# YANG-MILLS EQUATIONS ON RIEMANN SURFACE

BOWEN LIU

Abstract.

## Contents

# 0. Preface

## 0.1. About this lecture.

### 0.2. **Notations.**

## 1. The Yang-Mills equations

In this section we assume $G$ is a compact Lie group, and $(M, g)$ is an oriented compact Riemannian manifold.

### 1.1. **The Yang-Mills functional.**

Let $P$ be a principal $G$-bundle with local trivializations $\{U_\alpha\}$ and $\rho\colon G \to \mathrm{GL}(V)$ be a linear representation of $G$. Note that any $s \in C^\infty(M, \Omega^k_M(P \times_\rho V))$ is given by data $\{(s_\alpha) \in \prod C^\infty(U_\alpha, \Omega^k_M(V)) \mid s_\alpha = \rho(g_{\alpha\beta})s_\beta\}$, so if we want to construct an inner product on $\Omega^k_M(P \times_\rho V)$, it suffices to construct a $\rho$-invariant inner product $\langle -, - \rangle$ on $V$ since we already have a Riemannian metric $g$.

The case that captivates our utmost interest is when $V = \mathfrak{g}$, as the curvature of a connection is a section of $\Omega^2_M(\operatorname{Ad}\mathfrak{g})$. In order to construct an inner product on $\Omega^k_M(\operatorname{Ad}\mathfrak{g})$, we need an inner product on $\mathfrak{g}$ which is invariant under the adjoint action. Since $G$ is compact, its Killing form is a non-degenerate inner product, that's what we're looking for! Thus we have a pointwise inner product on the bundle $\Omega^k_M(\operatorname{Ad}\mathfrak{g})$, denoted by $\langle -, - \rangle$, and define a global inner product on $\Omega^k_M(\operatorname{Ad}\mathfrak{g})$ as

$$(\alpha, \beta) := \int_M \langle \alpha, \beta \rangle \operatorname{vol}$$

where $\alpha, \beta \in C^\infty(M, \Omega^k_M(\operatorname{Ad}\mathfrak{g}))$.

**Definition 1.1.1** (Hodge star operator)**.** The Hodge star operator

$$\begin{aligned} \star\colon C^\infty(M, \Omega^k_M(\operatorname{Ad}\mathfrak{g})) &\to C^\infty(M, \Omega^{n-k}_M(\operatorname{Ad}\mathfrak{g})) \\ \beta &\mapsto \star\beta \end{aligned}$$

where $\star\beta$ is given by

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \operatorname{vol}, \quad \forall \alpha \in C^\infty(M, \Omega^k_M(\operatorname{Ad}\mathfrak{g}))$$

Having established these preliminary results, we now proceed to introduce the Yang-Mills functional.

**Definition 1.1.2** (Yang-Mills functional)**.** The Yang-Mills functional is the map $YM\colon \mathcal{A}(P) \to \mathbb{R}$ given by

$$YM(\omega) := \|F_\omega\|^2 = \int_M \langle F_\omega, F_\omega \rangle \operatorname{vol}$$

where $F_\omega$ is the curvature of connection $\omega$.

*Remark* 1.1.1. By using Hodge star operator, Yang-Mills functional can be rewritten as follows

$$YM(\omega) = \int_M F_\omega \wedge \star F_\omega$$

The advantages of writing Yang-Mills functional in this way is that we can use some properties of Hodge operator to simplify our computations.

**Proposition 1.1.1.** Yang-Mills functional is gauge invariant, that is for any gauge transformation $\Phi \in \mathcal{G}(P)$, one has $YM(\Phi^*\omega) = YM(\omega)$ holds for connection $\omega$.

*Proof.* On each local trivialization $U_\alpha$, the curvature of $\Phi^*\omega$ is given by $\mathrm{Ad}(\phi^{-1}) \circ F_\alpha$, where $\phi$ is given by $\Phi|_{U_\alpha}(x, g) = (x, \phi(x)g)$. Thus Yang-Mills functional is gauge invariant since inner product $\langle \text{-}, \text{-} \rangle$ is adjoint invariant. $\square$

**Definition 1.1.3** (Yang-Mills connection)**.** A Yang-Mills connection is a connection $A \in \mathcal{A}(P)$ which is a local extremum of Yang-Mills functional.

