# Curvatures of Left-invariant Metrics on Lie Groups

Bowen Liu

Mathematics Department of Tsinghua University

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In Riemannian geometry, it's natrual to ask the following questions:

- Given a smooth manifold $M$, does there exist a metric on $M$ with certain curvature properties? For example, Hopf's conjecture.
- Conversely, given certain curvature properties, does there exist obstruction for manifolds? For example, Myers' theorem, Cartan-Hadamard theorem and so on.

- In this talk, we will consider above questions in the category of Lie groups with left-invariant metrics, and the main reference is [Mil76].

But why left-invariant metrics?

- It's easy to compute: The left-invariant metric on Lie group is the same thing as an inner product on its Lie algebra $\mathfrak{g}$, and it turns out the curvature information is encoded in the structure of Lie algebra.
- It contains lots of examples and unknown questions: Not every Lie group admits bi-invariant metrics, but every Lie group admits left-invariant metrics, and there are still many questions about left-invariant are unknown.

- Along the slides, we always assume $G$ is an $n$-dimensional real Lie group, and $\mathfrak{g}$ is the associated Lie algebra, consisting of all left-invariant vector fields.
- Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathfrak{g}$. Then the structure constants $\alpha_{ijk}$ is defined by

$$[e_i, e_j] = \sum_{k=1}^{n} \alpha_{ijk} e_k.$$

In other words, $\alpha_{ijk} = \langle [e_i, e_j], e_k \rangle$. It's worth mentioning that the structure constants depends on the choice of basis, but the Lie algebra structure doesn't.

## Lemma

*Let $G$ be a Lie group equipped with a left-invariant metric. Then with structure constants $\alpha_{ijk}$ as above, the sectional curvature $\kappa(e_1, e_2)$ is given by*

$$\kappa(e_1, e_2) = \sum_k \Big\{ \frac{1}{2}\alpha_{12k}(-\alpha_{12k} + \alpha_{2k1} + \alpha_{k12}) - \frac{1}{4}(\alpha_{12k} - \alpha_{2k1} + \alpha_{k12})(\alpha_{12k} + \alpha_{2k1} - \alpha_{k12}) - \alpha_{k11}\alpha_{k22} \Big\}$$

## Proof.

See Proof 52 in Appendix. □

- This explicit expression shows that the curvature can be computed completely from information about Lie algebra, together with its metric.
- Furthermore, the curvature depends continuously on the structure constants $\alpha_{ijk}$ and vanishes whenever they vanish.

## Lemma

*If the linear transformation* $\mathrm{ad}(u)$ *is skew-symmetric, then*

$$\kappa(u, v) \geq 0$$

*for all* $v$*, where equality holds if and only if* $u$ *is orthogonal to* $[v, \mathfrak{g}]$*.*

## Proof.

Without lose of generality, we may assume $u, v$ are orthonormal and $e_1, \dots, e_n$ is an orthonormal basis with $e_1 = u, e_2 = v$. Note that

$$\mathrm{ad}(u)e_i = [u, e_i] = \sum_k \alpha_{1ij} e_j.$$

Then the statement of $\mathrm{ad}(u)$ is skew-symmetric means that $\alpha_{1ij}$ is skew in the last two indices. This show $\kappa(u, v) = \sum_k (\alpha_{2k1})^2/4$, and the equality holds if and only if all $\alpha_{2k1} = 0$. □

- The following lemma shows the "skew-symmetric" is natural, since it comes from bi-invariant metric on connected Lie groups.

## Lemma

*A left-invariant metric on a connected Lie group is also right-invariant if and only if* $\mathrm{ad}(u)$ *is skew-symmetric for all* $u \in \mathfrak{g}$.

## Lemma

*A connected Lie group admits a bi-invariant metric if and only if it's isomorphic to a product of a compact group and an abelian group.*

## Corollary

*Every compact Lie group admits a left-invariant (and in fact a bi-invariant) metric with non-negative sectional curvature.*

- Conversely, there is no satisfied description for Lie groups which possesse a left-invariant metric with non-negative sectional curvature. However, if we sharpen the inequality and require positive sectional curvature, then Wallach shown in [Wal72] examples are scarce indeed.

