HODGE THOERY AND COMPLEX ALGEBRAIC GEOMETRY

BOWEN LIU

Contents

1. Overview

In this course, we will introduce two parts:

I Kähler manifold and Hodge decomposition;
II Hodge theory in algebra geometry.

For the first part, if X is a compact complex manifold, we can consider the following structures:

(1) Topology: $H^*_B(X,\mathbb{Z})$, singular cohomology, where B means "Betti".
(2) C^∞-structure: $H^*_{dR}(X,\mathbb{R}) = H^*(X,\Omega^{\cdot}_{X,\mathbb{R}})$, de Rham cohomology. In fact, de Rham theorem implies that
$$H^*_{dR}(X,\mathbb{R}) \cong H^*_B(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$$
(3) Complex structure: For $x \in X$, $T_{X,x}$ is tangent space at x, its real dimension is $2n$, that is $\dim_{\mathbb{R}} T_{X,x} = 2n$. And there is a linear map $J_x : T_{X,x} \to T_{X,x}$, such that $J_x^2 = -\operatorname{id}$. If we complexificate $T_{X,x}$, then we can decompose it into
$$T_{X,x} \otimes \mathbb{C} = T^{1,0}_{X,x} \oplus T^{0,1}_{X,x}$$
where $T^{1,0}_{X,x}$ is the eigenspace belonging to eigenvalue $\sqrt{-1}$, and $T^{0,1}_{X,x}$ is the eigenspace belonging to eigenvalue $-\sqrt{-1}$.

If we consider its dual, we will get bundle/sheaf of differential forms, and we can also decompose them as follows
$$\Omega^1_{X,\mathbb{C}} = \Omega^1_{X,\mathbb{R}} \otimes \mathbb{C} = \Omega^{1,0}_X \oplus \Omega^{0,1}_X$$
and
$$\Omega^k_{X,\mathbb{C}} = \Omega^k_{X,\mathbb{R}} \otimes \mathbb{C} = \bigoplus_{p+q=k} \Omega^{p,q}_X$$
where $\Omega^{p,q}_X = \bigwedge^p \Omega^{1,0}_X \otimes \wedge^q \Omega^{0,1}_X$

Since we have such decomposition for differential forms, it's natural to ask if there is a similar decomposition for de Rham cohomology? that is, do we have
$$H^k_{dR}(X,\mathbb{R}) = \bigoplus_{p+q=k} H^{p,q}(X)$$
such that $\overline{H^{p,q}(X)} = H^{q,p}(X)$?

The Hodge decomposition says it's true for compact Kähler manifolds. It's a very beautiful result, connecting "Topology" and "Complex geometry", since de Rham cohomology reflects the topology information and
$$H^{p,q}(X) \cong H^q_{Dol}(X,\Omega^p_X)$$
where "Dol" means Dolbealt cohomology.

Here is some examples of Kähler manifolds, in fact, almost every interesting manifold is Kähler manifold:

Example 1.0.1. Riemann surfaces, complex torus, projective manifolds are Kähler manifolds.

We also need to know an example that is not Kähler manifold:

Example 1.0.2 (Hopf surface)**.** Consider $\mathbb{Z}$ acts on $\mathbb{C}^2\backslash\{0\}$ by $mz = \lambda^m z, m \in \mathbb{Z}$ for some $\lambda \in (0,1)$, then we define Hopf surface as follows

$$S = \mathbb{C}^2\backslash\{0\}/\mathbb{Z}$$

As we can see, S is diffeomorphic to $S^3 \times S^1$,then $\dim_{\mathbb{C}} H^1(S,\mathbb{C}) = 1$, so S can not be a Kähler manifold by Hodge's decomposition, since for a Kähler manifold, $\dim_{\mathbb{C}} H^1$ must be even.

Remark 1.0.3. By Chow's theorem/GAGA[1], a compact complex manifold X admitting an embedding into projective space can be defined by polynomial equations, i.e., X is a projective variety, so here comes the forth structure, and that's the second part of this course, we want to apply Hodge theory in algebraic geometry.

(4) Algebraic structure.

2. Complex manifold

2.1. **Manifolds and vector bundles.**

2.1.1. *Definitions and Examples.*

Definition 2.1.1 (complex manifold)**.** *A complex manifold consists of* $(X, \{U_i, \phi_i\}_{i\in I})$*, where* X *is a connected, Hausdorff topological space,* $\{U_i\}_{i\in I}$ *is an open cover of* X *such that the index set* I *is countable, and* ϕ_i *is a homeomorphism from* U_i *to an open subset* V_i *of* $\mathbb{C}^n$*, such that*

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is biholomorphic.

Definition 2.1.2 (transition function)**.** *Such* $\phi_i \circ \phi_j^{-1}$ *is called transition function;* n *is called the dimension of* X*, denoted by* $\dim_{\mathbb{C}} C$*;* $\{U_i, \phi_i\}_{i\in I}$ *is called complex atlas.*

Definition 2.1.3 (atlas)**.** *Two atlas are equivalent, if the union of them is still an atlas.*

Definition 2.1.4 (complex structure)**.** *A complex structure is an equivalence class of a complex atlas.*

Remark 2.1.5. Replace $\mathbb{C}^n$ by $\mathbb{R}^n$, and biholomorphism is replaced by homeomorphism or diffeomorphism, then we get topological manifold or smooth manifold.

[1]Chow's theorem/GAGA: Every compact complex manifold X admitting embedding $X \hookrightarrow \mathbb{P}^n$ is defined by homogenous polynomials.

Remark 2.1.6. $V_i \subset \mathbb{C}^n$ usually can not be the whole $\mathbb{C}^n$. For example, there is no non-constant holomorphism from $\mathbb{C}$ to unit disk $\mathbb{D}$. More generally, X is called Brody hyperbodic if there is no non-constant holomorphism from $\mathbb{C}$ to X.

Example 2.1.7. Projective space $\mathbb{P}^n$ is a complex manifold. Atlas are $U_i = \{[z] \in \mathbb{P}^n \mid z_j \neq 0\}, 0 \leq i \leq n, \phi_i : U_i \to \mathbb{C}^n$ is defined as follows

$$[z] \mapsto (\frac{z_0}{z_i}, \ldots, \frac{\widehat{z_i}}{z_i}, \ldots, \frac{z_n}{z_i})$$

Transition functions are calculated as follows, for $i < j$

$$\phi_i \circ \phi_j^{-1} : (u_1, \ldots, u_n) \mapsto (\frac{u_1}{u_i}, \ldots, \frac{\widehat{u_i}}{u_i}, \ldots, \frac{u_{j-1}}{u_i}, \frac{1}{u_i}, \frac{u_{j+1}}{u_i}, \ldots, \frac{u_n}{u_i})$$

In fact, $\mathbb{P}^n$ is a compact complex manifold, since $\mathbb{P}^n$ is diffeomorphic to S^{2n+1}/S^1.

Example 2.1.8. Grassmannian manifold

$$G(r, n) = \{r\text{-dimensional subspace of } \mathbb{C}^n\}$$

Atlas: given $T_i \subset \mathbb{C}^n$ of dimension $n - r$, set $U_i = \{S \subset \mathbb{C}^n \text{of dimension r} \mid S \cap T_i = 0\}$. Choose $S_i \in U_i$, define

$$\phi_i : U_i \to \mathrm{Hom}(S_i, T_i) \cong \mathbb{C}^{r(n-r)}$$

as $S \mapsto f$, such that S is the graph of f.

Example 2.1.9. Complex torus is $\mathbb{C}^n/\Lambda$ where Λ is a free abelian subgroup of $\mathbb{C}^n$ with rank $2n$, called a lattice.

Definition 2.1.10 (holomorphic map)**.** *Let X, Y be complex manifolds of dimension n, m, with atlas $(U_i, \phi_i : U_i \to V_i)$ and $(M_j, \psi_j : M_j \to N_j)$ respectively. A continous map $f : X \to Y$ is called holomorphic, if for any two charts, we have*

$$\psi_j \circ f \circ \phi_i^{-1} : V_i \to \psi_j(f(U_i) \cap M_j)$$

is holomorphic.

Definition 2.1.11 (holomorphic function)**.** *A holomorphic function on X is a holomorphic map $f : X \to \mathbb{C}$.*

Example 2.1.12. Let $S = (\mathbb{C}^2 \backslash \{0\})/\mathbb{Z}$ be Hopf surface, then

$$f : S \to \mathbb{P}^1 = (\mathbb{C}^2 \backslash \{0\})/\mathbb{C}^*$$

is a holomorphic map. The fibers of f are biholomorphic to 1-dimensional complex torus.

Proposition 2.1. *If X is a compace complex manifold, then every holomorphic function on X is constant.*

Proof. Standard conclusion in complex analysis. $\square$

Definition 2.1.13 (immersion/submersion). *A holomorphic map $f : X \to Y$ is called an immersion(resp submersion), if for all $x \in X$, there exists $(x \in U_i, \phi_i), (f(x) \in M_j, \psi_j)$, such that*

$$J_{\psi_j \circ f \circ \phi_i^{-1}}(\phi_i(x))$$

has the max rank $\dim X$(*resp* $\dim Y$)

Definition 2.1.14 (embedding). *$f : X \to Y$ is an embedding, if it is immersion and $f : X \to f(X) \subset Y$ is homeomorphism.*

Definition 2.1.15 (submanifold). *A closed connected subset Y of X is called a submanifold, if for all $x \in Y$, there exists $x \in U \subset X$ and a holomorphic submersion $f : U \to \mathbb{D}^k$ such that*

$$U \cap Y = f^{-1}(0)$$

where k is the codimension of Y in X.