**Notation 1.1.1.** $\mathcal{A}_{YM}(P)$, or briefly $\mathcal{A}_{YM}$ denotes the set of all Yang-Mills connections.

1.2. **The variational problem.** Let's see how to use a second-order partial differential equation to characterize Yang-Mills connection. Recall that $\mathcal{A}(P)$ is an affine space modelled on $\Omega^1_M(\mathrm{Ad}\,\mathfrak{g})$, so the tangent space to $\mathcal{A}(P)$ at any point is isomorphic to $\Omega^1_M(\mathrm{Ad}\,\mathfrak{g})$.

Given $\omega \in \mathcal{A}(P)$ and $\tau \in C^\infty(M, \Omega^1_M(\mathrm{Ad}\,\mathfrak{g}))$, the directional derivative of Yang-Mills functional at $\omega$ in the direction $\tau$ is given by

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} YM(\omega + t\tau)$$

And Yang-Mills condition states that this vanishes for all $\tau$. In order to see what this means, firstly we need the following lemma.

**Lemma 1.2.1.** Given $\omega \in \mathcal{A}(P)$ and $\tau \in C^\infty(M, \Omega^1_M(\mathrm{Ad}\,\mathfrak{g}))$, then

$$F_{\omega+\tau} = F_\omega + \mathrm{d}_\omega \tau + \frac{1}{2}\tau \wedge \tau$$

where $\mathrm{d}_\omega$ is connection induced by $\omega$ on $\Omega^1_M(\mathrm{Ad}\,\mathfrak{g})$.

*Proof.* On local trivialization $U_\alpha$ one has

$$\begin{aligned}
(F_{\omega+\tau})_\alpha &= \mathrm{d}(A_\alpha + \tau_\alpha) + \frac{1}{2}(A_\alpha + \tau_\alpha) \wedge (A_\alpha + \tau_\alpha) \\
&= (F_\omega)_\alpha + \mathrm{d}\tau_\alpha + \frac{1}{2}(A_\alpha \wedge \tau_\alpha + \tau_\alpha \wedge A_\alpha) + \frac{1}{2}\tau_\alpha \wedge \tau_\alpha \\
&\overset{(1)}{=} (F_\omega)_\alpha + \mathrm{d}\tau_\alpha + A_\alpha \wedge \tau_\alpha + \frac{1}{2}\tau_\alpha \wedge \tau_\alpha \\
&\overset{(2)}{=} (F_\omega)_\alpha + \mathrm{d}_\omega \tau_\alpha + \frac{1}{2}\tau_\alpha \wedge \tau_\alpha
\end{aligned}$$

where

(1) holds from both $A_\alpha, \tau_\alpha$ are 1-form valued in $\mathfrak{g}$.
(2) holds from the definition of $\mathrm{d}_\omega$.

$\square$

**Proposition 1.2.1** (first variation formula)**.** Let $\omega$ be a Yang-Mills connection. Then

$$\mathrm{d}_\omega^* F_\omega = 0$$

*Proof.* A direct computation shows

$$\begin{aligned} YM(\omega + t\tau) &= \int_M \langle F_{\omega+t\tau}, F_{\omega+t\tau} \rangle \operatorname{vol} \\ &= \int_M \langle F_\omega + \frac{t^2}{2}(\tau \wedge \tau) + t\mathrm{d}_\omega \tau, F_\omega + \frac{t^2}{2}(\tau\wedge\tau) + t\mathrm{d}_\omega\tau \rangle \operatorname{vol} \end{aligned}$$

The coefficient of linear term is

$$\int_M \langle F_\omega, \mathrm{d}_\omega \tau \rangle + \langle \mathrm{d}_\omega \tau, F_\omega \rangle \operatorname{vol} = 2\int_M \langle \mathrm{d}_\omega \tau, F_\omega \rangle \operatorname{vol}$$

