A BRIEF INTRODUCTION TO HODGE THEORY

BOWEN LIU

ABSTRACT. In this talk we give a brief introduction to Hodge theory as preliminaries for [Fil16], such as (polarized) Hodge structures, variation of Hodge structures and differential geometry of Hodge bundles.

CONTENTS

1. Hodge structures

1.1. **Hodge structures.**

Let (X, ω) a compact Kähler manifold. The classical Hodge theory says that there is a decomposition on the k-th cohomology as follows

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where $H^{p,q}(X)$ is the Dolbeault cohomology. The Hodge structure generalized this structure.

1.1.1. *Objects.*

Definition 1.1.1. An **(effective) $\mathbb{Z}$-Hodge structure** of weight k consists of the following data:

1. a finitely generated abelian group $V_{\mathbb{Z}}$;
2. a decomposition
$$V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}$$
such that $\overline{V^{p,q}} = V^{q,p}$;
3. $V^{p,q} = 0$ unless $p, q \geq 0$.

Definition 1.1.2. The **Deligne torus** $\mathbb{S}$ is the real algebraic group

$$\mathbb{S} = \{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a^2 + b^2 \neq 0 \}.$$

Its real point are naturally isomorphic to $\mathbb{C}^*$ and its complex points are isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$.

Proposition 1.1.1. A Hodge structure on $V_{\mathbb{Z}}$ is the same as an algebraic representation of the Deligne torus $\mathbb{S}$ on $V_{\mathbb{Z}}$.

Definition 1.1.3. Let $(V_{\mathbb{Z}}, V^{p,q})$ be a $\mathbb{Z}$-Hodge structure of weight k. The **Hodge filtration** F^p is defined by

$$F^p = \bigoplus_{p' \geq p} V^{p',q}.$$

It's a decreasing filtration which satisfies

$$V_{\mathbb{C}} = F^p \oplus \overline{F^{k-p+1}}. \tag{1.1}$$

Remark 1.1.1. Let $V_{\mathbb{Z}}$ be a finitely generated abelian group and F^p be a filtration satisfies (1.1). Then it determines a Hodge structure by

$$V^{p,q} = F^p \cap \overline{F^q}.$$

In other words, a Hodge structure of weight k is equivalent to a filtration F^p satisfying (1.1).

Example 1.1.1. Let V, W be two Hodge structures of weight k and l respectively. Then

(1) V^* is a Hodge structure of weight $-k$;
(2) $V \otimes W$ is a Hodge structure of weight $k+l$;
(3) $\mathrm{Hom}(V, W)$ is a Hodge structure of weight $-k+l$;
(4) $V^{\otimes n}, \mathrm{Sym}^n V$ and $\bigwedge^n V$ are Hodge structures of weight nk.

1.1.2. *Morphisms.*

Definition 1.1.4. Let $(V_{\mathbb{Z}}, V^{p,q}), (W_{\mathbb{Z}}, W^{p,q})$ be two Hodge structures of weight k and $k+2r$ and $\phi\colon V_{\mathbb{Z}} \to W_{\mathbb{Z}}$ be a morphism of abelian groups. Then ϕ is called a **morphism of Hodge structure of type** (r,r), if its $\mathbb{C}$-linear extension $\phi_{\mathbb{C}}$ satisfies

$$\phi_{\mathbb{C}}(V^{p,q}) \subseteq W^{p+r,q+r}.$$

Proposition 1.1.2. Let ϕ be a morphism between Hodge structures. Then $\ker\phi, \operatorname{im}\phi$ and $\operatorname{coker}\phi$ are Hodge structures.

Example 1.1.2. Let $f\colon X \to Y$ be a holomorphic map between compact Kähler manifolds. Then $f^*\colon H^k(Y,\mathbb{Z}) \to H^k(X,\mathbb{Z})$ is a morphism of Hodge structure of type $(0,0)$.

Example 1.1.3. Let X, Y be two compact Kähler manifolds such that $\dim X = n, \dim Y = m$ and $m = n + r$. Then

$$\begin{array}{ccc} H^k(X,\mathbb{Z}) & \longrightarrow & H^{k+2r}(Y,\mathbb{Z}) \\ \big\downarrow & & \big\uparrow \\ H_{2n-k}(X,\mathbb{Z}) & \xrightarrow{f_*} & H_{2n-k}(Y,\mathbb{Z}). \end{array}$$

This gives a morphism of Hodge structure of type (r,r), which is called **Gysin pushforward**.

1.2. **Polarization.** Let (X,ω) be a complex Kähler n-manifold. There is an intersection form Q on $H^k(X,\mathbb{R})$ given by

$$Q(\alpha,\beta) = (-1)^{\frac{k(k-1)}{2}} \int_X \omega^{n-k} \wedge \alpha \wedge \beta.$$

The induced Hermitian form $H(\alpha,\beta) := Q(\alpha,\overline{\beta})$ on $H^k(X,\mathbb{C})$ satisfies the following properties:

(1) The Hodge decomposition is orthogonal with respect to H.
(2) $(\sqrt{-1})^{p-q} H(\alpha,\overline{\alpha}) > 0$ for $0 \neq \alpha \in H^{p,q}(X)$.

