contradiction to $\text{genus}(X) \geq 1$.

For $D \in \text{Div}(X)$, let's estimate the upper bound of the dimension of the \mathbb{C} -vector space $\Gamma(X, \mathcal{O}(D))$.

Lemma 6.3.4. For any $D \in \text{Div}(X)$, and $p \in X$, then either $\Gamma(X, \mathcal{O}(D - p)) = \Gamma(X, \mathcal{O}(D))$ or $\Gamma(X, \mathcal{O}(D - p))$ has codimension 1 in $\Gamma(X, \mathcal{O}(D))$ holds.

Proof. Let $n = -D(p)$, and choose a local coordinate z centered at p . For any $f \in \Gamma(X, \mathcal{O}(D))$, the Laurent series of f at p must have the following form

$$cz^n + \text{higher order terms}$$

Consider $\alpha: \Gamma(X, \mathcal{O}(D)) \rightarrow \mathbb{C}$, which is defined by $f \mapsto c$.

- (1) If $\alpha \neq 0$, then it's a surjective linear map. If $f \in \ker \alpha$, then $\text{ord}_p(f) \geq n+1$, and thus $\text{ord}_p(f) + D(p) - 1 \geq 0$, which implies $f \in \Gamma(X, \mathcal{O}(D - p))$. By the same argument one can show $\Gamma(X, \mathcal{O}(D - p)) \subseteq \ker \alpha$, and thus $\ker \alpha = \Gamma(X, \mathcal{O}(D - p))$. This shows $\Gamma(X, \mathcal{O}(D - p))$ has codimension 1 in $\Gamma(X, \mathcal{O}(D))$.
- (2) If $\alpha \equiv 0$, then $\Gamma(X, \mathcal{O}(D - p)) = \Gamma(X, \mathcal{O}(D))$.

□

Theorem 6.3.1. For any $D \in \text{Div}(X)$, write $D = P - N$ such that $P, N \geq 0$ and $\text{Supp}(P) \cap \text{Supp}(N) = \emptyset$. Then

$$\ell(D) \leq 1 + \deg(P).$$

Proof. Let's prove it by induction on $\deg(P)$. If $\deg(P) = 0$, that is, $P = 0$, then one has $\Gamma(X, \mathcal{O}_X(P)) \cong \mathbb{C}$. Thus $\ell(D) \leq \ell(P) = 1 = 1 + \deg(P)$.

Assume induction hypothesis holds for $\deg(P) = k - 1$. Let $D = P - N$ be a divisor with $\deg(P) = k$, such that $P, N \geq 0$ and $\text{Supp}(P) \cap \text{Supp}(N) = \emptyset$. Since $\text{Supp}(P) \neq \emptyset$, we choose $q \in \text{Supp}(P)$, and write $D - q = (P - q) - N$. Then $\text{Supp}(P - q) \cap \text{Supp}(N) = \emptyset$ and $\deg(P - q) = k - 1$. Then by induction, one has

$$\ell(D - q) \leq 1 + \deg(P - q) = 1 + k - 1 = k,$$

and by Lemma 6.3.4, one has

$$\ell(D) \leq \ell(D - q) + 1 \leq k + 1 = \deg(P) + 1.$$

This completes the proof. □

Corollary 6.3.1. For any $D \in \text{Div}(X)$, $\Gamma(X, \mathcal{O}(D))$ is a finite-dimensional⁷ \mathbb{C} -vector space.

6.3.2. *Global sections of $\Omega^1(D)$.* Let $\mathcal{M}^{(1)}(X)$ be the set of all meromorphic 1-forms on X . For $D \in \text{Div}(X)$, the global sections of $\Omega^1(D)$ is defined by

$$\Gamma(X, \Omega^1(D)) = \{\omega \in \mathcal{M}^{(1)}(X) \mid \text{div}(\omega) + D \geq 0\}.$$

Example 6.3.4. $\Gamma(X, \Omega^1(0))$ consists of all holomorphic 1-forms, and sometimes is denoted by $\Omega^1(X)$. Not like holomorphic functions, there may be many non-trivial holomorphic 1-forms on X . Later (in) we will see its dimension is closely related to the genus of X .

Lemma 6.3.5. For $D_1, D_2 \in \text{Div}(X)$. If $D_1 \sim D_2$, one has $\Gamma(X, \Omega^1(D_1)) \cong \Gamma(X, \Omega^1(D_2))$

Proof. The same as Lemma 6.3.3. □

Theorem 6.3.2. Let K be the canonical divisor on X . Then for any $D \in \text{div}(X)$, one has

$$\Gamma(X, \Omega^1(D)) \cong \Gamma(X, \mathcal{O}(K + D)).$$

Proof. Suppose the canonical divisor K is given by meromorphic 1-form ω , that is, $K = \text{div}(\omega)$. For any $f \in \Gamma(X, \mathcal{O}(K + D))$, one has

$$\text{div}(f\omega) = \text{div}(f) + \text{div}(\omega) \geq -(K + D) + K = -D.$$

Thus $f\omega \in \Gamma(X, \Omega^1(D))$. This gives a linear map

$$\begin{aligned} \mu_\omega: \Gamma(X, \mathcal{O}(K + D)) &\rightarrow \Gamma(X, \Omega^1(D)) \\ f &\mapsto f\omega. \end{aligned}$$

It's clear that μ_ω is injective, and thus it suffices to show μ_ω is surjective. For any $\theta \in \Gamma(X, \Omega^1(D))$, by Lemma 6.2.4, there exists meromorphic function f such that $\theta = f\omega$. Note that

$$-D \leq \text{div}(\theta) = \text{div}(f) + \text{div}(\omega) = \text{div}(f) + K.$$

This shows $\text{div}(f) + (D + K) \geq 0$, that is, $f \in \Gamma(X, \mathcal{O}(D + K))$, as desired. □

6.4. Linear system and morphisms to projective space.

6.4.1. Linear system.

Definition 6.4.1 (complete linear system). For $D \in \text{Div}(X)$, the complete linear system of D is defined by

$$|D| = \{E \in \text{Div}(X) \mid E \geq 0, E \sim D\}.$$

Lemma 6.4.1.

$$\begin{aligned} S: \mathbb{P}(\Gamma(X, \mathcal{O}(D))) &\rightarrow |D| \\ [f] &\mapsto \text{div}(f) + D \end{aligned}$$

is bijective.

⁷Later we will introduce Riemann-Roch theorem to compute its dimension.

Proof. It's clear S is well-defined and by definition it's surjective. Now let's show the injectivity. For $f_1, f_2 \in \Gamma(X, \mathcal{O}(D)) \setminus \{0\}$, if $S(f_1) = S(f_2)$, then $\text{div}(f_1/f_2) = 0$. This shows f_1/f_2 is a holomorphic function, and thus f_1/f_2 is constant. Then $f_1 = f_2$ in $\mathbb{P}(\Gamma(X, \mathcal{O}(D)))$. \square

Corollary 6.4.1.

- (1) If $\deg(D) < 0$, then $|D| = \emptyset$.
- (2) If $D_1 \sim D_2$, then $|D_1| = |D_2|$.
- (3) $\ell(D) \geq 1$ if and only if $|D| \neq \emptyset$.

Proof. (1) follows from Lemma 6.3.2, and (2) follows from Lemma 6.3.3. For (3). It suffices to note that $\ell(D) \geq 1$ if and only if $\mathbb{P}(\Gamma(X, \mathcal{O}(D))) \neq \emptyset$. \square

6.4.2. *Linear system of holomorphic maps to projective space.*

Definition 6.4.2 (holomorphic maps to projective space). A map $\Phi: X \rightarrow \mathbb{P}^n$ is holomorphic at $p \in X$ if there are holomorphic functions f_0, \dots, f_n defined near p , not all zero at p , such that $\Phi(q) = [f_0(q) : \dots : f_n(q)]$ for every q near p . The map Φ is a holomorphic map if it's holomorphic at all points of X .

Remark 6.4.1. Note that if X is a compact Riemann surface, there is no global defined holomorphic function, and thus one cannot expect to use the same holomorphic functions f_i at all points of X to define a holomorphic map Φ .

However, one can use meromorphic functions to construct holomorphic maps to projective space, and it turns out every holomorphic map can be defined in this way. Let X be a Riemann surface and $f = \{f_0, \dots, f_N\}$ is a set of meromorphic functions on X . Now we define

$$\begin{aligned} \Phi_f: X &\rightarrow \mathbb{P}^N \\ p &\mapsto [f_0(p), \dots, f_N(p)]. \end{aligned}$$

In apriori, Φ_f is defined for p such that p is not a pole of any f_i and p is not a zero of every f_i , and Φ_f is holomorphic at all points where it's defined.

Lemma 6.4.2. If the set of meromorphic functions $f = \{f_0, \dots, f_N\}$ is not all identically zero, then the map $\Phi_f: X \rightarrow \mathbb{P}^N$ is defined on all of X .

Suppose $\Phi: X \rightarrow \mathbb{P}^N$ be a holomorphic map defined by meromorphic functions $f = \{f_0, \dots, f_N\}$ on X . If we denote $D = -\min_f \{\text{div}(f_i)\}$, then for each i , one has

$$\text{ord}_p(f_i) \geq -D(p).$$

Therefore $\{f_i\} \subseteq \Gamma(X, \mathcal{O}(D))$, and if we use V_f to denote the linear subspace generated by $\{f_i\}$, then it gives a linear system

$$|\Phi| = \{\text{div}(g) + D \mid g \in V_f\}.$$

6.4.3. Base locus of linear systems.

Definition 6.4.3 (base locus). Let Q be a linear system on X . A point p is a base point of Q if every divisor $E \in Q$ contains p , and the set of all base points of Q is called its base locus.

Definition 6.4.4 (base-point-free). A linear system Q is said to be base-point-free if it has no base point.

Notation 6.4.1. For convenience, for a divisor D , the base locus of D is the base locus of the complete linear system $|D|$, and D is said to be base-point-free if $|D|$ is base-point-free.

Lemma 6.4.3. Let $D \in \text{Div}(X)$ and $Q \subseteq |D|$ be a linear system defined by the subspace $V \subseteq \Gamma(X, \mathcal{O}(D))$. Then $p \in X$ is a base point of Q if and only if

$$V \subseteq \Gamma(X, \mathcal{O}(D - p)).$$

In particular, p is a base point of $|D|$ if and only if

$$\ell(D) = \ell(D - p).$$

Proof. Note that the linear system Q is given by $\{\text{div}(f) + D \mid f \in V\}$, and thus p is a base point of Q if and only if for every $f \in V$, one has

$$\text{ord}_p(f) + D(p) \geq 1.$$

In other words, for every $f \in V$, one has

$$\text{ord}_p(f) \geq -D(p) + 1,$$

which is equivalent to $f \in \Gamma(X, \mathcal{O}(D - p))$, as desired. \square

Proposition 6.4.1. A divisor D is base-point-free if and only if

$$\ell(D - p) = \ell(D) - 1$$

for all $p \in X$.

