CHERN INEQUALITIES IN HIGHER DIMENSION

BOWEN LIU

ABSTRACT. It's a lecture note for studying the paper [Miy87].

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0. Conventions

- (1) An (algebraic) variety over a field k is an integral seperated scheme of finite type over k.
- (2) A subvariety of a variety is a closed subscheme which is a variety.
- (3) A curve, surface or a threefold means a variety of dimension 1, 2 or 3.
- (4) A point on a scheme will always be a closed point.

1. Preliminaries

In this section, unless otherwise specified, X always denote a variety of dimension n over an algebraically closed field k.

1.1. Torsion-freeness and relexivity.

$1.1.1.\ Torsion\mbox{-} freeness.$

Definition 1.1.1. An \mathscr{O}_X -module \mathscr{F} is said to be **locally free sheaf** if there is an open covering $\{U_i\}$ of X such that $\mathscr{F}|_{U_i} \cong \mathscr{O}_{U_i}^{\oplus r}$ holds for every U_i .

Definition 1.1.2. An \mathscr{O}_X -module \mathscr{F} is said to be **coherent sheaf** if

- (1) \mathscr{F} is of finite type.
- (2) For every open subset $U \subseteq X$ and every morphism $\alpha \colon \mathscr{O}_U^r \to \mathscr{F}|_U$, the kernel of α is of finite type.

Definition 1.1.3. A coherent sheaf \mathscr{F} on X is **torsion-free** if a stalk \mathscr{F}_x is a torsion-free $\mathscr{O}_{X,x}$ -module for every $x \in X$.

Definition 1.1.4. A coherent subsheaf \mathscr{F} of a torsion-free sheaf \mathscr{E} is said to be **saturated** if the quotient \mathscr{E}/\mathscr{F} is again torsion-free.

Proposition 1.1.1. Let X, Y be two varieties and $f: X \to Y$ be a dominant morphism. Then for any torsion-free \mathscr{O}_X -module \mathscr{F} , the direct image $f_*\mathscr{F}$ is a torsion-free \mathscr{O}_Y -module.

Proof. See Proposition 8.4.5 in [GD71]. \Box

Proposition 1.1.2. Let X be a normal variety. Then every torsion-free sheaf is locally free outside a set of codimension two.

Proof. See Proposition 5.1.7 in [Ish14].

Corollary 1.1.1. Every torsion-free sheaf on a smooth curve is locally free.

1.1.2. Reflexivity.

Definition 1.1.5. A coherent \mathscr{O}_X -module \mathscr{F} is said to be **reflexive** if the canonical homomorphism $\mathscr{F} \to \mathscr{F}^{**}$ is an isomorphism.

Proposition 1.1.3. Every locally free sheaf is reflexive, and every reflexive sheaf is torsion-free.

Proof. It follows from the definitions.

Proposition 1.1.4. The dual sheaf of any coherent sheaf is reflexive.

Proof. See Proposition 5.5.18 in [Kob87].

Theorem 1.1.1. Let S be a smooth surface and \mathscr{E} be a torsion-free on S. Then \mathscr{E}^{**} is a locally free sheaf.

1.2. Chow ring.

1.2.1. *Cycles*.

Definition 1.2.1. A k-cycle on X is a \mathbb{Z} -linear combination of irreducible subvarieties of dimension k. The group of all such is denoted by $Z_k(X)$, and a **Weil divisor** on X is an (n-1)-cycle.

Remark 1.2.1. There is a cycle map from the group of Cartier divisor to the group of Weil divisor. In general it's neither injective nor surjective, but it's injective when X is normal and an isomorphism when X is smooth.

Definition 1.2.2. A k-cycle α on X is defined to be **rationally equivalent to zero** if there are finitely many (k+1)-dimensional irreducible subvarieties $W_i \subseteq X$ and non-zero rational functions. $f_i \in \mathbb{C}(W_i)$ such that

$$\alpha = \sum_{i} [\operatorname{div}_{W_i}(f_i)],$$

where $\operatorname{div}_{W_i}(f_i)$ is the divisor of the rational functions f_i on f_i on f_i on f_i

Definition 1.2.3. The group of k-cycles modulo rational equivalences is defined to be $A_k(X)$, which is said to be the k-th **Chow group**.

Example 1.2.1. $A_{n-1}(X)$ is the group of Weil divisors modulo linear equivalence.

Definition 1.2.4. The group of cycles of codimension k modulo rational equivalence is defined to be $A^k(X) := A_{n-k}(X)$.

1.2.2. The intersection pairing.

Theorem 1.2.1. Let X be a smooth variety. There is a unique intesection product $A^r(X) \times A^s(X) \to A^{r+s}(X)$ for each r, s satisfying the axioms listed below

- (1) The intersection pairing makes makes $A^*(X)$ into a commutative associated graded ring with identity. It's called the **Chow ring** of X.
- (2) For any morphism $f: X \to Y$, $f^*: A^*(Y) \to A^*(X)$ is a ring homomorphism. If $g: Y \to Z$ is another morphism, then $f^* \circ g^* = (g \circ f)^*$.
- (3) If $f: X \to Y$ is a proper morphism, $f_*: A^*(X) \to A^*(Y)$ is a homomorphism of graded groups. If $g: Y \to Z$ is another proper morphism, then $g_* \circ f_* = (g \circ f)_*$.
- (4) If $f: X \to Y$ is a proper morphism, $x \in A^*(X)$ and $y \in A^*(Y)$, then

$$f_*(x.f^*y) = f_*(x), y.$$

This is said to be the **projection formula**.

(5) If Y, Z are cycles on X, and if $\Delta \colon X \to X \times X$ is the diagonal morphism, then

$$Y.Z = \Delta^*(Y \times Z).$$

¹Note that the subvariety W_i may fail to be normal, so this requires a more general definition of $\operatorname{div}_{W_i}(f_i)$ than the usual one.