## Theorem

SU(2) *is the only simply-connected Lie group admits a left-invariant metric with positive sectional curvature.*

- On the other hand, it's natural to ask on which Lie groups it admits a left-invariant metric with respect to which it's flat, that is all sectional curvatures vanish. A simple example is that if the Lie algebra $\mathfrak{g}$ is abelian. In fact, we have the following result.

## Theorem

*A Lie group with left-invariant metric is flat if and only if its Lie algebra splits as an orthogonal direct sum $\mathfrak{b} \oplus \mathfrak{u}$, where $\mathfrak{b}$ is an abelian subalgebra, $\mathfrak{u}$ is an abelian ideal, and $\mathrm{ad}(b)$ is skew-symmetric for every $b \in \mathfrak{b}$.*

- Those with non-positive sectional curvature have been classified by Azencott and Wilson in [AW76]. Since the statements are complicated, we just give a qualitive result as follows.

### Theorem

*If a connected Lie group $G$ has a left-invariant metric with non-positive sectional curvatures, then it's solvable.*
*If $G$ is unimodular, then any left-invariant metric with non-positive sectional curvatures must actually be flat.*

## Example

Suppose the Lie algebra $\mathfrak{g}$ with $\dim \mathfrak{g} \geq 2$ has the property that the bracket $[x, y]$ is always equal to a linear combination of $x$ and $y$. Then in fact one has

$$[x, y] = \ell(x)y - \ell(y)x,$$

where $\ell$ is a well-defined linear mapping from $\mathfrak{g}$ to the real number. Choosing any positive definite metric, the sectional curvatures are constant

$$K = -\|\ell\|^2.$$

Thus, in the non-abelian case $\ell \neq 0$, every possible metric on $\mathfrak{g}$ has constant negative sectional curvature.

## Lemma

*If the linear transformations* $\mathrm{ad}(x)$ *is skew-symmetric, then* $\mathrm{Ric}(x) \geq 0$*, where the equality holds if and only if* $x$ *is orthogonal to the commutator ideal* $[\mathfrak{g}, \mathfrak{g}]$*.*

## Proof.

This follows immediately from Lemma 2. $\square$

## Corollary

*If the linear transformations* $\mathrm{ad}(x)$ *is skew-symmetric and* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$*, then Ricci curvature is positive.*

- The criterion for positive Ricci curvature is classical and elegant.

**Theorem**

*A connected Lie group $G$ admits a left-invariant metric with all positive Ricci curvature if and only if it's compact with finite fundamental group.*

**Proof.**

In one direction this follows from the theorem of Myers which asserts that any complete Riemannian manifold with positive Ricci curvature is compact with finite fundamental group.

## Continuation.

Conversely, if $G$ is compact we can choose a bi-invariant metric, so that each $\operatorname{ad}(x)$ is skew-symmetric. If $G$ has finite fundamental group, then its universal covering $\widetilde{G}$ is also compact.
Here we claim $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. For otherwise, $\mathfrak{g}$ admits a non-trivial abelianization which is an abelian Lie algebra, and thus there would exist a non-trivial Lie algebra holds from $\mathfrak{g}$ to the abelian Lie algebra $\mathbb{R}$.
Since $\widetilde{G}$ is simply-connected, this would induce a non-trivial homomorphism from $\widetilde{G}$ to the additive Lie group $\mathbb{R}$, but any non-trivial subgroup of $\mathbb{R}$ is non-compact, contradicting the hypothesis that $\widetilde{G}$ is compact. Now, using Lemma 10, it follows that $G$ has positive Ricci curvature. $\square$

## Lemma

*If $u$ is orthogonal to the commutator ideal $[\mathfrak{g}, \mathfrak{g}]$, then $\mathrm{Ric}(u) \leq 0$, where the equality holds if and only if $\mathrm{ad}(u)$ is skew-symmetric.*

## Lemma

*If a connected Lie group $G$ has a left-invariant metric with non-negative Ricci curvature, then it's unimodular.*