Proof. It follows from Lemma 6.3.4 and Lemma 6.4.3. \square

Corollary 6.4.2. Let B be the base locus of D . Then $D - B$ is base-point-free, and

$$\Gamma(X, \mathcal{O}(D - B)) \rightarrow \Gamma(X, \mathcal{O}(D))$$

is an isomorphism.

Corollary 6.4.3. A divisor D is base-point-free if and only if for every $p \in X$, there exists a basis $\{f_0, f_1, \dots, f_N\}$ of $\Gamma(X, \mathcal{O}(D))$ such that $\text{ord}_p f_0 = -D(p)$ and $\text{ord}_p f_i > -D(p)$ for $1 \leq i \leq N$.

6.4.4. Holomorphic maps and linear systems.

Theorem 6.4.1. Let Q be a base-point-free linear system on a compact Riemann surface X . Then there exists a holomorphic map $\Phi: X \rightarrow \mathbb{P}^N$ such that $|\Phi| = Q$. Moreover, Φ is unique up to the choice of coordinates in \mathbb{P}^N .

6.4.5. *Criterion for ampleness.* Let D be a divisor on a compact Riemann surface X . By Corollary 6.4.2 one can always remove the base locus of D without changing the complete linear system $|D|$. Thus without loss of generality we may always assume $|D|$ is base-point-free, and thus it induces a holomorphic map $\Phi_D: X \rightarrow \mathbb{P}^N$.

Definition 6.4.5 (very ample). A base-point-free divisor D on a compact Riemann surface X is called very ample if Φ_D is an embedding. D is called a very ample divisor.

Proposition 6.4.2. A divisor D is very ample if and only if for every p and q in X , one has

$$\ell(D - p - q) = \ell(D) - 2.$$

7. RIEMANN-ROCH THEOREM

In this section, X always denotes a compact Riemann surface with genus g and K is the canonical divisor on X .

7.1. Statement and applications.

Theorem 7.1.1 (Riemann-Roch). For any $D \in \text{Div}(X)$, one has

$$\ell(D) - \ell(K - D) = 1 - g + \deg D.$$

Now let's show some applications of Riemann-Roch theorem before giving a proof.

7.1.1. Hodge decomposition.

Corollary 7.1.1. $\ell(K) = g$.

Proof. By Example 6.3.1 one has $\ell(0) = 1$. □

Proposition 7.1.1.

$$\Gamma(X, \mathcal{O}(K)) \rightarrow \Gamma(X, \Omega^1)$$

is an isomorphism.

7.1.2. The uniqueness of complex structure on S^2 .

Corollary 7.1.2 (Riemann inequality). $\ell(D) \geq 1 - g + \deg(D)$.

Historically, Riemann found this inequality and his student Roch made it into an equality. However, the following lemma shows that in a very generic case, Riemann inequality is always an equality.

Lemma 7.1.1. If $\deg(D) \geq 2g - 1$, then $\ell(D) = 1 - g + \deg(D)$.

Proof. If $\deg(D) \geq 2g - 1$, then $\deg(K - D) < 0$, since

$$\deg(K - D) = \deg(K) - \deg(D) = 2g - 2 - \deg(D).$$

Then by Lemma 6.3.2 one has $\ell(K - D) = 0$. □

Lemma 7.1.2. If there exists $p \in X$ such that $\ell(p) > 1$, then X is isomorphic to \mathbb{P}^1 .

Proof. If $\ell(p) > 1$ for some $p \in X$, there exists a non-constant meromorphic function $f \in \Gamma(X, \mathcal{O}(p))$. Suppose $\Phi: X \rightarrow \mathbb{P}^1$ is the holomorphic map corresponding to f . By the same argument used in Example 6.3.3 one has Φ is an isomorphism. □

Corollary 7.1.3. Any compact Riemann surface X with genus zero is isomorphic to \mathbb{P}^1 .

Proof. For any $p \in X$, $\deg(p) = 1 \geq 2 \times 0 - 1$, and thus $\ell(p) = \deg(p) + 1 - 0 = 2 > 1$. Then by Lemma 7.1.2 one has X is isomorphic to \mathbb{P}^1 . □

Corollary 7.1.4. The complex structure on topological sphere S^2 is unique.

7.1.3. Genus one curve.

Proposition 7.1.2. Let X be a compact Riemann surface with genus one. Then X is isomorphic to a complex torus, that is, $X \cong \mathbb{C}/L$, where $L = \mathbb{Z}w_1 + \mathbb{Z}w_2$ is a lattice.

Proof. By Riemann-Roch theorem one has $\ell(K) = 1$. Then there exists a non-trivial holomorphic 1-form η . Suppose $\eta = f(z)dz$, where $f(z)$ is a holomorphic function. \square

Moreover, one has $\ell(p) = 1, \ell(2p) = 2$. Then there exists $f \in \Gamma(X, \mathcal{O}(2p))$, it gives $f: X \rightarrow \mathbb{P}^1$, and f has only one pole at p , double. This shows the degree of f is 2. By Riemann-Hurwitz one has

$$2 - 2g = 4 - \deg R_F$$

and thus it has four ramification points p_1, p_2, p_3, p_4 , and suppose $f(p_i) = q_i$. On the other hand, consider

$$\{y^2 = (x - x_1)(x - x_2)(x - x_3)\} = \bar{C}$$

and \tilde{C} is the normalization. If genus of C is one, then X is a cubic curve.

7.1.4. *Canonical map.* For the case $g = 2$, one has $\ell(0) = 1$ and $\ell(K) = 2$, and $\ell(p) = 1, \ell(K - p) = 1$, and $\ell(p + q)$ has different cases

Lemma 7.1.3. Let X be a compact Riemann surface with $\text{genus}(X) \geq 1$ and K be a canonical divisor. Then the complete linear system $|K|$ is base point free.

Proof. For any $p \in X$, one has $\ell(p) = 1$ for every $p \in X$, otherwise $X \cong \mathbb{P}^1$ by Lemma 7.1.2. Then by Riemann-Roch theorem, one has

$$\ell(p) - \ell(K - p) = \deg(p) + 1 - g.$$

This shows $\ell(K - p) = g - 1 < g = \ell(K)$, for all $p \in X$. By Proposition 6.4.1 one has $|K|$ is base-point-free. \square

Definition 7.1.1 (canonical map).

7.1.5. Hyperelliptic curve.

Definition 7.1.2 (hyperelliptic). A compact Riemann surface X is called hyperelliptic if there exists a holomorphic map $\Phi: X \rightarrow \mathbb{P}^1$ such that $\deg(\Phi) = 2$.

Lemma 7.1.4. Let X be a hyperelliptic curve. Then there exists an involution, called hyperelliptic involution, on X which has $2\text{genus}(X) + 2$ fixed points.

Proof. Suppose $\Phi: X \rightarrow \mathbb{P}^1$ is a holomorphic map with degree 2. Then by Riemann-Hurwitz formula, one has

$$2\text{genus}(X) - 2 = \deg(\Phi)(2 \times 0 - 2) + B(\Phi)$$

This shows $B(F) = 2g + 2$. In other words, Φ has exactly $2g + 2$ ramification points $x_1, \dots, x_{2g+2} \in X$, since $\deg(\Phi) = 2$, and $2g + 2$ ramification values $b_i = \Phi(x_i) \in \mathbb{P}^1$.

For any $z \in \mathbb{P}^1 \setminus \{b_1, \dots, b_{2g+2}\}$, one has $\Phi^{-1}(z)$ contains 2 points. Now we define the involution $T: X \rightarrow X$ as follows: For each ramification point x_i , $T(x_i) = x_i$, and $T(p) = q$ if $\Phi(p) = \Phi(q)$ and $p \neq q$. It's clear that the fixed points of T are $\{x_1, \dots, x_{2g+2}\}$. \square

Lemma 7.1.5. A compact Riemann surface X is hyperelliptic if and only if there exists $D \in \text{Div}(X)$ such that $D \geq 0$, $\deg(D) = 2$ and $\ell(D) \geq 2$.

Proof. If X is hyperelliptic, then there exists a holomorphic map $\Phi: X \rightarrow \mathbb{P}^1$ with degree 2. Suppose f is the non-constant meromorphic function corresponding to Φ , and let $D = \text{div}_\infty(f) \geq 0$. Then $\deg(D) = \deg(\Phi) = 2$. Moreover,

$$\text{div}(f) = \text{div}_0(f) - \text{div}_\infty(f) \geq -D.$$

This shows $f \in L(D)$, and thus $\ell(D) \geq 2$.

Conversely, given divisor D such that $D \geq 0$, $\deg(D) = 2$ and $\ell(D) \geq 2$, we choose meromorphic function $f \in \Gamma(X, \mathcal{O}(D))$ with corresponding holomorphic map $\Phi: X \rightarrow \mathbb{P}^1$. Then

$$1 \leq \deg(\Phi) = \deg(\text{div}_\infty(f)) \leq \deg(D) = 2.$$

This shows $\deg(\Phi) = 1$ or 2 .

- (1) If $X \not\cong \mathbb{P}^1$, then $\deg(\Phi) = 2$, and thus X is hyperelliptic.
- (2) If $X \cong \mathbb{P}^1$, then X is hyperelliptic by considering $z \mapsto z^2$.

\square

Theorem 7.1.2. If $\text{genus}(X) \leq 2$, then X is hyperelliptic.

Proof. Let $D \in \text{Div}(X)$ with $D \geq 0$, $\deg(D) = 2$. By Riemann-Roch inequality one has

$$\ell(D) \geq \deg(D) + 1 - g = 3 - g.$$

This shows $\ell(D) \geq 2$ if $g \leq 1$, and thus X is hyperelliptic by Lemma 7.1.5.

If $g = 2$, there exists a non-zero holomorphic 1-form ω since $\dim \Gamma(X, \Omega^1) = g = 2$. For $K = \text{div}(\omega)$, one has $\deg(K) = 2g - 2 = 2$. Then by Riemann-Roch theorem one has $\ell(K) = 2$, and thus X is hyperelliptic by Lemma 7.1.5. \square

Theorem 7.1.3. There exists hyperelliptic curve of any genus.

