HODGE THOERY AND COMPLEX ALGEBRAIC GEOMETRY

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Contents

0. Overview

In this course, we will introduce two parts:

I Kähler manifold and Hodge decomposition;
II Hodge theory in algebra geometry.

For the first part, if X is a compact complex manifold, we can consider the following structures:

(1) Topology: $H_B^*(X,\mathbb{Z})$, singular cohomology, where B means "Betti".
(2) C^∞-structure: $H_{dR}^*(X,\mathbb{R}) = H^*(X,\Omega^\bullet_{X,\mathbb{R}})$, de Rham cohomology. In fact, de Rham theorem implies that
$$H_{dR}^*(X,\mathbb{R}) \cong H_B^*(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$$
(3) Complex structure: For $x \in X$, we use $T_{X,x}$ to denote its tangent space at x, its real dimension is $2n$. And there is a linear map $J_x : T_{X,x} \to T_{X,x}$ for any $x \in M$, such that $J_x^2 = -\mathrm{id}$. If we complexificate $T_{X,x}$, then we can decompose it into
$$T_{X,x} \otimes_{\mathbb{R}} \mathbb{C} = T_{X,x}^{1,0} \oplus T_{X,x}^{0,1}$$
where $T_{X,x}^{1,0}$ is the eigenspace belonging to eigenvalue $\sqrt{-1}$, and $T_{X,x}^{0,1}$ is the eigenspace belonging to eigenvalue $-\sqrt{-1}$.

If we consider its dual, we will get bundle/sheaf of differential forms, and we can also decompose them as follows
$$\Omega^1_{X,\mathbb{C}} = \Omega^1_{X,\mathbb{R}} \otimes \mathbb{C} = \Omega_X^{1,0} \oplus \Omega_X^{0,1}$$
and
$$\Omega^k_{X,\mathbb{C}} = \Omega^k_{X,\mathbb{R}} \otimes \mathbb{C} = \bigoplus_{p+q=k} \Omega_X^{p,q}$$
where $\Omega_X^{p,q} = \bigwedge^p \Omega_X^{1,0} \otimes \wedge^q \Omega_X^{0,1}$

Since we have such decomposition for differential forms, it's natural to ask if there is a similar decomposition for de Rham cohomology? that is, do we have
$$H_{dR}^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$
such that $\overline{H^{p,q}(X)} = H^{q,p}(X)$?

The Hodge decomposition says it's true for compact Kähler manifolds. It's a very beautiful result, connecting "Topology" and "Complex geometry", since de Rham cohomology reflects the topology information and
$$H^{p,q}(X) \cong H^q_{Dol}(X,\Omega_X^p)$$
where "Dol" means Dolbeault cohomology, reflects the holomorphic information of a complex manifold.

Here is some examples of Kähler manifolds. In fact, almost every interesting manifold is Kähler manifold:

Example 0.0.1. Riemann surfaces, complex torus, projective manifolds are Kähler manifolds.

We also need to know an example that is not Kähler manifold:

Example 0.0.2 (Hopf surface)**.** Consider $\mathbb{Z}$ acts on $\mathbb{C}^2\backslash\{0\}$ by $mz = \lambda^m z, m \in \mathbb{Z}$ for some $\lambda \in (0,1)$, then we define Hopf surface as follows

$$S = \mathbb{C}^2\backslash\{0\}/\mathbb{Z}$$

As we can see, S is diffeomorphic to $S^3 \times S^1$,then $\dim_{\mathbb{C}} H^1(S, \mathbb{C}) = 1$, so S can not be a Kähler manifold by Hodge's decomposition, since for a Kähler manifold, $\dim_{\mathbb{C}} H^1$ must be even.

Remark 0.0.3. By Chow's theorem/GAGA, a compact complex manifold X admitting an embedding into projective space can be defined by polynomial equations, i.e. X is a projective variety, so here comes the forth structure, and that's the second part of this course, we want to apply Hodge theory in algebraic geometry.

(4) Algebraic structure.

