TOPCIS IN COMPLEX ALGEBRAIC GEOMETRY

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0. Preface

0.1. Introduction.

In this lecture, the object we're most interested in is the complex variety.

Definition 0.1 (complex variety)**.** A complex algebraic variety or simply a complex variety is a quasi-projective[1] variety over $\mathbb{C}$.

Definition 0.2 (non-singular)**.** A complex variety X is non-singular if the sheaf of Kähler differentials $\Omega_{X/\mathbb{C}}$ is locally free.

Given any non-singular projective complex variety X, one can show that $X \subseteq \mathbb{CP}^n$ is a submanifold by using inverse function theorem. Conversely, Chow showed that

Theorem 0.1 (Chow)**.** Any compact complex submanifold[2] of complex projective space must be a complex variety.

Chow's theorem implies that there is a deep connection between complex manifolds and complex varieties, and thus techniques from complex differential geometry may be used to solve some questions in algebraic geometry, such as corollaries of Calabi-Yau theorem. On the other hand, motivated by Chow's theorem, it's natural to ask whether a compact complex manifold can be (holomorphically) embedded into complex projective space or not.

Theorem 0.2 (Riemann)**.** Any compact Riemann surface can be embedded into $\mathbb{CP}^N$.

Theorem 0.3 (Kodaira)**.** A compact complex manifold with a positive holomorphic line can be embedded into $\mathbb{CP}^N$.

Remark 0.1. In fact, Riemann's result can be obtained from Kodaira's embedding. Given a Hermitian holomorphic line bundle (L, h), its Chern curvature $\sqrt{-1}\Theta_h$ represents the first Chern class $c_1(L)$, and $\partial\overline{\partial}$-lemma shows that any real $(1,1)$-form which represents $c_1(L)$ can be realized as the Chern curvature of some Hermitian metric h. Thus if we want to see whether a holomorphic line bundle is positive or not, it suffice to compute its first Chern class, and there always exists holomorphic line with positive first Chern class[3].

The Kähler manifold is an important object in the complex differential geometry, which lies in the intersection of Riemannian manifold, complex manifold and symplectic manifold, and has many elegant properties. One of the most profound results is the Hodge decomposition.

[1] $X \subseteq \mathbb{CP}^n$ is a projective variety if it's the zero-locus of some (finite) family of homogeneous polynomials, that generate a prime ideal, and it's called quasi-projective if it's an open subset of a projective variety.

[2] In fact, "submanifold" can be replaced by analytic subvariety, that is, we allow some singularities.

[3] For holomorphic line bundle L over Riemann surface, the "positivity" of first Chern class is determined by its degree, that is, holomorphic line bundle with positive degree has positive first Chern class.

Theorem 0.4 (Hodge)**.** Let (X, ω) be a compact Kähler manifold. Then there is a decomposition

$$H^n(X) \cong \bigoplus_{p+q=n} H^{p,q}(X),$$

where $H^{p,q}(X)$ is the Dolbeault cohomology of X.

Remark 0.2. The Hodge decomposition is independent of the choice of Kähler form ω, but for the proof, we need to use theory of harmonic forms and Kähler identities.

The Hodge decomposition has lots of consequences in algebraic geometry. Let X be a non-singular projective complex variety. The algebraic de Rham complex is defined by

$$\Omega^\bullet_{X/\mathbb{C}} \colon \mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega_{X/\mathbb{C}} \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Omega^n_{X/\mathbb{C}},$$

where $n = \dim X$, and the algebraic de Rham cohomology is defined by the hypercohomology of above complex as follows

$$H^k_{alg}(X) = \mathbb{H}^k(\Omega^\bullet_{X/\mathbb{C}}),$$

where $k \in \mathbb{Z}_{\geq 0}$. Note that there is a natural filteration on algebraic de Rham complex

$$\Omega^\bullet_{X/\mathbb{C}} = F^0\Omega^\bullet_{X/\mathbb{C}} \supseteq F^1\Omega^\bullet_{X/\mathbb{C}} \supseteq \cdots \supseteq F^n\Omega^\bullet_{X/\mathbb{C}} = \{0\},$$

where

$$F^p\Omega^\bullet_{X/\mathbb{C}} \colon 0 \to \cdots \to 0 \to \Omega^p_{X/\mathbb{C}} \to \cdots \to \Omega^n_{X/\mathbb{C}}.$$

This filteration gives the Hodge to de Rham spectral sequence.

Theorem 0.5 (E_1-degeneration)**.** Let X be a non-singular projective complex variety. The Hodge to de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_{X/\mathbb{C}}) \Longrightarrow H^{p+q}_{alg}(X)$$

degenerates at E_1-page, and

$$\dim_{\mathbb{C}} H^p(X, \Omega^q_{X/\mathbb{C}}) = \dim_{\mathbb{C}} H^q(X, \Omega^p_{X/\mathbb{C}}).$$

Remark 0.3. The inequality

$$\dim H^k_{alg}(X) \leq \sum_{p+q=k} H^q(X, \Omega^p_{X/\mathbb{C}})$$

always holds, and the equality holds if and only if the Hodge to de Rham spectral sequence degenerates at E_1-page.

There are several important developments in Kähler geometry after Hodge and Kodaira, such as the solution to Calabi conjecture given by Shing-Tung Yau, and the connection between stable vector bundles and Hermitian-Yang-Mills metrics proved by Uhlenbeck-Yau.

Theorem 0.6 (Calabi-Yau)**.** Let (X,ω) be a compact Kähler manifold and $\mathcal{H}i$ be a real $(1,1)$-form that represents the first Chern class. Then there exists a unique $\omega_h \in [\omega]$ such that $\mathrm{Ric}(\omega_h) = \mathcal{H}i$.

Corollary 0.1. There exists a unique Ricci-flat Kähler metric on compact Kähler manifold with vanishing first Chern class.

Now let's introduce some algebraic consequence of Calabi-Yau theorem.

Theorem 0.7. Let X be a non-singular projective complex variety with ample canonical bunle K_X. Then

$$(-1)^n\left(c_1^n(X) - \frac{2(n+1)}{n}c_1^{n-2}(X)c_2(X)\right) \leq 0.$$

Moreover, the equality holds if and only if X is a locally symmetric variety of ball type.

Corollary 0.2. If X is a locally symmetric variety of ball type, then X^σ is again a locally symmetric variety of ball type for any $\sigma \in \mathrm{Aut}(\mathbb{C})$.

Theorem 0.8. Let X be a non-singular projective complex variety with $c_1(X) = 0$. Then for any ample line bundle L on X,

$$c_2(X) \cdot L^{n-2} \geq 0.$$

Moreover, the equality holds if and only if X is an abelian variety.

To state Uhlenbeck-Yau's theorem, we need the following preparations.

Definition 0.3 (slope)**.** Let (X,ω) be a compact Kähler and E be a holomorphic vector bundle. The slope of E with respect to ω is defined by

$$\mu_\omega(E) = \frac{\deg_\omega(E)}{\operatorname{rk} E},$$

where

$$\deg_\omega(E) = \int_X c_1(E) \cdot \omega^{n-1}.$$

Definition 0.4 (stability)**.** Let (X,ω) be a compact Kähler and E be a holomorphic vector bundle.