Theorem 7.1.4. If X is hyperelliptic with $g \geq 2$, then hyperelliptic involution is the unique involution on X which has $2g + 2$ fixed points.

Theorem 7.1.5. K is very ample if and only if X is hyperelliptic.

Proof. K is not very ample if and only if $\ell(K - p - q) = \ell(K) - 1$ for every p, q . \square

7.1.6. *Genus three curve.* The canonical map of genus three curve is $\Phi_{|K|}: C \rightarrow \mathbb{P}^2$. Note that $\deg \Phi_{|K|}^*(H) = 2g - 2 = 4$. On the other hand, the projective plane curve of degree 4 also has genus 3.

Proposition 7.1.3. Let X be a compact Riemann surface of genus 3. Then X is not hyperelliptic if and only if a non-singular quatic curve.

Proof. Consider

$$R_k: \text{Sym}^k(\mathbb{C}^{n+1}) \rightarrow \Gamma(X, \mathcal{O}(kD))$$

$$p(x_0, \dots, x_n) \mapsto p(f_0, \dots, f_n).$$

If k is large, then $\ell(kD) - \ell(K - kD) = 1 - g + k \deg(D)$. □

7.2. **Proof of the theorem.**

8. HOMEWORK

8.1. Homework 1.

Exercise 8.1.1. Prove that when $\omega_1, \omega_2 \in \mathbb{C}$ are \mathbb{R} -linearly independent, then

- (1) $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is discrete.
- (2) $\mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is Hausdorff.
- (3) $\mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is a covering map.

Proof. For (1). Choose $0 < \epsilon < \min\{|w_1|/2, |w_2|/2, |w_1 - w_2|/2\}$. Then for any two elements u, v in $\mathbb{Z}w_1 + \mathbb{Z}w_2$, one has $B_\epsilon(u) \cap B_\epsilon(v) = \emptyset$, and thus $\mathbb{Z}w_1 + \mathbb{Z}w_2$ is discrete.

For (2). Let L denote the lattice $\mathbb{Z}w_1 + \mathbb{Z}w_2$ and $\pi: \mathbb{C} \rightarrow \mathbb{C}/L$ be the canonical projection. Suppose \mathbb{C}/L is equipped with the quotient topology, that is, $U \subseteq \mathbb{C}/L$ is an open subset if and only if $\pi^{-1}(U)$ is open in \mathbb{C} . It's easy to show $\pi: \mathbb{C} \rightarrow \mathbb{C}/L$ is an open map, since for any open subset $U \subseteq \mathbb{C}$, one has

$$\pi^{-1}(\pi(U)) = \bigcup_{w \in L} w + U.$$

For $u, v \in \mathbb{C}/L$, we choose $\tilde{u}, \tilde{v} \in \mathbb{C}$ such that $\pi(\tilde{u}) = u$ and $\pi(\tilde{v}) = v$. Since \mathbb{C} is Hausdorff, there exists open neighborhoods \tilde{U}, \tilde{V} of \tilde{u}, \tilde{v} such that $\tilde{U} \cap \tilde{V} = \emptyset$. Moreover, we may assume $\pi|_{\tilde{U}}$ and $\pi|_{\tilde{V}}$ are injective by shrinking \tilde{U}, \tilde{V} when necessary. Then $\pi(\tilde{U})$ and $\pi(\tilde{V})$ are open neighborhoods of u, v respectively such that $\pi(\tilde{U}) \cap \pi(\tilde{V}) = \emptyset$. This shows \mathbb{C}/L with quotient topology is Hausdorff.

For (3). For $u \in \mathbb{C}/L$, the preimages of u is discrete since L is discrete. For each preimage \tilde{u}_i , we choose $\epsilon > 0$ small sufficiently such that $B_\epsilon(\tilde{u}_i) \cap B_\epsilon(u_j) = \emptyset$ for $i \neq j$ and $\pi|_{B_\epsilon(\tilde{u}_i)}$ is injective for all i . If we denote $U = \pi(B_\epsilon(\tilde{u}_i))$, then $\pi: B_\epsilon(\tilde{u}_i) \rightarrow U$ is a homeomorphism for each i and by construction $B_\epsilon(\tilde{u}_i) \cap B_\epsilon(u_j) = \emptyset$ for $i \neq j$. This shows $\pi: \mathbb{C} \rightarrow \mathbb{C}/L$ is a covering map. \square

Exercise 8.1.2. Let V be a complex vector space of dimension n , with \mathbb{C} -basis e_1, \dots, e_n , and $T: V \rightarrow V$ is a \mathbb{C} -linear transformation. Suppose T has matrix representation $X = A + \sqrt{-1}B$ where $A, B \in M_n(\mathbb{R})$ under (complex) basis e_1, \dots, e_n . Prove

- (1) $e_1, \dots, e_n, \sqrt{-1}e_1, \dots, \sqrt{-1}e_n$ is an \mathbb{R} -basis of V .
- (2) T has matrix

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

under the \mathbb{R} -basis above when T is viewed as an \mathbb{R} -linear transformation.

(3)

$$\det \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = |\det X|^2.$$

Proof. For (1). Since e_1, \dots, e_n are \mathbb{C} -linearly independent and $1, \sqrt{-1}$ are \mathbb{R} -linearly independent, one has $e_1, \dots, e_n, \sqrt{-1}e_1, \dots, \sqrt{-1}e_n$ are \mathbb{R} -linearly independent. On the other hand, since e_1, \dots, e_n is a \mathbb{C} -basis, then any element $v \in V$ can be expressed as $v = v_1e_1 + \dots + v_ne_n$, where $v_i \in \mathbb{C}$. If we write $v_i = a_i + \sqrt{-1}b_i$ with $a_i, b_i \in \mathbb{R}$, then

$$v = a_1e_1 + \dots + a_ne_n + \sqrt{-1}b_1e_1 + \dots + \sqrt{-1}b_ne_n.$$

This shows V as a \mathbb{R} -vector space is spanned by $e_1, \dots, e_n, \sqrt{-1}e_1, \dots, \sqrt{-1}e_n$.

For (2). Since T has matrix representation $X = A + \sqrt{-1}B$ under \mathbb{C} -basis e_1, \dots, e_n , one has

$$\begin{aligned} T(e_i) &= \sum_{j=1}^n X_{ij}e_j = \sum_{j=1}^n (A_{ij}e_j + B_{ij}\sqrt{-1}e_j) \\ T(\sqrt{-1}e_i) &= \sum_{j=1}^n X_{ij}\sqrt{-1}e_j = \sum_{j=1}^n (-B_{ij}e_j + A_{ij}\sqrt{-1}e_j). \end{aligned}$$

This shows T has matrix

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

under the \mathbb{R} -basis $e_1, \dots, e_n, \sqrt{-1}e_1, \dots, \sqrt{-1}e_n$.

For (3). By elementary operations, one has

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \rightarrow \begin{pmatrix} A + \sqrt{-1}B & B \\ -B + \sqrt{-1}A & A \end{pmatrix} \rightarrow \begin{pmatrix} A + \sqrt{-1}B & B \\ 0 & A + \sqrt{-1}B \end{pmatrix}$$

Since the elementary operations don't change the determinant, this shows the desired result. \square

Exercise 8.1.3 (implicit function theorem). Let $f(z, w): \mathbb{C}^2 \rightarrow \mathbb{C}$ be holomorphic function of two variables and $X = \{(z, w) \in \mathbb{C}^2 \mid f(z, w) = 0\}$ be its zero locus. Let $p = (z_0, w_0)$ be a point of X and $\partial f / \partial z(p) \neq 0$. Then there exists a function $g(w)$ defined and holomorphic in a neighborhood of w_0 such that, near p , X is equal to the graph $z = g(w)$.

Proof. If we write $z = a + \sqrt{-1}b, w = c + \sqrt{-1}d$ and $f(z, w) = u + \sqrt{-1}v$, then u, v are smooth functions of a, b, c, d . Moreover, the Cauchy-Riemann equations give

$$\frac{\partial f}{\partial z} = \frac{\partial u}{\partial a} + \sqrt{-1}\frac{\partial v}{\partial a} = \frac{\partial v}{\partial b} - \sqrt{-1}\frac{\partial u}{\partial b} = A + \sqrt{-1}B.$$

Then

$$\frac{\partial(u, v)}{\partial(a, b)} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

and $\det \frac{\partial(u, v)}{\partial(a, b)} = A^2 + B^2 \neq 0$ if and only if $A + \sqrt{-1}B \neq 0$. Then the classical implicit function theorem implies the zero locus

$$\begin{cases} u = 0 \\ v = 0 \end{cases}$$

is locally given by

$$\begin{cases} a = a(c, d) \\ b = b(c, d). \end{cases}$$

In other words, $z = g(w)$. Now it suffices to compute $\partial g / \partial \bar{w}$ to show g is holomorphic. Again by Cauchy-Riemann equations

$$\frac{\partial f}{\partial w} = \frac{\partial u}{\partial c} + \sqrt{-1} \frac{\partial v}{\partial c} = \frac{\partial v}{\partial d} - \sqrt{-1} \frac{\partial u}{\partial d} = C + \sqrt{-1}D.$$

Then by chain rule one has

$$\begin{aligned} \frac{\partial(a, b)}{\partial(c, d)} &= \left(\frac{\partial(u, v)}{\partial(a, b)} \right)^{-1} \frac{\partial(u, v)}{\partial(c, d)} \\ &= \begin{pmatrix} A & B \\ -B & A \end{pmatrix}^{-1} \begin{pmatrix} C & D \\ -D & C \end{pmatrix} \\ &= \frac{1}{A^2 + B^2} \begin{pmatrix} AC + BD & AD - BC \\ BC - AD & BD + AC \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial g}{\partial \bar{w}} &= \frac{1}{2} \left(\frac{\partial}{\partial c} + \sqrt{-1} \frac{\partial}{\partial d} \right) (a + \sqrt{-1}b) \\ &= \frac{1}{2} \left(\frac{\partial a}{\partial c} + \sqrt{-1} \frac{\partial b}{\partial c} + \sqrt{-1} \frac{\partial a}{\partial d} - \frac{\partial b}{\partial d} \right) \\ &= 0 \end{aligned}$$

□

Exercise 8.1.4. Let x_1, \dots, x_n be distinct points on \mathbb{C} and

$$f(x, y) = y^d - (x - x_1) \cdots (x - x_n).$$

Prove that $C = \{f(x, y) = 0\}$ defines a Riemann surface in \mathbb{C}^2 . (Question to think about: what is the topological shape of C ?)

