RIEMANNIAN GEOMETRY

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Contents

0. Preface

0.1. **To readers.** This note is divided into several parts:

1. In the **First** part, we firstly introduce Levi-Civita connection on a (real) vector bundle E equipped with a metric, and in Riemannian geometry we mostly concern about tangent bundle. Holding a connection on E, one can construct connection on its dual bundle E^*, tensor product $E \otimes E^*$ and so on. When E is chosen to be tangent bundle, a section of tensor products is sometimes called a tensor, and tensor computation is a powerful tool of Riemannian geometry so we collect some basic properties and operations about tensor together in section 2.

   However, tensor computation may be quite tough in general. To give a neat local computation for tensor, we introduce geodesic in section 3 in order to introduce normal coordinates. By the way we introduce some other properties about geodesic such as global existence of geodesic and geodesics on Lie group.

   In section 4, we introduce curvature using two different views: curvature form and curvature tensor and prove Bianchi identities in these two views. We also introduce some other important curvatures such as sectional curvature, Ricci curvature and scalar curvature.

   Section 5 is about Hodge theory. Thanks to normal coordinates, we can only define Hodge star operator on an orthnormal basis, which will largely reduce complexity of computation. However, a shortage is that we don't know how does our Laplace-Beltrami operator rely on our metric. We will show a special case Laplace-Beltrami operator defined on smooth function, and introduce so-called conformal Laplacian.
2. In the **Second** part, we firstly solve the following question: "Given two points p, q, what' the length-minimizing curve connecting p, q?". To answer this question, we need to consider the arc-length functional, and
   (a) First variation formula implies geodesics are critical points of arc-length functional;
   (b) Second variation formula implies if a geodesic contains no interior conjugate points, then it's locally minimum of arc-length functional.

   Along the way we develop the tools of index form and Jacobi fields, which are also quite important in the third part and fourth part.
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0.2. **Some notations and conventions.**

0.2.1. *Conventions.*

1. We always use Einstein summation.
2. When we say M is a smooth manifold, we assume it's a real smooth manifold.

0.2.2. *Notations about smooth manifolds.*

1. For a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, we use $\frac{\partial f}{\partial x^i}$ to denote its partial derivative with respect to x^i, where x^i are coordinates of $\mathbb{R}^n$.
2. For a smooth manifold M, we use TM, T^*M to denote its tangent space and cotangent space respectively, and we also use Ω^k_M to denote the bundle of k-forms, that is $\bigwedge^k T^*M$.
3. We always use X, Y, Z to denote vector fields, ω to denote 1-forms and φ, ψ to denote k-forms.
4. For a smooth map $f : N \to M$ between smooth manifolds, we use $\mathrm{d}f$ or f_* to denote its differential.
5. Given a vector bundle $E \to M$ over a smooth manifold M, we use $C^\infty(M, E)$ to denote the set of all smooth sections of E.

0.2.3. *Notations about Riemannian manifolds.*

1. We use (M, g) to denote a Riemannian manifold, where M is a smooth manifold, and g is its Riemannian metric. If there is no ambiguity, we will omit g.
2. For a Riemannian metric g, we sometimes use $\langle \text{-}, \text{-} \rangle_g$ to denote it, or directly $\langle \text{-}, \text{-} \rangle$ if there is no ambiguity.

Part 1. Basic settings

1. Levi-Civita connection

Connection is a very basic conception in realm of geometry of vector bundles, and there are too many definitions of it which seem to be different. This part is divided into four parts:

1. In the first section, we will introduce one approach to connection in two different ways, the first one is often used in complex geometry and the second is given by Do carmo in [Car92];
2. In the second section, we will give another characterization of connection using parallel transport, and we will see all these approaches are same in fact.
3. In the third section, we will put more restrictions on our connection, such as compatibility with metric and torsion-free;
4. In the fourth section, we will construct many new connections from a given connection, which play an important role in our later discuss.

1.1. Connection.

1.1.1. *First definition.* When I first learn Riemannian geometry or complex geometry, I'm quite confused about why we need connection, and why we define it like this? In fact, given a vector bundle $\pi : E \to M$, connections on E are arised to take "derivative" of a section $s : M \to E$ in a given direction.

It's quite natural to ask such a question, since when we learn calculus, we already know how to take derivative of a smooth function $f : M \to \mathbb{R}^m$ to obtain a 1-form, that is a section of T^*M. In another point of view, any smooth function $f : M \to \mathbb{R}^m$ can be regarded as a section of trivial vector bundle $M \times \mathbb{R}^m$, as follows

$$x \mapsto (x, f(x))$$

and we can also regard its derivative $\mathrm{d}f$ as a section of $T^*M \otimes (M \times \mathbb{R}^m)$. So taking derivative can be seen as the following operator:

$$\nabla : C^\infty(M, M \times \mathbb{R}^m) \to C^\infty(M, T^*M \otimes (M \times \mathbb{R}^m))$$

In general, we can define a connection as follows:

Definition 1.1.1 (connection)**.** A connection ∇ on a vector bundle E on a smooth manifold M is a linear operator

$$\nabla : C^\infty(M, E) \to C^\infty(M, T^*M \otimes E)$$

satisfying Leibniz rule $\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s$, where $s \in C^\infty(M, E)$.

Remark 1.1.1 (local form). We can locally write a section s of E as $s^\alpha e_\alpha$, then Leibniz rule implies

$$\nabla(s^\alpha e_\alpha) = \mathrm{d}s^\alpha e_\alpha + s^\alpha \nabla e_\alpha$$

If we write ∇e_α explicitly as follows

$$\nabla e_\alpha = \omega_\alpha^\beta e_\beta$$

where ω_α^β are 1-forms. So connection locally looks like $\mathrm{d} + \omega$, where ω is a 1-form valued matrix.

Now let's see how does ω change with change of local basis. Suppose there is another local basis $\widetilde{e}_\alpha$, which is related by $\widetilde{e}_\alpha = g_\alpha^\beta e_\beta$, then

$$\begin{aligned}\nabla \widetilde{e}_\alpha &= \nabla(g_\alpha^\beta e_\beta)\\ &= g_\alpha^\beta \nabla e_\beta + \mathrm{d} g_\alpha^\beta e_\beta\\ &= g_\alpha^\beta \omega_\beta^\gamma e_\gamma + \mathrm{d} g_\alpha^\beta e_\beta\end{aligned}$$

So if we write in matrix notation, we have

$$\begin{aligned}\nabla \widetilde{e} &= g\omega e + \mathrm{d} g e\\ &= (g\omega g^{-1} + \mathrm{d} g g^{-1})\widetilde{e}\end{aligned}$$

which implies $\widetilde{\omega} = g\omega g^{-1} + \mathrm{d} g g^{-1}$.

1.1.2. *Second definition.* The following is the definition given by Do carmo in [Car92].

Definition 1.1.2 (connection)**.** A connection ∇ on a vector bundle E on a smooth manifold M is a mapping

$$\begin{aligned}\nabla : C^\infty(M, TM) \times C^\infty(M, E) &\to C^\infty(M, E)\\ (X, s) &\mapsto \nabla_X s\end{aligned}$$

satisfying the following properties:

1. $\nabla_{fX+gY} s = f\nabla_X s + g\nabla_Y s$
2. $\nabla_X(s + s') = \nabla_X s + \nabla_X s'$
3. $\nabla_X(fs) = f\nabla_X s + X(f)s$

where $X, Y \in C^\infty(M, TM)$, $f, g \in C^\infty(M)$ and $s, s' \in C^\infty(M, E)$.

Remark 1.1.2 (local form). For a given point $p \in M$ and choose a local basis $\{\frac{\partial}{\partial x^i}\}$ of TM and a local basis $\{e_\alpha\}$ of E, then we can write a vector field X and a section s of E as

$$X = X^i \frac{\partial}{\partial x^i}, \quad e = s^\alpha e_\alpha$$

Then

$$\begin{aligned}\nabla_X s &= \nabla_{X^i \frac{\partial}{\partial x^i}} s^\alpha e_\alpha\\ &= X^i \nabla_{\frac{\partial}{\partial x^i}} s^\alpha e_\alpha\\ &= X^i s^\alpha \nabla_{\frac{\partial}{\partial x^i}} e_\alpha + X^i \frac{\partial s^\alpha}{\partial x^i} e_\alpha\\ &= X^i s^\alpha \nabla_{\frac{\partial}{\partial x^i}} e_\alpha + X(s^\alpha) e_\alpha\end{aligned}$$

If we write $\nabla_{\frac{\partial}{\partial x^i}} s_\alpha = \Gamma_{i\alpha}^\beta e_\beta$, we can write

$$\nabla_X s = (X^i s^\alpha \Gamma_{i\alpha}^\beta + X(s^\beta))e_\beta$$

So as we can see, $\Gamma_{i\alpha}^\beta$, which is sometimes called Christoffel symbol, completely determines our connection ∇.

Remark 1.1.3 (The equivalence between two definitions). Locally a connection in definition 1.1.1 is a 1-form valued matrix ω, and write it as $\omega_\alpha^\beta = \Gamma_{j\alpha}^\beta \mathrm{d}x^j$. Then

$$\begin{aligned}\nabla e_\alpha &= \omega_\alpha^\beta e_\beta \\ &= \Gamma_{i\alpha}^\beta \mathrm{d}x^i e_\beta\end{aligned}$$

So if want to define $\nabla_{\frac{\partial}{\partial x^i}} e_\alpha$, ∇e_α need to "eat" a vector field, and luckily $\mathrm{d}x^j$ can eat one, so we can define it as follows

$$\begin{aligned}\nabla_{\frac{\partial}{\partial x^i}} e_\alpha :=& \Gamma_{j\alpha}^\beta \mathrm{d}x^j (\frac{\partial}{\partial x^i}) e_\beta \\ =& \Gamma_{i\alpha}^\beta e_\beta\end{aligned}$$

From this we can see these two definitions are same.

Remark 1.1.4 (connection and covariant derivative). Some authors may also use terminology "covariant derivative", here we make a clearify: Here we give two definitions of connection ∇ on a vector bundle E. Given a section s of E and a vector field X, we call $\nabla_X s$ the covariant derivative of s with respect to X. In fact, you can see connection and covariant derivative the same thing, just different terminology.

1.2. **Parallel transport.** In this section we fix a vector bundle E over M with connection ∇ , $\gamma : I \to M$ is a smooth curve. With this setting, we can define what is parallel transport along a smooth curve $\gamma(t)$.

Firstly, we can define a connection on pullback bundle $\gamma^* E$ over γ as follows

$$\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma^* s := \nabla_{\gamma_*(\frac{\mathrm{d}}{\mathrm{d}t})} s$$

where $s \in C^\infty(M, E)$.

Remark 1.2.1 (local form). Locally we have

$$\begin{aligned}\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma^* s &= \nabla_{\frac{\mathrm{d}\gamma^i}{\mathrm{d}t} \frac{\partial}{\partial x^i}} s^\alpha e_\alpha \\ &= \frac{\mathrm{d}\gamma^i}{\mathrm{d}t} \nabla_{\frac{\partial}{\partial x^i}} s^\alpha e_\alpha \\ &= \frac{\mathrm{d}\gamma^i}{\mathrm{d}t} (\frac{\partial s^\alpha}{\partial x^i} e_\alpha + s^\alpha \Gamma_{i\alpha}^\beta e_\beta)\end{aligned}$$

Definition 1.2.1 (parallel)**.** A section s of $\gamma^* E$ is called parallel along γ, if $\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} s = 0$.

From local form we can see $\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} s = 0$ is a system of ODEs locally, which can always be solved uniquely in a sufficiently short interval if we given a initial value, that's how we define parallel transport.

Definition 1.2.2 (parallel transport)**.** For $t_0, t \in I$, parallel transport $P^{\gamma}_{t_0,t}$ is an isomorphism between vector spaces[1] defined by

$$\begin{aligned} P^{\gamma}_{t_0,t} : E_{\gamma(t)} &\to E_{\gamma(t)} \\ s_0 &\mapsto s(t) \end{aligned}$$

where s is the unique parallel section along γ satisfying $s(t_0) = s_0$.

Remark 1.2.2 (parallel frame). A useful tool is parallel frame: Fix a basis $\{e_\alpha\}$ of $E_{\gamma(t_0)}$, we can use parallel transport to give a family of basis $\{e_\alpha(t)\}$ of $E_{\gamma(t)}$ along γ such that $e_\alpha(0) = e_\alpha$.

Proposition 1.2.1. For any section s of E along γ and $t_0, t \in I$, we have

$$\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} P^{\gamma}_{t,t_0} s(t) = P^{\gamma}_{t,t_0} \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} s(t)$$

Proof. Assume $\{e_\alpha(t)\}$ is a parallel frame along γ. With respect to this parallel frame we can write $s(t)$ as

$$s(t) = s^\alpha(t) e_\alpha(t)$$

Thus

$$\begin{aligned} \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} P^{\gamma}_{t,t_0} s(t) &= \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} (s^\alpha(t) e_\alpha(t_0)) \\ &= \frac{\mathrm{d}s^\alpha}{\mathrm{d}t}(t) e_\alpha(t_0) \\ P_{t,t_0} \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} s(t) &= P^{\gamma}_{t,t_0} (\frac{\mathrm{d}s^\alpha}{\mathrm{d}t}(t) e_\alpha(t)) \\ &= \frac{\mathrm{d}s^\alpha}{\mathrm{d}t}(t) e_\alpha(t_0) \end{aligned}$$

□

Remark 1.2.3. In fact, connection and parallel transport are the same things.

1.3. **Compatibility and torsion-free.**

1.3.1. *Compatibility with metric.* Now consider a vector bundle E with a metric g, which can be locally written as $g_{\alpha\beta} e^\alpha \otimes e^\beta$. So if there is a connection ∇ on E, so it's natural to ask it to be compatible with our metric.

Definition 1.3.1 (compatibility)**.** A connection ∇ on vector bundle E is compatible with metric g, if for any two section s, t of E, we have

$$\mathrm{d}g(s,t) = g(\nabla s, t) + g(s, \nabla t)$$

[1] Its inverse is P^{γ}_{t,t_0}.

Remark 1.3.1 (local form). Locally we can compute it as

$$\begin{aligned} \mathrm{d}g_{\alpha\beta} &= \mathrm{d}g(e_\alpha, e_\beta) \\ &= g(\nabla e_\alpha, e_\beta) + g(e_\alpha, \nabla e_\beta) \\ &= \omega_\alpha^\gamma g_{\gamma\beta} + g_{\alpha\gamma}\omega_\beta^\gamma \end{aligned}$$

So in matrix notation we have[2]

$$\mathrm{d}g = \omega g + g\omega^t$$

In particular we have

$$\frac{\partial}{\partial x^i} g_{\alpha\beta} = \Gamma_{i\alpha}^\gamma g_{\gamma\beta} + \Gamma_{i\beta}^\gamma g_{\alpha\gamma}$$

for all i, α, β.

Proposition 1.3.1. A connection ∇ is compatible with metric if and only if for arbitrary curve $\gamma: I \to M$ and two parallel sections s_1, s_2 along γ we have $g(s_1, s_2)$ is constant.

Proof. It's clear if ∇ is compatible with metric g, then and two sections s, t are parallel along γ, we have

$$\mathrm{d}g(s_1, s_2) = g(\nabla s_1, s_2) + g(s_1, \nabla s_2) = 0$$

which implies $g(s, t)$ is constant.

Conversely, let $\{e_\alpha(t)\}$ be a parallel orthnormal frame with respect to g along γ and write

$$s_1(t) = s_1^\alpha(t) e_\alpha, \quad s_2(t) = s_2^\alpha(t) e_\alpha(t)$$

Then we have

$$\begin{aligned} g(\nabla s_1, s_2) + g(s_1, \nabla s_2) &= \sum_\alpha \frac{\mathrm{d}s_1^\alpha}{\mathrm{d}t} s_2^\alpha + s_1^\alpha \frac{\mathrm{d}s_2^\alpha}{\mathrm{d}t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\sum_\alpha s_1^\alpha s_2^\alpha) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} g(s_1, s_2) \end{aligned}$$

□

1.3.2. *Torsion-free.* Now let's choose our vector bundle E to be tangent bundle of a Riemannian manifold (M, g).

Definition 1.3.2 (torsion-free)**.** A connection ∇ of TM is torsion-free if

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

where X, Y are vector fields.

[2]Here we need to pay more attention, although as a number $g_{\alpha\gamma}\omega_\beta^\gamma = \omega_\beta^\gamma g_{\alpha\gamma}$, we can not write this matrix notation as $\mathrm{d}g = \omega g + \omega g^t$, since $\omega_\beta^\gamma g_{\gamma\alpha}$ is (β, α)-entry of ωg^t, but $\mathrm{d}g_{\alpha\beta}$ and $g_{\alpha\gamma}\omega_\beta^\gamma$ are (α, β)-entries of $g\omega^t$.

Remark 1.3.2 (local form). If we choose $X = \frac{\partial}{\partial x^i}, Y = \frac{\partial}{\partial x^j}$, then we have

$$\begin{aligned}\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} &= (\Gamma^k_{ij} - \Gamma^k_{ji}) \frac{\partial}{\partial x^k} \\ &= 0\end{aligned}$$

which is equivalent to say Γ^k_{ij} is symmetric in i and j.

1.3.3. *Levi-Civita connection.* An interesting thing is that there is only one connection of TM on a Riemannian manifold (M, g) which is both compatible with Riemannian metric and torsion-free. It suffices to see such connection is completely determined, in other words, Γ^k_{ij} is completely determined by compatibility and torsion-free.

Note that compatibility implies

$$\begin{aligned}Xg(Y, Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Yg(Z, X) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ Zg(X, Y) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y)\end{aligned}$$

Adding first two equations, substract the third and use torsion-free condition, we will see

$$Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) = g([X, Z], Y) + g([Y, Z], X) + g([X, Y], Z) + 2g(Z, \nabla_Y X)$$

thus

$$g(Z, \nabla_Y X) = \frac{1}{2}(Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g([X, Z], Y) - g([Y, Z], X) - g([X, Y], Z))$$

which implies $\nabla_X Y$ is uniquely determined.

Remark 1.3.3 (local form). Firstly, compatibility implies

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma^l_{ki} g_{lj} + \Gamma^l_{kj} g_{il}$$

By permuting i, j, k we obtain the following two equations

$$\begin{aligned}\frac{\partial g_{jk}}{\partial x^i} &= \Gamma^l_{ij} g_{lk} + \Gamma^l_{ik} g_{jl} \\ \frac{\partial g_{ki}}{\partial x^j} &= \Gamma^l_{jk} g_{li} + \Gamma^l_{ji} g_{kl}\end{aligned}$$

By the symmetry of Γ^l_{ij} in i, j and symmetry of g_{ij}, we have

$$2\Gamma^l_{ij} g_{lk} = \frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}$$

If we use (g^{ij}) to denote the inverse matrix of (g_{ij}), then we have

$$\Gamma^l_{ij} = \frac{1}{2} g^{kl} (\frac{g_{kj}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k})$$

which implies Christoffel symbol is completely determined by Riemannian metric and its partial derivatives.

1.4. **Induced connection.** Given a vector bundle E, you can construct many new vector bundles by algebraic method, such as considering its dual bundle E^*, tensor product $E \otimes E$ and so on. Now let's see if we already have a connection ∇ defined on E, let's construct some new connections on new vector bundles.

1.4.1. *Induced connection on dual bundle.* Firstly let's consider how to induce a connection on dual bundle E^*. If s is a section of E, and use s^* to denote its dual section, its natural to ask

$$\mathrm{d}(s, s^*) = (\nabla s, s^*) + (s, \nabla s^*)$$

Here we still use ∇ to denote the induced connection on E^*. So if $\{e_\alpha\}$ is a local basis of E and $\{e^\alpha\}$ is the dual basis of E^*, then

$$\begin{aligned} 0 &= (\omega_\alpha^\gamma e_\gamma, e^\beta) + (e_\alpha, (\omega^*)^\beta_\gamma e^\gamma) \\ &= \omega_\alpha^\beta + (\omega^*)^\beta_\alpha \end{aligned}$$

which implies induced connection on E^* locally looks like $(-\omega_\alpha^\beta)$.[3]

Remark 1.4.1 (Another characterization for torsion-free)*.* If we consider connection ∇ defined on TM, locally given by Christoffel symbol Γ_{ij}^k, then induced connection on T^*M locally looks like

$$\nabla \mathrm{d}x^k = -\Gamma_{ij}^k \mathrm{d}x^i \otimes \mathrm{d}x^j$$

that is $\nabla \mathrm{d}x^k \in C^\infty(M, T^*M \otimes T^*M)$.

Given a section s of T^*M, we can obtain a 2-form $\mathrm{d}s \in C^\infty(M, \bigwedge^2 T^*M)$. Note that $\bigwedge^2 T^*M$ is just the skew-symmetrization of $T^*M \otimes T^*M$, so it's natural to require the skew-symmetrization of ∇s is $\mathrm{d}s$.

If we write this down in a local basis $\{\mathrm{d}x^i\}$ of T^*M, we have

$$\nabla \mathrm{d}x^k = -\Gamma_{ij}^k \mathrm{d}x^i \otimes \mathrm{d}x^j$$

But $\mathrm{d}^2 x^k = 0$, so condition for torsion-free is equivalent to skew-symmetrization of $\nabla \mathrm{d}x^k = 0$, that is $-\Gamma_{ij}^k \mathrm{d}x^i \wedge \mathrm{d}x^j = 0$, which is equivalent to say Γ_{ij}^k is symmetric in i, j.

Remark 1.4.2*.* If we use notations in [Car92], ∇ induces a map

$$\begin{aligned} \nabla : C^\infty(M, TM) \times C^\infty(M, T^*M) &\to C^\infty(M, T^*M) \\ (X, \omega) &\mapsto \nabla_X \omega \end{aligned}$$

By definition we have for any vector field Y,

$$\mathrm{d}\omega(Y) = \nabla\omega(Y) + \omega(\nabla Y)$$

which implies

$$\nabla_X \omega(Y) = X(\omega(Y)) - \omega(\nabla_X Y)$$

[3]However, there is one thing to be care about, the upper index is row index and lower index is column index, not the same as ω_α^β. Or in other words, if a connection on E locally looks like ω, then connection induced on E^* locally looks like $-\omega^t$.

1.4.2. *Induced connection on tensor product.* For any two vector bundles E, F over M, we use ∇ to denote connections on them in order to save symbols. We can define a connection ∇ on $E\otimes F$ as follows: Take s, f as sections of E and F, then

$$\nabla(s\otimes f)=\nabla s\otimes f+s\otimes\nabla f\in C^{\infty}(M, T^*M\otimes(E\otimes F))$$

In particular, there is an induced connection ∇ on $\operatorname{End}E$, since we have $\operatorname{End}E\cong E\otimes E^*$. In this case, we can write it more explicitly as follows: Locally we have a basis $\{e_\alpha\}$ of E and a basis $\{e^\beta\}$ of E^*. Thus

$$\nabla(e_\alpha\otimes e^\beta)=\omega_\alpha^\gamma e_\gamma\otimes e^\beta+e_\alpha\otimes(-\omega_\gamma^\beta e^\gamma)$$

So in general a section of $E\otimes E^*$ locally takes form $s=s_\beta^\alpha e_\alpha\otimes e^\beta$, then

$$\begin{aligned}\nabla(s_\beta^\alpha e_\alpha\otimes e^\beta)&=\mathrm{d}s_\beta^\alpha e_\alpha\otimes e^\beta+s_\beta^\alpha(\nabla e_\alpha\otimes e^\beta+e_\alpha\otimes\nabla e^\beta)\\&=\mathrm{d}s_\beta^\alpha e_\alpha\otimes e^\beta+s_\beta^\alpha\omega_\alpha^\gamma e_\gamma\otimes e^\beta-s_\beta^\alpha\omega_\gamma^\beta e_\alpha\otimes e^\gamma\\&=(\mathrm{d}s_\beta^\alpha+s_\beta^\alpha\omega_\alpha^\gamma-\omega_\gamma^\beta s_\beta^\alpha)e_\alpha\otimes e^\beta\end{aligned}$$

Thus in matrix notation we have

$$\nabla s=\mathrm{d}s+s\omega-\omega s$$

However, there is another way to induce a connection on $E\otimes E^*$ as follows: For any section s of $E\otimes E^*$, we have a function $s(e^\alpha, e_\beta)$, so it's natural to ask

$$\mathrm{d}s(e^\alpha, e_\beta)=\nabla s(e^\alpha, e_\beta)+s(\nabla e^\alpha, e_\beta)+s(e^\alpha,\nabla e_\beta)$$

Locally if we write $s=s_\beta^\alpha e_\alpha\otimes e^\beta$, then

$$\begin{aligned}\mathrm{d}(s_\beta^\alpha)&=(\nabla s)_\beta^\alpha+s(-\omega_\gamma^\alpha e^\gamma, e_\beta)+s(e^\alpha,\omega_\beta^\gamma e_\gamma)\\&=(\nabla s)_\beta^\alpha-s_\beta^\gamma\omega_\gamma^\alpha+\omega_\beta^\gamma s_\gamma^\alpha\end{aligned}$$

which implies these two ways to induce are same!

Remark 1.4.3. If we use notations in [Car92], a connection ∇ on TM induces a connection (still denoted by ∇) on $T^*M\otimes TM$ as follows

$$\begin{aligned}\nabla: C^{\infty}(M, TM)\times C^{\infty}(M, T^*M\otimes TM)&\to C^{\infty}(M, T^*M\otimes TM)\\(X, T)&\mapsto\nabla_X T\end{aligned}$$

where $\nabla_X T$ is defined as

$$\nabla_X T(\frac{\partial}{\partial x^i},\mathrm{d}x^j)=XT(\frac{\partial}{\partial x^i},\mathrm{d}x^j)-T(\nabla_X\frac{\partial}{\partial x^i},\mathrm{d}x^j)-T(\frac{\partial}{\partial x^i},\nabla_X\mathrm{d}x^j)$$

2. Tensor

2.1. Induced connection on tensor.

Definition 2.1.1 (tensor)**.** A section of $\bigotimes^s TM\otimes\bigotimes^r T^*M$ is called a (s,r)-tensor.

Example 2.1.1. A smooth function f is a $(0,0)$-tensor.

Example 2.1.2. A vector field X is a $(1,0)$-tensor.

Example 2.1.3. A 1-form ω is a $(0,1)$-tensor.

Example 2.1.4. The Riemannian metric g is a $(0,2)$-tensor.

Definition 2.1.2 (connection on tensor)**.** For a (s,r)-tensor T, ∇T is a $(s,r+1)$-tensor, which is defined by

$$\begin{aligned}\nabla T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}}) :=&\frac{\partial}{\partial x^i}T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}})\\ &-\sum_{l=1}^{s}T(\mathrm{d}x^{j_1},\dots,\nabla_{\frac{\partial}{\partial x^i}}\mathrm{d}x^{j_l},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}})\\ &-\sum_{m=1}^{r}T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^{i_1}},\dots,\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^{i_m}},\dots,\frac{\partial}{\partial x^{i_r}})\end{aligned}$$

Definition 2.1.3 (covariant derivative of tensor)**.** For a (s,r)-tensor T, the covariant derivative of T with respect to vector field X, which is a (s,r)-tensor, is defined as

$$\nabla_X T:=\nabla T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},X,\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}})$$

Remark 2.1.1 (local form)*.* If we write a (s,r)-tensor T locally as

$$T^{j_1\dots j_s}_{i_1\dots i_r}\frac{\partial}{\partial x^{j_1}}\otimes\dots\otimes\frac{\partial}{\partial x^{j_s}}\otimes\mathrm{d}x^{i_1}\otimes\dots\otimes\mathrm{d}x^{i_r}$$

and $(s,r+1)$-tensor ∇T locally as

$$\nabla_i T^{j_1\dots j_s}_{i_1\dots i_r}\frac{\partial}{\partial x^{j_1}}\otimes\dots\otimes\frac{\partial}{\partial x^{j_s}}\otimes\mathrm{d}x^{i}\otimes\mathrm{d}x^{i_1}\otimes\dots\otimes\mathrm{d}x^{i_r}$$

Then by definition we have

$$\nabla_i T^{j_1\dots j_s}_{i_1\dots i_r}=\frac{\partial T^{j_1\dots j_s}_{i_1\dots i_r}}{\partial x^i}+\sum_{l=1}^{s}\Gamma^{j_l}_{iq}T^{j_1\dots j_{l-1}qj_{l+1}\dots j_s}_{i_1\dots i_r}-\sum_{m=1}^{r}\Gamma^{q}_{ii_m}T^{j_1\dots j_s}_{i_1\dots i_{m-1}qi_{m+1}\dots i_r}$$

Example 2.1.5. Consider $(0,0)$-tensor f, that is a smooth function. Then ∇f is a $(0,1)$-tensor, given by

$$\nabla f=\nabla_i f\mathrm{d}x^i$$

by our definition $\nabla_i f=\frac{\partial f}{\partial x^i}$, it coincides with our usual notations.

Inductively, we can define $\nabla^2 T$ to be $\nabla(\nabla T)$, which is a $(s, r+2)$-tensor, and locally write it as

$$\nabla^2 T = \nabla_k \nabla_i T^{j_1 \dots j_s}_{i_1 \dots i_r} \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_s}} \otimes \mathrm{d}x^k \otimes \mathrm{d}x^i \otimes \mathrm{d}x^{i_1} \otimes \dots \otimes \mathrm{d}x^{i_r}$$

Now there is a natural question: Note that $\nabla_k \nabla_i T$ is a (s, r)-tensor, and if we regard $\nabla_i T$ as a (s, r)-tensor and take covariant derivative of it with respect to $\frac{\partial}{\partial x^k}$, does we obtain the same thing?

Example 2.1.6. For $(0, 0)$-tensor f, by definition we have $\nabla^2 f$ is $\nabla(\nabla_i f \mathrm{d}x^i)$, which is called the Hessian of f, denoted by $\operatorname{Hess} f$. More explicitly

$$\begin{aligned}
\operatorname{Hess} f &= \nabla(\nabla_i f \mathrm{d}x^i) \\
&= \frac{\partial \nabla_i f}{\partial x^k} \mathrm{d}x^k \otimes \mathrm{d}x^i - \nabla_i f \Gamma^i_{kj} \mathrm{d}x^k \otimes \mathrm{d}x^j \\
&= (\frac{\partial^2 f}{\partial x^k \partial x^i} - \Gamma^j_{ki} \frac{\partial f}{\partial x^j}) \mathrm{d}x^k \otimes \mathrm{d}x^i
\end{aligned}$$

that is $\nabla_k \nabla_i f = \frac{\partial^2 f}{\partial x^k \partial x^i} - \Gamma^j_{ki} \frac{\partial f}{\partial x^j}$.

However, if we regard $\nabla_i f$ as a $(0, 0)$-tensor, that is a smooth function, and take covariant derivative with respect to $\frac{\partial}{\partial x^k}$, we will obtain $\frac{\partial^2 f}{\partial x^k \partial x^i}$, and it's clear that it doesn't equal $\nabla_k \nabla_i f$, unless Christoffel symbol Γ^j_{ki} vanishes.

Proposition 2.1.1.

$$\begin{aligned}
\nabla_k \nabla_i T^{j_1 \dots j_s}_{i_1 \dots i_r} = &\nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^i}} T(\mathrm{d}x^{j_1}, \dots, \mathrm{d}x^{j_s}, \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_r}}) \\
&- \nabla_{\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}} T(\mathrm{d}x^{j_1}, \dots, \mathrm{d}x^{j_s}, \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_r}})
\end{aligned}$$

Proof. By definition, we have

$$\begin{aligned}
\nabla_k \nabla_i T^{j_1 \dots j_s}_{i_1 \dots i_r} = &\nabla^2 T(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_r}}, \mathrm{d}x^{j_1}, \dots, \mathrm{d}x^{j_s}) \\
= &\nabla_{\frac{\partial}{\partial x^k}} \nabla T(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_r}}, \mathrm{d}x^{j_1}, \dots, \mathrm{d}x^{j_s}) \\
= &\frac{\partial}{\partial x^k} \nabla T(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_r}}, \mathrm{d}x^{j_1}, \dots, \mathrm{d}x^{j_s}) \\
&- \nabla T(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_r}}, \mathrm{d}x^{j_1}, \dots, \mathrm{d}x^{j_s}) \\
&- \sum_{l=1}^{r} \nabla T(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{i_1}}, \dots, \nabla_{\frac{\partial}{\partial x^k}} X_{i_l}, \dots, \frac{\partial}{\partial x^{i_r}}, \mathrm{d}x^{j_1}, \dots, \mathrm{d}x^{j_s}) \\
&- \sum_{m=1}^{s} \nabla T(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_r}}, \mathrm{d}x^{j_1}, \dots, \nabla_{\frac{\partial}{\partial x^k}} \mathrm{d}x^{j_m}, \dots, \mathrm{d}x^{j_s})
\end{aligned}$$

That is,

$$\begin{aligned}
\nabla_k\nabla_i T^{j_1\dots j_s}_{i_1\dots i_r} =& \frac{\partial}{\partial x^k}\nabla_{\frac{\partial}{\partial x^i}} T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}})\\
&-\sum_{l=1}^{r}\nabla_{\frac{\partial}{\partial x^i}} T(\frac{\partial}{\partial x^{i_1}},\dots,\nabla_{\frac{\partial}{\partial x^k}} X_{i_l},\dots,\frac{\partial}{\partial x^{i_r}},\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s})\\
&-\sum_{m=1}^{s}\nabla_{\frac{\partial}{\partial x^i}} T(\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}},\mathrm{d}x^{j_1},\dots,\nabla_{\frac{\partial}{\partial x^k}}\mathrm{d}x^{j_m},\dots,\mathrm{d}x^{j_s})\\
&-\nabla_{\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^i}} T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}})\\
=&\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^i}} T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}})\\
&-\nabla_{\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^i}} T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}})
\end{aligned}$$

$\square$

Example 2.1.7. Consider $(0,0)$-tensor f in the this proposition. Note that

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^i}} f &= \nabla_{\frac{\partial}{\partial x^k}}(\nabla_i f)\\
&= \frac{\partial \nabla_i f}{\partial x^k}\\
\nabla_{\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^i}} f &= \nabla_{\Gamma^j_{ki}\frac{\partial}{\partial x^j}} f\\
&= \Gamma^j_{ki}\nabla_{\frac{\partial}{\partial x^j}} f\\
&= \Gamma^j_{ki}\nabla_j f
\end{aligned}$$

Thus

$$\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^i}} f - \nabla_{\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^i}} f = \frac{\partial \nabla_i f}{\partial x^k} - \Gamma^j_{ki}\nabla_j f$$

coincides with $\nabla_k\nabla_i f$.

Remark 2.1.2 (Another characterization of compatibility)*.* Note that we can regard our Riemannian metric g as a $(0,2)$-tensor. Recall our definition for compatibility is for any two vector fields X, Y we have

$$\mathrm{d}g(X,Y) = g(\nabla X, Y) + g(X, \nabla Y)$$

Or more explicit for vector field Z, we have

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

However, by definition of ∇g we have

$$\nabla_Z g(X,Y) = Zg(X,Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)$$

which shows that our compatibility implies $\nabla g = 0$.

2.2. **Type change of tensor.** In general, for a (s,r)-tensor, we can change its type into any type of $(s-k, r+k)$ for all k such that $s-k \geq 0, r+k \geq 0$, since TM is canonically isomorphic to T^*M, which is called music isomorphism. More explicitly, for any section $\frac{\partial}{\partial x^i}$ of TM, it gives an section of T^*M by

$$X \mapsto g(\frac{\partial}{\partial x^i}, X)$$

which can be written as

$$\begin{aligned} g(\frac{\partial}{\partial x^i}, X) &= g(\frac{\partial}{\partial x^i}, \mathrm{d}x^j(X)\frac{\partial}{\partial x^j}) \\ &= \mathrm{d}x^j(X) g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \\ &= g_{ij}\mathrm{d}x^j(X) \end{aligned}$$

So $\frac{\partial}{\partial x^i}$ can be regarded as a section $g_{ij}\mathrm{d}x^j$ of T^*M; Similarly we can regard $\mathrm{d}x^j$ as a section of $TM = T^{**}M$ by

$$\omega \mapsto g(\mathrm{d}x^j, \omega)$$

which can be written as

$$\begin{aligned} g(\mathrm{d}x^j, \omega) &= g(\mathrm{d}x^j, \omega(\frac{\partial}{\partial x^i})\mathrm{d}x^i) \\ &= \omega(\frac{\partial}{\partial x^i}) g^{ij} \end{aligned}$$

Thus $\mathrm{d}x^j$ can be regarded as $g^{ij}\frac{\partial}{\partial x^i}$, a section of TM.

In a summary, we have the following so-called music isomorphism

$$\begin{aligned} \flat : TM &\to T^*M & \sharp : T^*M &\to TM \\ \frac{\partial}{\partial x^i} &\mapsto g_{ij}\mathrm{d}x^j & \mathrm{d}x^j &\mapsto g^{ij}\frac{\partial}{\partial x^i} \end{aligned}$$

Example 2.2.1 (dual vector field)**.** For a smooth function f, ∇f is a $(0,1)$-tensor, locally written as

$$\nabla f = \frac{\partial f}{\partial x^i}\mathrm{d}x^i$$

Then we can change its type into $(1,0)$, that is

$$\nabla f = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}$$

More generally, for a 1-form ω, locally looks like $\omega_i \mathrm{d}x^i$, then we can change it into a $(1,0)$-tensor, called its dual vector field as follows

$$X_\omega = g^{ij}\omega_i\frac{\partial}{\partial x^j}$$

Example 2.2.2 (Induced metric on T^*M)**.** Recall that a Riemannian metric g is a $(0,2)$-tensor, locally written as

$$g = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$$

Then we can change its type into $(2,0)$, that is

$$g_{ij}g^{ik}g^{jl}\frac{\partial}{\partial x^k}\otimes\frac{\partial}{\partial x^i}=\delta^k_j g^{jl}\frac{\partial}{\partial x^k}\otimes\frac{\partial}{\partial x^i}=g^{ij}\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j}$$

that is a metric on T^*M.

2.3. **Tensor product of tensor.** Given a (s,r)-tensor T and a (s',r')-tensor S, we can obtain a $(s+s',r+r')$-tensor by putting them together directly, that is $T\otimes S$. More explicitly,

$$\begin{aligned}&T\otimes S(\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}},\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^{i'_1}},\dots,\frac{\partial}{\partial x^{i'_r}},\mathrm{d}x^{j'_1},\dots,\mathrm{d}x^{j'_s})\\ &:=T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}})S(\frac{\partial}{\partial x^{i'_1}},\dots,\frac{\partial}{\partial x^{i'_r}},\mathrm{d}x^{j'_1},\dots,\mathrm{d}x^{j'_s})\end{aligned}$$

Furthermore, we can define a induced connection ∇ as

$$\nabla(T\otimes S)=\nabla T\otimes S+T\otimes\nabla S$$

Remark 2.3.1 (Induced metric)*.* The case we're concern about is tensor product of metric tensor, that is $g\otimes g$. More explicitly, if we regard g as a $(2,0)$-tensor, then $g\otimes g$ is a $(4,0)$-tensor. More explicitly, locally take $S=S_{ij}\mathrm{d}x^i\otimes\mathrm{d}x^j, T=T_{kl}\mathrm{d}x^k\otimes\mathrm{d}x^l$, then

$$\begin{aligned}g\otimes g(S,T)&=S_{ij}T_{kl}g\otimes g(\mathrm{d}x^i\otimes\mathrm{d}x^j,\mathrm{d}x^k\otimes\mathrm{d}x^l)\\ &=S_{ij}T_{kl}g^{ik}g^{jl}\end{aligned}$$

Such metric on $T^*M\otimes T^*M$ is called induced metric, we still use g to denote it in convenience. In general we also have induced metric on $\bigotimes^s T^*M$, which will be used later in Hodge theory.

If our connection ∇ is compatible with Riemannian metric, that is $\nabla g=0$, it's easy to see induced connection is also compatible with induced metric, that is

$$Xg(T,S)=g(\nabla_X T,S)+g(T,\nabla_X S)$$

2.4. **Trace of tensor.** Let's see a simple example: For a (s,r)-tensor T such that $s+r=2$, we can define its trace, since for a $(1,1)$-tensor, we can just realize it as a matrix, thus we can take its trace. For example, if $T=T^i_j\frac{\partial}{\partial x^i}\otimes\mathrm{d}x^j$, then trace of T, denoted by $\operatorname{tr}_g T$, is defined as T^i_i.

If T is not in type $(1,1)$, then we change it into type $(1,1)$ and then take trace:

1. If $T=T_{ij}\mathrm{d}x^i\otimes\mathrm{d}x^j$, then $T=g^{ik}T_{ij}\frac{\partial}{\partial x^k}\otimes\mathrm{d}x^j$, thus $\operatorname{tr}_g T=g^{ij}T_{ij}$.
2. If $T=T^{ij}\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j}$, then $T=g_{kj}T^{ij}\frac{\partial}{\partial x^i}\otimes\mathrm{d}x^k$, thus $\operatorname{tr}_g T=g_{ij}T^{ij}$.

In general, if a tensor of type (r,s) with $r+s=2n$, we can change its type into (n,n) and take trace n times to obtain a number. Later we will see we obtain Ricci curvature by taking trace of curvature, and we obtain scalar curvature by taking trace of Ricci curvature.

Remark 2.4.1 ((Scalar) Laplacian). For a smooth function $f : M \to \mathbb{R}$, we have $\nabla^2 f$ as a $(0,2)$-form, locally looks like

$$\nabla_i \nabla_j f \mathrm{d}x^i \otimes \mathrm{d}x^j$$

Then we have its trace as

$$\operatorname{tr}_g \nabla^2 f = g^{ij} \nabla_i \nabla_j f$$

That's called (Scalar Laplacian) of f, denoted by $\Delta_g f$. Later we will learn more properties of it.

Remark 2.4.2. If g is induced metric on $(0,2)$-tensor, then for any $(0,2)$-tensor T, we have

$$\begin{aligned} g(g, T) &= g(g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j, T_{kl} \mathrm{d}x^k \otimes \mathrm{d}x^l) \\ &= g_{ij} T_{kl} g^{ik} g^{jl} \\ &= \delta_j^k g^{jl} T_{kl} \\ &= g^{kl} g_{kl} \\ &= \operatorname{tr}_g T \end{aligned}$$

Proposition 2.4.1 (magic formula)**.** For a $(0,2)$-tensor T, we have

$$X(\operatorname{tr}_g T) = g(g, \nabla_X T)$$

Proof. From above remark we can see $\operatorname{tr}_g T = g(g, T)$, then ∇ is compatible with metric completes the proof. □

Remark 2.4.3 (local form). Locally we have

$$\nabla_i (g^{jk} T_{jk}) = g^{jk} (\nabla_i T_{jk})$$

that is, g^{kl} can "pass through" taking derivative, which is called "magic formula".

3. Geodesic I: Normal coordinate

3.1. Geodesic.

Definition 3.1.1 (geodesic)**.** A smooth curve $\gamma:(-\varepsilon,\varepsilon)\to M$ is called a geodesic, if

$$\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\gamma_*(\frac{\mathrm{d}}{\mathrm{d}t})=0$$

Remark 3.1.1 (local form). Locally we have

$$\begin{aligned}\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\gamma_*(\frac{\mathrm{d}}{\mathrm{d}t}):&=\nabla_{\gamma_*(\frac{\mathrm{d}}{\mathrm{d}t})}\gamma_*(\frac{\mathrm{d}}{\mathrm{d}t})\\&=\nabla_{\frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\frac{\partial}{\partial x^i}}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\partial}{\partial x^j}\\&=(\frac{\mathrm{d}^2\gamma^j}{\mathrm{d}t}\frac{\partial}{\partial x^j}+\frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\Gamma_{ij}^k\frac{\partial}{\partial x^k})\\&=(\frac{\mathrm{d}^2\gamma^k}{\mathrm{d}t}+\frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\Gamma_{ij}^k)\frac{\partial}{\partial x^k}\end{aligned}$$

Thus condition for geodesic is a system of ODEs locally. So if we fix $\gamma(0),\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}(0)$, then the existence and uniqueness of geodesic follow from standard result of ODEs.