(1) E is μ_ω-stable if for all subbundle $F \subseteq E$, one has

$$\mu_\omega(F) < \mu_\omega(E).$$

(2) E is μ_ω-semistable if for all subbundle $F \subseteq E$, one has

$$\mu_\omega(F) \leq \mu_\omega(E).$$

Definition 0.5 (Hermitian-Yang-Mills metric)**.** Let (X,ω) be a compact Kähler and E be a holomorphic vector bundle. A Hermitian metric h on E is called Hermitian-Yang-Mills if

$$\wedge_\omega \Theta_h = \lambda \, \mathrm{id}_E,$$

where $\lambda \in \mathbb{R}$.

Theorem 0.9 (Uhlenbeck-Yau)**.** Let (X, ω) be a compact Kähler and E be a μ_ω-stable holomorphic bundle. Then there exists a unique Hermitian-Yang-Mills metric on E.

It also has lots of algebraic consequences.

Theorem 0.10. Let X be a non-singular projective complex variety and H be a line bundle. Let E be a μ_H-semistable vector bundle. Then

$$\left(c_1^2(E) - \frac{\mathrm{rk}(E)+1}{\mathrm{rk}(E)} c_2(E)\right) \cdot H^{n-2} \geq 0$$

Theorem 0.11. Let X be a non-singular projective complex variety and H be a line bundle. Let E be a μ_H-semistable vector bundle. Then

$$H^p(X, \Omega_X^q \otimes E \otimes H) = 0$$

for all $p + q > \dim X$.

In particular, above vanishing theorem generalizes the classical Kodaira vanishing theorem, which is a consequence of Hodge theory.

0.2. **Outlines.**

0.2.1. *Part I: Hodge theory.*

(1) Existence of harmonic forms.
(2) Kähler condition and Hodge package.
(3) Kodaira's vanishing theorem.
(4) Cartier descent theorem.
(5) de Rham decomposition theorem of Deligne-Illusie's theorem.
(6) Hodge symmetry.

0.2.2. *Part II: Non-abelian Hodge theory.*

(1) Existence of Hermitian-Yang-Mills metrics.
(2) Higgs bundle and the variant.
(3) Non-abelian Hodge theory.
(4) Ogus-Vologodsky theorem.
(5) Higgs-de Rham flow.

Part 1. Hodge theory

1. Existence of harmonic forms

Let (M,g) be an oriented, compact Riemannian manifold with volumn form vol. Then there is a L^2-metric on $\mathcal{A}^n(M) := \Gamma(M, \Omega_M^n)$ defined by

$$\begin{aligned}(\text{-},\text{-})_{L^2}\colon \mathcal{A}^n(M)\times\mathcal{A}^n(M) &\to \mathbb{C}\\ (\omega,\tau) &\mapsto \int_X \langle\omega,\tau\rangle_g \operatorname{vol}.\end{aligned}$$

The space of harmonic forms is defined by

$$\mathcal{H}^n(M) := \{\omega\in\mathcal{A}^n(M \mid \Delta_g(\omega)=0\}$$

Theorem 1.0.1 (Hodge)**.** Let (M,g) be a compact Riemannian manifold. Then $\mathcal{H}^n(M)\cong H^n_{dR}(M)$. In other words, every de Rham class has a unique harmonic representative.

Remark 1.0.1. The Hodge theorem gives a split of following exact sequence.

$$\begin{tikzcd} 0 \arrow[r] & B^n(M) \arrow[r] & Z^n(M) \arrow[r] & H^n_{dR}(M) \arrow[r] & 0 \\ & & \mathcal{H}^n(M) \arrow[u, hook] \arrow[ur, "\cong"] & & \end{tikzcd}$$

One of advantages of above split is that we can regard element of de Rham cohomology as a closed form with certain properties, not an equivalent class.

1.1. Approach I: Elliptic operators.

Let E,F be smooth complex vector bundles over a smooth manifold M of dimension d.

1.1.1. *Differential operators.*

Definition 1.1.1 (differential operator)**.** A differential operator of order k is a $\mathbb{C}$-linear map

$$P\colon \Gamma(M,E)\to\Gamma(M,F),$$

such that on local trivialization U of E,F with coordinates $x^1,\dots,x^d$, P is given by a matrix (p_{ij}), where

$$p_{ij} = \sum_{|I|\leq k} P_{I,ij}\frac{\partial}{\partial x^I},\quad P_{I,ij}\in\mathcal{A}^0(U).$$

Notation 1.1.1. The set of all differential operators from E to F of order k is denoted by $\operatorname{Diff}_k(E,F)$.

Example 1.1.1. Let M be a smooth manifold of dimension 1 and $E=F=M\times\mathbb{C}$. For convenience we suppose trivialization of M is given by two charts as follows

$$\begin{aligned} x\colon U&\to\mathbb{R}\\ y\colon V&\to\mathbb{R}\,.\end{aligned}$$

Let P be a differential operator of order k locally given by ∂_x^k on the trivialization U. By chain rule and Leibniz's rule one has

$$\partial_y^k = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^k \partial_x^k \quad (\text{mod lower order terms}).$$

More generally, for $P \in \mathrm{Diff}_k(E, F)$, which is given by p_{ij} on the local trivialization U. If we define

$$P_{ij}^k = \sum_{|I|=k} P_{I,ij} \frac{\partial}{\partial x^I}.$$

Again by chain rule and Leibniz's rule one can see the coefficients of matrix P^k are transformed like the sections of the $\mathrm{Sym}^k TM$ and similarly by a change of trivialization of the bundles E and F, the matrix transforms like a section of $\mathrm{Hom}(E, F)$. It gives a section of $\mathrm{Sym}^k TM \otimes \mathrm{Hom}(E, F)$, which is called the symbol of P, and denoted by σ_P.

Example 1.1.2. Consider the differential operator $\mathrm{d}\colon \mathcal{A}^n(M) \to \mathcal{A}^{n+1}(M)$, its symbol is

$$\sigma_{\mathrm{d},\omega}(\text{-}) = \omega \wedge \text{-}.$$

Definition 1.1.2. A differential operator $P \in \mathrm{Diff}_k(E, F)$ is said to be elliptic if for all $x \in M$ and $0 \neq \omega \in \Omega_{M,x}$, the homomorphism

$$\sigma_P(\omega)\colon E_x \to F_x$$

is an injective.

1.1.2. *Adjoint operator.* Now suppose (M, g) is an oriented, compact Riemannian manifold with volumn form vol and $P \in \mathrm{Diff}_k(E, F)$ is a differential operator from smooth vector bundles E to F. There exists a formal adjoint operator for P, which is a differential operator from F to E, and satisfies

$$(\alpha, P\beta)_{L^2} = (P^*\alpha, \beta)_{L^2},$$

where $\alpha \in \Gamma(M, F)$ and $\beta \in \Gamma(M, E)$. Moreover, the symbol of P^* equals to the adjoint of the symbol of P, that is,

$$\sigma_{P^*,\omega} = (\sigma_{P,\omega})^*.$$

In particular, if E and F are of equal rank, then P is elliptic if and only if P^* is.

