ALGEBRAIC CURVES

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Contents

0. Motivations

0.1. **Meromorphic functions.** Let $U \subseteq \mathbb{C}$ be an open subset with coordinate $\{z\}$. In complex analysis we learnt that a meromorphic function f is a function that is holomorphic on all of U except for a set of isolated points, which are poles of the function. In other words, a meromorphic function can be regarded as a function $f\colon U \to \mathbb{C} \cup \{\infty\}$.

Topologically speaking, $\mathbb{C} \cup \{\infty\}$ is S^2, and in fact there is a complex manifold structure on it. More precisely, we can glue two pieces of complex plane via $w = 1/z$ to obtain a complex manifold called Riemann sphere

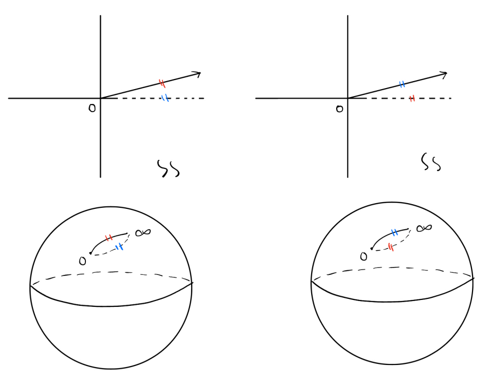
$$\mathbb{P}^1 = \mathbb{C} \cup_{\mathbb{C}^*} \mathbb{C},$$

and topologically $\mathbb{P}^1$ is exactly $\mathbb{C} \cup \{\infty\}$. By using this viewpoint, meromorphic function on U is exactly the same thing as holomorphic map from U to the Riemann sphere, and thus it gives us a lovely way to study meromorphic functions by using theories of holomorphic maps between Riemann surfaces, such as the number (counted with multiplicity) of zeros is equal to the number (counted with multiplicity) of poles.

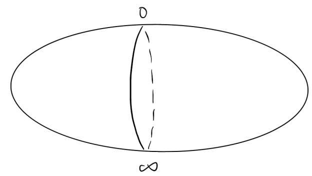
0.2. **Multivalueness of holomorphic functions.** For complex number $z = \rho e^{\sqrt{-1}\theta}$, where $\rho \in [0, \infty)$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, one has

$$(\sqrt{\rho} e^{\sqrt{-1}\theta/2})^2 = (\sqrt{\rho} e^{\sqrt{-1}\theta/2+\pi})^2 = z.$$

This shows there are two candidates for $\sqrt{z}$, and this phenomenon is called multivalueness of holomorphic function. If we define square root as $\sqrt{z} = \sqrt{\rho} e^{\sqrt{-1}\theta/2}$, then it's only well-defined on $\mathbb{C} \setminus [0, \infty)$, since it will "jump" when passing through the two sides of $[0, \infty)$, and $\mathbb{C} \setminus [0, \infty)$ is called a single value component of $\sqrt{z}$.



The ideal to solve this phenomenon is that, when passing the segment $[0, \infty)$, $\sqrt{z}$ should come into another single value component. In other words, if we want to make square root $\sqrt{z}$ defined on the whole complex plane, it should be no longer a function from $\mathbb{C}$ to $\mathbb{C}$, but a function from $\mathbb{C}$ to an object we obtained from gluing two single value components together. This construction also gives the Riemann sphere.

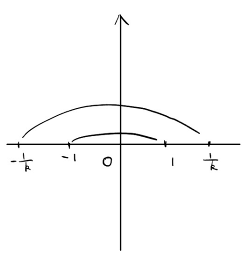


Similarly, $f(z) = \sqrt{1-z^2}$ is well-defined on $\mathbb{C}\setminus[-1,1]$, and it gives a well-defined function from $\mathbb{C}$ to something obtained by gluing two copies of $\mathbb{C}\setminus[-1,1]$, which is also the Riemann sphere.