## Proof.

Suppose on contrary that $G$ were not unimodular. Then the unimodular kernel $\mathfrak{u} = \{x \in \mathfrak{g} \mid \operatorname{tr}\operatorname{ad}(x) = 0\}$, which is an ideal containing $[\mathfrak{g}, \mathfrak{g}]$, doesn't equal to $\mathfrak{g}$. Choosing a unit vector $b$ orthogonal to $\mathfrak{u}$, we would have $\operatorname{tr}\operatorname{ad}(b) \neq 0$. Hence $\mathrm{ad}(b)$ could not be skew-symmetric, and it would follow by Lemma 13 that $\mathrm{Ric}(b) < 0$, a contradiction. $\square$

- In general, there is no obstruction for the existence of negative Ricci curvature for manifolds with dimension $\geq 3$. See [GY86] and [Loh94].
- But for left-invariant metrics on Lie group, we will see later that simple group $\mathrm{SL}(2, \mathbb{R})$ and the unimodular solvable group $E(1, 1)$ both admit non-flat left-invariant metrics with non-positive Ricci curvature. It seems unlikely that any higher dimensional simple group admits such a metric.

## Theorem

*If the Lie algebra of $G$ contains linearly independent vectors $x, y, z$ such that*

$$[x, y] = z,$$

*then there exists a left-invariant metric such that* $\mathrm{Ric}(x) < 0$ *and* $\mathrm{Ric}(z) > 0$.

## Proof.

Choose a fixed basis $b_1, \ldots, b_n$ with $b_1 = x, b_2 = y, b_3 = z$. For any real number $\varepsilon > 0$, consider an auxiliary basis $e_1, \ldots, e_n$ defined by $e_1 = \varepsilon b_1, e_2 = \varepsilon b_2, e_i = \varepsilon^2 b_i$ for $i \geq 3$. Define a left-invariant metric by requiring that $e_1, \ldots, e_n$ should be orthonormal. Let $\mathfrak{g}_\varepsilon$ denote the Lie algebra $\mathfrak{g}$ equipped with this particular metric and particular orthonormal basis.

## Continuation.

Setting $[e_i, e_j] = \sum \alpha_{ijk} e_k$, the structure constants $\alpha_{ijk}$ are clearly functions of $\varepsilon$. Now consider the limit $\varepsilon \to 0$. Then each $\alpha_{ijk}$ tends to a well-defined limit, and thus we obtain a limit Lie algebra $\mathfrak{g}_0$ with prescribed metric and prescribed orthonormal basis. Furthermore, the bracket operator in $\mathfrak{g}_0$ is given by

$$[e_1, e_2] = -[e_2, e_1] = e_3,$$

with $[e_i, e_j] = 0$ otherwise. Applying Lemma 10 and Lemma 13 it follows that

$$\mathrm{Ric}(e_1) < 0 < \mathrm{Ric}(e_3)$$

are satisfied in $\mathfrak{g}_0$. But these Ricci curvatures must vary continuously as we vary the structure constants, so it follows that $\mathrm{Ric}(e_1) < 0 < \mathrm{Ric}(e_3)$ whenever $\varepsilon$ is sufficiently close to zero. $\square$

- According to Eliasson in [Eli71], any smooth manifold of dimension $\geq 3$ admits a Riemannian metric of negative scalar curvature.
- However, metrics of non-negative scalar curvature do not always exist. From [Lic63], one cannot have a metric with non-negative scalar curvature, except possibly identically zero, on a compact spin manifold whose $\widehat{A}$-genus is not zero.
- Moreover, Kazdan and Warner showed in [KW75] there are no topological obstructions to scalar curvature which are negative somewhere.

- For left-invariant metrics, the situation of negative scalar curvature can be described as follows.

**Theorem**

*If the Lie group is solvable, then every left-invariant metric on $G$ is either flat, or else has negative scalar curvature.*

**Theorem**

*If the Lie algebra of $G$ is not abelian, then $G$ possesses a left-invariant metric of negative scalar curvature.*

- There remains the question as to which Lie groups admit left-invariant metrics of positive scalar curvature.

## Theorem (Wallach)

*Let $G$ be a connected Lie group. If the universal covering of $G$ is not homomorphic to Euclidean space, then $G$ admits a left-invariant metric of positive scalar curvature.*

- To prove this, we need the following basic result of Lie groups.