Example 2.1.16 (regular value theorem). Let X, Y be complex manifold with dimension n, m, If $y \in Y$ such that $\text{rank} J_{f(x)}$ reaches maximum m for all $x \in f^{-1}(y)$, then $f^{-1}(y)$ is a submanifold of codimension m.

Definition 2.1.17 (projective manifold). *A projective manifold*[2] *X is a submanifold of $\mathbb{P}^N$ of the form*

$$X = \{[z] \in \mathbb{P}^N \mid f_1(z) = \cdots = f_m(z) = 0\}$$

where f_i is a homogenous polynomial in $\mathbb{C}[z_0, \ldots, z_n]$

Remark 2.1.18. Here we always assume $(f_1, \ldots, f_m) \subset \mathbb{C}[z_0, \ldots, z_n]$ is a prime ideal, so the case that X is defined by polynomials like $f^2 = 0$ is not allowed, what's more, the condition that X is a manifold implies the following cases won't happen:

1. $f_1 f_2 = 0$;
2. X has a singular point.

Definition 2.1.19 (complete intersection). *Let $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{P}^n$ be the projection, then X is a submanifold of codimension k if and only if*

$$J = (\frac{\partial f_i}{\partial z_j})_{\substack{1 \leq i \leq m \\ 0 \leq j \leq N}}$$

has rank k, for all $x \in \pi^{-1}(X)$. Then X is called a complete intersection, if $m = k$.

Example 2.1.20. Consider $C \subset \mathbb{P}^n$ defined by

$$xw - yz = y^2 - xz = z^2 - yw = 0$$

is not a complete intersection, called twisted cubic.

[2] Chow's theorem claims that every submanifold of $\mathbb{P}^n$ must be defined by a set of homogenous polynomials, so we can use this property to define a projective manifold, in convenient.

Example 2.1.21. Plücker embedding

$$\Phi : G(r, V) \hookrightarrow \mathbb{P}(\wedge^r V)$$

defined by $S \subset V$ with basis $s_1, \ldots, s_r$ is mapped to $[s_1 \wedge \cdots \wedge s_r]$. Check Φ is well-defined embedding.

2.1.2. *Vector bundle.*

Definition 2.1.22 (complex vector bundle)**.** *Let X be a differential manifold, E is a complex vector bundle of rank r on X*

1. (*Via total space*) *E is a differential manifold with surjective map $\pi : E \to X$, such that*
   (1) *For all $x \in X$, fiber E_x is a $\mathbb{C}$-vector space of dimension r.*
   (2) *For all $x \in X$, there exists $x \in U \subset X$ and $\pi^{-1}(U)$ is diffeomorphic to $U \times \mathbb{C}^r$ via h, such that*

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\pi"] \arrow[dr, "h"'] & \arrow[d, phantom, "\circlearrowleft"] & U \\
 & U \times \mathbb{C}^r \arrow[ur, "p_1"'] \arrow[r, "p_2"] & \mathbb{C}^r
\end{tikzcd}$$

   and for all $y \in U$, $E_y \xrightarrow{p_2 \circ h} \mathbb{C}^r$ is a $\mathbb{C}$-vector space isomorphism.
   (U, h) is called a local trivialization.

   Remark 2.1.23. Consider two local trivialization $(U_\alpha, h_\alpha), (U_\beta, h_\beta)$, then $h_\alpha \circ h_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$, this induces

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{\text{diff}} \mathrm{GL}(r, \mathbb{C})$$

   such $g_{\alpha\beta}$ are called transition function, such that

$$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = \mathrm{id} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma$$
$$g_{\alpha\alpha} = \mathrm{id} \quad \text{on } U_\alpha$$

2. (*Via transition function*) *E is the data of*
   (1) *open covering $\{U_\alpha\}$*
   (2) *transition functions $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{\text{diff}} \mathrm{GL}(r, \mathbb{C})\}$, satisfies*

$$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = \mathrm{id} \quad \textit{on } U_\alpha \cap U_\beta \cap U_\gamma$$
$$g_{\alpha\alpha} = \mathrm{id} \quad \textit{on } U_\alpha$$

Remark 2.1.24. The two definitions above are equivalent. The first definition implies the second clearly. The converse is a little bit complicated

Definition 2.1.25 (holomorphic vector bundle)**.** *X is a complex manifold, $\pi : E \to X$ is a complex vector bundle, given by $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{C})$, E is called holomorphic if $g_{\alpha\beta}$ is holomorphic.*

Exercise 2.1.26. Show that the total space of a holomorphic vector bundle E is a complex manifold.

Definition 2.1.27 (morphism between vector bundles)**.** *ϕ is a diffeomorphic/holomorphic morphism of vector bundle on X of rank k, if $\phi : E \to F$ is diffeomorphic/holomorphic map and fiberwise $\mathbb{C}$-linear of rank k.*

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi_1"'] & \arrow[d, phantom, "\circlearrowleft"] & F \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

Example 2.1.28. X is a differential/complex manifold, then $X \times \mathbb{C}^r$ is the trivial rank r complex/holomorphic vector bundle on X.

Example 2.1.29. *E, F are complex/holomorphic vector bundles on X, then $E \oplus F, E \otimes F, \mathrm{Hom}(E, F).E^* = \mathrm{Hom}(E, \mathbb{C}), \mathrm{Sym}^k E, \bigwedge^k E, \det E$ are complex/holomorphic vector bundles.*

Example 2.1.30. holomorphic line bundle L is a rank 1 vector bundle, i.e.,

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{\text{holo}} \mathbb{C}^*$$

Exercise 2.1.31. L is a trivial line bundle $X \times \mathbb{C}$ if and only if up to refinement, there exists $s_\alpha : U_\alpha \to \mathbb{C}^*$, such that $g_{\alpha\beta} = s_\alpha / s_\beta$ on $U_\alpha \cap U_\beta$

Definition 2.1.32 (picard group)**.** *X is a complex manifold, then*

$$\mathrm{Pic}(X) = (\{\textit{holomorphic line bundles on } X\}/\textit{isomorphism}, \otimes)$$

called the Picard group of X.

Example 2.1.33. Line bundle on $\mathbb{P}^n$

$$\begin{tikzcd} L = \{([l], x) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid x \in l\} \subset \mathbb{P}^n \times \mathbb{C}^{n+1} \arrow[d, "\pi"] \\ \mathbb{P}^n \end{tikzcd}$$

is called tautological line bundle.

Consider open covers

$$U_i = \{[l] = [l_1, \dots, l_n] \in \mathbb{P}^n \mid l_i \neq 0\}$$

there is a map $U_i \to \pi^{-1}(U_i)$, defined as

$$[l] \mapsto ([l], (\frac{l_0}{l_i}, \dots, 1, \dots, \frac{l_n}{l_i}))$$

and local trivialization $h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ defined as

$$([l], x) \mapsto ([l], \lambda)$$

where

$$x = \lambda(\frac{l_0}{l_i}, \dots, 1, \dots, \frac{l_n}{l_i})$$

so we can calculate transition function

$$\begin{aligned} h_i \circ h_j^{-1} : (U_i \cap U_j) \times \mathbb{C} &\longrightarrow (U_i \cap U_j) \times \mathbb{C} \\ ([l], \lambda_j) \mapsto ([l], \lambda_j(\frac{l_0}{l_j}, \dots, \frac{l_n}{l_j})) &\mapsto ([l], \lambda_i) \end{aligned}$$

such that

$$\lambda_j(\frac{l_0}{l_j}, \dots, \frac{l_n}{l_j}) = \lambda_i(\frac{l_0}{l_i}, \dots, \frac{l_n}{l_i})$$

which implies

$$\lambda_i = \lambda_j \frac{l_i}{l_j}$$

so we can see transition function $g_{ij} = l_i/l_j \in \mathbb{C}^*$. This line bundle L will be denoted by $\mathcal{O}_{\mathbb{P}^n}(-1)$.

Definition 2.1.34 (line bundles on $\mathbb{P}^n$)**.** *We can define*

$$\mathcal{O}_{\mathbb{P}^n}(-k) = \mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes k}, \quad k \in \mathbb{N}^+$$

$$\mathcal{O}_{\mathbb{P}^n}(k) = (\mathcal{O}_{\mathbb{P}^n}(-k))^*, \quad k \in \mathbb{N}^+$$

$$\mathcal{O}_{\mathbb{P}^n}(0) = \mathbb{P}^n \times \mathbb{C}, \quad \textit{trivial line bundle.}$$

In fact, line bundle listed above contain all possible line bundles over $\mathbb{P}^n$.

Example 2.1.35. More generally, consider

$$\begin{array}{c} E = \{([S], x) \in G(r, n) \times \mathbb{C}^n \mid x \in S\} \subset G(r, n) \times \mathbb{C}^n \\ \Big\downarrow{\scriptstyle \pi} \\ G(r, n) \end{array}$$

Definition 2.1.36 (section)**.** *$\pi : E \to X$ is a complex/holomorphic vector bundle. A (global) section of E is a differential/holomorphic map $s : X \to E$, such that $\pi \circ s = \mathrm{id}_X$, denoted by $C^\infty(X, E)$ / $\Gamma(X, E)$.*

Example 2.1.37. Global holomorphic sections of trivial holomorphic vector bundle are exactly holomorphic functions $f : X \to \mathbb{C}^r$.