(6) If Y and Z are subvarieties of X which intersec properly (meaning that every irreducible component of $Y \cap Z$ has codimension equal to codim $Y + \operatorname{codim} Z$), then

$$Y.Z = \sum i(Y, Z; W_j)W_j,$$

where the sum runs over the irreducible components W_j of $Y \cap Z$, and where the integer $i(Y, Z; W_j)$ depends only on a neighborhood of the generic point of W_j on X, which is said to be the **local intersection** multiplicity of Y and Z along W_j .

(7) If Y is a subvariety of X, and Z is an effective Cartier divisor meeting Y properly, then Y.Z is just the cycle associated to the Cartier divisor $Y \cap Z$ on Y, which is defined by restricting the local equation of Z to Y.

Proof. See appendix A.1 in [Har77].

- 1.3. Chern classes. In this section we assume X is a smooth variety.
- 1.3.1. Chern classes of locally free sheaf.

Definition 1.3.1. A locally free sheaf \mathscr{E} of rank r on X has **Chern classes** $c_i(\mathscr{E}) \in A^i(X)$ for all $0 \le i \le r$, which is defined by

$$\sum_{i=0}^{r} (-1)^{i} \pi^* c_i(\mathscr{E}) \xi^{r-i} = 0$$

in $A^r(\mathbb{P}(\mathscr{E}))$, where $\xi \in A^1(\mathbb{P}(\mathscr{E}))$ be the class of the divisor corresponding to $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ and $\pi \colon \mathbb{P}(\mathscr{E}) \to X$ be the projection.

Definition 1.3.2. Let $\mathscr E$ be a locally free sheaf of rank r on X. The **total** Chern class is

$$c(\mathscr{E}) = c_0(\mathscr{E}) + \dots + c_r(\mathscr{E}) \in A^*(X).$$

Proposition 1.3.1.

- (1) $c_0(\mathscr{E}) = 1$ for any \mathscr{E} and $c_1(\mathscr{O}_X) = 1$ for any X.
- (2) If $f: X \to Y$ is a morphism and \mathscr{E} is locally free on Y, then $c_i(f^*\mathscr{E}) = f^*(c_i(\mathscr{E}))$.
- (3) If $0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$ is an exact sequence, then $c(\mathscr{F}) = c(\mathscr{E})c(\mathscr{G})$.
- (4) $c_i(\mathscr{E}^{\vee}) = (-1)^i c_i(\mathscr{E})$, where \mathscr{E}^{\vee} is the dual of \mathscr{E} .
- (5) $c_1(\bigwedge^r \mathscr{E}) = c_1(\mathscr{E})$ when \mathscr{E} has rank r.
- (6) If D is a Cartier divisor on X, then

$$c_1(\mathscr{O}_X(D)) = D.$$

Proof. See appendix A.3 in [Har77].

1.3.2. Chern classes of coherent sheaf. Let F(X) be the free abelian group generated by the set of coherent sheaves (up to isomorphisms, otherwise it's not a set) on X, that is, an element of F(X) is a formal linear combination $\sum_{i} n_{i}\mathscr{F}_{i}$, where $n_{i} \in \mathbb{Z}$ and \mathscr{F}_{i} is coherent. Let

(E)
$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

be an exact sequence of sheaves, and we associate the element $Q(E) = \mathscr{F} - \mathscr{F}' - \mathscr{F}''$ of F(X) to this exact sequence.

Definition 1.3.3. The group of classes of sheaves K(X) on X is defined to be the quotient of F(X) by the subgroup generated by the Q(E), where E runs over all short exact sequences.

Definition 1.3.4. Let $F_1(X)$ be the free group generated by the set of locally free sheaves (up to isomorphisms), and $K_1(X)$ be the quotient of $F_1(X)$ by the subgroup generated by the Q(E), where E runs over all short exact sequences of locally free sheaves.

Theorem 1.3.1 ([BS58]). Let X be a smooth quasi-projective variety. Then the homomorphism $\epsilon \colon K_1(X) \to K(X)$ is a bijection.

Corollary 1.3.1. The definition of Chern classes can be extended to arbitrary coherent sheaves.

1.4. Cones of divisors and curves.

1.4.1. The cones of divisors.

Definition 1.4.1. Let D_1, D_2 be two Cartier divisors on X. Then D_1 is numerically equivalent to D_2 if $D_1 \cdot C = D_2 \cdot C$ for all irreducible curves C

Definition 1.4.2. The **Néron-Severi group** $N^1(X)$ is the quotient group of Cartier divisors by numerically equivalent, and

$$N^1(X)_{\mathbb{Q}} = N_1(X) \otimes_{\mathbb{Z}} \mathbb{Q}, \quad N^1(X)_{\mathbb{R}} = N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Theorem 1.4.1. The Néron-Severi group $N^1(X)$ is a free abelian group of finit rank, and the rank of $N^1(X)$ is said to be the **Picard number**.

Definition 1.4.3. Let C and C' be two 1-cycles on X. Then C and C' are **numerically equivalent** if they have the same intersection number with every Cartier divisor. The quotient group of $Z_1(X)$ by numerically equivalent relation is denoted by $N_1(X)$, and

$$N_1(X)_{\mathbb{Q}} = N_1(X) \otimes_{\mathbb{Z}} \mathbb{Q}, \quad N_1(X)_{\mathbb{R}} = N_1(X) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Remark 1.4.1. The intersection pairing

$$N^1(X) \times N_1(X) \to \mathbb{Z}$$

is by definition non-degenerate.

Definition 1.4.4. The **cone of effective curves** $NE(X) \subseteq N_1(X)_{\mathbb{R}}$ is the cone spanned by non-negative linear combinations of curves, and $\overline{NE}(X)$ is the **cone of pseudo-effective curves**, where $N_1(X)_{\mathbb{R}}$ is endowed with its usual topology as a \mathbb{R} -vector space.

1.4.2. Nef cones and ample cones.

Definition 1.4.5. A Cartier divisor on X is **nef (numerically effective)** if it has non-negative intersection with every irreducible curve on X.

Definition 1.4.6. The ample classes in $N^1(X)_{\mathbb{R}}$ forms an open cone NA(X), which is said to be **ample cone**.

Definition 1.4.7. The nef classes in $N^1(X)_{\mathbb{R}}$ forms a closed cone Nef(X), which is said to be **nef cone**.

