

# RIEMANN SURFACE

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## 1. RIEMANN SURFACE

## 1.1. Definitions and Examples.

**Definition 1.1.1.** If  $X$  is a surface, a (almost) complex structure is a smooth map  $J : TX \rightarrow TX$ , such that for any  $p \in X$ ,  $J_p : T_p X \rightarrow T_p X$  is a linear map with  $J_p^2 = -\text{id}$ .

**Remark 1.1.2.** If  $X$  admits a complex structure, then  $X$  is orientable.

**Example 1.1.3.** Assume  $X$  has a Riemann metric, and  $X$  is orientable. For any  $v \in T_p X$ , define  $J(v)$  to be the tangent vector obtained by rotating  $v$  by  $\pi/2$  counterclockwise.

**Corollary 1.1.4.** Any orientable surface admits a complex structure.

**Example 1.1.5.** If  $X = \mathbb{C}$ , then  $T_q X \cong \mathbb{C}, \forall q \in X$ , choose  $v \in T_q X$ , define  $J(v) = iv$ , then  $J$  is a complex structure on  $X$ .

**Definition 1.1.6.** Assume  $X$  is a topological space. A complex chart on  $X$  is an open subset  $U \subset X$  together with a homeomorphism  $\varphi : U \rightarrow V \subset \mathbb{C}$ , where  $V$  is an open subset. If  $p \in U$ , and  $\varphi(p) = 0$ , then  $(U, \varphi)$  is called a chart centered at  $p$ . For  $q \in U$ ,  $z = \varphi(q)$  is called a local coordinate of  $q$ .

**Definition 1.1.7.** If  $(U_1, \varphi_1), (U_2, \varphi_2)$  are two charts on  $X$ , we say they're compatible if  $U_1 \cap U_2 = \emptyset$  or

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2)$$

is holomorphic.

**Definition 1.1.8.** An atlas is a collection of compatible charts  $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ , such that  $\bigcup_{\alpha \in I} U_\alpha = X$ . Two atlas  $\mathcal{A}, \mathcal{B}$  are equivalent if every chart in  $\mathcal{A}$  and every chart in  $\mathcal{B}$  is compatible.

**Definition 1.1.9.** A complex structure on  $X$  is an equivalent class of atlas on  $X$ .

**Remark 1.1.10.** Given an atlas  $\mathcal{A}$  on  $X$ , we can use charts in  $\mathcal{A}$  to define  $J : TX \rightarrow TX$  such that  $J^2 = -\text{id}$ .

**Definition 1.1.11.** A Riemann surface is a second countable, connected, Hausdorff topological space  $X$  together with a complex chart on  $X$ .

**Example 1.1.12.** Every open subset of  $\mathbb{C}$  is a Riemann surface.

**Example 1.1.13.**  $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ , consider

$$U_1 = S^2 \setminus \{(0, 0, 1)\} \xrightarrow{\varphi_1} \mathbb{C}$$

where  $\varphi_1$  is defined as  $\varphi_1(x, y, z) = \frac{x}{1-z} + i \frac{y}{1-z} = w$ . Similarly consider

$$U_2 = S^2 \setminus \{(0, 0, -1)\} \xrightarrow{\varphi_2} \mathbb{C}$$

where  $\varphi_2$  is defined as  $\varphi_2(x, y, z) = \frac{x}{1+z} - i \frac{y}{1+z} = w'$ . Note that  $ww' = \frac{x^2 + y^2}{1 - z^2} = 1$ . And it's easy to see the transition function is  $T(w) = \frac{1}{w}$ . So  $\{U_1, U_2\}$  is an atlas of  $S^2$ .

