ALGEBRAIC CURVES

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Contents

0. Preface

0.1. **Notations.**

(1) X, Y always denote Riemann surfaces.
(2) C always denotes the algebraic plane curve.
(3) $\Phi, \Psi \colon X \to Y$ always denote the holomorphic map between Riemann surfaces.
(4) f, g sometimes denote functions (smooth,holomorphic or meromorphic), sometimes denote polynomials, and sometimes denote the convergent power series.
(5) F, G always denote polynomials, and most of time they denote homogenous polynomials given by polynomials f, g.
(6) f_x always denote the partial derivative of f with respect to variable x.

0.2. **Motivations.**

0.2.1. *Meromorphic functions.* Let $U \subseteq \mathbb{C}$ be an open subset with coordinate $\{z\}$. In complex analysis we learnt that a meromorphic function f is a function that is holomorphic on all of U except for a set of isolated points, which are poles of the function. In other words, a meromorphic function can be regarded as a function $f\colon U \to \mathbb{C} \cup \{\infty\}$.

Topologically speaking, $\mathbb{C} \cup \{\infty\}$ is S^2, and in fact there is a complex manifold structure on it. More precisely, we can glue two pieces of complex plane via $w = 1/z$ to obtain a complex manifold called Riemann sphere

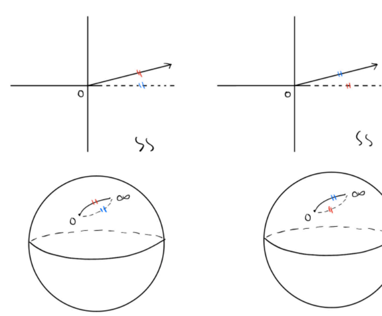
$$\mathbb{P}^1 = \mathbb{C} \cup_{\mathbb{C}^*} \mathbb{C},$$

and topologically $\mathbb{P}^1$ is exactly $\mathbb{C} \cup \{\infty\}$. By using this viewpoint, meromorphic function on U is exactly the same thing as holomorphic map from U to the Riemann sphere, and thus it gives us a lovely way to study meromorphic functions by using theories of holomorphic maps between Riemann surfaces, such as the number (counted with multiplicity) of zeros is equal to the number (counted with multiplicity) of poles.

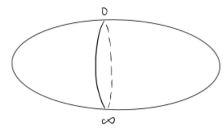
0.2.2. *Multivalueness of holomorphic functions.* For complex number $z = \rho e^{\sqrt{-1}\theta}$, where $\rho \in [0, \infty)$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, one has

$$(\sqrt{\rho} e^{\sqrt{-1}\theta/2})^2 = (\sqrt{\rho} e^{\sqrt{-1}(\theta/2+\pi)})^2 = z.$$

This shows there are two candidates for $\sqrt{z}$, and this phenomenon is called multivalueness of holomorphic function. If we define square root as $\sqrt{z} = \sqrt{\rho} e^{\sqrt{-1}\theta/2}$, then it's only well-defined on $\mathbb{C} \setminus [0, \infty)$, since it will "jump" when passing through the two sides of $[0, \infty)$, and $\mathbb{C} \setminus [0, \infty)$ is called a single value component of $\sqrt{z}$.



The ideal to solve this phenomenon is that, when passing the segment $[0, \infty)$, $\sqrt{z}$ should come into another single value component. In other words, if we want to make square root $\sqrt{z}$ defined on the whole complex plane, it should be no longer a function from $\mathbb{C}$ to $\mathbb{C}$, but a function from $\mathbb{C}$ to an object we obtained from gluing two single value components together. This construction also gives the Riemann sphere.

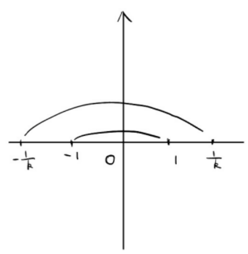


Similarly, $f(z) = \sqrt{1-z^2}$ is well-defined on $\mathbb{C}\setminus[-1,1]$, and it gives a well-defined function from $\mathbb{C}$ to something obtained by gluing two copies of $\mathbb{C}\setminus[-1,1]$, which is also the Riemann sphere.

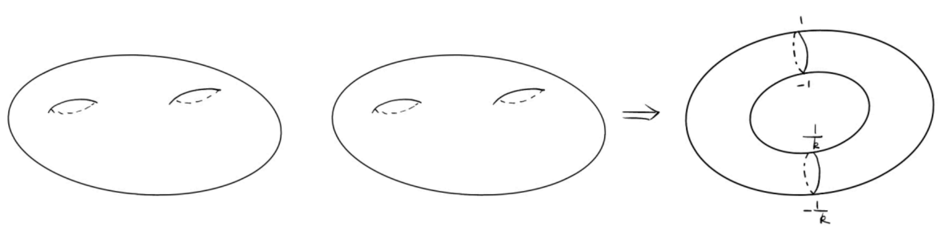
Now let's consider a more complicated example. For

$$f(z) = \sqrt{(1-z^2)(1-k^2z^2)},$$

where $k \neq \pm 1$, it gives a well-defined function on $\mathbb{C}$ minus two line segments connecting $-1, 1$ and $-1/k, 1/k$.



If we want to obtain a function defined on $\mathbb{C}$, we should glue two copies of above single value components. This gives a new Riemann surface called complex torus.



0.2.3. *Abelian integrals.*

Example 0.2.1 (arc-length of ellipse)**.** For ellipse given by $(x/a)^2+(y/b)^2 = 1$, by using parameterization

$$x = a\cos\theta$$
$$y = b\sin\theta,$$

it's easy to see arc-length is given by

$$\begin{aligned}\int_{\theta_0}^{\theta_1}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\mathrm{d}\theta &= a\int_{\theta_0}^{\theta_1}\sqrt{1-k^2\sin^2\theta}\mathrm{d}\theta\\ &\overset{z=\sin\theta}{=}\int_{z_0}^{z_1}\frac{\sqrt{1-k^2z^2}}{\sqrt{1-z^2}}\mathrm{d}z\\ &=\int_{z_0}^{z_1}\frac{1-k^2z^2}{\sqrt{(1-k^2z^2)(1-z^2)}}\mathrm{d}z,\end{aligned}$$

where $k=\sqrt{1-b^2/a^2}$. For $k=0$, since $\arcsin z$ is a primitive function of $1/\sqrt{1-z^2}$, one has

$$\int_{z_0}^{z_1}\frac{1}{\sqrt{1-z^2}}\mathrm{d}z=\arcsin z_1-\arcsin z_0.$$

The classical theory of "addition formula" gives

$$\sin(\alpha+\beta)=\sin\alpha\sqrt{1-\sin^2\beta}+\sqrt{1-\sin^2\alpha}\sin\beta.$$

In terms of integration

$$\int_0^{z_1}\frac{1}{\sqrt{1-t^2}}\mathrm{d}t+\int_0^{z_2}\frac{1}{\sqrt{1-t^2}}\mathrm{d}t=\int_0^{z_1\sqrt{1-z_2^2}+z_2\sqrt{1-z_1^2}}\frac{1}{\sqrt{1-t^2}}\mathrm{d}t.$$

For analogue of above case, if we define ellipse sine sn as

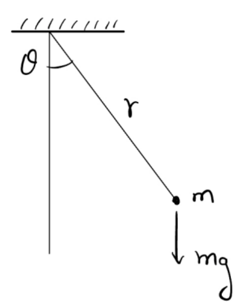
$$\int_0^{\arcsin z}\frac{1}{\sqrt{1-k^2\sin^2 t}}\mathrm{d}t=\mathrm{sn}^{-1}(z),$$

one can also show it satisfies some addition formula

$$\mathrm{sn}(\alpha+\beta)=\frac{\mathrm{sn}\,\alpha\sqrt{(1-\mathrm{sn}^2\beta)(1-k^2\,\mathrm{sn}^2\beta)}+\mathrm{sn}\,\beta\sqrt{(1-\mathrm{sn}^2\alpha)(1-k^2\,\mathrm{sn}^2\alpha)}}{1-k^2\,\mathrm{sn}^2\alpha\,\mathrm{sn}^2\beta}.$$

However, the ellipse sine cannot be expressed as an elementary function, and this is closely related to the fact that $y^2=(1-z^2)(1-k^2z^2)$ is not a Riemann sphere.

Example 0.2.2 (simple pendulum)**.** Suppose there is an object with mass m is released at $\theta=\alpha$ with zero initial velocity, and the length of pendulum is r.



The conservation of energy gives the following equation

$$\frac{1}{2}mr^2(\frac{\mathrm{d}\theta}{\mathrm{d}t})^2 = mgr\cos\theta - mgr\cos\alpha.$$

In other words,

$$(\frac{\mathrm{d}\theta}{\mathrm{d}t})^2 = 2\frac{g}{r}(\cos\theta - \cos\alpha) = 4\frac{g}{r}(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}). \tag{0.1}$$

An approximation with θ sufficiently small, one has

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \sqrt{\frac{g}{r}(\alpha^2 - \theta^2)}.$$

This shows

$$t = \int_0^\theta \sqrt{\frac{r}{g}}\frac{1}{\sqrt{\alpha^2 - s^2}}\mathrm{d}s.$$

Thus the period of the simple pendulum is given by

$$T = 4\int_0^\alpha \sqrt{\frac{r}{g}}\frac{1}{\sqrt{\alpha^2 - s^2}}\mathrm{d}s = 2\pi\sqrt{\frac{r}{g}}.$$

However, if we don't use the approximation, and use substitution

$$\sin\varphi = \frac{\sin\frac{\theta}{2}}{\sin\frac{\alpha}{2}}$$

in (0.1), one has

$$(\frac{\mathrm{d}\varphi}{\mathrm{d}t})^2 = \frac{g}{r}(1 - \sin^2\frac{\alpha}{2}\sin^2\varphi).$$

Then

$$t = \sqrt{\frac{r}{g}}\int_0^\varphi \frac{1}{\sqrt{1 - k^2\sin^2 s}}\mathrm{d}s,$$

where $k = \sin\dfrac{\alpha}{2}$, and thus explicit formula for the period of simple pendulum is

$$T = 4\sqrt{\frac{r}{g}}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2\sin^2 s}}\mathrm{d}s.$$

This is exactly ellipse integral.

Remark 0.2.1 (general case). Let f be a polynomial of two variables and $y = \Phi(X)$ be a solution of equation $f(x, y) = 0$. Then

$$\int R(x, f(x)) = 0$$

can be expressed as elementary function if and only if $\deg f = 0, 1, 2$, and in fact $\deg f$ is closely related to the topology of algebraic curves.

1. Riemann surface and algebraic curves

1.1. **Riemann surface.**

1.1.1. *Definitions.*

Definition 1.1.1 (complex atlas)**.** Let X be a topological space. A complex atlas on X consists of the following data:
(1) $\{U_i\}_{i\in I}$ is an open covering of X.
(2) For each $i\in I$, there exists a homeomorphism $\varphi_i\colon U_i\to\varphi_i(U_i)\subseteq\mathbb{C}$.
(3) For $i,j\in I$, if $U_i\cap U_j\neq\varnothing$, then the transition function
$$\varphi_{ij}:=\varphi_i\circ\varphi_j^{-1}\colon \varphi_j(U_i\cap U_j)\to\varphi_i(U_i\cap U_j)$$
is holomorphic.

Remark 1.1.1. If $\{U_i,\varphi_i\}$ is a complex atlas on a topological space, then all transition functions φ_{ij} are not only holomorphic, but biholomorphic with inverse φ_{ji}.

Definition 1.1.2 (complex structure)**.** Two complex atlas $\mathscr{A},\mathscr{B}$ are equivalent if $\mathscr{A}\cup\mathscr{B}$ is also a complex atlas, and a complex structure is an equivalent class of atlas on X.

Definition 1.1.3 (Riemann surface)**.** A Riemann surface is a connected, second countable, Hausdorff topological space X together with a complex structure on X.

Remark 1.1.2. A Riemann surface X is a complex manifold with $\dim_{\mathbb{C}} X=1$, and it's called a surface since $\dim_{\mathbb{R}} X=2$.

1.1.2. *Examples.*

Example 1.1.1 (Riemann sphere)**.** Let $S^2=\{(x,y,z)\in\mathbb{R}^3\mid x^2+y^2+z^2=1\}$ be 2-sphere and $\{U_1=S^2\setminus(0,0,1),U_2=S^2\setminus(0,0,-1)\}$ be an open covering of S^2. Consider
$$\begin{aligned}\varphi_1\colon U_1&\to\mathbb{C}\\(x_1,x_2,x_3)&\mapsto\frac{x_1}{1-x_3}+\sqrt{-1}\frac{x_2}{1-x_3},\end{aligned}$$
and
$$\begin{aligned}\varphi_2\colon U_1&\to\mathbb{C}\\(x_1,x_2,x_3)&\mapsto\frac{x_1}{1+x_3}-\sqrt{-1}\frac{x_2}{1+x_3}.\end{aligned}$$
A direct computation shows that
$$(\frac{x_1}{1-x_3}+\sqrt{-1}\frac{x_2}{1-x_3})(\frac{x_1}{1+x_3}-\sqrt{-1}\frac{x_2}{1+x_3})=\frac{x_1^2+x_2^2}{1-x_3^2}=1,$$
and thus the transition function $\varphi_2\circ\varphi_1^{-1}(z)=1/z$. This shows $\{U_1,U_2\}$ is a complex atlas of S^2. It's clear as a topological space S^2 is connected, second countable and Hausdorff, and thus S^2 is a Riemann surface, called Riemann sphere.

Remark 1.1.3. There is another construction of Riemann sphere, given by gluing two complex planes together on $\mathbb{C}^*$, and the gluing data on $\mathbb{C}^*$ is given by $z \sim 1/w$. One thing to mention is that it's not clear object constructed in this way is Hausdorff. For example, if we glue two complex planes together on $\mathbb{C}^*$ by using gluing data $z \sim w$, then the object obtained is not Hausdorff.

Example 1.1.2 (complex projective line)**.** The complex projective line $\mathbb{P}^1 = \mathbb{C}^2 \setminus (0,0)/\sim$, where $(x,y) \sim (z,w)$ if and only if $(\lambda x, \lambda y) = (z,w)$ for some $\lambda \in \mathbb{C}^*$, and the equivalent class for (x,y) is denoted by $[x,y]$, called the homogenous coordinate. The quotient topology on $\mathbb{P}^1$ which makes it second countable, Hausdorff and compact. Consider

$$U_0 = \{[z,w] \mid z \neq 0\} \xrightarrow{\varphi_0} \mathbb{C}$$

where φ_0 is defined as $\varphi_1([z,w]) = z/w$. Similarly consider

$$U_1 = \{[z,w] \mid w \neq 0\} \xrightarrow{\varphi_1} \mathbb{C}$$

where φ_1 is defined as $\varphi_1([z,w]) = w/z$. For $z \in \varphi_1(U_0 \cap U_1)$, one has

$$z \xrightarrow{\varphi_1^{-1}} [z:1] = [1:\frac{1}{z}] \xrightarrow{\varphi_0} \frac{1}{z}.$$

This shows the transition function $\varphi_{01}(z) = 1/z$, which is holomorphic, and thus $\{(U_0,\varphi_0),(U_1,\varphi_1)\}$ gives a complex atlas on $\mathbb{P}^1$.

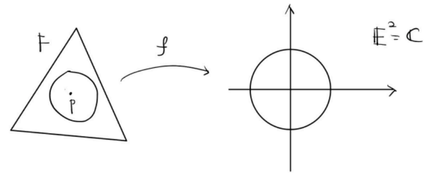
Remark 1.1.4 (complex projective space)*.* The complex projective space $\mathbb{P}^n$ is defined by $\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{0\}/\sim$, where $(x_0, x_1, \dots, x_n) \sim (y_0, y_1, \dots, y_n)$ if and only if there exists $\lambda \in \mathbb{C}^*$ such that $y_i = \lambda x_i$ holds for all $i = 0, 1, \dots, n$, and the equivalent class $[x_0 : x_1 : \cdots : x_n]$ is call the homogenous coordinate of $\mathbb{P}^n$. There is a canonical affine open covering $\{(U_i, \varphi_i)\}$ of $\mathbb{P}^n$ defined by

$$U_i = \{[x_0 : x_1 : \cdots : x_n] \mid x_i \neq 0\} \xrightarrow{\varphi_i} \mathbb{C}^n,$$

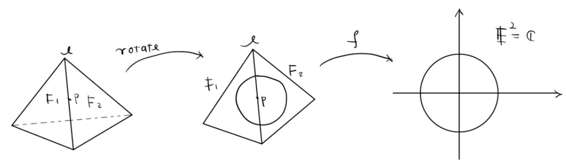
where $\varphi_i([x_0 : x_1 : \cdots : x_n]) = (x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i)$, and it makes $\mathbb{P}^n$ to be a complex n-manifold.

Example 1.1.3. Let P be a convex polyhedra in Euclidean 3-dimensional space $\mathbb{E}^3$. Topologically P is S^2, and let's construct a complex atlas on it.

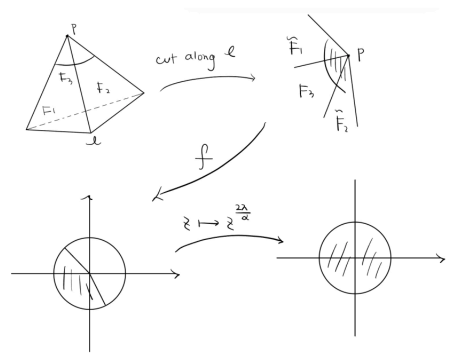
(1) Suppose p is the interior point of some face F. Since F can be isometrically embedded into $\mathbb{E}^2$, we choose an orientation-preserving, isometric embedding f which maps an open neighborhood U of p into $\mathbb{E}^2 = \mathbb{C}$.



(2) Suppose p is the interior point of some edge $l = F_1 \cap F_2$. Firstly we rotate F_2 along l to the plane of F_1, and then choose an orientation-preserving, isometric embedding f which maps an open neighborhood U of p into $\mathbb{E}^2 = \mathbb{C}$.



(3) Suppose p is an vertex which is the intersection of three faces F_1, F_2 and F_3. Firstly we cut it along some edge $l = F_1 \cap F_2$, and then rotate F_1, F_2 to the plane of F_3. Then we use some orientation-preserving, isometric embedding f to map it into $\mathbb{E}^2$, and finally composite it with $z \mapsto z^{2\pi/\alpha}$.



Exercise 1.1.1. Prove that above constructions give a complex atlas on convex polyhedra.

Remark 1.1.5. All of above three examples give complex structure on topological sphere S^2, a non-trivial fact is that all of them are the "same" complex structure for S^2 (See Corollary 9.2.3).

Example 1.1.4 (complex torus)**.** For non-zero $w_1, w_2 \in \mathbb{C}$ such that w_1, w_2 are $\mathbb{R}$-linearly independent, $L = \mathbb{Z}\, w_1 + \mathbb{Z}\, w_2$ is a discrete subgroup of $(\mathbb{C}, +)$. Let $\pi\colon \mathbb{C} \to T = \mathbb{C}/L$ be the natural projection. Then T equipped with the quotient topology is a connected, Hausdorff and second countable topological space. For $p \in T$, suppose z_0 is an inverse image of p. If we choose $\varepsilon \in \mathbb{R}_{>0}$ such that

$$B_{2\varepsilon}(0) \cap L = \{0\},$$

then $B_\varepsilon(z_0) \xrightarrow{\pi} \pi(B_\varepsilon(z_0)) \subseteq T$ is injective, and thus $\pi^{-1}\colon \pi(B_\epsilon(z_0)) \to B_\epsilon(z_0) \subseteq \mathbb{C}$ is a homeomorphism. Then $\{\pi(B_\varepsilon(\pi^{-1}(p))\}_{p \in T}$ gives an open covering of T, and together with π^{-1} it gives a complex atlas of T.

Remark 1.1.6. It's clear complex structure constructed above depends on the choice of w_1, w_2, but it's not obvious to see whether w_1, w_2 and w_1', w_2' give the same complex structure or not. In fact, they give the same complex structure if and only if they differ some elements in $\mathrm{SL}(2, \mathbb{Z})$, and all complex structure on torus are arisen in this way in fact (See Proposition 9.2.1).

1.1.3. *Morphisms.*

Definition 1.1.4 (holomorphic map)**.** Let X, Y be two Riemann surfaces and $\Phi\colon X \to Y$ be a continous map. For $p \in X$, Φ is called holomorphic at x, if there exists a chart (U, φ) of x, and a chart (V, ψ) of $\Phi(x)$, such that

$$\psi \circ \Phi \circ \varphi^{-1}\colon \varphi\left(U \cap \Phi^{-1}(V)\right) \to \psi\left(V \cap \Phi(U)\right)$$

is holomorphic at $\varphi(x)$. Moreover, Φ is called holomorphic in an open subset $W \subseteq X$, if Φ is holomorphic at any point in W.

Remark 1.1.7. It's clear the definition of holomorphic map is independent of the choice of charts, since change of coordinate is biholomorphic.

Definition 1.1.5 (isomorphism)**.** Let $\Phi\colon X \to Y$ be a holomorphic map between Riemann surfaces. Φ is called an isomorphism if it's bijective and holomorphic.

Proposition 1.1.1. Let $\Phi\colon X \to Y$ be a holomorphic map between Riemann surfaces. Φ is an isomorphism if and only if Φ has an two-side inverse Ψ, and Ψ is holomorphic.

Proposition 1.1.2. The complex projective space is isomorphic to Riemann sphere.

Theorem 1.1.1 (open map theorem)**.** Let $\Phi\colon X \to Y$ be a non-constant holomorphic map between Riemann surfaces. Then Φ is open.

Corollary 1.1.1. Let $\Phi\colon X \to Y$ be a non-constant holomorphic map between Riemann surfaces and X is compact. Then $\Phi(X) = Y$, and thus Y is compact.

Proof. By open map theorem, $\Phi(X)$ is an open subset of Y, and $\Phi(X)$ is compact in Y, since continous function maps compact set to compact set. Then $\Phi(X)$ is both open and closed in Y, and thus $\Phi(X) = Y$ since Y is assumed to be connected. □

Theorem 1.1.2. Let $\Phi\colon X \to Y$ be a non-constant holomorphic map between Riemann surfaces. Then for any $p \in Y$, $\Phi^{-1}(y)$ is a discrete set. In particular, if X is compact, then $\Phi^{-1}(y)$ is a non-empty finite set.

1.1.4. *Meromorphic functions.*

Definition 1.1.6 (singularity)**.** Let X be a Riemann surface and f be a holomorphic function defined on $U \setminus \{x\}$ where $U \subseteq X$ is an open subset. The point x is called a removbale singularity/pole/essential singularity, if there exists a chart (U, φ) of x, such that $f \circ \varphi^{-1}\colon \varphi(U) \to \mathbb{C}$ has $\varphi(x)$ as a removbale singularity/pole/essential singularity.

Remark 1.1.8.

(1) If $|f(x)|$ is bounded in a punctured neighborhood of x, then x is a removable singularity, and we can cancel the singularity by defining $f(x) = \lim_{y\to x} f(y)$.
(2) If $\lim_{y\to x} |f(y)| = \infty$, then x is a pole.
(3) If $\lim_{y\to x} |f(y)|$ doesn't exist, then p is a essential singularity.

Definition 1.1.7 (meromorphic function)**.** Let X be a Riemann surface and f be a holomorphic function defined on $U \setminus \{x\}$ where $U \subseteq X$ is an open subset.

(1) f is called a meromorphic function at x if x is either a removbale singularity or a pole, or f is holomorphic at x.
(2) f is called a meromorphic function on W, if it's meromorphic at any point $x \in W$.

Remark 1.1.9. If f, g are meromorphic on W, then $f \pm g, fg$ are meromorphic on W. If in addition, $g \not\equiv 0$, then f/g is also meromorphic on W. In other words, the set of meromorphic functions on W forms a field, which is called meromorphic function field.

Example 1.1.5. Consider f, g are two polynomials with $g \not\equiv 0$, then f/g is a meromorphic function on Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$. In fact, all meromorphic functions on S^2 are in this form.

Theorem 1.1.3 (discreteness of singularities and zeros)**.** Let X be a Riemann surface and $W \subseteq X$ be an open subset. If f is a meromorphic function on W, then set of singularities and zeros of f is discrete, unless $f \equiv 0$.

Corollary 1.1.2. Let X be a compact Riemann surface.

(1) If f is a non-zero meromorphic function, then f has finitely many poles and zeros on X.
(2) If f, g are two meromorphic functions on an open subset $W \subseteq X$, and f agrees with g on a set with limit point in W, then $f \equiv g$.

1.2. **Algebraic curves.**

1.2.1. *Affine plane curves.* Let $V \subseteq \mathbb{C}$ be a connected open subset and g be a holomorphic function defined on U. The graph X of g, as a subset of $\mathbb{C}^2$ is defined by

$$\{(z, g(z)) \mid z \in U\}.$$

Given X the subspace topology, and let $\pi\colon X \to U$ be the projection to the first factor. Note that π is a homeomorphism, whose inverse sends the point $z \in U$ to the ordered pair $(z, g(z))$. This makes X a Riemann surface.

A generalization of the graph of holomorphic function is that we consider "Riemann surface" which is locally a graph, but perhaps not globally. The tools we use is implicit function theorem in fact.

Theorem 1.2.1 (The implicit function theorem)**.** Let $f(z,w)\colon \mathbb{C}^2 \to \mathbb{C}$ be holomorphic function of two variables and $X = \{(z,w) \in \mathbb{C}^2 \mid f(z,w) = 0\}$ be its zero loucs. Let $p = (z_0, w_0)$ be a point of X and $\partial f/\partial z(p) \neq 0$. Then there exists a function $g(w)$ defined and holomorphic in a neighborhood of w_0 such that, near p, X is equal to the graph $z = g(w)$.

Method one. If we write $z = a + \sqrt{-1}b, w = c + \sqrt{-1}d$ and $f(z,w) = u + \sqrt{-1}v$, then u, v are smooth functions of a, b, c, d. Moreover, the Cauchy-Riemann equations give

$$\frac{\partial f}{\partial z} = \frac{\partial u}{\partial a} + \sqrt{-1}\frac{\partial v}{\partial a} = \frac{\partial v}{\partial b} - \sqrt{-1}\frac{\partial u}{\partial b} = A + \sqrt{-1}B.$$

Then

$$\frac{\partial(u,v)}{\partial(a,b)} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

and $\det \dfrac{\partial(u,v)}{\partial(a,b)} = A^2 + B^2 \neq 0$ if and only if $A + \sqrt{-1}B \neq 0$. Then the classical implicit function theorem implies the zero loucs

$$\begin{cases} u = 0 \\ v = 0 \end{cases}$$

is locally given by

$$\begin{cases} a = a(c,d) \\ b = b(c,d). \end{cases}$$

In other words, $z = g(w)$. Now it suffices to compute $\partial g/\partial \overline{w}$ to show g is holomorphic. Again by Cauchy-Riemann equations

$$\frac{\partial f}{\partial w} = \frac{\partial u}{\partial c} + \sqrt{-1}\frac{\partial v}{\partial c} = \frac{\partial v}{\partial d} - \sqrt{-1}\frac{\partial u}{\partial d} = C + \sqrt{-1}D.$$

Then by chain rule one has

$$\begin{aligned} \frac{\partial(a,b)}{\partial(c,d)} &= \left(\frac{\partial(u,v)}{\partial(a,b)}\right)^{-1} \frac{\partial(u,v)}{\partial(c,d)} \\ &= \begin{pmatrix} A & B \\ -B & A \end{pmatrix}^{-1} \begin{pmatrix} C & D \\ -D & C \end{pmatrix} \\ &= \frac{1}{A^2+B^2} \begin{pmatrix} AC+BD & AD-BC \\ BC-AD & BD+AC \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial g}{\partial \overline{w}} &= \frac{1}{2}\left(\frac{\partial}{\partial c} + \sqrt{-1}\frac{\partial}{\partial d}\right)(a + \sqrt{-1}b) \\ &= \frac{1}{2}\left(\frac{\partial a}{\partial c} + \sqrt{-1}\frac{\partial b}{\partial c} + \sqrt{-1}\frac{\partial a}{\partial d} - \frac{\partial b}{\partial d}\right) \\ &= 0 \end{aligned}$$

□

Method two. Firstly let's recall some basic facts in complex analysis: For a holomorphic function f defined on U, the integral

$$\frac{1}{2\pi\sqrt{-1}}\oint_{\partial U}\frac{f'(z)}{f(z)}\mathrm{d}z$$

counts the number of zeros of $f(z)$ in U with multiplicity, and the integral

$$\frac{1}{2\pi\sqrt{-1}}\oint_{\partial U}z\frac{f'(z)}{f(z)}\mathrm{d}z$$

is the summation of zeros of $f(z)$ in U. Now let's prove the implicit function theorem by using above observation. Fix $w=w_0$, the holomorphic function $f(z,w_0)$ has a zero at $z=z_0$, and we may choose an open neighborhood U of z_0 such that z_0 is the only zero of $f(z,w_0)$ in U since holomorphic function has discrete zeros. Consider the integral

$$\frac{1}{2\pi\sqrt{-1}}\oint_{\partial U}\frac{f_z(z,w)}{f(z,w)}\mathrm{d}z=N(w),$$

which is well-defined on sufficiently small neighborhood D_{w_0} of w_0. It gives a continous, integer-valued function with $N(w_0)=1$. This shows $N(w)=1$ for all $w\in D_{w_0}$, and thus $f(z,w)$ has only one zero for every $w\in D_{w_0}$. Moreover, this zero point z is given by

$$\frac{1}{2\pi\sqrt{-1}}\oint_{\partial U}z\frac{f_z(z,w)}{f(z,w)}\mathrm{d}z=g(w),$$

which is holomorphic with respect to w. □

Definition 1.2.1 (affine plane curve)**.** An affine plane curve is the loucs of zeros in $\mathbb{C}^2$ of a (non-trivial) polynomial $f(x,y)$.

Definition 1.2.2 (non-singular)**.**

(1) A polynomial $f(x,y)$ is non-singular at root x if either $\partial f/\partial x$ or $\partial f/\partial y$ is not zero at x, otherwise it's called singular.
(2) The affine plane curve X defined by $f(x,y)$ is non-singular is non-singular at $p\in X$ if f is non-singular at x.
(3) The curve X is non-singular if it's non-singular at each of its points.

Example 1.2.1. The affine plane curve $C\subseteq\mathbb{C}^2$ defined by x^2+y^2-1 is non-singular.

Given a non-singular affine plane curve C, by the implicit function theorem, one has C is locally a graph, and thus it gives a complex structure of C. To be precise, suppose C is defined by the non-singular polynomial $f(x,w)$. Let $p=(x_0,y_0)\in C$ with $\partial f/\partial x(p)\neq 0$, then there exists a holomorphic function $g(x)$ such that in an open neighborhood U of p, C is the graph $w=g(x)$. Thus the projection $\pi\colon U\to\mathbb{C}$, which maps $(x,y)\to x$ is a homeomorphism from U to its image, which is an open subset in $\mathbb{C}$. This gives a complex chart of C.

A straightforward computation shows that complex charts given as above are compatible with each other, and thus it gives a complex structure on C. Moreover, C is second countable and Hausdorff, as a subspace of $\mathbb{C}^2$. The only thing we need to check is C is connected. However, if f is an arbitrary non-singular polynomial, the affine plane curve defined by f may not be connected. For example, consider

$$f(x, y) = (x + y)(x + y - 1).$$

Then the affine plane curve defined by above non-singular polynomial is the union of two complex planes which do not meet. Later in Section 3.2.3 we will show that the plane curve defined by an irreducible polynomial must be connected. Thus we have the following theorem.

Theorem 1.2.2. A non-singular affine plane curve defined by an irreducible polynomial is a Riemann surface.

1.2.2. *Projective plane curve.*

Definition 1.2.3 (projective plane curve)**.** Let F be a homogenous polynomial in $\mathbb{C}[x, y, z]$. A projective plane curve C defined by F is the zero loucs of F, that is,

$$C = \{[x : y : z] \in \mathbb{P}^2 \mid F(x, y, z) = 0\}.$$

Remark 1.2.1 (relations between affine plane curve and projective plane curve)*.* Given a projective plane curve C given by homogenous polynomial F. Consider

$$\begin{aligned} \varphi_0 \colon U_0 = \mathbb{C}^2 &\to \mathbb{P}^2 \\ (y, z) &\mapsto [1 : y : z] \end{aligned}$$

Then $\varphi_0^{-1}(U_0 \cap C) = \{(y, z) \in \mathbb{C}^2 \mid F(1, y, z) = 0\}$ is an affine plane curve, and similarly there are other affine plane curves given by $\varphi_0^{-1}(U_1 \cap C)$ and $\varphi_0^{-1}(U_2 \cap C)$. Conversely, given an affine plane curve C defined by $f \in \mathbb{C}[y, z]$. Consider the homogenous polynomial $F(x, y, z)$ defined by

$$F(x, y, z) = x^d f(\frac{y}{x}, \frac{z}{x})$$

where $d = \deg f$. Then F defines a projective plane curve such that the affine plane curve it gives on affine chart U_0 is exactly C.

Definition 1.2.4 (non-singular)**.** A projective plane curve C is non-singular if the affine plane curves $\varphi_i^{-1}(U_i \cap C)$ are non-singular for $i = 0, 1, 2$, where $\varphi_i \colon U_i \to \mathbb{P}^2$ are standard affine covering of $\mathbb{P}^2$.

Proposition 1.2.1. A projective plane curve $C = \{[x : y : z] : F(x, y, z) = 0\} \subseteq \mathbb{P}^2$ is non-singular if and only if

$$F = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$$

has no solution in $\mathbb{P}^2$.

Proof. Since F is a homogenous polynomial, it satisfies the Euler's formula

$$dF = x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z},$$

where $d = \deg F$. Now let's start our proof as follows:

(1) Suppose $F = \partial F/\partial x = \partial F/\partial y = \partial F/\partial z = 0$ has a solution (a, b, c) with $a \neq 0$. Then

$$\begin{aligned}\frac{\partial F}{\partial y}(1, \frac{b}{a}, \frac{c}{a}) &= \frac{1}{a^{d-1}}\frac{\partial F}{\partial y}(a, b, c) = 0\\ \frac{\partial F}{\partial z}(1, \frac{b}{a}, \frac{c}{a}) &= \frac{1}{a^{d-1}}\frac{\partial F}{\partial z}(a, b, c) = 0\\ F(1, \frac{b}{a}, \frac{c}{a}) &= \frac{1}{a^{d}}F(a, b, c) = 0.\end{aligned}$$

This shows the affine plane curve $\varphi_0^{-1}(U_0 \cap C)$ is singular, and thus C is singular.

(2) Conversely, if the projective plane curve defined by F is singular, without lose of generality we may assume $X_0 := \varphi_0^{-1}(U_0 \cap C)$ is singular. Then there exists a solution $(b, c) \in \mathbb{C}^2$ such that

$$F(1, b, c) = \frac{\partial F}{\partial y}(1, b, c) = \frac{\partial F}{\partial z}(1, b, c) = 0.$$

By Euler's formula one has

$$\frac{\partial F}{\partial x}(1, b, c) = dF(1, b, c) - b\frac{\partial F}{\partial y} - c\frac{\partial F}{\partial z} = 0.$$

As a consequence, $(1, a, b)$ is a solution of $F = \partial F/\partial x = \partial F/\partial y = \partial F/\partial z = 0$.

□

Theorem 1.2.3. Any non-singular projective plane curve C is a compact Riemann surface.

Proof. Later we will show that a non-singular homogenous polynomial must be irreducible (See Proposition 3.2.1). Then the three affine charts of C are non-singular affine plane curve defined by irreducible polynomials, and thus Riemann surfaces by Theorem 1.2.2. A straightforward computation shows that the complex structures on each affine charts are compatible, and thus C is a Riemann surface. Moreover, it's compact since $\mathbb{P}^2$ is compact and C is a closed subset of $\mathbb{P}^2$. □

Remark 1.2.2. One way to understand projective plane curve is to regard it as a compactifications of affine plane curve.

Example 1.2.2 (Fermat curve)**.** $x^d + y^d = z^d$ gives a non-singular projective plane curve.

Example 1.2.3. The polynomial $f(x,y) = y^2-(1-x^2)(1-k^2x^2), k \neq 0, \pm 1$ gives a non-singular affine plane curve C. Now we consider the compactification of C. Let $F(x,y,z)$ be the homogenous polynomial given by $f(x,y)$, that is,

$$F(x,y,z) = z^2y^2 - (z^2-x^2)(z^2-k^2x^2).$$

$F(x,y,z)$ gives a projective plane curve, and the affine plane curve it gives on the affine chart U_2 is exactly C, so it suffices to see the affine plane curves it gives on the other affine charts.

(1) The affine plane curve it gives on the affine chart U_1 is defined by

$$f(x,1,z) = z^2 - (z^2-x^2)(z^2-k^2x^2).$$

In this case there is a new point $[0:1:0]$, which is singular.

(2) The affine plane curve it gives on the affine chart U_0 is defined by

$$f(1,y,z) = z^2y^2 - (z^2-1)(z^2-k^2).$$

But in this case, there is no more new point since there is no solution satisfying $z=0$.

In a summary, the compactification of the affine plane curve C adds a singular point, and later we will see how to handle with singularities by resolutions.

1.2.3. *Quadratic.* A homogenous polynomial F of degree 2 can be written as

$$F = (x,y,z)A\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where $A \in M_{3\times 3}(\mathbb{C})$ is a symmetric matrix. In this section we will see the projective plane curve C defined by F is determined by the rank of A.

Proposition 1.2.2. If $\operatorname{rk} A = 3$, then F is non-singular, and C is isomorphic to $\mathbb{P}^1$.

Method one. If $\operatorname{rk} A = 3$, then there exists $P \in \mathrm{GL}(3,\mathbb{C})$ such that

$$P^TAP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

This shows that afer a suitable change of coordinate, we may assume the projective plane curve C defined by F is $\{[x:y:z] \mid x^2+y^2-z^2=0\} \subseteq \mathbb{P}^2$. The following map gives an isomorphism between C and $\mathbb{P}^1$.

$$\begin{aligned} \Phi \colon \mathbb{P}^1 &\to C \\ [1:t] &\mapsto [1-t^2 : 2t : 1+t^2]. \end{aligned}$$

$\square$

Method two. Consider the following holomorphic embedding

$$\begin{aligned}\Phi\colon \mathbb{P}^1 &\to \mathbb{P}^2\\ [t_0 : t_1] &\mapsto [t_0^2 : t_0t_1 : t_1^2].\end{aligned}$$

Note that the image of Φ is a projective plane curve defined by the equation $xz = y^2$. On the other hand, after a suitable change of coordinate we may also assume C is defined by this equation since there also exists $P \in \mathrm{GL}(3, \mathbb{C})$ such that

$$P^TAP = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

□

Proposition 1.2.3. If $\operatorname{rk} A = 2$, then C is isomorphic to the union of two $\mathbb{P}^1$.

Proof. If $\operatorname{rk} A = 2$, then there exists $P \in \mathrm{GL}(3, \mathbb{C})$ such that

$$P^TAP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This shows the projective plane curve C is defined by $x^2 + y^2 = (x + \sqrt{-1}y)(x - \sqrt{-1}y)$, which is the union of two $\mathbb{P}^1$ which intersects at $[0 : 0 : 1]$. In particular, it's singular. □

Proposition 1.2.4. If $\operatorname{rk} A = 1$, then C is isomorphic to a double line.

Proof. If $\operatorname{rk} A = 1$, then there exists $P \in \mathrm{GL}(3, \mathbb{C})$ such that

$$P^TAP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This shows the projective plane curve C is defined by $x^2 = 0$, which is a singular projective plane curve called double line. □

2. Ramification

Topologically speaking a Riemann surface is an orientable 2-dimensional real manifold without boundary. In particular, the topology of a compact Riemann surface can be classified by its genus. So there is a natural question: Given a non-singular projective plane curve C defined by the homogenous polynomial $F(x,y,z)=y^2z-x(x-z)(x-\lambda z), \lambda\neq 0,1$, topologically C is a closed orientable surface, is there any way to compute its genus?

Consider the following map

$$\begin{aligned}\Phi\colon C\setminus[0:1:0]&\to\mathbb{P}^1\\ [x:y:z]&\mapsto[x:z].\end{aligned}$$

It's clear that Φ is well-defined holomorphic map. If we desire to extend F to a holomorphic map $\widetilde{\Phi}$ defined on C, we need to consider the behavior of C around $[0:1:0]$. On affine chart $U_1=\{[x:1:z]\mid x,z\in\mathbb{C}\}$, it gives an affine plane curve defined by

$$f(x,z)=z-x(x-z)(x-\lambda z).$$

A direct computation shows that

$$\left.\frac{\partial f}{\partial z}\right|_{(0,0)}=1,\quad \left.\frac{\partial f}{\partial x}\right|_{(0,0)}=0.$$

Then by implicit function theorem, C is given by $[x:1:z(x)]$ locally around $[0:1:0]$, and

$$z'(0)=-\left.\frac{\partial p}{\partial x}\right|_{(0,0)}/\left.\frac{\partial p}{\partial z}\right|_{(0,0)}=0/1=0.$$

Thus $x=0$ is a removable singularity of $z(x)/x$, so it's reasonable to define $\widetilde{\Phi}([0:1:0])=[1:0]$ to give an extension of Φ since for $x\neq 0$,

$$\Phi([x:1:z(x)])=[x:z(x)]=[1:\frac{z(x)}{x}].$$

There are four special points for $\widetilde{\Phi}\colon C\to\mathbb{P}^1$, listed as follows

$$\begin{aligned}[0:1:0]&\mapsto[1:0]\\ [0:0:1]&\mapsto[0:1]\\ [z:0:1]&\mapsto[z:1]\\ [\lambda z:0:1]&\mapsto[\lambda z:1].\end{aligned}$$

These points are called ramification points/values of $\widetilde{\Phi}$, and besides these points, $\widetilde{\Phi}$ is a double covering. Such a holomorphic map is called a ramification covering, and in this section we will show that all holomorphic maps between Riemann surfaces are ramification coverings. Moreover, we introduce the Riemann-Hurwitz formula, which gives a method to compute the genus of the ramification covering of a given space.

2.1. **Ramification covering.**

Theorem 2.1.1 (local normal form)**.** Let $\Phi\colon X \to Y$ be a non-constant holomorphic map. Then there are local coordinates (U, φ) and (V, ψ) of p and $\Phi(p)$ respectively, such that

$$\psi \circ \Phi \circ \varphi^{-1}(z) = z^k$$

holds for all $z \in \varphi(U \cap \Phi^{-1}(V))$.

Proof. Firstly we fix a local coordinate (V, ψ) of $\Phi(p)$, and choose a local coordinate (U_1, φ_1) of p such that $\Phi(U) \subset V$. If we denote $\psi \circ \Phi \circ \varphi_1^{-1} = T$, then $T(0) = 0$. Suppose the Taylor expansion of T at $w = 0$ is

$$T(w) = \sum_{k=m}^{\infty} a_k w^k, \quad a_m \neq 0.$$

Then $T(w) = w^m S(w)$, where $S(w)$ is a holomorphic function with $S(0) \neq 0$, and thus there exists a holomorphic function $R(w)$ such that $R^m(w) = S(w)$.

Then $T(w) = (wR(w))^m = (\eta(w))^m$, where $\eta(0) = 0, \eta'(0) = R(0) \neq 0$. By inverse function theorem, there exists a sufficiently small neighborhood $U \subseteq U_1$ of p such that η is invertible in $\varphi_1(U)$, and thus this gives a new local coordinate of p as

$$U_1 \supseteq U \xrightarrow{\varphi_1} \varphi_1(U) \xrightarrow{\eta} \eta \circ \varphi_1(U) \subset \mathbb{C}.$$

If we define $\varphi = \eta \circ \varphi_1$, then with respect to (U, φ) and (V, ψ), the local representation of Φ is given by

$$\psi \circ \Phi \circ \varphi^{-1}(z) = \psi \circ \Phi \circ \varphi_1^{-1} \circ \eta^{-1}(z) = T(\eta^{-1}(z)) = z^m.$$

□

Definition 2.1.1 (multiplicity)**.** Let $\Phi\colon X \to Y$ be a holomorphic map between Riemann surfaces. If its local normal form at point $p \in X$ is given by $z \mapsto z^k$, then k is called the multiplicity[1] of Φ at p, denoted by $\mathrm{mult}_p(\Phi)$.