Theorem 1.4.2. Let X be a projective variety.

- (1) The closure of the ample cone is the nef cone;
- (2) The interior of nef cone is the ample cone.

Theorem 1.4.3. Let X be a projective variety.

(1) The pseudo-effective cone is the closed cone dual to the nef cone, that is.

$$\overline{NE}(X) = \{ \gamma \in N_1(X)_{\mathbb{R}} \mid D \cdot \gamma \ge 0, \quad \forall \ D \in \overline{NA}(X) \}.$$

(2)

$$NA(X) = \{ \gamma \in N^1(X)_{\mathbb{R}} \mid D \cdot \gamma > 0, \quad \forall \ D \in \overline{NE}(X) - \{0\} \}.$$

Proof. See Theorem 1.4.28 and Theorem 1.4.29 in [Laz04].

1.5. Asymptotic Riemann-Roch.

Theorem 1.5.1. Let X be a projective variety of dimension n and D be a Cartier divisor on X. Then

$$\chi(X, \mathcal{O}(mD)) = \frac{D^n}{n!} m^n + O(m^{n-1}).$$

More generally, for any coherent sheaf \mathscr{F} on X,

$$\chi(X,\mathscr{F}\otimes\mathscr{O}_X(mD))=\operatorname{rank}\mathscr{F}\cdot\frac{D^n}{n!}m^n+O(m^{n-1}).$$

Proof. See Theorem 1.1.24 in [Laz04].

2. Techniques

2.1. **Semistable sheaves.** Let X be a smooth projective variety of dimension n over an algebraically closed field k of arbitrary characteristic.

Definition 2.1.1. The average first Chern class of a torsion-free sheaf $\mathscr E$ is

$$\delta(\mathscr{E}) = \frac{c_1(\mathscr{E})}{\operatorname{rank}\mathscr{E}} \in A^1(X)_{\mathbb{Q}}.$$

Definition 2.1.2. For a given (n-1)-tuple $\mathfrak{A} = (h_1, \ldots, h_{n-1})$ of nef \mathbb{Q} -Cartier divisors, the **average degree** with respect to \mathfrak{A} is the rational number $\delta_{\mathfrak{A}}(\mathscr{E}) = \delta(\mathscr{E})h_1 \ldots h_{n-1}$.

Definition 2.1.3. A torsion-free sheaf $\mathscr E$ is said to be **semistable** if

$$\delta_{\mathfrak{A}}(\mathscr{F}) \leq \delta_{\mathfrak{A}}(\mathscr{E})$$

for every non-zero subsheaf \mathscr{F} of \mathscr{E} .

Notation 2.1. If $\mathfrak{A} = ([H], \dots, [H])$, we use the terminology H-semistable instead of \mathfrak{A} -semistable.

Theorem 2.1.1 ([HN75]). Let \mathscr{E} be a torsion-free sheaf on X and $\mathfrak{A} \in \overline{NA}(X)^{n-1}_{\mathbb{O}}$. Then there exists a unique filtration $\Sigma_{\mathfrak{A}}$

$$0 = \mathscr{E}_0 \subsetneq \mathscr{E}_1 \subsetneq \cdots \subsetneq \mathscr{E}_s = \mathscr{E}$$

that has the following properties

- (1) $\operatorname{Gr}_i(\Sigma_{\mathfrak{A}}) = \mathscr{E}_i/\mathscr{E}_{i+1}$ is a torsion-free \mathfrak{A} -semistable sheaf;
- (2) $\delta_{\mathfrak{A}}(Gr_i(\Sigma_{\mathfrak{A}}))$ is a strictly decreasing function in *i*.

Definition 2.1.4. The coherent subsheaf \mathscr{E}_1 in the above \mathfrak{A} -semistable filtration is said to be the **maximal** \mathfrak{A} -destabilizing subsheaf of \mathscr{E} , which is characterized by the following properties:

- (1) $\delta_{\mathfrak{A}}(\mathscr{E}_1) \geq \delta_{\mathfrak{A}}(\mathscr{F})$ for every coherent subsheaf \mathscr{F} of \mathscr{E} ;
- (2) If $\delta_{\mathfrak{A}}(\mathscr{E}_{1}) = \delta_{\mathfrak{A}}(\mathscr{F})$ for $\mathscr{F} \subset \mathscr{E}$, then $\mathscr{F} \subset \mathscr{E}_{1}$.

Remark 2.1.1. The \mathfrak{A} -semistable filtration of the dual sheaf \mathscr{E}^* is essentially the same as that of \mathscr{E} , with each entry substituted by the duals of the quotient $\mathscr{E}/\mathscr{E}_{s-i}$.

Theorem 2.1.2. Let $\mathscr{E}_1^{\mathfrak{A}} \subset \mathscr{E}$ denote the maximal \mathfrak{A} -destabilizing subsheaf for $\mathfrak{A} \in \overline{NA}(X)^{n-1}_{\mathbb{Q}}$.

- (1) Let L be a closed affine segment joining $\mathfrak{A}, \mathfrak{C} \in \overline{NA}(X)^{n-1}_{\mathbb{Q}}$ and $\mathfrak{B} = (1-t)\mathfrak{A}+t\mathfrak{C}$ be a rational point on L. Then $\delta_{\mathfrak{A}}(\mathscr{E}_{1}^{\mathfrak{B}})=\delta_{\mathfrak{A}}(\mathscr{E}_{1}^{\mathfrak{A}})$ whenever $0 < t < \epsilon$, where ϵ is a positive constant depends continously on \mathfrak{C} provided \mathscr{E} and \mathfrak{A} is fixed.
- (2) Let $K \subset \overline{NA}(X)_Q^{n-1}$ be a compact subset and $\mathfrak{A} \in \overline{NA}(X)_{\mathbb{Q}}^{n-1}$ is away from K. Let $\mathfrak{A}\sharp K$ stands the union of the segments joining \mathfrak{A} and K. Then there exists an open neighborhood $U \subset N^1(X)_{\mathbb{Q}}$ of \mathfrak{A} such that $\delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{B}}) = \delta_{\mathfrak{A}}(\mathscr{E}_1^{\mathfrak{A}})$ for every $\mathfrak{B} \in U \cap (\mathfrak{A}\sharp K) \cap \overline{NA}(X)_{\mathbb{Q}}^{n-1}$.

