SPECTRAL SEQUENCES AND APPLICATIONS

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Contents

Part 1. Spectral Sequences

1. Exact couples

A simple way to construct spectral sequence is through exact couples.

Definition 1.1 (exact couple)**.** An exact couple is an exact sequence of abelian groups of the form

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & B \arrow[ul, "k"] & \end{tikzcd}$$

where i, j and k are group homomorphisms.

From an exact couple, we can define a homomorphism $\mathrm{d} : B \to B$ by $\mathrm{d} = j \circ k$, then $\mathrm{d}^2 = 0$, so the homology group $H(B) = \ker \mathrm{d} / \operatorname{im} \mathrm{d}$ is well-defined.

Furthermore, from this exact couple, we can define a new exact couple, called derived couple,

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & B' \arrow[ul, "k'"] & \end{tikzcd}$$

by making the following definitions.

1. $A' = i(A)$ and $B' = H(B)$;
2. i' is induced from i, that is $i'(ia) = i(ia)$;
3. For $a' = ia$ for some $a \in A$, then $j'a' = [ja]$. To show j' is well defined, we need to check the following things
   a. ja is a cycle. Indeed, $\mathrm{d}(ja) = jkja = 0$;
   b. The homology class $[ja]$ is independent of the choice of a. Indeed, if $a' = i\overline{a}$ for some other $\overline{a} \in A$. Then $a - \overline{a} = kb$ for some $b \in B$, since $a - \overline{a} \in \ker i = \operatorname{im} k$. Thus
   $$ja - j\overline{a} = jkb = \mathrm{d}b$$
   that is $[ja] = [j\overline{a}]$.
4. k' is induced from k. Let $[b] \in H(B)$, then $\mathrm{d}b = jkb = 0$ implies $kb \in \ker j = \operatorname{im} i$, so there exists $a \in A$ such that $kb = ia$. Define
   $$k'[b] := kb \in i(A) = A'$$
   Note that we also need to check k' is well-defined: take another $b' \in [b]$, that is $b' - b = \mathrm{d}b''$ for some $b'' \in B$. Then
   $$kb' = kb + k\mathrm{d}b'' = kb + kjkb'' = kb$$

As we have already defined these homomorphisms i', j' and k', it suffices to check above diagram is an exact sequence. Let's check step by step:

1. $\operatorname{im} j' = \ker k'$: Take $j'a' \in \operatorname{im} j'$, then $k'j'a' = k'j'(ia) = k'[jia] = kjia = 0$; Conversely , if $[b] \in B'$ such that $k'[b] = kb = 0$, that is $b \in \ker k = \operatorname{im} j$. So there exists $a \in A$ such that $b = ja$, so $[b] = [ja] = j'a'$, where $a' = ia$.
2. $\operatorname{im} k' = \ker i'$: Take $k'[b] = kb \in \operatorname{im} k'$, then $i'kb = ikb = 0$; Conversely , if $ia \in A'$ such that $i'ia = iia = 0$, so there exists $b \in B$ such that $ia = kb$. Furthermore, such b must be a cycle, since $jkb = jia = 0$. So $ia = kb = k'[b]$.
3. $\operatorname{im} i' = \ker j'$: Take $iia \in \operatorname{im} i'$, then $j'(iia) = [jia] = 0$; Conversely , if $ia \in A'$ such that $j'ia = [ja] = [0]$, that is there exists $b \in B$ such that $\mathrm{d}b = jkb = ja$, that is $a - kb \in \ker j = \operatorname{im} i$. So there exists $a' \in A$ such that $a - kb = ia'$. So $a - ia' \in \operatorname{im} k = \ker i$, that is $ia = iia'$. This completes the proof.

2. The Spectral Sequence of a Filtered Complex

In this section we fix a differential graded complex $K = \bigoplus_{k\in\mathbb{Z}} C^k$ with a differential operator $D : C^k \to C^{k+1}$.

Definition 2.1 (filtration)**.** A sequence of subcomplexes

$$K = K_0 \supset K_1 \supset K_2 \supset K_3 \supset \dots$$

is called a filtration on K.

Notation 2.1. We usually extend the filtration to negative indices by defining $K_p = K$ for $p < 0$.

Definition 2.2 (filtered complex)**.** A complex K with a filteration $\{K_p\}_{p\in\mathbb{Z}_{\geq 0}}$ is called a filtered complex and the associated graded complex is defined as

$$GK = \bigoplus_{p=0}^{\infty} K_p/K_{p+1}$$

Consider

$$A = \bigoplus_{p\in\mathbb{Z}} K_p$$

A is again a differential complex with operator D. Define $i : A \to A$ to be the inclusion $K_{p+1} \hookrightarrow K_p$ and define B to be the quotient, then we obtain a short sequence

$$0 \to A \xrightarrow{i} A \xrightarrow{j} B \to 0$$

and it induces a long exact sequence

$$\cdots \to H^k(A) \xrightarrow{i_1} H^k(A) \xrightarrow{j_1} H^k(B) \xrightarrow{k_1} H^{k+1}(A) \to \dots$$

In other words, we can write it as an exact couple as follows

$$\begin{tikzcd} A_1 \arrow[rr, "i"] & & A_1 \arrow[dl, "j_1"] \\ & B_1 \arrow[ul, "k_1"] & \end{tikzcd}$$

where $A_1 = H(A), B_1 = H(B)$ and $i = i_1$. We suppress the subcript of i_1 to avoid cumbersome notation later. This exact couple gives rise to a sequence of exact couples:

$$\begin{tikzcd} A_r \arrow[rr, "i"] & & A_r \arrow[dl, "j_r"] \\ & B_r \arrow[ul, "k_r"] & \end{tikzcd}$$

Example 2.1. Let's see a simple example: Consider the filtered complex terminates after K_3, that is

$$\cdots = K_{-1} = K_0 \supset K_1 \supset K_2 \supset K_3 \supset 0$$

Then by definition, A_1 is the direct sum of all terms in the following sequence

$$\ldots \xleftarrow{\cong} H(K) \xleftarrow{\cong} H(K) \xleftarrow{i} H(K_1) \xleftarrow{i} H(K_2) \xleftarrow{i} H(K_3) \leftarrow 0$$

And by definition of A_2, it equals iA_1, so it's the direct sum of all terms in the following sequence

$$\ldots \xleftarrow{\cong} H(K) \xleftarrow{\cong} H(K) \supset iH(K_1) \xleftarrow{i} iH(K_2) \xleftarrow{i} iH(K_3) \leftarrow 0$$

Note that $iH(K_1) \subset H(K)$, and $i : H(K) \to H(K)$ is identity map, thus $iiH(K_1) = iH(K_1)$. So A_3 is the direct sum of all terms in the following sequence

$$\ldots \xleftarrow{\cong} H(K) \xleftarrow{\cong} H(K) \supset iH(K_1) \supset iiH(K_2) \xleftarrow{i} iiH(K_3) \leftarrow 0$$

Similarly we have A_4 is the sum of

$$\ldots \xleftarrow{\cong} H(K) \xleftarrow{\cong} H(K) \supset iH(K_1) \supset iiH(K_2) \supset iiiH(K_3) \supset 0$$

Since all terms appearing in A_4 is in $H(K)$, then i is identity on A_4. So A's are stationary after A_4 and we define

$$A_4 = A_5 = \cdots = A_\infty$$

Furthermore, since $\ker\{i : A_4 \to A_5\} = \operatorname{im} k_4$, thus $k_4 = 0$. Therefore after the fourth stage all the differential of the exact couple are zero, since $\mathrm{d} = jk$. So B's are also stationary, that is

$$B_4 = B_5 = \cdots = B_\infty$$

In the exact couple

$$\begin{tikzcd} A_\infty \arrow[rr, "i_\infty"] & & A_\infty \arrow[dl, "j_r"] \\ & B_\infty \arrow[ul, "k_\infty = 0"] & \end{tikzcd}$$

A_∞ is the direct sum of groups

$$\ldots \xleftarrow{\cong} H(K) \xleftarrow{\cong} H(K) \supset iH(K_1) \supset iiH(K_2) \supset iiiH(K_3) \supset 0$$

So if we let above sequence be a filteration of $H(K)$, then B_∞ is the associated graded complex of the filtered complex $H(K)$.

Now let's come back to general case. The sequence of subcomplexes

$$\cdots = K = K \supset K_1 \supset K_2 \supset K_3 \supset \ldots$$

induces a sequence in cohomology

$$\ldots \xleftarrow{\cong} H(K) \xleftarrow{\cong} H(K) \xleftarrow{i} H(K_1) \xleftarrow{i} H(K_2) \xleftarrow{i} H(K_3) \leftarrow \ldots$$

Note that i are of course no longer inclusions. Let F_p be the image of $H(K_p)$ in $H(K)$. For example, $F_3 = iiiH(K_3)$. There exists a sequence of inclusions

$$H(K) = F_0 \supset F_1 \supset F_2 \supset F_3 \supset \ldots$$

making $H(K)$ into a filtered complex. This filtration is called the induced filteration on $H(K)$.

Definition 2.3 (length of filtration)**.** A filtration K_p on the filtered complex K is said to have length l if $K_l \neq 0$ and $K_p = 0$ for $p > l$.

So as we can see from simple example we have computed, if the filtration of K has finite length, then A_r and B_r are stationary and the stationary value B_∞ is the associated graded complex $\bigoplus F_p/F_{p+1}$ of the filtered complex $H(K)$.

It's customary to write E_r for B_r, and there is a differential d_r on E_r such that $H_{\mathrm{d}_r}(E_r) = E_{r+1}$, and that's definition of a spectral sequence.

Definition 2.4 (spectral sequence)**.** A sequence of differential complex $\{E_r, \mathrm{d}_r\}$ in which each E_r is the homology of its predecessor E_r is called a spectral sequence.

Definition 2.5 (convegence of spectral sequence)**.** A spectral sequence $\{E_r, \mathrm{d}_r\}$ is said to converge to some filtered group H, if E_∞ is equal to the associated graded group of H.

Let's summarize what we have done: For a differential complex K and a filteration $\{K_p\}$ of K, if the filtration is finite length, then the spectral sequence we obtained from this filtration will converge to $H(K)$.

However, it's quit strong requirement for a filteration to be finite length. Suppose filtered complex $K = \bigoplus_n K^n$, then a filteration $\{K_p\}$ on K induces a filteration on K^n for each n, that is $K_p^n := K_p \cap K^n$. And we can prove the same result, only asking $\{K_p^n\}$ to be finite length for each n.

Theorem 2.1. Let $K = \bigoplus_n K^n$ be a graded filtered complex with filtration $\{K_p\}$ and let $H_D^*(K)$ be the cohomology of K with filtration given by $\{K_p\}$. Suppose for each n we have $\{K_p^n\}$ is finite length. Then the short exact sequence of complex

$$0 \to \bigoplus K_{p+1} \to \bigoplus K_p \to \bigoplus K_p/K_{p+1} \to 0$$

induces a spectral sequence which converges to $H_D^*(K)$.

Proof. The ideal here is that since it's a convegence between two graded groups, so it suffices to treat the convegence question one dimension at a time, then it's reduced to the ungraded situation.