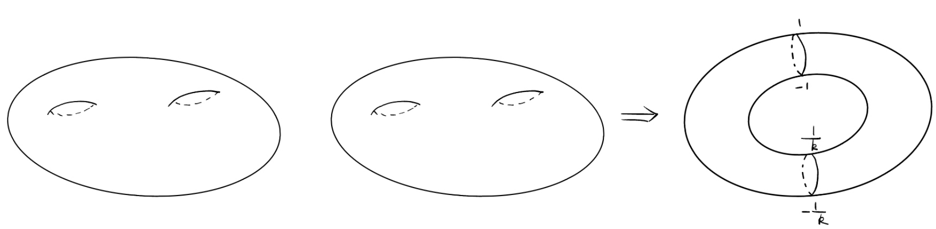
Now let's consider a more complicated example. For

$$f(z) = \sqrt{(1-z^2)(1-k^2z^2)},$$

where $k \neq \pm 1$, it gives a well-defined function on $\mathbb{C}$ minus two line segments connecting $-1, 1$ and $-1/k, 1/k$.



If we want to obtain a function defined on $\mathbb{C}$, we should glue two copies of above single value components. This gives a new Riemann surface called complex torus.



0.3. **Abelian integrals.**

0.3.1. *Arc-length of ellipse.* For ellipse given by $(x/a)^2 + (y/b)^2 = 1$, by using parameterization

$$x = a\cos\theta$$
$$y = b\sin\theta,$$

it's easy to see arc-length is given by

$$\begin{aligned}\int_{\theta_0}^{\theta_1}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\mathrm{d}\theta &= a\int_{\theta_0}^{\theta_1}\sqrt{1-k^2\sin^2\theta}\mathrm{d}\theta\\ &\overset{z=\sin\theta}{=}\int_{z_0}^{z_1}\frac{\sqrt{1-k^2z^2}}{\sqrt{1-z^2}}\mathrm{d}z\\ &=\int_{z_0}^{z_1}\frac{1-k^2z^2}{\sqrt{(1-k^2z^2)(1-z^2)}}\mathrm{d}z,\end{aligned}$$

where $k=\sqrt{1-b^2/a^2}$. For $k=0$, since $\arcsin z$ is a primitive function of $1/\sqrt{1-z^2}$, one has

$$\int_{z_0}^{z_1}\frac{1}{\sqrt{1-z^2}}\mathrm{d}z=\arcsin z_1-\arcsin z_0.$$

The classical theory of "addition formula" gives

$$\sin(\alpha+\beta)=\sin\alpha\sqrt{1-\sin^2\beta}+\sqrt{1-\sin^2\alpha}\sin\beta.$$

In terms of integration

$$\int_0^{z_1}\frac{1}{\sqrt{1-t^2}}\mathrm{d}t+\int_0^{z_2}\frac{1}{\sqrt{1-t^2}}\mathrm{d}t=\int_0^{z_1\sqrt{1-z_2^2}+z_2\sqrt{1-z_1^2}}\frac{1}{\sqrt{1-t^2}}\mathrm{d}t.$$

For analogue of above case, if we define ellipse sine sn as

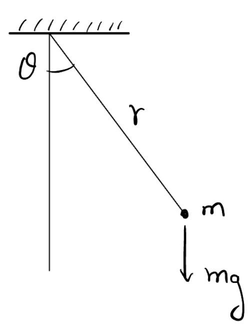
$$\int_0^{\arcsin z}\frac{1}{\sqrt{1-k^2\sin^2 t}}\mathrm{d}t=\mathrm{sn}^{-1}(z),$$

one can also show it satisfies some addition formula

$$\mathrm{sn}(\alpha+\beta)=\frac{\mathrm{sn}\,\alpha\sqrt{(1-\mathrm{sn}^2\beta)(1-k^2\,\mathrm{sn}^2\beta)}+\mathrm{sn}\,\beta\sqrt{(1-\mathrm{sn}^2\alpha)(1-k^2\,\mathrm{sn}^2\alpha)}}{1-k^2\,\mathrm{sn}^2\alpha\,\mathrm{sn}^2\beta}.$$

However, the ellipse sine cannot be expressed as an elementary function, and this is closely related to the fact that $y^2=(1-z^2)(1-k^2z^2)$ is not a Riemann sphere.

0.3.2. *Simple pendulum.* Suppose there is an object with mass m is released at $\theta=\alpha$ with zero initial velocity, and the length of pendulum is r.