**Example 1.1.14.**  $\mathbb{CP}^1 = \{\text{complex 1-dimensional subspaces of } \mathbb{C}^2\}$ , is called a 1-dimensional projective space. Given a point  $(0,0) \neq (z,w) \in \mathbb{C}^2$ , exists a unique point  $[z,w] \in \mathbb{CP}^1$ , called the homogenous coordinate of  $\mathbb{CP}^1$ . Consider

$$U_1 = \{[z,w] \mid z \neq 0\} \xrightarrow{\varphi_1} \mathbb{C}$$

where  $\varphi_1$  is defined as  $\varphi_1([z,w]) = z/w$ . Similarly consider

$$U_2 = \{[z,w] \mid w \neq 0\} \xrightarrow{\varphi_2} \mathbb{C}$$

where  $\varphi_2$  is defined as  $\varphi_2([z,w]) = w/z$ . It's easy to check  $\{U_1, U_2\}$  is a atlas of  $\mathbb{CP}^1$ .

In fact,  $\mathbb{CP}^1$  is a Riemann surface which is isomorphic to  $S^2$ .

**Example 1.1.15.** Given two nonzero  $w_1, w_2 \in \mathbb{C}$ , with  $w_1 \neq aw_2$  for any  $a \in \mathbb{C}$ . Define lattice:

$$L = \mathbb{Z}w_1 + \mathbb{Z}w_2$$

In fact,  $L$  is a subgroup of  $\mathbb{C}$  with respect to operation “+”.

Then  $T = \mathbb{C}/L$  is a Riemann surface called complex torus. Consider the projection  $\pi : \mathbb{C} \rightarrow T$ . For  $p \in T$ , find one of its inverse image of  $\pi$ , denoted by  $z_0$ . Choose  $\varepsilon \in \mathbb{R}^+$  small enough such that

$$B_{2\varepsilon} \cap L = \{0\}$$

Consider

$$B_\varepsilon(z_0) \xrightarrow{\pi} \pi(B_\varepsilon(z_0)) \subset T$$

and the condition on  $\varepsilon$  implies  $\pi|_{B_\varepsilon}$  is injective. So let  $\{\pi(B_\varepsilon(z_0))\}$  be a open cover of  $T$ , and  $\pi^{-1}$  is the parametrization, this is an atlas of  $T$ .

**Remark 1.1.16.** The complex structure of complex torus depends on  $w_1, w_2$ . In fact, all complex structure of complex torus forms a Riemann surface of genus one. \*

## 1.2. Holomorphic function and Properties.

**Definition 1.2.1.** If  $X$  is a Riemann surface,  $W \subset X$  is a open subset. The function  $f : W \rightarrow \mathbb{C}$  is a complex valued function on  $W$ .  $f$  is called holomorphic at  $p \in W$ , if there exists a chart  $(U, \varphi)$  of  $p$  such that  $f \circ \varphi^{-1} : \varphi(U) \rightarrow \mathbb{C}$  is holomorphic at  $\varphi(p)$ .  $f$  is called holomorphic on  $W$ , if it is holomorphic at any  $p \in W$ .

**Theorem 1.2.2** (Maximum modulus theorem). For a Riemann surface  $X$ ,  $W \subset X$  is an open subset, and  $f$  is a holomorphic function on  $W$ . If there exists a point  $p \in W$ , such that  $|f(p)| \geq |f(x)|$  for all  $x \in W$ , then  $f$  must be a constant.

*Proof.* Clear. □

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\*The space consists of all complex structure of a Riemann surface is called the moduli space of it.

**Corollary 1.2.3.** *If  $X$  is a compact Riemann surface, then any global holomorphic function  $f$  must be constant.*

So, it's boring to consider holomorphic functions on a compact Riemann surface. In order to get something interesting, we need to consider meromorphic functions.

**Definition 1.2.4.** *If  $X$  is a Riemann surface, let  $f$  be a holomorphic function defined on  $U \setminus \{p\}$  where  $U \subset X$  is an open subset.  $p$  is called a removable singularity/pole/essential singularity, if there exists a chart  $(U, \varphi)$  of  $p$ , such that  $f \circ \varphi^{-1} : \varphi(U) \rightarrow \mathbb{C}$  has  $\varphi(p)$  as a removable singularity/pole/essential singularity.*

**Remark 1.2.5.** We have the following criterions:

1. If  $|f(x)|$  is bounded in a punctured neighborhood of  $p$ , then  $p$  is a removable singularity. And we can cancel the singularity by defining  $f(p) = \lim_{x \rightarrow p} f(x)$ .
2. If  $\lim_{x \rightarrow p} |f(x)| = \infty$ , then  $p$  is a pole.
3. If  $\lim_{x \rightarrow p} |f(x)|$  doesn't exist, then  $p$  is an essential singularity.