Fix a number n and consider n-th grade and let $\ell(n)$ be the length of $\{K_p^n\}_{p\in\mathbb{Z}}$, we have the following sequence

$$\dots \xleftarrow{\cong} H^n(K) \xleftarrow{i} H^n(K_1) \xleftarrow{i} H^n(K_2) \xleftarrow{i} \dots \xleftarrow{i} H^n(K_{l(n)}) \xleftarrow{i} 0 \xleftarrow{i} \dots$$

Use F_p^n to denote the image of $H^n(K_p)$ in $H^n(K)$. If $r \geq \ell(n)+1$, then for all p

$$i^r H^n(K_p) = F_p^n$$

so we have

$$i : i^r H^n(K_{p+1}) \to i^r H^n(K_p)$$

is an inclusion, since both of them are in $H^n(K)$. By definition we have

$$A_r^n = \bigoplus_p i^r H^n(K_p)$$

and i_r sends $i^r H^n(K_{p+1})$ to $i^r H^n(K_p)$. It follows that

$$i_r : A_r^n \to A_r^n$$

is an inclusion thus $k_r : B_r^{n-1} \to A_r^n$ is the zero map. So we have $A_k^n = A_r^n$ and $B_k^{n-1} = B_r^{n-1}$ for all $k \geq r$, that is $A_\infty^n = A_r^n = \bigoplus F_p^n$ and $B_\infty^n = B_r^n = \bigoplus_p F_p^n / F_{p+1}^n$. Thus

$$B_\infty = \bigoplus_n B_\infty^n = \bigoplus_{n,p} F_p^n / F_{p+1}^n = \bigoplus_p F_p / F_{p+1}$$

that is associated graded complex of $H_D^*(K)$, as desired. $\square$

3. The Spectral Sequence of a Double Complex

3.1. **Basic setting.** Now for a double complex $K = \bigoplus_{p,q\geq 0} K^{p,q}$ with differential d and δ, we can make it into a complex, called total complex with differential D by

$$K = \bigoplus_{k=0}^{\infty} C^k$$

where $C^k = \bigoplus_{p+q=k} K^{p,q}$ and $D = \delta + (-1)^p \mathrm{d} = \delta + D''$. There is a natural filtration on K as follows

$$K_p = \bigoplus_{i\geq p, q\geq 0} K^{i,q}$$

The direct sum $A = \bigoplus_{p\geq 0} K_p$ is also a double complex, and we can also make it into a single complex $A = \bigoplus_{k\geq 0} A^k$ by summing the bidegrees.

Note that

$$A^k = \bigoplus_p A^k \cap K_p$$

and inclusion $i : A^k \to A^k$ is given by

$$i : A^k \cap K_{p+1} \to A^k \cap K_p$$

This gives an inclusion $i : A \to A$ and the quotient is denoted by B, where B is also a double complex, we can also make it into a single complex $B = \bigoplus_{k \geq 0} B^k$ by summing the bidegrees. We can write this short exact sequence as follows

$$0 \to \bigoplus_{k,p} A^k \cap K_p \to \bigoplus_{k,p} A^k \cap K_p \to \bigoplus_{k,p} B^k \cap (K_p/K_{p+1}) \to 0$$

where the differential of these complexes are listed as follows:

1. A inherits the differential operator $D = \delta + (-1)^p \mathrm{d}$ from K;
2. $B = \bigoplus K_p/K_{p+1}$ also inherits the differential operator D, but D on B is just $(-1)^p \mathrm{d}$, since any element in K_p is mapped into K_{p+1} by δ. Therefore

$$E_1 = H_D(B) = H_{\mathrm{d}}(K)$$

Remark 3.1. From above section, we obtain a spectral sequence which converges $H_D(K)$, since our filtration is finite on each degree n. However, we want to show a more refinement theorem, since in this case our complex comes from a double complex, which has a more subtle structure. In order to do this, we need to compute the explicit formula of d_r.

Notation 3.1. We will denote the class of b in E_r, if it's well-defined, by $[b]_r$.

3.2. **Explicit formula of** d_r.

3.2.1. *Case of* d_1. Note that

$$B^k = \bigoplus_p B^k \cap (K_p/K_{p+1})$$

So if we want to compute $k_1 : H^k(B) \to H^{k+1}(A)$, it suffices to compute

$$k_1 : H^k(B) \cap (K_p/K_{p+1}) \to H^{k+1}(A) \cap K_{p+1}$$

for each p.

Remark 3.2 (**characterization of elements in** E_1). Any element $[b]_1 \in H^k(B) \cap (K_p/K_{p+1})$ is $b + K_{p+1} \in B^k \cap (K_p/K_{p+1})$ such that $b \in K^{p,k-p}$ and $\mathrm{d}b = 0$. So you can regard $E_1^{p,q}$ as $H_{\mathrm{d}}^{p,q}(K)$.

Now we fix p and consider

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A^{k+1}\cap K_{p+1} & \longrightarrow & A^{k+1}\cap K_p & \longrightarrow & B^{k+1}\cap K_p/K_{p+1} & \longrightarrow & 0 \\
 & & {\scriptstyle D}\uparrow & & {\scriptstyle D}\uparrow & & {\scriptstyle \mathrm{d}}\uparrow & & \\
0 & \longrightarrow & A^{k}\cap K_{p+1} & \longrightarrow & A^{k}\cap K_p & \longrightarrow & B^{k}\cap K_p/K_{p+1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & &
\end{array}$$

In order to get $k_1[b]_1$, where $[b]_1 \in E_1^{p,k-p}$, we need to chase diagram as follows

1. Choose $b \in A^k \cap K_p$ to represent $[b]_1$[1];
2. $Db = \delta b + (-1)^p \mathrm{d}b = \delta b \in A^{k+1} \cap K_p$, since $\mathrm{d}b = 0$;
3. Take inverse of $\delta b \in A^{k+1} \cap K_p$ under i, we obtain $\delta b \in A^{k+1} \cap K_{p+1}$.

Thus $k_1[b]_1 = [\delta b]_1 \in H^{k+1}(A) \cap K_{p+1}$. By definition of d_1 we can see

$$\begin{aligned}
\mathrm{d}_1 : H^k(B) \cap (K_p/K_{p+1}) &\to H^{k+1}(B) \cap (K_{p+1}/K_{p+2}) \\
[b]_1 &\mapsto [\delta b]_1
\end{aligned}$$

By characterization of elements in E_1, we can regard $\mathrm{d}_1[b]_1$ as $\delta b \in K^{p+1,k-p}$ with $\mathrm{d}(\delta b) = 0$, and $[\delta b]_1 = 0 \in E_1$ is equivalent to say there exists $c \in K^{p+1,k-p-1}$ such that $\delta b = -D''c$.

Remark 3.3 (**characterization of elements in** E_2). For an element of $[b]_2 \in E_2$, it can be represented by an element $b \in K$ with a zig-zag of length 2

$$\begin{array}{ccc}
0 & & \\
{\scriptstyle \mathrm{d}}\uparrow & & \\
b & \xrightarrow{\delta} & \delta b \\
 & & {\scriptstyle D''}\uparrow \\
 & & c
\end{array}$$

In other words, $E_2 = H_\delta H_{\mathrm{d}}(K)$.

For $[b]_2 \in E_2^{p,q}$, by definition of derived couple, we have

$$\mathrm{d}_2[b]_2 = j_2 k_2 [b]_2 = j_2[k_1[b]_1]_2$$

In order to compute $j_2[k_1[b]_1]_2$, we need to find $a \in K$ such that $k_1[b]_1 = i[a]_1$, then $j_2[k_1[b]_1]_2 = [j_1 a]_2$. Since $k_1[b]_1 \in A^{k+1} \cap K_{p+1}$, we have $a \in A^{k+1} \cap K_{p+2}$.

To find such a we use not b but $b + c$ in $A^k \cap K_p$ to represent $[b]_1$, that's possible since b and $b + c$ have the same image under the projection $K_p \to$

[1] It's clear the choice isn't unique, any element taking form $b + c$, where $c \in A^k \cap K_{p+1}$ also can represent $b + K_{p+1}$.

K_p/K_{p+1}, since $c \in A^k \cap K_{p+1}$. Then

$$k_1[b]_1 = D(b+c) = \delta b + Dc = \delta b + \delta c + D''c = i(\delta c) \in A^{k+1} \cap K_{p+1}$$

where $\delta c \in A^{k+1} \cap K_{p+2}$. So

$$\mathrm{d}_2[b]_2 = [\delta c]_2$$

Thus differential d_2 is given by the delta of the tail of the zig-zag which extends b. By characterization of E_2, you can regard it as an element in $H_\delta H_\mathrm{d}(K)$. Now let's check well-defineness:

1. $\delta c \in H_\delta H_\mathrm{d}(K)$: $\delta(\delta c) = 0$ is clear; $\mathrm{d}\delta c = \delta \mathrm{d}c = (-1)^p \delta\delta b = 0$, since $(-1)^p \mathrm{d}c = \delta b$.
2. $\mathrm{d}_2[b]_2$ is independent of the choice of c: Any two possible c and c' differs something lies in $\ker \mathrm{d}$. Assume $c' = c + x$ where $x \in \ker \mathrm{d}$, then it suffices to show $[\delta x]_2 = 0$, and that's tautological.

Remark 3.4 (**characterization of elements in** E_3). For an element of $[b]_3 \in E_3$, it can be represented by an element $b \in K$ with a zig-zag of length 3

$$\begin{tikzcd}
0 & & \\
b \arrow[u, "\mathrm{d}"] \arrow[r, "\delta"] & \delta b & \\
& c_1 \arrow[u, "D''"] \arrow[r, "\delta"] & \delta c_1 \\
& & c_2 \arrow[u, "D''"]
\end{tikzcd}$$

Notation 3.2. We say that an element b in K lives to E_r if it represents a cohomology class in E_r, or equivalently, b is a cocycle in $E_1, E_2, \ldots, E_{r-1}$. And we already see there is a zig-zag description for d_1 and d_2.

Remark 3.5 (**characterization of elements in** E_r). Generally, an element $b \in K$ lives to E_r if it can be extended to a zig-zag of length r

$$\begin{tikzcd}
0 & & \\
b \arrow[u, "\mathrm{d}"] \arrow[r, "\delta"] & \delta b & \\
& c_1 \arrow[u, "D''"] \arrow[dr, "\cdots"] & \\
& & c_{r-2} \arrow[r, "\delta"] & \delta c_{r-2} \\
& & & c_{r-1} \arrow[u, "D''"]
\end{tikzcd}$$

The differential d_r on E_r is given by δ of the tail of zig-zag:

$$\mathrm{d}_r[b]_r = [\delta c_{r-1}]_r$$

Thus the bidegrees (p,q) of the double complex persist in the spectral sequence

$$E_r = \bigoplus_{p,q} E_r^{p,q}$$

and d_r shifts the bidegrees by $(r, -r+1)$.

