ALGEBRAIC GEOMETRY

BOWEN LIU

Contents

Part 1. Preliminaries

1. Category theory

1.1. Category.

1.1.1. *Category and Functors.*

1.1.2. *Morphisms.*

Definition 1.1.1 (monomorphism)**.** A morphism $f\colon A \to B$ in $\mathcal{C}$ is called a monomorphism (or injective) if for any two morphisms $\alpha, \beta\colon C \to A$ satisfying $f \circ \alpha = f \circ \beta$, we have $\alpha = \beta$.

Definition 1.1.2 (epimorphism)**.** A morphism $f\colon A \to B$ in $\mathcal{C}$ is called a epimorphism (or surjective) if for any two morphisms $\alpha, \beta\colon A \to C$ satisfying $\alpha \circ f = \beta \circ f$, we have $\alpha = \beta$.

Definition 1.1.3 (bijective)**.** A morphism is called bijective if it's both monomorphism and epimorphism.

Definition 1.1.4 (isomorphism)**.** A morphism is called an isomorphism if it admits two-sided inverse.

Remark 1.1.1. Any isomorphism is bijective, but in general a bijective morphism may not be an isomorphism. For example, in the category of topological spaces, it's easy to construct a morphism (continuous map) which is a bijective map, but it's not an isomorphism.

1.1.3. *Categorical objects.*

Definition 1.1.5 (direct product)**.** Let $\{A_i\}_{i\in I}$ be a family of objects in category $\mathcal{C}$. The direct product of A_i is tuple $(\prod_{i\in I} A_i, p_i)$, where $\prod_{i\in I} A_i$ is an object in $\mathcal{C}$, and $p_i\colon \prod_{i\in I} A_i \to A_i$ is a family of morphisms called projections, such that the following universal property: For any object C and any family of morphisms $f_i\colon C \to A_i$, there exists a unique morphism $f\colon C \to \prod_{i\in I} A_i$ such that $p_i \circ f = f_i$ for all $i \in I$.

$$\begin{tikzcd}
\prod_{I\in I} A_i \arrow[d, "p_i"'] & C \arrow[l, dashed, "f"'] \arrow[dl, "f_i"] \\
A_i &
\end{tikzcd}$$

Definition 1.1.6 (direct sum)**.** Let $\{A_i\}_{i\in I}$ be a family of objects in category $\mathcal{C}$. The direct sum of A_i is tuple $(\bigoplus_{i\in I} A_i, k_i)$, where $\bigoplus_{i\in I} A_i$ is an object in $\mathcal{C}$, and $k_i\colon A_i \to \bigoplus_{i\in I} XIAOWEIJIANGA_i$ is a family of morphisms called projections, such that the following universal property: For any object C and any family of morphisms $f_i\colon A \to C$, there exists a unique morphism $f\colon \bigoplus_{i\in I} A_i \to C$ such that $f \circ k_i = f_i$ for all $i \in I$.

$$
\begin{tikzcd}
\bigoplus_{i\in I} A_i \arrow[r, dashed, "f"] & C \\
A_i \arrow[u, "k_i"] \arrow[ur, "f_i"'] &
\end{tikzcd}
$$

1.2. **Abelian category.**

1.2.1. *Additive category.*

Definition 1.2.1 (additive category)**.** A category $\mathcal{C}$ is called an additive category if for any objects A, B, C in $\mathcal{C}$,

(1) the direct product of A and B exists;
(2) $\operatorname{Hom}(A, B)$ is an abelian group, and $0 \in \operatorname{Hom}(A, B)$ is called zero morphism;
(3) the map
$$
\begin{aligned}
\operatorname{Hom}(A, B) \times \operatorname{Hom}(B, C) &\to \operatorname{Hom}(A, C) \\
(f, g) &\mapsto g \circ f
\end{aligned}
$$
is bilinear.

Definition 1.2.2. Let $\mathcal{C}$ be an additive category and $f\colon A \to B$ be a morphism in $\mathcal{C}$.

(1) A morphism $K \to A$ is the kernel of f if the composite $K \to A \to B$ is 0, and for any morphism $K' \to A$ such that the composite $K' \to A \to B$ is 0, there exists a unique morphism $K' \to K$ such that the diagram
$$
\begin{tikzcd}
K' \arrow[d] \arrow[dr] & \\
K \arrow[r] & A
\end{tikzcd}
$$
commutes. For convenience we often denote K by $\ker f$ and call it the kernel of f.
(2) A morphism $B \to C$ is the cokernel of f if the composite $A \to B \to C$ is 0, and for any morphism $B \to C'$ such that the composite $A \to B \to C$ is 0, there exists a unique morphism $C \to C'$ such that the diagram
$$
\begin{tikzcd}
B \arrow[r] \arrow[dr] & C \arrow[d] \\
& C'
\end{tikzcd}
$$
commutes. For convenience we often denote C by $\operatorname{coker} f$ and call it the cokernel of f.
(3) The image of f is defined to be the kernel of the cokernel of f, and the coimage of f is defined to be the cokernel of the kernel of f.

Remark 1.2.1. A kernel is neccessarily a monomorphism, and a cokernel is neccessarily an epimorphism.

Remark 1.2.2. There is a natural morphism $\operatorname{coim} f \to \operatorname{im} f$ induced by universal property

$$\begin{tikzcd} \ker f \arrow[r] & A \arrow[r, "f"] \arrow[d] & B \arrow[r] & \operatorname{coker} f \\ & \operatorname{coim} f \arrow[r, dashed] & \operatorname{im} f \arrow[u] & \end{tikzcd}$$

Definition 1.2.3 (zero object)**.** Let $\mathcal{C}$ be an additive category. A zero object 0 in $\mathcal{C}$ is an object such that $\operatorname{Hom}(0,0) = \{0\}$.

1.2.2. *Abelian category.*

Definition 1.2.4 (abelian category)**.** An abelian category $\mathcal{C}$ is an additive category with zero objects such that for every morphism f in $\mathcal{C}$, the kernel and the cokernel of f exist, and the canonical morphism $\operatorname{coim} f \to \operatorname{im} f$ is an isomorphism.

Proposition 1.2.1. In abelian category, a bijective morphism is an isomorphism.

Definition 1.2.5 (exact)**.** In an abelian category, a sequence of morphisms

$$A \xrightarrow{u} B \xrightarrow{v} C$$

is called exact if $v \circ u = 0$ and the canonical morphism from $\operatorname{coim} u \to \ker v$ is an isomorphism.

Definition 1.2.6 (short exact sequence)**.** An exact sequence of the form

$$0 \to A \to B \to C \to 0$$

is called a short exact sequence.

Definition 1.2.7 (split)**.** A short exact sequence $0 \to A \to B \to C \to 0$ is called split if it's isomorphic to

$$0 \to A \to A \oplus C \to C \to 0,$$

where $A \to A \oplus C$ and $A \oplus C \to C$ are the canonical morphisms.

2. Sheaf and Cohomology

2.1. Sheaves.
Along this section, X denotes a topological space.

2.1.1. *Definitions and Examples.*

Definition 2.1.1 (sheaf)**.** A presheaf of abelian group $\mathscr{F}$ on X consisting of the following data:

(1) For any open subset U of X, $\mathscr{F}(U)$ is an abelian group.
(2) If $U \subseteq V$ are two open subsets of X, then there is a group homomorphism $r_{UV}\colon \mathscr{F}(U) \to \mathscr{F}(V)$. Moreover, above data satisfy
   I $\mathscr{F}(\varnothing) = 0$.
   II $r_{UU} = \mathrm{id}$.
   III If $W \subseteq U \subseteq V$ are open subsets of X, then $r_{UW} = r_{VW} \circ r_{UV}$.

Moreover, $\mathscr{F}$ is called a sheaf if it satisfies the following extra conditions

IV Let $\{V_i\}_{i\in I}$ be an open covering of open subset $U \subseteq X$ and $s \in \mathscr{F}(U)$. If $s|_{V_i} := r_{UV_i}(s) = 0$ for all $i \in I$, then $s = 0$.
V Let $\{V_i\}_{i\in I}$ be an open covering of open subset $U \subseteq X$ and $s_i \in \mathscr{F}(V_i)$. If $s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j}$ for all $i, j \in I$, then there exists $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$ for all $i \in I$.

Example 2.1.1 (constant presheaf)**.** For an abelian group G, the constant presheaf assign each open subset U the group G itself, but in general it's not a sheaf.

Definition 2.1.2 (morphism of presheaves)**.** A morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$ between presheaves consisting of the following data:

(1) For any open subset U of X, there is a group homomorphism $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$.
(2) If $U \subseteq V$ are two open subsets of X, then the following diagram commutes

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{G}(U) \\ \downarrow{\scriptstyle r_{UV}} & & \downarrow{\scriptstyle r_{UV}} \\ \mathscr{F}(V) & \xrightarrow{\varphi(V)} & \mathscr{G}(V) \end{array}$$

Notation 2.1.1. For convenience, for $s \in \mathscr{F}(U)$, we often write $\varphi(s)$ instead of $\varphi(U)(s)$.

Remark 2.1.1. The morphisms between sheaves are defined as morphisms of presheaves.

Definition 2.1.3 (isomorphism)**.** A morphism of presheaves $\varphi\colon \mathscr{F} \to \mathscr{G}$ is called an isomorphism if it has two-sided inverse, that is, there exists a morphism of presheaves $\psi\colon \mathscr{G} \to \mathscr{F}$ such that $\psi\varphi = \mathrm{id}_{\mathscr{F}}$ and $\varphi\psi = \mathrm{id}_{\mathscr{G}}$.

Remark 2.1.2. A morphism of presheaves $\varphi\colon \mathscr{F} \to \mathscr{G}$ is an isomorphism if and only if for every open subset $U \subseteq X$, $\varphi(U) \to \mathscr{G}(U)$ is an isomorphism of abelian groups.

2.1.2. *Stalks.*

Definition 2.1.4 (stalks)**.** For a presheaf $\mathscr{F}$ and $p \in X$, the stalk at p is defined as

$$\mathscr{F}_p = \varinjlim_{p \in U} \mathscr{F}(U)$$

Remark 2.1.3 (alternative definition). In order to avoid language of direct limit, we give a more useful but equivalent description of stalk: For $p \in U \cap V$, $s_U \in \mathscr{F}(U)$ and $s_V \in \mathscr{F}(V)$ are equivalent if there exists $x \in W \subseteq U \cap V$ such that $s_U|_W = s_V|_W$. An element $s_p \in \mathscr{F}_p$, which is called a germ, is an equivalence class $[s_U]$, and for $s \in \mathscr{F}(U)$, the germ given by s is denoted by $s|_p$.

Notation 2.1.2.

(1) For $s \in \mathscr{F}(U)$ and $p \in U$, $s|_p$ denotes the equivalent class it gives.
(2) For $s_p \in \mathscr{F}_p$, $s \in \mathscr{F}(U)$ denotes the section such that $s|_p = s_p$.

Definition 2.1.5 (morphisms on stalks)**.** Given a morphism of sheaves $\varphi\colon \mathscr{F} \to \mathscr{G}$, it induces a morphism of abelian groups $\varphi_p\colon \mathscr{F}_p \to \mathscr{G}_p$ as follows:

$$\begin{aligned} \varphi_p\colon \mathscr{F}_p &\to \mathscr{G}_p \\ s_p &\mapsto \varphi(s)|_p. \end{aligned}$$

Remark 2.1.4. It's neccessary to check the φ_p is well-defined since there are different choices s such that $s|_p = s_p$.

Proposition 2.1.1. Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism between sheaves. Then φ is an isomorphism if and only if the induced map $\varphi_p\colon \mathscr{F}_p \to \mathscr{G}_p$ is an isomorphism for every $p \in X$.

Proof. It's clear if φ is an isomorphism between sheaves, then it induces an isomorphism between stalks. Conversely, it suffices to show $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$ is an isomorphism for every open subset $U \subseteq X$.

(1) Injectivity: For $s, s' \in \mathscr{F}(U)$ such that $\varphi(s) = \varphi(s')$, by passing to stalks one has $\varphi_p(s|_p) = \varphi_p(s'|_p)$ for every $p \in U$, and thus $s|_p = s'|_p$ since φ_p is an isomorphism. By definition of stalks there exists an open subset $V_p \subseteq U$ containing p such that s agrees with s' on V_p. Then it gives an open covering $\{V_p\}$ of U, and by axiom (IV) one has $s = s'$ on U.
(2) Surjectivity: For $t \in \mathscr{G}(U)$, by passing to stalks there exists $s_p \in \mathscr{F}_p$ such that $\varphi_p(s_p) = t|_p$ for every $p \in U$ since φ_p is surjective. By definition of stalks there exists an open subset $V_p \subseteq U$ containing p and $s \in \mathscr{F}(V_p)$ such that $\varphi(s) = t$ on V_p. This gives a collection of sections defined on an open covering $\{V_p\}$ of U, and by injectivity we proved above one has these sections agree with each other on the intersections. Then by axiom (V) there exists a section $s \in \mathscr{F}(U)$ such that $\varphi(s) = t$.

$\square$

2.1.3. *Sheafification.* In Example 2.1.1, we come across a presheaf that is not a sheaf. To obtain a sheaf from a presheaf, we require a process known as sheafification. One approach to defining sheafification is through its universal property.

Definition 2.1.6 (sheafification)**.** Given a presheaf $\mathscr{F}$ there is a sheaf $\mathscr{F}^+$ and a morphism $\theta\colon \mathscr{F} \to \mathscr{F}^+$ with the property that for any sheaf $\mathscr{G}$ and any morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$ there is a unique morphism $\overline{\varphi}\colon \mathscr{F}^+ \to \mathscr{G}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\varphi"] \arrow[d, "\theta"'] & \mathscr{G} \\
\mathscr{F}^+ \arrow[ur, "\overline{\varphi}"] &
\end{tikzcd}$$

The universal property shows that if the sheafification exists, then it's unique up to a unique isomorphism. One way to give an explicit construction of sheafification is to glue stalks together in a suitable way. Let $\mathscr{F}^+(U)$ be a set of functions

$$f\colon U \to \coprod_{p\in U} \mathscr{F}_p$$

such that $f(p) \in \mathscr{F}_p$ and for every $p \in U$ there is an open subset $V_p \subseteq U$ containing p and $t \in \mathscr{F}(V_p)$ such that $t|_q = f(q)$ for all $q \in V_p$.

Proposition 2.1.2. $\mathscr{F}^+$ is the sheafication of $\mathscr{F}$.

Proof. Firstly let's show $\mathscr{F}^+$ is a sheaf: It's clear $\mathscr{F}^+$ is a presheaf, so it suffices to check conditions (IV) and (V) in the definition. Let $U \subseteq X$ be an open subset and $\{V_i\}$ be an open covering of U.

(1) If $s \in \mathscr{F}^+(U)$ such that $s|_{V_i} = 0$ for all i, then s must be zero: It suffices to show $s(p) = 0$ for all $p \in U$. For any $p \in U$, then there exists an open subset V_i contains p, hence $s(p) = s|_{V_i}(p) = 0$.
(2) Suppose there exists a collection of sections $\{s_i \in \mathscr{F}^+(V_i)\}_{i\in I}$ such that

$$s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j}$$

holds for all $i, j \in I$. Now we construct $s \in \mathscr{F}^+(U)$ as follows: For $p \in U$ and V_i containing p, we define $s(p) = s_i(p)$. This is well-defined since s_i agree on the intersections, so it remains to show $s \in \mathscr{F}^+(U)$. It's clear $s(p) \in \mathscr{F}_p$. For $p \in U$, there exists V_i containing p, and thus there exists $W_i \subseteq V_i$ containing p and $t \in \mathscr{F}(W_i)$ such that $t|_q = s_i(q) = s(q)$ for all $q \in V_p$.

There is a canonical morphism $\theta\colon \mathscr{F} \to \mathscr{F}^+$ as follows: For open subset $U \subseteq X$, and $s \in \mathscr{F}(U)$, $\theta(s)$ is defined by

$$\begin{aligned}\theta(s)\colon U &\to \coprod_{p\in U} \mathscr{F}_p \\ p &\mapsto s|_p.\end{aligned}$$

Note that if $\mathscr{F}$ is a sheaf, the canonical morphism $\theta\colon \mathscr{F} \to \mathscr{F}^+$ is an isomorphism.

(1) Injectivity: If $s \in \mathscr{F}(U)$ such that $s|_p = 0$ for all $p \in U$, then there exists an open covering $\{V_i\}_{i\in I}$ of U such that $s|_{V_i} = 0$, by axiom (IV) of sheaf one has $s = 0$.
(2) Surjectivity: For $f \in \mathscr{F}^+(U)$ and $p \in U$, there exists $p \in V_p \subseteq U$ and $t \in \mathscr{F}(V_p)$ such that $f(p) = t|_p$ by construction of $\mathscr{F}^+$. Then glue these sections together to get our desired s such that $\theta(s) = f$.

Finally let's show $\mathscr{F}^+$ statisfies the universal property of sheafification. A morphism of presheaves $\varphi\colon \mathscr{F} \to \mathscr{G}$ induces a map on stalks

$$\varphi_p\colon \mathscr{F}_p \to \mathscr{G}_p.$$

For $f \in \mathscr{F}^+(U)$, the composite of f with the map

$$\coprod_{p\in U} \varphi_p\colon \coprod_{p\in U} \mathscr{F}_p \to \coprod_{p\in U} \mathscr{G}_p$$

gives a map $\widetilde{\varphi}(f)\colon U \to \coprod_{p\in U} \mathscr{G}_p$, and in fact $\widetilde{\varphi}(f) \in \mathscr{G}^+(U)$: For $p \in U$, $\widetilde{\varphi}(f)(p) \in \mathscr{G}_p$ since $f(p) \in \mathscr{F}_p$ and $\varphi_p\colon \mathscr{F}_p \to \mathscr{G}_p$. If for all $q \in V_p$ we have $t|_q = f(q)$, then

$$\widetilde{\varphi}(f)(q) = \varphi_q(f(q)) = \varphi_q(t|_q) = \varphi(t)|_q.$$

Since $\mathscr{G}$ is a sheaf, the canonical morphism $\theta'\colon \mathscr{G} \to \mathscr{G}^+$ is an isomorphism, so we can define $\overline{\varphi}\colon = \theta'^{-1} \circ \widetilde{\varphi}$. Now let's show $\varphi = \overline{\varphi} \circ \theta = \theta'^{-1} \circ \widetilde{\varphi} \circ \theta$. It's easy to show they coincide on each stalk since $\varphi_p = \theta_p'^{-1} \circ \widetilde{\varphi}_p \circ \theta_p$, and thus $\varphi = \overline{\varphi} \circ \theta$ by Proposition 2.1.1. Furthermore, uniqueness follows from the fact that $\overline{\varphi}_p$ is uniquely determined by φ_p. □

Remark 2.1.5. From the construction, one can see the stalk of $\mathscr{F}^+$ at p is exactly $\mathscr{F}_p$.

Remark 2.1.6. The sheafification can be described in a more fancy language: Since we have sheaf of abelian groups on X as a category, denote it by $\underline{Ab}_X$, and presheaf is a full subcategory of $\underline{Ab}_X$, there is a natural inclusion functor ι from category of sheaf to category of presheaf. The sheafification is the adjoint functor of ι.

Example 2.1.2 (constant sheaf)**.** For an abelian group G, the associated constant sheaf $\underline{G}$ is the sheafication of the constant presheaf. By the construction of sheafification, $\underline{G}$ can be explicitly expressed as

$$\underline{G}(U) = \{\text{locally constant function } f\colon U \to G\}$$

2.1.4. *Exact sequence of sheaf.* Given a morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$ between sheaves of abelian groups, there are the following presheaves

$$U \mapsto \ker\varphi(U)$$
$$U \mapsto \operatorname{im}\varphi(U)$$
$$U \mapsto \operatorname{coker}\varphi(U),$$

since $\varphi(U)\colon \mathscr{F}(U)\to\mathscr{G}(U)$ is a group homomorphism.

Proposition 2.1.3. Kernel of a morphism between sheaves is a sheaf.

Proof. Let $\{V_i\}_{i\in I}$ be an open covering of U.

(1) For $s\in\ker\varphi(U)$, if $s|_{V_i}=0$, then $s=0$ since s is also in $\mathscr{F}(U)$.
(2) If there exists $s_i\in\ker\varphi(V_i)$ such that $s_i|_{V_i\cap V_j}=s_j|_{V_i\cap V_j}$, then they glue together to get $s\in\mathscr{F}(U)$. Note that

$$\varphi(U)(s)|_{V_i}=\varphi(V_i)(s|_{V_i})=\varphi(V_i)(s_i)=0$$

Then $s\in\ker\varphi(U)$.

□

But the image of morphism may not be a sheaf. Although we can prove the first requirement in the same way, the proof for the second requirement fails: If there exists $s_i\in\operatorname{im}\varphi(V_i)$, and we can glue them together to get a $s\in\mathscr{G}(U)$, but s may not be the image of some $t\in\mathscr{F}(U)$. The cokernel fails to be a sheaf for the same reason.

Definition 2.1.7 (image and cokernel)**.** Let $\varphi\colon\mathscr{F}\to\mathscr{G}$ be a morphism between sheaves of abelian groups. Then the image and cokernel of φ is defined to be the sheafification of the following presheaves

$$U\mapsto\operatorname{im}\varphi(U)$$
$$U\mapsto\operatorname{coker}\varphi(U)$$

respectively.

Definition 2.1.8 (exact)**.** For a sequence of sheaves:

$$\cdots\to\mathscr{F}^{i-1}\xrightarrow{\varphi^{i-1}}\mathscr{F}^{i}\xrightarrow{\varphi^{i}}\mathscr{F}^{i+1}\to\dots$$

It's called exact at $\mathscr{F}^i$, if $\ker\varphi^i=\operatorname{im}\varphi^{i-1}$. If a sequence is exact at everywhere, then it's an exact sequence of sheaves.

Definition 2.1.9 (short exact sequence)**.** An exact sequence of sheaves is called a short exact sequence if it looks like

$$0\to\mathscr{F}\xrightarrow{\varphi}\mathscr{G}\xrightarrow{\psi}\mathscr{H}\to 0$$

Proposition 2.1.4. Let $\varphi\colon\mathscr{F}\to\mathscr{G}$ be a morphism between sheaves of abelian groups. Then for any $p\in X$, one has

$$(\ker\varphi)_p=\ker\varphi_p$$
$$(\operatorname{im}\varphi)_p=\operatorname{im}\varphi_p.$$

Proof. For (1). It's clear $(\ker\varphi)_p\subseteq\ker\varphi_p$. Conversely, if $s_p\in\ker\varphi_p$, then $\varphi_p(s_p)=0\in\mathscr{G}_p$. In other words, there exists an open subset U containing p and $s\in\mathscr{F}(U)$ such that $s|_p=s_p$ and $\varphi(s)|_p=0$, which implies there is another open subset V containing p such that $\varphi(s)|_V=0$. Hence $\varphi(s|_V)=0$, that is, $s|_V\in\ker\varphi(V)$. Thus $s_p=(s|_V)|_p\in(\ker\varphi)_p$.