Theorem 3.1.1. Let (M,g) be a Riemannian manifold. For any $p\in M, v\in T_pM$, there exists $\varepsilon>0$ and a geodesic $\gamma:(-\varepsilon,\varepsilon)\to M$ such that

$$\gamma(0)=p,\gamma'(0)=v$$

Moreover, any two such geodesics agree on their common domain.

Remark 3.1.2. Since any geodesic γ with $\gamma(0)=p,\gamma'(0)=v$ will agree on their common domain, thus by gluing these geodesics together we can obtain a unique geodesic $\gamma:I\to M$ which can not be extended to a geodesic defined on a larger interval. This unique geodesic is denoted by $\gamma_v(t)$.

Lemma 3.1.1. For each $p\in M, v\in T_pM, c,t\in\mathbb{R}$,

$$\gamma_{cv}(t)=\gamma_v(ct)$$

whenever either side is defined.

Definition 3.1.2 (exponential map)**.** Let (M,g) be a Riemannian manifold. For any $p\in M$ we define $V_p\subseteq T_pM$ by

$$V_p:=\{v\in T_pM\mid\gamma_v(1)\text{ is defined}\}$$

The exponential map at p is the map

$$\begin{aligned}\exp_p:V_p&\to M\\v&\mapsto\gamma_v(1)\end{aligned}$$

Remark 3.1.3. Although V_p may not be the whole T_pM, it always at least contains a small neighborhood of $0 \in T_pM$ from Lemma 3.1.1. In fact, later we will see Hopf-Rinow theorem implies if M is complete as a metric space, then $V_p = T_pM$ for any $p \in M$.

Theorem 3.1.2. The exponential map $\exp_p$ maps a neighborhood $0 \in T_pM$ diffeomorphically onto a neighborhood of $p \in M$.

Proof. Note that

$$(\mathrm{d}\exp_p)_0 : T_0(T_pM) \to T_pM$$

Since T_pM is a vector space, we can identify it with T_0T_pM. Thus $(\mathrm{d}\exp_p)_0$ then becomes a map from T_pM onto itself. To see what we need, it suffices to check $(\mathrm{d}\exp_p)_0$ is identity map. For all $v \in T_pM$,

$$\begin{aligned}(\mathrm{d}\exp_p)_0(v) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \exp_p(0+tv) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \gamma_{tv}(1) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \gamma_v(t) \\ &= \gamma_v'(0) \\ &= v\end{aligned}$$

□

Remark 3.1.4 (normal coordinates). Fix a basis $\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p$ of T_pM which is orthnormal with respect to Riemannian metric g, we have the following linear isomorphism

$$\begin{aligned}\Phi : T_pM &\to \mathbb{R}^n \\ v^i \frac{\partial}{\partial x^i}\bigg|_p &\mapsto (v^1, \dots, v^n)\end{aligned}$$

Then Theorem 3.1.2 implies there exists a neighborhood U of p which is mapped by $\Phi \circ \exp_p^{-1}$ diffeomorphically onto a neighborhood of $0 \in \mathbb{R}^n$. Thus $(r := \Phi \circ \exp_p^{-1}, U)$ gives a local coordinates of M with center p, which is called normal coordinates.

Given a coordinate (ϕ, U), we give a more explicit formula for function f on manifold, that is $f(x) := f(\phi^{-1}(x))$. In particular, if we consider normal coordinates, we have the following characterization for Riemannian metric and Christoffel symbols.

Theorem 3.1.3. In normal coordinates we have

$$\begin{aligned}g_{ij}(0) &= \delta_{ij} \\ \Gamma_{ij}^k(0) &= 0\end{aligned}$$

Proof. Note that

$$\begin{aligned} g_{ij}(0) &= \langle \mathrm{d}(\exp_p \circ \Phi^{-1})_0 e_i, \mathrm{d}(\exp_p \circ \Phi^{-1})_0 e_j \rangle_p \\ &= \langle (\mathrm{d}\exp_p)_0 \left.\frac{\partial}{\partial x^i}\right|_p, (\mathrm{d}\exp_p)_0 \left.\frac{\partial}{\partial x^j}\right|_p \rangle_p \\ &= \langle \left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p \rangle_p \\ &= \delta_{ij} \end{aligned}$$

where $e_i = (0, \dots, \underbrace{1}_{i-\text{th}}, \dots, 0) \in \mathbb{R}^n$.

For Christoffel symbol: For arbitrary $v = (v^1, \dots, v^n) \in \mathbb{R}^n$, consider geodesic $\gamma(t) = \exp_p(t\Phi^{-1}(v))$ with $\gamma(0) = p$ and $\gamma'(t) = \Phi^{-1}(v)$. In normal coordinates γ looks like $\gamma(t) = (tv^1, \dots, tv^n)$, thus geodesic equation simplifies to

$$\Gamma^k_{ij}(tv)v^iv^j = 0$$

Evaluating this expression at $t = 0$ shows $\Gamma^k_{ij}(0)v^iv^j = 0$ for arbitrary index k and every v. Now take $v = \frac{1}{2}(e_i + e_j)$ to conclude $\Gamma^k_{ij}(0) = 0$ for all i, j, k. □

Corollary 3.1.1. In Riemannian normal coordinates we have for Taylor expression of $g_{ij} : T_pM \to \mathbb{R}$ around zero as

$$g_{ij}(x) = \delta_{ij} + O(|x|^2)$$

Proof. Note that

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma^l_{ki}(0)g_{lj}(0) + \Gamma^l_{kj}(0)g_{il}(0) = 0$$

□

3.2. **Global existence of geodesic.**

3.2.1. *Geodesic on compact Riemannian manifold.*

Theorem 3.2.1. Let M be a compact Riemannian manifold, $p, q \in M$. Then there exists a geodesic in every homotopy class of curves from p to q, and this geodesic may be chosen as a shortest curve in its homotopy class. In particular, every homotopy class of closed curves in M contains a curve which is shortest and geodesic.

3.2.2. *Geodesic on complete Riemannian manifold.* In this section we want to address the question whether result of Theorem 3.2.1 continue to hold for a more general class of Riemannian manifold, since in Euclidean space, they do hold for non-compact cases, but they do not hold for any proper open subset of Euclidean space. It turns out that completeness will be the right condition.

Definition 3.2.1 (geodesically complete)**.** A Riemannian manifold (M, g) is geodesically complete if for all $p \in T_pM$, the exponential map is defined on T_pM.

Theorem 3.2.2 (Hopf-Rinow)**.** Let M be a Riemannian manifold, the following statements are equivalent:

1. M is complete as a metric space;
2. The closed and bounded subsets of M is compact;
3. M is geodesically complete;
4. Any two points $p, q \in M$ can be joined by a geodesic of shortest length.

3.3. **Geodesics on Lie group.** In this section we give a quick review of Lie groups, such as left-invariant vector fields and integral curves. Then we consider the invariant metrics on Lie groups G and we show that geodesics are exactly integral curves (or one parameter subgroup) of G, and that's why we define exponential map of Lie group by integral curves.

3.3.1. *A quick review of Lie group.*

Definition 3.3.1 (Lie group)**.** A Lie group G is a smooth manifold which is also endowed with a group structure such that the multiplication map and the inverse map are smooth.

Since the multiplication map is smooth, then for any $g \in G$, there are two smooth maps L_g, R_g, defined by

$$\begin{aligned} L_g(h) &= gh \\ R_g(h) &= hg \end{aligned}$$

Furthermore, they're also diffeomorphisms with inverse $L_{g^{-1}}, R_{g^{-1}}$, since inverse maps are also smooth.

Definition 3.3.2 (invariant vector field)**.** A vector field X on a Lie group G is called left-invariant, if

$$(L_g)_* X = X$$

for arbitrary $g \in G$.

Remark 3.3.1*.* It's clear there is the following isomorphism

$$\begin{aligned} \{\text{Left-invaraint vector fields}\} &\to \mathfrak{g} := T_eG \\ X &\mapsto X_e \end{aligned}$$

where X_e is its value in T_eG. Furthermore, since Lie bracket of two left-invariant vector fields is still left-invariant, thus there is a natural Lie bracket on $\mathfrak{g}$.

Definition 3.3.3 (Lie algebra)**.** The tangent space T_eG of a Lie group G equipped with Lie bracket is called Lie algebra of G, denoted by $\mathfrak{g}$.

Definition 3.3.4 (adjoint representation)**.** The adjoint representation is defined as follows

$$\begin{aligned} \mathrm{Ad}: G &\to \mathrm{GL}(\mathfrak{g}) \\ g &\mapsto (R_{g^{-1}} \circ L_g)_* \end{aligned}$$

Definition 3.3.5 (integral curve)**.** Let X be a vector field of G and $g \in G$, then an integral curve of X through the point p is a smooth curve $\gamma : I \subseteq \mathbb{R} \to G$ such that

$$\begin{aligned} \gamma(0) &= g \\ \gamma'(t) &= X(\gamma(t)) \end{aligned}$$

Definition 3.3.6 (complete vector field)**.** A vector field X is called complete, if its integral curve is defined for all $t \in \mathbb{R}$.

Proposition 3.3.1. Every left-invariant vector field on a Lie group G is complete.

Proof. Let X be a left-invariant vector field, γ the unique integral curve for X such that $\gamma(0) = e$, defined on $(-\varepsilon, \varepsilon)$. Then $\gamma_g := L_g\gamma$ is an integral curve for X such that $\gamma_g(0) = g$. Indeed,

$$\begin{aligned} \gamma_g'(t) &= \mathrm{d}(L_g)_{\gamma(t)}(\gamma'(t)) \\ &= \mathrm{d}(L_g)_{\gamma(t)}(X(\gamma(t))) \\ &= X(L_g\gamma(t)) \\ &= X(\gamma_g(t)) \end{aligned}$$

In particular, for $t_0 \in (-\varepsilon, \varepsilon)$, the curve $t \mapsto \gamma(t_0)\gamma(t)$ is an integral curve for X starting at $\gamma(t_0)$. By uniqueness, this curve coincides with $\gamma(t_0 + t)$ for all $t \in (-\varepsilon, \varepsilon) \cap (-\varepsilon - t_0, \varepsilon - t_0)$. Define

$$\widetilde{\gamma}(t) = \begin{cases} \gamma(t), & t \in (-\varepsilon, \varepsilon) \\ \gamma(t_0)\gamma(t), & t \in (-\varepsilon - t_0, \varepsilon - t_0) \end{cases}$$

Repeat above operations to get our desired extension. □

Remark 3.3.2. From this proof we can see integral curve of left-invariant vector fields through identity e is just a Lie group homomorphism $\gamma : \mathbb{R} \to G$, such homomorphism is called a one parameter subgroup.

3.3.2. *Riemannian geometry of Lie group.*

Definition 3.3.7 (left-invariant metric)**.** A Riemannian metric h on a Lie group G is called left-invariant if

$$L_g^* h = h$$

for arbitrary $g \in G$.

Remark 3.3.3. Similarly we can define a right invariant metric, and a Riemannian metric which is both left-invariant and right invariant is called bi-invariant metric.

Proposition 3.3.2. There is a bijective correspondence between left-invariant metrics on a Lie group G, and inner products on the Lie algebra $\mathfrak{g}$ of G.

Proof. Given an inner product $\langle -,- \rangle_e$ on Lie algebra $\mathfrak{g}$, then we have an inner product on G defined as follows

$$\langle X_g, Y_g \rangle := \langle (L_{g^{-1}})_* X_g, (L_{g^{-1}})_* Y_g \rangle_e$$

where X, Y are two vector fields on G. It's left-invariant, since

$$\begin{aligned}\langle (L_h)_* X_g, (L_h)_* Y_g \rangle &= \langle (L_{(hg)^{-1}})_* (L_h)_* X_g, (L_{(hg)^{-1}})_* (L_h)_* Y_g \rangle_e \\ &= \langle (L_{g^{-1}})_* X_g, (L_{g^{-1}})_* Y_g \rangle_e\end{aligned}$$

Conversely, if we have a left-invariant inner product $\langle -,- \rangle$ on G, then it's clear we have an inner product on $\mathfrak{g}$, by just considering its value at identity. Furthermore, these two constructions are inverse to each other, this completes the proof. $\square$

Proposition 3.3.3. There is a bijective correspondence between bi-invariant metrics on a Lie group G, and Ad-invariant inner products on the Lie algebra $\mathfrak{g}$ of G.

Proof. Given a Ad-invariant inner product $\langle -,- \rangle_e$ on the Lie algebra $\mathfrak{g}$, by Proposition 3.3.2, there is a left-invariant metric $\langle -,- \rangle$ on G, it suffices to check it's also right-invariant:

$$\begin{aligned}\langle (R_h)_* X_g, (R_h)_* Y_g \rangle &= \langle (L_{(hg)^{-1}})_* (R_h)_* X_g, (L_{(hg)^{-1}})_* (R_h)_* Y_g \rangle_e \\ &= \langle \mathrm{Ad}(h^{-1})(L_{g^{-1}})_* X_g, \mathrm{Ad}(h^{-1})(L_{g^{-1}})_* Y_g \rangle_e \\ &= \langle (L_{g^{-1}})_* X_g, (L_{g^{-1}})_* Y_g \rangle_e \\ &= \langle X_g, Y_g \rangle\end{aligned}$$

Conversely, if we start with a bi-invariant metric, then it's restriction to the Lie algebra is a Ad-invariant, since $\mathrm{Ad}(g)$ is exactly the differential of $L_g \circ R_{g^{-1}}$. $\square$

Remark 3.3.4. In particular, if G is a compact connected Lie group, then it admits a bi-invariant metric, since its Killing form is a Ad-invariant inner product on $\mathfrak{g}$.

Lemma 3.3.1. If h is a left-invariant metric on a Lie group G, ∇ is the Levi-Civita connection, then for all left-invariant vector fields X, Y, Z, we have

$$h(X, \nabla_Y Y) = h(Y, [X, Y])$$

Proof. Recall that

$$h(X, \nabla_Y Z) = \frac{1}{2}(Yh(Z,X) + Zh(X,Y) - Xh(Y,Z) - h([Y,X],Z) - h([Z,X],Y) - h([Y,Z],X))$$

But $Yh(Z,X) = Zh(X,Y) = Xh(Y,Z) = 0$ since h is left-invariant and X, Y, Z is left-invariant, that is

$$h(X, \nabla_Y Z) = \frac{1}{2}\{h(Z,[X,Y]) + h(Y,[X,Z]) + h(X,[Z,Y])\}$$

Now take $Y = Z$ to conclude. □

Proposition 3.3.4. If h is a bi-invariant metric on a Lie group G, then for all left-invariant vector fields X, Y, Z, we have

$$h([X, Y], Z) = h(X, [Y, Z])$$

Proof. Let y_t be the flow of Y, then

$$[X, Y] = \lim_{t\to 0} \frac{1}{t}((y_t)_*(X) - X)$$

On the other hand, since Y is left-invariant, that is $L_g \circ y_t = y_t \circ L_g$, giving

$$y_t(g) = y_t(L_g(e)) = L_g y_t(e) = g y_t(e) = R_{y_t(e)}(g)$$

Thus $(y_t)_* = (R_{y_t(e)})_*$ and

$$[X, Y] = \lim_{t\to 0} \frac{1}{t}((R_{y_t(e)})_*(X) - X)$$

Note that h is bi-invariant, thus

$$\begin{aligned} h(X, Z) &= h((R_{y_t(e)})_*(L_{y_t^{-1}(e)})_* X, (R_{y_t(e)})_*(L_{y_t^{-1}(e)})_* Z) \\ &= h((R_{y_t(e)})_* X, (R_{y_t(e)})_* Z) \end{aligned}$$

Differentiating the expression above with respect to t and setting $t = 0$ we conclude

$$0 = h([X, Y], Z) + h(X, [Z, Y])$$

This completes the proof. □

Theorem 3.3.1. For every left-invariant vector field X on G, then $\nabla_X X = 0$.

Proof. From Lemma 3.3.1, we have

$$h(Y, \nabla_X X) = h(X, [Y, X])$$

where h is a bi-invariant metric. From Proposition 3.3.4, we have

$$h(X, [Y, X]) = h([Y, X], X) = -h(X, [Y, X])$$

that is $h(Y, \nabla_X X) = 0$ for arbitrary vector field Y, which implies $\nabla_X X = 0$. □

Corollary 3.3.1. If X, Y are left-invariant vector field, then $\nabla_X Y = \frac{1}{2}[X, Y]$.

Proof. Note that

$$\begin{aligned} 0 &= \nabla_{X+Y}(X + Y) \\ &= \nabla_X Y + \nabla_Y X + \nabla_X X + \nabla_Y Y \\ &= \nabla_X Y + \nabla_Y X \\ &= 2\nabla_X Y - [X, Y] \end{aligned}$$

Division by two finally yields

$$\nabla_X Y = \frac{1}{2}[X, Y]$$

□

Theorem 3.3.2. The geodesics on G are precisely the integral curves of left-invariant vector fields.

Proof. Let $X \in \mathfrak{g}$ be a left-invariant vector field, and $\gamma : \mathbb{R} \to G$ its integral curve. Then

$$\begin{aligned}\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\gamma_*(\frac{\mathrm{d}}{\mathrm{d}t}) &= \nabla_{\gamma_*(\frac{\mathrm{d}}{\mathrm{d}t})}\gamma_*(\frac{\mathrm{d}}{\mathrm{d}t}) \\ &= \nabla_X X \\ &= 0\end{aligned}$$

which implies integral curves of left-invariant vector fields are geodesics.

Furthermore, since geodesics are unique, we have geodesics are precisely integral curves of left-invariant vector fields. □

Corollary 3.3.2. The exponential map for the Lie group coincides with the exponential map of the Levi-Civita connection.

3.4. **Geodesic convex neighborhood.**

Recall that for any $p \in M$, there exists a normal neighborhood U, that is $(U, \exp_p^{-1})$ is a diffeomorphism. Or in other words, any two points $p_1, p_2 \in U$, there exists a geodesic connecting these two points. However, this geodesic may not lie in U.

Definition 3.4.1 (strongly convex)**.** A subset S of M is strongly convex if for any two points $p_1, p_2 \in \overline{S}$, there exists a geodesic connecting p_1, p_2 whose interior is contained in S.

Proposition 3.4.1 (convex neighborhood)**.** For any $p \in M$, there exists a strongly convex normal neighborhood.

Proof. See Proposition 4.2 of page 76 in [Car92]. □

Remark 3.4.1. Convex neighborhood is a technique tool which will be used in topology. Note that convex set is contractible and the intersection of convex sets is still convex, thus from Proposition 3.4.1, we know that there exists a open covering of M such that

1. U_α is contractible;
2. For any finite intersection $\bigcap_{i=1}^r U_{\alpha_i}$, it's still contractible.

Such covering is called a "good cover" in [RB82], and it's widely used in Mayer-Vietoris argument.

4. Curvature

4.1. **Curvature form.**

Let (M,g) be a Riemannian manifold, $\nabla : C^\infty(M,E) \to C^\infty(M, T^*M\otimes E)$ a connection of vector bundles E on M. Now we're going to extend connection to something called exterior derivative defined on sections of vector bundle valued k-forms as follows

$$\begin{aligned} \mathrm{d}^\nabla : C^\infty(M, \Omega_M^k \otimes E) &\to C^\infty(M, \Omega_M^{k+1}\otimes E) \\ \omega\otimes e &\mapsto \mathrm{d}\omega\otimes e + (-1)^k \omega\wedge\nabla e \end{aligned}$$

Note that d^∇ on $C^\infty(M,E)$ is exactly ∇. If we use Ω to denote $\mathrm{d}^\nabla\circ\mathrm{d}^\nabla$, let's see Ω locally:

$$\begin{aligned} \Omega(s^\alpha e_\alpha) &= \mathrm{d}^\nabla(\mathrm{d}s^\alpha e_\alpha + s^\alpha\omega_\alpha^\beta e_\beta) \\ &= -\mathrm{d}s^\alpha\wedge\omega_\alpha^\beta e_\beta + \mathrm{d}(s^\alpha\omega_\alpha^\beta)e_\beta - s^\alpha\omega_\alpha^\beta\wedge\omega_\beta^\gamma e_\gamma \\ &= s^\alpha(\mathrm{d}\omega_\alpha^\beta - \omega_\alpha^\gamma\wedge\omega_\gamma^\beta)e_\beta \\ \Omega(e_\alpha) &= \mathrm{d}^\nabla(\omega_\alpha^\beta e_\beta) \\ &= \mathrm{d}\omega_\alpha^\beta e_\beta - \omega_\alpha^\beta\wedge\nabla e_\beta \\ &= \mathrm{d}\omega_\alpha^\beta e_\beta - \omega_\alpha^\beta\wedge\omega_\beta^\gamma e_\gamma \\ &= (\mathrm{d}\omega_\alpha^\beta - \omega_\alpha^\gamma\wedge\omega_\gamma^\beta)e_\beta \end{aligned}$$

This shows smooth functions commutes with Ω. This is a quite good property, from this we can conclude:

1. $\Omega(e_\alpha)$ completely determines Ω locally, thus we can say Ω locally looks like $\mathrm{d}\omega - \omega\wedge\omega$;
2. Ω is a global section of $\Omega_M^2\otimes\operatorname{End}E$, that is it's compatible with change of basis. Indeed, for two local basis $e, \widetilde{e}$ such that $\widetilde{e} = ge$, we will see

$$\begin{aligned} g\nabla^2 e &= \nabla^2 ge \\ &= \nabla^2\widetilde{e} \\ &= (\mathrm{d}\widetilde{\omega} - \widetilde{\omega}\wedge\widetilde{\omega})\widetilde{e} \\ &= (\mathrm{d}\widetilde{\omega} - \widetilde{\omega}\wedge\widetilde{\omega})ge \end{aligned}$$

   which implies

$$g^{-1}(\mathrm{d}\widetilde{\omega} - \widetilde{\omega}\wedge\widetilde{\omega})g = \mathrm{d}\omega - \omega\wedge\omega$$

Definition 4.1.1 (curvature form)**.** For a connection ∇ of a vector bundle E on M, its curvature form $\Omega\in C^\infty(M, \Omega_M^2\otimes\operatorname{End}E)$ is defined as above.

Remark 4.1.1 (local form). We can give a more explicit expression of Ω using Christoffel symbol: If we locally write Ω as

$$\Omega_\alpha^\beta = \Omega_{ij\alpha}^\beta \mathrm{d}x^i\wedge\mathrm{d}x^j$$

Then $\Omega = \mathrm{d}\omega - \omega \wedge \omega$ can be written as

$$\begin{aligned}
\Omega_{ij\alpha}^{\beta}\mathrm{d}x^i \wedge \mathrm{d}x^j &= \Omega_\alpha^\beta \\
&= \mathrm{d}\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta \\
&= \mathrm{d}(\Gamma_{i\alpha}^\beta \mathrm{d}x^i) - (\Gamma_{i\alpha}^\gamma \mathrm{d}x^i) \wedge (\Gamma_{j\gamma}^\beta \mathrm{d}x^j) \\
&= (-\partial_j \Gamma_{i\alpha}^\beta - \Gamma_{i\alpha}^\gamma \Gamma_{j\gamma}^\beta)\mathrm{d}x^i \wedge \mathrm{d}x^j
\end{aligned}$$

4.2. **Curvature tensor.** In Do carmo [Car92], he defines the curvature of a connection ∇ as follows:

$$\begin{aligned}
R : TM \times TM \times E &\to E \\
(X, Y, s) &\mapsto R(X, Y)s
\end{aligned}$$

where $R(X,Y)s = \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s$. It's easy to check R we defined above is a tensor, that is a section of $T^*M \otimes T^*M \otimes \operatorname{End} E$.

Remark 4.2.1 (local form)*.* Locally we write R as

$$R = R_{ij\alpha}^\beta \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes e^\alpha \otimes e_\beta$$

To see $R_{ij\alpha}^\beta$, it suffices to compute

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} e_\alpha &= \nabla_{\frac{\partial}{\partial x^i}}(\Gamma_{j\alpha}^\beta e_\beta) \\
&= \partial_i \Gamma_{j\alpha}^\beta e_\beta + \Gamma_{j\alpha}^\beta \Gamma_{i\beta}^\gamma e_\gamma \\
&= (\partial_i \Gamma_{j\alpha}^\beta + \Gamma_{j\alpha}^\gamma \Gamma_{i\gamma}^\beta) e_\beta
\end{aligned}$$

Thus

$$R_{ij\alpha}^\beta e_\beta = (\partial_i \Gamma_{j\alpha}^\beta - \partial_j \Gamma_{i\alpha}^\beta + \Gamma_{j\alpha}^\gamma \Gamma_{i\gamma}^\beta - \Gamma_{i\alpha}^\gamma \Gamma_{j\gamma}^\beta) e_\beta$$

or in other words,

$$R_\alpha^\beta = (\partial_i \Gamma_{j\alpha}^\beta - \partial_j \Gamma_{i\alpha}^\beta + \Gamma_{j\alpha}^\gamma \Gamma_{i\gamma}^\beta - \Gamma_{i\alpha}^\gamma \Gamma_{j\gamma}^\beta)\mathrm{d}x^i \otimes \mathrm{d}x^j$$

Recall that our curvature form Ω is a section of $\Omega_M^2 \otimes \operatorname{End} E$, and you can regard it as a section of $T^*M \otimes T^*M \otimes \operatorname{End} E$, that is

$$\begin{aligned}
\Omega_\alpha^\beta &= (-\partial_j \Gamma_{i\alpha}^\beta - \Gamma_{i\alpha}^\gamma \Gamma_{j\gamma}^\beta)\mathrm{d}x^i \wedge \mathrm{d}x^j \\
&= (\partial_i \Gamma_{j\alpha}^\beta - \partial_j \Gamma_{i\alpha}^\beta + \Gamma_{j\alpha}^\gamma \Gamma_{i\gamma}^\beta - \Gamma_{i\alpha}^\gamma \Gamma_{j\gamma}^\beta)\mathrm{d}x^i \otimes \mathrm{d}x^j
\end{aligned}$$

So if you regard curvature form as a tensor, then it's exactly curvature tensor we defined here.

If we take E to be tangent bundle, then we can regard R as a $(1,3)$-tensor, locally looks like

$$R_{ijk}^r \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \frac{\partial}{\partial x^r}$$

However, we always use its $(0,4)$ type, that is

$$R_{ijkl} = g_{rl} R_{ijk}^r$$

Now let's give a more explicit expression about R_{ijkl}. By definition we directly have

$$
\begin{aligned}
R_{ijkl} &= R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}) \\
&= \langle \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \rangle \\
&= \partial_i \langle \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \rangle - \langle \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^l} \rangle - (\partial_j \langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \rangle - \langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}, \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l} \rangle) \\
&= \underbrace{\partial_i \langle \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \rangle - \partial_j \langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \rangle}_{\text{part I}} + \underbrace{\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}, \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l} \rangle - \langle \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^l} \rangle}_{\text{part II}}
\end{aligned}
$$

For part II, we have

$$g_{rs}(\Gamma^r_{ik}\Gamma^s_{jl} - \Gamma^r_{jk}\Gamma^s_{il})$$

For part I, note that

$$
\begin{aligned}
\partial_i(\Gamma^r_{jk} g_{rl}) &= \partial_i(\frac{1}{2} g^{rs}(\partial_j g_{ks} + \partial_k g_{js} - \partial_s g_{jk}) g_{rl}) \\
&= \partial_i(\frac{1}{2} \delta^s_l (\partial_j g_{ks} + \partial_k g_{js} - \partial_s g_{jk})) \\
&= \frac{1}{2} \partial_i (\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk})
\end{aligned}
$$

Thus we have part I is

$$\partial_i(\Gamma^r_{jk} g_{rl}) - \partial_j(\Gamma^r_{ik} g_{rl}) = \frac{1}{2}(\partial_i \partial_k g_{jl} + \partial_j \partial_l g_{ik} - \partial_i \partial_l g_{jk} - \partial_j \partial_k g_{il})$$

So we have an explicit expression for R_{ijkl}

$$R_{ijkl} = \frac{1}{2}(\partial_i \partial_k g_{jl} + \partial_j \partial_l g_{ik} - \partial_i \partial_l g_{jk} - \partial_j \partial_k g_{il}) + g_{rs}(\Gamma^r_{ik}\Gamma^s_{jl} - \Gamma^r_{jk}\Gamma^s_{il})$$

From this expression, we can see in general curvature depends on two order partial derivatives of metric. Furthermore, there are some skew symmetries and symmetries of R_{ijkl}.

1. $R_{ijkl} = -R_{jikl}$;
2. $R_{ijkl} = -R_{ijlk}$;
3. $R_{ijkl} = R_{klij}$.

Proposition 4.2.1. In Riemannian normal coordinates we have

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{iklj}(0) x^k x^l + O(|x|^3)$$

Proof. Recall we already have

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma^m_{ki} g_{mj} + \Gamma^m_{kj} g_{mi}$$

We differential the equation with respect to x^l, evaluate at 0 and use the fact that Christoffel symbol vanishes to have

$$\frac{\partial^2 g_{ij}}{\partial x^l \partial x^k} = \frac{\partial \Gamma^m_{ki}}{\partial x^l}(0) g_{mj}(0) + \frac{\partial \Gamma^m_{kj}}{\partial x^l}(0) g_{mi}(0)$$

Here we claim

$$\frac{\partial \Gamma^k_{ij}}{\partial x^l}(0) + \frac{\partial \Gamma^k_{li}}{\partial x^j}(0) + \frac{\partial \Gamma^k_{jl}}{\partial x^i}(0) = 0$$

Indeed, in normal coordinates we have

$$0 = \Gamma^k_{ij}(tx) x^i x^j$$

Then differential this with respect to t and evaluate at $t = 0$ we have

$$0 = \frac{\partial \Gamma^k_{ij}}{\partial x^l}(0) x^i x^j x^l$$

which implies

$$\sum_{\sigma \in S_3} \frac{\partial \Gamma^k_{\sigma(i)\sigma(j)}}{\partial x^{\sigma(l)}}(0) = 0$$

Then use symmetry of Christoffel symbol in term i, j to conclude.

From $R^l_{ijk}(0) = \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j}$ we have

$$\begin{aligned} R_{ijkl}(0) &= (\frac{\partial \Gamma^m_{jk}}{\partial x^i}(0) - \frac{\partial \Gamma^m_{ik}}{\partial x^j}(0)) g_{ml}(0) \\ &= -(\frac{\partial \Gamma^m_{ij}}{\partial x^k}(0) + \frac{\partial \Gamma^m_{ki}}{\partial x^j}(0) + \frac{\partial \Gamma^m_{ik}}{\partial x^j}(0)) g_{ml}(0) \\ &= -(\frac{\partial \Gamma^m_{ij}}{\partial x^k}(0) + 2\frac{\partial \Gamma^m_{ki}}{\partial x^j}(0)) g_{ml}(0) \end{aligned}$$

Thus we have

$$\begin{aligned} 2R_{ikjl}(0) x^k x^l =& -(R_{iklj}(0) + R_{jlki}(0)) x^k x^l \\ =& (\frac{\partial \Gamma^m_{ik}}{\partial x^l}(0) + 2\frac{\partial \Gamma^m_{il}}{\partial x^k}(0)) g_{mj}(0) x^k x^l \\ &+ (\frac{\partial \Gamma^m_{jl}}{\partial x^k}(0) + 2\frac{\partial \Gamma^m_{jk}}{\partial x^l}(0)) g_{mi}(0) x^k x^l \\ =& 3\frac{\partial g_{ij}}{\partial x^k \partial x^l}(0) x^k x^l \end{aligned}$$

Thus we get for the second term in the Taylor expansion

$$\begin{aligned} \frac{1}{2}\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}(0) x^k x^l &= \frac{1}{3} R_{ikjl}(0) x^k x^l \\ &= -\frac{1}{3} R_{iklj}(0) x^k x^l \end{aligned}$$

that is

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{iklj}(0) x^k x^l + O(|x|^3)$$

□

Corollary 4.2.1. In Riemannian normal coordinates we have

1. $g^{ij}=\delta_{ij}+\frac{1}{3}R_{iklj}(0)x^kx^l+O(|x|^3)$
2. $\det(g_{ij})=1-\frac{1}{3}R_{kl}x^kx^l+O(|x|^3)$
3. $\sqrt{\det(g_{ij})}=1-\frac{1}{6}R_{kl}x^kx^l+O(|x|^3)$

Proof. For (1). Note that g^{ij} gives a Riemannian metric on T^*M, and Levi-Civita connection ∇ on T^*M with respect to g^{ij} is exactly the induced connection from the one on TM. Note that

$$\nabla \mathrm{d}x^k=-\Gamma_{ij}^k\mathrm{d}x^i\otimes\mathrm{d}x^j$$

where Γ_{ij}^k is the Christoffel symbol for Levi-Civita connection on TM, we have curvature form in this case differs a sign since

$$R_{ijk}^l(0)=\frac{\partial\Gamma_{jk}^l}{\partial x^i}-\frac{\partial\Gamma_{ik}^l}{\partial x^j}$$

Thus all computations are same as proof above, but result differs a sign in curvature.

For (2). By Jacobi's formula, we have

$$\frac{\partial\det(g_{ij})}{\partial x^k}=\det(g_{ij})g^{ij}\frac{\partial g_{ij}}{\partial x^k}$$

Thus $\frac{\partial\det(g_{ij})}{\partial x^k}(0)=0$, since first-order partial derivatives of g_{ij} vanishes. Furthermore, since first-order partial derivatives of g^{ij} also vanishes, we have

$$\begin{aligned}\frac{1}{2}\frac{\partial^2\det(g_{ij})}{\partial x^l\partial x^k}&=\det(g_{ij})g^{ij}\frac{1}{2}\frac{\partial^2 g_{ij}}{\partial x^l\partial x^k}\\&=\det(g_{ij})g^{ij}(-\frac{1}{3}R_{iklj}x^kx^l)\\&=-\frac{1}{3}\det(g_{ij})R_{kl}x^kx^l\end{aligned}$$

which implies

$$\det(g_{ij})=1-\frac{1}{3}R_{kl}x^kx^l+O(|x|^3)$$

For (3). It follows from (2) directly. □

4.3. **Bianchi identities.** There are two famous Bianchi identities in Riemannian geometry, in Do carmo [Car92] they are stated as follows

1. First Bianchi identity: $R(X,Y,Z,W)+R(Y,Z,X,W)+R(Z,X,Y,W)=0$;
2. Second Bianchi identity: $\nabla_XR(Y,Z,W,R)+\nabla_YR(Z,X,W,R)+\nabla_ZR(X,Y,W,R)=0$.

4.3.1. *First Bianchi.* Locally we have first Bianchi identity as

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0$$

In order to compute we use $(1,3)$ type as follows

$$R^r_{ijk} + R^r_{jki} + R^r_{kij} = 0$$

since we have

$$R^r_{ijk} = \underbrace{\partial_i \Gamma^r_{jk} - \partial_j \Gamma^r_{ik}}_{\text{part I}} + \underbrace{\Gamma^s_{jk}\Gamma^r_{is} - \Gamma^s_{ik}\Gamma^r_{js}}_{\text{part II}}$$

1. For the first part, if we permuting i, j, k, we have
$$\partial_i \Gamma^r_{jk} - \partial_j \Gamma^r_{ik} + \partial_j \Gamma^r_{ki} - \partial_k \Gamma^r_{ji} + \partial_k \Gamma^r_{ij} - \partial_i \Gamma^r_{kj} = 0$$
since $\Gamma^r_{ij} = \Gamma^r_{ji}$ by torsion-free.
2. For the second part, if we permuting i, j, k, we have
$$\Gamma^s_{jk}\Gamma^r_{is} - \Gamma^s_{ik}\Gamma^r_{js} + \Gamma^s_{ki}\Gamma^r_{js} - \Gamma^s_{ji}\Gamma^r_{ks} + \Gamma^s_{ij}\Gamma^r_{ks} - \Gamma^s_{kj}\Gamma^r_{is} = 0$$
by the same reason.

Thus we obtain first Bianchi identity, which is just a consequence of torsion-free.

Remark 4.3.1. If we consider connection on arbitrary vector bundle E, there is no first Bianchi identity, since e_α is just a section of E, not a section of TM, so $R(e_\alpha, \cdot)$ or $R(\cdot, e_\alpha)$ is nonsense.

4.3.2. *Second Bianchi.* In fact, we can write second Bianchi identity for arbitrary vector bundle E as follows

$$\nabla_X R(Y, Z, s, t) + \nabla_Y R(Z, X, s, t) + \nabla_Z R(X, Y, s, t) = 0$$

where $s, t \in C^\infty(M, E), X, Y, Z \in C^\infty(M, TM)$. It's clear that it's equivalent to

$$\nabla_i R_{jk\alpha\beta} + \nabla_j R_{ki\alpha\beta} + \nabla_k R_{ij\alpha\beta} = 0$$

To prove it, here we choose normal coordinates, that is $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$. Then

$$\nabla_{\frac{\partial}{\partial x^i}} g(\nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^m}) = g(\nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^m})$$

By permuting i, j, k we have

$$\begin{aligned}
&\nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l} \\
&+\nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^l} \\
&+\nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^l} \\
&= R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \nabla_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_l} + R(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}) \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_l} + R(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_i}) \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_l} \\
&= 0
\end{aligned}$$

This completes the computation of second Bianchi identity.

From another aproach, recall that our curvature form Ω is a section of $\Omega^2_M \otimes \operatorname{End} E$, which can be written as $\Omega^\alpha_\beta e_\alpha \otimes e^\beta$ locally. Then we have $\nabla\Omega$ can be written as

$$\nabla\Omega = \mathrm{d}\Omega + \Omega \wedge \omega - \omega \wedge \Omega$$

However, $\nabla\Omega = 0$, since

$$\begin{aligned}\nabla\Omega &= \mathrm{d}\Omega + \Omega \wedge \omega - \omega \wedge \Omega \\ &= \mathrm{d}(\mathrm{d}\omega - \omega \wedge \omega) + (\mathrm{d}\omega - \omega \wedge \omega) \wedge \omega - \omega \wedge (\mathrm{d}\omega - \omega \wedge \omega) \\ &= \mathrm{d}^2\omega - \mathrm{d}\omega \wedge \omega + \omega \wedge \mathrm{d}\omega + \mathrm{d}\omega \wedge \omega - \omega \wedge \omega \wedge \omega - \omega \wedge \mathrm{d}\omega + \omega \wedge \omega \wedge \omega \\ &= 0\end{aligned}$$

If we back to local form, we have

$$\mathrm{d}\Omega^\beta_\alpha + \Omega^\gamma_\alpha \wedge \omega^\beta_\gamma - \omega^\gamma_\alpha \wedge \Omega^\beta_\gamma = 0$$

More explicitly, if we write $\Omega^\beta_\alpha = \Omega^\beta_{ij\alpha}\mathrm{d}x^i \wedge \mathrm{d}x^j$, we obtain

$$(\partial_k\Omega^\beta_{ij\alpha} + \Omega^\gamma_{ij\alpha}\Gamma^\beta_{k\gamma} - \Gamma^\gamma_{k\alpha}\Omega^\beta_{ij\gamma})\mathrm{d}x^k \wedge \mathrm{d}x^i \wedge \mathrm{d}x^j = 0$$

In other words

$$\begin{aligned}&\partial_k\Omega^\beta_{ij\alpha} + \Omega^\gamma_{ij\alpha}\Gamma^\beta_{k\gamma} - \Gamma^\gamma_{k\alpha}\Omega^\beta_{ij\gamma} \\ +&\partial_i\Omega^\beta_{jk\alpha} + \Omega^\gamma_{jk\alpha}\Gamma^\beta_{i\gamma} - \Gamma^\gamma_{i\alpha}\Omega^\beta_{jk\gamma} \\ +&\partial_j\Omega^\beta_{ki\alpha} + \Omega^\gamma_{ki\alpha}\Gamma^\beta_{j\gamma} - \Gamma^\gamma_{j\alpha}\Omega^\beta_{ki\gamma} = 0\end{aligned}$$

Note that $2\Omega^\beta_{ij\alpha} = R^\beta_{ij\alpha}$, and

$$\nabla_k R^\beta_{ij\alpha} = \partial_k R^\beta_{ij\alpha} + \Gamma^\beta_{k\gamma}R^\gamma_{ij\alpha} - \Gamma^\gamma_{k\alpha}R^\beta_{ij\gamma}$$

So $\nabla\Omega = 0$ locally looks like

$$\nabla_k R^\beta_{ij\alpha} + \nabla_i R^\beta_{jk\alpha} + \nabla_j R^\beta_{ki\alpha} = 0$$

This shows two Bianchi identities are same.

4.4. **Other curvatures.**

4.4.1. *Sectional curvature.* Closely related to curvature is sectional curvature that we're going to define, which is used to characterize a two dimensional subspace of tangent space.

Fix $p \in M$ and let x, y are two linearly independent tangent vectors in T_pM, then sectional curvature for these two vectors are defined as

$$K_p(x,y) = \frac{R(x,y,y,x)}{g(x,x)g(y,y) - g(x,y)^2}$$

In order to show it's a invariant defined for a two dimensional subspace, we need to check if $\operatorname{span}_{\mathbb{R}}\{x,y\} = \operatorname{span}_{\mathbb{R}}\{z,w\}$, then

$$K_p(x,y) = K_p(z,w)$$

Indeed, if we write

$$\begin{cases} z = ax + by \\ w = cx + dy \end{cases}$$

Then by symmetry and skew symmetry properties of R we have

$$\begin{aligned} R(z,w,w,z) &= R(ax+by, cx+dy, cx+dy, ax+by) \\ &= R(ax,dy,dy,ax) + R(ax,dy,cx,by) + R(by,cx,dy,ax) + R(by,cx,cx,by) \\ &= a^2d^2R(x,y,y,x) - abcdR(x,y,y,x) - abcdR(x,y,y,x) + b^2c^2R(x,y,y,x) \\ &= (ad-bc)^2R(x,y,y,x) \end{aligned}$$

And by the same computations we have

$$g(z,z)g(w,w) - g(z,w)^2 = (ad-bc)^2\{g(x,x)g(y,y) - g(x,y)^2\}$$

Thus

$$K_p(x,y) = K_p(z,w)$$

So the following definition is well-defined:

Definition 4.4.1 (sectional curvature)**.** The sectional curvature $K_p(\sigma)$ for two dimensional subspace $\sigma \subseteq T_pM$ is defined as

$$K_p(\sigma) := K_p(x,y)$$

where $\{x,y\}$ is a basis of σ.

Definition 4.4.2 (isotropic)**.** A Riemannian manifold (M,g) is called isotropic, if for each point $p \in M$, the sectional curvature $K_p(\sigma)$ is independent of σ.

Definition 4.4.3 (constant sectional curvature)**.** A Riemannian manifold (M,g) has constant sectional curvature, if $K_p(\sigma)$ is constant for arbitrary $\sigma \subset T_pM, p \in M$.

Remark 4.4.1. By definition, we can see if a Riemannian manifold has constant sectional curvature, then it must be isotropic; Conversely, we will see if the dimension of a Riemannian manifold ≥ 3, then isotropic is equivalent to constant sectional curvature.

Lemma 4.4.1.

$$\begin{aligned} -6R(X,Y,Z,W) = \frac{\partial^2}{\partial s \partial t}\bigg|_{s=t=0} &\{R(X+sZ, Y+tW, Y+tW, X+sZ) \\ &- R(X+sW, Y+tZ, Y+tZ, X+sW)\} \end{aligned}$$

where X,Y,Z,W are vector fields.

Proof. It suffices to compute coefficients of st of $R(X+sZ, Y+tW, Y+tW, X+sZ)$ and exchange Z with W to obtain coefficients of st of $R(X+sW, Y+tZ, Y+tZ, X+sW)$.