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

The filtration on $H(K)$

$$H(K) = F_0 \supset F_1 \supset F_2 \supset \dots$$

induces a filteration on each component $H^n(K)$ as follows

$$H^n(K) = (F_0 \cap \underbrace{H^n) \supset (F_1}_{E_\infty^{0,n}} \cap \underbrace{H^n) \supset (F_2}_{E_\infty^{1,n-1}} \cap H^n) \supset \dots \supset (F_n \cap \underbrace{H^n) \supset 0}_{E_\infty^{n,0}}$$

In a summary, we have proven the following refinement:

Theorem 3.1. Given a double complex $K = \bigoplus K^{p,q}$ there is a spectral sequence $\{E_r, \mathrm{d}_r\}$ converging to the total cohomology $H_D(K)$ such that E_r has a bigrading with

$$\mathrm{d}_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

and

$$\begin{aligned} E_1^{p,q} &= H_{\mathrm{d}}^{p,q}(K) \\ E_2^{p,q} &= H_\delta^{p,q} H_{\mathrm{d}}(K) \end{aligned}$$

Furthermore, the associated graded complex of the total cohomology is given by

$$GH_D^n(K) = \bigoplus_{p+q=n} E_\infty^{p,q}(K)$$

Remark 3.6. There is another filtration, that is $K_q = \bigoplus_{j\geq q, p\geq 0} K^{p,j}$. This gives a second spectral sequence $\{E'_r, \mathrm{d}'_r\}$ converging to the total cohomology $H_D(K)$, but with

$$\begin{aligned} E'_1 &= H_\delta(K) \\ E'_2 &= H_{\mathrm{d}} H_\delta(K) \end{aligned}$$

and

$$\mathrm{d}'_r : E_r'^{p,q} \to E_r'^{p-r+1,q+r}$$

Example 3.1 (Revisit generalized Mayer-Vietoris principle)**.** Given a smooth manifold M and an open covering $\mathfrak{U}$ of it, consider double complex $C^*(\mathfrak{U}, \Omega^*)$, then there is only one column in E'_1-page, therefore the E'_2-page degenrates, which implies generalized Mayer-Vietoris principle. Furthermore, if we take good cover, the E_2-page also degenrates, which implies

$$H_{dR}^*(M) \cong H^*(\mathfrak{U}, \mathbb{R})$$

3.3. **Extension problem.** Since the dimension is the only invariant of a vector space, the associated graded vector space GV of a filtered vector-space V is isomorphic to V itself. In particular, if a double complex K is a vector space, then

$$H_D^n(K) \cong GH_D^n(K) \cong \bigoplus_{p+q=n} E_\infty^{p,q}$$

However, the same thing fails in the realm of abelian groups. For example: the two group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_4$ filtered by

$$\mathbb{Z}_2 \subset \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

and

$$\mathbb{Z}_2 \subset \mathbb{Z}_4$$

have isomorphic associated graded groups, but $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_4$. In other words, in a short exact sequence of abelian groups

$$0 \to A \to B \to C \to 0$$

A and C do not determine B uniquely. The ambiguity is called the extension problem.

Proposition 3.1. In a short exact sequence of abelian groups

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

if C is free, then there exists a homomorphism $s : C \to B$ such that $g \circ s$ is identity on C.

Proof. Since C is free, then it suffices to define a suitable s on the generators $\{c_i\}$ of C and it automatically extends to C linearly. Take c_i and choose any preimage of c_i, denoted by b_i, then s is defined by $c_i \mapsto b_i$. Clearly $s \circ g$ is identity on C, but note that such s is not unique. □

Corollary 3.1. Under the hypothesis of the proposition,

1. The map $(f, s) : A \oplus C \to B$ is an isomorphism;
2. For any abelian group G the induced sequence

   $$0 \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G) \to 0$$

   is exact;
3. For any abelian group G the sequence

   $$0 \to A \otimes G \to B \otimes G \to C \otimes G \to 0$$

   is exact.

Proof. For (1). Since (f, s) is a group homomorphism, it suffices to check it's both injective and surjective. It's easy to see (f, s) is injective, since f and s are injective; For $b \in B$, if $b \in \operatorname{im} f$, that is $b = f(a)$ for some $a \in A$, then $(a, 0)$ is mapped to b. If $b \notin \operatorname{im} f = \ker g$, then consider $g(b) \in C$. Although $sg(b)$ may not equal to b, we have $sg(b) - b \in \ker g = \operatorname{im} f$, so there exists $a \in A$ such that $f(a) + sg(b) = b$, this completes the proof of surjectivity.

For (2). Since it's known to all $\mathrm{Hom}(-, G)$ is a left exact functor, then it suffices to show $\mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$ is surjective. Take any $k : A \to G$, then consider the composition of following maps

$$B \xrightarrow{(f,s)^{-1}} A \oplus C \xrightarrow{p_1} A \xrightarrow{k} G$$

it's a map in $\mathrm{Hom}(B, G)$ such that it extends k.

For (3). Since it's known to all $-\otimes G$ is a right exact functor, then it suffices to show $A \otimes G \to B \otimes G$ is injective, and the proof is quite similar as above. $\square$

Remark 3.7. If you are quite familiar with homological algebra, you will know that:

1. The failure of $\mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$ to be exact is measured by $\mathrm{Ext}(C, G)$, and it's zero by the property of Ext, since C is a free abelian group;
2. The failure of $A \otimes G \to B \otimes G$ to be injective is measured by $\mathrm{Tor}(C, G)$, and it's zero for the same reason.

Part 2. **Applications in cohomology theory**

4. Leray spectral sequence

Now let's focus on a special spectral sequence we're concerned about, that is Leray spectral sequence.

4.1. **Basic setting.** Let $\pi : E \to M$ be a fiber bundle with fiber F over a manifold M. Given a good cover $\mathfrak{U}$ of M, $\pi^{-1}\mathfrak{U}$ is a cover on E and we can form the double complex

$$K = C^*(\pi^{-1}\mathfrak{U}, \Omega^*)$$

with E_1-page and E_2-page as follows

$$\begin{aligned} E_1^{p,q} &= H_{\mathrm{d}}^{p,q}(K) = \prod_{\alpha_0 < \cdots < \alpha_p} H^q(\pi^{-1}U_{\alpha_0 \dots \alpha_p}) = C^p(\mathfrak{U}, \mathscr{H}^q) \\ E_2^{p,q} &= H_\delta^p(\mathfrak{U}, \mathscr{H}^q) \end{aligned}$$

where $\mathscr{H}^q$ is the presheaf $U \mapsto H^q(\pi^{-1}U)$ on M. By theorem 3.1 we have the spectral sequence of K converges to $H_D^*(K)$, which is equal to $H^*(E)$ by generalized Mayer-Vietoris principle, since $\pi^{-1}\mathfrak{U}$ is a cover of E.

$\mathscr{H}^q$ is a locally constant sheaf, since $\mathfrak{U}$ is a good cover, then. So if M is simply connected, then there is no monodromy, that is $\mathscr{H}^q$ is a constant sheaf $\underbrace{\mathbb{R} \oplus \cdots \oplus \mathbb{R}}_{\dim H^q(F)}$, thus

$$E_2^{p,q} = H^p(M) \otimes H^q(F)$$

Example 4.1 (Orientability and the Euler class of sphere bundle)**.** Let $\pi : E \to M$ be a S^n-bundle over a manifold M and let $\mathfrak{U}$ be a good cover of M. Then the E_2-page of Leray spectral sequence is

$$E_2^{p,q} = H^p(\mathfrak{U}, \mathscr{H}^q(S^n))$$

However, since only n-th and 0-th cohomology of S^n don't vanish, so there are only two non-zero rows in E_2-page, thus $\mathrm{d}_2 = \cdots = \mathrm{d}_{n-1} = 0$, that is

$$E_n = E_2 = H_\delta H_{\mathrm{d}}(K) = H^*(\mathfrak{U}, \mathscr{H}^*(S^n))$$

Let $\sigma \in E_1^{0,n}$ be the local angular forms on the sphere bundle E, it's clear that $\mathrm{d}_1\sigma = 0$ if and only if E is orientable. So if E is orientable, σ lives to E_2, and it lives to E_n.

Up to a sign $\mathrm{d}_n\sigma \in H^{n+1}(\mathfrak{U}, \mathscr{H}^0(S^n)) \cong H^{n+1}(M)$, so whether σ lives to $E_{n+1} = \cdots = E_\infty = H^*(E)$ or not depends on $\mathrm{d}_n\sigma = 0 \in H^{n+1}(M)$ or not, that is there is a global angular form on E if and only if the Euler class $e(E)$ of E vanishes.

Example 4.2 (Orientability of simply-connected manifold)**.** Let M be a simply-connected manifold of dimension n and $S(T_M)$ is the S^{n-1}-sphere bundle of its tangent bundle. $H^1(M) = 0$ since M is simply-connected, thus let $\sigma \in E_1^{0,n-1}$ be the local angular forms on $S(T_M)$, we must have $\mathrm{d}_1\sigma = 0$, since $E_2^{1,n-1} = H^1(M) \otimes H^{n-1}(S^{n-1})$, thus $S(T_M)$ is orientable, that is T_M is orientable, which implies M is orientable.

Example 4.3 (The cohomology of $\mathbb{CP}^2$)**.** Consider Hopf fiberation of $\mathbb{CP}^2$, that is

$$\begin{tikzcd} S^1 \arrow[r] & S^5 \arrow[d] \\ & \mathbb{CP}^2 \end{tikzcd}$$

Since $\mathbb{CP}^2$ is simply-connected, thus

$$E_2^{p,q} = H^p(\mathbb{CP}^2) \otimes H^q(S^1)$$

that is E_2-page looks like

$$\begin{tikzcd} \mathbb{R} \arrow[drr] & A \arrow[drr] & B \arrow[drr] & C \arrow[drr] & D & 0 \\ \mathbb{R} & A & B & C & D & 0 \end{tikzcd}$$

Since d_3 moves down two steps, then $\mathrm{d}_3 = 0$, similarly for $\mathrm{d}_4 = \cdots = 0$. So the spectral sequence degenerates at the E_3 page and $E_3 = E_\infty = H^*(S^5)$, that is E_3 page looks like

$$\begin{array}{cccccc} 0 & 0 & 0 & 0 & \mathbb{R} & 0 \\ \\ \mathbb{R} & 0 & 0 & 0 & 0 & 0 \end{array}$$

This means

$$0 \to A, \quad \mathbb{R} \to B, \quad A \to C, \quad B \to D, \quad C \to 0$$

are isomorphisms. Thus

$$H^k(\mathbb{CP}^2) = \begin{cases} \mathbb{R} & k = 0, 2, 4 \\ 0 & \text{otherwise} \end{cases}$$

Remark 4.1. By same argument you can compute cohomology of $\mathbb{CP}^n$.

4.2. **Product structure.** If a double complex K has a product structure relative to which its differential D is an antiderivation, the same is true of all the groups E_r and their operator d_r, since E_r is the homology of E_{r-1} and d_r is induced from D. With product structures, we have

Theorem 4.1. Let K be a double complex with a product structure relative to which D is an antiderivation. There exists a spectral sequence

$$\{E_r, \mathrm{d}_r : E_r^{p,q} \to E_r^{p+r, q-r+1}\}$$

converging to $H_D(K)$ with the following properties:

1. The $E_2^{p,q}$ term is $H_\delta^{p,q} H_{\mathrm{d}}(K)$;
2. Each E_r, being the homology of E_{r-1}, inherits a product structure from E_{r-1}. Relative to this product, d_r is an antiderivation.

Remark 4.2. Although both E_∞ and $H_D(K)$ inherit their ring structure from K, they're generally not isomorphic as rings.

