ANALYTIC COMPLEX GEOMETRY

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Contents

0. Basic notations

1. M denotes a smooth real manifold, with tangent bundle TM and cotangent bundle T^*M.
2. ${}^s\mathcal{E}^p(M)$ denotes the space of C^s-global sections of $\bigwedge^p T^*M$, and $\mathcal{E}^p(M)$ denotes the space of smooth global sections of $\bigwedge^p T^*M$.
3. X denotes a smooth complex manifold, with tangent bundle TX and cotangent bundle T^*X.

1. Currents

In this section, M is assumed to be an oriented smooth real manifold with dimension m.

1.1. **Currents on smooth manifold.** Firstly we want to give a topology on the space of ${}^s\mathcal{E}^p(M)$ to make it to be a topological vector space. For $u \in {}^s\mathcal{E}^p(M)$, on coordinate open set $\Omega \subset M$ it can be written as

$$u = \sum_{|I|=p} u_I \mathrm{d}x^I$$

To each $L \Subset \Omega$ and every integer $s \in \mathbb{N}$, we associate a seminorm

$$P_{L,\Omega} = \sup_{x \in L} \max_{|\alpha| \leq s, |I|=p} |D^\alpha u_I(x)|$$

Since our manifolds are suppose to be Hausdorff, then M can be covered by countable coordinate set, that is $M = \bigcup_{k=1}^\infty \Omega_k$, and consider exhaustion for ech k

$$L_{k_1} \Subset L_{k_2} \Subset \cdots \Subset \Omega_k$$

then seminorms $\{P_{L_{k_m},\Omega_k}\}$ gives a topology on ${}^s\mathcal{E}^p(M)$. More explicitly, for $\{u_l\}_{l=1}^\infty \in {}^s\mathcal{E}^p(M)$, $u_l \to 0$ if for arbitrary Ω_k and $P_{L_{k_m}}$, one has $P_{L_{k_m},\Omega_k}(u_l) \to 0$ as $l \to \infty$.

Let $K \Subset M$, ${}^s\mathcal{D}^p(K)$ is the subspace of elements $u \in {}^s\mathcal{E}^p(M)$ with compact support in K, together with induced topology. The ${}^s\mathcal{D}^p(M)$ denotes the set of all elements of ${}^s\mathcal{E}^p(M)$ with compact support, that is

$${}^s\mathcal{D}^p(M) = \bigcup_{K \Subset M} {}^s\mathcal{D}^p(K)$$

A sequence $u_l \to 0$ in ${}^s\mathcal{D}^p(M)$ if there exists $K \Subset M$ such that $\operatorname{supp} u_l \subset K$ for all $l \geq 1$ and $u_l \to 0$ in ${}^s\mathcal{E}^p(M)$.

Remark 1.1.1. Similarly one can define $\mathcal{D}^p(K), \mathcal{D}^p(M)$, in particular, if $p = 0$ and $M = \mathbb{R}^n$, then $\mathcal{D}^0(\mathbb{R}^n)$ is exactly the space of test functions.

Definition 1.1.1 (current)**.** The space of current of dimension p or degree $m-p$, denoted by $\mathcal{D}'_p(M) = \mathcal{D}'^{m-p}(M)$, is the space of linear functionals on $\mathcal{D}^p(M)$ such that the restriction on any $\mathcal{D}^p(K)$ is continous, where $K \Subset M$.

Notation 1.1.1. For a current $T \in \mathcal{D}'_p(M)$, $\langle T, u \rangle$ denotes the pairing between a current T and test form $u \in \mathcal{D}^p(M)$.

Remark 1.1.2. If a current T extends continuously to ${}^s\mathcal{D}^p(M)$, then T is called of order s.

Definition 1.1.2. For a current $T \in \mathcal{D}'_p(M)$, the support of T, denoted by $\operatorname{supp}(T)$, is the smallest closed set A such that $T|_{\mathcal{D}^p(M \setminus A)} = 0$.

The following two basic examples explains the terminology used for dimension and degree.

Example 1.1.1. Let $Z \subseteq M$ be an oriented closed submanifold with dimension p. The current of integration $[Z]$ is given by

$$\langle [Z], u \rangle := \int_Z u$$

where $u \in \mathcal{D}^p(M)$. It's clear that $[Z]$ is a current with $\operatorname{supp}[Z] = Z$, and its dimension is exactly the dimension of Z as a manifold.

Example 1.1.2. Let f be a p-form with L^1_{loc} coefficients, the T_f given by

$$\langle T_f, u \rangle = \int_M f \wedge u$$

where $u \in \mathcal{D}^{m-p}(M)$, is a current of degree p.