Proof. Note that there is no common zero of $f(x, y)$ and $\partial f / \partial x$ since x_1, \dots, x_n are distinct points, so the affine plane curve defined by $f(x, y)$ is non-singular, and thus it's a Riemann surface. □

Remark 8.1.1. Now let's consider the singularity of its compactification. Suppose $n \geq d$, and consider the homogenous polynomial defined by $f(x, y)$ as follows

$$F(x, y, z) = z^{n-d} y^d - (x - x_1 z) \cdots (x - x_n z).$$

By setting $z = 0$ we found a new point $[0 : 1 : 0]$. It suffices to see it's singular or not. A direct computation shows

$$\frac{\partial F}{\partial x} = -(x - x_2z) \dots (x - x_nz) - \dots - (x - x_1z) \dots (x - x_{n-1}z)$$

$$\frac{\partial F}{\partial y} = dz^{n-d}y^{d-1}$$

$$\frac{\partial F}{\partial z} = (n-d)z^{n-d-1}y^d + x_1(x - x_2z) \dots (x - x_nz) + \dots + x_n(x - x_1z) \dots (x - x_{n-1}z).$$

Then

- (1) If $n > d + 1$, then it's singular.
- (2) If $n = d + 1$ or $n = d$, it's non-singular.

Now we suppose $n < d$, and then the homogenous polynomial defined $f(x, y)$ is given by

$$F(x, y, z) = y^d - z^{d-n}(x - x_1z) \dots (x - x_nz).$$

By setting $z = 0$ we find a new point $[1 : 0 : 0]$. It suffices to see it's singular or not. A direct computation shows

$$\frac{\partial F}{\partial x} = -z^{d-n}((x - x_2z) \dots (x - x_nz) + \dots + (x - x_1z) \dots (x - x_{n-1}z))$$

$$\frac{\partial F}{\partial y} = dy^{d-1}$$

$$\begin{aligned} \frac{\partial F}{\partial z} = & (n-d)z^{d-n-1}(x - x_1z) \dots (x - x_nz) \\ & + x_1z^{d-n}(x - x_2z) \dots (x - x_nz) + \dots + x_nz^{d-n}(x - x_1z) \dots (x - x_{n-1}z). \end{aligned}$$

Then

- (1) If $n < d - 1$, then it's singular.
- (2) If $n = d - 1$, then it's non-singular.

In a summary, only when $n = d - 1, d, d + 1$, the compactification is non-singular, otherwise it's singular.

8.2. Homework 2.

Exercise 8.2.1. Consider the affine plane curve

$$C = \{y^2 = x^3 + ax + b\}, \quad a, b \in \mathbb{C}.$$

- (1) Find the equation for the corresponding projective plane curve in \mathbb{P}^2 .
- (2) When is C smooth?
- (3) When C is not smooth, find the singular points.

Proof. For (1). The corresponding projective plane curve in \mathbb{P}^2 is defined by

$$F(x, y, z) = zy^2 - x^3 - axz^2 - bz^3.$$

For (2). For $f(x, y) = y^2 - x^3 - ax - b$, a direct computation shows

$$\begin{aligned} \frac{\partial f}{\partial x} &= -3x^2 - a, \\ \frac{\partial f}{\partial y} &= 2y. \end{aligned}$$

Note that C is non-singular if and only if for every point $(x, y) \in C$, at least one of above derivatives is non-zero. In other words, the singularities the solutions of the following systems of equations

$$f(x, y) = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0.$$

Note that above systems of equations is equivalent to

$$\begin{cases} x^3 + ax + b = 0 \\ 3x^2 + a = 0 \end{cases}$$

This shows C is non-singular if and only if $x^3 + ax + b$ has three different roots.

For (3). If C is non-singular, the singularities are given by the roots of $x^3 + ax + b$ with multiplicity > 1 . \square

Exercise 8.2.2. For a projective plane curve defined by a linear equation, we call it a projective line. Show that for any two distinct points on \mathbb{P}^2 , there is a unique projective line passing through them. Prove also that any two distinct projective lines intersect at one point.

Proof. For points $p, q \in \mathbb{P}^2$, without lose of generality we may assume $p = [x : y : 1]$ and $q = [z : w : 1]$. In the affine piece $U_2 = \{[z_0 : z_1 : z_2] \mid z_2 \neq 0\}$, it's clear that there exists a line, given by $az_0 + bz_1 + c = 0$, connecting the points (x, y) and (z, w) . Then the p, q is connected by the projective line defined by

$$az_0 + bz_1 + cz_2 = 0.$$

Conversely, suppose l_1, l_2 are two projective lines given by

$$\begin{aligned} az_0 + bz_1 + cz_2 &= 0 \\ ez_0 + fz_1 + gz_2 &= 0. \end{aligned}$$

Consider the corresponding lines in affine piece U_2 , that is,

$$az_0 + bz_1 + c = 0$$

$$ez_0 + fz_1 + g = 0.$$

There are two cases:

- (1) If $af \neq be$, then there exists a unique intersection of l_1, l_2 in U_2 . For $z_2 = 0$, points in l_1, l_2 are given by $[a/b : 1 : 0]$ and $[e/f : 1 : 0]$, so l_1 and l_2 cannot intersect at $z_2 = 0$ since $af \neq be$.
- (2) If $af = be$, then there exists no intersection of l_1, l_2 in U_2 , and the unique intersection are at $z_2 = 0$.

□

Exercise 8.2.3. We say $p_1, \dots, p_n \in \mathbb{P}^2$ are in general position if no three are colinear (i.e. lie on a projective line). Show that for four points in \mathbb{P}^2 in general position $\{p_1, \dots, p_4\}$ and $\{q_1, \dots, q_4\}$, there exists a $g \in \text{GL}(3, \mathbb{C})$ such that $gp_i = q_i, 1 \leq i \leq 4$.

Proof. Without loss of generality we assume $\{q_1, \dots, q_4\}$ are

$$\{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1]\}.$$

Now if we regard $\{p_1, \dots, p_4\}$ as four vectors in \mathbb{C}^3 , then there exists the following relations

$$ap_1 + bp_2 + cp_3 = p_4,$$

where $a, b, c \in \mathbb{C}$, since any four vectors in \mathbb{C}^3 are \mathbb{C} -linearly dependent. Moreover, since $\{p_1, \dots, p_4\}$ are colinear, one has $a, b, c \in \mathbb{C}^*$ and p_1, p_2, p_3 forms a basis of \mathbb{C}^3 . Then consider $g \in \text{GL}(3, \mathbb{C})$ defined by

$$\begin{cases} ap_1 \mapsto e_1 \\ bp_2 \mapsto e_2 \\ cp_3 \mapsto e_3, \end{cases}$$

where $\{e_1, e_2, e_3\}$ is the standard basis of \mathbb{C}^3 . Then

$$g(p_4) = g(ap_1 + bp_2 + cp_3) = [1 : 1 : 1]$$

as desired. □

Exercise 8.2.4. Given 5 points in \mathbb{P}^2 in general position, show that there exists a unique smooth conic passing through them (By conic we mean a projective plane curve defined by a degree-2 equation).

Proof. Suppose the five points are given by homogenous coordinates $\{[x_i : y_i : z_i]\}_{i=1}^5$. Then

$$\det \begin{pmatrix} x^2 & xy & y^2 & xz & yz & z^2 \\ x_1^2 & x_1y_1 & y_1^2 & x_1z_1 & y_1z_1 & z_1^2 \\ x_2^2 & x_2y_2 & y_2^2 & x_2z_2 & y_2z_2 & z_2^2 \\ x_3^2 & x_3y_3 & y_3^2 & x_3z_3 & y_3z_3 & z_3^2 \\ x_4^2 & x_4y_4 & y_4^2 & x_4z_4 & y_4z_4 & z_4^2 \\ x_5^2 & x_5y_5 & y_5^2 & x_5z_5 & y_5z_5 & z_5^2 \end{pmatrix} = 0$$

is a conic passing through them. \square

Exercise 8.2.5. Consider

$$C := \{x^3 + y^3 = z^3\}$$

and

$$\begin{aligned} F: C &\rightarrow \mathbb{P}^1, \\ [x : y : z] &\mapsto [x : z]. \end{aligned}$$

How many critical points are there and what are their multiplicities?

Proof. For $[x : z] \in \mathbb{P}^1$ with $x^3 \neq z^3$, it's clear there are three different values for y such that

$$y^3 = z^3 - x^3.$$

On the other hand, the points $[1 : 1], [1 : e^{\frac{2\pi\sqrt{-1}}{3}}], [1 : e^{\frac{4\pi\sqrt{-1}}{3}}] \in \mathbb{P}^1$ are the ramification value of above projection, with multiplicity 3. \square

Exercise 8.2.6. Let $F: X \rightarrow Y$ and $G: Y \rightarrow Z$ be two holomorphic maps between Riemann surfaces such that X, Y are connected, F, G are not constant maps. Prove that

$$\text{mult}_p(G \circ F) = \text{mult}_p F \cdot \text{mult}_{F(p)} G$$

Proof. Suppose $\text{mult}_p F = m$ and $\text{mult}_{F(p)} G = n$. Recall that the multiplicity is defined by the local normal form of holomorphic map. In other words, there exists an open neighborhood U of p with coordinate u , open neighborhood V of $F(p)$ with coordinate v and open neighborhood W of $G \circ F(p)$ with coordinate w , such that F is locally given by

$$u \mapsto v = u^m,$$

and G is locally given by

$$v \mapsto w = v^n.$$

Then $G \circ F$ is locally given by

$$u \mapsto w = u^{mn}.$$

Note that the multiplicity is independent of the choice of the local coordinates, and thus $\text{mult}_p(G \circ F) = mn = \text{mult}_p F \cdot \text{mult}_{F(p)} G$ as desired. \square

Exercise 8.2.7. Consider maps between \mathbb{C} defined by

$$\begin{aligned} F: \mathbb{C} &\rightarrow \mathbb{C} \\ z &\mapsto z^3(z^2 - 2z + a)^2, \end{aligned}$$

where

$$a = \frac{34 \pm 6\sqrt{21}}{7}.$$

Find the critical values of F and the corresponding multiplicities on critical points.