This gives a polarization.

Definition 1.2.1. Let $(V_{\mathbb{Z}}, V^{p,q})$ be a $\mathbb{Z}$-Hodge structure. The **Weil operator** C associated to $V_{\mathbb{Z}}$ acts by $(\sqrt{-1})^{p-q}$ on $V^{p,q}$.

Definition 1.2.2. A **polarized $\mathbb{Z}$-Hodge structure** of weight k is $\mathbb{Z}$-Hodge structure $(V_{\mathbb{Z}}, V^{p,q})$ of weight k together with a morphism of Hodge structure $Q\colon V_{\mathbb{Z}} \otimes V_{\mathbb{Z}} \to \mathbb{Z}$ of type $(-k,-k)$ such that

$$\begin{aligned} H\colon V_{\mathbb{C}} \otimes V_{\mathbb{C}} &\to \mathbb{C} \\ (\alpha,\beta) &\mapsto Q(\mathsf{C}\alpha,\overline{\beta}) \end{aligned}$$

is a positive definite Hermitian form.

2. Variation of Hodge structures

2.1. **Local system and flat connection.** In this section we always assume X is a complex manifold.

Definition 2.1.1. A sheaf $\mathcal{V}$ on X is called a locally constant sheaf of rank r valued in $\mathbb{C}$, if for each point $x \in X$, there is an open subset U containing x such that $\mathcal{V}|_U$ is constant sheaf $\underline{\mathbb{C}^r}$.

Remark 2.1.1. In other words, there exists an open covering $\{U_\alpha\}$ such that $\mathcal{V}|_{U_\alpha}$ is isomorphic to constant sheaf $\underline{\mathbb{C}^r}$. Then the local system $\mathcal{V}$ is completely determined by the transition functions $g_{\alpha\beta}\colon U_{\alpha\beta} \to \mathrm{GL}_n(\mathbb{C})$, which are locally constant functions.

Definition 2.1.2. Let $\mathcal{E}$ be a locally free sheaf on X. A **connection** is a $\mathbb{C}$-linear map

$$\nabla\colon \mathcal{E} \to \mathcal{A}^1_X \otimes \mathcal{E}$$

satisfying the following condition

$$\nabla(\varphi \otimes e) = \mathrm{d}\varphi \otimes e + \varphi \nabla e$$

for all sections e of $\mathcal{E}$ and φ of $\mathcal{O}_X$.

Remark 2.1.2. The definition of ∇ extends to $\nabla\colon \mathcal{A}^p_X \otimes \mathcal{E} \to \mathcal{A}^{p+1}_X \otimes \mathcal{E}$ by defining

$$\nabla(\omega \otimes e) = \mathrm{d}\omega \otimes e + (-1)^p \omega \wedge \nabla e$$

for all sections ω of $\mathcal{A}^p_X$ and sections e of $\mathcal{E}$.

Remark 2.1.3. Let $\{e_\alpha\}$ be a local frame of $\mathcal{E}$. For any section $s = s^\alpha e_\alpha$ of $\mathcal{E}$, one has

$$\nabla(s^\alpha e_\alpha) = \mathrm{d}s^\alpha e_\alpha + s^\alpha \nabla e_\alpha.$$

Thus the connection ∇ is completely determined by

$$\nabla e_\alpha = \omega_\alpha^\beta e_\beta,$$

where ω_α^β are 1-forms, which forms a (smooth) 1-form valued matrix ω.

Definition 2.1.3. A connection ∇ is **integrable** if its curvature $\nabla^2\colon \mathcal{E} \to \mathcal{A}^2_X \otimes \mathcal{E}$ vanishes.

Remark 2.1.4. Let $\{e_\alpha\}$ be a local frame of $\mathcal{E}$. For any section $s = s^\alpha e_\alpha$ of $\mathcal{E}$, one has

$$\begin{aligned}
\nabla^2(s^\alpha e_\alpha) &= \nabla(\mathrm{d}s^\alpha \otimes e_\alpha + s^\alpha \omega_\alpha^\beta \otimes e_\beta) \\
&= -\mathrm{d}s^\alpha \wedge \omega_\alpha^\beta \otimes e_\beta + \mathrm{d}(s^\alpha \omega_\alpha^\beta) \otimes e_\beta - s^\alpha \omega_\alpha^\beta \wedge \omega_\beta^\gamma \otimes e_\gamma \\
&= s^\alpha (\mathrm{d}\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta) \otimes e_\beta, \\
\nabla^2(e_\alpha) &= \nabla(\omega_\alpha^\beta \otimes e_\beta) \\
&= \mathrm{d}\omega_\alpha^\beta \otimes e_\beta - \omega_\alpha^\beta \wedge \nabla e_\beta \\
&= \mathrm{d}\omega_\alpha^\beta \otimes e_\beta - \omega_\alpha^\beta \wedge \omega_\beta^\gamma \otimes e_\gamma \\
&= (\mathrm{d}\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta) \otimes e_\beta.
\end{aligned}$$

This shows ∇^2 is a global section of $\mathcal{A}_X^2 \otimes \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E})$, which is locally given by $\mathrm{d}\omega - \omega \wedge \omega$.

