

Uniqueness of the Kähler structure of \mathbb{CP}^n

Bowen Liu

Mathematics Department of Tsinghua University

2023/06/05



- 1 Overview
- 2 Theorem A
- 3 Theorem B
- 4 Closing remarks

- 1 Overview
- 2 Theorem A
- 3 Theorem B
- 4 Closing remarks

- In this talk we mainly focus on the following two theorems, which show the uniqueness of the Kähler structure of \mathbb{CP}^n .

Theorem (Hirzebruch, Kodaira, Yau)

If a Kähler manifold M is homeomorphic to \mathbb{CP}^n , then M is biholomorphic to it.

Theorem (Yau)

If a compact Kähler surface M is homotopy equivalent to \mathbb{CP}^2 , then M is biholomorphic to it.

- To prove these two theorems, the following lemma motivates us it suffices to construct a holomorphic line bundle with some properties.

Lemma

If M is a compact Kähler n -manifold and L is a positive holomorphic line bundle over M with $\int_M c_1(L)^n = 1$ and $\dim H^0(M, L) = n + 1$, then M is biholomorphic to \mathbb{CP}^n .

Rough idea of proof

- If M is a compact Kähler manifold whose cohomology groups are the same as the ones of \mathbb{CP}^n , then
 - $c_1: \text{Pic}(M) \rightarrow H^2(M, \mathbb{Z})$ is an isomorphism, which allows us to construct holomorphic line bundle L with a given cohomology class as its first Chern class.
 - The holomorphic Euler characteristic satisfies

$$\chi(M, \mathcal{O}) = \sum_{p=0}^n (-1)^p \dim H^{0,p}(M) = 1.$$

- By using Hirzebruch-Riemann-Roch theorem one can conclude

$$\chi(M, L) = n + 1.$$

- By using Kodaira vanishing theorem one can conclude $H^k(M, L) = 0$ for $k > 0$. In particular, one has $\dim H^0(M, L) = n + 1$.

- 1 Overview
- 2 Theorem A**
- 3 Theorem B
- 4 Closing remarks

- In this section, M is a Kähler manifold which is homeomorphic to \mathbb{CP}^n .
- Choose a Kähler form $\tilde{\omega}$ on M . Its cohomology class lies in $H^2(M, \mathbb{R}) \cong \mathbb{R}$, so we can rescale $\tilde{\omega}$ to get another Kähler form ω whose cohomology class generates $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$. In particular, $\int_M \omega^n = 1$.
- Since c_1 is an isomorphism, there exists a holomorphic line bundle L whose first Chern class is $[\omega]$.

Lemma

For any holomorphic line bundle L over M we have

$$\chi(M, L) = \int_M e^{c_1(L) + \frac{c_1(M)}{2}} \left(\frac{\omega/2}{\sinh(\omega/2)} \right)^{n+1}.$$

Corollary

$c_1(M)$ equals either $(n+1)[\omega]$ or $-(n+1)[\omega]$, with the latter only possibly occurring when n is even.

Proof.

The reduction mod 2 of $c_1(M)$ is the second Stiefel-Whitney class $w_2(M) \in H^2(M, \mathbb{Z})$, which is a topological invariant. Hence it is equal to $w_2(\mathbb{CP}^n)$ which equals $c_1(\mathbb{CP}^n) \equiv n+1 \pmod{2}$. On the other hand since $c_1(M)$ and $[\omega]$ both belong to $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$, we have $c_1(M) = (n+1+2s)[\omega]$ for some $s \in \mathbb{Z}$.

Continuation.

By Lemma 4 one has

$$\chi(M, \mathcal{O}) = \int_M e^{\frac{n+1+2s}{2}\omega} \left(\frac{\omega/2}{\sinh(\omega/2)} \right)^{n+1} = \int_M e^{s\omega} \left(\frac{\omega}{1 - e^{-\omega}} \right)^{n+1}.$$

By residue theorem a direct computation shows

$$\int_M e^{s\omega} \left(\frac{\omega}{1 - e^{-\omega}} \right)^{n+1} = \binom{n+s}{n}.$$

Since $\chi(M, \mathcal{O}) = 1$, one has $\binom{n+s}{n} = 1$, which can be rewritten as

$$n! = (s+n) \dots (s+1).$$

So if n is odd this implies $s = 0$, while if n is even, s is either 0 or $-n-1$. This completes the proof. □

Proof of Theorem 1.

Case I: Assume first $c_1(M) = (n+1)[\omega]$, which implies that M is a Fano manifold. Then $c_1(K_M) = -c_1(M) = -(n+1)c_1(L)$ and so $K_M = -(n+1)L$ since c_1 is an isomorphism. Then Serre duality gives $H^k(M, L) = H^{n-k}(M, K_M - L)$ and $K_M - L = -(n+2)L$ is negative, so $H^k(M, L) = 0$ if $k > 0$ by Kodaira vanishing. Hence one has

$$\dim H^0(M, L) = \chi(M, L) = \int_M e^{c_1(L) + \frac{c_1(M)}{2}} \left(\frac{\omega/2}{\sinh(\omega/2)} \right)^{n+1} = n+1,$$

and Lemma 3 implies M is biholomorphic to \mathbb{CP}^n .