However, things are not tooo bad. If we consider Leray spectral sequence to fiber bundle (E, M, F), and equip the double complex $C^*(\pi^{-1}\mathfrak{U}, \Omega^*)$ with the following product structure

$$\begin{aligned} \cup : C^p(\pi^{-1}\mathfrak{U}, \Omega^q) \otimes C^r(\pi^{-1}\mathfrak{U}, \Omega^s) &\to C^{p+r}(\pi^{-1}\mathfrak{U}, \Omega^{q+s}) \\ \omega \otimes \eta &\mapsto \omega \cup \eta \end{aligned}$$

where

$$\omega \cup \eta(\pi^{-1}U_{\alpha_0 \ldots \alpha_{p+r}}) := (-1)^{qr} \omega(\pi^{-1}U_{\alpha_0 \ldots \alpha_p}) \wedge \eta(\pi^{-1}U_{\alpha_p \ldots \alpha_{p+r}})$$

Remark 4.3. Here we need sign $(-1)^{qr}$ to make the differential operator D into an antiderivation with respect to this product, that is[2]

$$D(\omega \cup \eta) = D\omega \cup \eta + (-1)^{\deg \omega} \omega \cup D\eta$$

If M is simply-connected, then E_2-page of Leray spectral sequence is isomorphic to $H^p(M) \otimes H^q(F)$. If we equip $H^p(M) \otimes H^q(F)$ with the following product structure

$$(a \otimes b)(c \otimes d) := (-1)^{\deg b \deg c}(ac \otimes bd)$$

Then $H_\delta^p(\mathfrak{U}, \mathscr{H}^q)$ is isomorphic[3] to $H^p(M) \otimes H^q(F)$ as rings.

[2] You can directly check this fact by yourself, or refer to Hatcher for a proof.

[3] In fact, it's almost clear from the definition: You can regard an element in $H_\delta^p(\mathfrak{U}, \mathscr{H}^q)$ as two parts, one eats an intersection of $(p+1)$-fold, and the other outputs a q-form, that's how you get this isomorphism.

Example 4.4 (cohomology ring of $\mathbb{CP}^2$)**.** Consider E_2-page

$$\begin{tikzcd}
\mathbb{R} \arrow[rrd, "\mathrm{d}_2"] & 0 & \mathbb{R} \arrow[rrd, "\mathrm{d}_2"] & 0 & \mathbb{R} & 0 \\
\mathbb{R} & 0 & \mathbb{R} & 0 & \mathbb{R} & 0
\end{tikzcd}$$

where two d_2 are isomorphisms. Let a be a generator of $H^1(S^1)$, then

$$\mathrm{d}_2(1 \otimes a) = 1 \otimes x$$

is a generator of

$$E_2^{2,0} = H^2(\mathbb{CP}^2) \otimes H^0(S^1)$$

where x is a generator of $H^2(\mathbb{CP}^2)$. Then $x \otimes a$ is a generator of

$$E_2^{2,1} = H^2(\mathbb{CP}^2) \otimes H^1(S^1)$$

Thus a generator of $E_2^{4,0} = H^4(\mathbb{CP}^2)$ is given by

$$\begin{aligned}
\mathrm{d}_2(x \otimes a) &= \mathrm{d}_2(x \otimes 1) \cdot (1 \otimes a) + (-1)^2 (x \otimes 1) \cdot \mathrm{d}_2(1 \otimes a) \\
&= (1 \otimes x)(1 \otimes x) \\
&= (1 \otimes x^2)
\end{aligned}$$

which implies x^2 is a generator of $H^4(\mathbb{CP}^2)$. So as a ring,

$$H^*(\mathbb{CP}^2) = \mathbb{R}[x]/(x^3)$$

where $|x| = 2$.

Remark 4.4. The same argument shows

$$H^*(\mathbb{CP}^n) = \mathbb{R}[x]/(x^{n+1})$$

where $|x| = 2$.

4.3. **Gysin sequence.** In special cases the spectral sequence simplifies to a long exact sequence. One special case is the cohomology of sphere bundle and the resulting sequence is called Gysin sequence.

Let $\pi : E \to M$ be an oriented sphere bundle with fiber S^k. By assumption of orientability, there is no monodromy of locally constant sheaf $\mathscr{H}^k$, thus the E_2-page of Leray spectral sequence is $H^p(M) \otimes H^q(S^k)$. Note that for arbitrary integer $n \geq k$, nothing in $E_2^{n-k,k}$ can be killed, thus there is an exact sequence

$$0 \to E_\infty^{n-k,k} \to E_2^{n-k,k}$$

and it can be extended to the following exact sequence

$$0 \to E_\infty^{n-k,k} \to E_2^{n-k,k} \xrightarrow{\mathrm{d}_{k+1}} E_2^{n+1,0} \to E_\infty^{n+1,0} \to 0$$

since d_{k+1} is the only possible non-trivial map.

On the other hand, the filtration on $H^n(E)$ becomes

$$\underbrace{H^n(E) \supset E_\infty^{n,0}}_{E_\infty^{n-k,k}} \supset 0$$

which gives another exact sequence

$$0 \to E_\infty^{n,0} \to H^n(E) \to E_\infty^{n-k,k} \to 0$$

Fit these two exact sequence together one has

$$\cdots \to H^n(E) \to H^{n-k}(M) \to H^{n+1}(M) \to H^{n+1}(E) \to \dots$$

To be explicit, you can find the map $H^{n-k}(M) \to H^{n+1}(M)$ is to wedge the Euler class of E.

4.4. **Other coefficients.** Since the de Rham cohomology is a cohomology theory with real coefficients, it's necessarily overlooks the torsion phenomena. In this section we give a quick review of singular (co)homology, and show that the preceding results on spectral sequences carry over to integer coefficients.

4.4.1. *Review of singular (co)homology.* In this section X is a topological space.

Definition 4.1 (singular q-simplex)**.** A singular q-simplex in X is a continous map $s : \Delta_q \to X$, where Δ_q is standard q-simplex.

Definition 4.2 (singular q-chain with $\mathbb{Z}$-coefficient)**.** A singular q-chain in X is a finite linear combination with integer coefficients of singular q-simplices.

Notation 4.1. All singular q-chains form an abelian group, denoted by $S_q(X; \mathbb{Z})$.

Definition 4.3 (boundary map)**.** The boundary map ∂ is defined as follows

$$\begin{aligned} \partial_q : S_n(X; \mathbb{Z}) &\to S_{q-1}(X; \mathbb{Z}) \\ \sigma &\mapsto \sum_i (-1)^i \sigma|[v_0, \dots, \widehat{v_i}, \dots, v_q] \end{aligned}$$

where we identify $[v_0, \dots, \widehat{v_i}, \dots, v_q]$ with Δ^{q-1}.

Definition 4.4 (singular homology group $\mathbb{Z}$-coefficient)**.** The q-th singular homology group $H_q(X; \mathbb{Z})$ is defined as

$$H_q(X; \mathbb{Z}) := \ker \partial_q / \operatorname{im} \partial_{q+1}$$

Lemma 4.1 (Poincaré lemma)**.** $H_q(\mathbb{R}^n; \mathbb{Z}) = 0$ for all $q > 0$.

Definition 4.5 (singular q-cochain with $\mathbb{Z}$-coefficient)**.** The group of singular q-cochains is defined as

$$S^q(X; \mathbb{Z}) := \operatorname{Hom}(S_q(X; \mathbb{Z}), \mathbb{Z})$$

with coboundary map d_q defined by

$$(\mathrm{d}_q \omega)(c) = \omega(\partial_{q+1} c)$$

where $\omega \in S^q(X), c \in S_q(X)$.

Definition 4.6 (singular cohomology group with $\mathbb{Z}$-coefficient)**.** The q-th singular cohomology group $H^q(X;\mathbb{Z})$ is defined as

$$H^q(X;\mathbb{Z}) := \ker \mathrm{d}_q / \operatorname{im} \mathrm{d}_{q-1}$$

Remark 4.5. Replacing $\mathbb{Z}$ with any arbitrary abelian group G, you can define singular (co)homology group with coefficients G.

Proposition 4.1. Given an open covering of X, the following sequence is exact

$$0 \leftarrow S_q^{\mathfrak{U}}(X;G) \leftarrow \bigoplus_{\alpha_0} S_q(U_{\alpha_0};G) \leftarrow \bigoplus_{\alpha_0<\alpha_1} S_q(U_{\alpha_0\alpha_1};G) \leftarrow \dots$$

where G is an arbitrary abelian group G and $S_q^{\mathfrak{U}}(X,G)$ is the group of $\mathfrak{U}$-small singular q-chain. Furthermore, there is a chain homotopy between $S_q(X;G)$ and $S_q^{\mathfrak{U}}(X;G)$.

Corollary 4.1. Given an open covering of X, the following sequence is exact

$$0 \to S^q_{\mathfrak{U}}(X;G) \to \bigoplus_{\alpha_0} S^q(U_{\alpha_0};G) \to \bigoplus_{\alpha_0<\alpha_1} S^q(U_{\alpha_0\alpha_1};G) \to \dots$$

where G is an arbitrary abelian group G and $S_q^{\mathfrak{U}}(X,G)$ is the group of $\mathfrak{U}$-small singular q-chain.

Theorem 4.2 (de Rham theorem)**.** The singular cohomology with coefficients $\mathbb{R}$ is isomorphic to de Rham cohomology on smooth manifold.

Proof. Consider the double complex $C^*(\mathfrak{U}, S^*(\mathfrak{U};\mathbb{R}))$, we can show Čech cohomology of constant sheaf $\mathbb{R}$ is isomorphic to singular cohomology with coefficients $\mathbb{R}$, and we also know Čech cohomology of constant sheaf $\mathbb{R}$ is isomorphic to de Rham cohomology. □

Remark 4.6. In fact, for a topological space X with good cover is cofinal, we can show Čech cohomology of constant sheaf G is isomorphic to singular cohomology with coefficients G.

Theorem 4.3 (Leray spectral sequence for singular cohomology with coefficients in a communicative ring A)**.** Let $\pi : E \to X$ be a fiber bundle with fiber F over a topological space X and $\mathfrak{U}$ an open covering of X. There is a spectral sequence converging to $H^*(E;A)$ with E_2-term

$$E_2^{p,q} = H^p(\mathfrak{U}, \mathscr{H}^q(F;A))$$

Each E_r in the spectral sequence can be given a product structure relative to which the differential d_r is an antiderivation. If X is simply-connected and has a good cover, then

$$E_2^{p,q} = H^p(X, H^q(F;A))$$

Furthermore, if $H^*(F;A)$ is a finitely generated free A-module, then

$$E_2 = H^*(X;A) \otimes H^*(F;A)$$

as algebras over A.

Remark 4.7. Of course there is Leray spectral sequence for singular homology with coefficients, just reverse arrows in above case, here we omit the statement of it.

5. Cohomology of some Lie groups

A crucial fact is that if G is a Lie group and H is a closed subgroup of G, then there exists the following fiberation

$$\begin{tikzcd} H \arrow[r] & G \arrow[d] \\ & G/H \end{tikzcd}$$

If we're familiar with G/H and H, then above fiberation is a good way to compute cohomology ring of G. In fact, we always use the view of group action to give an explicit description of G/H.

5.1. Cohomology rings of $\mathrm{U}(n)$ and $\mathrm{SU}(n)$.

Note that $\mathrm{U}(n)$ acts on S^{2n-1} with stablizer $\mathrm{U}(n-1)$, that is $\mathrm{U}(n)/\mathrm{U}(n-1) = S^{2n-1}$, thus we have the following fiberation:

$$\begin{tikzcd} \mathrm{U}(n-1) \arrow[r] & \mathrm{U}(n) \arrow[d] \\ & S^{2n-1} \end{tikzcd}$$

The same fiberation still holds if we replace $\mathrm{U}(n)$ by $\mathrm{SU}(n)$.