**Definition 1.2.6.**  *$f$  is called a meromorphic function at  $p$  if  $p$  is either a removable singularity or a pole, or  $f$  is holomorphic at  $p$ ;  $f$  is called a meromorphic function on  $W$ , if it's meromorphic at any point  $p \in W$ .*

**Remark 1.2.7.** If  $f, g$  are meromorphic on  $W$ , then  $f \pm g, fg$  are also meromorphic on  $W$ . If in addition,  $g \neq 0$ , then  $f/g$  is also meromorphic on  $W$ .

**Example 1.2.8.** Consider  $f, g$  are two polynomials in variable  $z$  with  $g \neq 0$ , then  $f/g$  is a meromorphic function on  $S^2 = \mathbb{C} \cup \{\infty\}$ . In fact, all meromorphic functions on  $S^2$  are in this form.

**Theorem 1.2.9** (Singularities and zeros). *Let  $X$  be a Riemann surface and  $W \subset X$  is an open subset,  $f$  is a meromorphic function on  $W$ , then set of singularities and zeros of  $f$  is discrete, unless  $f \equiv 0$ .*

**Corollary 1.2.10.** *If  $X$  is compact,  $f \neq 0$ , then  $f$  has finitely many poles and zeros on  $X$ . As a consequence, if  $f, g$  are two meromorphic functions on an open subset  $W \subset X$ , and  $f$  agrees with  $g$  on a set with limit point in  $W$ , then  $f \equiv g$ .*

**Definition 1.2.11.** *Let  $X, Y$  be two Riemann surfaces,  $F : X \rightarrow Y$ . For a point  $p \in X$ ,  $f$  is called holomorphic at  $p$ , if there exists a chart  $(U, \varphi)$  of  $p$ , and a chart  $(V, \psi)$  of  $F(p)$ , such that*

$$\psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \rightarrow \psi(V \cap F(U))$$

*is holomorphic at  $\varphi(p)$ ;  $F$  is called holomorphic in  $W$ , if  $F$  is holomorphic at any point in  $W$ .*

**Remark 1.2.12.**  $\psi \circ F \circ \varphi^{-1}$  is called the local representation of  $F$  at  $p$ .

**Example 1.2.13.** Any meromorphic function on  $X$  can be seen as a holomorphic map from  $X$  to  $S^2$ ; Conversely, we can construct a meromorphic function from a holomorphic map from  $X$  to  $S^2$ .

**Definition 1.2.14.** Two Riemann surfaces are called biholomorphic or isomorphic to each other, if there are two holomorphic map  $F : X \rightarrow Y, G : Y \rightarrow X$ , such that  $F \circ G = G \circ F = \text{id}$ .

**Example 1.2.15.**  $S^2$  is biholomorphic to  $\mathbb{RP}^2$ .

**Theorem 1.2.16** (Open mapping theorem).  $F : X \rightarrow Y$  is a non-constant holomorphic map, then  $F$  is an open map.

**Corollary 1.2.17.** If  $X$  is compact, and  $Y$  is connected,  $F : X \rightarrow Y$  is a non-constant holomorphic map, then  $Y$  is compact and  $F(X) = Y$ .

*Proof.* By open mapping theorem,  $F(X)$  is an open subset of  $Y$ , and  $F(X)$  is compact in  $Y$ , since continuous function maps compact set to compact set. Then  $F(X)$  is both open and closed in  $Y$ , then  $F(X) = Y$ .  $\square$

### 1.3. Ramification covering.

**Theorem 1.3.1.**  $F : X \rightarrow Y$  is a non-constant holomorphic function on Riemann surfaces, then for any  $p \in Y$ ,  $F^{-1}(p)$  is a discrete set. Furthermore, if  $X$  is compact, then  $F^{-1}(p)$  only contains finite many points.

So we wonder what's exact number of  $F^{-1}(p)$ , the local normal form tells you answer.