Definition 2.1.4. A locally free sheaf together with an integrable connection is called a **flat bundle**.

Proposition 2.1.1. Let ∇ be a integrable connection on locally free sheaf $\mathcal{E}$ on X. Then the horizontal section $\mathcal{E}^{\nabla=0}$ is a local system.

Proposition 2.1.2. Let $\mathcal{L}$ be a local system on X. Then the locally free sheaf $\mathcal{E} := \mathcal{O}_X \otimes \mathcal{L}$ together with canonical connection $\nabla_{\mathrm{can}}(f \otimes s) := \mathrm{d}f \otimes s$ is a flat bundle.

Theorem 2.1.1. The functor $(\mathcal{E}, \nabla) \mapsto \mathcal{E}^{\nabla=0}$ is an equivalence between category of flat bundles and the category of the complex local system with quasi-inverse $\mathcal{L} \mapsto (\mathcal{O}_X \otimes \mathcal{L}, \nabla_{\mathrm{can}})$.

Proposition 2.1.3. Let $\mathcal{L}$ be a local system on X. Then

$$H^*(X, \mathcal{L}) \cong \mathbb{H}^*(X, \mathcal{A}_X^\bullet \otimes \mathcal{L}).$$

Proof. Note the following complex of sheaves

$$0 \to \mathcal{L} \to \mathcal{A}_X^\bullet \otimes \mathcal{L}$$

gives a resolution of $\mathcal{L}$ by coherent sheaves. □

2.2. **Abstract variation of Hodge structures.** In this section we always assume S is a complex manifold.

Definition 2.2.1. A **variation of Hodge structure of weight** k on S consists of the following data:

(1) a local system $\mathbb{V}_\mathbb{Z}$ of finitely generated abelian groups on S;
(2) a finite decreasing filtration $\{\mathcal{F}^p\}$ of the holomorphic vector bundle $\mathcal{V} := \mathbb{V}_\mathbb{Z} \otimes \mathcal{O}_X$ by holomorphic subbundles (**the Hodge filtration**).

These data should satisfy the following conditions:

(a) for each $s \in S$ the filtration $\{\mathcal{F}^p(s)\}$ of $\mathbb{V}_s \simeq \mathbb{V}_{\mathbb{Z},s} \otimes_\mathbb{Z} \mathbb{C}$ defines a Hodge structure of weight k on the finitely generated abelian group $\mathbb{V}_{\mathbb{Z},s}$;
(b) the Gauss-Manin connection $\nabla^{GM} \colon \mathcal{V} \to \Omega_S^1 \otimes \mathcal{V}$ whose sheaf of horizontal sections is $\mathbb{V}_\mathbb{C}$ satisfies the **Griffiths' transversality condition**

$$\nabla^{GM}(\mathcal{F}^p) \subseteq \Omega_S^1 \otimes \mathcal{F}^{p-1}.$$

The notion of **a morphism of variations of Hodge structure** is defined in the obvious way.

Example 2.2.1. Given two variations $\mathbb{V}, \mathbb{V}'$ of Hodge structure over S of weights k and k', there is an obvious structure of variation of Hodge structure on the underlying local systems of $\mathbb{V} \otimes \mathbb{V}'$ and $\mathrm{Hom}\,(\mathbb{V}, \mathbb{V}')$ of weights $k + k'$ and $k - k'$ respectively.

Definition 2.2.2. A **polarized variation of Hodge structure** of weight k is a variation of Hodge structures $\mathbb{V}$ of weight k together with a bilinear pairing $I(\text{-},\text{-})$ satisfying

(1) The pairing is flat, that is, preserved by the Gauss-Manin connection ∇.
(2) On each fiber of $\mathcal{V}$, the pairing induces a polarization of the Hodge structure.

Remark 2.2.1. The intersection pairing $I(\text{-},\text{-})$ on a polarization variation of Hodge structure is flat for the Gauss-Manin connection, but the Hodge metric $Q(\text{-},\text{-})$ may not. Since the Hodge metric Q is expressed in terms of I and the Weil operator C, the compatiblity with the Weil operator implies the compatiblity with the Hodge metric.

2.3. **Variation of Hodge structures coming from smooth families.** In this section we will explain the motivation of variation of Hodge structures and the Griffiths transversality condition is inspired by the geometric case naturally.

Let $f\colon X \to S$ be a family[1] of compact Kähler manifolds. Then $R^k f_*\underline{\mathbb{C}}$ is a local system on S such that for each point $s \in S$, one has $(R^k f_*\underline{\mathbb{C}})_s \cong H^k(X_s, \mathbb{C})$. The flat bundle corresponding to the local system $R^k f_*\underline{\mathbb{C}}$ is the relative de Rham cohomology

$$H^k_{dR}(X/S) := \mathcal{O}_S \otimes R^k f_*\underline{\mathbb{C}}$$

together with the Gauss-Manin connection ∇^{GM}.