For (2). It's clear $(\operatorname{im}\varphi)_p \subseteq \operatorname{im}\varphi_p$ since the sheafication doesn't change stalk. Conversely, if $s_p \in \operatorname{im}\varphi_p$, then there exists $t_p \in \mathscr{F}_p$ such that $\varphi_p(t_p) = s_p$. Suppose $t \in \mathscr{F}(U)$ is a section of some open subset U containing p such that $t|_p = t_p$. Then $\varphi(t)|_p = \varphi_p(t_p) = s_p$. In other words, s_p is in the stalk of the image presheaf at p, but the sheafication doesn't change stalk, so we have $s_p \in (\operatorname{im}\varphi)_p$. □

Corollary 2.1.1. The sequence of sheaves

$$\dots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \dots$$

is exact if and only if the sequence of abelian groups are exact

$$\dots \to \mathscr{F}^{i-1}_p \xrightarrow{\varphi^{i-1}_p} \mathscr{F}^i_p \xrightarrow{\varphi^i_p} \mathscr{F}^{i+1}_p \to \dots$$

for all $p \in X$.

Corollary 2.1.2. The the sequence of sheaves

$$0 \to \mathscr{F} \to \mathscr{G}$$

is exact if and only if for any open subset U, the following sequence of abelian groups is exact

$$0 \to \mathscr{F}(U) \to \mathscr{G}(U).$$

Method one. For any open subset $U \subseteq X$, one has

$$\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$$

is injective, since by definition we have for any open subset $U \subseteq X$, $\ker\varphi(U) = 0$, that is injectivity. □

Method two. Or from another point of view, for each $p \in U$, we have

$$\varphi_p\colon \mathscr{F}_p \to \mathscr{G}_p$$

is injective. That is $\ker\varphi_p = 0$. So we obtain $(\ker\varphi(U))_p = 0$ for all $p \in U$. But for a section $s \in \mathscr{F}(U)$ if we have $s|_p = 0$, then we must have $s = 0$. So we obtain $\ker\varphi(U) = 0$. □

Example 2.1.3 (exponential sequence)**.** Let X be a complex manifold and $\mathcal{O}_X$ be its holomorphic function sheaf. Then

$$0 \to 2\pi\sqrt{-1}\underline{\mathbb{Z}} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

is an exact sequence of sheaves, called exponential sequence.

Proof. The difficulty is to show exp is surjective on stalks at $p \in X$. That is we need to construct logarithms of functions $g \in \mathcal{O}_X^*(U)$ for U, a neighborhood of p. We may choose U is simply-connected, then define

$$\log g(q) = \log g(p) + \int_{\gamma_q} \frac{\mathrm{d}g}{g}$$

for $q \in U$, where γ_q is a path from p to q in U, and the definition is independent of the choice of γ_q since U is simply-connected. □

Remark 2.1.7. In fact, U is simply-connected is crucial for constructing logarithm. If we consider $X = \mathbb{C}$ and $U = \mathbb{C}\setminus\{0\}$, then

$$\exp\colon \mathcal{O}_X(U) \to \mathcal{O}_X^*(U)$$

cannot be surjective.

2.2. **Derived functor formulation of Sheaf Cohomology.** The category $\underline{Ab}_X$: sheaves of abelian groups on X. In this section we will introduce sheaf cohomology by considering it as a derived functor.

Given an exact sequence of sheaf as follows

$$0 \to \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}''.$$

By taking its section over open subset U, we obtain a sequence of abelian groups

$$0 \to \mathscr{F}'(U) \xrightarrow{\phi(U)} \mathscr{F}(U) \xrightarrow{\psi(U)} \mathscr{F}''(U).$$

Above sequence is not only exact at $\mathscr{F}'(U)$, but also is exact at $\mathscr{F}(U)$. In other words, the functor given by taking section over open subset is a left exact functor.

(1) Firstly let's show $\ker\psi(U) \supseteq \operatorname{im}\phi(U)$. For $s \in \mathscr{F}'(U)$, if we want to show $\psi\circ\phi(s) = 0$, it suffices to show $(\psi\circ\phi(s))|_p = 0$ for all $p \in U$ since $\mathscr{F}''$ is a sheaf. For any $p \in U$, by considering stalk at p we obtain an exact sequence of abelian groups

$$0 \to \mathscr{F}'_p \xrightarrow{\phi_p} \mathscr{F}_p \xrightarrow{\psi_p} \mathscr{F}''_p.$$

Then we obtain $\psi_p\circ\phi_p(s|_p) = 0$, which implies $(\psi\circ\phi(s))|_p = 0$.

(2) Conversely, Given $s \in \ker\psi(U)$, we have $s|_p \in \ker\psi_p$ for any $p \in U$. By exactness of stalks, there exists $t_p \in \mathscr{F}'_p$ such that $\phi_p(t_p) = s|_p$. Thus there exists an open subset V_i containing p and $t_i \in \mathscr{F}'(V_i)$ such that $\phi(t_i) = s|_{V_i}$. Now it suffices to show these t_i can be glued together to obtain $t \in \mathscr{F}(U)$, and since $\mathscr{F}$ is a sheaf, it suffices to check these t_i agree on intersections $V_i\cap V_j$. Note that $\phi(t_i - t_j|_{V_i\cap V_i}) = s|_{V_i\cap V_j} - s|_{V_i\cap V_j} = 0$, then these t_i agree on intersections since ϕ is injective.

Remark 2.2.1. From above argument, we can see that

$$0 \to \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}''$$

is exact if and only if for any open subset $U \subseteq X$

$$0 \to \mathscr{F}'(U) \xrightarrow{\phi(U)} \mathscr{F}(U) \xrightarrow{\psi(U)} \mathscr{F}''(U)$$

is exact.

In homological algebra, we always consider the derived functor of a left or right-exact functor. In particular, the functor of taking global section is a left exact functor, and its right derived functor defines the cohomology of a sheaf. Before we come into the definition of derived functor, firstly let's define the injective resolution of a sheaf.

Definition 2.2.1 (injective)**.** A sheaf $\mathcal{I}$ is injective if $\mathrm{Hom}(-,\mathcal{I})$ is an exact functor.

Definition 2.2.2 (injective resolution)**.** Let $\mathscr{F}$ be a sheaf. An injective resolution of $\mathscr{F}$ is an exact sequence

$$0 \to \mathscr{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \dots$$

where $\mathcal{I}^i$ are injective for all i.

Theorem 2.2.1. Every sheaf admits an injective resolution.

Theorem 2.2.2. Let $\mathscr{F} \to \mathcal{I}^\bullet$ and $\mathscr{G} \to \mathcal{G}^\bullet$ are two resolutions and $\phi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Then there exists a morphism $\widetilde{\phi}\colon \mathcal{I}^\bullet \to \mathcal{G}^\bullet$ which lifts ϕ, which is unique up to homotopy.

Definition 2.2.3 (sheaf cohomology)**.** Let $\mathscr{F}$ be a sheaf of abelian groups. Then

$$H^p(X,\mathscr{F}) := H^p(\mathcal{I}^\bullet(X)).$$

Remark 2.2.2. The Theorem 2.2.2 shows that the definition of sheaf cohomology is independent of the choice of injective resolution.

Example 2.2.1. By definition, the 0-th cohomology is exact the global section

$$H^0(X,\mathscr{F}) := \ker\left\{\mathcal{I}^0(X) \to \mathcal{I}^1(X)\right\}.$$

Thus $H^0(X,\mathscr{F}) = \mathscr{F}(X)$, the global sections of sheaf.

Example 2.2.2. If $\mathscr{F}$ is a injective sheaf, then $H^i(X,\mathscr{F}) = 0$ for all $i > 0$, since the sheaf cohomology of injective sheaf can be computed by using the following special injective resolution

$$0 \to \mathscr{F} \xrightarrow{\mathrm{id}} \mathscr{F} \to 0 \to 0 \to \dots$$

Theorem 2.2.3 (zig-zag)**.** If

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$

is a short sequence of sheaves, then there is an induced long exact sequence of abelian groups

$$0 \to H^0(X,\mathscr{F}) \to H^0(X,\mathscr{G}) \to H^0(X,\mathscr{H}) \to H^1(X,\mathscr{F}) \to H^1(X,\mathscr{G}) \to \dots$$

Definition 2.2.4 (direct image)**.** Let $f\colon X \to Y$ be continuous map between topological spaces and $\mathscr{F}$ be a sheaf of abelian groups on X. The direct image of $\mathscr{F}$, denoted by $f_*\mathscr{F}$, is a sheaf on Y defined by

$$f_*\mathscr{F}(U) := \mathscr{F}(f^{-1}(U)).$$

Proposition 2.2.1. $f_*\colon \underline{Ab}_X \to \underline{Ab}_Y$ is a left exact functor.

Proof. Given an exact sequence of sheaves on X

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''.$$

Then we need to show

$$0 \to f_*\mathscr{F}' \to f_*\mathscr{F} \to f_*\mathscr{F}''$$

is also an exact sequence of sheaves on Y. By Remark 2.2.1 it suffices to show that for any open subset $V \subseteq Y$, we have the following exact sequence

$$0 \to f_*\mathscr{F}'(V) \to f_*\mathscr{F}(V) \to f_*\mathscr{F}''(V),$$

and that's exactly

$$0 \to \mathscr{F}'(f^{-1}(V)) \to \mathscr{F}(f^{-1}(V)) \to \mathscr{F}''(f^{-1}(V)).$$

Since f is continuous, then $f^{-1}(V)$ is an open subset in X, and thus above sequence of abelian is exact since $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ is exact. □

2.3. **Acyclic resolution.** In practice it may be difficult for us to choose an injective resolution, so we usual other resolutions to compute sheaf cohomology.

Definition 2.3.1 (acyclic sheaf)**.** A sheaf $\mathscr{F}$ is acyclic if $H^i(X, \mathscr{F}) = 0$ for all $i > 0$.

Example 2.3.1. Injective sheaf is acyclic.

Definition 2.3.2 (acyclic resolution)**.** Let $\mathscr{F}$ be a sheaf. An acyclic resolution of $\mathscr{F}$ is an exact sequence

$$0 \to \mathscr{F} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \mathcal{A}^2 \to \dots$$

where $\mathcal{A}^i$ is acyclic for all i.

Proposition 2.3.1. The cohomology of sheaf $\mathscr{F}$ can be computed using acyclic resolution.

In fact, it's a quite homological techniques, called dimension shifting, so we will state this technique in language of homological algebra. Let's see a baby version of it.

Example 2.3.2. Let $\mathcal{F}$ be a left exact functor and $0 \to A \to M_1 \to B \to 0$ be an exact sequence with M_1 is $\mathcal{F}$-acyclic. Then $R^{i+1}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$ for $i > 0$, and $R^1\mathcal{F}(A)$ is the cokernel of $\mathcal{F}(M_1) \to \mathcal{F}(B)$.

Proof. By considering the long exact sequence induced by $0 \to A \to M^1 \to B \to 0$, one has

$$R^i\mathcal{F}(M^1) \to R^i\mathcal{F}(B) \to R^{i+1}\mathcal{F}(A) \to R^{i+1}\mathcal{F}(M^1)$$

(1) If $i > 0$, then $R^i\mathcal{F}(M^1) = R^{i+1}\mathcal{F}(M^1) = 0$ since M^1 is $\mathcal{F}$-acyclic, and thus $R^{i+1}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$ for $i > 0$.

(2) If $i = 0$, then
$$0 \to \mathcal{F}(M^1) \to \mathcal{F}(B) \to R^1\mathcal{F}(A) \to 0$$
implies $R^1\mathcal{F}(A) = \operatorname{coker}\{\mathcal{F}(M^1) \to \mathcal{F}(B)\}$.

□

Now let's prove dimension shifting in a general setting.

Lemma 2.3.1 (dimension shifting)**.** If
$$0 \to A \to M^1 \to M^2 \to \cdots \to M^m \to B \to 0$$
is exact with M^i is $\mathcal{F}$-acyclic, then $R^{i+m}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$ for $i > 0$, and $R^m\mathcal{F}(A)$ is the cokernel of $\mathcal{F}(M^m) \to \mathcal{F}(B)$.

Proof. Prove it by induction on m. For $m = 1$, we already see it in Example 2.3.2. Assume it holds for $m < k$, then for $m = k$, let's split $0 \to A \to M^1 \to M^2 \to \cdots \to M^k \xrightarrow{\mathrm{d}_k} B \to 0$ into two exact sequences
$$0 \to A \to M^1 \to M^2 \to \cdots \to M^{k-1} \to \ker \mathrm{d}_k \to 0$$
$$0 \to \ker \mathrm{d}_k \to M^k \xrightarrow{\mathrm{d}_k} B \to 0.$$
Then by induction hypothesis, for $i > 0$ we have
$$R^{i+k-1}\mathcal{F}(A) \cong R^i\mathcal{F}(\ker \mathrm{d}_k)$$
$$R^{i+1}\mathcal{F}(\ker \mathrm{d}_k) \cong R^i\mathcal{F}(B).$$
Combine these two isomorphisms together we obtain $R^{i+k}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$ for $i > 0$, as desired. For $i = 0$, it suffices to let $i = 1$ in $R^{i+k-1}\mathcal{F}(A) \cong R^i\mathcal{F}(\ker \mathrm{d}_k)$, then we obtain
$$R^k\mathcal{F}(A) = R^1\mathcal{F}(\ker \mathrm{d}_k) = \operatorname{coker}\{\mathcal{F}(M^k) \to \mathcal{F}(B)\}.$$
This completes the proof. □

Corollary 2.3.1. If $0 \to A \to M^\bullet$ is a $\mathcal{F}$-acyclic resolution, then $R^i\mathcal{F}(A) = H^i(\mathcal{F}(M^\bullet))$.

Proof. Truncate the resolution as
$$0 \to A \to M^0 \to M^1 \to \ldots M^{i-1} \to B \to 0$$
$$0 \to B \to M^i \to M^{i+1} \to \ldots$$
Since we already have $R^i\mathcal{F}(A) = \operatorname{coker}\{\mathcal{F}(M^{i-1}) \to \mathcal{F}(B)\}$, and $\mathcal{F}$ is left exact, one has
$$\mathcal{F}(B) = \ker\{\mathcal{F}(M^i) \to \mathcal{F}(M^{i+1})\}.$$
Thus we obtain
$$R^i\mathcal{F}(A) = \operatorname{coker}\{\mathcal{F}(M^{i-1}) \to \ker\{\mathcal{F}(M^i) \to \mathcal{F}(M^{i+1})\}\} = H^i(\mathcal{F}(M^\bullet)).$$

□

2.4. **Examples about acyclic sheaf.**

2.4.1. *Flabby sheaf.* First kind of acyclic sheaf is flabby[1] sheaf.

Definition 2.4.1 (flabby)**.** A sheaf $\mathscr{F}$ is flabby if for all open $U \subseteq V$, the restriction map $\mathscr{F}(V) \to \mathscr{F}(U)$ is surjective.

Now let's see some examples about flabby sheaves.

Example 2.4.1. A constant sheaf on an irreducible topological space is flabby.

Proof. Note that the constant presheaf on a irreducible topological space is a sheaf in fact, and it's easy to see this constant presheaf is flabby. □

In particular, we have

Example 2.4.2. Let X be an algebraic variety. Then constant sheaf $\mathbb{Z}_X$ is flabby.

Example 2.4.3. If $\mathscr{F}$ is a flabby sheaf on X, and $f\colon X \to Y$ is a continuous map, then $f_*\mathscr{F}$ is a flabby sheaf on Y.

Proof. For $V \subseteq W$ in Y, it suffices to show $f_*\mathscr{F}(W) \to f_*\mathscr{F}(V)$ is surjective, and that's

$$\mathscr{F}(f^{-1}W) \to \mathscr{F}(f^{-1}V)$$

it's surjective since $\mathscr{F}$ is flabby. □

Example 2.4.4. An injective sheaf is flabby.

Proof. Let $\mathcal{I}$ be an injective sheaf and $V \subseteq U$ be open subsets. Now we define sheaf $\underline{\mathbb{Z}}_U$ on X by

$$\underline{\mathbb{Z}}_U := \begin{cases} \underline{\mathbb{Z}}(W) & W \subseteq U \\ 0 & \text{otherwise} \end{cases}$$

where $\underline{\mathbb{Z}}$ is constant sheaf valued in $\mathbb{Z}$, and similarly we define sheaf $\underline{\mathbb{Z}}_V$. By construction one has $\underline{\mathbb{Z}}_U(W) = \underline{\mathbb{Z}}_V(W)$ unless $W \subseteq U$ and $W \not\subseteq V$. Thus we obtain an exact sequence

$$0 \to \underline{\mathbb{Z}}_V \to \underline{\mathbb{Z}}_U.$$

Applying the functor $\mathrm{Hom}(-, \mathcal{I})$, which is exact, we obtain an exact sequence

$$\mathrm{Hom}(\underline{\mathbb{Z}}_U, \mathcal{I}) \to \mathrm{Hom}(\underline{\mathbb{Z}}_V, \mathcal{I}) \to 0.$$

Now let's explain why we need such a weird sheaf $\underline{\mathbb{Z}}_U$. In fact, we will prove $\mathrm{Hom}(\underline{\mathbb{Z}}_U, \mathcal{I}) = \mathcal{I}(U)$. Indeed since $\varphi\colon \underline{\mathbb{Z}}_U \to \mathcal{I}$ is a sheaf morphism. Then if $W \not\subseteq U$, then $\varphi(U)$ must be zero. If $W = U$, then the group of sections of $\underline{\mathbb{Z}}_U(U)$ over any connected component is simply $\mathbb{Z}$ and hence $\varphi(U)$ on this connected component is determined by the image of $1 \in \mathbb{Z}$. Thus $\varphi(U)$ can be thought of an element of $\mathcal{I}(U)$. Now on any proper open subset of U, φ is determined by restriction maps. Hence $\mathrm{Hom}(\underline{\mathbb{Z}}_U, \mathcal{I}) = \mathcal{I}(U)$, as desired.

[1] Some authors also call this flasque.

The same argument shows $\mathrm{Hom}(\mathbb{Z}_U, \mathcal{I}) = \mathcal{I}(V)$, and thus we obtain an exact sequence
$$\mathcal{I}(U) \to \mathcal{I}(V) \to 0,$$
which shows $\mathcal{I}$ is flabby. $\square$

Our goal is to prove a flabby sheaf is acyclic, but we still need some property of flabby sheaves.

Proposition 2.4.1. If $0 \to \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and $\mathscr{F}'$ is flabby, then for any open subset U, the sequence
$$0 \to \mathscr{F}'(U) \xrightarrow{\phi(U)} \mathscr{F}(U) \xrightarrow{\psi(U)} \mathscr{F}''(U) \to 0$$
is exact.

Proof. It suffices to show $\mathscr{F}(U) \to \mathscr{F}''(U) \to 0$ is exact. Here we only gives a sketch of the proof. Since we have exact sequence on stalks for each $p \in U$ as follows
$$0 \to \mathscr{F}'_p \xrightarrow{\phi_p} \mathscr{F}_p \xrightarrow{\psi_p} \mathscr{F}''_p \to 0$$
Then for each $s \in \mathscr{F}''(U)$, there exists $t_p \in \mathscr{F}_p$ such that $\psi_p(t_p) = s|_p$, so there exists open subset $V_p \subseteq U$ containing p and $t \in \mathscr{F}(V_p)$ such that $\psi(t) = s|_{V_p}$. If we can glue these t together then we get a section in $\mathscr{F}(U)$ and is mapped to s, which completes the proof. However, they may not equal on the intersection. But things are not too bad, consider another point q and $t' \in \mathscr{F}(V_q)$ such that $\psi(t') = s|_{V_q}$, $(t'-t)|_{V_p \cap V_q} \in \ker \psi(V_p \cap V_q) = \operatorname{im} \phi(V_p \cap V_q)$. So there exists $t'' \in \mathscr{F}'(V_p \cap V_q)$ such that
$$\phi(t'') = (t'-t)|_{V_p \cap V_q}$$
Now since $\mathscr{F}'$ is flabby, then there exists $t''' \in \mathscr{F}(V_p)$ such that $t'''|_{V_p \cap V_q} = t''$. And consider $t + \phi(t''') \in \mathscr{F}(V_p)$, which will coincide with t' on $V_p \cap V_q$. After above corrections, we can glue t after correction together. $\square$

Proposition 2.4.2. If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ and $\mathscr{F}$ are flabby, then $\mathscr{F}''$ is flabby.

Proof. Take $V \subseteq U$ and consider the following diagram
$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathscr{F}'(U) \arrow[r] \arrow[d] & \mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{F}''(U) \arrow[r] \arrow[d, dashed] & 0 \arrow[d] \\
0 \arrow[r] & \mathscr{F}'(V) \arrow[r] & \mathscr{F}(V) \arrow[r] & \mathscr{F}''(V) \arrow[r] & 0
\end{tikzcd}$$
Then the desired result follows from five lemma. $\square$

Proposition 2.4.3. A flabby sheaf is acyclic.

Proof. Let $\mathscr{F}$ be a flabby sheaf. Since there are enough injective objects in the category of sheaf of abelian groups, there is an exact sequence
$$0 \to \mathscr{F} \to \mathcal{I} \to \mathscr{Q} \to 0$$

with $\mathcal{I}$ is injective. By Example 2.4.4 we have $\mathcal{I}$ is flabby, and thus by Proposition 2.4.2 we have $\mathscr{Q}$ is flabby. Consider the long exact sequence induced from above short exact sequence

$$\mathscr{F}(X) \to \mathcal{I}(X) \to \mathscr{Q}(X) \to H^1(X, \mathscr{F}) \to H^1(X, \mathcal{I}) \to \dots$$

Note that $H^1(X, \mathcal{I}) = 0$ since $\mathcal{I}$ is injective, and thus acyclic. Then $H^1(X, \mathscr{F}) = \operatorname{coker}\{\mathcal{I}(X) \to \mathscr{Q}(X)\}$. But Proposition 2.4.1 shows that $\mathcal{I}(X) \to \mathscr{Q}(X)$ is surjective since $\mathscr{F}$ is flabby, so $H^1(X, \mathscr{F}) = 0$.