Definition 2.1.2 (ramification point and ramification value)**.** Let $\Phi\colon X \to Y$ be a holomorphic map between Riemann surfaces. A point $p \in X$ is called a ramification point if $\mathrm{mult}_p(\Phi) > 1$, and the image of ramification point is called a ramification value.

Lemma 2.1.1. Let $\Phi\colon X \to Y$ be a non-constant holomorphic map between Riemann surfaces. A point $p \in X$ is a ramification point if there exists some local representation of Φ, denoted by T, such that $T'(0) = 0$.

Corollary 2.1.1. The set of ramification points of a holomorphic map is a discrete set.

[1]Sometimes this number is also called ramification of F at p.

Theorem 2.1.2. Let $\Phi\colon X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces and define

$$d_q(\Phi) = \sum_{p\in\Phi^{-1}(q)} \mathrm{mult}_p(\Phi).$$

Then $d_q(\Phi)$ is independent of $q \in Y$, which is called the degree of Φ, and denoted by $\deg(\Phi)$.

Proof. Suppose $X = Y = \mathbb{D}$ are unit disks and $F\colon \mathbb{D} \to \mathbb{D}$ is a holomorphic map defined by $z \mapsto z^m$. Then it's easy to show $d_q(\Phi) = m$, for all $q \in \mathbb{D}$, since for $q = 0$, there is only one preimage of multiplicity m and for $q \neq 0$, there are m preimages of multiplicity 1.

Let's consider the general case. For $q \in Y$, since X is compact, $\Phi^{-1}(q)$ only consists of finitely many points, denoted by $\{p_1, \dots, p_k\}$. Fix a local coordinate (V, ψ) centered at $q \in Y$, for any $i = 1, \dots, k$, there is a local coordinate (U_i, φ_i) centered at $p_i \in X$ such that

$$\psi \circ \Phi \circ \varphi_i^{-1}(z) = z^{m_i}, \quad z \in \varphi_i(U_i),$$

where $m_i = \mathrm{mult}_{p_i}(\Phi)$. If we choose another neighborhood $q \in W \subseteq V$ such that $\Phi^{-1}(W) \subseteq \bigcup_{i=1}^k U_i$, then for any $q \in W$, from the trivial case discussed before one has

$$d_q(\Phi) = \sum_{i=1}^k m_i.$$

This shows $d_q(\Phi)$ is a locally constant function, and thus $d_q(\Phi)$ is constant since Y is connected. □

Corollary 2.1.2. A holomorphic map between compact Riemann surfaces is an isomorphism if and only if it has degree one.

Corollary 2.1.3. X is a compact Riemann surface, and f is a meromorphic function on X, then the number (counted with multiplicity) of zeros is equal to the number (counted with multiplicity of poles.

Proof. Note that meromorphic function f on X is equivalent to the holomorphic map Φ from X to S^2. Then the number of zeros is the multiplicity of Φ at zero and the number of poles is the multiplicity of Φ at ∞. □

2.2. **Riemann-Hurwitz formula.** In this section we talk about Riemann-Hurwitz formula, which computes the genus from a given ramification covering. Before that we recall some basic facts in topology. Let X be a compact oriented surface, the Euler number of X can be defined by the triangulation of X as follows: Suppose a triangulation of X is given with v vertices, e edges and t tirangles. Then the Euler characterisitic of X is defined by $v - e + t$. On the other hand, the Euler number can also be defined as

$$\chi(X) := \sum_i (-1)^i \dim H_i(X; \mathbb{R}),$$

where $H_i(X;\mathbb{R})$ is the i-th singular holology of X. The genus of X is defined by

$$\chi(X) = 2 - 2g_X.$$

Theorem 2.2.1 (Riemann-Hurwitz formula)**.** Let $\Phi\colon X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces. Then

$$\chi(X) = \deg(\Phi)\chi(Y) - \sum_{p\in X}(\mathrm{mult}_p(\Phi) - 1)$$

Proof. Choose a triangulation Δ of Y such that its vertex are exactly ramification values of F. Let v, e, t denote the number of vertices, edges and triangles of Δ respectively. Suppose Δ' is the triangulation of X obtained by pulling back Δ through F, and use v', e' and t' to denote the number of vertices, edges and triangles of Δ' respectively.

It's clear we have the following relations

$$t' = td, \quad e' = ed$$

where $d = \deg(\Phi)$. For $q \in Y$, note that

$$|\Phi^{-1}(q)| = \sum_{p\in\Phi^{-1}(q)} 1 = d + \sum_{p\in\Phi^{-1}(q)} (1 - \mathrm{mult}_p(\Phi)).$$

Then the relation between v and v' is given by

$$\begin{aligned}
v' &= \sum_{\text{vertex } q \text{ of } \Delta} |\Phi^{-1}(q)| \\
&= \sum_{\text{vertex } q \text{ of } \Delta} \left(d + \sum_{p\in\Phi^{-1}(q)} (1 - \mathrm{mult}_p(\Phi)) \right) \\
&= vd + \sum_{\text{vertex } q \text{ of } \Delta} \left(\sum_{p\in\Phi^{-1}(q)} (1 - \mathrm{mult}_p(\Phi)) \right) \\
&= vd + \sum_{p\in X} (1 - \mathrm{mult}_p(\Phi)).
\end{aligned}$$

Thus by the relation between Euler number and triangulation, we obtain the desired conclusion. □

Remark 2.2.1. Since the set of ramification points is finite, then $\sum_{p\in X}(\mathrm{mult}_p(\Phi) - 1)$ is a finite number, and for convenience we denote it by $B(\Phi)$. It describes how many ramification points of Φ are there on X.

Definition 2.2.1 (ramified holomorphic map)**.** A holomorphic map Φ is called ramified if $B(\Phi) > 0$.

Definition 2.2.2 (unramified holomorphic map)**.** A holomorphic map Φ is called unramified if $B(\Phi) = 0$.

Remark 2.2.2. A unramified holomorphic map is a covering map, and thus ramified holomorphic map is sometimes called ramified covering map.

Corollary 2.2.1. Let $\Phi\colon X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces. Then

(1) If Y is Riemann sphere and $\deg(\Phi) > 1$, then Φ must be ramified.
(2) If $g_X = g_Y = 1$, then Φ must be unramified.
(3) If $g_X = g_Y > 1$, then Φ must be an isomorphism.

Proof.

(1) Since Riemann sphere has genus zero, one has
$$B(\Phi) = 2(\deg(\Phi) - 1) + 2g_X > 0.$$
(2) By Riemann-Hurwitz formula we have
$$0 = 0 + B(\Phi).$$
(3) By Riemann-Hurwitz formula we have
$$(1 - \deg(\Phi))(2g_X - 2) = B(\Phi).$$
Then $\deg(\Phi) = 1$, since $\deg(\Phi) \geq 1$, $2g_X - 2 > 0$ and $B(\Phi) \geq 0$.

□

2.2.1. *Genus of projective plane curve.* Now we're going to use Riemann-Hurwitz formula to compute the genus of projective plane curves. Firstly consider the example at the beginning of this section, that is, the non-singular projective plane curve C is defined by homogenous polynomial
$$F(x,y,z) = y^2z - x(x-z)(x-\lambda z),$$
where $\lambda \neq 0,1$. The ramification covering $\widetilde{\Phi}\colon C \to \mathbb{P}^1$ has degree 2, and the ramification values are $[1:0],[0:1],[z:1],[\lambda z:1]$. Then by Riemann-Hurwitz formula one has
$$\chi(C) = 2 \times 2 - 4$$
This shows the genus of C is 1.

Example 2.2.1 (Fermat curve)**.** Let C be the projective plane curve defined by the homogenous polynomial $F(x,y,z) = x^d + y^d - z^d$. A direct computation shows C is non-singular, and thus it gives a Riemann surface. Consider the holomorphic map
$$\Phi\colon C \to \mathbb{P}^1$$
$$[x:y:z] \mapsto [x:z].$$
Note that
$$y^d = z^d - x^d = (x - \alpha_1 z)\dots(x - \alpha_d z),$$
where $\alpha_1, \dots, \alpha_d \in \mathbb{C}$ are different d-th unit roots. Then Φ is a ramification covering of degree d, and has d ramification values. Then by Riemann-Hurwitz formula,
$$\chi(C) = 2 \times d - d(d-1).$$

This shows the genus of C is $(d-1)(d-2)/2$.

Remark 2.2.3. In general, for a non-singular projective plane curve C defined by a homogenous polynomial of degree d, the genus of C is $(d-1)(d-2)/2$, and this is called Plücker's formula or genus-degree formula. Moreover, if C is singular, then the genus of the normalization of C is

$$\frac{(d-1)(d-2)}{2} - \delta,$$

where $\delta > 0$ is related to the type of singularities of C.

3. Bezout theorem

3.1. **Statement and proof.** Let C, C' be a non-singular projective plane curves defined by a homogenous polynomials F, G such that F, G has no common divisors[2]. In this section we will show how to count the number of the intersections of C and C'.

Definition 3.1.1 (intersection number)**.** The intersection number at point $p \in C \cap C'$ is the order of zero of G at p on some affine chart on C. The intersection number of C and C' is the summation of intersection numbers of all intersections $p \in C \cap C'$.

Remark 3.1.1. Note that the change of affine charts does not change the vanishing order of a polynomial. This shows the intersection number of an intersection is well-defined. For convenience, the intersection number at point p is denoted by $(C, C')_p$, and the intersection number of C and C' is denoted by (C, C'), that is,

$$(C, C') = \sum_{p \in C \cap C'} (C, C')_p.$$

A standard exercise to show $(C, C')_p = (C', C)_p$.

Theorem 3.1.1 (Bezout theorem)**.** Let C, C' be two non-singular projective plane curves defined by homogenous polynomials F, G such that F, G has no common divisors. Then the intersection number

$$(C, C') = ed,$$

where $\deg F = e, \deg G = d$.

Proof. Let L be a linear homogenous polynomial such that $L \nmid F$ and H be the projective line defined by L. Consider the holomorphic map

$$\Phi \colon C \to \mathbb{P}^1$$
$$[x : y : z] \mapsto [L^d : G].$$

Since C is compact, by Corollary 1.1.1 one has Φ is surjective.

(1) Suppose Φ is a non-constant holomorphic map. Note that the order of zeros of Φ equals (C, H^d), and the order of poles of Φ equals to (C, C'). Then
$$(C, H^d) = (C, C').$$
since both order of zeros and order of poles are degree of Φ. By definition one has
$$(C, H^d) = d(C, H).$$
Now it suffices to show a projective plane curve defined by a homogenous polynomial with degree e intersects a projective line e times, which is straightforward.

[2]Since both F and G are irreducible, this assumption exclude the trivial cases $F \mid G$ and $G \mid F$.

(2) If Φ is a constant holomorphic map, then there exists a constant $\lambda \in \mathbb{C}^*$ such that $G = \lambda L^d$. Again one has
$$(C, L^d) = (C, \lambda H^d) = (C, C'),$$
since $\lambda \neq 0$.

□

3.2. **Applications.**

3.2.1. *Plücker formula.* In this section we will prove Plücker formula as a consequence of Bezout theorem, but before that we prove a technique lemma.

Lemma 3.2.1. Let C be a projective plane curve of degree d. Then there exists an affine coordinate $[x : y : 1] \subseteq \mathbb{P}^2$ such that C is given by the following equation
$$f(x, y) = y^d + a_1(x)y^{d-1} + \cdots + a_d(x) = 0,$$
where $a_i(x) \in \mathbb{C}[x]$ with $\deg a_i(x) \leq i$, or $a_j(x) = 0$.

Proof. Let $[z : w : 1]$ be an arbitrary affine coordinate of $\mathbb{P}^2$ and C is defined by
$$p'(z, w) = 0, \quad \deg p' = d.$$
If p' is not of the necessary form, then consider the following coordinate transformation
$$\begin{aligned} z &= x + \lambda y \\ w &= y. \end{aligned}$$
Consider the coefficient $b(\lambda)$ of the term involving y^n in $f'(x + \lambda y, y)$. It's clear $b(\lambda)$ is a non-zero polynomial in λ, and hence can equal 0 for only a finite number of values of λ. Then we choose λ such that $b(\lambda) \neq 0$, and for such a chosen λ, we consider
$$f(x, y) = \frac{1}{b(\lambda)} f'(x + \lambda y, y).$$
Then in affine coordinate $[x : y : 1]$, the equation of C is
$$f(x, y) = 0,$$
which satisfies our desire. □

Corollary 3.2.1 (Plücker formula)**.** Let $C \subseteq \mathbb{P}^2$ be a non-singular projective plane curve of degree d. Then the genus of C is $(d - 1)(d - 2)/2$.

Proof. By Lemma 3.2.1, without lose of generality we may assume C is defined by the non-singular homogenous polynomial F with
$$F(x, y, z) = y^d - a_1(x, z)y^{d-1} - \cdots - a_d(x, z).$$
Then consider the following holomorphic map
$$\begin{aligned} \Phi \colon C &\to \mathbb{P}^1 \\ [x : y : z] &\mapsto [x : z], \end{aligned}$$

which is a ramification covering in fact. Now by Riemann-Hurwitz formula it suffices to compute the ramification data of Φ. On affine charts $U_2 = \{[x : y : 1]\} \subseteq \mathbb{P}^2$, C is defined by

$$f(x, y) = y^d - a_1(x, 1)y^{d-1} - \cdots - a_d(x, 1) = 0.$$

(1) If $f_y(x_0, y_0) \neq 0$, then by implicit function theorem, around the point $[x_0 : y_0 : 1]$, the affine plane curve $C \cap U_2$ is given by $[x : y(x) : 1]$, and thus Φ is a local diffeomorphism at this point.
(2) If $f_y(x_0, y_0) = 0$, then $f_x(x_0, y_0) \neq 0$ since f is non-singular. By implicit function theorem again, around $[x_0 : y_0 : 1]$, there exists a local coordinate $y \mapsto [x(y) : y : 1]$, and Φ is given by $y \mapsto x(y)$. By chain rule one has
$$x'(y) = -(f_x(x(y), y))^{-1} f_y(x(y), y).$$
This shows the order of zero of $x'(y)$ equals to the order of zero of $f_y(x(y), y)$.

This shows $B(\Phi) = \sum_{p \in C}(\mathrm{mult}_p(\Phi) - 1)$ is exactly the intersection number of F and F_y, and since both F and F_y are non-singular homogenous polynomial, by Bezout theorem one has

$$B(\Phi) = d(d - 1).$$

By Rmann-Hurwitz formula, the genus of C is $(d - 1)(d - 2)/2$. □

3.2.2. *Non-singular homogenous polynomial is irreducible.* Another application of Bezout theorem is that any non-singular homogenous polynomial is irreducible.

Proposition 3.2.1. Let F be a non-singular homogenous polynomial. Then F is irreducible.

Proof. On contrary we suppose $F = F_1F_2$. By chain rule of derivative it's easy to see both F_1 and F_2 are non-singular. Then by Bezout theorem, F_1 and F_2 have at least a common zero, which contradicts to P is non-singular, since P is singular at the common zero of F_1 and F_2, which can be shown by chain rule of derivatives again. □

3.2.3. *Connectness of irreducible plane curve.* In this section, we will prove the connectness of plane curves as we mentioned before. In fact, we will prove the following stronger theorem.

Theorem 3.2.1. Let F be an irreducible homogenous polynomial and C be the projective plane curve defined by F. Then the set of singularities S is finite, and $C \setminus S$ is connected.

Before starting the proof, we prepare some basic facts we will use.

Lemma 3.2.2. If R is a UFD and

$$f = a_0x^m + a_1x^{m-1} + \cdots + a_m,$$
$$g = b_0x^n + b_1x^{n-1} + \cdots + b_n$$

are polynomials in $R[x]$ with $a_0 \neq 0, b_0 \neq 0$. Then f, g has a non-trivial common divisor if and only if there exists $F, G \in R[x]$ with $\deg F < m, \deg G < n$ such that

$$f \cdot G = F \cdot g.$$

Proof. On one hand, if f, g has a non-trivial common divisor h, then

$$\begin{aligned} f &= h \cdot F \\ g &= h \cdot G. \end{aligned}$$

This shows $f \cdot G = F \cdot g$, where $\deg F < \deg f \leq m$ and $\deg G < \deg g \leq n$.

On the other hand, if $f \cdot G = F \cdot g$ with $\deg F < m$ and $\deg G < n$, then all factors of f cannot be all factors of F since $\deg f > \deg F$. Hence there exists a non-trivial divisor of f which is also a divisor of g since $R[x]$ is UFD by Gauss lemma. □

Suppose

$$\begin{aligned} F(x) &= A_0 x^{m-1} + \cdots + A_{m-1} \\ G(x) &= B_0 x^{n-1} + \cdots + B_{n-1}. \end{aligned}$$

Then $f \cdot G = F \cdot g$ if and only if

$$\begin{cases} a_0 B_0 = b_0 A_0 \\ a_1 B_0 + a_0 B_1 = b_1 A_0 + b_0 A_1 \\ \vdots \\ a_m B_{m-1} = b_n A_{m-1}. \end{cases} \tag{3.1}$$

Thus $f \cdot G = F \cdot g$ has non-zero solutions F, G if and only if (3.1) has a non-zero solution $(A_0, \ldots A_{m-1}, B_0, \ldots, B_{n-1})$. Then by basic theory of systems of linear equations, (3.1) has a non-zero solution if and only if the following determinant equals to zero.

$$\det \begin{pmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 & b_1 & b_0 & \cdots & 0 \\ a_2 & a_1 & \ldots & 0 & b_2 & b_1 & \ldots & 0 \\ \vdots & \vdots & \ldots & a_0 & \vdots & \vdots & \ldots & b_0 \\ a_m & a_{m-1} & \cdots & \vdots & b_n & b_{n-1} & \cdots & \vdots \\ 0 & a_m & \ldots & \vdots & 0 & b_n & \ldots & \vdots \\ \vdots & \vdots & \ldots & a_{m-1} & \vdots & \vdots & \ldots & b_{n-1} \\ 0 & 0 & \cdots & a_m & 0 & 0 & \cdots & b_n \end{pmatrix} \tag{3.2}$$

Definition 3.2.1 (resultant)**.** If R is a ring and

$$\begin{aligned} f &= a_0 x^m + a_1 x^{m-1} + \cdots + a_m, \\ g &= b_0 x^n + b_1 x^{n-1} + \cdots + b_n \end{aligned}$$

are polynomials in $R[x]$. The resultant of f, g is defined as the determinant in (3.2), and denoted by $\mathscr{R}(f, g)$.

Theorem 3.2.2. If R is a UFD and

$$f = a_0x^m + a_1x^{m-1} + \cdots + a_m,$$
$$g = b_0x^n + b_1x^{n-1} + \cdots + b_n$$

are polynomials in $R[x]$ with $a_0 \neq 0$, then

(1) f, g have a non-trivial common divisor if and only if $\mathscr{R}(f, g) = 0$;
(2) there exists polynomial $\alpha, \beta \in R[x]$, with $\deg \alpha < n, \deg \beta < m$ such that
$$\alpha(x)f(x) + \beta(x)g(x) = \mathscr{R}(f, g).$$

Definition 3.2.2 (discriminant)**.** Let R be a ring and $f \in R[x]$. The discriminant of p is defined by $\mathscr{D}(f) := \mathscr{R}(f, f')$, where f' is the formal derivative of f.

Corollary 3.2.2. Let R be a UFD and $f \in R[x]$. Then f has a multiple root if and only if $\mathscr{D}(f) = 0$.

Now let's start the proof of Theorem 3.2.1.

Proof. Firstly let's shows F has only finitely many singularities. By Lemma 3.2.1, without lose of generality we may assume C is defined by

$$f(x, y) = y^d + a_1(x)y^{d-1} + \cdots + a_d(x)$$

on some affine chart. If we regard $f(x, y)$ and $f_y(x, y)$ as elements in $\mathbb{C}[x][y]$, then $R(f, f_y) \in \mathbb{C}[x]$, which is a non-zero polynomial since $f(x, y)$ is irreducible. By Theorem 3.2.2 there exists $\alpha, \beta \in \mathbb{C}[x, y]$ such that

$$\alpha(x, y)f(x, y) + \beta(x, y)f_y(x, y) = \mathscr{R}(f, f_y)(x).$$

If point (x_0, y_0) such that $f(x_0, y_0) = f_y(x_0, y_0) = 0$, then

$$\mathscr{R}(f, f_y)(x_0) = 0.$$

This shows $f(x, y) = f_y(x, y) = 0$ has finitely many solutions, and thus C only has finitely many singularities on this affine chart. On the other hand, the infty line $z = 0$ only intersects with C finitely many times, and thus there are at most finitely many singularities on $z = 0$. As a consequence, C has only finitely many singularities.

To prove $C^* = C \setminus S$ is connected, it suffices to show C^* is connected on the affine chart $U_2 = \{[x : y : 1]\}$ since

$$C^* \cap U_2 \subseteq C^* \subseteq C = \overline{C^* \cap U_2},$$

and a basic fact in point set topology says that if a set is connected, so is its closure. For convenience, in the following proof we still use C to denote the affine plane curve $C \cap U_2$. Now consider the ramification covering

$$\Phi \colon C \to \mathbb{C}$$
$$(x, y) \mapsto x.$$

If we define $B = \{x_0 \in \mathbb{C} \mid \mathscr{R}(f, f_y)(x_0) = 0\} \subseteq \mathbb{C}^1$, then the argument in the proof of Corollary 3.2.1 can be used here to show $\Phi \colon C \setminus \Phi^{-1}(B) \to \mathbb{C} \setminus B$

is a local diffeomorphism. Thus $\Phi\colon C \setminus \Phi^{-1}(B) \to \mathbb{C} \setminus B$ is a covering map on each component of $C \setminus \Phi^{-1}(B)$ since Φ is a proper.

For each point $x_0 \notin B$, the fiber $\Phi^{-1}(x_0)$ are exactly the d distinct solutions of $y^d + a_1(x_0)y^{d-1} + \cdots + a_d(x_0) = 0$ has d distinct solutions, denoted by $\{y_1(x_0), \ldots, y_d(x_0)\}$. By the basic theory of covering space, there is an action of the fundamental group $\pi_1(\mathbb{C} \setminus B, x_0)$ on the fiber $\Phi^{-1}(x_0)$. To be precise, given $[\gamma] \in \pi_1(\mathbb{C} \setminus B, x_0)$, we choose arbitrary representive $\gamma \in [\gamma]$ and consider its lift $\widetilde{\gamma}$, which is independent of the choice of γ. If $y_i(x_0)$ and $y_j(x_0)$ are endpoints of $\widetilde{\gamma}$, then $[\gamma] \cdot y_i(x_0) = y_j(x_0)$. Thus it's clear to see the number of connected components of $C \setminus \Phi^{-1}(B)$ equals to the number of orbits of $\Phi^{-1}(x_0)$ under the $\pi_1(\mathbb{C} \setminus B, x_0)$-action.

Suppose $\{y_1(x_0), \ldots, y_l(x_0)\}$ is an orbit of $\pi_1(\mathbb{C} \setminus B, x_0)$-action. Then for any $x \notin B$, we choose a path $\gamma\colon [0,1] \to \mathbb{C} \setminus B$ connecting x_0 and x. Then γ has l different liftings ending at points $y_1(x), \ldots, y_l(x)$, which can be extended as holomorphic functions defined on an open neighborhood of x. If we define

$$\begin{aligned}
\sigma_1(x) &= \sum_i y_i(x) \\
\sigma_2(x) &= \sum_{i<j} y_i(x) y_j(x) \\
&\vdots \\
\sigma_l(x) &= y_1(x) \ldots y_d(x),
\end{aligned}$$

then $\sigma_i(x)$ does not depend on the choice of paths connecting x_0 and x, and thus $\sigma_i(x)$ are holomorphic functions defined over $\mathbb{C} \setminus B$. By Rouché's theorem, one can see these $\sigma_i(x)$ has polynomial growth, that is, there exists constants C and N such that

$$|\sigma_i(x)| < C|x|^N$$

holds for all $i = 1, \ldots, l$. Then by Riemann extension theorem one has $\sigma_i(x)$ are defined on $\mathbb{C}$, and they are polynomials of x in fact. Note that

$$(y - y_1(x)) \ldots (y - y_l(x)) \mid f(x, y).$$

Then

$$g(x,y) = y^d - \sigma_1(x) y^{d-1} + \sigma_2(x) y^{d-2} + \cdots + (-1)^l \sigma_l(x) \in \mathbb{C}[x, y]$$

also divides $f(x,y)$. But since $f(x,y)$ is irreducible, one has $f = g$, and thus the $\pi_1(\mathbb{C} \setminus B, x_0)$-action is transitive as desired. $\square$

4. Differential forms

4.1. Differential forms, differential operators and integrations.

4.1.1. *Differential forms.* Firstly let's consider the differential forms defined on an open subset $U \subseteq \mathbb{C}$. Suppose $\{z\}$ is the coordinate on $\mathbb{C}$. Then

(1) A smooth 1-form is of the form $f\mathrm{d}z + g\mathrm{d}\overline{z}$, where f, g are smooth functions, and the set of all smooth 1-forms defined on U is denoted by $\mathcal{A}^1(U)$.
(2) A smooth 1-form is a $(1,0)$-form, if it's of the form $f\mathrm{d}z$, where f is a smooth function, and the set of all $(1,0)$-form defined on U is denoted by $\mathcal{A}^{1,0}(U)$.
(3) A smooth 1-form is a $(0,1)$-form, if it's of the form $f\mathrm{d}\overline{z}$, where f is a smooth function, and the set of all $(0,1)$-form defined on U is denoted by $\mathcal{A}^{0,1}(U)$.
(4) A smooth 1-form is a holomorphic 1-form, if it's of the form $f\mathrm{d}z$, where f is a holomorphic function, and the set of all holomorphic 1-form defined on U is denoted by $\Omega^1_X(U)$.
(5) A smooth 2-form is of the form $f\mathrm{d}z \wedge \mathrm{d}\overline{z}$, where f is a smooth function, and the set of all smooth 2-forms defined on U is denoted by $\mathcal{A}^2(U)$.
(6) A holomorphic 2-form is of the form $f\mathrm{d}z \wedge \mathrm{d}\overline{z}$, where f is a holomorphic function, and the set of all holomorphic 2-forms defined on U is denoted by $\Omega^2(U)$.

Remark 4.1.1. It's clear $\mathcal{A}^1(U) = \mathcal{A}^{1,0}(U) \oplus \mathcal{A}^{0,1}(U)$.

If we want to define differential forms on Riemann surfaces, a natural idea is to define them on each coordinate chart, and glue them together in a suitable way, so we need to know what will happen under the holomorphic change of coordinate.

Suppose $\Phi \colon U \to V$ is a holomorphic function between open subsets $U, V \subseteq \mathbb{C}$ and $\theta = f\mathrm{d}w + g\mathrm{d}\overline{w}$ is a smooth 1-form on V. Then the pullback of θ is defined by

$$\Phi^*(\theta) = f(\Phi(z))\Phi'(z)\mathrm{d}z + g(\Phi(z))\overline{\Phi'(z)}\mathrm{d}\overline{z}.$$

Similarly, if $\theta = f\mathrm{d}w \wedge \mathrm{d}\overline{w}$ is a smooth 2-form, then the pullback is defined by

$$\Phi^*(\theta) = f(\Phi(z))|\Phi'(z)|^2\mathrm{d}z \wedge \mathrm{d}\overline{z}.$$

In fact, pullback is a contravariant functor.

Definition 4.1.1 (k-form)**.** A smooth (holomorphic) k-form θ on a Riemann surface X assigns to any local coordinate $\varphi \colon U \to V$ a smooth (holomorphic) k-form α, and assignments are compatible[3] with the charts.

[3]This means if $U' \xrightarrow{\varphi'} V'$ is another local coordinate assigned with k-form β, then

$$\Phi^*(\beta) = \alpha,$$

where $\Phi = \varphi' \circ \varphi^{-1}(z)$.

Definition 4.1.2 ($(1,0)$-form and $(0,1)$-form)**.** A smooth 1-form θ on a Riemann surface X is called

(1) a $(1,0)$-form, if it can be represented as $f\mathrm{d}z$ locally, where f is a smooth function;
(2) a $(0,1)$-form, if it can be represented as $f\mathrm{d}\overline{z}$ locally, where f is a smooth function.

Definition 4.1.3 (holomorphic 1-form)**.** A holomorphic 1-form θ on a Riemann surface X is a differential $(1,0)$-form which can be locally represented as $f(z)\mathrm{d}z$, where f is a holomorphic function.

4.1.2. *Differential operators.* Given a smooth function f defined on an open subset $U\subseteq\mathbb{C}$, one has

$$\mathrm{d}f=\frac{\partial f}{\partial z}\mathrm{d}z+\frac{\partial f}{\partial\overline{z}}\mathrm{d}\overline{z}.$$

The operators ∂ and $\overline{\partial}$ on smooth functions as follows

$$\begin{aligned}\partial f&:=\frac{\partial f}{\partial z}\mathrm{d}z\\ \overline{\partial}f&:=\frac{\partial f}{\partial\overline{z}}\mathrm{d}\overline{z}.\end{aligned}$$

For a smooth 1-form $\theta=f\mathrm{d}z+g\mathrm{d}\overline{z}$, similarly one has

$$\mathrm{d}\theta=\frac{\partial f}{\partial\overline{z}}\mathrm{d}\overline{z}\wedge\mathrm{d}z+\frac{\partial g}{\partial z}\mathrm{d}z\wedge\mathrm{d}\overline{z}=(\frac{\partial g}{\partial z}-\frac{\partial f}{\partial\overline{z}})\mathrm{d}z\wedge\mathrm{d}\overline{z}.$$

Thus we can define the operators ∂ and $\overline{\partial}$ on smooth 1-form $\theta=f\mathrm{d}z+g\mathrm{d}\overline{z}$ as follows

$$\begin{aligned}\partial\theta&:=\partial g\wedge\mathrm{d}\overline{z}\\ \overline{\partial}\theta&:=\overline{\partial}f\wedge\mathrm{d}z.\end{aligned}$$

In a summary, we have constructed differential operators d,∂ and $\overline{\partial}$ on open subset $U\subseteq\mathbb{C}$, and above constructions can also be paralled to the Riemann surface X.

Theorem 4.1.1.

(1) $\mathrm{d}=\partial+\overline{\partial}$.
(2) $\mathrm{d}^2=\partial^2=\overline{\partial}^2=0$.
(3) $\partial\overline{\partial}=-\overline{\partial}\partial$.
(4) A $(1,0)$-form θ is holomorphic if and only if $\mathrm{d}\theta=\overline{\partial}\theta=0$.
(5) d,∂ and $\overline{\partial}$ satisfy the Leibniz rule, and commute with pullback.

4.1.3. *Integrations of differential forms.* Let θ be a smooth 1-form on a Riemann surface X and γ be a piecewise smooth curve on X. Suppose the curve γ is divided into $\gamma=\gamma_1\cup\cdots\cup\gamma_n$, such that $\gamma_i\colon[a_i,b_i]\to U_i$, where (U_i,φ_i) is a local coordinate. If θ is given by $f_i\mathrm{d}z_i+g_i\mathrm{d}\overline{z}_i$ in the local chart (U_i,φ_i), then the integration of θ along γ is defined by

$$\int_\gamma\theta=\sum_{i=1}^n\int_{\gamma_i}\theta:=\sum_{i=1}^n\int_{a_i}^{b_i}\{f\cdot z_i'(t)+g\cdot\overline{z_i'(t)}\}\mathrm{d}t.$$

Similarly, if η is a 2-form and D is a region on X, we also divide D into $D = D_1 \cup \cdots \cup D_n$ such that each D_i lies in some local chart (U_i, φ_i). If we write $z_i = x_i + \sqrt{-1}y_i$, then

$$\mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i = (\mathrm{d}x_i + \sqrt{-1}\mathrm{d}y_i) \wedge (\mathrm{d}x_i - \sqrt{-1}\mathrm{d}y_i) = -2\sqrt{-1}\mathrm{d}x_i \wedge \mathrm{d}y_i.$$

Thus if η is given locally by

$$f \mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i,$$

then the integration is defined by

$$\int_D \eta = \sum_{i=1}^n \int_{D_i} \eta := \sum_{i=1}^n \int_{\varphi_i(D_i)} -2\sqrt{-1} f \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

Theorem 4.1.2 (Stokes)**.** Let X be a Riemann surface and θ be a smooth 1-form. If D is a compact reigon with piecewise smooth boundary ∂D, then

$$\int_D \mathrm{d}\theta = \int_{\partial D} \theta.$$

4.2. **Holomorphic 1-form and meromorphic 1-form.**

4.2.1. *Holomorphic 1-form.*

Example 4.2.1. Consider the non-singular affine plane curve C defined by $f(x, y) = y^2 - x(x-1)(x-\lambda) = 0$. Then $\mathrm{d}x/y$ is a holomorphic 1-form on C.

(1) For point (x, y) with $y \neq 0$, $\mathrm{d}x/y$ is a well-defined holomorphic 1-form.
(2) For point (x, y) with $y = 0$, since C is non-singular, at this point one has $f_x \neq 0$. Note that $f(x, y) = 0$ holds on C, and thus one has $f_x \mathrm{d}x + f_y \mathrm{d}y = 0$ holds on C, which implies
$$\frac{\mathrm{d}x}{2y} = -\frac{\mathrm{d}y}{f_x}.$$

This shows $\mathrm{d}x/y$ is always a well-defined holomorphic 1-form on C.

More generally, arguments shown in above example can be used to prove the following proposition.

Proposition 4.2.1. Let C be a non-singular affine plane curve defined by $f(x, y) = 0$. Then

$$\omega = \frac{\mathrm{d}x}{f_y} = \frac{\mathrm{d}y}{f_x}$$

is a holomorphic 1-form on C.

Proposition 4.2.2. Let C be a non-singular projective plane curve defined by $F(x, y, z) = 0$ with $\deg F \geq 3$. Then the holomorphic 1-form

$$\omega = \frac{\mathrm{d}x}{F_y(x, y, 1)}$$

on the affine piece $\{z = 1\}$ extends to a holomorphic 1-form on C.

Proof. Firstly we extend the holomorphic 1-form as follows

$$\omega = \frac{\mathrm{d}(x/z)}{F_y(x/z, y/z, z/z)}.$$

Then on the affine piece defined by $x = 1$, one has

$$\omega = -\frac{z^{d-3}\mathrm{d}z}{F_y(1, y, z)} = \frac{z^{d-3}\mathrm{d}z}{F_z(1, y, z)}.$$

Thus if $d \geq 3$, the extension of ω is a holomorphic 1-form defined on C. $\square$

Remark 4.2.1. More generally, if $g(x, y) \in \mathbb{C}[x, y]$ is a polynomial, then by the same argument one can show that the holomorphic 1-form

$$\omega = \frac{g(x, y)\mathrm{d}x}{F_y(x, y, 1)}$$

defined on affine piece also extends to a holomorphic 1-form on C if $\deg g \leq d - 3$. Note that the dimension of vector space consisting of homogenous polynomial with degree $d - 3$ in three variables is $(d - 1)(d - 2)/2$. On the other hand, by genus formula one has $g = (d - 1)(d - 2)/2$ and later (in Lemma 9.1.1) we will show the dimension of vector space consisting of all holomorphic 1-forms is also genus. In other words, we have gave an explicit basis of holomorphic 1-forms on non-singular projective plane curve.

4.2.2. *Meromorphic* 1-*forms.*

Definition 4.2.1 (meromorphic 1-form)**.** A meromorphic 1-form θ on a Riemann surface X is a smooth $(1, 0)$-form which can be locally represented as $f(z)\mathrm{d}z$, where f is a meromorphic function.

Recall that given a meromorphic meromorphic function f on a Riemann surface X, for $p \in X$, we can chosoe a local coordinate z centered at p, and consider the Laurent series of $f \circ \varphi^{-1}(z)$ as

$$f(z) = \sum_{n=m}^{\infty} c_n z^n, \quad c_m \neq 0.$$

The order of f at p is defined by m and denoted by $\mathrm{ord}_p(f)$.

Lemma 4.2.1. $\mathrm{ord}_p(f)$ is independent of the choice of local coordinate.

Proof. A meromorphic function f on a Riemann surface X corresponds to a holomorphic map $\Phi \colon X \to \mathbb{P}^1$. If p is a zero point of f, then $\mathrm{ord}_p(f) = \mathrm{mult}_p(\Phi)$, and if p is a pole of f, then $\mathrm{ord}_p(f) = -\mathrm{mult}_p(\Phi)$. $\square$

Let θ be a meromorphic 1-form on Riemann surface X, in local coordinate z centered at p, we can write

$$\theta = f(z)\mathrm{d}z$$

so we can define $\mathrm{ord}_p(\theta) = \mathrm{ord}_p(f)$, and clearly it's independent of the choice of local coordinate.

Example 4.2.2. Let $X = \mathbb{P}^1$ and $\{(\mathbb{C}, z), (\mathbb{C}, w)\}$ be an atlas of $\mathbb{P}^1$, where the transition is given by $w = 1/z$. Consider 1-form θ which locally looks like $\mathrm{d}z$ on $(\mathbb{C}, z)$. Using holomorphic change of coordinate, one has θ looks like

$$\theta = \frac{-1}{z^2}\mathrm{d}z$$

on $(\mathbb{C}, w)$. This shows θ gives a meromorphic 1-form $\mathbb{P}^1$, and

$$\mathrm{ord}_p(\theta) = \begin{cases} 0, & p \in \mathbb{P}^1 \setminus \{\infty\} \\ -2, & p = \infty \end{cases}$$

Then

$$\sum_{p \in \mathbb{P}^1} \mathrm{ord}_p(\theta) = -2.$$

Example 4.2.3. Let $X = \mathbb{P}^1$ and $\{(\mathbb{C}, z), (\mathbb{C}, w)\}$ be an atlas of $\mathbb{P}^1$, where the transition is given by $w = 1/z$. Consider the meromorphic 1-form θ which is given by a rational function $r(z)$ on $(\mathbb{C}, z)$, where

$$r(z) = c\prod_{j=1}^{n}(z - \lambda_i)^{a_j},$$

where $c \neq 0, a_i \in \mathbb{Z}, \lambda_j \neq \lambda_j \in \mathbb{C}$. Using holomorphic change of coordinate, one has θ looks like

$$\theta = c\prod_{j=1}^{n}(\frac{1}{w} - \lambda_j)^{a_j}(-\frac{1}{w^2})\mathrm{d}w$$

on $(\mathbb{C}, w)$. This shows θ gives a meromorphic 1-form $\mathbb{P}^1$, and

$$\mathrm{ord}_p(\theta) = \begin{cases} a_j, & p = \lambda_j \\ -2 - \sum_{j=1}^{n} a_j, & p = \infty. \end{cases}$$

Then

$$\sum_{p \in \mathbb{P}^1} \mathrm{ord}_p(\theta) = -2.$$

Remark 4.2.2. This shows for a meromorphic 1-form θ on the projective line $\mathbb{P}^1$, one always has

$$\sum_{p \in \mathbb{P}^1} \mathrm{ord}_p(\theta) = -2.$$

Later we will see it's not a coincidence (in Theorem 4.4.1).

4.3. **Residue theorem.** Let θ be a meromorphic 1-form on a Riemann surface X. Suppose θ is locally given by $f\mathrm{d}z$, where f is a meromorphic function. The order of f lose too many information given by the coefficient of its Laurent series and we want to keep track coefficients which are invariant under the holomorphic change of local coordinate. Luckily, there exists such an invariant, that is -1-th coefficient of Laurent series c_{-1}.

Definition 4.3.1 (residue)**.** The residue of a meromorphic 1-form θ is defined by $\mathrm{Res}_p(\theta) = c_{-1}$.

The following lemma shows that the residue is independent of the choice of local coordinate, and gives a formula to compute it.

Lemma 4.3.1. Let D be any compact region in Riemann surface X such that $p \in D\backslash\partial D$, ∂D is piecewise smooth, and θ cannot have poles in $D\backslash\{p\}$. Then

$$\mathrm{Res}_p(\theta) = \frac{1}{2\pi\sqrt{-1}}\int_{\partial D}\theta.$$

Proof. Choose $D' \subseteq D$ such that $p \in D' \setminus \partial D'$, $\partial D'$ is smooth, and D' is contained in a local chart with local coordinate z centered at p. In this local chart, we can write θ as

$$\theta = (\sum_{n=m}^{\infty} c_n z^n)\mathrm{d}z.$$

Then

$$\int_{\partial D}\theta - \int_{\partial D'}\theta = \int_{D\backslash D'}\mathrm{d}\theta = 0,$$

where the last equality holds since θ is holomorphic in $D \setminus D'$. As a consequence,

$$\int_{\partial D}\theta = \int_{\partial D'}\theta = \int_{\varphi(\partial D')}(\sum_{n=m}^{\infty} c_n z^n)\mathrm{d}z = 2\pi\sqrt{-1}c_{-1} = 2\pi\sqrt{-1}\,\mathrm{Res}_p(\theta).$$

□

Theorem 4.3.1 (residue theorem)**.** Let X be a compact Riemann surface and θ be a meromorphic 1-form on X. Then

$$\sum_{p\in X}\mathrm{Res}_p(\theta) = 0$$

Proof. Since X is compact, there are only finitely many poles of θ, denoted by $\{p_1, \ldots, p_k\}$. For each $1 \leq j \leq k$, we can choose a neighborhood D_j of p_j which plays the role of D' in Lemma 4.3.1. Then

$$\sum_{p\in X}\mathrm{Res}(\theta) = \sum_{j=1}^{k}\mathrm{Res}_{p_j}(\theta) = \frac{1}{2\pi\sqrt{-1}}\sum_{j=1}^{k}\int_{\partial D_j}\theta = \frac{1}{2\pi\sqrt{-1}}\int_{D\backslash\bigcup_{j=1}^{k} D_j}\mathrm{d}\theta = 0.$$

□

Corollary 4.3.1. Let X be a compact Riemann surface and f be a meromorphic function on X. Then

$$\sum_{p\in X}\mathrm{ord}_p(f) = 0$$

Proof. It suffices to note that

$$\operatorname{ord}_p(f) = \operatorname{Res}_p(\frac{\mathrm{d}f}{f}).$$

□

4.4. **Poincaré-Hopf theorem.** Let M be a real closed 2-manifold and σ be a smooth 1-form with isolated zeros. Suppose σ is locally given by $\sigma = u\mathrm{d}x + v\mathrm{d}y$ on an open neighborhood U of zero p such that $U \setminus \{p\}$ contains no zero of σ. Then the index of σ at p, denoted by $\operatorname{Ind}_p(\sigma)$, is defined by the degree of the following map

$$\begin{aligned}\Phi \colon S^1(\epsilon) &\to S^1\\ (x, y) &\mapsto \frac{(u, v)}{\sqrt{u^2+v^2}},\end{aligned}$$

where $S^1(\epsilon)$ is the sphere of radius ϵ contained in U. The Poincaré-Hopf theorem[4] says that

$$\sum_{i=1}^{k} \operatorname{Ind}_{p_i}(\sigma) = \chi(M),$$

where $\{p_1, \dots, p_k\}$ are all zeros of σ. Moreover, Poincaré-Hopf theorem still holds if σ is smooth except finitely many singularities, by adding the index of these singularities. In this section we will use Poincaré-Hopf theorem to show that the phenomenon we have seen in Example 4.2.2 and Example 4.2.3 are not coincidences.