Proposition 2.3.1 ([Del70])**.** Let $f\colon X \to S$ be a family of complex manifolds and $\mathbb{V}$ be a local system of complex vector spaces on X. There is a natural isomorphism

$$\mathcal{O}_S \otimes R^k f_*\mathbb{V} \cong R^k f_*(\Omega^\bullet_{X/S} \otimes \mathbb{V}),$$

where $\Omega^\bullet_{X/S} = \Omega^\bullet_X / f^*\Omega^\bullet_S$ is the relative de Rham complex.

Corollary 2.3.1. Let $f\colon X \to S$ be a family of complex manifolds. Then

$$H^k_{dR}(X/S) \cong R^k f_*\Omega^\bullet_{X/S}.$$

Remark 2.3.1. In this viewpoint, the Hodge filtration on $R^k f_*\Omega^\bullet_{X/S}$ is described as follows

$$\mathcal{F}^p := \operatorname{Im}\left\{R^k f_*\sigma^{\geq p}\Omega^\bullet_{X/S} \to R^k f_*\Omega^\bullet_{X/S}\right\},$$

where $\sigma^{\geq p}$ is the stupid filtration.

Proposition 2.3.2. Let $f\colon X \to S$ be a family of compact Kähler manifolds. The Gauss-Manin connection ∇^{GM} satisfies the Griffiths transversality, that is,

$$\nabla^{GM}(\mathcal{F}^p) \subseteq \Omega^1_S \otimes \mathcal{F}^{p-1}.$$

[1] In other words, f is a proper holomorphic submersion between complex manifolds such that every fiber of f a compact Kähler manifold.

Proof. See Corollary 10.31 in [PS08]. □

Corollary 2.3.2. Let $f\colon X \to S$ be a family of compact Kähler manifolds. Then the local system $R^k f_*\underline{\mathbb{C}}$ underlies a variation of Hodge structures of weight k.

3. Differential geometry of Hodge bundles

3.1. **General setting.** In this section, we consider a speical connection on a holomorphic vector bundle E on a complex manifold X equipped with a Hermitian metric. Recall that we can define connection for any (smooth) vector bundle over X, which is a $\mathbb{C}$-linear map between (smooth) sections satisfying the Leibniz rules.

There is a canonical connection, called Chern connection, on Hermitian vector bundle E, which is uniquely defined by the following two conditions:

(1) It's compatible with the Hermitian metric.
(2) It's compatible with the holomorphic structure.

3.1.1. *Hermitian vector bundle.*

Definition 3.1.1. Let E be a complex vector bundle. A **Hermitian metric** h on E is a smooth section of $E^* \otimes \overline{E}^*$.

Remark 3.1.1. Let $\{e_\alpha\}$ be a local frame of E. Then a (positive definite) Hermitian metric is determined by a (positive definite) Hermitian matrix $(h_{\alpha\overline{\beta}})$, that is

$$h = h_{\alpha\overline{\beta}} e^\alpha \otimes \overline{e}^\beta,$$

where $h_{\alpha\overline{\beta}} = h(e_\alpha, \overline{e}_\beta)$.

Definition 3.1.2. A complex vector bundle E together with a Hermitian metric h is called a **Hermitian vector bundle** (E, h).

Remark 3.1.2. Let L be a Hermitian line bundle. A Hermitian metric h is locally given by $e^{-2\varphi}$, where φ is a smooth function, which is called **metric weight**. Suppose $\{g_{\alpha\beta}\}$ is the transition function of L with respect to open covering $\{U_\alpha\}$. Then h is given by a collection $\{h_\alpha \in C^\infty(U_\alpha)\}$ such that $h_\alpha = |g_{\alpha\beta}|^{-2} h_\beta$. In other words, a Hermitian metric is a collection of metric weights $\{\varphi_\alpha \in C^\infty(U_\alpha)\}$ such that

$$\varphi_\alpha = \varphi_\beta + \log |g_{\alpha\beta}|.$$

Definition 3.1.3. For a Hermitian vector bundle (E, h) over complex manifold X, there is a **sesquilinear map**

$$\begin{aligned} \mathcal{A}^p_X(E) \times \mathcal{A}^q_X(E) &\to \mathcal{A}^{p+q}_X \\ (s, t) &\mapsto \{s, t\}, \end{aligned}$$

which is locally given by

$$\{s^\alpha e_\alpha, t^\beta e_\beta\} = h_{\alpha\overline{\beta}} s^\alpha \wedge \overline{t^\beta}.$$

Definition 3.1.4. A connection ∇ on a Hermitian vector bundle (E, h) is **compatible with the metric**, if

$$\mathrm{d}\langle s, t\rangle = \{\nabla s, t\} + \{s, \nabla t\},$$

where s, t are smooth sections of E.