Now let's prove $H^k(X, \mathscr{F}) = 0$ for $k > 0$ by induction on k, and above argument shows it's true for $k = 1$. Assume this holds for $k < n$, and consider

$$\dots \to H^{n-1}(X, \mathscr{Q}) \to H^n(X, \mathscr{F}) \to H^n(X, \mathcal{I}) \to H^n(X, \mathscr{Q}) \to \dots$$

By induction hypothesis, we can reduce above sequence to

$$\dots \to 0 \to H^n(X, \mathscr{F}) \to 0 \to H^n(X, \mathscr{Q}) \to \dots$$

which implies $H^n(X, \mathscr{F}) = 0$. This completes the proof. $\square$

2.4.2. *Soft sheaf.* The second kind of acyclic sheaves is called soft sheaves, which is quit similar to flabby.

Definition 2.4.2 (soft)**.** A sheaf $\mathscr{F}$ over X is soft if for any closed subset $S \subseteq X$ the restriction map $\mathscr{F}(X) \to \mathscr{F}(S)$ is surjective.

Remark 2.4.1. For closed subset S, the section over it is defined by

$$\mathscr{F}(S) := \varinjlim_{S \subseteq U} \mathscr{F}(U)$$

Parallel to Proposition 2.4.1 and Proposition 2.4.2, soft sheaf has the following properties:

Proposition 2.4.4. If $0 \to \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and $\mathscr{F}'$ is soft, then the following sequence

$$0 \to \mathscr{F}'(X) \xrightarrow{\phi(X)} \mathscr{F}(X) \xrightarrow{\psi(X)} \mathscr{F}''(X) \to 0$$

is exact.

Proposition 2.4.5. If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ and $\mathscr{F}$ are soft, then $\mathscr{F}''$ is soft.

Proposition 2.4.6. A soft sheaf is acyclic.

So you may wonder, what's the difference between flabby and soft since the definitions are quite similar, and both of them are acyclic. Clearly by definition of sections over a closed subset, we know that every flabby sheaf is soft, but converse fails

Example 2.4.5. The sheaf of smooth functions on a smooth manifold is soft but not flabby.

Lemma 2.4.1. If $\mathscr{M}$ is a sheaf of modules over a soft sheaf of rings $\mathscr{R}$, then $\mathscr{M}$ is a soft sheaf.

Proof. Let $s \in \mathscr{M}(K)$ for some closed subset $K \subseteq X$. Then s extends to some open neighborhood U of K. Let $\rho \in \mathscr{R}(K \cup (X \setminus U))$ be defined by

$$\rho = \begin{cases} 1, & \text{on } K \\ 0, & \text{on } X \setminus U \end{cases}$$

Since $\mathscr{R}$ is soft, then ρ extends to a section over X, then $\rho \circ s$ is the desired extension of s. $\square$

2.4.3. *Fine sheaf.* Another important kind of acyclic sheaves, which behaves like sheaf of differential forms Ω^k_X is called fine sheaf. Recall what is a partition of unity: Let $U = \{U_i\}_{i\in I}$ be a locally finite open covering of topological space X. A partition of unity subordinate to U is a collection of continuous functions $f_i \colon U_i \to [0,1]$ for each $i \in I$ such that its support lies in U_i, and for any $x \in X$

$$\sum_{i\in I} f_i(x) = 1.$$

Definition 2.4.3 (fine sheaf)**.** A fine sheaf $\mathscr{F}$ on X is a sheaf of $\mathscr{A}$-modules, where $\mathscr{A}$ is a sheaf of rings such that for every locally finite open covering $\{U_i\}_{i\in I}$ of X, there is a partition of unity

$$\sum_{i\in I} \rho_i = 1$$

where $\rho_i \in \mathscr{A}(X)$ and $\mathrm{supp}(\rho_i) \subseteq U_i$.

Remark 2.4.2. For a sheaf $\mathscr{F}$ on X and a section $s \in \mathscr{F}(X)$, its support is defined as

$$\mathrm{supp}(s) := \overline{\{x \in X : s|_x \neq 0\}}.$$

Proposition 2.4.7. A fine sheaf is acyclic.

Proof. Let $\mathscr{F}$ be a sheaf of $\mathscr{A}$-modules and a fine sheaf. And choose a injective resolution

$$0 \to \mathscr{F} \xrightarrow{\mathrm{d}} \mathcal{I}^0 \xrightarrow{\mathrm{d}} \mathcal{I}^0 \xrightarrow{\mathrm{d}} \mathcal{I}^1 \xrightarrow{\mathrm{d}} \dots$$

such that $\mathcal{I}^i$ are injective sheaves of $\mathscr{A}$-modules. Let $s \in \mathcal{I}^p(X)$ such that $\mathrm{d}s = 0$. Then by exactness of injective resolution we have X is covered by open subsets U_i such that for each i there is an element $t_i \in \mathcal{I}^{p-1}(U_i)$ such that $\mathrm{d}t_i = s|_{U_i}$. By passing to a refinement we may assume that the cover $\{U_i\}$ is locally finite. Let $\{\rho_i\}$ be a partition of unity subordinate to $\{U_i\}$. Then we have $t = \sum \rho_i t_i \in \mathscr{I}^{p-1}(X)$ such that $\mathrm{d}t = s$. This completes the proof. $\square$

Example 2.4.6. Let M be a smooth manifold and $\pi\colon E \to M$ be a vector bundle. Then the sheaf of smooth sections of E is a $C^\infty(M)$-module sheaf, which is a fine sheaf. In particular, the sheaf of tangent bundle, sheaf of differential forms Ω_M and k-forms Ω^k_M are fine sheaves.

Remark 2.4.3. As a consequence, it's meaningless to compute cohomology of sheaf of differential k-forms, or any other vector bundle over a smooth manifold. But in complex version, something interesting happens: Let $(X, \mathcal{O}_X)$ be a complex manifold and $\pi\colon E \to X$ be a holomorphic vector bundle. Then the sheaf of holomorphic sections of E is not a fine sheaf since there is no partition of unity may not be holomorphic, so the cohomology of holomorphic vector bundle is not trivial, and that's what Dolbeault cohomology computes.

For fine sheaf and soft sheaf, we have

Lemma 2.4.2. Fine sheaf is soft.

Proof. Let $\mathscr{F}$ be a fine sheaf, $S \subseteq X$ closed and $s \in \mathscr{F}(S)$. Let $\{U_i\}$ be an open covering of S and $s_i \in \mathscr{F}(U_i)$ such that

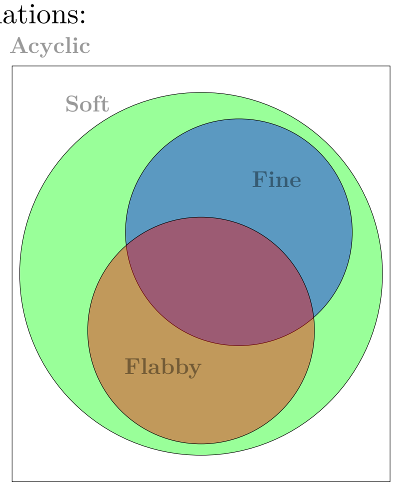
$$s_i|_{S\cap U_i} = s|_{S\cap U_i}.$$

Let $U_0 = X - S$, and $s_0 = 0$. Then $\{U_i\} \coprod \{U_0\}$ is an open covering of X. Without lose of generality, we assume this open covering is locally finite and choose a partition of unity $\{\rho_i\}$ subordinate to it. Then

$$\overline{s} := \sum_i \rho_i(s_i)$$

is a section in $\mathscr{F}(X)$ which extends s. □

Remark 2.4.4. Until now, we have shown that soft, fine and flabby sheaves are acyclic. Lemma 2.4.2 shows fine sheaf is soft, and by definition a flabby sheaf is soft. The Example 2.4.5 shows that soft sheaf may not be flabby, and constant sheaf on an irreducible space is flabby but not fine. In a summary, we have the following relations:



2.5. **Proof of de Rham theorem using sheaf cohomology.** As we already know, for constant sheaf $\underline{\mathbb{R}}$ over a smooth manifold M, we have the following fine resolution

$$0 \to \underline{\mathbb{R}} \xrightarrow{i} \Omega^0_M \xrightarrow{\mathrm{d}} \Omega^1_M \xrightarrow{\mathrm{d}} \Omega^2_M \xrightarrow{\mathrm{d}} \dots$$

And de Rham cohomology computes the sheaf cohomology of $\underline{\mathbb{R}}$. de Rham theorem implies that de Rham cohomology equals to the singular cohomology with real coefficient. So if we can give constant sheaf another resolution using singular cochains, we may derive the de Rham cohomology.

We state this in a general setting: Let X be a topological manifold, and a constant sheaf $\underline{G}$ over X, where G is an abelian group. Let $S^p(U, G)$ be the group of singular cochains in U with coefficients in G, and let δ denote the coboundary operator.

Let $\mathscr{S}^p(G)$ be the sheaf over X generated by the presheaf $U \mapsto S^p(U, G)$, with induced differential mapping $\mathscr{S}^p(G) \xrightarrow{\delta} \mathscr{S}^{p+1}(G)$.

Similar to Poincaré lemma, we have for a unit ball U in Euclidean space, we have the following sequence

$$\dots \to S^{p-1}(U, G) \xrightarrow{\delta} S^p(U, G) \xrightarrow{\delta} S^{p+1}(U, G) \to \dots$$

is exact. So we have the following resolution of the constant sheaf $\underline{G}$

$$0 \to \underline{G} \to \mathscr{S}^0(G) \xrightarrow{\delta} \mathscr{S}^1(G) \xrightarrow{\delta} \mathscr{S}^2(G) \to \dots$$

Remark 2.5.1. If M is a smooth manifold, then we can consider smooth chains, that is $f\colon \Delta^p \to U$, where f is a smooth function. The corresponding results above still hold, and we have a resolution by smooth cochains with coefficients in G:

$$0 \to \underline{G} \to \mathscr{S}^\bullet_\infty(G)$$

So if we choose $G = \mathbb{R}$, then it suffices to show $0 \to \underline{\mathbb{R}} \to \mathscr{S}^\bullet_\infty(\mathbb{R})$ is an acyclic resolution, then we obtain de Rham theorem.

First, note that $\mathscr{S}^p_\infty$ is a $\mathscr{S}^0_\infty$-module, given by cup product on open subsets. Then by Lemma 2.4.1 and the fact $\mathscr{S}^0_\infty$ is soft we know that it's a soft resolution. This completes the proof.

2.6. **Hypercohomology.** In homological algebra, the hypercohomology is a generalization of cohomology functor which takes as input not objects in abelian category but instead chain complexes of objects.

One of the motivations for hypercohomology is to generalize the zig-zag lemma, that is, the short exact sequence of sheaves induces a long exact sequence of cohomology groups. It turns out hypercohomology gives techniques for constructing a similar cohomological associated long exact sequence from an arbitrary long exact sequence

$$0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \dots \to \mathscr{F}_k \to 0$$

Now let's give the definition of hypercohomology: Let $\mathscr{F}^\bullet\colon \dots \to \mathscr{F}^{i-1} \to \mathscr{F}^i \to \mathscr{F}^{i+1} \to \cdots$ be a complex of sheaves of abelian groups, which is bounded from below, that is, $\mathscr{F}^n = 0$ for $n \ll 0$. Then $\mathscr{F}^\bullet$ admits an injective resolution $\mathscr{F}^\bullet \to \mathcal{I}^\bullet$. In other words,

$$\begin{tikzcd}
\dots \arrow[r] & \mathscr{F}^{i-1} \arrow[r] \arrow[d] & \mathscr{F}^i \arrow[r] \arrow[d] & \mathscr{F}^{i+1} \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \mathcal{I}^{i-1} \arrow[r] & \mathcal{I}^i \arrow[r] & \mathcal{I}^{i+1} \arrow[r] & \dots
\end{tikzcd}$$

such that

(1) All $\mathcal{I}^i$ are injective sheaves.
(2) The induced homomorphism $H^i(\mathscr{F}^\bullet) \to H^i(\mathcal{I}^\bullet)$ is an isomorphism.

The hypercohomology of $\mathscr{F}^\bullet$ is defined by

$$\mathbb{H}^i(X, \mathscr{F}^\bullet) := H^i(\Gamma(X, \mathcal{I}^\bullet))$$

Definition 2.6.1. For a sheaf $\mathscr{F}$, $\mathscr{F}^\bullet[n]$ is a sheaf of complex defined by

$$(\mathscr{F}^\bullet[n])^i = \begin{cases} \mathscr{F} & i = n \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.6.1. Let $\mathscr{F}$ be a sheaf and $0 \to \mathscr{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \dots$ be an injective resolution of $\mathscr{F}$. Then

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\
0 \arrow[r] & \mathcal{I}^0 \arrow[r] & \mathcal{I}^1 \arrow[r] & \mathcal{I}^2 \arrow[r] & \dots
\end{tikzcd}$$

is an injective resolution of $\mathscr{F}^\bullet[0]$. Indeed, $\mathcal{I}^i$ are injective for all $i \geq 0$, and

$$H^i(\mathcal{I}^\bullet) = \begin{cases} \mathscr{F}, & n = 0 \\ 0, & \text{otherwise} \end{cases} = H^i(\mathscr{F}^\bullet[0])$$

So by definition of hypercohomology, we have $\mathbb{H}^i(X, \mathscr{F}^\bullet[0]) = H^i(\Gamma(X, \mathcal{I}^\bullet)) = H^i(X, \mathscr{F}^\bullet)$. In general, one has

$$\mathbb{H}^i(X, \mathscr{F}^\bullet[n]) \cong H^{i+n}(X, \mathscr{F}).$$

Theorem 2.6.1 (zig-zag)**.** Let $0 \to \mathscr{F}^\bullet \to \mathscr{G}^\bullet \to \mathscr{H}^\bullet \to 0$ be a short exact sequence of complexes of sheaves which are bounded from below. Then there is an induced long exact sequence

$$\dots \to \mathbb{H}^{i-1}(X, \mathscr{H}^\bullet) \to \mathbb{H}^i(X, \mathscr{F}^\bullet) \to \mathbb{H}^i(X, \mathscr{G}^\bullet) \to \mathbb{H}^i(X, \mathscr{H}^\bullet) \to \mathbb{H}^{i+1}(X, \mathscr{F}^\bullet) \to \dots$$

Part 2. Schemes

3. Schemes and Morphisms

3.1. **Schemes.** Throughout this lecture, all rings are assumed to be commutative with identity element, and all homomorphisms of rings are assumed to map 1 to 1.

3.1.1. *Ringed space.*

Definition 3.1.1 (ringed space)**.** A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X.

Definition 3.1.2 (locally ringed space)**.** A ringed space $(X, \mathcal{O}_X)$ is a locally ringed space if for every $p \in X$, the stalk $\mathcal{O}_{X,p}$ is a local ring.

Definition 3.1.3 (morphisms between ringed space)**.** Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two ringed space. A morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$ consisting of a continuous map $f\colon X \to Y$ and a morphism of sheaves $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$.

Remark 3.1.1. Let $(f, f^\sharp)$ be a morphism between ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$. For every point $p \in X$, there is a homomorphism $(f_*\mathcal{O}_X)_{f(p)} \to \mathcal{O}_{X,p}$ defined by

$$(f_*\mathcal{O}_X)_{f(p)} = \varinjlim_{f(p)\in V} (f_*\mathcal{O}_X)(V) = \varinjlim_{p\in f^{-1}(V)} \mathcal{O}_X(f^{-1}(V)) \to \varinjlim_{p\in U} \mathcal{O}_X(U) = \mathcal{O}_{X,p}.$$

On the other hand, the morphism of sheaves $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ induces a homeomorphism between stalks

$$(f^\sharp)_p\colon \mathcal{O}_{Y,f(p)} \to (f_*\mathcal{O}_X)_{f(p)}.$$

By composing above two homomorphisms, there is a homomorphism

$$f^\sharp_p\colon \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}.$$

Definition 3.1.4 (morphisms between locally ringed space)**.** A morphism $(f, f^\sharp)$ between locally ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a morphism between ringed spaces, and for each $p \in X$, the morphism

$$f^\sharp_p\colon \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$$

is a local homomorphism of local rings.

Definition 3.1.5 (isomorphism)**.** A isomorphism of locally ringed space is a morphism with a two-side inverse.

3.1.2. *Schemes.*

Definition 3.1.6 (prime spectrum)**.** Let A be a ring. The spectrum of A is a locally ringed space, consisting of the following data:

(1) A topological space $\operatorname{Spec} A$, which is the set of all prime ideals of A, equipped with Zariski topology, that is, all closed subsets of $\operatorname{Spec} A$ are of the form $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{a} \subseteq \mathfrak{p}\}$.

(2) A structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$, which is defined as follows: For every open subset U of $\operatorname{Spec} A$, $\mathcal{O}_{\operatorname{Spec} A}(U)$ consists of mappings $s\colon U \to \coprod_{\mathfrak{p}\in\operatorname{Spec} A} A_{\mathfrak{p}}$ satisfying the following two conditions:
   (a) For every $\mathfrak{p} \in U$, one has $s(\mathfrak{p}) \in A_{\mathfrak{p}}$.
   (b) For every $\mathfrak{p} \in U$, there exists a neighborhood $U_{\mathfrak{p}}$ of $\mathfrak{p}$ contained in U and $a, f \in A$ such that for every $\mathfrak{q} \in U_{\mathfrak{p}}$, one has $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = a/f$ in $A_{\mathfrak{q}}$.

Definition 3.1.7 (affine scheme)**.** A locally ringed space that is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A is called an affine scheme.

Definition 3.1.8 (scheme)**.** A scheme $(X, \mathcal{O}_X)$ is a locally ringed space for which there exists an open covering $\{U_i\}_{i\in I}$ of X such that each $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme.

Proposition 3.1.1. Let A be a ring.

(1) For every $\mathfrak{p} \in \operatorname{Spec} A$, there is a canonical isomorphism $\mathcal{O}_{\mathfrak{p}} \cong A_{\mathfrak{p}}$.
(2) For every $f \in A$, there is a canonical isomorphism $\mathcal{O}(D(f)) \cong A_f$. In particular, $\mathcal{O}(\operatorname{Spec} A) \cong A$.

Proposition 3.1.2.

(1) Let $\phi\colon A \to B$ be a homeomorphism of rings. Then ϕ induces a canonical morphism of locally ringed spaces
$$(f, f^\sharp)\colon (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}).$$
(2) Any morphism $(f, f^\sharp)$ between $(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B})$ and $(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B})$ is obtained this way.

Proposition 3.1.3. For any $f \in A$, there is a canonical isomorphism of locally ringed spaces
$$(D(f), \mathcal{O}_{\operatorname{Spec} A}|_{D(f)}) \cong (\operatorname{Spec} A_f, \mathcal{O}_{\operatorname{Spec} A_f}).$$

Corollary 3.1.1. Let $(X, \mathcal{O}_X)$ be a scheme and U be an open subset of X. Then $(U, \mathcal{O}_X|_U)$ is a scheme.

Proposition 3.1.4. Let X be a scheme and A be a ring. Then there is a one-to-one correspondence between the set of morphisms of schemes from X to $\operatorname{Spec} A$ and the set of homomorphisms of rings from A to $\mathcal{O}_X(X)$.

3.2. **Proj construction.** In this section we fix a graded ring S with decompostion $S = \bigoplus_{d=0}^{\infty} S_d$.

Proposition 3.2.1. An ideal of S is called a homogeneous ideal if it satisfies one of the following equivalent conditions:

(1) $\mathfrak{a} = \bigoplus_d (\mathfrak{a} \cap S_d)$.
(2) If $a \in \mathfrak{a}$ and $a = \sum_d a_d$ with $a_d \in S_d$, then $a_d \in \mathfrak{a}$.
(3) $\mathfrak{a}$ is generated by homogeneous elements as an additive subgroup of S.
(4) $\mathfrak{a}$ is generated by homogeneous elements an an ideal of S.

Proposition 3.2.2. Let $\mathfrak{a}$ be a homogeneous ideal of S. If for any homogeneous elements f and g in S such that $fg \in \mathfrak{a}$, one has either $f \in \mathfrak{a}$ or $g \in \mathfrak{a}$, then $\mathfrak{a}$ is a prime ideal.

Proposition 3.2.3. Let $\mathfrak{a}, \mathfrak{b}$ be homogeneous ideals of S. Then
(1) $\mathfrak{a} + \mathfrak{b}, \mathfrak{a}\mathfrak{b}, \mathfrak{a} \cap \mathfrak{b}$ are homogeneous ideals.
(2) $\sqrt{\mathfrak{a}}$ is a homogeneous ideal.

Proposition 3.2.4. Let $S_+ = \bigoplus_{d=1}^{\infty} S_d$ and Proj S be the set of all homogeneous prime ideals of S not containing S_+. For any homogeneous ideal a of S, define

$$V_+(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{a} \subseteq \mathfrak{p}\}.$$

Then
(1) $V_+(0) = \operatorname{Proj} S$ and $V_+(S) = \varnothing$.
(2) $\bigcap_{i \in I} V_+(\mathfrak{a}_i) = V_+(\sum_{i \in I} \mathfrak{a}_i)$ for any family of homogeneous ideals $\{\mathfrak{a}_i\}_{i \in I}$ of S.
(3) $V_+(\mathfrak{a}) \cap V_+(\mathfrak{b}) = V_+(\mathfrak{a}\mathfrak{b}) = V_+(\mathfrak{a} \cap \mathfrak{b})$ for any homogeneous ideals $\mathfrak{a}$ and $\mathfrak{b}$ of S.

In particular, $\operatorname{Proj} S$ is a topological space so that closed subsets are of the form $V_+(\mathfrak{a})$ for homogeneous ideals a of S. This topology is called Zariski topology of $\operatorname{Proj} S$.

Now let's define the structure sheaf on $\operatorname{Proj} S$. For each $\mathfrak{p} \in \operatorname{Proj} S$, consider the ring

$$S_{(\mathfrak{p})} = \{\frac{a}{t} \in S_{\mathfrak{p}} \mid a \text{ and } t \text{ are homogeneous of the same degree }\}.$$

For open subset $U \subseteq \operatorname{Proj} S$, $\mathcal{O}_{\operatorname{Proj} S}(U)$ is defined to be the set of functions $s \colon U \to \coprod_{\mathfrak{p} \in U} S_{(\mathfrak{p})}$ such that
(1) For every $\mathfrak{p} \in U$, one has $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$.
(2) For every $\mathfrak{p} \in U$, there exists a neighborhood U_p of p contained in U and homogeneous elements $a, f \in S$ of the same degree such that for every $\mathfrak{q} \in U_p$, one has $f \notin q$ and $s(\mathfrak{q}) = a/f$ in $S_{(\mathfrak{q})}$.