1.1.3. *Fundamental decomposition theorem for self-adjoint elliptic operator.* Let (X, g) be a Riemannian manifold and E be a Hermitian vector bundle on (X, g). For self-adjoint, elliptic differential operator $L \in \mathrm{Diff}_k(E)$, there exists a linear mapping $H_L, G_L\colon \Gamma(X, E) \to \Gamma(X, E)$ such that

(1) $H_L(\Gamma(X, E)) = \mathcal{H}_L$, and $\dim \mathcal{H}_L < \infty$.
(2) $L \circ G_L + H_L = G_L \circ L + H_L = \mathrm{id}_E$.
(3) The following decomposition is orthogonal with respect to L^2-norm

$$\begin{aligned}\Gamma(X, E) &= \mathcal{H}_L \oplus G_L \circ L(\Gamma(X, E)) \\ &= \mathcal{H}_L \oplus L \circ G_L(\Gamma(X, E)).\end{aligned}$$

1.1.4. *Elliptic complex and generalized Laplacian operator.* Let $E_0, E_1, \dots, E_N$ be a sequence of smooth complex vector bundles on smooth manifold M. Consider the complex

$$\Gamma(M,E_0) \xrightarrow{L_0} \Gamma(M,E_1) \xrightarrow{L_1} \Gamma(M,E_2) \xrightarrow{L_2} \dots \xrightarrow{L_{N-1}} \Gamma(M,E_N), \tag{1.1}$$

where L_i is a differential operator of order k such that $L_i \circ L_{i-1} = 0$ for all i.

Definition 1.1.3 (elliptic complex)**.** The complex (1.1) is called elliptic if for all $x \in X$ and $0 \neq \omega \in \Omega_{M,x}$, the sequence

$$(E_0)_x \xrightarrow{\sigma_{L_0,\omega}} (E_1)_x \xrightarrow{\sigma_{L_1,\omega}} (E_2)_x \xrightarrow{\sigma_{L_2,\omega}} \dots \xrightarrow{\sigma_{L_{N-1},\omega}} (E_{N-1})_x$$

is exact.

Example 1.1.3.

$$\mathcal{A}^0(M) \xrightarrow{\mathrm{d}} \mathcal{A}^1(M) \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \mathcal{A}^n(M)$$

is an elliptic complex.

Definition 1.1.4 (Laplacian operator)**.** The Laplacian operator of the complex (1.1) is defined by

$$\Delta_i = L_i^* \circ L_i + L_{i-1} \circ L_{i-1}^*,$$

which is a differential operator from E_i to E_i.

Lemma 1.1.1. Let U, V, W be finite-dimensional vector spaces. If we have a diagram of finite-dimensional vector spaces

$$\begin{tikzcd} U \arrow[r, "A"] \arrow[d, no head] & V \arrow[r, "B"] \arrow[d, no head] & W \arrow[d, no head] \\ U & V \arrow[l, "A^*"] & W, \arrow[l, "B^*"] \end{tikzcd}$$

which is exact at V. Then

(1) $V = \operatorname{im} A \oplus \operatorname{im} B^*$.
(2) AA^* is injective on $\operatorname{im} A$ and is zero on $\operatorname{im} B^*$.
(3) BB^* is injective on $\operatorname{im} B^*$ and is zero on $\operatorname{im} A$.
(4) $AA^* + BB^* \colon V \to V$ is an isomorphism.

Proof. For (1). Suppose $v \in \operatorname{im} A \cap \operatorname{im} B^*$, written as $Au = B^*w$. Then

$$\langle Au, B^*w \rangle = \langle BAu, w \rangle = 0.$$

This shows $\operatorname{im} A \cap \operatorname{im} B^* = 0$. On the other hand, note that

$$\begin{aligned} \dim \operatorname{im} A &= \dim \ker B \\ &= \dim V - \dim \operatorname{im} B \\ &= \dim V - \dim \operatorname{im} B^*. \end{aligned}$$

For (2). □

Corollary 1.1.1. If the complex (1.1) is elliptic, then Δ_i is elliptic for all i.

Example 1.1.4. $\Delta_{\mathrm{d}} = \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$ is a self-adjoint elliptic operator of order 2.

Theorem 1.1.1. Consider $E = \bigoplus_{i=0}^N E_i$ and $L = \bigoplus_{i=0}^N L_i$.

(1) The following decomposition is orthogonal

$$\Gamma(X, E) = \mathcal{H}(E) \oplus LL^*G(\Gamma(X, E)) \oplus L^*LG(\Gamma(X, E)).$$

(2) The following commutation relations are valid:

$$\begin{aligned} HG = GH = H\Delta = \Delta H = 0. \\ LH = HL = L^*H = HL^* = 0. \\ L\Delta = \Delta L, L^*G = GL^*. \\ LG = GL, L^*G = GL^*. \end{aligned}$$

(3) $\dim_{\mathbb{C}} \mathcal{H}(E) < \infty$, and for each i, there is a canonical isomorphism

$$\mathcal{H}(E_i) \cong H^i(E).$$

1.2. **Approach II: Heat equation.** Let (M, g) be a compact Riemannian manifold and $\omega(t)\colon \mathbb{R}_{\geq 0} \to \mathcal{A}^n(M)$. The heat equation is given by

$$\begin{cases} (\frac{\partial}{\partial t} + \Delta)(\omega(t)) = 0, \\ \omega(0) = \omega_0. \end{cases} \tag{1.2}$$

Now let's explain why the heat equation can be used to prove Hodge theory. The idea is to use heat equation to flow what we have to something we desired, and this philosophy is frequently used in solving other problems, such as Ricci flow, Kähler-Einstein flow and so on.

Suppose $\omega(t)$ is a solution defined on $\mathbb{R}$ for heat equation (1.2). Roughly speaking we desire

$$\lim_{t\to\infty} \frac{\partial}{\partial t}\omega(t) = 0,$$

and thus $\omega_\infty := \lim_{t\to\infty} \omega(t)$ is expected to be a harmonic form. On the other hand, we desire the flow doesn't change the cohomology class, that is, $[\omega_0] = [\omega(t)]$. If so, for $\alpha \in H^n_{dR}(M)$, we pick an arbitary representative ω_0 and consider the heat equation with initial value ω_0. If there exists a unique solution $\omega(t)$ defined on $\mathbb{R}$, then ω_∞ is the unique harmonic representative of α. This proves the Hodge theorem.

Before we come into deep theories about partial differential equations, let's consider a baby example.

Example 1.2.1. Let $M = S^1$ equipped with the metric induced from $\mathbb{R}^2$. Then the heat equation is

$$(\partial_t - \partial_\theta^2) f(t, \theta) = 0,$$

where θ is the coordinate on S^1. Let $f_0(\theta)$ be the initial value with Fourier expansion

$$f_0(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{\sqrt{-1}n\theta}.$$

and

$$f(t,\theta) = \sum_{n=\infty} a_n(t) e^{\sqrt{-1}n\theta}.$$

Then the heat equation is given by

$$(\partial_t - (\partial_\theta)^2)(f(t,\theta)) = \sum_{n=\infty}^{\infty} (a_n'(t) + a_n(t)n^2) e^{\sqrt{-1}n\theta}.$$

This shows

$$a_n(t) = a_n e^{-n^2 t}.$$

Thus the solution is given by

$$\lim_{t\to\infty} f(t,\theta) = a_0.$$

1.2.1. *Existence and uniqueness of the heat equation.*

Theorem 1.2.1. For any $\omega_0 \in \mathcal{A}^n(M)$, there exists a unique smooth map

$$\omega(t)\colon \mathbb{R}_{\geq 0} \to \mathcal{A}^n(M)$$

such that

$$\begin{cases} (\partial_t + \Delta_g)(\omega(t)) = 0 \\ \omega(0) = 0 \end{cases}$$

Consider

$$\begin{aligned} H_t\colon \mathcal{A}^n(M) &\to \mathcal{A}^n(M) \\ \omega_0 &\mapsto \omega(t) \end{aligned}$$

Then

(1) $H_0 = \mathrm{id}$.
(2) $H_{t+s} = H_s \circ H_t = H_t \circ H_s$.
(3) $H_t \circ \Delta = \Delta \circ H_t$, and $H_t \circ \mathrm{d} = \mathrm{d} \circ H_t$.