Let $\mathrm{d}_\omega^*$ be the formal adjoint[1] of $\mathrm{d}_\omega$. Then

$$0 = \int_M \langle \mathrm{d}_\omega \tau, F_\omega \rangle \operatorname{vol} = \int_M \langle \tau, \mathrm{d}_\omega^* F_\omega \rangle \operatorname{vol}$$

holds for arbitrary $\tau$. This shows $\mathrm{d}_\omega^* F_\omega = 0$. □

**Definition 1.2.1** (Yang-Mills equations)**.** A connection $\omega \in \mathcal{A}(P)$ is called satisfying Yang-Mills equations, if

$$\begin{cases} \mathrm{d}_\omega F_\omega = 0 \\ \mathrm{d}_\omega^* F_\omega = 0 \end{cases}$$

*Remark* 1.2.1. In fact, the first equation is exactly the Bianchi identity, which is automatically holds.

**Example 1.2.1.** In the case that $G = U(1)$, we have that the curvature of a connection $A$ can be identified as a section of $\Omega_M^2$. Indeed, the curvature form takes value in the bundle $\operatorname{Ad}\mathfrak{g}$, but here $G = U(1)$ is abelian, thus the adjoint action on $\mathfrak{u}(1)$ is trivial, so

$$\operatorname{Ad}\mathfrak{g} = M \times \mathfrak{u}(1) = M \times \mathbb{R}$$

is trivial bundle. Furthermore, $\omega$ is a Yang-Mills connection if and only if $F_\omega$ is a harmonic 2-form, that is $\Delta F_\omega = 0$, where $\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$. Indeed, thanks to $U(1)$ is abelian again, $\mathrm{d}_\omega$ can be reduced to $\mathrm{d}$, since for arbitrary form $\beta$, we have[2] $\omega \wedge \beta = 0$. This shows the Yang-Mills equations in this case is

$$\begin{cases} \mathrm{d}F_\omega = 0 \\ \mathrm{d}^* F_\omega = 0 \end{cases}$$

[1]In fact, the form adjoint of $\mathrm{d}_\omega$ can be explicitly written as $\mathrm{d}_\omega^* = (-1)^{2n+1} \star \mathrm{d}_\omega \star$.

[2]This follows from in the definition of wedge product of forms valued in Lie algebra we used Lie bracket, and abelian Lie algebra has trivial Lie bracket.

On the other hand, a form is harmonic if and only if it satisfies above equations since

$$\begin{aligned} 0 &= \int_M \langle \Delta F_\omega, F_\omega \rangle \,\mathrm{vol} \\ &= \int_M \langle \mathrm{d}\mathrm{d}^* F_\omega, F_\omega \rangle + \langle \mathrm{d}^* \mathrm{d} F_\omega, F_\omega \rangle \,\mathrm{vol} \\ &= \int_M \|\mathrm{d}^* F_\omega\|^2 + \|\mathrm{d} F_\omega\|^2 \,\mathrm{vol} \end{aligned}$$

## 2. Narasimhan-Seshadri theorem

Recall that the Yang-Mills functional is gauge invariant, so if a connection $\omega$ solves the Yang-Mills equations, so does any gauge transformed $\Phi^*\omega$. In other words, the gauge group acts on $\mathcal{A}_{YM}$, and it's natural to consider the quotient $\mathcal{A}_{YM}/\mathcal{G}$. In general it is infinite dimensional, and the topology of this space may be quite bad (It may be neither Hausdorff nor a smooth manifold). However, with the imposition of certain restrictions, there is a good result, and that's Narasimhan-Seshadri theorem.

### 2.1. **The Kempf-Ness theorem.**

To get a picture of what to expect, we firstly study a finite dimensional analogue.

#### 2.1.1. *Baby version.*

Consider the following setting:

(1) Let $V$ be a complex vector space with a Hermitian inner product.
(2) Let $S^1 \to \mathrm{U}(V)$ be an action of circle by unitary matrices.
(3) Let $\mathbb{C}^* \to \mathrm{GL}(V)$ be the complexification of this action.

**Example 2.1.1.** Consider $\lambda \in \mathbb{C}^*$ acting on $\mathbb{C}^2$ by $(x, y) \mapsto (\lambda x, \lambda^{-1}y)$. The orbits are

(1) the conics $xy = c \neq 0$,
(2) the axes $y = 0, x \neq 0$ and $x = 0, y \neq 0$,
(3) the origin.