Remark 2.1.38. In fact, global holomorphic sections are very rare, as we can seen from the above example, if X is a compact complex manifold, then all global holomorphic functions are only constant.

Definition 2.1.39 (subbundle)**.** *$\pi : E \to X$ is a complex/holomorphic vector bundle. $F \subset E$ is called a subbundle of rank s, if*

1. *For all $x \in X$, $F \cap E_x$ is a subvector space of dimension s.*
2. *$\pi|_F : F \to X$ induces a complex/holomorphic vector bundle.*

Example 2.1.40. $\mathcal{O}_{\mathbb{P}^n}(-1) \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$, is a subbundle.

Exercise 2.1.41.

$$\Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(k)) = \begin{cases} 0, & k < 0 \\ \mathbb{C}, & k = 0 \\ \text{homogeneous polynomials in } n+1 \text{ variables of deg } k, & k > 0 \end{cases}$$

Proof. Let's see what happened for $k = -1$, the tautological line bundle. Since we have $\mathscr{O}_{\mathbb{P}^n}(-1)$ is a subbundle of trivial bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$. So we have a global section $\sigma \in \Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(-1))$ must be a global section of $\Gamma(\mathbb{P}^n, \mathbb{P}^n \times \mathbb{C}^{n+1})$. However, since $\mathbb{P}^n$ is a compact complex manifold, we have that global sections $\Gamma(\mathbb{P}^n, \mathbb{P}^n \times \mathbb{C}^{n+1})$ must be constant, i.e. for any $x \in \mathbb{P}^n$, $\sigma(x) = v$ is a constant. However, $v \in [l]$, for all $[l] \in \mathbb{P}^n$, which forces $v = 0$. Similarly we will get the result for case $k < 0$.

And for case $k = 0$, global sections are exactly constant so we have $\Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(0)) = \mathbb{C}$.

Now consider what will when $k > 0$. Take $k = 1$ for an example □

Example 2.1.42. For a morphism between vector bundles $\phi : E \to F$, $\operatorname{Ker}\phi \subset E, \operatorname{Im}\phi \subset F$ are subbundles.

Definition 2.1.43 (exact). *A sequence of vector bundles*

$$S \xrightarrow{\phi} E \xrightarrow{\psi} Q$$

is called exact at E if $\operatorname{Ker}\psi = \operatorname{Im}\phi$;

Definition 2.1.44 (pullback). *$f : X \to Y$ is a differential/holomorphic map, $\pi : E \to Y$ is a vector bundle, define*

$$f^*E = \{(x, v) \in X \times E \mid f(x) = \pi(v)\} \subset X \times E$$

is called the pullback of π.

2.2. **Episode: sheaves.** Why we need sheaves here? As we have seen in the last section, the global sections of holomorphic vector bundle are very rare, but there are many local sections, we need to keep these information and learn the connection between global and local systemically. Sheaf is a power language for us to manage global and local at the same time. However, there is nothing more that sheaf can give, it's just a different language, as we can see in Exercise 4.5.

Definition 2.2.1 (sheaf). *X is a topological space. A sheaf*[3] *of abelian group $\mathscr{F}$ on X is the data of:*

1. *For any open subset U of X, $\mathscr{F}(U)$ is an abelian group.*
2. *If $U \subset V$ are two open subsets of X, then there is a group homomorphism $r_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$, such that*
   (1) *$\mathscr{F}(\varnothing) = 0$*
   (2) *$r_{UU} = \mathrm{id}$*
   (3) *If $W \subset U \subset V$, then $r_{UW} = r_{VW} \circ r_{UV}$*

[3] A sheaf which fails to meet (4), (5) is called a presheaf.

(4) $\{V_i\}$ *is an open covering of* $U \subset X$, *and* $s \in \mathscr{F}(U)$. *If* $s|_{V_i} := r_{UV_i}(s) = 0, \forall i$, *then* $s = 0$.
(5) $\{V_i\}$ *is an open covering of* $U \subset X$, *and* $s_i \in \mathscr{F}(V_i)$ *such that* $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, *then there exists* $s \in \mathscr{F}(U)$ *such that* $s|_{V_i} = s_i$.

Example 2.2.2 (sheaf of sections of a holomorphic vector bundle)**.** If $\pi : E \to X$ is a holomorphic vector bundle, then define

$$\mathscr{F}(U) = \Gamma(U, E|_U), \quad \forall U \subset_{\text{open}} X$$

This $\mathscr{F}$ will be denoted by $\mathcal{O}_X(E)$. In particular, E is a trivial vector bundle, then $\mathcal{O}_X(E) = \mathcal{O}_X$, the sheaf of holomorphic function, also called the structure sheaf of X.

Definition 2.2.3 (morphism of sheaves on X)**.** $\phi : \mathscr{F} \to \mathscr{G}$ *is called a morphism of sheaves, if for any open subset* U *of* X, *there is a group homomorphism* $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$, *such that if* $U \subset V$ *are two open subsets of* X, *the the following diagram commutes*

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\phi(U)"] \arrow[d, "r_{UV}"] & \mathscr{G}(U) \arrow[d, "r_{UV}"] \\
\mathscr{F}(V) \arrow[r, "\phi(V)"] & \mathscr{G}(V)
\end{tikzcd}$$

Example 2.2.4 (locally free sheaves)**.** A sheaf is called locally free, if there exists covering $\{U_\alpha\}$ such that $\mathscr{F}|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}^{\oplus r}$ of rank r.

For $r = 1$, it is called invertible sheaf.

Exercise 2.2.5. There are correspondences:

$$\{\text{holomorphic vector bundles}\} \xleftrightarrow{1-1} \{\text{locally free sheaves}\}$$

$$\{\text{holomorphic line bundles}\} \xleftrightarrow{1-1} \{\text{invertible sheaves}\}$$

Proof. It suffices to prove the first correspondence. If we have a holomorphic vector bundle $\pi E \to X$. Then consider the sheaf of sections $\mathcal{O}_X(E)$, We claim it's a locally free sheaf. Since we have local trivialization of holomorphic vector bundle $\{U_\alpha\}$. Then consider what's $\mathcal{O}_X(E)|_{U_\alpha}$. Since $E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^r$, then holomorphic sections of $U_\alpha \times \mathbb{C}^r \to U_\alpha$ are just holomorphic functions $f : U \to \mathbb{C}^r$, i.e. $\mathcal{O}_X(E|_{U_\alpha}) = \mathcal{O}_{U_\alpha}^{\oplus r}$. So sheaf $\mathcal{O}_X(E)$ is a locally free sheaf.

Conversely, if we have a locally free sheaf, how can we get a holomorphic vector bundle? □

2.3. **Tangent bundle.**

Definition 2.3.1 (tangent bundle)**.** X *is a differential manifold,* $\dim_{\mathbb{R}} X = n$, *and* $\{U_\alpha, \phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n\}$ *is a atlas of* X. *The* (*real*) *tangent bundle*

$T_{X,\mathbb{R}}$ is defined through transition functions

$$\begin{aligned} g_{\alpha\beta} : U_\alpha \cap U_\beta &\overset{diff}{\to} \mathrm{GL}(n, \mathbb{R}) \\ x &\mapsto J_{\phi_\alpha \circ \phi_\beta^{-1}}(\phi_\beta(x)) \end{aligned}$$

Then $T_{X,\mathbb{C}} = T_{X,\mathbb{R}} \otimes \mathbb{C}$ is a complex vector bundle, called the complexified tangent vector bundle.

Remark 2.3.2. The following statement may be a little bit boring, I write it down just to make myself more clear and to get familiar with two definition of vector bundle.

The tangent bundle $T_{X,\mathbb{R}}$ can be defined as the set

$$T_{X,\mathbb{R}} = \coprod_{x \in X} T_{X,x}$$

and note that there is a natural projection $\pi : T_{X,\mathbb{R}} \to X$, sending $v \in T_{X,x}$ to $x \in X$. Now we want to give a chart on $T_{X,\mathbb{R}}$ to make it into a differential manifold. Let $\{(U_i, \phi_i = (x_i^1, \dots, x_i^n)\}$ be a chart of X, then we can define a chart on X by considering $\{(\pi^{-1}(U_i), \widetilde{\phi}_i)\}$, where $\widetilde{\phi}_i$ is defined through

$$\widetilde{\phi}_i(v) = (\phi_i(\pi(v)), (\mathrm{d}x_i^1)_{\pi(v)}(v), \dots, (\mathrm{d}x_i^n)_{\pi(v)}(v)) \subset \mathbb{R}^n \times \mathbb{R}^n$$

note that such $\widetilde{\varphi}_i$ is bijective. And it's easy to equip $T_{X,\mathbb{R}}$ with a topology such that $\widetilde{\varphi}_i$ is diffeomorphism.