The conservation of energy gives the following equation

$$\frac{1}{2}mr^2(\frac{\mathrm{d}\theta}{\mathrm{d}t})^2 = mgr\cos\theta - mgr\cos\alpha.$$

In other words,

$$(\frac{\mathrm{d}\theta}{\mathrm{d}t})^2 = 2\frac{g}{r}(\cos\theta - \cos\alpha) = 4\frac{g}{r}(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}). \tag{0.1}$$

An approximation with θ sufficiently small, one has

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \sqrt{\frac{g}{r}(\alpha^2 - \theta^2)}.$$

This shows

$$t = \int_0^\theta \sqrt{\frac{r}{g}}\frac{1}{\sqrt{\alpha^2 - s^2}}\mathrm{d}s.$$

Thus the period of the simple pendulum is given by

$$T = 4\int_0^\alpha \sqrt{\frac{r}{g}}\frac{1}{\sqrt{\alpha^2 - s^2}}\mathrm{d}s = 2\pi\sqrt{\frac{r}{g}}.$$

However, if we don't use the approximation, and use substitution

$$\sin\varphi = \frac{\sin\frac{\theta}{2}}{\sin\frac{\alpha}{2}}$$

in (0.1), one has

$$(\frac{\mathrm{d}\varphi}{\mathrm{d}t})^2 = \frac{g}{r}(1 - \sin^2\frac{\alpha}{2}\sin^2\varphi).$$

Then

$$t = \sqrt{\frac{r}{g}}\int_0^\varphi \frac{1}{\sqrt{1 - k^2\sin^2 s}}\mathrm{d}s,$$

where $k = \sin\frac{\alpha}{2}$, and thus explicit formula for the period of simple pendulum is

$$T = 4\sqrt{\frac{r}{g}}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2\sin^2 s}}\mathrm{d}s.$$

This is exactly ellipse integral.

0.3.3. *General.* Let P be a polynomial of two variables and $y = f(x)$ be a solution of equation $P(x, y) = 0$. Then

$$\int R(x, f(x)) = 0$$

can be expressed as elementary function if and only if $\deg P = 0, 1, 2$, and in fact $\deg P$ is closely related to the topology of algebraic curves.

1. Riemann Surface

1.1. Definitions and Examples.

1.1.1. *Definitions.*

Definition 1.1.1 (complex atlas)**.** Let X be a topological space. A complex atlas on X consists of the following data:
(1) $\{U_i\}_{i\in I}$ is an open covering of X.
(2) For each $i \in I$, there exists a homeomorphism $\varphi_i\colon U_i \to \varphi_i(U_i) \subseteq \mathbb{C}$.
(3) For $i, j \in I$, if $U_i \cap U_j \neq \varnothing$, then the transition function

$$\varphi_2 \circ \varphi_1^{-1}\colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is holomorphic.

Definition 1.1.2 (complex structure)**.** Two complex atlas $\mathscr{A}, \mathscr{B}$ are equivalent if $\mathscr{A} \cup \mathscr{B}$ is also a complex atlas, and a complex structure is an equivalent class of atlas on X.

Definition 1.1.3 (Riemann surface)**.** A Riemann surface is a connected, second countable, Hausdorff topological space X together with a complex structure on X.

1.1.2. *Examples.*

Example 1.1.1 (Riemann sphere)**.** Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ be 2-sphere and $\{U_1 = S^2 \setminus (0, 0, 1), U_2 = S^2 \setminus (0, 0, -1)\}$ be an open covering of S^2. Consider

$$\begin{aligned} \varphi_1\colon U_1 &\to \mathbb{C} \\ (x_1, x_2, x_3) &\mapsto \frac{x_1}{1 - x_3} + \sqrt{-1}\frac{x_2}{1 - x_3}, \end{aligned}$$

and

$$\begin{aligned} \varphi_2\colon U_1 &\to \mathbb{C} \\ (x_1, x_2, x_3) &\mapsto \frac{x_1}{1 + x_3} - \sqrt{-1}\frac{x_2}{1 + x_3}. \end{aligned}$$

A direct computation shows that

$$\left(\frac{x_1}{1 - x_3} + \sqrt{-1}\frac{x_2}{1 - x_3}\right)\left(\frac{x_1}{1 + x_3} - \sqrt{-1}\frac{x_2}{1 + x_3}\right) = \frac{x_1^2 + x_2^2}{1 - x_3^2} = 1,$$

and thus the transition function $\varphi_2 \circ \varphi_1^{-1}(z) = 1/z$. This shows $\{U_1, U_2\}$ is a complex atlas of S^2, and thus S^2 is a Riemann surface, called Riemann sphere.

