YANG-MILLS EQUATIONS ON RIEMANN SURFACE

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Abstract.

Contents

0. Preface

0.1. **About this lecture.**

Part 1. GIT quotient and symplectic quotient: the Kempf-Ness theorem

In this section, we mainly follow [**?**].

1. Geometric invariant theory

1.1. **Introduction.**

Many objects we want to take a quotient always have some sort of geometric structures, and we desire the quotients still preserve geometric structure.

Example 1.1.1. Suppose G is a Lie group and M is a smooth manifold, the quotient X/G will not always have the structure of a smooth manifold (For example, the presence of non-closed orbits, usually gives a non-Hausdorff quotient). However, if G acts properly and freely, then M/G has a smooth manifold structure, such that natural projection $\pi\colon M \to M/G$ is a smooth map.

Geometric invariant theory (GIT) is the study of such question in the context of algebraic geometry.

Example 1.1.2. Let $M_n(\mathbb{C})$ be the group of all $n \times n$ matrices over $\mathbb{C}$, then it can be given a geometric structure by regarding it as an affine variety. Consider the conjugate action of $\mathrm{GL}_n(\mathbb{C})$ on $M_n(\mathbb{C})$. Can we regard $M_n(\mathbb{C})/\mathrm{GL}_n(\mathbb{C})$ as a variety?

The answer of above question is yes, but good thing does not happen always, consider

Example 1.1.3. Let $\mathbb{C}^\times$ acts on $\mathbb{C}^2$ by $\lambda(x,y) := (\lambda x, \lambda y)$. The $\mathbb{C}^\times$-orbits are $\{(\lambda x, \lambda y) \mid \lambda \in \mathbb{C}^\times, (x,y) \neq (0,0)\}$ as well as the origin $\{(0,0)\}$. Now suppose that the set of orbits is a variety, then every point must be closed

So we need to be more careful when we constructing quotients in the category of varieties. As we have seen in smooth manifold, we can guess

1. only certain types of group (compared with Lie group) are allowed, see Definition 1.3.2;
2. only certain types of group actions (compared with properly and freely) are allowed, see Definition 1.3.3.

1.2. **Good categorical quotient.**

Definition 1.2.1 (G-invariant morphism)**.** A morphism $f\colon X \to Y$ is called G-invariant morphism, if it is constant on orbits.

Definition 1.2.2 (categorical quotient)**.** In any category, we call a G-invariant morphism $\pi\colon X \to Y$ is categorical quotient of X by G, when for any G-invariant morphism $f\colon X \to Z$, we have that f factors uniquely through π, that is

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Z \\ & Y \arrow[ur, dotted, "\overline{f}"'] & \end{tikzcd}$$

Remark 1.2.1. Since categorical quotient is defined by its universal property, so it is unique when it exists.

However, for a quotient in the category of varieties, simple being a categorical quotient may not have a good geometric properties, so we need to define good categorical quotient. If G acts on a variety X, then we can get an action on the regular functions on X as follows: For $f \in \mathcal{O}(U), U \subset X$ and $g \in G$, we define

$$gf(x) = f(g^{-1}x)$$

Definition 1.2.3 (G-invariant ring)**.** For a ring on which G acts, the subring of G-invariant elements is

$$R^G = \{f \in R \mid gf = f \text{ for all } g \in G\}$$

Definition 1.2.4 (good categorical quotient)**.** A surjective G-invariant map of varieties $p : X \to Y$ is called a good categorical quotient of X by G, if the following three properties holds

1. For all open $U \subset Y$, $p^*\colon \mathcal{O}(U) \to \mathcal{O}(p^{-1}(U))^G$ is an isomorphism.
2. If $W \subseteq X$ is closed and G-invariant, then $p(W) \subset Y$ is closed.
3. If $V_1, V_2 \subseteq X$ are closed, G-invariants, and $V_1 \cap V_2 = \varnothing$, then $p(V_1) \cap p(V_2) = \varnothing$.

Proposition 1.2.1. A good categorical quotient is a categorical quotient.