Let $W_0(\mathcal{A}^n(M))$ be the L^2-complete of $\mathcal{A}^n(M)$. The H_t extends to

$$H_t\colon W_0(\mathcal{A}^n(M)) \to W_0(\mathcal{A}^n(M)).$$

Moreover, H_t is a compact, self-adjoint operator.

Theorem 1.2.2 (spectral decomposition)**.** Let $T\colon H \to H$ be a compact, self-adjoint operator on countable dimension Hilbert space. Then there is an orthogonal eigenvalue decomposition

$$H = \bigoplus_{n=0}^{\infty} \langle v_n \rangle.$$

Moreover, $Tv_n = \gamma_n v_n$ such that $\lim_{n\to\infty} \gamma_n = 0$.

Proposition 1.2.1.

2. Kähler identities and Hodge package

Let X be a compact complex n-manifold. Before Hodge, we only know a little about $\bigoplus_{k=0}^{2n} H^n(X, \mathbb{Q})$. One thing is Poincaré duality, that is,

$$H^k(X, \mathbb{Q}) \otimes H^{2n-k}(X, \mathbb{Q}) \xrightarrow{\cup} H^{2n}(X, \mathbb{Q}) \cong \mathbb{Q}$$

is a perfect pairing. But after Hodge, $\bigoplus_{k=0}^{2n} H^k(X, \mathbb{Q})$ turns out to be a rich and rigid object.

Let (X, ω) be a compact Kähler n-manifold and L be its Lefschetz operator with dual operator Λ. The Kähler identities are given by

$$\begin{aligned} [L, \Lambda] &= (2k-n)\,\mathrm{id} \quad \text{ on } \mathcal{A}^k(X) \\ [\Lambda, \partial] &= \sqrt{-1} \cdot \overline{\partial}^* \\ \left[\Lambda, \overline{\partial}\right] &= -\sqrt{-1} \cdot \partial^* \\ \Delta_{\mathrm{d}} &= 2\Delta_{\partial} = 2\Delta_{\overline{\partial}}. \end{aligned}$$

3. Kodaira vanishing

Theorem 3.1 (Kodaira-Akizuki-Nakano vanishing)**.** Let X be a non-singular projective complex variety with dimension d and L be an ample line bundle on X. Then

$$H^q(X, \Omega_X^p \otimes L) = 0$$

for all $p + q > d$.

Theorem 3.2 (Kodaira vanishing)**.** Let X be a non-singular projective complex variety and L be an ample line bundle on X. Then

$$H^q(X, K_X \otimes L) = 0$$

for all $q > 0$.

In birational geometry, the following vanishing theorem is also extremely useful.

Theorem 3.3 (Kawamata-Viehweg vanishing)**.** Let X be a non-singular projective complex variety and $D = \sum_i a_i D_i$ be an effective $\mathbb{Q}$-divisor, where $a_i \in [0, 1) \cap \frac{1}{N}\mathbb{Z}$ for some integer $N \geq 1$. For the line bundle L such that $L^{\otimes N} \otimes \mathcal{O}_X(-ND)$ is ample[4], it holds that

$$H^q(X, K_X \otimes L) = 0$$

for all $q > 0$.

3.1. **Differential geometry method.** Let (X, ω) be a compact Kähler manifold and (E, h) be a Hermitian holomorphic vector bundle over X equipped with Chern connection ∇_h. For the following operators

$$\begin{aligned}
\partial_E &\colon \mathcal{A}^{p,q}(X, E) \to \mathcal{A}^{p+1,q}(X, E) \\
\overline{\partial}_E &\colon \mathcal{A}^{p,q}(X, E) \to \mathcal{A}^{p,q+1}(X, E) \\
L &\colon \mathcal{A}^{p,q}(X, E) \to \mathcal{A}^{p+1,q+1}(X, E) \\
\Lambda &\colon \mathcal{A}^{p,q}(X, E) \to \mathcal{A}^{p-1,q-1}(X, E),
\end{aligned}$$

there are also Kähler identities

$$\begin{aligned}
[\overline{\partial}_E^*, L] &= \sqrt{-1}\partial_E \\
[\partial_E^*, L] &= -\sqrt{-1} \cdot \overline{\partial}_E \\
[\Lambda, \overline{\partial}_E] &= -\sqrt{-1}\partial_E^* \\
[\Lambda, \partial_E] &= \sqrt{-1} \cdot \overline{\partial}_E^*,
\end{aligned}$$

and

$$[L, \Lambda] = (p + q - n)\,\mathrm{id}$$

holds on E-valued (p, q)-forms.

[4]In fact, nef and big.

Proposition 3.1.1 (Bochner-Kodaira-Nakano identity)**.** Let (X,ω) be a compact Kähler manifold and (E,h) be a Hermitian holomorphic vector bundle. Then

$$\Delta_{\overline{\partial}_E} = [\sqrt{-1}\Theta_h, \Lambda] + \Delta_{\partial_E}.$$

Proof. Direct computation shows

$$\begin{aligned}\Delta_{\overline{\partial}_E} &= [\overline{\partial}_E, \overline{\partial}_E^*]\\ &= -\sqrt{-1}[\overline{\partial}_E, [\Lambda, \partial_E]]\\ &= -\sqrt{-1}[\Lambda, [\partial_E, \overline{\partial}_E]] - \sqrt{-1}[\partial_E, [\overline{\partial}_E, \Lambda]]\\ &= -\sqrt{-1}[\Lambda, \Theta_h] - \sqrt{-1}[\partial_E, \sqrt{-1}\partial_E^*]\\ &= [\sqrt{-1}\Theta_h, \Lambda] + \Delta_{\partial_E}.\end{aligned}$$

□

Corollary 3.1.1 (Bochner-Kodaira-Nakano inequality)**.** Let (X,ω) be a compact Kähler manifold and (E,h) a Hermitian holomorphic vector bundle. Then for $\alpha \in \mathcal{A}^{p,q}(X,E)$, one has

$$([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) \leq (\Delta_{\overline{\partial}_E}\alpha, \alpha)$$

In particular, if α is $\Delta_{\overline{\partial}_E}$-harmonic, then $([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) \leq 0$.

Proof. Direct computation shows

$$\begin{aligned}(\Delta_{\overline{\partial}_E}\alpha, \alpha) - ([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) &= (\Delta_{\partial_E}\alpha, \alpha)\\ &= \|\partial_E\alpha\|^2 + \|\partial_E^*\alpha\|^2 \geq 0.\end{aligned}$$

□

Theorem 3.1.1 (Kodaira-Akizuki-Nakano vanishing)**.** Let X be a compact n-manifold, (L,h) a positive Hermitian holomorphic line bundle. Then

$$H^{p,q}(X,L) = 0$$

for $p+q>n$.