Proposition 5.1. The cohomology ring of $\mathrm{U}(n)$ is $\Lambda[x_1, \dots, x_{2n-1}]$, where $|x_i| = i, 1 \leq i \leq 2n-1$.

Proof. Note that $\mathrm{U}(1) = S^1$, thus cohomology ring of $\mathrm{U}(1)$ is $\Lambda[x_1]$, where $|x_1| = 1$. Apply Leray spectral sequence fiberation

$$\begin{tikzcd} \mathrm{U}(n-1) \arrow[r] & \mathrm{U}(n) \arrow[d] \\ & S^{2n-1} \end{tikzcd}$$

we have E_2-page has only two columns, that is $p = 0$ and $p = 2n-1$. Furthermore by induction we have cohomology ring of $\mathrm{U}(n-1)$ is $\Lambda[x_1, \dots, x_{2n-3}]$, where $|x_i| = i, 1 \leq i \leq 2n-3$. Although there may toooo many non-zero rows of E_2-page, but it suffices to check d_2 on those generators, that is the ones on $p = 0, q = 0, 1, 3, \dots, 2n-3$.

By dimension reasons, it's clear this spectral sequence degenerates at E_2-page, which implies cohomology group structure of $\mathrm{U}(n)$ is clear. If we choose a generator of $E_2^{2n-1,0}$, denoted by x_{2n-1}, then we can write the generator of $E_2^{2n-1,i}$ through product $E_2^{0,i} \times E_2^{2n-1,0} \to E_2^{2n-1,i}$. This show cohomology ring of $\mathrm{U}(n)$ is exactly $\Lambda[x_1, \dots, x_{2n-1}]$. $\square$

Proposition 5.2. The cohomology ring of $\mathrm{SU}(n)$ is $\Lambda[x_3, \dots, x_{2n-1}]$, where $n \geq 2, |x_i| = i, 1 \leq i \leq 2n-1$.

Proof. Note that $\mathrm{SU}(2) = S^3$, thus cohomology ring of $\mathrm{SU}(2)$ is $\Lambda[x_3]$, where $|x_3| = 3$. Apply Leray spectral sequence fiberation

$$\begin{tikzcd} \mathrm{SU}(n-1) \arrow[r] & \mathrm{SU}(n) \arrow[d] \\ & S^{2n-1} \end{tikzcd}$$

The same argument shows the desired result. $\square$

5.2. **Cohomology of** $\mathrm{SO}(4)$**.** In this section we need to following fact.

Proposition 5.2.1. For a compact orientable manifold M, the integral $\int_M e(TM)$ is equal to the Euler number of it, that is $\sum(-1)^q H^q(M)$.

Example 5.1 (The cohomology of the unit tangent bundle of a sphere)**.** The unit tangent bundle $S(T_{S^2})$ to the S^2 is a fiber bundle with fiber S^1, that is

$$\begin{tikzcd} S^1 \arrow[r] & S(T_{S^{n-1}}) \arrow[d] \\ & S^2 \end{tikzcd}$$

The E_2-page of the Leray spectral sequence is $E_2^{p,q} = H^p(S^2) \otimes H^q(S^1)$, that is

$$\begin{tikzcd} \mathbb{Z} \arrow[drr, "\mathrm{d}_2"] & 0 & \mathbb{Z} \\ \mathbb{Z} & 0 & \mathbb{Z} \end{tikzcd}$$

In order to compute E_3, it suffices to compute above $\mathrm{d}_2 : E_2^{0,1} \to E_2^{2,0}$, and we know it defines the Euler class of $S(T_{S^2})$. By Proposition 5.2.1, we have Euler class of $S(T_{S^2})$ is twice the generator of $H^2(S^2)$, then d_2 is multiplication by 2. So E_3-page is

$$\begin{matrix} 0 & 0 & \mathbb{Z} \\ & & \\ \mathbb{Z} & 0 & \mathbb{Z}_2 \end{matrix}$$

For dimension reasons $\mathrm{d}_3 = \mathrm{d}_4 = \cdots = 0$, so $E_3 = E_\infty$. thus

$$H^k(S(T_{S^2})) = \begin{cases} \mathbb{Z} & k = 0, 3 \\ \mathbb{Z}_2 & k = 2 \\ 0 & \text{otherwise} \end{cases}$$

Remark 5.1. A point in $S(T_{S^2})$ is specified by a unit vector in $\mathbb{R}^3$ and another unit vector orthogonal to it, which can be completed to a unique orthnormal basis with positive determinant. Therefore $S(T_{S^2}) \cong \mathrm{SO}(3)$ and we have computed the cohomology of $\mathrm{SO}(3)$.

Remark 5.2. In fact, $\mathrm{SO}(3)$ comes in a different guise as $\mathbb{RP}^3$.

Example 5.2 (The cohomology of $\mathrm{SO}(4)$)**.** The $\mathrm{SO}(n)$ acts on S^{n-1} transitively with stablizer $\mathrm{SO}(n-1)$. Therefore $\mathrm{SO}(n)/\mathrm{SO}(n-1) = S^{n-1}$. Thus we can use Leray spectral sequence to

$$\begin{tikzcd} \mathrm{SO}(3) \arrow[r] & \mathrm{SO}(4) \arrow[d] \\ & S^3 \end{tikzcd}$$

The E_2-page is

$$\begin{array}{cccc} \mathbb{Z} & 0 & 0 & \mathbb{Z} \\ \mathbb{Z}_2 & 0 & 0 & \mathbb{Z}_2 \\ 0 & 0 & 0 & 0 \\ \mathbb{Z} & 0 & 0 & \mathbb{Z} \end{array}$$

It's easy to see $\mathrm{d}_2 = \mathrm{d}_3 = \cdots = 0$, which implies the cohomology of $\mathrm{SO}(4)$ is

$$H^k(\mathrm{SO}(4)) = \begin{cases} \mathbb{Z} & k = 0, 6 \\ \mathbb{Z}_2 & k = 2, 5 \\ \mathbb{Z} \oplus \mathbb{Z} & k = 3 \\ 0 & \text{otherwise} \end{cases}$$

since there is no extension problem.

Example 5.3. Consider a manifold

$$W := \mathrm{SU}(3)/\mathrm{SO}(3)$$

where $\mathrm{SO}(3)$ is embedded as a closed subgroup of $\mathrm{SU}(3)$. Hence there is a fiber bundle:

$$\begin{tikzcd} \mathrm{SO}(3) \arrow[r] & \mathrm{SU}(3) \arrow[d] \\ & W \end{tikzcd}$$

It's clear that W is simply-connected, and we know $\mathrm{SO}(3)$ is diffeomorphic to $\mathbb{RP}^3$. If we consider $\mathbb{Z}_2$ coefficient, then the cohomology ring of $\mathbb{RP}^3$ is $\mathbb{Z}_2[x]/(x^4)$ and the cohomology ring of $\mathrm{SU}(3)$ is $\Lambda[x_3, x_5]$, where $|x_i| = i$. The E_2-page is

$$\begin{tikzcd}
\mathbb{Z}_2 & A & B & C \\
\mathbb{Z}_2 \arrow[dr, "\mathrm{d}_2"] & A & B & C \\
\mathbb{Z}_2 \arrow[dr, "\mathrm{d}_2"] & A & B \arrow[dr, "\mathrm{d}_2"] & C \\
\mathbb{Z}_2 & A & B & C
\end{tikzcd}$$

where $A = H^2(W), B = H^3(W), C = H^5(W)$, since W is simply-connected, then $H^1(W) = H^4(W) = 0$, by Poincaré duality.

In E_2-page, we have the following observations:

1. $\mathrm{d}_2 : E_2^{0,1} \to E_2^{2,0}$ is isomorphism, since if d_2 is not isomorphism, and by dimension reason $\mathrm{d}_3 : E_3^{0,1} = 0$, which implies $E_\infty^{0,1} \neq 0$, a contradiction to $H^1(\mathrm{SU}(3)) = 0$.
2. $\mathrm{d}_2 : E_2^{3,1} \to E_2^{5,0}$ is isomorphism by the same reason.
3. $\mathrm{d}_2 : E_2^{0,2} \to E_2^{2,1}$ is zero. Indeed, if $a \in H^1(\mathbb{RP}^3)$ is a generator, then a^2 is a generator of $H^2(\mathbb{RP}^3)$, but
$$\mathrm{d}_2((1 \otimes a^2)) = 0$$
since we consider $\mathbb{Z}_2$ coefficient. In particular, we have this spectral doesn't converge in E_2-page.

In E_3-page,

$$\begin{tikzcd}
\mathbb{Z}_2 \arrow[ddrr, "\mathrm{d}_3"] & & & \\
0 & \mathbb{Z}_2 & 0 & \\
\mathbb{Z}_2 & 0 & B & 0
\end{tikzcd}$$

and it's clear $\mathrm{d}_3 : E_3^{0,2} \to E_3^{3,0}$ is an isomorphism, thus[4] $B = \mathbb{Z}_2$. Untill now we have computed the cohomology group of W.

Now let's consider the cohomology ring structure of W.

1. Since $\mathrm{d}_2 : E_2^{0,1} \to E_2^{2,0}$ is an isomorphism, then we can use $x_2 \otimes 1$ to denote $\mathrm{d}_2(1 \otimes a)$, then x_2 is a generator of $H^2(W)$;
2. Similarly we use $x_3 \otimes 1$ to denote $\mathrm{d}_3(1 \otimes a^2)$, and x_3 is also a generator of $H^3(W)$ by the same reason.

[4]In fact you can directly use Poincaré duality to conclude $B = C = \mathbb{Z}_2$.

3. Since $\mathrm{d}_2 : E_2^{3,1} \to E_2^{5,0}$ is an isomorphism, and $x_3 \otimes a$ is a generator of $E_2^{3,1}$, then
$$\begin{aligned} \mathrm{d}_3(x_3 \otimes a) &= \mathrm{d}_2(x_3 \otimes 1) \cdot (1 \otimes a) + (-1)^3 (x_3 \otimes 1) \cdot \mathrm{d}_3(1 \otimes a) \\ &= (x_2 x_3 \otimes 1) \end{aligned}$$
which implies $x_2 x_3$ is a generator of $H^5(W)$.

All in all, the cohomology ring of W is $\Lambda[x_2, x_3]$.

5.3. **A glimpse of characteristice class.**

Definition 5.1 (classification space)**.** Let G be a Lie group, a space BG is called a classification space for G if there is a natural isomorphism
$$\{\text{Isomorphism classes of } G\text{-principle bundles over } X\} \Longleftrightarrow [X, BG]$$

Example 5.4 (Narasimhan)**.** $B\,\mathrm{U}(n)$ is infinite Grassmannian $G_n(\mathbb{C}^\infty)$.

Proposition 5.3. The cohomology ring of $B\,\mathrm{U}(n)$ with integer coefficients is $\mathbb{Z}[c_1, \ldots, c_n]$.