Proof. If $f\colon X \to Z$ is a G-invariant morphism, then the image of $f^*\colon \mathcal{O}(Z) \to \mathcal{O}(X)$ must be embeded in $\mathcal{O}(X)^G$. If $p : X \to Y$ is a good categorical quotient, then by definition $p^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)^G$ is an isomorphism, thus

$$\begin{tikzcd} \mathcal{O}(Z) \arrow[rr, "f^*"] \arrow[dr, dotted, "\overline{f^*}"'] & \arrow[d, phantom, "\circlearrowleft"] & \mathcal{O}(X)^G \arrow[r, hook] & \mathcal{O}(X) \\ & \mathcal{O}(Y) \arrow[ur, hook, two heads, "p^*"'] & & \end{tikzcd}$$

So f^* can factor through $\mathcal{O}(Y)$, and this factoring is unique since p^* is an isomorphism. By the anti-equivalence of category, the dual $f = \overline{f} \circ p$ is a unique factoring of f through p. $\square$

Notation 1.2.1. $X//G$ denotes the good categorical quotient, or GIT quotient, of a variety X by a group G.

Let's first construct GIT quotient in affine case, and it can serves as a guide for projective case, since every projective variety admits an affine covering. It's natural to define $X//G = \operatorname{Spec} \mathcal{O}(X)^G$ in affine cases, since

$X = \operatorname{Spec} \mathcal{O}(X)$, so G-invariant regular functions may representate the quotient we desire, but for this we hope $\mathcal{O}(X)^G$ is finitely generated.

Historically, whether the ring of invariants is finitely generated or not is knowns as Hilbert's 14-th problem. Let R be a ring, Hilbert showed that the invariant rings R^G are always finitely generated when $G = \mathrm{GL}_n(\mathbb{C})$. However, Nagata gave an counterexample that R^G is not finitely generated, and proved that for any reductive group G, R^G is finitely generated, see [**?**].

1.3. **Reductive groups.** Now we focus on the reductive group which we can use to construct GIT quotient. We will define when a linear algebraic group is reductive and give some properties of it.

Definition 1.3.1 (algebraic group)**.** A (linear) algebraic group is a subgroup of $\mathrm{GL}_n(k)$ which is an affine variety, that is an irreducible algebraic set.

Example 1.3.1. The set of unitary matrices with determinant 1

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc - 1 = 0 \right\}$$

is an algebraic group[1].

Example 1.3.2. $k^\times$ is also an algebraic group, by the embedding $\lambda \to \lambda I$.

Example 1.3.3. $\mathrm{GL}_n(k)$ is an algebraic group[2].

Definition 1.3.2 (reductive)**.** A linear algebraic group G over k is reductive if every representation $\rho : G \to \mathrm{GL}_n(k)$ has a decomposition as a direct sum of irreducible representations.

Lemma 1.3.1. Let G be a reductive group acting rationally on an affine variety X, then $\mathcal{O}(X)^G$ is finitely generated.

Proof. See [**?**]. □

Now let's see some examples about reductive groups.

Proposition 1.3.1 (Maschke)**.** Let G be a finite group, then G is reductive.

Proposition 1.3.2. The multiplicative group $\mathbb{C}^\times$ is reductive.

Proof. Let $\rho : \mathbb{C}^\times \to \mathrm{GL}_n(\mathbb{C})$ be a representation of $\mathbb{C}^\times$, we will show ρ has a decomposition as a direct sum of irreducible representations. Assume ρ is not irreducible. Let $\langle \text{-}, \text{-} \rangle$ denote the standard inner product on $V = \mathbb{C}^n$, then define

$$\langle x, y \rangle := \int_0^{2\pi} \langle \rho(e^{i\theta})x, \rho(e^{i\theta})y \rangle \mathrm{d}\theta$$

[1]In general, special linear group $\mathrm{SL}(n)$ is always an algebraic group by considering the irreducible polynomial $\det -1$.

[2]We can check this by introducing a new variable T and consider irreducible polynomial $T \cdot \det -1$ with $n^2 + 1$ variables.