It's clear to see the quotient topology on the orbit space is non-Hausdorff since the axes come arbitrarily close to the origin. But $\mathbb{C}^2 \setminus \{\text{axes}\}/\mathbb{C}^*$ is homeomorphic to $\mathbb{C}$, and thus Hausdorff.

The reason for the orbit space fail to be Hausdorff is that there exists non-closed sets, so generally we want to form a Hausdorff quotients by considering only closed orbits.

**Definition 2.1.1** (stable)**.** A point $v \in V$ is stable if its orbit under $\mathbb{C}^*$ is closed.

**Theorem 2.1.1** (Kempf-Ness)**.** A point $v$ is stable if and only if the function $\|\cdot\|^2$ restricted to its orbit attains its minimum.

We can think this function as a function $p_v \colon \mathbb{C}^* \to \mathbb{R}$ given by $p_v(g) = \|g(v)\|^2$. Note that since the norm is $\mathrm{U}(V)$-invariant, the function $p_v$ is $S^1$-invariant and descends to a function defined on $(\mathbb{C}^*/S^1, \times) \xrightarrow{\log} (\mathbb{R}, +)$.

$$p_v(x) = \|e^x(v)\|^2$$

In Example 2.1.1 one has

$$e^x(v_1, v_2) = (e^{-x}v_1, e^x v_2)$$

Thus

$$p_v(x) = \|v_1\|^2 e^{-2x} + \|v_2\|^2 e^{2x}$$

By taking derivative one has

$$\frac{\mathrm{d}p_v}{\mathrm{d}x} = -2x\|v_1\|^2e^{-2x} + 2x\|v_2\|^2e^{2x}$$

If both $v_1$ and $v_2$ are non-zero, then it obtains its minimum at

$$\frac{1}{2}\left(\log(\|v_1\|) - \log(\|v_2\|)\right)$$

and if $v = 0$, then it obtains its minimum at $x = 0$. Furthermore, the minimum is not obtained along two punctured axes, so in Example 2.1.1 the stable points are $\{(x, y) \mid xy \neq 0\} \cup \{(0, 0)\}$, and definitely the orbit space of stable points are Hausdorff.

*Proof of Theorem 2.1.1.*

**Lemma 2.1.1.** If $v$ is not stable, then $p_v$ does not attains its minimum.

*Proof.* □

□

To understand the space of stable points it's therefore important to understand the critical points of $p_v$. Suppose $p_v(x)$ is written as $\sum_{m=1}^{n} \|v_m\|^2e^{2j_mx}$. Then

$$\frac{\mathrm{d}p_v}{\mathrm{d}x} = 2\sum_{m=1}^{n} j_m\|v_m\|^2e^{2j_mx}$$

Suppose $v$ is stable and that the minimum occurs at $x = x_0$. Without lose of generality we may assume $x = 0$ since by replacing $v$ by $v' = e^{-x_0}v$ one has

$$\frac{\mathrm{d}p_{v'}}{\mathrm{d}x} = \sum_{m=1}^{n} j_m\|e^{-j_mx_0}v_m\|^2e^{2j_mx_0} = \sum_{m=1}^{n} j_m\|v_m\|^2$$

Therefore the orbit of a stable vector contains a zero of the function

$$\mu = \sum_{m=1}^{n} j_m\|v_m\|^2$$

In fact it contains a whole $S^1$ since $\mu$ is $S^1$-invariant, and this shows the following correspondence.

**Theorem 2.1.2.** Let $V^s$ denote the space of stable vectors under the action of $\mathbb{C}^*$. Then

$$V^s/\mathbb{C}^* = \mu^{-1}(0)/S^1$$

2.1.2. *Moment map.*

2.2. **The moment map in Yang-Mills theory.** Let $M$ be a Riemann surface and $P$ be a principal $G$-bundle over $M$. Then the space of connections $\mathcal{A}(P)$ has a natural symplectic form.

**Proposition 2.1** (Atiyah-Bott)**.** The following bilinear form

$$Q(\alpha,\beta)=\int_M \alpha\wedge\beta$$

where $\alpha,\beta\in C^\infty(M,\Omega^1_M(\operatorname{Ad}P))$, is a symplectic form defined on $\mathcal{A}(P)$.