It's easy to see coefficients of st of $R(X+sZ, Y+tW, Y+tW, X+sZ)$ is

$$R(Z,W,Y,X) + R(Z,Y,W,X) + R(X,W,Y,Z) + R(X,Y,W,Z)$$

So coefficients of st of $R(X+sZ,Y+tW,Y+tW,X+sZ)$ is

$$R(W,Z,Y,X)+R(W,Y,Z,X)+R(X,Z,Y,W)+R(X,Y,Z,W)$$

Thus the right hand of our desired identity is

$$-4R(X,Y,Z,W)-(R(Y,Z,W,X)+R(W,Y,Z,X))-(R(W,X,Y,Z)+R(W,Y,Z,X))$$

By first Bianchi identity we have

$$\begin{aligned}R(Y,Z,W,X)+R(W,Y,Z,X)&=R(Y,Z,W,X)+R(Z,X,W,Y)\\&=R(X,Y,Z,W)\\R(W,X,Y,Z)+R(W,Y,Z,X)&=R(Y,Z,W,X)+R(Z,X,W,Y)\\&=R(X,Y,Z,W)\end{aligned}$$

This completes the proof. □

Notation 4.4.1. For convenience, we use $R_0(X,Y,Z,W)$ to denote

$$R_0(X,Y,Z,W)=g(X,W)g(Y,Z)-g(X,Z)g(Y,W)$$

where X,Y,Z,W are vector fields. Then we can write sectional curvature as

$$K_p(\sigma)=\frac{R(x,y,y,x)}{R_0(x,y,y,x)}$$

where $\sigma\subset T_pM$ is spanned by x,y.

Proposition 4.4.1. A Riemannian manifold has constant sectional curvature K_p at point $p\in M$ if and only if $R=K_pR_0$, where K_p is a constant(may depend on p), R is curvature tensor.

Proof. If $R=K_pR_0$, then for a arbitrary x,y, we have

$$K_p(x,y)=\frac{R(x,y,y,x)}{R_0(x,y,y,x)}=K_p$$

Conversely, if $K(\sigma)$ is constant at point $p\in M$, that is for arbitrary x,y we have

$$\frac{R(x,y,y,x)}{R_0(x,y,y,x)}=K_p$$

If we denote

$$F(s,t)=R(x+sz,y+tw,y+tw,x+sz)-R(x+sw,y+tz,y+tz,x+sw)$$

$$F_0(s,t)=R_0(x+sz,y+tw,y+tw,x+sz)-R_0(x+sw,y+tz,y+tz,x+sw)$$

we still have $F(s,t)=K_pF_0(s,t)$. By Lemma 4.4.1, we have

$$R(x,y,z,w)=-\frac{1}{6}\left.\frac{\partial^2}{\partial s\partial t}\right|_{s=t=0}F(s,t)$$

and it's easy to see

$$R_0(x,y,z,w)=-\frac{1}{6}\left.\frac{\partial^2}{\partial s\partial t}\right|_{s=t=0}F_0(s,t)$$

This completes the proof. □

Corollary 4.4.1. A Riemannian manifold is isotropic if and only if $R = KR_0$, where K is a smooth function.

Corollary 4.4.2. A Riemannian manifold has constant sectional curvature K if and only if $R = KR_0$, where K is a constant.

Remark 4.4.2. An important corollary is that curvature tensor of Riemannian manifold with constant sectional curvature K is quite simple, since

$$R_{ijkl} = K(g_{il}g_{jk} - g_{ik}g_{jl})$$

that is, curvature is completely determined by zero order partial derivatives of metric, not two order in general.

Remark 4.4.3. Suppose the dimension of Riemannian manifold (M, g) is 2, and $\{e_1, e_2\}$ is a basis of T_pM. Then

$$K_p = K_p(e_1, e_2) = \frac{R(e_1, e_2, e_2, e_1)}{|e_1|^2|e_2|^2 - |g(e_1, e_2)|^2}$$

is exactly Gauss curvature we learnt in theory of surface.

4.4.2. *Ricci curvature and scalar curvature.*

Definition 4.4.4 (Ricci curvature)**.** For a Riemannian manifold (M, g), the Ricci curvature is defined to be

$$\mathrm{Ric}(X, Y) := \mathrm{tr}_g(Z \mapsto R(Z, X)Y)$$

where X, Y are vector fields.

Remark 4.4.4 (local form)*.* The trace of above endomorphism is exactly R^i_{ijk}, and it can be written as

$$g^{il}R_{ijkl}$$

In other words, Ricci curvature tensor is the contracted tensor of curvature with respect to the first and fourth index.

Definition 4.4.5 (Ricci curvature in one direction)**.** For a point $p \in M$, and $x \in T_pM$, Ricci curvature in the direction x is defined as

$$\mathrm{Ric}_p(x) := \mathrm{Ric}(x, x)$$

Remark 4.4.5. For $x \in T_pM$, we can write it as $x = x^i e_i$, where $\{e_1, \ldots, e_n\}$ is a basis of T_pM, then

$$\mathrm{Ric}_p(x) = R_{jk}x^j x^k$$

Definition 4.4.6 (scalar curvature)**.** For a Riemannian manifold (M, g), the scalar curvature S at $p \in M$ is defined as

$$S(p) := \sum_{i=1}^{n} \mathrm{Ric}_p(e_i)$$

where $\{e_1, \ldots, e_n\}$ is an orthnormal basis of T_pM.

Remark 4.4.6 (local form). Locally we have

$$S = g^{jk} R_{jk}$$

Proposition 4.4.2 (contracted Bianchi identity)**.**

$$g^{jk}\nabla_k R_{ij} = \frac{1}{2}\nabla_i S$$

where R_{ij} is Ricci curvature and S is scalar curvature.

Proof. Direct computation shows

$$\begin{aligned} g^{jk}\nabla_k R_{ij} &= g^{jk}\nabla_k g^{pq} R_{pijq} \\ &= g^{jk} g^{pq}\nabla_k R_{pijq} \\ &= g^{jk} g^{pq}(-\nabla_p R_{ikjq} - \nabla_i R_{kpjq}) \\ &= -g^{pq}\nabla_p R_{iq} + \nabla_i S \\ &= -g^{jk}\nabla_k R_{ij} + \nabla_i S \end{aligned}$$

This completes the proof. □

Proposition 4.1. The scalar curvature S at $p \in M$ is given by

$$S(p) = \frac{1}{\alpha_n}\int_{S^{n-1}} \mathrm{Ric}_p(x) \mathrm{d}S^{n-1}$$

where α_n is the volume of n-dimension unit ball in $\mathbb{R}^{n+1}$ and $\mathrm{d}S^{n-1}$ is the area elements in S^{n-1}.

Proof. Choose an orthnormal basis $\{e_1, \ldots, e_n\}$ in T_pM and write $x = x^i e_i$, then

$$\begin{aligned} \mathrm{Ric}_p(x) &= \mathrm{Ric}_p(x^i e_i) \\ &= (x^i)^2 \, \mathrm{Ric}_p(e_i) \end{aligned}$$

Since $|x| = 1$, then the vector $\mu = (x^1, \ldots, x^n)$ is a unit normal vector on S^{n-1}. Denoting $V = (x^1 \, \mathrm{Ric}_p(e_1), \ldots, x^n \, \mathrm{Ric}_p(e_n))$, then Stokes theorem implies

$$\begin{aligned} \frac{1}{\alpha_n}\int_{S^{n-1}} (x^i)^2 \, \mathrm{Ric}_p(e_i) \mathrm{d}S^{n-1} &= \frac{1}{\alpha_n}\int_{S^{n-1}} \langle V, \mu\rangle \mathrm{d}S^{n-1} \\ &= \frac{1}{\alpha_n}\int_{B^n} \operatorname{div} V \mathrm{d}B^n \\ &= \operatorname{div} V \\ &= \sum_{i=1}^{n} \mathrm{Ric}_p(e_i) \\ &= S(p) \end{aligned}$$

where B^n is unit ball in T_pM with $\partial B^n = S^{n-1}$. □

Theorem 4.4.1. Let (M, g) be a Riemannian manifold, then for all $p \in M$ and r sufficiently small, the volume of the geodesic ball $B(p, r)$ is

$$\mathrm{vol}(B(p,r)) = \alpha_n r^n (1 - \frac{S(p)}{6(n+2)} r^2 + O(r^3))$$

where α_n is the volume of n-dimension unit ball in $\mathbb{R}^{n+1}$.

Proof. Note that we have

$$\sqrt{\det(g_{ij})} = \delta_{ij} - \frac{1}{6} R_{jk}(p) x^j x^k + O(|x^3|)$$

Directly computation shows

$$\begin{aligned}
\mathrm{Vol}(B(p,r)) &= \int_0^r \int_{S^{n-1}(t)} \sqrt{\det g}\mathrm{d}S\mathrm{d}t \\
&= \int_0^r \int_{S^{n-1}(t)} (1 - \frac{1}{6}\operatorname{Ric}_p(x) + O(|x|^3))\mathrm{d}S\mathrm{d}t \\
&= \alpha_n r^n - \frac{\alpha_n}{6} \int_0^r t^{n+1}\mathrm{d}t + O(r^{n+3}) \\
&= \alpha_n r^n - \frac{\alpha_n S(p) r^{n+2}}{6(n+2)} + O(r^{n+3}) \\
&= \alpha_n r^n (1 - \frac{S(p)}{6(n+2)} r^2 + O(r^3))
\end{aligned}$$

where we use the fact $\alpha_n = \omega_{n-1}/n$. □

4.4.3. *Einstein manifold.*

Definition 4.4.7 (Einstein manifold)**.** A Riemannian manifold (M, g) is called Einstein manifold, if its Ricci curvature satisfies $R_{ij} = \lambda g_{ij}$ for some $\lambda \in \mathbb{R}$.

Lemma 4.4.2 (Schur's lemma)**.** Let (M, g) be a Riemannian manifold with $\dim M \geq 3$, suppose $R_{ij} = f g_{ij}$, where $f \in C^\infty(M)$, then (M, g) is an Einstein manifold.

Proof. If $R_{ij} = f g_{ij}$, then contracted Bianchi identity shows

$$\begin{aligned}
\frac{n}{2}\nabla_i f &= g^{jk}\nabla_k f g_{ij} \\
&= \nabla_i f
\end{aligned}$$

for arbitrary i, which implies f is constant, since $n \geq 3$. □

Corollary 4.4.3. For a Riemannian manifold (M, g) with $\dim M \geq 3$, it is isotropic if and only if it has constant sectional curvature.

Proof. By Remark 4.4.2, it suffices to show if M is isotropic then it has constant sectional curvature. If M is isotropic, then there exists a smooth function K such that

$$R_{ijkl} = K(g_{il}g_{jk} - g_{ik}g_{jl})$$

Consider its Ricci curvature, that is

$$R_{jk} = (n-1)Kg_{jk}$$

Then Schur's lemma implies $(n-1)K$ is constant, that is K is constant. □

Proposition 4.4.3. Let (M, g) be an Einstein manifold of 3-dimension, then (M, g) is of constant sectional curvature.

Proof. At any point $p \in M$, we choose normal basis at this point, that is $g_{ij} = \delta_{ij}$, thus

$$R_{11} = g^{ij}R_{i11j} = R_{2112} + R_{3113} = \lambda$$

Similarly we have

$$R_{1221} + R_{3223} = \lambda$$
$$R_{1331} + R_{2332} = \lambda$$

Thus we can conclude

$$R_{1221} = R_{1331} = R_{2332} = \frac{\lambda}{2}$$

that is

$$R_{ijkl} = \frac{\lambda}{2}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})$$

□

Remark 4.4.7. In fact, it's a special case of Ricci curvature controls curvature. For a n-dimensional Riemannian manifold, it's easy to see R_{jk} has $n(n+1)/2$ independent components. But for R_{ijkl}, this counting problem becomes a little bit complicated, it has

$$\frac{n^2(n^2-1)}{12}$$

independent components. Indeed, since R_{ijkl} is skew symmetric in ij and kl, this means that these pair of indices can take

$$m = \binom{n}{2} = \frac{n(n-1)}{2}$$

R_{ijkl} is also symmetric when you swap ij with kl, this means there would be

$$\frac{m(m+1)}{2} = \frac{n^4 - 2n^3 + 3n^2 - 2n}{8}$$

choices. However, these are not independent, since there is first Bianchi identity

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0$$

, and it provides

$$\binom{n}{4} = \frac{n^4 - 6n^3 + 11n^2 - 6n}{24}$$

relations between these components, thus the number of independent components of R_{ijkl} is

$$\frac{n^4 - 2n^3 + 3n^2 - 2n}{8} - \frac{n^4 - 6n^3 + 11n^2 - 6n}{24} = \frac{n^4 - n^2}{12} = \frac{n^2(n^2-1)}{12}$$

Therefore curvature is fully determined by the Ricci curvature if and only if

$$\frac{n^2(n^2-1)}{12} \leq \frac{n(n+1)}{2}$$

or in other words, $n \leq 3$.

4.5. **Examples.** Now let's compute some examples of Riemannian manifold to see their curvatures.

Example 4.5.1 (Euclidean space)**.** Riemannian metric on Euclidean space $\mathbb{R}^n$ is given by

$$g = \delta_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$$

Thus $R_{ijkl} = 0, R_{jk} = 0$ and $S = 0$.

Example 4.5.2 (Sphere)**.** Let $\mathbb{S}^n(K)$ denote n-dimensional sphere with radius K. There is a natural inclusion $f : \mathbb{S}^n(K) \hookrightarrow (\mathbb{R}^{n+1}, g_0)$, and we can use f to pullback g_0 to obtain a metric on $\mathbb{S}^n(K)$, denoted by $g = f^*g_0$. Given a local chart (U, φ, x^i), we can write

$$f(x^1, \dots, x^n) = (x^1, \dots, x^n, \sqrt{K^2 - \sum_{i=1}^n (x^i)^2})$$

For any $\frac{\partial}{\partial x^i}$, we have

$$\begin{aligned} \mathrm{d}f(\frac{\partial}{\partial x^i}) &= \frac{\partial f^j}{\partial x^i}\frac{\partial}{\partial x^j} \\ &= \frac{\partial}{\partial x^i} - \frac{x^i}{\sqrt{K^2 - \sum_{i=1}^n (x^i)^2}}\frac{\partial}{\partial x^{n+1}} \end{aligned}$$

Thus for any two $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}$ we have

$$\begin{aligned} g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) &= g_0(\mathrm{d}f\frac{\partial}{\partial x^i}, \mathrm{d}f\frac{\partial}{\partial x^j}) \\ &= g_0(\frac{\partial}{\partial x^i} - \frac{x^i}{\sqrt{K^2 - \sum_{i=1}^n (x^i)^2}}\frac{\partial}{\partial x^{n+1}}, \frac{\partial}{\partial x^j} - \frac{x^j}{\sqrt{K^2 - \sum_{i=1}^n (x^i)^2}}\frac{\partial}{\partial x^{n+1}}) \\ &= \delta_{ij} + \frac{x^i x^j}{K^2 - \sum_{i=1}^n (x^i)^2} \end{aligned}$$

which implies

$$g_{ij} = \delta_{ij} + \frac{x^i x^j}{T^2}, \quad T^2 = K^2 - \sum (x^i)^2$$

Thus we have

$$g^{ij} = \delta^{ij} - \frac{x^i x^j}{K^2}$$

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\delta_{ki}x^j + \delta_{kj}x^i}{T^2} + \frac{2x^i x^j x^k}{T^4}$$

So Christoffel symbol can be computed as

$$\begin{aligned}
\Gamma_{ij}^k &= \sum_l \frac{1}{2} g^{kl}\Big(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\Big) \\
&= \sum_l \frac{1}{2}\Big(\delta^{kl} - \frac{x^k x^l}{K^2}\Big)\Big(\frac{\delta_{ij}x^l + \delta_{il}x^j}{T^2} + \frac{2x^i x^j x^l}{T^4} + \frac{\delta_{ji}x^l + \delta_{jl}x^i}{T^2} + \frac{2x^i x^j x^l}{T^4} - \frac{\delta_{li}x^j + \delta_{kj}x^i}{T^2} - \frac{2x^i x^j x^l}{T^4}\Big) \\
&= \sum_l \frac{x^l}{T^2}\Big(\delta_{ij} + \frac{x^i x^j}{T^2}\Big)\Big(\delta^{kl} - \frac{x^k x^l}{K^2}\Big) \\
&= \frac{g_{ij}}{T^2} x^k \Big(1 - \frac{\sum_{l=1}^n (x^l)^2}{K^2}\Big) \\
&= \frac{x^k}{K^2} g_{ij}
\end{aligned}$$

Thus curvature can be written as[4]

$$\begin{aligned}
R_{ijkl} &= \frac{1}{2}(\partial_i\partial_k g_{jl} + \partial_j\partial_l g_{ik} - \partial_i\partial_l g_{jk} - \partial_j\partial_k g_{il}) + g_{rs}(\Gamma_{ik}^r\Gamma_{jl}^s - \Gamma_{jk}^r\Gamma_{il}^s) \\
&= \frac{1}{K^2}(g_{il}g_{jk} - g_{ik}g_{il})
\end{aligned}$$

So Ricci curvature and scalar curvature can be computed as follows

$$\begin{aligned}
R_{jk} &= g^{il}R_{ijkl} \\
&= \frac{1}{K^2} g^{il}(g_{il}g_{jk} - g_{ik}g_{jl}) \\
&= \frac{1}{K^2}(n g_{jk} - \delta_k^l g_{jl}) \\
&= \frac{n-1}{K^2} g_{jk} \\
S &= g^{jk}R_{jk} \\
&= \frac{n(n-1)}{K^2}
\end{aligned}$$

Example 4.5.3 (Poincaré disk)**.** Let $\mathbb{B}^n = \{x \in \mathbb{R}^n \mid |x| < 1\}$ with a metric

$$g = \frac{4\delta_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j}{(1-|x|^2)^2}$$

$$R_{ijkl} = -(g_{il}g_{jk} - g_{ik}g_{jl})$$

[4]Here I omit a huge computation, and I suggest you compute it by yourself. Maybe first it's quite tough for you to do this in first time, but you should try.

Three examples we compute above all have constant sectional curvature, in fact we have

Theorem 4.5.1 (Hopf)**.** Let (M, g) be a complete, simply-connected, n-dimensional Riemannian manifold with constant sectional curvature. Then (M, g) is isometric to either $\mathbb{R}^n$, S^n or $\mathbb{B}^n$ with standard metric.

5. Hodge theory on Riemannian manifold

For convenience, in this section we assume (M, g) is a compact oriented Riemannian manifold of dimension n.

5.1. Inner product on Ω^k_M.

Before we talk about Hodge theory on (M, g), let's recall some basic facts about differential k-forms. For a k-form φ, locally it can be written as

$$\varphi = \sum_{1\leq i_1<\dots<i_k\leq n} \varphi_{i_1\dots i_k}\mathrm{d}x^{i_1}\wedge\dots\wedge\mathrm{d}x^{i_k}$$

where $\varphi_{i_1\dots i_k} := \varphi(\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_k}})$, is skew-symmetric. If we don't want our indices are arranged, we can write

$$\varphi = \frac{1}{k!}\varphi_{i_1\dots i_k}\mathrm{d}x^{i_1}\wedge\dots\wedge\mathrm{d}x^{i_k}$$

Here we mean summation runs over any arbitrary different k indices. It's clear this two expressions are same, since both $\varphi_{i_1\dots i_k}$ and $\mathrm{d}x^{i_1}\wedge\dots\wedge\mathrm{d}x^{i_k}$ are skew-symmetric.

Notation 5.1.1. We always write $\varphi_{i_1\dots i_k}\mathrm{d}x^{i_1}\wedge\dots\wedge\mathrm{d}x^{i_k}$ as $\varphi_I\mathrm{d}x^I$.

Recall that we already have a induced metric g on $\bigotimes^k T^*M$, and Ω^k_M is a subbundle of $\bigotimes^k T^*M$. Thus we can define a metric on Ω^k_M as follows

Definition 5.1.1. Let φ, ψ be two k-forms, define

$$\langle\varphi,\psi\rangle := \frac{1}{k!}g(\varphi,\psi)$$

where g is induced metric on $\bigotimes^k T^*M$.

Lemma 5.1.1. For $\varphi = \varphi_I\mathrm{d}x^I, \psi = \psi_J\mathrm{d}x^J$, then

$$\langle\varphi,\psi\rangle = \varphi_I\psi_J g^{IJ}$$

where

$$g^{IJ} = \frac{1}{k!}g(dx^I, dx^J) = \det\begin{pmatrix} g^{i_1j_1} & \dots & g^{i_1j_k} \\ \dots & \dots & \dots \\ g^{i_kj_1} & \dots & g^{i_kj_k}\end{pmatrix}$$

Proof. It suffices to compute

$$g(dx^I, dx^J) = k!\det\begin{pmatrix} g^{i_1j_1} & \dots & g^{i_1j_k} \\ \dots & \dots & \dots \\ g^{i_kj_1} & \dots & g^{i_kj_k}\end{pmatrix}$$

Indeed, by definition we have

$$\mathrm{d}x^I = \mathrm{d}x^{i_1}\wedge\dots\wedge\mathrm{d}x^{i_k} = \sum_{\sigma\in S_k}(-1)^{|\sigma|}e_{i_{\sigma(1)}}\otimes\dots\otimes e_{i_{\sigma(k)}}$$

Then

$$\begin{aligned}
g(\mathrm{d}x^I, \mathrm{d}x^J) &= \sum_{\sigma,\tau}(-1)^{|\sigma|}(-1)^{|\tau|}g(\mathrm{d}x^{i_{\sigma(1)}}\otimes\cdots\otimes\mathrm{d}x^{i_{\sigma(k)}}, \mathrm{d}x^{j_{\tau(1)}}\otimes\cdots\otimes\mathrm{d}x^{j_{\tau(k)}})\\
&= \sum_{\sigma,\tau}(-1)^{|\sigma|}(-1)^{|\tau|}g^{i_{\sigma(1)}j_{\tau(1)}}\dots g^{i_{\sigma(k)}j_{\tau(k)}}\\
&= \sum_{\sigma,\tau}(-1)^{|\sigma\tau^{-1}|}g^{i_{\sigma\tau^{-1}(1)}j_1}\dots g^{i_{\sigma\tau^{-1}(k)}j_k}\\
&= \sum_{\sigma}\sum_{\rho}(-1)^{|\rho|}g^{i_{\rho(1)}j_1}\dots g^{i_{\rho(k)}j_k}\\
&= \sum_{\sigma}\det(g^{i_pj_q})\\
&= k!\det(g^{i_pj_q})
\end{aligned}$$

□

Remark 5.1.1. Note that here we don't assume φ_I, ψ_I is skew-symmetric, they can be arbitrary functions.

Corollary 5.1.1. For two k-forms φ, ψ, locally write them as

$$\begin{aligned}
\varphi &= \sum_{1\le i_1<\dots<i_k\le n}\varphi_{i_1\dots i_k}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\\
\psi &= \sum_{1\le j_1<\dots<j_k\le n}\varphi_{j_1\dots j_k}\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_k}
\end{aligned}$$

with φ_I, ψ_J is skew-symmetric, then

$$\langle\varphi,\psi\rangle = \sum_{\substack{1\le i_1<\dots i_k\le n\\ 1\le j_1<\dots j_k\le n}}\varphi_{i_1\dots i_k}\psi_{j_1\dots j_k}\det\begin{pmatrix} g^{i_1j_1} & \dots & g^{i_1j_k}\\ \dots & \dots & \dots\\ g^{i_kj_1} & \dots & g^{i_kj_k}\end{pmatrix}$$

Example 5.1.1. Let φ, ψ be two 2-forms, locally write them as

$$\varphi = \varphi_{i_1i_2}\mathrm{d}x^{i_1}\wedge\mathrm{d}x^{i_2}, \quad \psi = \psi_{j_1j_2}\mathrm{d}x^{j_1}\wedge\mathrm{d}x^{j_2}$$

where $i_1 < i_2, j_1 < j_2$. Then

$$\begin{aligned}
\langle\varphi,\psi\rangle &= \frac{1}{2}\varphi_{i_1i_2}\psi_{j_1j_2}g(\mathrm{d}x^{i_1}\wedge\mathrm{d}x^{i_2}, \mathrm{d}x^{j_1}\wedge\mathrm{d}x^{j_2})\\
&= \frac{1}{2}\varphi_{i_1i_2}\psi_{j_1j_2}g(\mathrm{d}x^{i_1}\otimes\mathrm{d}x^{i_2}-\mathrm{d}x^{i_2}\otimes\mathrm{d}x^{i_1}, \mathrm{d}x^{j_1}\otimes\mathrm{d}x^{j_2}-\mathrm{d}x^{j_2}\otimes\mathrm{d}x^{j_1})\\
&= \frac{1}{2}\varphi_{i_1i_2}\psi_{j_1j_2}(g^{i_1j_1}g^{i_2j_2}-g^{i_1j_2}g^{i_2j_1}-g^{i_2j_1}g^{i_1j_2}+g^{i_2j_2}g^{i_1j_1})\\
&= \varphi_{i_1i_2}\psi_{j_1j_2}(g^{i_1j_1}g^{i_2j_2}-g^{i_1j_2}g^{i_2j_1})\\
&= \varphi_{i_1i_2}\psi_{j_1j_2}\det\begin{pmatrix} g^{i_1j_1} & g^{i_1j_2}\\ g^{i_2j_1} & g^{i_2j_2}\end{pmatrix}
\end{aligned}$$

Definition 5.1.2 (volume form)**.** A form vol locally looks like $\sqrt{\det g}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, where $\sqrt{\det g} = \sqrt{\det(g_{ij})}$, is called a volume form.

Proposition 5.1.1.

$$\langle \mathrm{vol}, \mathrm{vol} \rangle = 1$$

Proof. Directly compute

$$\begin{aligned} \langle \sqrt{\det g}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n, \sqrt{\det g}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \rangle &= \det(g_{ij}) \det(g^{ij}) \\ &= 1 \end{aligned}$$

□

Definition 5.1.3 (inner product on Ω^k_M)**.** For two k-forms φ, ψ, their inner product is defined as

$$(\varphi, \psi) := \int_M \langle \varphi, \psi \rangle \, \mathrm{vol}$$

Definition 5.1.4 (formal adjoint)**.** For a k-form φ and a $(k+1)$-form ψ, if there exists $\mathrm{d}^* : C^\infty(M, \Omega^{k+1}_M) \to C^\infty(M, \Omega^k_M)$ such that

$$(\mathrm{d}\varphi, \psi) = (\varphi, \mathrm{d}^*\psi)$$

Then d^* is called formal adjoint of d.

Remark 5.1.2. Later we will see such d^* do exists.

Definition 5.1.5 (Laplace-Beltrami operator)**.** The Laplacian operator $\Delta : C^\infty(M, \Omega^k_M) \to C^\infty(M, \Omega^k_M)$ is defined as

$$\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$$

Definition 5.1.6 (harmonic)**.** A k-form α is called harmonic, if $\Delta\alpha = 0$. The space of all harmonic forms is denoted by $\mathcal{H}^k(M)$

Lemma 5.1.2. A k-form α is harmonic if and only if $\mathrm{d}\alpha = 0$ and $\mathrm{d}^*\alpha = 0$.

Proof. Note that

$$\begin{aligned} (\alpha, \Delta\alpha) &= (\alpha, \mathrm{d}\mathrm{d}^*\alpha) + (\alpha, \mathrm{d}^*\mathrm{d}\alpha) \\ &= \|\mathrm{d}^*\alpha\|^2 + \|\mathrm{d}\alpha\|^2 \end{aligned}$$

□

5.2. **Hodge star operator.** Although we have defined an inner product on Ω^k_M, it's still quite difficult to compute. Hodge star operator gives us an effective method to compute.

5.2.1. *Baby case.* Recall that for a $\mathbb{F}$-vector space V with inner product $\langle\ ,\ \rangle$, $\{e_1, \ldots, e_n\}$ is a orthnormal basis of V. For any $0 \leq k \leq n$, there is a natural basis of $\bigwedge^k V$, consisting of $\{e_I := e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$. We also use I^c to denote $[n] - I = \{i'_1, \ldots, i'_{n-k}\}$.

Remark 5.2.1. Here are two special cases:
1. For $k = 0$, we regard $\bigwedge^0 V^k$ as base field $\mathbb{F}$, and $e_I = 1$.
2. For $k = n$, we use vol to denote its basis $e_1 \wedge \cdots \wedge e_n$.

Furthermore, there is an inner product on $\bigwedge^k V$, induced by $\langle\ ,\ \rangle$, given by

$$\langle e_{i_1} \wedge \cdots \wedge e_{i_k}, e_{j_1} \wedge \cdots \wedge e_{j_k} \rangle = \det \begin{pmatrix} \langle e_{i_1}, e_{j_1} \rangle & \ldots & \langle e_{i_1}, e_{j_k} \rangle \\ \ldots & \ldots & \ldots \\ \langle e_{i_k}, e_{j_1} \rangle & \ldots & \langle e_{i_k}, e_{j_k} \rangle \end{pmatrix}$$

Definition 5.2.1 (Hodge star)**.** Hodge star operator is defined as

$$\begin{aligned} \star : \bigwedge^{k} V &\to \bigwedge^{n-k} V \\ e_I &\mapsto \operatorname{sign}(I, I^c) e_{I^c} \end{aligned}$$

where $\operatorname{sign}(I, I^c)$ is the sign of the permutation $(i_1, \ldots, i_k, i'_1, \ldots, i'_{n-k})$.

Example 5.2.1. It's clear $\star 1 = \mathrm{vol}$ and $\star\,\mathrm{vol} = 1$.

Proposition 5.2.1.

$$\star^2 = (-1)^{k(n-k)} \operatorname{id}, \quad \text{on } \bigwedge^{k} V$$

Proof. It suffices to check on basis e_I as follows

$$\begin{aligned} \star^2 e_I &= \star(\operatorname{sign}(I, I^c) e_{I^c}) \\ &= \operatorname{sign}(I, I^c) \operatorname{sign}(I^c, I) e_I \\ &= (-1)^{k(n-k)} e_I \end{aligned}$$

□

Proposition 5.2.2. For $u \in \bigwedge^k V, v \in \bigwedge^{n-k} V$, we have

$$\star(u \wedge v) = (-1)^{k(n-k)} \langle u, \star v \rangle$$

Proof. It suffices to check on basis $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}, e_J = e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}$. Furthermore, it's clear $e_I \wedge e_J = 0$, if $J \neq I^c$, so we may assume $J = I^c$.

$$\begin{aligned} \star(e_I \wedge e_{I^c}) &= \star(\operatorname{sign}(I, I^c) \,\mathrm{vol}) \\ &= \operatorname{sign}(I, I^c) \\ \langle e_I, \star e_{I^c} \rangle &= \langle e_I, \operatorname{sign}(I, I^c) e_I \rangle \\ &= \operatorname{sign}(I, I^c) \langle e_I, e_I \rangle \\ &= \operatorname{sign}(I, I^c) \end{aligned}$$

□

Corollary 5.2.1. For $u, v \in \bigwedge^k V$, we have

$$\begin{aligned} u \wedge \star v = v \wedge \star u &= \langle u, v\rangle \operatorname{vol} \\ \langle \star u, \star v\rangle &= \langle u, v\rangle \end{aligned}$$

Proof. For the first part:

$$\star(u \wedge \star v) = (-1)^{k(n-k)}\langle u, \star^2 v\rangle = \langle u, v\rangle$$

which implies $u \wedge \star v = \langle u, v\rangle$. Since $\langle u, v\rangle = \langle v, u\rangle$, we obtain $u \wedge \star v = v \wedge \star u$.

For the second part:

$$\begin{aligned} \langle \star u, \star v\rangle &= (-1)^{k(n-k)} \star (\star u \wedge v) \\ &= (-1)^{2k(n-k)} \star (v \wedge \star u) \\ &= (-1)^{3k(n-k)}\langle v \wedge \star^2 u\rangle \\ &= (-1)^{4k(n-k)}\langle v, u\rangle \\ &= \langle u, v\rangle \end{aligned}$$

□

Remark 5.2.2. Here are two remarks about this corollary:

1. First part gives us a method to compute inner product, some authors also use this method to denote Hodge star operator;
2. Second part implies that Hodge star operator is an isometry between $\bigwedge^k V$ and $\bigwedge^{n-k} V$.

Corollary 5.2.2 (almost self-adjoint)**.** For $u \in \bigwedge^k V, v \in \bigwedge^{n-k} V$, we have

$$\langle u, \star v\rangle = (-1)^{k(n-k)}\langle \star u, v\rangle$$

Proof.

$$\begin{aligned} \langle u, \star v\rangle &= \langle \star u, \star^2 v\rangle \\ &= (-1)^{k(n-k)}\langle \star u, v\rangle \end{aligned}$$

□

Remark 5.2.3. This corollary implies the adjoint operator of $\star$ is $(-1)^{k(n-k)}\star$, so here I call it almost self-adjoint.

Since locally we always can choose a orthnormal basis $\{\xi_1, \dots, \xi_n\}$ of TM thus there is a dual basis $\{\xi^1, \dots, \xi^n\}$ which is also orthnormal on T^*M locally. So we can define Hodge star operator on Riemannian manifold locally as follows

$$\begin{aligned} \star : \Omega^k_M &\to \Omega^{n-k}_M \\ v_I \xi^I &\mapsto v_I \operatorname{sign}(I, I^c)\xi^{I_c} \end{aligned}$$

Theorem 5.2.1. Properties of Hodge star operator:

1. $\star 1 = \operatorname{vol}, \star \operatorname{vol} = 1$;
2. $\star^2 = (-1)^{k(n-k)}$ on Ω^k_M;

3. If u is a k-form and v a $(n-k)$-form, then

$$\star(u \wedge v) = (-1)^{k(n-k)} \langle u, \star v \rangle$$
$$\langle u, \star v \rangle = (-1)^{k(n-k)} \langle \star u, v \rangle$$

4. For any two k-forms u, v, then

$$u \wedge \star v = v \wedge \star u = \langle u, v \rangle \operatorname{vol} = \langle v, u \rangle \operatorname{vol}$$
$$\langle \star u, \star v \rangle = \langle u, v \rangle$$

5. $\mathrm{d}^* = (-1)^{nk+n+1} \star \mathrm{d} \star$ on Ω_M^k

Remark 5.2.4. (4) allows us to give a new expression for inner product (φ, ψ), where φ, ψ are two k-forms, that is

$$(\varphi, \psi) := \int_M \langle \varphi, \psi \rangle \operatorname{vol} = \int_M \varphi \wedge \star \psi$$

Proof. It suffices to check (5), other cases we have already solved in the case of linear algebra. Take any $(k-1)$-form α and k-form β, we need to show

$$(\mathrm{d}\alpha, \beta) = (\alpha, \mathrm{d}^* \beta)$$

that is to show

$$\int_M \mathrm{d}\alpha \wedge \star \beta = \int_M \alpha \wedge \star \mathrm{d}^* \beta$$

By Stokes theorem and Leibniz rule we have

$$0 = \int_M \mathrm{d}(\alpha \wedge \star \beta) = \int_M \mathrm{d}\alpha \wedge \star \beta + (-1)^{k-1} \int_M \alpha \wedge \mathrm{d} \star \beta$$

Since $\star^2 = (-1)^{(n-k+1)(k-1)}$ on $(n-k+1)$-forms, then

$$(-1)^{k-1} \int_M \alpha \wedge \mathrm{d} \star \beta = (-1)^{k-1+(n-k+1)(k-1)} \int_M \alpha \wedge \star^2 \mathrm{d} \star \beta$$

Therefore

$$\begin{aligned}
(\mathrm{d}\alpha, \beta) &= \int_M \mathrm{d}\alpha \wedge \star \beta \\
&= (-1)^{k+(n-k+1)(k-1)} \int_M \alpha \wedge \star \star \mathrm{d} \star \beta \\
&= (-1)^{nk+k+1} \int_M \alpha \wedge \star(\star \mathrm{d} \star \beta)
\end{aligned}$$

which implies

$$\mathrm{d}^* \beta = (-1)^{nk+k+1} \star \mathrm{d} \star \beta$$

□

5.2.2. *General case.* Although above definition gives us a neat way to compute Hodge star, it lost information about how does Hodge star depend on our Riemannian metric, and it's fatal when we not only consider computations about linear algebra, but taking derivatives.

Proposition 5.2.3.

$$\star(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}) = \frac{\sqrt{\det g}}{(n-k)!} g^{i_1 j_1} \dots g^{i_k j_k} \varepsilon_{j_1 \dots j_n} \mathrm{d}x^{j_{k+1}} \wedge \cdots \wedge \mathrm{d}x^{j_n}$$

where $\{\mathrm{d}x^1, \dots, \mathrm{d}x^n\}$ is a local basis of T^*M and $\varepsilon_{j_1 \cdots j_n}$ is Levi-Civita symbol.

5.2.3. *Some computations.*

Example 5.2.2. For a 1-form ω, we write it in normal coordinates as $\omega_i \mathrm{d}x^i$, then

$$\begin{aligned}
\mathrm{d}^*\omega &= -\star \mathrm{d} \star (\omega_i \mathrm{d}x^i) \\
&= -\star \mathrm{d}(\sum_{i=1}^n (-1)^{i-1} \omega_i \mathrm{d}x^1 \wedge \dots \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n) \\
&= -\star (\sum_{i=1}^n \frac{\partial \omega_i}{\partial x^i} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n) \\
&= -\sum_{i=1}^n \frac{\partial \omega_i}{\partial x^i}
\end{aligned}$$

Example 5.2.3. For a n-form ω, we write it in normal coordinates as $f\,\mathrm{vol}$, then

$$\begin{aligned}
\mathrm{d}^*\omega &= (-1)^n \star \mathrm{d} \star (f\,\mathrm{vol}) \\
&= (-1)^n \star \mathrm{d}f \\
&= (-1)^n \star (\frac{\partial f}{\partial x^i} \mathrm{d}x^i) \\
&= \sum_{i=1}^n (-1)^{n+i-1} \frac{\partial f}{\partial x^i} \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n
\end{aligned}$$

Example 5.2.4. For a smooth function f, let's see $\Delta f := (\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})f$ as follows

$$\begin{aligned}
\Delta f &= (\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})f \\
&= \mathrm{d}^*\mathrm{d}f \\
&= \mathrm{d}^*(\frac{\partial f}{\partial x^i} \mathrm{d}x^i) \\
&= -\sum_{i=1}^n \frac{\partial^2 f}{\partial x^i \partial x^i}
\end{aligned}$$

So you can see why we call $\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$ Laplacian.

Definition 5.2.2 (divergence)**.** Given a Riemannian manifold (M,g). For any vector field X, its divergence $\operatorname{div}(X)$ is defined as $\operatorname{tr}\nabla X$.

Remark 5.2.5 (local form). If we locally write X as $X^i\frac{\partial}{\partial x^i}$, then

$$\nabla X=\nabla_i X^j \mathrm{d}x^i\otimes\frac{\partial}{\partial x^j}$$

Then

$$\operatorname{tr}\nabla X=\nabla_i X^i$$

Lemma 5.2.1.

$$\operatorname{div}(X)\operatorname{vol}=\mathscr{L}_X\operatorname{vol}$$

Proof. Cartan's magic formula shows that

$$\mathscr{L}_X=i_X\circ\mathrm{d}+\mathrm{d}\circ i_X$$

So

$$\begin{aligned}\mathscr{L}_X\operatorname{vol}&=(i_X\circ\mathrm{d}+\mathrm{d}\circ i_X)\operatorname{vol}\\&=\mathrm{d}\circ i_X\operatorname{vol}\\&=\mathrm{d}((-1)^{i-1}X^i\sqrt{\det g}\mathrm{d}x^1\wedge\cdots\wedge\widehat{\mathrm{d}x^i}\wedge\cdots\wedge\mathrm{d}x^n)\\&=\frac{1}{\sqrt{\det g}}\frac{\partial(X^i\sqrt{\det g})}{\partial x^i}\operatorname{vol}\\&=\frac{1}{\sqrt{\det g}}(\frac{\partial X^i}{\partial x^i}\sqrt{\det g}+X^i\frac{\partial\sqrt{\det g}}{\partial x^i})\operatorname{vol}\\&=(\frac{\partial X^i}{\partial x^i}+X^i\frac{\partial\log\sqrt{\det g}}{\partial x^i})\operatorname{vol}\\&=(\frac{\partial X^i}{\partial x^i}+\frac{1}{2}X^i\frac{\partial\log\det g}{\partial x^i})\operatorname{vol}\end{aligned}$$

Note that Jacobi's formula implies: For a function $(a_{ij}(t))$ valued in $\mathrm{GL}(n,\mathbb{R})$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\det(a_{ij}(t))=\det(a_{ij}(t))a^{ij}(t)\frac{\mathrm{d}a_{ij}(t)}{\mathrm{d}t}$$

where $(a^{ij}(t))$ is the inverse matrix of $(a_{ij}(t))$.