Part 1. Preliminaries

1. Complex manifold

1.1. Manifolds and vector bundles.

1.1.1. *Definitions and Examples.*

Definition 1.1.1 (complex manifold)**.** *A complex manifold consists of* $(X, \{U_i, \phi_i\}_{i\in I})$, *where X is a connected, Hausdorff topological space, $\{U_i\}_{i\in I}$ is an open cover of X such that the index set I is countable, and ϕ_i is a homeomorphism from U_i to an open subset V_i of $\mathbb{C}^n$, such that*

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is biholomorphic.

Definition 1.1.2 (transition function)**.** *Such $\phi_i \circ \phi_j^{-1}$ is called transition function; n is called the dimension of X, denoted by* $\dim_{\mathbb{C}} C$*; $\{U_i, \phi_i\}_{i\in I}$ is called complex atlas.*

Definition 1.1.3 (atlas)**.** *Two atlas are equivalent, if the union of them is still an atlas.*

Definition 1.1.4 (complex structure)**.** *A complex structure is an equivalence class of a complex atlas.*

Remark 1.1.5. Replace $\mathbb{C}^n$ by $\mathbb{R}^n$, and biholomorphism is replaced by homeomorphism or diffeomorphism, then we get topological manifold or smooth manifold.

Remark 1.1.6. $V_i \subset \mathbb{C}^n$ usually can not be the whole $\mathbb{C}^n$. For example, there is no non-constant holomorphism from $\mathbb{C}$ to unit disk $\mathbb{D}$. More generally, X is called Brody hyperbodic if there is no non-constant holomorphism from $\mathbb{C}$ to X.

Example 1.1.7. Projective space $\mathbb{P}^n$ is a complex manifold. Atlas are $U_i = \{[z] \in \mathbb{P}^n \mid z_j \neq 0\}, 0 \leq i \leq n, \phi_i : U_i \to \mathbb{C}^n$ is defined as follows

$$[z] \mapsto (\frac{z_0}{z_i}, \dots, \frac{\widehat{z_i}}{z_i}, \dots, \frac{z_n}{z_i})$$

Transition functions are calculated as follows, for $i < j$

$$\phi_i \circ \phi_j^{-1} : (u_1, \dots, u_n) \mapsto (\frac{u_1}{u_i}, \dots, \frac{\widehat{u_i}}{u_i}, \dots, \frac{u_{j-1}}{u_i}, \frac{1}{u_i}, \frac{u_{j+1}}{u_i}, \dots, \frac{u_n}{u_i})$$

In fact, $\mathbb{P}^n$ is a compact complex manifold, since $\mathbb{P}^n$ is diffeomorphic to S^{2n+1}/S^1.

Example 1.1.8. Grassmannian manifold

$$G(r, n) = \{r\text{-dimensional subspace of } \mathbb{C}^n\}$$

Atlas: given $T_i \subset \mathbb{C}^n$ of dimension $n - r$, set $U_i = \{S \subset \mathbb{C}^n \text{of dimension r} \mid S \cap T_i = 0\}$. Choose $S_i \in U_i$, define

$$\phi_i : U_i \to \mathrm{Hom}(S_i, T_i) \cong \mathbb{C}^{r(n-r)}$$

as $S \mapsto f$, such that S is the graph of f.

Example 1.1.9. Complex torus is $\mathbb{C}^n/\Lambda$ where Λ is a free abelian subgroup of $\mathbb{C}^n$ with rank $2n$, called a lattice.

Definition 1.1.10 (holomorphic map)**.** *Let X, Y be complex manifolds of dimension n, m, with atlas $(U_i, \phi_i : U_i \to V_i)$ and $(M_j, \psi_j : M_j \to N_j)$ respectively. A continous map $f : X \to Y$ is called holomorphic, if for any two charts, we have*

$$\psi_j \circ f \circ \phi_i^{-1} : V_i \to \psi_j(f(U_i) \cap M_j)$$

is holomorphic.