**Theorem 1.3.2** (Local normal form).  $F : X \rightarrow Y$  is a non-constant holomorphic function on  $X$ , then there is a local representation of  $F$  at  $p \in X$ , such that

$$\psi \circ F \circ \varphi^{-1}(z) = z^k, \quad \forall z \in \varphi(U \cap F^{-1}(V))$$

$k$  is called the multiplicity<sup>†</sup> of  $F$  at  $p$ , denoted by  $\text{mult}_p(F)$ . In fact,  $k$  is independent of the choice of charts.

*Proof.* Fix a chart  $(U_2, \varphi_2)$  of  $F(p)$ , choose an arbitrary local chart  $(U, \psi)$  of  $p$  such that  $F(U) \subset U_2$ , denote  $\varphi_2 \circ F \circ \psi^{-1} = T$ , then  $T(0) = 0$ . Consider the Taylor expansion of  $T$  at  $w = 0$  has

$$T(w) = \sum_{k=m}^{\infty} a_k w^k, \quad a_m \neq 0$$

So  $T(w) = w^m S(w)$ , where  $S(w)$  is a holomorphic function with  $S(0) \neq 0$ , then there exists a holomorphic function  $R(w)$  such that  $R^m(w) = S(w)$ .

Then  $T(w) = (wR(w))^m = (\eta(w))^m$ , so  $\eta(0) = 0, \eta'(0) = R(0) \neq 0$ , so  $\eta$  is invertible near  $w = 0$  by inverse function theorem. So there exists another chart of  $p \in U_1 \subset U$ , with

$$U \supset U_1 \xrightarrow{\psi} V \xrightarrow{\eta} V_1 \subset \mathbb{C}$$

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<sup>†</sup>Sometimes this number is also called ramification of  $F$  at  $p$ .

then we can define a local chart  $(U_1, \varphi_1 = \eta \circ \psi)$ , and check

$$\varphi_2 \circ F \circ \varphi_1^{-1}(z) = \varphi_2 \circ F \circ \psi^{-1} \circ \eta^{-1}(z) = T(w) = (\eta(w))^m = z^m$$

What's more, we can see from the local normal form that for any  $q \in Y, q \neq F(p)$  and  $q$  lies in a small neighborhood of  $p$  such that  $F^{-1}(q)$  lies in a small neighborhood of  $p$ , then  $F^{-1}(q)$  consists of exactly  $k$  points. So the ramification index is independent of the charts we choose.  $\square$

**Definition 1.3.3.**  $p$  is called a ramification point of a holomorphic map  $F : X \rightarrow Y$ , if  $\text{mult}_p(F) > 1$ , such  $F(p)$  is called a ramification value.

**Lemma 1.3.4.**  $p$  is a ramification point of a holomorphic map  $F : X \rightarrow Y$  if  $T'(w) = 0$ , for any local representation of  $F$ .

**Corollary 1.3.5.** The set of ramification points of a holomorphic map is a discrete set.

**Theorem 1.3.6.** Assume  $X, Y$  are complex Riemann surface,  $F : X \rightarrow Y$  is non-constant holomorphic function, for  $q \in Y$ , let

$$d_q(F) = \sum_{p \in F^{-1}(q)} \text{mult}_p(F)$$

then  $d_q(F)$  is independent of  $q \in Y$ , and denoted by  $\deg(F)$ .

*Proof.* Consider  $F : \mathbb{D} \rightarrow \mathbb{D}$ , defined by  $z \mapsto z^m$ , it's easy to check  $d_q(F) = m$ , for all  $q \in \mathbb{D}$ .