So

$$\frac{\partial\log\det g}{\partial x^i}=\frac{1}{\det g}\frac{\partial\det g}{\partial x^i}=g^{jk}\frac{\partial g_{jk}}{\partial x^i}=g^{jk}(\Gamma^l_{ij}g_{lk}+\Gamma^l_{ik}g_{jl})=2\Gamma^j_{ij}$$

Thus

$$\begin{aligned}\mathscr{L}_X\operatorname{vol}&=(\frac{\partial X^i}{\partial x^i}+\frac{1}{2}X^i\frac{\partial\log\det g}{\partial x^i})\operatorname{vol}\\&=(\frac{\partial X^i}{\partial x^i}+\Gamma^j_{ij}X^i)\operatorname{vol}\\&=(\frac{\partial X^i}{\partial x^i}+\Gamma^i_{ij}X^j)\operatorname{vol}\\&=\nabla_i X^i\operatorname{vol}\end{aligned}$$

$\square$

Proposition 5.2.4. If X_ω is the dual vector field of 1-form ω, then

$$\mathrm{d}^*\omega = -\operatorname{div}(X_\omega)$$

Proof. In normal coordinates we have $X_\omega = \omega_i \frac{\partial}{\partial x^i}$, and its clear

$$\operatorname{div} X_\omega = \sum_{i=1}^n \frac{\partial \omega_i}{\partial x^i}$$

from Lemma 5.2.1. Thus $-\operatorname{div}(X_\omega)$ coincides result we have seen in Example 5.2.2. This completes the proof. □

5.3. **Conformal Laplacian.** For a smooth function u, we can write Laplace-Beltrami operator Δu as follows

$$\begin{aligned}\Delta u &= \mathrm{d}^*\mathrm{d}u \\ &= -\operatorname{div}(g^{ij}\frac{\partial u}{\partial x^i}\frac{\partial}{\partial x^j}) \\ &= -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^j}(\sqrt{\det g}g^{ij}\frac{\partial u}{\partial x^i})\end{aligned}$$

Thus Laplace-Beltrami Δ_g with respect to g

$$\Delta_g = -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^j}(\sqrt{\det g}g^{ij}\frac{\partial}{\partial x^i})$$

So if we consider conformal transformation $\widetilde{g} = e^{2f}g$ for some smooth function f, we have

$$\begin{aligned}\widetilde{g}_{ij} &= e^{2f}g_{ij} \\ \widetilde{g}^{ij} &= e^{-2f}g^{ij} \\ \det\widetilde{g} &= e^{2nf}\det g \\ \sqrt{\widetilde{g}} &= e^{nf}\sqrt{\det g}\end{aligned}$$

Thus

$$\begin{aligned}\Delta_{\widetilde{g}} &= -\frac{1}{e^{nf}\sqrt{\det g}}\frac{\partial}{\partial x^j}(e^{nf}\sqrt{\det g}e^{-2f}g^{ij}\frac{\partial}{\partial x^i}) \\ &= -\frac{e^{-nf}}{\sqrt{\det g}}\frac{\partial}{\partial x^j}(e^{(n-2)f}\sqrt{\det g}g^{ij}\frac{\partial}{\partial x^i}) \\ &= -\frac{e^{-2f}}{\sqrt{\det g}}\frac{\partial}{\partial x^j}(\sqrt{\det g}g^{ij}\frac{\partial}{\partial x^i}) - \frac{(n-2)e^{-2f}}{\sqrt{\det g}}\frac{\partial f}{\partial x^j}\sqrt{\det g}g^{ij}\frac{\partial}{\partial x^i} \\ &= -e^{-2f}\Delta_g - (n-2)e^{-2f}g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^i}\end{aligned}$$

So we have

$$\Delta_{\widetilde{g}} = -e^{-2f}\Delta_g$$

when $n=2$. It's a kind of conformal invariance. However this fails in higher dimension. Let's consider the following so-called conformal Laplacian when $n>3$

$$\begin{aligned} L: C^\infty(M) &\to C^\infty(M) \\ u &\mapsto -\frac{4(n-1)}{n-2}\Delta_g u + Su \end{aligned}$$

where S is scalar curvature. Let's show

$$\widetilde{L}u = e^{-\frac{n+2}{2}f}L(e^{\frac{n-2}{2}f}u)$$

where $\widetilde{L}$ is the conformal Laplacian after conformal transformation. Divide computations into several parts:

(1)

$$\begin{aligned} \nabla^2(e^{\frac{n-2}{2}f}u) =& \nabla(\frac{n-2}{2}e^{\frac{n-2}{2}f}\frac{\partial f}{\partial x^i}u\mathrm{d}x^i + e^{\frac{n-2}{2}f}\frac{\partial u}{\partial x^i}\mathrm{d}x^i) \\ =& e^{\frac{n-2}{2}f}\nabla^2 u + \frac{n-2}{2}e^{\frac{n-2}{2}f}\frac{\partial f}{\partial x^j}\frac{\partial u}{\partial x^i}\mathrm{d}x^i\otimes\mathrm{d}x^j \\ &+ (\frac{(n-2)^2}{4}e^{\frac{n-2}{2}f}u\frac{\partial f}{\partial x^j}\frac{\partial f}{\partial x^i} + \frac{n-2}{2}e^{\frac{n-2}{2}f}\frac{\partial f}{\partial x^i}\frac{\partial u}{\partial x^j})\mathrm{d}x^i\otimes\mathrm{d}x^j + \frac{n-2}{2}e^{\frac{n-2}{2}f}u\nabla^2 f \end{aligned}$$

(2)

$$\begin{aligned} \Delta_g(e^{\frac{n-2}{2}f}u) =& \operatorname{tr}_g\nabla^2(e^{\frac{n-2}{2}f}u) \\ =& e^{\frac{n-2}{2}f}\Delta_g u + \frac{n-2}{2}e^{\frac{n-2}{2}f}g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial u}{\partial x^i} \\ &+ g^{ij}(\frac{(n-2)^2}{4}e^{\frac{n-2}{2}f}u\frac{\partial f}{\partial x^j}\frac{\partial f}{\partial x^i} + \frac{n-2}{2}e^{\frac{n-2}{2}f}\frac{\partial f}{\partial x^i}\frac{\partial u}{\partial x^j}) + \frac{n-2}{2}e^{\frac{n-2}{2}f}u\Delta_g f \end{aligned}$$

(3)

$$\begin{aligned} e^{-\frac{n+2}{2}f}L(e^{\frac{n-2}{2}f}u) =& -\frac{4(n-1)}{n-2}e^{-2f}\Delta_g u - 4(n-1)e^{-2f}g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial u}{\partial x^i} \\ &- g^{ij}(n-2)(n-1)e^{-2f}u\frac{\partial f}{\partial x^j}\frac{\partial f}{\partial x^i} - 2(n-1)e^{-2f}u\Delta_g f + e^{-2f}Su \\ =& -\frac{4(n-1)}{n-2}e^{-2f}\Delta_g u - 4(n-1)e^{-2f}g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial u}{\partial x^i} \\ &- (n-2)(n-1)e^{-2f}u|\mathrm{d}f|^2 - 2(n-1)e^{-2f}u\Delta_g f + e^{-2f}Su \end{aligned}$$

(4)

$$-\frac{4(n-1)}{n-2}\Delta_{\widetilde{g}}u = -\frac{4(n-1)}{n-2}e^{-2f}\Delta_g u - 4(n-1)e^{-2f}g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial u}{\partial x^i}$$

(5) Note that

$$\widetilde{S} = e^{-2f}S - 2(n-1)e^{-2f}\Delta_g f - (n-2)(n-1)e^{-2f}|\mathrm{d}f|^2$$

This completes the computation. In particular, in (2) if we take $u = 1$ we have

$$-\frac{4(n-1)}{n-2}\Delta_g(e^{\frac{n-2}{2}f}) = -(n-2)(n-1)e^{\frac{n-2}{2}f}|\mathrm{d}f|^2 - 2(n-1)e^{\frac{n-2}{2}f}\Delta_g f$$

Thus we have

$$\widetilde{S} = e^{-\frac{n+2}{2}f}(-\frac{4(n-1)}{n-2}\Delta_g e^{\frac{n-2}{2}f} + Se^{\frac{n-2}{2}f}) = e^{-\frac{n+2}{2}f}L(e^{\frac{n-2}{2}f})$$

So if we put $e^{2f} = \varphi^{\frac{4}{n-2}}$, we have

$$\widetilde{S} = \varphi^{-\frac{n+2}{n-2}}L\varphi$$

So it's clear g is conformal to $\widetilde{g}$ with constant scalar curvature λ if and only if φ is a smooth positive solution to the Yamabe equation

$$L\varphi = \lambda\varphi^{\frac{n+2}{n-2}}$$

5.4. **Hodge theorem and corollaries.**

Theorem 5.4.1 (Hodge theorem)**.** Consider the Laplace operator $\Delta : C^\infty(M, \Omega^k_M) \to C^\infty(M, \Omega^k_M)$, then

1. $\dim_{\mathbb{R}} \mathcal{H}^k(M) < \infty$;
2. There is an orthogonal direct sum decomposition

$$C^\infty(M, \Omega^k_M) = \mathcal{H}^k(M) \oplus \operatorname{im}\Delta$$

Proof. See Appendix A. □

Corollary 5.4.1. More explicitly, we have the following orthogonal decomposition

$$C^\infty(M, \Omega^k_M) = \mathcal{H}^k(M) \oplus \mathrm{d}(C^\infty(M, \Omega^{k-1}_M)) \oplus \mathrm{d}^*(C^\infty(M, \Omega^{k+1}_M))$$

Proof. It suffices to check $\mathrm{d}(C^\infty(M, \Omega^{k-1}_M))$ is orthogonal to $\mathrm{d}^*(C^\infty(M, \Omega^{k+1}_M))$. Take $\mathrm{d}\alpha$ and $\mathrm{d}^*\beta$, where α is a $k-1$-form and β is a $k+1$-form. Then

$$(\mathrm{d}\alpha, \mathrm{d}^*\beta) = (\mathrm{d}^2\alpha, \beta) = 0$$

□

Corollary 5.4.2.

$$\ker \mathrm{d} = \mathcal{H}^k(M) \oplus \mathrm{d}(C^\infty(M, \Omega^{k-1}_M))$$
$$\ker \mathrm{d}^* = \mathcal{H}^k(M) \oplus \mathrm{d}^*(C^\infty(M, \Omega^{k+1}_M))$$

Proof. Clear from above corollary. □

Corollary 5.4.3. The natural map $\mathcal{H}^k(M) \to H^k(M, \mathbb{R})$ is an isomorphism. In other words, every element in $H^k(M, \mathbb{R})$ is represented by a unique harmonic form.

Proof. Clear from above corollary. □

Corollary 5.4.4. $\star : \mathcal{H}^k(M) \to \mathcal{H}^{n-k}(M)$ is an isomorphism.

Proof. It suffices to show $*$ maps harmonic forms to harmonic forms, since we already have $\star$ maps k-forms to k-forms By Lemma 5.1.2, we just need to show $\mathrm{d}\star\alpha = \mathrm{d}^*\star\alpha = 0$ for a harmonic form α. Directly compute as follows

$$\mathrm{d}\star\alpha = (-1)^{\bullet_1}\star\star\mathrm{d}\star\alpha = (-1)^{\bullet_2}\star\mathrm{d}^\star\alpha = 0$$
$$\mathrm{d}^*\star\alpha = (-1)^{\bullet_3}\star\mathrm{d}\star\star\alpha = (-1)^{\bullet_4}\star\mathrm{d}\alpha = 0$$

Here we use $\bullet, \bullet'$ to denote the power of (-1), since it's not neccessary for us to know what exactly it is. □

Remark 5.4.1. In fact, above corollary follows from the following identity

$$\Delta\star = \star\Delta$$

which can be directly checked. In other words, Hodge star commutes with Laplacian Δ. Here gives a method of computation: From what we have done in the proof, we will see

$$\star\mathrm{d}^*\mathrm{d} = (-1)^{\bullet_2}\mathrm{d}\star\mathrm{d} = (-1)^{\bullet_2+\bullet_4}\mathrm{d}\mathrm{d}^*\star$$
$$\star\mathrm{d}\mathrm{d}^* = (-1)^{\bullet_4}\mathrm{d}^*\star\mathrm{d}^* = (-1)^{\bullet_2+\bullet_4}\mathrm{d}^*\mathrm{d}\star$$

So all we need to do is to figure out the precise number of $\bullet_2, \bullet_4$ and show that $\bullet_2 + \bullet_4$ is even.

Corollary 5.4.5 (Poincaré duality)**.** $H^k(M,\mathbb{R}) \cong H^{n-k}(M,\mathbb{R})$.

Proof. Clear from Corollary 5.4.3 and Corollary 5.4.4. □

6. Bochner technique

6.1. **Hessian of smooth function.** Let (M,g) be a Riemannian manifold, ∇ is a Levi-Civita connection. Given a smooth function $f: M\to\mathbb{R}$, $\operatorname{Hess} f := \nabla^2 f$ is a $(0,2)$-tensor.

Remark 6.1.1 (local form). Locally we have

$$\begin{aligned}
\operatorname{Hess} f &= \nabla(\frac{\partial f}{\partial x^j}\mathrm{d}x^j)\\
&= \frac{\partial^2 f}{\partial x^i\partial x^j}\mathrm{d}x^i\otimes\mathrm{d}x^j + \frac{\partial f}{\partial x^j}\nabla\mathrm{d}x^j\\
&= \frac{\partial^2 f}{\partial x^i\partial x^j}\mathrm{d}x^i\otimes\mathrm{d}x^j - \frac{\partial f}{\partial x^j}\Gamma_{ik}^j\mathrm{d}x^i\otimes\mathrm{d}x^k\\
&= (\frac{\partial^2 f}{\partial x^i\partial x^j} - \Gamma_{ij}^k\frac{\partial f}{\partial x^k})\mathrm{d}x^i\otimes\mathrm{d}x^j
\end{aligned}$$

that is

$$\nabla_i\nabla_j f = \frac{\partial^2 f}{\partial x^i\partial x^j} - \Gamma_{ij}^k\frac{\partial f}{\partial x^k}$$

Definition 6.1.1 (scalar Laplacian)**.**

$$\Delta f = \operatorname{tr}_g(\operatorname{Hess} f) = g^{ij}\nabla_i\nabla_j f$$

Remark 6.1.2. It's easy to check scalar Laplacian and Laplace-Beltrami differ a sign, that is

$$\Delta f = -\Delta_g f = -(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)f$$

Theorem 6.1.1. Let $f:(M,g)\to\mathbb{R}$

1. $p\in M$ is a local minimum(maximum), then $\nabla f(p)=0$;
2. $p\in M$ is a local minimum, then

$$\begin{cases}\operatorname{Hess} f(p)\geq 0\\ \Delta_g f(p)\geq 0\end{cases}$$

3. $p\in M$ is a local maximum, then

$$\begin{cases}\operatorname{Hess} f(p)\leq 0\\ \Delta_g f(p)\leq 0\end{cases}$$

Lemma 6.1.1. Let $f: M\to\mathbb{R}$ be a smooth function, then

$$\frac{1}{2}\Delta_g|\nabla f|_g^2 = |\operatorname{Hess} f|_g^2 + \operatorname{Ric}(\nabla f,\nabla f) + g(\nabla\Delta_g f,\nabla f)$$

Proof. We write ∇f as its dual vector field, that is $\nabla f = g^{ij}\nabla_i f\frac{\partial}{\partial x^j}$. Thus

$$\begin{aligned}
|\nabla f|_g^2 &= g(\nabla f,\nabla f)\\
&= g(g^{ij}\nabla_i f\frac{\partial}{\partial x^j}, g^{kl}\nabla_k f\frac{\partial}{\partial x^l})\\
&= g^{ij}g^{kl}g_{jl}\nabla_i f\nabla_k f\\
&= g^{ij}\nabla_i f\nabla_j f
\end{aligned}$$

By magic formula we have

$$\begin{aligned}\frac{1}{2}\Delta_g|\nabla f|_g^2 &= \frac{1}{2}g^{kl}\nabla_k\nabla_l(g^{ij}\nabla_i f\nabla_j f)\\ &= \frac{1}{2}g^{kl}g^{ij}\nabla_k\nabla_l(\nabla_i f\nabla_j f)\\ &= g^{kl}g^{ij}\nabla_l\nabla_i f\cdot\nabla_k\nabla_j f + g^{kl}g^{ij}\nabla_k\nabla_l\nabla_i f\cdot\nabla_j f\\ &= |\operatorname{Hess} f|_g^2 + g^{kl}g^{ij}\nabla_k\nabla_l\nabla_i f\cdot\nabla_j f\end{aligned}$$

Note that

$$\begin{aligned}g^{kl}g^{ij}\nabla_k\nabla_l\nabla_i f\cdot\nabla_j f &= g^{kl}g^{ij}\nabla_k\nabla_i\nabla_l f\cdot\nabla_j f\\ &= g^{kl}g^{ij}(\nabla_i\nabla_k\nabla_l f - R^s_{kil}\nabla_s f)\cdot\nabla_j f\\ &= g^{ij}\nabla_i(g^{kl}\nabla_k\nabla_l f)\nabla_j f + g^{ij}R^s_i\nabla_s f\nabla_j f\\ &= g^{ij}\nabla_i(\Delta_g f)\nabla_j f + \operatorname{Ric}(\nabla f,\nabla f)\\ &= g(\nabla\Delta_g f,\nabla f) + \operatorname{Ric}(\nabla f,\nabla f)\end{aligned}$$

□

6.2. **Killing field and harmonic form.**

6.2.1. *Killing field.*

Definition 6.2.1 (Killing vector field)**.** A vector field X on a Riemannian manifold (M,g) is called a Killing vector field, if $\mathscr{L}_X g = 0$.

Remark 6.2.1 (Lie derivative). Recall that for any 1-form ω and vector field X, Y, we have

$$\begin{aligned}\mathscr{L}_X Y &= [X,Y]\\ (\mathscr{L}_X\omega)(Y) &= X(\omega(Y)) = \omega([X,Y])\end{aligned}$$

More generally, for two 1-form ω_1,ω_2, we have

$$\mathscr{L}_X(\omega_1\otimes\omega_2) = (\mathscr{L}_X\omega_1)\otimes\omega_2 + \omega_1\otimes(\mathscr{L}_X\omega_2)$$

Theorem 6.2.1. The followings are equivalent:

1. X is a Killing field;
2. For any two vector fields Y, Z, we have
$$\langle\nabla_Y X, Z\rangle + \langle Y,\nabla_Z X\rangle = 0$$
3. $\mathscr{L}_X$ commutes with Laplacian Δ on smooth functions.

Proof. To see (1) is equivalent to (2). Note that

$$\begin{aligned}\mathscr{L}_X\langle Y,Z\rangle &= X\langle Y,Z\rangle - \langle\mathscr{L}_X Y,Z\rangle - \langle Y,\mathscr{L}_X Z\rangle\\ &= \langle\nabla_X Y,Z\rangle + \langle Y,\nabla_X Z\rangle - \langle[X,Y],Z\rangle - \langle Y,[X,Z]\rangle\\ &= \langle\nabla_Y X,Z\rangle + \langle Y,\nabla_Z X\rangle\end{aligned}$$

so by definition we have (1) is equivalent to (2).

To see (1) is equivalent to (3). Note that Lie derivative commutes with d, so it suffices to check $\mathscr{L}_X$ commutes with hodge star $\star$ if and only if X is Killing, since $\Delta = \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$ and d^* can be expressed by d and $\star$.

□

Remark 6.2.2. For (2) locally we have

$$g_{kj}\nabla_i X^j = -g_{ij}\nabla_k X^j$$

Thus X is a Killing vector if and only if ∇X is a skew-symmetric $(1,1)$-tensor, that is $\nabla_i X^j$ is skew-symmetric in i, j.

Corollary 6.2.1. If X is a Killing field, then for arbitrary vector field Y we have

$$\langle \nabla_Y X, Y\rangle = 0$$

Proof. Set $Y = Z$ in $\langle \nabla_Y X, Z\rangle + \langle Y, \nabla_Z X\rangle = 0$. □

Corollary 6.2.2. If X is parallel, that is $\nabla X = 0$, then X is Killing.

Proof. A zero matrix must be skew-symmetric. □

Corollary 6.2.3. If X is Killing, then $\operatorname{div} X = \nabla_i X^i = 0$.

Proof. The trace of a skew-symmetric matrix is zero. □

Lemma 6.2.1. Suppose X is a Killing field, and $f = \frac{1}{2}|X|_g^2$. Then

1. $\nabla f = -\nabla_X X$;
2. $\operatorname{Hess} f(V, V) = \langle \nabla_V X, \nabla_V X\rangle - R(V, X, X, V)$ holds for any vector field V;
3. $\Delta_g f = |\nabla X|_g^2 - \operatorname{Ric}(X, X)$.

Proof. For (1). By direct computation we have

$$\begin{aligned}
\nabla f &= \langle \nabla X, X\rangle \\
&= \langle \nabla_k X^i \mathrm{d}x^k \otimes \frac{\partial}{\partial x^i}, X^j \frac{\partial}{\partial x^j}\rangle \\
&= g_{ij} X^j \nabla_k X^i \mathrm{d}x^k \\
&= -g_{ik} X^j \nabla_k X^i \mathrm{d}x^k \\
\nabla_X X &= X^j \nabla_k X^i \frac{\partial}{\partial x^i} \\
&= g_{ik} X^j \nabla_k X^i \mathrm{d}x^k
\end{aligned}$$

For (2). By direct computation we have

$$\begin{aligned}
\operatorname{Hess} f(V,V) &= V^pV^q\nabla_p\nabla_q f \\
&= \frac{1}{2}V^pV^q\nabla_p\nabla_q(g_{ij}X^iX^j) \\
&= g_{ij}V^pV^q\nabla_pX^i\nabla_qX^j + V^pV^q\nabla_p g_{ij}\nabla_qX^iX^j \\
&= g(\nabla_VX,\nabla_VX) - V^pV^q\nabla_p g_{iq}\nabla_jX^iX^j \\
&= |\nabla_VX|_g^2 - V^pV^qX^ig_{iq}(\nabla_j\nabla_pX^i + R^i_{pjm}X^m) \\
&= |\nabla_VX|_g^2 - R(V,X,X,V)
\end{aligned}$$

Note that $g_{iq}\nabla_j\nabla_pX^iX^j = 0$, since this expression is skew symmetric in p,q. And (3) holds from (2) directly. □

Theorem 6.2.2 (Bochner)**.** Let (M,g) be a compact Riemannian manifold with $\operatorname{Ric}(g) < 0$, then (M,g) has no non-trivial Killing field.

Proof. Let X be a Killing field, then by above lemma we have

$$\frac{1}{2}\Delta_g|X|_g^2 = |\nabla X|_g^2 - \operatorname{Ric}(X,X)$$

Suppose $|X|_g^2$ attains its maximum at some point $p\in M$, then $\Delta_g|X|_g^2 \leq 0$. Together with $\operatorname{Ric}(X,X)<0$, we will have

$$|\Delta X|_g^2 = \operatorname{Ric}(X,X) + \frac{1}{2}\Delta_g|X|_g^2 < 0$$

A contradiction. □

Remark 6.2.3. Above proof still holds, if $\operatorname{Ric}(g)\leq 0$ everywhere and $\operatorname{Ric}(g) < 0$ at some point. This curvature condition is called quasi-negative.

6.2.2. *Harmonic* 1*-form.* To some extend, Killing field is dual to harmonic 1-form. Let's explain this in more detail.

Lemma 6.2.2. For a harmonic 1-form α, locally written as $\alpha_i\mathrm{d}x^i$, we have

$$\begin{aligned}
\nabla_i\alpha_j &= \nabla_j\alpha^i \\
g^{ij}\nabla_j\alpha_i &= 0
\end{aligned}$$

Proof. Recall α is harmonic if and only if

$$\begin{aligned}
\mathrm{d}\alpha &= 0 \\
\mathrm{d}^*\alpha &= 0
\end{aligned}$$

It's clear

$$\mathrm{d}(\alpha_j\mathrm{d}x^j) = \nabla_i\alpha_j\mathrm{d}x^i\wedge\mathrm{d}x^j = 0$$

implies $\nabla_i\alpha_j = \nabla_j\alpha_i$. Similarly explicit expression for d^* implies the second identity. □

Remark 6.2.4. Recall Killing field implies $g_{ij}\nabla_kX^j$ is skew-symmetric in i,k, we can see both Killing field and harmonic 1-form implies some (skew)symmetries.

Lemma 6.2.3. If α is a harmonic 1-form, then

$$\frac{1}{2}\Delta_g|\alpha|^2 = |\nabla\alpha|^2 + \mathrm{Ric}(X_\alpha, X_\alpha)$$

where X_α is the dual vector field of α.

Proof. Routine computation as follows:

$$\begin{aligned}
\frac{1}{2}\Delta_g|\alpha|_g^2 &= \frac{1}{2}g^{kl}\nabla_k\nabla_l(g^{ij}\alpha_i\alpha_j) \\
&= |\nabla\alpha|^2 + g^{kl}g^{ij}\nabla_k\nabla_l\alpha_i\cdot\alpha_j \\
&= |\nabla\alpha|_g^2 + g^{kl}g^{ij}\nabla_k\nabla_i\alpha_l\cdot\alpha_j \\
&= |\nabla\alpha|_g^2 + g^{kl}g^{ij}(\nabla_i\nabla_k\alpha_l - R_{kil}^s\alpha_s)\alpha_j \\
&= |\Delta\alpha|_g^2 - g^{kl}g^{ij}R_{kil}^s\alpha_s\cdot\alpha_j \\
&= |\Delta\alpha|_g^2 + \mathrm{Ric}(X_\alpha, X_\alpha)
\end{aligned}$$

□

Theorem 6.2.3 (Bochner)**.** Let (M, g) be a compact Riemannian manifold with $\mathrm{Ric}(g) > 0$, then $b_1(M) = 0$, that is M has no non-trivial harmonic 1-form.

Remark 6.2.5. It's a kind of vanishing theorem.

Part 2. Variation formulas

7. Geodesic II: Variation formulas

In this section, we fix the following notations:

1. $I = [a, b] \subset \mathbb{R}$ is a closed interval;
2. For two different points $p, q \in M$, where (M, g) is a Riemannian manifold, the space of smooth curves from p to q is denoted as
$$\mathscr{L}_{p,q} = \{\text{smooth curve } \gamma : [a, b] \to M, \text{ with } \gamma(a) = p, \gamma(b) = q\}$$
3. For $\gamma \in \mathscr{L}_{p,q}$, we define $\gamma'(t) := \gamma_*(\frac{\mathrm{d}}{\mathrm{d}t}) \in C^\infty(I, \gamma^*TM)$. Note that γ^*TM is equipped with pullback connection $\widehat{\nabla}$ and pullback metric $\widehat{g}$.
4. Consider the following functionals defined on $\mathscr{L}_{p,q}$:
$$L(\gamma) = \int_a^b |\gamma'(t)|\mathrm{d}t$$
$$E(\gamma) = \frac{1}{2}\int_a^b |\gamma'(t)|^2\mathrm{d}t$$
The former is called arc-length functional and the latter is called energy functional.

7.1. First variation formula.

Definition 7.1.1 (variation)**.** Given $\gamma \in \mathscr{L}_{p,q}$, a variation of γ is a smooth map
$$\alpha : [a, b] \times (-\varepsilon, \varepsilon) \to M$$
such that

1. $\alpha(\text{-}, s) \in \mathscr{L}_{p,q}$ for any $s \in (-\varepsilon, \varepsilon)$;
2. $\alpha(t, 0) = \gamma(t)$ for any $t \in [a, b]$.

Remark 7.1.1. For pullback bundle α^*TM, we use $\overline{\nabla}$ and $\overline{g}$ to denote connection and metric pulled back from the ones on TM. By definition we have the restriction of $\overline{\nabla}$ on γ^*TM is exactly $\widehat{\nabla}$, and the restriction of $\overline{g}$ on γ^*TM is $\widehat{g}$.

Definition 7.1.2 (variation vector field)**.** For a variation α of $\gamma \in \mathscr{L}_{p,q}$, $\alpha_*(\frac{\partial}{\partial s})\big|_{s=0} \in C^\infty(I, \gamma^*TM)$ is called variation vector field of variation α.

Remark 7.1.2. Note that
$$\begin{cases} \alpha(a, s) = p \\ \alpha(b, s) = q \end{cases}$$
for any $s \in (-\varepsilon, \varepsilon)$. Thus we have,
$$\begin{cases} \alpha_*(\frac{\partial}{\partial s})(a, s) = 0 \\ \alpha_*(\frac{\partial}{\partial s})(b, s) = 0 \end{cases}$$
for any $s \in (-\varepsilon, \varepsilon)$. In particular it holds for $s = 0$, that's a variation vector field vanishes at endpoints, a crucial fact we need in later computation.

Lemma 7.1.1. Let X be a smooth vector field along γ with $X(a) = X(b) = 0$. Then there exists a variation α of γ such that the variation vector field is exactly X, that is

$$\alpha_*(\frac{\partial}{\partial s})\Big|_{s=0} = X$$

Proof. See Proposition 2.2 in Page193 of [Car92]. □

Remark 7.1.3. Thanks to this technical lemma, we always call a vector field along γ a variation vector field, if it it vanishes at endpoints.

Theorem 7.1.1 (First variation formula)**.** Let $\gamma : [a, b] \to (M, g)$ be a unit-speed curve, α a normal variation of γ and V the variation vector field. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} L(\alpha(\text{-}, s)) \overset{(1)}{=} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} E(\alpha(\text{-}, s)) &\overset{(2)}{=} \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma'(t)\rangle \mathrm{d}t \\ &\overset{(3)}{=} -\int_a^b \langle V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma'(t)\rangle \mathrm{d}t\end{aligned}$$

Proof. Note that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} L(\alpha(\text{-}, s)) &= \int_a^b \frac{1}{2|\gamma'(t)|} \frac{\partial}{\partial s}\Big|_{s=0} |\alpha_*(\frac{\partial}{\partial t})|^2 \mathrm{d}t \\ &= \frac{1}{|\gamma'(t)|} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} E(\alpha(\text{-}, s))\end{aligned}$$

Since γ is unit-speed, this show equality marked by (1).

Note that

$$\begin{aligned}0 &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} \langle V, \gamma'(t)\rangle \mathrm{d}t \\ &= \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma'(t)\rangle \mathrm{d}t + \int_a^b \langle V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma'(t)\rangle \mathrm{d}t\end{aligned}$$

This shows the equality marked by (3).

For equality marked by (2), we compute as follows

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}E(\alpha(\text{-},s)) &= \frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{2}\int_a^b |\alpha_*(\frac{\partial}{\partial t})|^2\mathrm{d}t \\
&= \frac{1}{2}\int_a^b \frac{\partial}{\partial s}|\alpha_*(\frac{\partial}{\partial t})|^2\mathrm{d}t \\
&= \frac{1}{2}\int_a^b 2\langle\overline{\nabla}_{\frac{\partial}{\partial s}}\alpha_*(\frac{\partial}{\partial t}),\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t \\
&\stackrel{\star}{=} \int_a^b \langle\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial s}),\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t \\
&= \int_a^b \frac{\partial}{\partial t}\langle\alpha_*(\frac{\partial}{\partial s}),\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}} - \langle\alpha_*(\frac{\partial}{\partial s}),\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t \\
&= -\int_a^b \langle\alpha_*(\frac{\partial}{\partial s}),\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t
\end{aligned}
$$

The hallmark of above computation is the equality marked by star, which can be seen from follows

$$
\overline{\nabla}_{\frac{\partial}{\partial s}}\alpha_*(\frac{\partial}{\partial t}) = B(\frac{\partial}{\partial s},\frac{\partial}{\partial t}) + \alpha_*(\nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial t})
$$

where B is second fundamental form, and it's symmetric, which can be seen in Appendix B. Thus

$$
\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} E(\alpha(\text{-},s)) = -\int_a^b \langle V,\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\gamma'(t)\rangle_{\widehat{g}}\mathrm{d}t
$$

since $\alpha_*(\frac{\partial}{\partial s})\big|_{s=0} = V$ and $\alpha_*(\frac{\partial}{\partial t})\big|_{s=0} = \gamma'(t)$. $\square$

Corollary 7.1.1. Given $\gamma \in \mathscr{L}_{p,q}$. The followings are equivalent:

1. γ is a critical point of energy functional $E : \mathscr{L}_{p,q} \to \mathbb{R}$;
2. γ has constant speed $|\gamma'(t)| = c > 0$ and γ is a critical point of arc-length functional $L : \mathscr{L}_{p,q} \to \mathbb{R}$;
3. γ is a geodesic.

Proof. From (3) to (2): Firstly a geodesic must have constant speed c, and $c > 0$ since p, q are distinct points. It's also a critical point of L since first variation formula implies

$$
\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} L(\alpha(\text{-},s)) = -\int_a^b \langle V,\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\gamma'(t)\rangle_{\widehat{g}}\mathrm{d}t = 0
$$

From (2) to (1): It's clear, since from above proof we have already seen for constant speed curve, the first variation of arc-length functional and energy functional only differs a scalar.

From (1) to (3): In order to show $\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\gamma'(t) = 0$, the key point is to choose an appropriate variation vector field V to conclude. $\square$

7.2. **Second variation formula.** We already know a geodesic γ is a critical point for energy functional or arc-length functional, so it left to determine whether it's local minimum or not.

To see this, we need to consider the following 2-dimensional variation

$$\alpha:[a,b]\times(-\varepsilon_1,\varepsilon_1)\times(-\varepsilon_2,\varepsilon_2)$$

such that

1. $\alpha(t,0,0)=\gamma(t)$
2. $\alpha(\text{-},s_1,s_2)\in\mathscr{L}_{p,q}$

7.2.1. *Second variation formula for energy.*

Theorem 7.2.1 (second variation formula for energy)**.** Let $\gamma:[a,b]\to(M,g)$ be a smooth curve. If α is a 2-dimensional variation of γ with variation fields V,W. Then

$$\begin{aligned}\frac{\partial^2}{\partial s_1\partial s_2}\Big|_{s_1=s_2=0}E(\alpha(\text{-},s_1,s_2))&=\int_a^b\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V,\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}W\rangle\mathrm{d}t\\&\quad-\int_a^b R(V,\gamma',\gamma',W)\mathrm{d}t-\int_a^b\langle\overline{\nabla}_{\frac{\partial}{\partial s_1}}\alpha_*(\frac{\partial}{\partial s_2})\Big|_{s_1=s_2=0},\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\gamma'(t)\rangle\mathrm{d}t\end{aligned}$$

Proof. By first variation formula we have

$$\frac{\partial}{\partial s_2}E(\alpha(\text{-},s_1,s_2))=-\int_a^b\langle\alpha_*(\frac{\partial}{\partial s_2}),\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t$$

Thus

$$\frac{\partial^2}{\partial s_1\partial s_2}E(\alpha(\text{-},s_1,s_2))=\underbrace{-\int_a^b\langle\overline{\nabla}_{\frac{\partial}{\partial s_1}}\alpha_*(\frac{\partial}{\partial s_2}),\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t}_{\text{part I}}\underbrace{-\int_a^b\langle\alpha_*(\frac{\partial}{\partial s_2}),\overline{\nabla}_{\frac{\partial}{\partial s_1}}\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t}_{\text{part II}}$$

For part II, we have

$$\overline{\nabla}_{\frac{\partial}{\partial s_1}}\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial t})=R(\alpha_*(\frac{\partial}{\partial s_1}),\alpha_*(\frac{\partial}{\partial t}))\alpha_*(\frac{\partial}{\partial t})+\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial s_1}}\alpha_*(\frac{\partial}{\partial t})$$

Thus we can write part II as

$$-\int_a^b\langle\alpha_*(\frac{\partial}{\partial s_2}),R(\frac{\partial}{\partial s_1},\frac{\partial}{\partial t})\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t\underbrace{-\int_a^b\langle\alpha_*(\frac{\partial}{\partial s_2}),\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial s_1}}\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t}_{\text{part III}}$$

For part III, we have

$$\begin{aligned}-\int_a^b\langle\alpha_*(\frac{\partial}{\partial s_2}),\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial s_1}}\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t&=-\int_a^b\langle\alpha_*(\frac{\partial}{\partial s_2}),\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial s_1})\rangle_{\overline{g}}\mathrm{d}t\\&=\int_a^b\langle\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial s_2}),\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial s_1})\rangle_{\overline{g}}\mathrm{d}t\end{aligned}$$

Now let's evaluate at $s_1=s_2=0$, then we have

1. Part I
$$-\int_a^b \langle \overline{\nabla}_{\frac{\partial}{\partial s_1}} \alpha_*(\frac{\partial}{\partial s_2})\Big|_{s_1=s_2=0}, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\gamma'(t)\rangle \mathrm{d}t$$
2. Part II
$$\int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W\rangle \mathrm{d}t - \int_a^b R(V,\gamma',\gamma',W)\mathrm{d}t$$

This completes the proof. □

Corollary 7.2.1. Let $\gamma : [a,b] \to (M,g)$ be a geodesic, then
$$\frac{\partial^2}{\partial s_1 \partial s_2}\Big|_{s_1=s_2=0} E(\alpha(\text{-}, s_1, s_2)) = \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W\rangle \mathrm{d}t - \int_a^b R(V,\gamma',\gamma',W)\mathrm{d}t$$

7.2.2. *Second variation formula for arc-length.*

Theorem 7.2.2 (second variation formula for arc-length)**.** Let $\gamma : [a,b] \to (M,g)$ be a unit-speed curve. If α is a 2-dimensional variation of γ with variation fields V, W. Then
$$\begin{aligned}\frac{\partial^2}{\partial s_1 \partial s_2}\Big|_{s_1=s_2=0} L(\alpha(\text{-}, s_1, s_2)) &= \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W\rangle \mathrm{d}t - \int_a^b R(V,\gamma',\gamma',W)\mathrm{d}t \\ &\quad - \int_a^b \langle \overline{\nabla}_{\frac{\partial}{\partial s_1}} \alpha_*(\frac{\partial}{\partial s_2})\Big|_{s_1=s_2=0}, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\gamma'(t)\rangle \mathrm{d}t \\ &\quad - \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma'\rangle \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W, \gamma'\rangle \mathrm{d}t\end{aligned}$$

Corollary 7.2.2. Let $\gamma : [a,b] \to (M,g)$ be a unit-speed geodesic. If α is a 2-dimensional variation of γ with variation fields V, W. Then
$$\begin{aligned}\frac{\partial^2}{\partial s_1 \partial s_2}\Big|_{s_1=s_2=0} L(\alpha(\text{-}, s_1, s_2)) &= \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W\rangle \mathrm{d}t - \int_a^b R(V,\gamma',\gamma',W)\mathrm{d}t \\ &\quad - \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma'\rangle \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W, \gamma'\rangle \mathrm{d}t \\ &= \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V^{\perp}, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W^{\perp}\rangle \mathrm{d}t - \int_a^b R(V^{\perp},\gamma',\gamma',W^{\perp})\mathrm{d}t\end{aligned}$$

where
$$V^{\perp} = V - \langle V, \gamma'\rangle\gamma', \quad W^{\perp} = W - \langle W, \gamma'\rangle\gamma'$$

Remark 7.2.1. So if we want to show a geodesic γ is a (locally) minimal geodesic, it suffices to check for any 2-dimensional variation α with variation vector fields V, W, we have
$$\frac{\partial^2}{\partial s_1 \partial s_2}\Big|_{s_1=s_2=0} L(\alpha(\text{-}, s_1, s_2)) \geq 0$$

Definition 7.2.1 (index form)**.** Suppose $\gamma : [a, b] \to (M, g)$ is a unit-speed geodesic. The index form I_γ is defined as

$$I_\gamma(V, W) = \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} W \rangle \mathrm{d}t - \int_a^b R(V, \gamma', \gamma', W) \mathrm{d}t$$

where V, W are vector fields along γ.

Thus a geodesic γ is locally minimal if and only if for any variation fields V, W we have index form $I_\gamma(V, W) \geq 0$. However, it's clear

$$I_\gamma(\gamma', \gamma') = 0$$

So if we want to obtain a kind of positive-definite property of index form, we must consider index form defined on normal vector fields along γ.

Definition 7.2.2 (normal vector field)**.** Let $\gamma : [a, b] \to (M, g)$ be a geodesic, a vector field V along γ is called normal, if V is perpendicular to γ'.

In the following section, we will study when the index form defined on the normal vector fields along γ is positive-definite, semipositive-definite or not.

8. Jacobi fields

8.1. First properties.

Definition 8.1.1 (Jacobi field)**.** A vector field J along geodesic γ is called a Jacobi field, if it satisfies

$$\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}J+R(J,\gamma')\gamma'=0$$

Notation 8.1.1. For convenience, we sometimes use the following notations

$$\begin{aligned}J'&=\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}J\\ J''&=\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}J\end{aligned}$$

Remark 8.1.1 (local form)*.* If we choose a parallel orthnormal vector fields $\{e_1,\dots,e_n\}$ along γ and write $J(t)=J^i(t)e_i(t)$, the condition for Jacobi fields becomes

$$\frac{\mathrm{d}^2J^k}{\mathrm{d}t^2}+\langle J^jR(e_j,\gamma')\gamma',e_k\rangle=0$$

Thus by standard results in ODEs, a Jacobi field J is completely determined by its initial conditions

$$J(0),J'(0)\in T_{\gamma(0)}M$$

Furthermore, you can see the set of Jacobi fields is a vector space with dimension $2n$.

Example 8.1.1. There is always a trivial Jacobi field along geodesic $\gamma:[a,b]\to(M,g)$, that is $J(t)=(t-a)\gamma'(t)$.

On a general Riemannian manifold, we can write down all Jacobi fields by using the following construction. However, we're more interested in Jacobi fields vanishes at one endpoint, let's write down an explicit construction for this case.

Lemma 8.1.1. Let $\gamma:[a,b]\to(M,g)$ be a geodesic and $\alpha:[a,b]\times(-\varepsilon,\varepsilon)\to(M,g)$ a variation consisting of geodesics of γ, then

$$J=\alpha_*(\frac{\partial}{\partial s})\Big|_{s=0}\in\gamma^*TM$$

is a Jacobi field.

Proof. Note that

$$\begin{aligned}\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial s})&=\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}\alpha_*(\frac{\partial}{\partial t})\\&=R(\frac{\partial}{\partial t},\frac{\partial}{\partial s})\alpha_*(\frac{\partial}{\partial t})+\nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial t})\\&=R(\frac{\partial}{\partial t},\frac{\partial}{\partial s})\alpha_*(\frac{\partial}{\partial t})\end{aligned}$$

Setting $s = 0$ we have

$$\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}J = R(\gamma', J)\gamma' = -R(J, \gamma')\gamma'$$

which implies J is a Jacobi field. □

Corollary 8.1.1. Let $\gamma : [0, 1] \to M$ be a geodesic with $\gamma(0) = p, \gamma'(0) = v$, where $v \in T_pM$, then for any $w \in T_pM$, consider the following variation of $\gamma(t)$ consisting of geodesics

$$\alpha(t, s) = \exp_p(t(v + sw))$$

Then $J(t) = \alpha_*(\frac{\partial}{\partial s})\big|_{s=0}$ is a Jacobi field along γ such that

$$J(0) = 0$$
$$J'(0) = w$$

Remark 8.1.2. In fact, for $\alpha(t, s) = \exp_p(t(v+sw))$, we can regard $t(v+sw)$ as a curve parametered by s in T_pM, that is it's a curve starting at tv with direction tw. So by definition we have

$$\alpha_*(\frac{\partial}{\partial s})\bigg|_{s=0} = (\mathrm{d}\exp_p)_{tv}(tw)$$

8.2. **Conjugate points.**

Definition 8.2.1 (conjugate points)**.** Let $p \neq q$ be two endpoints of a geodesic γ. p and q are called conjugate along γ if there exists a non-zero Jacobi field J along γ which vanishes at endpoints.

Notation 8.2.1. The conjugate set of p, denoted by $\mathrm{conj}(p)$ is defined as

$$\mathrm{conj}(p) := \{q \in M \mid p \text{ and } q \text{ are conjugate along some geodesic.}\}$$

Remark 8.2.1. There are at most $n - 1$ linearly independent Jacobi fields along γ such that $J(a) = J(b) = 0$. Indeed, by Remark 8.1.1, there are at most n linearly independent Jacobi fields such that $J(a) = 0$. However, trivial Jacobi field $J(t) = (t - a)\gamma'(t)$ never vanishes at $t = b$.

Theorem 8.2.1. Let (M, g) be a Riemannian manifold, $p \in M$ and $v \in V_p \subset T_pM$. Let $\gamma_v : [0, 1] \to M$ be the geodesic $\gamma_v(t) = \exp_p(tv)$ and $q = \gamma_v(1)$. Then $(\mathrm{d}\exp_p)_v$ is not injective if and only if q is conjugate to p along γ_v.

Proof. For any $w \in T_pM$, consider Jacobi field given by

$$J(t) = (\mathrm{d}\exp_p)_{tv}(tw)$$

So if $w \neq 0$ lies in the kernel of $(\mathrm{d}\exp_p)_v$, then $J(0) = J(1) = 0$, that is p is conjugate to q. Conversely, if p and q are conjugate along γ, then there exists a Jacobi field J such that $J(0) = J(1) = 0$, then it's clear

$$J(t) = (\mathrm{d}\exp_p)_{tv}(tw)$$

where $0 \neq w = J'(0) \in T_pM$. Thus

$$(\mathrm{d}\exp_p)_v(w) = J(1) = 0$$

which implies $(\mathrm{d}\exp_p)_v$ is not injective. □

Corollary 8.2.1. Let (M, g) be a complete Riemannian manifold, $p \in M$. If the conjugate locus $\mathrm{conj}(p) = \varnothing$, then $\exp_p : T_pM \to M$ is a local diffeomorphism.

Proof. Since M is complete, then $\exp_p : T_pM \to M$ is surjective. Furthermorem, since the conjugate locus $\mathrm{conj}(p) = \varnothing$, so for arbitrary $v \in T_pM$, we have $(\mathrm{d}\exp_p)_v$ is non-degenerated, which implies $\exp_p$ is a local diffeomorphism at $v \in T_pM$. □

Example 8.2.1. For $p \in \mathbb{S}^n$, we have $\mathrm{conj}(p) = \{-p\}$.

Example 8.2.2. For $p \in S^1 \times \mathbb{R}$, we have $\mathrm{conj}(p) = \varnothing$.

8.3. **Jacobi field as a null space.**

Lemma 8.3.1. Let $\gamma : [a, b] \to (M, g)$ be a unit-speed geodesic with no conjugate points, then there exist Jacobi fields $J_2, \dots, J_n$ along γ such that
1. $J_i(a) = 0, i \geq 2$ and $\{\gamma'(b), J_2(b), \dots, J_n(b)\}$ is an orthnormal basis of $T_{\gamma(b)}M$;
2. $\langle J_i(t), \gamma'(t)\rangle \equiv 0$ for any $t \in [a, b]$;
3. $\{\gamma'(t), J_2(t), \dots, J_n(t)\}$ are linearly independent for $t \in (a, b]$.

Proof. For (1). Suppose $\{\gamma'(b), e_2, \dots, e_n\}$ is an orthnormal basis of $T_{\gamma(b)}M$, since there is no conjugate points along γ, there exists a unique Jacobi field J_i such that

$$J_i(a) = 0, J_i(b) = e_i$$

for each $i = 2, \dots, n$.