This form has the following property: $\langle \rho(g)x, \rho(g)y\rangle = \langle x; y\rangle$, where $x, y \in V, g = e^{i\psi} \in S^1 = \{z \in \mathbb{C}^\times : |z| = 1\}$. Indeed,

$$\begin{aligned}
\langle \rho(e^{i\psi})x, \rho(e^{i\psi})y\rangle &= \int_0^{2\pi} \langle \rho(e^{i\theta}\rho(e^{i\psi}))x, \rho(e^{i\theta})\rho(e^{i\psi})y\rangle \mathrm{d}\theta \\
&= \int_0^{2\pi} \langle \rho(e^{i(\theta+\psi)})x, \rho(e^{i(\theta+\psi)})y\rangle \mathrm{d}\theta \\
&\overset{\phi=\theta+\psi}{=} \int_0^{2\pi} \langle \rho(e^{i\phi})x, \rho(e^{i\phi})y\rangle \mathrm{d}\phi \\
&= \langle x, y\rangle
\end{aligned}$$

And also note that $\langle$-,-$\rangle$ is an inner product. If ρ is not irreducible, then there exists some $\mathbb{C}^\times$-invariant subspace U of V, let $W = U^\perp$ be the orthogonal complement of U with respect to $\langle$-,-$\rangle$. Then we can see W is S^1-invariant as follows

$$\begin{aligned}
\langle u, \rho(g)w\rangle &= \langle \rho(g^{-1})u, \rho(g^{-1})\rho(g)w\rangle \\
&= \langle \rho(g^{-1})u, w\rangle \\
&= 0
\end{aligned}$$

where $w \in W, u \in U, g \in S$. The last equality holds since U is S^1-invariant. What we need to do is to show W is $\mathbb{C}^\times$-invaraint.

Let N be the subset of $\mathbb{C}^\times$ which leaves W invariant, it contains S obviously. We will show that this set is closed in the Zariski topology. If we can do this, since all Zariski closed subset in $\mathbb{C}^\times$ are finite sets and whole space, so we can conclude $N = \mathbb{C}^\times$, as desired.

Let $W = \operatorname{span}\{e_1, \dots, e_r\}$, and extends this basis to a basis $\{e_1, \dots, e_n\}$ of V. Then we can regard W as solutions of equations

$$\langle v, e_i\rangle = 0, \quad i = r+1, \dots, n$$

these define polynomials which take the coordinate of v as variables, which we call it f_i, so we can see W as a zero set of $\{f_{r+1}, \dots, f_n\}$.

For each $i \in \{1, \dots, r\}, j \in \{r+1, \dots, n\}$, consider the set $\{T \in \mathrm{GL}(V) \mid f_j(Te_i) = 0\}$. If we fix i, j, this set is the zero set of a polynomial in the coordinates of T. So it's a closed set in $\mathrm{GL}(V)$, with respect to Zariski topology. Then we have $\{T \in \mathrm{GL}(V) \mid Te_i \in W\} = \bigcap_{j=r+1}^n \{T \in \mathrm{GL}(V) \mid f_j(Te_i) = 0\}$ is closed, so

$$\{T \in \mathrm{GL}(V) \mid Te_i \in W, \forall i \in \{1, \dots, r\}\} = \bigcap_{i=1}^r \{T \in \mathrm{GL}(V) \mid Te_i \in W\}$$

is closed, thus we have

$$\begin{aligned}
\{T \in \mathrm{GL}(V) \mid Tw \in W, \forall w \in W\} &= \{T \in \mathrm{GL}(V) : T(\lambda_1 e_1 + \ldots + \lambda_r e_r) \in U \text{ for all } \lambda_i \in \mathbb{C}\} \\
&= \{T \in \mathrm{GL}(V) : \lambda_1 (Te_1) + \ldots + \lambda_r (Te_r) \in U \text{ for all } \lambda_i \in \mathbb{C}\} \\
&= \{T \in \mathrm{GL}(V) : Te_i \in W \text{ for each } i \in \{1, 2, \dots, r\}\}
\end{aligned}$$

is closed with respect to Zariski topology, so $N = \rho^{-1}(\{T \in \mathrm{GL}(V) \mid Tw \in W, \forall w \in W\})$ is closed, as we desired. $\square$

Remark 1.3.1. In fact, many classical groups such as $\mathrm{GL}_n(\mathbb{C}), \mathrm{SL}_n(\mathbb{C})$ are reductive, now we give a proof of $\mathbb{C}^\times$ is a reductive group.