Remark 3.1.3. If $\{e_\alpha\}$ is a local frame of E, then

$$\begin{aligned}\mathrm{d}h_{\alpha\overline{\beta}} &= \mathrm{d}\langle e_\alpha, \overline{e}_\beta\rangle \\ &= \{\nabla e_\alpha, \overline{e}_\beta\} + \{e_\alpha, \nabla\overline{e}_\beta\} \\ &= \omega_\alpha^\gamma h_{\gamma\overline{\beta}} + \overline{\omega_\beta^\gamma} h_{\alpha\overline{\gamma}}.\end{aligned}$$

In the matrix notation, we have

$$\mathrm{d}h = \omega h + h\overline{\omega}^T.$$

3.1.2. *Compatiblity with complex structure.* Let $E \to X$ be a complex vector bundle with connection ∇. Then we can decompose $\nabla = \nabla^{1,0} + \nabla^{0,1}$ by composing the projection as follows

$$\begin{tikzcd} & & \mathcal{A}_X^{1,0}(E) \\ \mathcal{A}_X^0(E) \arrow[r, "\nabla"] & \mathcal{A}_X^1(E) \arrow[ur] \arrow[dr] & \\ & & \mathcal{A}_X^{0,1}(E) \end{tikzcd}$$

For convenience, we use $\nabla^{0,1}$ to denote the composition $\mathcal{A}_X^0(E) \xrightarrow{\nabla} \mathcal{A}_X^1(E) \to \mathcal{A}_X^{0,1}(E)$. On the other hand, there is a natural operator $\overline{\partial}_E\colon \mathcal{A}_X^0(E) \to \mathcal{A}_X^{0,1}(E)$, which is locally defined by

$$\overline{\partial}_E(s^\alpha \otimes e_\alpha) = \overline{\partial}s^\alpha \otimes e_\alpha.$$

Definition 3.1.5. A connection ∇ on a holomorphic vector bundle E over a complex manifold X is said to be **compatible with holomorphic structure** if $\nabla^{0,1} = \overline{\partial}_E$.

Remark 3.1.4. Let $\{e_\alpha\}$ be a holomorphic local form of E and denote

$$\nabla e_\alpha = (\Gamma_{i\alpha}^\beta \mathrm{d}z^i + \Gamma_{\overline{i}\alpha}^\beta \mathrm{d}\overline{z}^i) \otimes e_\beta.$$

Then

$$0 = \nabla^{0,1} e_\alpha = \Gamma_{\overline{i}\alpha}^\beta \mathrm{d}\overline{z}^i \otimes e_\beta.$$

This shows ∇ is compatible with holomorphic structure if and only if $\Gamma_{\overline{i}\alpha}^\beta = 0$.

Remark 3.1.5. Let E be a holomorphic vector bundle and ∇ be a connection which is compatible with the holomorphic structure. Then

$$\nabla\colon \mathcal{E} \to \Omega_X^1 \otimes \mathcal{E},$$

where $\mathcal{E}$ is the locally free sheaf given by the holomorphic section of E and Ω_X^1 is the locally free sheaf of holomorphic 1-forms.

3.1.3. *Chern connection.*

Theorem 3.1.1. Let X be a complex manifold and (E, h) a Hermitian holomorphic vector bundle. Then there exists a unique connection called **Chern connection** such that it's compatible with holomorphic structure and metric.

Proof. If metric connection ∇ is compatible with holomorphic structure, then the following three equations are equivalent

$$\begin{aligned}
\mathrm{d}h &= \omega h + h\overline{\omega}^t \\
\partial h &= \omega h \\
\overline{\partial} h &= h\overline{\omega}^t,
\end{aligned}$$

since ω is a $(1,0)$-valued matrix. This shows the Chern connection is determined by $\omega = (\partial h)h^{-1}$ uniquely. □

Corollary 3.1.1. Let E be a complex vector bundle on a complex manifold X and h, h' are two Hermitian metrics on E which are same up to a sign. Then the Chern connection of (E, h) is the same as the one of (E, h').

Remark 3.1.6. The Chern connection is locally determined by

$$\frac{\partial h_{\alpha\overline{\beta}}}{\partial z^i} = \Gamma^{\gamma}_{i\alpha} h_{\gamma\overline{\beta}}.$$

Definition 3.1.6. Let X be a complex manifold and (E, h) be a Hermitian holomorphic vector bundle. The **Chern curvature** Θ_h of (E, h) is defined as the curvature of Chern connection with respect to h.

Corollary 3.1.2. Let X be a complex manifold and (E, h) a Hermitian vector bundle equipped with Chern connection ∇ locally given by ω. Then

(1) $\partial\omega = \omega \wedge \omega$.
(2) $\Theta_h = \overline{\partial}\omega$.
(3) $\overline{\partial}\Theta_h = 0$.

Proof. For (1). Since $\omega = (\partial h)h^{-1}$, then directly computation shows

$$\begin{aligned}
\partial\omega &= -\partial h \wedge \partial(h^{-1}) \\
&= -\partial h \wedge (-h^{-1}\partial h h^{-1}) \\
&= (\partial h)h^{-1} \wedge (\partial h)h^{-1} \\
&= \omega \wedge \omega.
\end{aligned}$$

For (2). The Chern curvature Θ_h locally looks like

$$\Theta_h = \mathrm{d}\omega - \omega \wedge \omega = \mathrm{d}\omega - \partial\omega = \overline{\partial}\omega.$$