For (2). Note that

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle J_i(t), \gamma'(t)\rangle &= \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} J_i, \gamma'\rangle \\ &= \langle R(J, \gamma')\gamma', \gamma'\rangle \\ &= 0\end{aligned}$$

Thus $\langle J_i(t), \gamma'(t)\rangle = \lambda t + \mu$. Note that $\langle J_i(a), \gamma'(a)\rangle = \langle J_i(b), \gamma'(b)\rangle = 0$, which implies $\langle J_i(t), \gamma'(t)\rangle \equiv 0$ on $[a, b]$.

For (3). Suppose there exists $c \in (a, b]$ and $\lambda_i \in \mathbb{R}$ such that

$$\sum_{i=2}^n \lambda_i J_i(c) = 0$$

which implies

$$W(t) = \sum_{i=2}^n \lambda_i J_i(t) \equiv 0$$

on $(a,c]$ since there is no conjugate points. By uniqueness we have $W(t)\equiv 0$ on $(a,b]$, thus we have $\lambda_i=0, i=2,\dots,n$ from (1). □

Theorem 8.3.1. Let $\gamma:[a,b]\to(M,g)$ be a unit-speed geodesic, then

1. If γ has no conjugate points, then index form I_γ is **positive-definite** on vector space consisting of normal variation fields;
2. If γ only has conjugate points as endpoints, then index form is **semipositive-definite** on vector space consisting of normal variation fields. Furthermore, Jacobi field is null space;
3. If γ has an interior conjugate point, then index form is **not positive-definite** on vector space consisting of normal variation fields.

Proof. For (1). Let $\{\gamma'(b),e_2,\dots,e_n\}$ be a orthnormal basis for $T_{\gamma(b)}M$, then there exist unique Jacobi fields J_i such that

$$J_i(a)=0, J_i(b)=e_i$$

where $i=2,\dots,n$. Then for any normal variation vector V along γ we write it as

$$V=\sum_{i=2}^{n}f_i(t)J_i(t)$$

Then it's clear $f_i(b)=0$ since $V(b)=0$ and $\{e_2,\dots,e_n\}$ is orthnormal. For index form we have

$$\begin{aligned}I_\gamma(V,V)=\sum_{i,j=2}^{n}\Bigg\{&\underbrace{\int_a^b f_if_j\langle J_i',J_j'\rangle+f_i'f_j\langle J_i,J_j'\rangle+f_if_j'\langle J_i',J_j\rangle\mathrm{d}t}_{\text{Part I}}\\&+\underbrace{\int_a^b\{f_i'f_j'\langle J_i,J_j\rangle-f_if_jR(J_i,\gamma',\gamma',J_j)\}\mathrm{d}t}_{\text{Part II}}\Bigg\}\end{aligned}$$

Note that

$$\langle J_i',J_j\rangle=\langle J_i,J_j'\rangle$$

Then Part I is

$$\int_a^b\{(f_if_j\langle J_i',J_j\rangle)'-f_if_j\langle J_i'',J_j\rangle\}\mathrm{d}t$$

Thus

$$\begin{aligned}I_\gamma(V,V)&=\sum_{i,j=2}^{n}f_if_j\langle J_i',J_j\rangle\big|_a^b+\int_a^b f_i'f_j'\langle J_i,J_j\rangle\mathrm{d}t\\&=\sum_{i,j=2}^{n}\int_a^b f_i'f_j'\langle J_i,J_j\rangle\mathrm{d}t\\&\geq 0\end{aligned}$$

Furthermore, $I_\gamma(V,V)=0$ if and only if $\sum_{i=2}^n f_i' J(t)=0$ if and only if $f_i'(t)=0, t\in[a,b]$, thus $f_i(t)\equiv 0$, that is $V=0$.

For (2). For any $c\in(a,b)$, we set $\gamma^c:[a,c]\to(M,g)$ and define I_{γ^c}. By (1) it's clear I_{γ^c} is positive-definite on the vector space consisting of normal variation fields along γ^c. By standard approximation argument we can show I_γ is semipositive-definite.

To see its null space: It's clear a normal variation Jacobi field V satisfies $I_\gamma(V,V)=0$; Conversely, if a normal variation field V satisfies $I_\gamma(V,V)=0$, then by a variation argument we have for arbitrary W we have

$$I_\gamma(V,W)=0$$

Take appropriate W to see V satisfies the equation for Jacobi fields.

For (3). If $\gamma(a)$ is conjugate to $\gamma(c)$ for some $c\in(a,b)$, then there exists a non-zero normal Jacobi field J_1 along $\gamma([a,c])$ such that $J_1(a)=J_1(c)=0$. Consider

$$J=\begin{cases} J_1(t) & t\in[a,c] \\ 0 & t\in[c,b]\end{cases}$$

It's easy to see $I_\gamma(J,J)=0$. Note that here our J may not be smooth. Let W be a smooth normal variation vector field along γ such that $W(c)=-\lim_{t\to c^-}\nabla_{\frac{\mathrm{d}}{\mathrm{d}t}}J_1$. It' clear $W(c)\neq 0$. Consider $J_\varepsilon=J+\varepsilon W$ and so[5]

$$I_\gamma(J_\varepsilon,J_\varepsilon)=2\varepsilon I_\gamma(J,W)+\varepsilon^2 I_\gamma(W,W)$$

And integration by parts we have

$$I_\gamma(J,W)=\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}J_1,W\rangle\Big|_a^c=-W(c)^2<0$$

So for sufficiently small ε we have $I_\gamma(J_\varepsilon,J_\varepsilon)<0$, and by approximation argument we can show there exists a smooth normal variation field such that $I_\gamma(V,V)<0$. □

Corollary 8.3.1. Let $\gamma:[a,b]\to(M,g)$ be a unit-speed geodesic with no conjugate points, and V,W are normal vector fields satisfying $V(a)=W(a), V(b)=W(b)$. If V is a Jacobi field, then $I_\gamma(V,V)\leq I_\gamma(W,W)$, and the equality holds if and only if $V=W$.

Proof. Since V,W agree at end points, then $V-W$ is a normal variation field, thus we have

$$0\leq I_\gamma(V-W,V-W)=I_\gamma(V,V)+I_\gamma(W,W)-2I_\gamma(V,W)$$

Since V is a Jacobi field, then integration by parts shows

$$I_\gamma(V,V)=\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V,V\rangle\Big|_a^b=\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V,W\rangle\Big|_a^b=I_\gamma(V,W)$$

Hence we get $I_\gamma(V,V)\leq I_\gamma(W,W)$, and the equality holds if and only if $V=W$. □

[5]Note that here our J and J_ε may not be smooth, so keep in mind here we already extend our index form I_γ to the one defined on piecewise smooth vector field.

Remark 8.3.1. From second variation formula, we can conclude that a geodesic γ is a **locally minimal geodesic** if and only if it has no interior conjugate points. However, it may not be **globally minimal geodesic**. Indeed, consider $M = S^1 \times \mathbb{R}$, it's clear there is no conjugate points for any geodesic on M, thus for geodesic $\gamma : [a, b] \to M$ starting at $(x, y) \in M$, it's locally minimal, but if there exists $c \in (a, b)$ such that $\gamma(c) \in \{-x\} \times \mathbb{R}$, then γ is not globally minimal.

9. Cut locus and injective radius

9.1. Cut locus.

Definition 9.1.1 (distance)**.** Let (M,g) be a complete Riemannian manifold, $p,q\in M$, the distance between p and q are the length of minimal geodesic connecting p,q, denoted by $\mathrm{dist}(p,q)$.

Definition 9.1.2 (cut time/point/locus)**.** Let (M,g) be a complete Riemannian manifold, $p\in M$ and $v\in T_pM$.

1. The cut time of (p,v) is defined as
$$t_{\mathrm{cut}}(p,v)=\sup\{c>0\mid \gamma_v|_{[0,c]}\text{ is a minimal geodesic}\}$$
2. Suppose $t_{\mathrm{cut}}(p,v)<\infty$, the cut point of p along γ along γ_v is $\gamma_v(t_{\mathrm{cut}}(p,v))\in M$;
3. The cut locus of p, denoted by $\mathrm{cut}(p)$ is the set
$$\mathrm{cut}(p)=\{q\in M\mid \exists v\in T_pM\text{ such that }q\text{ is a cut point of }p\text{ along }\gamma_v.\}$$

Remark 9.1.1. Here are some remarks:

1. It's possibly for $t_{\mathrm{cut}}(p,v)$ to be $+\infty$. For example, just let M be Euclidean space with standard metric;
2. The cut point (if it exists) occurs at or before the first conjugate point along every geodesic;
3. It's clear that $t_{\mathrm{cut}}(p,v)$ depends on the $|v|$, but $\gamma_v(t_{\mathrm{cut}}(p,v))$ is independent of $|v|$. So when we consider cut points of p along some geodesic γ, we always assume γ is unit-speed.

Theorem 9.1.1. Let (M,g) be a complete Riemannian manifold, $p\in M, v\in T_pM$ with unit length. Let $c=t_{\mathrm{cut}}(p,v)\in(0,\infty]$, then

1. If $0<b<c$ and b is finite, then $\gamma_v|_{[0,b]}$ has no conjugate point and it is the unique minimal unit-speed geodesic between endpoints;
2. If $c<\infty$, then $\gamma_v|_{[0,c]}$ is a minimal geodesic.
3. In the case of (2), one or both of the following holds:
   (a) $\gamma_v(c)$ is conjugate to p along γ_v;
   (b) There are two or more different unit-speed minimal geodesic connecting p and $\gamma_v(c)$.

Proof. (1) and (2) is clear. For (3). Assume $\gamma_v(c)$ is not conjugate to p along γ_v, we shall prove the existence of another unit-speed minimal geodesic from p to $\gamma_v(c)$.

Firstly we choose a sequence $\{b_i\}$ descending to c. Note that $\gamma_v:[0,b_i]\to M$ is not a minimal geodesic, thus there exists a unit-speed minimal geodesic $\gamma_i:[0,a_i]\to M$ connecting p and $\gamma_v(b_i)$. In particular we have

1. $\gamma_i(a_i)=\gamma_v(b_i)$;
2. $a_i<b_i$.

If we denote $\omega_i=\gamma_i'(0)\in T_pM$, by compactness of unit sphere on T_pM and the fact $\{a_i\}$ is bounded, we can find a subsequence of $\{\gamma_i\}$ such that

ω_i converging to some $w \in T_pM$ with $|w| = 1$, and $\lim_{i\to\infty} a_i = a$. For convenience we still denote this subsequence by $\{\gamma_i\}$.

On one hand $\gamma_i(a_i) = \exp_p(a_i w_i)$ converges to $\exp_p(aw)$; On the other hand, $\gamma_i(a_i) = \gamma_v(b_i)$ implies $\exp_p(cv) = \gamma_v(c) = \exp_p(aw)$. Furthermore,

$$c = \mathrm{dist}(p, \gamma_v(c)) = \lim_{i\to\infty} \mathrm{dist}(p, \gamma_v(b_i)) = \lim_{i\to\infty} \mathrm{dist}(p, \gamma_i(a_i)) = \lim_{i\to\infty} a_i = a$$

So it suffices to check $v \neq w$.

By assumption we have $\gamma_v(c)$ is not conjugate to p, thus cv is not a critical point of $\exp_p$, that is $\exp_p$ is injective in $B_\varepsilon(cv)$, where $\varepsilon > 0$ is sufficiently small. On one hand we have $a_i w_i \neq b_i v$ since $a_i < b_i$; On the other hand we have

$$\exp_p(b_i v) = \gamma_v(b_i) = \gamma_i(a_i) = \exp_p(a_i w_i)$$

Thus injectivity implies $a_i w_i \notin B_\varepsilon(cv)$ for sufficiently large i, since in this case $b_i v \in B_\varepsilon(cv)$. Taking limits we have

$$aw \neq cv$$

that is $w \neq v$. □

Example 9.1.1. Consider the following cases:

1. $M = \mathbb{S}^n$, then $\mathrm{cut}(p) = \mathrm{conj}(p) = \{-p\}$. In this case both $(a), (b)$ hold in Theorem 9.1.1;
2. $M = \mathbb{S}^1 \times \mathbb{R}$, then $\mathrm{cut}(p) = \{-p\} \times \mathbb{R}$. In this case (a) fails and (b) holds in Theorem 9.1.1;
3.

Definition 9.1.3 (injective radius)**.** Let (M, g) be a Riemannian manifold, $p \in M$. The injective radius of p is defined as

$$\mathrm{inj}(p) := \sup\{\rho > 0 : \exp_p \text{ is defined on } B(0,\rho) \subset T_pM \text{ and injective}\}$$

The injectivity radius of M is

$$\mathrm{inj}(M) := \inf_{p\in M} \mathrm{inj}(p)$$

Theorem 9.1.2. Let (M, g) be a complete Riemannian manifold, then

$$\mathrm{inj}(p) = \begin{cases} \mathrm{dist}(p, \mathrm{cut}(p)) & \mathrm{cut}(p) \neq \varnothing \\ \infty & \mathrm{cut}(p) = \varnothing \end{cases}$$

Proof. See Proposition 10.36 in Page312 of [Lee18]. □

Proposition 9.1.1. Let (M, g) be a complete Riemannian manifold and $p \in M$. Suppose there exists some point $q \in \mathrm{cut}(p)$ such that $\mathrm{dist}(p, q) = \mathrm{dist}(p, \mathrm{cut}(p))$, then

1. Either q is a conjugate point of p along some minimizing geodesic from p to q, or there are exactly two minimizing geodesics from p to q, say $\gamma_1, \gamma_2 : [0, b] \to M$, such that $\gamma_1'(b) = -\gamma_2'(b)$.

2. If in addition that $\mathrm{inj}(p) = \mathrm{inj}(M)$, and q is not conjugate to p along any minimizing geodesic, then there is a closed unit-speed geodesic $\gamma : [0, 2b] \to M$ such that $\gamma(0) = \gamma(2b) = p$ and $\gamma(b) = q$ where $b = \mathrm{dist}(p, q)$.

Proof. For (1). Suppose q is not conjugate to p along any minimizing geodesic, then by Theorem 9.1.1 there are at least two unit-speed minimal geodesics $\gamma_1(t), \gamma_2(t)$ such that $\gamma_1(b) = \gamma_2(b) = q$. Suppose $\gamma_1'(b) \neq -\gamma_2'(b)$, then there exists unit vector $X_q \in T_qM$ such that

$$\langle X_q, \gamma_1'(b)\rangle < 0, \quad \langle X_q, \gamma_2'(b)\rangle < 0$$

Since q is not conjugate to p along γ_1, there exists a neighborhood U_1 of $b\gamma_1'(0)$ in T_pM such that $\exp_p|_{U_1}$ is diffeomorphism. Now choose a sufficiently small s and let

$$\xi_1(s) = (\exp_p|_{U_1})^{-1} \exp_q(sX_q)$$

Consider the following variation of γ_1 consisting of geodesics:

$$\alpha_1(t, s) = \exp(\frac{t}{b}\xi_1(s))$$

It's clear $\alpha_1(t, 0) = \gamma_1(t)$, since $\xi_1(0) = (\exp_p|_{U_1})^{-1} \exp_q(0) = (\exp_p|_{U_1})^{-1}(q) = b\gamma_1'(0)$. Then the first variation formula yields

$$\left.\frac{\mathrm{d}L(\gamma_s)}{\mathrm{d}s}\right|_{s=0} = \langle X_q, \gamma_1'(b)\rangle < 0$$

which implies for sufficiently small s we have $L(\alpha_1(t, s)) < L(\gamma_1(t))$. For γ_2 we can do the same construction and the same argument implies for sufficiently small s we have $L(\alpha_2(t, s)) < L(\gamma_2(t))$. Thus for each sufficiently small s we have two geodesics $\alpha_1(t, s), \alpha_2(t, s)$ from p to $\exp_q(sX_q)$. However,

$$\mathrm{d}(p, \exp_q(sX_q)) \leq L(\alpha_1(t, s)) < L(\gamma_1(t)) = \mathrm{dist}(p, q) = \mathrm{inj}(p) \tag{9.1}$$

A contradiction to the definition of injective radius. So any two different minimizing geodesics γ_1, γ_2 from p to q satisfy $\gamma_1'(b) = -\gamma_2'(b)$, which implies there are exactly two minimizing geodesics from p to q.

For (2). By (1) we know that there exists exactly two geodesics γ_1, γ_2 such that $\gamma_1(b) = \gamma_2(b) = q$ with $\gamma_1'(b) = \gamma_2'(b)$. Consider the loop $\gamma = \gamma_1 \circ \gamma_2^{-1}$, then it's a unit-speed geodesic such that $\gamma(0) = \gamma(2b) = p, \gamma(b) = q$, where $b = \mathrm{dist}(p, q)$, since we have already shown $\gamma_1'(b) = -\gamma_2'(b)$. To show γ is a closed geodesic, it suffices to show $\gamma'(2b) = \gamma'(0)$, that is equivalent to show $(\gamma_1^{-1})'(b) = (\gamma_2^{-1})'(b)$. Note that in the proof of (1), condition of $\mathrm{dist}(p, q) = \mathrm{dist}(p, \mathrm{cut}(p)) = \mathrm{inj}(p)$ is used in inequality (9.1), and in fact we only need $\mathrm{dist}(p, q) \leq \mathrm{inj}(p)$, strict equality is not neccessary. So if $\mathrm{inj}(p) = \mathrm{inj}(M)$, thus

$$\mathrm{dist}(q, p) = \mathrm{dist}(p, q) = \mathrm{inj}(p) \leq \mathrm{inj}(q)$$

Then (1) implies $(\gamma_1^{-1})'(b) = (\gamma_2^{-1})'(b)$. $\square$

Part 3. Topology of Riemannian manifold

10. Topology of non-positive sectional curvature manifold

10.1. Cartan-Hadamard manifold.

Definition 10.1.1 (Cartan-Hadamard manifold)**.** A simply-connected, complete Riemannian manifold with non-positive sectional curvature is called Cartan-Hadamard manifold.

10.1.1. *Expansion property of exponential map of Cartan-Hadamard manifold.* In this section we explore some properties of Cartan-Hadamard manifold using Jacobi fields.

Proposition 10.1.1. Let $p \in M$ and $\gamma : [0,1] \to M$ be a geodesic such that $\gamma(0) = p, \gamma'(0) = v$. Then for any $w \in T_pM$ with $|w| = 1$, let $J(t)$ be the Jacobi field along γ given by

$$J(t) = (\mathrm{d}\exp_p)_{tv}(tw)$$

Then we have the following Taylor expansions about $t = 0$

$$\begin{aligned}|J(t)|^2 &= t^2 - \frac{1}{3}R(J', \gamma', \gamma', J')(0)t^4 + O(t^4)\\ |J(t)| &= t - \frac{1}{6}R(J', \gamma', \gamma', J')(0)t^3 + O(t^3)\end{aligned}$$

Proof. For (1). Since $J(0) = 0, J'(0) = w$, the first three coefficients are given as

$$\begin{aligned}\langle J, J\rangle(0) &= 0\\ \langle J, J\rangle'(0) &= 2\langle J, J'\rangle(0) = 0\\ \langle J, J\rangle''(0) &= 2\langle J', J'\rangle(0) + 2\langle J'', J\rangle(0) = 2\\ \langle J, J\rangle'''(0) &= 6\langle J', J''\rangle(0) + 2\langle J''', J\rangle(0) = 0\\ &= 6\langle J', R(J, \gamma')\gamma'\rangle(0) = 0\\ \langle J, J\rangle''''(0) &= 8\langle J', J'''\rangle(0) + 6\langle J'', J''\rangle(0) + 2\langle J'''', J\rangle(0)\\ &= 8\langle J', J'''\rangle(0) + 6\langle R(J, \gamma')\gamma', R(J, \gamma')\gamma'\rangle(0)\\ &= 8\langle J', J'''\rangle(0)\end{aligned}$$

So we need to compute J'''. For arbitrary vector field W along γ, direct computation shows

$$\begin{aligned}\langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} R(J, \gamma')\gamma', W\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\langle R(J, \gamma')\gamma', W\rangle - \langle R(J, \gamma')\gamma, W'\rangle\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\langle R(W, \gamma')\gamma', J\rangle - \langle R(J, \gamma')\gamma, W'\rangle\\ &= \langle R(W, \gamma')\gamma', J'\rangle - \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} R(W, \gamma')\gamma', J\rangle - \langle R(J, \gamma')\gamma, W'\rangle\\ &= \langle R(J', \gamma')\gamma', W\rangle - \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} R(W, \gamma')\gamma', J\rangle - \langle R(J, \gamma')\gamma, W'\rangle\end{aligned}$$

Setting $t=0$ we obtain

$$\langle J', J'''\rangle(0) = -\langle J'(0), \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} R(J,\gamma')\gamma'\Big|_{t=0}\rangle = -R(J',\gamma',\gamma',J')(0)$$

So we have

$$|J(t)|^2 = t^2 - \frac{1}{3}R(J',\gamma',\gamma',J')(0)t^4 + O(t^4)$$

For (2). It follows directly from (1). □

Theorem 10.1.1. Let (M,g) be a simply-connected complete Riemannian manifold. The followings are equivalent:

1. M is Cartan-Hadamard manifold;
2. For any $p\in M$ and $v,w\in T_pM$, we have
$$|(\mathrm{d}\exp_p)_v w| \geq |w|$$
3. For any $p\in M, T>0$ and $v,w\in T_pM$, we have
$$|v-w| \leq \frac{\mathrm{dist}(\exp_p(tv),\exp_p(tw))}{t}$$
holds for arbitrary $t>0$.

Proof. From (1) to (2). For all $p\in M$ and $v,w\in T_pM$, consider geodesic $\exp_p(tv)$ and Jacobi field

$$J(t) = (\mathrm{d}\exp_p)_{tv}(tw)$$

along it. If M has non-positive sectional curvature, direct computation shows

$$|J(t)|'' = \frac{|J^2||J'|^2 - \langle J,J'\rangle^2}{|J|^3} - \frac{R(J,\gamma',\gamma',J)}{|J|} \geq 0$$

for all $t>0$. Thus consider

$$f(t) = |J(t)| - t|w|$$

It's clear $f''(t)\geq 0$ and $f'(0)=0$, thus $f(t)\geq 0$ for all $t>0$ since $f(0)=0$. In particular, set $t=1$ we have

$$|(\mathrm{d}\exp_p)_v(w)| - |w| \geq 0$$

From (2) to (1). If M has sectional curvature $K(\sigma)>0$ at $p\in M$, where σ is the plane spanned by v,w with $|v|=|w|=1$. Then consider geodesic $\exp_p(tv)$ and Jacobi field

$$J(t) = (\mathrm{d}\exp_p)_{tv}(tw)$$

along it. Then by Proposition 10.1.1 we have $|J(t)|''<0$ for sufficiently small t. If we set $f(t)=|J(t)|-t|w|$, then we can see $f(0)=0, f'(0)=0$ and $f''(0)<0$ for sufficiently small t. In particular, we have

$$|(\mathrm{d}\exp_p)_{\varepsilon v}(\varepsilon w)| - |\varepsilon w| = f(\varepsilon) < 0$$

where $\varepsilon>0$ is sufficiently small. This leads to a contradiction.

From (2) to (3). For arbitrary $t > 0$. Let $\gamma(s) : [0,1] \to M$ be a geodesic connecting $\exp_p(tv), \exp_p(tw)$ and choose a curve $v(s) \in T_pM$ such that

$$\exp_p(v(s)) = \gamma(s)$$

for all $s \in [0,1]$. Hence $v(0) = tv, v(1) = tw$. Then

$$\begin{aligned}
\mathrm{dist}(\exp_p(tv), \exp_p(tw)) &= \int_0^1 |\gamma'(s)| \mathrm{d}s \\
&= \int_0^1 |(\mathrm{d}\exp_p)_{v(s)}(v'(s))| \mathrm{d}s \\
&\geq |\int_0^1 v'(s) \mathrm{d}s| \\
&= t|v - w|
\end{aligned}$$

This shows

$$|v - w| \leq \frac{\mathrm{dist}(\exp_p(tv), \exp_p(tw))}{t}$$

holds for arbitrary $t > 0$.

From (3) to (2). Note that

$$\begin{aligned}
|(\mathrm{d}\exp_p)_v(w)| &= \lim_{t \to 0} \frac{\exp_p(v + tw) - \exp_p(v)}{t} \\
&= \lim_{t \to 0} \frac{\exp_p(tv' + tw) - \exp_p(tv')}{t} \\
&\geq |v' + w - v'| \\
&= |w|
\end{aligned}$$

□

Corollary 10.1.1. Let (M, g) be a Cartan-Hadamard manifold with $a, b, c \in M$. Such points determine a unique geodesic triangle T with vertices a, b, c. Let α, β, γ be the angles of the vertices a, b, c respectively, and let A, B, C be the lengths of the side opposite the vertices a, b, c respectively. Then

1. $A^2 + B^2 - 2AB\cos\gamma \leq C^2 (< C^2, \text{if } K < 0)$;
2. $\alpha + \beta + \gamma \leq \pi (< \pi, \text{if } K < 0)$

Proof. See Lemma 3.1 in Page259 of Do carmo[Car92]. □

So you find that the exponential map of simply-connected complete Riemannian manifold with non-positive sectional curvature has a property of "expansion".

10.1.2. *Complete Riemannian manifold with non-positive sectional curvature is $K(G, 1)$.*

Lemma 10.1.1. If (M, g) is a complete Riemannian manifold with sectional curvature $K \leq 0$, then for any $p \in M$, the conjugate locus $\mathrm{conj}(p) = \varnothing$. In particular, $\exp_p : T_pM \to M$ is a local diffeomorphism.

Proof. Suppose q is conjugate to p along $\gamma : [0,1] \to M$, and without lose of generality we may assume there is no conjugate point for $t \in (0,1)$. Let J be a Jacobi field along γ with $J(0) = J(1) = 0$, then

$$\begin{aligned}(\frac{1}{2}|J|^2)' &= (g(J', J))' \\ &= g(J'', J) + g(J', J') \\ &= -R(J, \gamma', \gamma', J) + |J'|^2 \\ &\geq |J'|^2\end{aligned}$$

Since $J'(0) \neq 0$, we have

$$\begin{aligned}g(J', J)(t) &\geq \int_0^t |J'|^2 + g(J'(0), J(0)) \\ &= \int_0^t |J'|^2 \\ &> 0\end{aligned}$$

which implies $(\frac{1}{2}|J|^2)' = g(J', J) > 0$, a contradiction to $J(1) = 0$. □

Lemma 10.1.2. Let M be a complete Riemannian manifold and let $f : M \to N$ be a local diffeomorphism onto a Riemannian manifold N which has the following property: For all $p \in M$ and for all $v \in T_pM$, we have $|\mathrm{d}f_p(v)| \geq |v|$. Then f is a covering map.

Proof. See Lemma 3.3 in Page150 of Do carmo[Car92]. □

Theorem 10.1.2 (Cartan-Hadamard)**.** If (M, g) is a complete Riemannian manifold with sectional curvature $K \leq 0$, then $\exp_p : T_pM \to M$ is a covering map.

Proof. Combine above two lemmas with Theorem 10.1.1. □

Corollary 10.1.2. Cartan-Hadamard manifold is diffeomorphic to $\mathbb{R}^n$.

Corollary 10.1.3. If (M, g) is a complete Riemannian manifold with $K \leq 0$, then $\pi_k(M) = 0, k \geq 2$, that is M is $K(\pi_1(M), 1)$.

Remark 10.1.1. Thoery in topology says if a finite dimension CW complex is a $K(G, 1)$ space, then its fundamental group is torsion-free. So if M is a complete Riemannian manifold with $K \leq 0$, we have $\pi(M)$ is torsion-free. We will prove this fact later by tools of Riemannian manifold, called Cartan's torsion-free theorem.

Corollary 10.1.4. If M and N are two compact Riemannian manifold and one of them is simply-connected, then $M \times N$ has no metric with non-positive sectional curvature.

Proof. If both of M and N are simply-connected, and $M \times N$ admits a metric with non-positive sectional curvature, then it's diffeomorphic to $\mathbb{R}^n$ for some positive integer n, a contradiction to compactness.

So suppose M is simply-connected and N is not simply-connected with universal covering $\widetilde{N}$, then there is a universal covering

$$\pi : M \times \widetilde{N} \to M \times N$$

If $M \times N$ admits a Riemannian metric g with non-positive sectional curvature, then $\pi^* g$ is a complete metric of non-positive sectional curvature on $M \times \widetilde{N}$, so we have $M \times \widetilde{N}$ is diffeomorphic to $\mathbb{R}^n$ for some n. M is orientable since it's simply-connected, thus $H^m(M) = \mathbb{Z}$, where $m = \dim M$, thus by Künneth formula $H^m(M \times \widetilde{N}) \neq 0$, a contradiction to Poincaré lemma. □

Remark 10.1.2. The condition simply-connected is crucial, for example $S^1 \times S^1$.

10.2. **Cartan's torsion-free theorem.**

Lemma 10.2.1. *Let (M, g) be a Cartan-Hadamard manifold, $p \in M$ and $v \in T_pM$. For all $q \in M$ we have*

$$2\operatorname{dist}(p,q)^2 + \operatorname{dist}(p_0,p)^2 + \operatorname{dist}(p_1,p)^2 \leq \operatorname{dist}(p_0,q)^2 + \operatorname{dist}(p_1,q)^2$$

where $p_0 = \exp_p(-v), p_1 = \exp_p(v)$.

Proof. Since $\exp_p : T_pM \to M$ is a diffeomorphism, there exists $w \in T_pM$ such that $q = \exp_p(w)$ with $\operatorname{dist}(p,q) = |w|$. So we have

$$\begin{aligned}
\operatorname{dist}(p_0,q) &= \operatorname{dist}(\exp_p(-v), \exp_p(w)) \geq |w+v| \\
\operatorname{dist}(p_1,q) &= |w-v| \\
\operatorname{dist}(p,q)^2 &= |w|^2 \\
&= \frac{|w+v|^2 + |w-v|^2}{2} - |v|^2 \\
&\leq \frac{\operatorname{dist}(p_0,q)^2 + \operatorname{dist}(p_1,q)^2}{2} - \frac{\operatorname{dist}(p_0,p)^2 + \operatorname{dist}(p_1,p)^2}{2}
\end{aligned}$$

□

Lemma 10.2.2 (Serre)**.** *Let (M, g) be a Cartan-Hadamard manifold, $p \in M$ and $B(p,r)$ be the closed ball of radius r. If $\Omega \subset M$ is non-empty bounded set and define*

$$r_\Omega = \inf\{r > 0 \mid \Omega \subset B(p,r)\}$$

There exists unique $p_\Omega \in M$ such that $\Omega \subset B(p_\Omega, r_\Omega)$.

Proof. Existence: Choose a sequence $r_i > r_\Omega$ and $p_i \in M$ such that

$$\Omega \subset B(p_i, r_i), \lim r_i = r_\Omega$$

Fix arbitrary $q \in \Omega$, one has $\operatorname{dist}(q, p_i) \leq r_i$ for each i, thus $\{p_i\}$ is bounded since we can choose $\{r_i\}$ is bounded, which has a convergent subsequence since M is complete. The limit of this convergent subsequence is p_Ω.

Uniqueness: Let $p_0, p_1 \in M$ such that

$$\Omega \subset B(p_0, r_\Omega) \cap B(p_1, r_\Omega)$$

Since $\exp_{p_0}$ is a diffeomorphism, there exists unique v_0 such that $p_1 = \exp_{p_0} v_0$. Set $p = \exp_{p_0}(v_0/2)$, for all $q \in \Omega$ we have

$$\begin{aligned}\operatorname{dist}(p,q)^2 &\leq \frac{\operatorname{dist}(p_0,q)^2 + \operatorname{dist}(p_1,q)^2}{2} - \frac{\operatorname{dist}(p_0,p_1)^2}{4} \\ &\leq r_\Omega^2 - \frac{\operatorname{dist}(p_0,p_1)^2}{4}\end{aligned}$$

By definition of r_Ω, we have $\operatorname{dist}(p_0,p_1) = 0$, hence $p_0 = p_1$. □

Theorem 10.2.1 (Cartan's fixed-point theorem)**.** Suppose (M,g) is a Cartan-Hadamard manifold and G is a compact Lie group acting smoothly and isometrically on M, then G has a fixed-point in M.

Proof. Let $p \in M$, consider its orbit

$$\Omega = \{gp \mid g \in G\}$$

it's a bounded since M is compact. Note

$$\Omega = g\Omega \subset B(gp_\Omega, r_\Omega)$$

Then by uniqueness of p_Ω, we have p_Ω is a fixed-point of G. □

Corollary 10.2.1. If (M,g) is a complete Riemannian manifold with $K \leq 0$, then $\pi_1(M)$ is torsion-free.

Proof. Consider the universal covering $\widetilde{M}$ of M, it's a Cartan-Hadamard manifold, and $\pi_1(M)$ acts on $\widetilde{M}$ freely. If there exists a torsion element φ, consider the finite group G generated by φ, it's a 0-dimension Lie group with discrete topology. By Cartan's fixed-point theorem there exists a fixed-point of G, which implies φ is identity, since $\pi_1(M)$-action is free. □

10.3. **Preissmann's Theorem.**

Definition 10.3.1 (axis)**.** Let (M,g) be a complete Riemannian manifold, $F: M \to M$ is an isometry. A non-trivial geodesic $r: \mathbb{R} \to M$ is called an axis of F if $F \circ r$ is a non-trivial translation of γ, that is there exists $c > 0$ such that

$$F(\gamma(t)) = \gamma(t+c)$$

Definition 10.3.2. An isometry with no fixed points that has an axis is said to be axial.

Lemma 10.3.1. $F: (M,g) \to (M,g)$ is a isometry of complete Riemannian manifold, if $\delta_F(p) = \operatorname{dist}(p, F(p))$ has a positive minimum, then F has a axis.

Proof. Suppose δ_F attains its minimum at some $p \in M$, $\gamma(t): [0,1] \to M$ is a minimum geodesic connecting p and $F(p)$, then $F \circ \gamma: [0,1] \to M$ is also a minimum geodesic connecting $F(p)$ and $F^2(p)$, since F is a isometry.

We claim these two geodesics form an angle π at point $F(p)$ and thus fit together an extension of γ to $[0,2]$. Indeed, for any $t\in[0,1]$,

$$\begin{aligned}\delta_F(p)&=\mathrm{dist}(p,F(p))\\&\leq\delta_F(\gamma(t))\\&=\mathrm{dist}(\gamma(t),F(\gamma(t)))\\&\leq\mathrm{dist}(\gamma(t),\gamma(1))+\mathrm{dist}(\gamma(1),F(\gamma(t)))\\&=\mathrm{dist}(\gamma(t),\gamma(1))+\mathrm{dist}(F(\gamma)(0),F(\gamma(t)))\\&=\mathrm{dist}(\gamma(t),\gamma(1))+\mathrm{dist}(\gamma(0),\gamma(t))\\&=\delta_F(p)\end{aligned}$$

Thus we have $(F\circ\gamma)(t)=\gamma(1+t)$ for $0\leq t\leq 1$. Repeating this argument to obtain a geodesic $\gamma:\mathbb{R}\to M$ with period 1, and it's an axis for F. $\square$

Lemma 10.3.2. Let (M,g) be a compact Riemannian manifold and $F:\widetilde{M}\to\widetilde{M}$ be a non-trivial deck transformation on the universal covering map $\pi:\widetilde{M}\to M$. Then

1. δ_F has a positive minimum and $\delta_F\geq 2\,\mathrm{inj}(M)$. In particular, F has an axis $\gamma:\mathbb{R}\to\widetilde{M}$;
2. $\pi\circ r$ is a closed geodesic in M whose length is minimal in the free homotopy class $[\pi\circ\gamma]$.

Lemma 10.3.3. Suppose (M,g) is a Cartan-Hadamard manifold with $K<0$, if $F:M\to M$ has an axis, then it's unique up to reparametrization.

Proof. Suppose $\gamma_1,\gamma_2:\mathbb{R}\to M$ are two axes of F, without lose of generality we may assume

$$\begin{aligned}F(\gamma_1(t))&=\gamma_1(t+1)\\F(\gamma_2(t))&=\gamma_2(t+1)\end{aligned}$$

Suppose γ_1,γ_2 do not intersect, then points $A=\gamma_1(0),B=\gamma_1(1)=F(A),C=\gamma_2(0)$ and $D=\gamma_2(1)=F(C)$ are all distinct. Let σ be a geodesic from A to C, then $F\circ\sigma$ is the geodesic from B to D. Furthermore, the geodesic quadrilateral $ABCD$ has angle sum 2π, since F preserves angles. However, according to Lemma 10.1.1, triangle $\triangle ABC$ and $\triangle BCD$ have angle sum strictly less than π, and

$$\begin{aligned}\angle ACD&\leq\angle ACB+\angle BCD\\\angle ABD&\leq\angle ABC+\angle CBD\end{aligned}$$

thus the angle sum of $ABCD$ is strictly less than 2π, a contradiction.

Hence γ_1 and γ_2 must intersect at some point $p=\gamma_1(t_1)=\gamma_2(t_2)$, then

$$\begin{aligned}F(p)=F(\gamma_1(t_1))&=\gamma_1(t_1+1)\\=F(\gamma_2(t_2))&=\gamma_2(t_2+1)\end{aligned}$$

is another intersection point. By the uniqueness of geodesic we have γ_1 is a reparametrization of γ_2. $\square$

Lemma 10.3.4. If H is a additive subgroup of $\mathbb{R}$, then either H is dense in $\mathbb{R}$ or $H \cong \mathbb{Z}$.

Proof. Let H be an additive subgroup of $\mathbb{R}$, it's clear $H \cap \mathbb{R}^{>0} \neq \varnothing$, consider

$$b := \inf\{h \in H \cap \mathbb{R}^{>0}\}$$

1. If $b > 0$: Let $h \in H$ and $k \in \mathbb{Z}$ such that
$$kb \leq |h| < (k+1)b$$
then we have $|h| - kb \in H$, and $0 \leq |h| - kb < (k+1)b - kb = b$. By the choice of b, we have $|h| - kb$, which implies $h = \pm kb$. In this case $H = b\mathbb{Z}$.
2. If $b = 0$: For arbitrary $r \in \mathbb{R}^{\geq 0}$ and $\varepsilon > 0$, there exists $h \in H \cap (0, \varepsilon]$ since $b = 0$ and $k \in \mathbb{N}$ such that
$$kh \leq r \leq (k+1)h$$
Thus
$$0 \leq r - kh \leq (k+1)h - kh = h \leq \varepsilon$$
which implies $|r - kh| \leq \varepsilon$, that is H is dense in $\mathbb{R}^{\geq 0}$. For the same argument you can show H is also dense in $\mathbb{R}^{\leq 0}$.

$\square$

Theorem 10.3.1 (Preissmann)**.** If (M, g) is a compact Riemannian manifold with negative sectional curvature, then any non-trivial abelian subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$.

Proof. Let $\pi : \widetilde{M} \to M$ be the universal covering, then $(\widetilde{M}, \pi^*g)$ is a Cartan-Hadamard manifold with negative sectional curvature. Now it suffices to show every non-trivial abelian subgroup H of $\mathrm{Aut}_\pi(\widetilde{M})$ is isomorphic to $\mathbb{Z}$, since $\pi_1(M) \cong \mathrm{Aut}_\pi(\widetilde{M})$. Let φ be a non-trivial deck transformation and $\gamma : \mathbb{R} \to \widetilde{M}$ is an axis of φ, that is there exists $c > 0$ such that

$$\varphi \circ \gamma(t) = \gamma(t + c)$$

for all $t \in \mathbb{R}$. If ψ is another non-trivial element of H, then for any $t \in \mathbb{R}$ we have

$$\varphi(\psi(\gamma(t))) = \psi\varphi(\gamma(t)) = \psi(\gamma(t + c))$$

which implies $\psi \circ \gamma$ is also an axis of φ. So by Lemma 10.3.3 we have $\psi \circ \gamma$ is a reparametrization of γ. Furthermore, they have the same speed since ψ is an isometry, thus $\psi(\gamma(t)) = \gamma(t + a)$ or $\psi(\gamma(t)) = \gamma(-t + a)$. The latter can't happen, otherwise $\psi(\gamma(\frac{a}{2})) = \gamma(\frac{a}{2})$, contradicts to $\mathrm{Aut}_\pi(\widetilde{M})$ acts on $\widetilde{M}$ freely.

Define $F : H \to \mathbb{R}$ by $F(\psi) = a$ such that $\psi(\gamma(t)) = \gamma(t + a)$. It's easy to see F is a group homomorphism with trivial kernel and $F(H)$ is an additive subgroup of $\mathbb{R}$. Consider

$$b := \inf\{h \in F(H) \cap \mathbb{R}^+\}$$

By Lemma 10.3.4, it suffices to show $b > 0$. If $b = 0$, then there exists $a \in (0, \mathrm{inj}(M))$ and $\psi \in H$ such that

$$a = F(\psi), \quad \psi(\gamma(t)) = \gamma(t + a)$$

Since $\pi \circ \psi = \pi$, thus we have $\pi(\gamma(t)) = \pi(\gamma(t + a))$. Set $t = 0$ one has

$$\pi(\gamma(a)) = \pi(\gamma(0))$$

A contradiction to $a < \mathrm{inj}(M)$ since $\pi \circ \gamma$ is a geodesic. □

Corollary 10.3.1. Suppose M and N are compact smooth manifold, then $M \times N$ doesn't admit a Riemannian metric with negative sectional curvature.

Proof. If $M \times N$ admits a Riemannian metric with negative sectional curvature, Cartan's torsion-free theorem implies $\pi_1(M \times N)$ is torsion-free, thus for arbitrary $\alpha \in \pi_1(M), \beta \in \pi_1(N)$, unless either M or N is simply-connected, $\pi_1(M \times N)$ will contain an abelian subgroup $\mathbb{Z} \times \mathbb{Z}$ generated by α, β, which contradicts to Preissmann's theorem.

So we may assume M is simply-connected, then consider the universal covering $M \times \widetilde{N}$ of $M \times N$, Cartan-Hadamard's theorem implies it's diffeomorphic to $\mathbb{R}^n$ for some positive integer n, but M is orientable since it's simply-connected, so $H^m(M) = \mathbb{Z}$ where $m = \dim M$. So by Künneth formula $H^n(M \times \widetilde{N}) \neq 0$, a contradiction to Poincaré lemma. □

Lemma 10.3.5. Let (M, g) be a complete Riemannian manifold with non-positive sectional curvature and $\pi : \widetilde{M} \to M$ the universal covering. If $\gamma : \mathbb{R} \to \widetilde{M}$ is a common axis for all elements of $\mathrm{Aut}_\pi(\widetilde{M})$, then M is not compact.

Proof. □

Theorem 10.3.2 (Preissmann)**.** If (M, g) is a compact Riemannian manifold with negative sectional curvature, then $\pi_1(M)$ is not abelian.

Proof. Suppose $\pi_1(M)$ is abelian, then let γ be the axis of some deck transformation, then it's the axis of all deck transformations since $\pi_1(M)$ is abelian, which implies M is non-compact, a contradiction. □

Theorem 10.3.3 (Byers)**.** If (M, g) is a compact Riemannian manifold with negative sectional curvature, then any non-trivial solvable subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$.

10.4. **Other facts.**

Theorem 10.4.1 (Yau)**.** Let (M, g) be a compact Riemannian manifold with non-positive sectional curvature. If $\pi_1(M)$ is solvable, then M is flat and isometric to some $\mathbb{R}^n/\Gamma$.

Theorem 10.4.2 (Farrell-Jones)**.** Let $(M_i, g_i), i = 1, 2$ be two compact Riemannian manifolds with non-positive sectional curvature. If $\pi_1(M_1) = \pi_1(M_2)$ then M_1 and M_2 are homeomorphic.

11. Topology of positive curvature manifold

11.1. Myers' theorem.

Theorem 11.1.1 (Myers)**.** Let (M,g) be a complete Riemannian manifold with $\mathrm{Ric}(g)\geq\frac{n-1}{R^2}g$ where $n=\dim M$, then
1. $\mathrm{diam}(M)\leq\pi R$;
2. M is compact.