Definition 1.1.11 (holomorphic function)**.** *A holomorphic function on X is a holomorphic map $f : X \to \mathbb{C}$.*

Example 1.1.12. Let $S = (\mathbb{C}^2 \backslash \{0\})/\mathbb{Z}$ be Hopf surface, then

$$f : S \to \mathbb{P}^1 = (\mathbb{C}^2 \backslash \{0\})/\mathbb{C}^*$$

is a holomorphic map. The fibers of f are biholomorphic to 1-dimensional complex torus.

Proposition 1.1. *If X is a compace complex manifold, then every holomorphic function on X is constant.*

Proof. Standard conclusion in complex analysis. □

Definition 1.1.13 (immersion/submersion)**.** *A holomorphic map* $f : X \to Y$ *is called an immersion(resp submersion), if for all* $x \in X$*, there exists* $(x \in U_i, \phi_i), (f(x) \in M_j, \psi_j)$*, such that*

$$J_{\psi_j \circ f \circ \phi_i^{-1}}(\phi_i(x))$$

has the max rank $\dim X$(*resp* $\dim Y$)

Definition 1.1.14 (embedding)**.** $f : X \to Y$ *is an embedding, if it is immersion and* $f : X \to f(X) \subset Y$ *is homeomorphism.*

Definition 1.1.15 (submanifold)**.** *A closed connected subset* Y *of* X *is called a submanifold, if for all* $x \in Y$*, there exists* $x \in U \subset X$ *and a holomorphic submersion* $f : U \to \mathbb{D}^k$ *such that*

$$U \cap Y = f^{-1}(0)$$

where k *is the codimension of* Y *in* X*.*

Example 1.1.16 (regular value theorem)**.** Let X, Y be complex manifold with dimension n, m, If $y \in Y$ such that $\mathrm{rank} J_{f(x)}$ reaches maximum m for all $x \in f^{-1}(y)$, then $f^{-1}(y)$ is a submanifold of codimension m.

Definition 1.1.17 (projective manifold)**.** *A projective manifold*[1] X *is a submanifold of* $\mathbb{P}^N$ *of the form*

$$X = \{[z] \in \mathbb{P}^N \mid f_1(z) = \cdots = f_m(z) = 0\}$$

where f_i *is a homogenous polynomial in* $\mathbb{C}[z_0, \ldots, z_n]$

Remark 1.1.18. Here we always assume $(f_1, \ldots, f_m) \subset \mathbb{C}[z_0, \ldots, z_n]$ is a prime ideal, so the case that X is defined by polynomials like $f^2 = 0$ is not allowed, what's more, the condition that X is a manifold implies the following cases won't happen:

1. $f_1 f_2 = 0$;
2. X has a singular point.

Definition 1.1.19 (complete intersection)**.** *Let* $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{P}^n$ *be the projection, then* X *is a submanifold of codimension* k *if and only if*

$$J = (\frac{\partial f_i}{\partial z_j})_{\substack{1 \le i \le m \\ 0 \le j \le N}}$$

has rank k*, for all* $x \in \pi^{-1}(X)$*. Then* X *is called a complete intersection, if* $m = k$*.*

Example 1.1.20. Consider $C \subset \mathbb{P}^n$ defined by

$$xw - yz = y^2 - xz = z^2 - yw = 0$$

is not a complete intersection, called twisted cubic.

[1] Chow's theorem claims that every submanifold of $\mathbb{P}^n$ must be defined by a set of homogenous polynomials, so we can use this property to define a projective manifold, in convenient.

Example 1.1.21. Plücker embedding

$$\Phi : G(r, V) \hookrightarrow \mathbb{P}(\wedge^r V)$$

defined by $S \subset V$ with basis $s_1, \dots, s_r$ is mapped to $[s_1 \wedge \cdots \wedge s_r]$. Check Φ is well-defined embedding.