Proof. Let X be endowed with the Kähler metric ω given by Chern curvature of L. Then there is an isomorphism $H^{p,q}(X,L) \cong \mathcal{H}^{p,q}(X,L)$. For $\alpha \in \mathcal{H}^{p,q}(X,L)$, by Corollary 3.1.1 one has

$$[\sqrt{-1}\Theta_h, \Lambda]\alpha \leq 0.$$

On the other hand,

$$([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) = 2\pi(p+q-n)\|\alpha\|^2 \geq 0.$$

Thus if $p+q>n$, one has $\alpha = 0$. This completes the proof. □

Corollary 3.1.2 (Kodaira vanishing)**.** Let X be a compact n-manifold and (L,h) be a positive holomorphic line bundle over X. Then

$$H^q(X, K_X \otimes L) = 0$$

for $q>0$.

3.2. **Algebraic geometry method.** An algebraic geometry proof of Kodaira's vanishing theorem uses the cyclic cover tricks and logarithmic differential forms. Here we give a brief introduction about these techniques, and a good reference is [EV92].

For convenience, X always denotes a non-singular projective complex variety of dimension d unless otherwise stated.

3.2.1. *Logarithmic differential forms.* Let D be a reduced normal crossing divisor[5] in X. The sheaf of differential k-forms with logarithmic singularities along D, denoted by $\Omega_X^k(\log D)$, is the subsheaf of $\Omega_X^k(*D)$[6] defined by the following condition:

- If α is a meromorphic differential forms on U, holomorphic on $U\setminus D\cap U$, then $\alpha\in\Omega_X^k(\log D)|_U$ if α admits a pole of order at most 1 along D, and the same holds for $\mathrm{d}\alpha$.

Lemma 3.2.1. Let $\{z_1,\dots,z_n\}$ be a local coordinate on an open subset U of X, in which $D\cap U$ is defined by the equation $z_1\dots z_r=0$. For convenience we denote

$$\delta_j=\begin{cases}\mathrm{d}z_j/z_j & j\leq r\\ \mathrm{d}z_j & j>r,\end{cases}$$

and for $I=\{j_1,\dots,j_k\}\subseteq\{1,\dots,n\}$ with $j_1<\dots<j_s$, we denote

$$\delta_I=\delta_{j_1}\wedge\dots\wedge\delta_{j_k}.$$

Then $\Omega_X^k(\log D)|_U$ is a sheaf of free $\mathcal{O}_U$-modules with basis $\{\delta_I\}_{|I|=k}$.

Proof. See Proposition 2.2 in [EV92]. □

Corollary 3.2.1.

(1) $\Omega_X^k(\log D)=\bigwedge^k\Omega_X^1(\log D)$.
(2) The sheaves $\Omega_X^k(\log D)$ are sheaves of locally free $\mathcal{O}_X$-modules.

Notations for local frames of logarithmic differential forms as Lemma 3.2.1 will be used along the way. For example, one can defined the following several maps by using local frames.

(1) The first one is

$$\alpha\colon\Omega_X^1(\log D)\to\bigoplus_{j=1}^r\mathcal{O}_{D_j}$$

which is locally defined by $\sum_{j=1}^n a_j\delta_j\mapsto\bigoplus_{j=1}^r a_j|_{D_j}$.

[5] A divisor $D=\sum_{j=1}^r D_j$ is called a reduced normal crossing divisor, if locally there exists coordinate $\{z_1,\dots,z_n\}$ on X such that D is defined by the equation $z_1\dots z_r=0$ for an integer r which naturally depends on the considered open set.

[6] $\Omega_X^k(*D)$ is the sheaf of meromorphic forms on X, holomorphic on $X\setminus D$.

(2) For $k \geq 1$, one has
$$\beta_1 \colon \Omega_X^k(\log D) \to \Omega_{D_1}^{k-1}(\log(D-D_1)|_{D_1})$$
which is given by: For local section
$$\varphi = \varphi_1 + \varphi_2 \wedge \frac{\mathrm{d}z_1}{z_1},$$
where φ_1 lies in the span of the δ_I with $1 \notin I$ and $\varphi_2 = \sum_{1\in I} a_I \delta_{I\setminus\{1\}}$, we define
$$\beta_1(\varphi) = \sum a_I \delta_{I\setminus\{1\}}|_{D_1}.$$

(3) Finally the natural restriction gives
$$\gamma_1 \colon \Omega_X^k(\log(D-D_1)) \to \Omega_{D_1}^k(\log(D-D_1)|_{D_1}).$$
Note that $\{z_1 \cdot \delta_I \mid 1 \in I\} \cup \{\delta_I \mid 1 \notin I\}$ gives a local frame of $\Omega_X^k(\log(D-D_1))$. Then γ_1 can be described as
$$\gamma_1(\sum_{1\in I} z_1 a_I \delta_I + \sum_{1\notin I} a_I \delta_I) = \sum_{1\notin I} a_I \delta_I|_{D_1}.$$

Remark 3.2.1. Similarly, β_i and γ_i are the corresponding map for the i-th component D_i.

Proposition 3.2.1. The following sequences of sheaves are exact.

(1)
$$0 \to \Omega_X^1 \to \Omega_X^1(\log D) \xrightarrow{\alpha} \bigoplus_{j=1}^r \mathcal{O}_{D_j} \to 0$$

(2)
$$0 \to \Omega_X^k(\log(D-D_1)) \to \Omega_X^k(\log D) \to \Omega_{D_1}^{k-1}(\log(D-D_1)|_{D_1}) \xrightarrow{\beta_1} 0$$

(3)
$$0 \to \Omega_X^k(\log D)(-D_1) \to \Omega_X^k(\log(D-D_1)) \to \Omega_{D_1}^k(\log(D-D_1)|_{D_1}) \xrightarrow{\gamma_1} 0$$

Proof. It follows from the definition of α, β_1 and γ_1. $\square$

Definition 3.2.1 (logarithmic connection)**.** Let $\mathcal{E}$ be a locally free coherent sheaf on X. A logarithmic connection is a $\mathbb{C}$-linear map $\nabla \colon \mathcal{E} \to \Omega_X^1(\log D) \otimes \mathcal{E}$ satisfying the Lebniz rule, that is
$$\nabla(f \cdot e) = f \cdot \nabla e + \mathrm{d}f \otimes e.$$
One defines
$$\nabla \colon \Omega_X^k(\log D) \otimes \mathcal{E} \to \Omega_X^{k+1}(\log D) \otimes \mathcal{E}$$
by the rule
$$\nabla(\omega \otimes e) = \mathrm{d}\omega \otimes e + (-1)^k \omega \wedge \nabla e.$$

Definition 3.2.2 (flat logarithmic connection)**.** A logarithmic connection ∇ is called flat[7] if it's curvature is zero, that is, $\nabla^2 = 0$.

[7]Sometimes is also called integrable.

Definition 3.2.3 (residue)**.** For a flat logarithmic connection

$$\nabla\colon \mathcal{E} \to \Omega^1_X(\log D) \otimes \mathcal{E},$$

the residue map along D_1 is defined to be the composed map

$$\mathrm{Res}_{D_1}(\nabla)\colon \mathcal{E} \xrightarrow{\nabla} \Omega^1_X(\log D) \otimes \mathcal{E} \xrightarrow{\beta_1 \otimes \mathrm{id}} \mathcal{O}_{D_1} \otimes \mathcal{E}.$$

Remark 3.2.2.