Proof. For (1). If $\mathrm{diam}(M)>\pi R$, then there exists $l>\pi R$ and a (locally) minimal geodesic $\gamma:[0,l]\to M$ of unit-speed, since M is complete. Choose a parallel orthnormal basis $\{e_1(t)=\gamma'(t),e_2(t),\dots,e_n(t)\}$ with $e_1(t),\dots,e_n(t)$ along γ and set

$$V_i(t)=\sin(\frac{\pi t}{l})e_i(t), i=2,\dots,n$$

It's clear $V_i(0)=V_i(l)=0$ for $i\geq 2$. Let $\alpha:(-\varepsilon,\varepsilon)\to M$ be a variation of γ with variation field $V(t)=\sum_{i=2}^n V_i(t)$, then by second variation formula we have

$$\begin{aligned}\frac{\mathrm{d}^2L(\alpha(t,s))}{\mathrm{d}s^2}\Big|_{s=0}&=\sum_{i=2}^n\int_0^l\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V_i,\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V_i\rangle\mathrm{d}t-\sum_{i=2}^n\int_0^l R(V_i,\gamma',\gamma',V_i)\mathrm{d}t\\&=\sum_{i=2}^n\int_0^l(\frac{\pi}{l})^2\cos^2(\frac{\pi t}{l})\mathrm{d}t-\sum_{i=2}^n\int_0^l\sin^2(\frac{\pi}{l})R(e_i,e_1,e_1,e_i)\mathrm{d}t\\&\leq(n-1)(\frac{\pi}{l})^2\int_0^l\cos^2(\frac{\pi t}{l})\mathrm{d}t-\frac{(n-1)}{R^2}\int_0^l\sin^2(\frac{\pi t}{l})\mathrm{d}t\\&<0\end{aligned}$$

A contradiction to γ is minimal. (2) follows from (1). ☐

Corollary 11.1.1. Let M be a complete Riemannian manifold with positive Ricci curvature, then the universal covering of M is compact. In particular, the fundamental group $\pi_1(M)$ is finite.

Proof. Endow the universal covering $\widetilde{M}$ with pullback metric, thus $\widetilde{M}$ is a complete Riemannian manifold with positive Ricci curvature, thus $\widetilde{M}$ is compact, which implies $\pi:\widetilde{M}\to M$ is a finite covering, thus $\pi_1(M)$ is finite, since $|\pi_1(M)|$ equals the number of sheets of covering. ☐

Corollary 11.1.2. Let (M,g) be a complete Riemannian manifold with sectional curvature $K\geq\frac{1}{R^2}$, then M is compact and $\mathrm{diam}(M)\leq\pi R$ and $\pi_1(M)$ is finite.

Remark 11.1.1. The estimate for the diameter given by Myers's theorem can't be improved. Indeed, the unit sphere $S^n\subset\mathbb{R}^{n+1}$ has constant sectional curvature $K=1$ and $\mathrm{diam}(S^n)=\pi$. A surprising theorem is that this example is unique in the following sense: Let (M,g) be a complete Riemannian manifold with dimension n, $\mathrm{Ric}(g)\geq\frac{n-1}{R^2}g$ and $\mathrm{diam}(M)=\pi R$, then M is isometric to sphere $S^n(R)$.

11.2. **Synge's theorem.**

Lemma 11.2.1. Let A be an orthogonal linear transformation of $\mathbb{R}^{n-1}$ and suppose $\det A = (-1)^n$. Then 1 is an eigenvalue of A.

Proof. If n is even, then $\det(\lambda I - A)$ is a polynomial of odd degree, therefore A has at least a real eigenvalue, and it must be ± 1 since A is orthogonal. Furthermore, since $\det A = 1$ and the product of complex eigenvalue is positive, there is at least a real eigenvalue which equals 1.

If n is odd, then $\det A = -1$. Because the product of complex eigenvalue is positive, there are at least two real eigenvalues, and one of them is 1. $\square$

Theorem 11.2.1 (Synge)**.** Let (M, g) be a compact Riemannian manifold with $K > 0$, then

1. If $\dim M$ is even and M is orientable, then M is simply-connected;
2. If $\dim M$ is odd, then M is orientable;
3. If M is even and M is not orientable, then $\pi_1(M) = \mathbb{Z}_2$.

Proof. Suppose $\pi : \widetilde{M} \to M$ is the universal covering with pullback metric $\widetilde{g} = \pi^* g$

1. If $\dim M$ is even, give $\widetilde{M}$ the pullback orientation;
2. If $\dim M$ is odd, give $\widetilde{M}$ arbitrary orientation.

Suppose the conclusions are not correct, thus $\pi_1(M)$ is non-trivial. Choose a non-trivial deck transformation $F : \widetilde{M} \to \widetilde{M}$ such that

1. If $\dim M$ is even, F is orientation preserving;
2. If $\dim M$ is odd, F is orientation reserving.

By Lemma, there exists an axis $\widetilde{\gamma} : \mathbb{R} \to \widetilde{M}$ for F and $\gamma = \pi \circ \widetilde{\gamma}$ is a closed geodesic in M that minimizes the length in $[\gamma]$,

$$F(\widetilde{\gamma}(t)) = \widetilde{\gamma}(t+1)$$

$\square$

Example 11.1. $\mathbb{RP}^2 \times \mathbb{RP}^2$ admits no metric with positive sectional curvature, since its fundamental group is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Conjecture 11.1 (Hopf conjecture)**.** Does $S^2 \times S^2$ admit a metric with positive sectional curvature?

11.3. **Other facts.**

Theorem 11.3.1. Let (M, g) be a simply-connected compact Riemannian manifold, then

1. (Hamilton) $\dim M = 3$, $\mathrm{Ric}(g) > 0$, then M is diffeomorphism to S^3.
2. (Hamilton) If $\dim M = 4$ with curvature operator > 0, then M is diffeomorphism to S^4.
3. (Böhm-Wilking) If curvature operator > 0, then M is diffeomorphism to S^n.
4. (Brendle-Schoen) If sectional curvature satisfies $\frac{1}{4} < K \leq 1$, then M is diffeomorphism to S^n.

12. Topology of constant sectional curvature manifold

12.1. Isometry.

Theorem 12.1.1. Let $(M,g),(\widetilde{M},\widetilde{g})$ be two Riemannian manifold, $\varphi: M\to\widetilde{M}$ is a bijective. Then the followings are equivalent:
1. φ is an isometry, that is
$$\mathrm{dist}_g(p,q)=\mathrm{dist}_{\widetilde{g}}(\varphi(p),\varphi(q))$$
2. φ is a diffeomorphism and $(\varphi_*)_p: T_pM\to T_{\varphi(p)}\widetilde{M}$ is a linear isometry for all $p\in M$;

Theorem 12.1.2. $\varphi:(M,g)\to(\widetilde{M},\widetilde{g})$ is a smooth map. Then the followings are equivalent:
1. φ is a local isometry;
2. For all $p\in M$, there exists $U\subset M$ containing p such that
$$\varphi|_U: U\to\varphi(U)$$
is an isometry.

Theorem 12.1.3 (Cartan-Ambrose-Hicks)**.** Let (M,g) and $(\widetilde{M},\widetilde{g})$ be two Riemannian manifold $p\in M,\widetilde{p}\in\widetilde{M}$ and $\Phi_0: T_pM\to T_{\widetilde{p}}\widetilde{M}$ some fixed linear isometry. Suppose $\delta\in(0,\min\{\mathrm{inj}_p(M),\mathrm{inj}_{\widetilde{p}}(\widetilde{M})\})$. Then the followings are equivalent:
1. There exists an isometry $\varphi: B(p,\delta)\to B(\widetilde{p},\delta)$ such that $\varphi(p)=\widetilde{p}$ and $(\varphi_*)_p=\Phi_0$;
2. If $v\in T_pM, |v|<\delta, \gamma(t)=\exp_p(tv), \widetilde{\gamma}(t)=\exp_{\widetilde{p}}(t\Phi_0 v)$ and
$$\Phi_t=P^{\widetilde{\gamma}}_{0,t}\circ\Phi_0\circ P^{\gamma}_{t,0}: T_{\gamma(t)}M\to T_{\widetilde{\gamma}(t)}\widetilde{M}$$
Then Φ_t satisfies
$$R(u,v,w,z)=\widetilde{R}(\Phi_t u,\Phi_t v,\Phi_t w,\Phi_t z)$$

Proof. From (1) to (2). If we can show $\Phi_t=(\varphi_*)_{\gamma(t)}$, then it's clear that Φ_t preserves curvature, since φ is an isometry. By definition of Φ_t, it suffices to show the following diagram commutes
$$\begin{tikzcd}
T_pM \arrow[r, "(\varphi_*)_p"] \arrow[d, "P^{\gamma}_{0,t}"] & T_{\widetilde{p}}\widetilde{M} \arrow[d, "P^{\widetilde{\gamma}}_{0,t}"] \\
T_{\gamma(t)}M \arrow[r, "(\varphi_*)_{\gamma(t)}"] & T_{\widetilde{\gamma}(t)}\widetilde{M}
\end{tikzcd}$$
since $(\varphi_*)_p=\Phi_0$. Note that
$$\varphi(\gamma(t))=\widetilde{\gamma}(t)$$
since they agree at $t=0$, so do their derivatives. So it's tautological that
$$P^{\varphi\circ\gamma}_{0,t}\circ(\varphi_*)_p(v)=(\varphi_*)_{\gamma(t)}\circ P^{\gamma}_{0,t}(v)$$

where $v=\gamma'(0)$, since $P^{\gamma}_{0,t}(v)=\gamma'(t), (\varphi_*)_{\gamma(t)}(\gamma'(t))=(\varphi\circ\gamma)'(t)=P^{\varphi\circ\gamma}_{0,t}\circ(\varphi_*)_p(v)$[6]. Now consider $w\in T_pM$ which is not parallel to $v=\gamma'(0)$. Since both $(\varphi_*)_{\gamma(t)}$ and parallel transport preserve angles, so $P^{\varphi\circ\gamma}_{0,t}\circ(\varphi_*)_p(w)$ and $(\varphi_*)_{\gamma(t)}\circ P^{\gamma}_{0,t}(w)$ has the same angle with $(\varphi_*)_{\gamma(t)}(\gamma'(t))$, and the they have the same length, so they're equal.

From (2) to (1). Define

$$\varphi=\exp_{\widetilde{p}}\circ\Phi_0\circ\exp_p^{-1}$$

It suffices to show for any $q\in B(p,\delta)$,

$$(\varphi_*)_q:T_qM\to T_{\varphi(q)}\widetilde{M}$$

is a linear isometry. For any $w\in T_qM$, there exists a geodesic $\gamma:[0,1]\to M$ with $\gamma(0)=p,\gamma(1)=q$ and a Jacobi field J such that $J(0)=0, J(1)=w$ along γ. We claim:

1. **Claim 1**: $\widetilde{J}(t)=\Phi_t(J(t))$ is a Jacobi field;
2. **Claim 2**: $\widetilde{J}(1)=(\varphi_*)_q(J(1))$.

From claim 2 we have

$$|(\varphi_*)_q(w)|=|\widetilde{J}(1)|=|J(1)|=|w|$$

which completes the proof. Now let's give proofs of these two claims.

1. **Proof of Claim 1**: Given an orthnormal $\{e_1(0)=\frac{\gamma'(0)}{|\gamma'(0)|},e_2(0),\dots,e_n(0)\}$ of T_pM and use parallel transport to obtain a parallel frame along γ. With respect to this frame we can write $J(t)=J^i(t)e_i(t)$, then $\widetilde{J}(t)=J^i(t)\widetilde{e}_i(t)$, where $\widetilde{e}_i(t)=\Phi_t(e_i(t))$. Furthermore, $\widetilde{e}_i(t)$ is also a parallel frame by definition of Φ_t. Then $\widetilde{J}(t)$ is a Jacobi field, since
$$\begin{aligned}&\frac{\mathrm{d}^2J^j}{\mathrm{d}t^2}+J^i(t)|\widetilde{\gamma}(t)|^2\widetilde{R}(\widetilde{e}_i(t),\widetilde{e}_1(t),\widetilde{e}_1(t),\widetilde{e}_j(t))\\ =&\frac{\mathrm{d}^2J^j}{\mathrm{d}t^2}+J^i(t)|\gamma(t)|^2R(e_i(t),e_1(t),e_1(t),e_j(t))\\ =&0\end{aligned}$$
holds for arbitrary j, where we use the fact Φ_t preserves the length and curvature, and $J(t)$ is a Jacobi field.
2. **Proof of Claim 2**:

□

Theorem 12.1.4. Let (M,g) be a connected manifold. Suppose φ and ψ are two local isometries from (M,g) to $(\widetilde{M},\widetilde{g})$. If there exists $p\in M$ such that

$$\varphi(p)=\psi(p)$$
$$(\varphi_*)_p=(\psi_*)_p$$

Then $\varphi=\psi$.

[6] These identities hold since both γ and $\varphi\circ\gamma$ are geodesics.

Proof. Suppose $\varphi|_V, \psi|_V$ is diffeomorphism and V is a geodesic ball, then

$$f := (\varphi^{-1} \circ \psi)|_V : V \to V$$

satisfies $f(p) = p, (f_*)_p = \mathrm{id}$. Given $q \in V$, there exists unique $v \in T_pM$ such that $\exp_p(v) = q$, then

$$f(q) = \exp_p \circ \mathrm{id} \circ \exp_p^{-1}(q) = q$$

which implies φ agrees with ψ in V. Consider the following set

$$A = \{p \in M \mid \psi(p) = \varphi(p)\}$$

Above argument shows it's open, and it's clearly closed, then $A = M$, since M is connected. This completes the proof. $\square$

12.2. **Hopf's theorem.**

Theorem 12.2.1 (Hopf)**.** Let (M, g) be a simply-connected complete Riemannian manifold with constant sectional curvature K, then (M, g) is isometric to $(\widetilde{M}, g_{\mathrm{can}})$, where

$$\widetilde{M} = \begin{cases} \mathbb{S}^n(\frac{1}{\sqrt{K}}), & K > 0 \\ \mathbb{R}^n, & K = 0 \\ \mathbb{H}^n(\frac{1}{\sqrt{-K}}), & K < 0 \end{cases}$$

Proof. Let M be a simply-connected complete Riemannian manifold with constant sectional curvature K.

1. If $K \leq 0$, let $\widetilde{M} = \mathbb{R}^n$ or $\mathbb{H}^n(r)$, where $r = \frac{1}{\sqrt{-K}}$. Fix $p \in M, \widetilde{p} \in \widetilde{M}$ and a linear isometry $\Phi_0 : T_pM \to T_{\widetilde{p}}\widetilde{M}$, then Cartan-Ambrose-Hicks's theorem implies
$$\varphi = \exp_{\widetilde{p}} \circ \Phi_0 \circ \exp_p^{-1}$$
maps $B(p, \delta)$ to $B(\widetilde{p}, \delta)$ is an isometry, where $0 < \delta < \min\{\mathrm{inj}_p(M), \mathrm{inj}_p(\widetilde{M})\}$. Thus it's a local isometry defined on $\widetilde{M}$. Furthermore, Cartan-Hadamard's theorem implies φ is a diffeomorphism, since $M, \widetilde{M}$ are simply-connected with non-positive sectional curvature. Combine these facts together we have φ is an isometry.
2. If $K > 0$, let $\widetilde{M} = \mathbb{S}^n(r)$, where $r = \frac{1}{\sqrt{K}}$. Fix $p \in M, \widetilde{p} \in \widetilde{M}$ and a linear isometry $\Phi_0 : T_pM \to T_{\widetilde{p}}\widetilde{M}$, then Cartan-Ambrose-Hicks's theorem implies
$$\varphi_1 = \exp_{\widetilde{p}} \circ \Phi_0 \circ \exp_p^{-1}$$
is an isometry defined on $M\backslash\{-p\}$, since $-p$ is the only cut point of p. Choose another $q \in M\backslash\{p, -p\}$, $\widetilde{q} = \varphi_1(q)$ and $\Psi_0 = (\mathrm{d}\varphi_1)_q : T_qM \to T_{\widetilde{q}}\widetilde{M}$, then Cartan-Ambrose-Hicks's theorem implies
$$\varphi_2 = \exp_{\widetilde{q}} \circ \Psi_0 \circ \exp_q^{-1}$$

is an isometry defined on $M\backslash\{-q\}$ by the same reason. Note that

$$\varphi_2(q) = \widetilde{q} = \varphi_1(q)$$
$$(\mathrm{d}\varphi_2)_q = \Psi_0 = (\mathrm{d}\varphi_1)_q$$

So by Theorem 12.1.4, we have the φ_1 agrees with φ_2 on $M\backslash\{-p,-q\}$. Thus

$$\varphi(x) = \begin{cases} \varphi_1(x), & x \in M\backslash\{-p\} \\ \varphi_2(x), & x \in M\backslash\{-q\} \end{cases}$$

is an isometry from $M \to \widetilde{M}$.

□

Notation 12.2.1. We usually use $S(n,k)$ to denote the complete, simply-connected Riemannian manifold of dimension n with constant sectional curvature k, and call them space forms.

Example 12.2.1. Let (M,g) be a complete Riemannian manifold with constant sectional curvature $K = 1$. If $\dim M = 2m$, then (M,g) is isometric to the sphere $(\mathbb{S}^{2m}, g_{\mathrm{can}})$ or the real projective space $(\mathbb{RP}^{2m}, g_{\mathrm{can}})$.

Proof. Note that Hopf's theorem implies (M,g) is isometric to $(\mathbb{S}^{2m}/\Gamma, g_{\mathrm{can}})$, where Γ is the fundamental group of M, and Synge's theorem implies if $\dim M$ is even and $K > 0$, then $\pi_1(M) = \{e\}$ or $\pi_1(M) = \mathbb{Z}_2$. Combine these two facts together we have M is isometric to $(\mathbb{S}^{2m}, g_{\mathrm{can}})$ or $(\mathbb{RP}^{2m}, g_{\mathrm{can}})$. □

Remark 12.2.1. In general, we have no ideal about what does $\pi_1(M)$ look like.

Proposition 12.2.1. Let (M,g) be a connected, simply connected, complete Riemannian manifold. The following are equivalent:

1. (M,g) has constant sectional curvature;
2. For every pair of point $p,q \in M$ and every linear isometry $\Phi_0 : T_pM \to T_qM$, there exists an isometry $\varphi : M \to M$ such that $\varphi(p) = q, (\varphi_*)_p = \Phi_0$.

Proof. (1) to (2) is already shown in the proof of Hopf's theorem. For (2) to (1). Firstly let's show for $p \in M$, sectional curvature $K_p(\sigma)$ at p is independent of σ, where σ is a 2-dimensional subspace of T_pM. For arbitrary two 2-dimensional subspaces σ_1, σ_2 of T_pM. By tricks of linear algebra it's easy to find an linear isometry Φ_0 such that $\Phi_0\sigma_1 = \sigma_2$, then there exists an isometry $\varphi : M \to M$ such that $(\varphi_*)_p = \Phi_0$. Since isometry preserves curvature, in particular we have

$$R(x,y,y,x) = R(\Phi_0 x, \Phi_0 y, \Phi_0 y, \Phi_0 x)$$

where $\{x,y\}$ is a basis of σ_1 and $\{\Phi_0 x, \Phi_0 y\}$ is a basis of σ_2, which implies $K_p(\sigma_1) = K_p(\sigma_2)$.

Now let's show for arbitrary $p, q \in M$, sectional curvature $K_p = K_q$[7]. For $p, q \in M$, 2-dimensional subspace σ_1, σ_2 of T_pM, T_qM respectively, we also can find a linear isometry $\Phi_0 : T_pM \to T_qM$ such that $\Phi_0\sigma_1 = \sigma_2$, and there also exists an isometry $\varphi : M \to M$ such that $(\varphi_*)_p = \Phi_0$, then the same argument shows $K_p = K_q$. $\square$

[7]In fact, if $\dim M \geq 3$, then Schur's lemma implies K_p is independent of p.

Part 4. Comparision theorems

13. Preparations

In this section we select some basic tools we will used in later computations.

13.1. Radial vector field.

In this section (M, g) is a Riemannian manifold and $p \in M$, we always assume (x^i, U, p) is a normal coordinate centered at p.

Definition 13.1.1 (radial distance function)**.** The radial distance function r defined on U is given by

$$r(q) := \sqrt{\sum_{i=1}^{n} (q^i)^2}$$

where $q = (q^1, \dots, q^n)$ in normal coordinates (x^i, U, p).

Definition 13.1.2 (radial vector field)**.** The radial vector field on $U \backslash \{p\}$ is defined as

$$\partial_r = \frac{x^i}{r} \frac{\partial}{\partial x^i}$$

Proposition 13.1.1. The geodesic starting at p with unit-speed is the integral curve of radial vector field ∂_r over $U \backslash \{p\}$.

Proof. We need to show for geodesic $\gamma : I \to U \subset M$ with $\gamma(0) = p, \gamma'(0) = v$, where $|v| = 1$, we have

$$\gamma'(b) = \partial_r|_{\gamma(b)}$$

where I is an open interval and $b \in I$.

In normal coordinates γ looks like $\gamma(t) = (tv^1, \dots, tv^n)$. If we denote $\gamma(b) = q = (q^1, \dots, q^n)$, then it's clear $v^i = q^i/b$. Furthermore, $r(q) = b$, since $|v| = 1$. Then in normal coordinates,

$$\gamma'(b) = v^i \frac{\partial}{\partial x^i}\bigg|_q = \frac{q^i}{b} \frac{\partial}{\partial x^i}\bigg|_q = \frac{q^i}{r(q)} \frac{\partial}{\partial x^i}\bigg|_q = \partial_r|_q$$

□

Theorem 13.1.1. For radial vector field ∂_r, we have the following properties:

1. $|\partial_r|^2 = 1$;
2. $g^{ij} \frac{\partial r}{\partial x^i} \frac{\partial}{\partial x^j} = \nabla r = \partial_r$.

Proof. For (1). It's clear, since we have already shown geodesic with unit-speed is integral curve of ∂_r.

For (2). We need the following lemma:

Lemma 13.1.1. Given a smooth function $f : M \to \mathbb{R}$ and X is a vector field, if

1. $Xf = |X|^2$;
2. X is perpendicular to the level set of f.

then $X = \nabla f$.

Now apply $X = \partial_r$ to $f = r$, we have

$$Xr = \frac{x^i}{r}\frac{\partial r}{\partial x^i} = \sum_{i=1}^n \frac{(x^i)^2}{r^2} = 1 = |\partial_r|^2$$

This shows the first condition in above lemma. For any $q \in U\backslash\{p\}$ we write it as $q = (q^1, \dots, q^n)$ in normal coordinates with $b = r(q)$. Given $w \in T_qM$ which is tangent to the level set of r, there exists $c(s) : (-\varepsilon, \varepsilon) \to M$ such that $c(0) = q, c'(0) = w$ with $\sum_{i=1}^n (c^i(s))^2 = b$, where c^i is the coordinates of c in normal coordinates. Taking derivative with respect to s we obtain

$$\sum_{i=1}^n 2c^i(s)(c^i(s))' = 0$$

We're almost there, since $w = (c^i(0))' \frac{\partial}{\partial x^i}\Big|_q, \partial_r|_q = \frac{c^j(0)}{b}\frac{\partial}{\partial x^j}\Big|_q$ and if metric at T_qM is standard, then we're done. However, we only know metric at T_pM is standard, so we may use parallel transport to transport w to T_pM and show they're perpendicular in T_pM, which implies they're perpendicular in T_qM, since geodesic is integral curve of ∂_r.

□

Corollary 13.1.1. In normal coordinates (x^i, U, p) centered at p, the following identities hold:

1. $g_{ij}x^j = x^i$;
2. $g_{im} = \delta_{im} - \frac{\partial g_{ij}}{\partial x^m}x^j$;
3. $\frac{\partial g_{ij}}{\partial x^m}x^j = \frac{\partial g_{mj}}{\partial x^i}x^j$;
4. $\frac{\partial g_{ij}}{\partial x^m}x^j x^i = \frac{\partial g_{mj}}{\partial x^i}x^j x^i = 0$;
5. $\Gamma^k_{ij}x^i x^j = 0$;
6. $\nabla_{\partial_r}\partial_r = 0$ on $U\backslash\{p\}$.

Proof. For (1). On one hand by Theorem 13.1.1 we have $\partial_r = \nabla r = g^{ij}\frac{x^i}{r}\frac{\partial}{\partial x^j}$; On the other hand by definition of ∂_r we have $\partial_r = \frac{x^j}{r}\frac{\partial}{\partial x^j}$, which implies

$$g^{ij}x^i = x^j$$

This shows (1).

For (2). Take partial derivatives of (1) with respect to x^m, we have

$$\frac{\partial g_{ij}}{\partial x^m}x^j + g_{ij}\delta_{jm} = \delta_{im}$$

This shows (2).

For (3). It follows from (2), since g_{im}, δ_{im} are symmetric in i, m.

For (4). It follows from (1) and (2), since

$$\frac{\partial g_{ij}}{\partial x^m}x^jx^i \overset{(2)}{=} (\delta_{im} - g_{im})x^i = x^m - g_{im}x^i \overset{(1)}{=} 0$$
$$\frac{\partial g_{mj}}{\partial x^i}x^jx^i \overset{(2)}{=} (\delta_{mi} - g_{mi})x^i = x^m - g_{im}x^i \overset{(1)}{=} 0$$

For (5). It follows from (4) and

$$\Gamma_{ij}^k = \frac{1}{2}g^{mk}(\frac{\partial g_{mj}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m})$$

For (6). Direct computation shows

$$\begin{aligned}\nabla_{\partial_r}\partial_r &= \frac{x^k}{r}\nabla_{\frac{\partial}{\partial x^k}}(g^{ij}\frac{x^i}{r}\frac{\partial}{\partial x^j}) \\ &= g^{ij}\frac{x^k}{r}\{\underbrace{(\frac{\delta_{ki}}{r} - \frac{x^kx^i}{r^3})\frac{\partial}{\partial x^j}}_{\text{part I}} + \underbrace{\frac{x^i}{r}\Gamma_{kj}^m\frac{\partial}{\partial x^m}}_{\text{part II}}\}\end{aligned}$$

By (1) and (5) we have

$$g^{ij}\frac{x^kx^i}{r}\Gamma_{kj}^m = \frac{1}{r}\Gamma_{kj}^m x^kx^j = 0$$

which implies part II is zero. For part I, we have

$$\frac{1}{r^2}(g^{ij}x^k\delta_{ki} - \frac{(x^k)^2}{r^2}g^{ij}x^i) = \frac{1}{r^2}(g^{ij}x^i - g^{ij}x^i) = 0$$

□

Remark 13.1.1. Note that we firstly establish the fact unit-speed geodesic is integral curve of ∂_r and show $\partial_r = \nabla r$, then we obtain lots of identities. In particular we have $\nabla_{\partial_r}\partial_r = 0$, which also implies unit-speed geodesic is integral curve of ∂_r. This shows over $U\backslash\{p\}$ the following statements are equivalent:

1. The unit-speed geodesic is integral curve of ∂_r;
2. $g^{ij}x^i = x^j$;
3. $\nabla_{\partial_r}\partial_r = 0$.

13.2. Jacobi fields on constant sectional curvature manifold.

Proposition 13.2.1. Let (M, g) be a Riemannian manifold with constant sectional curvature k and $\gamma : [0, b] \to M$ be a unit-speed geodesic. Then the normal Jacobi field with $J(0) = 0$ is of the form

$$J(t) = m\,\mathrm{sn}_k(t)E(t)$$

where

1. The constant m is determined by $J'(0) = mE(0)$;

2.

$$\mathrm{sn}_k(t) = \begin{cases} t, & k=0 \\ \frac{\sin(\sqrt{k}t)}{\sqrt{k}}, & k>0 \\ \frac{\sinh(\sqrt{-k}t)}{\sqrt{-k}}, & k<0 \end{cases}$$

3. $E(t)$ is a normal parallel vector field along γ with $|E(t)| = 1$

Proof. Since (M, g) has constant sectional curvature k, thus $R_{ijkl} = k(g_{il}g_{jk} - g_{ik}g_{jl})$, so for any normal vector field J along γ we have

$$\begin{aligned} R(J, \gamma', \gamma', W) &= k(\langle J, W\rangle\langle\gamma', \gamma'\rangle - \langle J, \gamma'\rangle\langle\gamma', W\rangle) \\ &= k\langle J, W\rangle \end{aligned}$$

which implies

$$R(J, \gamma')\gamma' = kJ$$

since γ is unit-speed and J is normal. Thus equation for Jacobi field can be written as

$$0 = J'' + kJ$$

Assume $J = u(t)E(t)$, then

$$[u''(t) + ku(t)]E(t) = 0$$

So if we want to find normal Jacobi fields J, it suffices to solve

$$\begin{cases} u''(t) + ku = 0 \\ u(0) = 0 \end{cases}$$

and it's clear $\mathrm{sn}_k(t)$ is solution of this ODE. □

13.3. **Polar decomposition of constant curvature metric.** Let $\pi : \mathbb{R}^n \backslash \{0\} \to S^{n-1}$ defined by $\pi(x) = x/|x|$. We can use π to pullback canonical metric on S^{n-1}, and still use $g_{S^{n-1}}$ to denote it.

Lemma 13.3.1. Let $\overline{g}$ be the Euclidean metric on $\mathbb{R}^n\backslash\{0\}$, then

$$\overline{g} = \mathrm{d}r \otimes \mathrm{d}r + r^2 g_{S^{n-1}}$$

where $r(x) = |x|$.

Theorem 13.3.1 (constant curvature metric in normal coordinates)**.** For $p \in S(n, k)$ and (x^i, U, p) is a normal coordinate centered at p, where U is a geodesic ball, then

$$(U\backslash\{p\}, g) \overset{\text{isotropic}}{=} ((0, \rho) \times S^{n-1}, \mathrm{d}r \otimes \mathrm{d}r + \mathrm{sn}_k^2(r) g_{S^{n-1}})$$

where r is radial distance function.

Proof. We use g_c to denote metric $\mathrm{d}r \otimes \mathrm{d}r + \mathrm{sn}_k^2(r) g_{S^{n-1}}$ on $(0, \rho) \times S^{n-1}$. By proof of (1) of Theorem 13.1.1, we have

$$g(\partial_r, \partial_r) = g_c(\frac{x^i}{r}\frac{\partial}{\partial x^i}, \frac{x^i}{r}\frac{\partial}{\partial x^i})$$

So it remains to show for each $q \in U\backslash\{p\}$ and $w \in T_qM$ such that $g(w, \partial_r|_q) = 0$, we have

$$g(w,w) = g_c(w^i \frac{\partial}{\partial x^i}\bigg|_q, w^i \frac{\partial}{\partial x^i}\bigg|_q) = \mathrm{sn}_k^2(r)|w|^2_{g_{S^{n-1}}}$$

where w can be written as $w^i \frac{\partial}{\partial x^i}\big|_q$ in normal coordinates. Let $b = \mathrm{dist}(p,q)$ and $\gamma : [0,b] \to U$ a unit-speed geodesic connecting p, q. In normal coordinates we can write it as

$$\gamma(t) = (\frac{tq^1}{b}, \dots, \frac{tq^n}{b})$$

Let J be a Jacobi field such that $J(0) = 0, J(b) = w$. On one hand,

$$|w|^2_g = |J(b)|^2_g = \mathrm{sn}_k^2(b)|J'(0)|^2_g = \mathrm{sn}_k^2(b)|J'(0)|^2_{\overline{g}}$$

where $\overline{g}$ is standard metric on T_pM, since

1. The metric on T_pM is standard metric in normal coordinates;
2. Jacobi field on constant sectional curvature space is of form $J(t) = |J'(0)|\,\mathrm{sn}_k(t)E(t)$.

On the other hand, suppose $J'(0) = a$, then we can write it as $J(t) = \alpha_*(\frac{\partial}{\partial s})\big|_{s=0}$, where

$$\alpha(s,t) = \exp_p(t(\gamma'(0) + sJ'(0)))$$

In normal coordinates we can write $\alpha(s,t)$ explicitly as

$$\alpha(s,t) = (\frac{tq^1}{b} + tsa^1, \dots, \frac{tq^n}{b} + tsa^n)$$

thus $J(t) = ta^i \frac{\partial}{\partial x^i}\big|_{\gamma(t)}$. We can conclude $a^i = \frac{w^i}{b}$ by setting $t = b$, in particular we have $J'(0) = \frac{w^i}{b}\frac{\partial}{\partial x^i}\Big|_p$. Then

$$\begin{aligned}
\mathrm{sn}_k^2(b)|J'(0)|^2_{\overline{g}} &= \mathrm{sn}_k^2(b)\frac{|w|^2_{\overline{g}}}{b^2} \\
&= \mathrm{sn}_k^2(b)\frac{|w|^2_{b^2 g_{S^{n-1}}}}{b^2} \\
&= \mathrm{sn}_k^2(b)|w|^2_{g_{S^{n-1}}}
\end{aligned}$$

where the second equality holds from polar decomposition of standard metric. $\square$

13.4. **A criterion for constant sectional curvature space.** Recall that for a smooth function $f : M \to \mathbb{R}$, its Hessian $\mathrm{Hess}\, f$ is a $(0,2)$-tensor, then its $(1,1)$-type $\mathscr{H}_f$ is given by

$$g(\mathscr{H}_f(X), Y) := \mathrm{Hess}\, f(X, Y)$$

where X, Y are two vector fields.

Proposition 13.4.1. Let (M,g) be a complete Riemannian manifold, U a geodesic ball around p and r the radial distance function on U. If $\gamma:[0,b]\to M$ is unit-speed geodesic with $\gamma(0)=p,\gamma'(0)=v\in T_pM$, and J is a normal Jacobi field along γ with $J(0)=0$. Then for all $t\in[a,b]$

$$\begin{aligned}\mathscr{H}_r(J(t))&=J'(t)\\ \mathscr{H}_r(\gamma'(t))&=0\end{aligned}$$

In particular, for any vector field W along γ with $W(0)=0$,

$$\begin{aligned}(\operatorname{Hess} r)(J(s),W(s))&=g(\mathscr{H}_r(J(s),W(s)))\\ &=g(J'(t),W(s))\\ &=\int_0^s\langle J'(t),W(t)\rangle'\mathrm{d}t\\ &=\int_0^s\langle J'(t),W'(t)\rangle-R(J,\gamma',\gamma',W)\mathrm{d}t\end{aligned}$$

Proof. Here we only prove the first identity, the second can be computed in the same method. Let $J'(0)=a$, then $J(t)=ta^i\frac{\partial}{\partial x^i}\big|_{\gamma(t)}$,

$$\begin{aligned}J'(t)&=\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}(ta^i\frac{\partial}{\partial x^i})\\ &=a^i\frac{\partial}{\partial x^i}+ta^i\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\frac{\partial}{\partial x^i}\\ &=a^i\frac{\partial}{\partial x^i}+ta^i\Gamma_{ij}^k(\gamma(t))\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\partial}{\partial x^k}\\ &=(a^k+ta^iv^j\Gamma_{ij}^k(\gamma(t)))\frac{\partial}{\partial x^k}\\ \mathscr{H}_r(J(t))&=\nabla_{J(t)}\partial_r\\ &=\nabla_{ta^i\frac{\partial}{\partial x^i}}(\frac{x^j}{r}\frac{\partial}{\partial x^j})\\ &=ta^i\nabla_{\frac{\partial}{\partial x^i}}(\frac{x^j}{r}\frac{\partial}{\partial x^j})\\ &=ta^i\frac{x^j}{r}\Gamma_{ij}^k\frac{\partial}{\partial x^k}+\sum_{i=1}^n ta^i(\frac{\delta_{ij}}{r}-\frac{x^ix^j}{r^3})\frac{\partial}{\partial x^j}\end{aligned}$$

Note that

$$\begin{aligned}r(\gamma(t))&=t\\ x^i&=tv^i\\ \sum_{i=1}^n a^iv^i&=0\end{aligned}$$

where the last equality holds since J is a normal vector field, then

$$0=\langle J(t),\gamma'(t)\rangle=\langle J(0),\gamma'(0)\rangle+\langle J'(0),\gamma'(0)\rangle t$$

implies $\langle J'(0),\gamma'(0)\rangle=\sum_{i=1}^n a^iv^i=0$. □

Corollary 13.4.1. Let $p \in U \subset S(n,k)$, then the following holds on $U \backslash \{p\}$

$$\mathscr{H}_r = \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\pi_r$$

where $\pi_r : T_qS(n,k) \to W \subset T_qS(n,k), W \oplus \{\partial_r\} = T_qS(n,k)$. In particular

$$\begin{aligned}\operatorname{Hess} r &= \mathrm{sn}_k'(r)\,\mathrm{sn}_k(r) g_{S^{n-1}} \\ &= \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}(g - \mathrm{d}r \otimes \mathrm{d}r) \\ \Delta r &= (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)} \\ \Delta r^2 &= 2 + 2(n-1)\frac{\mathrm{sn}_k'(r) r}{\mathrm{sn}_k(r)}\end{aligned}$$

Proof. Let $E(t)$ be a normal parallel vector field along unit-speed geodesic $\gamma : [0,b] \to M$ with $|E(t)| = 1$, and consider

$$J(t) = c\,\mathrm{sn}_k(t)E(t), J(0) = 0$$

Then

$$\begin{aligned}c\,\mathrm{sn}_k'(t)E(t) &= J'(t) \\ &= \mathscr{H}_r(J(t)) \\ &= \mathscr{H}_r(c\,\mathrm{sn}_k(t)E(t)) \\ &= c\,\mathrm{sn}_k(t)\mathscr{H}_r(E(t))\end{aligned}$$

□

Proposition 13.4.2. Let (M,g) be a Riemannian manifold and U a geodesic ball around $p \in M$, r radius distant function. If

$$\mathscr{H}_r = \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\pi_r$$

holds in $U \backslash \{p\}$, then (M,g) has constant sectional curvature k in U.

Proof. Let $\gamma : [0,b] \to U$ be a unit-speed geodesic $r(0) = p$, J is a normal Jacobi vector field along γ with $J(0) = 0$, then $\mathscr{H}_r(J) = J'$ implies

$$J'(t) = \frac{\mathrm{sn}_k'(t)}{\mathrm{sn}_k(t)}J(t)$$

that is

$$(\frac{J(t)}{\mathrm{sn}_k(t)})' = 0$$

So we can write every normal Jacobi fields as $J(t) = m\,\mathrm{sn}_k(t)E(t)$, where E is normal a parallel vector field with $|E| = 1$, then

$$\begin{aligned} R(\gamma'(t), E(t), E(t), \gamma'(t)) &= -\frac{\langle J''(t), J(t)\rangle}{m^2\,\mathrm{sn}_k^2(t)} \\ &= -\frac{\mathrm{sn}_k''(t)}{\mathrm{sn}_k(t)} \\ &= k \end{aligned}$$

□

14. Comparision theorems based on sectional curvature

14.1. Rauch comparision. In this section, we will see the following philosophy: "The larger sectional curvature is, the smaller the distance is."

Theorem 14.1.1 (Rauch comparision)**.** Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be two Riemannian manifold with $\dim M \leq \dim \widetilde{M}$. Suppose $\gamma : [0, b] \to M$ and $\widetilde{\gamma} : [0, b] \to \widetilde{M}$ are two unit-speed geodesics such that

1. For all $t \in [0, b]$, and any planes $\Sigma \subset T_{\gamma(t)}M, \gamma'(t) \in \Sigma, \widetilde{\Sigma} \subset T_{\widetilde{\gamma}(t)}\widetilde{M}, \widetilde{\gamma}'(t) \in \widetilde{\Sigma}$, we have $K_{\gamma(t)}(\Sigma) \leq K_{\widetilde{\gamma}(t)}(\widetilde{\Sigma})$;
2. $\widetilde{\gamma}(0)$ has no conjugate points along $\widetilde{\gamma}|_{[0,b]}$.

Then for any Jacobi fields $J(t)$ and $\widetilde{J}(t)$ with

1. $$\begin{cases} J(0) = c\gamma'(0) \\ \widetilde{J}(0) = c\widetilde{\gamma}'(0) \end{cases}$$
2. $|J'(0)| = |\widetilde{J}'(0)|$;
3. $\langle J'(0), \gamma'(0)\rangle = \langle \widetilde{J}'(0), \widetilde{\gamma}'(0)\rangle$.

we have $|J(t)| \geq |\widetilde{J}(t)|$ for all $t \in [0, b]$.

Proof. Firstly we consider the following simple case:

1. $J(0) = \widetilde{J}(0) = 0$;
2. $|J'(0)| = |\widetilde{J}'(0)|$;
3. $\langle J'(0), \gamma'(0)\rangle = \langle \widetilde{J}'(0), \widetilde{\gamma}'(0)\rangle = 0$.

Since $\widetilde{\gamma}(0)$ has no conjugate points along $\widetilde{\gamma}|_{[0,b]}$, then $\frac{|J(t)|^2}{|\widetilde{J}(t)|^2}$ is well-definedd for all $t \in (0, b]$, and standard calculus implies

$$\lim_{t\to 0} \frac{|J|^2}{|\widetilde{J}^2|} = \lim_{t\to 0} \frac{\langle J'(t), J(t)\rangle}{\langle \widetilde{J}'(t), \widetilde{J}(t)\rangle} = \lim_{t\to 0} \frac{|J'|^2}{|\widetilde{J}'|^2} = 1$$

So it suffices to show in $(0, b]$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{|J|^2}{|\widetilde{J}|^2}\right) \geq 0$$

Direct computation shows above inequality is equivalent to:

$$\frac{\langle J'(t), J(t)\rangle}{|J(t)|^2} \geq \frac{\langle \widetilde{J}'(t), \widetilde{J}(t)\rangle}{|\widetilde{J}(t)|^2}$$

holds for arbitrary $t \in (0, b]$. For arbitrary $s \in (0, b]$, we can define the following Jacobi fields by scaling $J(t)$:

$$W_s(t) = \frac{J(t)}{|J(s)|}, \quad \widetilde{W}_s(t) = \frac{\widetilde{J}(t)}{|\widetilde{J}(s)|}$$

Then

$$\frac{\langle J'(s), J(s)\rangle}{|J(s)|^2} = \langle W_s'(s), W_s(s)\rangle$$

So it suffices to show

$$\langle W_s'(s), W_s(s)\rangle \geq \langle \widetilde{W}_s'(s), \widetilde{W}_s(s)\rangle$$

holds for arbitrary $s \in (0, b]$. Direct computation shows:

$$\begin{aligned}
\langle W_s'(s), W_s(s)\rangle &= \int_0^s (\langle W_s(t), W_s(t)\rangle)' \mathrm{d}t \\
&= \int_0^s \langle W_s'(t), W_s'(t)\rangle \mathrm{d}t + \int_0^s \langle W_s''(t), W_s(t)\rangle \mathrm{d}t \\
&= \int_0^s \langle W_s'(t), W_s'(t)\rangle \mathrm{d}t - \int_0^s R(W_s(t), \gamma'(t), \gamma'(t), W(t)) \mathrm{d}t
\end{aligned}$$

Choose a parallel orthnormal frame $\{e_1(t), \dots, e_n(t)\}$ with $e_1(t) = \gamma'(t), e_2(t) = W_s(t)$. With respect to this frame we write

$$W_s(t) = \lambda^i(t) e_i(t)$$

Similarly we choose a parallel orthogonal frame $\{\widetilde{e}_1(t), \dots, \widetilde{e}_n(t)\}$ and construct the following vector field

$$\widetilde{V}(t) = \lambda^i(t)\widetilde{e}_i(t)$$

Then it's clear we have

$$\int_0^s \langle W_s'(t), W_s'(t)\rangle \mathrm{d}t = \int_0^s \langle \widetilde{V}'(t), \widetilde{V}'(t)\rangle \mathrm{d}t$$

and our curvature condition implies

$$\int_0^s R(W_s(t), \gamma'(t), \gamma'(t), W_s(t)) \mathrm{d}t \leq \int_0^s \widetilde{R}(\widetilde{V}(t), \gamma'(t), \gamma'(t), \widetilde{V}(t)) \mathrm{d}t$$

Thus we have

$$\begin{aligned}
\langle W_s'(s), W_s(s)\rangle &\leq \int_0^s \langle \widetilde{V}'(t), \widetilde{V}'(t)\rangle \mathrm{d}t - \int_0^s R(\widetilde{V}(t), \gamma'(t), \gamma'(t), \widetilde{V}(t)) \mathrm{d}t \\
&= \widetilde{I}(\widetilde{V}, \widetilde{V})
\end{aligned}$$

where $\widetilde{I}$ is index form on $\widetilde{M}$. According to Corollary 8.3.1, we have

$$\widetilde{I}(\widetilde{V}, \widetilde{V}) \geq \widetilde{I}(\widetilde{W}_s, \widetilde{W}_s)$$

since $\widetilde{W}_s$ is a Jacobi field. This shows the desired result.