(3) If $\mathfrak{A} \in NA(X)^{n-1}_{\mathbb{Q}}$, then there exists an open neighborhood $U \subset NA(X)^{n-1}_{\mathbb{Q}}$ of \mathfrak{A} such that $\delta_{\mathfrak{A}}(\mathscr{E}_{1}^{\mathfrak{B}}) = \delta_{\mathfrak{A}}(\mathscr{E}_{1}^{\mathfrak{A}})$ for every $\mathfrak{B} \in U$.

Proof. For simplicity, we show the case n=2 only, and the proof is quite similar for higher dimensions.

(1). Suppose $\mathfrak{C} = H \in \overline{NA}(X)_{\mathbb{Q}}$. If $\mathscr{E}^*(H)$ is globally generated, that is, there exists a surjective morphism $\mathscr{O}_X^{\oplus N} \to \mathscr{E}^*(H)$ for some integer N. By taking dual we have an injective morphism $\mathscr{E} \to \mathscr{O}_X^{\oplus N}(H)$, and thus

$$\delta_{\mathfrak{C}}(\mathscr{E}_{1}^{\mathfrak{B}}) \leq c,$$

where c is a constant depending on \mathscr{E} , and on \mathfrak{C} continuously. If H is ample, then there exists some integer m such that mH is globally generated, and thus in this case $\delta_{\mathfrak{C}}(\mathscr{E}_1^{\mathfrak{B}}) \leq c$ for some constant c depending on \mathscr{E} , and on \mathfrak{C} continuously. Finally if $H \in \overline{NA}(X)_{\mathbb{Q}}$, we also have the same result, as it's a limit of ample divisors. Furthermore, we put $c' = \delta_{\mathfrak{C}}(\mathscr{E}_1^{\mathfrak{A}})$. By the definition of the maximal destabilizing sheaves, we get

$$\delta_{\mathfrak{B}}(\mathscr{E}_{1}^{\mathfrak{A}}) \leq \delta_{\mathfrak{B}}(\mathscr{E}_{1}^{\mathfrak{B}}).$$

As $\delta_{\mathfrak{B}}$ is a linear function in $\mathfrak{B} = (1-t)\mathfrak{A} + t\mathfrak{C}$, this inequality is rewritten as

$$(1-t)\delta_{\mathfrak{A}}(\mathscr{E}_{1}^{\mathfrak{A}}) + t\delta_{\mathfrak{C}}(\mathscr{E}_{1}^{\mathfrak{A}}) \leq (1-t)\delta_{\mathfrak{A}}(\mathscr{E}_{1}^{\mathfrak{B}}) + t\delta_{\mathfrak{C}}(\mathscr{E}_{1}^{\mathfrak{B}}).$$

Hence

$$\begin{split} \delta_{\mathfrak{A}}(\mathscr{E}_{1}^{\mathfrak{B}}) &\leq \delta_{\mathfrak{A}}(\mathscr{E}_{1}^{\mathfrak{A}}) \leq \delta_{\mathfrak{A}}(\mathscr{E}_{1}^{\mathfrak{B}}) + \frac{t}{1-t} (\delta_{\mathfrak{C}}(\mathscr{E}_{1}^{\mathfrak{B}}) - \delta_{\mathfrak{C}}(\mathscr{E}_{1}^{\mathfrak{A}})) \\ &\leq \delta_{\mathfrak{A}}(\mathscr{E}_{1}^{\mathfrak{B}}) + \frac{t}{1-t} (c-c'). \end{split}$$

Note that $\delta(\mathscr{E}_1^{\mathfrak{A}}), \delta(\mathscr{E}_1^{\mathfrak{B}}) \in (1/r!)A^1(X)_{\mathbb{Z}}$ and $\mathfrak{A} \in (1/m)N^1(X)_{\mathbb{Z}}$ for some positive integer m. Therefore, if

$$\frac{t}{1-t}(c-c') < \frac{1}{r!m},$$

then $\delta_{\mathfrak{A}}(\mathscr{E}_{1}^{\mathfrak{A}}) = \delta_{\mathfrak{A}}(\mathscr{E}_{1}^{\mathfrak{B}}).$

- (2). Let U be the open ball centered at \mathfrak{A} with radius r, where $r = \inf_{\mathfrak{C} \in K} \epsilon(\mathscr{E}, \mathfrak{A}, \mathfrak{C}) d(\mathfrak{A}, \mathfrak{C})$, d standing for Euclidean metric.
 - (3). Let $K \subset NA(X)^{n-1}_{\mathbb{Q}}$ be a sphere centered at \mathfrak{A} and apply (2). \square

Corollary 2.1.1. Let $K \subset \overline{NA}(X)^{n-1}_{\mathbb{Q}}$ be a compact subset and $\mathfrak{A} \in \overline{NA}(X)^{n-1}_{\mathbb{Q}}$ is away from K. Then the \mathfrak{B} -semistable filtration is a refinement of \mathfrak{A} -semistable filtration for all $\mathfrak{B} \in \mathfrak{A} \sharp K$ sufficiently near \mathfrak{A} .

Proof. By (2) of above theorem, we have $\mathscr{E}_1^{\mathfrak{B}} \subseteq \mathscr{E}_1^{\mathfrak{A}}$ for all $\mathfrak{B} \in \mathfrak{A} \sharp K$ sufficiently near \mathfrak{A} . If \mathscr{E} is semistable, it's clear that the \mathfrak{B} -semistable filtration of \mathscr{E} is a refinement of \mathfrak{A} -semistable filtration of \mathscr{E} .

Corollary 2.1.2. Let \mathscr{E} be a torsion-free sheaf on X.