Example 1.1.2 (projective space)**.** The complex projective space $\mathbb{P}^1 = \mathbb{C}^2 \setminus (0, 0)/\sim$, where $(x, y) \sim (z, w)$ if and only if $(\lambda x, \lambda y) = (z, w)$ for some $\lambda \in \mathbb{C}^*$, and $[x, y]$, called the homogenous coordinate, denotes this equivalent class. There is a quotient topology on $\mathbb{P}^1$ which makes it second countable, Hausdorff and compact. Consider

$$U_1 = \{[z, w] \mid z \neq 0\} \xrightarrow{\varphi_1} \mathbb{C}$$

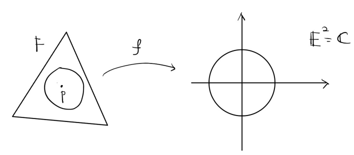
where φ_1 is defined as $\varphi_1([z,w]) = z/w$. Similarly consider

$$U_2 = \{[z,w] \mid w \neq 0\} \xrightarrow{\varphi_2} \mathbb{C}$$

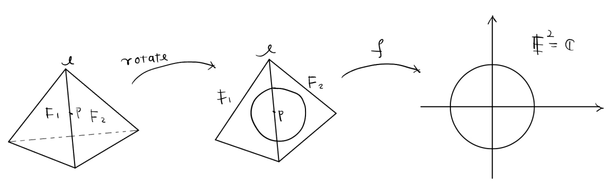
where φ_2 is defined as $\varphi_2([z,w]) = w/z$. It's easy to check $\{U_1, U_2\}$ is a atlas of $\mathbb{P}^1$.

Example 1.1.3. Let P be a convex polyhedra in Euclidean 3-dimensional space $\mathbb{E}^3$. Topologically P is S^2, and let's construct a complex atlas on it.

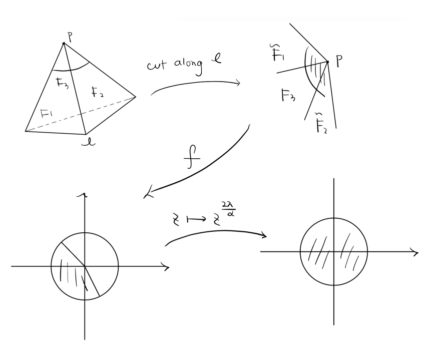
(1) Suppose p is the interior point of some face F. Since F can be isometrically embedded into $\mathbb{E}^2$, we choose an orientation-preserving, isometric embedding f which maps an open neighborhood U of p into $\mathbb{E}^2 = \mathbb{C}$.



(2) Suppose p is the interior point of some edge $l = F_1 \cap F_2$. Firstly we rotate F_2 along l to the plane of F_1, and then choose an orientation-preserving, isometric embedding f which maps an open neighborhood U of p into $\mathbb{E}^2 = \mathbb{C}$.



(3) Suppose p is an vertex which is the intersection of three faces F_1, F_2 and F_3. Firstly we cut it along some edge $l = F_1 \cap F_2$, and then rotate F_1, F_2 to the plane of F_3. Then we use some orientation-preserving, isometric embedding f to map it into $\mathbb{E}^2$, and finally composite it with $z \mapsto z^{2\pi/\alpha}$.



Exercise 1.1.1. Prove that above constructions give a complex atlas on convex polyhedra.

Remark 1.1.1. All of above three examples give complex structure on topological sphere S^2, and we will see all of them are the "same" after we define the isomorphism between Riemann surfaces. In fact, there is only one complex structure on S^2.