For general case, we consider the following decomposition

$$J(t) = J_1(t) + \langle J(t), \gamma'(t)\rangle \gamma'(t)$$
$$\widetilde{J}(t) = \widetilde{J}_1(t) + \langle \widetilde{J}(t), \widetilde{\gamma}'(t)\rangle \widetilde{\gamma}'(t)$$

Then it's clear $J_1(t)$ and $\widetilde{J}_1(t)$ satisfy requirement of our simple case, that is for $t \in [0,1]$ we have

$$|J_1(t)| \geq |\widetilde{J}_1(t)|$$

Furthermore,

$$\langle J(t), \gamma'(t)\rangle = \langle \widetilde{J}(t), \widetilde{\gamma}'(t)\rangle$$

always holds, since

$$\begin{aligned}\langle J(t), \gamma'(t)\rangle &= \langle J(0), \gamma'(0)\rangle + \langle J'(0), \gamma'(0)\rangle t \\ &\overset{*}{=} \langle \widetilde{J}(0), \widetilde{\gamma}'(0)\rangle + \langle \widetilde{J}'(0), \widetilde{\gamma}'(0)\rangle t \\ &= \langle \widetilde{J}(t), \widetilde{\gamma}'(t)\rangle\end{aligned}$$

where $*$ holds from our assumption. □

Corollary 14.1.1. Let (M, g) be a Riemannian manifold, U a geodesic ball containing $p \in M$, $\gamma : [0, b] \to U$ a unit-speed geodesic with $\gamma(0) = p$ and J a Jacobi field along γ with $J(0) = 0$.

1. If the sectional curvature $K \leq k$ in U, then $|J(t)| \geq \mathrm{sn}_k(t)|J'(0)|$, for all $t \in [0, d']$, where
$$d' = \begin{cases} d, & k \leq 0 \\ \min\{d, \pi R\}, & k = \frac{1}{R^2} > 0 \end{cases}$$
2. If the sectional curvature $K \geq k$ in U, then
$$|J(t)| \leq \mathrm{sn}_k(t)|J'(0)|$$
for all $t \in [0, b]$.

Proof. Apply Rauch comparision between M and constant sectional curvature Riemannian manifold. □

Remark 14.1.1. In particular, from above corollary, we immediately have the following corollary when $K \leq k$:

1. If $k \leq 0$, then M has no conjugate point along any geodesic;
2. If $k = \frac{1}{R^2} > 0$, then there is no conjugate point along any geodesic shorter than πR.

Corollary 14.1.2 (metric comparision)**.** Let (M, g) be a Riemannian manifold with $\dim M = n$, U a geodesic ball containing $p \in M$. For all $k \in \mathbb{R}$, we use g_k to denote the metric $\mathrm{d}r \otimes \mathrm{d}r + \mathrm{sn}_k(r) g_{S^{n-1}}$ on $U \backslash \{p\}$.

1. If the sectional curvature K such that $K \leq k$ for all $q \in U\backslash\{p\}$, then for $w \in T_q M$ we have
$$g(w, w) \geq g_k(w, w)$$

2. If the sectional curvature K such that $K \geq k$ for all $q \in U\backslash\{p\}$, then for $w \in T_qM$ we have
$$g(w,w) \leq g_k(w,w)$$

Corollary 14.1.3. Let (M,g) and $(\widetilde{M},\widetilde{g})$ be two Riemannian manifolds with $K \leq \widetilde{K}$. Fix $p \in M, \widetilde{p} \in \widetilde{M}$, linear isometry $\Phi_0 : T_pM \to T_{\widetilde{p}}\widetilde{M}$ and $0 \leq \delta < \min(\mathrm{inj}(p), \mathrm{inj}(\widetilde{p}))$. Then for any smooth curve $\gamma : [0,1] \to \exp_p(B(0,\delta))$ and $\widetilde{\gamma}(t) = \exp_{\widetilde{p}} \circ \Phi_0 \circ \exp_p^{-1}(\gamma(t))$, we have
$$L(\gamma) \geq L(\widetilde{\gamma})$$

Proof. Let $c(s) = \exp_p^{-1} \circ \gamma(s)$ and $\widetilde{c}(s) = \exp_{\widetilde{p}}^{-1} \circ \widetilde{\gamma}(s)$, then $\widetilde{c}(s) = \Phi_0(c(s))$. Consider the following variations
$$\alpha(t,s) = \exp_p(tc(s))$$
$$\widetilde{\alpha}(t,s) = \exp_{\widetilde{p}}(t\widetilde{c}(s))$$
and Jacobi fields
$$J_s(t) = \alpha_*(\frac{\partial}{\partial s})(t,s)$$
$$\widetilde{J}_s(t) = \widetilde{\alpha}_*(\frac{\partial}{\partial s})(t,s)$$
A crucial obeservation is for arbitrary $s_0 \in [0,1]$, we have
$$J_{s_0}(1) = \gamma'(s_0)$$
$$\widetilde{J}_{s_0}(1) = \widetilde{\gamma}'(s_0)$$
So it suffices to prove $|J_{s_0}(1)| \geq |\widetilde{J}_{s_0}(1)|$ holds for arbitrary $s_0 \in [0,1]$, that is we need to use Rauch comparision to Jacobi fields $J_{s_0}(t), \widetilde{J}_{s_0}(t)$ along γ_{s_0} and $\widetilde{\gamma}_{s_0}$, where $\gamma_{s_0}(t) = \alpha(t,s_0)$ and $\widetilde{\gamma}_{s_0}(t) = \widetilde{\alpha}(t,s_0)$. Check requirements as follows:

1. $J_{s_0}(0) = \widetilde{J}_{s_0}(0) = 0$;
2. $J'_{s_0}(0) = c'(s_0), \widetilde{J}'_{s_0}(0) = \widetilde{c}'(s_0)$, and $\widetilde{c}(s_0) = \Phi_0(c(s_0))$ implies $|J'_{s_0}(0)| = |\widetilde{J}'_{s_0}(0)|$, since Φ_0 is linear isometry;
3. $\langle \widetilde{J}'_{s_0}(0), \widetilde{\gamma}'_{s_0}(0)\rangle = \langle \Phi_0(c'(s_0)), \Phi_0(c(s_0))\rangle = \langle c'(s_0), c(s_0)\rangle = \langle J'_{s_0}(0), \gamma'_{s_0}(0)\rangle$.

□

Corollary 14.1.4. Let (M,g) be an Riemannian manifold with $\dim M = n$, $0 < c_1 \leq K \leq c_2$. Let γ be any geodesic in M and d the distance along γ between two consective conjugate point, then
$$\frac{\pi}{\sqrt{c_2}} \leq d \leq \frac{\pi}{\sqrt{c_1}}$$

Proof. Without lose of generality, we assume $\gamma : [0,d] \to M$ is a unit-speed geodesic with $\gamma(0) = p, \gamma(d) = q$ and p, q are two consective conjugate point along γ.

1. Apply Rauch comparision to (M, g) and $(\mathbb{S}^n(\frac{\pi}{\sqrt{c_2}}), g_{\text{can}})$, we have
$$|J(t)| \geq |\widetilde{J}(t)|$$
for $t \in [0, d]$, where $J(t)$ is a Jacobi field along γ and $\widetilde{J}(t)$ is a Jacobi field along some unit-speed geodesic in $\mathbb{S}^n(\frac{\pi}{\sqrt{c_2}})$. So for arbitrary $t < \frac{\pi}{\sqrt{c_2}}$, we must have
$$|J(t)| \geq |\widetilde{J}(t)| > 0$$
which implies $d \geq \frac{\pi}{\sqrt{c_2}}$.
2. Apply Rauch comparision to (M, g) and $(\mathbb{S}^n(\frac{\pi}{\sqrt{c_1}}), g_{\text{can}})$, we have
$$|J(t)| \leq |\widetilde{J}(t)|$$
for $t \in [0, d]$, where $J(t), \widetilde{J}(t)$ are defined the same as before. Suppose $d > \frac{\pi}{\sqrt{c_1}}$, then take $t = \frac{\pi}{\sqrt{c_1}}$, we have
$$0 < |J(t)| \leq |\widetilde{J}(t)| = 0$$
A contradiction.

□

Theorem 14.1.2. Let (M, g) be a compact Riemannian manifold with sectional curvature $K \leq c, c > 0$. If we define
$$l(M, g) := \inf\{L(\gamma) \mid \gamma \text{ is a smooth closed geodesic in } M\}$$
Then either $\operatorname{inj}(M) \geq \frac{\pi}{\sqrt{c}}$ or $\operatorname{inj}(M) = \frac{l(M,g)}{2}$.

Proof. By compactness of M, there exists $p, q \in M, q \in \operatorname{cut}(p)$ such that $\operatorname{dist}(p, q) = \operatorname{inj}(M) = \operatorname{inj}(p)$. Let $\gamma : [0, d] \to M$ be a minimal geodesic connecting p and q, that is $d = \operatorname{dist}(p, q) = \operatorname{inj}(M)$.
1. If p and q are not conjugate along γ, then by Corollary 14.1.4 we have $\operatorname{inj}(M) = d \geq \frac{\pi}{\sqrt{c}}$.
2. If p and q are not conjugate along γ, then by Proposition 9.1.1 there exists a unit-speed closed geodesic $\gamma : [0, 2d] \to M$ with $\gamma(0) = p, \gamma(d) = q$, where $d = \operatorname{dist}(p, q) = \operatorname{inj}(M)$. On one hand by definition of $l(M, g)$ one has $2d \geq l(M, g)$; On the other hand, $l(M, g) \geq 2d$, since $\operatorname{dist}(p, q) = q$. Thus in this case $\operatorname{inj}(M) = \frac{l(M,g)}{2}$.

□

14.2. **Hessian comparision.**

Theorem 14.2.1 (Hessian comparision)**.** Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be two Riemannian manifolds with $\dim M = \dim \widetilde{M}$. Let $U \subset M, \widetilde{U} \subset \widetilde{M}$ be geodesic balls around $p \in M$ and $\widetilde{p} \in M$ respectively. Suppose
$$\gamma : [0, b] \to U, \gamma(0) = p, \gamma(b) = q$$
$$\widetilde{\gamma} : [0, b] \to U, \widetilde{\gamma(0)} = \widetilde{p}, \widetilde{\gamma}(b) = \widetilde{q}$$
two unit-speed geodesics such that

1. For all $t \in [0,b]$, and any planes $\Sigma \subset T_{\gamma(t)}M, \gamma'(t) \in \Sigma, \widetilde{\Sigma} \subset T_{\widetilde{\gamma}(t)}\widetilde{M}, \widetilde{\gamma}'(t) \in \widetilde{\Sigma}$, we have $K_{\gamma(t)}(\Sigma) \geq K_{\widetilde{\gamma}(t)}(\widetilde{\Sigma})$.

Then for any $X \in T_qM, \widetilde{X} \in T_{\widetilde{q}}\widetilde{M}, |X| = |\widetilde{X}| = 1$ and $X \perp \gamma'(b), \widetilde{X} \perp \widetilde{\gamma}'(b)$, we have

1. $\operatorname{Hess} r(X,X) \leq \operatorname{Hess} \widetilde{r}(\widetilde{X},\widetilde{X})$;
2. $\Delta r(\gamma(t)) \leq \widetilde{\Delta}\widetilde{r}(\widetilde{\gamma}(t))$ for all $t \in (0,b]$;
3. Moreover, the equality holds if and only if $K_\Sigma(\gamma(t)) = \widetilde{K}_{\widetilde{\Sigma}}(\widetilde{\gamma}(t))$.

Proof. For (1). Let $\{e_1(t),\dots,e_n(t)\}$ be a parallel orthnormal basis along γ such that $e_n(t) = \gamma'(t)$ and $\{\widetilde{e}_1(t),\dots,\widetilde{e}_n(t)\}$ a parallel orthnormal basis along $\widetilde{\gamma}$ suc that $\widetilde{e}_n(t) = \widetilde{\gamma}'(t)$. Without lose of generality we may assume $\langle X, e_i(b)\rangle = \langle \widetilde{X}, \widetilde{e}_i(b)\rangle$ for $i = 1,\dots,n-1$, it's just a trick of linear algebra.

Consider Jacobi field

$$\begin{cases} J(0) = 0, J(b) = X \\ \widetilde{J}(0) = 0, \widetilde{J}(b) = \widetilde{X} \end{cases}$$

With respect to $\{\widetilde{e}_i(t)\}$ we write $\widetilde{J}(t)$ as $\lambda^i(t)\widetilde{e}_i(t)$, and construct $V(t) = \lambda^i(t)e_i(t)$. Then

$$\begin{aligned} \operatorname{Hess} r(X,X) &= \operatorname{Hess} r(J(b),J(b)) \\ &\overset{(a)}{=} \int_0^b \langle J'(t),J'(t)\rangle - R(J,\gamma',\gamma',J)\mathrm{d}t \\ &\overset{(b)}{\leq} \int_0^b \langle V'(t),V'(t)\rangle - R(V,\gamma',\gamma',V)\mathrm{d}t \\ &\overset{(c)}{\leq} \int_0^b \langle \widetilde{J}'(t),\widetilde{J}'(t)\rangle - \widetilde{R}(\widetilde{J},\widetilde{\gamma},\widetilde{\gamma},\widetilde{J})\mathrm{d}t \\ &= \operatorname{Hess} \widetilde{r}(\widetilde{J}(b),\widetilde{J}(b)) \\ &= \operatorname{Hess} \widetilde{r}(\widetilde{X},\widetilde{X}) \end{aligned}$$

where

1. (a) holds from Proposition 13.4.1;
2. (b) holds from Corollary 8.3.1;
3. (c) holds from our curvature condition.

For (2) and (3). They directly follow from (1) and proof of (1). □

Corollary 14.2.1 (Hessian and Laplace comparision)**.** Let (M,g) be a Riemannian manifold with $\dim M = n$ and U a geodesic ball containing $p \in M$.

1. If sectional curvature $K \leq k$ in $U\backslash\{p\}$, then

$$\mathscr{H}_r \geq \frac{\operatorname{sn}_k'(r)}{\operatorname{sn}_k(r)}\pi_r, \quad \Delta r \geq (n-1)\frac{\operatorname{sn}_k'(r)}{\operatorname{sn}_k(r)}$$

holds in $U\backslash\{p\}$. Moreover, if equality holds, g has constant sectional curvature k in U.

2. If sectional curvature $K \geq k$ in $U\backslash\{p\}$, then
$$\mathscr{H}_r \leq \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\pi_r, \quad \Delta r \leq (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}$$
holds in $U_0\backslash\{p\}$, where
$$U_0 = \begin{cases} U, & k \leq 0 \\ U \cap B(p, \pi R), & k = \frac{1}{R^2} > 0 \end{cases}$$

Proof. For (1). Apply Hessian comparision to (M, g) and space form $S(n, k)$, then we directly have
$$\mathscr{H}_r \geq \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\pi_r$$
since the latter is exactly the $\mathscr{H}_r$ of $S(n, k)$. By taking trace we obtain $\Delta r \geq (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}$, since π_r is a projection onto a subspace with codimension 1. Furthermore, if equality holds, the Proposition 13.4.2 implies g has constant sectional curvature k in U.

For (2). The same as above. □

15. Comparision theorems based on Ricci curvature

15.1. Local Laplacian comparision.

Theorem 15.1.1 (Local Laplacian comparision)**.** Let (M, g) be a Riemannian manifold with $\dim M = n$ and U a geodesic ball containing $p \in M$. If there exists $k \in \mathbb{R}$ such that
$$\mathrm{Ric}(g) \geq (n-1)kg$$
Then
$$\Delta r \leq (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}$$
holds in $U_0\backslash\{p\}$, where
$$U_0 = \begin{cases} U, & k \leq 0 \\ U \cap B(p, \pi R), & k = \frac{1}{R^2} > 0 \end{cases}$$
Moreover, if equality holds, then g has constant sectional curvature.

15.1.1. *Proof via Jacobi fields.*

Proof of Theorem 15.1.1 via Jacobi fields. For arbitrary $q \in U_0\backslash\{p\}$, choose a minimal unit-speed geodesic $\gamma : [0, b] \to M$ with $\gamma(0) = p, \gamma(d) = q$, and $\{e_1(t), \ldots, e_n(t)\}$ is a parallel orthnormal frame along γ with $e_n(t) = \gamma'(t)$. Then
$$\Delta r = \sum_{i=1}^{n} \mathrm{Hess}\, r(e_i, e_i)$$

Let $J_i(t), i=1,\dots,n$ be Jacobi fields such that $J_i(0)=0, J_i(d)=e_i(d)$, then at point q we have

$$\Delta r = \sum_{i=1}^{n-1} \mathrm{Hess}(J_i(d), J_i(d)) = \sum_{i=1}^{n-1} I(J_i, J_i)$$

Now let $\widetilde{M}$ be the space form $S(n,k)$ and $\widetilde{U}$ a geodesic ball containing $\widetilde{p}\in\widetilde{M}$. Repeat the same things as above we have

$$(n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)} = \widetilde{\Delta}\widetilde{r} = \sum_{i=1}^{n-1}\widetilde{I}(\widetilde{J}_i, \widetilde{J}_i)$$

A crucial obeservation is that $\widetilde{J}_i(t) = f(t)\widetilde{e}_i(t)$, and the **key point** is that $f(t)$ is independent of i. Denote $V_i(t) = f(t)e_i(t)$, then

$$\begin{aligned}
\Delta r &= \sum_{i=1}^{n-1} I(J_i, J_i) \\
&\leq \sum_{i=1}^{n-1} I(V_i, V_i) \\
&= \sum_{i=1}^{n-1}\int_0^b \langle V_i'(t), V_i'(t)\rangle - R(V_i,\gamma',\gamma',V_i)\mathrm{d}t \\
&\overset{*}{=} \sum_{i=1}^{n-1}\int_0^b \langle V_i'(t), V_i'(t)\rangle - f^2(t)R(e_i,e_n,e_n,e_i)\mathrm{d}t \\
&= \sum_{i=1}^{n-1}\int_0^b \langle V_i'(t), V_i'(t)\rangle - \int_0^b f^2(t)\,\mathrm{Ric}(e_n,e_n)\mathrm{d}t \\
&\leq \sum_{i=1}^{n-1}\int_0^b \langle \widetilde{J}_i(t), \widetilde{J}_i(t)\rangle - \int_0^b (n-1)kf^2(t)\mathrm{d}t \\
&= \sum_{i=1}^{n-1}\widetilde{I}(\widetilde{J}_i, \widetilde{J}_i) \\
&= (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}
\end{aligned}$$

The key point is used in equality marked $*$. $\square$

15.1.2. *Proof via Bochner's technique.*

Lemma 15.1.1. Let (M,g) be a Riemannian manifold, (x^i, U, p) a normal coordinate centered at p, then

$$\Delta r = \partial_r \log(r^{n-1}\sqrt{\det g})$$

in $U\backslash\{p\}$. Moreover, along any unit-speed geodesic $\gamma : [0,b] \to U$ with $\gamma(0) = p$, if we define $f(t) := (\Delta r)(\gamma(t))$, then

$$f(t) = \frac{n-1}{t} + O(1)$$

Proof. Direct computation shows

$$\begin{aligned}
\Delta r &= \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}(g^{ij}\sqrt{\det g}\frac{\partial r}{\partial x^j}) \\
&= \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}(g^{ij}\sqrt{\det g}\frac{x^j}{r}) \\
&= \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}(\frac{x^i}{r}\sqrt{\det g}) \\
&= \frac{\partial}{\partial x^i}(\frac{x^i}{r}) + \frac{1}{\sqrt{\det g}}\frac{x^i}{r}\frac{\partial}{\partial x^i}(\sqrt{\det g}) \\
&= \frac{n-1}{r} + \frac{1}{\sqrt{\det g}}\partial_r(\sqrt{\det g}) \\
&= \partial_r \log(r^{n-1}\sqrt{\det g})
\end{aligned}$$

Moreover, for unit-speed geodesic $\gamma : [0,b] \to U$, we have

$$f(t) = \frac{n-1}{r(\gamma(t))} + \partial_r(\log\sqrt{\det g})\Big|_{\gamma(t)}$$

Then note that
1. $r(\gamma(t)) = t$, since γ is unit-speed geodesic.
2. Jacobi's formula implies

$$\partial_r(\log\sqrt{\det g})\Big|_{\gamma(t)} = \frac{1}{2}g^{ij}\frac{\partial g_{ij}}{\partial x^k}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t} = O(1)$$

we obtain the desired results. □

Lemma 15.1.2 (Riccati comparision theorem)**.** If $f : (0,b) \to \mathbb{R}$ is a smooth function satisfying
1. $f(t) = \frac{1}{t} + O(1)$;
2. $f' + f^2 + k \leq 0$.

Then

$$f(t) \leq \frac{\mathrm{sn}_k'(t)}{\mathrm{sn}_k(t)}$$

for all $t \in (0,b)$, where $k > 0, b \leq \frac{\pi}{\sqrt{k}}$.

Proof. Consider $f_k(t) = \frac{\mathrm{sn}_k'(t)}{\mathrm{sn}_k(t)}$, it's a smooth function defined on $(0,b)$ satisfying
1. $f_k(t) = \frac{1}{t} + O(1)$
2. $f_k' + f_k^2 + k = 0$

Choose a smooth function $F : (0,b) \to \mathbb{R}$ satisfying

1. $F(t) = 2\log t + O(1)$;
2. $F'(t) = f + f_k$

Then

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(e^F(f-f_k)) = e^F(f^2 - f_k^2 + f' - f_k') \leq 0\\ &\lim_{t\to 0} e^F(f-f_k) = 0\end{aligned}$$

Then we have $f(t) \leq f_k(t)$ holds for all $t \in (0, b)$. $\square$

Lemma 15.1.3.

$$|\operatorname{Hess} r|^2 \geq \frac{(\Delta r)^2}{(n-1)}$$

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal frame with $e_1 = \partial_r$. Then

$$\begin{aligned}|\operatorname{Hess} r|^2 &= \sum_{i,j=1}^n (\langle \nabla_{e_i}\partial_r, e_j\rangle)^2\\ &= \sum_{i,j=2}^n (\langle \nabla_{e_i}\partial_r, e_j\rangle)^2\\ &\geq \frac{1}{n-1}\sum_{i=2}^n (\langle \nabla_{e_i}\partial_r, e_i\rangle)^2\\ &= \frac{1}{n-1}(\Delta r)^2\end{aligned}$$

The inequality

$$|A|^2 \geq \frac{1}{k}|\operatorname{tr}(A)|^2$$

for a $k\times k$ matrix A is a direct consequence of the Cauchy-Schwarz inequality. $\square$

Proof of Theorem 15.1.1 via Bochner's technique. Recall Bochner's technique says

$$\frac{1}{2}\Delta|\nabla f|^2 = |\operatorname{Hess} f|^2 + \operatorname{Ric}(\nabla f, \nabla f) + g(\nabla\Delta f, \nabla f)$$

Set $f = r$ we have

$$\begin{aligned}0 &= |\operatorname{Hess} r|^2 + \operatorname{Ric}(\nabla r, \nabla r) + g(\nabla\Delta r, \nabla r)\\ &\overset{(1)}{\geq} |\operatorname{Hess} r|^2 + \partial_r(\Delta r) + (n-1)k\\ &\overset{(2)}{\geq} \partial_r(\frac{\Delta r}{n-1}) + \frac{(\Delta r)^2}{(n-1)^2} + k\end{aligned}$$

where

1. (1) holds from $\partial_r = \nabla_r$ and lower bounded of Ricci;
2. (2) holds from Lemma 15.1.3.

Consider
$$f(t) = (\frac{\Delta r}{n-1}(\gamma(t)))$$
Then Riccati comparision implies
$$\frac{\Delta r}{n-1} \leq \frac{\mathrm{sn}_k'(t)}{\mathrm{sn}_k(t)}$$
Furthermore, if
$$|\operatorname{Hess} r|^2 = \frac{(\Delta r)^2}{n-1}$$
then
$$\operatorname{Hess} r = \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}(g - \mathrm{d}r \otimes \mathrm{d}r)$$
□

15.2. **Maximal principle.**

Proposition 15.2.1. Let (M, g) be a Riemannian manifold and f, h be two smooth functions on M. If there is a point p such that $f(p) = h(p)$ and $f(x) \geq h(x)$ for all x near p, then
$$\nabla f(p) = \nabla h(p), \quad \operatorname{Hess} f|_p \geq \operatorname{Hess} h|_p, \quad \Delta f(p) \geq \Delta h(p).$$

Proof. Firstly let's consider the case $(M, g) \subset (\mathbb{R}^n, g_{\mathrm{can}})$, it's a simple calculus since we can use Taylor expansion. To be explicit, for all x near p, we have
$$f(x) = f(p) + \nabla f(p)^T(x-p) + \frac{1}{2}(x-p)^T \operatorname{Hess} f|_p(x-p) + O(|x|^3)$$
where ∇f is a n column vector and $\operatorname{Hess} f$ is a $n \times n$ matrix in this case. Similarly we have
$$h(x) = h(p) + \nabla h(p)^T(x-p) + \frac{1}{2}(x-p)^T \operatorname{Hess} h|_p(x-p) + O(|x|^3)$$
Then consider
$$f(x)-h(x) = (\nabla f-\nabla h)(p)^T(x-p)+\frac{1}{2}(x-p)^T \operatorname{Hess}(f-h)|_p(x-p)+O(|x|^3)$$
Since $f(x) - h(x) \geq 0$ for all x near p, then we must have
$$\nabla f(p) = \nabla h(p)$$
$$\operatorname{Hess} f|_p \geq \operatorname{Hess} h|_p$$
By taking trace we have
$$\Delta f(p) \geq \Delta h(p)$$
For general case, take $\gamma : (-\varepsilon, \varepsilon) \to M$ to be a geodesic with $\gamma(0) = p$, then use previous case on $f \circ \gamma, h \circ \gamma$ to obtain
$$\nabla_{\gamma'(0)} f(p) = \nabla_{\gamma'(0)} h(p)$$
$$\operatorname{Hess} f_p(\gamma'(0), \gamma'(0)) \geq \operatorname{Hess} h_p(\gamma'(0), \gamma'(0))$$

Then it's clear this Proposition holds if we let $v=\gamma'(0)$ run over all $v\in T_pM$. $\square$

Definition 15.2.1 (barrier sense)**.** Let (M,g) be a Riemannian manifold and $f\in C^0(M)$. Suppose f_q is a C^2 function defined in a neighborhood of U of $q\in M$.

1. f_q is called a lower barrier function of f at q if
$$f_q(q)=f(q),\quad f_q(x)\leq f(x),\quad x\in U.$$
2. $$\Delta f(q)\geq c$$
in the barrier sense if for all $\varepsilon>0$, there exists a lower barrier function $f_{q,\varepsilon}$ of f at q such that
$$\Delta f_{q,\varepsilon}(q)\geq c-\varepsilon$$
3. $$\Delta f(q)\leq c$$
in the barrier sense if for all $\varepsilon>0$, there exists a upper barrier function $f_{q,\varepsilon}$ of f at q such that
$$\Delta f_{q,\varepsilon}(q)\leq c+\varepsilon$$

Definition 15.2.2 (distribution sense)**.** Let (M,g) be a Riemannian manifold and $f\in C^0(M)$.
$$\Delta f\leq h$$
in distribution sense, if
$$\int f\Delta\varphi\leq\int h\varphi$$
holds for all $\varphi\geq 0\in C_c^\infty(M)$

Theorem 15.2.1 (maximal principle)**.** Let (M,g) be a Riemannian manifold and $f\in C^0(M)$.

1. If $\Delta f\geq 0$ in the barrier sense or distribution sense, then if f has a local(global) maximum, then it's local(global) constant;
2. If $\Delta f\leq 0$ in the barrier sense or distribution sense, then if f has a local(global) minimal, then it's local(global) constant;
3. $\Delta f=0$ implies $f\in C^\infty(M)$.

Proof. Here we only prove (1) for barrier sense: First, suppose that $\Delta f(x)>0$ everywhere. Then f can't have any local maxima at all. For if f has a local maximum at $p\in M$, then there would exist a smooth support function $f_\varepsilon(x)$ with

(1) $f_\varepsilon(p)=f(p)$,
(2) $f_\varepsilon(x)\leq f(x)$ for all x near p,
(3) $\Delta f_\varepsilon(p)>0$.

Here (1) and (2) imply that f_ε must also have a local maximum at p. But this implies that $\nabla^2 f_\varepsilon(p) \leq 0$, which contradicts (3).

Next just assume that $\Delta f \geq 0$ and let $p \in M$ be a local maximum for f. For sufficiently small $r < \mathrm{inj}(p)$ we therefore have a function $f : (B(p,r), g) \to \mathbb{R}$ with $\Delta f \geq 0$ and a global maximum at p. If f is constant on $B(p,r)$, then we are done; otherwise, we can assume (by possibly decreasing r) that f is not equal to $f(p)$ on $S(p,r) = \{x \in M : \mathrm{dist}(p,x) = r\}$. Then define $V = \{x \in S(p,r) : f(x) = f(p)\}$. Now construct a smooth function $h = e^{\alpha\varphi} - 1$ such that

$$\begin{aligned} h &< 0 \quad \text{on} \quad V \\ h(p) &= 0 \\ \Delta h &> 0 \quad \text{on} \quad \bar{B}(p,r) \end{aligned}$$

This function is found by first selecting an open disc $U \subset S(p,r)$ that contains V. We can then find φ such that

$$\begin{aligned} &\varphi(p) = 0 \\ &\varphi < 0 \text{ on } \quad U \\ &\nabla\varphi \neq 0 \text{ on } \quad \bar{B}(p,r) \end{aligned}$$

In an appropriate coordinate system $\left(x^1, \ldots, x^n\right)$ we can simply assume that U looks like the lower half-plane: $x^1 < 0$ and then define $\varphi = x^1$. Then choose α so big that $\Delta h = \alpha e^{\alpha\varphi}\left(\alpha|\nabla\varphi|^2 + \Delta\varphi\right) > 0$ on $\bar{B}(p,r)$.

Now consider the function $f_\delta = f + \delta h$ on $\bar{B}(p,r)$. Provided that δ is very small, this function has a local maximum in the interior $B(p,r)$, since

$$f_\delta(p) = f(p) > \max\{f(x) + \delta h(x) = f_\delta(x) : x \in \partial B(p,r)\}$$

On the other hand, we can also show that f_δ has positive Laplacian, thus giving a contradiction with the first part of the proof. To see that the Laplacian is positive, select f_ε a support function from below for f at $q \in B(p,r)$. Then $f_s + \delta h$ is a support function from below for f_δ at q. The Laplacian of this support function is estimated by

$$\Delta\left(f_\varepsilon + \delta h\right)(q) \geq -\varepsilon + \delta\Delta h(q),$$

which for given δ must become positive as $\varepsilon \to 0$. □

15.3. **Global Laplacian comparision.**

15.3.1. *In the barrier sense.*

Lemma 15.3.1. Let (M,g) be a complete Riemannian manifold, $p, q \in M$ and $\gamma : [0,b] \to M$ a unit-speed minimal geodesic connecting p, q. Then for any $0 < \varepsilon < b$, $\gamma|_{[\varepsilon,b]}$ is the unique minimal geodesic connecting $\gamma(\varepsilon)$ and q.

Proposition 15.3.1. Let (M,g) be a complete Riemannian manifold and $p,q\in M$. Let $\gamma:[0,b]\to M$ be a unit-speed minimal geodesic with $\gamma(0)=p$ and $\gamma(b)=q$. For any small $\varepsilon>0$, we define

$$r_\varepsilon(x)=\varepsilon+\operatorname{dist}(\gamma(\varepsilon),x),\quad x\in M.$$

Then

1. $q\notin\operatorname{cut}(\gamma(\varepsilon))$ and in particular, r_ε is smooth at q.
2. r_ε is an upper barrier function of $r(x)=\operatorname{dist}(p,x)$ at point q.

Proof. For (1). If $q\in\operatorname{cut}(\gamma(\varepsilon))$, then by definition there exists a minimal geodesic $\widetilde{\gamma}$ connecting $\gamma(\varepsilon)$ and q which is no longer minimizing after q. By Lemma 15.3.1, we have $\widetilde{\gamma}$ is exactly $\gamma|_{[\varepsilon,b]}$. By Theorem 9.1.1, there are two cases:

1. q is conjugate to $\gamma(\varepsilon)$ along γ. That's impossible, since γ is the minimal geodesic connecting p and q;
2. There exists at least two different minimal geodesics connecting $\gamma(\varepsilon)$ and q, by Lemma 15.3.1, that's also impossible.

This shows $q\notin\operatorname{cut}(\gamma(\varepsilon))$. In particular, γ_ε is smooth at q.

For (2). Firstly note that $\gamma(b)=q$, then

$$r(q)=\operatorname{dist}(p,q)=\operatorname{dist}(\gamma(0),\gamma(b))\overset{I}{=}\operatorname{dist}(\gamma(0),\gamma(\varepsilon))+\operatorname{dist}(\gamma(\varepsilon),\gamma(b))\overset{II}{=}r_\varepsilon(q)$$

where

I holds since γ is a minimal geodesic;
II holds since γ is unit-speed minimal geodesic, then $\operatorname{dist}(\gamma(0),\gamma(\varepsilon))=\varepsilon$.

By triangle inequality, one has

$$r(q')=\operatorname{dist}(p,q')\leq\varepsilon+\operatorname{dist}(\gamma(\varepsilon),q)=r_\varepsilon(q')$$

for all q' near q. Combining these two facts together we have r_ε is an upper barrier function of r. □

Theorem 15.3.1 (global Laplacian comparision)**.** Let (M,g) be a complete Riemannian manifold with

$$\operatorname{Ric}(g)\geq(n-1)kg$$

Then for $q\in M$

$$\Delta r(q)\leq(n-1)\frac{\operatorname{sn}_k'(r(q))}{\operatorname{sn}_k(r(q))}$$

in the barrier sense.

Proof. We consider the following three cases:

1. If $q\in M\backslash\{p\}\cup\operatorname{cut}(p)$, it's exactly smooth case we have proven;
2. If $q=p$, it's clear, since the right hand is infinite;
3. For arbitrary $q\in\operatorname{cut}(p)$, there exists a unit-speed $\gamma:[0,b]\to M$ with $\gamma(0)=p,\gamma(b)=q$. Then for each $\gamma>0$, define

$$\gamma_\varepsilon(x)=\varepsilon+\operatorname{dist}(\gamma(\varepsilon),x)$$

Then by Proposition 15.3.1 we have $\gamma_\varepsilon(x)$ is an upper barrier of $r(x)$ and γ_ε is smooth at q. Thus we have

$$\begin{aligned}
\Delta\gamma_\varepsilon(q) &= \Delta\operatorname{dist}(\gamma(\varepsilon), q) \\
&\leq (n-1)\frac{\mathrm{sn}_k'(\gamma_\varepsilon(q)-\varepsilon)}{\mathrm{sn}_k(\gamma_\varepsilon(q)-\varepsilon)} \\
&= (n-1)\frac{\mathrm{sn}_k'(\gamma(q)-\varepsilon)}{\mathrm{sn}_k(\gamma(q)-\varepsilon)}
\end{aligned}$$

which descends to $(n-1)\frac{\mathrm{sn}_k'(\gamma(q))}{\mathrm{sn}_k(\gamma(q))}$ as $\varepsilon \to 0$ by monotonicity. This completes the proof.

∎

15.3.2. *In the distribution sense.*

Proposition 15.1. Let (M, g) be a Riemannian manifold and $f : M \to \mathbb{R}$ be a Lipschitz function. Then for any $\varphi \in C_0^\infty(M, \mathbb{R})$, one has

$$-\int_M \langle\nabla\varphi, \nabla f\rangle d\mathrm{vol}_g = \int_M \Delta\varphi \cdot f d\mathrm{vol}_g.$$

Proof. Let $f : M \to \mathbb{R}$ be Lipschitz function, then by a partition-of-unity procedure one may express f as a locally finite sum $\sum_{\alpha\in I} f_\alpha$ subordinate to open covering $\{U_\alpha\}_{\alpha\in I}$, that is f_α has compact support in U_α. Without lose of generality, one chooses locally finite open covering U_α by geodesic balls. Then each f_α can be considered to be a function with compact support on a euclidean space.

Rademacher's theorem says if U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ is a Lipschitz function, then f is differentiable almost everywhere in U. So in this viewpoint, we can see Lipschitz functions on a Riemannian manifold is almost everywhere differentiable.

The followings are routine calculus to show integration by parts holds:

$$\begin{aligned}
\operatorname{div}(f\nabla\varphi) &= \nabla_k(f\nabla\varphi)^k \\
&= \frac{\partial(f\nabla\varphi)^k}{\partial x^k} + \Gamma_{ks}^k(f\nabla\varphi)^s \\
&= \frac{\partial(fg^{ik}\frac{\partial\varphi}{\partial x^i})}{\partial x^k} + \Gamma_{ks}^k f g^{is}\frac{\partial\varphi}{\partial x^i} \\
&= \underbrace{g^{ik}\frac{\partial f}{\partial x^k}\frac{\partial\varphi}{\partial x^i}}_{\text{part I}} + \underbrace{f(\frac{\partial g^{ik}}{\partial x^k}\frac{\partial\varphi}{\partial x^i} + g^{ik}\frac{\partial^2\varphi}{\partial x^k\partial x^i} + g^{is}\Gamma_{ks}^k\frac{\partial\varphi}{\partial x^i})}_{\text{part II}}
\end{aligned}$$

We have the following observations:

1. Part I equals

$$
\begin{aligned}
g^{ik}\frac{\partial f}{\partial x^k}\frac{\partial \varphi}{\partial x^i} &= g_{lj}g^{lk}\frac{\partial f}{\partial x^k}g^{ji}\frac{\partial \varphi}{\partial x^i} \\
&= \langle g^{lk}\frac{\partial f}{\partial x^k}\frac{\partial}{\partial x^l}, g^{ji}\frac{\partial \varphi}{\partial x^i}\frac{\partial}{\partial x^j}\rangle \\
&= \langle \nabla f, \nabla \varphi\rangle
\end{aligned}
$$

2. Note

$$
\begin{aligned}
\frac{\partial g^{ik}}{\partial x^k} + g^{is}\Gamma^k_{ks}\frac{\partial \varphi}{\partial x^i} &= -g^{is}g^{kt}\frac{\partial g_{st}}{\partial x^k} + \frac{1}{2}g^{is}g^{kt}(\frac{\partial g_{kt}}{\partial x^s} + \frac{\partial g_{st}}{\partial x^k} - \frac{\partial g_{ks}}{\partial x^t}) \\
&= -\frac{1}{2}g^{is}g^{kt}(\frac{\partial g_{ks}}{\partial x^t} + \frac{\partial g_{st}}{\partial x^k} - \frac{\partial g_{kt}}{\partial x^s}) \\
&= -g^{kt}\Gamma^i_{kt}
\end{aligned}
$$

where $\frac{\partial g^{ik}}{\partial x^k} = -g^{is}g^{kt}\frac{\partial g_{st}}{\partial x^k}$ holds from the fact $g^{ik}g_{kt} = \delta^i_t$, then take partial derivative with respect to x^k to conclude.

3. From (2) and local expression of Δ, it's clear part II equals $f\Delta\varphi$.

Thus we have

$$
\operatorname{div}(f\nabla\varphi) = \langle \nabla\varphi, \nabla f\rangle + f\Delta\varphi
$$

Then divergence theorem completes the proof. □

Theorem 15.3.2 (global Laplacian comparision II)**.** Let (M, g) be a complete Riemannian manifold with

$$
\operatorname{Ric}(g) \geq (n-1)kg
$$

Then for $x \in M$

$$
\Delta r(x) \leq (n-1)\frac{\operatorname{sn}_k'(r(x))}{\operatorname{sn}_k(r(x))}
$$

in the distribution sense.

Proof. For fixed $p \in M$, the domain $\Sigma(p)$ of injective radius $\operatorname{inj}(p)$ is a star-shaped open subset of T_pM and $M = \exp_p(\Sigma(p)) \cup \operatorname{cut}(p)$. The boundary of $\Sigma(p)$ is locally a graph of continuous function and so there exists a family of star-shaped domains $\{U_j\}$ with smooth boundaries such that

$$
U_j \subset U_{j+1} \subset \cdots \subset \Sigma(p), \quad \Sigma(p) = \bigcup U_j
$$

If we set $\Omega = \exp_p(\Sigma(p))$, then $\Omega = \bigcup \Omega_j$, where $\Omega_j = \exp_p(U_j)$. Since each U_j is star-shaped, by Gauss lemma, on each boundary $\partial\Omega_j$, one has $\frac{\partial r}{\partial v} = g(\nabla r, v) \geq 0$ where v is the outer normal vector on $\partial\Omega_j$.

Therefore for each $\varphi \in C_c^\infty(M)$ with $\varphi \geq 0$, one has

$$\begin{aligned}
\int_M r\Delta\varphi \,\mathrm{vol} &\overset{(1)}{=} -\int_M \langle \nabla r, \nabla\varphi\rangle \,\mathrm{vol} \\
&\overset{(2)}{=} -\lim_j \int_{\Omega_j\setminus\{p\}} \langle \nabla r, \nabla\varphi\rangle \\
&\overset{(3)}{=} \lim_j \Big(\int_{\Omega_j\setminus\{p\}} \Delta r\varphi \,\mathrm{vol} - \int_{\partial\Omega_j} \varphi\frac{\partial r}{\partial v}\Big) \\
&\overset{(4)}{\leq} \lim_j \int_{\Omega_j\setminus\{p\}} \Delta r\varphi \,\mathrm{vol} \\
&\overset{(5)}{\leq} \lim_j \int_{\Omega_j\setminus\{p\}} (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\varphi \,\mathrm{vol} \\
&\overset{(6)}{=} \int_{\Omega\setminus\{p\}} (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k'(r)} \,\mathrm{vol} \\
&\overset{(7)}{=} \int_M (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\varphi \,\mathrm{vol}
\end{aligned}$$

where

(1) holds from the fact r is Lipschitz and Proposition 15.1;
(2) and (6) holds from dominated convergence theorem;
(3) holds from Stokes theorem;
(4) holds from $\varphi \geq 0$ and $\frac{\partial r}{\partial v} \geq 0$;
(5) holds from Local Laplacian comparision theorem, that is Theorem 15.1.1;
(7) holds from the fact $\mathrm{cut}(p)$ is zero-measure.