1.1.2. *Vector bundle.*

Definition 1.1.22 (complex vector bundle)**.** *Let X be a differential manifold, E is a complex vector bundle of rank r on X*

1. (*Via total space*) *E is a differential manifold with surjective map $\pi : E \to X$, such that*
   (1) *For all $x \in X$, fiber E_x is a $\mathbb{C}$-vector space of dimension r.*
   (2) *For all $x \in X$, there exists $x \in U \subset X$ and $\pi^{-1}(U)$ is diffeomorphic to $U \times \mathbb{C}^r$ via h, such that*

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\pi"] \arrow[dr, "h"'] & \arrow[d, phantom, "\circlearrowleft"] & U \\
& U \times \mathbb{C}^r \arrow[ur, "p_1"'] \arrow[r, "p_2"] & \mathbb{C}^r
\end{tikzcd}$$

   and for all $y \in U$, $E_y \xrightarrow{p_2 \circ h} \mathbb{C}^r$ is a $\mathbb{C}$-vector space isomorphism.
   (U, h) is called a local trivialization.

   Remark 1.1.23. Consider two local trivialization $(U_\alpha, h_\alpha), (U_\beta, h_\beta)$, then $h_\alpha \circ h_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$, this induces

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{\text{diff}} \mathrm{GL}(r, \mathbb{C})$$

   such $g_{\alpha\beta}$ are called transition function[2], such that

$$\begin{aligned} g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} &= \mathrm{id} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma \\ g_{\alpha\alpha} &= \mathrm{id} \quad \text{on } U_\alpha \end{aligned}$$

2. (*Via transition function*) *E is the data of*
   (1) *open covering $\{U_\alpha\}$*
   (2) *transition functions $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{diff} \mathrm{GL}(r, \mathbb{C})\}$, satisfies*

$$\begin{aligned} g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} &= \mathrm{id} \quad \textit{on } U_\alpha \cap U_\beta \cap U_\gamma \\ g_{\alpha\alpha} &= \mathrm{id} \quad \textit{on } U_\alpha \end{aligned}$$

Remark 1.1.24. The two definitions above are equivalent. The first definition implies the second clearly. The converse is a little bit complicated.

If we already have an open covering and a set of transition functions, the vector bundle E is defined to be the quotient of the disjoint union $\coprod_{U_\alpha} (U \times \mathbb{C}^r)$ by the equivalence relation that puts $(p', v') \in U_\beta \times \mathbb{C}^r$ equivalent to $(p, v) \in U_\alpha \times \mathbb{C}^r$ if and only if $p = p'$ and $v' = g_{\alpha\beta}(p)v$. To connect this

[2]Note that here "diff" means for each $x \in U_\alpha \cap U_\beta$, we have $g_{\alpha\beta}(x)$ is a smooth function on $\mathbb{C}^r$.

definition with the previous one, define the map π to send the equivalence class of any given (p, v) to p.

Definition 1.1.25 (holomorphic vector bundle). *X is a complex manifold, $\pi : E \to X$ is a complex vector bundle, given by $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{C})$, E is called holomorphic if $g_{\alpha\beta}(x)$ is holomorphic function on $\mathbb{C}^r$ for each $x \in U_\alpha \cap U_\beta$.*

Exercise 1.1.26. Show that the total space of a holomorphic vector bundle E is a complex manifold.

Proof. Since we already have a complex structure on X, we need to pull it back to E using π and use the holomorphic transition functions to show it indeed gives a complex structure on E. $\square$

Definition 1.1.27 (morphism between vector bundles). *ϕ is a diffeomorphic/holomorphic morphism of vector bundle on X of rank k, if $\phi : E \to F$ is diffeomorphic/holomorphic map and fiberwise $\mathbb{C}$-linear of rank k.*

$$\begin{tikzcd}
E \arrow[rr, "\phi"] \arrow[dr, "\pi_1"] & \arrow[d, phantom, "\circlearrowleft"] & F \arrow[dl, "\pi_2"] \\
& X &
\end{tikzcd}$$

Example 1.1.28. X is a differential/complex manifold, then $X \times \mathbb{C}^r$ is the trivial rank r complex/holomorphic vector bundle on X.