Theorem 4.4.1. Let X be a compact Riemann surface and θ be a meromorphic 1-form on X. Then

$$\sum_{p \in X} \operatorname{ord}_p(\theta) = -\chi(X) = 2g - 2.$$

Proof. Consider the 1-form $\sigma = \operatorname{Re}(\theta)$, which is a smooth 1-form besides the poles of θ, and the zeros of σ are exactly the one of θ. For any zero or pole $p \in X$ of θ, without lose of generality we may assume θ is of the form $z^m \mathrm{d}z$ locally. Then

$$\sigma = r^m \left(\cos(m\theta)\mathrm{d}x - \sin(m\theta)\mathrm{d}y\right)$$

where $r = |z|$. Thus the index at point p is

$$\begin{aligned}\operatorname{Ind}_p(\sigma) &= \frac{1}{2\pi}\int_0^{2\pi} \cos(m\theta)\mathrm{d}\sin(-m\theta) + \sin(-m\theta)\mathrm{d}\cos(m\theta)\\ &= \frac{1}{2\pi}\int_0^{2\pi} -m\left(\sin(-m\theta)^2 + \cos(-m\theta)^2\right)\mathrm{d}\theta\\ &= -\operatorname{ord}_p(\theta).\end{aligned}$$

[4]See page35 of [Mil65].

Thus by Poincaré-Hopf theorem one has

$$\sum_{p\in X}\operatorname{ord}_p(\theta)=-\chi(X).$$

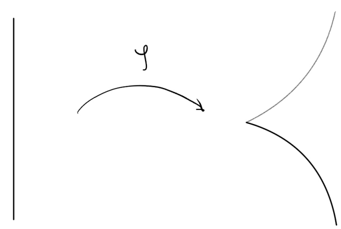
$\square$

Remark 4.4.1. In fact, above theorem is equivalent to the Riemann-Hurwitz formula, which is left an exercise in homework.

5. Normalization

In this section we will deal with singularities of algebraic curves. Roughly speaking, after resoluting all singularities of a curve C, we should obtain a Riemann surface, which is "isomorphic" to C besides these singularities. Before the formal definitions, let's see some examples of singularities we have already seen.

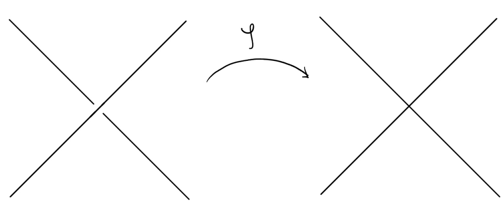
Example 5.1. The affine plane curve C defined by $x^2-y^2=0$ has a singular point $(0,0)$. Geometrically speaking, there are two projective line intersect at the point $(0,0)$, which cause the singularity. Thus one way to solve the singularity is to "split" these two lines.



Formally speaking, we should consider the disjoint union of two copy of $\mathbb{C}$, which is mapped to C as follows

$$\begin{aligned}\Phi\colon\ \mathbb{C}\coprod\mathbb{C} &\to C\\ \{t_1\},\{t_2\} &\mapsto (t_1,t_1),(t_2,-t_2).\end{aligned}$$

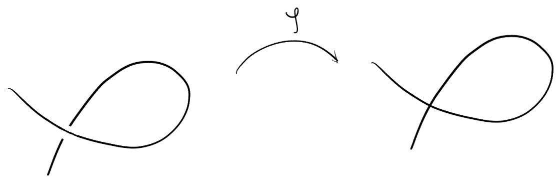
Example 5.2. The affine plane curve C defined by $x^2-y^3=0$ has a singular point $(0,0)$. To solve this singularity, geometrically thinking we should pull this curve "straightly", which can be seen as



Formally speaking, we should consider the parameterization

$$\begin{aligned}\Phi\colon\ \mathbb{C} &\to C\\ t &\mapsto (t^3,t^2).\end{aligned}$$

Example 5.3. The affine plane curve defined by $y^2-x^2(x-1)=0$ has a singular point $(0,0)$. From the following picture we can see that if we want to solve the singularity, we should also "split" the two part which intersect at $(0,0)$, as what we have done in the Example 5.1.



5.1. **Weierstrass preparation theorem.** Denote

$$\mathbb{C}\{x\} = \{\sum_{k=1}^{\infty} a_k x^k \mid \text{convergent series with positive convergence radius.}\}$$

$$\mathbb{C}\{x,y\} = \{\sum_{k=1}^{\infty} a_{kl} x^k y^l \mid \text{convergent series with positive convergence radius.}\}$$

They are called germs of holomorphic functions.

Definition 5.1.1 (Weierstrass polynomial)**.** An element $f(x,y) \in \mathbb{C}\{x,y\}$ is called a Weierstrass polynomial if $f(x,y) = y^d + a_1(x)y^{d-1} + \cdots + a_d(x) \in \mathbb{C}\{x,y\}$, where $a_i(x) \in \mathbb{C}\{x\}$ and $a_i(0) = 0$.

Theorem 5.1.1 (Weierstrass preparation theorem)**.** If $f \in \mathbb{C}\{x,y\}$ such that $f(0,y)$ is not identically zero, then there exist a unique $u \in \mathbb{C}\{x,y\}^*$ and a unique Weierstrass polynomial w such that $f = uw$.

Proof. Firstly we may assume $f(0,0) = 0$, otherwise $f \in \mathbb{C}\{x,y\}^*$ and there is nothing to prove. If so, then $f(0,y)$ has an isolated zero at $y = 0$, that is, there exists $\epsilon > 0$ such that

$$\{f(0,y) = 0\} \cap \{|y| \leq \epsilon\} = \{y = 0\}.$$

By continuity we may choose $\rho > 0$ sufficiently small such that $f(x,y) \neq 0$ on $\{|x| < \rho, |y| = \epsilon\}$. Then the number of zeros of $f(x,y)$ in $|y| \leq \epsilon$ for a fixed x is computed by

$$n(x) = \frac{1}{2\pi\sqrt{-1}} \int_{|y|=\epsilon} \frac{f_y(x,y)}{f(x,y)} \mathrm{d}y,$$

which is an integer-valued holomorphic function, and thus $n(x) \equiv m$ is a constant.

For all $|x| < \rho$, suppose $y_1(x), \ldots, y_m(x)$ are zeros of $f(x,y)$ contained in $\{|y| \leq \epsilon\}$. Then we claim that $w(x,y) = (y - y_1(x)) \cdots (y - y_n(x))$ is a Weierstrass polynomial. Indeed, note that

$$\sigma_k(x) := \sum_{i=1}^{m} y_i^k(x) = \frac{1}{2\pi\sqrt{-1}} \int_{|y|=\epsilon} y^k \frac{f_y(x,y)}{f(x,y)} \mathrm{d}y$$

are holomorphic, and thus if we write

$$w(x,y) = y^m + a_1(x)y^{m-1} + \cdots + a_m(x),$$

then $a_i(x)$ are polynomials of $\sigma_1(x), \dots, \sigma_m(x)$. This shows $a_i(x) \in \mathbb{C}\{x\}$, and $a_i(0) = 0$ for all i since $y_i(0) = 0$ for all i.

For convenience we denote $D = \{|x| < \rho, |y| \leq \epsilon\}$. By definition $u(x,y) = f(x,y)/w(x,y)$ is well-defined in $D \setminus \{w = 0\}$. For fixed $|x| < \rho$, by construction $w(x,y)$ and $f(x,y)$ have the same zeros in y. Therefore $u(x,y) \neq 0$ on D and $u(x,y)$ is holomorphic in variable y for each x. Now for given y_0 with $|y_0| < \epsilon$, one has

$$u(x, y_0) = \frac{1}{2\pi\sqrt{-1}} \int_{|y|=\epsilon} \frac{u(x,y)}{y - y_0} \mathrm{d}y.$$

This shows $u(x,y)$ is holomorphic in variables x and y, and thus one has $u(x,y)$ is holomorphic. Moreover, since u has no zeros, it has a non-zero constant term $u(0,0)$, and thus $u \in \mathbb{C}\{x,y\}^*$.

Finally let's see the uniqueness. If $f = u'w'$ in D, then

$$w' = y^d + c_1(x)y^{d-1} + \cdots + c_d(x) = (y - y_1(x)) \dots (y - y_m(x)) = w.$$

This shows $w = w'$ and thus $u = u'$. □

Corollary 5.1.1. $\mathbb{C}\{x,y\}$ is UFD.

Proof. Firstly note that $\mathbb{C}\{x\}$ is UFD, since for $f \in \mathbb{C}\{x\}$, one has

$$f = x^\mu g,$$

where $g \in \mathbb{C}\{x\}$ is a unit. Then by Gauss lemma one has $\mathbb{C}\{x\}[y]$ is UFD. Now for $f \in \mathbb{C}\{x,y\}$, suppose $f = x^\mu g$ with $g(0,y) \neq 0$. Since μ is unique, it suffices to show the unique factorization for g. By Weierstrass preparation theorem there is a decomposition

$$g(x,y) = uw.$$

Since Weierstrass polynomial w belongs to $\mathbb{C}\{x\}[y]$ which is UFD, there is a unique decomposition

$$w = w_1^{p_1} \dots w_k^{p_k},$$

where $w_i \in \mathbb{C}\{x\}[y]$ is monic irreducible. Now we need to show each w_i is irreducible in $\mathbb{C}\{x,y\}$. If not, suppose $w_i = a_i b_i$ in $\mathbb{C}\{x,y\}$. Then $w_i(0,y) \not\equiv 0$ implies both $a_i(0,y) \not\equiv 0$ and $b_i(0,y) \not\equiv 0$, and again by Weierstrass preparation theorem one has

$$a_i = u_i' w_i'$$
$$b_i = u_i'' w_i'',$$

where $w_i', w_i'' \in \mathbb{C}\{x\}[y]$. Since the decomposition in Weierstrass preparation theorem is unique, one has $u_i' u_i'' = 1$ and $w_i = w_i' w_i''$ in $\mathbb{C}\{x\}[y]$, a contradiction. Thus we obtain a decomposition of g into

$$g = u w_1^{p_1} \dots w_k^{p_k},$$

where u is a unit in $\mathbb{C}\{x,y\}$ and w_i are irreducible in $\mathbb{C}\{x,y\}$.

Now let's prove the uniqueness of decomposition of g. Suppose g is decomposed as

$$g = u w_1^{p_1} \dots w_k^{p_k} = v \widetilde{w}_1^{q_1} \dots \widetilde{w}_l^{q_l}.$$

Since $g(0,y) \not\equiv 0$, then again by Weierstrass preparation theorem we may decomposition these w_i and $\widetilde{w}_j$ into

$$\begin{aligned} w_i &= u_i w_i' \\ \widetilde{w}_j &= v_j \widetilde{w}_j'. \end{aligned}$$

By the uniqueness of the decomposition in Weierstrass preparation and the factorization in $\mathbb{C}\{x\}[y]$, one has $\{w_i\}$ and $\{\widetilde{w}_j\}$ are the same up to ordering, and $l = k$. This completes the proof. □

Remark 5.1.1. Although $y^2 - x^2(x-1)$ is irreducible in $\mathbb{C}[x,y]$, it's reducible in $\mathbb{C}\{x\}[y]$, that is,

$$(y - x\sqrt{x-1})(y + x\sqrt{x-1}).$$

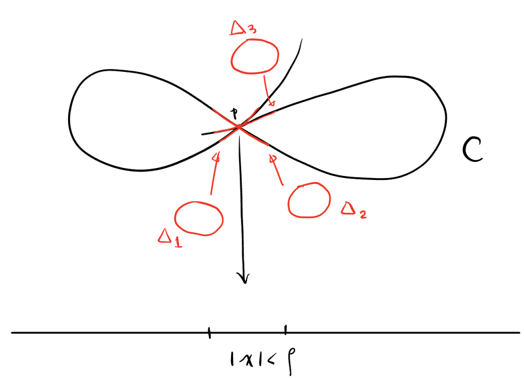
In the following section we will see such local decomposition gives the local resolution of singularities.

5.2. **Resolution of singularities.** Let C be an irreducible projective plane curve with singularities $\mathrm{Sing}(C)$. A normalization of C is a compact Riemann surface $\widetilde{C}$ together with a continous map $\Phi\colon \widetilde{C} \to C$ such that Φ is surjective and

$$\Phi\colon \widetilde{C} \setminus \Phi^{-1}(\mathrm{Sing}(C)) \to C \setminus \mathrm{Sing}(C)$$

is an isomorphism. In this section we will use unique factorization of $\mathbb{C}\{x,y\}$ to construct the normalization of C. Firstly let's give a rough ideal about what we're going to do.

The idea is to find sufficiently small $D = \{|x| < \rho, |y| < \epsilon\}$ such that $C \cap D$ decomposed into several pieces[5] $C_1 \cup \cdots \cup C_l$, where each C_i is homeomorphic to a disk and the union attaches them only at their centers. If we have constructed homeomorphisms φ_i from disk Δ_i to C_i for each i and repeat this procedure for all singularities, then we may construct the normalization $\widetilde{C}$ by adding these C_i to $C \setminus \mathrm{Sing}(C)$ in a suitable way.



[5]In fact, suppose $f = uw_1 \dots w_l$ in $\mathbb{C}\{x,y\}$, where $w_1, \dots, w_l$ are distinct irreducible Weierstrass polynomials and $u \in \mathbb{C}\{x,y\}^*$. Then each C_i is the zero loucs $\{w_i = 0\}$.

In the following sections we will explain above procedures in detail. The construction of homeomorphism for each singularity is called local resolution and adding these C_i to $C \setminus \mathrm{Sing}(C)$ is called the global resolution.

5.2.1. *Local resolution of singularities.* Let C be an irreducible projective plane curve with singularities $\mathrm{Sing}(C)$. Suppose p is a singularity of C, and without lose of generality we may assume $p = [1 : 0 : 0]$ by after a suitable $\mathrm{PGL}(3, \mathbb{C})$ transformation. Moreover, we may put the affine equation of C in the following form

$$f(x, y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x) = 0,$$

where $a_i(x) \in \mathbb{C}[x]$. Since $f(x, y)$ is irreducible, it has no multiple divisors in $\mathbb{C}[x][y]$, so $\mathscr{R}(f, f_y) \neq 0$ in $\mathbb{C}[x]$. Regardless of whether we are in $\mathbb{C}[x][y]$ or in $\mathbb{C}\{x\}[y]$, the resultant $\mathscr{R}(f, f_y)$ is the same, which implies $f(x, y)$ has no multiple divisors in $\mathbb{C}\{x\}[y]$. Then $f(x, y)$ is decomposed into the product of distinct irreducible factors in $\mathbb{C}\{x\}[y]$ as follows

$$f = f_1 \dots f_l.$$

Moreover, from $f(0, y) = y^n$, one can see every f_i must satisfy $f_i(0, y) \neq 0$. Thus by Weierstrass preparation theorem one has in $\mathbb{C}\{x, y\}$ one has the following decomposition

$$f_i = u_i w_i,$$

where $u_i \in \mathbb{C}\{x, y\}^*$ and w_i is a Weierstrass polynomial. Thus $f(x, y)$ is decomposed into the product of irreducible Weierstrass polynomials in $\mathbb{C}\{x, y\}$ as follows

$$f = uw_1 \dots w_i.$$

In order to avoid messy notations, in the following discussion we use w to denote one of the irreducible Weierstrass polynomials appeared in above decomposition.

Note that $\mathscr{R}(w, w_y)(x) \not\equiv 0$ implies $\mathscr{R}(w, w_y)(x)$ can only have isolated zeros. And since $w(0, y) = y^k$ has multiple roots, then $\mathscr{R}(w, w_y)(0) = 0$. Then there exists sufficiently small $\rho > 0$ such that for each $x \neq 0$ in

$$D = \{x \in \mathbb{C} \mid |x| < \rho\},$$

one has $\mathscr{R}(w, w_y)(x) \neq 0$. Then

$$w(x, y) = \prod_{\nu=1}^{k} (y - y_\nu(x)),$$

where $y_\nu(x)$'s are roots of $w(x, y)$. Moreover, for $0 \neq x \in D$, $\mathscr{R}(w, w_y)(x) \neq 0$, so that

$$w_y(x, y) \neq 0.$$

Then by the implicit function theorem, every $y_\nu(x)$ is locally a holomorphic function, and it can be uniquely analytically extended to a holomorphic function defined on $D \setminus \{x \in \mathbb{R}_{\geq 0}\}$, still denoted by $y_\nu(x)$. Now analytically

extend $y_\nu(x)$ across the cut line, the $y_\nu^*(x)$ obtained after this continuation must still satisfy
$$f(x, y_\nu^*(x)) = 0,$$
and thus $y_\nu^*(x)$ is one of the $y_1(x), \dots, y_k(x)$. In other words, the monodromy is given by a permutaion τ of $\{1, \dots, k\}$. By the same argument used in the proof of connectness of irreducible projective plane curve, one can show that the permutation τ is a k-cycle.

Theorem 5.2.1. Notations as above, and denote $\Delta = \{t \in \mathbb{C} \mid |t| < \rho^{1/k}\}$. Then the map
$$\begin{aligned} \varphi\colon \Delta &\to \mathbb{C}^2 \\ t &\mapsto (t^k, y_\nu(t^k)), \end{aligned}$$
is a well-defined holomorphic map and φ is injective from Δ onto
$$C^\Delta = \{(x, y) \in \mathbb{C}^2 \mid |x| < \rho, |y| < \epsilon, w(x, y) = 0\}.$$
Furthermore, φ is a biholomorphic from $\Delta \setminus \{0\}$ onto $C^\Delta \setminus \{(0, 0)\}$.

Proof. As t wraps one around the origin of Δ, t^k wraps around the origin k times. This shows when t wraps one around the origin once, $y_\nu(t^k)$ remains unchanged since the monodromy is given by a k-cycle. In this way, $y_\nu(t^k)$ defines a single-valued holomorphic function for $0 < |t| < \rho^{1/k}$, which can be extended to a holomorphic function defined on Δ by Riemann extension theorem. From this one can see $\phi\colon \Delta \to \mathbb{C}^2$ is a well-defined holomorphic map.

To see φ is injective: If $(t_1^k, y_\nu(t_1^k)) = (t_2^k, y_\nu(t_2^k))$, then $t_2 = (\xi_k)^\ell t_1$ for some $\ell \in \mathbb{Z}$, where ξ_k is the k-th unit root, and thus
$$y_\nu((\xi_k)^\ell t_1^k) = y_\nu(t_1^k).$$
Note that only when the variable x wraps around the origin km times does the value $y_\nu(x)$ remains unchanged. Therefore one has $k \mid \ell$, which implies ϕ is injective. Moreover, as t varies in Δ, one can see $y_\nu(t^k)$ passes all possible values of $y_1(x), \dots, y_k(x)$ for $|x| \le \rho$, which shows ϕ maps Δ onto C^Δ.

By implicit function theorem, there is a Riemann surface structure on $C^\Delta \setminus \{(0, 0)\}$, and thus φ is biholomorphic since it's holomorphic and both injective and surjective. $\square$

5.2.2. *Global resolution of singularities.* Let C be an irreducible projective plane curve with singularities $\mathrm{Sing}(C)$ and $C^* = C \setminus \mathrm{Sing}(C)$. For the singularity p, according to the method of the proceeding section, there exists m open discs Δ_i together with l holomorphic maps φ_i such that
$$\varphi_i\colon \Delta_i \setminus \{0\} \to C^*$$
is a biholomorphic map onto the image set. Now we use these φ_i to glue discs Δ_i to C^* to get the following Riemann surface
$$\widetilde{C} = C^* \bigcup_{\varphi_1} \Delta_1 \bigcup_{\varphi_2} \Delta_1 \cdots \bigcup_{\varphi_l} \Delta_l.$$

Repeat this procedure for each singularity and then we obtain the normalization $\widetilde{C}$ of C.

5.3. **Blow-up.** In this section we will introduce a method to determine the irreducible decomposition of a polynomial $f(x,y)$ in $\mathbb{C}\{x,y\}$. Without lose of generality we may assume $f(0,0)=0$ and write

$$f(x,y) = f_m(x,y) + f_{m+1}(x,y) + \dots,$$

where f_i are homogenous polynomials of degree i, and m is the smallest integer such that $f_d \neq 0$.

(1) If $m=1$, then $(0,0)$ is a non-singular point. In this case, $\{f_1(x,y)=0\}$ is the (unique) tangent line of $f(x,y)=0$ at $(0,0)$.
(2) If $m \geq 2$, then $\{f_m(x,y)=0\}$ is the union of lines, and this set is called the tangent cone of $f(x,y)=0$.

After a suitable linear transformation, we may assume the tangent cone of $f(x,y)=0$ at $(0,0)$ does not contain $\{x=0\}$. Thus

$$f(x,y) = (y-\alpha_1 x)\dots(y-\alpha_m x) + f_{m+1}(x,y) + \dots, \tag{5.1}$$

where $a_i \in \mathbb{C}$.

Definition 5.3.1 (ordinary singularity)**.** If $\{f_m = 0\}$ is the union of m distinct lines, then $(0,0)$ is called an ordinary m-tuple singularity.

Example 5.3.1. If $f(x,y) = y^2 + x^2(x-1)$, then the tangent cone of $f(x,y)=0$ at $(0,0)$ is the union of lines $y = \pm x$, and thus $(0,0)$ is an ordinary 2-tuple singularity.

Example 5.3.2. If $f(x,y) = y^2 - x^3$, then the tangent cone of $f(x,y)=0$ at $(0,0)$ is $y^2=0$, that is, the union of double copies of $y=0$. In this case $(0,0)$ is not an ordinary singularity. The singularity of this type is called a cusp.

Proposition 5.3.1. If $f(x,y)$ has ordinary m-tuple singularity at $(0,0)$, then $f(x,y)$ decomposes as a product of m irreducible factors in $\mathbb{C}\{x,y\}$ as follows

$$f(x,y) = u\prod_{i=1}^{m}(y - xh_i(x)),$$

where $u \in \mathbb{C}\{x,y\}^*$.

Proof. It's left as an exercise in homework, and here we list the key steps of the proof as follows.

(1) Denote by $w = y/x$,

$$g(x,w) = \frac{f(x,xw)}{x^m} \in \mathbb{C}\{x,y\}.$$

Prove that g converges in a product of discs

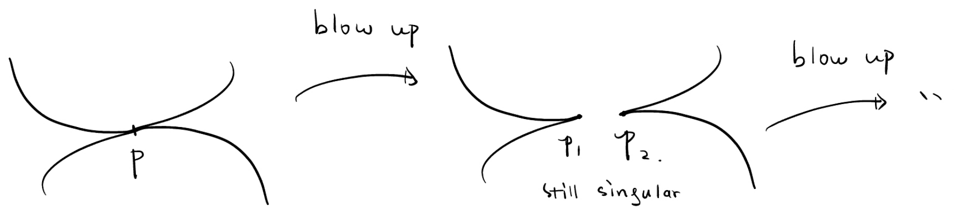
$$D_{\rho_1} \times D_{\rho_2} = \{(x,w) \mid |x| < \rho_1, |w| < \rho_2\}$$

that contains $(0,\alpha_i)$.

(2) Prove that $g(0,\alpha_i)=0$ and $\partial g/\partial w(0,\alpha_i)\neq 0$ and hence $g(x,w)=0$ has a solution $w=h_i(x)$ near $(0,\alpha_i)$ with $h_i(x)\in\mathbb{C}\{x\}$ and $h_i(0)=\alpha_i$.
(3) Prove that $\prod(y-xh_i(x)) \mid f(x,y)$ and $f(x,y)$ equals to the product of m irreducible factors up to units in $\mathbb{C}\{x,y\}$.

□

The procedure by introducing a new variable w such that $y=xw$ to consider the polynomial $g(x,w)$ in the proof of above proposition is called "blow up". Note that locally around $x=0$, there is no difference between $f(x,y)=0$ and $g(x,w)=0$ except $x=0$, but $g(0,w)$ may has lots of solutions, denoted by $\alpha_1,\dots,\alpha_k$. (You can think that one point is blowed up to several points geometrically.)



If $g(x,w)$ is non-singular at all these points, then $g(x,w)=0$ can be viewed as a local normalization of $f(x,y)=0$ at $(0,0)$. Otherwise we may need to blow up again along those singularities. In particular, Proposition 5.3.1 shows that if $(0,0)$ is an ordinary m-tuple singularity, then after blowing up once, you can get the desired local normalization.

Example 5.3.3. For $f(x,y)=x^2-y^2$, we know that $f(x,y)=0$ is the union of two line at origin and thus $(0,0)$ is a singularity. By considering

$$g(x,w)=\frac{f(x,xw)}{x^2}=\frac{x^2-x^2w^2}{x^2}=1-w^2.$$

One can see that for $x=0$, $g(0,w)$ has solutions $w=\pm1$, and that's exactly $y=\pm x$.

Example 5.3.4. For $f(x,y)=y^2-x^2+x^3-y^3=0$, one has

$$g(x,w)=\frac{f(x,xw)}{x^2}=\frac{x^2w^2-x^2+x^3-x^3w^3}{x^2}=w^2-1+x-xw^3.$$

Then $x=0$ has solutions $w=\pm1$. A direct computation shows $\partial g/\partial w=2w-3xw^2$, and thus

$$\left.\frac{\partial g}{\partial w}\right|_{x=0,w=\pm1}=\pm1\neq 0.$$

By implicit function theorem there exists $w_1(x), w_2(x) \in \mathbb{C}\{x\}$ such that $w_1(0) = 1$ and $w_2(0) = -1$. Then $f(x, y) = 0$ is parameterized by

$$y = xw_1(x)$$
$$y = xw_2(x)$$

locally around $(0, 0)$.

Example 5.3.5. For $f(x, y) = y^2 - y^3 + x^3 = 0$, one has

$$g(x, w) = \frac{f(x, xw)}{x^2} = \frac{x^2w^2 - x^3w^3 + x^3}{x^2} = w^2 - xw^3 + x = 0.$$

Then $g(0, w)$ has solution $w = 0$. Note that

$$\partial g/\partial x = w^3 + 1$$
$$\partial g/\partial w = 2w - 3xw^2.$$

Then

$$\left.\frac{\partial g}{\partial x}\right|_{x=0,w=0} = 1 \neq 0,$$

By implicit function theorem there exists $x = x(w)$ with $x(0) = 0$. Note that

$$x'(0) = -\left.\frac{\partial g/\partial w}{\partial g/\partial x}\right|_{w=0} = \frac{0}{1} = 0,$$

and $x''(0) \neq 0$. Then $f(x, y) = 0$ is parameterized by

$$\begin{cases} x(w) = w^2(c + \dots) \\ y(w) = w^3(c + \dots) \end{cases}$$

locally around $(0, 0)$.

Example 5.3.6. For $y^2 - y^3 - x^4 = 0$. $y = xw$, one has

$$g_1(x, w) = \frac{f(x, xw)}{x^2} = \frac{x^2w^2 - x^3y^3 - x^4}{x^2} = w^2 - xw^3 - x^2 = 0.$$

Then $g_1(0, w) = 0$ has solution $w = 0$. Since

$$\left.\frac{\partial g_1}{\partial x}\right|_{(0,0)} = 0, \quad \left.\frac{\partial g_1}{\partial w}\right|_{(0,0)} = 0,$$

one has $(0, 0)$ is still a singular point of $g_1(x, w)$, so we may blow it up again by setting $w = xt$. By doing this, one has

$$g_2(x, t) = \frac{g_1(x, xt)}{x^2} = \frac{x^2t^2 - x^4t^3 - x^2}{x^2} = t^2 - x^2t^3 - 1.$$

Then $g_2(0, t)$ has solutions $t = \pm 1$. Note that

$$\left.\frac{\partial g_2}{\partial t}\right|_{(0,\pm 1)} = \pm 2 \neq 0.$$

Then by implicit function theorem there exists $t_1(x), t_2(x) \in \mathbb{C}\{x\}$ such that $t_1(0) = 1$ and $t_2(0) = -1$, and thus $f(x, y) = 0$ is parameterized by

$$y = x^2 t_1(x)$$
$$y = x^2 t_2(x)$$

locally around $(0, 0)$.

5.4. **Bezout formula for singular curve.** In Theorem 3.1.1 we have shown the Bezout theorem under the assumption of C, C' are non-singular projective plane curve. In general case Bezout theorem still holds, if we use the right definition for the intersection number.

Definition 5.4.1 (intersection number)**.** Let C, C' be irreducible projective plane curves defined by homogenous polynomials F, G such that F, G has no common divisors, and $p \in C \cap C'$. If U is an open neighborhood of p such that the local decomposition of C into irreducibles as

$$C \cap U = C_1 \cup \cdots \cup C_k,$$

with local normalizations $\varphi_i \colon \Delta_i \to C_i$, then the intersection number at p is defined by

$$(C, C')_p = \sum_{i=1}^{k} \mathrm{ord}_{t=0}(G(\varphi_i(t))).$$

The intersection number of C and C', denoted by (C, C'), is the summation

$$(C, C') = \sum_{p \in C \cap C'} (C, C')_p.$$

Remark 5.4.1*.* The definition given above is compatible with the intersection number in the non-singular case, since if C is non-singular, then locally there is only one piece and the local normalization is a biholomorphism.

Theorem 5.4.1 (Bezout)**.** Let C, C' be irreducible projective plane curves defined by homogenous polynomials F, G such that F, G has no common divisors. Then the intersection number

$$(C, C') = ed,$$

where $\deg F = e$ and $\deg G = d$.

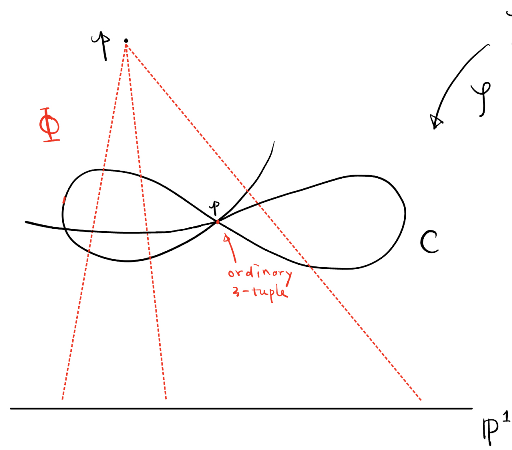
5.5. **Plücker formula for singular curve.**

5.5.1. *Riemann-Hurwitz approach.*

Theorem 5.5.1. Let C be an irreducible projective plane curve defined by the homogenous polynomial F of degree d with $\mathrm{Sing}(C) = \{p_1, \ldots, p_k\}$, and $\varphi \colon \widetilde{C} \to C$ be its normalization. Suppose each p_i is an ordinary m_i-tuple singularity. Then

$$g_{\widetilde{C}} = \binom{d-1}{2} - \sum_{i=1}^{k} \binom{m_i}{2} \geq 0$$

Proof. Choose a point $p \notin C$ and not lies in the tangent cone of C for any singularity. The point p defines a projection Φ from C to the projective line $\mathbb{P}^1$, and the composition $\widetilde{\Phi} = \Phi \circ \varphi$ is a holomorphic map from $\widetilde{C}$ to $\mathbb{P}^1$.



Now it suffices to figure out the ramification data of $\widetilde{\Phi}$ and use Riemann-Hurwitz formula to compute the genus of $\widetilde{C}$.

(1) If $q \in C \setminus \mathrm{Sing}(C)$, then

$$\mathrm{mult}_{\varphi^{-1}(q)} \widetilde{\Phi} - 1 = \mathrm{mult}_q \Phi - 1 = (F, F_y)_q$$

(2) If $q \in \mathrm{Sing}(C)$, then for each $q_i \in \pi^{-1}(q)$, one has $\mathrm{mult}_{q_i} \widetilde{\Phi} = 1$ since $\widetilde{C}$ is locally defined by irreducible linear function.

This shows the ramification data

$$B(\widetilde{\Phi}) = \sum_{q \in C \setminus \mathrm{Sing}(C)} (F, F_y)_q.$$

Now let's figure out the intersection number of singular points. Suppose $p \in \mathrm{Sing}(C)$ is a m-tuple singularity. For convenience we may assume $p = [0 : 0 : 1]$ and denote $f(x, y) = F(x, y, 1)$. Since p is an ordinary m-tuple singularity, by Proposition 5.3.1 one has

$$f(x, y) = (y - y_1(x)) \dots (y - y_m(x)),$$

where $y_i(x) = a_i x + o(x^2)$. Then there are m local normalizations at point p, which are given by $\varphi_i(t) = (t, y_i(t))$. This shows the intersection number at singularity p is given by

$$\begin{aligned}(f, f_y)_{(0,0)} &= \sum_{i=1}^{m} \operatorname{ord}_{t=0} f_y(\varphi_i(t)) \\ &= \sum_{i=1}^{m} \operatorname{ord}_{t=0} \prod_{j\neq i}(y_i(t) - y_j(t)) \\ &= \sum_{i=1}^{m} m-1 \\ &= m(m-1).\end{aligned}$$

Then by Bezout theorem (Theorem 5.4.1), one has

$$B(\widetilde{\Phi}) = d(d-1) - \sum_{i=1}^{k} m_i(m_i-1).$$

By Riemann-Hurwitz formula one has

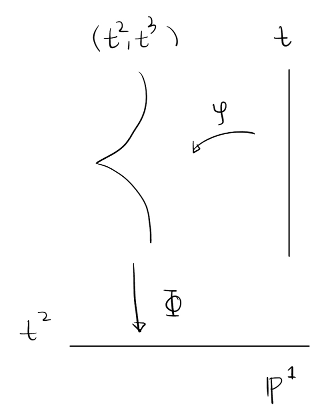
$$2g_{\widetilde{C}} - 2 = -2d + d(d-1) - \sum_{i=1}^{k} m_i(m_i-1),$$

and thus

$$g_{\widetilde{C}} = \binom{d-1}{2} - \sum_{i=1}^{k}\binom{m_i}{2}.$$

□

Remark 5.5.1. Above argument shows that the genus of the normalization $\widetilde{C}$ only depends on the degree of C and the type of singularities. For example, suppose $p \in C$ is a cusp, that is, locally it's given by $y^2 = x^3$, and its local normalization is given by $\varphi\colon t \mapsto (t^2, t^3)$.



The preimage $\varphi^{-1}(0)$ is a ramification point of $\widetilde{\Phi}$, with $\operatorname{mult}_{\varphi^{-1}(0)}\widetilde{\Phi}=2$, and thus

$$B(\widetilde{\Phi})=\sum_{q\in C\setminus\operatorname{Sing}(C)}(f,f_y)_q+\sum_{q\text{ is a cusp}}1.$$

On the other hand, the intersection number at the cusp is

$$(f,f_y)_p=\operatorname{ord}_{t=0}2t^3=3.$$

Then by the same argument one can see that one more cusp will decrease the genus of $\widetilde{C}$ by 1.

In general, for each type singularity we may define a δ-invariant, such that one more such type singularity will decrease the genus of normalization by δ. As we have seen, the δ-invariant for m-tuple singularity is $\binom{m}{2}$ and the δ-invariant for cusp is 1.

5.5.2. *Poincaré-Hopf approach.* In this section we introduce another approach to compute the genus of the normalization by Poincaré-Hopf theorem, from which it's relatively easy to compute the δ-invariance of singularity.

Suppose C is a projective plane curve defined by homogenous polynomial F. Consider

$$\eta=\frac{\mathrm{d}x}{F_y(x,y,1)}=-\frac{\mathrm{d}y}{F_x(x,y,1)}.$$

If F is non-singular, then η has no zeros or poles on the affine piece $C\cap\{[x:y:1]\}$. On $\{z=0\}$, a direct computation shows

$$\eta=-\frac{z^{d-3}\mathrm{d}z}{F_y(1,y,z)}=\frac{z^{d-3}\mathrm{d}z}{F_x(x,1,z)}.$$

Thus η gives a meromorphic 1-form on C, and by using Bezout theorem one has

$$\sum_{p\in C}\operatorname{ord}_p(\theta)=(d-3)d.$$

Then Poincaré-Hoft theorem implies that $g_C=(d-1)(d-2)/2$.

Now let's generalize above arguments to the case C is singular. Suppose $f(x,y)$ has singularity a $(0,0)$ with multiplicity m, and $x=0$ is not in the tangent cone of f at $(0,0)$. Now consider the blow up at the singularity $(0,0)$, that is,

$$g(x,w)=\frac{f(x,xw)}{x^m}.$$

A direct computation by chain rule shows that

$$x^{m-1}g_w(x,w)=f_y(x,xw).$$

Thus

$$\eta=\frac{\mathrm{d}x}{f_y(x,y)}=\frac{\mathrm{d}x}{x^{m-1}g_w(x,xw)}=x^{-(m-1)}\frac{\mathrm{d}x}{g_w(x,w)}.$$

If $g(0,w)=0$ has solutions $\alpha_1,\dots,\alpha_k$, and $g(x,w)$ is non-singular at these points, then $g(x,w)=0$ gives the normalization of C, denoted by $\widetilde{C}$.

Then by Poincaré-Hopf theorem one has

$$\begin{aligned}2g_{\widetilde{C}}-2&=\sum_{p\in\widetilde{C}}(x^{-(m-1)},x)_p+\sum_{p\in\widetilde{C}}\mathrm{ord}_p(\frac{\mathrm{d}x}{g_w(x,w)})\\&=-m(m-1)+d(d-3).\end{aligned}$$

Otherwise we repeat above procedures again to each singularity $(0,w_i)$, and by induction we have the δ-invariance for singularity $(0,0)$ is

$$\delta=\binom{m}{2}+\binom{m_1}{2}+\cdots+\binom{m_k}{2}+\dots.$$

Example 5.5.1. The δ-invariance of ordinary m-tuple singularity is $\binom{m}{2}$, since after blowing up once, it's already non-singular. This coincides with previous result.

Example 5.5.2. For $f(x,y)=y^n-x^m$, with $\gcd(m,n)=1$. Without lose of generality we may assume $n<m$. Then

$$g(x,w)=\frac{f(x,xw)}{x^n}=\frac{x^nw^n-x^m}{x^n}=w^n-x^{m-n}.$$

This shows

$$\delta(n,m)=\delta(n,m-n)+\binom{n}{2}.$$

Thus by induction one has the δ-invariance of singularity $(0,0)$ is $(m-1)(n-1)/2$.

6. Divisors

In this section, we always assume X is a compact Riemann surface.

6.1. **Divisors.**

Definition 6.1.1 (divisors)**.** A divisor on X is a formal sum $D = \sum_{p\in X} D(p)\cdot p$, where $D(p) \in \mathbb{Z}$ such that $D(p) \neq 0$ for only finitely many p.

Notation 6.1.1. $\mathrm{Div}(X)$ denotes the free abelian group generated by divisors on X.

Definition 6.1.2 (degree)**.** For $D \in \mathrm{Div}(X)$, the degree of D is defined by

$$\deg(D) = \sum_{p\in X} D(p).$$

Remark 6.1.1. The degree gives a group homomorphism $\deg\colon \mathrm{Div}(X) \to \mathbb{Z}$. The kernel of deg is denoted by $\mathrm{Div}^0(X)$, that is,

$$\mathrm{Div}^0(X) := \{D \in \mathrm{Div}(X) \mid \deg(D) = 0\}.$$

6.1.1. *Principal divisor.*

Definition 6.1.3 (principal divisor)**.** If $f \not\equiv 0$ is a meromorphic function on X, the principal divisor corresponding to f is

$$\mathrm{div}(f) := \sum_{p\in X} \mathrm{ord}_p(f)\cdot p.$$

Notation 6.1.2. $\mathrm{PDiv}(X)$ denotes the set of all principal divisors on X.

Lemma 6.1.1. Suppose f, g are meromorphic functions on X and $g \not\equiv 0$.

1. $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$.
2. $\mathrm{div}(1/g) = -\mathrm{div}(g)$.
3. $\mathrm{div}(f/g) = \mathrm{div}(f) - \mathrm{div}(g)$.

Corollary 6.1.1. $\mathrm{PDiv}(X)$ is a subgroup of $\mathrm{Div}(X)$.

Example 6.1.1 (divisor of zeros or poles)**.** Let $f \not\equiv 0$ be a meromorphic function on X, the zero divisor is defined by

$$\mathrm{div}_0(f) := \sum_{\substack{p\in X\\ \mathrm{ord}_p(f)>0}} \mathrm{ord}_p(f)\cdot p,$$

and the pole divisor is defined by

$$\mathrm{div}_\infty(f) := -\sum_{\substack{p\in X\\ \mathrm{ord}_p(f)<0}} \mathrm{ord}_p(f)\cdot p.$$

It's clear

$$\mathrm{div}(f) = \mathrm{div}_0(f) - \mathrm{div}_\infty(f).$$

Definition 6.1.4 (linearly equivalent)**.** For $D_1, D_2 \in \mathrm{Div}(X)$, if $D_1 - D_2$ is a principal divisor, then D_1, D_2 are called linearly equivalent, and denoted by $D_1 \sim D_2$.

Example 6.1.2. $\mathrm{div}_0(f)$ is linearly equivalent to $\mathrm{div}_\infty(f)$.

Lemma 6.1.2. If $f \not\equiv 0$ is a meromorphic function on X, then

$$\deg(\mathrm{div}(f)) = 0$$

Proof. It follows from Corollary 2.1.3. □

Corollary 6.1.2.

$$\mathrm{PDiv}(X) \subseteq \mathrm{Div}^0(X).$$

It's natural to ask whether $\mathrm{PDiv}(X) = \mathrm{Div}^0(X)$ or not. The following theorem shows that the statement holds for $X = \mathbb{P}^1$, and for higher genus case, this statement fails even for torus.

Theorem 6.1.1. $\mathrm{PDiv}(\mathbb{P}^1) = \mathrm{Div}^0(\mathbb{P}^1)$.

Proof. For $D \in \mathrm{Div}^0(\mathbb{P}^1)$, we may write is as

$$D = \sum_{i=1}^{n} e_i \cdot \lambda_j + e_\infty \cdot \infty, \quad \lambda_i \in \mathbb{C},$$

where $e_\infty = -\sum_{i=1}^n e_i$. Then for the meromorphic function given by $f = \prod_{i=1}^n (z - \lambda_i)^{e_i}$, one has $\mathrm{div}(f) = D$. □

Corollary 6.1.3. For $D_1, D_2 \in \mathrm{Div}(\mathbb{P}^1)$, $D_1 \sim D_2$ if and only if $\deg(D_1) = \deg(D_2)$.

Theorem 6.1.2. Let $X = \mathbb{C}/L$ be the complex torus, where $L = \mathbb{Z}w_1 + \mathbb{Z}w_2$ is a lattice. Then

$$\mathrm{Div}^0(X)/\mathrm{PDiv}(X) \cong X.$$

Proof. Consider the following group homomorphism

$$\begin{aligned} A\colon \mathrm{Div}(X) &\to X \\ \sum_{p\in X} n_p \cdot p &\mapsto \sum_{p \in X} n_p p, \end{aligned}$$

where $\sum_{p\in X} n_p p$ is the addition structure of X. It's clear that $A|_{\mathrm{Div}^0(X)}$ is surjective, since for any $p \in X$, one has

$$A(p - 0) = p,$$

and $p - 0$ is a divisor with degree zero. Now it suffices to show that $\ker A = \mathrm{PDiv}(X)$, which is left as an exercise. □

Remark 6.1.2. It's a special case of the Abel-Jacobi theorem, and A is the Abel-Jacobi map.

6.1.2. *Canonical divisor.*

Definition 6.1.5 (canonical divisor)**.** Let θ be a meromorphic 1-form on X. The canonical divisor K given by θ is defined by

$$K := \sum_{p \in X} \operatorname{ord}_p(\theta) \cdot p.$$

Lemma 6.1.3. If f is a meromorphic function, and θ is a meromorphic 1-form, then $f\theta$ is also a meromorphic 1-form, and

$$\operatorname{div}(f\theta) = \operatorname{div}(f) + \operatorname{div}(\theta).$$

Conversely, we have

Lemma 6.1.4. If θ_1, θ_2 are meromorphic 1-form, then there exists a meromorphic function f such that

$$\theta_1 = f\theta_2$$

Proof. Suppose locally meromorphic 1-forms θ_1, θ_2 are given by

$$\theta_1 = f_1 \mathrm{d}z, \quad \theta_2 = f_2 \mathrm{d}z.$$

Then we can define a meromorphic function f locally by f_1/f_2. The construction is independent of the choice of local charts, since factors coming from the change of charts with cancel with each other, as one of them is on the denominator and the other one is on the numerator. □

Corollary 6.1.4. The difference of any two canonical divisors is a principal divisor.