Now I need to calculate transition function to confirm myself as follows: For two charts $(U, \phi = (x_1, \dots, x_n)), (V, \psi = (y_1, \dots, y_n))$, then calculate

$$\widetilde{\psi} \circ \widetilde{\phi}^{-1} : \phi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$$

Note that

$$\widetilde{\phi}^{-1}(r_1, \cdots, r_n, u_1, \cdots, u_n) = \sum_i u_i \frac{\partial}{\partial x_i}|_{\phi^{-1}(r_1, \cdots, r_n)} \in T_{\phi^{-1}(r_1, \cdots, r_n)} M$$

But

$$\mathrm{d}y_j(\sum_i u_i \frac{\partial}{\partial x_i}) = \sum_i u_i(\frac{\partial}{\partial x_i}(y_j)) = \sum_i \frac{\partial y_i}{\partial x_j} u_i$$

Thus transition functions are

$$\begin{aligned} \widetilde{\psi} \circ \widetilde{\phi}^{-1}(r_1, \cdots, r_n, u_1, \cdots, u_n) &= (\psi \circ \phi^{-1}(r), (\sum_i \frac{\partial y_1}{\partial x_i}(r) u_i, \dots, \sum_i \frac{\partial y_n}{\partial x_i}(r) u_i)) \\ &= (\psi \circ \phi^{-1}(r), (\frac{\partial y_j}{\partial x_i}(r))(\begin{matrix} u_1 \\ \vdots \\ u_n \end{matrix})) \end{aligned}$$

So transition function $g_{\alpha\beta}$ are exactly Jacobian of $\varphi_\alpha \circ \varphi_\beta^{-1}$.

Definition 2.3.3 (holomorphic tangent bundle)**.** *X is a complex manifold, $\dim_{\mathbb{C}} X = n$, and $\{U_\alpha, \phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^n\}$ is a atlas of X. The holomorphic tangent bundle T_X is defined through transition functions*

$$\begin{aligned} g_{\alpha\beta} : U_\alpha \cap U_\beta &\to \mathrm{GL}(n, \mathbb{C}) \\ z &\mapsto J^{holo}_{\phi_\alpha \circ \phi_\beta^{-1}}(\phi_\beta(x)) \end{aligned}$$

where J^{holo} is holomorphic Jacobian.

Remark 2.3.4. Clearly, $T_X \neq T_{X,\mathbb{C}}$, even they don't have the same rank! For example, if X is a n-dimensional complex manifold, then

$$\dim T_X = n \neq 2n = \dim T_{X,\mathbb{C}}$$

Later we will see the relationship between them.

Remark 2.3.5 (sheaf viewpoint)**.** $\mathcal{O}_X$ is the sheaf of holomorphic function, then define the stalk at x is

$$\mathcal{O}_{X,x} := \varinjlim_{x \in U \subset X} \mathcal{O}_X(U)$$

The elements of $\mathcal{O}_{X,x}$ are called germs.

For a tangent vector, we can take derivation in this direction, so

$$\text{tangent vector} \longrightarrow \text{derivation } D : \mathcal{O}_{X,x} \to \mathbb{C}$$

where a derivation satisfies

1. $\mathbb{C}$-linear
2. Leibniz rule $D(fg) = D(f)g + fD(g)$

In fact, the above correspondence is $1-1$, i.e., $T_{X,x} \cong$ space of derivations of $\mathcal{O}_{X,x}$.

Definition 2.3.6 (cotangent bundle/(anti)canonical bundle)**.** $\Omega_X = T_X^*$ *is called holomorphic cotangent bundle; $K_X = \det \Omega_X$ is called canonical bundle; $K_X^* = \det T_X$ is called the anticanonical bundle.*

Example 2.3.7. Tangent bundle of $\mathbb{P}^n$: we can calculate it through Euler sequence[4]

$$0 \to \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \xrightarrow{\psi} T_{\mathbb{P}^n} \to 0$$

[4]Hint: consider $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n$. Basis of tangent vector at $z \in \mathbb{C}^{n+1}\backslash\{0\}$ is $\{\frac{\partial}{\partial z_0}, \dots, \frac{\partial}{\partial z_n}\}$, but these are not tangent vector for $\mathbb{P}^n$. However, $z_i \frac{\partial}{\partial z_i}$ will descend to tangent vector at $[z] \in \mathbb{P}^n$. Recall $z_0, \dots, z_n$ form basis of $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and span $(\mathcal{O}_{\mathbb{P}^n(1)})_{[z]}$, so can define

$$\begin{aligned} \psi : (\mathcal{O}_{\mathbb{P}^n}(1))^{\oplus n+1}_{[z]} &\to T_{\mathbb{P}^n,[z]} \\ (0, \dots, \underbrace{z_i}_{j-th}, \dots, 0) &\to z_i \frac{\partial}{\partial z_j} \end{aligned}$$

But $\sum_{i=0}^n z_i \frac{\partial}{\partial z_i}$ tangent to the fibers of π, so descends to zero at $[z] \in \mathbb{P}^n$, so we can define ϕ as

$$\begin{aligned} \phi : \mathcal{O}_{\mathbb{P}^n,[z]} &\to (\mathcal{O}_{\mathbb{P}^n}(1))^{\oplus n+1}_{[z]} \\ 1 &\to (z_0, \dots, z_n) \end{aligned}$$

In fact, $\sum_{i=0}^n z_i \frac{\partial}{\partial z_i}$ is Euler vector field, that's why this sequence is called Euler sequence.

Example 2.3.8. For Grassmannian manifold $G(r,n)$, we have

$$0 \to E \to G(r,n) \otimes \mathbb{C}^n \to Q \to 0$$

Show that

$$T_{G(r,n)} \cong \operatorname{Hom}(E,Q)$$

Example 2.3.9. Let $\pi : L \to X$ is a holomorphic line bundle, given $s \in \Gamma(X,L)$, suppose that $D = \{x \in X \mid s(x) = 0\}$ is a smooth submanifold of codimensional 1.

$$0 \to T_D \to T_X|_D \to L|_D \to 0$$

Then we can get

$$K_X^*|_D \cong K_D^* \otimes L|_D$$

which is called adjunct formula

2.4. **Almost complex structure and integrable theorem.** Now let us talk about some linear algebra:

Consider a $2n$-dimensional real vector space V, a complex structure on V is a $\mathbb{R}$-linear transformation $J : V \to V$ such that $J^2 = -\operatorname{id}$. We can regard V as a complex vector space, by

$$(a+bi)v = av + bJ(v), \quad a, b \in \mathbb{R}$$

If we consider $V \otimes \mathbb{C}$, then J can extend to $V \otimes \mathbb{C}$, by $J(v \otimes \alpha) = J(v) \otimes \alpha$, then we can decompose $V \otimes \mathbb{C}$ into

$$V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$$

such that $\overline{V^{1,0}} = V^{0,1}$, where conjugate means $\overline{v \otimes \alpha} = v \otimes \overline{\alpha}$.

Such definition may be a little abstract for someone, let's calculate in a concrete example to show how it works.

Example 2.4.1. Let $V = \mathbb{C}^n \cong \mathbb{R}^{2n}$, stand coordinate in $\mathbb{C}^n$ is $(z_1, \dots, z_n)$, and in $\mathbb{R}^{2n}$ we always write $(x_1, y_1, \dots, x_n, y_n)$. Note that complex structure on $\mathbb{C}^n$ maps $z_i = x_i + \sqrt{-1}y_i$ to $\sqrt{-1}(x_i + \sqrt{-1}y_i) = -y_i + \sqrt{-1}x_i$.

So we can define $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by $(x_1, y_1, \dots, x_n, y_n) \mapsto (-y_1, x_1, \dots, -y_n, x_n)$. If we complexify $\mathbb{R}^{2n}$ into a complex vector space $\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$, with a basis $\{e_1, f_1, \dots, e_n, f_n\}$, J is defined by $e_j \mapsto f_j, f_j \mapsto -e_j$, then eigenspace decomposition is as follows

$$V^{1,0} = \{\frac{1}{2}(e_j - if_j)\}, \quad V^{0,1} = \{\frac{1}{2}(e_j + if_j)\}$$

such that conjugation $\overline{e_j - if_j} = e_j + if_j$

Definition 2.4.2 (almost complex structure)**.** *X is a differential manifold of $\dim_{\mathbb{R}} X = 2n$. An almost complex structure on X is a complex structure on $T_{X,\mathbb{R}}$ i.e., an isomorphism of differential vector bundles $J : T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$ such that $J^2 = -\operatorname{id}$*

Example 2.4.3. X is a complex manifold, and $T_{X,\mathbb{R}}$ is its (real) tangent bundle if we just see X as a differential manifold. Locally we have

$$T_{X,\mathbb{R}}|_U \cong U \times \mathbb{C}^n$$

for open subset U in X. So we get $J : T_{X,\mathbb{R}}|_U \to T_{X,\mathbb{R}}|_U$, defined by multiplying i, and this is well-defined, explained as following:

For two charts $(U_1, \varphi_1 = (z_1 = x_1 + iy_1, , \ldots, z_n = x_n + iy_n)), (U_2, \varphi_2 = (g_1 = u_1 + iv_1, \ldots, g_n = u_n + iv_n))$ with $U_1 \cap U_2 \neq \varnothing$, there are two ways to define J on $U_1 \cap U_2$

$$\begin{tikzcd}
U_1 \cap U_2 \times \mathbb{C}^n \arrow[ddd, "\times i"] & & & U_1 \cap U_2 \times \mathbb{C}^n \arrow[ddd, leftrightarrow, "\times i"] \\
& T_X|_{U_1 \cap U_2} \arrow[ul, "\widetilde{\varphi}_1"] \arrow[d, "J"] & T_X|_{U_1 \cap U_2} \arrow[l] \arrow[ur, "\widetilde{\varphi}_2"'] \arrow[d, "J"] & \\
& T_X|_{U_1 \cap U_2} \arrow[dl, "\widetilde{\varphi}_1"] & T_X|_{U_1 \cap U_2} \arrow[l] \arrow[dr, "\widetilde{\varphi}_2"] & \\
U_1 \cap U_2 \times \mathbb{C}^n & & & U_1 \cap U_2 \times \mathbb{C}^n
\end{tikzcd}$$

We need to check transition function commutes with J, calculated in a local chart as follows: For a 2×2 part, Jacobian of $\varphi_2 \circ \varphi_1^{-1}$ is

$$\begin{pmatrix} \dfrac{\partial u_j}{\partial x_k} & \dfrac{\partial u_j}{\partial y_k} \\ \dfrac{\partial v_j}{\partial x_k} & \dfrac{\partial v_j}{\partial y_k} \end{pmatrix} \overset{\text{C-R}}{=} \begin{pmatrix} \dfrac{\partial v_j}{\partial y_k} & \dfrac{\partial u_j}{\partial y_k} \\ -\dfrac{\partial u_j}{\partial y_k} & \dfrac{\partial v_j}{\partial y_k} \end{pmatrix}$$

and J is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

So they commute with each other.

