YANG-MILLS EQUATIONS ON RIEMANN SURFACE

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Abstract.

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0.1. **About this lecture.**

Part 1. Yang-Mills equations on Riemann surface

1. Moment map in Yang-Mills theory

In first lecture, we have already established the foundations of Yang-Mills equations in a general stage, or in other words, in the stage of Riemannian manifold (M, g).

As we have seen, when the dimension of underlying space is one, all curvature forms are trivial, so there is nothing interesting. Thus the first "non-trivial" theory arises when our underlying space is of dimension two.

This prototype theory merits a good deal of study due to the richness of structures naturally occurring on such manifold, such as a complex structure associated to the almost complex structure determined by the Hodge star operator $* : \Omega_M^p \to \Omega_M^{2-p}$. Furthermore, smooth hermitian vector bundle E over Riemann surface have inherent holomorphic structures due to the vacuous integrability conditions on connections on E, in other words, this gives a correspondence between unitary connections and holomorphic structure $\overline{\partial}_E$ on E. Thus the study of Yang-Mills connections on Riemann surface can be put into a complex analytic framework.

Using such ideal, we give a description of Kempf-Ness theorem which relates symplectic quotient and GIT quotient. In this section, if the underlying space is a Riemann surface, we will see there is a parallel story for the action of gauge group $\mathscr{G}$ on the space of connections $\mathscr{A}(P)$. We will complexify the action of $\mathscr{G}$ and state a theorem analogous to Kempf-Ness theorem, which is known as Narasimhan-Seshadri theorem.

1.1. The moment map.

Let M be a Riemann surface, and P is a principal G-bundle over M, the space of connections $\mathscr{A}(P)$ has a natural symplectic form.

Proposition 1.1 (Atiyah-Bott)**.** The following bilinear form

$$Q(\alpha, \beta) = \int_M \alpha \wedge \beta$$

where $\alpha, \beta \in C^\infty(M, \Omega_M^1(\operatorname{ad} P))$, is a symplectic form defined on $\mathscr{A}(P)$.

Proof. Let's check step by step:

1. It's clear that Q is a 2-form, since $\mathscr{A}(P)$ is affine modelled on $C^\infty(M, \Omega_M^1(\operatorname{ad} P))$.
2.
3.

□

Remark 1.1.1. Note that this integral do makes sense since the real dimension of M is two.

Lemma 1.1.1. For $\phi \in C^\infty(M, \Omega_M^0(\operatorname{ad} P))$, $\nabla\phi$ is the Hamiltonian vector field of function $f \colon \nabla \to -\int_M F_\nabla \wedge \phi$.

Proof. By definition we need to check

$$\mathrm{d}f = \iota_{\nabla\phi} Q$$

Take arbitrary $\tau \in C^\infty(M, \Omega^1_M(\operatorname{ad} P))$, integration by parts shows

$$\begin{aligned} Q(\nabla\phi, \tau) &= \int_M \nabla\phi \wedge \tau \\ &= -\int_M \phi \wedge \nabla\tau \\ &= -\int_M \nabla\tau \wedge \phi \end{aligned}$$

Note that $F_{\nabla+\varepsilon\tau} = F_\nabla + \varepsilon\nabla\tau + O(\varepsilon^2)$, then

$$\begin{aligned} \mathrm{d}f(\tau) &= \lim_{\varepsilon\to 0} \frac{-\int_M F_{\nabla+\varepsilon\tau}\wedge\phi + \int_M F_\nabla \wedge \phi}{\varepsilon} \\ &= -\int_M \nabla\tau\wedge\phi \end{aligned}$$

This completes the proof. □

Remark 1.1.2. In our case the $(\operatorname{Lie}\mathscr{G})^* = C^\infty(M, \Omega^2_M(\operatorname{ad} P))$ and the moment map is just

$$\nabla \mapsto -F_\nabla$$

The Yang-Mills functional is just the norm of the moment map.

1.2. **Complexifying the action of gauge group.** Let M be a Riemann surface, our ultimate goal is to relate moduli spaces of holomorphic vector bundles over M to Yang-Mills connections. Firstly, we want to consider $\mathscr{A}(P)$ as a space of holomorphic vector bundles.