Corollary 6.1.5. The canonical divisors have the same degree. Moreover, the degree of canonical divisor is $2g_X - 2$.

Proof. It follows from Poincaré-Hopf theorem. □

6.1.3. *Pullback of divisors.* Let $\Phi\colon X \to Y$ be a non-constant holomorphic map. For point $q \in Y$, we regard it as a divisor and define the pullback of it as follows

$$\Phi^*(q) := \sum_{p \in \Phi^{-1}(q)} \operatorname{mult}_p(\Phi) \cdot p.$$

Then for any divisor $D \in \operatorname{Div}(Y)$, its pullback is defined by

$$\Phi^*(D) = \sum_{q \in Y} D(q) \cdot \Phi^*(q).$$

Moreover, for any $q \in Y$, one has

$$\deg(\Phi^*(q)) = \sum_{p \in \Phi^{-1}(q)} \operatorname{mult}_p(\Phi) = \deg(\Phi).$$

Then since taking degree is a group homomorphism, one has

$$\deg(\Phi^*(D)) = \sum_{q \in Y} D(q) \deg(\Phi^*(q)) = \deg(\Phi) \deg(D).$$

Lemma 6.1.5. Let $\Phi\colon X\to Y$ be a non-constant holomorphic map.
(1) $\Phi^*\colon \mathrm{Div}(Y)\to\mathrm{Div}(X)$ is a group homomorphism.
(2) $\Phi^*(\mathrm{PDiv}(Y))\subseteq\mathrm{PDiv}(X)$.

Proof. It's clear that Φ^* is a group homomorphism. For (2). Let $f\not\equiv 0$ be a meromorphic function on Y. Then we claim that

$$\Phi^*(\mathrm{div}(f))=\mathrm{div}(f\circ\Phi).$$

To see this, for any $p\in X$, we have

$$\begin{aligned}\Phi^*(\mathrm{div}(f))(p)&=\mathrm{mult}_p(\Phi)\mathrm{div}(f)(\Phi(p))\\&=\mathrm{mult}_p(\Phi)\,\mathrm{ord}_{\Phi(p)}(f)\\&=\mathrm{ord}_p(f\circ\Phi)\\&=\mathrm{div}(f\circ\Phi)(p).\end{aligned}$$

□

Corollary 6.1.6. Let $\Phi\colon X\to Y$ be a non-constant holomorphic map. If $D_1\sim D_2$ on Y, then $\Phi^*(D_1)\sim\Phi^*(D_2)$ on X.

Definition 6.1.6 (ramification divisor)**.** Let $\Phi\colon X\to Y$ be a holomorphic map between compact Riemann surfaces. The ramification divisor is defined by

$$R_\Phi:=\sum_{p\in X}(\mathrm{mult}_p(\Phi)-1)\cdot p.$$

Remark 6.1.3. Recall that Riemann-Hurwitz formula says that

$$2g_X-2=\deg(\Phi)(2g_Y-2)+\sum_{p\in X}(\mathrm{mult}_p(\Phi)-1).$$

If θ is a non-zero meromorphic 1-form on Y, then $\Phi^*(\theta)$ is also a meromorphic 1-form, and thus

$$\begin{aligned}\deg(\mathrm{div}(\theta))&=2g_Y-2\\\deg(\mathrm{div}(\Phi^*\theta))&=2g_X-2.\end{aligned}$$

Then the Riemann-Hurwitz formula can be written as

$$\deg(\mathrm{div}(\Phi^*(\theta)))=\deg(\Phi)\deg(\mathrm{div}(\theta))+\deg(R_\Phi)=\deg(\Phi^*(\mathrm{div}(\theta)))+\deg(R_\Phi).$$

As a consequence, the pullback of a canonical divisor $\mathrm{div}(\theta)$ by a holomorphic map Φ is still a canonical divisor if and only if Φ is not ramified.

6.1.4. *Partial order of divisors.*

Definition 6.1.7 (effective divisors)**.** A divisor D on X is called effective divisor if $D(p)\geq 0$ for all $p\in X$, and denote it by $D\geq 0$. Similarly, we denote $D>0$ if $D(p)>0$ for all $p\in X$.

Remark 6.1.4. For any divisor D, it can be written as a difference of two effective divisors as follows
$$D = \sum_{\substack{p\in X\\ D(p)\geq 0}} D(p)\cdot p - \sum_{\substack{p\in X\\ D(p)<0}} -D(p)\cdot p.$$

Definition 6.1.8 (partial order)**.** For two divisors $D_1, D_2 \in \mathrm{Div}(X)$, we say $D_1 \geq D_2$ if $D_1 - D_2 \geq 0$.

6.2. **Global sections associated to divisors.**

6.2.1. *Global sections of $\mathcal{O}_X(D)$.* Given $D \in \mathrm{Div}(X)$, consider the following set[6]
$$\Gamma(X, \mathcal{O}_X(D)) := \{f \in \mathcal{M}(X) \mid \mathrm{div}(f) + D \geq 0\}.$$
Moreover, if $f \equiv 0$, we define $\mathrm{ord}_p(f) = \infty$, and thus $0 \in \Gamma(X, \mathcal{O}_X(D))$.

Remark 6.2.1. $\Gamma(X, \mathcal{O}_X(D))$ consists of meromorphic functions with poles not too bad.

(1) If $D(p) = -n < 0$, then p must be a zero of f with order $\geq n$;
(2) If $D(p) = n > 0$, then p may be a pole, but its order at least won't be larger than n.

It's clear that there is a $\mathbb{C}$-vector space structure on $\Gamma(X, \mathcal{O}_X(D))$. Later (in Corollary 6.2.1) we will show that $\Gamma(X, \mathcal{O}_X(X))$ is a finite-dimensional $\mathbb{C}$-vector space, and we use $\ell(D)$ to denote its dimension for convenience. Before that, let's see some basic properties and examples.

Lemma 6.2.1. For $D_1, D_2 \in \mathrm{Div}(X)$. If $D_1 \leq D_2$, then $\Gamma(X, \mathcal{O}_X(D_1)) \subseteq \Gamma(X, \mathcal{O}_X(D_2))$.

Lemma 6.2.2. If $\deg(D) < 0$, then $\Gamma(X, \mathcal{O}_X(D)) = \{0\}$.

Proof. If $f \in \Gamma(X, \mathcal{O}_X(D))$ and $f \not\equiv 0$, then one has $\mathrm{div}(f) + D \geq 0$. By taking degree one has
$$0 = \deg(\mathrm{div}(f)) \geq -\deg(D) > 0.$$
A contradiction. □

Lemma 6.2.3. If $D_1 \sim D_2$ are two linearly equivalent divisors, then $\Gamma(X, \mathcal{O}_X(D_1)) \cong \Gamma(X, \mathcal{O}_X(D_2))$ as $\mathbb{C}$-vector spaces.

Proof. Since $D_1 \sim D_2$, there exists a meromorphic function h such that $D_1 = D_2 + \mathrm{div}(h)$. For any $f \in \Gamma(X, \mathcal{O}_X(D_1))$, then
$$\mathrm{div}(fh) = \mathrm{div}(f) + \mathrm{div}(h) \geq -D_1 + D_1 - D_2 = -D_2.$$

[6]In algebraic geometry, a (Cartier) divisor corresponds to a line bundle $\mathcal{O}_X(D)$, and here $\Gamma(X, \mathcal{O}_X(D))$ is exactly the global section of line bundle $\mathcal{O}_X(D)$. In section 7.1 we will introduce sheaves and discuss it in detail.

Thus one can define a linear map

$$\begin{aligned}\mu_h\colon \Gamma(X,\mathcal{O}_X(D_1)) &\to \Gamma(X,\mathcal{O}_X(D_2))\\ f &\mapsto fh.\end{aligned}$$

with inverse $\mu_{h^{-1}}\colon \Gamma(X,\mathcal{O}_X(D_2)) \to \Gamma(X,\mathcal{O}_X(D_1))$. This shows $\Gamma(X,\mathcal{O}_X(D_1)) \cong \Gamma(X,\mathcal{O}_X(D_2))$. ☐

Example 6.2.1. If $D=0$, then $\Gamma(X,\mathcal{O}_X(0))$ consists of holomorphic function, and since X is compact, one has

$$\Gamma(X,\mathcal{O}_X(0)) \cong \mathbb{C}\,.$$

In particular, if $D \in \mathrm{PDiv}(X)$, then $\Gamma(X,\mathcal{O}_X(D)) \cong \Gamma(X,\mathcal{O}_X(0)) \cong \mathbb{C}$.

Example 6.2.2. Suppose D is a divisor on $\mathbb{P}^1$ with $\deg(D) \geq 0$, which is written by

$$D = \sum_{i=1}^{n} e_i \cdot \lambda_i + e_\infty \cdot \infty.$$

Consider the function

$$f_D(z) = \prod_{i=1}^{n} (z-\lambda_i)^{-e_i}.$$

Then we claim that

$$\Gamma(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(D)) = \{g(z)f_D(z) \mid g(z) \text{ is a polynomial of degree at most } \deg(D)\}.$$

For a polynomial $g(z)$ of degree d, one has $\mathrm{div}(g) \geq -d\cdot\infty$, and note that

$$\mathrm{div}(f_D) = \sum_i e_i \cdot \lambda_i + (\sum_i e_i)\cdot\infty.$$

Then

$$\begin{aligned}\mathrm{div}(g(z)f_D(z)) + D &= \mathrm{div}(g) + \mathrm{div}(f_D) + D\\ &\geq (\sum_i e_i + e_\infty - d)\cdot\infty\\ &\geq 0.\end{aligned}$$

Conversely, for any function $h \in \Gamma(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(D))$, we define $g = h/f_D$. A direct computation shows

$$\begin{aligned}\mathrm{div}(g) &= \mathrm{div}(h) - \mathrm{div}(f_D)\\ &\geq -D - \mathrm{div}(f_D)\\ &= (-\sum_i e_i - e_\infty)\cdot\infty\\ &= -\deg(D)\cdot\infty.\end{aligned}$$

This shows that g can admit no poles in the finite part $\mathbb{C}$, and can have a pole of order at most $\deg(D)$. This forces g to be a polynomial of degree at most $\deg(D)$. In particular, one has

$$\ell(\mathcal{O}_{\mathbb{P}^1}(D)) = \begin{cases} 0, & \deg(D) < 0 \\ 1 + \deg(D), & \deg(D) \geq 0 \end{cases}$$

Example 6.2.3. Suppose X is a compact Riemann surface with genus ≥ 1 and D is a divisor given by a point p. Then we claim that

$$\Gamma(X, \mathcal{O}_X(p)) = \mathbb{C}.$$

Note that if $f \in \Gamma(X, \mathcal{O}_X(p))$, then f is only allowed to have a simple pole at p. If f is non-constant, then it gives a holomorphic map $\Phi\colon X \to \mathbb{P}^1$ with $\Phi^{-1}(\infty) = \{p\}$. This shows $\deg(\Phi) = 1$, and thus it's an isomorphism, a contradiction to $g_X \geq 1$.

For $D \in \mathrm{Div}(X)$, let's estimate the upper bound of the dimension of the $\mathbb{C}$-vector space $\Gamma(X, \mathcal{O}_X(D))$.

Lemma 6.2.4. For any $D \in \mathrm{Div}(X)$, and $p \in X$, then either $\Gamma(X, \mathcal{O}_X(D-p)) = \Gamma(X, \mathcal{O}_X(D))$ or $\Gamma(X, \mathcal{O}_X(D-p))$ has codimension 1 in $\Gamma(X, \mathcal{O}_X(D))$ holds.

Proof. Let $n = -D(p)$, and choose a local coordinate z centered at p. For any $f \in \Gamma(X, \mathcal{O}_X(D))$, the Laurent series of f at p must have the following form

$$cz^n + \text{higher order terms}$$

Consider $\alpha\colon \Gamma(X, \mathcal{O}_X(D)) \to \mathbb{C}$, which is defined by $f \mapsto c$.

(1) If $\alpha \not\equiv 0$, then it's a surjective linear map. If $f \in \ker \alpha$, then $\mathrm{ord}_p(f) \geq n+1$, and thus $\mathrm{ord}_p(f) + D(p) - 1 \geq 0$, which implies $f \in \Gamma(X, \mathcal{O}_X(D-p))$. By the same argument one can show $\Gamma(X, \mathcal{O}_X(D-p)) \subseteq \ker \alpha$, and thus $\ker \alpha = \Gamma(X, \mathcal{O}_X(D-p))$. This shows $\Gamma(X, \mathcal{O}_X(D-p))$ has codimension 1 in $\Gamma(X, \mathcal{O}_X(D))$.

(2) If $\alpha \equiv 0$, then $\Gamma(X, \mathcal{O}_X(D-p)) = \Gamma(X, \mathcal{O}_X(D))$.

□

Theorem 6.2.1. For any $D \in \mathrm{Div}(X)$, write $D = P - N$ such that $P, N \geq 0$ and $\mathrm{Supp}(P) \cap \mathrm{Supp}(N) = \varnothing$. Then

$$\ell(D) \leq 1 + \deg(P).$$

Proof. Let's prove it by induction on $\deg(P)$. If $\deg(P) = 0$, that is, $P = 0$, then one has $\Gamma(X, \mathcal{O}_X(P)) \cong \mathbb{C}$. Thus $\ell(D) \leq \ell(P) = 1 = 1 + \deg(P)$.

Assume induction hypothesis holds for $\deg(P) = k-1$. Let $D = P - N$ be a divisor with $\deg(P) = k$, such that $P, N \geq 0$ and $\mathrm{Supp}(P) \cap \mathrm{Supp}(N) = \varnothing$. Since $\mathrm{Supp}(P) \neq \varnothing$, we choose $q \in \mathrm{Supp}(P)$, and write $D - q = (P - q) - N$. Then $\mathrm{Supp}(P-q) \cap \mathrm{Supp}(N) = \varnothing$ and $\deg(P-q) = k-1$. Then by induction, one has

$$\ell(D-q) \leq 1 + \deg(P-q) = 1 + k - 1 = k,$$

and by Lemma 6.2.4, one has

$$\ell(D) \leq \ell(D-q)+1 \leq k+1 = \deg(P)+1.$$

This completes the proof. $\square$

Corollary 6.2.1. For any $D \in \mathrm{Div}(X)$, $\Gamma(X, \mathcal{O}_X(D))$ is a finite-dimensional[7] $\mathbb{C}$-vector space.

6.2.2. *Global sections of* $\Omega^1_X(D)$. Let $\mathcal{M}^{(1)}(X)$ be the set of all meromorphic 1-forms on X. For $D \in \mathrm{Div}(X)$, the global sections of $\Omega^1_X(D)$ is defined by

$$\Gamma(X, \Omega^1_X(D)) = \{\omega \in \mathcal{M}^{(1)}(X) \mid \mathrm{div}(\omega) + D \geq 0\}.$$

Example 6.2.4. $\Gamma(X, \Omega^1_X(0))$ consists of all holomorphic 1-forms, and sometimes it's denoted by $\Gamma(X, \Omega^1_X)$ or $\Omega^1_X(X)$. Not like holomorphic functions, there may be many non-trivial holomorphic 1-forms on X. Later we will see (in Lemma 9.1.1) its dimension equals to the genus of X.

Lemma 6.2.5. For $D_1, D_2 \in \mathrm{Div}(X)$. If $D_1 \sim D_2$, one has $\Gamma(X, \Omega^1_X(D_1)) \cong \Gamma(X, \Omega^1_X(D_2))$

Proof. The same as Lemma 6.2.3. $\square$

Theorem 6.2.2. Let K be the canonical divisor on X. Then for any $D \in \mathrm{Div}(X)$, one has

$$\Gamma(X, \Omega^1_X(D)) \cong \Gamma(X, \mathcal{O}_X(K+D)).$$

Proof. Suppose the canonical divisor K is given by meromorphic 1-form ω, that is, $K = \mathrm{div}(\omega)$. For any $f \in \Gamma(X, \mathcal{O}_X(K+D))$, one has

$$\mathrm{div}(f\omega) = \mathrm{div}(f) + \mathrm{div}(\omega) \geq -(K+D) + K = -D.$$

Thus $f\omega \in \Gamma(X, \Omega^1_X(D))$. This gives a linear map

$$\begin{aligned} \mu_\omega \colon \Gamma(X, \mathcal{O}_X(K+D)) &\to \Gamma(X, \Omega^1_X(D)) \\ f &\mapsto f\omega. \end{aligned}$$

It's clear that μ_ω is injective, and thus it suffices to show μ_ω is surjective. For any $\theta \in \Gamma(X, \Omega^1_X(D))$, by Lemma 6.1.4, there exists meromorphic function f such that $\theta = f\omega$. Note that

$$-D \leq \mathrm{div}(\theta) = \mathrm{div}(f) + \mathrm{div}(\omega) = \mathrm{div}(f) + K.$$

This shows $\mathrm{div}(f) + (D+K) \geq 0$, that is, $f \in \Gamma(X, \mathcal{O}_X(D+K))$, as desired. $\square$

6.3. **Linear system and morphisms to projective space.**

[7] Later we will introduce Riemann-Roch theorem to compute its dimension.

6.3.1. *Motivations.* Let X be a compact Riemann surface. If there exists a non-singular projective plane curve $C \subseteq \mathbb{P}^2$ of degree d such that X is biholomorphic to C as Riemann surfaces, then the genus of X is given by $(d-1)(d-2)/2$. In other words, the genus of X cannot be arbitrary integers, and thus not every compact Riemann surface can be embedded holomorphically into $\mathbb{P}^2$, so it's natural to ask whether there exists some $\mathbb{P}^N$ such that X can be embedded into $\mathbb{P}^N$ holomorphically?

Definition 6.3.1 (projective curve)**.** A compact Riemann surface X is called a (non-singular) projective curve, if X can be embedded into some projective space $\mathbb{P}^N$ holomorphically.

Note that if X is a isomorphic to a projective plane curve C defined by a homogenous polynomial F, then we can say the degree of X is the degree of F. In general, for a projective curve $X \subseteq \mathbb{P}^N$, one can also define its degree.

Firstly, fix a homogenous polynomial $G(x_0, \dots, x_N)$ which is not identically zero on X, we're going to define the intersection divisor $\operatorname{div}(G)$ on X, which records the points (with multiplicity) where $G = 0$. Fix a point $p \in X$ where G vanishes, and choose a homogenous polynomial H of the same degree as G, which does not vanish at p.

In this case G/H is a meromorphic function on X, which vanishes at p. Then $\operatorname{div}(G)(p)$ is defined to be the order of this meromorphic function at p, and for points q where $G \neq 0$, we set $\operatorname{div}(G)(q) = 0$. It's easy to see that $\operatorname{div}(G)$ is independent of the choice of H, and thus a well-defined divisor on X. In particular, if G has degree one, it's called the hyperplane divisor on X, and the degree of X is defined to be the degree of hyperplane divisor. It's well-defined, since the difference between any two hyperplane divisors is a principal divisor.

6.3.2. *Linear system.*

Definition 6.3.2 (complete linear system)**.** For $D \in \operatorname{Div}(X)$, the complete linear system of D is defined by

$$|D| = \{E \in \operatorname{Div}(X) \mid E \geq 0, E \sim D\}.$$

Lemma 6.3.1.

$$\begin{aligned} S\colon \mathbb{P}(\Gamma(X, \mathcal{O}_X(D))) &\to |D| \\ [f] &\mapsto \operatorname{div}(f) + D \end{aligned}$$

is bijective.

Proof. It's clear S is well-defined and by definition it's surjective. Now let's show the injectivity. For $f_1, f_2 \in \Gamma(X, \mathcal{O}_X(D)) \setminus \{0\}$, if $S(f_1) = S(f_2)$, then $\operatorname{div}(f_1/f_2) = 0$. This shows f_1/f_2 is a holomorphic function, and thus f_1/f_2 is constant. Then $f_1 = f_2$ in $\mathbb{P}(\Gamma(X, \mathcal{O}_X(D)))$. □

Corollary 6.3.1.

(1) If $\deg(D) < 0$, then $|D| = \varnothing$.
(2) If $D_1 \sim D_2$, then $|D_1| = |D_2|$.

(3) $\ell(D) \geq 1$ if and only if $|D| \neq \varnothing$.

Proof. (1) follows from Lemma 6.2.2, and (2) follows from Lemma 6.2.3. For (3). It suffices to note that $\ell(D) \geq 1$ if and only if $\mathbb{P}(\Gamma(X, \mathcal{O}_X(D))) \neq \varnothing$. $\square$

Definition 6.3.3 (linear system)**.** A linear system is a subset of a complete linear system $|D|$, which corresponds (via the map S) to a linear subspace of $\mathbb{P}(\Gamma(X, \mathcal{O}_X(D))$.

6.3.3. *Linear system of holomorphic maps to projective space.*

Definition 6.3.4 (holomorphic maps to projective space)**.** A map $\Phi\colon X \to \mathbb{P}^N$ is holomorphic at $p \in X$ if there are holomorphic functions $f_0, \dots, f_N$ defined near p, not all zero at p, such that $\Phi(q) = [f_0(q) : \dots f_N(q)]$ for every q near p. The map Φ is a holomorphic map if it's holomorphic at all points of X.

Note that if X is a compact Riemann surface, there is no global defined holomorphic function, and thus one cannot expect to use the same holomorphic functions f_i at all points of X to define a holomorphic map Φ.

However, one can use meromorphic functions to construct holomorphic maps to projective space, and it turns out every holomorphic map can be defined in this way. Let X be a Riemann surface and $f = \{f_0, \dots, f_N\}$ is a set of meromorphic functions on X. Now we define

$$\begin{aligned} \Phi_f\colon X &\to \mathbb{P}^N \\ p &\mapsto [f_0(p), \dots, f_N(p)]. \end{aligned}$$

In apriori, Φ_f is only defined for p such that p is not a pole of any f_i and p is not a zero of every f_i, and Φ_f is holomorphic at all points where it's defined.

Lemma 6.3.2. If the set of meromorphic functions $f = \{f_0, \dots, f_N\}$ is not all identically zero, then the map $\Phi_f\colon X \to \mathbb{P}^N$ is defined on all of X.

Proof. $\square$

Example 6.3.1 (linear system given by morphism)**.** Suppose $\Phi\colon X \to \mathbb{P}^N$ be a holomorphic map defined by meromorphic functions $f = \{f_0, \dots, f_N\}$ on X. If we denote $D = -\min_i\{\mathrm{div}(f_i)\}$, then for each i, one has

$$\mathrm{ord}_p(f_i) \geq -D(p).$$

Therefore $\{f_i\} \subseteq \Gamma(X, \mathcal{O}_X(D))$, and if we use V_f to denote the linear subspace generated by $\{f_i\}$, then it gives a linear system

$$|\Phi| = \{\mathrm{div}(g) + D \mid g \in V_f\}.$$

6.3.4. *Base locus of linear systems.*

Definition 6.3.5 (base locus)**.** Let Q be a linear system on X. A point p is a base point of Q if every divisors $E \in Q$ contains p, and the set of all base points of Q is called its base locus.

Definition 6.3.6 (base-point-free)**.** A linear system Q is said to be base-point-free if it has no base point.

Notation 6.3.1. For convenience, for a divisor D, the base locus of D is the base locus of the complete linear system $|D|$, and D is said to be base-point-free if $|D|$ is base-point-free.

Lemma 6.3.3. Let $D \in \mathrm{Div}(X)$ and $Q \subseteq |D|$ be a linear system defined by the subspace $V \subseteq \Gamma(X, \mathcal{O}_X(D))$. Then $p \in X$ is a base point of Q if and only if

$$V \subseteq \Gamma(X, \mathcal{O}_X(D-p)).$$

In particular, p is a base point of $|D|$ if and only if

$$\ell(D) = \ell(D-p).$$

Proof. Note that the linear system Q is given by $\{\mathrm{div}(f) + D \mid f \in V\}$, and thus p is a base point of Q if and only if for every $f \in V$, one has

$$\mathrm{ord}_p(f) + D(p) \geq 1.$$

In other words, for every $f \in V$, one has

$$\mathrm{ord}_p(f) \geq -D(p) + 1,$$

which is equivalent to $f \in \Gamma(X, \mathcal{O}_X(D-p))$, as desired. $\square$

Proposition 6.3.1. A divisor D is base-point-free if and only if

$$\ell(D-p) = \ell(D) - 1$$

for all $p \in X$.

Proof. It follows from Lemma 6.2.4 and Lemma 6.3.3. $\square$

Corollary 6.3.2. Let B be the base locus of D. Then $D-B$ is base-point-free, and

$$\Gamma(X, \mathcal{O}_X(D-B)) \to \Gamma(X, \mathcal{O}_X(D))$$

is an isomorphism.

Corollary 6.3.3. A divisor D is base-point-free if and only if for every $p \in X$, there exists a basis $\{f_0, f_1, \ldots, f_N\}$ of $\Gamma(X, \mathcal{O}_X(D))$ such that $\mathrm{ord}_p f_0 = -D(p)$ and $\mathrm{ord}_p f_i > -D(p)$ for $1 \leq i \leq N$.

Proof. If for every $p \in X$, there exists $f_0 \in \Gamma(X, \mathcal{O}_X(D))$ such that $\mathrm{ord}_p f_0 = -D(p)$, then $f \notin \Gamma(X, \mathcal{O}_X(D-p))$, and thus $\ell(D-p) = \ell(D) - 1$. This shows D is base-point-free.

Conversely, if D is base-point-free, then there exists $f_0 \in \Gamma(X, \mathcal{O}_X(D)) \setminus \Gamma(X, \mathcal{O}_X(D-p))$, that is, $\mathrm{ord}_p f_0 = -D(p)$. Suppose $f_1, \ldots, f_r$ is a basis of $\Gamma(X, \mathcal{O}_X(D-p))$. Then $f_0, f_1, \ldots, f_r$ is a basis of $\Gamma(X, \mathcal{O}_X(D))$, and $\mathrm{ord}_p f_i \geq -D(p) + 1 > -D(p)$. $\square$

6.3.5. *Hyperplane divisor of holomorphic maps to projective space.*

Lemma 6.3.4. Suppose the homogenous coordinate of $\mathbb{P}^N$ are $[x_0 : \cdots : x_N]$, and H is defined by the linear equation $L = \sum_i a_i x_i = 0$. Let the holomorphic map $\Phi\colon X \to \mathbb{P}^N$ be defined by $\Phi = [f_0 : \cdots : f_N]$ and set $D = \min_i\{\operatorname{div}(f_i)\}$. If $\Phi(X)$ is not contained in hyperplane H, then

$$\Phi^*(H) = \operatorname{div}(\sum_i a_i f_i) + D.$$

Corollary 6.3.4. Let $\Phi\colon X \to \mathbb{P}^N$ be a holomorphic map. Then the set of hyperplane divisors $\{\Phi^*(H)\}$ forms the linear system $|\Phi|$ of the map.

6.3.6. *Holomorphic maps and linear systems.*

Theorem 6.3.1. Let Q be a base-point-free linear system on a compact Riemann surface X. Then there exists a holomorphic map $\Phi\colon X \to \mathbb{P}^N$ such that $|\Phi| = Q$. Moreover, Φ is unique up to the choice of coordinates in $\mathbb{P}^N$.

6.3.7. *Criterion for ampleness.* Let D be a divisor on a compact Riemann surface X. By Corollary 6.3.2 one can always remove the base locus of D without changing the complete linear system $|D|$. Thus without lose of generality we may always assume $|D|$ is base-point-free, and thus it induces a holomorphic map $\Phi_D\colon X \to \mathbb{P}^N$.

Definition 6.3.7 (very ample)**.** A base-point-free divisor D on a compact Riemann surface X is called very ample if Φ_D is an embedding is called a very ample divisor.

Proposition 6.3.2. Φ_D is injective if and only if $\ell(D-p-q) = \ell(D) - 2$ for every $p \neq q \in X$.

Proof. For $p \in X$, by Corollary 6.3.3, choose a basis $f_0, \dots, f_N$ of $\Gamma(X, \mathcal{O}_X(D))$ such that $\operatorname{ord}_p f_0 = -D(p)$ and $\operatorname{ord}_p f_i > -D(p)$ for $i = 1, \dots, N$. Then for any $p \neq q \in X$, $\Phi_D(p) = \Phi_D(q)$ if and only if $\Phi_D(q) = [1 : 0 : \cdots : 0]$, which is equivalent to $\operatorname{ord}_q f_0 < \operatorname{ord}_q f_i$ for $1 \leq i \leq N$. Since q is not a base point of $|D|$, this happens if and only if $\operatorname{ord}_q f_0 = -D(q)$ and $\operatorname{ord}_q f_i > -D(q)$ for $1 \leq i \leq N$, which is equivalent to say $f_1, \dots, f_N$ is a basis of $\Gamma(X, \mathcal{O}_X(D-q))$. Therefore, $\Phi_D(p) = \Phi_D(q)$ if and only if

$$\Gamma(X, \mathcal{O}_X(D-p-q)) = \Gamma(X, \mathcal{O}_X(D-p)) = \Gamma(X, \mathcal{O}_X(D-q)).$$

Using above observation, it's easy to prove this proposition:

(1) If Φ_D is injective, then $\Gamma(X, \mathcal{O}_X(D-p-q)) \neq \Gamma(X, \mathcal{O}_X(D-p))$, and thus $\ell(D-p-q) = \ell(D-p) - 1 = \ell(D) - 2$ since $|D|$ is base-point-free.
(2) If $\ell(D-p-q) = \ell(D) - 2$ for every $p \neq q \in X$, then we must have
$$\Gamma(X, \mathcal{O}_X(D-p-q)) \neq \Gamma(X, \mathcal{O}_X(D-p)),$$
otherwise $\ell(D) - \ell(D-p-q) \leq 1$.

□

Proposition 6.3.3. Φ_D seperates tangent directions at $p \in X$ if and only if $\ell(D-2p) = \ell(D) - 2$.

Proof. For $p \in X$, by Corollary 6.3.3, choose a basis $f_0, \dots, f_N$ of $\Gamma(X, \mathcal{O}_X(D))$ such that $\operatorname{ord}_p f_0 = -D(p)$ and $\operatorname{ord}_p f_i > -D(p)$ for $i = 1, \dots, N$. Around $\Phi_D(p) = [1 : 0 : \cdots : 0]$, Φ_D seperates the tangent directions if and only if at least one of f_i satisfies $\operatorname{ord}_p f_i = -D(p) + 1$. Thus Φ_D seperates the tangent directions if and only if

$$\Gamma(X, \mathcal{O}_X(D-p)) \neq \Gamma(X, \mathcal{O}_X(D-2p)),$$

which is equivalent to $\ell(D-2p) = \ell(D) - 2$, since $|D|$ is base-point-free. $\square$

As a summary, we have proven the following result.

Theorem 6.3.2. A base-point-free divisor D is very ample if and only if for every $p, q \in X$, one has

$$\ell(D-p-q) = \ell(D) - 2.$$

7. Sheaves and its cohomology

7.1. **Sheaves.** Unless otherwise specified, X denotes a topological space along this section.

7.1.1. *Definitions and examples.*

Definition 7.1.1 (sheaf)**.** A presheaf of abelian group $\mathscr{F}$ on X consisting of the following data:

(1) For any open subset U of X, $\mathscr{F}(U)$ is an abelian group.
(2) If $U \subseteq V$ are two open subsets of X, then there is a group homomorphism $r_{UV} \colon \mathscr{F}(U) \to \mathscr{F}(V)$. Moreover, above data satisfy:
   I $\mathscr{F}(\varnothing) = 0$.
   II $r_{UU} = \mathrm{id}$.
   III If $W \subseteq U \subseteq V$ are open subsets of X, then $r_{UW} = r_{VW} \circ r_{UV}$.

Moreover, $\mathscr{F}$ is called a sheaf if it satisfies the following extra conditions

IV Let $\{V_i\}_{i\in I}$ be an open covering of open subset $U \subseteq X$ and $s \in \mathscr{F}(U)$. If $s|_{V_i} := r_{UV_i}(s) = 0$ for all $i \in I$, then $s = 0$.
V Let $\{V_i\}_{i\in I}$ be an open covering of open subset $U \subseteq X$ and $s_i \in \mathscr{F}(V_i)$. If $s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j}$ for all $i, j \in I$, then there exists $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$ for all $i \in I$.

Notation 7.1.1. Given a (pre)sheaf $\mathscr{F}$ on X, we also use $\Gamma(X, \mathscr{F})$ to denote $\mathscr{F}(U)$, and the elements in $\Gamma(U, \mathscr{F})$ are called sections of $\mathscr{F}$ over U. In particular, the elements in $\Gamma(X, \mathscr{F})$ are called global section of $\mathscr{F}$.

Example 7.1.1 (constant presheaf)**.** For an abelian group G, the constant presheaf assign each open subset U the group G itself, but in general it's not a sheaf.

Example 7.1.2. Let X be a Riemann surface and $\mathcal{O}_X(U)$ be the set of all holomorphic functions $f\colon U \to \mathbb{C}$. This gives a sheaf $\mathcal{O}_X$, which is called sheaf of holomorphic functions on X.

Example 7.1.3. Let X be a compact Riemann surface and D be a divisor on X. Let $\Gamma(X, \mathcal{O}_X(D))$ be the set of all meromorphic functions on U which satisfy the condition that

$$\operatorname{ord}_p(f) \geq -D(p)$$

for all $p \in U$. This gives a sheaf $\mathcal{O}_X(D)$, which is called sheaf of meromorphic functions with poles bounded by D.

Example 7.1.4 (skyscraper sheaf)**.** For an abelian group G, the skyscraper sheaf G_p is given by

$$G_p(U) = \begin{cases} \{0\}, & p \notin U \\ G, & p \in U. \end{cases}$$

7.1.2. *Morphisms and stalks.*

Definition 7.1.2 (morphism of presheaves)**.** A morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$ between presheaves consisting of the following data:

(1) For any open subset $U \subseteq X$, there is a group homomorphism $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$.
(2) If $U \subseteq V$ are two open subsets of X, then the following diagram commutes

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{G}(U) \\ \Big\downarrow{\scriptstyle r_{UV}} & & \Big\downarrow{\scriptstyle r_{UV}} \\ \mathscr{F}(V) & \xrightarrow{\varphi(V)} & \mathscr{G}(V) \end{array}$$

Notation 7.1.2. For convenience, for $s \in \mathscr{F}(U)$, we often write $\varphi(s)$ instead of $\varphi(U)(s)$.

Remark 7.1.1. The morphisms between sheaves are defined as morphisms of presheaves.

Definition 7.1.3 (isomorphism)**.** A morphism of presheaves $\varphi\colon \mathscr{F} \to \mathscr{G}$ is called an isomorphism if it has two-sided inverse, that is, there exists a morphism of presheaves $\psi\colon \mathscr{G} \to \mathscr{F}$ such that $\psi\varphi = \mathrm{id}_{\mathscr{F}}$ and $\varphi\psi = \mathrm{id}_{\mathscr{G}}$.

Remark 7.1.2. A morphism of presheaves $\varphi\colon \mathscr{F} \to \mathscr{G}$ is an isomorphism if and only if for every open subset $U \subseteq X$, $\varphi(U) \to \mathscr{G}(U)$ is an isomorphism of abelian groups.

Definition 7.1.4 (stalks)**.** For a presheaf $\mathscr{F}$ and $p \in X$, the stalk at p is defined as

$$\mathscr{F}_p = \varinjlim_{p \in U} \mathscr{F}(U)$$

Remark 7.1.3 (alternative definition). In order to avoid language of direct limit, we give a more useful but equivalent description of stalk: For $p \in U \cap V$, $s_U \in \mathscr{F}(U)$ and $s_V \in \mathscr{F}(V)$ are equivalent if there exists $p \in W \subseteq U \cap V$ such that $s_U|_W = s_V|_W$. An element $s_p \in \mathscr{F}_p$, which is called a germ, is an equivalence class $[s_U]$.

Notation 7.1.3.

(1) For $s \in \mathscr{F}(U)$ and $p \in U$, $s|_p$ denotes the equivalent class it gives.
(2) For $s_p \in \mathscr{F}_p$, $s \in \mathscr{F}(U)$ denotes the section such that $s|_p = s_p$.

Definition 7.1.5 (morphisms on stalks)**.** Given a morphism of sheaves $\varphi\colon \mathscr{F} \to \mathscr{G}$, it induces a morphism of abelian groups $\varphi_p\colon \mathscr{F}_p \to \mathscr{G}_p$ as follows:

$$\begin{aligned} \varphi_p\colon \mathscr{F}_p &\to \mathscr{G}_p \\ s_p &\mapsto \varphi(s)|_p. \end{aligned}$$

Remark 7.1.4. It's necessary to check the φ_p is well-defined since there are different choices s such that $s|_p = s_p$.

Proposition 7.1.1. Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism between sheaves. Then φ is an isomorphism if and only if the induced map $\varphi_p\colon \mathscr{F}_p \to \mathscr{G}_p$ is an isomorphism for every $p \in X$.

Proof. It's clear if φ is an isomorphism between sheaves, then it induces an isomorphism between stalks. Conversely, it suffices to show $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$ is an isomorphism for every open subset $U \subseteq X$.

(1) Injectivity: For $s, s' \in \mathscr{F}(U)$ such that $\varphi(s) = \varphi(s')$, by passing to stalks one has $\varphi_p(s|_p) = \varphi_p(s'|_p)$ for every $p \in U$, and thus $s|_p = s'|_p$ since φ_p is an isomorphism. By definition of stalks there exists an open subset $V_p \subseteq U$ containing p such that s agrees with s' on V_p. Then it gives an open covering $\{V_p\}$ of U, and by axiom (IV) one has $s = s'$ on U.
(2) Surjectivity: For $t \in \mathscr{G}(U)$, by passing to stalks there exists $s_p \in \mathscr{F}_p$ such that $\varphi_p(s_p) = t|_p$ for every $p \in U$ since φ_p is surjective. By definition of stalks there exists an open subset $V_p \subseteq U$ containing p and $s \in \mathscr{F}(V_p)$ such that $\varphi(s) = t$ on V_p. This gives a collection of sections defined on an open covering $\{V_p\}$ of U, and by injectivity we proved above one has these sections agree with each other on the intersections. Then by axiom (V) there exists a section $s \in \mathscr{F}(U)$ such that $\varphi(s) = t$.

□

7.1.3. *Sheafification.* In Example 7.1.1, we come across a presheaf that is not a sheaf. To obtain a sheaf from a presheaf, we require a process known as sheafification. One approach to defining sheafification is through its universal property.

Definition 7.1.6 (sheafification)**.** Given a presheaf $\mathscr{F}$ there is a sheaf $\mathscr{F}^+$ and a morphism $\theta\colon \mathscr{F} \to \mathscr{F}^+$ with the property that for any sheaf $\mathscr{G}$ and any morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$ there is a unique morphism $\overline{\varphi}\colon \mathscr{F}^+ \to \mathscr{G}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\varphi"] \arrow[d, "\theta"] & \mathscr{G} \\
\mathscr{F}^+ \arrow[ur, "\overline{\varphi}"] &
\end{tikzcd}$$

The universal property shows that if the sheafification exists, then it's unique up to a unique isomorphism. One way to give an explicit construction of sheafification is to glue stalks together in a suitable way. Let $\mathscr{F}^+(U)$ be a set of functions

$$f\colon U \to \coprod_{p \in U} \mathscr{F}_p$$

such that $f(p) \in \mathscr{F}_p$ and for every $p \in U$ there is an open subset $V_p \subseteq U$ containing p and $t \in \mathscr{F}(V_p)$ such that $t|_q = f(q)$ for all $q \in V_p$.

Proposition 7.1.2. $\mathscr{F}^+$ is the sheafication of $\mathscr{F}$.

Proof. Firstly let's show $\mathscr{F}^+$ is a sheaf: It's clear $\mathscr{F}^+$ is a presheaf, so it suffices to check conditions (IV) and (V) in the definition. Let $U \subseteq X$ be an open subset and $\{V_i\}$ be an open covering of U.

(1) If $s \in \mathscr{F}^+(U)$ such that $s|_{V_i} = 0$ for all i, then s must be zero: It suffices to show $s(p) = 0$ for all $p \in U$. For any $p \in U$, then there exists an open subset V_i contains p, hence $s(p) = s|_{V_i}(p) = 0$.
(2) Suppose there exists a collection of sections $\{s_i \in \mathscr{F}^+(V_i)\}_{i\in I}$ such that
$$s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j}$$
holds for all $i, j \in I$. Now we construct $s \in \mathscr{F}^+(U)$ as follows: For $p \in U$ and V_i containing p, we define $s(p) = s_i(p)$. This is well-defined since s_i agree on the intersections, so it remains to show $s \in \mathscr{F}^+(U)$. It's clear $s(p) \in \mathscr{F}_p$. For $p \in U$, there exists V_i containing p, and thus there exists $W_i \subseteq V_i$ containing p and $t \in \mathscr{F}(W_i)$ such that $t|_q = s_i(q) = s(q)$ for all $q \in V_p$.

There is a canonical morphism $\theta\colon \mathscr{F} \to \mathscr{F}^+$ as follows: For open subset $U \subseteq X$, and $s \in \mathscr{F}(U)$, $\theta(s)$ is defined by
$$\begin{aligned} \theta(s)\colon U &\to \coprod_{p\in U} \mathscr{F}_p \\ p &\mapsto s|_p. \end{aligned}$$
Note that if $\mathscr{F}$ is a sheaf, the canonical morphism $\theta\colon \mathscr{F} \to \mathscr{F}^+$ is an isomorphism.

(1) Injectivity: If $s \in \mathscr{F}(U)$ such that $s|_p = 0$ for all $p \in U$, then there exists an open covering $\{V_i\}_{i\in I}$ of U such that $s|_{V_i} = 0$, by axiom (IV) of sheaf one has $s = 0$.
(2) Surjectivity: For $f \in \mathscr{F}^+(U)$ and $p \in U$, there exists $p \in V_p \subseteq U$ and $t \in \mathscr{F}(V_p)$ such that $f(p) = t|_p$ by construction of $\mathscr{F}^+$. Then glue these sections together to get our desired s such that $\theta(s) = f$.

Finally let's show $\mathscr{F}^+$ statisfies the universal property of sheafification. A morphism of presheaves $\varphi\colon \mathscr{F} \to \mathscr{G}$ induces a map on stalks
$$\varphi_p\colon \mathscr{F}_p \to \mathscr{G}_p.$$
For $f \in \mathscr{F}^+(U)$, the composite of f with the map
$$\coprod_{p\in U} \varphi_p\colon \coprod_{p\in U} \mathscr{F}_p \to \coprod_{p\in U} \mathscr{G}_p$$
gives a map $\widetilde{\varphi}(f)\colon U \to \coprod_{p\in U} \mathscr{G}_p$, and in fact $\widetilde{\varphi}(f) \in \mathscr{G}^+(U)$: For $p \in U$, $\widetilde{\varphi}(f)(p) \in \mathscr{G}_p$ since $f(p) \in \mathscr{F}_p$ and $\varphi_p\colon \mathscr{F}_p \to \mathscr{G}_p$. If for all $q \in V_p$ we have $t|_q = f(q)$, then
$$\widetilde{\varphi}(f)(q) = \varphi_q(f(q)) = \varphi_q(t|_q) = \varphi(t)|_q.$$
Since $\mathscr{G}$ is a sheaf, the canonical morphism $\theta'\colon \mathscr{G} \to \mathscr{G}^+$ is an isomorphism, so we can define $\overline{\varphi} := \theta'^{-1} \circ \widetilde{\varphi}$. Now let's show $\varphi = \overline{\varphi} \circ \theta = \theta'^{-1} \circ \widetilde{\varphi} \circ \theta$. It's easy to show they coincide on each stalk since $\varphi_p = \theta_p'^{-1} \circ \widetilde{\varphi}_p \circ \theta_p$, and thus

$\varphi = \overline{\varphi} \circ \theta$ by Proposition 7.1.1. Furthermore, uniqueness follows from the fact that $\overline{\varphi}_p$ is uniquely determined by φ_p. $\square$

Remark 7.1.5. From the construction, one can see the stalk of $\mathscr{F}^+$ at p is exactly $\mathscr{F}_p$.