*Proof.* Let's check step by step:

1. It's clear that $Q$ is a 2-form, since $\mathcal{A}(P)$ is affine modelled on $C^\infty(M,\Omega^1_M(\operatorname{Ad}P))$.
2. 
3. 

□

*Remark* 2.2.1. Note that this integral do makes sense since the real dimension of $M$ is two.

**Lemma 2.2.1.** For $\phi\in C^\infty(M,\Omega^0_M(\operatorname{Ad}P))$, $\nabla\phi$ is the Hamiltonian vector field of function $f\colon\nabla\to-\int_M F_\nabla\wedge\phi$.

*Proof.* By definition we need to check

$$\mathrm{d}f=\iota_{\nabla\phi}Q$$

Take arbitrary $\tau\in C^\infty(M,\Omega^1_M(\operatorname{Ad}P))$, integration by parts shows

$$\begin{aligned}Q(\nabla\phi,\tau)&=\int_M\nabla\phi\wedge\tau\\&=-\int_M\phi\wedge\nabla\tau\\&=-\int_M\nabla\tau\wedge\phi\end{aligned}$$

Note that $F_{\nabla+\varepsilon\tau}=F_\nabla+\varepsilon\nabla\tau+O(\varepsilon^2)$, then

$$\begin{aligned}\mathrm{d}f(\tau)&=\lim_{\varepsilon\to0}\frac{-\int_M F_{\nabla+\varepsilon\tau}\wedge\phi+\int_M F_\nabla\wedge\phi}{\varepsilon}\\&=-\int_M\nabla\tau\wedge\phi\end{aligned}$$

This completes the proof. □

*Remark* 2.2.2. In our case the $(\operatorname{Lie}\mathcal{G})^*=C^\infty(M,\Omega^2_M(\operatorname{Ad}P))$ and the moment map is just

$$\nabla\mapsto-F_\nabla$$

The Yang-Mills functional is just the norm of the moment map.

2.3. **Complexifying the action of gauge group.** Let $M$ be a Riemann surface, our ultimate goal is to relate moduli spaces of holomorphic vector bundles over $M$ to Yang-Mills connections. Firstly, we want to consider $\mathcal{A}(P)$ as a space of holomorphic vector bundles.

**Definition 2.3.1** (holomorphic vector bundle)**.** A holomorphic vector bundle is a complex vector bundle $\pi\colon E \to X$ such that the total space $E$ is a complex manifold and $\pi$ is holomorphic.

**Proposition 2.2.** If $P$ is a principal $\mathrm{U}(n)$-bundle over a Riemann surface $M$ and $\nabla$ is a $\mathrm{U}(n)$-connection then $\mathrm{Ad}(P)$ inherits the structure of a holomorphic vector bundle over $M$ such that $\nabla^{0,1} = \overline{\partial}$.

## 3. Stability of holomorphic vector bundles

In thise section, the guiding problem is to classify holomorphic vector bundles on a Riemann surface with genus $g$, denoted by $\Sigma_g$. For the case $g = 0, 1$, there are complete classification results for holomorphic vector bundles on $\Sigma_g$, due to Grothendieck for the case of the Riemann sphere [Gro57], and due to Atiyah for the case of elliptic curves [Ati57]. So in the following discussion, we always assume $g \geq 2$.

### 3.1. Stable bundle.

**Definition 3.1.1** (degree)**.** Let $\pi\colon E \to X$ be a holomorphic vector bundle, its degree is defined as

$$\deg(E) := \int_X c_1(E)$$

where $c_1(E) \in H^2(X, \mathbb{Z})$ is the first Chern class of $E$.

**Definition 3.1.2** (slope)**.** Let $\pi\colon E \to X$ be a holomorphic vector bundle, its slope is defined as

$$\mu(E) := \frac{\deg(E)}{\mathrm{rk}(E)}$$

*Remark* 3.1.1. One thing to note is that the slope of a holomorphic vector bundle is independent of the holomorphic structure, since both the degree and rank are topological invariants.