Example 1.1.4 (complex torus)**.** For non-zero $w_1, w_2 \in \mathbb{C}$ such that w_1, w_2 are $\mathbb{R}$-linearly independent, $L = \mathbb{Z}\, w_1 + \mathbb{Z}\, w_2$ is a discrete subgroup of $(\mathbb{C}, +)$. Then $T = \mathbb{C}\,/L$ equipped with quotient topology is a topological manifold. Let $\pi\colon\ \mathbb{C} \to T$ be the natural projection. For $p \in T$, suppose z_0 is an inverse image of p. For $\varepsilon \in \mathbb{R}_{>0}$ such that

$$B_{2\varepsilon}(0) \cap L = \{0\},$$

one has $B_\varepsilon(z_0) \xrightarrow{\pi} \pi(B_\varepsilon(z_0)) \subseteq T$ is injective, and thus $\pi^{-1}\colon \pi(B_\epsilon(z_0)) \to B_\epsilon(z_0) \subseteq \mathbb{C}$ is a homeomorphism. By choosing different $p \in T$, $\{\pi(B_\varepsilon(z_0))\}$ gives an open covering of T, and together with π^{-1} it gives a complex atlas of T.

Remark 1.1.2. It's clear complex structure constructed above depends on the choice of w_1, w_2, but it's not obvious to see whether w_1, w_2 and w_1', w_2' give the same complex structure or not. Moreover, all complex structure on torus are arisen in this way.

1.2. **Morphisms.**

Definition 1.2.1 (holomorphic map)**.** Let X, Y be two Riemann surfaces and $F\colon X \to Y$ be a continous map. For $p \in X$, F is called holomorphic at p, if there exists a chart (U, φ) of p, and a chart (V, ψ) of $F(p)$, such that

$$\psi \circ F \circ \varphi^{-1}\colon \varphi\left(U \cap F^{-1}(V)\right) \to \psi\left(V \cap F(U)\right)$$

is holomorphic at $\varphi(p)$. Moreover, F is called holomorphic in $W \subseteq X$, if F is holomorphic at any point in W.

Exercise 1.2.1. Show that the definition of holomorphic map is independent of the choice of charts.

Definition 1.2.2 (isomorphism)**.** Let $F\colon X \to Y$ be a holomorphic map between Riemann surfaces. F is called an isomorphism if it's bijective and holomorphic.

Proposition 1.2.1. Let $F\colon X \to Y$ be a holomorphic map between Riemann surfaces. F is an isomorphism if and only if F has an two-side inverse G, and G is holomorphic.

Proposition 1.2.2. The complex projective space is isomorphic to Riemann sphere.

1.3. **Algebraic curves.**

1.3.1. *Affine plane curves.* Let $V \subseteq \mathbb{C}$ be a connected open subset and g be a holomorphic function defined on V. The graph X of g, as a subset of $\mathbb{C}^2$ is defined by

$$\{(z, g(z)) \mid z \in V\}.$$

Given X the subspace topology, and let $\pi\colon X \to V$ be the projection to the first factor. Note that π is a homeomorphism, whose inverse sends the point $z \in V$ to the ordered pair $(z, g(z))$. This makes X a Riemann surface.

A generalization of the graph of holomorphic function is that we consider "Riemann surface" which is locally a graph, but perhaps not globally. The tools we use is implicit function theorem in fact.

Theorem 1.3.1 (The implicit function theorem)**.** Let $f(z, w)\colon \mathbb{C}^2 \to \mathbb{C}$ be holomorphic function of two variables and $X = \{(z, w) \in \mathbb{C}^2 \mid f(z, w) = 0\}$ be its zero loucs. Let $p = (z_0, w_0)$ be a point of X and $\partial f/\partial z(p) \neq 0$. Then there exists a function $g(w)$ defined and holomorphic in a neighborhood of w_0 such that, near p, X is equal to the graph $z = g(w)$.