Proof. The functoriality of the universal bundle yields that for any subgroup $H < G$, there is a filteration
$$\begin{tikzcd} G/H \arrow[r] & BG \arrow[d] \\ & BH \end{tikzcd}$$
In particular, if we consider $\mathrm{U}(n-1)$ as a subgroup of $\mathrm{U}(n-1)$, then we have the following filteration
$$\begin{tikzcd} S^{2n-1} \cong \mathrm{U}(n)/\,\mathrm{U}(n-1) \arrow[r] & B\,\mathrm{U}(n) \arrow[d] \\ & B\,\mathrm{U}(n-1) \end{tikzcd}$$
Apply Leray spectral sequence this this fiberation and use the fact that the cohomology ring of $\mathbb{CP}^\infty$ is $\mathbb{Z}[c_1]$ to conclude. □

Definition 5.2. The generators $c_1, \cdots, c_n$ of $H^*(B\,\mathrm{U}(n); \mathbb{Z})$ are called the universal Chern classes of $\mathrm{U}(n)$-bundles.

Definition 5.3. The i-th Chern class of the $\mathrm{U}(n)$-bundle $\pi : E \to X$ with classifying map $f_\pi : X \to B\,\mathrm{U}(n)$ is defined as
$$c_i(\pi) := f_\pi^*(c_i) \in H^{2i}(X; \mathbb{Z})$$

Remark 5.3. Note that if π is a $\mathrm{U}(n)$-bundle, then by definition we have that $c_i(\pi) = 0$, if $i > n$.

Definition 5.4. The total Chern class of a $\mathrm{U}(n)$-bundle $\pi : E \to X$ is defined by

$$c(\pi) = c_0(\pi) + c_1(\pi) + \cdots c_n(\pi) = 1 + c_1(\pi) + \cdots c_n(\pi) \in H^*(X; \mathbb{Z}),$$

as an element in the cohomology ring of the base space.

Proposition 5.4 (Functoriality of Chern classes)**.** If $f : Y \to X$ is a continuous map, and $\pi : E \to X$ is a $\mathrm{U}(n)$-bundle, then $c_i(f^*\pi) = f^* c_i(\pi)$, for any i.

Proof. We have a commutative diagram

$$\begin{tikzcd}
f^*E \arrow[r, "\widetilde{f}"] \arrow[d, "f^*\pi"] & E \arrow[r] \arrow[d, "\pi"] & E\,\mathrm{U}(n) \arrow[d, "\pi_{\mathrm{U}(n)}"] \\
Y \arrow[r, "f"] & X \arrow[r, "f_\pi"] & B\,\mathrm{U}(n)
\end{tikzcd}$$

which implies that $f_\pi \circ f$ classifies the $U(n)$-bundle $f^*\pi$ on Y. Therefore,

$$\begin{aligned}
c_i\left(f^*\pi\right) &= \left(f_\pi \circ f\right)^* c_i \\
&= f^*\left(f_\pi^* c_i\right) \\
&= f^* c_i(\pi)
\end{aligned}$$

□

6. Path fiberation

Recall that for a fiber bundle (E, X, F), where E, X, F are topological spaces and X admits a good cover, then the E_2-page of Leray's spectral sequence is

$$E_2^{p,q} = H^p(\mathfrak{U}, \mathscr{H}^q(F))$$

where $\mathscr{H}^q(F)$ is a locally constant sheaf. Now suppose $\pi : E \to X$ is just a map, not necessarily locally trivial, we can still obtain a spectral sequence with E_2-page $H^p(\mathfrak{U}, \mathscr{H}^q(F))$ which converges to $H_D(E)$ as long as $\pi : E \to X$ has the property that

Property 6.1. $H^q(\pi^{-1}U) \cong H^q(F)$ for some fixed F and for all contractible open subset U.

An important example is path fiberation.

6.1. **Basic setting.**

Let X be a topological space with a base point $*$ and $[0,1]$ the unit interval with base point 0. The path space of X is defined to be the space $P(X)$ consisting of all the paths in X with initial point $*$, that is

$$P(X) := \{\text{maps } \mu : [0,1] \to X \mid \mu(0) = *\}$$

The path space $P(X)$ is equipped with compact open topology, that is a topology basis consists of all base-point preserving maps $\mu : [0,1] \to X$ such that $\mu(K) \subset U$ for a fixed compact set K in $[0,1]$ and a fixed open set U in X.

There is a natural projection $\pi : P(X) \to X$, defined by $\pi(\mu) = \mu(1)$. Now we claim $\pi : P(X) \to X$ has property 6.1. Indeed, for arbitrary contractible open set U containing p, there is a natural inclusion

$$i : \pi^{-1}(p) \to \pi^{-1}(U)$$

Since U is contractible, then we can get a map

$$\phi : \pi^{-1}(U) \to \pi^{-1}(p)$$

It's clear $i \circ \phi = \mathrm{id}$, and $\phi \circ i$ is homotopic to id, which implies $\pi^{-1}(U)$ has the same homotopy type as $\pi^{-1}(p)$. Furthermore, if p and q are in the same path component of X, then a fixed path from p to q gives a homotopy equivalence $\pi^{-1}(p) \cong \pi^{-1}(q)$. Thus all fibers have the homotopy type of $\pi^{-1}(*)$, which is loop space ΩX of X. To be explicit,

$$\Omega X = \{\mu : [0,1] \to X \mid \mu(0) = \mu(1) = *\}$$

Thus $\pi : P(X) \to X$ has the property 6.1, that is $H^q(\pi^{-1}U) \cong H^*(\Omega X)$. Furthermore, path space PX is always contractible, since there exists a homotopy H from arbitrary path γ to constant one given by

$$\begin{aligned} H : PX \times I &\to PX \\ (\gamma, t) &\mapsto (1-t) \cdot \gamma \end{aligned}$$

Proposition 6.1. Let $\pi : E \to X$ be a path fiberation. If X is simply-connected and E is path connected, then the fibers are path connected.

Proof. Trivially the $E_2^{0,0}$ term survives to E_∞, hence

$$E_2^{0,0} = E_\infty^{0,0} = H^0(E) = \mathbb{Z}$$

since E is path connected. On the other hand,

$$E_2^{0,0} = H^0(X, H^0(F)) = H^0(F)$$

which implies F is path connected. □

In fact there is a more general class of maps satisfying property 6.1, which is called fiberation. To be explicit, a map $\pi : E \to X$ is called a fiberation if it satisfies the following property:

Property 6.2 (covering homotopy property)**.** Given a map $f : Y \to E$ from any topological space Y into E and a homotopy $\overline{f}_t$ of $\overline{f} = \pi \circ f$, there is a homotopy f_t of f such that $\pi \circ f_t = \overline{f}_t$.

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d] & E \arrow[d, "\pi"] \\ Y \times I \arrow[ur, dashed, "f_t"] \arrow[r, "\overline{f}_t"] & X \end{tikzcd}$$

Proposition 6.2. For fiberations we have the following properties:

1. Any two fibers of a fiberation over an arcwise-connected space have the same homotopy type;
2. For every contractible open set U, the inverse image $\pi^{-1}U$ has the homotopy type of the fiber F_a, where a is any point in U.

Proof. Here we only explain some key ideas of proof of (1), which will be used in later.

1. A path γ from a to b may be regarded as a homotopy of the point a;
2. Let $\overline{g} : F_a \times I \to X$ be given by $(y, t) \mapsto \gamma(t)$, then covering homotopy property implies there exists a map $g : F_a \times I \to E$ that covers $\overline{g}$. Furthermore, $g_1 := g|_{F_a \times \{1\}}$ is a map from F_a to F_b, since $\gamma(1) = b$. Thus a path from a to b induces a map from F_a to the fiber F_b.
3. The **key point** is that homotopic paths from a to b in X induces homotopic maps from F_a to F_b.
4. If (3) holds, given $a, b \in X$ and a path γ from a to b, let $u : F_a \to F_b$ be a map induced by γ and $v : F_a \to F_b$ a map induced by γ^{-1}. Since $\gamma^{-1} \circ \gamma^{-1}$ is homotopic to the constant map to a, the composition $v \circ u$ is homotopic to identity on F_a, which implies F_a and F_b have the same homotopy type.

□

Remark 6.1. In fact, we can slightly change the proof to see if $\overline{f}_t, \overline{g}_t : Y \times I \to X$ are two homotopic homotopies, then their lifts $f_t, g_t : Y \times I \to E$ are also homotopic.

6.2. **The cohomology ring of ΩS^n.**

6.2.1. *The cohomology group structure.* In this section, we compute the integer cohomology groups of the loop space $\Omega S^n, n \geq 2$.

Example 6.1 (The cohomology group of ΩS^2)**.** Since S^2 is simply-connected, thus the spectral sequence of the path fiberation

$$\begin{tikzcd} \Omega S^2 \arrow[r] & PS^2 \arrow[d] \\ & S^2 \end{tikzcd}$$

has E_2-page $H^p(S^2, H^q(\Omega S^2)) = H^p(S^2) \otimes H^q(\Omega S^2)$, thus only two non-zero columns at $p = 0, 2$. By dimensional reason, $\mathrm{d}_3 = \mathrm{d}_4 = \cdots = 0$, thus $E_3 = E_\infty$. Furthermore, since PS^2 is contractible, we have all non-zero d_2 are isomorphisms. Thus $\mathrm{d}_2 : E_2^{0,1} \to E_2^{2,0}$ is an isomorphism, that is $H^1(\Omega S^2) = \mathbb{Z}$, but then

$$E_2^{2,1} = H^2(S^2) \otimes H^1(\Omega S^2) = \mathbb{Z}$$

by the same reason $E_2^{0,2} = \mathbb{Z}$. Step by step we find $H^q(\Omega S^2) = \mathbb{Z}$ in every dimension q.

Example 6.2 (The cohomology group of ΩS^3)**.** Since S^3 is simply-connected, thus the spectral sequence of the path fiberation

$$\begin{tikzcd} \Omega S^3 \arrow[r] & PS^3 \arrow[d] \\ & S^3 \end{tikzcd}$$

has E_2-page $H^p(S^3, H^q(\Omega S^3)) = H^p(S^3) \otimes H^q(\Omega S^3)$, thus only two non-zero columns at $p = 0, 3$. By dimensional reason, $\mathrm{d}_2 = \mathrm{d}_4 = \cdots = 0$, thus $E_3 = E_\infty$. Furthermore, since PS^3 is contractible, we have all non-zero d_3 are isomorphisms. Thus $\mathrm{d}_3 : E_2^{0,2} \to E_2^{3,0}$ is an isomorphism, that is $H^2(\Omega S^3) = \mathbb{Z}$, but then

$$E_2^{3,2} = H^3(S^3) \otimes H^2(\Omega S^3) = \mathbb{Z}$$

by the same reason $E_2^{0,4} = \mathbb{Z}$. Step by step we find $H^q(\Omega S^2) = \mathbb{Z}$ in every even dimension q.

Example 6.3. In general

$$H^k(\Omega S^n) = \begin{cases} \mathbb{Z}, & k = n-1, 2(n-1), \dots \\ 0, & \text{otherwise} \end{cases}$$

6.2.2. *The cohomology ring structure.* In this section, we compute the integer cohomology rings of the loop space $\Omega S^n, n \geq 2$.