Definition 1.3.3 (rationally)**.** For a reductive alegbraic group, we say that G acts rationally on a variety X if it acts by a morphism of varieties $G \times X \to X$.

The following lemma is used in the construction of GIT quotient. It allows us to find a G-invariant function which separates disjoint G-invariant sets.

Lemma 1.3.2. Let G be a reductive group acting rationally on an affine variety $X \subset \mathbb{A}^n$. Let Z_1, Z_2 be two closed G-invariant subsets of X with $Z_1 \cap Z_2 = \varnothing$. Then there exists a G-invariant function $F \in \mathcal{O}(X)^G$ such that $F(Z_1) = 1, F(Z_2) = 0$.

Proof. See [**?**]. $\square$

1.4. **The affine quotient.** We now have enough tools to construct the quotient of an affine variety by a reductive group. For an affine variety X, the quotient of X by a reductive group G is just $\operatorname{Spec} \mathcal{O}(X)^G$. We will prove that this construction satisfies the required conditions being a good categorical quotient in Definition 1.2.4.

Theorem 1.4.1. Let X be an affine variety and G a reductive group acting rationally on X. Let $p^* \colon \mathcal{O}(X)^G \to \mathcal{O}(X)$ denotes the natural inclusion. Then the dual of this map, $p \colon X \to Y := \operatorname{Spec} \mathcal{O}(X)^G$ is a good categorical quotient.

Now we give a concrete example to show how powerful the GIT construction is, and gives the answer to the Example 1.1.1 we mentioned at first.

Example 1.4.1. Consider the set X of 2×2 matrices over $\mathbb{C}$, embedded in $\mathbb{C}^4$ by

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} \mapsto (w, x, y, z)$$

It is an affine variety obviously, and consider the general linear group acts on it by conjugate action, then as the theorem above implies

$$X//G = \operatorname{Spec} k[w, z, y, z]^G$$

We know that there are two important invariants under conjugate action, that is, determinant and trace. In this case they are $\det = wz - xy$ and $\mathrm{tr} = w + z$, so we have an obvious inclusion

$$k[wz - xy, w + z] \subset k[w, x, y, z]^G$$

We will show that we in fact have equality.

Let $\lambda \in \mathbb{C}^\times$ be arbitrary and consider the matrix $A = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$. For all matrices $M = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$, we can calculate as follows

$$\begin{aligned} A^{-1}MA &= \begin{pmatrix} 0 & -\frac{1}{\lambda} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \\ &= \begin{pmatrix} z & \frac{y}{\lambda} \\ \lambda x & w \end{pmatrix} \end{aligned}$$

Let $f \in k[w,x,y,z]^G$, i.e. we require f satisfy that $f(w,x,y,z) = A.f(M) = f(A.M) = f(A^{-1}MA) = f(z, \frac{y}{\lambda}, \lambda x, w)$. That is

$$f(w,x,y,z) = f\left(z, \frac{y}{\lambda}, \lambda x, w\right)$$

From this equality, we can make the following observations

1. x must appear in the form xy to cancel λ in $A.f$.
2. z and w must appear in an symmetric way, i.e. must in the forms of $z+w$ or zw.