**Definition 3.1.3** (slope stability)**.** Let $\pi\colon E \to X$ be a holomorphic vector bundle, it's

1. stable if for every non-trivial holomorphic subbundle $F$, $\mu(F) < \mu(E)$;
2. semi-stable if for every non-trivial holomorphic subbundle $F$, $\mu(F) \leq \mu(E)$;
3. unstable if it's not semi-stable.

*Remark* 3.1.2. For slope stability, we have the following remarks:

($a$) It's clear that all holomorphic line bundles are stable, since they don't have non-trivial subbundles;
($b$) A semi-stable vector bundle with coprime rank and degree is actually stable, since
($c$) While the slope is a topological invariant, slope stability is not, since here we only consider holomorphic subbundles, which depends on the holomorphic structure.

**Proposition 3.1.1.** Let $E \to \Sigma_g$ be a holomorphic vector bundle, it's

1. stable if and only if for every non-trivial holomorphic subbundle $F$, $\mu(E/F) > \mu(E)$;
2. semi-stable if and only if for every non-trivial holomorphic subbundle $F$, $\mu(E/F) \geq \mu(E)$.

*Proof.* Denote $r, r', r''$ the ranks of $E, F, E/F$ respectively, and $d, d', d''$ their degrees respectively. From exact sequence

$$0 \to E \to E \to E/F \to 0$$

one has $r = r' + r''$ and $d = d' + d''$, thus

$$\frac{d'}{r'} < \frac{d' + d''}{r' + r''} \iff \frac{d'}{r'} < \frac{d''}{r''} \iff \frac{d' + d''}{r' + r''} < \frac{d''}{r''}$$

and likewise with the case semi-stable. □

A philosophy is that semi-stable bundles don't admit too many subbundles, since any subbundle they may have is of slope no greater than their own. This turns out to have many interesting consequences we're going to show, for example, the category of semi-stable bundles is abelian.

**Lemma 3.1.1.** If $\varphi\colon E \to E'$ is a non-zero homomorphism of holomorphic vector bundles over $\Sigma_g$, then

$$\mu(E/\ker\varphi) \leq \mu(\operatorname{im}\varphi)$$

**Proposition 3.1.2.** Let $E, E'$ be two semi-stable bundles such that $\mu(E) > \mu(E')$, then any homomorphism $\varphi\colon E \to E'$ is zero.

*Proof.* If $\varphi$ is non-zero, since $E$ is semi-stable, then

$$\mu(\operatorname{im}\varphi) \overset{(1)}{\geq} \mu(E/\ker\varphi) \overset{(2)}{\geq} \mu(E) > \mu(E')$$

where

(1) holds from Lemma 3.1.1;
(2) holds from Proposition 3.1.1.

which contradicts to the semi-stablity of $E'$. □

**Proposition 3.1.3.** Let $\varphi\colon E \to E'$ be a non-zero homomorphism of semi-stable holomorphic of slope $\mu$, then $\ker\varphi$ and $\operatorname{im}\varphi$ are semi-stable bundles of slope $\mu$, and the natural map $E/\ker\varphi \to \operatorname{im}\varphi$ is an isomorphism.

**Corollary 3.1.1.** The category of semi-stable bundles of slope $\mu$ is abelian, and the simple object[3] in this category is the stable bundles of slope $\mu$.

*Proof.* By Proposition 3.1.3 one has the category of semi-stable bundles of slope $\mu$ is abelian. A stable bundle $E$ is simple in this category, since it admits no non-trivial subbundles with slope $\mu$; Conversely, if a semi-stable bundle $E$ is simple, then any non-trivial subbundle $F$ satisfies $\mu(F) \leq \mu(E)$ since $E$ is semi-stable and $\mu(F) \neq \mu(E)$ since $E$ is simple, this shows $E$ is stable. □

**Proposition 3.1.4.** Let $E, E'$ be two stable vector bundles over $\Sigma_g$ with same slopes, and $\varphi\colon E \to E'$ be a non-zero homomorphism, then $\varphi$ is an isomorphism.

[3] Recall a simple object in an abelian category is an object with no non-trivial subobject.