## Theorem (Iwasawa)

*Let $G$ be a connected Lie group. Then*

1. *Every compact subgroup is contained in a maximal compact subgroup $H$, which is necessary a connected Lie group.*
2. *This maximal compact subgroup is unique up to conjugation.*
3. *As a topological space, $G$ is homeomorphic with the product of $H$ and some Euclidean space $\mathbb{R}^n$.*

## Proof.

See [Iwa49]. □

## Corollary

*The universal covering of a connected Lie group $G$ is homeomorphic to a Euclidean space if and only if every compact subgroup of $G$ is abelian.*

## Proof.

If every compact subgroup of $G$ is abelian, then by Theorem 19, $G$ is homeomorphic to the product of an abelian Lie group and some Euclidean space $\mathbb{R}^n$. Note that any abelian (real) Lie group must be $(S^1)^k \times \mathbb{R}^m$. This shows the universal covering of $G$ is homeomorphic to a Euclidean space.
Conversely, if there exists a non-abelian compact subgroup, then the maximal compact subgroup $H$ is also non-abelian. Note that the universal covering of any connected compact non-abelian Lie group is not homeomorphic to the Euclidean space, and thus $G$ cannot be homeomorphic to some Euclidean. $\square$

## Proof of Theorem 18.

Since the universal covering of $G$ is not Euclidean, there exists a compact non-abelian subgroup $H$, and by Iwasawa's theorem we may assume $H$ is connected. Since $H$ is compact, we can construct an inner product on $\mathfrak{g}$ which is invariant under $\mathrm{Ad}(H)$. Let $e_1, \ldots, e_m$ be an orthonormal basis for the Lie algebra of $H$, and extend to an orthonormal basis $e_1, \ldots, e_n$ for $\mathfrak{g}$. Since inner product on $\mathfrak{g}$ is $\mathrm{Ad}(H)$-invariant, we see that $\mathrm{ad}(e_1), \ldots, \mathrm{ad}(e_m)$ must be skew-symmetric.

Fixing any $\varepsilon > 0$, consider a new basis $e'_1, \ldots, e'_n$ defined by

$$e'_1 = e_1, \ldots, e'_m = e_m, \quad e'_{m+1} = \varepsilon e_{m+1}, \ldots, e'_n = \varepsilon e_n.$$

Choose a new inner product so that basis $e'_1, \ldots, e'_n$ is orthogonal. The symbol $\mathfrak{g}_\varepsilon$ will denote the Lie algebra provided with this new inner product, and with this specified orthonormal basis.

## Continuation.

It's clear the structure constants of $\mathfrak{g}_\varepsilon$ are continuous functions of $\varepsilon$, so there is a well-defined limit algebra $\mathfrak{g}_0$ with prescribed inner product and prescribed orthonormal basis. Evidently $\mathfrak{g}_0$ splits as an orthogonal direct sum $\mathfrak{h} \oplus \mathfrak{u}$, where $\mathfrak{h}$ is the subalgebra spanned by $e'_1, \ldots, e'_m$ and $\mathfrak{u}$ is the abelian ideal spanned by $e'_{m+1}, \ldots, e'_n$. Applying Lemma 26 we see that $\nabla_u = 0$ for all $u \in \mathfrak{u}$, so

$$R(x, u) = \nabla_x \nabla_u - \nabla_u \nabla_x - \nabla_{[x,u]} = 0,$$

and thus $\kappa(x, u) = 0$ for all $x \in \mathfrak{g}$. In particular, the Ricci curvature $\mathrm{Ric}(u) = 0$ for all $u \in \mathfrak{u}$. On the other hand, by Lemma 10 we have $\mathrm{Ric}(b) \geq 0$ for $b \in \mathfrak{h}$, and the equality does not always hold since $\mathfrak{h}$ is not abelian. Therefore the scalar curvature $\rho = \sum_i \mathrm{Ric}(e'_i)$ of the limit algebra $\mathfrak{g}_0$ is positive. It follows by continuity that $\rho(\mathfrak{g}_\varepsilon) > 0$ whenever $\varepsilon$ is sufficiently small. □

- In this section we study 3-dimensional Lie algebra, and a useful tool is the "cross product operation".
- If $u, v$ are elements of a 3-dimensional vector space which is provided with an inner product and a preferred orientation, then the cross product $u \times v$ is defined. This product is bilinear and skew-symmetric as a function of $u$ and $v$. The vector $u \times v$ is orthogonal to both $u$ and $v$ and has length $\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2$. Its direction is determined by the requirement that the triple $u, v, u \times v$ is positively oriented.