3.2.2. $\mathbb{Q}$-*divisors.*

Definition 3.2.4 ($\mathbb{Q}$-divisor)**.** Let $\mathrm{Div}(X)$ be the group of divisors on X and $\mathrm{Div}_{\mathbb{Q}}(X) = \mathrm{Div}(X) \otimes_Z \mathbb{Q}$. A $\mathbb{Q}$-divisor $\Delta \in \mathrm{Div}_{\mathbb{Q}}(X)$ is a sum

$$\Delta = \sum_{j=1}^r \alpha_j D_j$$

where D_j are irreducible divisors and $\alpha_j \in \mathbb{Q}$.

Definition 3.2.5 (integeral part)**.** For a $\mathbb{Q}$-divisor $\Delta = \sum_{j=1}^r \alpha_j D_j$, its integeral part

$$[\Delta] := \sum_{j=1}^r [a_j] D_j$$

where $[\alpha]$ denotes the integeral part of α.

Definition 3.2.6. Let $\mathcal{L}$ be an invertible sheaf, $D = \sum_{j=1}^r \alpha_j D_j$ be an effective divisor and N be a positive natural number such that $\mathcal{L}^N = \mathcal{O}_X(D)$. Then we will write

$$\mathcal{L}^{(i,D)} = \mathcal{L}^i \otimes \mathcal{O}_X(-[\frac{i}{N}D])$$

and

$$\mathcal{A} = \bigoplus_{i=0}^{N-1} \mathcal{L}^{(i,D)^{-1}}$$

Remark 3.2.3. If there is no ambiguous, then we write $\mathcal{L}^{(i)}$ for convenience.

Theorem 3.2.1. Let $\mathcal{L}$ be an invertible sheaf, $D = \sum_{j=1}^r \alpha_j D_j$ be an effective divisor and N be a positive natural number such that $\mathcal{L}^N = \mathcal{O}_X(D)$. Then for $i = 0, 1, \dots, N-1$, the sheaf $\mathcal{L}^{(i)^{-1}}$ has a flat logarithmic connection

$$\nabla^{(i)}\colon \mathcal{L}^{(i)^{-1}} \to \Omega^1_X(\log D^{(i)}) \otimes \mathcal{L}^{(i)^{-1}}$$

where $D^{(i)} = \sum_{j=1, \frac{i\alpha_j}{N} \notin \mathbb{Z}}^r D_j$, and $\nabla^{(i)}$ satisfies

(1) The residue of $\nabla^{(i)}$ along D_j is given by multiplication with

$$(i \cdot \alpha_j - N \cdot [\frac{i \cdot \alpha_j}{N}]) \cdot N^{-1}$$

(2)
(3)

3.2.3. *Cyclic coverings.* Let $\mathcal{L}$ be an invertible sheaf, $D = \sum_{j=1}^{r} \alpha_j D_j$ be an effective divisor and N be a positive natural number such that $\mathcal{L}^N = \mathcal{O}_X(D)$. Let $s \in H^0(X, \mathcal{L}^N)$ be a section whose zero divisor is D. Then the dual of $s\colon \mathcal{O}_X \to \mathcal{L}^N$ gives a $\mathcal{O}_X$-algebra structure on

$$\mathcal{A}' = \bigoplus_{i=0}^{N-1} \mathcal{L}^{-i}$$

Let $Y' = \operatorname{Spec} \mathcal{A}' \to X$ be the spectrum of the $\mathcal{O}_X$-algebra $\mathcal{A}'$ and $\pi\colon Y \to X$ be the finite morphism obtained by normalizing $Y' \to X$. Then Y is called cyclic covering obtained by taking n-th root out of D.

The inclusion

$$\mathcal{L}^{-i} \to \mathcal{L}^{(i)^{-1}} = \mathcal{L}^{-i} \otimes \mathcal{O}_X([\frac{i}{N} D])$$

gives a morphism of $\mathcal{O}_X$-module $\phi\colon \mathcal{A}' \to \mathcal{A}$.

Lemma 3.2.2. $\mathcal{A}$ has a structure of $\mathcal{O}_X$-algebra such that ϕ is a homomorphism of algebras.

Now let ξ_N be a fixed primitive N-th root of unit and $G = \langle \sigma \rangle$ be the cyclic group of order N. Then G acts on $\mathcal{A}$ by $\mathcal{O}_X$-algebra homomorphisms by $\sigma(l) = \xi_N^i \cdot l$, where l is a local section of $\mathcal{L}^{(i)^{-1}}$, and it's clear the invariants under this G-action are $\mathcal{A}^G = \mathcal{O}_X$.

Proposition 3.2.2. The cyclic group G acts on Y and on $\pi_* \mathcal{O}_Y$. One has $Y/G = X$ and the decomposition

$$\pi_* \mathcal{O}_Y = \bigoplus_{i=0}^{N-1} \mathcal{L}^{(i)^{-1}}$$

is the decomposition in eigenspaces.

Proposition 3.2.3. $\operatorname{Spec} \mathcal{A} \to X$ is finite and $\operatorname{Spec} \mathcal{A}$ is normal.

3.2.4. *Cyclic cover.* Now suppose X is a non-singular projective complex variety with dimension d and L is an ample line bundle. Take $N \gg 1$ large sufficiently such that there exists $s \in \Gamma(X, L^{\otimes N})$, and the divisor $D = \operatorname{div}(s)$ is non-singular. Then it gives a N-cyclic cover $\pi\colon Y \to X$.

Proposition 3.2.4. Notations as above. Then
(1) $\pi^* \Omega_X^n(\log D) = \Omega_Y^n(\log \pi^* D)$.
(2) For each $1 \le i \le N-1$, $\mathrm{d}\colon \mathcal{O}_Y \to \Omega_Y$ induces a flat log connection

$$\nabla^i\colon L^{-i} \to L^{-i} \otimes \Omega_X^1(\log D)$$

such that

$$\bigoplus_{i=1}^{N-1} \nabla^i = \pi_* \mathrm{d}\colon \pi_* \mathcal{O}_Y \to \pi_*(\Omega_Y^1(\log \pi^* D))$$

is an eigen-decomposition with respect to the action of cyclic group $\mathbb{Z}/N\mathbb{Z}$.

(3) $\pi_*\Omega_Y^n = \Omega_X^n \oplus \bigoplus_{i=1}^{N-1} \Omega_X^n(\log D)\otimes L^{-i}$.

Proposition 3.2.5. The Hodge to de Rham spectral sequence associated to the log de Rham complex for $1\leq i\leq N-1$

$$L^{-i}\xrightarrow{\nabla^i} L^{-i}\otimes\Omega_X^1(\log D)\xrightarrow{\nabla^i} L^{-i}\otimes\Omega_X^2(\log D)\to\dots$$

degenerates at E_1-page.

3.2.5. *An Algebraic proof of Kodaira's vanishing theorem.* In this section we present an algebraic proof of Kodaira's vanishing theorem by Serre vanishing, E_1-degeneration and techniques introduced above.

