ALGEBRAIC GEOMETRY

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Contents

1. Properties of Schemes

1.1. Reduced, irreducible and integral scheme.

Definition 1.1.1. Let $(X, \mathcal{O}_X)$ be a scheme. Then it's

(1) connected if X is connected.
(2) irreducibe if X is irreducible.
(3) reduced if for every open subset U of X, $\mathcal{O}_X(U)$ is reduced.
(4) integral if for every open subset U of X, $\mathcal{O}_X(U)$ is an integral domain.
(5) locally integral if $\mathcal{O}_{X,P}$ is an integral domain for every $P \in X$.

Proposition 1.1.1. A scheme $(X, \mathcal{O}_X)$ is integral if and only if it's irreducibe and reduced.

Proposition 1.1.2. Let $(X, \mathcal{O}_X)$ be an integral scheme and ξ be its generic point. Then $\mathcal{O}_{X,\xi}$ is a field.

Proposition 1.1.3. A scheme $(X, \mathcal{O}_X)$ is reduced if and only if $\mathcal{O}_{X,P}$ is reduced for every $P \in X$.

Proposition 1.1.4. Let $(X, \mathcal{O}_X)$ be a scheme such that X is a noetherian topological space. Then $(X, \mathcal{O}_X)$ is locally integral if and only if it's reduced and its irreducibe component are disjoint.

1.2. Affine criterion.

Definition 1.2.1. Let $(X, \mathcal{O}_X)$ be a scheme. For any section $f \in \mathcal{O}_X(X)$, X_f is defined to be the subset of X consisting of thoes $P \in X$ such that the germ of f at P is a unit in $\mathcal{O}_{X,P}$.

Proposition 1.2.1. Let $(X, \mathcal{O}_X)$ be a scheme.

(1) For every $f \in \mathcal{O}_X(X)$, X_f is open. It's empty if and only if there exists an open covering $\{U_i\}_{i\in I}$ of X such that each $f|_{U_i}$ is nilpotent.
(2) For any $f, g \in \mathcal{O}_X(X)$, we have $X_f \cap X_g = X_{fg}$.
(3) Let $(\varphi, \varphi^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes and $f \in \mathcal{O}_Y(Y)$. Then $\varphi^{-1}(Y_f) = X_{\varphi^\sharp(f)}$.
(4) Suppose X can be covered by finitely many affine open subschemes $\{U_i\}_{i\in I}$ such that $U_i \cap U_j$ can be covered by finitely many affine open subschemes for all $i, j \in I$. Let $A = \mathcal{O}_X$. Then for any $f \in A$, we have $\mathcal{O}_X(X_f) = A_f$.

Proposition 1.2.2. A scheme $(X, \mathcal{O}_X)$ is affine if and only if there exist finitely many sections $f_1, \ldots, f_n \in \mathcal{O}_X(X)$ generating the unit ideal of $\mathcal{O}_X(X)$ such that each open subscheme $(X_{f_i}, \mathcal{O}_X|_{X_{f_i}})$ is affine.

1.3. Noetherian scheme.

Definition 1.3.1. A scheme $(X, \mathcal{O}_X)$ is called locally noetherian if it can be covered by affine open subschemes $\{U_i = \operatorname{Spec} A\}_{i\in I}$ such that each A_i is noetherian, and it's called noetherian if it's quasi-compact and locally noetherian.

Remark 1.3.1. If $(X, \mathcal{O}_X)$ is a noetherian scheme, then X is a noetherian topological space, but the converse is not true.

Proposition 1.3.1. Let $(X, \mathcal{O}_X)$ be a locally noetherian scheme. Then for any affine open subscheme $U = \operatorname{Spec} A$ of X, A is noetherian. In particular, an affine scheme $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is locally noetherian if and only if A is noetherian.