Case II: Assume $c_1(M) = -(n+1)[\omega] < 0$, it suffices to show the following identity

$$(2(n+1)c_2(M) - nc_1^2(M)) [\omega]^{n-2} = 0.$$

Continuation.

Since by Chern number inequality theorem of Yau, M has constant holomorphic sectional curvature, and thus by Uniformization theorem one has M is biholomorphic to the unit ball in \mathbb{C}^n , a contradiction.

To compute $c_2(M)$, note that $p_1(M) = p_1(TM) = -c_2(TM \otimes \mathbb{C})$, $TM \otimes \mathbb{C} \cong TM \oplus \overline{TM}$ and Chern classes satisfy $c_k(\overline{TM}) = (-1)^k c_k(TM)$, so

$$\begin{aligned} p_1(M) &= -c_2(TM \oplus \overline{TM}) \\ &= -c_2(TM) - c_2(\overline{TM}) - c_1(TM)c_1(\overline{TM}) \\ &= -2c_2(M) + c_1^2(M). \end{aligned}$$

On the other hand, $p_1(M) = (n+1)[\omega]^2$. Thus

$$2c_2(M) = (n+1)^2[\omega]^2 - (n+1)[\omega]^2 = n(n+1)[\omega]^2.$$

- 1 Overview
- 2 Theorem A
- 3 Theorem B**
- 4 Closing remarks

Proof of Theorem 2.

Let $\tau(M)$ denote the signature of M , that is the signature of its intersection form. The signature is a topological invariant up to sign, and so

$$\tau(M) = \pm \tau(\mathbb{CP}^2) = \pm 1.$$

Hirzebruch's signature theorem gives

$$\tau(M) = \frac{1}{3} \int_M p_1(M) = \frac{1}{3} \int_M (c_1^2(M) - 2c_2(M)) = \pm 1.$$

Chern-Gauss-Bonnet's theorem gives

$$\int_M c_2(M) = \chi(M) = \chi(\mathbb{CP}^2) = 3.$$



Continuation.

As before we see that $\chi(M, \mathcal{O}) = 1$ and Hirzebruch-Riemann-Roch gives

$$\chi(M, \mathcal{O}) = \frac{K_M^2 + \chi(M)}{12} = \frac{K_M^2 + 3}{12},$$

which gives $\int_M c_1^2(M) = K_M^2 = 9$. Let ω be as before, and $c_1(M) = \lambda[\omega]$ for some $\lambda \in \mathbb{Z}$. Then $\lambda = \pm 3$. Here it suffices to show in case $\lambda = 3$, $\dim H^0(M, L) = 3$, and the case $\lambda = -3$ leads the same contradiction as before. By Hirzebruch-Riemann-Roch formula one has

$$\chi(M, L) = 1 + \frac{L^2 - K_M \cdot L}{2} = 3.$$

Serre duality and Kodaira vanishing gives

$H^1(M, L) = H^2(M, L) = 0$ as before. So

$\dim H^0(M, L) = \chi(M, L) = 3$. This completes the proof.



- ① Overview
- ② Theorem A
- ③ Theorem B
- ④ Closing remarks

- Libgober-Wood proved that a compact Kähler n -manifold with $n \leq 6$ which is homotopy equivalent to \mathbb{CP}^n must be biholomorphic to it.
- The Kähler condition in Theorem 2 can be replaced by complex, so a natural question is that whether the Kähler hypothesis is really necessary in Theorem 1, and one can also ask whether a compact complex manifold diffeomorphic to \mathbb{CP}^n must be biholomorphic to it. If it's true when $n = 3$, then there is no complex structure on S^6 .

Lemma

If there exists a compact complex manifold M diffeomorphic to S^6 , then there exists a compact complex manifold \tilde{M} diffeomorphic to \mathbb{CP}^3 but not biholomorphic to it.

Proof.

Let M be a compact complex manifold diffeomorphic to S^6 , and let \tilde{M} be its blowup at one point $p \in M$, which is a compact complex manifold which is diffeomorphic to the connected sum $S^6 \sharp \overline{\mathbb{CP}^3}$, where $\overline{\mathbb{CP}^3}$ is the smooth manifold obtained from \mathbb{CP}^3 by reversing orientation. So it's clear \tilde{M} is diffeomorphic to \mathbb{CP}^3 , and if \tilde{M} was biholomorphic to \mathbb{CP}^3 , one has

$$\int_{\tilde{M}} c_1(\tilde{M})^3 = \int_{\mathbb{CP}^3} c_1(\mathbb{CP}^3)^3 = 64$$

Continuation.

On the other hand, if we let $\pi: \tilde{M} \rightarrow M$ be the blowup map and $E = \pi^{-1}(p)$ be its exceptional divisor, then one has

$$c_1(\tilde{M}) = \pi^* c_1(M) - 2[E]$$

where $[E]$ is the Poincaré duality of E . Since $b_2(M) = 0$, one has $c_1(M) = 0$, and so

$$\begin{aligned} \int_{\tilde{M}} c_1(\tilde{M})^3 &= -8 \int_{\tilde{M}} [E]^3 \\ &= -8 \int_E [E]^2 \\ &= -8 \int_{\mathbb{CP}^2} c_1(\mathcal{O}(-1))^2 = -8 \end{aligned}$$

Therefore \tilde{M} is not biholomorphic to \mathbb{CP}^3 , as desired.

Thanks!