Proof. Note that the critical points of F are zero locs of $\partial F/\partial z = 0$, and a direct computation shows

$$\begin{aligned}\frac{\partial F}{\partial z} &= 3z^2(z^2 - 2z + a)^2 + 2z^3(z^2 - 2z + a)(2z - 2) \\ &= z^2(z^2 - 2z + a)(3(z^2 - 2z + a) + 2z(2z - 2)) \\ &= z^2(z^2 - 2z + a)(7z^2 - 10z + 3a).\end{aligned}$$

- (1) It's clear $z_0 = 0$ is a critical points of F with multiplicity 3, and thus $F(0) = 0$ is a critical value.
- (2) If z_1, z_2 are two solutions of $z^2 - 2z + a$, then $F(z_1) = F(z_2) = 0$, and the corresponding multiplicities on critical points z_1, z_2 are 2.
- (3) If z_3, z_4 are two solutions of $7z^2 - 10z + 3a$, then

$$\begin{aligned}z_3 &= \frac{10 + \sqrt{100 - 4 \times 7 \times 3a}}{14} \\ z_4 &= \frac{10 - \sqrt{100 - 4 \times 7 \times 3a}}{14}\end{aligned}$$

□

8.3. Homework 3.

Exercise 8.3.1. For which $\lambda \in \mathbb{C}$ we have

$$C := \{[x : y : z] \in \mathbb{P}^2 \mid x^3 + y^3 + z^3 + 3\lambda xyz = 0\}$$

is smooth? For such λ , compute the degree and critical values for

$$F: C \rightarrow \mathbb{P}^1, \quad [x : y : z] \mapsto [x : z]$$

and find the genus of C .

Proof. Let $P(x, y, z) = x^3 + y^3 + z^3 + 3\lambda xyz$ and consider the following equations

$$(8.1) \quad \begin{cases} \frac{\partial P}{\partial x} = 3x^2 + 3\lambda yz = 0 \\ \frac{\partial P}{\partial y} = 3y^2 + 3\lambda xz = 0 \\ \frac{\partial P}{\partial z} = 3z^2 + 3\lambda xy = 0 \end{cases}$$

It's clear that if (x, y, z) is a solution of (8.1), then it's also a solution of $P(x, y, z) = 0$. Thus if we want to find for which λ the curve C will be singular, it suffices to find non-zero solutions of (8.1). If (x, y, z) is a solution of (8.1) with $x \neq 0$, then both y and z are non-zero, otherwise we will obtain $x = 0$. Note that

$$0 = xz^2 + \lambda x^2 y = xz^2 - \lambda^2 y^2 z = xz^2 + \lambda^3 xz^2.$$

This shows that if $1 + \lambda^3 \neq 0$, then the curve C must be non-singular. Now if C is non-singular, then the genus formula implies

$$\text{genus}(C) = \frac{(3-1)(3-2)}{2} = 1.$$

□

Exercise 8.3.2. Assume affine plane curves $C_1 = \{f = 0\}, C_2 = \{g = 0\} \subseteq \mathbb{C}^2$ are smooth at $p = (0, 0)$. Define the intersection number $(C_1, C_2)_p$ of C_1, C_2 at p to be $\text{mult}_p G$, where

$$\begin{aligned} G: C_1 &\rightarrow \mathbb{C} \\ (x, y) &\mapsto g(x, y). \end{aligned}$$

Prove

$$(C_1, C_2)_p = (C_2, C_1)_p.$$

Proof.

□

Exercise 8.3.3. In many branches of mathematics, we use perturbation method to solve equations. For example, if we want to solve the quadratic equation

$$x^5 - x = \frac{1}{2},$$

we may start by solving

$$x^5 - x = 0.$$

We have five solutions $x = 0, \pm 1, \pm\sqrt{-1}$. For the solution $x_1 = 0$, we introduce a parameter t and try to solve $x^5 - x = t$ by power series

$$x_1(t) = a_0 + a_1t + \cdots + a_k t^k + \cdots$$

where $x_1(t) = a_0 = 0$ recursively by comparing the coefficients of Taylor expansion of both sides

$$x_1^5(t) - x_1(t) = t.$$

What is the convergence radius of $x_1(t)$?

Proof. Consider the affine plane curve $C \subseteq \mathbb{C}^2$ defined by $x^5 - x - t = 0$ and the following holomorphic map

$$\begin{aligned} F: C &\rightarrow \mathbb{C} \\ (x, t) &\mapsto t. \end{aligned}$$

For $t = 0$, there are 5 preimages of 0, which are $0, \pm 1$ and $\pm\sqrt{-1}$. Furthermore, if the equation $x^5 - x - t = 0$ has no multiple roots, then F is also a covering map. In other words, if

$$|t| < \frac{4}{5} \times \left(\frac{1}{5}\right)^{\frac{1}{4}},$$

then F is a covering map. For the curve $t \mapsto t$ in \mathbb{C} , the curve $x_1(t)$ constructed in the exercise is a lifting of this curve starting from the preimage $x_1 = 0$. Thus $|t| = \frac{4}{5} \times \left(\frac{1}{5}\right)^{\frac{1}{4}}$ is the maximal radius of lifting. \square

Exercise 8.3.4. Find two smooth conic curves (conic means degree two) in \mathbb{P}^2 which meet at one point with multiplicity 4.

Proof. By Bezout theorem if two smooth conic curves intersect at one point, then this point must have multiplicity 4, so it suffices to construct two conic curves which intersect at one point.

Consider the curve C_1 defined by $x^2 - yz = 0$ and the curve C_2 defined by $y^2 - 4xy + 6yz - 4xz + z^2 = 0$. Then the only intersection is $[1 : 1 : 1]$, which has multiplicity 4. \square

Exercise 8.3.5. Let X, Y be two compact connected Riemann surfaces of genus $\text{genus}(X) > \text{genus}(Y)$. Prove that every holomorphic map from Y to X is constant.

Proof. In fact we prove the following equivalent statement: If there exists a non-constant holomorphic map $F: Y \rightarrow X$ between compact Riemann surfaces, then $\text{genus}(Y) \geq \text{genus}(X)$.

Now let's begin the proof: If $\text{genus}(X) = 0$, it's trivial. Otherwise, by Hurwitz formula we have

$$2 \text{genus}(Y) - 2 = \deg(F)(2 \text{genus}(X) - 2) + B(F) \geq 2 \text{genus}(X) - 2$$

since $\deg(F) \geq 1$ and $B(F) \geq 0$. \square

Exercise 8.3.6. Try to define smooth algebraic curve in

$$\mathbb{P}^1 \times \mathbb{P}^1 = \{[x_1 : y_1], [x_2 : y_2]\}$$

by considering homogeneous polynomial of bidegree (d_1, d_2) in $\mathbb{C}[x_1, y_1, x_2, y_2]$, in which

$$\deg x_1 = \deg y_1 = (1, 0),$$

$$\deg x_2 = \deg y_2 = (0, 1).$$

Proof. Note that a homogenous polynomial P of bidegree (d_1, d_2) can be regarded as a section s of line bundle $\mathcal{O}(d_1) \otimes \mathcal{O}(d_2)$, and the non-singular algebraic curve C defined by P is exactly the zero divisor of s . By adjunction formula one has

$$K_C \cong \mathcal{O}(d_1 - 2) \otimes \mathcal{O}(d_2 - 2),$$

and thus

$$2 \operatorname{genus}(C) - 2 = K_C \cdot C = (d_1 - 2)d_2 + (d_2 - 2)d_1.$$

This shows $\operatorname{genus}(C) = (d_1 - 1)(d_2 - 1)$. □

8.4. Homework 4.

Exercise 8.4.1. Let X be a compact Riemann surface. Prove that

$$\mathcal{O}(X) = \{\text{constant functions}\}.$$

Proof. Suppose $f: X \rightarrow \mathbb{C}$ be a non-constant holomorphic map. Then by open map theorem one has f is an open map, and thus $f(X) \subseteq \mathbb{C}$ is open. On the other hand, since X is compact and then $f(X)$ is compact in \mathbb{C} . Thus $f(X) \subseteq \mathbb{C}$ is both open and closed in \mathbb{C} , which implies $f(X) = \mathbb{C}$. But \mathbb{C} is not compact, which leads to a contradiction. \square

Exercise 8.4.2. Let w_1, w_2 be \mathbb{R} -linearly independent complex numbers, and $C = \mathbb{C}/L$, where $L = \mathbb{Z}w_1 + \mathbb{Z}w_2$.

- (1) Prove that $\omega = dz$ defines a holomorphic 1-form on \mathbb{C} , where z is the coordinate of \mathbb{C} .
- (2) Compute $\dim_{\mathbb{C}} \Omega^1(C)$.

Proof. For (1). If we write $z = x + \sqrt{-1}y$, then firstly

$$dz = dx + \sqrt{-1}dy$$

is a 1-form on \mathbb{C} , and it's holomorphic since

$$\frac{\partial}{\partial \bar{z}} dz = \frac{1}{2} \left(\frac{\partial}{\partial x} + \sqrt{-1} \frac{\partial}{\partial y} \right) (dx + \sqrt{-1}dy) = 0.$$

Moreover, above computation also shows that up to constants dz is the only holomorphic 1-form on \mathbb{C} .

For (2). If ω is a holomorphic 1-form on C , then it can be extended to be a holomorphic 1-form on \mathbb{C} . Thus $\dim_{\mathbb{C}} \Omega^1(C) \leq 1$. On the other hand, since points in \mathbb{C} which are identified in the torus only differs constants, dz descends to a holomorphic 1-form on C . This shows $\dim_{\mathbb{C}} \Omega^1(C) = 1$. \square

Remark 8.4.1. More generally, for a compact Riemann surface Σ_g with genus g , the Hodge decomposition implies that $\dim_{\mathbb{C}} \Omega^1(\Sigma_g) = g$.

Exercise 8.4.3. For any two points $p \neq q \in \mathbb{P}^1$, construct a meromorphic 1-form ω with $\text{ord}_p \omega = 0$ and $\text{ord}_q \omega = -1$.

Proof. Without loss of generality we may assume $q = \infty$ and $p = [\lambda : 1]$. Then the meromorphic 1-form $\omega = 1/(z - \lambda)dz$ on the affine piece $\{[z : 1]\}$ gives a meromorphic 1-form on \mathbb{P}^1 such that $\text{ord}_p \omega = 0$ and $\text{ord}_q \omega = -1$. \square

Exercise 8.4.4. Show that

$$\{([x_0 : x_1], [y_0 : y_1]) \mid (x_0^2 + x_1^2)(y_0^2 + y_1^2) = x_0 x_1 y_0 y_1\}$$

is a smooth curve in $\mathbb{P}^1 \times \mathbb{P}^1$, and compute its genus.