Remark 7.1.6. The sheafification can be described in a more fancy language: Since we have sheaf of abelian groups on X as a category, denote it by $\underline{Ab}_X$, and presheaf is a full subcategory of $\underline{Ab}_X$, there is a natural inclusion functor ι from category of sheaf to category of presheaf. The sheafification is the adjoint functor of ι.

Example 7.1.5 (constant sheaf)**.** For an abelian group G, the associated constant sheaf $\underline{G}$ is the sheafication of the constant presheaf. By the construction of sheafification, $\underline{G}$ can be explicitly expressed as

$$\underline{G}(U) = \{\text{locally constant function } f\colon U \to G\}$$

7.1.4. *Exact sequence of sheaf.* Given a morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$ between sheaves of abelian groups, there are the following presheaves

$$\begin{aligned} U &\mapsto \ker\varphi(U) \\ U &\mapsto \operatorname{im}\varphi(U) \\ U &\mapsto \operatorname{coker}\varphi(U), \end{aligned}$$

since $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$ is a group homomorphism.

Proposition 7.1.3. The kernel of a morphism between sheaves is a sheaf.

Proof. Let $\{V_i\}_{i\in I}$ be an open covering of U.

(1) For $s \in \ker\varphi(U)$, if $s|_{V_i} = 0$, then $s = 0$ since s is also in $\mathscr{F}(U)$.
(2) If there exists $s_i \in \ker\varphi(V_i)$ such that $s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j}$, then they glue together to get $s \in \mathscr{F}(U)$. Note that

$$\varphi(U)(s)|_{V_i} = \varphi(V_i)(s|_{V_i}) = \varphi(V_i)(s_i) = 0$$

Then $s \in \ker\varphi(U)$.

$\square$

But the image of morphism may not be a sheaf. Although we can prove the first requirement in the same way, the proof for the second requirement fails: If there exists $s_i \in \operatorname{im}\varphi(V_i)$, and we can glue them together to get a $s \in \mathscr{G}(U)$, but s may not be the image of some $t \in \mathscr{F}(U)$. The cokernel fails to be a sheaf for the same reason.

Definition 7.1.7 (image and cokernel)**.** Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism between sheaves of abelian groups. Then the image and cokernel of φ is defined to be the sheafification of the following presheaves

$$\begin{aligned} U &\mapsto \operatorname{im}\varphi(U) \\ U &\mapsto \operatorname{coker}\varphi(U) \end{aligned}$$

respectively.

Definition 7.1.8 (exact)**.** For a sequence of sheaves:

$$\dots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \dots$$

It's called exact at $\mathscr{F}^i$, if $\ker \varphi^i = \operatorname{im} \varphi^{i-1}$. If a sequence is exact at everywhere, then it's an exact sequence of sheaves.

Definition 7.1.9 (short exact sequence)**.** An exact sequence of sheaves is called a short exact sequence if it looks like

$$0 \to \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{H} \to 0$$

Proposition 7.1.4. Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism between sheaves of abelian groups. Then for any $p \in X$, one has

$$(\ker \varphi)_p = \ker \varphi_p$$
$$(\operatorname{im} \varphi)_p = \operatorname{im} \varphi_p.$$

Proof. For (1). It's clear $(\ker \varphi)_p \subseteq \ker \varphi_p$. Conversely, if $s_p \in \ker \varphi_p$, then $\varphi_p(s_p) = 0 \in \mathscr{G}_p$. In other words, there exists an open subset U containing p and $s \in \mathscr{F}(U)$ such that $s|_p = s_p$ and $\varphi(s)|_p = 0$, which implies there is another open subset V containing p such that $\varphi(s)|_V = 0$. Hence $\varphi(s|_V) = 0$, that is, $s|_V \in \ker \varphi(V)$. Thus $s_p = (s|_V)|_p \in (\ker \varphi)_p$.

For (2). It's clear $(\operatorname{im} \varphi)_p \subseteq \operatorname{im} \varphi_p$ since the sheafication doesn't change stalk. Conversely, if $s_p \in \operatorname{im} \varphi_p$, then there exists $t_p \in \mathscr{F}_p$ such that $\varphi_p(t_p) = s_p$. Suppose $t \in \mathscr{F}(U)$ is a section of some open subset U containing p such that $t|_p = t_p$. Then $\varphi(t)|_p = \varphi_p(t_p) = s_p$. In other words, s_p is in the stalk of the image presheaf at p, but the sheafication doesn't change stalk, so we have $s_p \in (\operatorname{im} \varphi)_p$. $\square$

Corollary 7.1.1. The sequence of sheaves

$$\dots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \dots$$

is exact if and only if the sequence of abelian groups are exact

$$\dots \to \mathscr{F}^{i-1}_p \xrightarrow{\varphi^{i-1}_p} \mathscr{F}^i_p \xrightarrow{\varphi^i_p} \mathscr{F}^{i+1}_p \to \dots$$

for all $p \in X$.

Corollary 7.1.2. The the sequence of sheaves

$$0 \to \mathscr{F} \to \mathscr{G}$$

is exact if and only if for any open subset U, the following sequence of abelian groups is exact

$$0 \to \mathscr{F}(U) \to \mathscr{G}(U).$$

Method one. For any open subset $U \subseteq X$, one has

$$\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$$

is injective, since by definition we have for any open subset $U \subseteq X$, $\ker \varphi(U) = 0$, that is injectivity. $\square$

Method two. Or from another point of view, for each $p \in U$, we have

$$\varphi_p \colon \mathscr{F}_p \to \mathscr{G}_p$$

is injective. That is $\ker \varphi_p = 0$. So we obtain $(\ker \varphi(U))_p = 0$ for all $p \in U$. But for a section $s \in \mathscr{F}(U)$ if we have $s|_p = 0$, then we must have $s = 0$, and thus $\ker \varphi(U) = 0$. $\square$

Example 7.1.6 (exponential sequence)**.** Let X be a Riemann surface and $\mathcal{O}_X$ be its holomorphic function sheaf. Then

$$0 \to 2\pi\sqrt{-1}\underline{\mathbb{Z}} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

is an exact sequence of sheaves, called exponential sequence.

Proof. The difficulty is to show exponential map is surjective on stalks at $p \in X$. That is we need to construct logarithms of functions $g \in \mathcal{O}_X^*(U)$ for U, a neighborhood of p. We may choose U is simply-connected, then define

$$\log g(q) = \log g(p) + \int_{\gamma_q} \frac{\mathrm{d}g}{g}$$

for $q \in U$, where γ_q is a path from p to q in U, and the definition of $\log g(q)$ is independent of the choice of γ_q since U is simply-connected. $\square$

Remark 7.1.7. In fact, U is simply-connected is crucial for constructing logarithm. If we consider $X = \mathbb{C}$ and $U = \mathbb{C} \setminus \{0\}$, then

$$\exp \colon \mathcal{O}_X(U) \to \mathcal{O}_X^*(U)$$

cannot be surjective.

Example 7.1.7. Let X be a compact Riemann surface and D be a divisor on X. For any point p, the sequence

$$0 \to \mathcal{O}_X(D-p) \to \mathcal{O}_X(D) \xrightarrow{ev_p} \mathbb{C}_p \to 0$$

is exact, where the evaluation map ev_p is given by sending $f = \sum_{n \geq -D(p)} c_n z^n$ to the coefficient $c_{-D(p)}$ on open subsets containing p, and is identically zero on open subsets not containing p.

7.2. **Čech cohomology.** In this section we talk about the Čech cohomology of sheaf $\mathscr{F}$ with repest to open covering $\mathfrak{U}$ on a topological space X. For convenience, we denote

$$U_{i_0 \dots i_n} := U_{i_0} \cap \dots \cap U_{i_n}.$$

The Čech cochain is defined by

$$0 \to C^0(\mathfrak{U}, \mathscr{F}) \xrightarrow{\delta} C^1(\mathfrak{U}, \mathscr{F}) \xrightarrow{\delta} C^2(\mathfrak{U}, \mathscr{F}) \to \dots,$$

where

$$C^n(\mathfrak{U}, \mathscr{F}) = \prod_{(i_0 \dots i_n)} \mathscr{F}(U_{i_0 \dots U_{i_n}}),$$

and the differential $\delta\colon C^n(\mathfrak{U},\mathscr{F})\to C^{n+1}(\mathfrak{U},\mathscr{F})$ is given by

$$\delta(f_{i_0\dots i_n})=(g_{i_0\dots i_{n+1}}),$$

where

$$g_{i_0\dots i_{n+1}}=\sum_{k=0}^{n+1}(-1)^k f_{i_0\dots\widehat{i_k}\dots i_{n+1}}|_{U_{i_0\dots i_{n+1}}}.$$

A routine computation shows above Čech cochain is a complex, and the cohomology of this complex, denoted by $\check{H}^*(\mathfrak{U},\mathscr{F})$, is the Čech cohomology of sheaf $\mathscr{F}$ with repest to open covering $\mathfrak{U}$. One natural question is what will happen when we change the choice of open covering.

Definition 7.2.1 (refinement)**.** Let $\mathfrak{U}=\{U_\alpha\}_{\alpha\in I}$ and $\mathfrak{V}=\{V_\beta\}_{\beta\in J}$ be two open coverings of X. We say that $\mathfrak{V}$ is a refinement of $\mathfrak{U}$ if for every open subset V_j, there exists an open subset U_i such that $V_j\subseteq U_i$.

Remark 7.2.1 (refining map)*.* Any choice of such a U_i for every V_j can be viewed as a function $r\colon J\to I$ on the index sets, and such a function is called a refining map.

Given $\mathfrak{U}=\{U_\alpha\}_{\alpha\in I}$ an open covering and $\mathfrak{V}=\{V_\beta\}_{\beta\in J}$ is a refinement of $\mathfrak{U}$. If $\phi\colon J\to I$ is a refining map, then it gives a map between n-cochains as follows

$$\phi^\sharp\colon C^n(\mathfrak{U},\mathscr{F})\to C^n(\mathfrak{V},\mathscr{F}),$$

given by

$$\phi^\sharp(\omega)(V_{\beta_0\dots\beta_n})=\omega(U_{\phi(\beta_0)\dots\phi(\beta_n)}),$$

where $\omega\in C^n(\mathfrak{U},\mathscr{F})$.

Lemma 7.2.1. Given $\mathfrak{U}=\{U_\alpha\}_{\alpha\in I}$ an open covering and $\mathfrak{V}=\{V_\beta\}_{\beta\in J}$ is a refinement of $\mathfrak{U}$. If ϕ,ψ are two refining maps $J\to I$, then there is a homotopy operator between $\phi^\#$ and $\psi^\#$. In other words, there exists a homeomorphism from $\check{H}^*(\mathfrak{U},\mathscr{F})\to\check{H}^*(\mathfrak{V},\mathscr{F})$, whihc is independent of the choice of the refining maps.

Proof. Define $K\colon C^q(\mathfrak{U},\mathscr{F})\to C^{q-1}(\mathfrak{V},\mathscr{F})$ by

$$(K\omega)(V_{\beta_0\dots\beta_{q-1}})=\sum(-1)^i\omega(U_{\phi(\beta_0)\dots\phi(\beta_i)\psi(\beta_i)\dots\psi(\beta_{q-1})}).$$

Now let's show[8]

$$\psi^\#-\phi^\#=\delta K+K\delta.$$

For any cochain $\omega\in C^q(\mathfrak{U},\mathscr{F})$ and $V_{\beta_0\dots\beta_q}$, it's easy to see

$$\psi^\#-\phi^\#(\omega)(V_{\beta_0\dots\beta_q})=\omega(U_{\psi(\beta_0)\dots\psi(\beta_q)})-\omega(U_{\phi(\beta_0)\dots\phi(\beta_q)}).$$

[8] An exercise you only check once in your whole life.

On the other hand, one has

$$\begin{aligned}\delta K(\omega)(V_{\beta_0\ldots\beta_q}) &= \sum_i (-1)^i K\omega(V_{\beta_0\ldots\widehat{\beta_i}\ldots\beta_q})\\ &= \underbrace{\sum_{i\leq j}(-1)^{i+j}\omega(U_{\phi(\beta_0)\ldots\widehat{\phi(\beta_i)}\ldots\phi(\beta_{j+1})\psi(\beta_{j+1})\ldots\psi(\beta_q)})}_{\text{part I}}\\ &\quad+\underbrace{\sum_{i>j}(-1)^{i+j}\omega(U_{\phi(\beta_0)\ldots\phi(\beta_j)\psi(\beta_j)\ldots\widehat{\psi(\beta_j)}\ldots\psi(\beta_q)})}_{\text{part II}}.\end{aligned}$$

By the same computation one has

$$\begin{aligned}K\delta\omega(V_{\beta_0\ldots\beta_q}) &= \sum_j (-1)^j \delta\omega(U_{\phi(\beta_0)\ldots\phi(\beta_j)\psi(\beta_j)\ldots\psi(\beta_q)})\\ &= \underbrace{\sum_{i<j}(-1)^{i+j}\omega(U_{\phi(\beta_0)\ldots\widehat{\phi(\beta_i)}\ldots\phi(\beta_j)\psi(\beta_j)\ldots\psi(\beta_q)})}_{\text{part III}}\\ &\quad+\underbrace{\sum_{i>j}(-1)^{i+j}\omega(U_{\phi(\beta_0)\ldots\phi(\beta_j)\psi(\beta_j)\ldots\widehat{\psi(\beta_{i-1})}\ldots\psi(\beta_q)})}_{\text{part IV}}\\ &\quad+\underbrace{\sum_j \omega(U_{\phi(\beta_0)\ldots\widehat{\phi(\beta_j)}\psi(\beta_j)\ldots\psi(\beta_q)})}_{\text{part V}}.\end{aligned}$$

Note that part I cancels with part III, since if you fix i, you will find j-th terms of part I and part III are equal but differ a sign. Similarly you can find part II and part IV almost cancel each other, but

$$\text{part II}+\text{part IV}=\underbrace{\sum_j -\omega(U_{\phi(\beta_0)\ldots\phi(\beta_j)\widehat{\psi(\beta_j)}\psi(\beta_{j+1})\ldots\psi(\beta_q)})}_{\text{part VI}},$$

and it's clear to see that

$$\text{part V}+\text{part VI}=\omega(U_{\psi(\beta_0)\ldots\psi(\beta_q)})-\omega(U_{\phi(\beta_0)\ldots\phi(\beta_q)})$$

as desired. This completes the proof. □

Thus for two different open covering $\mathfrak{U},\mathfrak{V}$ such that $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, there is a natural homomorphism

$$f_{\mathfrak{U}\mathfrak{V}}\colon H^*(\mathfrak{U},\mathscr{F})\to H^*(\mathfrak{V},\mathscr{F}).$$

Furthermore, if there are three open covering such that $\mathfrak{C}$ is a refinement of $\mathfrak{V}$, and $\mathfrak{V}$ is a refinement of $\mathfrak{U}$. then we have

$$f_{\mathfrak{U}\mathfrak{C}}=f_{\mathfrak{U}\mathfrak{V}}f_{\mathfrak{V}\mathfrak{C}}.$$

So if we give a partial order on set of all open coverings, that is $\mathfrak{U} < \mathfrak{V}$, if $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, we obtain a direct system $\{H^*(\mathfrak{U}, \mathscr{F}), f_{\mathfrak{U}\mathfrak{V}}\}$. The direct limit of this direct system

$$\check{H}^*(X, \mathscr{F}) := \varinjlim_{\mathfrak{U}} \check{H}^*(\mathfrak{U}, \mathscr{F}).$$

is called Čech cohomology of X valued in the sheaf $\mathscr{F}$.

Remark 7.2.2. In fact, the definition of Čech cohomology makes sense for any presheaf $\mathscr{F}$, and if X is a paracompact topological space[9](In particular, any manifold), then the $\check{H}(X, \mathscr{F}) = \check{H}(X, \mathscr{F}^+)$.

Notation 7.2.1. For convenience, we denote

$$H^i(X, \mathscr{F}) = \check{H}^i(X, \mathscr{F}),$$

and

$$h^i(X, \mathscr{F}) := \dim H^i(X, \mathscr{F}).$$

In particular, if X is a compact Riemann surface and D is a divisor, then

$$h^0(X, \mathcal{O}_X(D)) = \ell(D).$$

7.3. Computations for Čech cohomology.

7.3.1. *The vanishing of H^1 for skyscraper sheaves.*

Theorem 7.3.1. Let X be a paracompact topological space and $\mathscr{F}$ be a skyscraper sheaf on X. Then $H^n(X, \mathscr{F}) = 0$ for $n \geq 1$.

7.3.2. *The vanishing of H^2 for $\mathcal{O}_X(D)$.*

Theorem 7.3.2. Let X be a compact Riemann surface and D be a divisor on X. Then $H^n(X, \mathcal{O}_X(D)) = 0$ for $n \geq 2$.

7.3.3. *The long exact sequence of cohomology.*

Theorem 7.3.3 (Zig-zag)**.** Let X be a paracompact topological space and

$$0 \to \mathscr{K} \to \mathscr{F} \to \mathscr{G} \to 0$$

is an exact sequence of sheaves on X. Then there is a long exact sequence of cohomology groups

$$\begin{tikzcd}
0 \arrow[r] & H^0(X, \mathscr{K}) \arrow[r] & H^0(X, \mathscr{F}) \arrow[r] & H^0(X, \mathscr{G}) \arrow[dll] & \\
& H^1(X, \mathscr{K}) \arrow[r] & H^1(X, \mathscr{F}) \arrow[r] & H^1(X, \mathscr{G}) \arrow[dll] & \\
& H^2(X, \mathscr{K}) \arrow[r] & H^2(X, \mathscr{F}) \arrow[r] & H^2(X, \mathscr{G}) \arrow[r] & \dots
\end{tikzcd}$$

7.4. Algebraic sheaves and Zariski cohomology.

[9]A topological space is called paracompact, if it's Hausdorff and every open covering has a locally finite refinement, and an open covering is called locally finite, if every point has a neighborhood which intersects only finite many of the open subsets in the covering.

8. Riemann-Roch theorem

8.1. First version of Riemann-Roch theorem.

Definition 8.1.1 (Euler characterisitic)**.** Let X be a compact Riemann surface and $\mathscr{F}$ is a locally free sheaf on X. The Euler characterisitic of $\mathscr{F}$ is defined by

$$\chi(X,\mathscr{F}) := h^0(X,\mathscr{F}) - h^1(X,\mathscr{F}) + h^2(X,\mathscr{F}).$$

Example 8.1.1. Let X be a compact Riemann surface and D is divisor on X. Then

$$\chi(X,\mathcal{O}_X(D)) = h^0(X,\mathcal{O}_X(D)) - h^1(X,\mathcal{O}_X(D)),$$

since $h^2(X,\mathcal{O}_X(D)) = 0$.

Theorem 8.1.1 (Riemann-Roch)**.** Let X be a compact Riemann surface and D be a divisor on X. Then

$$\chi(X,\mathcal{O}_X(D)) = \chi(\mathcal{O}) + \deg(D).$$

Proof. By using the short exact of sheaves

$$0 \to \mathcal{O}_X(D-p) \to \mathcal{O}_X(D) \xrightarrow{ev_p} \mathbb{C}_p \to 0,$$

there is a long exact sequence of cohomology groups as follows

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathcal{O}_X(D-p)) \arrow[r] & H^0(X,\mathcal{O}_X(D)) \arrow[r] & H^0(X,\mathbb{C}_p) \arrow[dll] \\
& H^1(X,\mathcal{O}_X(D-p)) \arrow[r] & H^1(X,\mathcal{O}_X(D)) \arrow[r] & H^1(X,\mathbb{C}_p) = 0.
\end{tikzcd}$$

This shows

$$h^0(X,\mathcal{O}_X(D-p)) - h^0(X,\mathcal{O}_X(D)) + 1 - h^1(X,\mathcal{O}_X(D-p)) + h^1(X,\mathcal{O}_X(D)) = 0,$$

that is

$$\chi(X,\mathcal{O}_X(D)) = \chi(X,\mathcal{O}_X(D-p)) + 1.$$

By induction one has

$$\chi(X,\mathcal{O}_X(D)) = \chi(X,\mathcal{O}) + \deg(D).$$

$\square$

Remark 8.1.1 (Hirzebruch–Riemann–Roch). Let X be a compact manifold and $\mathscr{E}$ be a sheaf of holomorphic vector bundle. Then

$$\chi(X,\mathscr{E}) = \int_X \mathrm{Ch}(\mathscr{E})\,\mathrm{Td}(X),$$

where $\mathrm{Ch}(\mathscr{E})$ is the Chern character of $\mathscr{E}$ and $\mathrm{Td}(X)$ is the Todd class of TX. In the curve case, the Todd class of tangent bundle is $1 + c_1(TX)/2$ and the Chern character of $\mathcal{O}_X(D)$ is $1 + c_1(\mathcal{O}_X(D))$. This shows $\chi(X,\mathcal{O}_X(D)) = c_1(\mathcal{O}_X(D)) + c_1(TX)/2 = \deg(D) + 1 - g$.

8.2. **Serre duality.**

Theorem 8.2.1 (Serre duality)**.** Let X be a compact Riemann surface and $\Omega^1_X(D)$ be the sheaf of meromorphic 1-forms with poles bounded by D. Then there is a a perfect pairing

$$H^1(X, \mathcal{O}_X(D)) \times H^0(X, \Omega^1_X(-D)) \to \mathbb{C}\,.$$

In particular, $h^1(X, \mathcal{O}_X(D)) = h^0(X, \mathcal{O}_X(K - D))$.

Corollary 8.2.1 (Riemann-Roch)**.** Let X be a compact Riemann surface and D be a divisor on X. Then

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g,$$

where K is the canonical divisor.

8.2.1. *Laurent tail divisor.* Let X be a Riemann surface and z_p is a local coordinate centered at point $p \in X$. A Laurent tail with respect to p is a function of the form

$$r_p(z_p) = \sum_{i=-n_p}^{k_p} a_i z_p^i,$$

where $a_i \in \mathbb{C}$.

Definition 8.2.1 (Laurent tail divisor)**.** A Laurent tail divisor on X is a formal finite sum

$$\sum_{p \in X} r_p(z_p) \cdot p,$$

where $r_p(z_p)$ is a Laurent tail with respect to p.

Let $\mathcal{T}_X$ be the sheaf of Laurent tail divisors, which is defined by setting

$$U \mapsto \mathcal{T}_X(U) := \{\sum_{p \in U} r_p(z_p) \cdot p \mid r_p(z_p) \text{ is a Laurent tail with respect to } p\}.$$

For any $D \in \mathrm{Div}(X)$, the Laurent tail sheaf associated to D is defined by setting

$$U \mapsto \mathcal{T}_X[D](U) = \{\sum_{p \in U} r_p(z_p) \mid k_p < -D(p) \text{ for all } p \in U\},$$

where k_p is the maximal order of the function r_p.

Consider the following morphism of sheaves

$$\begin{aligned} \alpha_D(U) \colon \mathcal{M}(U) &\to \mathcal{T}_X[D](U) \\ f &\mapsto \sum_{p \in U} r_p(z_p) p, \end{aligned}$$

where $r_p(z_p)$ is obtained from the Laurent series of f in z_p by cutting off all terms with degree $\geq -D(p)$.

Lemma 8.2.1. $\ker \alpha_D = \mathcal{O}_X(D)$.

Proof. For any open subset $U \subseteq X$ and $f \in \mathcal{M}(U)$, if $\alpha_D(U)(f) = \sum_p r_p(z_p)p$, then

$$\begin{aligned} f \in \Gamma(U, \mathcal{O}_X(D)) &\iff \operatorname{div}(f) \geq -D \\ &\iff \operatorname{ord}_p(f) \geq -D(p) \\ &\iff r_p(z_p) = 0 \\ &\iff \alpha_D(U)(f) = 0. \end{aligned}$$

□

Remark 8.2.1. In other words, there is the following exact sequence of sheaves

$$0 \to \mathcal{O}_X(D) \to \mathcal{M} \xrightarrow{\alpha_D} \mathcal{T}_X[D] \to 0.$$

This induces a long exact sequence in cohomology, that is,

$$0 \to H^0(X, \mathcal{O}_X(D)) \to H^0(X, \mathcal{M}_X) \xrightarrow{\alpha_D} H^0(X, \mathcal{T}_X[D]) \to H^1(X, \mathcal{O}_X(D)) \to 0$$

This follows

$$H^1(X, \mathcal{O}_X(D)) = \operatorname{coker} \alpha_D.$$

8.2.2. *Proof of Serre duality.* For $\omega \in \Gamma(X, \Omega^1_X(-D))$, consider the following residue map

$$\begin{aligned} \operatorname{Res}_\omega \colon \mathcal{T}_X[D](X) &\to \mathbb{C} \\ \sum_p r_p(z_p)p &\mapsto \sum_p \operatorname{Res}_p(r_p(z_p)\omega). \end{aligned}$$

The following lemma shows that the residue map can descend to $H^1(X, \mathcal{O}_X(D)) = \mathcal{T}_X[D](X)/\mathcal{M}(X)$.

Lemma 8.2.2. For $f \in \mathcal{M}(X)$, one has $\operatorname{Res}_\omega(\alpha_D(f)) = 0$.

Proof. Suppose the Laurent series of f at p is $\sum_k a_k z_p^k$, and ω is locally given by

$$(\sum_{n=D(p)}^{\infty} c_n z_p^n)\mathrm{d}z_p.$$

Then $\operatorname{Res}_p(f\omega)$ equals to the coefficient of z_p^{-1} in $(\sum_k a_k z_p^k)(\sum_{n=D(p)}^{\infty} c_n z_p^n)\mathrm{d}z_p$, which is $\sum_{n=D(p)}^{\infty} a_{-n-1}c_n$. Thus only a_k with $k < -D(p)$ can contribute to $\operatorname{Res}_p(f\omega)$. On the other hand, by definition of α_D, we have

$$\operatorname{Res}_p(f\omega) = \operatorname{Res}_p(r_p(z_p)w)$$

where $\alpha_D(f) = \sum_p r_p(z_p)p$. By residue theorem, we have

$$\operatorname{Res}_p(\alpha_D(f)) = \sum_p \operatorname{Res}_p(f\omega) = 0$$

□

As a consequence, one has the following linear map

$$\mathrm{Res}_\omega\colon H^1(X, \mathcal{O}_X(D)) \to \mathbb{C}\,.$$

In other words, we have

$$\begin{aligned}\mathrm{Res}\colon \Gamma(X, \Omega^1_X(-D)) &\to H^1(X, \mathcal{O}_X(D))^* \\ \omega &\mapsto \mathrm{Res}_\omega\,.\end{aligned}$$

Theorem 8.2.2 (Serre duality)**.** The residue map Res is an isomorphism.

Proof of injectivity. For any $0 \neq \omega \in \Gamma(X, \Omega^1_X(-D))$, we fix $p \in X$ and let $k = \mathrm{ord}_p(\omega) \geq D(p)$. Let

$$Z = \frac{1}{z_p^{k+1}} p \in \mathcal{T}_X[D](X).$$

Near p, we write ω as

$$(\sum_{n=k}^{\infty} c_k z_p^k)\mathrm{d}z_p, \quad c_k \neq 0$$

then

$$\mathrm{Res}_\omega(Z) = c_k \neq 0$$

So $\mathrm{Res}_\omega \neq 0$, that's injectivity. □

It remains to show it's surjective, which is still a long way to prove it, and let's make some preparations first. For $f \in \mathcal{M}(X)$ and $D \in \mathrm{Div}(X)$, we define multiplicative map

$$\begin{aligned}\mu_f = \mu_f^D \colon \mathcal{T}_X[D](X) &\to \mathcal{T}_X[D - \mathrm{div}(f)](X) \\ \sum_p r_p(z_p) \cdot p &\mapsto \text{suitable truncation of } \sum_p (f r_p(z_p)) \cdot p\end{aligned}$$

Exercise 8.2.1. If $f \not\equiv 0$, then μ_f is an isomorphism with inverse $\mu_{\frac{1}{f}}$.

Exercise 8.2.2. For $f, g \in \mathcal{M}(X)$ and $D \in \mathrm{Div}(X)$, one has

$$\mu_f(\alpha_D(g)) = \alpha_{D-\mathrm{div}(f)}(fg).$$

In other words, the following diagram commutes

$$\begin{tikzcd} \mathcal{M}(X) \arrow[r, "f"] \arrow[d, "\alpha_D"] & \mathcal{M}(X) \arrow[d, "\alpha_{D-\mathrm{div}(f)}"] \\ \mathcal{T}_X[D](X) \arrow[r, "\mu_f"] & \mathcal{T}_X[D](X) \end{tikzcd}$$

As a consequence, deduce that

$$\mu_f(\mathrm{im}\,\alpha_D) \subseteq \mathrm{im}(\alpha_{D-\mathrm{div}(f)}).$$

Remark 8.2.2. For any $\varphi \in H^1(X, \mathcal{O}_X(D))^*$, the composite

$$\widetilde{\varphi}\colon \mathcal{T}_X[D](X) \xrightarrow{\pi} H^1(X, \mathcal{O}_X(D)) \xrightarrow{\varphi} \mathbb{C},$$

satisfies $\widetilde{\varphi}|_{\operatorname{im}\alpha_D} = 0$; Conversely, any linear map $\widetilde{\varphi}\colon \mathcal{T}_X[D](X) \to \mathbb{C}$ such that $\widetilde{\varphi}|_{\operatorname{im}\alpha_D} = 0$ gives a linear map $\varphi\colon H^1(X, \mathcal{O}_X(D)) \to \mathbb{C}$. By Exercise 8.2.2 one has

$$\widetilde{\varphi} \circ \mu_f|_{\operatorname{im}(\alpha_{D+\operatorname{div}(f)})} = 0,$$

and thus $\widetilde{\varphi} \circ \mu_f$ induces a linear map $H^1(X, \mathcal{O}_X(D + \operatorname{div}(f))) \to \mathbb{C}$.

Lemma 8.2.3. For any $A \in \operatorname{Div}(X)$ and two non-zero $\varphi_1, \varphi_2 \in H^1(X, \mathcal{O}_X(A))^*$, there exists $B \in \operatorname{Div}(X)$ with $B > 0$ together with non-zero meromorphic functions $f_1, f_2 \in H^0(X, \mathcal{O}_X(B))$ such that

$$\widetilde{\varphi}_1 \circ t_A^{A-B-\operatorname{div}(f_1)} \circ \mu_{f_1} = \widetilde{\varphi_2} \circ t_A^{A-B-\operatorname{div}(f_2)} \circ \mu_{f_2}.$$

In other words, the following diagram commutes

$$\begin{tikzcd}
 & \mathcal{T}_X[A-B-\operatorname{div}(f_1)](X) \arrow[r, "t"] & \mathcal{T}_X[A](X) \arrow[dr, "\widetilde{\varphi}_1"] & \\
\mathcal{T}_X[A-B](X) \arrow[ur, "\mu_{f_1}"] \arrow[dr, "\mu_{f_2}"'] & & & \mathbb{C} \\
 & \mathcal{T}_X[A-B-\operatorname{div}(f_2)](X) \arrow[r, "t"] & \mathcal{T}_X[B](X) \arrow[ur, "\widetilde{\varphi}_2"'] &
\end{tikzcd}$$

Proof. Suppose that no such divisor B and non-zero meromorphic functions f_1, f_2 exist. Then for every positive definite B, it turns out that

$$\begin{aligned}\Gamma(X, \mathcal{O}_X(B)) \times \Gamma(X, \mathcal{O}_X(B)) &\to H^1(X, \mathcal{O}_X(A-B))^* \\ (f_1, f_2) &\mapsto \widetilde{\varphi}_1 \circ t_A^{A-B-\operatorname{div}(f_1)} \circ \mu_{f_1} - \widetilde{\varphi}_2 \circ t_A^{A-B-\operatorname{div}(f_2)} \circ \mu_{f_2}\end{aligned}$$

is injective. In particular, one has $2\ell(B) \leq h^1(X, \mathcal{O}_X(A-B))$, and by the first version of Riemann-Roch Theorem (Theorem 8.1.1), one has

$$\begin{aligned}h^1(X, \mathcal{O}_X(A-B)) &= \dim \ell(A-B) - \deg(A-B) - 1 + h^1(X, \mathcal{O}_X) \\ &\leq \dim L(A) - \deg(A) - 1 + h^1(X, \mathcal{O}_X) + \deg(B) \\ &= a + \deg(B).\end{aligned}$$

where a is constant. On the other hand, one has

$$\begin{aligned}\ell(B) &= h^1(X, \mathcal{O}_X(B)) + \deg(B) - 1 + h^1(X, \mathcal{O}_X) \\ &\geq 1 - h^1(X, \mathcal{O}_X) + \deg(B) \\ &= b + \deg(B)\end{aligned}$$

where b is constant. This leads to the following inequalities

$$a + \deg(B) \geq h^1(X, \mathcal{O}_X(A-B)) \geq 2\ell(B) \geq 2b + 2\deg(B),$$

and it cannot hold for sufficiently large $\deg(B)$, which is a contradiction. $\square$

Lemma 8.2.4. Let $D_1 \in \mathrm{Div}(X)$ be a divisor and and $\omega \in \Gamma(X, \Omega_X^1(-D_1))$. Suppose there is another divisor $D_2 \geq D_1$ such that $\mathrm{Res}_\omega \colon \mathcal{T}_X[D_1](X) \to \mathbb{C}$ satisfies

$$\mathrm{Res}_\omega |_{\ker t_{D_2}^{D_1}} = 0.$$

Then $\omega \in \Gamma(X, \Omega_X^1(-D_2))$, and the following diagram commutes

$$\begin{tikzcd} \mathcal{T}_X[D_1](X) \arrow[rr, "t_{D_2}^{D_1}"] \arrow[dr, "\mathrm{Res}_\omega"'] & & \mathcal{T}_X[D_2](X) \arrow[dl, "\mathrm{Res}_\omega"] \\ & \mathbb{C} & \end{tikzcd}$$

Proof. Suppose $\omega \notin \Gamma(X, \Omega_X^1(-D_2))$. Then there exists $p \in X$ such that

$$D_1(p) \leq k = \mathrm{ord}_p(\omega) < D_2(p).$$

Let's consider the Laurent tail divisor $Z = z_p^{-k-1} p \in \mathcal{T}_X[D_1](X)$. Then $t_{D_2}^{D_1}(Z) = 0$, but for $\omega = (\sum_{n=k}^{\infty} c_n z_p^n) \mathrm{d} z_p$, one has

$$\mathrm{Res}_\omega(Z) = c_k \neq 0,$$

which is a contradiction.

For the half part, given any $Z = \sum_p r_p(z) p \in \mathcal{T}_X[D_1](X)$, $\mathrm{Res}_\omega(Z)$ only depends on terms in r_p with order which is less than $-D_2(p) \leq -D_1(p)$. This proves that the diagram commutes. □

Finally, let's complete the proof of Serre duality.

Proof of surjectivity. Let ω be any meromorphic 1-form on X and $K = \mathrm{div}(\omega)$ be the canonical divisor. For any $0 \neq \varphi \in H^1(X, \mathcal{O}_X(D))^*$, we pick $A \in \mathrm{Div}(X)$ such that $A \leq D$ and $A \leq K$, so that $\omega \in \Gamma(X, \Omega_X^1(-A))$. Let's set $\varphi_A := \widetilde{\varphi} \circ t_D^A$. By Lemma 8.2.3, it turns out that there exists a divisor $B > 0$ and non-zero meromorphic functions $f_1, f_2 \in \Gamma(X, \mathcal{O}_X(B))$ such that

$$\varphi_A \circ t_A^{A-B-\mathrm{div}(f_1)} \circ \mu_{f_1} = \mathrm{Res}_\omega \circ t_A^{A-B-\mathrm{div}(f_2)} \circ \mu_{f_2}.$$

For the right hand side, one has

$$\begin{tikzcd} \mathcal{T}_X[A-B](X) \arrow[r, "\mu_{f_2}"] & \mathcal{T}_X[A-B-\mathrm{div}(f_2)](X) \arrow[r] & \mathcal{T}_X[A](X) \arrow[d, "\mathrm{Res}_\omega"] \\ & & \mathbb{C}\,. \end{tikzcd}$$

Since

$$\begin{aligned} \mathrm{div}(\omega) &\geq A \geq A - B - \mathrm{div}(f_2) \\ \mathrm{div}(f_2\omega) &\geq A - B, \end{aligned}$$

we can add two more arrows in the above diagram such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{T}_X[A-B](X) \arrow[r, "\mu_{f_2}"] \arrow[drr, "\mathrm{Res}_{f_2\omega}"'] & \mathcal{T}_X[A-B-\mathrm{div}(f_2)](X) \arrow[r] \arrow[dr, "\mathrm{Res}_\omega"] & \mathcal{T}_X[A](X) \arrow[d, "\mathrm{Res}_\omega"] \\
& & \mathbb{C}\,.
\end{tikzcd}$$

In other words, one has

$$\varphi_A \circ t^{A-B-\mathrm{div}(f_1)} \circ \mu_{f_1} = \mathrm{Res}_{f_2\omega}\,.$$

By composing $\mu_{f_1}^{-1}$, one has

$$\varphi_A \circ t_A^{A-B-\mathrm{div}(f_1)} = \mathrm{Res}_{\frac{f_2}{f_1}\omega}\,.$$

If we define $\widetilde{\omega} = f_2\omega/f_1$, then $\mathrm{Res}_{\widetilde{\omega}}|_{\ker t_A^{A-B-\mathrm{div}(f_1)}} = 0$. By Lemma 8.2.4, $\widetilde{\omega} \in \Gamma(X, \Omega_X^1(-A))$, and hence $\mathrm{Res}_{\widetilde{\omega}} = \varphi_A$.

By definition, the map φ_A is the composite of $\widetilde{\varphi}$ and t_D^A, hence $\mathrm{Res}_{\widetilde{\omega}}|_{\ker t_D^A} = 0$. By Lemma 8.2.4 again, $\widetilde{\omega} \in \Gamma(X, \Omega_X^1(-D))$ such that $\mathrm{Res}_{\widetilde{\omega}} = \widetilde{\varphi}$. This completes the proof. $\square$

9. Applications of Riemann-Roch theorem

In this section, X always denotes a compact Riemann surface with genus g and K is the canonical divisor on X.

Theorem 9.1 (Riemann-Roch)**.** For any $D \in \mathrm{Div}(X)$, one has

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g.$$

9.1. Hodge decomposition.

Lemma 9.1.1. $\ell(K) = g$.

Proof. By Example 6.2.1 one has $\ell(0) = 1$. □

Theorem 9.1.1 (Hodge decomposition)**.** $H^1_{dR}(X) \cong \Gamma(X, \Omega^1) \oplus \overline{\Gamma(X, \Omega^1)}$.

9.2. Curves of genus zero and one.

9.2.1. *The uniqueness of complex structure on S^2.*

Corollary 9.2.1 (Riemann inequality)**.** $\ell(D) \geq \deg(D) + 1 - g$.

Historically, Riemann found this inequality and his student Roch made it into an equality. However, the following lemma shows that in a very generic case, Riemann inequality is always an equality.

Lemma 9.2.1. If $\deg(D) \geq 2g - 1$, then $\ell(D) = \deg(D) + 1 - g$.

Proof. If $\deg(D) \geq 2g - 1$, then $\deg(K - D) < 0$, since

$$\deg(K - D) = \deg(K) - \deg(D) = 2g - 2 - \deg(D).$$

Then by Lemma 6.2.2 one has $\ell(K - D) = 0$. □

Lemma 9.2.2. If there exists $p \in X$ such that $\ell(p) > 1$, then X is isomorphic to $\mathbb{P}^1$.

Proof. If $\ell(p) > 1$ for some $p \in X$, there exists a non-constant meromorphic function $f \in \Gamma(X, \mathcal{O}_X(p))$. Suppose $\Phi\colon X \to \mathbb{P}^1$ is the holomorphic map corresponding to f. By the same argument used in Example 6.2.3 one has Φ is an isomorphism. □

Corollary 9.2.2. Any compact Riemann surface X with genus zero is isomorphic to $\mathbb{P}^1$.

Proof. For any $p \in X$, $\deg(p) = 1 \geq 2\times 0 - 1$, and thus $\ell(p) = \deg(p) + 1 - 0 = 2 > 1$. Then by Lemma 9.2.2 one has X is isomorphic to $\mathbb{P}^1$. □

Corollary 9.2.3. The complex structure on topological sphere S^2 is unique.

9.2.2. *Genus one curve.*

Proposition 9.2.1. Let X be a compact Riemann surface with genus one. Then X is isomorphic to a complex torus, that is, $X \cong \mathbb{C}/L$, where $L = \mathbb{Z}w_1 + \mathbb{Z}w_2$ is a lattice.

Proof. By Lemma 9.1.1 one has $\ell(K) = 1$, and thus there exists a holomorphic 1-form η. Choose a basis α, β of $H_1(X, \mathbb{Z})$ and define $w_1 = \int_\alpha \eta, w_2 = \int_\beta \eta$. Now let's prove w_1, w_2 are $\mathbb{R}$-linearly independent: If $aw_1 + bw_2 = 0$ for some $a, b \in \mathbb{R}$, that is, $\int_{a\alpha+b\beta} \eta = 0$, then

$$\overline{\int_{a\alpha+b\beta} \eta} = \int_{a\alpha+b\beta} \overline{\eta} = 0.$$

By Theorem 9.1.1, one has $\eta, \overline{\eta}$ is a basis of $H^1_{dR}(X)$, and thus $a\alpha + b\beta = 0$, which implies $a = b = 0$. This shows $L = \mathbb{Z}w_1 + \mathbb{Z}w_2$ is a lattice.

Fix a point $p_0 \in X$ and for every $p \in X$, we choose a path γ_p connecting p_0 and p. Then it gives a map

$$\begin{aligned} \Phi \colon X &\to \mathbb{C}/L \\ p &\mapsto \int_{\gamma_p} \eta. \end{aligned}$$

It's well-defined since for different choice of paths γ_p and γ'_p connecting p_0 and p, one has $\gamma_p - \gamma'_p = a\alpha + b\beta$ for some $a, b \in \mathbb{Z}$, and thus $\int_{\gamma_p} \eta - \int_{\gamma'_p} \eta = aw_1 + bw_2 \in L$. Since η has no zeros, one has Φ is a local isomorphism, and thus a covering map since Φ is proper. In other words,

$$\begin{tikzcd} & \mathbb{C} \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "\Phi"] & & \mathbb{C}/L, \end{tikzcd}$$

where $\Phi_*(\pi_1(X))$ is a subgroup of $\pi_1(\mathbb{C}/L) = L$. In other words, $X \cong \mathbb{C}/\mathbb{Z}\widetilde{w}_1 + \mathbb{Z}\widetilde{w}_2$. If we take holomorphic 1-form $\eta = \mathrm{d}z$ on X, it's clear that $w_1 = \widetilde{w}_1$ and $w_2 = \widetilde{w}_2$, and thus $X \cong \mathbb{C}/L$. □

Above proposition shows that every genus one compact Riemann surface is a complex torus. In the following of this section, we will give an algebraic description for genus one compact Riemann surface, which turns out to be a plane cubic curve, and it's called an elliptic curve.

Proposition 9.2.2. Let X be a compact Riemann surface of genus one. Then it's isomorphic to a non-singular projective plane cubic curve.