- (1) The \mathfrak{A} -semistability of \mathscr{E} is a closed condition for $\mathfrak{A} \in NA(X)^{n-1}_{\mathbb{O}}$.
- (2) The length of the \mathfrak{A} -semistability of \mathscr{E} is a lower semicontinous in $\mathfrak{A} \in NA(X)^{n-1}_{\mathbb{O}}$, while rank $\mathscr{E}^{\mathfrak{A}}_{1}$ is upper semicontinous.
- (3) $\delta_{\mathfrak{A}}(\mathscr{E}_{1}^{\mathfrak{A}})$ is continous, piecewise multilinear function on $NA(X)_{\mathbb{Q}}^{n-1}$ and continous on any rational segment of $\overline{NA}(X)_{\mathbb{Q}}^{n-1}$.
- 2.2. A numerical criterion for semistability on curves. Throught this section, the ground field k is always an algebraically closed field with characteristic 0 except Lemma 2.2.1, and C is a smooth complete curve.
- 2.2.1. Preliminaries on projective bundle. Let \mathscr{E} be a locally free sheaf of rank r on C and $\pi \colon \mathbb{P}(\mathscr{E}) \to C$ the associated projective bundle with tautological line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$.

Definition 2.2.1. The **normalized hyperplane class** $\lambda_{\mathscr{E}}$ is the numerical class of $c_1(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)) - \pi^*\delta(\mathscr{E}) \in N^1(\mathbb{P}(\mathscr{E}))_{\mathbb{Q}}$.

Proposition 2.2.1. The class of relative anti-canonical divisor $-K_{\mathbb{P}(\mathscr{E})} + \pi^* K_C$ equals $r \lambda_{\mathscr{E}}$.

Proposition 2.2.2. The normalized hyperplane class $\lambda_{\mathscr{E}}$ is uniquely determined by two properties:

- (1) $\lambda_{\mathscr{E}}^{r} = 0$.
- (2) $\lambda_{\mathscr{E}}$ on each fiber is numerically equivalent to the hyperplane.

Proposition 2.2.3. The Néron-Severi group of $\mathbb{P}(\mathscr{E})$ is

$$N^{1}(\mathbb{P}(\mathscr{E})) = \mathbb{R} \, \lambda_{\mathscr{E}} + \pi^{*} N^{1}(X),$$

and the group of numerically equivalent 1-cycles is

$$N_1(\mathbb{P}(\mathscr{E})) = \lambda_{\mathscr{E}}^{r-2} N^1(\mathbb{P}(\mathscr{E})).$$

2.2.2. Criterion.

Lemma 2.2.1. Let f be a seperable surjective k-morphism of a smooth complete curve C' onto C. Then a locally free sheaf $\mathscr E$ is semistable if and only if $f^*\mathscr E$ is semistable.

Proof. Firstly let's prove "if" part. Let $\mathscr{G} \subseteq \mathscr{E}$ be a non-zero subsheaf. Then $\delta(f^*\mathscr{G}) \leq \delta(f^*\mathscr{E})$ as $f^*\mathscr{E}$ is semistable, and thus $\delta(\mathscr{G}) \leq \delta(\mathscr{F})$.

Conversely, suppose \mathscr{E} is semistable. Without lose of generality we may assume f is a Galois morphism with Galois group G, which acts on $f^*\mathscr{E}$. If $f^*\mathscr{E}$ is not semistable and \mathscr{F}_1 be the maximal destabilizing subbundle of $f^*\mathscr{E}$. For any $g \in G$, we have $g^*\mathscr{F}_1 = \mathscr{F}_1$ as the maximal destabilizing subsheaf is unique. Hence there exists a subbundle \mathscr{E}_1 of \mathscr{E} such that $f^*\mathscr{E}_1 = \mathscr{F}_1$, and by "if" part \mathscr{E}_1 is semistable. On the other hand, $\mathscr{E}_1 = \mathscr{E}$ by semistability, and thus $\mathscr{F}_1 = f^*\mathscr{E}$. This completes the proof.

Theorem 2.2.1. The following conditions are equivalent:

(1) \mathscr{E} is semistable;

- (2) $\lambda_{\mathscr{E}}$ is nef;
- (3) $\overline{NA}(\mathbb{P}(\mathscr{E})) = \mathbb{R}_+ \lambda_{\mathscr{E}} + \mathbb{R}_+ \pi^* d$, where d is a positive generator of $N^1(C)_{\mathbb{Z}} \cong$
- (4) $\overline{NE}(\mathbb{P}(\mathscr{E})) = \mathbb{R}_+ \lambda_{\mathscr{E}}^{r-1} + \mathbb{R}_+ \lambda_{\mathscr{E}}^{r-2} \pi^* d;$ (5) Every effective divisor on $\mathbb{P}(\mathscr{E})$ is nef.

Proof. (1) to (2). Suppose $\lambda_{\mathscr{E}}$ is not nef, that is, there exists an irreducible curve $C' \subset \mathbb{P}(\mathscr{E})$ with $C'\lambda_{\mathscr{E}} < 0$. It's clear that C' is mapped surjective onto C

(2) to (4). If $\lambda_{\mathscr{E}}^{r-2}(a\lambda_{\mathscr{E}}+b\pi^*d)$ is pseudo-effective and $\lambda_{\mathscr{E}}$ is nef, then

$$b = \lambda_{\mathscr{E}}^{r-1}(a\lambda_{\mathscr{E}} + b\pi^*d) \ge 0.$$

On the other hand, $\lambda_{\mathscr{E}}^{r-1}$ is pseudo-effective since $\lambda_{\mathscr{E}}$ is nef, and thus $a \geq 0$. The equivalent between (3) and (4) is straightforward since the nef cone is the closed cone dual to the pseudo-effective cone (Theorem 1.4.3).