- Let $G$ be a connected 3-dimensional Lie group with left-invariant metric. Choose an orientation for the Lie algebra of $G$, so that the cross product is defined.

## Lemma

*The bracket product operation in Lie algebra $\mathfrak{g}$ is related to the cross product operation by the formula*

$$[u, v] = L(u \times v),$$

*where $L$ is a uniquely defined linear mapping from $\mathfrak{g}$ to itself. The Lie group $G$ is unimodular if and only if $L$ is symmetric.*

## Proof.

Let $\mathfrak{g}$ be a 3-dimensional Lie algebra with an inner product and preferred orientation. Choose an oriented orthonormal basis $e_1, e_2, e_3$, define the linear transformation $L\colon \mathfrak{g} \to \mathfrak{g}$ by $L(e_1) = [e_2, e_3], L(e_2) = [e_3, e_1], L(e_3) = [e_1, e_2]$. Then the identity $L(e_i \times e_j) = [e_i, e_j]$ is true for all basis elements, hence $L(x \times y) = [x, y]$ for all $x$ and $y$. Setting

$$L(e_i) = \sum \alpha_{ij} e_j.$$

Note that

$$\begin{aligned} \operatorname{tr}\operatorname{ad}(e_1) &= -\alpha_{23} + \alpha_{32} \\ \operatorname{tr}\operatorname{ad}(e_2) &= -\alpha_{31} + \alpha_{13} \\ \operatorname{tr}\operatorname{ad}(e_3) &= -\alpha_{12} + \alpha_{21}. \end{aligned}$$

Thus $\mathfrak{g}$ is unimodular if and only if $(\alpha_{ij})$ is symmetric, or in other words if and only if $L$ is symmetric. □

- Now let's specialize to the unimodular case. If $L$ is symmetric, then there exists an orthonormal basis $e_1, e_2, e_3$ consisting of eigenvectors, $Le_i = \lambda_i e_i$. Replacing $e_1$ by $-e_1$ if necessary, we may assume $e_1, e_2, e_3$ is positively oriented. The bracket product operation is then given by $[e_1, e_2] = L(e_3) = \lambda_3 e_3$, with similar expressions for the other $[e_i, e_j]$. Thus we obtain the following normal form

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3,$$

for the bracket product operation in a 3-dimensional unimodular Lie algebra.

- For convenience, we denote $\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i$. Then the curvature properties are described as follows.

## Theorem

*The orthonormal basis $e_1, e_2, e_3$, chosen as before, diagonalizes the Ricci quadratic form, the principal Ricci curvatures being given by*

$$\mathrm{Ric}(e_1) = 2\mu_2\mu_3, \quad \mathrm{Ric}(e_2) = 2\mu_1\mu_3, \quad \mathrm{Ric}(e_3) = 2\mu_1\mu_2.$$

*As a consequence, the scalar curvature is given by* $s = 2(\mu_2\mu_3 + \mu_1\mu_3 + \mu_1\mu_2)$.

## Corollary

*In the 3-dimensional unimodular case, the determinant of the Ricci quadratic form is always non-negative. If this determinant is zero, then at least two of the principal Ricci curvatures must be zero.*

- There are now just six distinct cases, which we tabulate as follows. By changing signs if necessary, we assume that at most one of the structure constants $\lambda_1, \lambda_2, \lambda_3$ is negative.

| Signs of $\lambda_1, \lambda_2, \lambda_3$ | Associated Lie group | Description |
|---|---|---|
| $+,+,+$ | SU(2) | compact, simple |
| $+,+,-$ | SL$(2,\mathbb{R})$ | noncompact, simple |
| $+,+,0$ | $E(2)$ | solvable |
| $+,-,0$ | $E(1,1)$ | solvable |
| $+,0,0$ | Heisenberg group | nilpotent |
| $0,0,0$ | $\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$ | abelian |