So we conclude $f \in k[xy, wz, z+w]$. Similarly consider matrix $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. And after the same calculation we can get

$$f(w,x,y,z) = f(w-x, w+x-y-z, y, y+z)$$

As we already have $f \in k[xy, wz, w+z]$, we can reformulate this requirement into

$$f(xy, wz, w+z) = f\left(wy + xy - y^2 - z, wy + wz - y^2 - yz, w+z\right)$$

We can see that this formular holds only when extra terms in $B.f$ must cancel with each other, which implies $f \in k[wz - xy, w+z]$, as desired. So we have the construction

$$\begin{aligned} X//G &= \operatorname{Spec} k[w,x,y,z]^G \\ &= \operatorname{Spec} k[wz - xy, w+z] \\ &= \operatorname{Spec} k[u,v] \\ &= \mathbb{C}^2 \end{aligned}$$

Remark 1.4.1. There is a high-dimensional analogous: if $\mathrm{GL}_n(\mathbb{C})$ acts on $M_n(\mathbb{C})$ by conjugate action, then

$$M_n(\mathbb{C})//\mathrm{GL}_n(\mathbb{C}) = \mathbb{C}^n$$

See [?] for more details.

1.5. **The projective quotient.** Now we construct projective quotient by gluing together affine quotients.

Let X be a projective variety, then X can be covered by some affine varieties X_{f_i}. In order to construct GIT quotient of X by G, it's natural for us to take quotient for every affine variety of G of the form $X_{f_i}//G = \mathrm{Spec}(\mathcal{O}(X_{f_i})^G)$, and cover the projective quotient by them. To do this, we need an action of G on the coordinates of X.

Our approach is to embed X in $\mathbb{P}^m$ for some m such that the action of G can be extended to a linear action on $\mathbb{A}^{m+1}$. This is called a linearisation of the action of G.

Definition 1.5.1. Let the group G act rationally on a projective variety X. Let $\varphi : X \hookrightarrow \mathbb{P}^m$ be an embedding of X that extends the group action, i.e. we have a rationally group action on $\mathbb{P}^m$ such that $\varphi(g.x) = g.\varphi(x)$. Let $\pi\colon \mathbb{A}^{m+1} \to \mathbb{P}^m$ be the natural projection. A linearisation of the action of G with respect to φ is a linear action of G on $\mathbb{A}^{m+1}$ that is compatible with the action of G on X in the following sense

1. For any $y \in \mathbb{A}^{m+1}, g \in G$

$$\pi(g.y) = g.(\pi(y))$$

2. For all $g \in G$, the map

$$\mathbb{A}^{m+1} \to \mathbb{A}^{m+1}, \quad y \mapsto g.y$$

   is linear.

We write φ_G for a linearisation of the action of G with respect to φ.

Remark 1.5.1. Note that such action induces an action of G on $\mathcal{O}(X)$. we have $\mathcal{O}(X) \cong k[x_0, \dots, x_m]/I$ for some homogeneous ideal I, since X is isomorphic to the image $\varphi(X) \subseteq \mathbb{P}^m$. Using the fact that G acts on $k[x_0, \dots, k_m]$ by $g.f(x_0, \dots, x_m) := f(g^{-1}.(x_0, \dots, x_m))$, we can know that G also acts on $\mathcal{O}(X)$, and it's well-defined, since $g.f' \in I$ for $f' \in I$.

Example 1.5.1. Let $\mathbb{C}^\times$ act on $\mathbb{P}^1$ by $\lambda.(x_0, x_1) = (x_0 : \lambda x_1)$. A linearisation can be given by the obvious action on $\mathbb{A}^2$ with $\lambda.(x_0, x_1) = (x_0, \lambda x_1)$.

The above example illustrates a quite important issue when we are constructing projective quotient: good categorical quotient may not exist. The only possible G-invariant morphism sends all orbits to a point, since $(1, 0), (0, 1)$ are both in the closure of $(1, t)$. But this fails to seperate closed orbits, so is not a good categorical quotient.

The solution to such problem is to take an open G-invariant subset which has a good categorical quotient. We desire this subset to be covered by G-invariant open affine subsets so that we can cover the quotient by gluing together affine quotients. This leads us to the notion of semistability,

Definition 1.5.2. Let G be a reductive group acting on a projective variety X which has an embedding $\varphi : X \to \mathbb{P}^m$. A point $x \in X$ is called semistable (with respect to the linearisation φ_G) if there exists some G-invariant homogeneous polynomial f of degree greater than 0 in $\mathcal{O}(X)$, such that $f(x) \neq 0$ and X_f is affine.