For (3). It follows from (2) directly. □

Remark 3.1.7. The Chern curvature can be expressed in terms of Christoffel symbol as follows

$$\Theta_h = \Theta^{\gamma}_{i\bar{j}\alpha} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j \otimes e^{\alpha} \otimes e_{\gamma},$$

where $\Theta^{\gamma}_{i\bar{j}\alpha} = -\partial \Gamma^{\gamma}_{i\alpha}/\partial \bar{z}^j$. In other type one has

$$\begin{aligned} \Theta_{i\bar{j}\alpha\bar{\beta}} &= h_{\gamma\bar{\beta}} \Theta^{\gamma}_{i\bar{j}\alpha} \\ &= -h_{\gamma\bar{\beta}} \partial_{\bar{j}} (h^{\gamma\bar{\delta}} \frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i}) \\ &= -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} + h^{\gamma\bar{\delta}} \frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j}. \end{aligned}$$

3.1.4. *Second variation formula.*

Lemma 3.1.1. Let (E, h) be a Hermitian holomorphic vector bundle and ∇ be the Chern connection with Chern curvature Θ. Suppose ϕ is a holomorphic section of E. Then we have the formula

$$\bar{\partial}\partial \log \|\phi\|^2 = \frac{\langle \Theta\phi, \phi\rangle}{\|\phi^2\|} + \frac{\langle \nabla\phi, \phi\rangle\langle \phi, \nabla\phi\rangle + \|\phi\|^2 \langle \nabla\phi, \nabla\phi\rangle}{\|\phi\|^4}.$$

Proof. Firstly note that $\partial\|\phi\|^2$ is the $(1,0)$-part of $\mathrm{d}\|\phi\|^2$, but on the other hand, by the compatiblity with the metric, one has

$$\mathrm{d}\langle \phi, \phi\rangle = \langle \nabla\phi, \phi\rangle + \langle \phi, \nabla\phi\rangle.$$

Then

$$\partial \log \|\phi\|^2 = \frac{\partial\|\phi\|^2}{\|\phi\|^2} = \frac{\langle \nabla\phi, \phi\rangle}{\|\phi\|^2}.$$

Next, we apply the chain rule

$$\bar{\partial}(\partial \log \|\phi\|^2) = \underbrace{\left(\bar{\partial} \frac{1}{\|\phi\|^2}\right) \langle \nabla\phi, \phi\rangle}_{\text{part I}} + \underbrace{\frac{1}{\|\phi\|^2} \bar{\partial} \langle \nabla\phi, \phi\rangle}_{\text{part II}}.$$

For part I, one has

$$\text{part I} = \frac{-1}{\|\phi\|^4} \bar{\partial}\|\phi\|^2 = -\frac{\langle \phi, \nabla\phi\rangle}{\|\phi\|^4}.$$

For part II, note that $\bar{\partial}\langle \nabla\phi, \phi\rangle$ is the $(1,1)$-part of $\mathrm{d}\langle \nabla\phi, \phi\rangle$. By the compatiblity with the metric, we have

$$\mathrm{d}\langle \nabla\phi, \phi\rangle = \langle \Theta\phi, \phi\rangle - \langle \nabla\phi, \nabla\phi\rangle.$$

Combining the above result, this completes the proof. □

3.1.5. *Second fundamental form.* Let (E,h) be a Hermitian holomorphic vector bundle over complex manifold X with rank r and S be a holomorphic subbundle of E with rank s. Then there is an exact sequence of holomorphic vector bundles

$$0\to S\to E\to Q\to 0,$$

where Q is the holomorphic quotient bundle, which is isomorphic to $S^\perp$ as complex vector bundle.

Suppose ∇^E is the Chern connection on E and define $\nabla^S:=\pi_S\circ\nabla^E$, where $\pi_S\colon E\to S$ is the orthogonal projection.

(1) It's clear ∇^S is compatible with holomorphic structure of S since ∇^E is the Chern connection of E, and S is a holomorphic subbundle of E.
(2) For sections s,t of S, one has

$$\begin{aligned}\mathrm{d}h(s,t)&=h(\nabla^E s,t)+h(s,\nabla^E t)\\&\overset{(a)}{=}h(\pi_S\circ\nabla^E s,t)+h(s,\pi_S\circ\nabla^E t)\\&=h(\nabla^S s,t)+h(s,\nabla^S t),\end{aligned}$$

where (a) holds from π_S is orthogonal projection.

This shows that ∇^S is the Chern connection of S with respect to Hermitian metric induced by the one on E.

Definition 3.1.7. The **second fundamental form** of the subbundle S of E is defined as

$$B=\nabla^E-\nabla^S\colon\mathcal{A}^0(S)\to\mathcal{A}^{1,0}(Q).$$

In other words, the second fundamental form $B\in\mathcal{A}^{1,0}(\mathrm{Hom}(S,Q))$.