So complex structure gives a almost complex structure(naturally), But the question is: Does every complex structure on a complex manifold can be induced from a almost complex structure on a even-dimensional differential manifold? Unfortunately, it's false in general, but we have the following theorem.

Theorem 2.4.4 (Newlander-Nirenberg)**.** *Let (X, J) be a complex manifold, J is induced by a almost complex structure on X is equivalent to*

$$[T_X^{1,0}, T_X^{1,0}] \subset T_X^{1,0}$$

which is called an integrable condition.

2.5. **Operator ∂ and $\overline{\partial}$.** We have $T_X \hookrightarrow T_{X,\mathbb{C}}$ as complex vector bundle, with image $T^{1,0}_{X,\mathbb{C}}$. In fact, if we consider locally, take $(z_1, \dots, z_n) \in V \subset \mathbb{C}^n$, $z_j = x_j + iy_j$, then

$$\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$$

moreover,

$$\frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$$

If we consider its dual space, we get the differential forms

$$\Omega^1_{X,\mathbb{C}} = \Omega^1_{X,\mathbb{R}} \otimes \mathbb{C} = \Omega^{1,0}_X \oplus \Omega^{0,1}_X$$

and take wedge product k times, then we get

$$\Omega^k_{X,\mathbb{C}} = \bigwedge^k \Omega^1_{X,\mathbb{C}} = \bigoplus_{p+q=k} \Omega^{p,q}_X, \quad \text{where } \Omega^{p,q}_X = \bigwedge^p \Omega^{1,0}_X \otimes \bigwedge^q \Omega^{0,1}_X$$

Remark 2.5.1. We look it locally, the dual of $\frac{\partial}{\partial z_j}$ is

$$\mathrm{d}z_j = \mathrm{d}x_j + i\mathrm{d}y_j \in \Omega^{1,0}_X$$

and the dual of $\frac{\partial}{\partial \overline{z_j}}$ is

$$\mathrm{d}\overline{z_j} = \mathrm{d}x_j - i\mathrm{d}y_j \in \Omega^{0,1}_X$$

For any $\alpha \in C^\infty(X, \Omega^1_{X,\mathbb{C}})$, locally we have

$$\alpha = \sum \alpha_j \mathrm{d}x_j + \sum \beta_j \mathrm{d}y_j$$

then we can decompose it into

$$\alpha = \sum \frac{1}{2}(\alpha_j - i\beta_j)\mathrm{d}z_j + \sum \frac{1}{2}(\alpha_j + i\beta_j)\mathrm{d}\overline{z_j}$$

where the first part lies in $\Omega^{1,0}_X$ and the later part lies in $\Omega^{0,1}_X$.

Definition 2.5.2 (differential k-form)**.** *A k-form α of type (p,q) is a differential section of $\Omega^{p,q}_X$, that is*

$$\alpha \in C^\infty(X, \Omega^{p,q}_X) \subset C^\infty(X, \Omega^k_{X,\mathbb{C}})$$

Example 2.5.3. For $\mathbb{C}^n \cong \mathbb{R}^{2n}$, we have[5]

$$w = \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}x_n = (\frac{i}{2})^n \mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\overline{z_n}$$

[5]By induction on n

2.6. **Exterior differential.** Recall: Let X be a differential manifold, with real dimension n, we have exterior differential

$$\mathrm{d}: C^\infty(X, \Omega^k_{X,\mathbb{R}}) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{R}})$$

such that

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \mathrm{d}\beta$$

Exterior differential has an important property:

$$\mathrm{d}^2 = 0$$

So we can consider such chain complex

$$0 \to C^\infty(X, \Omega^0_{X,\mathbb{R}}) \to C^\infty(X, \Omega^1_{X,\mathbb{R}}) \to C^\infty(X, \Omega^2_{X,\mathbb{R}}) \to \cdots \to C^\infty(X, \Omega^n_{X,\mathbb{R}}) \to 0$$

with de Rham cohomology group

$$H^k(X, \mathbb{R}) := Z^k(X, \mathbb{R}) / B^k(X, \mathbb{R})$$

The following theorem implies that the de Rham cohomology is just a topological data.

Theorem 2.6.1 (comparision theorem)**.** *$H^k(X, \mathbb{R})$ computes the singular cohomology of X with real coefficient.*

Theorem 2.6.2 (Poincaré lemma)**.** *Let $X = B(x_0, r_0) \subset \mathbb{R}^n$ is a open ball, then $H^k(X, \mathbb{R}) = 0, \forall k > 0$.*

Remark 2.6.3. Poincaré lemma implies that for small enough open set, the cohomology groups are trivial, so only for global differential forms, de Rham cohomology tells interesting information.

Now Let X be a complex manifold, with complex dimension n, then

$$\mathrm{d}: C^\infty(X, \Omega^k_{X,\mathbb{C}}) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{C}})$$

Example 2.6.4. For $\alpha \in C^\infty(X, \Omega^0_{X,\mathbb{C}})$, then

$$\mathrm{d}\alpha \in C^\infty(X, \Omega^1_{X,\mathbb{C}}) = C^\infty(X, \Omega^{1,0}_X) \oplus C^\infty(X, \Omega^{0,1}_X)$$

Locally, we have

$$\begin{aligned}
\mathrm{d}\alpha &= \sum \frac{\partial \alpha}{\partial x_j} \mathrm{d}x_j + \sum \frac{\partial \alpha}{\partial y_j} \mathrm{d}y_j \\
&= \sum \frac{1}{2}\left(\frac{\partial \alpha}{\partial x_j} - i \frac{\partial \alpha}{\partial y_j}\right) \mathrm{d}z_j + \sum \frac{1}{2}\left(\frac{\partial \alpha}{\partial x_j} + i \frac{\partial \alpha}{\partial y_j}\right) \mathrm{d}\overline{z_j} \\
&= \sum \frac{\partial \alpha}{\partial z_j} \mathrm{d}z_j + \sum \frac{\partial \alpha}{\partial \overline{z_j}} \mathrm{d}\overline{z_j}
\end{aligned}$$

More generally, for $\alpha \in C^\infty(X, \Omega^{p,q}_X)$, then locally

$$\alpha = \sum_{|J|=p, |K|=q} \alpha_{J,K} \mathrm{d}z_J \wedge \mathrm{d}\overline{z_K}$$

then

$$\mathrm{d}\alpha = \sum_{|J|=p,|K|=q} \frac{\partial \alpha_{J,K}}{\partial z_l} \mathrm{d}z_l \wedge \mathrm{d}z_J \wedge \mathrm{d}\overline{z_K} + \sum_{|J|=p,|K|=q} \frac{\partial \alpha_{J,K}}{\partial \overline{z_l}} \mathrm{d}\overline{z_l} \wedge z_J \wedge \overline{z_K}$$

Definition 2.6.5 (partial operator)**.** *For $\alpha \in C^\infty(X, \Omega_X^{p,q})$, we can define partial operator and its conjugation $\partial\alpha, \overline{\partial}\alpha$ as follows*

$$\mathrm{d}\alpha = \partial\alpha + \overline{\partial}\alpha$$

where $\partial\alpha \in C^\infty(X, \Omega_X^{p+1,q}), \overline{\partial}\alpha \in C^\infty(X, \Omega_X^{p,q+1})$

More generally, if $\alpha \in C^\infty(X, \Omega_{X,\mathbb{C}}^k)$, write $\alpha = \sum \alpha^{p,q}$, then

$$\partial\alpha = \sum_{p,q} \partial\alpha^{p,q}, \quad \overline{\partial}\alpha = \sum_{p,q} \overline{\partial}\alpha^{p,q}$$

Remark 2.6.6. We have the following relations

1. Leibniz rule

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \partial\beta$$

2. [6]

$$\partial^2 = \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0$$

So we can do the same thing for ∂ by consider the following chain complex

$$0 \to C^\infty(X, \Omega_X^{p,0}) \xrightarrow{\overline{\partial}} C^\infty(X, \Omega_X^{p,1}) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} C^\infty(X, \Omega_X^{p,n}) \to 0$$

Definition 2.6.7 (Dolbeault cohomology)**.**

$$H^{p,q} := Z^{p,q}(X)/B^{p,q}(X) = H^q_{\overline{\partial}}(C^\infty(X, \Omega_X^{p,*}))$$

Key question: Since we have $C^\infty(X, \Omega_{X,\mathbb{C}}^k) = \bigoplus_{p+q=k} C^\infty(X, \Omega_X^{p,q})$, could we have the following decomposition?