Definition 1.2.1 (holomorphic vector bundle)**.** A holomorphic vector bundle is a complex vector bundle $\pi\colon E \to X$ such that the total space E is a complex manifold and π is holomorphic.

Proposition 1.2. If P is a principal $\mathrm{U}(n)$-bundle over a Riemann surface M and ∇ is a $\mathrm{U}(n)$-connection then $\operatorname{ad}(P)$ inherits the structure of a holomorphic vector bundle over M such that $\nabla^{0,1} = \overline{\partial}$.

2. Stability of holomorphic vector bundles

In thise section, the guiding problem is to classify holomorphic vector bundles on a Riemann surface with genus g, denoted by Σ_g. For the case $g = 0, 1$, there are complete classification results for holomorphic vector bundles on Σ_g, due to Grothendieck for the case of the Riemann sphere [Gro57], and due to Atiyah for the case of elliptic curves [Ati57]. So in the following discussion, we always assume $g \geq 2$.

2.1. **Stable bundle.**

Definition 2.1.1 (degree)**.** Let $\pi\colon E \to X$ be a holomorphic vector bundle, its degree is defined as

$$\deg(E) := \int_X c_1(E)$$

where $c_1(E) \in H^2(X, \mathbb{Z})$ is the first Chern class of E.

Definition 2.1.2 (slope)**.** Let $\pi\colon E \to X$ be a holomorphic vector bundle, its slope is defined as

$$\mu(E) := \frac{\deg(E)}{\operatorname{rank}(E)}$$

Remark 2.1.1. One thing to note is that the slope of a holomorphic vector bundle is independent of the holomorphic structure, since both the degree and rank are topological invariants.

Definition 2.1.3 (slope stability)**.** Let $\pi\colon E \to X$ be a holomorphic vector bundle, it's

1. stable if for every non-trivial holomorphic subbundle F, $\mu(F) < \mu(E)$;
2. semi-stable if for every non-trivial holomorphic subbundle F, $\mu(F) \leq \mu(E)$;
3. unstable if it's not semi-stable.

Remark 2.1.2. For slope stability, we have the following remarks:

(a) It's clear that all holomorphic line bundles are stable, since they don't have non-trivial subbundles;
(b) A semi-stable vector bundle with coprime rank and degree is actually stable, since
(c) While the slope is a topological invariant, slope stability is not, since here we only consider holomorphic subbundles, which depends on the holomorphic structure.

Proposition 2.1.1. Let $E \to \Sigma_g$ be a holomorphic vector bundle, it's

1. stable if and only if for every non-trivial holomorphic subbundle F, $\mu(E/F) > \mu(E)$;
2. semi-stable if and only if for every non-trivial holomorphic subbundle F, $\mu(E/F) \geq \mu(E)$.

Proof. Denote r, r', r'' the ranks of $E, F, E/F$ respectively, and d, d', d'' their degrees respectively. From exact sequence

$$0 \to E \to E \to E/F \to 0$$

one has $r = r' + r''$ and $d = d' + d''$, thus

$$\frac{d'}{r'} < \frac{d'+d''}{r'+r''} \iff \frac{d'}{r'} < \frac{d''}{r''} \iff \frac{d'+d''}{r'+r''} < \frac{d''}{r''}$$

and likewise with the case semi-stable. □

A philosophy is that semi-stable bundles don't admit too many subbundles, since any subbundle they may have is of slope no greater than their own. This turns out to have many interesting consequences we're going to show, for example, the category of semi-stable bundles is abelian.

Lemma 2.1.1. If $\varphi : E \to E'$ is a non-zero homomorphism of holomorphic vector bundles over Σ_g, then

$$\mu(E/\ker\varphi) \leq \mu(\operatorname{im}\varphi)$$

Proposition 2.1.2. Let E, E' be two semi-stable bundles such that $\mu(E) > \mu(E')$, then any homomorphism $\varphi : E \to E'$ is zero.