Proof. Consider the de Rham complex

$$\mathcal{O}_Y\xrightarrow{\mathrm{d}}\Omega_Y^1\xrightarrow{\mathrm{d}}\Omega_Y^2\xrightarrow{\mathrm{d}}\cdots\xrightarrow{\mathrm{d}}\Omega_Y^d.$$

The E_1-degeneration implies

$$E_1^{p,q}=H^q(Y,\Omega_Y^p)\xrightarrow{\mathrm{d}=0}H^q(Y,\Omega_Y^{p+1})=E_1^{p+1,q}.$$

Then

$$\pi_*\mathcal{O}_Y\xrightarrow{\pi_*\mathrm{d}}\pi_*\Omega_Y^1\xrightarrow{\pi_*\mathrm{d}}\pi_*\Omega_Y^2\xrightarrow{\pi_*\mathrm{d}}\cdots\xrightarrow{\pi_*\mathrm{d}}\pi_*\Omega_Y^d$$

also has E_1-degeneration since π is a finite morphism.

By Proposition 3.2.5 one has

$$E_1^{0,n}=H^n(X,L^{-1})\xrightarrow{\nabla_1}H^n(X,L^{-1}\otimes\Omega_X(\log D))=E_1^{1,n}$$

is a zero map. Hence

$$\begin{tikzcd}
H^n(X,L^{-1}) \arrow[rr, "0"] \arrow[dr, "\nabla_1"'] & & H^n(D,L^{-1}|_D) \\
& H^n(X,L^{-1}\otimes\Omega_X(\log D)) \arrow[ur, "\mathrm{Res}"'] &
\end{tikzcd}$$

On the other hand,

$$L^{-1}\xrightarrow{\nabla_1}L^{-1}\otimes\Omega_X(\log D)\xrightarrow{\mathrm{Res}}L^{-1}|_D$$

is just the restriction map up to the factor $1/N$. Thus the long exact sequence of

$$0\to L^{-1}\otimes\mathcal{O}_X(-D)\to L^{-1}\to L^{-1}|_D\to 0$$

reads

$$\cdots\to H^n(X,L^{-1}\otimes\mathcal{O}_X(-D))\to H^n(X,L^{-1})\xrightarrow{0}H^n(D,L^{-1}|_D)\to\cdots.$$

Note that $L^{-1}\otimes\mathcal{O}_X(-D)=L^{-N-1}$ and by Serre vanishing theorem

$$H^n(X,L^{-N-1})=0$$

for $n<\dim X$ and $N\gg 1$. Therefore

$$H^n(X,L^{-1})=0$$

for $n<\dim X$ as desired. $\square$

4. Cartier descent theorem

Let k be an algebraically closed field with $\mathrm{char} k = p > 0$ and $F_k \colon k \to k$ be the Frobenius map, that is, $x \mapsto x^p$. Suppose X is a smooth variety over k and X' is the base change of X given by the Frobenius map, that is, there is the following commucative diagram

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} k \arrow[r, "F_k"] & \operatorname{Spec} k \end{tikzcd}$$

On the other hand, the Frobenius map $F_k \colon k \to k$ induces a homomorphism $F_X \colon X \to X$, called the absolute Frobenius morphism, which also satisfies the universal property of of fiber product. Thus it induces a morphism $F_{X/k} \colon X \to X'$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[dr, dotted] \arrow[drr, "F_X"] \arrow[ddr, "\alpha"'] & & \\ & X' \arrow[r, "\pi"'] \arrow[d, "\alpha'"] & X \arrow[d, "\alpha"] \\ & \operatorname{Spec} k \arrow[r, "F_k"] & \operatorname{Spec} k \end{tikzcd}$$

The morphism $F_{X/k} \colon X \to X'$ is called the relative Frobenius morphism.

Definition 4.1 (k-connection)**.** A k-connection on X/k is a pair (V, ∇), where

(1) V is a (quasi)-coherent $\mathcal{O}_X$-module.
(2) $\nabla \colon V \to V \otimes \Omega_{X/k}$ is k-linear such that

$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s$$

Remark 4.1*.* There is another viewpoint to understand the k-connection (V, ∇) on X/k, that is, think it as a "quasi-representation" as follows

$$\nabla \colon T_{X/k} \to \mathrm{End}_k(V).$$

Over $T_{X/k}$, there is the following Lie bracket

$$[\text{-},\text{-}] \colon T_{X/k} \times T_{X/k} \to T_{X/k},$$

which is given by $[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$. The curvature is given by the following difference

$$\begin{aligned} \Theta_\nabla \colon \bigwedge^2 T_{X/k} &\to \mathrm{End}_k(V) \\ D_1 \wedge D_2 &\mapsto [D_1, D_2] - \nabla_{[D_1, D_2]}, \end{aligned}$$

which measure the failure of ∇ to be a Lie algebra representation. Moreover, one can show that $\Theta_\nabla \colon \bigwedge^2 T_{X/k} \to \mathrm{End}_{\mathcal{O}_X}(V)$.

In the case of char$k = p > 0$, there is so called p-curvature which doesn't appear in the case char$k = 0$. Firstly, the p-th power map $D \mapsto D^p := \underbrace{D \circ \cdots \circ D}_{p \text{ times}}$ gives a map between $T_{X/k} \to T_{X/k}$ since char$k = p$. Then the p-curvature of a k-connection (V, ∇) over X/k is given by

$$\begin{aligned}\Psi_\nabla \colon T_{X/k} &\mapsto \mathrm{End}_k(V)\\ D &\mapsto (\nabla_D)^p - \nabla_{D^p}.\end{aligned}$$

Proposition 4.1. For any $D \in T_{X/k}$, $\Psi_\nabla(D)$ is $\mathcal{O}_X$-linear, that is,

$$\Psi_\nabla \colon T_{X/k} \to \mathrm{End}_{\mathcal{O}_X}(V).$$

Proof. For any $f \in \mathcal{O}_X$ and $s \in V$, one has

$$\begin{aligned}(\nabla_D)^p(fs) &= \sum_{i=0}^{p} \binom{p}{i} D^i(f)(\nabla_D)^{p-i}(s)\\ &= D^p(f)s + f\nabla_D^p(s).\end{aligned}$$

On the other hand, it's clear

$$\nabla_{D^p}(fs) = \nabla^p(f)s + f\nabla_{D^p}(s).$$

Thus it follows

$$\Psi_\nabla(D)(fs) = f((\nabla_D)^p - \nabla_{D^p})(s) = f\Psi_\nabla(D)(s).$$

$\square$

Proposition 4.2. Let (V, ∇) be a k-conection over X/k. If the curvature Θ_∇ vanishes, then

(1) $\Psi_\nabla \colon T_{X/k} \to \mathrm{End}_{\mathcal{O}_X}(V)$ is additive.

(2) Ψ_∇ is F_X-linear, that is,

$$\Psi_\nabla(fD) = f^p\Psi_\nabla(D).$$

(3) Ψ_∇ is integrable, that is, $\Psi_\nabla \wedge \Psi_\nabla = 0$.

In this not an easy result, whose proof relies on the following difficult algebra result.

Lemma 4.1. Let R be an associated ring with char$R = p > 0$. For $a, b \in R$,

(1) $(a+b)^p = a^p + b^p + \sum_{i=1}^{p-1} s_i(a,b)$, where

$$(\mathrm{ad}(ta+b))^p(a) = \sum_{i=0}^{p-1} is_i(a,b)t^i.$$

(2) If $\{a^{(n)}\}_{n\geq 1}$ are mutually commutative, then

$$(ab)^p = a^pb^p + a(a^{p-1})^{(p-1)}b,$$

where

$$a^{(n)} := (\mathrm{ad}\, b)^n(a)$$

Proof. See $\square$

Now let's begin the proof of Proposition 4.2.