*Proof.* Since $\varphi\colon E \to E'$ is a non-zero homomorphism between stable bundles with same slopes, then by Proposition 3.1.3 one has $\ker\varphi$ is either 0 or has slope $\mu(E)$, but $E$ is actually stable, then $\ker\varphi$ must be 0, and since $\varphi$ is strict, this shows $\varphi$ is injective. Likewise, $\operatorname{im}\varphi \neq 0$ and has slope $\mu(E')$, then it must be $E'$ since $E'$ is stable. Then again by $\varphi$ is strict, $\operatorname{im}\varphi = E'$ impiles $\varphi$ is surjective. Therefore $\varphi$ is an isomorphism. □

**Proposition 3.1.5.** If $E$ is a stable bundle over $\Sigma_g$, then $\operatorname{End} E = \mathbb{C}$. In particular, $\operatorname{Aut} E = \mathbb{C}^*$.

*Proof.* Let $\varphi$ be a non-zero endomorphism of $E$, by Proposition 3.1.4 one has $\varphi$ is an automorphism, so $\operatorname{End} E$ is a field, which contains $\mathbb{C}$ as its subfield of scalar endomorphisms. For any $\varphi \in \operatorname{End} E$, by Cayley-Hamilton theorem one has $\varphi$ is algebraic over $\mathbb{C}$, and since $\mathbb{C}$ is algebraic closed, this shows $\operatorname{End} E \cong \mathbb{C}$. □

**Corollary 3.1.2.** A stable bundle is indecomposable, that is it's not isomorphic to a direct sum of non-trivial subbundles.

*Proof.* The automorphism group of $E = E_1 \oplus E_2$ contains $\mathbb{C}^* \times \mathbb{C}^*$, so by Proposition 3.1.5 it can't be stable. □

**Theorem 3.1.1** (Jordan-Hölder filteration)**.** Any semi-stable bundle of slope $\mu$ over $\Sigma_g$ admits a filteration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_k = E$$

by holomorphic subbundles such that for each $1 \leq i \leq k$, one has

1. $E_i/E_{i-1}$ is stable;
2. $\mu(E_i/E_{i-1}) = \mu(E)$.

**Proposition 3.1.6** (Seshadri)**.** Any two Jordan-Hölder filterations

$$S : 0 = E_0 \subset E_1 \subset \cdots \subset E_k = E$$

and

$$S' : 0 = E'_0 \subset E'_1 \subset \cdots \subset E'_l = E$$

of a semi-stable bundle $E$ have same length, and the associated graded objects

$$\operatorname{gr}(S) : 0 = E_1/E_0 \oplus \cdots \oplus E_k/E_{k-1}$$

and

$$\operatorname{gr}(S') : 0 = E'_1/E'_0 \oplus \cdots \oplus E'_k/E'_{k-1}$$

satisfy $E_i/E_{i-1} \cong E'_i/E'_{i-1}$ for all $1 \leq i \leq k$.

**Definition 3.1.4** (poly-stable bundle)**.** A holomorphic vector bundle $E$ over $\Sigma_g$ is called poly-stable if it is isomorphic to a direct sum $E_1 \oplus \cdots \oplus E_k$ of stable bundles of the same slope.

**Example 3.1.1.** A stable bundle is poly-stable.

**Example 3.1.2.** The graded object associated to any Jordan-Hölder filteration of a semi-stable bundle $E$ is a poly-stable, and by Proposition 3.1.6, it's unique up to isomorphism, this isomorphic class is denoted by $\mathrm{gr}(E)$.

**Definition 3.1.5** ($S$-equivalence class)**.** The graded isomorphism class $\mathrm{gr}(E)$ associated to a semi-stable bundle $E$ is called the $S$-equivalence class of $E$. If $\mathrm{gr}(E) \cong \mathrm{gr}(E')$, $E$ and $E'$ are called $S$-equivalent, and denoted by $E \sim_S E'$.

**Definition 3.1.6.** The set $\mathcal{M}_{\Sigma_g}(r,d)$ of $S$-equivalence classes of semi-stable bundles of rank $r$ and degree $d$ over $\Sigma_g$ is called its moduli set, it contains the set $\mathcal{N}_{\Sigma_g}(r,d)$ of isomorphism classes of stable bundles of rank $r$ and degree $d$.