2. Properties of Morphisms

2.1. Quasi-compact, affine, finite type and finite.

Definition 2.1.1. Let $f\colon X \to Y$ be a morphism of schemes. Then it's

(1) quasi-compact if there exists a covering of Y by affine open subschemes $\{V_i\}_{i\in I}$ such that each $f^{-1}(V_i)$ is quasi-compact.
(2) affine if there exists a covering of Y by affine open subschemes $\{V_i\}_{i\in I}$ such that each $f^{-1}(V_i)$ is affine.
(3) locally of finite type if if there exists a covering of Y by affine open subschemes $\{V_i = \operatorname{Spec} B_i\}_{i\in I}$ such that each $f^{-1}(V_i)$ can be covered by affine open subschemes $\{U_{ij} = \operatorname{Spec} A_{ij}\}_{j\in J_i}$ for some finitely generated B_i-algebra A_{ij}.
(4) finite type if it's quasi-compact and locally of finite type.
(5) finite if there exists a covering of Y by affine open subschemes $\{V_i = \operatorname{Spec} B_i\}_{i\in I}$ such that each $f^{-1}(V_i) = \operatorname{Spec} A_i$ for some finitely generated B_i-module A_i.

Proposition 2.1.1. Let $f\colon X \to Y$ be a morphism of schemes.

(1) f is quasi-compact if and only if for every open quasi-compact subset V of Y, $f^{-1}(V)$ is quasi-compact.
(2) f is affine if and only if for every affine open subscheme V of Y, $f^{-1}(V)$ is affine.
(3) f is locally of finite type if and only if for every affine open subscheme $V = \operatorname{Spec} B$ of Y and every affine open subscheme $U = \operatorname{Spec} A$ of X such that $f(U) \subseteq V$, the B-algebra A is finitely generated.
(4) f is of finite type if and only if for every affine open subscheme $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by finitely many affine open subschemes $\{U_j = \operatorname{Spec} A_j\}_{j\in J}$ such that each A_j is a finitely generated B-algebra.
(5) f is finite if and only if for every affine open subscheme $V = \operatorname{Spec} B$ of Y, $f^{-1}(V) = \operatorname{Spec} A$ for some finitely generated B-module A.

2.2. Open immersion and closed immersion.

Definition 2.2.1. A morphism $(f, f^\sharp)\colon (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is called an open immersion if it induces an isomorphism of $(Z, \mathcal{O}_Z)$ with an open subscheme of $(X, \mathcal{O}_X)$.

Definition 2.2.2. A morphism $(f, f^\sharp)\colon (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is called a closed immersion if it induces a homeomorphism of Z with a closed subset of X, and $f^\sharp\colon \mathcal{O}_X \to f_*\mathcal{O}_Z$ is surjective.

Definition 2.2.3. A morphism $Z \to X$ is called an immersion if it can be written as a composite $Z \to U \to X$ such that $U \to X$ is an open immersion and $Z \to U$ is a closed immersion.

Definition 2.2.4. A subset Z of X is called locally closed if it's the intersection of an open subset with a closed subset.

Proposition 2.2.1. Let $(f, f^\sharp)\colon (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a morphism of schemes.

(1) $(f, f^\sharp)\colon (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is an open immersion if and only if f induces a homeomorphism of Z with an open subset of X and $f^\sharp_P\colon \mathcal{O}_{X,f(P)} \to \mathcal{O}_{Z,P}$ is an isomorphism for every $P \in Z$.
(2) $(f, f^\sharp)\colon (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is an immersion if and only if f induces a homeomorphism of Z with a locally closed subset of X and $f^\sharp_P\colon \mathcal{O}_{X,f(P)} \to \mathcal{O}_{Z,P}$ is an epimorphism.
(3) Immersions are monomorphisms in the category of schemes. Moreover, the composite of immersions is an immersion, so are open immersion and closed immersion.

2.3. **Fiber product.** In this section S always is a scheme.

Definition 2.3.1.

(1) An S-scheme is a scheme X together with a morphism $X \to S$.
(2) An S-morphism from an S-scheme X to an S-scheme Y is a morphism $X \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

commutes.