For general case, for  $q \in Y$ , let  $F^{-1}(q) = \{p_1, \dots, p_k\} \subset X$ . Fix a chart  $(U_2, \varphi_2)$  centered at  $q \in Y$ , for any  $i = 1, \dots, k$ , we can find local chart  $(U_{1,q}, \psi_i)$  centered at  $p_i \in X$ , such that

$$\varphi_2 \circ F \circ \psi_i^{-1}(z) = z^{m_i}, \quad z \in \psi_i(U_{1,i})$$

where  $m_i = \text{mult}_{p_i}(F)$ . Choose  $q \in W \subset U_2$  such that  $F^{-1}(W) \subset \bigcup_{i=1}^k U_{1,i}$ , then for any  $q \in W$

$$d_q(F) = \sum_{i=1}^k m_i$$

which can be seen from trivial case we discuss firstly. Then  $d_q(F)$  is a locally constant function, then  $d_q(F)$  must be global constant, since  $Y$  is connected.  $\square$

**Corollary 1.3.7.**  $X$  is a compact Riemann surface, and  $f$  is a meromorphic function on  $X$ , then the number (with multiplicity) of zeros is equal to the number (with multiplicity) of poles.

*Proof.* Note that meromorphic function on  $X$  is equivalent to the holomorphic map from  $X$  to  $S^2$ .  $\square$

**1.4. Hurwitz Formula.** Now let us forget the complex structure of Riemann surface, and recall some facts about topological invariants.

Let  $X$  be a compact oriented surface, we can say the genus of  $X$  is the number of “holes” which  $X$  has, informally. We can use genus to classify all oriented compact surfaces: any two surfaces which have the same genus are diffeomorphic to each other.

We can also define Euler characteristic of  $X$ , as

$$\chi(X) := \sum_i (-1)^i \dim H_i(X)$$

And there is a connection between genus of  $X$  and  $\chi(X)$ ,

$$\chi(X) = 2 - 2 \text{genus}(X)$$

so we can also use  $\chi(X)$  to classify oriented compact surface.

**Theorem 1.4.1** (Hurwitz Formula). *Let  $X, Y$  be two compact Riemann surfaces, and  $F : X \rightarrow Y$  be a non-constant holomorphic map, then*

$$2 \text{genus}(Y) - 2 = \deg(F)(2 \text{genus}(X) - 2) + \sum_{p \in X} (\text{mult}_p(F) - 1)$$

*Note that the set of ramification points is finite, then  $\sum_{p \in X} (\text{mult}_p(F) - 1)$  is a finite sum, and denoted by  $B(F)$ .*

*Proof.* Choose a triangulation of  $Y$  such that its vertex are exactly ramification values of  $F$ . Let  $v$  denote the number of vertices of  $\Delta$ ,  $c$  and  $t$  denote the number of edges and triangles of  $\Delta$ , where  $\Delta$  denotes a triangulation of  $Y$ . We can get a triangulation  $\Delta'$  of  $X$ , by pulling back  $\Delta$  through  $F$ , and use  $v', c'$  and  $t'$  to denote the same thing in  $\Delta'$ .

Then we have the following obvious relations

$$t' = td, \quad e' = ed$$

where  $d = \deg(F)$ . The relation between  $v$  and  $v'$  is a little bit complicated, consider  $q \in Y$ , then

$$|F^{-1}(q)| = \sum_{p \in F^{-1}(q)} 1 = d + \sum_{p \in F^{-1}(q)} (1 - \text{mult}_p(F))$$

then

$$\begin{aligned} v' &= \sum_{\text{vertex } q \text{ of } \Delta} |F^{-1}(q)| \\ &= \sum_{\text{vertex } q \text{ of } \Delta} (d + \sum_{p \in F^{-1}(q)} (1 - \text{mult}_p(F))) \\ &= vd + \sum_{\text{vertex } q \text{ of } \Delta} \sum_{p \in F^{-1}(q)} (1 - \text{mult}_p(F)) \\ &= vd + \sum_{p \in X} (1 - \text{mult}_p(F)) \end{aligned}$$

Then by the relation between Euler characterisitic and triangulation, we get the desired conclusion.  $\square$

**Definition 1.4.2.** *A holomorphic map  $F$  is called ramified if  $B(F) > 0$ , this is equivalent to  $F$  has at least one ramification point; A holomorphic map  $F$  is called unramified if  $B(F) = 0$ , this is equivalent to  $F$  is a covering map.*

**Corollary 1.4.3.** *Let  $X, Y$  be two compact Riemann surfaces, and  $F : X \rightarrow Y$  is a non-constant holomorphic map, then consider*

1. *If  $Y = S^2$ , and  $\deg(F) > 1$ , then  $F$  must be ramified.*
2. *If  $\text{genus}(X) = \text{genus}(Y) = 1$ , then  $F$  must be unramified.*
3.  *$\text{genus}(X) \geq \text{genus}(Y)$ .*
4. *If  $\text{genus}(X) = \text{genus}(Y) > 1$ , then  $F$  must be an isomorphism.*

*Proof.* All of them are simple applications of Hurwitz Formula.