**Theorem 3.1.2** (Mumford-Seshadri)**.** Let $g \geq 2, r \geq 1$ and $d \in \mathbb{Z}$.

1. The set $\mathcal{N}_{\Sigma_g}(r,d)$ admits a structure of smooth, complex quasi-projective variety of dimension $r^2(g-1)+1$;
2. The set $\mathcal{M}_{\Sigma_g}(r,d)$ admits a structure of complex projective variety of dimension $r^2(g-1)+1$;
3. $\mathcal{N}_{\Sigma_g}(r,d)$ is an open dense subvariety of $\mathcal{M}_{\Sigma_g}(r,d)$.

In particular, when $r$ and $d$ are coprime, $\mathcal{M}_{\Sigma_g}(r,d) = \mathcal{N}_{\Sigma_g}(r,d)$ is a smooth complex projective variety.

*Proof.* See [Mum62] and [Ses67]. □

### 3.2. **The Harder-Narasimhan filteration.**

**Theorem 3.2.1** (Harder-Narasimhan)**.** Any holomorphic vector bundle $E$ over $\Sigma_g$ has a unique filteration

$$0 = E_0 \subset E_1 \subset \dots E_k = E$$

by holomorphic subbundles such that

1. for all $1 \leq i \leq k$, $E_i/E_{i-1}$ is semi-stable;
2. the slope $\mu_i := \mu(E_i/E_{i-1})$ of successive quotients satisfies

$$\mu_1 > \mu_2 > \cdots > \mu_k$$

This filteration is called Harder-Narasimhan filteration.

*Proof.* See [HN75]. □

*Remark* 3.2.1. If we denote $r = \mathrm{rk}\, E, d = \deg E, r_i = \mathrm{rk}(E_i/E_{i-1})$ and $d_i = \deg(E_i/E_{i-1})$, one has

$$r_1 + \cdots + r_k = r, \quad d_1 + \cdots + d_k = d$$

The $k$-tuple

$$\vec{\mu} := (\underbrace{\mu_1, \dots, \mu_1}_{r_1 \text{ times}}, \dots, \underbrace{\mu_k, \dots, \mu_k}_{r_k \text{ times}})$$

is called the Harder-Narasimhan type of $E$. It's equivalent to the data of the $k$-tuple $(r_i, d_i)_{1\leq i\leq k}$. In the plane of coordinates $(r, d)$, the polygonal line

$$P_{\vec{\mu}} := \{(0,0), (r_1, d_1), (r_1 + r_2, d_1 + d_2), \dots, (r_1 + \cdots + r_k, d_1 + \cdots + d_k)\}$$

defines a convex polygon called the Harder-Narasimhan polygon of $E$. The slope of the line from $(0,0)$ to $(r_1, d_1)$ is $\mu_1$, that is the slope of $E_1/E_0$, and perhaps that's why it's called slope. It's indeed convex, since $\mu_1 > \cdots > \mu_k$. A vector bundle is semi-stable if and only if it is it own Harder-Narasimhan filteration, and if and only if its Harder-Narasimhan filteration is a single line from $(0,0)$ to $(r, d)$.

## 4. Narasimhan-Seshadri theorem

## 5. $G$-equivariant cohomology

## References

[Ati57] M. F. Atiyah. Vector bundles over an elliptic curve. *Proceedings of the London Mathematical Society*, s3-7(1):414–452, 1957.

[Gro57] A. Grothendieck. Sur la classification des fibres holomorphes sur la sphere de riemann. *American Journal of Mathematics*, 79(1):121–138, 1957.

[HN75] G. Harder and M. S. Narasimhan. On the cohomology groups of moduli spaces of vector bundles on curves. *Mathematische Annalen*, 212(3):215–248, 1975.

[Mum62] David Mumford. Projective invariants of projective structures and applications. In *Proc. Internat. Congr. Mathematicians (Stockholm, 1962)*, pages 526–530, 1962.

[Ses67] C. S. Seshadri. Space of unitary vector bundles on a compact riemann surface. *Annals of Mathematics*, 85(2):303–336, 1967.

Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, P.R. China,

*Email address*: `liubw22@mails.tsinghua.edu.cn`