Example 1.1.29. E, F are complex/holomorphic vector bundles on X, then $E \oplus F, E \otimes F, \mathrm{Hom}(E, F), E^* = \mathrm{Hom}(E, \mathbb{C}), \mathrm{Sym}^k E, \bigwedge^k E, \det E$ are complex/holomorphic vector bundles.

Though all of these are basic constructions of linear algebra, we need to make it clear here in order to avoid further confusion.

If E, F are given by transition functions $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$. WLOG, we may assume they use the same open covering $\{U_\alpha\}$, otherwise we can take their common refinement.

Then, for direct sum, we can define transition functions as

$$\begin{aligned} g''_{\alpha\beta} : U_\alpha \cap U_\beta &\to \mathrm{GL}(2r, \mathbb{C}) \\ x &\mapsto \mathrm{diag}(g_{\alpha\beta}(x), g'_{\alpha\beta}(x)) \end{aligned}$$

and similarly for tensor $E \otimes F$, we can define transition functions as $g_{\alpha\beta} \otimes g'_{\alpha\beta}$.

Now if we can define the dual vector bundle E^*, then in fact we can define $\mathrm{Hom}(E, F)$ as

$$\mathrm{Hom}(E, F) = E^* \otimes F$$

For dual vector bundle defined by $\{g_{\alpha\beta}\}$, in fact the transition functions are $\{(g_{\alpha\beta}^{-1})^T\}$, i.e. the transpose of the inverse. But it may be difficult to understand why? In fact, you will find it's just a fact in linear algebra.

Let's back to the definition via total space, it's natural to define the dual vector bundle of E, by defining all fibers to be the dual space of E_x. To elaborate, E^* is, first of all, the set of pairs $\{(p,l) \mid p \in X, \text{ and } l : E_x \to \mathbb{C} \text{ is a linear map.}\}$, and π maps (p,l) to $p \in X$. Furthermore, it's important to know what is the trivialization. If (U_α, h_α) is the trivialization of vector bundle E, defined by $E_p \ni (p,e) \mapsto (p, \lambda_\alpha(e)) \in U \times \mathbb{C}^r$, then we can define the trivialization of E^* as

$$\begin{aligned} h_\alpha^* : \pi^{-1}(U) &\to U \times \mathbb{C}^r \\ (p,l) &\mapsto (p, \lambda_\alpha^*(l)) \end{aligned}$$

where $\lambda_\alpha^*(l)$ can be seen as a functional on $\mathbb{C}^r$, such that $\lambda_\alpha^*(l)(\lambda_\alpha(e)) = l(e)$. It's quite natural to require that.

So in the language of linear algebra, if you have a matrix $A : \mathbb{C}^r \to \mathbb{C}^r$, then it induces a matrix of dual spaces $A' : (\mathbb{C}^r)^* \to (\mathbb{C}^r)^*$, then facts in linear algebra tells you $A' = (A^{-1})^T$, that's why here the relationship between $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ is transpose of inverse.

Remark 1.1.30. We should always hold such an ideal, all information of a vector bundle is encoded in its transition functions. So if the transition are trivial, i.e. identity matrix, then the vector bundle is just trivial one, or product bundle. So from the relationship between transition functions of vector bundle and its dual, if the vector bundle is a line bundle, i.e. $g_{\alpha\beta} \in \mathbb{C}\backslash\{0\}$, then

$$(g_{\alpha\beta}^{-1})^T g_{\alpha\beta} = g_{\alpha\beta}^{-1} g_{\alpha\beta} = \mathrm{id}$$

So the vector bundle $\mathrm{End}(L) = L^* \otimes L$ is the trivial bundle, but in general $\mathrm{End}(E)$ is not trivial. We will use this fact later.