Remark 2.3.1. For any scheme X, there is a unique morphism $X \to \operatorname{Spec}\mathbb{Z}$, so the category of schemes coincides with the category of $\operatorname{Spec}\mathbb{Z}$-schemes.

Definition 2.3.2. Let X and Y be S-schemes. The product in the category of S-schemes is called the fiber product of X and Y over S, which is a S-scheme denoted by $X \times_S Y$.

Proposition 2.3.1. For S-schemes X and Y, their fiber product over S exists and unique up to unique isomorphism.

2.4. **Seperated morphism.**

2.4.1. *Seperated.*

Definition 2.4.1. Let $f\colon X \to Y$ be a morphism of schemes. The diagonal morphism $\Delta_{X/Y}\colon X \to X \times_Y X$ to be the unique morphism satisfying

$$p \circ \Delta_{X/Y} = q \circ \Delta_{X/Y} = \mathrm{id}_X$$

Definition 2.4.2. Let $f\colon X \to Y$ be a morphism of schemes. It's called separated of $\Delta_{X/Y}$ is a closed immersion.

Definition 2.4.3. A scheme X is called separated if the canonical morphism $X \to \operatorname{Spec}\mathbb{Z}$ is separated.

Proposition 2.4.1. Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be a morphism of affine schemes. Then f is separated.

Proposition 2.4.2. Let $f\colon X \to Y$ be a morphism of schemes.

(1) The diagonal morphism $\Delta\colon X \to X \times_Y X$ is an immersion.
(2) $f\colon X \to Y$ is separated if and only if $\Delta_{X/Y}(X)$ is a closed subset of $X \times_Y X$.

2.4.2. *Quasi-seperated.*

Definition 2.4.4. A morphism $f\colon X \to Y$ of schemes is called quasi-separated if the diagonal morphism is quasi-compact, and a scheme X is quasi-seperated if the canonical morphism is quasi-seperated.

2.5. **Proper morphism.**

Definition 2.5.1. A morphism $f\colon X \to Y$ of schemes is proper if f satisfies

(1) f is of finite type.
(2) f is separated.
(3) For any morphism $Y' \to Y$, the base change $f'\colon X \times_Y Y' \to Y'$ of f is a closed map on the underlying topological spaces, and such a property is called universally closed.

2.6. **Projective morphism.**

Definition 2.6.1. For any scheme Y, the projective space over Y is the Y-scheme $\mathbb{P}^n_Y := \mathbb{P}^n_{\mathbb{Z}} \times Y$.

Definition 2.6.2. A morphism $f\colon X \to Y$ of schemes is projective if f can factorized as a composite

$$X \to \mathbb{P}^n_Y \to Y$$

such that $X \to \mathbb{P}^n_Y$ is a closed immersion and $\mathbb{P}^n_Y \to Y$ is the projection. It's called quasi-projective if it can be factorized as above with $X \to \mathbb{P}^n_Y$ being an immersion.

Proposition 2.6.1. Projective morphism is proper.

Proposition 2.6.2.

(1) Closed immersions are projective.
(2) Composites of projective morphisms are projective.
(3) Let $f\colon X \to Y$ and $Y' \to Y$ be morphism of schemes and let $f'\colon X \times_Y Y' \to Y'$ be the base change of f. If f is projective, then f' is projective.
(4) Let $f\colon X \to Y$ and $f'\colon X' \to Y'$ be projective S-morphisms between S-schemes. Then $f \times f'\colon X \times_S X' \to Y \times_S Y'$ is projective.
(5) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphism of schemes. If gf is projective and g is separated, then f is projective.

Proposition 2.6.3 (Segre embedding)**.** There exists a closed immersion

$$\mathbb{P}^m_S \times_S \mathbb{P}^n_S \to \mathbb{P}^{(m+1)(n+1)-1}_S$$

which is an S-morphism.

3. Coherent Sheaves

References

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