Method one. □

Method two. For each $p \in X$, since $\deg(K - 2p) = -2 < 0$, one has $\ell(K - 2p) = 0$, and thus by Riemann-Roch theorem one has

$$\ell(2p) = 1 - 1 + 2 = 2.$$

Then there exists a non-constant meromorphic function $f \in \Gamma(X, \mathcal{O}_X(2p))$, which gives a holomorphic map $\Phi\colon X \to \mathbb{P}^1$. Moreover, p is the only double pole of Φ. This shows the degree of Φ is 2. By Riemann-Hurwitz theorem one has

$$2 - 2g = 4 - B(\Phi)$$

and thus it has four ramification points p_1, p_2, p_3, p_4. Without lose of generality, we may assume they're $[0:1], [1:1], [\lambda:1]$ and $[1:0]$. On the other hand, consider the affine plane curve

$$C = \{y^2 = x(x-1)(x-\lambda)\}$$

with compactification $\overline{C} \subseteq \mathbb{P}^2$, and $\widetilde{C}$ is the normalization of $\overline{C}$. Note that both X and $\widetilde{C}$ are double cover (in the sense of topological space) of $\mathbb{P}^1$ besides four points.

$$\begin{tikzcd}
X \setminus \{p_1, p_2, p_3, p_4\} \arrow[dr, "2:1"'] & & \widetilde{C} \setminus \{p_1, p_2, p_3, p_4\} \arrow[dl, "2:1"] \\
& \mathbb{P}^1 &
\end{tikzcd}$$

Then by Riemann existence theorem, X is isomorphic to $\widetilde{C}$. $\square$

Remark 9.2.1. In fact, $\widetilde{C}$ constructed as above are called double cover of $\mathbb{P}^1$, and by Riemann existence theorem they're the same thing as hyperelliptic curve, as we'll introduce in the following section.

9.3. **Hyperelliptic curve and double cover of $\mathbb{P}^1$.**

9.3.1. *Hyperelliptic curve.*

Definition 9.3.1 (hyperelliptic)**.** A compact Riemann surface X is called hyperelliptic if there exists a holomorphic map $\Phi\colon X \to \mathbb{P}^1$ such that $\deg(\Phi) = 2$.

Lemma 9.3.1. Let X be a hyperelliptic curve. Then there exists an involution, called hyperelliptic involution, on X which has $2g+2$ fixed points.

Proof. Suppose $\Phi\colon X \to \mathbb{P}^1$ is a holomorphic map with degree 2. Then by Riemann-Hurwitz formula, one has

$$2g - 2 = \deg(\Phi)(2 \times 0 - 2) + B(\Phi)$$

This shows $B(\Phi) = 2g+2$. In other words, Φ has exactly $2g+2$ ramification points $x_1, \ldots, x_{2g+2} \in X$, since $\deg(\Phi) = 2$, and $2g+2$ ramification values $b_i = \Phi(x_i) \in \mathbb{P}^1$.

For any $z \in \mathbb{P}^1 \backslash \{b_1, \ldots, b_{2g+2}\}$, one has $\Phi^{-1}(z)$ contains 2 points. Now we define the involution $T\colon X \to X$ as follows: For each ramification point x_i, $T(x_i) = x_i$, and $T(p) = q$ if $\Phi(p) = \Phi(q)$ and $p \neq q$. It's clear that the fixed points of T are $\{x_1, \ldots, x_{2g+2}\}$. $\square$

Remark 9.3.1. In fact, $2g+2$ is a sharp upper bound for the number of fixed points of a non-trivial automorphism.

Proposition 9.3.1. If X is a compact Riemann surface with genus g and $T \in \operatorname{Aut}(X)$ is not identity, then T has at most $2g+2$ fixed points.

Proof. Suppose $\operatorname{Fix}(T)$ is the set of fixed points of T. For $p \notin \operatorname{Fix}(T)$, by Riemann inequality one has

$$\ell((g+1)p) \geq \deg((g+1)p) + 1 - g = 2.$$

Thus there exists a non-constant $f \in \Gamma(X, \mathcal{O}_X((g+1)p))$ such that

$$\operatorname{div}_\infty(f) = rp,$$

where $1 \leq r \leq g+1$. For $h = f - f \circ T \in \mathcal{M}(X)$, one has

$$\operatorname{div}_\infty(h) = rp + rq,$$

where $q = T^{-1}(p)$. This shows

$$\deg(\operatorname{div}_0(h)) = \deg(\operatorname{div}_\infty(h)) = 2r \leq 2g+2.$$

Since every fixed point of T is a zero of h, one has

$$|\operatorname{Fix}(T)| \leq \deg(\operatorname{div}_0(h)) \leq 2g+2.$$

□

Lemma 9.3.2. A compact Riemann surface X is hyperelliptic if and only if there exists an effective divisor D such that $\deg(D) = 2$ and $\ell(D) \geq 2$.

Proof. If X is hyperelliptic, then there exists a holomorphic map $\Phi \colon X \to \mathbb{P}^1$ with degree 2. Suppose f is the non-constant meromorphic function corresponding to Φ, and let $D = \operatorname{div}_\infty(f) \geq 0$. Then $\deg(D) = \deg(\Phi) = 2$. Moreover,

$$\operatorname{div}(f) = \operatorname{div}_0(f) - \operatorname{div}_\infty(f) \geq -D.$$

This shows $f \in L(D)$, and thus $\ell(D) \geq 2$.

Conversely, given an effective divisor D such that $\deg(D) = 2$ and $\ell(D) \geq 2$, we choose a non-constant meromorphic function $f \in \Gamma(X, \mathcal{O}_X(D))$ with corresponding holomorphic map $\Phi \colon X \to \mathbb{P}^1$. Then

$$1 \leq \deg(\Phi) = \deg(\operatorname{div}_\infty(f)) \leq \deg(D) = 2.$$

This shows $\deg(\Phi) = 1$ or 2.

(1) If $X \ncong \mathbb{P}^1$, then $\deg(\Phi) = 2$, and thus X is hyperelliptic.
(2) If $X \cong \mathbb{P}^1$, then X is hyperelliptic by considering $z \mapsto z^2$.

□

Theorem 9.3.1. If X is a compact Riemann surface with genus $g \leq 2$, then X is hyperelliptic.

Proof. Let D be an effective divisor with $\deg(D) = 2$. By Riemann-Roch inequality one has

$$\ell(D) \geq \deg(D) + 1 - g = 3 - g.$$

This shows $\ell(D) \geq 2$ if $g \leq 1$, and thus X is hyperelliptic by Lemma 9.3.2.

If $g = 2$, there exists a non-zero holomorphic 1-form ω since $\dim \Gamma(X, \Omega^1) = g = 2$. For $K = \operatorname{div}(\omega)$, one has $\deg(K) = 2g - 2 = 2$. Then by Riemann-Roch theorem one has $\ell(K) = 2$, and thus X is hyperelliptic by Lemma 9.3.2. □

9.3.2. *Double cover of* $\mathbb{P}^1$.

Theorem 9.3.2. There exists hyperelliptic curve of any genus.

9.4. Canonical map.

Lemma 9.4.1. Let X be a compact Riemann surface with genus $g \geq 1$ and K be the canonical divisor. Then the complete linear system $|K|$ is base-point-free.

Proof. For any $p \in X$, one has $\ell(p) = 1$ for every $p \in X$, otherwise $X \cong \mathbb{P}^1$ by Lemma 9.2.2. Then by Riemann-Roch theorem, one has

$$\ell(p) - \ell(K - p) = \deg(p) + 1 - g.$$

This shows $\ell(K - p) = g - 1 < g = \ell(K)$, for all $p \in X$. By Proposition 6.3.1 one has $|K|$ is base-point-free. □

Definition 9.4.1 (canonical map)**.** The holomorphic map $\Phi_K \colon X \to \mathbb{P}^{g-1}$ given by canonical divisor K is called the canonical map.

Proposition 9.4.1. Let X be a compact Riemann surface with genus $g \geq 3$. Then canonical map is an embedding if and only if X is not hyperelliptic.

Proof. Note that the canonical map fails to be an embedding if and only if the canonical divisor K is not very ample, and by Theorem 6.3.2, it's equivalent to $\ell(K - p - q) \neq \ell(K) - 2$ for every $p, q \in X$. This can only happen if $\ell(K - p - q) = \ell(K) - 1 = g - 1$ since $|K|$ is base-point-free. On the other hand, by Riemann-Roch theorem one has

$$\begin{aligned} \ell(K - p - q) &= \deg(K - p - q) + 1 - g + \ell(p + q) \\ &= g - 3 + \ell(p + q). \end{aligned}$$

Thus the canonical map fails to be an embedding if and only if there exists $p, q \in X$ such that $\ell(p + q) = 2$.

(1) If there exists $p, q \in X$ such that $\ell(p + q) = 2$, then any non-constant meromorphic function $f \in \Gamma(X, \mathcal{O}_X(p + q))$ gives a holomorphic map $X \to \mathbb{P}^1$ of degree 2, and thus X is hyperelliptic.
(2) If X is hyperelliptic and $\Phi \colon X \to \mathbb{P}^1$ is a holomorphic map of degree 2, then the preimage divisor $p + q$ of ∞ has degree 2 and $\ell(p + q) = 2$.

□

9.4.1. *Finding equations for projective curve.* Let $D \in \mathrm{Div}(X)$ be a base-point-free divisor and $\{f_0, \dots, f_N\}$ be a basis of $\Gamma(X, \mathcal{O}_X(D))$. Then it gives a holomorphic map into projective spaces as follows

$$\begin{aligned} \phi_{|D|} \colon X &\to \mathbb{P}^N \\ x &\mapsto [f_0 : \cdots : f_N]. \end{aligned}$$

A natural question is to find out the defining equations of X. Consider the following map

$$\begin{aligned} R_k \colon \operatorname{Sym}^k(\mathbb{C}^{n+1}) &\to \Gamma(X, \mathcal{O}_X(kD)) \\ p(x_0, \dots, x_n) &\mapsto p(f_0, \dots, f_N). \end{aligned}$$

Roughly speaking one has

$$\dim\ker R_k \geq \binom{k+N}{N} - \ell(kD).$$

If k is sufficiently large, by Riemann-Roch theorem one has

$$\dim\ker R_k \geq \binom{k+N}{N} - k\deg(D) + g - 1. \tag{9.1}$$

In other words, there are lots of equations in the kernel of R_k.

9.4.2. *Genus three curve.* Let X be a compact Riemann surface of genus 3, which is not hyperelliptic. Then by canonical embedding Φ_K, it's embedded into $\mathbb{P}^2$ as a non-singular curve of degree 4.

Proposition 9.4.2. Let X be a compact Riemann surface of genus 3. Then X is not hyperelliptic if and only if X is a non-singular quartic curve.

Proof. If X is not hyperelliptic, then the canonical map $\Phi_K \colon X \to \mathbb{P}^2$ is an embedding, and by (9.1) one has

$$\dim\ker R_4 \geq \binom{4+2}{2} - 4\times 4 + 3 - 1 = 1.$$

Thus there exists a homogenous quartic polynomial F vanishing on X, and this polynomial is irreducible since no polynomial of degree less than four can vanish on X.

Therefore, every polynomial vanishing on X is a multiple of the quartic polynomial F, and thus F is the defining function of X.

Conversely, suppose X is a non-singular quartic curve. Consider the holomorphic 1-form η defined by

$$\eta = \frac{\mathrm{d}x}{f_y} = -\frac{\mathrm{d}y}{f_x}.$$

Moreover, $p(x,y)\eta$ is holomorphic if and only if $\deg P \leq d - 3 = 1$. This shows $\{\eta, x\eta, y\eta\}$ is a basis of $\Gamma(X, \Omega^1)$, and thus the canonical embedding of C is

$$\begin{aligned} C &\to \mathbb{P}^2 \\ [x : y : 1] &\mapsto [1 : x : y]. \end{aligned}$$

□

Remark 9.4.1. The dimension of non-hyperelliptic curves of genus 3 is

$$\binom{4+2}{2} - 1 - (3^2 - 1) = 6,$$

while the dimension of hyperelliptic is $2 \times 3 - 1 = 5$.

9.4.3. *Genus four curve.* Consider the canonical map

$$\phi_{|K|} \colon X \to \mathbb{P}^3$$

Then $2 \deg K = 2(2g-2) = 12 > 2g - 1$. Then

$$\dim \ker R_2 \geq \binom{3+2}{3} - (12 - 4 + 1) = 10 - 9 = 1$$

and

$$\dim \ker R_3 \geq \binom{3+3}{3} - (18 - 4 + 1) = 20 - 15 = 5.$$

There exists $0 \neq Q \in \ker R_2$, now we claim $\ker R_2 = \mathrm{span} Q$. If Q_1, Q_2 are linear independent in $\ker R_2$. Then $\phi_{|K|} \subseteq Q_1 \cap Q_2$. For a hyperplane $H \subseteq \mathbb{P}^3$. By Bezout theorem one has $\sharp Q_1 \cap Q_2 \cap H \leq \deg Q_1 \deg Q_2 = 4$, while

$$\deg \phi^*_{|K|}(H) = 6,$$

a contradiction. This shows $\ker R_2 = \mathrm{span}_{\mathbb{C}} Q$. On the other hand,

$$(\ker R_2)\, \mathbb{C}[x_0, x_1, x_2, x_3]_1 \subseteq \ker R_3.$$

Pick $P \in \ker R_3 \setminus \mathrm{span}\{x_i Q\}$. Now we claim that $C = \{Q = 0\} \cap \{P = 0\}$.

Step one, classify $\{Q = 0\}$. Note that Q is of degree 2, then it's determined by the rank.

Remark 9.4.2. dimension of non-hyperelliptic and non trigonal of genus 4 is

$$\binom{3+1}{3} \times \binom{3+1}{3} - 1 - \dim \mathrm{PGL}(2, \mathbb{C}) \times \mathrm{PGL}(2, \mathbb{C}) = 9$$

dimension of hyperelliptic of genus 4 is

$$2g - 1 = 7$$

10. Abel-Jacobi theorem

10.1. **Abel-Jacobi map.** Let X be a compact Riemann surface with genus g and $\Omega^1(X)$ be the space of all holomorphic 1-forms on X, which is a $\mathbb{C}$-vector space of dimension g by Lemma 9.1.1.

For any $[c] \in H_1(X, \mathbb{Z})$, consider the following map

$$\begin{aligned} \int_{[c]}: \Omega^1(X) &\to \mathbb{C} \\ \omega &\mapsto \int_c \omega \end{aligned}$$

It's well-defined by Stokes theorem. This gives a linear functional on $\Omega^1_X(X)$, that is, $\int_{[c]} \in \Omega^1_X(X)^*$, which is called a period of X.

Definition 10.1.1 (period group)**.** Let Λ to denote the set of all periods of X, which forms a subgroup of $\Omega^1(X)^*$, called the period group of X.

Remark 10.1.1. More precisely, suppose $\{\alpha_i, \beta_j\}_{i,j=1}^g$ is a $\mathbb{Z}$-basis of $H_1(X, \mathbb{Z})$. Then Λ is generated by $\{\int_{\alpha_i}, \int_{\beta_j}\}_{i,j=1}^g$.

Lemma 10.1.1. Λ is a lattice in $\Omega^1(X)^*$.

Proof. For $a_i, b_j \in \mathbb{R}$, if $\int_{\sum_i a_i\alpha_i + \sum_j b_j\beta_j} \eta = 0$ holds for every holomorphic 1-form η. Then by taking conjugates one has

$$\int_{\sum_i a_i\alpha_i + \sum_j b_j\beta_j} \overline{\eta} = 0,$$

and thus $\sum a_i\alpha_i + b_j\beta_j = 0$ in $H_1(X, \mathbb{R})$. This shows $a_i = b_j = 0$ for all i, j, since $\{\alpha_i, \beta_j\}_{i,j=1}^g$ is a $\mathbb{R}$-basis of $H_1(X, \mathbb{R})$. $\square$

Definition 10.1.2 (Jacobian)**.** The Jacobian of X is defined as

$$\mathrm{Jac}(X) := \Omega^1(X)^*/\Lambda$$

Example 10.1.1. $\mathrm{Jac}(\mathbb{P}^1) = \{0\}$.

Example 10.1.2. If X is a compact Riemann surface of genus one, then $\mathrm{Jac}(X) = X$.

Now let's define the Abel-Jacobi map, which relates X to its Jacobian. Fix a base point $x \in X$. For any $p \in X$, we choose a path γ_p from x to p, and define

$$\begin{aligned} \int_{\gamma_p}: \Omega^1(X) &\to \mathbb{C} \\ \omega &\mapsto \int_{\gamma_p} \omega. \end{aligned}$$

It's clear that $\int_{\gamma_p} \in \Omega^1(X)^*$, but it depends on the choice of γ_p. If we choose another path γ_p', then

$$\int_{\gamma_p} - \int_{\gamma_p'} = \int_{\gamma \cup (-\gamma_p')} \in \Lambda.$$

In other words,

$$\int_{\gamma_p} \equiv \int_{\gamma_p'} \pmod{\Lambda}.$$

Definition 10.1.3 (Abel-Jacobi map)**.** The Abel-Jacobi map A is defined as follows

$$\begin{aligned} A\colon \operatorname{Div}(X) &\to \operatorname{Jac}(X) \\ \sum_p n_p \cdot p &\mapsto \sum_p n_p \int_{\gamma_p}. \end{aligned}$$

Remark 10.1.2. Note that the Abel-Jacobi map defined above may depend on the choice of base point, but if we restrict the Abel-Jacobi map on $\operatorname{Div}^0(X)$, and denoted it by A_0, then it's independent of the choice of base point.

Lemma 10.1.2. $A_0\colon \operatorname{Div}^0(X) \to \operatorname{Jac}(X)$ is independent of the choice of the base point.

Proof. Let x' be another base point, and use A_0' to denote the Abel-Jacobi map corresponding to x'. Choose any path α from x to x', one has

$$\begin{aligned} A(p) - A'(p) &= \int_{\gamma_p} - \int_{\gamma_p'} \\ &= \int_{\gamma_p \cup (-\gamma_p') \cup (-\alpha)} + \int_\alpha \\ &\equiv \int_\alpha \pmod{\Lambda}. \end{aligned}$$

Given any $D \in \operatorname{Div}^0(X)$, then

$$\begin{aligned} A_0(D) - A_0'(D) &= \sum_p n_p (A(p) - A'(p)) \\ &\equiv \sum_p n_p \int_\alpha \pmod{\Lambda} \\ &\equiv 0 \pmod{\Lambda}. \end{aligned}$$

This completes the proof. ☐

Theorem 10.1.1 (Abel-Jacobi)**.** $\ker A_0 = \operatorname{PDiv}(X)$.

Corollary 10.1.1. If $g_X \geq 1$, then $A\colon X \to \operatorname{Jac}(X)$ is injective.

Proof. If not, then there exist $p \neq p' \in X$ such that $A(p) = A(p')$. For degree zero divisor $D = p - p'$, one has

$$A_0(D) = A(p) - A(p') = 0 \in \mathrm{Jac}(X).$$

Then $D \in \ker A_0 = \mathrm{PDiv}(X)$ by Abel-Jacobi theorem. In other words, there exists a meromorphic function f such that $D = \mathrm{div}(f)$. Let $\Phi\colon X \to \mathbb{P}^1$ be the holomorphic map corresponding to f. Then $\Phi^{-1}(\infty) = p'$, and the multiplicity of p' is 1. This shows the degree of Φ is exactly 1, and thus Φ is an isomorphism, a contradiction to $g_X \geq 1$. □

By using Abel-Jacobi theorem, one can give an another proof of every genus one compact Riemann surface is torus.

Theorem 10.1.2. Let X be a compact Riemann surface with genus one. Then $X \cong \mathbb{C}/L$, where $L \subseteq$ is a lattice.

Proof. Since the genus of X is one, one has $\Omega^1(X)^* \cong \mathbb{C}$, and thus $\mathrm{Jac}(X) \cong \mathbb{C}/L$ for some lattice $L \subseteq \mathbb{C}$. In particular, $\mathrm{Jac}(X)$ is a compact Riemann surface.

On one hand, by Corollary 10.1.1, one has the Abel-Jacobi map A is injective. On the other hand, by Corollary 1.1.1 one has A is surjective, since X is compact. This shows $X \cong \mathrm{Jac}(X) = \mathbb{C}/L$. □

10.2. **Proof of necessity in Abel-Jacobi theorem.** Let $\Phi\colon X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces of degree d and $B(\Phi)$ be the set of ramification values. Choose an open neighborhood U of q such that $U \cap B(\Phi) = \varnothing$. Then $\Phi^{-1}(U) = \bigcup_{i=1}^{d} V_i$, where $V_i \cap V_j = \varnothing$, and $\Phi_i := \Phi|_{V_i}$ is an isomorphism.

For any function f and 1-form θ on X, we can define the trace of them on U as follows

$$\begin{aligned} \mathrm{tr}(f)|_U &= \sum_{i=1}^{d} f \circ \Phi_i^{-1} \\ \mathrm{tr}(\theta)|_U &= \sum_{i=1}^{d} (\Phi_i^{-1})^*(\theta). \end{aligned}$$

Theorem 10.2.1. If f and θ are meromorphic, then $\mathrm{tr}(f)$ and $\mathrm{tr}(\theta)$ can be extended to globally defined meromorphic function and meromorphic 1-forms on Y. Moreover, if f and θ are holomorphic, then $\mathrm{tr}(f)$ and $\mathrm{tr}(\theta)$ are holomorphic.

Proof. Firstly let's consider an easy case, that is, the preimage of q contains only one point p. Suppose w is a local coordinate centered at p and z is a local coordinate centered at q such that locally Φ is given by $z = w^d$.

Suppose f has the Laurent series $f = \sum_n c_n w^n$ at p and $\xi = \exp(2\pi\sqrt{-1}/d)$ is the d-th unit root. For any $z \neq 0$, one has preimages of $z = w^d$ are $\xi^i w$, for $i = 0, \ldots, d-1$. Hence,

$$\begin{aligned}\operatorname{tr}(f)(z) &= \sum_{j=0}^{d-1} f(w\xi^j) \\ &= \sum_{j=0}^{d-1} \sum_n c_n (w\xi^j)^n \\ &= \sum_n c_n (\sum_{j=0}^{d-1} \xi^{jn}) w^n.\end{aligned}$$

A direct computation shows that

$$(\xi^n - 1) \sum_{j=0}^{d-1} \xi^{jn} = \xi^{dn} - 1 = 0.$$

Thus

$$\sum_{j=0}^{d-1} \xi^{jn} = \begin{cases} 0, & \xi^n \neq 1 \\ d, & \xi^n = 1. \end{cases}$$

On the other hand, note that $\xi^n = 1$ if and only if $n = kd$ for some $k \in \mathbb{Z}$. Thus one has

$$\begin{aligned}\operatorname{tr}(f)(z) &= \sum_k c_{kd} d w^{kd} \\ &= \sum_k c_{kd} d (w^d)^k \\ &= \sum_k c_{kd} d z^k,\end{aligned}$$

which is a meromorphic function in a neighborhood of $z = 0$. Moreover, if f is holomorphic at $w = 0$, then $k \geq 0$, and thus $\operatorname{tr}(f)$ is also holomorphic.

Similarly, let's see the case of 1-form θ. Suppose θ is written as $\theta = (\sum_n c_n w^n) \mathrm{d}w$ at p. Then

$$\theta = (\sum_n c_n w^n) \frac{1}{d w^{d-1}} \mathrm{d}z,$$

since $\mathrm{d}z = dw^{d-1}\mathrm{d}w$. For $z \neq 0$, one has

$$\begin{aligned}
\operatorname{tr}(\theta) &= \sum_{j=0}^{d-1}\sum_{n}\frac{c_n}{d}(w\xi^j)^{n-d+1}\mathrm{d}z \\
&= \sum_{n}\frac{c_n}{d}(\sum_{j=0}^{d-1}\xi^{j(n-d+1)})w^{n-d+1}\mathrm{d}z \\
&= \sum_{k} c_{kd+d-1}w^{dk}\mathrm{d}z \\
&= \sum_{k} c_{kd+d-1}z^{k}\mathrm{d}z
\end{aligned}$$

This shows $\operatorname{tr}(\theta)$ defines a meromorphic 1-form near $z = 0$, and if θ is holomorphic, then $\operatorname{tr}(\theta)$ is holomorphic.

For the general case, suppose the preimage of ramification values of q are $\{p_1, \dots, p_n\}$. Then we choose an open neighborhood U of q such that $\Phi^{-1}(U) = V_1 \cup \cdots \cup V_n$ such that $p_i \in V_i$ and $V_i \cap V_j \neq \varnothing$. Then on each $V_i \to U$, it reduces to previous case. □

Corollary 10.2.1. If θ is a meromorphic 1-form on X, then for any $q \in Y$

$$\operatorname{Res}_q(\operatorname{tr}(\theta)) = \sum_{p\in\Phi^{-1}(q)}\operatorname{Res}_p(\theta).$$

Proof. It suffices to consider the case the preimage of q is only one point. In this case, from the proof of Theorem 10.2.1, one has the residue of $\operatorname{tr}(\theta)$ is c_{kd+d-1} when $k = 1$, and that's exactly c_{-1}. □

Let γ be a piecewise smooth curve in Y such that $\Phi^{-1}(\gamma)$ doesn't contain poles of θ. Then there are no poles of $\operatorname{tr}(\theta)$ on γ, and thus $\int_\gamma \operatorname{tr}(\theta)$ is well-defined. Away from ramification values of Φ, γ can be lifted to exactly d non-intersecting curves in X. By taking closures of these curves, we obtain curves $\gamma_1, \dots, \gamma_d \subseteq X$, and then we define the pullback of γ by $\Phi^*(\gamma) = \gamma_1 + \cdots + \gamma_d$.

Lemma 10.2.1.

$$\int_\gamma \operatorname{tr}(\theta) = \int_{\Phi^*(\gamma)}\theta := \sum_{i=1}^{d}\int_{\gamma_i}\theta$$

Proof. Since by removing finitely many points does not affect the result of integral, so without lose of generality we may assume γ is a path not through any ramification values. Let U be an open neighborhood of γ, which contains no ramification values, and thus

$$\Phi^{-1}(U) = V_1 \cup \cdots \cup V_d$$

such that $V_i \cap V_j \neq \varnothing$ and $\gamma_i \subseteq V_i$. Then

$$\begin{aligned}\int_{\Phi^*(\gamma)} \theta &= \sum_{i=1}^{d} \int_{\gamma_i} \theta \\ &= \sum_{i=1}^{d} \int_{\Phi(\gamma_i)} (\Phi_i^{-1})^* \theta \\ &= \int_{\gamma} \sum_{i=1}^{d} (\Phi_i^{-1})^* \theta \\ &= \int_{\gamma} \operatorname{tr}(\theta).\end{aligned}$$

□

Proof of necessity in Theorem 10.1.1. For any $D \in \mathrm{PDiv}(X)$, there exists a meromorphic function f such that $\operatorname{div}(f) = D$. Let $\Phi \colon X \to \mathbb{P}^1$ be the holomorphic map corresponding to f with degree d. Given a path γ in $\mathbb{P}^1$ from ∞ to 0, which contains no ramification values of Φ except 0 and ∞, one has $\Phi^*(\gamma) = \gamma_1 + \cdots + \gamma_d$, where γ_i is a curve from a pole q_i of f to a zero p_i of f. Then $D = \sum_{i=1}^{d}(p_i - q_i)$.

Fix a base point $x \in X$, and use α_i, β_i to denote the path from x to p_i and q_i respectively. Then by definition one has $A_0(D) = \sum_{i=1}^{d}(\int_{\alpha_i} - \int_{\beta_i})$. Let $\eta = \alpha_i - \gamma_i - \beta_i$. Then

$$\begin{aligned}A_0(D) &= \sum_{i=1}^{d}(\int_{\eta} + \int_{\gamma_i}) \pmod{\Lambda} \\ &= \sum_{i=1}^{d} \int_{\gamma_i} \pmod{\Lambda}.\end{aligned}$$

For any holomorphic 1-form θ on X, one has

$$A_0(D)(\theta) = \sum_{i=1}^{d} \int_{\gamma_i} \theta = \int_{\Phi^*(\gamma)} \theta = \int_{\gamma} \operatorname{tr}(\theta) = 0,$$

since $\operatorname{tr}(\theta)$ is holomorphic. This shows $A_0(D) = 0$, as desired. □

10.3. **Proof of sufficiency in Abel-Jacobi theorem.**

10.3.1. *Riemann bilinear relations.* Let X be a compact Riemann surface of genus g, and the homology group $H_1(X, \mathbb{Z})$ is generated by $\{\alpha_i, \beta_j \mid i, j = 1, \ldots, g\}$. For any closed smooth 1-form ω on X, consider

$$\begin{aligned}A_i(\omega) &= \int_{\alpha_i} \omega, \quad i = 1, \ldots, g \\ B_i(\omega) &= \int_{\beta_j} \omega, \quad i = 1, \ldots, g.\end{aligned}$$

On the other hand, let $\mathcal{P}$ be the polygon with $4g$ sides $\{\alpha_i, \beta_j, \alpha_i', \beta_j'\}_{i,j=1}^g$ such that X is obtained by identifying α_i, α_i' and β_j, β_j'. For any closed 1-form ω on X, it can be considered as a closed 1-form on $\mathcal{P}$. Fix a base point x in the interior of $\mathcal{P}$, and define

$$f_\omega(p) = \int_x^p \omega,$$

where integration along any path from x to p inside $\mathcal{P}$. Since ω is closed, this integration is independent of the choice of path. Thus f_ω is a well-defined function on a neighborhood of V, and $\mathrm{d}f_\omega = \omega$.

Lemma 10.3.1. Let ω, θ be closed smooth 1-forms on X. Then

$$\int_X \omega \wedge \theta = \int_{\partial\mathcal{P}} f_\omega \theta = \sum_{i=1}^g A_i(\omega)B_i(\theta) - A_i(\theta)B_i(\omega)$$

Proof. Firstly, $\int_X \omega\wedge\theta = \int_{\mathcal{P}} \omega\wedge\theta = \int_{\partial\mathcal{P}} f_\omega\theta$ follows from the Stokes theorem. For any $p \in \alpha_i$, we use $p' \in \alpha_i'$ to denote the point glued to p, and α_p is a curve from p to p'. Since α_p is homotopic to β_i, one has

$$\begin{aligned} f_\omega(p) - f_\omega(p') &= \int_x^p \omega - \int_x^{p'} \omega \\ &= -\int_{\alpha_p} \omega \\ &= -\int_{\beta_i} \omega \\ &= -B_i(\omega). \end{aligned}$$

Similarly for $q \in \beta_i$, $q' \in \beta_i'$, which is the point glued to p, one has

$$f_\omega(q) - f_\omega(q') = A_i(\omega)$$

Since θ is a closed smooth 1-form on X, its values along α_i and α_i' are same, and similarly for β_j and β_j'. Then

$$\begin{aligned} \int_{\partial\mathcal{P}} f_\omega\theta &= \sum_{i=1}^g (\int_{\alpha_i} + \int_{\beta_i} - \int_{\alpha_i'} - \int_{\beta_i'}) f_\omega\theta \\ &= \sum_{i=1}^g \int_{p\in\alpha_i} (f_\omega(p) - f_\omega(p'))\theta + \int_{q\in\beta_i} (f_\omega(q) - f_\omega(q'))\theta \\ &= \sum_{i=1}^g -B_i(\omega)A_i(\theta) + A_i(\omega)B_i(\theta). \end{aligned}$$

This completes the proof. □

Lemma 10.3.2. Let ω be a holomorphic 1-form on X which is not identically zero. Then

$$\mathrm{Im}\sum_{i=1}^{g} A_i(\omega)B_i(\overline{\omega}) < 0.$$

Proof. Suppose ω is locally given by $\omega = f\mathrm{d}z$, where f is a holomorphic function. Then

$$\begin{aligned}\omega \wedge \overline{\omega} &= |f(z)|^2\mathrm{d}z \wedge \mathrm{d}\overline{z}\\ &= -2\sqrt{-1}|f(z)|^2\mathrm{d}x \wedge \mathrm{d}y.\end{aligned}$$

This shows $\sqrt{-1}\int_X \omega \wedge \overline{\omega} > 0$, since $|f(z)|^2 \geq 0$ and not identically zero. By Lemma 10.3.1, one has

$$\sqrt{-1}\sum_{i=1}^{g}\{A_i(\omega)B_i(\overline{\omega}) - A_i(\overline{\omega})B_i(\omega)\} = \sqrt{-1}\int_X \omega \wedge \overline{\omega} > 0.$$

Since $\int_\gamma \overline{\omega} = \overline{\int_\gamma \omega}$ holds for any $\gamma \in H_1(X, \mathbb{Z})$, one has

$$\begin{aligned}A_i(\overline{\omega}) &= \overline{A_i(\omega)}\\ B_i(\overline{\omega}) &= \overline{B_i(\omega)}.\end{aligned}$$

Thus

$$\mathrm{Im}\sum_{i=1}^{g} A_i(\omega)B_i(\overline{\omega}) = \frac{1}{2}\mathrm{Im}\sum_{i=1}^{g}\{A_i(\omega)B_i(\overline{\omega}) - A_i(\overline{\omega})B_i(\omega)\} < 0.$$

□

Corollary 10.3.1. Let ω be a holomorphic 1-form on X.

(1) If $A_i(\omega) = 0$ for all $i = 1, \ldots, g$, then $\omega = 0$.
(2) If $B_i(\omega) = 0$ for all $i = 1, \ldots, g$, then $\omega = 0$.

Proof. If $A_i(\omega) = 0$ for all $i = 1, \ldots, g$ and $\omega \neq 0$, then by Lemma 10.3.2 one has

$$\mathrm{Im}\sum_{i=1}^{g} A_i(\omega)B_i(\overline{\omega}) < 0,$$

which gives a contradiction. The proof still holds for (2). □

Definition 10.3.1 (period matrices)**.** Let $\{\omega_1, \ldots, \omega_g\}$ be a basis of $\Omega^1_X(X)$. Then the period matrices of X are defined by

$$A = (A_i(\omega_j))_{g\times g}, \quad B = (B_i(\omega_j))_{g\times g}$$

Then A, B are called period matrices of X.

Remark 10.3.1. It's clear that A, B depends on the choice of basis $\{\omega_1, \ldots, \omega_g\}$ and generators $\{\alpha_i, \beta_j\}_{i,j=1}^g$ of $H_1(X, \mathbb{Z})$.

Lemma 10.3.3. Both A and B are invertible.

Proof. If A is not invertible, then there exists $0 \neq c = (c_1, \dots, c_g)^T \in \mathbb{C}^g$ such that $Ac = 0$. Let $\omega = \sum_{j=1}^g c_j\omega_j \in \Omega^1_X(X)$. Then

$$A_i(\omega) = \sum_{j=1}^g c_j A_i(\omega_j) = 0, \quad \text{for all } i = 1, \dots, g.$$

By Corollary 10.3.1, one has $\omega = 0$, a contradiction to the fact $\{\omega_1, \dots, \omega_g\}$ is a basis. Thus A is invertible, and by the same argument one can show B is invertible. □

Lemma 10.3.4 (first Riemann bilinear relation)**.** A^TB is a symmetric matrix.

Proof. For any $1 \leq j, k \leq g$, one has $\omega_j \wedge \omega_k = 0$, since both of them are $(1,0)$-form. Thus

$$0 = \int_X \omega_j \wedge \omega_k = \sum_{i=1}^g \{A_i(\omega_j)B_i(\omega_k) - A_i(\omega_k)B_i(\omega_k)\},$$

and that's exactly (j,k)-th entry of $A^TB - B^TA$. This shows $A^TB - B^TA = 0$, that is, A^TB is symmetric. □

Lemma 10.3.5 (second Riemann bilinear relation)**.** $\sqrt{-1}(A^T\overline{B} - B^T\overline{A})$ is a positive definite Hermitian matrix.

Proof. By Lemma 10.3.2 one has

$$\sqrt{-1}(\sum_{i=1}^g \{A_i(\omega)B_i(\overline{\omega}) - A_i(\overline{\omega})B_i(\omega)\}) > 0$$

for any holomorphic 1-form ω. Then for any $0 \neq c = (c_1, \dots, c_g)^T \in \mathbb{C}^g$, applying above equation to $\omega = \sum_{j=1}^g c_j\omega_j$, one has

$$\begin{aligned} 0 &< \sqrt{-1}\sum_{i=1}^g \sum_{k,l=1}^g c_j\overline{c_k}\{A_i(\omega)B_i(\overline{\omega}) - A_i(\overline{\omega})B_i(\omega)\} \\ &= \sqrt{-1}c^T(A^T\overline{B} - B^T\overline{A})\overline{c} \end{aligned}$$

This completes the proof. □

Remark 10.3.2. If $\{\omega'_1, \dots, \omega'_g\}$ is another basis of $\Omega^1(X)$, then there exists a matrix $M = (m_{ij}) \in \mathrm{GL}(g, \mathbb{C})$ such that

$$\omega_i = \sum_{j=1}^g m_{ij}\omega'_j.$$

Let A', B' be the period matrices with respect to $\{\omega'_1, \dots, \omega'_g\}$. Then

$$A_i(\omega_j) = \sum_k m_{jk}A_i(\omega'_k).$$

In other words, one has

$$A = A'M^T,$$

and similarly one has $B = B'M^T$. Thus we may choose a basis $\{\omega_1, \dots, \omega_g\}$ of $\Omega^1_X(X)$ such that $A = I$, and such basis is called normalized basis. In this case, the first Riemann relation is equivalent to B is symmetric, and the second Riemann bilinear relation is equivalent to $\mathrm{Im}(B)$ is positive definite.

Corollary 10.3.2. The $2g$ rows of any period matrices of A and B are $\mathbb{R}$-linear independent.

Proof. If suffices to prove for any $\alpha, \beta \in \mathbb{R}^n$, then

$$\alpha^T A + \beta^T B = 0 \implies \alpha = \beta = 0.$$

Without lose of generality we may assume $A = I$, and thus one has

$$0 = \alpha^T + \beta^T B = 0.$$

In particular, one has

$$\beta^T \mathrm{Im}(B) = 0.$$

Since $\mathrm{Im}(B)$ is positive definite, one has $\beta = 0$, so is α. □

10.3.2. *Proof of sufficiency in Abel-Jacobi theorem.*

Theorem 10.3.1. For any compact Riemann surface X, given finite set of distinct point $\{p_i\}$ on X and a corresponding set of complex numbers $\{\gamma_i\}$ with $\sum_i \gamma_i = 0$, then there exists a meromorphic 1-form ω on X such that the poles of ω are exactly $\{p_i\}$, all thoes poles are simple poles with residue $\{\gamma_i\}$.

Proof. If $g = 0$, then $X = \mathbb{C} \cup \{\infty\}$, we can construct as follows

$$\omega = \sum_i \frac{\gamma_i}{z - p_i} \mathrm{d}z$$

Now assume $g \geq 1$, let's see a lemma firstly. Note that this lemma has no requirement on genus.

Lemma 10.3.6. If Q is a linear system without base point, for any finite set of points $\{p_1, \dots, p_n\}$, there exists a divisor $E \in Q$ such that $p_i \notin \mathrm{Supp}(E)$ for all $i = 1, \dots, n$.

Proof. Assume $Q \subset |D|$ for some divisor D, $V \subset L(D)$ is the space corresponding to Q. Since p_i is not base point of Q, then $V \not\subset L(D - p_i)$ for all i. So $V \backslash \bigcup_{i=1}^n L(D - p_i)$ is non-empty. Choose $f \in V \backslash \bigcup_{i=1}^n L(D - p_i)$. Then $\mathrm{ord}_{p_i}(f) = -D(p_i)$ for all i. Let $E = \mathrm{div}(f) + D \in Q$, we have $E(p_i) = 0$ and $p_i \notin \mathrm{Supp}(E)$ for all i. This completes the proof of claim. □

Since $g \geq 1$, then complete linear system of canonical divisor K is base point free. So we may choose a canonical divisor $K \geq 0$, such that $p_i \notin \mathrm{Supp}(K)$ for all i. Let ω_0 be the meromorphic 1-form corresponding to K, since $K \geq 0$, then ω is holomorphic. We want to find $f \in \mathcal{M}(X)$ such that $\omega = f\omega_0$, which satisfies our requirements. Choose local coordinate z_i centered at p_i. In this coordinate, ω_0 can be written as

$$\omega_0 = (c_i + z_i g_i(z_i))\mathrm{d}z_i$$

where g_i is a holomorphic function. Since $p_i \notin \mathrm{Supp}(\omega_0)$, then $c_i \neq 0$. Consider Laurent tail divisor $Z = \sum_i \dfrac{\gamma_i}{c_i} z_i^{-1} \cdot p_i$. Since

$$-K(p_i) = 0 > -1, \quad \text{for all } i$$

Then $Z \in T[K](X)$. Let $\alpha_K \colon \mathcal{M}(X) \to T[K](X)$ be the divisor map and $H^1(K) = \mathrm{coker}(\alpha_K)$. Let $\pi \colon T[K](X) \to H^1(K)$ be the projection. $Z \in \mathrm{im}(\alpha_K)$ if and only if $\pi(Z) = 0$, and if and only if $\theta(\pi(Z)) = 0, \forall \theta \in H^1(K)^*$. By Serre duality, the residue map

$$\mathrm{Res}\colon L^{(1)}(K) \to H^1(K)^*$$

is an isomorphism.

Note that $\omega_0 \in L^{(1)}(-K)$ since $\mathrm{div}(\omega_0) = K$, and $\dim L^{(1)}(-K) = \dim L(0) = 1$, then $L^{(1)}(-K) = \{a\omega_0 \mid a \in \mathbb{C}\}$. So Res. Indeed,

$$\begin{aligned}
\mathrm{Res}_{\omega_0}(Z) &= \sum_i \mathrm{Res}_{z_i=0}\{\frac{\gamma_i}{c_i} z_i^{-1}(c_i + z_i g_i(z_i)\}\mathrm{d}z_i \\
&= \sum_i \gamma_i \\
&= 0
\end{aligned}$$

So there exists $f \in \mathcal{M}(X)$ such that $\alpha_K(f) = Z$. Let $\omega = f\omega_0$. If $q \neq p_i$, then $\alpha_K(f)(q) = 0$, so

$$\mathrm{ord}_{p_i}(f) \geq -K(q)$$

So

$$\mathrm{ord}_q(\omega) = \mathrm{ord}_q(f) + \mathrm{ord}_q(\omega_0) \geq 0$$

□

Lemma 10.3.7. Let $D \in \mathrm{Div}^0(X)$ such that $A_0(D) = 0 \in \mathrm{Jac}(X)$ where A_0 is the Abel-Jacobi map. Then there exists a meromorphic 1-form ω on X such that

1. $\mathrm{Supp}(D) =$ set of poles of ω and ω only has simple poles;
2. $\mathrm{Res}_p(\omega) = D(p)$;
3. periods of ω are integral multiples of $2\pi\sqrt{-1}$.

Proof. Since $\sum_{p\in X} D(p) = 0$, then by Theorem 10.3.1, there exists a meromorphic 1-form θ on X satisfying (1) and (2). Let $\{\omega_1, \ldots, \omega_n\}$ be a basis of $\Omega^1_X(X)$. Let $\omega = \theta - \sum_{i=1}^g c_i\omega_i$ with $c_i \in \mathbb{C}$. Then ω still satisfies (1) and (2), but the difficultly is to find suitable c_i such that ω satisfies (3).