Remark 1.5.2. Write $X^{\mathrm{as}}(\varphi_G)$ for the set of semistable points of X with respect to φ_G, or just X^{as} when it's not ambiguous.

For Example7.4.3, the set of semistable points of X with respect to φ_G is $X^{\mathrm{as}} = X_{x_0} = \mathbb{P}^1 \backslash \{(0:1)\}$. On this subset, the map to a point $p : X^{\mathrm{as}} \to \mathbb{P}^0$ is indeed a good categorical quotient.

Theorem 1.5.1. Let G be a reductive group acting rationally on a projective variety X embedded in $\mathbb{P}^m$ with a linearisation φ_G. Let R be the coordinate ring of X, then there is a good categorical quotient

$$p : X^{\mathrm{as}}(\varphi_G) \to X^{\mathrm{as}(\varphi_G)}//G \cong \operatorname{Proj} R^G$$

2. Symplectic quotient

A good reference to this section is [**?**].

2.1. **A quick review to symplectic geometry.** Let M be a smooth manifold admitting a Lie group G action, such manifold is often called a G-manifold. There is a one to one correspondence

$$\{\text{action of } \mathbb{R} \text{ on } M\} \longleftrightarrow \{\text{complete vector fields over } M\}$$

given by $\psi \mapsto X_p = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \psi(t,p)$. In particular, let X be an element of Lie algebra $\mathfrak{g}$, there is a complete vector field given by

$$\sigma(X) := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \exp(-tX)p$$

which is called fundamental field of X.

Definition 2.1.1 (symplectic manifold)**.** A symplectic manifold M is an even-dimensional manifold with a non-degenerate closed 2-form ω, which is called symplectic form.

Definition 2.1.2 (symplectomorphic)**.** A diffeomorphism between two symplectic manifolds $f\colon (M,\omega_M) \to (N,\omega_N)$ is called a symplectomorphic if

$$f^*\omega_N = \omega_M$$

Remark 2.1.1. The group consists of symplectomorphic of (M,ω) is denoted by $\mathrm{Sympl}(M,\omega)$, which is a subgroup of $\mathrm{Diff}(M)$.

Example 2.1.1 (standard symplectic manifold)**.** Consider $\mathbb{R}^{2n}$, there is a natural symplectic form given by

$$\omega_{\mathbb{R}^{2n}} = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}y^i$$

$(\mathbb{R}^{2n},\omega_{\mathbb{R}^{2n}})$ is called standard symplectic manifold. It's clear $(\mathbb{R}^{2n},\omega_{\mathbb{R}^{2n}})$ is symplectomorphic to $(\mathbb{C}^n,\omega_{\mathbb{C}^n})$, where $\omega_{\mathbb{C}^n} = \frac{\sqrt{-1}}{2}\sum_{i=1}^n \mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i$.

Theorem 2.1.1 (Darboux)**.** Let (M,ω) be a symplectic $2n$-manifold, around every $x \in M$, there exists a local coordinate $(x^1,\dots,x^n,y^1,\dots,y^n)$, which is sometimes caleed Darboux coordinate, such that

$$\omega = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}y^i$$

that is (M,ω) is locally symplectomorphic to the standard symplectic manifold $(\mathbb{R}^{2n},\omega_{\mathbb{R}^{2n}})$.

Proof. A good reference for the proof is □

Remark 2.1.2. In Hamiltonian mechanics, the manifold M is a cotangent bundle T^*U, the coordinates $x = (x^1,\dots,x^n)$ parameterize a point in U (the position), and the coordinates $y = (y^1,\dots,x^n)$ parameterize a point in the cotangent space T_xU (the momentum).