1. By Hurwitz Formula we have

$$B(F) = 2(\deg(F) - 1) + 2 \text{genus}(X) > 0$$

2. By Hurwitz Formula we have

$$0 = 0 + B(F)$$

3. If  $\text{genus}(Y) = 0$ , it's trivial. Otherwise, we have

$$2 \text{genus}(X) - 2 \geq 2 \text{genus}(Y) - 2 + B(F)$$

since  $\deg F \geq 1$ .

4. By Hurwitz Formula we have

$$(1 - \deg(F))(2 \text{genus}(X) - 2) = B(F)$$

Then  $\deg(F) = 1$ , since  $\deg(F) \geq 1$ ,  $2 \text{genus}(X) - 2 > 0$  and  $B(F) \geq 0$ .  $\square$

**Remark 1.4.4.** From above corollary, we can see that genus, as a topological invariants, controls geometric properties heavily.

**1.5. Holomorphic map between complex torus.** Consider a lattice  $L = \mathbb{Z}w_1 + \mathbb{Z}w_2$ , let  $X$  denote the complex torus  $X = \mathbb{C}/L$ , a Riemann surface with genus 1. Moreover, there is a group structure on  $X$ , induced by  $(\mathbb{C}, +)$  through natural projection  $\pi : \mathbb{C} \rightarrow X$ , defined as follows

$$[z_1] + [z_2] := [z_1 + z_2]$$

So, inversion map

$$[z] \mapsto [-z]$$

gives an automorphism.

For  $a \in \mathbb{C}$ , we can define a transformation

$$T_a : X \rightarrow X, \quad [z] \mapsto [z + a]$$

which is also an automorphism.



So, as we can see, there are too many automorphism on  $X$ , let  $\text{Aut}(X)$  denote all automorphisms on  $X$ , which forms a group which can reflect the symmetry of  $X$ .

Obviously, we have the following inclusion

$$\text{Aut}(X) \supset \{T_{[a]} \mid [a] \in X\} \cup \{\text{inversion}\}$$

In fact, we will see later that  $\text{Aut}(X)$  is a complex manifold with  $\dim_{\mathbb{C}} \text{Aut}(X) = 1$ , but for now, we can only conclude that  $\dim_{\mathbb{C}} \text{Aut}(X) \geq 1$ .

Before we come to see what is the automorphism group of  $X$ , we consider a more general case, holomorphic map between complex torus.

Assume  $L, M$  are two different lattices in  $\mathbb{C}$ ,  $X = \mathbb{C}/L, Y = \mathbb{C}/M$  are two complex torus.

Let  $F : X \rightarrow Y$  be a non-constant holomorphic map, after composing some translation  $T_a$ , we can assume that  $F([0]) = [0]$ . Since  $\text{genus}(X) = \text{genus}(Y)$ , then by Hurwitz formula  $F$  must be a covering map.

Let  $\pi_X : \mathbb{C} \rightarrow X, \pi_Y : \mathbb{C} \rightarrow Y$  are natural projection. In fact, they're universal covering map.

Consider

$$\mathbb{C} \xrightarrow{\pi_X} X \xrightarrow{F} Y$$

then  $F \circ \pi_X$  is also a universal covering of  $Y$ . By uniqueness of universal covering, there exists a holomorphic map<sup>‡</sup>  $G : \mathbb{C} \rightarrow \mathbb{C}$  such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{G} & \mathbb{C} \\ \downarrow \pi_X & & \downarrow \pi_Y \\ X & \xrightarrow{F} & Y \end{array}$$

Since  $F([0]) = [0]$ , then  $G(0) \in M$ . After composing with a translation in  $\mathbb{C}$  with respect to  $-G(0)$ , we can assume  $G(0) = 0$ .