Proof. Suppose U_0, U_1 and V_0, V_1 are affine pieces for the first factor and second factor of $\mathbb{P}^1 \times \mathbb{P}^1$ respectively. Then $\{U_i \times V_j\}$ gives an atlas for $\mathbb{P}^1 \times \mathbb{P}^1$, and it suffices to check the curve C defined by $(x_0^2 + x_1^2)(y_0^2 + y_1^2) = x_0 x_1 y_0 y_1$ is smooth on each affine piece.

Since the symmetry between x_0, x_1 and y_0, y_1 , it suffices to check the curve C is smooth on the affine piece $\{[1 : x]\} \times \{[1 : y]\}$. On this affine piece C is defined by

$$F(x, y) = (x^2 + 1)(y^2 + 1) - xy = 0.$$

Now it suffices to show the following system of equations has no solution

$$\begin{cases} p(x, y) = (x^2 + 1)(y^2 + 1) - xy = 0 \\ \frac{\partial p}{\partial x} = 2x(y^2 + 1) - y \\ \frac{\partial p}{\partial y} = 2y(x^2 + 1) - x \end{cases}$$

If (x_0, y_0) is a solution, then one has $4y_0^2x_0 = y_0$.

- (1) If $y_0 = 0$, then by the second equation one has $x_0 = 0$, which contradicts to the first equation.
- (2) If $y_0 \neq 0$, then one has $4x_0y_0 = 1$. By the first equation one has

$$(x_0^2 + 1)(y_0^2 + 1) = \frac{1}{4}.$$

On the other hand, the symmetry between x_0, y_0 implies $x_0 = y_0 = \pm\frac{1}{2}$, which is a contradiction.

This shows C is smooth. For the genus of C , as it's shown in the last exercise of Homework3, one has $\text{genus}(C) = (2 - 1)(2 - 1) = 1$. \square

Exercise 8.4.5. Let C be a smooth conic in \mathbb{P}^2 , $A, B, C, D, E, F \in C$ are six distinct points.

- (1) Prove that line \overline{AB} connecting A, B intersect with C at exactly two points A, B .
- (2) Let f be the product of lines $\overline{AB}, \overline{CD}, \overline{EF}$, g be the product of lines $\overline{BC}, \overline{DE}, \overline{FA}$. Choose $P \notin C \setminus \{A, B, C, D, E, F\}$. Prove that there exists $\lambda \in \mathbb{C}$ such that $f + \lambda g$ vanishes on P .
- (3) If $C = \{h = 0\}$, prove that $h \mid f + \lambda g$.
- (4) If $\overline{AB} \cap \overline{DE} = G, \overline{CD} \cap \overline{AF} = H, \overline{EF} \cap \overline{BC} = K$. Prove that G, H, K are colinear (on the line $(f + \lambda g)/h$).

Proof. For (1). It follows from Bezout theorem that \overline{AB} intersects with C on at most two points, since C is a smooth conic, and \overline{AB} is a smooth line. On the other hand, since $A, B \in C$, then the two intersection are exactly points A, B .

For (2). By evaluating at point P , we can regard $f(P) + \lambda g(P)$ as a linear function of λ , which must admit a zero in \mathbb{C} , which is also unique.

For (3). If $h \nmid f + \lambda g$, then by Bezout theorem there are at most 2×3 intersections between C and $\{f + \lambda g = 0\}$, but it already has seven intersections A, B, C, D, E, F, P .

For (4). Firstly by definition it's clear $f + \lambda g$ vanishes on points G, H, K , and since $(f + \lambda g)/h$ has degree one, it defines a line. Thus G, H, K are colinear on this line. \square

Exercise 8.4.6. Let R be a UFD and $f_1, f_2, g \in R[X]$ with $\deg f_1 = m, \deg g = n$. Prove that

- (1) $\mathcal{R}(f_1, g) = (-1)^{mn} \mathcal{R}(g, f_1)$.
- (2) $\mathcal{R}(f_1 f_2, g) = \mathcal{R}(f_1, g) \mathcal{R}(f_2, g)$.

Proof. For (1). It reduces to a problem of linear algebra, that is, for $A \in M_{(m+n) \times m}(R)$ and $B \in M_{(m+n) \times n}(R)$, one has

$$\det(A \mid B) = (-1)^{mn} \det(B \mid A).$$

□

8.5. Homework 5.

Exercise 8.5.1. Assume for any compact Riemann surface X , $\mathcal{M}^{(1)}(X) \neq 0$. Prove Riemann–Hurwitz by Poincaré–Hopf theorem.

Proof. Suppose $F: X \rightarrow Y$ is a holomorphic map between Riemann surfaces. For meromorphic 1-form $0 \neq \theta \in \mathcal{M}^{(1)}(Y)$, one has $F^*\theta$ is a meromorphic 1-form on X . Thus by Poincaré–Hopf theorem one has

$$\begin{aligned} \sum_{q \in Y} \text{ord}_q \theta &= -\chi(Y) = 2 \text{genus}(Y) - 2 \\ \sum_{p \in X} \text{ord}_p F^*\theta &= -\chi(X) = 2 \text{genus}(X) - 2. \end{aligned}$$

Thus it suffices to show

$$\sum_{p \in X} \text{ord}_p(F^*\theta) = \deg(F) \left(\sum_{q \in Y} \text{ord}_q(\theta) \right) + \sum_{p \in X} (\text{mult}_p F - 1),$$

Firstly let's establish the following lemma.

Lemma 8.5.1. Notations as above. For any $p \in X$,

$$\text{ord}_p(F^*\theta) + 1 = (\text{ord}_{F(p)}(\theta) + 1) \cdot \text{mult}_p(F)$$

Proof. Choose local coordinate w centered at p and local coordinate z at $F(p)$ such that F is given by

$$z = w^n,$$

where $n = \text{mult}_p(F)$. If $k = \text{ord}_{F(p)}(\theta)$, then θ is given by

$$\theta = \left(\sum_{j=k}^{\infty} c_j z^j \right) dz, \quad c_k \neq 0.$$

Thus

$$\begin{aligned} F^*(\theta) &= (c_k (w^n)^k + \text{higher order terms}) n w^{n-1} dw \\ &= (n c_k w^{n(k+1)-1} + \text{higher order terms}) dw. \end{aligned}$$

This shows

$$\text{ord}_p(F^*(\theta)) + 1 = (\text{ord}_{F(p)}(\theta) + 1) \cdot \text{mult}_p(F).$$

□

Note that $F: X \rightarrow Y$ is a non-constant holomorphic map, and thus it's surjective by Corollary 1.1.1. Then by above Lemma one has

$$\begin{aligned} \sum_{p \in X} \text{ord}_p(F^*\theta) &= \sum_{p \in X} \{(\text{ord}_{F(p)}(\theta) + 1) \cdot \text{mult}_p(F) - 1\} \\ &= \left(\sum_{p \in X} \text{ord}_{F(p)}(\theta) \right) \cdot \text{mult}_p(F) + \sum_{p \in X} (\text{mult}_p F - 1) \\ &= \deg(F) \left(\sum_{q \in Y} \text{ord}_q(\theta) \right) + \sum_{p \in X} (\text{mult}_p F - 1). \end{aligned}$$

□

Exercise 8.5.2. Let $f(x, y) = x^3 - x^2 + y^2$. Prove that

- (1) $f(x, y)$ is irreducible in $\mathbb{C}[x, y]$.
- (2) $f(x, y)$ is reducible in $\mathbb{C}\{x\}[y]$.
- (3) Is $f(x, y)$ reducible in $\mathbb{C}\{y\}[x]$?

Proof. For (1). If $f(x, y)$ is irreducible in $\mathbb{C}[x, y]$, then the only possible decomposition must be of the form

$$f(x, y) = (y + g(x))(y + h(x)).$$

This gives the equalities

$$\begin{cases} g(x) + h(x) = 0 \\ g(x)h(x) = x^3 - x^2. \end{cases}$$

However, there is no polynomial $g(x) = -h(x)$ such that

$$g^2(x) = x^2 - x^3.$$

This shows $f(x, y)$ is irreducible in $\mathbb{C}[x, y]$.

For (2). In $\mathbb{C}\{x\}[y]$ one has

$$f(x, y) = (y - x\sqrt{1-x})(y + x\sqrt{1-x}).$$

For (3). If $f(x, y)$ is irreducible in $\mathbb{C}\{y\}[x]$, then we may write it as

$$f(x, y) = (g(y) + p(x))(h(y) - p(x)),$$

where $p(x) \in \mathbb{C}[x]$. However, there is no polynomial $p(x) \in \mathbb{C}[x]$ such that

$$p^2(x) = x^2 - x^3.$$

Thus $f(x, y)$ is irreducible in $\mathbb{C}\{y\}[x]$.

□

Exercise 8.5.3 (Miranda IV.3 E). Let τ be a complex number with strictly positive imaginary part. Let h be a meromorphic function on \mathbb{C} which is $(\mathbb{Z} + \mathbb{Z}\tau)$ -periodic; in other words, $h(z + 1) = h(z + \tau) = h(z)$ for all z . For any point p in \mathbb{C} , let γ_p be the path which is the counterclockwise boundary of the parallelogram with vertices $p, p + 1, p + \tau + 1, p + \tau, p$ (in that order).

Assume p is chosen so that there are no zeroes or poles of h on γ_p . Show that

$$\frac{1}{2\pi\sqrt{-1}} \int_{\gamma_p} z \frac{h'(z)}{h(z)} dz$$

is an element of the lattice $(\mathbb{Z} + \mathbb{Z}\tau)$.