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

Example 2.6.8. What is $H^{p,0}(X)$? Since $B^{p,0} = 0$, then

$$H^{p,0}(X) = Z^{p,0}(X) = \{\alpha \in C^\infty(X, \Omega_X^{p,0} \mid \overline{\partial}\alpha = 0\}$$

Locally $\alpha = \sum_{|J|=p} \alpha_J \mathrm{d}z_J$, then

$$\overline{\partial}\alpha = \sum_{|J|=p} \frac{\partial \alpha_J}{\partial \overline{z_k}} \mathrm{d}\overline{z_k} \wedge \mathrm{d}z_J = 0 \implies \frac{\partial \alpha_J}{\partial \overline{z_k}} = 0$$

That is, α_J is holomorphic function. Since $\Omega_X^{p,0} \cong \Omega_X^p$ as complex vector bundle, we have $H^{p,0}(X) = \Gamma(X, \Omega_X^p)$.

[6] Hint:consider $\mathrm{d}^2 = (\partial + \overline{\partial})^2 = 0$

Example 2.6.9. For a holomorphic map $f: X \to Y$ between complex manifold, then

$$f^*: C^\infty(Y, \Omega^k_{Y,\mathbb{C}}) \to C^\infty(X, \Omega^k_{X,\mathbb{C}})$$

Then[7]

$$f^*: C^\infty(Y, \Omega^{p,q}_{Y,\mathbb{C}}) \to C^\infty(X, \Omega^{p,q}_{X,\mathbb{C}})$$

and

$$f^*: H^{p,q}(Y) \to H^{p,q}(X)$$

Example 2.6.10. Dolbeault cohomology of a holomorphic vector bundle[8] $E \to X$, define

$$\overline{\partial_E}: C^\infty(X, \Omega^{0,q}_X \otimes E) \to C^\infty(X, \Omega^{0,q+1}_X \otimes E)$$

satisfies $\overline{\partial_E}^2 = 0$, so we can construct a chain complex and define its cohomology, denoted by

$$H^q(X, E) = H^q_{\overline{\partial_E}}(C^\infty(X, \Omega^{0,*}_X \otimes E))$$

and we can calculate

$$H^0(X, E) = \Gamma(X, E)$$

Theorem 2.6.11 (Dolbeault lemma)**.** *Let $X = D(z_0, r_0) \subset \mathbb{C}^n$ be a polydisk, then*

$$H^{p,q}(X) = 0, \quad \forall p \geq 0, q > 0$$

2.7. **Čech cohomology.** Let X be a topological space, and $\mathcal{U} = (U_i)_{i\in I}$ be an open covering, such that I is countable and an ordered set. For all $i_0, \dots, i_p \in I$ write

$$U_{i_0\dots i_p} = U_{i_0} \cap \dots \cap U_{i_p}$$

Let $\mathscr{F}$ be a sheaf of abelian group, define a chain complex $C^*(\mathcal{U}, \mathscr{F})$ as

$$0 \to C^0(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta} \dots$$

where

$$C^p = \prod_{i_0 < \dots < i_p} \mathscr{F}(U_{i_0\dots i_p})$$

and δ is defined as

$$(\delta\alpha)_{i_0\dots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0\dots\widehat{i_k}\dots i_{p+1}}|_{U_{i_0\dots i_{p+1}}}$$

Exercise 2.7.1 (Once and only once exercise in your whole life)**.** Check that $\delta \circ \delta = 0$

So we can define Čech cohomology as

$$\check{H}^q(\mathcal{U}, \mathscr{F}) := H^q_\delta(C^*(\mathcal{U}, \mathscr{F}))$$

---

[7]Check this, we need back to definition, a holomorphic map induces a tangent map $T_f: T_{X,\mathbb{C}} \to f^*T_{Y,\mathbb{C}}$,and consider its dual we get cotangent map $\Omega_f: f^*\Omega_{Y,\mathbb{C}} \to \Omega_{X,\mathbb{C}}$

[8]In previous, $E = \Omega^p_X$

Example 2.7.2. We consider

$$\check{H}^0(\mathcal{U}, \mathscr{F}) = \{\alpha \in C^0(\mathcal{U}, \mathscr{F}) \mid \delta\alpha = 0\}$$

then if $\alpha = \prod_{i_0} \alpha_{i_0}$, then $\delta\alpha = 0$ implies

$$\alpha_i|_{U_i \cap U_j} = \alpha_j|_{U_i \cap U_j}$$

then we have

$$\check{H}^0(\mathcal{U}, \mathscr{F}) = \mathscr{F}(X)$$

However, we want out definition is independent of open cover, so

Definition 2.7.3. *We define Čech cohomology as*

$$\check{H}^q(X, \mathscr{F}) = \varinjlim_{\mathcal{U}} \check{H}^q(\mathcal{U}, \mathscr{F})$$

Remark 2.7.4. In other words, $\alpha = \alpha' \in \check{H}^q(X, \mathscr{F})$ is equivalent to there exists a common refinement $\mathcal{U}''$ such that

$$\alpha = \alpha' \in \check{H}^q(\mathcal{U}'', \mathscr{F})$$

Why we want to introduce Čech cohomology here? In fact, it provides a method to compute de Rham cohomology and Dolbeault cohomology we defined before. However, we will use a cohomology theory to unify all cohomology theory later.

Recall that if X is a complex manifold, $E \to X$ is a holomorphic vector bundle. Then we can define a sheaf of holomorphic sections, defined by

$$\mathcal{O}_X(E) : U \mapsto \Gamma(U, E|_U)$$

then we get a Čech cohomology of this sheaf

$$\check{H}^q(X, \mathcal{O}_X(E)) = \varinjlim_{\mathcal{U}} \check{H}^q(\mathcal{U}, \mathcal{O}_X(E))$$

Theorem 2.7.5 (comparision). *We have the following isomorphism*

$$\check{H}^q(X, \mathcal{O}_X(E)) \cong H^q(X, E) = H^q_{\overline{\partial}_E}(C^\infty(X, \Omega_X^{0,*} \otimes E))$$

In particular, let $E = \Omega_X^p$, we have

$$\check{H}^q(X, \mathcal{O}_X(\Omega_X^p)) \cong H^{p,q}(X) = H^q_{\overline{\partial}}(C^\infty(X, \Omega_X^{p,*}))$$

Remark 2.7.6. In fact, Theorem 2.7.5 uses the sequence of sheaves

$$0 \to E \to \Omega_X^{0,0} \otimes E \xrightarrow{\overline{\partial_E}} \Omega_E^{0,1} \otimes E \xrightarrow{\overline{\partial_E}} \cdots \xrightarrow{\overline{\partial_E}} \Omega_X^{0,n} \otimes E \to 0$$

And Dolbeault lemma implies the above sequence is exact. That's what behind the comparision theorem, and tells us why Dolbeault cohomology is about holomorphic information, since here we use $\mathcal{O}_X(\Omega_X^p)$, sheaf of holomorphic sections.

Similarly, in de Rham cohomology, we have the same story. There is a sequence of sheaves

$$0 \to \underline{\mathbb{C}} \xrightarrow{i} \Omega^0_{X,\mathbb{C}} \xrightarrow{\mathrm{d}} \Omega^1_{X,\mathbb{C}} \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n_{X,\mathbb{C}} \to 0$$

where $\underline{\mathbb{C}}$ is the sheaf of locally constant functions, i.e.

$$\underline{\mathbb{C}} : U \mapsto \{\text{locally constant functions } f : U \to \mathbb{C}\}$$

Then Poincaré lemma implies the above sequence is also exact. Parallel to Theorem 2.7.5, we will get

$$\check{H}^q(X, \underline{\mathbb{C}}) \cong H^k(X, \mathbb{C}) = H^k_{dR}(C^\infty(X, \Omega^*_{X,\mathbb{C}}))$$

This also explain why de Rham cohomology is just a topological information, since here we just use $\underline{\mathbb{C}}$, a pure topological information.

Theorem 2.7.7 (Leray)**.** *Let $\mathcal{U}$ be a covering such that for all $i_0 \dots i_k \in I$, and for all $q > 0$, we have*

$$H^q(U_{i_0\dots i_k}, E|_{U_{i_0\dots i_k}}) = 0$$

then $\mathcal{U}$ is called acyclic for E. Then

$$\check{H}^q(\mathcal{U}, \mathcal{O}_X(E)) \cong H^q(X, E)$$

Remark 2.7.8. This provides us a practical way to compute Čech cohomology.

Example 2.7.9. Consider $\mathcal{O}_X^\times \subset \mathcal{O}_X$, the sheaf of invertible holomorphic functions.

Then we have

$$\check{H}^1(X, \mathcal{O}_X^\times) \cong \operatorname{Pic}(X)$$

3. Geometry of vector bundles

3.1. **Connections.**

Definition 3.1.1 (connection)**.** *X is a differential manifold, and $\pi : E \to X$ is a complex vector bundle. A connection on E is a $\mathbb{C}$-linear operator*

$$D : C^\infty(X, E) \to C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes E)$$

satisfying the Leibiz rule

$$D(f\sigma) = \mathrm{d}f \otimes \sigma + fD(\sigma)$$

Remark 3.1.2. In fact, if we ask D to satisfy the Leibniz rule, it induces

$$D : C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes E) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{C}} \otimes E)$$

for any k, satisfying

$$D(\tau \wedge \sigma) = D\tau \wedge \sigma + (-1)^{\deg \tau} \tau \wedge D\sigma$$

Remark 3.1.3. Let's see what's going on in local pointview.