Theorem 3.2.1.
(1) For every $\mathfrak{p} \in \operatorname{Proj} S$, there is a canonical isomorphism $\mathcal{O}_{\operatorname{Proj} S, \mathfrak{p}} \cong S_{(\mathfrak{p})}$.
(2) For every homomorphism element $f \in S_+$, let

$$D_+(f) = \operatorname{Proj} S \setminus V_+((f)) = \{\mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p}\}.$$

Then $D_+(f)$ is open in $\operatorname{Proj} S$, and open subsets of this type form a basis for the topology of Proj S. Moreover, there is an isomorphism of locally ringed space

$$(D_+(f), \mathcal{O}_{\operatorname{Proj} S}|_{D_+(f)}) \cong (\operatorname{Spec} S_{(f)}, \mathcal{O}_{\operatorname{Spec} S_{(f)}}).$$

In particular, $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ is a scheme.

4. Properties of schemes

4.1. Irreducible, noetherian topological space.

4.1.1. *Irreducible topological space.*

Definition 4.1.1 (irreducible topological space)**.** A topological space X is called irreducible if X is not the union of two proper closed subsets. A subset $Y \subseteq X$ is called irreducible if Y is a irreducible topological space equipped with induced topological.

Proposition 4.1.1. Let X be a topological space and $Y \subseteq X$ be a subset equipped with induced topological. If Y is irreducible, then $\overline{Y}$ is also irreducible.

Proposition 4.1.2. Let A be a ring. A closed subset of $\operatorname{Spec} A$ is irreducible if and only if it's of the form $V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ of A.

Proposition 4.1.3. Let X be a scheme. For any irreducible closed subset Y of X, there exists a unique point $y \in Y$ such that $Y = \overline{\{y\}}$

4.1.2. *Noetherian topological space.*

Definition 4.1.2 (noetherian topological space)**.** A topological space X is called a noetherian topological space if the family of closed subsets of X satisfies the descending chain conditions.

Example 4.1.1. If A is a noetherian ring, then $\operatorname{Spec} A$ is a noetherian topological space.

Proposition 4.1.4. Suppose X is a noetherian topological space.

(1) For every closed subset Y of X, there is a decompostion $Y = Y_1 \cup \cdots \cup Y_n$ into closed irreducible subsets Y_i such that $Y_i \not\subset Y_j$ whenever $i \neq j$, where Y_i are called irreducible component of Y.
(2) An irreducible closed subset Y is an irreducible component of X if and only if Y is maximal among the family of irreducible closed subset of X.

Corollary 4.1.1. Let A be a noetherian ring. Then there is a one to one correspondence between the family of irreducible components of $\operatorname{Spec} A$ and the family of minimal prime ideals of A.

4.2. Reduced, irreducible and integral scheme.

Definition 4.2.1. Let $(X, \mathcal{O}_X)$ be a scheme. Then it's

(1) connected if X is connected.
(2) irreducibe if X is irreducible.
(3) reduced if for every open subset U of X, $\mathcal{O}_X(U)$ is reduced.
(4) integral if for every open subset U of X, $\mathcal{O}_X(U)$ is an integral domain.
(5) locally integral if $\mathcal{O}_{X,P}$ is an integral domain for every $P \in X$.

Proposition 4.2.1. A scheme $(X, \mathcal{O}_X)$ is integral if and only if it's irreducibe and reduced.

Proposition 4.2.2. Let $(X, \mathcal{O}_X)$ be an integral scheme and ξ be its generic point. Then $\mathcal{O}_{X,\xi}$ is a field.

Proposition 4.2.3. A scheme $(X, \mathcal{O}_X)$ is reduced if and only if $\mathcal{O}_{X,P}$ is reduced for every $P \in X$.

Proposition 4.2.4. Let $(X, \mathcal{O}_X)$ be a scheme such that X is a noetherian topological space. Then $(X, \mathcal{O}_X)$ is locally integral if and only if it's reduced and its irreducibe component are disjoint.

4.3. **Affine criterion.**

Definition 4.3.1. Let $(X, \mathcal{O}_X)$ be a scheme. For any section $f \in \mathcal{O}_X(X)$, X_f is defined to be the subset of X consisting of thoes $P \in X$ such that the germ of f at P is a unit in $\mathcal{O}_{X,P}$.

Proposition 4.3.1. Let $(X, \mathcal{O}_X)$ be a scheme.

(1) For every $f \in \mathcal{O}_X(X)$, X_f is open. It's empty if and only if there exists an open covering $\{U_i\}_{i \in I}$ of X such that each $f|_{U_i}$ is nilpotent.
(2) For any $f, g \in \mathcal{O}_X(X)$, we have $X_f \cap X_g = X_{fg}$.
(3) Let $(\varphi, \varphi^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes and $f \in \mathcal{O}_Y(Y)$. Then $\varphi^{-1}(Y_f) = X_{\varphi^\sharp(f)}$.
(4) Suppose X can be covered by finitely many affine open subschemes $\{U_i\}_{i \in I}$ such that $U_i \cap U_j$ can be covered by finitely many affine open subschemes for all $i, j \in I$. Let $A = \mathcal{O}_X$. Then for any $f \in A$, we have $\mathcal{O}_X(X_f) = A_f$.

Proposition 4.3.2. A scheme $(X, \mathcal{O}_X)$ is affine if and only if there exist finitely many sections $f_1, \dots, f_n \in \mathcal{O}_X(X)$ generating the unit ideal of $\mathcal{O}_X(X)$ such that each open subscheme $(X_{f_i}, \mathcal{O}_X|_{X_{f_i}})$ is affine.

4.4. **Noetherian scheme.**

Definition 4.4.1. A scheme $(X, \mathcal{O}_X)$ is called locally noetherian if it can be covered by affine open subschemes $\{U_i = \operatorname{Spec} A\}_{i \in I}$ such that each A_i is noetherian, and it's called noetherian if it's quasi-compact and locally noetherian.

Remark 4.4.1. If $(X, \mathcal{O}_X)$ is a noetherian scheme, then X is a noetherian topological space, but the converse is not true.

Proposition 4.4.1. Let $(X, \mathcal{O}_X)$ be a locally noetherian scheme. Then for any affine open subscheme $U = \operatorname{Spec} A$ of X, A is noetherian. In particular, an affine scheme $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is locally noetherian if and only if A is noetherian.

5. Properties of Morphisms

5.1. Quasi-compact, affine, finite type and finite.

Definition 5.1.1. Let $f\colon X \to Y$ be a morphism of schemes. It's called

(1) quasi-compact if there exists a covering of Y by affine open subschemes $\{V_i\}_{i\in I}$ such that each $f^{-1}(V_i)$ is quasi-compact.
(2) affine if there exists a covering of Y by affine open subschemes $\{V_i\}_{i\in I}$ such that each $f^{-1}(V_i)$ is affine.
(3) locally of finite type if if there exists a covering of Y by affine open subschemes $\{V_i = \operatorname{Spec} B_i\}_{i\in I}$ such that each $f^{-1}(V_i)$ can be covered by affine open subschemes $\{U_{ij} = \operatorname{Spec} A_{ij}\}_{j\in J_i}$ for some finitely generated B_i-algebra A_{ij}.
(4) finite type if it's quasi-compact and locally of finite type.
(5) finite if there exists a covering of Y by affine open subschemes $\{V_i = \operatorname{Spec} B_i\}_{i\in I}$ such that each $f^{-1}(V_i) = \operatorname{Spec} A_i$ for some finitely generated B_i-module A_i.

Proposition 5.1.1. Let $f\colon X \to Y$ be a morphism of schemes.

(1) f is quasi-compact if and only if for every open quasi-compact subset V of Y, $f^{-1}(V)$ is quasi-compact.
(2) f is affine if and only if for every affine open subscheme V of Y, $f^{-1}(V)$ is affine.
(3) f is locally of finite type if and only if for every affine open subscheme $V = \operatorname{Spec} B$ of Y and every affine open subscheme $U = \operatorname{Spec} A$ of X such that $f(U) \subseteq V$, the B-algebra A is finitely generated.
(4) f is of finite type if and only if for every affine open subscheme $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by finitely many affine open subschemes $\{U_j = \operatorname{Spec} A_j\}_{j\in J}$ such that each A_j is a finitely generated B-algebra.
(5) f is finite if and only if for every affine open subscheme $V = \operatorname{Spec} B$ of Y, $f^{-1}(V) = \operatorname{Spec} A$ for some finitely generated B-module A.

5.2. Birational morphism.

Proposition 5.2.1. Let $X \to S$ and $Y \to S$ be two morphisms and assume $Y \to S$ is locally of finite type.

(1) Let $f, g\colon X \to Y$ be two morphisms making the following diagram commutes

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

Let $x \in X$ such that $f(x) = g(x) = y$ and such that $f^\sharp_x = g^\sharp_x$. Then there exists an open neighborhood U of x in X such that $f|_U = g|_U$.

(2) Suppose S is locally noetherian. Let $x \in X$ and $y \in Y$ be two points such that their images in S are the same point $s \in S$, and let $\phi \colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ be a homomorphism making the following diagram commutes

$$\begin{tikzcd} \mathcal{O}_{X,x} & & \mathcal{O}_{Y,y} \arrow[ll, "\phi"'] \\ & \mathcal{O}_{S,s} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Then there exists an open neighborhood U of x in X and a morphism $f \colon U \to Y$ such that $f(x) = y$, $f^\sharp_x = \phi$ and the following diagram commutes

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

(3) Suppose S is locally noetherian, $X \to S$ is also locally of finite type, and $f \colon X \to Y$ is a morphism making the diagram in (1) commutes. Assume $f(x) = y$ and $f^\sharp_x \colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is an isomorphism. Then there exist open neighborhoods U of x in X and V of y in Y such that f induces an isomorphism from U to V.

Definition 5.2.1 (dominant and birational morphism)**.** Let X and Y be integral schemes.

(1) A morphism $f \colon X \to Y$ is dominant if $\overline{f(X)} = Y$.
(2) A dominant morphism $f \colon X \to Y$ is called birational if $f^\sharp_\xi \colon \mathcal{O}_{Y,\eta} \to \mathcal{O}_{X,\xi}$ is an isomorphism, where ξ, η are generic points of X and Y respectively.

Corollary 5.2.1. Let S be a locally noetherian scheme and let X and Y be two integral schemes. Suppose we have the following commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

such that $X \to S$ and $Y \to S$ are locally of finite type and f is a birational morphism. Then there exists non-empty open subsets $U \subseteq X$ and $V \subseteq Y$ such that f induces an isomorphism from U to V.

5.3. **Open immersion and closed immersion.**

Definition 5.3.1 (open immersion)**.** A morphism $(f, f^\sharp) \colon (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is called an open immersion if it induces an isomorphism of $(Z, \mathcal{O}_Z)$ with an open subscheme of $(X, \mathcal{O}_X)$.

Definition 5.3.2 (closed immersion)**.** A morphism $(f, f^\sharp) \colon (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is called a closed immersion if it induces a homeomorphism of Z with a closed subset of X, and $f^\sharp \colon \mathcal{O}_X \to f_*\mathcal{O}_Z$ is surjective.

Definition 5.3.3 (immersion)**.** A morphism $Z \to X$ is called an immersion if it can be written as a composite $Z \to U \to X$ such that $U \to X$ is an open immersion and $Z \to U$ is a closed immersion.

Definition 5.3.4. A subset Z of X is called locally closed if it's the intersection of an open subset with a closed subset.

Proposition 5.3.1. Let $(f, f^\sharp)\colon (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a morphism of schemes.

(1) $(f, f^\sharp)\colon (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is an open immersion if and only if f induces a homeomorphism of Z with an open subset of X and $f^\sharp_P\colon \mathcal{O}_{X,f(P)} \to \mathcal{O}_{Z,P}$ is an isomorphism for every $P \in Z$.
(2) $(f, f^\sharp)\colon (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is an immersion if and only if f induces a homeomorphism of Z with a locally closed subset of X and $f^\sharp_P\colon \mathcal{O}_{X,f(P)} \to \mathcal{O}_{Z,P}$ is an epimorphism.
(3) Immersions are monomorphisms in the category of schemes. Moreover, the composite of immersions is an immersion, so are open immersion and closed immersion.

5.4. **Fibred product.** In this section S always is a scheme.

Definition 5.4.1.

(1) An S-scheme is a scheme X together with a morphism $X \to S$.
(2) An S-morphism from an S-scheme X to an S-scheme Y is a morphism $X \to Y$ such that the diagram
$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$
commutes.

Remark 5.4.1. For any scheme X, there is a unique morphism $X \to \operatorname{Spec}\mathbb{Z}$, so the category of schemes coincides with the category of $\operatorname{Spec}\mathbb{Z}$-schemes.

Definition 5.4.2. Let X and Y be S-schemes. The product in the category of S-schemes is called the fibred product of X and Y over S, which is a S-scheme denoted by $X \times_S Y$.

Proposition 5.4.1. For S-schemes X and Y, their fibred product over S exists and unique up to unique isomorphism.

5.5. **Seperated morphisms.**

Definition 5.5.1 (diagonal morphism)**.** Let $f\colon X \to Y$ be a morphism of schemes. The diagonal morphism $\Delta_{X/Y}\colon X \to X \times_Y X$ to be the unique morphism satisfying
$$p \circ \Delta_{X/Y} = q \circ \Delta_{X/Y} = \mathrm{id}_X$$

Definition 5.5.2 (separated)**.** Let $f\colon X \to Y$ be a morphism of schemes. It's called separated of $\Delta_{X/Y}$ is a closed immersion.

Definition 5.5.3 (separated)**.** A scheme X is called separated if the canonical morphism $X \to \operatorname{Spec} \mathbb{Z}$ is separated.

Proposition 5.5.1. Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be a morphism of affine schemes. Then f is separated.

Proposition 5.5.2. Let $f\colon X \to Y$ be a morphism of schemes.

(1) The diagonal morphism $\Delta\colon X \to X \times_Y X$ is an immersion.
(2) $f\colon X \to Y$ is separated if and only if $\Delta_{X/Y}(X)$ is a closed subset of $X \times_Y X$.

Proposition 5.5.3.

(1) A morphism $f\colon X \to Y$ of schemes is separated if and only if there exists an open covering $\{V_i\}_{i \in I}$ of Y such that $f^{-1}(V_i) \to V_i$ is separated.
(2) Immersions are separated.
(3) The composite of two separated morphisms is separated.
(4) Seperated morphisms is stable under base change.
(5) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of schemes. If $g \circ f$ is separated, then f is separated.

Definition 5.5.4 (quasi-seperated)**.** A morphism $f\colon X \to Y$ of schemes is called quasi-separated if the diagonal morphism is quasi-compact, and a scheme X is quasi-seperated if the canonical morphism is quasi-seperated.

5.6. **Proper and projective morphisms.**

5.6.1. *Proper morphisms.*

Definition 5.6.1 (universally closed)**.** A morphism $f\colon X \to Y$ of schemes is called universally closed, if for any morphism $Y' \to Y$, the base change $f'\colon X \times_Y Y' \to Y'$ of f is a closed map on the underlying topological spaces.

Definition 5.6.2 (proper)**.** A morphism $f\colon X \to Y$ of schemes is proper if f is finite type, separated and universally closed.

Proposition 5.6.1.

(1) A morphism $f\colon X \to Y$ of schemes is proper if and only if there exists an open covering $\{V_i\}_{i \in I}$ of Y such that $f^{-1}(V_i) \to V_i$ is proper.
(2) Closed immersions are proper.
(3) The composite of two proper morphisms is proper.
(4) Seperated morphisms is stable under base change.
(5) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of schemes. If $g \circ f$ is proper and g is seperated, then f is proper.

5.6.2. *Projective morphisms.*

Definition 5.6.3. For any scheme Y, the projective space over Y is the Y-scheme $\mathbb{P}^n_Y := \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y$.

Definition 5.6.4 (projective morphisms)**.** A morphism $f: X \to Y$ of schemes is projective if f can factorized as a composite

$$X \to \mathbb{P}^n_Y \to Y$$

such that $X \to \mathbb{P}^n_Y$ is a closed immersion and $\mathbb{P}^n_Y \to Y$ is the projection. It's called quasi-projective if it can be factorized as above with $X \to \mathbb{P}^n_Y$ being an immersion.

Proposition 5.6.2. Projective morphism is proper.

Proposition 5.6.3.

(1) Closed immersions are projective.
(2) Composites of projective morphisms are projective.
(3) Let $f: X \to Y$ and $Y' \to Y$ be morphism of schemes and let $f': X \times_Y Y' \to Y'$ be the base change of f. If f is projective, then f' is projective.
(4) Let $f: X \to Y$ and $f': X' \to Y'$ be projective S-morphisms between S-schemes. Then $f \times f': X \times_S X' \to Y \times_S Y'$ is projective.
(5) Let $f: X \to Y$ and $g: Y \to Z$ be morphism of schemes. If $g \circ f$ is projective and g is separated, then f is projective.

Proposition 5.6.4 (Segre embedding)**.** There exists a closed immersion

$$\mathbb{P}^m_S \times_S \mathbb{P}^n_S \to \mathbb{P}^{(m+1)(n+1)-1}_S$$

which is an S-morphism.

6. Coherent Sheaves

6.1. Basic definitions and examples.

Let $(X, \mathcal{O}_X)$ be a ringed space.

Definition 6.1.1 ($\mathcal{O}_X$-module)**.** A sheaf of $\mathcal{O}_X$-module (or $\mathcal{O}_X$-module) is a sheaf $\mathcal{F}$ such that

(1) every open subset $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module;
(2) for every inclusion of open subsets $V \subseteq U$, the restriction $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structure via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

Definition 6.1.2 (morphism of $\mathcal{O}_X$-module)**.** Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. A morphism of $\mathcal{O}_X$-modules is a morphism of sheaves $\phi \colon \mathcal{F} \to \mathcal{G}$ such that for each open subset $U \subseteq X$, $\phi(U) \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.

Notation 6.1.1. The set of morphisms between $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ is denoted by $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, which is a $\mathcal{O}_X(X)$-module.

Example 6.1.1. Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules.

(1) The sheaf hom $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the $\mathcal{O}_X$-module
$$U \mapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$
(2) The tensor $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the $\mathcal{O}_X$-module associated to the presheaf
$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

Remark 6.1.1. For any $p \in X$, one has
$$(\mathcal{F} \otimes \mathcal{G})_p = \mathcal{F}_p \otimes \mathcal{G}_p.$$

Example 6.1.2. Let $\{\mathcal{F}_i\}_{i \in I}$ be a family of $\mathcal{O}_X$-modules.

(1) The direct sum in the category of $\mathcal{O}_X$-module is the sheaf associated to the presheaf sheaf
$$U \mapsto \bigoplus_{i \in I} \mathcal{F}_i(U).$$
(2) The direct product in the category of $\mathcal{O}_X$-module is the sheaf associated to the presheaf sheaf
$$U \mapsto \prod_{i \in I} \mathcal{F}_i(U).$$

Example 6.1.3. Let I be a direct set.

(1) For a direct system $(\mathcal{F}_i, \phi_{ij})$ of $\mathcal{O}_X$-modules, its direct limit in the category of $\mathcal{O}_X$-modules is the sheaf associated to the presheaf
$$U \to \varinjlim_i \mathcal{F}_i(U).$$
(2) For a inverse system $(\mathcal{F}_i, \phi_{ij})$ of $\mathcal{O}_X$-modules, its direct limit in the category of $\mathcal{O}_X$-modules is the sheaf associated to the presheaf
$$U \to \varprojlim_i \mathcal{F}_i(U).$$

Definition 6.1.3. Let $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.

(1) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The direct image $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module, and it becomes a $\mathcal{O}_Y$-module via the morphism $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$.

(2) Let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. The inverse image of $\mathcal{G}$ is defined to be $\mathcal{O}_X$-module
$$f^*\mathcal{G} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}.$$

Part 3. Homework

7. Homework

7.1. Homework-1.

Exercise 7.1. 1. A filtered abelian group is a pair $(A, F^\bullet A)$ such that A is an abelian group and

$$\cdots \supset F^i A \supset F^{i+1} A \supset \cdots$$

is a decreasing family of subgroups of A with indices $i \in \mathbb{Z}$. A homomorphism $f\colon (A, F^\bullet A) \to (B, F^\bullet B)$ of filtered abelian groups is a homomorphism $f\colon A \to B$ of abelian groups such that $f(F^i A) \subset F^i B$ for all $i \in \mathbb{Z}$.

(1) Prove that filtered abelian groups form an additive category with zero objects and every morphism has kernel and cokernel.
(2) Given an example of a morphism f such that the canonical morphism $\operatorname{coim} f \to \operatorname{im} f$ is not an isomorphism.

Proof. For (1). Suppose $(A, F^\bullet A)$ and $(B, F^\bullet B)$ are filtered abelian groups. The direct product of $(A, F^\bullet A)$ and $(B, F^\bullet B)$ is given by $(A\oplus, F^\bullet(A \oplus B))$, where the filtration of $A \oplus B$ is given by $F^i(A \oplus B) = F^i A \oplus F^i B$, and it's clear morphisms between $(A, F^\bullet A)$ and $(B, F^\bullet B)$ form an abelian group such that the composition is bilinear. This shows the category of filtered abelian groups is additive, and the zero object in this category is zero group with trivial filtration.

Suppose $f\colon (A, F^\bullet A) \to (B, F^\bullet B)$ is a morphism between filtered abelian groups. Since f is also a group homomorphism between abelian groups, it has kernel and cokernel in the category of abelian groups. More precisely, $\ker f \subset A$ and $\operatorname{coker} f = B/\operatorname{im} f$. Then the filtrations on A and B induce filtrations on $\ker f$ and $\operatorname{coker} f$ respectively, and thus it gives kernel and cokernel in the category of filtered abelian groups.