1.2. **Exterior derivative and wedge product on currents.**

1.2.1. *Exterior derivative.* As we have seen in Example 1.1.2, currents generalize the ideal of forms, and in this viewpoint, many of the operations for forms can also be extended to currents.

Let $T \in \mathcal{D}'^p(M)$, the exterior derivative $\mathrm{d}T$ is given by

$$\langle \mathrm{d}T, u \rangle := (-1)^{p+1} \langle T, \mathrm{d}u \rangle$$

where $u \in \mathcal{D}^{m-p-1}(M)$. The continuity of the linear functional $\mathrm{d}T$ follows from the exterior derivative d is continous, thus $\mathrm{d}T$ is a current of degree $p+1$.

Remark 1.2.1. If $T \in \mathcal{E}^p(M)$,

$$\langle \mathrm{d}T, u \rangle = \int_M \mathrm{d}T \wedge u = \int_M \mathrm{d}(T \wedge u) + (-1)^{p+1} T \wedge \mathrm{d}u = (-1)^{p+1} \int_M T \wedge \mathrm{d}u$$

That's why we define exterior derivative like this.

Example 1.2.1. Consider current T_f given by p-form with L^1_{loc} coefficients, then

$$\begin{aligned}
\langle T_{\mathrm{d}f}, u \rangle &= \int_M \mathrm{d}f \wedge u \\
&= \int_M \mathrm{d}(f \wedge u) + (-1)^{p+1} f \wedge \mathrm{d}u \\
&= (-1)^{p+1} \int_M f \wedge \mathrm{d}u \\
&= \langle \mathrm{d}T_f, u \rangle
\end{aligned}$$

This shows $T_{\mathrm{d}f} = \mathrm{d}T_f$, and that's why exterior derivative is defined like this.

Example 1.2.2. Consider current $T = [Z]$ given by a oriented closed submanifold of M with dimension p, then

$$\begin{aligned}\langle \mathrm{d}T, u\rangle &= (-1)^{m-p+1}\langle T, \mathrm{d}u\rangle \\ &= (-1)^{m-p+1}\int_Z \mathrm{d}u \\ &= (-1)^{m-p+1}\int_{\partial Z} u\end{aligned}$$

that is $\mathrm{d}T = (-1)^{m-p+1}[\partial Z]$.

1.2.2. *Wedge product.* Let $T \in {\mathcal{D}'}^p(M)$, $g \in \mathcal{E}^q(M)$, the wedge product $T\wedge g$ is a current of degree $p + q$, given by

$$\langle T \wedge g, u\rangle := \langle T, g \wedge u\rangle$$

where $u \in \mathcal{D}^{m-p-q}(M)$.

Proposition 1.2.1. Let $T \in {\mathcal{D}'}^p(M)$, $g \in \mathcal{E}^q(M)$, then

$$\mathrm{d}(T \wedge g) = \mathrm{d}T \wedge g + (-1)^p T \wedge \mathrm{d}g$$

1.3. **Mass of currents.** Let $T \in \mathcal{D}'_p(M)$, where (M, g) is a Riemannian manifold, the norm of T is defined as

$$\|T\| = \sup_{\substack{\|u\|(x)\leq 1, x\in M \\ u\in\mathcal{D}^p(M)}} \langle T, u\rangle$$

Exercise 1.1. If $T = [Z]$, then $\|T\| = \operatorname{vol} Z$.

For open subset $V \subseteq M$, then

$$\|T\|_V := \sup_{\substack{\|u\|(x)\leq 1, x\in U \\ \operatorname{supp}\ u\subseteq V, u\in\mathcal{D}^p(M)}} \langle T, u\rangle$$

Theorem 1.3.1 (Banach-Alaogulu theorem)**.** Let $\{T_k\}_{k=1}^{\infty}$ be a sequence of $\mathcal{D}'_p(M)$, assume $\sup_{k\geq 1} \|T_k\|_V < \infty$ for every $V \Subset M$, then $\{T_k\}$ is weak-star compact in the following sense: There exists a subsequence $\{T_{k_l}\}$ and a current T such that

$$\langle T_{k_l}, u\rangle \to \langle T, u\rangle$$

for all $u \in \mathcal{D}^p(M)$.

1.4. **Currents on complex manifold.** Let X be a complex n-manifold. $\bigwedge^{p,q} T^*X$, the space of (p,q)-forms, and $\mathcal{E}^{p,q}(X)$ the space of $C^\infty(X, \bigwedge^{p,q} T^*X)$, with C^∞_{loc} topology.