- (3) and (4) to (5). Since $\lambda_{\mathscr{E}}$ is nef, $\lambda_{\mathscr{E}} + \epsilon \pi^* d$ is ample for any positive real number ϵ . Assume $a\lambda_{\mathscr{E}} + b\pi^*d$ is an effective divisor. Then the 1-cycles $(a\lambda_{\mathscr{E}} + b\pi^*d)(\lambda_{\mathscr{E}} + \epsilon\pi^*d)^{r-2}$ is effective, and thus their limit $(a\lambda_{\mathscr{E}} + b\pi^*d)\lambda_{\mathscr{E}}^{r-2}$ is pseudo-effective. Then by (4) one has $a, b \ge 0$, and thus $a\lambda_{\mathscr{E}} + b\pi^*d$ is nef by (3).
- (5) to (1). Suppose that \mathscr{E} is unstable and let \mathscr{E}_1 be the maximal destabilizing subbundle. Let α be a rational number with $\delta(\mathcal{E}_1) > \alpha > \delta(\mathcal{E})$. Then by the Riemann-Roch theorem,

$$H^{0}(C, \operatorname{Sym}^{N} \mathscr{E}_{1}(-N\alpha d)) \subseteq H^{0}(C, \operatorname{Sym}^{N} \mathscr{E}(-N\alpha d))$$

$$\cong H^{0}(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(N) \otimes \pi^{*}\mathscr{O}_{C}(-N\alpha d)))$$

is non-trivial for sufficiently large N. Then $N\{\lambda_{\mathscr{E}} + (\delta(\mathscr{E}) - \alpha)\pi^*d\}$ is effective but clearly not nef.

2.2.3. Semipositive and semistability.

Definition 2.2.2. Let D be a \mathbb{Q} -Cartier divisor on C. A \mathbb{Q} -torsion-free sheaf $\mathscr{F} = \mathscr{E}(D)$ is said to be **ample** or **semipositive** if $\xi + \pi^*D$ is ample or nef, where $\xi = c_1(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1))$.

Definition 2.2.3. A \mathbb{Q} -torsion-free sheaf \mathscr{F} is said to be **negative** or **seminegative** if \mathscr{F}^* is ample or semipositive.

Proposition 2.2.4. The direct sums, tensor products, symmetric products and exterior products of ample (or semipositive) Q-torsion-free sheaves are all ample (or semipositive).

Theorem 2.2.2. Let \mathscr{E} be a vector bundle on C. Then \mathscr{E} is semistable if and only if $\mathscr{E}(-\delta(E))$ is semipositive.

Proof. It follows from Theorem 2.2.1.

Corollary 2.2.1. Let \mathscr{E} be a vector bundle on C. Then \mathscr{E} is semistable if and only if $\mathscr{E}(-\delta(E))$ is seminegative.

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Corollary 2.2.2.

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- (1) The Q-vector bundle $\mathscr{E}(-D)$ is seminegative if and only if deg $D \ge \deg \delta(\mathscr{E}_1)$, where \mathscr{E}_1 is the maximal destabilizing subsheaf of \mathscr{E} .
- (2) The Q-vector bundle $\mathscr{E}(-D)$ is negative if and only if deg $D > \deg \delta(\mathscr{E}_1)$, where \mathscr{E}_1 is the maximal destabilizing subsheaf of \mathscr{E} .
- (3) The \mathbb{Q} -vector bundle $\mathscr{E}(D)$ is semipositive if and only if deg $D \ge \deg \delta((\mathscr{E}^*)_1)$.
- (4) The Q-vector bundle $\mathscr{E}(D)$ is positive if and only if deg $D > \deg \delta((\mathscr{E}^*)_1)$.

Proof. For simplicity we only prove the first statement, and the proof is quite similar for others. Let $\mathscr{E}_1 \subset \cdots \subset \mathscr{E}_s = \mathscr{E}$ be the semistable filtration of \mathscr{E} . Since $\mathscr{G}_i = \mathscr{E}_i/\mathscr{E}_{i-1}$ is semistable and $\deg \delta(\mathscr{G}_i)$ is decreasing in i, one has $\mathscr{G}_i(-\delta(\mathscr{E}_1))$ is seminegative for all i, and thus $\mathscr{E}(-\delta(\mathscr{E}_1))$ is seminegative. If $\deg D \ge \deg \delta(\mathscr{E}_1)$, then $\mathscr{E}(-D)$ is also seminegative.

Conversely, if deg D is smaller than deg $\delta(\mathscr{E}_1)$ for a \mathbb{Q} -divisor D, then $\mathscr{E}(-D)$, containing an ample \mathbb{Q} -vector bundle $\mathscr{E}_1(-D)$, is never seminegative.

Corollary 2.2.3. A semistable vector bundle \mathscr{E} on C is ample (resp. semipositive, seminegative, negative) if and only if its degree is positive (resp. semipositive, seminegative, negative).

Proof. Take
$$D = 0$$
 in Corollary 2.2.2.

Corollary 2.2.4. Let \mathscr{E} and \mathscr{F} be semistable bundles on C. Then $\mathscr{E} \otimes \mathscr{F}$ and $\mathscr{H}_{em}(\mathscr{E},\mathscr{F})$ are also semistable.

Proof. It follows from the semipositive bundle tensor with semipositive bundle is still semipositive. \Box

Corollary 2.2.5. Let \mathscr{E} and \mathscr{F} be two vector bundles. Then $\mathscr{H}om(\mathscr{E},\mathscr{F})$ is negative if and only if $\deg \delta(\mathscr{F}_1) + \deg \delta((\mathscr{E}^*)_1) < 0$. As a consequence, $\mathscr{H}om(\mathscr{E}_1,\mathscr{E}/\mathscr{E}_1)$ is negative.

Proof. For the first part, note that $\mathscr{H}om(\mathscr{E},\mathscr{F}) = \mathscr{E}^* \otimes \mathscr{F}$ and take D = 0 in Corollary 2.2.2. For the half part, it suffices to note $(\mathscr{E}/\mathscr{E}_1)_1 = \mathscr{E}_2/\mathscr{E}_1$. \square

Proposition 2.2.5. Let \mathscr{E} be a vector bundle on C. The following conditions are equivalent:

- (1) \mathscr{E} is semistable;
- (2) $\mathscr{E}(-D)$ is negative with D is a \mathbb{Q} -divisor of degree $\delta(\mathscr{E}) + (1/2r!)$.

Proof. The implication (1) to (2) follows from Corollary 2.2.1.