Proposition 3.1.1.

$${}^S\Theta={}^E\Theta|_S+B^*\wedge B.$$

Proof. It suffices to check pointwisely. For $p\in X$, suppose $\{e_\alpha\}_{1\leq\alpha\leq r}$ is a holomorphic local frame of E such that $\{e_\alpha\}_{1\leq\alpha\leq s}$ is a holomorphic local frame of S, and assume $h_{\alpha\overline{\beta}}(p)=\delta_{\alpha\overline{\beta}}$. By the formula (3.1.6) of Chern connection, for $1\leq\alpha\leq s$, one has

$$\begin{aligned}\nabla^E e_\alpha(p)&=\sum_{\beta=1}^r\frac{h_{\alpha\overline{\beta}}}{\partial z^i}(p)\mathrm{d}z^i\otimes\overline{e}^\beta\\ \nabla^S e_\alpha(p)&=\sum_{\beta=1}^s\frac{h_{\alpha\overline{\beta}}}{\partial z^i}(p)\mathrm{d}z^i\otimes\overline{e}^\beta,\end{aligned}$$

and thus

$$Be_\alpha(p)=\sum_{\beta=s+1}^r\frac{h_{\alpha\overline{\beta}}}{\partial z^i}(p)\mathrm{d}z^i\otimes\overline{e}^\beta.$$

This shows

$$B(p)=\sum_{\alpha=1}^s\sum_{\beta=s+1}^r\frac{h_{\alpha\overline{\beta}}}{\partial z^i}(p)\mathrm{d}z^i\otimes e^\alpha\otimes\overline{e}^\beta,$$

and thus its conjugate transpose is

$$B^*(p)=\sum_{\beta=s+1}^{r}\sum_{\alpha=1}^{s}\frac{h_{\beta\overline{\alpha}}}{\partial\overline{z}^j}(p)\mathrm{d}\overline{z}^j\otimes e^\beta\otimes e_\alpha.$$

Then

$$\begin{aligned}B^*\wedge Be_\alpha(p)&=B^*(\sum_{\gamma=s+1}^{r}\frac{h_{\alpha\overline{\gamma}}}{\partial z^i}(p)\mathrm{d}z^i\otimes e_\gamma)\\&=-\sum_{\beta=1}^{s}\sum_{\gamma=s+1}^{r}\frac{h_{\alpha\overline{\gamma}}}{\partial z^i}\frac{h_{\gamma\overline{\beta}}}{\partial\overline{z}^j}(p)\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^j\otimes\overline{e}^\beta,\end{aligned}$$

which implies

$$B^*\wedge B(p)=-\sum_{\alpha,\beta=1}^{s}\left\{\sum_{\gamma=s+1}^{r}\frac{\partial h_{\alpha\overline{\gamma}}}{\partial z^i}\frac{\partial h_{\gamma\overline{\beta}}}{\partial\overline{z}^j}(p)\right\}\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^j\otimes e^\alpha\otimes\overline{e}^\beta.$$

On the other hand, by using formula (3.1.7), a direct computation shows

$${}^E\Theta|_S(p)-{}^S\Theta(p)=\sum_{\alpha,\beta=1}^{s}\left\{\sum_{\gamma=s+1}^{r}\frac{\partial h_{\alpha\overline{\gamma}}}{\partial z^i}\frac{\partial h_{\gamma\overline{\beta}}}{\partial\overline{z}^j}(p)\right\}\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^j\otimes e^\alpha\otimes\overline{e}^\beta.$$

This shows

$${}^S\Theta={}^E\Theta|_S+B^*\wedge B.$$

□

Proposition 3.1.2.

$${}^Q\Theta={}^E\Theta|_Q+B\wedge B^*.$$

3.1.6. *Positivity.*

Definition 3.1.8. A real $(1,1)$-form $\omega=\sqrt{-1}h_{i\overline{j}}\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^j$ is **positive** if the Hermitian matrix $h_{i\overline{j}}$ is positive definite.

Definition 3.1.9. Let (E,h) be a Hermitian holomorphic vector bundle on X. A form $\Theta\in\mathcal{A}^{1,1}_X\otimes\mathrm{End}(E)$ is **positive** if for any non-zero section e of E, one has $h(\Theta e,e)$ is positive.

Proposition 3.1.3. Let (E,h) be a Hermitian holomorphic vector bundle and $S\subseteq E$ be a holomorphic subbundle. Then $B\wedge B^*$ is positive and $B^*\wedge B$ is negative, where B is the second fundamental form.

Corollary 3.1.3. The curvature decreases in holomorphic subbundles and increases in holomorphic quotient bundles.

3.2. **Curvature of Hodge bundles.** Consider a variation of polarized Hodge structures of weight k over some fixed complex manifold. This data consists of a flat bundle $H_{\mathbb{C}}$ together with the Gauss-Manin connection ∇^{GM}, and there is a filtration by holomorphic subbundles

$$\cdots \subset \mathcal{F}^p \subset \mathcal{F}^{p-1} \subset \cdots \subset H_{\mathbb{C}}.$$

Denote the quotient subbundle by

$$\mathcal{H}^{p,q} := \mathcal{F}^p/\mathcal{F}^{p+1},$$

where $p+q=k$. The polarization provides the indefinite forms $I(\text{-},\text{-})$ on $H_{\mathbb{C}}$, and a definite metric

$$Q(\text{-},\text{-}) := I(\mathsf{C}\text{-},\text{-}),$$

where C is the Weil operator. In particular, restricted to $\mathcal{H}^{p,q}$, the definite and indefinite metrics agree up to sign.