For (2). Suppose $A = \mathbb{Z} \oplus \mathbb{Z}$ with filtration $\mathbb{Z} \oplus \mathbb{Z} \supset \mathbb{Z} \supset \{0\}$ and $B = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ with filtration $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \supset \mathbb{Z} \oplus \mathbb{Z} \supset \{0\}$. For homomorphism given by

$$\begin{aligned} A &\to B \\ (a, b) &\mapsto (a, b, 0), \end{aligned}$$

the coimage is exactly A with filtration $\mathbb{Z} \oplus \mathbb{Z} \supset \mathbb{Z} \supset \{0\}$, but the image is $\mathbb{Z} \oplus \mathbb{Z}$ with filtration $\mathbb{Z} \oplus \mathbb{Z} \supset \mathbb{Z} \oplus \mathbb{Z} \supset \{0\}$. □

Exercise 7.2. Let $0 \to A \to B \to C \to 0$ be a sequence of morphisms in an abelian category. Prove the following statements are equivalent:

(1) The sequence is a short exact sequence.
(2) $B \to C$ is an epimorphism and $A \to B$ is its kernel.
(3) $A \to B$ is a monomorphism and $B \to C$ is its cokernel.

Proof. Firstly let's show the following lemma:

Lemma 7.1.1. Suppose $B \xrightarrow{v} C \to 0$ is a sequence of morphisms in an abelian category. Then the following statements are equivalent:

(a) $B \to C \to 0$ is exact.
(b) the cokernel of v is $C \to 0$.
(c) v is an epimorphism.

Proof.

(a) to (b): If $B \to C \to 0$ is exact, then $\operatorname{coim} v = \operatorname{im} v$ is the kernel of $C \to 0$, that is the $\operatorname{im} v = C \to C$. On the other hand, $\operatorname{im} v$ is the kernel of cokernel v. Thus the cokernel of v is $C \to 0$.
(b) to (a): If the cokernel of v is $C \to 0$, then $\operatorname{coim} v = \operatorname{im} v = \ker(\operatorname{coker} v) = \ker\{C \to 0\}$, that is $B \to C \to 0$ is exact.
(b) to (c): If the cokernel of v is $C \to 0$ and $\alpha, \beta \colon C \to D$ are morphisms such that $\alpha \circ v = \beta \circ v$. Then $(\alpha - \beta) \circ v = 0$, and thus by universal property of cokernel there exists the following commutative diagram

$$\begin{tikzcd} B \arrow[r, "v"] & C \arrow[r] \arrow[d, "\alpha-\beta"'] & 0 \arrow[dl, dashed] \\ & D & \end{tikzcd}$$

This shows $\alpha = \beta$, that is, v is an epimorphism.
(c) to (b): If v is an epimorphism and $f \colon C \to D$ is a morphism such that $f \circ v = 0$, then $f = 0$ since v is an epimorphism, and thus every morphism f such that $f \circ v = 0$ factors through $C \to 0$, that is, the cokernel of v is $C \to 0$.

□

Remark 7.1.1. By the same argument one can see a sequence of morphisms $0 \to A \xrightarrow{u} B$ in abelian category is exact if and only if u is a monomorphism, also if and only if $0 \to A$ is the kernel of u.

Now suppose $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$ is an exact sequence in abelian category. Then we claim u is the kernel of v: Since $v \circ u = 0$, by the universal property of kernel there exists the following diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "u"] \arrow[d] \arrow[dr, dashed, "\overline{u}"] & B \\ & \operatorname{coim}\{0 \to A\} \arrow[r] & \ker v \arrow[u] \end{tikzcd}$$

Note that $\overline{u}$ is an epimorphism, since $A \to \operatorname{coim}\{0 \to A\}$ is an epimorphism and $\operatorname{coim}\{0 \to A\} \to \ker v$ is an isomorphism. Moreover, $\overline{u}$ is a monomorphism since u is a monomorphism: If $\alpha, \beta \colon D \to A$ such that $\overline{u} \circ \alpha = \overline{u} \circ \beta$, then we compose them with $\ker v \to B$, one has $u \circ \alpha = u \circ \beta$, and thus $\alpha = \beta$. Then $\overline{u}$ is both monomorphism and epimorphism, and since the category is abelian, one has $\overline{u}$ is an isomorphism, and thus u is the kernel of v. By the same argument, it's easy to see v is the cokernel of u.

In a summary, above arguments show that (1) implies (2) and (3). To see (2) implies (1), it suffices to show $0 \to A \to B \to C$ is exact, since $v\colon B \to C$ is epimorphism already implies $B \to C \to 0$ is exact. Firstly, since u is the kernel of v, then it's monomorphism, and thus $0 \to A \to B$ is exact. By previous lemma one has $0 \to A$ is the kernel of u, and thus $\operatorname{coim} u = A \to A$. On the other hand, kernel of v is u. This shows the coimage of u is exactly the kernel of v, that is $A \to B \to C$ is exact.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "u"] \arrow[d, "\cong"] & B \arrow[r, "v"] & C \\
 & \operatorname{coim} u = A \arrow[r, dashed] & \ker v = A \arrow[u, "u"] &
\end{tikzcd}$$

□

Exercise 7.3. Let A and B be objects in an abelian category. Prove that the canonical sequence

$$0 \to A \xrightarrow{i_1} A \oplus B \xrightarrow{p_2} B \to 0$$

is exact.

Proof. By Exercise 7.2 it suffices to show i_1 is a monomorphism and cokernel of i_1 is p_2. By definition there exists $p_1\colon A \oplus B \to A$ such that $p_1 \circ i_1 = \mathrm{id}_A$ and $i_2\colon B \to A \oplus B$ such that $p_2 \circ i_2 = \mathrm{id}_B$. Moreover, $p_2 \circ i_1 = p_1 \circ i_2 = 0$ and $i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}_{A\oplus B}$.

(1) Suppose $\alpha, \beta\colon C \to A$ are morphisms such that $i_1 \circ \alpha = i_1 \circ \beta$. Then $p_1 \circ i_1 \circ \alpha = p_1 \circ i_1 \circ \beta$ implies $\alpha = \beta$, and thus i_1 is a monomorphism.
(2) Suppose $\alpha\colon C \to A \oplus B$ is a morphism such that $p_2 \circ \alpha = 0$. Then

$$i_1 \circ p_1 \circ \alpha = (i_1 \circ p_1 + i_2 \circ p_2) \circ \alpha = \alpha.$$

Thus we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i_1"] & A \oplus B \arrow[r, "p_2"] & B \arrow[r] & 0 \\
 & C \arrow[u, "p_1\circ\alpha"] \arrow[ur, "\alpha"'] & & &
\end{tikzcd}$$

This shows $i_1\colon A \to A \oplus B$ satisfies the universal property of kernel.

□

Exercise 7.4. Let I be a category whose objects form a set, and let F be a covariant functor from I to the category of Abelian groups. For each $i \in I$, let $k_i\colon F(i) \to \bigoplus_{i\in I} F(i)$ be the canonical monomorphism. Let H be the subgroup of $\bigoplus_{i\in I} F(i)$ generated by

$$k_i(x_i) - k_j(F(i \to j)(x_i)),$$

where $i \to j$ goes over all morphisms in I, and x_i goes over all elements $F(i)$. Set

$$\varinjlim_{i\in I} F(i) = \left(\bigoplus_{i\in I} F(i)\right)/H.$$

Let $\phi_i\colon F(i) \to \varinjlim_{i\in I} F(i)$ be the composite of k_i with the projection $\bigoplus_{i\in I} F(i) \to \left(\bigoplus_{i\in I} F(i)\right)/H$. Then we have $\phi_j \circ F(i \to j) = \phi_i$ for every morphism $i \to j$ in I. If A is an abelian group and $\psi_i\colon F(i) \to A$ $(i \in I)$ is a family of homomorphisms such that $\psi_j \circ F(i \to j) = \psi_i$ for all morphisms $i \to j$ in I, then there exists one and only one homomorphism $\psi\colon \varinjlim_{i\in I} F(i) \to A$ such that $\psi \circ \phi_i = \psi_i$ for all i.

Proof. Firstly let's show the existence: Note that by universal property of direct sum, there exists a morphism $\phi\colon \bigoplus_i F(i) \to A$, such that $\psi_i = \phi \circ k_i$, where $k_i\colon F(i) \to \bigoplus_i F(i)$ is canonical inclusion. Moreover, for any element $k_i(x_i) - k_j(F(i \to j)(x_i)) \in H$, one has

$$\phi(k_i(x_i) - k_j(F(i \to j)(x_i))) = \psi_i(x_i) - \psi_j \circ F(i \to j)(x_i) = 0.$$

This shows $H \subseteq \ker \phi$, and thus we obtain a morphism $\psi\colon \varinjlim_{i\in I} F(i) \to A$ induced by ϕ, and it's clear $\psi_i = \psi \circ \phi_i$.

$$\begin{tikzcd}
F(i) \arrow[d] \arrow[dr, "\psi_i"] \arrow[dd, bend right=60, "\phi_i"'] & \\
\bigoplus_i F(i) \arrow[r, "\phi"] \arrow[d] & A \\
\varinjlim_{i\in I} F(i) \arrow[ur, "\psi"] &
\end{tikzcd}$$

Before we begin to prove the uniqueness, we claim any element of $\varinjlim_{i\in I} F(i)$ can be written in the form $\phi_i(x_i)$ for some $i \in I$ and some $x_i \in F(i)$: For any element $x \in \varinjlim_{i\in I} F(i) = \bigoplus_{i\in I} F(i)/H$, we write it as

$$x = \sum_{j=1}^{n} \phi_i(x_j), \quad x_j \in F(j).$$

It suffices to check the case of $n = 2$: Since I is a directed set, there exists $k \in I$ such that $k \geq 1, k \geq 2$. Then

$$\phi_1(x_1) + \phi_2(x_2) = \phi_k \circ F(1 \to k)(x_1) + \phi_k \circ F(2 \to k)(x_2).$$

Then x can be written as $\phi_k\left(F(1 \to k)(x_1) + F(1 \to k)(x_2)\right)$ as desired.

Let's show the uniqueness: If $\psi'\colon \varinjlim_{i\in I} F(i) \to A$ is another morphism such that $\psi_i = \psi' \circ \phi_i$ for all $i \in I$. By above claim, we know each element can be written as $\phi_i(x_i)$ for $x_i \in F(i)$. So it suffices to check $\psi(\phi_i(x_i)) = \psi'(\phi_i(x_i))$, which is clear

$$\psi(\phi_i(x_i)) = \psi_i(x_i) = \psi'(\phi_i(x_i)).$$

□

Exercise 7.5. Let $(A_i, \phi_{ji})_{i\in\mathbb{N}}$ be an inverse system of abelian groups over the direct set $(\mathbb{N}, \leq)$ of natural numbers. Consider the homomorphism

$$f\colon \prod_{i\in\mathbb{N}} A_i \to \prod_{i\in\mathbb{N}} A_i, \quad (a_i) \mapsto (a_i - \phi_{i+1,i}(a_{i+1})).$$

Define $\varprojlim^1_i A_i = \operatorname{coker} f$. Let $u\colon (A'_i, \phi_{ji})_{i\in\mathbb{N}} \to (A_i, \phi_{ji})_{i\in\mathbb{N}}$ and $v\colon (A_i, \phi_{ji})_{i\in\mathbb{N}} \to (A''_i, \phi_{ji})_{i\in\mathbb{N}}$ be morphisms of inverse systems of abelian groups such that the sequences

$$0 \to A'_i \xrightarrow{u_i} A_i \xrightarrow{v_i} A''_i \to 0$$

are exact for all i.

Prove that we have an exact sequence $0 \to \varprojlim_i A'_i \to \varprojlim_i A_i \to \varprojlim_i A''_i \to \varprojlim^1_i A'_i \to \varprojlim^1_i A_i \to \varprojlim^1_i A''_i \to 0$.

Proof. Consider the following commutative diagram consisting of exact sequences

$$\begin{tikzcd}
& \ker f' \arrow[d] & \ker f \arrow[d] & \ker f'' \arrow[d] & \\
0 \arrow[r] & \prod_{i\in I} A'_i \arrow[r, "u"] \arrow[d, "f'"] & \prod_{i\in I} A_i \arrow[r, "v"] \arrow[d, "f"] & \prod_{i\in I} A''_i \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & \prod_{i\in I} A'_i \arrow[r, "u"] \arrow[d] & \prod_{i\in I} A_i \arrow[r, "v"] \arrow[d] & \prod_{i\in I} A''_i \arrow[r] \arrow[d] & 0 \\
& \operatorname{coker} f' & \operatorname{coker} f & \operatorname{coker} f'' &
\end{tikzcd}$$

Since $\ker f \cong \varprojlim_i A_i$, the snake lemma yields the desired result. □

7.2. **Homework-2.**

7.2.1. *Part I.* In the following, we work with morphisms in an abelian category $\mathcal{C}$.

Exercise 7.6. Let $f\colon A \to B$ and $g\colon B \to C$ be morphisms.

(1) Suppose f and g are monomorphisms. Prove $g \circ f$ is a monomorphism.
(2) Suppose $g \circ f$ is a monomorphism. Prove f is a monomorphism.

Proof. For (1). Suppose $\alpha, \beta\colon D \to A$ are arbitrary morphisms such that $g \circ f \circ \alpha = g \circ f \circ \beta$. Then $f \circ \alpha = f \circ \beta$ since g is a monomorphism, and thus $\alpha = \beta$ since f is also a monomorphism.

For (2). Suppose $\alpha, \beta\colon D \to A$ are arbitrary morphisms such that $f \circ \alpha = f \circ \beta$. By composing g one has

$$g \circ f \circ \alpha = g \circ f \circ \beta,$$

and thus $\alpha = \beta$ since $g \circ f$ is a monomorphism. □

Exercise 7.7. Let $f\colon A \to B$ be a morphism in $\mathcal{C}$. Recall that we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ \operatorname{coim} f \arrow[r, "\cong"] & \operatorname{im} f \arrow[u] \end{tikzcd}$$

Moreover $A \to \operatorname{coim} f$ is an epimorphism and $\operatorname{im} f \hookrightarrow B$ is a monomorphism. Suppose we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"'] & B \\ C \arrow[r, "\cong"] & D \arrow[u, "\psi"'] \end{tikzcd}$$

such that $\phi\colon A \to C$ is an epimorphism, $\psi\colon D \hookrightarrow B$ is a monomorphism, and $C \cong D$ is an isomorphism. Prove that there exist isomorphisms $\operatorname{coim} f \xrightarrow{\cong} C$ and $D \xrightarrow{\cong} \operatorname{im} f$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, two heads, "\phi"] \arrow[dr] & C \arrow[r, "\cong"] & D \arrow[r, hook, "\psi"] & B \\ & \operatorname{coim} f \arrow[u, "\cong"] \arrow[r, "\cong"] & \operatorname{im} f \arrow[u, "\cong"] \arrow[ur] & \end{tikzcd}$$

Thus $\phi\colon A \to C$ can be identified with $\phi\colon A \to \operatorname{coim} f$, and $\psi\colon D \hookrightarrow B$ can be identified with $\operatorname{im} f \hookrightarrow B$.

Proof. For convenience we denote the kernel of f by $\tau\colon \ker f \to A$, denote the isomorphism between C and D by g, and denote canonical morphism from A to $\operatorname{coim} f$ by u.

Note that $\psi \circ g \circ \phi \circ \tau = f \circ \tau = 0$. Then $\phi \circ \tau = 0$ since ψ is a monomorphism and g is an isomorphism. By universal property of cokernel there is a morphism from $\operatorname{coim} f \to C$, denoted by α. Since $\alpha \circ u = \phi$ and both ϕ and u are epimorphisms, one has α is an epimorphism. By the same argument one can see there exists a morphism $\beta\colon \operatorname{im} f \to D$ which is a monomorphism. Since $\mathcal{C}$ is an abelian category, there is canonical isomorphism between $\operatorname{coim} f$ and $\operatorname{im} f$, and thus α is a monomorphism and β is an epimorphism. This shows both α and β are isomorphisms in $\mathcal{C}$, since $\mathcal{C}$ is an abelian category.

$$
\begin{tikzcd}
0 \arrow[r] & \ker f \arrow[r, "\tau"] & A \arrow[r, "f"] \arrow[dl, "u"'] \arrow[d, "\phi"] & B \arrow[r] & \operatorname{coker} f \arrow[r] & 0 \\
\operatorname{coim} f \arrow[r, dashed, "\alpha"] \arrow[rrrr, bend right, "\cong"'] & C \arrow[r, "g"] & D \arrow[u, "\psi"] & & \operatorname{im} f \arrow[ll, dashed, "\beta"'] \arrow[ull, "v"']
\end{tikzcd}
$$

□

Exercise 7.8. Define the opposite category $\mathcal{C}^\circ$ of $\mathcal{C}$ as follows:

(a) $\mathcal{C}^\circ$ has the same objects as $\mathcal{C}$. For any object A in $\mathcal{C}$, we denote the corresponding object in $\mathcal{C}^\circ$ by A°.

(b) For any objects A and B in $\mathcal{C}$, we define
$$\operatorname{Hom}_{\mathcal{C}^\circ}(A^\circ, B^\circ) = \operatorname{Hom}_{\mathcal{C}}(B, A).$$
For any morphism $\phi\colon A \to B$ in $\mathcal{C}$, we denote by $\phi^\circ\colon B^\circ \to A^\circ$ the corresponding morphism in $\mathcal{C}^\circ$.

Then

(1) Prove that $\mathcal{C}^\circ$ is an abelian category.

(2) Suppose
$$A \xrightarrow{\phi} B \xrightarrow{\psi} C$$
is an exact sequence in $\mathcal{C}$. Prove that
$$C^\circ \xrightarrow{\psi^\circ} B^\circ \xrightarrow{\phi^\circ} A^\circ$$
is an exact sequence in $\mathcal{C}^\circ$.

Proof. For (1). Firstly, let's see $\mathcal{C}$ is an additive category. For objects A°, B° and C° of $\mathcal{C}^\circ$, by definition $\operatorname{Hom}_{\mathbb{C}^\circ}(A^\circ, B^\circ) = \operatorname{Hom}_{\mathcal{C}}(A, B)$ is an abelian group, and the composition
$$\operatorname{Hom}_{\mathcal{C}^\circ}(A^\circ, B^\circ) \times \operatorname{Hom}_{\mathcal{C}^\circ}(B^\circ, C^\circ) \to \operatorname{Hom}_{\mathcal{C}^\circ}(A^\circ, C^\circ)$$
is bilinear. Moreover, the direct sum of A°, B° in $\mathcal{C}^\circ$ is the product of A, B in $\mathcal{C}$, which also exists. Secondly, let's show $\mathcal{C}^\circ$ is an abelian category. For morphism $f^\circ\colon B^\circ \to A^\circ$ in $\mathcal{C}^\circ$ corresponding to $f\colon A \to B$ in $\mathcal{C}$, we're going to show the kernel of f° is the cokernel of f. For arbitrary morphism $\alpha^\circ\colon C^\circ \to B^\circ$ such that $f^\circ \circ \alpha = 0$, by universal property of kernel, there exists the following commutative diagram

$$\begin{tikzcd}
\ker f^\circ \arrow[r] & B^\circ \arrow[r, "f^\circ"] & A^\circ \\
C^\circ \arrow[u, dashed] \arrow[ur, "\alpha^\circ"] & &
\end{tikzcd}$$

This corresponds to the following commutative diagram in category $\mathcal{C}$

$$\begin{tikzcd}
\ker f^\circ \arrow[d, dashed] & B \arrow[l] \arrow[dl, "\alpha"'] & A \arrow[l, "f"'] \\
C & &
\end{tikzcd}$$

Then by uniqueness of cokernel, one has $\ker f^\circ$ is exactly the cokernel of f. Similarly one can show the cokernel of f° is exactly the kernel of f. This shows for any morphism $f^\circ\colon B^\circ \to A^\circ$, it has kernel and cokernel since $\mathcal{C}$ is an abelian category. Moreover, by the same argument it's easy to see $\operatorname{coim} f^\circ$ is isomorphic to $\operatorname{im} f$, and $\operatorname{im} f^\circ$ is isomorphic to $\operatorname{coim} f$, and thus

$$\operatorname{coim} f^\circ \cong \operatorname{im} f^\circ,$$

since $\mathcal{C}$ is an abelian category.

For (2). Note that $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ is exact if and only if $\ker\psi = \operatorname{coim}\phi$, and since $\mathcal{C}$ is an abelian category, it's equivalent to $\ker\psi = \operatorname{im}\phi$. By arguments in the proof of (1) it's equivalent to $\operatorname{coker}\psi^\circ = \operatorname{coim}\phi^\circ$. □

7.2.2. *Part II.*

Exercise 7.9. Let X be a topological space, A an abelian group endowed with the discrete topology, and $\mathcal{F}$ the sheaf so that $\mathcal{F}(U)$ is the group of continuous maps from U to A for every open subset U of X. Prove that $\mathcal{F}$ is isomorphic to the sheaf associated to the constant presheaf $U \mapsto A$.

Proof. Firstly note that if A is equipped with discrete topology, then continuous map f from U to A is locally constant since every point $a \in A$ is an open subset, and thus its preimage $f^{-1}(a)$ is an open subset in U. On the other hand, by the construction of constant sheaf associated to the constant presheaf, the sections of it over U are also locally constant maps from U to A. This shows $\mathcal{F}$ is exactly the sheafication of constant presheaf. □

Exercise 7.10. For every open subset U of the complex plane $\mathbb{C}$, let $\mathcal{O}(U)$ be the ring of holomorphic functions on U, and let $\mathcal{O}^*(U)$ be the group of units in $\mathcal{O}(U)$. Prove that the morphism $\mathcal{O} \to \mathcal{O}^*$ defined by

$$\begin{aligned}\mathcal{O}(U) &\to \mathcal{O}^*(U)\\ f &\mapsto e^{2\pi\sqrt{-1}f}\end{aligned}$$

is an epimorphism in the category of sheaves of abelian groups, but not an epimorphism in the category of presheaves. Here we regard $\mathcal{O}$ as a sheaf of abelian groups with respect to addition of functions. Prove that the kernel of this morphism is isomorphic to the sheaf associated to the constant presheaf $U \mapsto \mathbb{Z}$.

Proof. For the first part, if we want to show $\mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$ is an exact sequence in the category of sheaves of abelian groups, it suffices to check for each $x \in \mathbb{C}$, the following sequence of stalks is exact

$$\mathcal{O}_x \xrightarrow{\exp} \mathcal{O}_x^* \to 0.$$

It holds since for any non-vanishing holomorphic function f, $\log f$ is well-defined on a sufficiently small neighborhood of x, which proves the surjectivity. On the other hand, $\mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$ is not an exact sequence in the category of presheaves of abelian groups, since

$$\mathcal{O}(\mathbb{C}^*) \xrightarrow{\exp} \mathcal{O}^*(\mathbb{C}^*) \to 0$$

fails to be an exact sequence.