Conversely, assume (2) and let \mathscr{E}_1 be the maximal destabilizing subsheaf. Then by Corollary 2.2.2 we have $\mathscr{E}(-D)$ is negative if and only if deg $D > \deg \delta(\mathscr{E}_1)$ so that

$$\delta(\mathscr{E}) \le \delta(\mathscr{E}_1) < \delta(\mathscr{E}) + \frac{1}{2r!}.$$

On the other hand, both $\deg \delta(\mathscr{E}_1)$ and $\deg \delta(\mathscr{E})$ sit in $(1/r!)\mathbb{Z}$. Hence we have $\deg \delta(\mathscr{E}_1) = \deg \delta(\mathscr{E})$, and thus $\mathscr{E}_1 \cong \mathscr{E}$.

2.3. Mumford-Mehta-Ramanathan's theorem.

Theorem 2.3.1. Let X be a complex normal projective variety of dimension n and $\mathscr E$ be a torsion-free sheaf. Let H_1, \ldots, H_{n-1} be ample Cartier divisors. Then for sufficiently large integers m_1, \ldots, m_{n-1} , the maximal destabilizing subsheaf $\mathscr F$ of $\mathscr E|_C$ extends to a saturated subsheaf of $\mathscr E$ on X if C is a general complete intersection curve of $|m_iH_i|$'s. (Such an extension of $\mathscr F$ is necessarily the maximal (H_1, \ldots, H_{n-1}) -destabilizing subsheaf of $\mathscr E$ and hence unique.)

2.4. The Bogomolov-Gieseker inequality for semistable sheaves.

Lemma 2.4.1. Let X be a normal projective variety of dimension n and $\mathfrak{A} \in NA(X)^{n-1}$. Let \mathscr{E} be an \mathfrak{A} -semistable torsion-free sheaf on X, with its first Chern class being a \mathbb{Q} -Cartier divisor. Let D be a non-zero effective Cartier divisor on X. Then

$$H^0(X, \operatorname{Sym}^{rt} \mathscr{E}(-tc_1(\mathscr{E}) - D)) = 0$$

for every positive integer t such that $tc_1(\mathscr{E})$ is an integral Cartier divisor.

Corollary 2.4.1. Let things be as Lemma 2.4.1 and L be a fixed Cartier divisor. Then $h^0(X, \operatorname{Sym}^{rt} \mathscr{E}(-tc_1(\mathscr{E}) + L))$ is bounded by a polynomial of degree r-1 in t.

Proof. For simplicity of the notation, put $\mathscr{F}^t = \operatorname{Sym}^{rt} \mathscr{E}(-tc_1(\mathscr{E}))$. The proof is by induction on the dimension n of X. If n = 1, let D be a reduced effective divisor of degree $d > \deg L$. Then there is a natural exact sequence

$$H^0(X, \mathscr{F}^t(-D)) \to H^0(X, \mathscr{F}^t(L)) \to H^0(D, \mathscr{F}^t(L))$$

of which the first term vanishes by Lemma 2.4.1, where the last term is a k-vector space of dimension $d\binom{rt+r-1}{rt}=d\binom{rt+r-1}{r-1}$. This completes the proof of n=1.

For $n \geq 2$, let $\mathfrak{A} = (H_1, \ldots, H_n)$ in $NA(X)^{n-1}$, where H_i is integral and ample. Let Y be a general hyperplane section in $|mH_i|$ for sufficiently large m such that $\mathscr{E}|_Y$ is (H_1, \ldots, H_{n-2}) -semistable on Y and Y - L is ample. (Note that such a number m, though possible very large, is independent of t.) Consider the exact sequence

$$H^0(X, \mathscr{F}^t(L-Y)) \to H^0(X, \mathscr{F}^t(L)) \to H^0(Y, \mathscr{F}^t(L)).$$

The first term vanishes by Lemma 2.4.1 and the dimension of the last term is bounded by a polynomial of degree r-1 by the induction hypothesis. This completes the proof.

Theorem 2.4.1 (The Bogomolov-Gieseker inequality). Let S be a smooth projective surface over k. If $\mathscr E$ is an H-semistable torsion-free sheaf of rank r on S, where H is an ample divisor, then

$$(r-1)c_1^2(\mathscr{E}) \le 2rc_2(\mathscr{E}).$$

Proof. From Corollary 2.4.1, it follows that neither $h^0(S, \operatorname{Sym}^{rt} \mathscr{E}(-tc_1(\mathscr{E})))$ nor $h^2(S, \operatorname{Sym}^{rt} \mathscr{E}(-tc_1(\mathscr{E}))) = h^0(S, \operatorname{Sym}^{rt} \mathscr{E}^*(-tc_1(\mathscr{E}^*)) + K_S)$ grows like t^{r+1} . Hence we obtain the inequality

$$\chi(S, \operatorname{Sym}^{rt} \mathscr{E}(-tc_1(\mathscr{E}))) \leq \operatorname{polynomial} \text{ of degree } r \text{ in } t.$$

On the other hand, by the asymptotic Riemann-Roch theorem (Theorem 1.5.1),

$$\chi(S, \operatorname{Sym}^{rt} \mathscr{E}(-tc_1(\mathscr{E}))) = \frac{t^{r+1}}{(r+1)!} \left\{ rc_1(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)) - \pi^* c_1(\mathscr{E}) \right\}^{r+1} + O(t^r)$$
$$= \frac{(rt)^{r+1}}{(r+1)!} \left\{ -c_2(\mathscr{E}) + \frac{r-1}{2r} c_1^2(\mathscr{E}) \right\} + O(t^r).$$

This completes the proof.

Corollary 2.4.2. Let $\mathscr E$ be a locally free sheaf of rank r on a smooth surface S. Let L be an ample integral divisor on S such that $\mathscr E(-\delta(\mathscr E)+L)$ is ample and $\mathscr E(-\delta(\mathscr E)-L)$ is negative (as $\mathbb Q$ -vector bundles). Assume the inequality $2rc_2(\mathscr E)<(r-1)c_1^2(\mathscr E)$ and put

$$\alpha = \frac{(r-1)c_1^2(\mathscr{E}) - 2rc_2(\mathscr{E})}{6r^2(r+1)L^2} \in \mathbb{Q}.$$

Then either $\operatorname{Sym}^t \mathscr{E}(-t\delta(\mathscr{E}))$ or $\operatorname{Sym}^t \mathscr{E}^*(-t\delta(\mathscr{E}^*))$ contains the ample line bundle $\mathscr{O}_S(t\alpha L)$, where t is any very large integer such that $t\delta(\mathscr{E})$ and $t\alpha$ are integral.