Definition 2.1.3. Let f be a smooth function over symplectic manifold (M, ω), then vector field X_f is defined as follows

$$\mathrm{d}f = \iota_{X_f}\omega$$

Remark 2.1.3 (local form). In Darboux coordinates, one has

$$\begin{aligned}\mathrm{d}f &= \sum_{i=1}^{n}\frac{\partial f}{\partial x^i}\mathrm{d}x^i + \sum_{i=1}^{n}\frac{\partial f}{\partial y^i}\mathrm{d}y^i \\ X_f &= \sum_{i=1}^{n}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i} - \sum_{i=1}^{n}\frac{\partial f}{\partial y^i}\frac{\partial}{\partial y^i}\end{aligned}$$

2.2. **Hamiltonian action.**

Definition 2.2.1 (symplectic action)**.** A symplectic action of a Lie group G over a symplectic manifold (M, ω) is a Lie group action on M which preserves ω.

Remark 2.2.1. If X is the vector field given rise from this action, then it's symplectic if and only if $\mathcal{L}_X\omega = 0$.

Let (M, ω) be a symplectic manifold, note that the non-degeneracy of ω gives an isomorphism $T_pM \to T_pM^*$ for each $p \in M$, that is we have the following one to one correspondence

$$\begin{aligned} C^\infty(M, TM) &\longleftrightarrow C^\infty(M, \Omega_M) \\ X &\mapsto \iota_X\omega \end{aligned}$$

Cartan's formula says $\mathcal{L}_X\omega = \iota_X\mathrm{d}\omega + \mathrm{d}(\iota_X\omega)$, then by closedness of ω one has $\iota_X\omega$ is closed if and only if $\mathcal{L}_X\omega = 0$, this yields the well-defineness of following definition.

Definition 2.2.2 (symplectic vector field)**.** A vector field X on a symplectic manifold (M, ω) is symplectic if the following equivalent conditions are satisfied

1. its associated 1-form is closed;
2. its associated $\mathbb{R}$-action is symplectic;
3. $\mathcal{L}_X\omega = 0$.

Remark 2.2.2. The symplectic vector field is just like Killing field in Riemannian geometry, and by the same reason one has symplectic vector fields are closed under Lie bracket, since

$$\mathcal{L}_{[X,Y]}\omega = \mathcal{L}_X\mathcal{L}_Y\omega - \mathcal{L}_Y\mathcal{L}_X\omega$$

Example 2.2.1. Let V be a complex vector space equipped with a hermitian product $\langle\text{-}, \text{-}\rangle$, there is a natural symplectic form given by its fundamental form, that is

$$\omega = -\operatorname{Im}\langle\text{-}, \text{-}\rangle$$

Indeed, (V,ω) is symplectomorphic to $(\mathbb{C}^n,\omega_{\mathbb{C}^n})$. Suppose furthermore there is a complex linear action of a Lie group G on V, and suppose $\langle -,- \rangle$ is G-invariant, then action of G is symplectic.

Definition 2.2.3 (Hamiltonian action)**.** Let G be a Lie group and (M,ω) a symplectic G-manifold, the action of G is Hamiltonian if there exists a map $\mu\colon M \to \mathfrak{g}^*$ such that

1. For every $X \in \mathfrak{g}$, if $\mu^X : M \to \mathbb{R}$ is given by $\mu^X(p) := \langle \mu(p), X\rangle$, then
$$\iota_{\sigma(X)}\omega = \mathrm{d}\mu^X$$
2. μ is equivariant with respect to the action of G on M and co-adjoint action of G on $\mathfrak{g}^*$.

Remark 2.2.3. The function μ above is called moment map and functions μ^X are called Hamiltonian functions.

Example 2.2.2. Let K be a compact Lie group acting on a vector space V, and $\langle -,- \rangle$ a K-invariant hermitian product[3]. In Example 2.2.1 we have seen there is a symplectic structure on V and action of K is symplectic with respect to it. Now we're going to show such an action is actually Hamiltonian.