Let $\{\alpha_i, \beta_j\}_{i,j=1}^g$ be a basis of $H_1(X, \mathbb{Z})$ such that $\mathrm{Supp}(D) \subseteq X \setminus \bigcup_i (\alpha_i \cup \beta_i)$. For $i = 1, \dots, g$, consider

$$\rho_k = \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^g \{A_i(\omega_k) B_i(\theta) - A_i(\theta) B_i(\omega_k)\}.$$

By Lemma 10.3.1 one has

$$\rho_k = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\mathcal{P}} f_{\omega_k}\theta = \sum_{p \in \mathcal{P}} \mathrm{Res}_p(f_{\omega_k}\theta) = \sum_{p \in \mathrm{Supp}(D)} f_{\omega_k}(p) D(p)$$

In other words,

$$\rho_k = \sum_p D(p) \int_x^p \omega_k,$$

where x is a fixed base point in interior of $\mathcal{P}$. If we consider the identification

$$\begin{aligned} \Omega^1(X)^* &\xrightarrow{\Phi} \mathbb{C}^g \\ \alpha &\mapsto (\alpha(\omega_1), \dots, \alpha(\omega_g)) \end{aligned}$$

and $\Lambda = \mathrm{span}_{\mathbb{Z}}\{\Phi(\int_{\alpha_i}), \Phi(\int_{\beta_i})\}$, then Φ induces isomorphism

$$\Phi\colon \mathrm{Jac}(X) \to \mathbb{C}^g / \Lambda.$$

Under such identification, it's clear to see

$$\Phi(A_0(D)) \equiv (\rho_1, \dots, \rho_g) \pmod{\Lambda}.$$

If $A_0(D) = 0$, then $(\rho_1, \dots, \rho_g) \in \Lambda$, and thus there exists $m_j, n_j \in \mathbb{Z}$ such that

$$(\rho_1, \dots, \rho_g) = \sum_{i=1}^g m_j (A_j(\omega_1), \dots, A_j(\omega_g)) - \sum_{i=1}^g n_j (B_j(\omega_1), \dots, B_j(\omega_g)).$$

On the other hand, by definition of ρ_k, we have

$$\rho_k = \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^g \{A_i(\omega_k) B_i(\theta) - A_i(\theta) B_i(\omega_k)\}.$$

Then we must have

$$\sum_{j=1}^g (B_j(\theta) - 2\pi\sqrt{-1} m_j) A_j(\omega_k) = \sum_{j=1}^g (A_j(\theta) - 2\pi\sqrt{-1} n_j) B_j(\omega_k)$$

for all $1 \leq k \leq g$. If we denote $\widetilde{b}_j = B_j(\theta) - 2\pi\sqrt{-1} m_j, \widetilde{a}_j = A_j(\theta) - 2\pi\sqrt{-1} n_j$, then above equations can be expressed as

$$A^T b = B^T a,$$

where $a = (\widetilde{a}_1, \dots, \widetilde{a}_g)^T, b = (\widetilde{b}_1, \dots, \widetilde{b}_g)^T$.

Consider linear transformations

$$\mathbb{C}^g \xrightarrow{\alpha} \mathbb{C}^{2g} \xrightarrow{\beta} \mathbb{C}^g,$$

where

$$\alpha = \begin{pmatrix} A \\ B \end{pmatrix}, \quad \beta = (B^T, -A^T).$$

Since A, B are invertible, then α is injective and β is surjective, and the first Riemann bilinear relation implies $\beta \circ \alpha(v) = (B^T A - A^T B)v = 0$. Then

$$\operatorname{im}\alpha \subseteq \ker\beta,$$

and the injectivity of α and surjectivity of β tells us $\operatorname{im}\alpha$ and $\ker\beta$ have the same dimension, so the following sequence is exact.

$$0 \to \mathbb{C}^g \xrightarrow{\alpha} \mathbb{C}^{2g} \xrightarrow{\beta} \mathbb{C}^g \to 0.$$

Since $\beta\begin{pmatrix} a \\ b \end{pmatrix} = 0$, there exists c such that $\alpha(c) = \begin{pmatrix} a \\ b \end{pmatrix}$, and if we define $\omega = \theta - \sum_{j=1}^g c_j\omega_j$, then periods of ω is

$$\begin{aligned} A_k(\omega) &= A_k(\theta) - \sum_j c_j A_k(\omega_j) \\ &= A_k(\theta) - (A_k(\theta) - 2\pi\sqrt{-1}n_k) \\ &= 2\pi\sqrt{-1}n_k, \end{aligned}$$

and

$$\begin{aligned} B_k(\omega) &= B_k(\theta) - \sum_j c_j B_k(\omega_j) \\ &= B_k(\theta) - (B_k(\theta) - 2\pi\sqrt{-1}m_k) \\ &= 2\pi\sqrt{-1}m_k. \end{aligned}$$

This completes the proof. □

Proof of sufficiency in Theorem 10.1.1. For $D \in \mathrm{Div}^0(X)$ such that $A_0(D) = 0 \in \mathrm{Jac}(X)$, we choose a meromorphic 1-form ω on X satisfying three conditions in Lemma 10.3.7.

Fix a base point $x \in X$ which is not a pole of ω. Define

$$f(p) := \exp(\int_x^p \omega),$$

where the integral is along any path from x to p not through poles of ω. Since period of ω are integral multiples of $2\pi\sqrt{-1}$ and residue of ω are integers. So $f(p)$ doesn't depend on the choice of path in the integral $\int_x^p \omega$. In other words, f is well-defined for p which is not a pole of ω, and f is holomorphic and non-zero at such points.

Since $\mathrm{Supp}(D)$ consists of poles of ω, one has f is holomorphic on $X \setminus \mathrm{Supp}(D)$. For $p \in \mathrm{Supp}(D)$ and $n = D(p)$. Choose a local coordinate z centered at p, we may write

$$\omega = (nz^{-1} + g(z))\mathrm{d}z,$$

where g is holomorphic since $\mathrm{Res}_p(\omega) = n$ and $\mathrm{ord}_p(\omega) = 1$. Thus around p we have

$$f(z) = \exp(\int_x^p \omega) = \exp(n \log z + h(z)) = z^n e^{h(z)}.$$

This shows f is a meromorphic function such that $D = \mathrm{div}(f)$, which completes the proof of Abel-Jacobi theorem. $\square$

11. Homework

11.1. Homework-1.

Exercise 11.1.1. Prove that when $\omega_1, \omega_2 \in \mathbb{C}$ are $\mathbb{R}$-linearly independent, then

(1) $\mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2$ is discrete.
(2) $\mathbb{C}\,/\,\mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2$ is Hausdorff.
(3) $\mathbb{C} \to \mathbb{C}\,/\,\mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2$ is a covering map.

Proof. For (1). Choose $0 < \epsilon < \min\{|w_1|/2, |w_2|/2, |w_1 - w_2|/2\}$. Then for any two elements u, v in $\mathbb{Z}\,w_1 + \mathbb{Z}\,w_2$, one has $B_\epsilon(u) \cap B_\epsilon(v) = \varnothing$, and thus $\mathbb{Z}\,w_1 + \mathbb{Z}\,w_2$ is discrete.

For (2). Let L denote the lattice $\mathbb{Z}\,w_1 + \mathbb{Z}\,w_2$ and $\pi\colon \mathbb{C} \to \mathbb{C}\,/L$ be the canonical projection. Suppose $\mathbb{C}\,/L$ is equipped with the quotient topology, that is, $U \subseteq \mathbb{C}\,/L$ is an open subset if and only if $\pi^{-1}(U)$ is open in $\mathbb{C}$. It's easy to show $\pi\colon \mathbb{C} \to \mathbb{C}\,/L$ is an open map, since for any open subset $U \subseteq \mathbb{C}$, one has

$$\pi^{-1}(\pi(U)) = \bigcup_{w \in L} w + U.$$

For $u, v \in \mathbb{C}\,/L$, we choose $\widetilde{u}, \widetilde{v} \in \mathbb{C}$ such that $\pi(\widetilde{u}) = u$ and $\pi(\widetilde{v}) = v$. Since $\mathbb{C}$ is Hausdorff, there exists open neighborhoods $\widetilde{U}, \widetilde{V}$ of $\widetilde{u}, \widetilde{v}$ such that $\widetilde{U} \cap \widetilde{V} = \varnothing$. Moreover, we may assume $\pi|_{\widetilde{U}}$ and $\pi|_{\widetilde{V}}$ are injective by shrinking $\widetilde{U}, \widetilde{V}$ when necessary. Then $\pi(\widetilde{U})$ and $\pi(\widetilde{V})$ are open neighborhoods of u, v respectively such that $\pi(\widetilde{U}) \cap \pi(\widetilde{V}) = \varnothing$. This shows $\mathbb{C}\,/L$ with quotient topology is Hausdorff.

For (3). For $u \in \mathbb{C}\,/L$, the preimages of u is discrete since L is discrete. For each preimage $\widetilde{u}_i$, we choose $\epsilon > 0$ small sufficiently such that $B_\epsilon(\widetilde{u}_i) \cap B_\epsilon(u_j) = \varnothing$ for $i \neq j$ and $\pi|_{B_\epsilon(\widetilde{u}_i)}$ is injective for all i. If we denote $U = \pi(B_\epsilon(\widetilde{u}_i))$, then $\pi\colon B_\epsilon(\widetilde{u}_i) \to U$ is a homeomorphism for each i and by construction $B_\epsilon(\widetilde{u}_i) \cap B_\epsilon(u_j) = \varnothing$ for $i \neq j$. This shows $\pi\colon \mathbb{C} \to \mathbb{C}\,/L$ is a covering map. ☐

Exercise 11.1.2. Let V be a complex vector space of dimension n, with $\mathbb{C}$-basis $e_1, \dots, e_n$, and $T\colon V \to V$ is a $\mathbb{C}$-linear transformation. Suppose T has matrix representation $X = A + \sqrt{-1}B$ where $A, B \in M_n(\mathbb{R})$ under (complex) basis $e_1, \dots, e_n$. Prove

(1) $e_1, \dots, e_n, \sqrt{-1}e_1, \dots, \sqrt{-1}e_n$ is an $\mathbb{R}$-basis of V.
(2) T has matrix

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

under the $\mathbb{R}$-basis above when T is viewed as an $\mathbb{R}$-linear transformation.

(3)

$$\det \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = |\det X|^2.$$

Proof. For (1). Since $e_1, \dots, e_n$ are $\mathbb{C}$-linearly independent and $1, \sqrt{-1}$ are $\mathbb{R}$-linearly independent, one has $e_1, \dots, e_n, \sqrt{-1}e_1, \dots, \sqrt{-1}e_n$ are $\mathbb{R}$-linearly independent. On the other hand, since $e_1, \dots, e_n$ is a $\mathbb{C}$-basis, then any element $v \in V$ can be expressed as $v = v_1 e_1 + \cdots + v_n e_n$, where $v_i \in \mathbb{C}$. If we write $v_i = a_i + \sqrt{-1}b_i$ with $a_i, b_i \in \mathbb{R}$, then

$$v = a_1 e_1 + \cdots + a_n e_n + \sqrt{-1}b_1 e_1 + \cdots + \sqrt{-1}b_n e_n.$$

This shows V as a $\mathbb{R}$-vector space is spanned by $e_1, \dots, e_n, \sqrt{-1}e_1, \dots, \sqrt{-1}e_n$.

For (2). Since T has matrix representation $X = A + \sqrt{-1}B$ under $\mathbb{C}$-basis $e_1, \dots, e_n$, one has

$$\begin{aligned} T(e_i) &= \sum_{j=1}^n X_{ij} e_j = \sum_{j=1}^n \left(A_{ij} e_j + B_{ij}\sqrt{-1}e_j \right) \\ T(\sqrt{-1}e_i) &= \sum_{j=1}^n X_{ij}\sqrt{-1}e_j = \sum_{j=1}^n \left(-B_{ij} e_j + A_{ij}\sqrt{-1}e_j \right). \end{aligned}$$

This shows T has matrix

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

under the $\mathbb{R}$-basis $e_1, \dots, e_n, \sqrt{-1}e_1, \dots, \sqrt{-1}e_n$.

For (3). By elementary operations, one has

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \longrightarrow \begin{pmatrix} A + \sqrt{-1}B & B \\ -B + \sqrt{-1}A & A \end{pmatrix} \longrightarrow \begin{pmatrix} A + \sqrt{-1}B & B \\ 0 & A + \sqrt{-1}B \end{pmatrix}$$

Since the elementary operations don't change the determinant, this shows the desired result. □

Exercise 11.1.3 (implicit function theorem)**.** Let $f(z, w) \colon \mathbb{C}^2 \to \mathbb{C}$ be holomorphic function of two variables and $X = \{(z, w) \in \mathbb{C}^2 \mid f(z, w) = 0\}$ be its zero loucs. Let $p = (z_0, w_0)$ be a point of X and $\partial f / \partial z(p) \neq 0$. Then there exists a function $g(w)$ defined and holomorphic in a neighborhood of w_0 such that, near p, X is equal to the graph $z = g(w)$.

Proof. If we write $z = a + \sqrt{-1}b, w = c + \sqrt{-1}d$ and $f(z, w) = u + \sqrt{-1}v$, then u, v are smooth functions of a, b, c, d. Moreover, the Cauchy-Riemann equations give

$$\frac{\partial f}{\partial z} = \frac{\partial u}{\partial a} + \sqrt{-1}\frac{\partial v}{\partial a} = \frac{\partial v}{\partial b} - \sqrt{-1}\frac{\partial u}{\partial b} = A + \sqrt{-1}B.$$

Then

$$\frac{\partial(u, v)}{\partial(a, b)} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

and $\det \dfrac{\partial(u,v)}{\partial(a,b)} = A^2 + B^2 \neq 0$ if and only if $A + \sqrt{-1}B \neq 0$. Then the classical implicit function theorem implies the zero loucs

$$\begin{cases} u = 0 \\ v = 0 \end{cases}$$

is locally given by

$$\begin{cases} a = a(c,d) \\ b = b(c,d). \end{cases}$$

In other words, $z = g(w)$. Now it suffices to compute $\partial g/\partial \overline{w}$ to show g is holomorphic. Again by Cauchy-Riemann equations

$$\frac{\partial f}{\partial w} = \frac{\partial u}{\partial c} + \sqrt{-1}\frac{\partial v}{\partial c} = \frac{\partial v}{\partial d} - \sqrt{-1}\frac{\partial u}{\partial d} = C + \sqrt{-1}D.$$

Then by chain rule one has

$$\begin{aligned} \frac{\partial(a,b)}{\partial(c,d)} &= \left(\frac{\partial(u,v)}{\partial(a,b)}\right)^{-1} \frac{\partial(u,v)}{\partial(c,d)} \\ &= \begin{pmatrix} A & B \\ -B & A \end{pmatrix}^{-1} \begin{pmatrix} C & D \\ -D & C \end{pmatrix} \\ &= \frac{1}{A^2+B^2} \begin{pmatrix} AC+BD & AD-BC \\ BC-AD & BD+AC \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial g}{\partial \overline{w}} &= \frac{1}{2}\left(\frac{\partial}{\partial c} + \sqrt{-1}\frac{\partial}{\partial d}\right)(a + \sqrt{-1}b) \\ &= \frac{1}{2}\left(\frac{\partial a}{\partial c} + \sqrt{-1}\frac{\partial b}{\partial c} + \sqrt{-1}\frac{\partial a}{\partial d} - \frac{\partial b}{\partial d}\right) \\ &= 0 \end{aligned}$$

□

Exercise 11.1.4. Let $x_1, \dots, x_n$ be distinct points on $\mathbb{C}$ and

$$f(x,y) = y^d - (x-x_1)\cdots(x-x_n).$$

Prove that $C = \{f(x,y) = 0\}$ defines a Riemann surface in $\mathbb{C}^2$.(Question to think about: what is the topological shape of C?)

Proof. Note that there is no common zero of $f(x,y)$ and $\partial f/\partial x$ since $x_1, \dots, x_n$ are distinct points, so the affine plane curve defined by $f(x,y)$ is non-singular, and thus it's a Riemann surface. □

Remark 11.1.1. Now let's consider the singularity of its compactification. Suppose $n \geq d$, and consider the homogenous polynomial defined by $f(x,y)$ as follows

$$F(x,y,z) = z^{n-d}y^d - (x - x_1 z)\dots(x - x_n z).$$

By setting $z = 0$ we found a new point $[0 : 1 : 0]$. It suffices to see it's singular or not. A direct computation shows

$$\begin{aligned}
\frac{\partial F}{\partial x} &= -(x - x_2 z)\dots(x - x_n z) - \dots - (x - x_1 z)\dots(x - x_{n-1} z)\\
\frac{\partial F}{\partial y} &= d z^{n-d} y^{d-1}\\
\frac{\partial F}{\partial z} &= (n-d) z^{n-d-1} y^d + x_1 (x - x_2 z)\dots(x - x_n z) + \dots + x_n (x - x_1 z)\dots(x - x_{n-1} z).
\end{aligned}$$

Then

(1) If $n > d + 1$, then it's singular.
(2) If $n = d + 1$ or $n = d$, it's non-singular.

Now we suppose $n < d$, and then the homogenous polynomial defined $f(x, y)$ is given by

$$F(x, y, z) = y^d - z^{d-n}(x - x_1 z)\dots(x - x_n z).$$

By setting $z = 0$ we find a new point $[1 : 0 : 0]$. It suffices to see it's singular or not. A direct computation shows

$$\begin{aligned}
\frac{\partial F}{\partial x} &= - z^{d-n}\left((x - x_2 z)\dots(x - x_n z) + \dots + (x - x_1 z)\dots(x - x_{n-1} z)\right)\\
\frac{\partial F}{\partial y} &= d y^{d-1}\\
\frac{\partial F}{\partial z} &= (n-d) z^{d-n-1}(x - x_1 z)\dots(x - x_n z)\\
&\quad + x_1 z^{d-n}(x - x_2 z)\dots(x - x_n z) + \dots + x_n z^{d-n}(x - x_1 z)\dots(x - x_{n-1} z).
\end{aligned}$$

Then

(1) If $n < d - 1$, then it's singular.
(2) If $n = d - 1$, then it's non-singular.

In a summary, only when $n = d - 1, d, d + 1$, the compactification is non-singular, otherwise it's singular.

11.2. **Homework-2.**

Exercise 11.2.1. Consider the affine plane curve

$$C = \{y^2 = x^3 + ax + b\}, \quad a, b \in \mathbb{C}.$$

(1) Find the equation for the corresponding projective plane curve in $\mathbb{P}^2$.
(2) When is C smooth?
(3) When C is not smooth, find the singular points.

Proof. For (1). The corresponding projective plane curve in $\mathbb{P}^2$ is defined by

$$F(x, y, z) = zy^2 - x^3 - axz^2 - bz^3.$$

For (2). For $f(x, y) = y^2 - x^3 - ax - b$, a direct computation shows

$$\begin{aligned}\frac{\partial f}{\partial x} &= -3x^2 - a,\\ \frac{\partial f}{\partial y} &= 2y.\end{aligned}$$

Note that C is non-singular if and only if for every point $(x, y) \in C$, at least one of above derivatives is non-zero. In other words, the singularities the solutions of the following systems of equations

$$f(x, y) = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0.$$

Note that above systems of equations is equivalent to

$$\begin{cases} x^3 + ax + b = 0 \\ 3x^2 + a = 0 \end{cases}$$

This shows C is non-singular if and only if $x^3 + ax + b$ has three different roots.

For (3). If C is non-singular, the singularities are given by the roots of $x^3 + ax + b$ with multiplicity > 1. □

Exercise 11.2.2. For a projective plane curve defined by a linear equation, we call it a projective line. Show that for any two distinct points on $\mathbb{P}^2$, there is a unique projective line passing through them. Prove also that any two distinct projective lines intersect at one point.

Proof. For points $p, q \in \mathbb{P}^2$, without lose of generality we may assume $p = [x : y : 1]$ and $q = [z : w : 1]$. In the affine piece $U_2 = \{[z_0 : z_1 : z_2] \mid z_2 \neq 0\}$, it's clear that there exists a line, given by $az_0 + bz_1 + c = 0$, connecting the points (x, y) and (z, w). Then the p, q is connected by the projective line defined by

$$az_0 + bz_1 + cz_2 = 0.$$

Conversely, suppose l_1, l_2 are two projective lines given by

$$\begin{aligned} az_0 + bz_1 + cz_2 &= 0 \\ ez_0 + fz_1 + gz_2 &= 0. \end{aligned}$$

Consider the corresponding lines in affine piece U_2, that is,

$$\begin{aligned} az_0 + bz_1 + c &= 0 \\ ez_0 + fz_1 + g &= 0. \end{aligned}$$

There are two cases:

(1) If $af \neq be$, then there exists a unique intersection of l_1, l_2 in U_2. For $z_2 = 0$, points in l_1, l_2 are given by $[a/b : 1 : 0]$ and $[e/f : 1 : 0]$, so l_1 and l_2 cannot intersect at $z_2 = 0$ since $af \neq be$.

(2) If $af = be$, then there exists no intersection of l_1, l_2 in U_2, and the unique intersection are at $z_2 = 0$.

□

Exercise 11.2.3. We say $p_1, \dots, p_n \in \mathbb{P}^2$ are in general position if no three are colinear (i.e. lie on a projective line). Show that for four points in $\mathbb{P}^2$ in general position $\{p_1, \dots, p_4\}$ and $\{q_1, \dots, q_4\}$, there exists a $g \in \mathrm{GL}(3, \mathbb{C})$ such that $gp_i = q_i, 1 \leq i \leq 4$.

Proof. Without lose of generality we assume $\{q_1, \dots q_4\}$ are

$$\{[1:0:0], [0:1:0], [0:0:1], [1:1:1]\}.$$

Now if we regard $\{p_1, \dots, p_4\}$ as four vectors in $\mathbb{C}^3$, then there exists the following relations

$$ap_1 + bp_2 + cp_3 = p_4,$$

where $a, b, c \in \mathbb{C}$, since any four vectors in $\mathbb{C}^3$ are $\mathbb{C}$-linearly dependent. Moreover, since $\{p_1, \dots, p_4\}$ are colinear, one has $a, b, c \in \mathbb{C}^*$ and p_1, p_2, p_3 forms a basis of $\mathbb{C}^3$. Then consider $g \in \mathrm{GL}(3, \mathbb{C})$ defined by

$$\begin{cases} ap_1 \mapsto e_1 \\ bp_2 \mapsto e_2 \\ cp_2 \mapsto e_3, \end{cases}$$

where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{C}^3$. Then

$$g(p_4) = g(ap_1 + bp_2 + cp_3) = [1:1:1]$$

as desired. □

Exercise 11.2.4. Given 5 points in $\mathbb{P}^2$ in general position, show that there exists a unique smooth conic passing through them (By conic we mean a projective plane curve defined by a degree-2 equation).

Proof. Suppose the five points are given by homogenous coordinates $\{[x_i : y_i : z_i]\}_{i=1}^5$. Then

$$\det \begin{pmatrix} x^2 & xy & y^2 & xz & yz & z^2 \\ x_1^2 & x_1y_1 & y_1^2 & x_1z_1 & y_1z_1 & z_1^2 \\ x_2^2 & x_2y_2 & y_2^2 & x_2z_2 & y_2z_2 & z_2^2 \\ x_3^2 & x_3y_3 & y_3^2 & x_3z_3 & y_3z_3 & z_3^2 \\ x_4^2 & x_4y_4 & y_4^2 & x_4z_4 & y_4z_4 & z_4^2 \\ x_5^2 & x_5y_5 & y_5^2 & x_5z_5 & y_5z_5 & z_5^2 \end{pmatrix} = 0$$

is a conic passing through them. □

Exercise 11.2.5. Consider

$$C := \{x^3 + y^3 = z^3\}$$

and

$$\begin{aligned} F\colon C &\to \mathbb{P}^1, \\ [x:y:z] &\mapsto [x:z]. \end{aligned}$$

How many critical points are there and what are their multiplicities?

Proof. For $[x:z] \in \mathbb{P}^1$ with $x^3 \neq z^3$, it's clear there are three different values for y such that

$$y^3 = z^3 - x^3.$$

On the other hand, the points $[1:1], [1:e^{\frac{2\pi\sqrt{-1}}{3}}], [1:e^{\frac{4\pi\sqrt{-1}}{3}}] \in \mathbb{P}^1$ are the ramification value of above projection, with multiplicity 3. □

Exercise 11.2.6. Let $F\colon X \to Y$ and $G\colon Y \to Z$ be two holomorphic maps between Riemann surfaces such that X, Y are connected, F, G are not constant maps. Prove that

$$\operatorname{mult}_p(G \circ F) = \operatorname{mult}_p F \cdot \operatorname{mult}_{F(p)} G$$

Proof. Suppose $\operatorname{mult}_p F = m$ and $\operatorname{mult}_{F(p)} G = n$. Recall that the multiplicity is defined by the local normal form of holomorphic map. In other words, there exists an open neighborhood U of p with coordinate u, open neighborhood V of $F(p)$ with coordinate v and open neighborhood W of $G \circ F(p)$ with coordinate w, such that F is locally given by

$$u \mapsto v = u^m,$$

and G is locally given by

$$v \mapsto w = v^n.$$

Then $G \circ F$ is locally given by

$$u \mapsto w = u^{mn}.$$

Note that the multiplicity is independent of the choice of the local coordinates, and thus $\operatorname{mult}_p(G \circ F) = mn = \operatorname{mult}_p F \cdot \operatorname{mult}_{F(p)} G$ as desired. □

Exercise 11.2.7. Consider maps between $\mathbb{C}$ defined by

$$\begin{aligned} F\colon\ \mathbb{C} &\to \mathbb{C} \\ z &\mapsto z^3(z^2 - 2z + a)^2, \end{aligned}$$

where

$$a = \frac{34 \pm 6\sqrt{21}}{7}.$$

Find the critical values of F and the corresponding multiplicities on critical points.

Proof. Note that the critical points of F are zero loucs of $\partial F/\partial z = 0$, and a direct computation shows

$$\begin{aligned}\frac{\partial F}{\partial z} &= 3z^2(z^2-2z+a)^2 + 2z^3(z^2-2z+a)(2z-2)\\ &= z^2(z^2-2z+a)\big(3(z^2-2z+a)+2z(2z-2)\big)\\ &= z^2(z^2-2z+a)(7z^2-10z+3a).\end{aligned}$$

(1) It's clear $z_0 = 0$ is a critical points of F with multiplicity 3, and thus $F(0) = 0$ is a critical value.
(2) If z_1, z_2 are two solutions of $z^2 - 2z + a$, then $F(z_1) = F(z_2) = 0$, and the corresponding multiplicities on critical points z_1, z_2 are 2.
(3) If z_3, z_4 are two solutions of $7z^2 - 10z + 3a$, then

$$\begin{aligned} z_3 &= \frac{10+\sqrt{100-4\times 7\times 3a}}{14}\\ z_4 &= \frac{10-\sqrt{100-4\times 7\times 3a}}{14}\end{aligned}$$

□

11.3. **Homework-3.**

Exercise 11.3.1. For which $\lambda \in \mathbb{C}$ we have

$$C := \{[x : y : z] \in \mathbb{P}^2 \mid x^3 + y^3 + z^3 + 3\lambda xyz = 0\}$$

is smooth? For such λ, compute the degree and critical values for

$$F\colon C \to \mathbb{P}^1, \quad [x : y : z] \mapsto [x : z]$$

and find the genus of C.

Proof. Let $P(x, y, z) = x^3 + y^3 + z^3 + 3\lambda xyz$ and consider the following equations

$$\begin{cases} \frac{\partial P}{\partial x} = 3x^2 + 3\lambda yz = 0 \\ \frac{\partial P}{\partial y} = 3y^2 + 3\lambda xz = 0 \\ \frac{\partial P}{\partial z} = 3z^2 + 3\lambda xy = 0 \end{cases} \tag{11.1}$$

It's clear that if (x, y, z) is a solution of (11.1), then it's also a solution of $P(x, y, z) = 0$. Thus if we want to find for which λ the curve C will be singular, it suffices to find non-zero solutions of (11.1). If (x, y, z) is a solution of (11.1) with $x \neq 0$, then both y and z are non-zero, otherwise we will obtain $x = 0$. Note that

$$0 = xz^2 + \lambda x^2 y = xz^2 - \lambda^2 y^2 z = xz^2 + \lambda^3 xz^2.$$

This shows that if $1 + \lambda^3 \neq 0$, then the curve C must be non-singular. Now if C is non-singular, then the genus formula implies

$$g_C = \frac{(3-1)(3-2)}{2} = 1.$$

□

Exercise 11.3.2. Assume affine plane curves $C_1 = \{f = 0\}, C_2 = \{g = 0\} \subseteq \mathbb{C}^2$ are smooth at $p = (0, 0)$. Define the intersection number $(C_1, C_2)_p$ of C_1, C_2 at p to be $\operatorname{mult}_p G$, where

$$\begin{aligned} G\colon C_1 &\to \mathbb{C} \\ (x, y) &\mapsto g(x, y). \end{aligned}$$

Prove

$$(C_1, C_2)_p = (C_2, C_1)_p.$$

Proof. □

Exercise 11.3.3. In many branches of mathematics, we use partubation method to solve equations. For example, if we want to solve the quadratic equation

$$x^5 - x = \frac{1}{2},$$

we may start by solving

$$x^5 - x = 0.$$

We have five solutions $x = 0, \pm 1, \pm\sqrt{-1}$. For the solution $x_1 = 0$, we introduce a parameter t and try to solve $x^5 - x = t$ by power series

$$x_1(t) = a_0 + a_1 t + \cdots + a_k t^k + \cdots$$

where $x_1(t) = a_0 = 0$ recursively by comparing the coefficients of Taylor expansion of both sides

$$x_1^5(t) - x_1(t) = t.$$

What is the convergence radius of $x_1(t)$?

Proof. Consider the affine plane curve $C \subseteq \mathbb{C}^2$ defined by $x^5 - x - t = 0$ and the following holomorphic map

$$\begin{aligned} F\colon C &\to \mathbb{C} \\ (x,t) &\mapsto t. \end{aligned}$$

For $t = 0$, there are 5 preimages of 0, which are $0, \pm 1$ and $\pm\sqrt{-1}$. Furthermore, if the equation $x^5 - x - t = 0$ has no multiple roots, then F is also a covering map. In other words, if

$$|t| < \frac{4}{5} \times \left(\frac{1}{5}\right)^{\frac{1}{4}},$$

then F is a covering map. For the curve $t \mapsto t$ in $\mathbb{C}$, the curve $x_1(t)$ constructed in the exercise is a lifting of this curve starting from the preimage $x_1 = 0$. Thus $|t| = \frac{4}{5} \times \left(\frac{1}{5}\right)^{\frac{1}{4}}$ is the maximal radius of lifting. $\square$

Exercise 11.3.4. Find two smooth conic curves (conic means degree two) in $\mathbb{P}^2$ which meet at one point with multiplicity 4.

Proof. By Bezout theorem if two smooth conic curves intersect at one point, then this point must have multiplicity 4, so it suffices to construct two conic curves which intersect at one point.

Consider the curve C_1 defined by $x^2 - yz = 0$ and the curve C_2 defined by $y^2 - 4xy + 6yz - 4xz + z^2 = 0$. Then the only intersection is $[1:1:1]$, which has multiplicity 4. $\square$

Exercise 11.3.5. Let X, Y be two compact connected Riemann surfaces of genus $g_X > g_Y$. Prove that every holomorphic map from Y to X is constant.

Proof. In fact we prove the following equivalent statement: If there exists a non-constant holomorphic map $F\colon Y \to X$ between compact Riemann surfaces, then $g_Y \geq g_X$.

Now let's begin the proof: If $g_X = 0$, it's trivial. Otherwise, by Hurwitz formula we have

$$2g_Y - 2 = \deg(F)(2g_X - 2) + B(F) \geq 2g_X - 2$$

since $\deg(F) \geq 1$ and $B(F) \geq 0$. $\square$

Exercise 11.3.6. Try to define smooth algebraic curve in

$$\mathbb{P}^1 \times \mathbb{P}^1 = \{[x_1 : y_1], [x_2 : y_2]\}$$

by considering homogeneous polynomial of bidegree (d_1, d_2) in $\mathbb{C}[x_1, y_1, x_2, y_2]$, in which

$$\begin{aligned} \deg x_1 = \deg y_1 = (1, 0), \\ \deg x_2 = \deg y_2 = (0, 1). \end{aligned}$$

Proof. Note that a homogenous polynomial P of bidegree (d_1, d_2) can be regarded as a section s of line bundle $\mathcal{O}(d_1) \otimes \mathcal{O}(d_2)$, and the non-singular algebraic curve C defined by P is exactly the zero divisor of s. By adjunction formula one has

$$K_C \cong \mathcal{O}(d_1 - 2) \otimes \mathcal{O}(d_2 - 2),$$

and thus

$$2g_C - 2 = K_C \cdot C = (d_1 - 2)d_2 + (d_2 - 2)d_1.$$

This shows $g_C = (d_1 - 1)(d_2 - 1)$. $\square$

11.4. **Homework-4.**

Exercise 11.4.1. Let X be a compact Riemann surface. Prove that

$$\mathcal{O}(X) = \{\text{constant functions}\}.$$

Proof. Suppose $f\colon X \to \mathbb{C}$ be a non-constant holomorphic map. Then by open map theorem one has f is an open map, and thus $f(X) \subseteq \mathbb{C}$ is open. On the other hand, since X is compact and then $f(X)$ is compact in $\mathbb{C}$. Thus $f(X) \subseteq$ is both open and closed in $\mathbb{C}$, which implies $f(X) = \mathbb{C}$. But $\mathbb{C}$ is not compact, which leads to a contradiction. $\square$

Exercise 11.4.2. Let w_1, w_2 be $\mathbb{R}$-linearly independent complex numbers, and $C = \mathbb{C}/L$, where $L = \mathbb{Z}\,w_1 + \mathbb{Z}\,w_2$.
(1) Prove that $\omega = \mathrm{d}z$ defines a holomorphic 1-form on $\mathbb{C}$, where z is the coordinate of $\mathbb{C}$.
(2) Compute $\dim_{\mathbb{C}} \Omega^1(C)$.

Proof. For (1). If we write $z = x + \sqrt{-1}y$, then firstly

$$\mathrm{d}z = \mathrm{d}x + \sqrt{-1}\mathrm{d}y$$

is a 1-form on $\mathbb{C}$, and it's holomorphic since

$$\frac{\partial}{\partial \overline{z}}\mathrm{d}z = \frac{1}{2}(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y})(\mathrm{d}x + \sqrt{-1}\mathrm{d}y) = 0.$$

Moreover, above computation also shows that up to constants $\mathrm{d}z$ is the only holomorphic 1-form on $\mathbb{C}$.

For (2). If ω is a holomorphic 1-form on C, then it can be extended to be a holomorphic 1-form on $\mathbb{C}$. Thus $\dim_{\mathbb{C}} \Omega^1(C) \leq 1$. On the other hand, since points in $\mathbb{C}$ which are be identified in the torus only differs constants, $\mathrm{d}z$ descends to a holomorphic 1-form on C. This shows $\dim_{\mathbb{C}} \Omega^1(C) = 1$. $\square$

Remark 11.4.1. More generally, for a compact Riemann surface Σ_g with genus g, the Hodge decomposition implies that $\dim_{\mathbb{C}} \Omega^1(\Sigma_g) = g$.

Exercise 11.4.3. For any two points $p \neq q \in \mathbb{P}^1$, construct a meromorphic 1-form ω with $\operatorname{ord}_p \omega =$ and $\operatorname{ord}_q \omega = -1$.

Proof. Without lose of generality we may assume $q = \infty$ and $p = [\lambda : 1]$. Then the meromorphic 1-form $\omega = 1/(z - \lambda)\mathrm{d}z$ on the affine piece $\{[z : 1]\}$ gives a meromorphic 1-form on $\mathbb{P}^1$ such that $\operatorname{ord}_p \omega = \operatorname{ord}_q \omega = -1$. $\square$

Exercise 11.4.4. Show that

$$\{([x_0 : x_1], [y_0 : y_1]) \mid (x_0^2 + x_1^2)(y_0^2 + y_1^2) = x_0x_1y_0y_1\}$$

is a smooth curve in $\mathbb{P}^1 \times \mathbb{P}^1$, and compute its genus.

Proof. Suppose U_0, U_1 and V_0, V_1 are affine pieces for the first factor and second factor of $\mathbb{P}^1 \times \mathbb{P}^1$ respectively. Then $\{U_i \times V_j\}$ gives an atlas for $\mathbb{P}^1 \times \mathbb{P}^1$, and it suffices to check the curve C defined by $(x_0^2 + x_1^2)(y_0^2 + y_1^2) = x_0x_1y_0y_1$ is smooth on each affine piece.

Since the symmetry between x_0, x_1 and y_0, y_1, it suffices to check the curve C is smooth on the affine piece $\{[1:x]\} \times \{[1:y]\}$. On this affine piece C is defined by

$$F(x,y) = (x^2+1)(y^2+1) - xy = 0.$$

Now it suffices to show the following system of equations has no solution

$$\begin{cases} p(x,y) = (x^2+1)(y^2+1) - xy = 0 \\ \frac{\partial p}{\partial x} = 2x(y^2+1) - y \\ \frac{\partial p}{\partial y} = 2y(x^2+1) - x \end{cases}$$

If (x_0, y_0) is a solution, then one has $4y_0^2 x_0 = y_0$.

(1) If $y_0 = 0$, then by the second equation one has $x_0 = 0$, which contradicts to the first equation.
(2) If $y_0 \neq 0$, then one has $4x_0y_0 = 1$. By the first equation one has

$$(x_0^2+1)(y_0^2+1) = \frac{1}{4}.$$

On the other hand, the symmetry between x_0, y_0 implies $x_0 = y_0 = \pm\frac{1}{2}$, which is a contradiction.

This shows C is smooth. For the genus of C, as it's shown in the last exercise of Homework3, one has $g_C = (2-1)(2-1) = 1$. □

Exercise 11.4.5. Let C be a smooth conic in $\mathbb{P}^2$, $A, B, C, D, E, F \in C$ are six distinct points.

(1) Prove that line $\overline{AB}$ connecting A, B intersect with C at exactly two points A, B.
(2) Let f be the product of lines $\overline{AB}, \overline{CD}, \overline{EF}$, g be the product of lines $\overline{BC}, \overline{DE}, \overline{FA}$. Choose $P \notin C \backslash \{A, B, C, D, E, F\}$. Prove that there exists $\lambda \in \mathbb{C}$ such that $f + \lambda g$ vanishes on P.
(3) If $C = \{h = 0\}$, prove that $h \mid f + \lambda g$.
(4) If $\overline{AB} \cap \overline{DE} = G, \overline{CD} \cap \overline{AF} = H, \overline{EF} \cap \overline{BC} = K$. Prove that G, H, K are colinear (on the line $(f+\lambda g)/h$).

Proof. For (1). It follows from Bezout theorem that $\overline{AB}$ intersects with C on at most two points, since C is a smooth conic, and $\overline{AB}$ is a smooth line. On the other hand, since $A, B \in C$, then the two intersection are exactly points A, B.

For (2). By evaluating at point P, we can regard $f(P) + \lambda g(P)$ as a linear function of λ, which must admit a zero in $\mathbb{C}$, which is also unique.

For (3). If $h \nmid f + \lambda g$, then by Bezout theorem there are at most 2×3 intersections between C and $\{f + \lambda g = 0\}$, but it already has seven intersections A, B, C, D, E, F, P.

For (4). Firstly by definition it's clear $f + \lambda g$ vanishes on points G, H, K, and since $(f + \lambda g)/h$ has degree one, it defines a line. Thus G, H, K are colinear on this line. □

Exercise 11.4.6. Let R be a UFD and $f_1, f_2, g \in R[X]$ with $\deg f_1 = m, \deg g = n$. Prove that

(1) $\mathscr{R}(f_1, g) = (-1)^{mn}\mathscr{R}(g, f_1)$.

(2) $\mathscr{R}(f_1 f_2, g) = \mathscr{R}(f_1, g)\mathscr{R}(f_2, g)$.

Proof. For (1). It reduces to a problem of linear algebra, that is, for $A \in M_{(m+n)\times m}(R)$ and $B \in M_{(m+n)\times n}(R)$, one has

$$\det(A \mid B) = (-1)^{mn} \det(B \mid A).$$

$\square$

11.5. **Homework-5.**

Exercise 11.5.1. Assume for any compact Riemann surface X, $\mathcal{M}^{(1)}(X) \neq 0$. Prove Riemann–Hurwitz by Poincaré-Hopf theorem.

Proof. Suppose $F\colon X \to Y$ is a holomorphic map between Riemann surfaces. For meromorphic 1-form $0 \neq \theta \in \mathcal{M}^{(1)}(Y)$, one has $F^*\theta$ is a meromorphic 1-form on X. Thus by Poincaré-Hopf theorem one has

$$\sum_{q\in Y} \operatorname{ord}_q \theta = -\chi(Y) = 2g_Y - 2$$
$$\sum_{p\in X} \operatorname{ord}_p F^*\theta = -\chi(X) = 2g_X - 2.$$

Thus it suffices to show

$$\sum_{p\in X} \operatorname{ord}_p(F^*\theta) = \deg(F)(\sum_{q\in Y} \operatorname{ord}_q(\theta)) + \sum_{p\in X}(\operatorname{mult}_p F - 1),$$

Firstly let's establish the following lemma.

Lemma 11.5.1. Notations as above. For any $p \in X$,

$$\operatorname{ord}_p(F^*\theta) + 1 = (\operatorname{ord}_{F(p)}(\theta) + 1) \cdot \operatorname{mult}_p(F)$$

Proof. Choose local coordinate w centered at p and local coordinate z at $F(p)$ such that F is given by

$$z = w^n,$$

where $n = \operatorname{mult}_p(F)$. If $k = \operatorname{ord}_{F(p)}(\theta)$, then θ is given by

$$\theta = (\sum_{j=k}^{\infty} c_j z^j)\mathrm{d}z, \quad c_k \neq 0.$$

Thus

$$\begin{aligned} F^*(\theta) &= (c_k(w^n)^k + \text{higher order terms})nw^{n-1}\mathrm{d}w \\ &= (nc_k w^{n(k+1)-1} + \text{higher order terms})\mathrm{d}w. \end{aligned}$$

This shows

$$\operatorname{ord}_p(F^*(\theta)) + 1 = (\operatorname{ord}_{F(p)}(\theta) + 1) \cdot \operatorname{mult}_p(F).$$

□

Note that $F\colon X \to Y$ is a non-constant holomorphic map, and thus it's surjective by Corollary 1.1.1. Then by above Lemma one has

$$\begin{aligned} \sum_{p\in X} \operatorname{ord}_p(F^*\theta) &= \sum_{p\in X}\{(\operatorname{ord}_{F(p)}(\theta) + 1) \cdot \operatorname{mult}_p(F) - 1\} \\ &= (\sum_{p\in X} \operatorname{ord}_{F(p)}(\theta)) \cdot \operatorname{mult}_p(F) + \sum_{p\in X}(\operatorname{mult}_p F - 1) \\ &= \deg(F)(\sum_{q\in Y} \operatorname{ord}_q(\theta)) + \sum_{p\in X}(\operatorname{mult}_p F - 1). \end{aligned}$$

□

Exercise 11.5.2. Let $f(x,y)=x^3-x^2+y^2$. Prove that
(1) $f(x,y)$ is irreducible in $\mathbb{C}[x,y]$.
(2) $f(x,y)$ is reducible in $\mathbb{C}\{x\}[y]$.
(3) Is $f(x,y)$ reducible in $\mathbb{C}\{y\}[x]$?

Proof. For (1). If $f(x,y)$ is irreducible in $\mathbb{C}[x,y]$, then the only possible decomposition must be of the form

$$f(x,y)=(y+g(x))(y+h(x)).$$

This gives the equalities

$$\begin{cases} g(x)+h(x)=0 \\ g(x)h(x)=x^3-x^2. \end{cases}$$

However, there is no polynomial $g(x)=-h(x)$ such that

$$g^2(x)=x^2-x^3.$$

This shows $f(x,y)$ is irreducible in $\mathbb{C}[x,y]$.

For (2). In $\mathbb{C}\{x\}[y]$ one has

$$f(x,y)=(y-x\sqrt{1-x})(y+x\sqrt{1-x}).$$

For (3). If $f(x,y)$ is irreducible in $\mathbb{C}\{y\}[x]$, then we may write it as

$$f(x,y)=(g(y)+p(x))(h(y)-p(x)),$$

where $p(x)\in\mathbb{C}[x]$. However, there is no polynomial $p(x)\in\mathbb{C}[x]$ such that

$$p^2(x)=x^2-x^3.$$

Thus $f(x,y)$ is irreducible in $\mathbb{C}\{y\}[x]$.