Locally around $x \in U \subset X$, then $\pi^{-1}(U) \cong U \times \mathbb{C}^r$, there is a basis $\{e_1, \dots, e_n\}$ for $\mathbb{C}^r$. For $\sigma \in C^\infty(U, E|_U)$, we have

$$\sigma = \sum_{j=1}^r s_j e_j, \quad s_j \in C^\infty(U)$$

By Leibniz rule, we have

$$D\sigma = \sum_{j=1}^{r} (\mathrm{d}s_j \otimes e_j + s_j De_j)$$

where $De_j \in C^\infty(U, \Omega^1_{U,\mathbb{C}} \otimes E)$. So we can write more explictly as

$$De_j = \sum_{i=1}^{r} a_{ij} \otimes e_i, \quad a_{ij} \in C^\infty(U, \Omega^1_{U,\mathbb{C}})$$

So we have

$$D\sigma = \sum_{j=1}^{r} (\mathrm{d}s_j \otimes e_j + \sum_{i=1}^{r} a_{ij} s_j \otimes e_i)$$

We can rewrite the above formula in frame of $\{e_1, \ldots, e_r\}$ as

$$D\sigma = Ds = \mathrm{d}s + As$$

where

$$\sigma = s = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}, \quad A = (a_{ij}) \in C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes \mathrm{End}(E|_U))$$

Here we chose a local trivialization of the vector bundle E, so we may wonder what will happen if we change our choice.

If $x \in U' \subset X$ is another trivialization, so $\pi^{-1}(U') \cong U' \times \mathbb{C}^r$, and $\{e'_1, \ldots, e'_r\}$ is another local frame. Then

$$D\sigma = \begin{cases} Ds = \mathrm{d}s + As \\ Ds' = \mathrm{d}s' + A's' \end{cases}$$

so we wonder the relationship between A and A'. Transition functions between U and U' are

$$g : U \cap U' \to \mathrm{GL}(r, \mathbb{C})$$

so we have $s = gs'$ and $Ds = gDs'$. We compute as follows

$$\begin{aligned} \mathrm{d}s &= \mathrm{d}(gs') = (\mathrm{d}g)s' + g(\mathrm{d}s') = g(g^{-1}(\mathrm{d}g)s' + \mathrm{d}s') \\ \mathrm{d}s + As &= g(g^{-1}(\mathrm{d}g)s' + \mathrm{d}s' + g^{-1}Ags') \\ &= g(\mathrm{d}s' + (g^{-1}\mathrm{d}g + g^{-1}Ag)s') \end{aligned}$$

Since we have

$$\mathrm{d}s + As = gDs' = g(\mathrm{d}s' + A's')$$

So we have

$$A' = g^{-1}\mathrm{d}g + g^{-1}Ag$$

You may feel quite uncomfortable since A' does not conjugate to A under the change of the frame, but if we apply D twice, something interesting may happen.

$$\begin{aligned} D^2\sigma &= D(\mathrm{d}s + As) = \mathrm{d}(\mathrm{d}s + As) + A \wedge (\mathrm{d}s + As) \\ &= \mathrm{d}^2 s + \mathrm{d}(As) + A \wedge \mathrm{d}s + A \wedge As \\ &= \mathrm{d}^2 s + (\mathrm{d}A)s - A \wedge \mathrm{d}s + A \wedge \mathrm{d}s + A \wedge As \\ &= (\mathrm{d}A + A \wedge A)s \end{aligned}$$

And we check what will happen if we choose another trivialization

$$\begin{aligned} D^2\sigma = D^2 s &= (\mathrm{d}A + A \wedge A)s = (\mathrm{d}A + A \wedge A)gs' \\ &= gD^2 s' = g(\mathrm{d}A' + A' \wedge A')s' \end{aligned}$$

so we have

$$\mathrm{d}A' + A' \wedge A' = g^{-1}(\mathrm{d}A + A \wedge A)g$$

that is, $\mathrm{d}A + A \wedge A$ behaves "well" under the change of frame, object with such property we always call it a "tensor"[9].

From discussion above, we can give the following definition

Definition 3.1.4 (curvature)**.** *There exists a global section $H_D \in C^\infty(X, \Omega^2_{X,\mathbb{C}} \otimes \mathrm{End}(E))$ such that*

$$D^2\sigma = H_D \wedge \sigma, \quad \forall \sigma \in C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes E)$$

such H_D is called the curvature tensor of connection D.

Definition 3.1.5 (Hermitian metric)**.** *X is a differential manifold, and $\pi : E \to X$ is a complex vector bundle. A Hermitian metric h on E is a hermitian inner product on each fiber E_x, such that for all open subset $U \subset X$, and $\xi, \eta \in C^\infty(U, E|_U)$, we have*

$$\begin{aligned} \langle \xi, \eta \rangle : U &\to \mathbb{C} \\ x &\mapsto \langle \xi(x), \eta(x) \rangle \end{aligned}$$

is a C^∞-function.

Example 3.1.6. Locally, for $x \in U \subset X$, we have $\pi^{-1}(U) \cong U \times \mathbb{C}^r$, and $\{e_1, \ldots, e_r\}$ is a local frame. Then our Hermitian metric is just a matrix

$$H = (h_{\lambda\mu})$$

where $h_{\lambda\mu} \in C^\infty(U)$, defined by

$$h_{\lambda\mu}(x) = \langle e_\lambda(x), e_\mu(x) \rangle$$

[9]But what is a "tensor"? Here I quote a motto said by Leonard Susskind, a well-known physicist. I'm quite impressed when I first heard it in my childhood. "Tensor is something which behaves like a tensor."

In our local frame, two sections $\xi, \eta \in C^\infty(U, E|_U)$ can be write as

$$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}$$

We have

$$h(\xi, \eta) = (\xi_1, \dots, \xi_n) H \begin{pmatrix} \overline{\eta_1} \\ \vdots \\ \overline{\eta_n} \end{pmatrix} = \xi^t H \overline{\eta}$$

And take another $x \in U' \subset X$, $\pi^{-1}(U') \cong U' \times \mathbb{C}^r$, with $\{e_1', \dots, e_r'\}$, with g is the transition function, we have

$$H' = g^t H \overline{g}$$

Proposition 3.1.7. *Every complex vector bundle admits a Hermitian metric*

Proof. Use partition of unity. □

Clearly a Hermitian metric h induces a pairing $\{\cdot, \cdot\}$

$$C^\infty(X, \Omega^p_{X,\mathbb{C}} \otimes E) \times C^\infty(X, \Omega^q_{X,\mathbb{C}} \otimes E) \to C^\infty(X, \Omega^{p+q}_{X,\mathbb{C}})$$

Locally, consider $x \in U \subset X$, $\pi^{-1}(U) \cong U \times \mathbb{C}^r$, and $\{e_1, \dots, e_r\}$ is a local frame. Then $\sigma \in C^\infty(X, \Omega^p_{X,\mathbb{C}} \otimes E)$ and $\eta \in C^\infty(X, \Omega^q_{X,\mathbb{C}} \otimes E)$ are in form

$$\sigma = \sum_{j=1}^r s_j \otimes e_j, \tau = \sum_{j=1}^r t_j \otimes e_j$$

where s_j are p-forms and t_j are q-forms, then the pairing is locally look like

$$\sum_{i,j=1}^r s_i \wedge t_j h_{ij} = s^t H \overline{t}$$

Definition 3.1.8 (Hermitian connection)**.** *(E, h) is a Hermitian vector bundle on X. A connection D on E is Hermitian if for all $\sigma \in C^\infty(X, \Omega^p_{X,\mathbb{C}} \otimes E), \eta \in C^\infty(X, \Omega^q_{X,\mathbb{C}} \otimes E)$,*

$$\mathrm{d}\{\sigma, \tau\} = \{D\sigma, \tau\} + (-1)^{\deg \sigma}\{\sigma, D\tau\}$$

Since we know that a connection locally looks like $D = \mathrm{d} + A\wedge$. Then let's compute in a local frame to show what condition A needs to satisfy for a Hermitian connection.

Take $\sigma = s = (s_1, \dots, s_n)^T, \tau = t = (t_1, \dots, t_n)^T$. WLOG, we assume $\{e_1, \dots, e_r\}$ is a orthonormal basis, i.e. H is identity matrix, then

$$\{\sigma, \tau\} = s^t \overline{t}$$

then we compute

$$
\begin{aligned}
\mathrm{d}\{\sigma,\tau\} &= (\mathrm{d}s)^t\wedge\bar{t}+(-1)^{\deg\sigma}s^t\wedge\mathrm{d}\bar{t}\\
\{D\sigma,\tau\} &= (\mathrm{d}s+A\wedge s)^t=(\mathrm{d}s)^t\wedge\bar{t}+(-1)^{\deg\sigma}s^t\wedge A^t\wedge\bar{t}\\
\{\sigma,D\tau\} &= s^t\wedge\overline{\mathrm{d}t+A\wedge t}=s^t\wedge\mathrm{d}\bar{t}+s^t\wedge\overline{A}\wedge\bar{t}
\end{aligned}
$$

then

$$
\mathrm{d}\{\sigma,\tau\}-\{D\sigma,\tau\}-\{\sigma,D\tau\}=(-1)^{\deg\sigma}s^t\wedge(A^t+\overline{A}^t)\wedge\bar{t}
$$

So D is a Hermitian connection if and only if $A^t+\overline{A}=0$.