Proof. For simplicity, put $\mathscr{F} = \mathscr{E}(-\delta(\mathscr{E}))$. Then we have

$$\chi(S,\operatorname{Sym}^t\mathscr{F})$$

Corollary 2.4.3. Let $\mathscr E$ be a torsion-free sheaf of rank r on a normal projective variety X of dimension n and H_1, \ldots, H_{n-2} be ample Cartier divisors. Let D be a nef Cartier divisor on X. Assume that $H_1 \ldots H_{n-2}D$ is not numerically trivial. If $\mathscr E$ is $(H_1, \ldots, H_{n-2}, D)$ -semistable, then

$$(r-1)c_1^2(\mathscr{E})H_1 \dots H_{n-2} \le 2rc_2(\mathscr{E})H_1 \dots H_{n-2}.$$

Corollary 2.4.4. Let $\mathscr E$ be a torsion-free sheaf of rank r on a normal projective variety X of dimension n and H_1, \ldots, H_{n-2} be ample Cartier divisors. If

$$\{(r-1)c_1^2(\mathscr{E}) - 2rc_2(\mathscr{E})\}H_1 \dots H_{n-2} > 0,$$

then \mathscr{E} is $(H_1, \ldots, H_{n-2}, D)$ -unstable for any non-zero nef divisor D.

2.5. Semistability in positive and mixed characteristic.

2.5.1. Semistability in positive characteristic. Let C be a smooth complete curve over an algebraically closed field k of characteristic p > 0.

Definition 2.5.1. A vector bundle \mathscr{E} on C is said to be **strongly semistable** if, for every positive integer s, $(F^s)^*\mathscr{E}$ is semistable, where $F^s \colon F^{-s}C \to C$ is the Frobenius k-morphism of degree $q = p^s$.

Proposition 2.5.1. If \mathscr{E} is strongly semistable on C, then $f^*\mathscr{E}$ is semistable for any surjective k-morphism $f: C' \to C$.

2.5.2. Semistability in mixed characteristic.

2.6. Generic semipositive theorem for cotangent bundle. From now on, all varieties are defined over an algebraically closed field k of characteristic 0. Let X be a normal projective variety of dimension n.

Definition 2.6.1. Let $\mathfrak{B} \in \overline{NA}(X)^{n-2}_{\mathbb{O}}$.

- (1) A torsion-free sheaf $\mathscr E$ on X is said to be **generically \mathfrak B-seminegative** if, for every numerically effective $\mathbb Q$ -Cartier divisor D on X, its maximal $(\mathfrak B, D)$ -destabilizing subsheaf $\mathscr E_1$ satisfies $\delta_{(\mathfrak B, D)}(\mathscr E_1) < 0$.
- (2) A torsion-free sheaf \mathscr{E} on X is said to be **generically \mathfrak{B}-semipositive** if \mathscr{E}^* is generically \mathfrak{B} -seminegative.

Lemma 2.6.1. Let \mathscr{E} be a torsion-free sheaf on X and

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_s = \mathcal{E}$$

be the (\mathfrak{B}, D) -semistable filtration of \mathscr{E} and put $\alpha_i = \delta_{(\mathfrak{B}, D)}(\mathscr{E}_i/\mathscr{E}_{i-1})$. Then $\alpha_1 > \cdots > \alpha_s \geq 0$ for every $D \in \overline{NA}(X)_{\mathbb{Q}}$ if \mathscr{E} is generically \mathfrak{B} -semipositive.

Proof. It follows from the definition.

Theorem 2.6.1. Let $\mathfrak{B} = (H_1, \dots, H_{n-2}) \in NA(X)^{n-2}_{\mathbb{Q}}$ and \mathscr{E} be a generically \mathfrak{B} -semipositive torsion-free sheaf on X. Then

$$c_2(\mathscr{E})H_1\ldots H_{n-2}\geq 0$$

holds.

Theorem 2.6.2. Let $\mathfrak{B} = (H_1, \ldots, H_{n-2}) \in NA(X)^{n-2}_{\mathbb{Q}}$. Then the torsion-free sheaf $\rho_*\Omega^1_{X'}$ is generically \mathfrak{B} -semipositive unless X is uniruled, where $\rho \colon X' \to X$ denotes an arbitrary resolution.

3. Results

3.1. Semipositivity of $3c_2 - c_1^2$.

Proposition 3.1.1. Let X be a non-uniruled, normal projective variety of dimension n with \mathbb{Q} -Cartier canonical divisor K_X which is nef. Let $\mathfrak{B} \in NA(X)^{n-2}_{\mathbb{Q}}$ such that $K_X^2|\mathfrak{B}|$ is positive. Then

$${3c_2(\mathscr{E}) - c_1(\mathscr{E})^2}|\mathfrak{B}| \ge 0,$$

where $\mathscr{E} = \rho_* \Omega^1_{X'}$ and $\rho \colon X' \to X$ is an arbitrary resolution.

3.2. Non-negativity of the Kodaira dimension of minimal three-folds.

3.2.1. The Gorenstein case.

Theorem 3.2.1. Let X be a normal projective Gorenstein threefold with only canonical singularities (X is Gorenstein if and only if K_X is a Cartier divisor). Assume K_X is nef. Then the Euler characteristic $\chi(X, \mathscr{O}_X)$ is nonnegative. In particular, either $h^0(X, \mathscr{O}_X(K_X))$ or $h^1(X, \mathscr{O}_X)$ is non-zero, and thus $\kappa(X) \geq 0$.

3.2.2. The K_X^2 is numerically non-trivial case.

Theorem 3.2.2. Let X be a normal projective Gorenstein threefold with only isolated singularities. Assume the \mathbb{Q} -Cartier divisor K_X is nef and K_X^2 is numerically non-trivial. Then $\kappa(X) \geq 0$.

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