Proof. Firstly we divide above integration into the following four parts

$$\frac{1}{2\pi\sqrt{-1}} \left(\underbrace{\int_p^{p+1} z \frac{h'(z)}{h(z)} dz}_A + \underbrace{\int_{p+1}^{p+\tau+1} z \frac{h'(z)}{h(z)} dz}_B + \underbrace{\int_{p+\tau+1}^{p+\tau} z \frac{h'(z)}{h(z)} dz}_C + \underbrace{\int_{p+\tau}^p z \frac{h'(z)}{h(z)} dz}_D \right).$$

Since h is $(\mathbb{Z} + \mathbb{Z}\tau)$ -periodic, one has

$$A + C = \int_p^{p+1} z \frac{h'(z)}{h(z)} dz + \int_{p+1}^p (z + \tau) \frac{h'(z + \tau)}{h(z + \tau)} dz = -\tau \int_p^{p+1} \frac{h'(z)}{h(z)} dz.$$

Now let's prove

$$\int_p^{p+1} \frac{h'(z)}{h(z)} dz \in 2\pi\sqrt{-1}\mathbb{Z}.$$

Since there is no zeros or poles of h on γ_p , we may choose a sufficiently small open neighborhood U of path $p \mapsto p + 1$ and write $h: U \rightarrow \mathbb{C}^*$. Consider the following commutative diagram

$$\begin{array}{ccc} [0, 1] & \xrightarrow{\gamma} & U \\ \tilde{h} \downarrow & & \downarrow h \\ \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^* \end{array}$$

Since $\exp: \mathbb{C} \rightarrow \mathbb{C}^*$ is the universal covering, there exists a lifting of $h \circ \gamma$, denoted by \tilde{h} . Moreover, one has

$$\int_p^{p+1} \frac{h'(z)}{h(z)} dz = \int_0^1 \tilde{h}(w) dw = \tilde{h}(1) - \tilde{h}(0).$$

Since \tilde{h} is a lifting of $h \circ \gamma$, and $h(p + 1) = h(p)$, one has $\exp(\tilde{h}(1)) = \exp(\tilde{h}(0))$, and thus

$$\tilde{h}(1) - \tilde{h}(0) \in 2\pi\sqrt{-1}\mathbb{Z}.$$

By the same argument one can show $B + D \in \mathbb{Z}$, and this completes the proof. \square

Exercise 8.5.4 (Miranda IV.3 F). Check by direct computation that if $r(z)$ is a rational function of z , then the meromorphic 1-form $r(z)dz$ on the Riemann sphere \mathbb{C}_∞ satisfies the residue theorem.

Proof. Without loss of generality we may assume the rational function $f(z)$ is of the form

$$r(z) = \frac{\alpha_1}{(z - \lambda_1)^{a_1}} + \cdots + \frac{\alpha_k}{(z - \lambda_k)^{a_k}} + \beta_1(z - \gamma_1)^{b_1} + \cdots + \beta_l(z - \gamma_l)^{b_l},$$

where $a_i, b_j > 0$ for all i, j . Then the summation of residues of meromorphic 1-form $\theta = r(z)dz$ of point except ∞ is given by

$$\sum_{p \in \mathbb{C}_\infty \setminus \{\infty\}} \text{Res}_p(\theta) = \sum_{i=1}^k \alpha_i \delta_{a_i 1}.$$

To see the residue at the infy point ∞ , note that

$$r\left(\frac{1}{z}\right)\left(-\frac{1}{z^2}\right)dz = -\sum_{i=1}^k \frac{\alpha_i z^{a_i-2}}{(1 - \lambda_i z)^{a_i}} - \sum_{j=1}^l \frac{\beta_j (1 - \gamma_j z)^{b_j}}{z^{b_j+2}}.$$

It's clear that $b_j + 2 > 1$ by definition. Thus the residue of θ at infy point ∞ is exactly

$$\text{Res}_\infty(\theta) = -\sum_{i=1}^k \alpha_i \delta_{a_i 1},$$

as desired. \square

Exercise 8.5.5 (Miranda IV.3 G). Check that if L is a lattice in \mathbb{C} and $h(z)$ is an L -periodic meromorphic function, then the meromorphic 1-form $\omega = h(z)dz$, considered as a form on the complex torus \mathbb{C}/L , satisfies the residue theorem.

Proof. If $h(z)$ is a L -periodic meromorphic function defined on \mathbb{C}/L , then there exists a meromorphic function $\tilde{h}(z)$ defined on \mathbb{C} such that $\tilde{h} = h \circ \pi$, where $\pi: \mathbb{C} \rightarrow \mathbb{C}/L$ is the canonical projection.

However, the summation of orders of a meromorphic function defined on \mathbb{C} is zero, and thus the summation of orders of a L -periodic meromorphic function defined on \mathbb{C}/L is zero. Then residue theorem holds since genus of a complex torus is 1. \square

Exercise 8.5.6. Let $f(x, y) = f_d(x, y) + f_{d+1}(x, y) + \cdots \in \mathbb{C}\{x, y\}$, where $f_i(x, y)$ are homogeneous with respect to (x, y) and $\deg f_i = i$ or $f_i = 0$. Prove that if $f_d(x, y)$ has d distinct linear factors, then $f(x, y)$ decomposes as product of d irreducible factors in $\mathbb{C}\{x, y\}$.

- (1) Reduce the question to $f_d(x, y) = \prod (y - \alpha_i x)$
- (2) Denote by $w = y/x$,

$$g(x, w) = \frac{f(x, xw)}{x^d} \in \mathbb{C}\{x, w\}.$$

Prove that g converges in a product of discs

$$D_{\rho_1} \times D_{\rho_2} = \{(x, w) \mid |x| < \rho_1, |w| < \rho_2\}$$

that contains $(0, \alpha_i)$.

- (3) Prove that $g(0, \alpha_i) = 0$ and $\frac{\partial g}{\partial w}(0, \alpha_i) \neq 0$ and hence $g(x, w) = 0$ has a solution $w = h_i(x)$ near $(0, \alpha_i)$ with $h_i(x) \in \mathbb{C}\{x\}$ and $h_i(0) = \alpha_i$.
- (4) Prove that $\prod (y - xh_i(x)) \mid f(x, y)$ and $f(x, y)$ is the product of m irreducible factors up to units in $\mathbb{C}\{x, y\}$.

Proof. For (1). Suppose $f_d(x, y)$ is decomposed into d distinct linear factors as follows

$$f_d(x, y) = \prod_{i=1}^d (\beta_i y - \alpha_i x).$$

Without lose of generality we may assume $x \nmid f_d(x, y)$, and thus we can reduce to the case $\beta_i = 1$ for all i by dividing $\prod_{i=1}^d \beta_i$.

For (2). For $f_k(x, y)$ with $k \geq d + 1$, it's clear

$$g_k(x, y) = \frac{f_k(x, xw)}{x^d} = 0$$

since the degree of x in $f_k(x, xw)$ is k , which is bigger than d , so it suffices to show $f_d(x, y)/x^d$ converges when (x, y) tends to $(0, \alpha_i)$. Since $f_d(x, y)$ is a homogenous polynomial of degree d with respect to (x, y) , one has

$$\frac{f_d(x, xw)}{x^d} = f_d(1, w).$$

This shows $g_d(x, y)$ converges in a sufficiently small product of discs $D_{\rho_1} \times D_{\rho_2}$, so does $g(x, y)$.

For (3). From the proof of (2) one can see $g(0, \alpha_i) = f_d(1, \alpha_i) = 0$. On the other hand, note that

$$\frac{\partial g}{\partial w} = \frac{\partial}{\partial w} \left(\frac{f(x, xw)}{x^d} \right) = \frac{1}{x^d} \frac{\partial y}{\partial w} \frac{\partial f(x, y)}{\partial y} = \frac{f_y(x, y)}{x^{d-1}}.$$

This shows

$$\frac{\partial g}{\partial w}(0, \alpha_i) = f_y(1, \alpha_i) \neq 0,$$

since these linear factors are distinct. Thus by the implicit function theorem, $g(x, w) = 0$ has a solution $w = h_i(x)$ with $w = h_i(x)$ near $(0, \alpha_i) \in \mathbb{C}\{x\}$ and $h_i(0) = \alpha_i$.

For (4). Now consider the following function

$$\alpha(x, y) = \frac{f(x, y)}{\prod_{i=1}^d (y - xh_i(x))}.$$

By construction of $h_i(x)$ one can see $\alpha(x, y)$ has a non-zero constant term $\alpha(0, 0)$, and thus $\alpha(x, y) \in \mathbb{C}\{x, y\}^*$. This shows $f(x, y)$ is decomposed into d irreducible linear factors in $\mathbb{C}\{x, y\}$. \square

8.6. Homework 6.

Exercise 8.6.1. Let x_1, \dots, x_n be distinct points on \mathbb{C} , and let

$$C = \{y^d = (x - x_1)^{a_1} \cdots (x - x_n)^{a_n}\} \subseteq \mathbb{C}^2$$

where $d, a_i \in \mathbb{Z}_{>0}$ and $\gcd(d, a_1, \dots, a_n) = 1$. Let $\overline{C} \subseteq \mathbb{C}^2$ be the corresponding projective plane curve. Prove \overline{C} is irreducible and compute the genus of the normalization of \overline{C} .

Proof. To prove \overline{C} is irreducible, it suffices to prove the polynomial

$$y^d = (x - x_1)^{a_1} \cdots (x - x_n)^{a_n}$$

is irreducible, since \overline{C} is the closure of C in \mathbb{P}^2 . Consider the projection $\Phi: C \rightarrow \mathbb{C}$ given by $(x, y) \mapsto x$. Then it's a d -covering from $C \setminus \Phi^{-1}(B)$ to $\mathbb{C} \setminus B$, where $B = \{x_1, \dots, x_n\}$. Since the base $\mathbb{C} \setminus B$ is connected, it suffices to show that the monodromy is transitive on each fiber. Note that the local monodromy at point x_i is given by $\xi_d^{a_i/d}$, where ξ_d is the d -th unit root. Since $\gcd(d, a_1, \dots, a_n) = 1$, there exists $k, k_1, \dots, k_n \in \mathbb{Z}$ such that

$$kd + k_1a_1 + \cdots + k_na_n = 1.$$

Thus by winding x_1 by k_1 times and winding x_2 by k_2 times and so on, one construct a monodromy given by

$$\xi_d^{k_1a_1 + \cdots + k_na_n} = \xi_d^{1-kd} = \xi_d.$$

Thus the monodromy acts on fiber transitively.

Now let's figure out the type of singularities of \overline{C} to compute the genus of the normalization of \overline{C} . Firstly, by blowing up finitely times, one can prove the following lemma, which is a generalization of Example 5.5.2.

Lemma 8.6.1. For $y^m = x^n$, the δ -invariance of $(0, 0)$ is

$$\delta(m, n) = \frac{(m-1)(n-1)}{2} - 1 + d,$$

where $d = \gcd(m, n)$.