For the half part, we need to prove

$$0 \to \underline{\mathbb{Z}} \xrightarrow{2\pi\sqrt{-1}} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^*$$

is an exact sequence in the category of sheaves of abelian groups. It suffices to show for any open subset $U \subseteq \mathbb{C}$, the following sequence of abelian groups is exact

$$0 \to \underline{\mathbb{Z}}(U) \xrightarrow{2\pi\sqrt{-1}} \mathcal{O}(U) \xrightarrow{\exp} \mathcal{O}^*(U).$$

If $u\colon U \to \mathbb{Z}$ is a locally constant function, then it's clear $\exp(2\pi\sqrt{-1}u) = 0$. Conversely, if $v\colon U \to \mathbb{C}$ is a holomorphic function such that $\exp v = 0$. Then for each $x \in U$, $v(x) = 2\pi\sqrt{-1}u(x)$, where $u\colon U \to \mathbb{Z}$ is a continuous function since v is continuous, and thus $v \in 2\pi\sqrt{-1}\underline{\mathbb{Z}}(U)$, since continuous integral-valued function is locally constant. □

Exercise 7.11. Let $\mathcal{C}$ be a category. For any object $X \in \operatorname{ob}\mathcal{C}$, let $\widetilde{X}\colon \mathcal{C} \to$ (Sets) be the contravariant functor from $\mathcal{C}$ to the category of sets defined by

$$\widetilde{X}(Y) = \operatorname{Hom}(Y, X).$$

A functor from $\mathcal{C}$ to the category of sets is called representable by X if it is isomorphic to $\widetilde{X}$. For any contravariant functor $G\colon \mathcal{C} \to$ (Sets), prove that we have a one-to-one correspondence

$$\begin{aligned} \operatorname{Hom}(\tilde{X}, G) &\to G(X) \\ \alpha &\to \alpha_X(\mathrm{id}_X), \end{aligned}$$

where $\operatorname{Hom}(\widetilde{X}, G)$ is the set of natural transformations from the functor $\widetilde{X}$ to the functor G. Prove the same result for covariant functors.

Proof. Let us first check this correspondence is surjective: For an object $s \in G(X)$, we define $\alpha = \alpha(s)\colon \widetilde{X} \to G$ as follows: For $X' \in \mathcal{C}$, let $\alpha_{X'}\colon \widetilde{X}(X') \to G(X')$ be the morphism of set which sends $f\colon X' \to C$ to $G(f)(s)$. Now let's show $\alpha\colon \widetilde{X} \to G$ is a natural transformation: For any morphism $g\colon X'' \to X'$ in $\mathcal{C}$, it suffices to show the following diagram commutes

$$\begin{array}{ccc} \widetilde{X}(X') & \xrightarrow{\alpha_{X'}} & G(X') \\ \big\downarrow{\scriptstyle \widetilde{X}(g)} & & \big\downarrow{\scriptstyle G(g)} \\ \widetilde{X}(X'') & \xrightarrow{\alpha_{X''}} & G(X'') \end{array}$$

For any element $f \in \widetilde{X}(X')$, that is, a morphism $f\colon X' \to X$, one has

$$G(f \circ g)(s) = G(g) \circ G(f)(s).$$

This shows above diagram commutes by the construction of α. Moreover, it's clear

$$\alpha_C(\mathrm{id}_X) = G(\mathrm{id}_X)(s) = s$$

as desired.

To see above correspondence is injective: If there are two natural transformation $\alpha, \eta\colon \widetilde{X} \to G$ such that $\alpha_X(\mathrm{id}_X) = \eta_X(\mathrm{id}_X)$, we need to show $\alpha = \eta$. In other words, it suffices to show for any $X' \in \mathcal{C}$, we have $\alpha_{X'} = \eta_{X'}$. For any morphism $g\colon X' \to X$, as α is a natural transformation, we have the following commutative diagram

$$\begin{array}{ccc} \widetilde{X}(X) & \xrightarrow{\alpha_{X}} & G(X) \\ \big\downarrow{\scriptstyle \widetilde{X}(g)} & & \big\downarrow{\scriptstyle G(g)} \\ \widetilde{X}(X') & \xrightarrow{\alpha_{X'}} & G(X') \end{array}$$

It follows that

$$G(g) \circ \alpha_X(\mathrm{id}_X) = \alpha_{X'} \circ \widetilde{X}(g)(\mathrm{id}_X) = \alpha_{X'}(g).$$

Similarly as η is a natural transformation, one has $(G(g) \circ \eta_X)(\mathrm{id}_X) = \eta_{X'}(g)$. Hence

$$\alpha_{X'}(g) = G(g) \circ \alpha_X(\mathrm{id}_X) = G(g) \circ \eta_X(\mathrm{id}_X) = \eta_{X'}(g).$$

By considering the opposite category, it's clear the same result holds for covariant functors. □

Exercise 7.12. Let $u\colon \mathcal{C} \to \mathcal{D}$ be a functor. Suppose that for each object $D \in \mathrm{ob}\, \mathcal{D}$, the functor

$$\begin{aligned} \mathcal{C} &\to (\mathrm{Sets}) \\ C &\mapsto \mathrm{Hom}(u(C), D) \end{aligned}$$

is representable by an object $v(D) \in \mathrm{ob}\, \mathcal{C}$. Then $v\colon \mathcal{D} \to \mathcal{C}$ is a functor right adjoint to u.

Proof. In other words, for any objects $C \in \mathcal{C}, D \in \mathcal{D}$, there is an one-to-one correspondence

$$\mathrm{Hom}(u(C), D) \cong \mathrm{Hom}(C, v(D)).$$

Thus by definition v is a right adjoint to u. □

Exercise 7.13. Let $u\colon \mathcal{C} \to \mathcal{D}$ be a functor.

(1) We say u is faithful (resp. fully faithful) if for any objects $C_1, C_2 \in \operatorname{ob} \mathcal{C}$, the map
$$\operatorname{Hom}(C_1, C_2) \to \operatorname{Hom}(u(C_1), u(C_2))$$
is injective (resp. bijective).

(2) We say u is essentially surjective if for any object D in $\mathcal{D}$, there exists an object C in $\mathcal{C}$ such that we have an isomorphism $u(C) \cong D$.

(3) We say u is an equivalence of categories if u is both fully faithful and essentially surjective.

Suppose u is an equivalence of categories. For any $D \in \operatorname{ob} \mathcal{D}$, choose an object $v(D) \in \operatorname{ob} \mathcal{C}$ such that $u \circ v(D) \cong D$. Prove that v is a functor that is both left and right adjoint to $u\colon \mathcal{D} \to \mathcal{C}$. It is called a quasi-inverse of u.

Proof. Firstly let's show v is a functor: If $f\colon D_1 \to D_2$ is a morphism in $\mathcal{D}$, then consider the following commutative diagram

$$\begin{tikzcd}
D_1 \arrow[r, "v"] \arrow[d, "f"] & v(D_1) \arrow[r, "u"] \arrow[d, dashed, "v(f)"] & D_1 \arrow[d, "f"] \\
D_2 \arrow[r, "v"] & v(D_2) \arrow[r, "u"] & D_2
\end{tikzcd}$$

Since u is an equivalence of categories, and thus it's fully faithfully, so there exists a morphism $v(f)\colon v(D_1) \to v(D_2)$ still making above diagram commutes, which shows v is a functor.

Now let's show v is the right adjoint of u, that is to show for any $C \in \mathcal{C}$ and $D \in \mathcal{D}$, there is a one-to-one correspondence $\operatorname{Hom}(u(C), D) = \operatorname{Hom}(C, v(D))$. Note that u is essentially surjective, so there exists C' such that $u(C') = D$, and thus
$$\operatorname{Hom}(u(C), D) = \operatorname{Hom}(u(C), u(C')) = \operatorname{Hom}(C, C').$$
On the other hand, one has
$$\operatorname{Hom}(C, v(D)) = \operatorname{Hom}(C, v \circ u(C')) = \operatorname{Hom}(u(C), u \circ v \circ u(C')) = \operatorname{Hom}(u(C), u(C')) = \operatorname{Hom}(C, C').$$
This shows v is the right adjoint of v, and by the same argument one can see v is the left adjoint of u. $\square$

7.3. **Homework-3.**

Exercise 7.14. Let A be a ring. For every open subset U of $\operatorname{Spec} A$, let S_U be the multiplicative subset $S_U = \bigcap_{\mathfrak{p}\in U}(A - \mathfrak{p})$, and let $\mathcal{P}(U) = S_U^{-1}A$. For every inclusion $V \subseteq U$ of open subsets, we have $S_U \subseteq S_V$ and hence we have a canonical homomorphism $\mathcal{P}(U) \to \mathcal{P}(V)$. This makes $\mathcal{P}$ a presheaf of rings on $\operatorname{Spec} A$. Prove that $\mathcal{O}_{\operatorname{Spec} A} \cong \mathcal{P}^+$.

Proof. It suffices to show for each point $\mathfrak{p} \in \operatorname{Spec} A$, one has

$$\mathcal{O}_{\operatorname{Spec} A,\mathfrak{p}} \cong \mathcal{P}^+_{\mathfrak{p}}.$$

Note that $\mathcal{O}_{\operatorname{Spec} A,\mathfrak{p}} \cong A_{\mathfrak{p}}$, so it suffices to show $\mathcal{P}^+_{\mathfrak{p}} \cong A_{\mathfrak{p}}$. But by the construction of $\mathcal{P}$, one has

$$\mathcal{P}^+_{\mathfrak{p}} = \mathcal{P}_{\mathfrak{p}} = \varinjlim_{\mathfrak{p}\in U} \mathcal{P}(U) = \varinjlim_{\mathfrak{p}\in U} S_U^{-1}A.$$

Now it suffices to show that $A_{\mathfrak{p}}$ satisfies the universal property of inverse limit $\varinjlim_{\mathfrak{p}\in U} S_U^{-1}A$, which follows from the universal property of localization. □

Exercise 7.15. Let S be a multiplicative subset of a ring A. Prove that the canonical morphism $\operatorname{Spec} S^{-1}A \to \operatorname{Spec} A$ induces an embedding on the underlying topological spaces.

Proof. Recall that the prime ideals in $S^{-1}A$ are in one to one correspondence with prime ideals in A which do not intersect with S, and the correspondence is given by pullback. This shows the canonical morphism $\phi\colon \operatorname{Spec} S^{-1}A \to \operatorname{Spec} A$ is bijective, and it's clear that ϕ is continuous, so it suffices to show that ϕ is closed.

Note that every ideal in $S^{-1}A$ is an extended ideal, that is, it's of the form $S^{-1}\mathfrak{a}$, where $\mathfrak{a} \subseteq A$ is an ideal. Then

$$\begin{aligned}\phi(V(S^{-1}\mathfrak{a})) &= \phi(\{S^{-1}\mathfrak{p} \mid S^{-1}\mathfrak{a} \subseteq S^{-1}\mathfrak{p},\ \mathfrak{p} \text{ is prime}\})\\ &= \{\mathfrak{p} \mid \mathfrak{a} \subseteq \mathfrak{p},\ \mathfrak{p} \text{ is prime}\}\\ &= V(\mathfrak{a}).\end{aligned}$$

This completes the proof. □

Exercise 7.16. Let x be a point in scheme X, and let $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ be the residue field at x. Construct a natural morphism $i\colon \operatorname{Spec} k(x) \to X$ with image x so that the homomorphism $\mathcal{O}_{X,x} \to k(x)$ induced by $i^\sharp$ is the canonical homomorphism.

Proof. If we want to construct a morphism from scheme $\operatorname{Spec} k(x) \to X$, it suffices to construct a continuous map between topological spaces and a morphism between structure sheaves.

(1) For the continuous map between topological spaces $\operatorname{Spec} k(x)$ and X, we simply send $\operatorname{Spec} k(x)$ to the point $x \in X$ since $\operatorname{Spec} k(x)$ is just a single point.

(2) For the morphism $i^\sharp$ between structure sheaves, it's defined as follows: For open subset $U \subseteq X$, $i^\sharp \colon \mathcal{O}_X(U) \to \mathcal{O}_{\operatorname{Spec} k(x)}(i^{-1}(U)) = k(x)$ is defined by
$$\mathcal{O}_X(U) \xrightarrow{\alpha} \mathcal{O}_{X,x} \to k(x),$$
where α is given by taking limit if $x \in U$, otherwise α is zero map.

Then above data gives a morphism between schemes $\operatorname{Spec} k(x)$ and X, and by definition the homomorphism $\mathcal{O}_{X,x} \to k(x)$ induced by $i^\sharp$ is canonical morphism. $\square$

7.4. **Homework-4.**

Exercise 7.17. Let X be a topological space and $\mathcal{C}_X$ be the sheaf of complex-valued functions on X. Prove that $(X, \mathcal{C}_X)$ is a locally ringed space. Moreover, for each $p \in X$, one has

$$\mathfrak{m}_p = \{f \in \mathcal{C}_{X,p} \mid f(p) = 0\},$$

and the residue field $k(p) = \mathbb{C}$.

Proof. To show $(X, \mathcal{C}_X)$ is a locally ringed space, it suffices to show that for each $p \in X$, every element in $\mathcal{C}_{X,p} \setminus \mathfrak{m}_p$ is a unit. Then $\mathfrak{m}_p$ is the unique maximal ideal and thus $\mathcal{C}_{X,p}$ is the local ring.

For $f \in \mathcal{C}_{X,p} \setminus \mathfrak{m}_p$, since $f(p) \neq 0$, we may construct a continuous function g defined on an open neighborhood U of p such that $g(p) = 1/f(p)$. Then g is an inverse of f in $\mathcal{C}_{X,p}$.

To see the residue field, it suffices to note that

$$0 \to \mathfrak{m}_p \to \mathcal{C}_{X,p} \to \mathbb{C} \to 0$$

is an exact sequence. $\square$

Exercise 7.18. Let $f \colon U \to X$ be an embedding of topological spaces. Then for any sheaf $\mathcal{F}$ defined on U and $p \in U$, prove that

$$(f_*\mathcal{F})_{f(p)} \cong \mathcal{F}_p.$$

Proof. Since f is a topological embedding, without lose of generality we may assume $U \subseteq X$ equipped with subspace topology and f is the inclusion map $i \colon U \hookrightarrow X$. By definition one has

$$\begin{aligned}(i_*\mathcal{F})_p &= \varinjlim_{p \in V \subseteq X} i_*\mathcal{F}(V) \\ &= \varinjlim_{p \in V \subseteq X} \mathcal{F}(i^{-1}(V)) \\ &= \varinjlim_{p \in V \subseteq X} \mathcal{F}(V \cap U).\end{aligned}$$

On the other hand, since U is equipped with subspace topology, every open subset of U containing p is exactly of the form $V \cap U$, where $V \subseteq X$ is an open subset containing p. This shows

$$\varinjlim_{p \in V \subseteq X} \mathcal{F}(V \cap U) = \varinjlim_{p \in U} \mathcal{F}(U) = \mathcal{F}_p$$

$\square$

7.5. **Homework-5.**

7.5.1. *Part I.*

Exercise 7.5.1. Let S be a graded ring.

(1) Let $\mathfrak{p}$ be a prime ideal of S, and let

$$\mathfrak{p}' = \bigoplus_d (\mathfrak{p} \cap S_d).$$

Prove that $\mathfrak{p}'$ is a homogeneous prime ideal of S.

(2) Let $\mathfrak{a}$ be a homogeneous ideal of S. Prove that $\sqrt{\mathfrak{a}}$ is the intersection of homogeneous prime ideals containing $\mathfrak{a}$.

Proof. For (1). Note that for any degree d, one has

$$\mathfrak{p}' \cap S_d = \mathfrak{p} \cap S_d.$$

This shows

$$\mathfrak{p}' = \bigoplus_d (\mathfrak{p} \cap S_d) = \bigoplus_d (\mathfrak{p}' \cap S_d),$$

and thus $\mathfrak{p}$ is a homogeneous ideal.

To see $\mathfrak{p}'$ is prime, it suffices to show if a, b are two homogeneous elements such that $ab \in \mathfrak{p}'$, then either a or b in $\mathfrak{p}$. Since both a and b are homogeneous, then ab is also homogeneous. If $ab \in \mathfrak{p} \cap S_d$, then either a or b in $\mathfrak{p}$ since $\mathfrak{p}$ is prime, and thus either a or b in some $\mathfrak{p} \cap S_{d'}$ since both a and b are homogeneous. This completes the proof of $\mathfrak{p}'$ is a prime homogeneous ideal.

For (2). Suppose I is the set of all homogeneous prime ideals of S containing $\mathfrak{a}$. Firstly one has $\sqrt{\mathfrak{a}} \subseteq \bigcap_{\mathfrak{p}' \in I} \mathfrak{p}'$ since $\sqrt{\mathfrak{a}}$ equals the intersection of all prime ideals containing $\mathfrak{a}$. On the other hand, for any prime ideal $\mathfrak{p}$ containing $\mathfrak{a}$, one has the homogeneous prime $\mathfrak{p}' = \bigoplus_d (\mathfrak{p} \cap S_d) \subseteq \mathfrak{p}$ also contains $\mathfrak{a}$, since

$$\mathfrak{a} = \bigoplus_d (\mathfrak{a} \cap S_d) \subseteq \bigoplus_p (\mathfrak{a} \cap S_d).$$

Thus $\sqrt{\mathfrak{a}} = \bigoplus_{\mathfrak{p}' \in I} \mathfrak{p}'$ as desired. $\square$

Exercise 7.5.2. Let $\phi \colon S \to T$ be a homomorphism of graded rings. Suppose there exists an integer $m \geq 1$ such that $\phi(S_d) \subset T_{md}$ for all d. Let I be the homogeneous ideal of T generated by $\phi(S_+)$, and let $U = \operatorname{Proj} T - V(T)$. Construct a morphism $f \colon U \to \operatorname{Proj} S$ of schemes so that $f(\mathfrak{q}) = \phi^{-1}(\mathfrak{q})$ for any $\mathfrak{q} \in U$ and that $f^\sharp_\mathfrak{q} \colon \mathcal{O}_{\operatorname{Proj} S, f(\mathfrak{q})} \to \mathcal{O}_{\operatorname{Proj} T, \mathfrak{q}}$ can be identified with the homomorphism $\phi_\mathfrak{q} \colon S_{(\phi^{-1}(\mathfrak{q}))} \to T_{(\mathfrak{q})}$.

Proof. Firstly let's construct the continuous map between the base topological spaces U and $\operatorname{Proj} S$. Since $\phi \colon S \to T$ is a homomorphism of graded rings, one has the pullback of a homogeneous ideal $\mathfrak{q} \subseteq T$ still is a homogeneous ideal of S. As a consequence, if $\mathfrak{q} \in U = \operatorname{Proj} T \backslash V(T)$, one has $\phi^{-1}(\mathfrak{q})$ is a homogeneous prime ideal, and $\phi^{-1}(\mathfrak{q})$ doesn't contain S_+, otherwise $\mathfrak{q}$

will contain T, which is a contradiction. This shows $f(\mathfrak{q}) := \phi^{-1}(\mathfrak{q})$ gives a well-defined map from U to $\operatorname{Spec} S$, which is also continuous.

Now let's construct the morphism between structure sheaves $\mathcal{O}_{\operatorname{Proj} S}$ and $f_*\mathcal{O}_U$, where $\mathcal{O}_U$ is $\mathcal{O}_{\operatorname{Spec} T}|_U$ in fact. For each open subset $V \subseteq \operatorname{Spec} S$, it suffices to construct a homomorphism of rings

$$\mathcal{O}_{\operatorname{Proj} S}(V) \to \mathcal{O}_U(f^{-1}(V)) = \mathcal{O}_{\operatorname{Proj} T}|_U(f^{-1}(V)),$$

which is compatible with the restriction map between different open subsets. Given $\mathfrak{q} \in \operatorname{Proj} T \setminus V(T)$, we will construct a homomorphism $\phi_{\mathfrak{q}} \colon S_{\phi^{-1}(\mathfrak{q})} \to T_{(\mathfrak{q})}$ as follows: For any element $s/t \in S_{\phi^{-1}(\mathfrak{q})}$, $\phi(s/t)$ is defined by $\phi(s)/\phi(t)$. Since $t \notin \phi^{-1}(\mathfrak{q})$, one has $\phi(t) \notin \mathfrak{q}$, and thus $\phi(s/t) \in T_{\mathfrak{q}}$. On the other hand, since ϕ is a ring homomorphism between graded rings, it maps elements of degree zero to the one of degree zero, and thus $\phi(s/t) \in T_{(\mathfrak{q})}$ as desired.

For any element $s \in \mathcal{O}_{\operatorname{Proj} S}(V)$, it's a map $s \colon V \to \coprod_{\mathfrak{p}} S_{\mathfrak{p}}$ satisfying some properties. Given $\mathfrak{q} \in f^{-1}(V)$ with $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$, that is, $f(\mathfrak{q}) = \mathfrak{p} \in V$. By composing

$$\mathfrak{q} \xrightarrow{f} \mathfrak{p} \xrightarrow{s_{\mathfrak{p}}} S_{(\mathfrak{p})} \xrightarrow{\phi_{\mathfrak{q}}} T_{(\mathfrak{q})},$$

one can construct a map $t \colon f^{-1}(V) \to \coprod_{\mathfrak{q} \in f^{-1}(V)} T_{(\mathfrak{q})}$. A routine check shows that t gives an element of $\mathcal{O}_U(f^{-1}(V))$, and this correspondence gives a morphism between sheaves $\mathcal{O}_{\operatorname{Proj} S} \to f_*\mathcal{O}_U$ such that the induced morphism on stalks is exactly $\phi_{\mathfrak{q}} \colon S_{(\phi^{-1}(\mathfrak{q}))} \to T_{(\mathfrak{q})}$. □

Exercise 7.5.3. Let A be a ring, I an ideal of A, and $S = \bigoplus_{d=0}^{\infty} I^d$. Then S is a graded ring. We call Proj S the blowing-up of $\operatorname{Spec} A$ along the ideal I. Prove that the inclusion $A = S_0 \hookrightarrow S$ induces a morphism of schemes $f \colon \operatorname{Proj} S \to \operatorname{Spec} A$ such that over the open subset $U = \operatorname{Spec} A - V(I)$, f induces an isomorphism $f^{-1}(U) \xrightarrow{\cong} U$.