$\mathcal{D}^{p,q}(X)$, the space of smooth (p,q)-forms with compact supports. Note that

$$\mathcal{E}^{p,q}(X) = \bigoplus_{p+q=k} \mathcal{E}^{p,q}(X)$$

so

$$\mathcal{D}^{p,q}(X) = \bigoplus_{p+q=k} \mathcal{D}^{p,q}(X)$$

Definition 1.4.1. The space of currents of bidimension (p,q) or bidegree $(n-p, n-q)$, denoted by

$$\mathcal{D}'_{p,q}(X) = \mathcal{D}'^{(n-p,n-q)}(X)$$

is the topological dual of $\mathcal{D}^{p,q}(X)$.

Let $\mathcal{E}^{p,q}$ be the locally free sheaf associated to $\bigwedge^{p,q} T^*X$, and it's a resolution of sheaf of holomorphic p-forms, that is

$$0 \to \Omega^p \to \mathcal{E}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1} \xrightarrow{\bar{\partial}} \dots$$

is an exact sequence of sheaves. That is to say, the Dolbeault cohomology computes the sheaf cohomology of Ω^p.

Similarly, we can also give a resolution via currents, that is

$$0 \to \Omega^p \to \mathcal{D}'^{p,0} \xrightarrow{\bar{\partial}} \mathcal{D}'^{p,1} \xrightarrow{\bar{\partial}} \dots$$

A non-trivial fact.

Remark 1.4.1. Twist a holomorphic vector bundle E, the same story goes.

2. Positive

2.1. **Positive** $(1,1)$ **current.**
Here we only consider the case of $(p,q) = (1,1)$. Let u be a smooth real $(1,1)$-form locally given by

$$u = \sqrt{-1}u_{ij}\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j$$

Then u is called positive if matrix $(u_{ij(x)})_{i\times j}$ is positive (semi-positive?)

Similarly, for a $(n-1,n-1)$-form v locally given by

$$v = v_{ij}\mathrm{d}\widehat{z^i \wedge \mathrm{d}\overline{z}^j}$$

where $\mathrm{d}\widehat{z^i \wedge \mathrm{d}\overline{z}^j}$ is a $(n-1,n-1)$-form such that

$$\mathrm{d}\widehat{z^i \wedge \mathrm{d}\overline{z}^j} \wedge \mathrm{d}z^i \wedge \mathrm{d}z^j = (\sqrt{-1})^n \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n \wedge \mathrm{d}\overline{z}^1 \wedge \cdots \wedge \mathrm{d}\overline{z}^n$$

and a $(n-1,n-1)$-form is called positive, if matrix $(v_{ij})_{i\times j}$ is positive.

Let $T \in \mathcal{D}'^{1,1}(X)$ be a real $(1,1)$-current, it's called positive, if

$$\langle T, v\rangle \geq 0$$

for any positive $(n-1,n-1)$-form $v \in \mathcal{D}^{n-1,n-1}(X)$.

2.2. **Pluri-subharmonic functions.**
Consider $u = \log|z|$, then

$$\frac{\sqrt{-1}}{\pi}\partial\overline{\partial}u = \delta_0$$

Definition 2.2.1. $u\colon \Omega \to [-\infty,\infty]$ is called pluri-subhamonic function, if

1. u is upper semi-continous;
2. For any complex line $L \subseteq \mathbb{C}^n$, $u|_{\Omega\cap L}$ is subharmonic.

Remark 2.2.1. subharmonic i.e. For all $a \in \Omega, \xi \in \mathbb{C}^n$ with $|\xi| \ll 1$, one has

$$u(a) \leq \int_0^{2\pi} u(a + e^{\sqrt{-1}\theta}\xi)\mathrm{d}\theta$$

Notation 2.2.1. The space of pluri-subhamonic functions on Ω is denoted by $\mathrm{Psh}(\Omega)$.

Proposition 2.2.1. Here are some basic properties of pluri-subhamonic functions

1. pluri-subhamonic function is subharmonic.
2. $u \in \mathrm{Psh}(\Omega)$, if Ω is connected, then $u \equiv -\infty$ or $u \in L^1_{\mathrm{loc}}(\Omega)$.
3. If $\{u_k\}$ is a sequence of pluri-subhamonic functions, u_k descends to u, then u is pluri-subhamonic.
4. Let $u \in \mathrm{Psh}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$ and $(\rho_\varepsilon)_{\epsilon>0}$ be a family of modifiers, then $u_\varepsilon := u * \rho_\varepsilon \in C^\infty(U_\varepsilon) \cap \mathrm{Psh}(U_\varepsilon)$, and u_ε descends to u as $\varepsilon \to 0$.
5. If $u \in C^2(\Omega)$, then $u \in \mathrm{Psh}(\Omega)$ if and only if $(\frac{\partial^2 u}{\partial z^i \partial \overline{z}^j})$ is semi-positive, that is $\sqrt{-1}\partial\overline{\partial}u \geq 0$.
6. (a) Let $u \in \mathrm{Psh}(\Omega) \cap L^1_{\mathrm{loc}}$, then $\sqrt{-1}\partial\overline{\partial}u$ is a positive $(1,1)$-current.
   (b) Given a distribution φ on Ω, then $\sqrt{-1}\partial\overline{\partial}\varphi \geq 0$ in the sense of current, then $\varphi = u$ for some $u \in \mathrm{Psh}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$.