$\square$

16. Splitting theorem

16.1. Volume comparision.

Lemma 16.1.1. Let (M,g) be a complete, connected Riemannian manifold and $p\in M$. For any $\delta\in\mathbb{R}^+$

$$\exp_p(B(0,\delta)\cap\Sigma(p))\subset B(p,\delta)\subset\exp_p(B(0,\delta)\cap\Sigma(p))\cup\mathrm{cut}(p)$$

In particular, under the map $\Phi:\mathbb{R}^+\times\mathbb{S}^{n-1}\to T_pM\backslash\{0\}$ given by $\Phi(\rho,\omega)=\rho\omega$

$$\begin{aligned}\mathrm{Vol}(B(p,\delta))&=\mathrm{Vol}(\exp_p(B(0,\delta))\cap\Sigma(p))\\&=\int_{B(0,\delta)\cap\Sigma(p)}\exp_p^*\mathrm{vol}\\&=\int_{B(0,\delta)}\chi_{\Sigma(p)}\exp_p^*\mathrm{vol}\\&=\int_{\mathbb{S}^{n-1}}\int_0^\delta\chi_{\Sigma(p)}\sqrt{\det g}\circ\Phi(\rho,\omega)\rho^{n-1}\mathrm{d}\rho\,\mathrm{vol}_{\mathbb{S}^{n-1}}\end{aligned}$$

Corollary 16.1.1. Let $p\in S(n,k)$

1. If $k\leq 0$, then for any $\delta\in\mathbb{R}^+$

$$\mathrm{Vol}(B(p,\delta))=\int_{\mathbb{S}^{n-1}}\int_0^\delta\mathrm{sn}_k^{n-1}(\rho)\mathrm{d}\rho\,\mathrm{vol}_{\mathbb{S}^{n-1}}$$

2. If $k=\frac{1}{R^2}\geq 0$, then for any $\delta\in\mathbb{R}^+$

$$\mathrm{Vol}(B(p,\delta))=\int_{\mathbb{S}^{n-1}}\int_0^\delta\chi_{B(0,\pi R)}\mathrm{sn}_k^{n-1}(\rho)\mathrm{d}\rho\,\mathrm{vol}_{\mathbb{S}^{n-1}}$$

Lemma 16.1.2. Let (M,g) be a Riemannian manifold, and (x^i,U,p) be a geodesic ball chart of radius b around $p\in M$.

1. If $K\leq k$, then for each fixed $\omega\in\mathbb{S}^{n-1}$ the volume density ratio

$$\lambda(\rho,\omega)=\frac{\rho^{n-1}\sqrt{\det g}\circ\Phi(\rho,\omega)}{\mathrm{sn}_k^{n-1}(\rho)}$$

is non-decreasing in $\rho\in(0,b_0)$ where

$$b_0=\begin{cases}b, & k\leq 0\\ \min\{b,\pi R\}, & k=\frac{1}{R^2}\end{cases}$$

Moreover, $\lim_{\rho\to 0}\lambda(\rho,\omega)=1$.

2. If $K\geq k$ or $\mathrm{Ric}(g)\geq(n-1)kg$, then for each fixed $\omega\in\mathbb{S}^{n-1}$ the volume density ratio $\lambda(\rho,\omega)$ is non-increasing in $\rho\in(0,b)$ and $\lim_{\rho\to 0}\lambda(\rho,\omega)=1$.

Proof. By Corollary 14.2.1 and Lemma 15.1.1

$$\partial_r\log(r^{n-1}\sqrt{\det g})=\Delta r\geq(n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}=\partial_r\log(\mathrm{sn}_k^{n-1}(r))$$

Hence $\log\left(\frac{r^{n-1}\sqrt{\det g}}{\mathrm{sn}_k^{n-1}(r)}\right)$ is a non-decreasing function of r along each radial geodesic γ, that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\log\left(\frac{r^{n-1}\sqrt{\det g}}{\mathrm{sn}_k^{n-1}(r)}\right)\circ\gamma(t)\right)\geq 0$$

Hence, $f(r)=\frac{r^{n-1}\sqrt{\det g}}{\mathrm{sn}_k^{n-1}(r)}$ is a non-decreasing function of r along each radial geodesic γ. It is easy to see that $r\circ\Phi=\rho$ (the exponential map is used in normal coordinates). Hence,

$$\lambda(\rho,\omega)=f\circ\Phi(\rho,\omega)$$

is nondecreasing in ρ for any fixed $\omega\in\mathbb{S}^{n-1}$. It is obvious that

$$\lim_{\rho\to 0}\sqrt{\det g}=\lim_{\rho\to 0}\frac{\rho^{n-1}}{\mathrm{sn}_k^{n-1}(\rho)}=1.$$

The proof of (2) is similar. $\square$

Part 5. Appendix

Appendix A. Hodge theorem

In this section, we mainly follow the Chapter 6 of [War10].

A.1. Introduction and proof of Hodge theorem.

We shall use Δ^* to denote the adjoint of Laplace-Beltrami operator on Ω^k_M. This operator is precisely Δ itself, since Laplace-Beltrami operator is self-adjoint, and we usually make no distinction between Δ and Δ^*. However, this distinction will be important for the form of the following definition.

An important question is to find a neccessary and sufficient condition for there to exist a solution ω of equation $\Delta\omega = \alpha$, where α is a given k-form. Suppose ω is a solution, then

$$(\Delta\omega, \varphi) = (\alpha, \varphi)$$

holds for all k-forms φ. Equivalently we have

$$(\omega, \Delta^*\varphi) = (\alpha, \varphi)$$

holds for all k-forms φ. In this viewpoint, we can regard a solution of $\Delta\omega = \alpha$ as a centain type of linear functional on $C^\infty(M, \Omega^k_M)$, namely solution ω determines a bounded linear functional l on $C^\infty(M, \Omega^k_M)$ by

$$l(\varphi) = (\omega, \varphi), \quad \varphi \in C^\infty(M, \Omega^k_M)$$

such that

$$l(\Delta^*\varphi) = (\alpha, \varphi)$$

holds for all k-forms φ.

Definition A.1.1 (weak solution)**.** A linear functional l on $C^\infty(M, \Omega^k_M)$ is called a weak solution of $\Delta\omega = \alpha$, if

$$l(\Delta^*\varphi) = (\alpha, \varphi)$$

holds for all k-forms φ.

We have seen that each ordinary solution of $\Delta\omega = \alpha$ determines a weak solution of it, it turns out that the major effort of this section will be to prove a regularity theorem which says that the converse of this is true, that is each weak solution determines an ordinary solution. The main step is to show if l is a weak solution of $\Delta\omega = \alpha$, then there exists a smooth form ω such that

$$l(\varphi) = (\alpha, \varphi), \quad \varphi \in C^\infty(M, \Omega^k_M)$$

Then ω is an ordinary solution follows from

$$(\Delta, \varphi) = (\omega, \Delta^*\varphi) = l(\Delta^*\varphi) = (\alpha, \varphi)$$

holds for all k-forms φ, which implies $\Delta\omega = \alpha$.

The key theorems we will prove are listed as follows:

Theorem A.1.1 (regularity theorem)**.** Let $\alpha \in C^\infty(M, \Omega^k_M)$, and l be a weak solution of $\Delta\omega = \alpha$, then there exists $\omega \in C^\infty(M, \Omega^k_M)$ such that

$$l(\varphi) = (\omega, \varphi)$$

holds for every k-forms φ. In particular, $\Delta\omega = \alpha$.

Theorem A.1.2. Let $\{\alpha_n\}$ be a sequence of smooth k-forms on M such that $\|\alpha_n\| \leq c$ and $\|\Delta\alpha_n\| \leq c$ for all n and for some constant $c > 0$. Then a subsequence of $\{\alpha_n\}$ is a Cauchy sequence in $C^\infty(M, \Omega^k_M)$.

Corollary A.1.1. There exists a constant $c > 0$ such that

$$\|\psi\| \leq c\|\Delta\psi\|$$

holds for all $\psi \in (\mathcal{H}^k)^\perp$

Proof. Suppose the contraty, then there exists a sequence $\psi_j \in (\mathcal{H}^k)^\perp$ with $\|\psi_j\| = 1$ and $\|\Delta\psi_j\| \to 0$. By Theorem A.1.2, there exists a subsection of $\{\psi_j\}$ which for convenience we can assume to be $\{\psi_j\}$ itself, is Cauchy. Thus for each $\varphi \in C^\infty(M, \Omega^k_M)$, $\lim_{j\to\infty}(\psi_j, \varphi)$ exists. Consider the linear functional l on $C^\infty(M, \Omega^k_M)$ defined by

$$l(\varphi) := \lim_{j\to\infty}(\psi_j, \varphi), \quad \varphi \in C^\infty(M, \Omega^k_M)$$

It's clear l is bounded, and

$$l(\Delta\varphi) = \lim_{j\to\infty}(\psi, \Delta\varphi) = \lim_{j\to\infty}(\Delta\psi_j, \varphi) = 0$$

holds for all $\varphi \in C^\infty(M, \Omega^k_M)$, which implies l is a weak solution of $\Delta\psi = 0$. By Theorem A.1.1, there exists a k-form ψ such that $l(\varphi) = (\psi, \varphi)$, where $\varphi \in C^\infty(M, \Omega^k_M)$. Consequently $\psi_j \to \psi$, and $\psi \in (\mathcal{H}^k)^\perp$ with $\|\psi\| = 1$. However, Theorem A.1.1 implies $\psi \in \mathcal{H}^k$, a contradiction. $\square$

Holding above results, we can prove Hodge theorem.

Theorem A.1.3 (Hodge theorem)**.** Consider the Laplace operator $\Delta : C^\infty(M, \Omega^k_M) \to C^\infty(M, \Omega^k_M)$, then

1. $\dim_{\mathbb{R}} \mathcal{H}^k < \infty$;
2. There is an orthogonal direct sum decomposition

$$C^\infty(M, \Omega^k_M) = \mathcal{H}^k \oplus \operatorname{im}\Delta$$

Proof. For (1). If $\mathcal{H}^k$ is not finite dimensional, then there exists an infinite orthnormal sequence. By Theorem A.1.2, this orthnormal sequence contains a Cauchy sequence, which is impossible. Thus $\mathcal{H}^k$ is finite dimensional.

For (2). Note that we naturally have the following orthogonal decomposition

$$C^\infty(M, \Omega^k_M) = (\mathcal{H}^k)^\perp \oplus \mathcal{H}^k$$

The theorem will be proved by showing that $(\mathcal{H}^k)^\perp = \operatorname{im}\Delta$. We use $\mathcal{H}$ to denote the projection from $C^\infty(M, \Omega^k_M)$ to $\mathcal{H}^k$, that is $\mathcal{H}(\alpha)$ is the harmonic part of α.

It's easy to see $\operatorname{im}\Delta \subset (\mathcal{H}^k)^\perp$, since for all $\omega \in C^\infty(M, \Omega^k_M)$ and $\alpha \in \mathcal{H}^k$, we have

$$(\Delta\omega, \alpha) = (\omega, \Delta\alpha) = 0$$

To see converse, for $\alpha \in (\mathcal{H}^k)^\perp$, we define a linear functional l on $\operatorname{im}\Delta$ by setting

$$l(\Delta\varphi) := (\alpha, \varphi)$$

for all $\varphi \in C^\infty(M, \Omega^k_M)$.

1. l is well-definedd, since if $\Delta\varphi_1 = \Delta\varphi_2$, then $\varphi_1 - \varphi_2 \in \mathcal{H}^k$, then $(\alpha, \varphi_1 - \varphi_2) = 0$;
2. l is bounded. Indeed, for $\varphi \in C^\infty(M, \Omega^k_M)$, let $\psi = \varphi - \mathcal{H}(\varphi)$. Then

$$\begin{aligned} |l(\Delta\varphi)| &= |l(\Delta\psi)| \\ &= |(\alpha, \psi)| \\ &\leq \|\alpha\|\|\psi\| \\ &\overset{*}{\leq} c\|\alpha\|\|\Delta\psi\| \\ &= c\|\alpha\|\|\Delta\varphi\| \end{aligned}$$

where $*$ holds from Corollary A.1.1.

By Hahn-Banach theorem, l extends to a bounded linear functional on $C^\infty(M, \Omega^k_M)$, thus l is a weak solution of $\Delta\omega = \alpha$. By Theorem A.1.1, there exists a k-form ω such that $\Delta\omega = \alpha$. Hence

$$(\mathcal{H}^k)^\perp = \operatorname{im}\Delta$$

This completes the proof of Hodge theorem. □

Appendix B. Second fundamental form

B.1. **Pullback connection.** In this section, we fix the following notations:

1. Vector bundle E equipped with a metric g over a smooth manifold M, endowed with Levi-Civita connection ∇^E;
2. $f : M \to N$ is a smooth map;
3. $\{\mathrm{d}x^i\}$ is used to denote a local basis of TM, $\{\mathrm{d}y^m\}$ is used to denote a local basis of TN and $\{e_\alpha\}$ is used to denote a local basis of E.

Definition B.1.1 (pullback vector bundle)**.** The pullback vector bundle f^*E over M is defined by the set

$$\widehat{E} = f^*E := \{(p, v) \in M \times E \mid f(p) = \pi(v)\}$$

endowed with subspace topology.

Remark B.1.1 (local form)*.* A local basis of $\widehat{E}$ can be written as

$$\widehat{e}_\alpha(x) := f^*e_\alpha(x) = e_\alpha(f(x))$$

where $x \in M$.

Definition B.1.2 (pullback connection)**.** The pullback connection $\widehat{\nabla}$ over $\widehat{E} \to M$ is defined as:

$$\begin{aligned} \widehat{\nabla} : C^\infty(M, \widehat{E}) &\to C^\infty(M, T^*M \otimes \widehat{E}) \\ f^*(s) &\mapsto f^*(\nabla s) \end{aligned}$$

where $s \in C^\infty(M, E)$.

Remark B.1.2 (local form)*.* If we take a local basis $\{\widehat{e}_\alpha\}$ of $\widehat{E}$, then

$$\begin{aligned} \widehat{\nabla}\widehat{e}_\alpha &= f^*(\nabla e_\alpha) \\ &= f^*(\Gamma^\beta_{m\alpha}\mathrm{d}y^m \otimes e_\beta) \\ &= \Gamma^\beta_{m\alpha}(f)\frac{\partial f^m}{\partial x^i}\mathrm{d}x^i \otimes \widehat{e}_\beta \end{aligned}$$

Note that we can also use f to pullback metric g on E to obtain a metric on $\widehat{E}$, denoted by $\widehat{g}$. Locally we can write

$$\begin{aligned} \widehat{g}_{\alpha\beta}\widehat{e}^\alpha \otimes \widehat{e}^\beta :&= f^*(g_{\alpha\beta}e^\alpha \otimes e^\beta) \\ &= g_{\alpha\beta}(f)\widehat{e}^\alpha \otimes \widehat{e}^\beta \end{aligned}$$

that is $\widehat{g}_{\alpha\beta} = g_{\alpha\beta}(f)$.

Lemma B.1.1. The pullback connection $\widehat{\nabla}$ is compatible with $\widehat{g}$, that is for any vector field X of M and section s, t of $\widehat{E}$, we have

$$X\widehat{g}(s, t) = \widehat{g}(\widehat{\nabla}_X s, t) + \widehat{g}(s, \widehat{\nabla}_X t)$$

Proof. Locally we take $X = \frac{\partial}{\partial x^i}, s = \widehat{e}_\alpha, t = \widehat{e}_\beta$, then

$$\begin{aligned}
\frac{\partial}{\partial x^i}\widehat{g}_{\alpha\beta} &= \frac{\partial}{\partial x^i} g_{\alpha\beta}(f) \\
&= \frac{\partial f^m}{\partial x^i}\frac{\partial}{\partial y^m} g_{\alpha\beta}(f) \\
&= \frac{\partial f^m}{\partial x^i}(\Gamma^\gamma_{m\alpha}(f) g_{\gamma\beta}(f) + \Gamma^\gamma_{m\beta}(f) g_{\alpha\gamma}(f)) \\
\widehat{g}(\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{e}_\alpha, \widehat{e}_\beta) &= \Gamma^\gamma_{m\alpha}(f)\frac{\partial f^m}{\partial x^i} g_{\gamma\beta}(f) \\
\widehat{g}(\widehat{e}_\alpha, \widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{e}_\beta) &= \Gamma^\gamma_{m\beta}(f)\frac{\partial f^m}{\partial x^i} g_{\alpha\gamma}(f)
\end{aligned}$$

This completes the proof. □

Definition B.1.3. The curvature tensor $\widehat{R}$ of pullback connection $\widehat{\nabla}$ on vector bundle $\widehat{E} \to M$ is given by

$$\widehat{R}(X, Y, s, t) = \widehat{g}(\widehat{\nabla}_X\widehat{\nabla}_Y s - \widehat{\nabla}_Y\widehat{\nabla}_X s, t)$$

Remark B.1.3 (local form).

$$\widehat{R}_{ij\alpha\beta} = R_{mn\alpha\beta}\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j}$$

where $R_{mn\alpha\beta}$ is curvature of ∇^E.

Proof.

$$\begin{aligned}
\widehat{R}_{ij\alpha\beta} &= \widehat{R}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \widehat{e}_\alpha, \widehat{e}_\beta) \\
&= \widehat{g}(\widehat{R}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\widehat{e}_\alpha, \widehat{e}_\beta) \\
&= \widehat{g}(\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{\nabla}_{\frac{\partial}{\partial x^j}}\widehat{e}_\alpha - \widehat{\nabla}_{\frac{\partial}{\partial x^j}}\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{e}_\alpha, \widehat{e}_\beta)
\end{aligned}$$

So it suffices to compute

$$\begin{aligned}
\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{\nabla}_{\frac{\partial}{\partial x^j}}\widehat{e}_\alpha &= \widehat{\nabla}_{\frac{\partial}{\partial x^i}}(\Gamma^\gamma_{m\alpha}(f)\frac{\partial f^m}{\partial x^j}\widehat{e}_\gamma) \\
&= \frac{\partial}{\partial x^i}(\Gamma^\gamma_{m\alpha}(f)\frac{\partial f^m}{\partial x^j})\widehat{e}_\gamma + \Gamma^\gamma_{m\alpha}(f)\frac{\partial f^m}{\partial x^j}\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{e}_\gamma \\
&= (\frac{\partial \Gamma^\gamma_{m\alpha}}{\partial y^n}\frac{\partial f^n}{\partial x^i}\frac{\partial f^m}{\partial x^j} + \Gamma^\gamma_{m\alpha}(f)\frac{\partial^2 f^m}{\partial x^i \partial x^j})\widehat{e}_\gamma + \frac{\partial f^m}{\partial x^j}\frac{\partial f^n}{\partial x^i}\Gamma^\gamma_{m\alpha}\Gamma^\delta_{n\gamma}\widehat{e}_\delta \\
&= \frac{\partial f^m}{\partial x^j}\frac{\partial f^n}{\partial x^i}(\frac{\partial \Gamma^\gamma_{m\alpha}}{\partial y^n} + \Gamma^\delta_{m\alpha}\Gamma^\gamma_{n\delta})\widehat{e}_\gamma + \Gamma^\gamma_{m\alpha}\frac{\partial^2 f^m}{\partial x^i \partial x^j}\widehat{e}_\gamma
\end{aligned}$$

Thus

$$\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{\nabla}_{\frac{\partial}{\partial x^j}}\widehat{e}_\alpha - \widehat{\nabla}_{\frac{\partial}{\partial x^j}}\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{e}_\alpha = \frac{\partial f^m}{\partial x^j}\frac{\partial f^n}{\partial x^i}R^\gamma_{mn\alpha}\widehat{e}_\gamma$$

that is

$$\begin{aligned}\widehat{R}_{ij\alpha\beta} &= \widehat{g}(\frac{\partial f^m}{\partial x^j}\frac{\partial f^n}{\partial x^i}R^{\gamma}_{mn\alpha}\widehat{e}_{\gamma}, \widehat{e}_{\beta}) \\ &= \frac{\partial f^m}{\partial x^j}\frac{\partial f^n}{\partial x^i}R^{\gamma}_{mn\alpha}g_{\gamma\beta} \\ &= \frac{\partial f^m}{\partial x^j}\frac{\partial f^n}{\partial x^i}R_{mn\alpha\beta}\end{aligned}$$

□

B.2. **Second fundamental form.** In this section, we fix the following notations:

1. $f : (M, g^M, \nabla^M) \to (N, g^N, \nabla^N)$ is a smooth map between two Riemannian manifolds.
2. Γ^k_{ij} is used to denote Christoffel symbol of ∇^M and Γ^l_{mn} is used to denote Christoffel symbol of ∇^N.
3. $\widehat{\nabla}$ is the connection on f^*TN induced by ∇^N.

Definition B.2.1 (second fundamental form)**.** The second fundamental form $B \in C^\infty(M, T^*M \otimes T^*M \otimes f^*TN)$ of f is defined as

$$B(X,Y) := \widehat{\nabla}_X f_* Y - f_*(\nabla^M_X Y) \in C^\infty(M, f^*TN)$$

where $X, Y \in C^\infty(M, TM)$.

Remark B.2.1 (local form). Suppose that $X = \frac{\partial}{\partial x^i}, Y = \frac{\partial}{\partial x^j}$, then one has

$$f_*(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}) = \Gamma^k_{ij} f_* \frac{\partial}{\partial x^k} = \Gamma^k_{ij}\frac{\partial f^m}{\partial x^k}\frac{\partial}{\partial y^m}$$

And

$$\begin{aligned}\widehat{\nabla}_{\frac{\partial}{\partial x^i}}(\frac{\partial f^m}{\partial x^j}\frac{\partial}{\partial y^m}) &= \frac{\partial^2 f^m}{\partial x^i \partial x^j}\frac{\partial}{\partial y^m} + \frac{\partial f^m}{\partial x^j}\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial y^m} \\ &= (\frac{\partial^2 f^l}{\partial x^i \partial x^j} + \frac{\partial f^m}{\partial x^j}\frac{\partial f^n}{\partial x^i}\Gamma^l_{nm})\frac{\partial}{\partial y^l}\end{aligned}$$

Therefore

$$\begin{aligned}B_{ij} :&= B(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \\ &= (\frac{\partial^2 f^l}{\partial x^i \partial x^j} + \frac{\partial f^m}{\partial x^j}\frac{\partial f^n}{\partial x^i}\Gamma^l_{mn} - \Gamma^k_{ij}\frac{\partial f^l}{\partial x^k})\frac{\partial}{\partial y^l}\end{aligned}$$

Remark B.2.2 (Geodesic). Consider a smooth regular curve $\gamma : [a,b] \to M$, we can regard it as $\gamma : [a,b], g, \nabla) \to (M, g^M, \nabla^M)$. Thus our second fundamental form in this case is

$$B_{ij} = (\frac{\mathrm{d}^2\gamma^k}{\mathrm{d}t^2} + \frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\Gamma^k_{ij})\frac{\partial}{\partial x^k}$$

So condition for geodesic is exactly second fundamental form is zero.

Remark B.2.3 (Hessian). Consider smooth function f, we can regard it as $f:(M,g^M,\nabla^M)\to(\mathbb{R},g,\nabla)$, where metric and connection on $\mathbb{R}$ are trivial. Thus our second fundamental form in this case is

$$B_{ij}=(\frac{\partial^2 f}{\partial x^i\partial x^j}-\Gamma_{ij}^k\frac{\partial f}{\partial x^k})\frac{\partial}{\partial y}$$

since $\Gamma_{mn}^l=0$. That's exactly our $\operatorname{Hess} f$, so second fundamental form generalizes our Hessian of smooth function;

Since Hessian of a smooth function is exactly $\nabla(\nabla f)$, where $\nabla f\in C^\infty(M,T^*M)$. In fact we can express our second fundamental form B as $\widetilde{\nabla}\mathrm{d}f$, where $\mathrm{d}f\in C^\infty(M,T^*M\otimes f^*TN)$ and $\widetilde{\nabla}$ is the connection on $T^*M\otimes f^*TN$ induced by ∇^M and pullback connection on f^*TN. Indeed, note that locally we have

$$\mathrm{d}f=\frac{\partial f^m}{\partial x^i}\mathrm{d}x^i\otimes\frac{\partial}{\partial y^m}$$

Then

$$\begin{aligned}
\widetilde{\nabla}\mathrm{d}f&=\widetilde{\nabla}(\frac{\partial f^m}{\partial x^i}\mathrm{d}x^i\otimes\frac{\partial}{\partial y^m})\\
&=\frac{\partial^2 f^m}{\partial x^j\partial x^i}\mathrm{d}x^j\otimes\mathrm{d}x^i\otimes\frac{\partial}{\partial y^m}-\frac{\partial f^m}{\partial x^i}\Gamma_{jk}^i\mathrm{d}x^j\otimes\mathrm{d}x^k\otimes\frac{\partial}{\partial y^m}+\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j}\Gamma_{mn}^l\mathrm{d}x^i\otimes\mathrm{d}x^j\frac{\partial}{\partial y^l}\\
&=(\frac{\partial^2 f^l}{\partial x^i\partial x^j}-\frac{\partial f^l}{\partial x^k}\Gamma_{ij}^k+\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j}\Gamma_{mn}^l)\mathrm{d}x^i\otimes\mathrm{d}x^j\otimes\frac{\partial}{\partial y^l}\\
&=B
\end{aligned}$$

as desired.

Appendix C. Harmonic map and its variation

In this section we fix a smooth map $f : (M, g) \to (N, h)$ between Riemannian manifolds with second fundamental form B. Keep in mind we regard $\mathrm{d}f$ as a section of $T^*M \otimes f^*TN$ and B as a section of $T^*M \otimes T^*M \otimes f^*TN$.

C.1. **Harmonic map and totally geodesic.**

Definition C.1.1 ((scalar) Laplacian)**.** The (scalar) Laplacian of f is defined as

$$\Delta_g f := \operatorname{tr}_g B \in C^\infty(M, f^*TN)$$

Definition C.1.2 (harmonic map)**.** f is called harmonic map if its scalar Laplacian $\Delta_g f = 0$.

Definition C.1.3 (totally geodesic)**.** f is called totally geodesic, if its second fundamental form $B = 0$.

Example C.1.1. For a geodesic $\gamma : [a, b] \to (M, g)$, if we endow $[a, b]$ with canonical metric, then γ is totally geodesic, thus it's harmonic.

Remark C.1.1. If γ is regular, that is $\gamma'(t) \neq 0$ for each $t \in [a, b]$, or in other words γ is immersion, there is an induced metric g_0 on $[a, b]$ given by

$$g_0 = g_{ij} \frac{\mathrm{d}\gamma^i}{\mathrm{d}t} \frac{\mathrm{d}\gamma^j}{\mathrm{d}t} \mathrm{d}t \otimes \mathrm{d}t$$

Furthermore, if γ is unit-speed, then

$$1 = |\gamma'(t)|^2 = g_{ij} \frac{\mathrm{d}\gamma^i}{\mathrm{d}t} \frac{\mathrm{d}\gamma^j}{\mathrm{d}t}$$

which implies g_0 is canonical metric on $[a, b]$.

Example C.1.2. For a smooth function $f : (M, g) \to \mathbb{R}$, if we endow $\mathbb{R}$ with canonical metric, then f is a harmonic map if and only if it's a harmonic function.

Lemma C.1.1. Let $\gamma : [a, b] \to M$ be a smooth curve and $\widetilde{\gamma} = f \circ \gamma$. If we use $\widehat{\nabla}$ and $\widetilde{\nabla}$ to denote the induced connection on γ^*TM and $\widetilde{\gamma}^*TN$ respectively, then

$$\widetilde{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \widetilde{\gamma}_*(\frac{\mathrm{d}}{\mathrm{d}t}) = f_*(\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma_*(\frac{\mathrm{d}}{\mathrm{d}t})) + \gamma^* B$$

Proof. Directly compute

$$\begin{aligned}
\widetilde{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \widetilde{\gamma}_*(\frac{\mathrm{d}}{\mathrm{d}t}) &= (\frac{\mathrm{d}^2\widetilde{\gamma}^l}{\mathrm{d}t^2} + \Gamma^l_{mn}(\widetilde{\gamma}) \frac{\mathrm{d}\widetilde{\gamma}^m}{\mathrm{d}t} \frac{\mathrm{d}\widetilde{\gamma}^n}{\mathrm{d}t}) \frac{\partial}{\partial y^l} \\
&= \{ \frac{\partial f^l}{\partial x^k} \frac{\mathrm{d}^2\gamma^k}{\mathrm{d}t^2} + (\frac{\partial^2 f^l}{\partial x^i \partial x^j} + \Gamma^l_{mn} \frac{\partial f^m}{\partial x^i} \frac{\partial f^n}{\partial x^j}) \frac{\mathrm{d}\gamma^i}{\mathrm{d}t} \frac{\mathrm{d}\gamma^j}{\mathrm{d}t} \} \frac{\partial}{\partial y^l} \\
&= \{ \frac{\partial f^l}{\partial x^k} (\frac{\mathrm{d}^2\gamma^k}{\mathrm{d}t^2} + \Gamma^k_{ij} \frac{\mathrm{d}\gamma^i}{\mathrm{d}t} \frac{\mathrm{d}\gamma^j}{\mathrm{d}t}) + (\frac{\partial^2 f^l}{\partial x^i \partial x^j} + \frac{\partial f^m}{\partial x^i} \frac{\partial f^n}{\partial x^j} \Gamma^l_{mn} - \Gamma^k_{ij} \frac{\partial f^l}{\partial x^k}) \frac{\mathrm{d}\gamma^i}{\mathrm{d}t} \frac{\mathrm{d}\gamma^j}{\mathrm{d}t} \} \frac{\partial}{\partial y^l} \\
&= f_*(\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma_*(\frac{\mathrm{d}}{\mathrm{d}t})) + \gamma^* B
\end{aligned}$$

□

Theorem C.1.1. The followings are equivalent:

1. f is totally geodesic;
2. f maps geodesics in M to geodesics in N.

Proof. Clear from above lemma. □

C.2. **First variation of smooth map.**

Definition C.2.1 (energy functional)**.** The energy density of smooth function $f:(M,g)\to(N,h)$ is

$$e(f)=|\mathrm{d}f|^2$$

The energy of f is

$$E(f)=\frac{1}{2}\int_M e(f)\,\mathrm{vol}$$

Remark C.2.1 (local form). We can locally write energy density as

$$\begin{aligned}\langle\mathrm{d}f,\mathrm{d}f\rangle&=\langle\frac{\partial f^m}{\partial x^i}\mathrm{d}x^i\otimes\frac{\partial}{\partial y^m},\frac{\partial f^n}{\partial x^j}\mathrm{d}x^j\otimes\frac{\partial}{\partial y^n}\rangle\\&=g^{ij}h_{mn}(f)\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j}\end{aligned}$$

Theorem C.2.1. The Euler-Lagrange equation of $E(f)$ is

$$\widehat{\nabla}^*\mathrm{d}f=0$$

where $\widehat{\nabla}^*$ is the formal adjoint operator of $\widehat{\nabla}$.

Proof. We fix the following notations in the proof:

1. Consider a smooth variation $F:M\times\mathbb{R}\to N$ of f, we also write $f_t(\text{-})=F(\text{-},t)$ for convenience;
2. Set $\overline{M}=M\times\mathbb{R}$ and there is a natural metric $\overline{g}=g\times g_{\mathbb{R}}$ on $\overline{M}$;
3. The pullback F^*TN bundle is denoted by W, and induced connection on W is denoted by ∇^W;
4. Fix $t\in\mathbb{R}$, $f_t:M\to N$, then $\mathrm{d}f_t$ is a section of $T^*M\otimes f_t^*TN$, and we can regard it as a section of $T^*\overline{M}\otimes W$.

Holding above notations, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(f_t)&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_M|\mathrm{d}f_t|^2\,\mathrm{vol}\\&=\int_M\langle\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial t}}\mathrm{d}f_t,\mathrm{d}f_t\rangle\,\mathrm{vol}\end{aligned}$$

Here we claim

$$\langle\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial t}}\mathrm{d}f_t,\mathrm{d}f_t\rangle\overset{1}{=}\langle\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial t}}\mathrm{d}F,\mathrm{d}f_t\rangle\overset{2}{=}\langle\nabla^W F_*(\frac{\partial}{\partial t}),\mathrm{d}f_t\rangle$$

1. For equation marked 1: Note that

$$\begin{aligned}\mathrm{d}F - \mathrm{d}f_t &= \frac{\partial F^m}{\partial x^i}\mathrm{d}x^i \otimes \frac{\partial}{\partial y^m} + \frac{\partial F^m}{\partial t}\mathrm{d}t \otimes \frac{\partial}{\partial y^m} - \frac{\partial f_t^m}{\partial x^i}\mathrm{d}x^i \otimes \frac{\partial}{\partial y^m} \\ &= \frac{\partial F^m}{\partial t}\mathrm{d}t \otimes \frac{\partial}{\partial y^m}\end{aligned}$$

since $\frac{\partial F^m}{\partial x^i} = \frac{\partial f_t^m}{\partial x^i}$. So we have

$$\begin{aligned}\nabla^{T^*\overline{M}\otimes W}(\mathrm{d}F - \mathrm{d}f_t) &= \frac{\partial^2 F^l}{\partial t^2}\mathrm{d}t \otimes \mathrm{d}t \otimes \frac{\partial}{\partial y^l} + \frac{\partial F^m}{\partial t}\mathrm{d}t \otimes (\frac{\partial F^n}{\partial t}\Gamma^l_{mn}\mathrm{d}t \otimes \frac{\partial}{\partial y^l} + \frac{\partial F^n}{\partial x^i}\Gamma^l_{mn}\mathrm{d}x^i \otimes \frac{\partial}{\partial y^l}) \\ &= (\frac{\partial^2 F^l}{\partial t^2} + \frac{\partial F^m}{\partial t}\frac{\partial F^n}{\partial t}\Gamma^l_{mn})\mathrm{d}t \otimes \mathrm{d}t \otimes \frac{\partial}{\partial y^l} + \frac{\partial F^m}{\partial t}\frac{\partial F^n}{\partial x^i}\Gamma^l_{mn}\mathrm{d}x^i \otimes \mathrm{d}t \otimes \frac{\partial}{\partial y^l}\end{aligned}$$

Thus we have

$$\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial t}}(\mathrm{d}F - \mathrm{d}f_t) = (\frac{\partial^2 F^l}{\partial t^2} + \frac{\partial F^m}{\partial t}\frac{\partial F^n}{\partial t}\Gamma^l_{mn})\mathrm{d}t \otimes \frac{\partial}{\partial y^l}$$

From above expression it's clear

$$\langle \nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial t}}(\mathrm{d}F - \mathrm{d}f_t), \mathrm{d}f_t\rangle = 0$$

since there is no $\mathrm{d}t$ in $\mathrm{d}f_t$, which implies equation marked 1 holds.

2. For equation marked 2: For arbitrary $X \in C^\infty(M, TM) \subset C^\infty(\overline{M}, T^*\overline{M})$, since second fundamental form is symmetric, thus

$$\begin{aligned}(\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial t}}\mathrm{d}F)(X) &= (\nabla^{T^*\overline{M}\otimes W}_X \mathrm{d}F)(\frac{\partial}{\partial t}) \\ &= \nabla^W_X F_*(\frac{\partial}{\partial t}) - F_*(\nabla^{\overline{M}}_X \frac{\partial t}{\partial t}) \\ &= \nabla^W_X F_*(\frac{\partial}{\partial t})\end{aligned}$$

Now let v be an arbitrary variation vector field, that is

$$v = F_*(\frac{\partial}{\partial t})\Big|_{t=0} \in C^\infty(M, f^*TN)$$

Hence when $t = 0$ we have

$$(\nabla^W F_*(\frac{\partial}{\partial t}))\Big|_{t=0} = \widehat{\nabla} v$$

where $\widehat{\nabla}$ is the induced connection on f^*TN. So we have first variation formula

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} E(f_t) &= \int_M \langle \widehat{\nabla} v, \mathrm{d}f\rangle \,\mathrm{vol} \\ &= \int_M \langle v, \widehat{\nabla}^* \mathrm{d}f\rangle \,\mathrm{vol} = 0\end{aligned}$$

where $\widehat{\nabla}^*$ is the formal adjoint operator of $\widehat{\nabla}$. since v is arbitrary, we deduce $\widehat{\nabla}^*\mathrm{d}f = 0$. $\square$

C.3. **Second variation formula of harmonic map.** Consider the following variation map of f

$$F: M \times (-\varepsilon_1, \varepsilon_1) \times (-\varepsilon_2, \varepsilon_2) \to N$$

with variation fields

$$v = F_*(\frac{\partial}{\partial t})\Big|_{s=t=0} \in C^\infty(M, f^*TN)$$

$$w = F_*(\frac{\partial}{\partial s})\Big|_{s=t=0} \in C^\infty(M, f^*TN)$$

For convenience we denote $F(\text{-}, s, t) = f_{s,t}(\text{-})$.

Theorem C.3.1 (second variation formula)**.** If $f : (M, g) \to (N, h)$ is a harmonic map, then the second variation of the harmonic map f along v, w is

$$\frac{\partial^2}{\partial s \partial t}\Big|_{s=t=0} E(f_{s,t}) = \int_M \langle \widehat{\nabla} v, \widehat{\nabla} w \rangle \,\mathrm{vol} - \int_M g^{ij} R_{pmnq} v^p w^q \frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j} \,\mathrm{vol}$$

Proof. In this proof, we still use the notations in proof of first variation formula. By first variation formula, we have

$$\frac{\partial}{\partial t} E(f_{s,t}) = \int_M \langle \nabla^W F_*(\frac{\partial}{\partial t}), \mathrm{d} f_{s,t} \rangle \,\mathrm{vol}$$

So

$$\begin{aligned}\frac{\partial^2}{\partial s \partial t} E(f_{s,t}) = & \underbrace{\int_M \langle \nabla^{T^*\overline{M} \otimes W}_{\frac{\partial}{\partial s}} \nabla^W F_*(\frac{\partial}{\partial t}), \mathrm{d} f_{s,t}) \rangle \,\mathrm{vol}}_{\text{part I}} \\ & + \underbrace{\int_M \langle \nabla^W F_*(\frac{\partial}{\partial t}), \nabla^{T^*\overline{M} \otimes W}_{\frac{\partial}{\partial s}} \mathrm{d} f_{s,t} \rangle \,\mathrm{vol}}_{\text{part II}}\end{aligned}$$

Note that

$$\begin{aligned}\nabla^{T^*\overline{M} \otimes W}_{\frac{\partial}{\partial s}} \mathrm{d} f_{s,t} &= \nabla^{T^*\overline{M} \otimes W}_{\frac{\partial}{\partial s}} (\frac{\partial F^m}{\partial x^i} \mathrm{d}x^i \otimes \frac{\partial}{\partial y^m}) \\ &= \frac{\partial^2 F^m}{\partial s \partial x^i} \mathrm{d}x^i \otimes \frac{\partial}{\partial y^m} + \frac{\partial F^m}{\partial x^i}\frac{\partial F^n}{\partial s} \Gamma^l_{mn} \mathrm{d}x^i \otimes \frac{\partial}{\partial y^l} \\ &= (\frac{\partial^2 F^l}{\partial s \partial x^i} + \frac{\partial F^m}{\partial x^i}\frac{\partial F^n}{\partial s} \Gamma^l_{mn}) \mathrm{d}x^i \otimes \frac{\partial}{\partial y^l} \\ \widehat{\nabla} w &= \widehat{\nabla}_{\frac{\partial}{\partial x^i}} (\frac{\partial F^n}{\partial s}\Big|_{t=s=0}) \mathrm{d}x^i \otimes \frac{\partial}{\partial y^n} + \frac{\partial F^m}{\partial s}\frac{\partial F^n}{\partial x^i}\Big|_{t=s=0} \Gamma^l_{mn} \mathrm{d}x^i \otimes \frac{\partial}{\partial y^l} \\ &= (\frac{\partial^2 F^l}{\partial s \partial x^i} + \frac{\partial F^m}{\partial x^i}\frac{\partial F^n}{\partial s}\Big|_{t=s=0} \Gamma^l_{mn}) \mathrm{d}x^i \otimes \frac{\partial}{\partial y^l}\end{aligned}$$

which implies setting $t = s = 0$ we have part II is

$$\int_M \langle \widehat{\nabla} v, \widehat{\nabla} w \rangle \,\mathrm{vol}$$

For part I, take arbitrary $X \in C^\infty(M, TM) \subset C^\infty(\overline{M}, T^*\overline{M})$, we have
Hence we obtain

$$\nabla^{T*\overline{M}\otimes W}_{\frac{\partial}{\partial s}} \nabla^W F_*(\frac{\partial}{\partial t})(X) = (\nabla^{T*\overline{M}\otimes W} \nabla^W F_*(\frac{\partial}{\partial t})(X))(\frac{\partial}{\partial s}, X)$$

Setting $t = s = 0$ we have
Hence

$$\begin{aligned}\frac{\partial^2}{\partial s \partial t}\bigg|_{t=s=0} E\left(f_{s,t}\right) = &\int_M \langle \widehat{\nabla}(\nabla^W_{\frac{\partial}{\partial s}} F_*(\frac{\partial}{\partial t})\bigg|_{s=t=0}), \mathrm{d}f\rangle \operatorname{vol} \\ &+ \int_M g^{ij} R_{pmqn} v^p w^q \frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j} \operatorname{vol} + \int_M \langle \widehat{\nabla} w, \widehat{\nabla} v\rangle \operatorname{vol}\end{aligned}$$

If f is harmonic, that is $\widehat{\nabla}^*\mathrm{d}f = 0$, we obtain the desired formula. □

C.4. **Bochner formula for harmonic map.** Recall that for a smooth function $f : (M, g) \to \mathbb{R}$,

$$\frac{1}{2}\Delta|\mathrm{d}f|^2 = |\operatorname{Hess} f|^2 + \operatorname{Ric}(\nabla f, \nabla f)$$

In this section we generalizes this formula to smooth map $f : (M, g) \to (N, h)$ between Riemannian manifolds, to get similar Bochner's theorem we have proven before.

Theorem C.4.1. Let $f : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds, then

$$\frac{1}{2}\Delta_g|\mathrm{d}f|^2 = |\widetilde{\nabla}\mathrm{d}f|^2 + \langle\widehat{\nabla}(\mathrm{d}f), \mathrm{d}f\rangle + g^{ik}g^{jl}R_{ij}\frac{\partial f^m}{\partial x^k}\frac{\partial f^n}{\partial x^l}h_{mn} - g^{kl}g^{ij}R_{mnpq}\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j}\frac{\partial f^p}{\partial x^k}\frac{\partial f^q}{\partial x^l}$$

Theorem C.4.2. Let $f : (M, g) \to (N, h)$ be a harmonic map between Riemannian manifolds. If

1. M is connected compact with positive Ricci curvature;
2. N has non-positive sectional curvature.

Then f is constant.

Proof. Suppose $|\mathrm{d}f|^2$ attains its maximum at some point $p \in M$, we have

$$\Delta_g|\mathrm{d}f|^2(p) \leq 0$$

Thus

$$\frac{1}{2}\Delta_g|\mathrm{d}f|^2 \geq g^{ik}g^{jl}R_{ij}\frac{\partial f^m}{\partial x^k}\frac{\partial f^n}{\partial x^l}h_{mn} - g^{kl}g^{ij}R_{mnpq}\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j}\frac{\partial f^p}{\partial x^k}\frac{\partial f^q}{\partial x^l}$$

WLOG we may assume $g_{ij}(p) = \delta_{ij}, h_{\alpha\beta}(f(p)) = \delta_{mn}$ by choosing normal coordinates. Then

$$\begin{aligned}\frac{1}{2}\Delta_g|\mathrm{d}f|^2 &\geq \sum_{i,j,m} R_{ij}\frac{\partial f^m}{\partial x^i}\frac{\partial f^m}{\partial x^j} - \sum_{i,j} R_{mnpq}\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^i}\frac{\partial f^p}{\partial x^j}\frac{\partial f^q}{\partial x^j} \\ &\geq 0\end{aligned}$$

since R_{ij} is positive, which implies $\mathrm{d}f \equiv 0$, thus f is constant since M is connected. □

Corollary C.4.1. If (M, g) be a connected Riemannian manifold and $f : (M, g) \to (N, h)$ is totally geodesic, then $\mathrm{d}f = 0$.

Corollary C.4.2. $f : (M, g) \to (N, h)$ is a harmonic map, then

1.
2.

Appendix D. Topology

D.1. The universal covering.

Definition D.1.1 (deck transformation)**.** $\widetilde{M}$ is the universal covering of M, the deck transformation group is defined as

$$\mathrm{Aut}_\pi(\widetilde{M}) = \{F : \widetilde{M} \to \widetilde{M} \text{ is diffeomorphism} \mid \pi \circ F = \pi\}$$

Furthermore, we have the following facts

1. $\mathrm{Aut}_\pi(\widetilde{M}) \cong \pi_1(M)$;
2. $\mathrm{Aut}_\pi(\widetilde{M})$ acts on $\widetilde{M}$ smoothly, freely and isometrically;
3. $\widetilde{M}/\mathrm{Aut}_\pi(\widetilde{M}) \cong M$.

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