Definition 1.1.31 (line bundle)**.** *A holomorphic line bundle L is a rank 1 vector bundle, i.e.,*

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{holo} \mathbb{C}^*$$

Exercise 1.1.32. L is a trivial line bundle $X \times \mathbb{C}$ if and only if up to refinement, there exists $s_\alpha : U_\alpha \to \mathbb{C}^*$, such that $g_{\alpha\beta} = s_\alpha / s_\beta$ on $U_\alpha \cap U_\beta$

Proof. □

Definition 1.1.33 (picard group)**.** *X is a complex manifold, then*

$$\mathrm{Pic}(X) = (\{\textit{holomorphic line bundles on } X\}/\textit{isomorphism}, \otimes)$$

called the Picard group of X.

Remark 1.1.34. Clearly the identity of this group is trivial line bundle, and from Remark 2.1.30 we can see that the inverse element of line bundle L is its dual bundle L^*.

Example 1.1.35. Line bundle on $\mathbb{P}^n$

$$\begin{array}{c} L = \{([l], x) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid x \in l\} \subset \mathbb{P}^n \times \mathbb{C}^{n+1} \\ \Big\downarrow{\scriptstyle\pi} \\ \mathbb{P}^n \end{array}$$

is called tautological line bundle.

Consider open covers

$$U_i = \{[l] = [l_1, \ldots, l_n] \in \mathbb{P}^n \mid l_i \neq 0\}$$

there is a map $U_i \to \pi^{-1}(U_i)$, defined as

$$[l] \mapsto ([l], (\frac{l_0}{l_i}, \ldots, 1, \ldots, \frac{l_n}{l_i}))$$

and local trivialization $h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ defined as

$$([l], x) \mapsto ([l], \lambda)$$

where

$$x = \lambda(\frac{l_0}{l_i}, \ldots, 1, \ldots, \frac{l_n}{l_i})$$

so we can calculate transition function

$$h_i \circ h_j^{-1} : (U_i \cap U_j) \times \mathbb{C} \longrightarrow (U_i \cap U_j) \times \mathbb{C}$$

$$([l], \lambda_j) \mapsto ([l], \lambda_j(\frac{l_0}{l_j}, \ldots, \frac{l_n}{l_j})) \mapsto ([l], \lambda_i)$$

such that

$$\lambda_j(\frac{l_0}{l_j}, \ldots, \frac{l_n}{l_j}) = \lambda_i(\frac{l_0}{l_i}, \ldots, \frac{l_n}{l_i})$$

which implies

$$\lambda_i = \lambda_j \frac{l_i}{l_j}$$

so we can see transition function $g_{ij} = l_i/l_j \in \mathbb{C}^*$. This line bundle L will be denoted by $\mathcal{O}_{\mathbb{P}^n}(-1)$.

Definition 1.1.36 (line bundles on $\mathbb{P}^n$)**.** *We can define*

$$\mathcal{O}_{\mathbb{P}^n}(-k) = \mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes k}, \quad k \in \mathbb{N}^+$$
$$\mathcal{O}_{\mathbb{P}^n}(k) = (\mathcal{O}_{\mathbb{P}^n}(-k))^*, \quad k \in \mathbb{N}^+$$
$$\mathcal{O}_{\mathbb{P}^n}(0) = \mathbb{P}^n \times \mathbb{C}, \quad \textit{trivial line bundle.}$$

In fact, line bundle listed above contain all possible line bundles over $\mathbb{P}^n$*.*

Example 1.1.37. More generally, consider

$$\begin{array}{c} E = \{([S], x) \in G(r,n) \times \mathbb{C}^n \mid x \in S\} \subset G(r,n) \times \mathbb{C}^n \\ \big\downarrow \pi \\ G(r,n) \end{array}$$

Definition 1.1.38 (section)**.** *$\pi : E \to X$ is a complex/holomorphic vector bundle. A* (*global*) *section of E is a differential/holomorphic map $s : X \to E$, such that $\pi \circ s = \mathrm{id}_X$, denoted by $C^\infty(X, E)$ / $\Gamma(X, E)$.*

Example 1.1.39. Global holomorphic sections of trivial holomorphic vector bundle are exactly holomorphic functions $f : X \to \mathbb{C}^r$.