For any  $z \in \mathbb{C}, l \in L$ , consider the difference  $w(z, l) := G(z + l) - G(z)$ . First note that  $w(z, l)$  is a holomorphic with respect to  $z$ . What's more,  $w(z, l)$  must lie in  $M$ . So  $w(z, l)$  must be a constant with respect to  $z$ , since  $M$  is discrete. So

$$\frac{\partial}{\partial z} w(z, l) = G'(z + l) - G'(z) = 0$$

That is,  $G'(z)$  is periodic with respect to  $L$ , so  $|G'(z)|$  is bounded. By Liouville's theorem, we have  $G'(z)$  is constant.

So  $G$  must have the form  $G(z) = \gamma z, \gamma \in \mathbb{C}$ , since we assume  $G(0) = 0$ . Since  $G(L) \subset G(M)$ , we have

$$\gamma L \subset M$$

Since  $G(z) = \gamma z$  is a group homomorphism, then  $F$  is also a group homomorphism between  $X$  and  $Y$ .

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<sup>‡</sup>Clearly,  $G$  is not unique.

Clearly,  $F$  is an isomorphism if and only if  $\gamma L = M$ .

We summarize as follows:

**Theorem 1.5.1.** *Any holomorphic map  $F : \mathbb{C}/L \rightarrow \mathbb{C}/M$  is induced by a linear map*

$$G : \mathbb{C} \rightarrow \mathbb{C}, \quad z \mapsto \gamma z + a, \quad \gamma, a \in \mathbb{C}$$

*moreover,  $F$  is a biholomorphic map if and only if  $\gamma L + a' = M$ , for some  $\gamma, a' \in \mathbb{C}$ .*

Now consider non-constant biholomorphic map  $F : X \rightarrow X$ , where  $X = \mathbb{C}/L$ . After composing some translation, we may assume  $F([0]) = [0]$ . Then  $F$  is induced by a map  $z \mapsto \gamma z$  such that  $\gamma L = L$ .

Note that this condition is a quite strong for  $\gamma$ . We list some facts as follows

1.  $|\gamma| = 1$ , otherwise the shortest length of non-zero element in  $L$  and  $\gamma L$  will be different.
2. There exists integers  $m \geq 1$  such that

$$\gamma^m = 1$$

otherwise  $L$  contains infy many points in a circle, a contradiction to the discreteness.

Note that  $\gamma = \pm 1$  is allowed,  $\gamma = 1$  is equivalent to  $F$  is identity and  $\gamma = -1$  is equivalent to the inversion. Assume  $\gamma \notin \mathbb{R}$ , choose  $w \in L \setminus \{0\}$ , such that

$$|w| \leq |v|, \quad \text{for all } v \in L \setminus \{0\}$$

We claim that:

**Lemma 1.5.2.**  $L = \mathbb{Z}w + \mathbb{Z}\gamma w$ , for  $w$  we choose above.

*Proof.* Let  $L' = \mathbb{Z}w + \mathbb{Z}\gamma w \subset L$ . If  $L' \neq L$ , we will find an element  $v \in L \setminus L'$ . Adding an element in  $L'$  if necessary, we may assume  $v$  lies in the parallelogram spanned by  $w$  and  $\gamma w$ . Then

$$|v - w| + |v - \gamma w| < |w| + |\gamma w| = 2|v|$$

So either  $|v - w|$  or  $|v - \gamma w|$  is less than  $|v|$ , a contradiction.  $\square$

Since  $\gamma L = L$ , then  $\gamma^2 w \in L = \mathbb{Z}w + \mathbb{Z}\gamma w$ , so

$$\gamma^2 w = mw + n\gamma w, \quad m, n \in \mathbb{Z}$$

After canceling  $w$  we have the quadratic equation that  $\gamma$  must satisfy

$$\gamma^2 = m + n\gamma$$

so we have

$$\gamma = \frac{1}{2}(n \pm \sqrt{n^2 + 4m})$$

Since  $\gamma \notin \mathbb{R}$ , we have  $n^2 + 4m < 0$ . And

$$|\gamma|^2 = \frac{1}{4}(n^2 - (n^2 + 4m)) = -m$$

so we must have  $m = -1$ . So  $n^2 < 4$  implies  $n = \pm 1, 0$ . Then all possible  $\gamma$  are listed as follows

$$\gamma = \begin{cases} \pm i, & n = 0 \\ \frac{1}{2}(\pm 1 \pm \sqrt{3}i), & n = \pm 1 \end{cases}$$