Proof of 4.2. For (1). For arbitary $D_1, D_2 \in T_{X/k}$, by using (1) of Lemma 4.1 one has

$$\begin{aligned}(\nabla_{D_1+D_2})^p &= (\nabla_{D_1} + \nabla_{D_2})^p \\ &= (\nabla_{D_1})^p + (\nabla_{D_2})^p + \sum_i s_i(D_1, D_2) \\ \nabla_{(D_1+D_2)^p} &= \nabla_{(\nabla_1^p + \nabla_2^p + \sum_i s_i(D_1,D_2))} \\ &= \nabla_{D_1^p} + \nabla_{D_2^p} + \sum_i \nabla_{s_i(D_1,D_2)}.\end{aligned}$$

Then

$$\Psi_\nabla(D_1+D_2) = (\nabla_{D_1+D_2})^p - \nabla_{(D_1+D_2)^p} = \Psi_\nabla(D_1) + \Psi_\nabla(D_2).$$

For (2). For arbitary $f \in \mathcal{O}_X$ and $D \in T_{X/k}$, by using (2) of Lemma 4.1, one has

$$\begin{aligned}(fD)^p &= f^pD^p + f(\mathrm{ad}(D))^{p-1}(f^{p-1})D \\ &= f^pD^p + f(D^{p-1}(f^{p-1}))D,\end{aligned}$$

since $\mathrm{ad}(D)(f^{p-1}) = D \circ f^{p-1} - f^{p-1}D = D(f^{p-1})$. Thus

$$\nabla_{(fD)^p} = f^p\nabla_{D^p} + f(D^{p-1}(f^{p-1}))\nabla_D.$$

Applying (2) of Lemma 4.1 again, one has

$$\begin{aligned}(\nabla_{fD})^p &= (f\nabla_D)^p \\ &= f^p(\nabla_D)^p + f(\mathrm{ad}(\nabla_D))^{p-1}(f^{p-1})\nabla_D \\ &= f^p(\nabla_D)^p + f(D^{p-1}(f^{p-1}))\nabla_D.\end{aligned}$$

This completes the proof of (2).

For (3) □

Remark 4.2. In other words,

$$\Psi_D \colon V \to V \otimes F^*_{X/k}\Omega_X$$

is a $\mathcal{O}_X$-linear morphism.

Holding notations as above, now we can state the main theorem of this section, which is a very basic theorem in geometry over field k with characteristic p.

Theorem 4.1 (Cartier)**.** There is a natural equivalent of categories between category of (quasi)-coherent $\mathcal{O}_X$-module and the category of flat k-connections (V, ∇) with vanishing p-curvatures. More explicitly, the correspondence is given by

(1) For (quasi)-coherent $\mathcal{O}_X$-module E, the flat k-connection $(F^*E, \nabla_{\mathrm{can}})$ is given by

$$\nabla_{\mathrm{can}}(e \otimes f) = \mathrm{d}f \otimes e.$$

(2) For flat k-conenction (V, ∇) with vanishing p-curvature, the corresponding (quasi)-coherent $\mathcal{O}_X$-module is the $\mathcal{O}_X$-submodule $V^{\nabla=0} \subseteq V$.

5. de Rham decomposition theorem of Deligne-Illusie

Let k be an algebraically closed field with char$k = p > 0$ and X be a non-singular variety over k with relative Frobenius morphism $F\colon X \to X'$.

$$F_*\Omega^\bullet_{X/k}\colon F\mathcal{O}_X \to F_*\Omega_{X/k} \to F_*\Omega^2_{X/k} \to \dots$$

Theorem 5.1 (Deligne-Illusie)**.** Let X/k be a non-singular variety such that X/k is $W_2(k)$-liftable and $\dim_k X < p$. Then

$$(F_*\Omega^\bullet_{X/k}, F_*\mathrm{d}) \cong \bigoplus_{i=0}^{d} \Omega^i_{X/k}[-i]$$

is a quasi-isomorphism.

Remark 5.1.

(1) The Witt ring $W_2(k)$ is the set $\{(a_0, a_1) \mid a_0, a_1 \in k\}$ equipped with the following operations

$$(a_0, a_1) + (b_0, b_1) = (a_0 + b_0, a_1 + b_1 - \sum_{i=1}^{p-1} \frac{\binom{p}{t}}{p} a_0^i b_0^{p-i})$$
$$(a_0, a_1)(b_0, b_1) = (a_0 a_1, b_0 a_1^p + b_1 a_0^p)$$

In particular, if $k = \mathbb{F}_p$, then $W_2(k) = \mathbb{Z}/p^2\,\mathbb{Z}$.

(2) $(p) = \{(0, a_1) \mid a_i \in k\}$ is a maximal ideal of $W_2(k)$, and the following exact sequence

$$0 \to pW_2(k) \to W_2(k) \to W_2(k)/p \to 0$$

(3) In fact, the operations on $W_2(k)$ makes the ghost polynomial $\Phi(a_0, a_1) = a_0^p + pa_1$ a ring homomorphism.

(4) X/k is $W_2(k)$-liftable, if there exists a flat morphism $\widetilde{X} \to W_2(k)$ such that $X \cong \widetilde{X}$ as follows

$$\begin{tikzcd}
X \arrow[r, "\cong"] \arrow[dr] & \widetilde{X} \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] \\
& \operatorname{Spec} k \arrow[r] & \operatorname{Spec} W_2(k)
\end{tikzcd}$$

There exists $\mathrm{ob}(\alpha) \in H^2(X, T_{X/k})$ such that $\mathrm{ob}(\alpha) = 0$ if and only if X is $W_2(k)$-liftable.

(5) The condition of $W_2(k)$-liftable cannot be removed. The first counterexample is given by Michel Raynaud.

(6) $\dim X = p$ is also ok, but $\dim X > p$ is not ok. See A. Petrov.

(7) Before the proof of Deligne-Illusie, there are many evidences, such as Cartier descent theorem. Let X/k be a non-singular variety. Then

$$H^i(F_*\Omega_{X/k}) \cong \Omega^i_{X'/k}$$

Or

$$\bigoplus_{i=0}^{d} H^i(F_*\Omega^\bullet_{X/k}) \cong \bigoplus_{i=0}^{d} \Omega^i_{X'/k}$$

such that

$$C_0^{-1}\colon \mathcal{O}'_X \to H^0(F_*\Omega^\bullet_{X/k}) = \ker\{F_*\mathrm{d}\colon F_*\mathcal{O} \to F_*\Omega_{X/k}\} \subseteq F_*\mathcal{O}_X$$

$$\begin{tikzcd} \Omega^1_{X'} \arrow[r] & H^1 \\ \mathcal{O}_{X'} \arrow[u] \arrow[ur] & \end{tikzcd}$$

Sending x to $x^{p-1}\mathrm{d}x$. To show

$$(x+y)^{p-1}\mathrm{d}(x+y) = x^{p-1}\mathrm{d}x + y^{p-1}\mathrm{d}y \pmod{B^1}$$
$$(xy)^{p-1}\mathrm{d}(xy) = (x^{p-1}\mathrm{d}x)y + (y^{p-1}\mathrm{d}y)x$$

6. Holomorphic foliations on schemes

References

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