Proof. For convenience we denote the inclusion $i \colon A \hookrightarrow S$. Firstly let's show the inclusion i gives a continuous map f between topological spaces between $\operatorname{Proj} S$ and $\operatorname{Spec} A$ with $\operatorname{im} f \subseteq U$. Given $\mathfrak{p} \in \operatorname{Proj} S$, one has $i^{-1}(\mathfrak{p})$ is a prime ideal of A, and $i^{-1}(\mathfrak{q})$ cannot contain the ideal I, otherwise $\mathfrak{q}$ contains all power of I, and thus it contains S_+, a contradiction.

(1) Note that the continuous map $f \colon \operatorname{Proj} S \to U$ is surjective, since for any prime ideal $\mathfrak{p} \subseteq A$, automatically it's a homogeneous prime ideal in S with $f(\mathfrak{p}) = \mathfrak{p}$.
(2) On the other hand, if homogeneous prime ideals $\mathfrak{p}, \mathfrak{q} \in S$ such that $i^{-1}(\mathfrak{p}) = i^{-1}(\mathfrak{q})$, then $\mathfrak{p} = i^{-1}(\mathfrak{p})S_+ = i^{-1}(\mathfrak{q})S_+ = \mathfrak{q}$. This shows f is injective.

(3) Finally let's show f is a closed map. Suppose $V_+(\mathfrak{a})$ is a closed subset of $\operatorname{Proj} S$, where $\mathfrak{a} \subseteq S$ is a homogeneous prime ideal. Then
$$\begin{aligned} f(V_+(\mathfrak{a})) &= f(\{\mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{a} \subseteq \mathfrak{p}\}) \\ &= \{i^{-1}(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Proj} S, \mathfrak{a} \subseteq \mathfrak{p}\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} A \mid i^{-1}(\mathfrak{a}) \subseteq \mathfrak{p}\} \\ &= V(i^{-1}(\mathfrak{a})). \end{aligned}$$

Above arguments shows that $f^{-1}(U) \xrightarrow{\cong} U$ as topological spaces.

To see there exists an isomorphism between structure sheaves $\mathcal{O}_U$ and $f_*\mathcal{O}_{f^{-1}(U)}$, it suffices to show $\mathcal{O}_{U,f(\mathfrak{p})} \cong \mathcal{O}_{f^{-1}(U),\mathfrak{p}}$ holds for each $\mathfrak{p} \in f^{-1}(U)$. On one hand, one has
$$\mathcal{O}_{U,f(\mathfrak{p})} = A_{i^{-1}(\mathfrak{p})}.$$
On the other hand, $\mathcal{O}_{f^{-1}(U),\mathfrak{p}} \cong S_{(\mathfrak{p})}$. Note that
$$S_{(\mathfrak{p})} = \{\frac{s}{t} \mid s \in S, t \in \mathfrak{p}, s, t \text{ are homogeneous and of the same degree.}\}$$
It's clear that there exists an inclusion $A_{i^{-1}(\mathfrak{p})} \hookrightarrow S_{(\mathfrak{p})}$. Conversely, for any $s/t \in S_{(\mathfrak{p})}$, it suffices to construct an element $a/b \in A_{i^{-1}(\mathfrak{p})}$ such that $a/b = s/t$ in $S_{(\mathfrak{p})}$. □

Exercise 7.5.4. Let $A = R[x_1, \dots, x_n]$ be a polynomial ring, let $I = (x_1, \dots, x_n)$ be the ideal of A generated by $x_1, \dots, x_n$, let $S = \bigoplus_{d=0}^{\infty} I^d$, let $T = A[y_1, \dots, y_n]$ be the graded ring so that T_d consists of homogeneous polynomials of degree d in the variables $y_1, \dots, y_n$ with coefficients in A, and let J be the homogeneous ideal of T generated by $x_iy_j - x_jy_i$ $(i, j = 1, \dots, n)$. Consider the epimorphism of graded rings $\phi\colon T \to S$ so that $\phi(a) = a \in S_0$ for any $a \in A$ and $\phi(y_i) = x_i \in S_1$. Prove that ϕ induces an isomorphism of schemes $\operatorname{Proj} S \xrightarrow{\cong} \operatorname{Proj} A[y_1, \dots, y_n]/J$.

Proof. Here we only give a proof for $n = 2$, and cases for more variables are similar. Firstly note that the kernel of the epimorphism of graded rings
$$\phi\colon T \to S$$
contains the ideal J, since $\phi(x_1y_2 - x_2y_1) = 0$. Conversely, if $\alpha = ay_1 + by_2$ is mapped to 0, then it must be in the ideal generated by $x_1y_2 - x_2y_1$ since x_1, x_2 are algebraically independent. This shows $\phi\colon T \to S$ induces an isomorphism between graded rings S and $A[y_1, \dots, y_n]/J$, that is, a ring isomorphism which preserves the degree. Thus it induces an isomorphism between schemes $\operatorname{Proj} S$ and $\operatorname{Proj} A[y_1, \dots, y_n]/J$. □

7.5.2. *Part II.*

Exercise 7.5.5. Let S be a graded ring and $\mathfrak{p}$ be a homogeneous prime ideal. Prove that

(1) If $f \in S \setminus \mathfrak{p}$, then $S_{(f)} \to S_{(\mathfrak{p})}$ is injective.
(2) $S_{(\mathfrak{p})}$ is a local ring.

Proof. For (1). Suppose $a/f^n = b/f^m$ in $S_{(\mathfrak{p})}$. Then there exists a homogeneous element $p \in \mathfrak{p}$ such that
$$p(af^m - bf^n) = 0.$$
If S is a integral domain, it's clear $(af^m - bf^n) = 0$, and thus $S_{(f)} \to S_{(\mathfrak{p})}$ is injective. In general case I don't know to show it.

For (2). Consider
$$\mathfrak{m} = \{\frac{s}{t} \mid s, t \in \mathfrak{p}, s, t \text{ are homogeneous and of the same degree.}\}.$$
Note that any element outside of $\mathfrak{m}$ is invertible, and thus $\mathfrak{m}$ is the only maximal ideal of the local ring $S_{(\mathfrak{p})}$. $\square$

Exercise 7.5.6. Prove that if $\varnothing \neq U \subseteq \operatorname{Spec} A$, then $0 \neq 1 \in \mathcal{O}_{\operatorname{Spec} A}(U)$.

Proof. Since $\varnothing \neq U$, we may assume there exists a non-zero ideal $\mathfrak{p} \in U$. If $0 = 1 \in \mathcal{O}_{\operatorname{Spec} A}$, that is, $\mathcal{O}_{\operatorname{Spec} A}(U)$ is a zero ring, then for any open subset $V \subseteq U$, one has $\mathcal{O}_{\operatorname{Spec} A}(V)$ is also zero ring. In particular, $\varinjlim_{\mathfrak{p} \in U} \mathcal{O}_{\operatorname{Spec} A}(U) = A_{\mathfrak{p}}$ is a zero ring, a contradiction. $\square$

7.6. **Homework-6.**

Exercise 7.6.1. Let X be a noetherian topological space and let $\mathcal{F}$ be a presheaf. Suppose that for every open subset U and every finite open covering $(U_i)_{i\in I}$, the following conditions hold:

(1) For any sections $s, t \in \mathcal{F}(U)$ such that $s|_{U_i} = t|_{U_i}$ for every $i \in I$, then $s = t$.
(2) Let $s_i \in \mathcal{F}(U_i)$ be sections such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for every pair $i, j \in I$. Then there exists a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for every $i \in I$.

Prove that $\mathcal{F}$ is a sheaf.

Proof. For a noetherian topological space X, any open subset $U \subseteq X$ is quasi-compact, that is, any open covering of U admits a finite subcovering. Thus it suffices to chech above two conditions for every finite open covering. □

Exercise 7.6.2. Let $(\mathcal{F}_i, \phi_{ij})_{i\in I}$ be a direct system of sheaves of abelian groups on X, and let $\varinjlim_i \mathcal{F}_i$ be the sheaf associated to the presheaf $U \mapsto \varinjlim_i \mathcal{F}_i(U)$. Prove that $\varinjlim_i \mathcal{F}_i$ is the direct limit of $(\mathcal{F}_i, \phi_{ij})_{i\in I}$ in the category of sheaves, and for every $P \in X$, we have

$$(\varinjlim_i \mathcal{F}_i)_P \cong \varinjlim_i \mathcal{F}_{i,P}.$$

Suppose furthermore that X is a noetherian topological space. Prove that the presheaf $U \mapsto \varinjlim_i \mathcal{F}_i(U)$ is a sheaf.

Proof. Firstly let's show $\varinjlim_i \mathcal{F}_i$ statisfies the universal property of the direct limit of $(\mathcal{F}_i, \phi_{ij})_{i\in I}$. Suppose $\mathcal{C}$ is a sheaf and $\psi_i \colon \mathcal{F}_i \to \mathcal{C}$ are morphisms such that $\psi_j\phi_{ij} = \psi_i$. Then For any open subset $U \subseteq X$, by the universal property of $\varinjlim_i \mathcal{F}_i(U)$, one has

$$\begin{tikzcd}
\mathcal{F}_i(U) \arrow[dd, "\phi_{ij}(U)"'] \arrow[dr] \arrow[drr, "\psi_i(U)"] & & \\
& \varinjlim_i \mathcal{F}_i(U) \arrow[r, dashed] & \mathcal{C}(U) \\
\mathcal{F}_j(U) \arrow[ur] \arrow[urr, "\psi_j(U)"'] & &
\end{tikzcd}$$

For convenience we denote $\psi(U)\colon \varinjlim_i \mathcal{F}(U) \to \mathcal{C}(U)$. Since all of $\phi_{ij}, \psi_i, \psi_j$ are morphisms between (pre)sheaves, it's clear that the collection of group homomorphisms $\{\psi(U)\}_{U\subseteq X}$ gives a morphism of presheaves $U \mapsto \varinjlim_i \mathcal{F}_i(U)$ and $\mathcal{C}$, and thus gives a morphism of sheaves $\varinjlim_i \mathcal{F}_i$ and $\mathcal{C}$.

Secondly, note that

$$\begin{tikzcd}
\mathcal{F}_i(U) \arrow[r] \arrow[d] \arrow[dr] & \varinjlim_i \mathcal{F}_i(U) \arrow[ddl, dashed] \\
\mathcal{F}_{i,P} \arrow[d] \arrow[dr] & \mathcal{F}_j(U) \arrow[u] \arrow[d] \\
\varinjlim_i \mathcal{F}_{i,P} & \mathcal{F}_{j,P} \arrow[l]
\end{tikzcd}$$

The morphism $\varinjlim_i \mathcal{F}_i(U) \to \varinjlim_i \mathcal{F}_{i,P}$ statisfies the universal property of taking stalk, and thus by the uniqueness one has

$$(\varinjlim_i \mathcal{F}_i)_P \cong \varinjlim_i \mathcal{F}_{i,P}.$$

Finally let's suppose X is a noetherian topological space and prove that $U \mapsto \varinjlim_i \mathcal{F}_i(U)$ is a sheaf. For every finite open covering $\{U_\alpha\}_{\alpha \in \mathscr{A}}$ of U. If $s, t \in \varinjlim_i \mathcal{F}_i(U)$ such that $s|_{U_\alpha} = t|_{U_\alpha}$ for all $\alpha \in \mathscr{A}$, then for each $\alpha \in \mathscr{A}$, there exists N_α such that for all $i > N_\alpha$, one has $s|_{U_\alpha} = t|_{U_\alpha}$ in $\mathcal{F}_i(U_\alpha)$. Since $\mathscr{A}$ is a finite index set, we may take $N > \max\{N_\alpha \mid \alpha \in \mathscr{A}\}$. Then for all $i > N$ and $\alpha \in \mathscr{A}$, one has $s|_{U_\alpha} = t|_{U_\alpha}$ in $\mathcal{F}_i(U_\alpha)$, and since $\mathcal{F}_i$ is a sheaf, one has $s = t$ in $\mathcal{F}_i(U)$ for all $i > N$, and thus $s = t$ in $\varinjlim_i \mathcal{F}_i(U)$. Similarly, one can check the other condition for $\varinjlim_i \mathcal{F}_i$ to be a sheaf by the same argument. □

Exercise 7.6.3. Let $(\mathcal{F}_i, \phi_{ij})_{i \in I}$ be an inverse system of sheaves of abelian groups on X. Prove that the presheaf $U \mapsto \varprojlim_i \mathcal{F}_i(U)$ is a sheaf and it is the inverse limit of $(\mathcal{F}_i, \phi_{ij})_{i \in I}$ in the category of sheaves.

Proof. By the same argument as above exercise one can show that $\varprojlim_i \mathcal{F}_i$ is the inverse limit in the category of sheaves, so here we only prove that $U \mapsto \varprojlim_i \mathcal{F}_i(U)$ is a sheaf.

Recall that a presheaf $\mathcal{F}$ is a sheaf if and only if for an open subset U and open covering $\{U_\alpha\}$ of U, the following sequence is exact

$$0 \to \mathcal{F}(U) \to \prod_\alpha \mathcal{F}(U_\alpha) \to \prod_{\alpha,\beta} \mathcal{F}(U_{\alpha\beta}).$$

Since the inverse limit is left exact, one has

$$0 \to \varprojlim_i \mathcal{F}_i(U) \to \prod_\alpha \varprojlim_i \mathcal{F}_i(U_\alpha) \to \prod_{\alpha,\beta} \varprojlim_i \mathcal{F}_i(U_{\alpha\beta}).$$

This completes the proof. □

7.7. **Homework-7.**

Exercise 7.7.1. Let $f\colon X \to Y$ be a morphism of schemes. Suppose that there exists an open affine covering $\{V_i = \operatorname{Spec} B_i\}_{i\in I}$ of Y such that each $f^{-1}(V_i)$ is covered by affine open subschemes $\{U_{ij} = \operatorname{Spec} A_{ij}\}_{j\in J_i}$ with A_{ij} being finitely generated B_i-algebras. Let $V = \operatorname{Spec} B$ be an affine open subscheme of Y and $U = \operatorname{Spec} A$ an affine open subscheme of X such that $f(U) \subseteq V$.

(1) Prove that $f^{-1}(V)$ can be covered by affine open subschemes $\{U_\lambda = \operatorname{Spec} A_\lambda\}_{\lambda\in\Lambda}$ such that A_λ are finitely generated B-algebras.
(2) Prove that there exist finitely many $a_1, \dots, a_n \in A$ such that $D(a_1), \dots, D(a_n)$ cover U, and each A_{a_i} is a finitely generated B-algebra.
(3) Prove that A is a finitely generated B-algebra.

Proof. For (1). Since $V \cap V_i$ is an open subset of $V_i = \operatorname{Spec} B_i$, then $V \cap V_i$ is a union of principal open subsets $D(f_{ik}) = \operatorname{Spec}(B_i)_{f_{ik}}$, and for each $f_{ik} \in B_i$, for convenience we still use f_{ik} to denote the image of f_{ik} under the morphism $B_i \to A_{ij}$. Then $(A_{ij})_{f_{ik}}$ is a finitely generated $(B_i)_{f_{ik}}$-algebra. After relabelling the index, in fact we have shown that V is covered by affine schemes $\operatorname{Spec} C_i$ such that each $f^{-1}(\operatorname{Spec} C_i)$ is a union of affine schemes $\operatorname{Spec} D_{ij}$, where D_{ij} is a finitely generated C_i-algebra.

For each point $\mathfrak{p} \in V$, suppose it lies in the affine scheme $\operatorname{Spec} C_i$. Then there exists a principal open subset $\operatorname{Spec} B_{f_\mathfrak{p}} \subseteq \operatorname{Spec} C_i$ which contains $\mathfrak{p}$. For convenience we still use $f_\mathfrak{p}$ to denote the image of $f_\mathfrak{p}$ under the morphism $B \to C_i \to D_{ij}$. Then each $(D_{ij})_{f_\mathfrak{p}}$ is a finitely generated $B_{f_\mathfrak{p}}$-algebra, and thus a finitely generated B-algebra. This completes the proof.

For (2). Since $f^{-1}(V)$ is covered by affine schemes $\operatorname{Spec} A_\lambda$, then $U = \operatorname{Spec} A$ is also covered by the intersection of $\operatorname{Spec} A \cap \operatorname{Spec} A_\lambda$. Moreover, since both A and A_λ are affine, then one can[2] pick a collection of open subset $U_{\lambda i}$ such that $U_{\lambda i}$ are simultaneously the principal open subsets of $\operatorname{Spec} A$ and $\operatorname{Spec} A_\lambda$. For convenience we write

$$U_{\lambda i} \cong \operatorname{Spec} A_{f_{\lambda i}} \cong \operatorname{Spec}(A_\lambda)_{g_{\lambda i}}.$$

Since A_λ are finitely generated B-algebra, so is $(A_\lambda)_{g_{\lambda i}}$, and thus each $A_{f_{\lambda i}}$ can be realized as a finitely generated B-algebra, which completes the proof.

For (3). Note that $\operatorname{Spec} A$ is covered by principal open subsets $D(a_1), \dots, D(a_n)$ if and only if $(a_1, \dots, a_n) = A$. Thus it reduces to the following lemma of commutative algebra.

Lemma 7.7.1. Let A be a B-algebra and $(a_1, \dots, a_n) = A$. If A_{a_i} is a finitely generated B-algebra for each i, then A is also a finitely generated B-algebra.

[2] See Proposition 5.3.1 in [Vak17]

Proof. For each $x \in A$, since A_{a_i} is finitely generated B-algebra, its image in A_{a_i} is equal to some

$$F^i(\frac{x_1^i}{a_i^{k_1^i}}, \ldots, \frac{x_{j_i}^i}{a_i^{k_{j_i}^i}}),$$

where $x_j^i/a_i^{k_j^i}$ are generators of A_{a_i} over B, and F^i is some polynomial with coefficients in B. After multiplying by a large power of a_i, there are n equations in A which are of the form

$$a_i^N x = \widetilde{F}^i(x_i^i, \ldots, x_{j_i}^i, a_i).$$

On the other hand, since $(a_1, \ldots, a_n) = 1$, there exists $m_1, \ldots, m_n \in A$ such that

$$m_1 a_1 + \cdots + m_n a_n = 1.$$

Exponentiate above equation to the nN-th power and multiply by x, one has

$$x = G(a_1, \ldots, a_n, m_1, \ldots, m_n, x_j^i),$$

where G is a polynomial with coefficients in B, since each monimial of m_i, there exists some a_j in the coefficients such that the power of a_j is $\geq N$, and thus $a_i^N x$ can be replaced by $\widetilde{F}^i$. This shows any $x \in A$ can be expressed as a polynomial of m_i, a_i, x_j^i with coefficients in B, and thue A is a finitely generated B-algebra. □

□

7.8. **Homework-8.**

Exercise 7.8.1. Let $(f,f^\sharp)\colon(Z,\mathcal{O}_Z)\to(X,\mathcal{O}_X)$ be a morphism of schemes.

(1) $(f,f^\sharp)\colon(Z,\mathcal{O}_Z)\to(X,\mathcal{O}_X)$ is an open immersion if and only if f induces a homeomorphism of Z with an open subset of X and $f^\sharp_P\colon\mathcal{O}_{X,f(P)}\to\mathcal{O}_{Z,P}$ is an isomorphism for every $P\in Z$.

(2) $(f,f^\sharp)\colon(Z,\mathcal{O}_Z)\to(X,\mathcal{O}_X)$ is an immersion if and only if f induces a homeomorphism of Z with a locally closed subset of X and $f^\sharp_P\colon\mathcal{O}_{X,f(P)}\to\mathcal{O}_{Z,P}$ is an epimorphism for every $P\in Z$.

(3) Immersions are monomorphisms in the category of schemes. Moreover, the composite of immersions is an immersion, so are open immersions and closed immersions.

Proof. For (1). Note that by definition one has $(f,f^\sharp)\colon(Z,\mathcal{O}_Z)\to(X,\mathcal{O}_X)$ is an open immersion if and only if it induces an isomorphism between $(Z,\mathcal{O}_Z)$ and an open subscheme of $(X,\mathcal{O}_X)$. Since f has already induced a homeomorphism of Z with an open subset of X, it suffices to show for every $P\in Z$, $f^\sharp_P\colon\mathcal{O}_{X,f(P)}\to\mathcal{O}_{Z,P}$ is an isomorphism if and only if $(f_*\mathcal{O}_Z)_{f(P)}\cong\mathcal{O}_{X,f(P)}$. In general it fails, but since f induces a homeomorphism, one has

$$(f_*\mathcal{O}_Z)_{f(P)}=\varinjlim_{P\in f^{-1}(V)}\mathcal{O}_Z(f^{-1}(V))=\varinjlim_{p\in U}\mathcal{O}_Z(U)=\mathcal{O}_{Z,P}$$

is an isomorphism. On the other hand, one has the following commutative diagram

$$\begin{tikzcd}
f^\sharp_P\colon\mathcal{O}_{X,f(P)} \arrow[rr] \arrow[dr] & & \mathcal{O}_{Z,P} \\
& (f_*\mathcal{O}_Z)_{f(P)} \arrow[ur] &
\end{tikzcd}$$

Thus $f^\sharp_P$ is an isomorphism if and only if $(f_*\mathcal{O}_Z)_{f(P)}\cong\mathcal{O}_{X,f(P)}$, and by the same argument one can show (2).

For (3). Since the composite of epimorphisms is an epimorphism, one has the composite of immersions is an immersion, so are open immersions and closed immersions.

Now let's show immersions are monomorphisms in the category of schemes. Suppose $\alpha,\beta\colon\widetilde{Z}\to Z$ are two morphisms between schemes such that $f\circ\alpha=f\circ\beta$, where f is an immersion. Firstly $\alpha=\beta$ as morphisms between topological spaces since f induces homeomorphism between underlying topological spaces. Moreover, on each stalk one has

$$\alpha^\sharp\circ f^\sharp=\beta^\sharp\circ f^\sharp,$$

and thus one has $\alpha^\sharp=\beta^\sharp$ on each stalk since $f^\sharp$ is an epimorphism. $\square$

Exercise 7.8.2. Let S be a graded ring and let $\mathfrak{a}$ be a homogeneous ideal of S. Prove that the canonical homomorphism $S\to S/\mathfrak{a}$ induces a closed immersion $\operatorname{Proj}S/\mathfrak{a}\to\operatorname{Proj}S$.