When  $n = 0$ ,  $L$  is called a square lattice. When  $\gamma = \pm 1$ ,  $L = \mathbb{Z}w + \mathbb{Z}w \cdot e^{\frac{\pi}{3}i}$ , is called a hexagonal lattice.

We summarize as follows

**Theorem 1.5.3.** *If we define  $\text{Aut}_0(X) = \{\text{automorphism } F : X \rightarrow X \mid F([0]) = [0]\}$ , then*

$$\text{Aut}_0(X) = \begin{cases} \mathbb{Z}_4, & L \text{ is a square lattice} \\ \mathbb{Z}_6, & L \text{ is a hexagonal lattice} \\ \mathbb{Z}_2, & \text{otherwise} \end{cases}$$

So we have

$$\dim_{\mathbb{C}} \text{Aut}(X) = 1$$

What's more, the three cases above are not isomorphic to each other, since Riemann surfaces which are isomorphic to each other have the same automorphism group.

**Remark 1.5.4.** The first example we meet, surfaces with the same topological structure but different complex structures.

**1.6. Moduli space of complex torus.** Since the above results show some different complex structures on a topological torus, we want to ask: How many different complex structures are there on a topological torus? And in general, how many different complex structures are there on a given Riemann surfaces? That leads to the conception of Moduli space.

For any lattice  $M = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ , if we let  $\gamma = \frac{\omega_2}{\omega_1}$ , then

$$L = \gamma M = \mathbb{Z} + \mathbb{Z}\frac{\omega_2}{\omega_1}$$

So it suffices to consider the complex torus of form  $X_\tau = \mathbb{C}/L_\tau$ , where

$$L_\tau = \mathbb{Z} + \mathbb{Z}\tau$$

Since  $L_{-\tau} = L_\tau$ , so we can assume that  $\text{Im } \tau > 0$ .

Let

$$\mathbb{H} = \{\tau \in \mathbb{C} \mid \text{Im } \tau > 0\}$$

Given  $\tau, \tau' \in \mathbb{H}$ , we want to ask when  $X_{\tau'}$  and  $X_\tau$  give the same complex structure on a topological torus. It is equivalent to that there exists  $\gamma \in \mathbb{C}$ , such that

$$\gamma L_\tau = L_{\tau'}$$

i.e.

$$\mathbb{Z}\gamma + \mathbb{Z}\gamma\tau = \mathbb{Z} + \mathbb{Z}\tau'$$

So there exists  $a, b, c, d \in \mathbb{Z}$ , such that

$$\begin{cases} \gamma = c + d\tau' \\ \gamma\tau = a + b\tau' \end{cases}$$

moreover,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible since it's a base change and its inverse matrix must have integral entries, so its determinant must be  $\pm 1$ .

So, it's the famous Mobius transformation

$$\tau = \frac{a + b\tau'}{c + d\tau'}$$

Since we require both  $\gamma$  and  $\gamma$  have positive imaginary part, we compute as follows

$$\tau = \frac{(a\tau' + b)(c\bar{\tau}' + b)}{|c\tau' + d|^2} \implies \operatorname{Im} \tau = \frac{ad - bc}{|c\tau' + d|^2} \operatorname{Im} \tau'$$

So we need  $A \in \operatorname{SL}_2(\mathbb{Z})$ .

We summarize as follows

**Theorem 1.6.1** (Moduli space of complex torus).  $X_\tau \cong X_{\tau'}$  if and only if there exists  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{SL}_2(\mathbb{Z})$  such that

$$\tau = \frac{a + b\tau'}{c + d\tau'}$$

What's more,  $A$  and  $-A$  give us the same information.

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