For $f(x, y) = y^d - (x - x_1)^{a_1} \cdots (x - x_n)^{a_n}$, a direct computation shows

$$\begin{aligned} \frac{\partial f}{\partial y} &= dy^{d-1} \\ \frac{\partial f}{\partial x} &= - \sum_{i=1}^n a_i (x - x_1)^{a_1} \cdots (x - x_i)^{a_i-1} \cdots (x - x_n)^{a_n}. \end{aligned}$$

Thus $(0, x_i)$ is a singularity of $f(x, y) = 0$ if and only if $a_i > 1$, and the δ -invariance for $(0, x_i)$ is $\delta(d, a_i)$. Let $F(x, y, z)$ be the homogenous polynomial corresponding to $f(x, y)$. Then

- (1) If $d > \sum_i a_i$, then $F(x, y, z) = y^d - z^{d-\sum_i a_i} (x - x_1 z)^{a_1} \dots (x - x_n z)^{a_n}$, and thus the infinity point is $[1 : 0 : 0]$. On the affine piece $\{x = 1\}$, the equation is given by

$$F(1, y, z) = y^d - z^{d-\sum_i a_i} (1 - x_1 z)^{a_1} \dots (1 - x_n z)^{a_n}.$$

To see $(0, 0)$ is a singularity of $F(1, y, z)$ or not, a direct computation shows that

$$\begin{aligned} \frac{\partial F(1, y, z)}{\partial y} &= dy^{d-1} \\ \frac{\partial F(1, y, z)}{\partial z} &= -\left(d - \sum_i a_i\right) z^{d-\sum_i a_i-1} (1 - x_1 z)^{a_1} \dots (1 - x_n z)^{a_n} \\ &\quad - z^{d-\sum_i a_i} \left(-\sum_i a_i x_i (1 - x_1 z)^{a_1} \dots (1 - x_i z)^{a_i-1} \dots (1 - x_n z)^{a_n}\right). \end{aligned}$$

Thus $(0, 0)$ is not a singularity of $F(1, y, z)$ if and only if $d = \sum_i a_i + 1$, and the δ -invariance for $(0, 0)$ is $\delta(d, d - \sum_i a_i)$.

- (2) If $d = \sum_i a_i$, then $F(x, y, z) = y^d - (x - x_1 z)^{a_1} \dots (x - x_n z)^{a_n}$. On the infinity line $\{z = 0\}$, the equation is given by $y^d = x^d$, and thus there are d points of \overline{C} on the infinity line, given by $\{[1, \xi_d^i : 0] \mid i = 1, \dots, d\}$, which are non-singular.
- (3) If $d < \sum_i a_i$, then $F(x, y, z) = y^d z^{\sum_i a_i - d} - (x - x_1 z)^{a_1} \dots (x - x_n z)^{a_n}$, and thus the infinity point is $[0 : 1 : 0]$. On the affine piece $\{y = 1\}$, the equation is given by

$$F(x, 1, z) = z^{\sum_i a_i - d} - (x - x_1 z)^{a_1} \dots (x - x_n z)^{a_n}.$$

To see $(0, 0)$ is a singularity of $F(x, 1, z)$ or not, a direct computation shows that

$$\begin{aligned} \frac{\partial F(x, 1, z)}{\partial x} &= -\sum_i a_i (x - x_1 z)^{a_1} \dots (x - x_i z)^{a_i-1} \dots (x - x_n z)^{a_n} \\ \frac{\partial F(x, 1, z)}{\partial z} &= \left(\sum_i a_i - d\right) z^{\sum_i a_i - d - 1} + \sum_i a_i x_i (x - x_1 z)^{a_1} \dots (x - x_i z)^{a_i-1} \dots (x - x_n z)^{a_n}. \end{aligned}$$

Thus $(0, 0)$ is not a singularity of $F(x, 1, z)$ if and only if $\sum_i a_i = d + 1$. To compute the δ -invariance, after once blow up one has

$$g(x, w) = \frac{F(x, 1, xw)}{x^{\sum_i a_i - d}} = w^{\sum_i a_i - d} - x^d (1 - x_1 w)^{a_1} \dots (1 - x_n w)^{a_n}.$$

Then it reduces to the standard model $w^{\sum_i a_i - d} = x^d$, and thus the δ -invariance for this case is

$$\binom{\sum_i a_i - d}{2} + \delta(d, \sum_i a_i - d).$$

As a consequence, the genus of the normalization of \overline{C} is

$$\begin{cases} (d-1)(d-2)/2 - \sum_{i=1}^n \delta(d, a_i) - \delta(d, d - \sum_i a_i), & d > \sum_i a_i + 1 \\ (d-1)(d-2)/2 - \sum_{i=1}^n \delta(d, a_i), & d = \sum_i a_i + 1 \\ (d-1)(d-2)/2 - \sum_{i=1}^n \delta(d, a_i), & d = \sum_i a_i \\ (\sum_i a_i - 1)(\sum_i a_i - 2)/2 - \sum_{i=1}^n \delta(d, a_i), & d = \sum_i a_i - 1 \\ (\sum_i a_i - 1)(\sum_i a_i - 2)/2 - \sum_{i=1}^n \delta(d, a_i) - \delta(d, \sum_i a_i - d) - \binom{\sum_i a_i - d}{2}, & d < \sum_i a_i - 1 \end{cases}$$

□

Exercise 8.6.2. A projective plane curve is called rational if it's irreducible and its normalization has genus zero. Find a rational curve for each degree d .

Proof. Consider the projective plane curve C defined by $y^d = x^{d-1}z$. On the affine piece $z = 1$, it's given by $y^d = x^{d-1}$. Then $(0, 0)$ is a singularity with δ -invariance

$$\frac{(d-1)(d-2)}{2}.$$

On the other hand, $[1 : 0 : 0]$ is not a singularity of $y^d = x^{d-1}z$. Thus by Bezout theorem, the genus of the normalization of C is

$$\frac{(d-1)(d-2)}{2} - \frac{(d-1)(d-2)}{2} = 0.$$

□

Exercise 8.6.3. Determine $y^2 - (x^2y^2 + x^4)$ is irreducible or not in $\mathbb{C}\{x, y\}$. This is an example of tacnode singularity.

Proof. Firstly consider the blow up $g(x, w)$, that is

$$g_1(x, w) = \frac{f(x, xw)}{x^2} = w^2 - x^2w^2 + x^2.$$

It's still singular at $(0, 0)$, so consider

$$g_2(x, t) = \frac{g_1(x, xt)}{x^2} = t^2 - x^2t^2 + 1.$$

Note that

$$\left. \frac{\partial g_2}{\partial t} \right|_{x=0, t=\pm 1} \neq 0.$$

Then by implicit function theorem, there exists $t_1(x), t_2(x) \in \mathbb{C}\{x\}$ such that $t_1(0) = 1$ and $t_2(0) = -1$, and thus in $\mathbb{C}\{x, y\}$, there is the following decomposition

$$y^2 - (x^2y^2 + x^4) = u(y - x^2t_1(x))(y - x^2t_2(x)),$$

where u is a unit in $\mathbb{C}\{x, y\}$.

□

Exercise 8.6.4. Compute the genus of the curve

$$C = \{x^2y^2 - z^2(x^2 + y^2) = 0\} \subseteq \mathbb{P}^2$$

Proof. For convenience we denote $F(x, y, z) = x^2y^2 - z^2(x^2 + y^2)$. Note that

$$\begin{aligned}\frac{\partial F(x, y, 1)}{\partial x} &= 2xy^2 - 2x \\ \frac{\partial F(x, y, 1)}{\partial y} &= 2x^2y - 2y.\end{aligned}$$

Then $(0, 0)$ is a singularity of $F(x, y, 1)$. On the infinity line $\{z = 0\}$, there are two points on C , that is, $[1 : 0 : 0]$ and $[0 : 1 : 0]$. Note that $(0, 0)$ is also a singularity for $F(x, 1, z)$, since

$$\begin{aligned}\frac{\partial F(x, 1, z)}{\partial x} &= 2x - 2xz^2 \\ \frac{\partial F(x, 1, z)}{\partial z} &= -2z(x^2 + 1).\end{aligned}$$

By the same argument one can show $[1 : 0 : 0]$ is also a singularity for $F(1, y, z)$. This shows there are three singularities of the projective plane curve defined by F , that is, $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$. Now it suffices to compute the δ -invariance for these singularities.

Note that by blow up $f(x, y) = F(x, y, 1)$ once, one has

$$g(x, w) = x^2w^2 - 1 - w^2,$$

which is non-singular at $(0, \pm\sqrt{-1})$, and thus the δ -invariance for $[0 : 0 : 1]$ is $\binom{2}{2} = 1$. For singularity $[0 : 1 : 0]$, the same computation shows that the δ -invariance of it is 1, so is the one of $[1 : 0 : 0]$. Then by Plücker formula one has the genus of C is

$$\frac{(4-1)(4-2)}{2} - 3 = 0.$$

□

Exercise 8.6.5. $C_1, C_2 \subseteq \mathbb{P}^2$ are curves of degree n . Assume C_1, C_2 intersect at n^2 distinct points. If mn of these points lie on an irreducible curve C_3 of degree m , then the remaining $(n-m)n$ points lie on a curve of degree $n-m$.

Proof. Suppose C_1, C_2, C_3 are defined by homogenous polynomials F_1, F_2, F_3 respectively. Suppose p is a point on C_3 which does not lie on $C_1 \cap C_2$. Then the curve C_4 of degree n , defined by

$$\lambda F_1 + \mu F_2 = 0,$$

where $\lambda = F_2(p), \mu = -F_1(p)$ intersects with C_3 at least $mn + 1$ points. Then C_3 must be a component of C_4 , otherwise it contradicts to the Bezout theorem. Thus there exists a homogenous polynomial G such that

$$\lambda F_1 + \mu F_2 = F_3 G,$$

where $\deg G = n - m$. Note that there are n^2 distinct points such that $F_1 = F_2 = 0$, and only mn of them such that $F_3 = 0$. This shows that there are $(n-m)n$ of them such that $G = 0$, which completes the proof. □

Exercise 8.6.6. If a degree n projective plane curve C has $\lfloor \frac{n}{2} \rfloor + 1$ singular points on a line L , then L is necessarily a component of C .

Proof. If L is not a component of C , then by Bezout theorem one has

$$\sum_{p \in C \cap L} (C, L)_p = n.$$

Note that for $p \in C \cap L$, if $(C, L)_p = 1$, then p must be a non-singular point since every linear polynomial is non-singular. Thus

$$\sum_{p \in C \cap L} (C, L)_p \geq 2 \times (\lfloor \frac{n}{2} \rfloor + 1) > n,$$

a contradiction. □

REFERENCES

- [Mil65] John W. Milnor. *Topology from the differentiable viewpoint*. University Press of Virginia, Charlottesville, VA, 1965. Based on notes by David W. Weaver.

YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, BEIJING, 100084,
P.R. CHINA,

Email address: `liubw22@mails.tsinghua.edu.cn`