Proof. If φ is non-zero, since E is semi-stable, then

$$\mu(\operatorname{im}\varphi) \overset{(1)}{\geq} \mu(E/\ker\varphi) \overset{(2)}{\geq} \mu(E) > \mu(E')$$

where

(1) holds from Lemma 2.1.1;
(2) holds from Proposition 2.1.1.

which contradicts to the semi-stablity of E'. □

Proposition 2.1.3. Let $\varphi : E \to E'$ be a non-zero homomorphism of semi-stable holomorphic of slope μ, then $\ker\varphi$ and $\operatorname{im}\varphi$ are semi-stable bundles of slope μ, and the natural map $E/\ker\varphi \to \operatorname{im}\varphi$ is an isomorphism.

Corollary 2.1.1. The category of semi-stable bundles of slope μ is abelian, and the simple object[1] in this category is the stable bundles of slope μ.

Proof. By Proposition 2.1.3 one has the category of semi-stable bundles of slope μ is abelian. A stable bundle E is simple in this category, since it admits no non-trivial subbundles with slope μ; Conversely, if a semi-stable bundle E is simple, then any non-trivial subbundle F satisfies $\mu(F) \leq \mu(E)$ since E is semi-stable and $\mu(F) \neq \mu(E)$ since E is simple, this shows E is stable. □

Proposition 2.1.4. Let E, E' be two stable vector bundles over Σ_g with same slopes, and $\varphi : E \to E'$ be a non-zero homomorphism, then φ is an isomorphism.

[1]Recall a simple object in an abelian category is an object with no non-trivial subobject.

Proof. Since $\varphi : E \to E'$ is a non-zero homomorphism between stable bundles with same slopes, then by Proposition 2.1.3 one has $\ker \varphi$ is either 0 or has slope $\mu(E)$, but E is actually stable, then $\ker \varphi$ must be 0, and since φ is strict, this shows φ is injective. Likewise, $\operatorname{im} \varphi \neq 0$ and has slope $\mu(E')$, then it must be E' since E' is stable. Then again by φ is strict, $\operatorname{im} \varphi = E'$ impiles φ is surjective. Therefore φ is an isomorphism. $\square$

Proposition 2.1.5. If E is a stable bundle over Σ_g, then $\operatorname{End} E = \mathbb{C}$. In particular, $\operatorname{Aut} E = \mathbb{C}^*$.

Proof. Let φ be a non-zero endomorphism of E, by Proposition 2.1.4 one has φ is an automorphism, so $\operatorname{End} E$ is a field, which contains $\mathbb{C}$ as its subfield of scalar endomorphisms. For any $\varphi \in \operatorname{End} E$, by Cayley-Hamilton theorem one has φ is algebraic over $\mathbb{C}$, and since $\mathbb{C}$ is algebraic closed, this shows $\operatorname{End} E \cong \mathbb{C}$. $\square$

Corollary 2.1.2. A stable bundle is indecomposable, that is it's not isomorphic to a direct sum of non-trivial subbundles.

Proof. The automorphism group of $E = E_1 \oplus E_2$ contains $\mathbb{C}^* \times \mathbb{C}^*$, so by Proposition 2.1.5 it can't be stable. $\square$

Theorem 2.1.1 (Jordan-Hölder filteration)**.** Any semi-stable bundle of slope μ over Σ_g admits a filteration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_k = E$$

by holomorphic subbundles such that for each $1 \leq i \leq k$, one has
1. E_i/E_{i-1} is stable;
2. $\mu(E_i/E_{i-1}) = \mu(E)$.

Proposition 2.1.6 (Seshadri)**.** Any two Jordan-Hölder filterations

$$S : 0 = E_0 \subset E_1 \subset \cdots \subset E_k = E$$

and

$$S' : 0 = E'_0 \subset E'_1 \subset \cdots \subset E'_l = E$$

of a semi-stable bundle E have same length, and the associated graded objects

$$\operatorname{gr}(S) : 0 = E_1/E_0 \oplus \cdots \oplus E_k/E_{k-1}$$

and

$$\operatorname{gr}(S') : 0 = E'_1/E'_0 \oplus \cdots \oplus E'_k/E'_{k-1}$$

satisfy $E_i/E_{i-1} \cong E'_i/E'_{i-1}$ for all $1 \leq i \leq k$.