Method one. If we write $z = a + \sqrt{-1}b, w = c + \sqrt{-1}d$ and $f(z, w) = u + \sqrt{-1}v$, then u, v are smooth functions of a, b, c, d. Moreover, the Cauchy-Riemann equations give

$$\frac{\partial f}{\partial z} = \frac{\partial u}{\partial a} + \sqrt{-1}\frac{\partial v}{\partial a} = \frac{\partial v}{\partial b} - \sqrt{-1}\frac{\partial u}{\partial b} = A + \sqrt{-1}B.$$

Then

$$\frac{\partial(u, v)}{\partial(a, b)} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

and $\det \dfrac{\partial(u, v)}{\partial(a, b)} = A^2 + B^2 \neq 0$ if and only if $A + \sqrt{-1}B \neq 0$. Then the classical implicit function theorem implies the zero loucs

$$\begin{cases} u = 0 \\ v = 0 \end{cases}$$

is locally given by

$$\begin{cases} a = a(c, d) \\ b = b(c, d). \end{cases}$$

In other words, $z = g(w)$. Now it suffices to compute $\partial g/\partial \overline{w}$ to show g is holomorphic. Again by Cauchy-Riemann equations

$$\frac{\partial f}{\partial w} = \frac{\partial u}{\partial c} + \sqrt{-1}\frac{\partial v}{\partial c} = \frac{\partial v}{\partial d} - \sqrt{-1}\frac{\partial u}{\partial d} = C + \sqrt{-1}D.$$

Then by chain rule one has

$$\begin{aligned}\frac{\partial(a,b)}{\partial(c,d)} &= \left(\frac{\partial(u,v)}{\partial(a,b)}\right)^{-1}\frac{\partial(u,v)}{\partial(c,d)} \\ &= \begin{pmatrix} A & B \\ -B & A\end{pmatrix}^{-1}\begin{pmatrix} C & D \\ -D & C\end{pmatrix} \\ &= \frac{1}{A^2+B^2}\begin{pmatrix} AC+BD & AD-BC \\ BC-AD & BD+AC\end{pmatrix}.\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial g}{\partial \overline{w}} &= \frac{1}{2}\left(\frac{\partial}{\partial c}+\sqrt{-1}\frac{\partial}{\partial d}\right)(a+\sqrt{-1}b) \\ &= \frac{1}{2}\left(\frac{\partial a}{\partial c}+\sqrt{-1}\frac{\partial b}{\partial c}+\sqrt{-1}\frac{\partial a}{\partial d}-\frac{\partial b}{\partial d}\right) \\ &= 0\end{aligned}$$

□

Method two. □

Definition 1.3.1 (affine plane curve)**.** An affine plane curve is the loucs of zeros in $\mathbb{C}^2$ of a (non-trivial) polynomial $f(z,w)$.

Definition 1.3.2 (non-singular)**.** A polynomial $f(z,w)$ is non-singular at root p if either $\partial f/\partial z$ or $\partial f/\partial w$ is not zero at p, otherwise it's called singular. The affine plane curve X defined by $f(z,w)$ is non-singular is non-singular at p if f is non-singular at p. The curve X is non-singular if it's non-singular at each of its points.

Example 1.3.1. The affine plane curve defined by z^2+w^2-1 is non-singular.

Theorem 1.3.2. *A non-singular affine plane curve defined by an irreducible polynomial is a Riemann surface.*

1.3.2. *Projective plane curve.*

Definition 1.3.3 (projective plane curve)**.** Let P be a homogenous polynomial in $\mathbb{C}[x,y,z]$. A projective plane curve C defined by P is the zero loucs of P, that is,

$$C=\{[x:y:z]\in\mathbb{P}^2 \mid P(x,y,z)=0\}.$$

Given a projective plane curve C given by homogenous polynomial P. Consider

$$\begin{aligned}\varphi_0\colon U_0=\mathbb{C}^2&\to\mathbb{P}^2 \\ (y,z)&\mapsto[1:y:z]\end{aligned}$$

Then $\varphi_0^{-1}(U_0\cap C)=\{(y,z)\in\mathbb{C}^2\mid P(1,y,z)=0\}$ is an affine plane curve, and similarly there are other affine plane curves given by $\varphi_0^{-1}(U_1\cap C)$ and $\varphi_0^{-1}(U_2\cap C)$.