Let's make it more beautiful. We define D^{adj}, adjoint connection, locally given by $-\overline{A}^t$ with respect to $H=I$, then we always have

$$
\mathrm{d}\{\sigma,\tau\}=\{D\sigma,\tau\}+(-1)^{\deg\sigma}\{\sigma,D^{adj}\tau\}
$$

Take $\dfrac{1}{2}(D+D^{adj})$, which is also a connection, locally looks like

$$
\frac{1}{2}(A-\overline{A}^t)
$$

is a Hermitian connection. So it's easy to get a Hermitian connection, just average A with its adjoint.

Proposition 3.1.9. *Every Hermitian vector bundle admits a Hermitian connection.*

Proof. Use partition of unity to show the existence of connection, and take average. □

3.2. **Connections and metrics on holomorphic vector bundles.** Recall that for a complex manifold X, we have

$$
\Omega^1_{X,\mathbb{C}}=\Omega^{1,0}_X\oplus\Omega^{0,1}_X
$$

Consider $E\to X$ is a complex vector bundle, and D is a connection, then we can decompose $D=D^{1,0}+D^{0,1}$ as

$$
\begin{tikzcd}
 & & C^\infty(X,\Omega^{1,0}_X\otimes E)\\
C^\infty(X,E) \arrow[r, "D"] & C^\infty(X,\Omega^1_{X,\mathbb{C}}\otimes E) \arrow[ur] \arrow[dr] & \\
 & & C^\infty(X,\Omega^{0,1}_X\otimes E)
\end{tikzcd}
$$

Locally, we have $D=\mathrm{d}+A$, then

$$
D^{1,0}=\partial+A^{1,0},\quad D^{0,1}=\overline{\partial}+A^{0,1}
$$

both $D^{1,0}$ and $D^{0,1}$ satisfy Leibniz rule.

Now consider X is a complex manifold, and $E \to X$ is a holomorphic vector bundle. Recall that we already have

$$\overline{\partial}_E : C^\infty(X, E) \to C^\infty(X, \Omega_X^{0,1} \otimes E)$$

We want to compare $D_E^{0,1}$ and $\overline{\partial}_E$

Theorem 3.2.1 (Chern connection)**.** *X is a complex manifold, (E, h) is a Hermitian holomorphic vector bundle, then there exists a unique Hermitian connection D_E such that $D_E^{0,1} = \overline{\partial}_E$. D_E is called the Chern connection of (E, h).*

Proof. Uniqueness: locally $x \in U \subset X$, $\{e_1, \ldots, e_r\}$ is holomorphic local frame. And smooth section $\sigma = s = (s_1, \ldots, s_n)^t$. Then

$$D_E \sigma = \mathrm{d}s + As$$
$$D_E^{0,1} \sigma = \overline{\partial} s + A^{0,1} s = \overline{\partial}_E \sigma$$

If s is a holomorphic section, then $\overline{\partial}_E \sigma = \overline{\partial} s = 0$, implies $A^{0,1} = 0$.

Since we have

$$\mathrm{d}H = A^t H + H \overline{A}$$

then

$$\overline{\partial} H = H \overline{A}$$

So A is uniquely determined by

$$A = \overline{H^{-1}} \partial \overline{H}$$

Existence: It suffices to prove we can glue A together to get a global connection, i.e. compatible with holomorphic change of frames.

Consider another holomorphic local chart $x \in U' \subset X$, with frame $\{e_1', \ldots, e_r'\}$. And the metric with respect to this new frame is H', we have

$$H' = g^t H \overline{g}$$

Then

$$\begin{aligned} A' = \overline{H'^{-1}} \partial \overline{H'} &= g^{-1} \overline{H^{-1} (g^t)^{-1}} \partial (\overline{g^t} \overline{H} g) \\ &= g^{-1} \overline{H^{-1} (g^t)^{-1}} ((\partial \overline{g^t}) + \overline{g^t} (\partial \overline{H}) g + \overline{g^t} \overline{H} \partial g) \\ &= g^{-1} \overline{H^{-1}} (\partial \overline{H} g) + g^{-1} \mathrm{d}g \\ &= g^{-1} \mathrm{d}g + g^{-1} A g \end{aligned}$$

As we desire. □

Corollary 3.2.2. *If X is a complex manifold, (E, h) is a Hermitian holomorphic vector bundle. D_E is Chern connection on it, and H_E is Chern curvature. A is the matrix of D_E with respect to holomorphic local frame, then*

1. *A is of type $(1, 0)$, with $\partial A = -A \wedge A$*
2. *locally $\overline{\partial} A$ of type $(1, 1)$*
3. $\overline{\partial} H_E = 0$

Proof. Locally we have $A = \overline{H}^{-1}\partial\overline{H}$, so it's of type $(1,0)$, and we compute

$$\begin{aligned}\partial A = \partial(\overline{H}^{-1}\partial\overline{H}) &= \partial\overline{H}^{-1}\wedge\partial\overline{H}\\ &= (-\overline{H}^{-1}\partial H\overline{H}^{-1})\wedge\partial\overline{H}\\ &= -(\overline{H}^{-1}\partial\overline{H})\wedge(\overline{H}^{-1}\partial\overline{H})\\ &= -A\wedge A\end{aligned}$$

Chern curvature locally looks like

$$H_E = \mathrm{d}A + A\wedge A = \mathrm{d}A - \partial A = \overline{\partial}A$$

, which is of type $(1,1)$. And clearly $\overline{\partial}H_E = 0$. $\square$

Exercise 3.2.3. (E,h) is a Hermitian holomorphic vector bundle, and $S\hookrightarrow E$ is a holomorphic subbundle. $S^\perp$ is defined by $(S^\perp)_x = (S_x)^\perp$ with respect to h. We have $E = S\oplus S^\perp$ as complex vector bundle.[10] We have a projection

$$P_s : C^\infty(X,\Omega^k_{X,\mathbb{C}}\otimes E)\to C^\infty(X,\Omega^k_{X,\mathbb{C}}\otimes S)$$

Show that $D_S = P_s\circ D_E$.

3.3. **Case of Line bundle.** Recall that if X is a differential manifold, and $\pi : X\to L$ is a complex line bundle, D is a connection on L. Something interesting happen in the case of line bundle. Since

$$H_D\in C^\infty(X,\Omega^2_{X,\mathbb{C}}\otimes\mathrm{End}(L))$$

where $\mathrm{End}(L)\cong L^*\otimes L\cong X\times\mathbb{C}$ is just trivial bundle. So in fact, $H_D\in C^\infty(X,\Omega^2_{X,\mathbb{C}})$.

In a local pointview, D is represented by 1-form A, then $H_D = \mathrm{d}A + A\wedge A = \mathrm{d}A$, then $\mathrm{d}H_D = 0$, i.e. H_D is a closed form, so we get

$$[H_D]\in H^2(X,\mathbb{C})$$

de Rham cohomology.

If we consider another connection $\widetilde{D}$, and $\widetilde{A}$, then we want to compare H_D and $H_{\widetilde{D}}$.

For all $\sigma\in C^\infty(X,\Omega^k_{X,\mathbb{C}}\otimes L)$, locally $\sigma = s$ with respect to $\{e_1,\dots,e_r\}$.

Then

$$\begin{aligned}D(\sigma)-\widetilde{D}(\sigma) &= (\mathrm{d}s + A\wedge s) - (\mathrm{d}s - \widetilde{A}\wedge s)\\ &= (A-\widetilde{A})\wedge s\\ &= B\wedge\sigma\end{aligned}$$

where $B = (A-\widetilde{A})\in C^\infty(X,\Omega^1_{X,\mathbb{C}})$.

Then

$$H_D - H_{\widetilde{D}} = \mathrm{d}B$$

[10] If we have a short exact sequence of holomorphic vector bundle

$$0\to S\to E\to Q\to 0$$

this exact sequence generally won's split. That's why we perfer short exact sequence rather than direct sum in algebraic geometry.

So different connections give the same cohomology class in $H^2(X, \mathbb{C})$. What a beautiful result!

Definition 3.3.1 (First Chern class)**.** *Let $\pi : E \to X$ be a complex line bundle, D is any connection. Then define*

$$c_1(L) := [\frac{i}{2\pi} H_D] \in H^2(X, \mathbb{C})$$

, which is called the first Chern class of line bundle.

Let's explain why we need coefficient here.

Lemma 3.3.2. *(E, h) is a Hermitian line bundle, and D is a Hermitian connection, then*

$$\frac{i}{2\pi} H_D \in C^\infty(X, \Omega^2_{X,\mathbb{R}})$$

hence $c_1(L) \in H^2(X, \mathbb{R})$.

Proof. Locally we have $\overline{A} = -A$

$$\overline{\frac{i}{2\pi} H_D} = -\frac{i}{2\pi}\overline{H_D} = -\frac{i}{2\pi}\overline{\mathrm{d}A} = -\frac{i}{2\pi}\mathrm{d}\overline{A} = \frac{i}{2\pi}\mathrm{d}A = \frac{i}{2\pi}H_D$$

□

Remark 3.3.3. However, why we need 2π here? Later we will see in fact $c_1(L) \in H^2(X, \mathbb{Z})$.

Exercise 3.3.4. Let $E \to X$ be a complex vector bundle of rank r, define

$$c_1(E) := c_1(\det E)$$

If $L \to X$ is a complex line bundle, Show that

$$c_1(E \otimes L) = c_1(E) + rc_1(L)$$

School of Mathematics, Shandong University, Jinan, 250100, P.R. China,
Email address: bowenl@mail.sdu.edu.cn