Definition 2.1.4 (poly-stable bundle)**.** A holomorphic vector bundle E over Σ_g is called poly-stable if it is isomorphic to a direct sum $E_1 \oplus \cdots \oplus E_k$ of stable bundles of the same slope.

Example 2.1.1. A stable bundle is poly-stable.

Example 2.1.2. The graded object associated to any Jordan-Hölder filteration of a semi-stable bundle E is a poly-stable, and by Proposition 2.1.6, it's unique up to isomorphism, this isomorphic class is denoted by $\mathrm{gr}(E)$.

Definition 2.1.5 (S-equivalence class)**.** The graded isomorphism class $\mathrm{gr}(E)$ associated to a semi-stable bundle E is called the S-equivalence class of E. If $\mathrm{gr}(E) \cong \mathrm{gr}(E')$, E and E' are called S-equivalent, and denoted by $E \sim_S E'$.

Definition 2.1.6. The set $\mathcal{M}_{\Sigma_g}(r, d)$ of S-equivalence classes of semi-stable bundles of rank r and degree d over Σ_g is called its moduli set, it contains the set $\mathcal{N}_{\Sigma_g}(r, d)$ of isomorphism classes of stable bundles of rank r and degree d.

Theorem 2.1.2 (Mumford-Seshadri)**.** Let $g \geq 2, r \geq 1$ and $d \in \mathbb{Z}$.

1. The set $\mathcal{N}_{\Sigma_g}(r, d)$ admits a structure of smooth, complex quasi-projective variety of dimension $r^2(g-1)+1$;
2. The set $\mathcal{M}_{\Sigma_g}(r, d)$ admits a structure of complex projective variety of dimension $r^2(g-1)+1$;
3. $\mathcal{N}_{\Sigma_g}(r, d)$ is an open dense subvariety of $\mathcal{M}_{\Sigma_g}(r, d)$.

In particular, when r and d are coprime, $\mathcal{M}_{\Sigma_g}(r, d) = \mathcal{N}_{\Sigma_g}(r, d)$ is a smooth complex projective variety.

Proof. See [Mum62] and [Ses67]. □

2.2. The Harder-Narasimhan filteration.

Theorem 2.2.1 (Harder-Narasimhan)**.** Any holomorphic vector bundle E over Σ_g has a unique filteration

$$0 = E_0 \subset E_1 \subset \dots E_k = E$$

by holomorphic subbundles such that

1. for all $1 \leq i \leq k$, E_i/E_{i-1} is semi-stable;
2. the slope $\mu_i := \mu(E_i/E_{i-1})$ of successive quotients satisfies

$$\mu_1 > \mu_2 > \cdots > \mu_k$$

This filteration is called Harder-Narasimhan filteration.

Proof. See [HN75]. □

Remark 2.2.1. If we denote $r = \operatorname{rank} E, d = \deg E, r_i = \operatorname{rank}(E_i/E_{i-1})$ and $d_i = \deg(E_i/E_{i-1})$, one has

$$r_1 + \cdots + r_k = r, \quad d_1 + \cdots + d_k = d$$

The k-tuple

$$\vec{\mu} := (\underbrace{\mu_1, \dots, \mu_1}_{r_1 \text{ times}}, \dots, \underbrace{\mu_k, \dots, \mu_k}_{r_k \text{ times}})$$

is called the Harder-Narasimhan type of E. It's equivalent to the data of the k-tuple $(r_i, d_i)_{1\leq i\leq k}$. In the plane of coordinates (r, d), the polygonal line

$$P_{\vec{\mu}} := \{(0,0), (r_1, d_1), (r_1 + r_2, d_1 + d_2), \ldots, (r_1 + \cdots + r_k, d_1 + \cdots + d_k)\}$$

defines a convex polygon called the Harder-Narasimhan polygon of E. The slope of the line from $(0,0)$ to (r_1, d_1) is μ_1, that is the slope of E_1/E_0, and perhaps that's why it's called slope. It's indeed convex, since $\mu_1 > \cdots > \mu_k$. A vector bundle is semi-stable if and only if it is it own Harder-Narasimhan filteration, and if and only if its Harder-Narasimhan filteration is a single line from $(0,0)$ to (r, d).

3. Narasimhan-Seshadri theorem

4. G-equivariant cohomology

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