Firstly, suppose K acts through a homomorphism $\rho : K \to \mathrm{GL}(V)$, then there is an induced representation of $\mathfrak{k}$, given by differential of ρ. To be explicit, for $\xi \in \mathfrak{k}$
$$\xi v := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \rho(\exp(t\xi))v = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \exp(t\mathrm{d}\rho(e)(\xi))v = \mathrm{d}\rho(e)(\xi)v$$
Now we're going to show the moment map is given by
$$\langle \mu(v), \xi\rangle := \frac{1}{2}\operatorname{Im}\langle v, \xi v\rangle$$
where $v \in V$ and $\xi \in \mathfrak{k}$ as follows:

1. Direct computation shows
$$\begin{aligned}\mathrm{d}\mu^\xi(v)(w) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \operatorname{Im}\langle v+tw, \xi v + t\xi w\rangle \\ &= \frac{1}{2}\operatorname{Im}\langle w, \xi v\rangle + \frac{1}{2}\operatorname{Im}\langle v, \xi w\rangle\end{aligned}$$
Note that $\mathfrak{k}$ acts on V as skew-hermitian matrices, so we have
$$\langle v, \xi w\rangle = -\overline{\langle w, \xi v\rangle}$$
This shows
$$\begin{aligned}\mathrm{d}\mu^\xi(v)(w) &= \frac{1}{2}\operatorname{Im}(\langle w, \xi v\rangle - \overline{\langle w, \xi v\rangle}) \\ &= -\operatorname{Im}\langle \xi v, w\rangle \\ &= \omega_v(\sigma(\xi), w)\end{aligned}$$

[3]Such hermitian product can be obtained from Haar's integral.

2. To see μ is K-equivariant:

$$\begin{aligned}\langle\mu(gv),\xi\rangle &= \frac{1}{2}\operatorname{Im}\langle\rho(g)v, \mathrm{d}\rho(e)(\xi)\rho(g)v\rangle \\ &= \frac{1}{2}\operatorname{Im}\langle v, \rho(g)^*\mathrm{d}\rho(e)(\xi)\rho(g)v\rangle\end{aligned}$$

Since $\rho(g)$ is unitary, then $\rho(g)^* = \rho(g)^{-1}$, then $\rho(g)^*\mathrm{d}\rho(e)(\xi)\rho(g)v = \mathrm{ad}_g(\xi)v$, which implies

$$\langle\mu(gv),\xi\rangle = \langle\mu(v), \mathrm{ad}_g(\xi)\rho\rangle$$

This completes the proof.

2.3. **Symplectic reduction.**

Theorem 2.3.1 (Meyer, Marsden-Weinstein)**.** Let (M,ω,G) be a Hamiltonian G-manifold with moment map μ. Suppose that the action of G is free and proper on $\mu^{-1}(0)$. Then

1. $M_{\mathrm{red}} := \mu^{-1}(0)/G$ is a manifold;
2. The projection $\pi\colon \mu^{-1}(0)\to\mu^{-1}(0)/G$ is a principal G-bundle;
3. There is a symplectic form ω_{red} on M_{red} such that $i^*\omega = \pi^*\omega_{\mathrm{red}}$, where $i\colon \mu^{-1}(0)\to M$ is natural inclusion.

3. The Kempf-Ness theorem

3.1. **Baby version.**

3.2. **Statement and proof of the Kempf-Ness theorem.**

Theorem 3.2.1 (Kempf-Ness)**.** Let G be the complexification of a compact real Lie group K acting on a finite dimensional complex vector space V through a representation $\rho\colon G \to \mathrm{GL}(V)$. Suppose that the action of K is Hamiltonian with respect to the symplectic form on V induced by a K-invariant hermitian product. Let $X \subseteq V$ be a smooth G-invariant affine variety, then

1. $\mu^{-1}(0) \subseteq X^{ps}$;
2. $X^{ps} \subseteq G\mu^{-1}(0)$;
3. Every G-orbit in X^{ps} contains only one K-oribit of $\mu^{-1}(0)$;
4. There is a bijection

$$X//G \cong X^{ps}/G \to \mu^{-1}(0)/K$$

References

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