## Corollary

*For any left-invariant metric on the Heisenberg group, the Ricci quadratic form has signature $(+,-,-)$ and the scalar curvature $s$ is negative. Furthermore, the principal Ricci curvatures satisfy*

$$|\operatorname{Ric}(e_1)| = |\operatorname{Ric}(e_2)| = |\operatorname{Ric}(e_1)| = \rho.$$

## Proof.

Taking $\lambda_2 = \lambda_3 = 0$ one has $\mu_1 = -\mu_2 = -\mu_3 = -\lambda_1/2$. Thus

$$\rho = 2\left(-\frac{1}{4}\lambda_1^2 - \frac{1}{4}\lambda_1^2 + \frac{1}{4}\lambda_1^2\right) = -\frac{1}{2}\lambda_1^2,$$

and $\operatorname{Ric}(e_1) = -\operatorname{Ric}(e_2) = -\operatorname{Ric}(e_3) = -\rho$. □

## Corollary

*Let $G$ be either $\mathrm{SL}(2,\mathbb{R})$ or $E(1,1)$. Then depending the choice of left-invariant metric the signature of the Ricci quadratic form can be either $(+,-,-)$ or $(0,0,-)$. However, the scalar curvature $\rho$ must always be strictly negative.*

## Proof.

If $\lambda_1 = 0$ while $\lambda_2$ and $\lambda_3$ have oppsite sign, then

$$u_1 = \frac{1}{2}(\lambda_2 + \lambda_3), \quad u_2 = \frac{1}{2}(\lambda_3 - \lambda_2), \quad u_3 = \frac{1}{2}(\lambda_2 - \lambda_3).$$

Thus

$$\rho = -\frac{1}{2}(\lambda_2 - \lambda_3)^2 < 0.$$

Since $\lambda_2$ and $\lambda_3$ has oppsite sign, then $\lambda_2 \neq \lambda_3$, and thus $\mathrm{Ric}(e_1) = 2u_2u_3 < 0$.

## Continuation.

If $\lambda_2 = -\lambda_3$, then $u_1 = 0$, and thus $\mathrm{Ric}(e_2) = \mathrm{Ric}(e_3) = 0$. In this case, the signature of Ricci quadratic form is $(-, 0, 0)$. If $\lambda_2 > 0 > \lambda_3$, then $u_1 > 0, u_2 < 0, u_3 > 0$. This shows the signature of Ricci quadratic form is $(+, -, -)$. Similarly for the case $\lambda_3 > 0 > \lambda_2$.

If the $\lambda_i$ are all non-zero with say $\lambda_1 < 0 < \lambda_2, \lambda_3$, then the computation $\partial\rho/\partial\lambda_1 = -\lambda_1 + \lambda_2 + \lambda_3$ shows that $\rho$ is monotone as a function of $\lambda_1$ for $\lambda_1 \leq 0$. Therefore

$$\rho(\lambda_1, \lambda_2, \lambda_3) < \rho(0, \lambda_2, \lambda_3) = -\frac{1}{2}(\lambda_2 - \lambda_3)^2 \leq 0.$$

By similar argument as above, more information about Ricci quadratic form can be derived. $\square$

- Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathfrak{g}$ and $\alpha_{ijk}$ be the structure constants of $\mathfrak{g}$.

## Lemma

*Let $\nabla$ be the Levi-Civita connection corresponding to the left-invariant metric on $G$. Then*

$$\nabla_{e_i} e_j = \sum_k \frac{1}{2}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij}) e_k.$$

## Proof.

It follows from the following formula

$$\langle \nabla_x y, z \rangle = \frac{1}{2}\left(\langle [x, y], z \rangle - \langle [y, z], x \rangle + \langle [z, x], y \rangle\right).$$

## Proof of Lemma 1.

Recall that the sectional curvature is given by

$$\kappa(u, v) = \langle R(u, v)v, u \rangle,$$

where $R(u, v)v = \nabla_u \nabla_v v - \nabla_v \nabla_u v - \nabla_{[u,v]} v$. Then inserting formula in Lemma 26 into the definition, we easily obtain the desired formula. $\square$

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