□

Exercise 11.5.3 (Miranda IV.3 E)**.** Let τ be a complex number with strictly positive imaginary part. Let h be a meromorphic function on $\mathbb{C}$ which is $(\mathbb{Z}+\mathbb{Z}\tau)$-periodic; in other words, $h(z+1)=h(z+\tau)=h(z)$ for all z. For any point p in $\mathbb{C}$, let γ_p be the path which is the counterclockwise boundary of the parallelogram with vertices $p,p+1,p+\tau+1,p+\tau,p$ (in that order). Assume p is chosen so that there are no zeroes or poles of h on γ_p. Show that

$$\frac{1}{2\pi\sqrt{-1}}\int_{\gamma_p} z\frac{h'(z)}{h(z)}\mathrm{d}z$$

is an element of the lattice $(\mathbb{Z}+\mathbb{Z}\tau)$.

Proof. Firstly we divide above integration into the following four parts

$$\frac{1}{2\pi\sqrt{-1}}\left(\underbrace{\int_p^{p+1} z\frac{h'(z)}{h(z)}\mathrm{d}z}_{A}+\underbrace{\int_{p+1}^{p+\tau+1} z\frac{h'(z)}{h(z)}\mathrm{d}z}_{B}+\underbrace{\int_{p+\tau+1}^{p+\tau} z\frac{h'(z)}{h(z)}\mathrm{d}z}_{C}+\underbrace{\int_{p+\tau}^{p} z\frac{h'(z)}{h(z)}\mathrm{d}z}_{D}\right).$$

Since h is $(\mathbb{Z}+\mathbb{Z}\tau)$-periodic, one has

$$A+C=\int_p^{p+1} z\frac{h'(z)}{h(z)}\mathrm{d}z+\int_{p+1}^{p}(z+\tau)\frac{h'(z+\tau)}{h(z+\tau)}\mathrm{d}z=-\tau\int_p^{p+1}\frac{h'(z)}{h(z)}\mathrm{d}z.$$

Now let's prove

$$\int_p^{p+1}\frac{h'(z)}{h(z)}\mathrm{d}z\in 2\pi\sqrt{-1}\,\mathbb{Z}\,.$$

Since there is no zeros or poles of h on γ_p, we may choose a sufficiently small open neighborhood U of path $p\mapsto p+1$ and write $h\colon U\to\mathbb{C}^*$. Consider the following commutative diagram

$$\begin{tikzcd}
{[0,1]} \arrow[r, "\gamma"] \arrow[d, dashed, "\widetilde{h}"'] & U \arrow[d, "h"] \\
\mathbb{C} \arrow[r, "\exp"] & \mathbb{C}^*
\end{tikzcd}$$

Since $\exp\colon\mathbb{C}\to\mathbb{C}^*$ is the universal covering, there exists a lifting of $h\circ\gamma$, denoted by $\widetilde{h}$. Moreover, one has

$$\int_p^{p+1}\frac{h'(z)}{h(z)}\mathrm{d}z=\int_0^1\widetilde{h}(w)\mathrm{d}w=\widetilde{h}(1)-\widetilde{h}(0).$$

Since $\widetilde{h}$ is a lifting of $h\circ\gamma$, and $h(p+1)=h(p)$, one has $\exp(\widetilde{h}(1))=\exp(\widetilde{h}(0))$, and thus

$$\widetilde{h}(1)-\widetilde{h}(0)\in 2\pi\sqrt{-1}\,\mathbb{Z}\,.$$

By the same argument one can show $B+D\in\mathbb{Z}$, and this completes the proof. □

Exercise 11.5.4 (Miranda IV.3 F)**.** Check by direct computation that if $r(z)$ is a rational function of z, then the meromorphic 1-form $r(z)\mathrm{d}z$ on the Riemann sphere $\mathbb{C}_\infty$ satisfies the residue theorem.

Proof. Without lose of generality we may assume the rational function $f(z)$ is of the form

$$r(z)=\frac{\alpha_1}{(z-\lambda_1)^{a_1}}+\cdots+\frac{\alpha_k}{(z-\lambda_k)^{a_k}}+\beta_1(z-\gamma_1)^{b_1}+\cdots+\beta_l(z-\gamma_l)^{b_l},$$

where $a_i,b_j>0$ for all i,j. Then the summation of residues of meromorphic 1-form $\theta=r(z)\mathrm{d}z$ of point except ∞ is given by

$$\sum_{p\in\mathbb{C}_\infty\setminus\{\infty\}}\mathrm{Res}_p(\theta)=\sum_{i=1}^k\alpha_i\delta_{a_i1}.$$

To see the residue at the infty point ∞, note that

$$r(\frac{1}{z})(-\frac{1}{z^2})\mathrm{d}z=-\sum_{i=1}^k\frac{\alpha_i z^{a_i-2}}{(1-\lambda_1 z)^{a_i}}-\sum_{j=1}^l\frac{\beta_j(1-\gamma_j z)^{b_j}}{z^{b_j+2}}.$$

It's clear that $b_j + 2 > 1$ by definition. Thus the residue of θ at infty point ∞ is exactly

$$\mathrm{Res}_\infty(\theta) = -\sum_{i=1}^{k} \alpha_i \delta_{a_i 1},$$

as desired. $\square$

Exercise 11.5.5 (Miranda IV.3 G)**.** Check that if L is a lattice in $\mathbb{C}$ and $h(z)$ is an L-periodic meromorphic function, then the meromorphic 1-form $\omega = h(z)\mathrm{d}z$, considered as a form on the complex torus $\mathbb{C}/L$, satisfies the residue theorem.

Proof. If $h(z)$ is a L-periodic meromorphic function defined on $\mathbb{C}/L$, then there exists a meromorphic function $\widetilde{h}(z)$ defined on $\mathbb{C}$ such that $\widetilde{h} = h \circ \pi$, where $\pi\colon \mathbb{C} \to \mathbb{C}/L$ is the canonical projection.

However, the summation of orders of a meromorphic function defined on $\mathbb{C}$ is zero, and thus the summation of orders of a L-periodic meromorphic function defined on $\mathbb{C}/L$ is zero. Then residue theorem holds since genus of a complex torus is 1. $\square$

Exercise 11.5.6. Let $f(x,y) = f_d(x,y) + f_{d+1}(x,y) + \cdots + \in \mathbb{C}\{x,y\}$, where $f_i(x,y)$ are homogeneous with respect to (x,y) and $\deg f_i = i$ or $f_i = 0$. Prove that if $f_d(x,y)$ has d distinct linear factors, then $f(x,y)$ decomposes as product of d irreducible factors in $\mathbb{C}\{x,y\}$.

(1) Reduce the question to $f_d(x,y) = \prod(y - \alpha_i x)$
(2) Denote by $w = y/x$,

$$g(x,w) = \frac{f(x,xw)}{x^d} \in \mathbb{C}\{x,y\}.$$

Prove that g converges in a product of discs

$$D_{\rho_1} \times D_{\rho_2} = \{(x,w) \mid |x| < \rho_1, |w| < \rho_2\}$$

that contains $(0,\alpha_i)$.
(3) Prove that $g(0,\alpha_i) = 0$ and $\frac{\partial g}{\partial w}(0,\alpha_i) \neq 0$ and hence $g(x,w) = 0$ has a solution $w = h_i(x)$ near $(0,\alpha_i)$ with $h_i(x) \in \mathbb{C}\{x\}$ and $h_i(0) = \alpha_i$.
(4) Prove that $\prod(y - xh_i(x)) \mid f(x,y)$ and $f(x,y)$ is the product of m irreducible factors up to units in $\mathbb{C}\{x,y\}$.

Proof. For (1). Suppose $f_d(x,y)$ is decomposed into d distinct linear factors as follows

$$f_d(x,y) = \prod_{i=1}^{d} (\beta_i y - \alpha_i x).$$

Without lose of generality we may assume $x \nmid f_d(x,y)$, and thus we can reduce to the case $\beta_i = 1$ for all i by dividing $\prod_{i=1}^d \beta_i$.

For (2). For $f_k(x,y)$ with $k \geq d+1$, it's clear

$$g_k(x,y) = \frac{f_k(x,xw)}{x^d} = 0$$

since the degree of x in $f_k(x, xw)$ is k, which is bigger than d, so it suffices to show $f_d(x,y)/x^d$ converges when (x,y) tends to $(0,\alpha_i)$. Since $f_d(x,y)$ is a homogenous polynomial of degree d with respect to (x,y), one has

$$\frac{f_d(x,xw)}{x^d} = f_d(1,w).$$

This shows $g_d(x,y)$ converges in a sufficiently small product of discs $D_{\rho_1} \times D_{\rho_2}$, so does $g(x,y)$.

For (3). From the proof of (2) one can see $g(0,\alpha_i) = f_d(1,\alpha_i) = 0$. On the other hand, note that

$$\frac{\partial g}{\partial w} = \frac{\partial}{\partial w}\left(\frac{f(x,xw)}{x^d}\right) = \frac{1}{x^d}\frac{\partial y}{\partial w}\frac{\partial f(x,y)}{\partial y} = \frac{f_y(x,y)}{x^{d-1}}.$$

This shows

$$\frac{\partial g}{\partial w}(0,\alpha_i) = f_y(1,\alpha_i) \neq 0,$$

since these linear factors are distinct. Thus by the implicit function theorem, $g(x,w) = 0$ has a solution $w = h_i(x)$ with $w = h_i(x)$ near $(0,\alpha_i) \in \mathbb{C}\{x\}$ and $h_i(0) = \alpha_i$.

For (4). Now consider the following function

$$\alpha(x,y) = \frac{f(x,y)}{\prod_{i=1}^{d}(y - xh_i(x))}.$$

By construction of $h_i(x)$ one can see $\alpha(x,y)$ has a non-zero constant term $\alpha(0,0)$, and thus $\alpha(x,y) \in \mathbb{C}\{x,y\}^*$. This shows $f(x,y)$ is decomposed into d irreducible linear factors in $\mathbb{C}\{x,y\}$. $\square$

11.6. **Homework-6.**

Exercise 11.6.1. Let $x_1, \ldots, x_n$ be distinct points on $\mathbb{C}$, and let

$$C = \{y^d = (x - x_1)^{a_1} \cdots (x - x_n)^{a_n}\} \subseteq \mathbb{C}^2$$

where $d, a_i \in \mathbb{Z}_{>0}$ and $\gcd(d, a_1, \ldots, a_n) = 1$. Let $\overline{C} \subseteq \mathbb{C}^2$ be the corresponding projective plane curve. Prove $\overline{C}$ is irreducible and compute the genus of the normalization of $\overline{C}$.

Proof. To prove $\overline{C}$ is irreducible, it suffices to prove the polynomial

$$y^d = (x - x_1)^{a_1} \ldots (x - x_n)^{a_n}$$

is irreducible, since $\overline{C}$ is the closure of C in $\mathbb{P}^2$. Consider the projection $\Phi \colon C \to \mathbb{C}$ given by $(x, y) \mapsto x$. Then it's a d-covering from $C \setminus \Phi^{-1}(B)$ to $\mathbb{C} \setminus B$, where $B = \{x_1, \ldots, x_n\}$. Since the base $\mathbb{C} \setminus B$ is connected, it suffices to show that the monodromy is transitive on each fiber. Note that the local monodromy at point x_i is given by $\xi_d^{a_i/d}$, where ξ_d is the d-th unit root. Since $\gcd(d, a_1, \ldots, a_n) = 1$, there exists $k, k_1, \ldots, k_n \in \mathbb{Z}$ such that

$$kd + k_1 a_1 + \cdots + k_n a_n = 1.$$

Thus by winding x_1 by k_1 times and winding x_2 by k_2 times and so on, one construct a monodromy given by

$$\xi_d^{k_1 a_1 + \cdots + k_n a_n} = \xi_d^{1 - kd} = \xi_d.$$

Thus the monodromy acts on fiber transitively.

Now let's figure out the type of singularities of $\overline{C}$ to compute the genus of the normalization of $\overline{C}$. Firstly, by blowing up finitely times, one can prove the following lemma, which is a generalization of Example 5.5.2.

Lemma 11.6.1. For $y^m = x^n$, the δ-invariance of $(0, 0)$ is

$$\delta(m, n) = \frac{(m-1)(n-1)}{2} - 1 + d,$$

where $d = \gcd(m, n)$.

For $f(x, y) = y^d - (x - x_1)^{a_1} \ldots (x - x_n)^{a_n}$, a direct computation shows

$$\frac{\partial f}{\partial y} = dy^{d-1}$$
$$\frac{\partial f}{\partial x} = -\sum_{i=1}^{n} a_i (x - x_1)^{a_1} \ldots (x - x_i)^{a_i - 1} \ldots (x - x_n)^{a_n}.$$

Thus $(0, x_i)$ is a singularity of $f(x, y) = 0$ if and only if $a_i > 1$, and the δ-invariance for $(0, x_i)$ is $\delta(d, a_i)$. Let $F(x, y, z)$ be the homogenous polynomial corresponding to $f(x, y)$. Then

(1) If $d > \sum_i a_i$, then $F(x,y,z) = y^d - z^{d-\sum_i a_i}(x-x_1z)^{a_1}\dots(x-x_nz)^{a_n}$, and thus the infinity point is $[1:0:0]$. On the affine piece $\{x=1\}$, the equation is given by

$$F(1,y,z) = y^d - z^{d-\sum_i a_i}(1-x_1z)^{a_1}\dots(1-x_nz)^{a_n}.$$

To see $(0,0)$ is a singularity of $F(1,y,z)$ or not, a direct computation shows that

$$\begin{aligned}\frac{\partial F(1,y,z)}{\partial y} =& dy^{d-1}\\ \frac{\partial F(1,y,z)}{\partial z} =& -(d-\sum_i a_i)z^{d-\sum_i a_i-1}(1-x_1z)^{a_1}\dots(1-x_nz)^{a_n}\\ &- z^{d-\sum_i a_i}\left(-\sum_i^n a_ix_i(1-x_1z)^{a_1}\dots(1-x_iz)^{a_i-1}\dots(1-x_nz)^{a_n}\right).\end{aligned}$$

Thus $(0,0)$ is not a singularity of $F(1,y,z)$ if and only if $d = \sum_i a_i + 1$, and the δ-invariance for $(0,0)$ is $\delta(d, d-\sum_i a_i)$.

(2) If $d = \sum_i a_i$, then $F(x,y,z) = y^d - (x-x_1z)^{a_1}\dots(x-x_nz)^{a_n}$. On the infinity line $\{z=0\}$, the equation is given by $y^d = x^d$, and thus there are d points of $\overline{C}$ on the infinity line, given by $\{[1,\xi_d^i:0] \mid i=1,\dots,d\}$, which are non-singular.

(3) If $d < \sum_i a_i$, then $F(x,y,z) = y^dz^{\sum_i a_i-d} - (x-x_1z)^{a_1}\dots(x-x_nz)^{a_n}$, and thus the infinity point is $[0:1:0]$. On the affine piece $\{y=1\}$, the equation is given by

$$F(x,1,z) = z^{\sum_i a_i-d} - (x-x_1z)^{a_1}\dots(x-x_nz)^{a_n}.$$

To see $(0,0)$ is a singularity of $F(x,1,z)$ or not, a direct computation shows that

$$\begin{aligned}\frac{\partial F(x,1,z)}{\partial x} =& -\sum_i a_i(x-x_1z)^{a_1}\dots(x-x_iz)^{a_i-1}\dots(x-x_nz)^{a_n}\\ \frac{\partial F(x,1,z)}{\partial z} =& (\sum_i a_i-d)z^{\sum_i a_i-d-1} + \sum_i a_ix_i(x-x_1z)^{a_1}\dots(x-x_iz)^{a_i-1}\dots(x-x_nz)^{a_n}.\end{aligned}$$

Thus $(0,0)$ is not a singularity of $F(x,1,z)$ if and only if $\sum_i a_i = d+1$. To compute the δ-invariance, after once blow up one has

$$g(x,w) = \frac{F(x,1,xw)}{x^{\sum_i a_i-d}} = w^{\sum_i a_i-d} - x^d(1-x_1w)^{a_1}\dots(1-x_nw)^{a_n}.$$

Then it reduces to the standard model $w^{\sum_i a_i-d} = x^d$, and thus the δ-invariance for this case is

$$\binom{\sum_i a_i-d}{2} + \delta(d, \sum_i a_i - d).$$

As a consequence, the genus of the normalization of $\overline{C}$ is

$$\begin{cases} (d-1)(d-2)/2 - \sum_{i=1}^n \delta(d,a_i) - \delta(d, d-\sum_i a_i), & d > \sum_i a_i + 1 \\ (d-1)(d-2)/2 - \sum_{i=1}^n \delta(d,a_i), & d = \sum_i a_i + 1 \\ (d-1)(d-2)/2 - \sum_{i=1}^n \delta(d,a_i), & d = \sum_i a_i \\ (\sum_i a_i - 1)(\sum_i a_i - 2)/2 - \sum_{i=1}^n \delta(d,a_i), & d = \sum_i a_i - 1 \\ (\sum_i a_i - 1)(\sum_i a_i - 2)/2 - \sum_{i=1}^n \delta(d,a_i) - \delta(d, \sum_i a_i - d) - \binom{\sum_i a_i - d}{2}, & d < \sum_i a_i - 1 \end{cases}$$

□

Exercise 11.6.2. A projective plane curve is called rational if it's irreducible and its normalization has genus zero. Find a rational curve for each degree d.

Proof. Consider the the projective plane curve C defined by $y^d = x^{d-1}z$. On the affine piece $z = 1$, it's given by $y^d = x^{d-1}$. Then $(0,0)$ is a singularity with δ-invariance

$$\frac{(d-1)(d-2)}{2}.$$

On the other hand, $[1:0:0]$ is not a singularity of $y^d = x^{d-1}z$. Thus by Bezout theorem, the genus of the normalization of C is

$$\frac{(d-1)(d-2)}{2} - \frac{(d-1)(d-2)}{2} = 0.$$

□

Exercise 11.6.3. Determine $y^2-(x^2y^2+x^4)$ is irreducible or not in $\mathbb{C}\{x,y\}$. This is an example of tacnode singularity.

Proof. Firstly consider the blow up $g(x,w)$, that is

$$g_1(x,w) = \frac{f(x,xw)}{x^2} = w^2 - x^2w^2 + x^2.$$

It's still singular at $(0,0)$, so consider

$$g_2(x,t) = \frac{g_1(x,xt)}{x^2} = t^2 - x^2t^2 + 1.$$

Note that

$$\left.\frac{\partial g_2}{\partial t}\right|_{x=0,t=\pm 1} \neq 0.$$

Then by implicit function theorem, there exists $t_1(x), t_2(x) \in \mathbb{C}\{x\}$ such that $t_1(0) = 1$ and $t_2(0) = -1$, and thus in $\mathbb{C}\{x,y\}$, there is the following decomposition

$$y^2 - (x^2y^2 + x^4) = u(y - x^2t_1(x))(y - x^2t_2(x)),$$

where u is a unit in $\mathbb{C}\{x,y\}$. □

Exercise 11.6.4. Compute the genus of the curve

$$C = \{x^2y^2 - z^2(x^2+y^2) = 0\} \subseteq \mathbb{P}^2$$

Proof. For convenience we denote $F(x,y,z) = x^2y^2 - z^2(x^2+y^2)$. Note that

$$\begin{aligned}\frac{\partial F(x,y,1)}{\partial x} &= 2xy^2 - 2x\\ \frac{\partial F(x,y,1)}{\partial y} &= 2x^2y - 2y.\end{aligned}$$

Then $(0,0)$ is a singularity of $F(x,y,1)$. On the infinity line $\{z=0\}$, there are two points on C, that is, $[1:0:0]$ and $[0:1:0]$. Note that $(0,0)$ is also a singularity for $F(x,1,z)$, since

$$\begin{aligned}\frac{\partial F(x,1,z)}{\partial x} &= 2x - 2xz^2\\ \frac{\partial F(x,1,z)}{\partial z} &= -2z(x^2+1).\end{aligned}$$

By the same argument one can show $[1:0:0]$ is also a singularity for $F(1,y,z)$. This shows there are three singularities of the projective plane curve defined by F, that is, $[1:0:0], [0:1:0]$ and $[0:0:1]$. Now it suffices to compute the δ-invariance for these singularities.

Note that by blow up $f(x,y) = F(x,y,1)$ once, one has

$$g(x,w) = x^2w^2 - 1 - w^2,$$

which is non-singular at $(0, \pm\sqrt{-1})$, and thus the δ-invariance for $[0:0:1]$ is $\binom{2}{2} = 1$. For singularity $[0:1:0]$, the same computation shows that the δ-invariance of it is 1, so is the one of $[1:0:0]$. Then by Plücker formula one has the genus of C is

$$\frac{(4-1)(4-2)}{2} - 3 = 0.$$

$\square$

Exercise 11.6.5. $C_1, C_2 \subseteq \mathbb{P}^2$ are curves of degree n. Assume C_1, C_2 intersect at n^2 distinct points. If mn of these points lie on an irreducible curve C_3 of degree m, then the remaining $(n-m)n$ points lie on a curve of degree $n-m$.

Proof. Suppose C_1, C_2, C_3 are defined by homogenous polynomials F_1, F_2, F_3 respectively. Suppose p is a point on C_3 which does not lie on $C_1 \cap C_2$, Then the curve C_4 of degree n, defined by

$$\lambda F_1 + \mu F_2 = 0,$$

where $\lambda = F_2(p), \mu = -F_1(p)$ intersects with C_3 at least $mn+1$ points. Then C_3 must be a component of C_4, otherwise it contradicts to the Bezout theorem. Thus there exists a homogenous polynomial G such that

$$\lambda F_1 + \mu F_2 = F_3 G,$$

where $\deg G = n - m$. Note that there are n^2 distinct points such that $F_1 = F_2 = 0$, and only mn of them such that $F_3 = 0$. This shows that there are $(n-m)n$ of them such that $G = 0$, which completes the proof. $\square$

Exercise 11.6.6. If a degree n projective plane curve C has $[\frac{n}{2}]+1$ singular points on a line L, then L is necessarily a component of C.

Proof. If L is not a component of C, then by Bezout theorem one has

$$\sum_{p\in C\cap L}(C,L)_p = n.$$

Note that for $p \in C\cap L$, if $(C,L)_p = 1$, then p must be a non-singular point since every linear polynomial is non-singular. Thus

$$\sum_{p\in C\cap L}(C,L)_p \geq 2\times([\frac{n}{2}]+1) > n,$$

a contradiction. □

11.7. **Homework-7.**

Exercise 11.7.1. Let $D \in \mathrm{Div}(X)$ and $|D|$ is base-point-free. Prove $|nD|$ is base-point-free for all $n \in \mathbb{Z}_{>0}$.

Proof. If $|D|$ is base-point-free, then $\mathrm{Supp} \bigcap_{E\in|D|} E = \varnothing$. As a consequence, one has $\mathrm{Supp} \bigcap_{E\in|D|} nE = \varnothing$. On the other hand, one has $\{nE \mid E \in |D|\} \subseteq |nD|$, and thus

$$\mathrm{Supp} \bigcap_{E\in|nD|} E \subseteq \mathrm{Supp} \bigcap_{E\in|D|} nE = \varnothing.$$

This shows $|nD|$ is base-point-free. □

Exercise 11.7.2. For $D \in \mathrm{Div}(X)$, prove that

(1) If $\deg D < 0$, then $\ell(D) = 0$.
(2) If $\deg D = 0$, then $\ell(D) = 0$ or 1.
(3) For $X = \mathbb{C} / \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, and use the fact that

$$\mathrm{Div}^0(X)/\,\mathrm{PDiv}(X) \simeq X$$

to find all divisors $D \in \mathrm{Div}^0(X)$ such that $\ell(D) = 0$ and all D such that $\ell(D) = 1$.

Proof. For (1). If $\deg(D) < 0$, then $\Gamma(X, \mathcal{O}(D)) = \{0\}$, and thus $\ell(D) = 0$.

For (2). If $\deg(D) = 0$ and $\ell(D) \neq 0$, then for any non-constant meromorphic function $f \in \Gamma(X, \mathcal{O}(D))$, one has

$$0 = \deg(\mathrm{div}(f) + D) \geq 0,$$

which implies $D = -\mathrm{div}(f)$ is a principal divisor, and thus $\ell(D) = 1$.

For (3). By (2), one has every divisor D with degree zero and $\ell(D) = 1$ is a principal divisor. Then for any $D \in \mathrm{Div}^0(X)$, if $D = p - 0$ for some $p \in X$, then $\ell(D) = 0$, otherwise $\ell(D) = 1$. □

Exercise 11.7.3. Let X be a smooth cubic curve, show that there exists $f \in \mathcal{M}(X)$ such that $\mathrm{div}(f)$ is divisable by 2 but f is not a square of a function in $\mathcal{M}(X)$.

Proof. Since X is a smooth cubic curve, one has $g_X = 1$. In particular, for any point $p \in X$, one has $\ell(p) = 1$, otherwise X is isomorphic to $\mathbb{P}^1$. Moreover, $\ell(2p) = 2$, since $2p$ is base-point-free (by the following exercise). Thus there exists a non-constant meromorphic function $f \in \Gamma(X, \mathcal{O}(2p))$ and $\mathrm{div}(f)$ is divisable by 2. On the other hand, if $f = g^2$ for some $g \in \mathcal{M}(X)$, then $g \in \Gamma(X, \mathcal{O}(p))$, which implies f is a constant, since $\ell(p) = 1$. □

Exercise 11.7.4. Let $D \in \mathrm{Div}(X)$.

(1) If $\deg(D) \geq 2g$, then $|D|$ is base-point-free.
(2) If $\deg(D) \geq 2g + 1$, then D is very ample.

Proof. For (1). If $\deg(D) \geq 2g$, then $\deg(K - D) \leq 2g - 2 - 2g = -2$, and thus $\ell(K - D) = 0$. By Riemann-Roch theorem, one has

$$\ell(D) = 1 - g + \deg(D).$$

On the other hand, since $\deg(D-p)=2g-1$, by the same argument one has

$$\ell(D-p)=1-g+\deg(D)-1.$$

This shows $\ell(D-p)=\ell(D)-1$ for every $p\in X$, and thus $|D|$ is base-point-free.

For (2). By the same arguments used in the proof of (1), one can show for every $p,q\in X$, one has

$$\ell(D-p-q)=\ell(D)-2.$$

This shows D is every ample. □

Exercise 11.7.5 (Theta function)**.**

(1) If $w_1,w_2\in\mathbb{C}$ are $\mathbb{R}$-linearly independent, then $X=\mathbb{C}\,/\,\mathbb{Z}\,w_1+\mathbb{Z}\,w_2$ is isomorphic to $\mathbb{C}\,/\,\mathbb{Z}+\mathbb{Z}\,\tau$ for some $\tau\in\mathbb{H}=\{\tau\in\mathbb{C}\mid \mathrm{Im}\tau>0\}$.

(2) If $z\in\mathbb{C},\ \mathrm{Im}\tau>0$, define

$$\theta(z)=\sum_{n=-\infty}^{\infty}e^{\pi\sqrt{-1}(n^2\tau+2nz)}.$$

Prove the series converges absolutely and uniformly on compact subsets of $\mathbb{C}$.

(3) Prove

$$\begin{aligned}\theta(z+1)&=\theta(z),\\ \theta(z+\tau)&=e^{-\pi\sqrt{-1}(\tau+2z)}\theta(z)\end{aligned}$$

(4) Consider the parallelogram with vertices $p,p+1,p+1+\tau,p+\tau$ and use integration of

$$\frac{1}{2\pi\sqrt{-1}}\int\frac{\theta'}{\theta}\mathrm{d}z$$

to conclude that θ has a simple zero inside this parallelogram for a generic p.

(5) For any $x\in\mathbb{C}$, let $\theta^{(x)}(z)=\theta(z-\frac{1}{2}-\frac{\tau}{2}-x)$. Prove that

$$\begin{aligned}\theta^{(x)}(z+1)&=\theta^{(x)}(z).\\ \theta^{(x)}(z+\tau)&=-e^{-2\pi\sqrt{-1}(z-x)}\theta^{(x)}(z).\end{aligned}$$

(6) Conclude that $\theta^{(x)}(z)$ has simple zeros at $x+m+n\tau$ with $m,n\in\mathbb{Z}$ and no other zeros.

(7) Let

$$R(z)=\frac{\prod_{i=1}^m\theta^{(x_i)}(z)}{\prod_{j=1}^n\theta^{(y_j)}(z)}$$

for $x_1,\dots,x_m,y_1,\dots,y_n\in\mathbb{C}$. Then $R(z+1)=R(z)$, and if $\sum_{i=1}^m x_i-\sum_{j=1}^n y_j\in\mathbb{Z}$, then $R(z+\tau)=R(z)$.

(8) Use (7) to prove for $X = \mathbb{C}\,/\,\mathbb{Z} + \mathbb{Z}\,\tau$,
$$\mathrm{PDiv}(X) = \ker A,$$
where A is the Abel-Jacobi map.

Proof. For (1). Given a lattice $L = \mathbb{Z}\,w_1 + \mathbb{Z}\,w_2$, multiplying by $1/w_1$ gives an isomorphism between
$$\mathbb{Z}\,w_1 + \mathbb{Z}\,w_2 \to \mathbb{Z} + \mathbb{Z}\,\tau,$$
where $\tau = w_2/w_1$, and we may assume $\mathrm{Im}\tau > 0$, since $\mathbb{Z} + \mathbb{Z}\,\tau = \mathbb{Z} + \mathbb{Z}(-\tau)$. $\square$

For (2). Notice that
$$e^{\pi\sqrt{-1}(n^2\tau+2nz)} = e^{\pi\sqrt{-1}\tau n^2}e^{2\pi\sqrt{-1}nz}.$$
Let $z = x + \sqrt{-1}y$ and $\tau = u + \sqrt{-1}v$, where $\mathrm{Im}\tau = v > 0$. Then
$$\begin{aligned}
|e^{\pi\sqrt{-1}\tau n^2}e^{2\pi\sqrt{-1}nz}| &= |e^{\pi\sqrt{-1}(u+\sqrt{-1}v)n^2}||e^{2\pi\sqrt{-1}n(x+\sqrt{-1}y)}| \\
&= |e^{\pi\sqrt{-1}un^2-\pi vn^2}||e^{2\pi\sqrt{-1}nx-2\pi ny}| \\
&= |e^{\pi\sqrt{-1}un^2}||e^{-\pi vn^2}||e^{2\pi\sqrt{-1}nx}||e^{-2\pi ny}| \\
&= e^{-\pi vn^2}e^{-2\pi ny} \\
&= e^{-\pi n(vn+2y)}.
\end{aligned}$$
For n large enough, $|n| \leqslant \pi n(vn + 2y)$, and thus $|e^{\pi\sqrt{-1}(n^2\tau+2nz)}| \leqslant e^{-|n|}$. As a result,
$$\sum_{n\in\mathbb{Z}} e^{\pi\sqrt{-1}(n^2\tau+2nz)}$$
converges absolutely and uniformly on compact subsets of $\mathbb{C}$.

For (3). It's clear that $\theta(z + 1) = \theta(z)$ since $e^{2\pi\sqrt{-1}} = 1$. For the other equality, it suffices to note that
$$\sum_{n\in\mathbb{Z}} e^{\pi\sqrt{-1}\{(n^2+2n)\tau+2nz\}} \overset{n=m-1}{=} \sum_{m\in\mathbb{Z}} e^{-\pi\sqrt{-1}(\tau+2z)}e^{\pi\sqrt{-1}(m^2\tau+2mz)}.$$

For (4). A direct computation shows that
$$\begin{aligned}
\theta\left(\frac{1+\tau}{2}\right) &= \sum_{n\in\mathbb{Z}} e^{\pi\sqrt{-1}(n^2\tau+n\tau+n)} \\
&= \sum_{n\in\mathbb{Z}} e^{\pi\sqrt{-1}(4n^2+2n)\tau} - \sum_{n\in\mathbb{Z}} e^{\pi\sqrt{-1}\{(2n+1)^2+2n+1\}\tau} \\
&\overset{\substack{\text{set } n=m-1 \\ \text{in second term}}}{=} \sum_{n\in\mathbb{Z}} e^{\pi\sqrt{-1}(4n^2+2n)\tau} - \sum_{m\in\mathbb{Z}} e^{\pi\sqrt{-1}(4m^2-2m)\tau} \\
&\overset{\substack{\text{set } m=-n \\ \text{in second term}}}{=} \sum_{n\in\mathbb{Z}} e^{\pi\sqrt{-1}(4n^2+2n)\tau} - \sum_{n\in\mathbb{Z}} e^{\pi\sqrt{-1}(4n^2+2n)\tau} \\
&= 0.
\end{aligned}$$

This shows $(1+\tau)/2$ is a zero of θ, and now we're going to show it's simple zero by considering the path γ consisting of four straight lines

$$\begin{cases}\gamma_1\colon p\to p+1\\ \gamma_2\colon p+1\to p+1+\tau\\ \gamma_3\colon p+1+\tau\to p+\tau\\ \gamma_4\colon p+\tau\to p,\end{cases}$$

where p is generic, that is, no zeros of θ is on these paths. For convenience we denote $f(z)=\theta'(z)/\theta(z)$. Since θ has no poles, it suffices to show the integration of $f(z)$ along $\gamma_1\to\gamma_2\to\gamma_3\to\gamma_4$ equals to $2\pi\sqrt{-1}$. Note that

$$\begin{aligned}&f(z+1)=f(z)\\ &f(z+\tau)=-2\pi\sqrt{-1}+f(z)\end{aligned}$$

Then

$$\begin{aligned}\int_\gamma f(z)\mathrm{d}z&=\int_{\gamma_1}f(z)-f(z+\tau)\mathrm{d}z+\int_{\gamma_2}f(z)-f(z+1)\mathrm{d}z\\ &=\int_{\gamma_1}2\pi\sqrt{-1}\mathrm{d}z\\ &=2\pi\sqrt{-1}.\end{aligned}$$

For (5) and (6). Note that $\theta^{(x)}$ is the translation of θ by $-(1+\tau)/2-x$. Then by the double periodicity of θ, one has

$$\begin{aligned}&\theta^{(x)}(z+1)=\theta^{(x)}(z)\\ &\theta^{(x)}(z+\tau)=-e^{-2\pi\sqrt{-1}(z-x)}\theta^{(x)}(z),\end{aligned}$$

and the simple zeros of $\theta^{(x)}$ are $m+n\tau+x$, since $(1+\tau)/2$ is a simple zero of θ, as shown in the proof of (4).

For (7). It's clear that $R(z+1)=R(z)$. For the second equality, a direct computation shows

$$\begin{aligned}R(z+\tau)&=(-1)^{m-n}\frac{\prod_{j=1}^m e^{-2\pi\sqrt{-1}(z-x_j)}\theta^{(x_i)}(z)}{\prod_{k=1}^n e^{-2\pi\sqrt{-1}(z-y_k)}\theta^{(y_k)}(z)}\\ &=(-1)^{m-n}e^{2\pi\sqrt{-1}\{(m-n)z+\sum_{j=1}^m x_j-\sum_{k=1}^n y_k\}}R(z)\end{aligned}$$

If $m=n$ and $\sum_{i=1}^m x_i-\sum_{j=1}^n y_j\in\mathbb{Z}$, then $R(z+\tau)=R(z)$.

For (8). Now it suffices to show $\ker A\subseteq\mathrm{PDiv}(X)$. For $D\in\mathrm{Div}^0(X)$, we write it as

$$D=\sum_{i=1}^n x_i-\sum_{j=1}^n y_j,$$

where we allow $x_i=x_{i'}$ for $i\neq i'$ and $y_j=y_{j'}$ for $j\neq j'$. If $A(D)=0$, then $\sum_{i=1}^n x_i-\sum_{j=1}^n y_j=0$, and thus by (7) one may construct a meromorphic function $R(z)$ on compact torus such that $\mathrm{div}(R(z))=D$.

11.8. **Homework-8.**

Exercise 11.8.1 (gonality)**.** Let C be an algebraic curve. Define

$$\operatorname{gon}(C) = \min\{\deg \Phi \mid \Phi\colon C \to \mathbb{P}^1 \text{ is a non-constant holomorphic map}\}.$$

Prove that

(1) If C is a non-singular projective plane curve of degree $d > 1$, then $\operatorname{gon}(C) \leq d - 1$.
(2) If C has genus g, then $\operatorname{gon}(C) \leq g + 1$.

Proof. For (1). Choose an arbitrary point $p \in C$ and then we can project C with center p to a line outside the point p. This gives a holomorphic map $C \to \mathbb{P}^1$ with degree $d - 1$, and thus $\operatorname{gon}(C) \leq d - 1$.

For (2). Choose an arbitrary point $p \in C$ and consider the divisor $D = (g + 1) \cdot p$. By Riemann inequality one has

$$\ell(D) \geq 1 - g + \deg(D) = 1 - g + g + 1 = 2.$$

In particular, there exists a non-constant $f \in \Gamma(X, \mathcal{O}_X(D))$, which gives a holomorphic map from $C \to \mathbb{P}^1$ with degree $g + 1$. As a consequence, one has $\operatorname{gon}(C) \leq g + 1$. $\square$

Exercise 11.8.2. Show that

$$\Phi\colon \mathbb{P}^2 \to \mathbb{P}^5$$
$$[x_0 : x_1 : x_2] \mapsto [x_0^2 : x_1^2 : x_2^2 : x_0x_1 : x_1x_2 : x_0x_2]$$

defines an embedding. Consider a non-singular projective plane curve C of degree 5. Prove that the canonical map of C into $\mathbb{P}^5$ is $\Phi|_C$, and C is not hyperelliptic.

Proof. It's easy to show that Φ is an embedding by considering the restriction of Φ onto affine pieces of $\mathbb{P}^2$. In fact, Φ is called the Veronese embedding.

Given a non-singular projective plane curve C of degree 5, there exists a natural holomorphic 1-form $\eta = \mathrm{d}x/f_y$, and

$$\{\eta, x^2\eta, y^2\eta, x\eta, y\eta, xy\eta\}$$

forms a $\mathbb{C}$-basis of $\Gamma(X, \Omega_X^1)$. As a consequence, the canonical map of C into $\mathbb{P}^5$ is exactly $\Phi|_C$, and thus C is not hyperelliptic since the canonical map is an embedding. $\square$

Exercise 11.8.3. Show that any non-singular projective plane curve C of degree $d \geq 4$ is not hyperelliptic.

Method one. By the same argument shown in the proof of above exercise, the canonical map of a projective plane curve C of degree $d \geq 4$, is exactly the composite of the inclusion $C \hookrightarrow \mathbb{P}^2$ and the Veronese embedding $\mathbb{P}^2 \to \mathbb{P}^{\binom{d-1}{2}-1}$. In particular, the canonical map is an embedding, and thus C is not hyperelliptic. $\square$

Method two. By adjunction formula one has the canonicald divisor of C is $\mathcal{O}_{\mathbb{P}^2}(d-3)|_C$, and thus it's very ample if $d \geq 4$. In particular, C is not hyperelliptic. □

Exercise 11.8.4. Let X be an algebraic curve of genus $g \geq 2$ and D a divisor on X with $\deg(D) > 0$.

(1) Show that if $\deg(D) \leq 2g-3$, then $\ell(D) \leq g-1$.
(2) Show that if $\deg(D) = 2g-2$, then $\ell(D) \leq g$.

Therefore we see that among divisors of degree $2g-2$, the canonical divisors have the most sections.

Proof. For (1). By Riemann-Roch theorem one has

$$\ell(D) = g - 2 + \ell(K - D).$$

If $\ell(K-D) \geq 1$, then $K - D$ is linearly equivalent to a degree one effective divisor. In other words, $K - D \sim p$ for some point $p \in X$. On the other hand, $\ell(p) = 1$ for any $p \in X$, otherwise $X \cong \mathbb{P}^1$, which contradicts to $g \geq 2$. As a consequence, we have shown that $\ell(K-D) \leq 1$, and thus $\ell(D) \leq g-1$.

For (2). By Riemann-Roch theorem one has

$$\ell(D) = g - 1 + \ell(K - D).$$

If $\ell(K-D) \geq 1$, then $K-D$ is linearly equivalent to an degree zero effective divisor, but the zero divisor is only effective divisor with degree zero, and thus $K \sim D$. In other words, we have shown that $\ell(K-D) \leq 1$, and the equality holds if and only if $D \sim K$. As a consequence, one has $\ell(D) \leq g$, and the equality holds if and only if $D \sim K$. □

Exercise 11.8.5. Let X be an algebraic curve of genus g.

(1) Show that if $g \geq 3$, then mK is very ample for every $m \geq 2$.
(2) Show that if $g = 2$, then mK is very ample for every $m \geq 3$.
(3) Show that if $g = 2$, then map Φ_{2K} maps X to a non-singular projective plane conic, and that this map has degree 2.

Proof. For (1). Note that $\deg(mK) = (2g-2)m > 2g+1$ holds for every $m \geq 2$ when $g \geq 3$, and thus mK is very ample for every $m \geq 2$.

For (2). Note that $\deg(mK) = 2m > 2g + 1 = 5$ holds for every $m \geq 3$, and thus mK is very ample for every $m \geq 3$.

For (3). Suppose $\{f, g\}$ is a $\mathbb{C}$-basis of $\Gamma(X, \mathcal{O}_X(K))$, since $\ell(K) = 2$. Then $\{f^2, fg, g^2\}$ forms a $\mathbb{C}$-basis of $\Gamma(X, \mathcal{O}_X(2K))$, and thus the image of Φ_{2K} is a non-singular projective plane conic, which is defined by $xz = y^2$, and thus $\deg(\Phi_{2K}) = 2$ follows from

$$4 = \deg(2K) = \deg(\Phi_{2K}^*(H)) = \deg(\Phi_{2K}) \times 2,$$

where $H \subseteq \mathbb{P}^2$ is a hyperplane divisor. □

Exercise 11.8.6.

(1) Suppose $C \subseteq \mathbb{P}^4$ is a canonical curve of genus 5. Show that C lies in at least three linearly independent second-degree hypersurfaces Q_1, Q_2, and Q_3.
(2) Suppose C is a non-hyperelliptic curve of genus $g = 5$ which is trigonal, that is, there exists a holomorphic map $\Phi\colon C \to \mathbb{P}^1$ with degree three. Let
$$\Phi^{-1}(t) = D_t = p_1(t) + p_2(t) + p_3(t) \in \operatorname{Div}(C).$$
Then prove that the image of $p_1(t), p_1(t)$ and $p_3(t)$ under the canonical embedding are always collinear.

Proof. For (1). For the canonical embedding $\Phi_K\colon C \to \mathbb{P}^4$, we consider the following map
$$R_2\colon \operatorname{Sym}^2(\mathbb{C}^5) \to \Gamma(C, \mathcal{O}_C(2K)).$$
Then
$$\dim\ker R_2 \geq \binom{6}{2} - 3g + 3 = 3.$$
In other words, C lies in at least three linearly independent second-degree hypersurfaces Q_1, Q_2, and Q_3.

For (2). For any point $t \in \mathbb{P}^1$, one has $\ell(p) = 2$, and thus $\ell(D_t) \geq 2$. Then by Riemann-Roch theorem one has
$$\ell(K - D_t) = 1 - g + 2g - 2 - 3 + \ell(D_t) \geq 3.$$
In other words, there exist linearly independent $f_1, f_2, f_3 \in \Gamma(X, \mathcal{O}_X(K - D_t)) \subseteq \Gamma(X, \mathcal{O}_X(K))$ such that the image of $p_1(t), p_1(t)$ and $p_3(t)$ under the canonical embedding are always on the line defined by $\{f_1 = 0\} \cap \{f_2 = 0\} \cap \{f_3 = 0\}$. $\square$

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