Proof. Firstly the canonical homomorphism $S \to S/\mathfrak{a}$ is a homomorphism of graded rings, and thus the pullback of homogeneous prime ideals are still homogeneous prime ideals. For any $\mathfrak{p}/\mathfrak{a} \in \operatorname{Proj} S/\mathfrak{a}$, the pullback of $\mathfrak{p}/\mathfrak{a}$ is $\mathfrak{p}$, which is a homogeneous prime ideal which contains $\mathfrak{a}$ and cannot contain S_+, otherwise $\mathfrak{p}/\mathfrak{a}$ contains $(S/\mathfrak{a})_+$, a contradiction. On the other hand, any homogeneous prime ideals $\mathfrak{p} \in V_+(\mathfrak{a})$ gives an element in $\operatorname{Proj} S/\mathfrak{a}$, and these two constructions are inverse to each other. Thus one has $\operatorname{Proj} S/\mathfrak{p} \cong V_+(\mathfrak{a})$ as sets. Moreover, it's also easy to show the canonical homomorphism $\operatorname{Proj} S/\mathfrak{a} \to V_+(\mathfrak{a})$ is a closed map and thus it's a homeomorphism with respect to Zariski topology.

Now it suffices to show for each $\mathfrak{p}/\mathfrak{a} \in \operatorname{Proj} S/\mathfrak{a}$, the canonical homomorphism $\mathcal{O}_{\operatorname{Proj} S,\mathfrak{p}} \to \mathcal{O}_{\operatorname{Proj} S/\mathfrak{a},\mathfrak{p}/\mathfrak{a}}$ is surjective. Note that

$$\begin{aligned}\mathcal{O}_{\operatorname{Proj} S,\mathfrak{p}} &\cong S_{(\mathfrak{p})}\\ \mathcal{O}_{\operatorname{Proj} S/\mathfrak{a},\mathfrak{p}/\mathfrak{a}} &\cong (S/\mathfrak{a})_{(\mathfrak{p}/\mathfrak{a})},\end{aligned}$$

and the canonical homomorphism is given by projection, which is surjective. This completes the proof. $\square$

7.9. **Homework-9.**

7.9.1. *Part I.*

Exercise 7.9.1. Finish step 3 of the proof of Proposition 1.3.20 as follows.

(1) Let

$$\begin{tikzcd} U_i \times_S Y \arrow[r, "q_i"] \arrow[d, "p_i"] & Y \arrow[d] \\ U_i \arrow[r] & S \end{tikzcd}$$

be the fibred product of U_i and Y. Prove that there exists one and only one isomorphism $\phi_{ij}\colon p_i^{-1}(U_i \cap U_j) \to p_j^{-1}(U_i \cap U_j)$ such that the following diagrams commute:

$$\begin{tikzcd} p_i^{-1}(U_i \cap U_j) \arrow[r, "\phi_{ij}"] \arrow[dr, "p_i"'] & p_j^{-1}(U_i \cap U_j) \arrow[d, "p_j"] \\ & U_i \cap U_j \end{tikzcd} \qquad \begin{tikzcd} p_i^{-1}(U_i \cap U_j) \arrow[r, "\phi_{ij}"] \arrow[dr, "q_i"'] & p_j^{-1}(U_i \cap U_j) \arrow[d, "q_j"] \\ & Y \end{tikzcd}$$

(2) Prove that $\phi_{ij} = \phi_{ji}^{-1}$ and $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ when restricted to $p_i^{-1}(U_i \cap U_j \cap U_k)$. So we can glue the schemes $U_i \times_S Y$ together to get a scheme Z.

(3) Suppose we have a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

and suppose X has an open covering $X = \bigcup_i U_i$ such that

$$\begin{tikzcd} p^{-1}(U_i) \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d] \\ U_i \arrow[r] & S \end{tikzcd}$$

are fibred product for all i. Prove that Z in the first diagram is the fibred product of X and Y over S.

Proof. For (1). By step 2 of the proof of Proposition 1.3.20, one has $p_i^{-1}(U_i \cap U_j)$ is the fibred product of $U_i \cap U_j$ and Y over S, so is $p_j^{-1}(U_i \cap U_j)$. Thus there exists one and only one isomorphism $\phi_{ij}\colon p_i^{-1}(U_i \cap U_j) \to p_j^{-1}(U_i \cap U_j)$ by the universal property of fibred product.

For (2). Note that $\phi_{ij} \circ \phi_{ji}$ is an isomorphism such that the following diagram commutes

$$\begin{tikzcd}
p_i^{-1}(U_i \cap U_j) \arrow[r, "\phi_{ij}\circ\phi_{ji}"] \arrow[dr, "p_i"'] & p_i^{-1}(U_i \cap U_j) \arrow[d, "p_i"] \\
& U_i \cap U_j,
\end{tikzcd}$$

so is the identity map. Then by the fact that the fibred product is unique up to a unique isomorphism, one has $\phi_{ij} \circ \phi_{ji} = \mathrm{id}$, that is, $\phi_{ij} = \phi_{ji}^{-1}$ as desired. The same argument shows that $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$.

For (3). In order to prove that Z satisfies the universal property of fibred product, we need to show for any commutative diagram

$$\begin{tikzcd}
W \arrow[r, "\beta"] \arrow[d, "\alpha"'] & Y \arrow[d] \\
X \arrow[r] & S,
\end{tikzcd}$$

there exists a unique morphism $W \to Z$ such that the following diagram commutes

$$\begin{tikzcd}
W \arrow[dr, dashed] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\
& Z \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d] \\
& X \arrow[r] & S.
\end{tikzcd}$$

Since $X = \bigcup_i U_i$, an observation is that the morphism $\alpha\colon W \to X$ is equivalent to a collection of morphisms $\{\alpha_i\colon X \to U_i\}$ which are compatible with each other. Thus for each i there exists a unique morphism $W \to p^{-1}(U_i)$ such that the following commutative diagram

$$\begin{tikzcd}
W \arrow[dr, dashed] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha_i"'] & & \\
& p^{-1}(U_i) \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d] \\
& U_i \arrow[r] & S
\end{tikzcd}$$

since $p^{-1}(U_i)$ is the fibred product of U_i and Y over S. By uniqueness the collection of morphisms $\{W \to \pi^{-1}(U_i)\}$ can be glued to a unique morphism from $W \to Z$, as desired. □

Exercise 7.9.2. Prove the isomorphism $X_i \times_S Y \cong X_i \times_{S_i} Y_i$ in step 6 in the proof of Proposition 1.3.20.

Proof. It suffices to note that given morphisms $f\colon Z \to X$ and $g\colon Z \to Y$ over S, the image of g must land inside Y_i. □

7.9.2. *Part II.*

Exercise 7.9.3. Let $\mathcal{C}$ be a category in which fibred product exists. Consider commutative diagrams

$$\begin{tikzcd} X'' \arrow[r, "f'"] \arrow[d] & X' \arrow[r, "g'"] \arrow[d] & X \arrow[d] \\ S'' \arrow[r, "f"] & S' \arrow[r, "g"] & S \end{tikzcd} \qquad \begin{tikzcd} X'' \arrow[r, "g'\circ f'"] \arrow[d] & X \arrow[d] \\ S'' \arrow[r, "g\circ f"] & S \end{tikzcd}$$

Suppose the second square is Cartesian. Prove that the first square is Cartesian if and only if the third one is Cartesian.

Proof. Suppose the first square is Cartesian and consider the following commutative diagram

$$\begin{tikzcd} W \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\ & X'' \arrow[r, "g'\circ f'"] \arrow[d] & X \arrow[d] \\ & S'' \arrow[r, "g\circ f"'] & S \end{tikzcd}$$

By composing α and f, one obtains the following commutative diagram

$$\begin{tikzcd} W \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "f\circ\alpha"'] \arrow[dr, dashed, "\phi"] & & \\ & X' \arrow[r, "g'"] \arrow[d] & X \arrow[d] \\ & S' \arrow[r, "g"'] & S \end{tikzcd}$$

where the morphism $\phi\colon W \to X'$ is induced by the assumption that the second square is Cartesian. On the other hand, consider the following commutative diagram

$$\begin{tikzcd} W \arrow[drr, bend left, "\phi"] \arrow[ddr, bend right, "\alpha"'] \arrow[dr, dashed, "\psi"] & & \\ & X'' \arrow[r, "f'"] \arrow[d] & X' \arrow[d] \\ & S'' \arrow[r, "f"'] & S', \end{tikzcd}$$

where the morphism $\psi\colon W \to X''$ is induced by the assumption that the first square is Cartesian, and that's exactly the morphism making the third square to be Cartesian.

Conversely, suppose the third square is Cartesian, and consider the following commutative diagram

$$\begin{tikzcd}
W \arrow[drr, bend left, "\beta"] \arrow[dr, bend right, "\alpha"'] & & \\
& X'' \arrow[r, "f'"] \arrow[d] & X' \arrow[d] \\
& S'' \arrow[r, "f"'] & S'.
\end{tikzcd}$$

Then by the assumption the third square is Cartesian, one has the following commutative diagram

$$\begin{tikzcd}
W \arrow[drr, bend left, "g'\circ\beta"] \arrow[dr, dashed, "\phi"] \arrow[ddr, bend right, "\alpha"'] & & \\
& X'' \arrow[r, "g'\circ f'"] \arrow[d] & X \arrow[d] \\
& S'' \arrow[r, "g\circ f"'] & S,
\end{tikzcd}$$

where the induced morphism $W \to X''$ is denoted by ϕ. In order to show the first square is Cartesian, it suffices to show the following diagram commutes

$$\begin{tikzcd}
W \arrow[drr, bend left, "\beta"] \arrow[dr, "\phi"] \arrow[ddr, bend right, "\alpha"'] & & \\
& X'' \arrow[r, "f'"] \arrow[d] & X' \arrow[d] \\
& S'' \arrow[r, "f"'] & S,
\end{tikzcd}$$

which follows from the second square is Cartesian. □

Exercise 7.9.4.

(1) Let $f\colon X \to S$ and $g\colon Y \to S$ be maps of sets. Prove that their fibred product is $(X \times_S Y, p, q)$, where

$$X \times_S Y = \{(x, y) \mid x \in X, y \in Y, f(x) = g(y)\},$$

and $p\colon X \times_S Y \to X, q\colon X \times_S Y \to Y$ are the projections $p(x, y) = x$ and $q(x, y) = y$ respectively.

(2) Use the description of the fibred product in (1) to prove Proposition 1.3.24 for the category of sets.

(3) Let $\mathcal{C}$ be a category and let

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

be a commutative diagram in $\mathcal{C}$. For every object $Z \in \operatorname{ob}\mathcal{C}$, it induces a commutative diagram

$$\begin{tikzcd} \operatorname{Hom}(Z, X') \arrow[r] \arrow[d] & \operatorname{Hom}(Z, X) \arrow[d] \\ \operatorname{Hom}(Z, S') \arrow[r] & \operatorname{Hom}(Z, S) \end{tikzcd}$$

in the category of sets. Prove that the first diagram is Cartesian in $\mathcal{C}$ if and only if the second diagram is Cartesian in the category of sets for every object $Z \in \operatorname{ob}\mathcal{C}$.

(4) Use (2) and (3) to prove Proposition 1.3.24 for every category $\mathcal{C}$ in which fibred product exists.

Proof. For (1). Suppose there exist morphisms (between sets) $\alpha\colon W \to X$ and $\beta\colon W \to Y$ such that the following diagram commutes

$$\begin{tikzcd} W \arrow[r, "\beta"] \arrow[d, "\alpha"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & S. \end{tikzcd}$$

Then one can construct

$$\begin{aligned} \varphi\colon W &\to X \times_S Y \\ w &\mapsto (\alpha(w), \beta(w)) \end{aligned}$$

such that the following diagram commutes

$$\begin{tikzcd} W \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\ & X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & S. \end{tikzcd}$$

Moreover, any morphism from $W \to X \times_S Y$ such that above diagram commutes must be of this form. This shows $X \times_S Y$ statisfies the universal property of fibred product.

For (2). Given the morphisms between sets as follows

$$\begin{tikzcd} X \arrow[dr, "f"] & & Y \arrow[dl, "g"'] \\ & T \arrow[d, "\phi"] & \\ & S. & \end{tikzcd}$$

By using the description of the fibred product in (1), the following diagram commutes

$$\begin{tikzcd} X \times_T Y \arrow[r] \arrow[d] & X \times_S Y \arrow[d] \\ T \arrow[r] & T \times_S T, \end{tikzcd}$$

since the morphisms in above diagram can be described as follows:

(a) $X \times_T Y \to X \times_S Y$ is given by $(x, y) \mapsto (x, y)$.
(b) $X \times_S Y \to T \times_S T$ is given by $(x, y) \mapsto (f(x), g(y))$.
(c) $T \mapsto T \times_S T$ is given by $t \mapsto (t, t)$.
(d) $X \times_T Y \to T$ is given by $(x, y) \mapsto f(x)$ or $(x, y) \mapsto g(y)$, since in this case $f(x) = g(y)$.

Moreover, given morphisms $\alpha\colon W \to T$ and $\beta\colon W \to X \times_S Y$ such that the following diagram commutes

$$\begin{tikzcd} W \arrow[dr, dotted] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\ & X \times_T Y \arrow[r] \arrow[d] & X \times_S Y \arrow[d] \\ & T \arrow[r] & T \times_S T, \end{tikzcd}$$

an observation is that the image of β lies in $X \times_T Y$, and thus there is a unique morphism from W to $X \times_T Y$ such that above diagram commutes, which is exactly β itself. This shows $X \times_T Y$ statisfies the universal property of fibred product.

For (3). Suppose

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ S' \arrow[r, "g"'] & S \end{tikzcd}$$

is Cartesian. For any $Z \in \operatorname{ob} \mathcal{C}$ and morphisms $\alpha\colon Z \to S'$ and $\beta\colon Z \to X$ such that $f \circ \beta = g \circ \alpha$, by the universal property there exists a unique mo $\phi\colon Z \to X'$. Conversely given morphisms $Z \to X'$, it's easy to construct morphisms $\alpha\colon Z \to X$ and $\beta\colon Z \to S'$ such that $f \circ \beta = g \circ \alpha$. This shows

$\mathrm{Hom}(Z,X') \cong \mathrm{Hom}(Z,S') \times_{\mathrm{Hom}(Z,S)} \mathrm{Hom}(Z,X)$ by the description of fibred product in (1). This shows

$$\begin{tikzcd} \mathrm{Hom}(Z,X') \arrow[r] \arrow[d] & \mathrm{Hom}(Z,X) \arrow[d] \\ \mathrm{Hom}(Z,S') \arrow[r] & \mathrm{Hom}(Z,S) \end{tikzcd}$$

is Cartesian, and by the same argument one can prove the converse statement.

For (4). By (3), it suffices to show that for each $Z \in \mathrm{ob}\,\mathcal{C}$, the following diagram is Cartesian

$$\begin{tikzcd} \mathrm{Hom}(Z,X\times_T Y) \arrow[r] \arrow[d] & \mathrm{Hom}(Z,X\times_S Y) \arrow[d] \\ \mathrm{Hom}(Z,T) \arrow[r] & \mathrm{Hom}(Z,T\times_S T). \end{tikzcd}$$

Note that by the proof of (3), one can see

$$\begin{aligned} \mathrm{Hom}(Z,T\times_S T) &= \mathrm{Hom}(Z,T)\times_{\mathrm{Hom}(Z,S)} \mathrm{Hom}(Z,T) \\ \mathrm{Hom}(Z,X\times_S Y) &= \mathrm{Hom}(Z,X)\times_{\mathrm{Hom}(Z,S)} \mathrm{Hom}(Z,Y) \\ \mathrm{Hom}(Z,X\times_T Y) &= \mathrm{Hom}(Z,X)\times_{\mathrm{Hom}(Z,T)} \mathrm{Hom}(Z,Y), \end{aligned}$$

and thus the desired result follows from (2). □

7.9.3. *Part III.*

Exercise 7.9.5. Let X, Y be S-schemes and $f, g\colon X \to Y$ be S-morphisms. Suppose $(f,g)\colon X \to Y\times_S Y$ is the morphism such that $p\circ(f,g) = f, q\circ(f,g) = g$ and K is the fibred product of X and Y over $Y\times_S Y$, which can be seen as follows

$$\begin{tikzcd} & K \arrow[r, "b"] \arrow[d, "\iota"'] & Y \arrow[d, "\Delta"] & \\ Z \arrow[r, "h"] \arrow[ur, dashed, "h'"] & X \arrow[r, "{(f,g)}"] & Y\times_S Y \arrow[r, "p"] \arrow[d, "q"] & Y \arrow[d] \\ & & Y \arrow[r] & S. \end{tikzcd}$$

(1) Prove that $\iota\colon K \to X$ is an immersion and $f\circ\iota = g\circ\iota$.
(2) Let $h\colon Z \to X$ be a morphism such that $f\circ h = g\circ h$. Prove that there is a unique morphism $h'\colon Z \to K$ such that $\iota\circ h' = h$.

Proof. For (1). If $\alpha,\beta\colon Z \to K$ are morphisms such that $\iota\circ\alpha = \iota\circ\beta$, then one also has $\Delta\circ b\circ\alpha = \Delta\circ b\circ\beta$. Then by the universal property of K one can see morphism from $Z \to K$ with such property must be unique, and thus $\alpha = \beta$. This shows ι is a monomorphism. For the half part, note that

$$p \circ (f, g) \circ \iota = f \circ \iota$$
$$q \circ (f, g) \circ \iota = g \circ \iota.$$

On the other hand, $(f, g) \circ \iota = \Delta \circ b$ and $p \circ \Delta = q \circ \Delta$.

For (2). Since $h \colon Z \to X$ statisfies $f \circ h = g \circ h$, one has $p \circ (f, g) \circ h = q \circ (f, g) \circ g$, which gives a morphism $Z \to Y$ satisfying desired commutative property, and thus by the universal property of K as a fibred product, there exists a morphism $h' \colon Z \to K$ such that $\iota \circ h' = h$. $\square$

7.10. **Homework-10.**

7.10.1. *Part I.*

Exercise 7.10.1. Let X be an S-scheme, $S' \to S$ a morphism, $X' = X \times_S S'$ the base change of $X, \Delta\colon X \to X \times_S X$ and $\Delta'\colon X' \to X' \times_{S'} X'$ the diagonal morphisms. Use the result in Exercise 7 on page 58 of the textbook to prove the following diagram is Cartesian:

$$\begin{array}{ccc} X' & \xrightarrow{\Delta'} & X' \times_{S'} X' \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta} & X \times_S X \end{array}$$

Proof. By (3) of Exercise 7 on page 58, it suffices to show for every scheme Z, the following diagram is Cartesian

$$\begin{array}{ccc} \mathrm{Hom}(Z, X') & \longrightarrow & \mathrm{Hom}(Z, X' \times_{S'} X) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(Z, X) & \longrightarrow & \mathrm{Hom}(Z, X \times_S X). \end{array}$$

On the other hand, one has

$$\mathrm{Hom}(Z, X \times_S X) = \mathrm{Hom}(Z, X) \times_{\mathrm{Hom}(Z,S)} \mathrm{Hom}(Z, X)$$
$$\mathrm{Hom}(Z, X' \times_{S'} X') = \mathrm{Hom}(Z, X') \times_{\mathrm{Hom}(Z,S')} \mathrm{Hom}(Z, X').$$

Thus it suffices to show the following diagram is Cartesian

$$\begin{array}{ccc} \mathrm{Hom}(Z, X') & \longrightarrow & \mathrm{Hom}(Z, X') \times_{\mathrm{Hom}(Z,S')} \mathrm{Hom}(Z, X') \\ \downarrow & & \downarrow \\ \mathrm{Hom}(Z, X) & \longrightarrow & \mathrm{Hom}(Z, X) \times_{\mathrm{Hom}(Z,S)} \mathrm{Hom}(Z, X), \end{array}$$

which is clear by the description of fibred product in the category of sets. □

Exercise 7.10.2. Let X and S be locally compact topological spaces, S is Hausdorff, and let $f\colon X \to S$ be a continuous map.

(1) Prove that a proper map is a closed map.
(2) Let S' be a locally compact topological space and let $g\colon S' \to S$ be a continuous map. For any proper map $f\colon X \to S$, prove the base change $f'\colon X \times_S S' \to S'$ of f is proper.

Proof. For (1). Let $V \subseteq X$ be a closed subset. It suffices to show $S \setminus f(V)$ is open. For $s \in S \setminus f(V)$, there exists an open neighborhood of U of s with compact closure since S is locally compact. Then $f^{-1}(U)$ is compact since f is proper. Let $E = V \cap f^{-1}(V)$. Then E is compact since it's a closed subset of a compact set, and hence $f(E)$. Again by S is Hausdorff, one has $f(E)$ is closed in S. Then $U \setminus f(E)$ is an open neighborhood of s which is disjoint from $f(V)$. This shows $S \setminus f(V)$ is open.

For (2). Note that the fibred product in the category of topological spaces can be described as $X \times_S S' = \{(x, y) \in X \times S' \mid f(x) = g(y)\}$ and $f' \colon X \times_S S' \to S'$ is given by $(x, y) \mapsto y$. Then for any compact subset $K \subseteq S'$, one has

$$(f')^{-1}(K) \subseteq f^{-1}g(K) \times K,$$

which is a compact subset since f is proper. □

7.10.2. *Part II.*

Exercise 7.10.3. The set

$$I = \{(i_0, \ldots, i_n) \mid i_0 + \cdots + i_n = d,\ i_0, \ldots, i_n \in \mathbb{Z}_{\geq 0}\}$$

has $\binom{d+n}{n}$ solutions. Prove that the homomorphism

$$\begin{aligned} \varphi \colon \mathbb{Z}[y_{i_0 \ldots i_n}]_{(i_0, \ldots, i_n) \in I} &\to \mathbb{Z}[x_0, \ldots, x_n] \\ y_{i_0 \ldots i_n} &\mapsto x_0^{i_0} \cdots x_n^{i_n} \end{aligned}$$

induces a morphism $\mathbb{P}^n_{\mathbb{Z}} \to \mathbb{P}^{\binom{d+n}{n}-1}_{\mathbb{Z}}$.

Proof. For convenience, we use $A = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_n$ to denote the graded ring $\mathbb{Z}[y_{i_0 \ldots i_n}]_{(i_0, \ldots, i_n) \in I}$. Note that for each degree n, one has $\varphi(A_n) \subseteq \mathbb{Z}[x_0, \ldots, x_n]_{nd}$. This shows ring homomorphism φ preserves the grade, and thus it induces a morphism between Proj. □

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Mathematics Department, Tsinghua University, Beijing, 100084, P.R. China,
Email address: liubw22@mails.tsinghua.edu.cn