Remark 1.1.40. In fact, global holomorphic sections are very rare, as we can seen from the above example, if X is a compact complex manifold, then all global holomorphic functions are only constant.

Definition 1.1.41 (subbundle)**.** *$\pi : E \to X$ is a complex/holomorphic vector bundle. $F \subset E$ is called a subbundle of rank s, if*

1. *For all $x \in X$, $F \cap E_x$ is a subvector space of dimension s.*
2. *$\pi|_F : F \to X$ induces a complex/holomorphic vector bundle.*

Remark 1.1.42. If F is a subbundle of E, then given a section of F, i.e. $\sigma : X \to F$ such that $\pi|_F \circ \sigma = \mathrm{id}_X$, then clearly we can extend it to a section of E.

Example 1.1.43. $\mathcal{O}_{\mathbb{P}^n}(-1) \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$, is a subbundle.

Exercise 1.1.44.

$$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \begin{cases} 0, & k < 0 \\ \mathbb{C}, & k = 0 \\ \text{homogeneous polynomials in } n+1 \text{ variables of deg } k, & k > 0 \end{cases}$$

Proof. Let's see what happened for $k = -1$, the tautological line bundle. Since we have $\mathcal{O}_{\mathbb{P}^n}(-1)$ is a subbundle of trivial bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$. So we have a global section $\sigma \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-1))$ must be a global section of $\Gamma(\mathbb{P}^n, \mathbb{P}^n \times \mathbb{C}^{n+1})$. However, since $\mathbb{P}^n$ is a compact complex manifold, we have that global sections $\Gamma(\mathbb{P}^n, \mathbb{P}^n \times \mathbb{C}^{n+1})$ must be constant, i.e. for any $x \in \mathbb{P}^n$, $\sigma(x) = v$ is a constant. However, $v \in [l]$, for all $[l] \in \mathbb{P}^n$, which forces $v = 0$. Similarly we will get the result for case $k < 0$.

And for case $k = 0$, global sections are exactly constant so we have $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(0)) = \mathbb{C}$.

Now consider what will happen when $k > 0$. Take $k = 1$ for an example. Life is like a seesaw, so is mathmatics. If something is defined concisely, it must be quite difficult to compute. Since sections of a trivial bundle is easy to compute, so in practice, we always compute the sections of the trivialization of a vector bundle, and glue them together to get a global one, that's what we always do.

For projective space $\mathbb{P}^n$, there exists a canonical affine cover $\{U_i\}$, where $U_i = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n \mid x_i = 0\} \cong \mathbb{C}^n$.

□

Example 1.1.45. For a morphism between vector bundles $\phi : E \to F$, $\operatorname{Ker}\phi \subset E, \operatorname{Im}\phi \subset F$ are subbundles.

Definition 1.1.46 (exact)**.** *A sequence of vector bundles*

$$S \xrightarrow{\phi} E \xrightarrow{\psi} Q$$

is called exact at E if $\operatorname{Ker}\psi = \operatorname{Im}\phi$;

Definition 1.1.47 (pullback)**.** *$f : X \to Y$ is a differential/holomorphic map, $\pi : E \to Y$ is a vector bundle, define*

$$f^*E = \{(x, e) \in X \times E \mid f(x) = \pi(e)\} \subset X \times E$$

is called the pullback of π.

Remark 1.1.48. To be somewhat more explicit, suppose $U \subset Y$ is a local trivialization, i.e. $\varphi_U : E|_U \to U \times \mathbb{C}^r$ with $\varphi_U(e) = (\pi(e), \lambda_U(e))$. Then we can define a local trivialization of f^*E on $f^{-1}(U) \subset X$, by

$$\begin{aligned} \varphi_U^* : f^*E|_{f^{-1}(U)} &\to f^{-1}(U) \times \mathbb{C}^r \\ (x, e) &\mapsto (x, \lambda_U(e)) \end{aligned}$$

and transition functions on $f^{-1}(U_\alpha) \cap f^{-1}(U_\beta)$ is given by $g_{\alpha\beta} \circ f$.