Note that ∇^{GM} is the Chern connection on $H_{\mathbb{C}}$ equipped with the indefinite metric $I(\text{-},\text{-})$. On the other hand, viewing $H_{\mathbb{C}}$ as the direct sum of the holomorphic bundles $\mathcal{H}^{p,q}$, each equipped with the definite metric $Q(\text{-},\text{-})$, there is also a Hodge connection ∇^{Hg}, which is defined as the Chern connection of $\bigoplus \mathcal{H}^{p,q}$ equipped with the definite metric.

Consider the second fundamental form (for the indefinite metric)

$$\sigma_p \colon \mathcal{F}^p \to \mathcal{H}_{\mathbb{C}}/\mathcal{F}^p \otimes \mathcal{A}_X^{1,0}.$$

The Griffiths transversality condition implies it must in fact map subspaces as follows

$$\sigma_p \colon \mathcal{H}^{p,q} \to \mathcal{H}^{p-1,q+1} \otimes \mathcal{A}_X^{1,0}.$$

Proposition 3.2.1.

$$\nabla^{GM} = \nabla^{Hg} + \sigma_\bullet + \sigma_\bullet^*,$$

where $\sigma_\bullet$ denotes $\bigoplus_p \sigma_p$ and similarly for $\sigma_\bullet^*$.

Proof. See Proposition 13.1.1 of [CMSP17]. □

Proposition 3.2.2.

$$\Theta_{\mathcal{H}^{p,q}} = \sigma_p^* \wedge \sigma_p + \sigma_{p+1} \wedge \sigma_{p+1}^*.$$

Proof. Note that the definite and indefinite metrics agree up to sign on $\mathcal{H}^{p,q}$, by Corollary 3.1.1 it suffices to prove the curvature for the indefinite metric. From the exact sequence

$$0 \to \mathcal{F}^p \to \mathcal{H}_{\mathbb{C}} \to \mathcal{H}_{\mathbb{C}}/\mathcal{F}^p \to 0,$$

we find using Proposition 3.1.1

$$\Theta_{\mathcal{F}_p} = \sigma_p^* \wedge \sigma_p.$$

Next, consider the exact sequence

$$0 \to \mathcal{F}^{p+1} \to \mathcal{F}^p \to \mathcal{H}^{p,q} \to 0.$$

Again Proposition 3.1.1 yields

$$\begin{aligned}\Theta_{\mathcal{H}^{p,q}} &= \Theta_{\mathcal{F}^p} + \sigma_{p+1} \wedge \sigma_{p+1}^* \\ &= \sigma_p^* \wedge \sigma_p + \sigma_{p+1} \wedge \sigma_{p+1}^*.\end{aligned}$$

This completes the proof. □

Proposition 3.2.3. Suppose e, e' are two smooth sections of $\mathcal{H}^{p,q}$. Then

$$Q(\Omega_{\mathcal{H}^{p,q}} e, e') = Q(\sigma_p e, \sigma_p e') + Q(\sigma_{p+1}^* e, \sigma_{p+1}^* e').$$

Proof. Note that on $\mathcal{H}^{p,q}$, we have

$$I(\text{-},\text{-}) = (\sqrt{-1})^{p-q} Q(\text{-},\text{-}) = (\sqrt{-1})^k \times (-1)^p Q(\text{-},\text{-}).$$

Then

$$\begin{aligned}(\sqrt{-1})^k Q(\Omega_{\mathcal{H}^{p,q}} e, e') &= (-1)^p I(\Omega_{\mathcal{H}^{p,q}} e, e') \\ &= (-1)^p \left(I(\sigma_p^* \wedge \sigma_p e, e'\right) + I(\sigma_{p+1} \wedge \sigma_{p+1}^* e, e')) \\ &= (-1)^{p+1} \left(I(\sigma_p e, \sigma_p e'\right) + I(\sigma_{p+1}^* e, \sigma_{p+1}^* e')) \\ &= (\sqrt{-1})^k \times (-1)^{p+1} \left((-1)^{p-1} Q(\sigma_p e, \sigma_p e'\right) + (-1)^{p+1} Q(\sigma_{p+1}^* e, \sigma_{p+1}^* e')) \\ &= (\sqrt{-1})^k Q(\sigma_p e, \sigma_p e') + (\sqrt{-1})^k Q(\sigma_{p+1}^* e, \sigma_{p+1}^* e').\end{aligned}$$

This completes the proof. □

Corollary 3.2.1. The bundle $\mathcal{H}^{0,k}$ has positive curvature.

Proof. In this case, the second fundamental form σ_0 vanishes, so

$$\Theta_{\mathcal{H}^{0,k}} = \sigma_1 \wedge \sigma_1^*.$$

This completes the proof. □

References

[CMSP17] James Carlson, Stefan Müller-Stach, and Chris Peters. *Period mappings and period domains*, volume 168 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2017.

[Del70] Pierre Deligne. *Équations différentielles à points singuliers réguliers*, volume Vol. 163 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1970.

[Fil16] Simion Filip. Semisimplicity and rigidity of the Kontsevich-Zorich cocycle. *Invent. Math.*, 205(3):617–670, 2016.

[PS08] Chris A. M. Peters and Joseph H. M. Steenbrink. *Mixed Hodge structures*, volume 52 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2008.