Appendix A. Topological vector spaces

In this appendix we mainly follows [Rud74].

A.1. **Basic definitions and first properties.**

All vector spaces are assumed to be over $\mathbb{R}$ or $\mathbb{C}$.

Definition A.1.1 (balance)**.** Let X be a vector space, a set $B \subset X$ is said to be balanced if $\alpha B \subset B$ for all scalars α with $|\alpha| < 1$.

Definition A.1.2 (invariant metric)**.** A metric d on a vector space X is called invariant, if

$$d(x+z, y+z) = d(x, y)$$

for all $x, y, z \in X$.

Definition A.1.3. A topological vector space is a vector space X with topology τ such that

1. every point of X is closed set;
2. the vector space operations are continuous with respect to τ.

Remark A.1.1. In the vector space context, the term local base always means a local base at 0, that is a collection $\mathscr{B}$ of neighborhoods of 0 such that every neighborhoods of 0 contains a member of $\mathscr{B}$.

Definition A.1.4 (types of topological vector space)**.** Let X be a topological vector space with topology τ.

1. X is locally convex if there is a local base $\mathscr{B}$ whose members are convex.
2. X is locally bounded if 0 has a bounded neighborhood.
3. X is locally compact if 0 has a neighborhood whose closure is compact.
4. X is metrizable if τ is compatible with some metric d.
5. X is a F-space if its topology is induced by a complete invariant metric d.
6. X is a Fréchet space if X is a locally convex F-space.
7. X is normable if there is a norm on X such that the metric induced by the norm is compatible with τ.
8. X has Heine-Borel property if every closed and bounded subset of X is compact.

Remark A.1.2. Here is a list of some relations between these properties of a topological vector space X.

1. If X is locally bounded, then X has a countable local base.
2. X is metrizable if and only if X has a countable local base.
3. X is normable if and only if X is locally convex and locally bounded.
4. X has finite dimension if and only if X is locally compact.
5. If a locally bounded space X has the Heine-Borel property, then X has finite dimension.

A.2. **Seminorms and local convexity.**

Definition A.2.1 (seminorm)**.** A seminorm on a vector space X is a real-valued function p on X such that

1. $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$;
2. $p(\alpha x) = |\alpha| p(x)$ for all $x \in X$ and scalars α;
3. $p(x) \neq 0$ if $x \neq 0$.

Definition A.2.2 (separating)**.** A family $\mathscr{P}$ of seminorms on X is said to be separating if to each $x \neq 0$ corresponds at least one $p \in \mathscr{P}$ with $p(x) \neq 0$.

Seminorms are closely to local convexity in two ways: In every locally convex space there exists a separating family of continous seminorms. Conversely, if $\mathscr{P}$ is a separating family of seminorms on a vector space X, then $\mathscr{P}$ can be used to define a locally convex topology on X with the property that every $p \in \mathscr{P}$ is continous.

Theorem A.2.1. Suppose $\mathscr{P}$ is a separating family of seminorms on a vector space X, associate to each $p \in \mathscr{P}$ and to each positive integer n the set

$$V(p,n) = \{x \colon p(x) < \frac{1}{n}\}$$

Let $\mathscr{B}$ be the collection of all finite intersections of the sets $V(p,n)$, then $\mathscr{B}$ is a convex balanced local base for a topology τ on X, which turns X into a locally convex space such that

1. every $p \in \mathscr{P}$ is continous;
2. a set $E \subset X$ is bounded if and only if every $p \in \mathscr{P}$ is bounded on E.

Remark A.2.1. If $\mathscr{P} = \{p_i \mid i = 1, 2, 3 \dots\}$ is a countable separating family of seminorms on X, then $\mathscr{P}$ induces a topology τ with a countable local base, thus it's metrizable. However, in this case, a compatible translation invariant metric can be defined directly in terms of $\{p_i\}$, that is

$$d(x,y) = \sum_{i=1}^{\infty} \frac{2^{-i} p_i(x-y)}{1 + p_i(x-y)}$$

References

[Rud74] W. Rudin. *Functional Analysis*. International series in pure and applied mathematics. Tata McGraw-Hill, 1974.

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