Conversely, given an affine plane curve C defined by $f \in \mathbb{C}[y,z]$. Consider the homogenous polynomial $P(x,y,z)$ defined by

$$P(x,y,z) = x^d f(\frac{y}{x}, \frac{z}{x})$$

where $d = \deg f$. Then P defines a projective plane curve, such that the affine plane curve it gives on affine piece U_0 is exactly C.

Definition 1.3.4 (non-singular)**.** A projective plane curve C is non-singular if the affine plane curves $\varphi_i^{-1}(U_i \cap C)$ are non-singular for $i = 0,1,2$, where $\varphi_i \colon U_i \to \mathbb{P}^2$ are standard affine covering of $\mathbb{P}^2$.

Proposition 1.3.1. A projective plane curve $C = \{P(x,y,z) = 0\} \subseteq \mathbb{P}^2$ is non-singular if and only if

$$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial P}{\partial y} = 0$$

has no solution in $\mathbb{P}^2$.

Proof. Since P is a homogenous polynomial, it satisfies the Euler's formula

$$dP = x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} + z\frac{\partial P}{\partial z},$$

where $d = \deg P$. Now let's start our proof as follows:

(1) Suppose $\partial P/\partial x = \partial P/\partial y = \partial P/\partial z = 0$ has a solution (a,b,c) with $a \neq 0$. Then

$$\frac{\partial P}{\partial y}(1, \frac{b}{a}, \frac{c}{a}) = \frac{1}{a^{d-1}}\frac{\partial P}{\partial y}(a,b,c) = 0$$
$$\frac{\partial P}{\partial z}(1, \frac{b}{a}, \frac{c}{a}) = \frac{1}{a^{d-1}}\frac{\partial P}{\partial z}(a,b,c) = 0.$$

Thus

$$P(1, \frac{b}{a}, \frac{c}{a}) = \frac{1}{a^d}P(a,b,c) = 0.$$

(2) Conversely, if the projective plane curve defined by P is singular, without lose of generality we may assume $X_0 := \varphi_i^{-1}(U_0 \cap C)$ is singular. Then there exists a solution $(b,c) \in \mathbb{C}^2$ such that

$$P(1,b,c) = \frac{\partial P}{\partial y}(1,b,c) = \frac{\partial P}{\partial z}(1,b,c) = 0.$$

By Euler's formula one has

$$\frac{\partial P}{\partial x}(1,b,c) = dP(1,b,c) - b\frac{\partial P}{\partial y} - c\frac{\partial P}{\partial z} = 0.$$

As a consequence, $(1,a,b)$ is a solution of $\partial P/\partial x = \partial P/\partial y = \partial P/\partial z = 0$.

□

Theorem 1.3.3. Any non-singular projective curve is a compact Riemann surface.

Example 1.3.2 (Fermat curve)**.** $x^d+y^d = z^d$ gives a non-singular projective plane curve.

Example 1.3.3. The polynomial $f(x, y) = y^2-(1-x^2)(1-k^2x^2), k \neq 0, \pm 1$ gives a non-singular affine plane curve. Let $P(x, y, z)$ be the homogenous polynomial given by $f(x, y)$, that is,

$$P(x, y, z) = z^2y^2 - (z^2 - x^2)(z^2 - k^2x^2).$$

Consider the affine plane curve it gives on the affine piece U_1

$$P(x, 1, z) = z^2 - (z^2 - x^2)(z^2 - k^2x^2) = 0.$$

By setting $z = 0$ we can see this compactification gives a new point $[0 : 1 : 0]$, which is singular. Similarly, we can consider the affine plane curve it gives on the affine piece U_2

$$P(1, y, z) = z^2y^2 - (z^2 - 1)(z^2 - k^2).$$

But in this case, there is no more new point.

Example 1.3.4 (quadratic)**.** A homogenous polynomial P of degree 2 can be written as

$$P = (x, y, z)A \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where A is a symmetric matrix. Then

(1) If $\operatorname{rk} A = 3$, then P is non-singular, and the projective plane curve defined by P is isomorphic to $\mathbb{P}^1$.
(2) If $\operatorname{rk} A = 2$, then C is the union of two lines.
(3) If $\operatorname{rk} A = 1$, then C is a double line.

References

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