Example 6.4 (The cohomology ring of ΩS^2)**.** Let u be a generator of $E_2^{2,0} = H^2(S^2)$ and x a generator of $H^1(\Omega S^2)$ such that $\mathrm{d}_2(1 \otimes x) = u \otimes 1$, then $u \otimes x$ is a generator of $H^2(S^2) \otimes H^1(\Omega S^2)$. Direct computation shows

$$\begin{aligned}
\mathrm{d}_2(1 \otimes x^2) &= \mathrm{d}_2(1 \otimes x) \cdot (1 \otimes x) - (1 \otimes x) \cdot \mathrm{d}_2(1 \otimes x) \\
&= (u \otimes 1) \cdot (1 \otimes x) - (1 \otimes x) \cdot (u \otimes 1) \\
&= u \otimes x - u \otimes x \\
&= 0
\end{aligned}$$

which implies $x^2 = 0$, since d_2 is an isomorphism. Let e be a generator of $H^2(\Omega S^2)$ such that $\mathrm{d}_2(1 \otimes e) = u \otimes x$ and $u \otimes e \in H^2(S^2) \otimes H^2(\Omega S^2)$, then

$$\begin{aligned}
\mathrm{d}_2(1 \otimes ex) &= \mathrm{d}_2(1 \otimes e) \cdot (1 \otimes x) + (1 \otimes e) \cdot \mathrm{d}_2(1 \otimes x) \\
&= (u \otimes x) \cdot (1 \otimes x) + (1 \otimes e) \cdot (u \otimes 1) \\
&= u \otimes e
\end{aligned}$$

implies ex is a generator of $H^3(\Omega S^2)$, since d_2 is an isomorphism. Similar computations shows

$$\begin{aligned}
\mathrm{d}_2(1 \otimes \frac{e^2}{2}) &= \frac{1}{2}\mathrm{d}_2(1 \otimes e) \cdot (1 \otimes e) + \frac{1}{2}(1 \otimes e) \cdot \mathrm{d}_2(1 \otimes e) \\
&= \frac{1}{2}(u \otimes x) \cdot (1 \otimes e) + \frac{1}{2}(1 \otimes e) \cdot (u \otimes x) \\
&= (u \otimes ex) \\
\mathrm{d}_2(1 \otimes \frac{e^2 x}{2}) &= \frac{1}{2}\mathrm{d}_2(1 \otimes e^2) \cdot (1 \otimes x) + \frac{1}{2}(1 \otimes e^2) \cdot \mathrm{d}_2(1 \otimes x) \\
&= (u \otimes ex) \cdot (1 \otimes x) + \frac{1}{2}(1 \otimes e^2)(u \otimes 1) \\
&= (u \otimes \frac{e^2}{2})
\end{aligned}$$

which implies $\frac{e^2}{2}$ is a generator of $H^4(\Omega S^2)$ and $\frac{e^2 x}{2}$ is a generator of $H^2(\Omega S^2)$. By induction we can show $\frac{e^k}{k!}$ is a generator of $H^{2k}(\Omega S^2)$ and $\frac{e^k x}{k!}$ is a generator of $H^{2k+1}(\Omega S^2)$.

The divided polynomial algebra $Z_\gamma(e)$ with generator e is the $\mathbb{Z}$-algebra with additive basis $\{1, e, e^2/2!, e^3/3!, \dots\}$, then

$$H^*(\Omega S^2) = \Lambda[x_1] \otimes Z_\gamma(e)$$

where $|x_1| = 1, |e| = 2$.

Remark 6.2. By the same argument one can show for n is even

$$H^*(\Omega S^n) = \Lambda[x_{n-1}] \otimes Z_\gamma(e)$$

where $|x_{n-1}| = n - 1, |e| = 2(n - 1)$.

Example 6.5 (The cohomology ring of ΩS^3)**.** Let u be a generator of $E_2^{3,0} = H^3(S^3)$ and e a generator of $H^2(\Omega S^3)$ such that $\mathrm{d}_2(1 \otimes e) = u \otimes 1$, then $u \otimes e$ is a generator of $H^3(S^3) \otimes H^2(\Omega S^3)$. The same computation as above case shows $\frac{e^2}{2}$ is a generator of $H^2(\Omega S^3)$, and by induction one has $\frac{e^k}{k!}$ is a $H^{2k}(\Omega S^3)$, which implies

$$H^*(\Omega S^3) = Z_\gamma(e)$$

where $|e| = 2$.

Remark 6.3. By the same argument one can show for n is odd

$$H^*(\Omega S^n) = Z_\gamma(e)$$

where $|e| = n - 1$.

Part 3. Applications in homotopy theory

7. Review of homotopy theory

7.1. First properties.

Let X be a topological space with base point $*$.

Definition 7.1 (q-th homotopy group)**.** For $q \geq 1$, the q-th homotopy group $\pi_q(X)$ of X is defined to be the homotopy classes of maps from q-cube I^q to X which send the faces $\dot{I}^q$ of I^q to the base point of X.

Remark 7.1. Equivalently, $\pi_q(X), q \geq 1$ may be regarded as the homotopy classes of base-point preserving maps from S^q to X.

Remark 7.2. $\pi_0(X)$ is defined to be the set of all path components of X, and for a manifold the path components are the same as the connected components. Although $\pi_0(X)$ is in general not a group, if G is a Lie group then $\pi_0(G)$ is a group.

Proposition 7.1. Basic properties:

1. $\pi_q(X \times Y) = \pi_q(X) \times \pi_q(Y)$;
2. $\pi_q(X)$ is abelian if $q \geq 1$;
3. If $\widetilde{X}$ is the universal covering of X, then $\pi_q(X) = \pi_q(\widetilde{X})$ for $q \geq 2$.
4. $\pi_{q-1}(\Omega X) = \pi_q(X)$ for $q \geq 2$.

Proof. For (4). Elements of $\pi_2(X)$ are given by maps of I^2 to X, which can be viewed as a map from I to ΩX, therefore $\pi_2(X) = \pi_1(\Omega X)$. The general case is similar. □

Example 7.1. The homotopy groups of S^1 is

$$\pi_q(S^1) = \begin{cases} \mathbb{Z}, & q = 1 \\ 0, & q > 1 \end{cases}$$

Theorem 7.1 (long exact sequence of homotopy)**.** Let $\pi : E \to X$ be a base-point preserving fiberation with fiber F, then there is an exact sequence of homotopy groups as follows

$$\cdots \to \pi_q(F) \xrightarrow{i_*} \pi_q(E) \xrightarrow{\pi_*} \pi_q(X) \xrightarrow{\partial} \pi_q(F) \to \cdots \to \pi_0(E) \to \pi_0(X) \to 0$$

Proof. Here we only gives the descriptions of these homomorphisms, readers may refer to other standard textbooks for exactness.

The maps i_*, π_* are induced by the inclusion $i : F \to E$ and projection $\pi : E \to X$ respectively, where we regard F as the fiber over the base-point $*$ of B. To describe ∂ we use the covering homotopy property of fiberation. A map $\alpha : I^q \to B$ representing an element of $\pi_q(X)$ can be regarded as a homotopy of $\alpha|_{I^{q-1}}$ in X. Note that $\alpha|_{I^{q-1}} : (t_1, \ldots, t_{q-1}, 0) \to * \in X$, then we take constant map $* : I^{q-1} \to E$ from I^{q-1} to the base-point of F as the map that covers $\alpha|_{I^{q-1}}$. By the covering homotopy property, there is a homotopy $\overline{\alpha} : I^q \to E$ which covers α such that $\overline{\alpha}_{I^{q-1}} = *$. Then $\partial[\alpha]$ is the homotopy class of the map $\overline{\alpha} : (t_1, \ldots, t_{q-1}, 1) \to F$. And the well-defineness follows from Remark 6.1. □

7.2. **Some homotopy groups of the sphere.**

7.2.1. *Hurewicz theorem.*

Theorem 7.2 (Hurewicz theorem)**.** Let X be a path-connected space, then $H_1(X)$ is the abelianization of $\pi_1(X)$.

Remark 7.3. So simply-connected space X must have $H_1(X) = 0$; Converse statement is not true, although it's quite difficult to give a simple example. For example, you can take an arbitrary perfect group[5] G (For example, $G = A_5$), then the space $K(G, 1)$, which will be defined later is what you want. However, in general you don't know what does it look like.

Theorem 7.3 (Hurewicz theorem)**.** Let X be a simply-connected path-connected CW complex. Then the first non-trivial homotopy group and homology group occur in the same dimension and are equal.

Proof. Let n denote the first dimension such that $H_n(X) \neq 0$, now let's prove by induction on n. Firstly consider the case $n = 2$. The E_2-page of homology spectral sequence of the path fiberation is

$$\begin{tikzcd}
H_1(\Omega X) & & \\
\mathbb{Z} & 0 & H_2(X) \arrow[ull]
\end{tikzcd}$$

Thus $\pi_2(X) = \pi_1(\Omega X) = H_1(\Omega X) = H_2(X)$.

Now let n be any positive integer ≥ 3, then in this case ΩX has the following properties:

[5] A group G such that its abelianization is trivial is called perfect group.

1. It's a CW complex[6];
2. It's simply-connected, since $\pi_1(\Omega X) = \pi_2(X) = H_2(X) = 0$;
3. The dimension of the first non-trivial homology group of ΩX is $n-1$, since $H_{q-1}(\Omega X) = H_q(X), q \geq 2$.

Then we can apply induction hypothesis to ΩX, one has

$$\pi_q(\Omega X) = H_q(\Omega X) = \begin{cases} 0, & q < n-1 \\ H_{n-1}(\Omega X), & q = n-1 \end{cases}$$

On the other hand, the E_2-page still implies $H_{q-1}(\Omega X) = H_q(X)$ for $2 \leq q \leq n$. Then

$$\pi_q(X) = \pi_{q-1}(\Omega X) = H_{q-1}(\Omega X) = \begin{cases} 0, & 2 \leq q < n \\ H_n(X), & q = n \end{cases}$$

□

Remark 7.4. Note that if we want to use Leray spectral sequence, X should admit a good cover. Fortunately, every CW complex admits a good cover.

Example 7.2. It follows from Hurewicz theorem that

$$\pi_q(S^n) = \begin{cases} 0, & q < n \\ \mathbb{Z}, & q = n \end{cases}$$

7.3. **Bott periodic theorem.**

Example 7.3 (some homotopy groups of $\mathrm{U}(n)$)**.** Consider the following fiberation

$$\begin{tikzcd} \mathrm{U}(n-1) \arrow[r] & \mathrm{U}(n) \arrow[d] \\ & S^{2n-1} \end{tikzcd}$$

Then homotopy exact sequence implies

$$\cdots \to \pi_q(S^{2n-1}) \to \pi_q(\mathrm{U}(n-1)) \to \pi_q(\mathrm{U}(n)) \to \pi_{q-1}(S^{2n-1}) \to \dots$$

Then for $q < 2n$, one has

$$\pi_q(\mathrm{U}(n-1)) = \pi_q(\mathrm{U}(n))$$

these mutally isomorphic groups are called q-th **stable homotopy groups** of the unitary group. They're denoted briefly by $\pi_q(\mathrm{U})$.

Remark 7.5. However, how to compute these stable homotopy groups? Bott has the following theorem:

Theorem 7.4 (Bott periodic theorem)**.** For $q \geq 1$,

$$\pi_{q-1}(\mathrm{U}) \cong \pi_{q+1}(\mathrm{U})$$

[6]Not a trivial fact, it's a theorem proved by Milnor: The loop space of a CW complex is still a CW complex.

From this theorem, it suffices to compute $\pi_0(\mathrm{U})$ and $\pi_1(\mathrm{U})$, and it's quite clear:

$$\pi_0(\mathrm{U}) = \pi_0(\mathrm{U}(1)) = 0$$
$$\pi_1(\mathrm{U}) = \pi_1(\mathrm{U}(1)) = \mathbb{Z}$$

Example 7.4 (some homotopy groups of $\mathrm{SU}(n)$)**.** For the same reason we have for $q < 2n$,

$$\pi_q(\mathrm{SU}(n-1)) = \pi_q(\mathrm{SU}(n))$$

and we also have q-th stable homotopy groups of the special unitary group, denoted by $\pi_q(\mathrm{SU})$. From the following fiberation

$$\begin{tikzcd} \mathrm{SU}(n) \arrow[r] & \mathrm{U}(n) \arrow[d, "\det"] \\ & S^1 \end{tikzcd}$$

we can conclude

$$\pi_q(\mathrm{U}(n)) = \begin{cases} \pi_q(\mathrm{SU}(n)), & q \geq 2 \\ \pi_1(\mathrm{SU}(n)) \oplus \mathbb{Z}, & q = 1 \end{cases}$$

for arbitrary $n \geq 1$. In particular, we have the isomorphisms between stable homotopy groups.

Example 7.5 (some homotopy groups of $\mathrm{O}(n)$)**.** Consider the following fiberation

$$\begin{tikzcd} \mathrm{O}(n-1) \arrow[r] & \mathrm{O}(n) \arrow[d] \\ & S^{n-1} \end{tikzcd}$$

Then homotopy exact sequence implies

$$\cdots \to \pi_q(S^{n-1}) \to \pi_q(\mathrm{O}(n-1)) \to \pi_q(\mathrm{O}(n)) \to \pi_{q-1}(S^{n-1}) \to \ldots$$

Then for $q < n$, one has

$$\pi_q(\mathrm{O}(n-1)) = \pi_q(\mathrm{O}(n))$$

and we can define q-th stable homotopy groups of special orthogonal groups, defined by $\pi_q(\mathrm{O})$. Similarly we also have the following theorem:

Theorem 7.5 (Bott periodic theorem)**.** For $q \geq 0$,

$$\pi_q(\mathrm{O}) \cong \pi_{q+8}(\mathrm{O})$$

7.4. **Some homotopy groups of Stiefel manifold.**

Example 7.6 (some homotopy groups of complex Stiefel manifold)**.** Recall that complex Stiefel manifold $V_k(\mathbb{C}^{n+k})$ is the set of all orthonormal k-frames in $\mathbb{C}^{n+k}$, and it have the following fiberation

$$\begin{tikzcd} V_{k-1}(\mathbb{C}^{n+k-1}) \arrow[r] & V_k(\mathbb{C}^{n+k}) \arrow[d] \\ & S^{2(n+k)-1} \end{tikzcd}$$

Thus homotopy exact sequence implies

$$\cdots \to \pi_q(S^{2(n+k)-1}) \to \pi_q(V_{k-1}(\mathbb{C}^{n+k-1})) \to \pi_q(V_k(\mathbb{C}^{n+k})) \to \pi_{q-1}(S^{2(n+k)-1}) \to \dots$$

Thus for $q < 2(n+k)$, one has

$$\pi_q(V_k(\mathbb{C}^{n+k})) = \pi_q(V_{k-1}(\mathbb{C}^{n+k-1}))$$

In particular, if $q < 2(n+1)$, one has

$$\pi_q(V_1(\mathbb{C}^{n+1})) = \pi_q(V_0(\mathbb{C}^n)) = 0$$

which implies if $q < 2(n+1)$, then

$$\pi_q(V_k(\mathbb{C}^{n+k})) = \pi_q(V_{k-1}(\mathbb{C}^{n+k-1})) = \cdots = \pi_q(V_0(\mathbb{C}^n)) = 0$$

Here we claim $\pi_{2n+2}(V_k(\mathbb{C}^{n+k}))$ is the first non-trivial homotopy group of Stiefel manifold. Consider $V_2(\mathbb{C}^{n+2})$, since it has the same $2n+2$-th homotopy group as $V_k(\mathbb{C}^{n+k})$. A crucial observation is that $V_2(\mathbb{C}^{n+2})$ is the unit tangent bundle of S^{2n+3}, thus we have the following fiberation:

$$\begin{tikzcd} S^{2n+2} \arrow[r] & V_2(\mathbb{C}^{n+2}) \arrow[d] \\ & S^{2n+3} \end{tikzcd}$$

and Gysin sequence implies

$$\cdots \to 0 \to H^{2n+2}(V_2(\mathbb{C}^{n+2})) \to H^0(S^{2n+3}) \xrightarrow{\wedge e} H^{2n+3}(S^{2n+3}) \to \dots$$

By Proposition 5.2.1, we have the Euler class of unit tangent bundle of S^{2n+3} is zero, which implies $H^{2n+2}(V_k(\mathbb{C}^{n+k})) = \mathbb{Z}$, thus by Poincaré duality and Hurewicz theorem we have $\pi_{2n+2}(V_k(\mathbb{C}^{n+k})) = \mathbb{Z}$.

Example 7.7 (some homotopy groups of real Stiefel manifold)**.** By the same argument, one can show if $q < n+k$, one has

$$\pi_q(V_k(\mathbb{R}^{n+k})) = \pi_q(V_{k-1}(\mathbb{R}^{n+k-1}))$$

In particular if $q < n$, one has

$$\pi_q(V_k(\mathbb{R}^{n+k})) = \pi_q(V_{k-1}(\mathbb{R}^{n+k-1})) = \cdots = \pi_q(V_0(\mathbb{R}^n))$$

Here we claim $\pi_n(V_k(\mathbb{R}^{n+k}))$ is the first non-trivial homotopy group of Stiefel manifold. Consider $V_2(\mathbb{R}^{n+2})$, since it has the same n-th homotopy group as $V_k(\mathbb{R}^{n+k})$. A crucial observation is that $V_2(\mathbb{R}^{n+2})$ is the unit tangent bundle of S^{n+1}, thus we have the following fiberation:

$$\begin{array}{ccc} S^n & \longrightarrow & V_2(\mathbb{R}^{n+2}) \\ & & \downarrow \\ & & S^{n+1} \end{array}$$

and Gysin sequence implies

$$\cdots \to 0 \to H^n(V_2(\mathbb{R}^{n+2})) \to H^0(S^{n+1}) \xrightarrow{\wedge e} H^{n+1}(S^{n+1}) \to \dots$$

By the same argument we have

$$\pi_n(V_2(\mathbb{R}^{n+2})) = \begin{cases} \mathbb{Z}, & n \text{ is odd} \\ \mathbb{Z}/2\mathbb{Z}, & n \text{ is even} \end{cases}$$

7.5. **Hopf invariant.**

7.5.1. *History.* In general, it's tough to compute $\pi_q(S^n)$ for $n \geq 2, q > n$. So the first non-trivial case is $\pi_3(S^2)$. Consider Hopf fiberation

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \\ & & \mathbb{CP}^1 = S^2 \end{array}$$

Then the exact sequence of homotopy groups implies

$$\cdots \to \pi_q(S^1) \to \pi_q(S^3) \to \pi_q(S^2) \to \pi_{q-1}(S^1) \to \dots$$

Use the fact that $\pi_q(S^1) = 0, q > 1$ one has

$$\pi_q(S^3) = \pi_q(S^2)$$

for $q > 1$. In particular one has $\pi_3(S^2) = \mathbb{Z}$.

In history $\pi_3(S^2)$ was first computed by Hopf in 1931 using a linking number argument which associates to each homotopy class of maps from S^3 to S^2 an integer now called the Hopf invariant. We first give an account of the Hopf invariant in the dual language of differential forms and then in terms of the linking number.

7.5.2. *The differential forms definition.*

Definition 7.2 (Hopf invariant)**.** Let $f : S^{2n-1} \to S^n$ be a smooth map and let α be a generator of $H^n_{dR}(S^n)$, then Hopf invariant of f is defined as

$$H(f) = \int_{S^{2n-1}} \omega \wedge \mathrm{d}\omega$$

where $f^*\alpha = \mathrm{d}\omega$.

Proposition 7.2. Properties of Hopf invariant:

1. The definition of Hopf invariant is independent of the choice of ω;
2. For odd n the Hopf invariant is 0;
3. Homotopic maps have the same Hopf invariant.

Proof. For (1). Let ω' be another $(n-1)$-form on S^{2n-1} such that $f^*\alpha = \mathrm{d}\omega'$. Then

$$\begin{aligned}\int_{S^{2n-1}} \omega \wedge \mathrm{d}\omega - \int_{S^{2n-1}} \omega' \wedge \mathrm{d}\omega' &= \int_{S^{2n-1}} (\omega - \omega') \wedge \mathrm{d}\omega \\ &= \pm \int_{S^{2n-1}} \mathrm{d}((\omega - \omega') \wedge \omega) \\ &= 0\end{aligned}$$

For (2). If n is odd, then ω is even-dimensional, thus

$$\omega \wedge \mathrm{d}\omega = \frac{1}{2}\mathrm{d}(\omega \wedge \omega)$$

For (3). From (2) we may assume n is even. Let $F : S^{2n-1} \times I \to S^n$ be a homotopy between $f_0, f_1 : S^{2n-1} \to S^n$. We use i_0 to denote the inclusion $i_0 : S^{2n-1} \to S_0 = S^{2n-1} \times \{0\} \subset S^{2n-1} \times I$ and similar for i_1. Then

$$\begin{aligned} F \circ i_0 &= f_0 \\ F \circ i_1 &= f_1 \end{aligned}$$

Let α be a generator of $H^n_{dR}(S^n)$, then $F^*\alpha = \mathrm{d}\omega$ for some $(n-1)$-form ω on $S^{2n-1} \times I$. Define $i_0^*\omega = \omega_0$ and $i_1^*\omega = \omega_1$, then

$$\begin{aligned} f_0^*\alpha = (F \circ i_0)^* = i_0^* \circ F^*\alpha = \omega_0 \\ f_1^*\alpha = (F \circ i_1)^* = i_1^* \circ F^*\alpha = \omega_1 \end{aligned}$$

Then

$$\begin{aligned} H(f_1) - H(f_2) &= \int_{S^{2n-1}} \omega_1 \wedge \mathrm{d}\omega_1 - \int_{S^{2n-1}} \omega_0 \wedge \mathrm{d}\omega_0 \\ &= \int_{S^{2n-1}} i_1^*(\omega \wedge \mathrm{d}\omega) - \int_{S^{2n-1}} i_0^*(\omega \wedge \mathrm{d}\omega) \\ &= \int_{S_1} - \int_{S_0} \omega \wedge \mathrm{d}\omega \\ &= \int_{S^{2n-1}} \mathrm{d}(\omega \wedge \mathrm{d}\omega) \\ &= \int_{S^{2n-1} \times I} F^*(\alpha \wedge \alpha) \\ &= 0 \end{aligned}$$

□

Thus Hopf invariant gives the following map

$$H : \pi_{2n-1}(S^n) \to \mathbb{R}$$

Furthermore, it gives a group homomorphism. Indeed, for two smooth maps $f, g : S^{2n-1} \to S^n$, it suffices to show

$$(fg)^*(\alpha) =$$

where α be a generator of $H^n_{dR}(S^n)$ and $\mathrm{d}\omega_f = f^*\alpha, \mathrm{d}\omega_g = g^*\alpha$. Then

$$H(fg) = \int_{S^{2n-1}}$$

7.5.3. *The intersection-theory definition.*

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