1.2. **Episode: sheaves.** Why we need sheaves here? As we have seen in the last section, the global sections of holomorphic vector bundle are very rare, but there are many local sections, we need to keep these information and learn the connection between global and local systemically. Sheaf is a power language for us to manage global and local at the same time. However, there is nothing more that sheaf can give, it's just a different language, as we can see in Exercise 4.5.

Definition 1.2.1 (sheaf)**.** *X is a topological space. A sheaf*[3] *of abelian group $\mathscr{F}$ on X is the data of:*

1. *For any open subset U of X, $\mathscr{F}(U)$ is an abelian group.*
2. *If $U \subset V$ are two open subsets of X, then there is a group homomorphism $r_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$, such that*
   (1) $\mathscr{F}(\varnothing) = 0$
   (2) $r_{UU} = \mathrm{id}$
   (3) *If $W \subset U \subset V$, then $r_{UW} = r_{VW} \circ r_{UV}$*
   (4) *$\{V_i\}$ is an open covering of $U \subset X$, and $s \in \mathscr{F}(U)$. If $s|_{V_i} := r_{UV_i}(s) = 0, \forall i$, then $s = 0$.*
   (5) *$\{V_i\}$ is an open covering of $U \subset X$, and $s_i \in \mathscr{F}(V_i)$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there exists $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$.*

Example 1.2.2 (sheaf of sections of a holomorphic vector bundle)**.** If $\pi : E \to X$ is a holomorphic vector bundle, then define

$$\mathscr{F}(U) = \Gamma(U, E|_U), \quad \forall U \subset_{\text{open}} X$$

This $\mathscr{F}$ will be denoted by $\mathcal{O}_X(E)$. In particular, E is a trivial vector bundle, then $\mathcal{O}_X(E) = \mathcal{O}_X$, the sheaf of holomorphic function, also called the structure sheaf of X.

Definition 1.2.3 (morphism of sheaves on X)**.** *$\phi : \mathscr{F} \to \mathscr{G}$ is called a morphism of sheaves, if for any open subset U of X, there is a group homomorphism $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$, such that if $U \subset V$ are two open subsets of X, the the following diagram commutes*

[3] A sheaf which fails to meet $(4), (5)$ is called a presheaf.

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\phi(U)} & \mathscr{G}(U) \\ \downarrow r_{UV} & & \downarrow r_{UV} \\ \mathscr{F}(V) & \xrightarrow{\phi(V)} & \mathscr{G}(V) \end{array}$$

Example 1.2.4 (locally free sheaves)**.** A sheaf is called locally free, if there exists covering $\{U_\alpha\}$ such that $\mathscr{F}|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}^{\oplus r}$ of rank r.

For $r = 1$, it is called invertible sheaf.

Exercise 1.2.5. There are correspondences:

$$\{\text{holomorphic vector bundles}\} \stackrel{1-1}{\longleftrightarrow} \{\text{locally free sheaves}\}$$

$$\{\text{holomorphic line bundles}\} \stackrel{1-1}{\longleftrightarrow} \{\text{invertible sheaves}\}$$

Proof. It suffices to prove the first correspondence. If we have a holomorphic vector bundle $\pi E \to X$. Then consider the sheaf of sections $\mathcal{O}_X(E)$, We claim it's a locally free sheaf. Since we have local trivialization of holomorphic vector bundle $\{U_\alpha\}$. Then consider what's $\mathcal{O}_X(E)|_{U_\alpha}$. Since $E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^r$, then holomorphic sections of $U_\alpha \times \mathbb{C}^r \to U_\alpha$ are just holomorphic functions $f : U \to \mathbb{C}^r$, i.e. $\mathcal{O}_X(E|_{U_\alpha}) = \mathcal{O}_{U_\alpha}^{\oplus r}$. So sheaf $\mathcal{O}_X(E)$ is a locally free sheaf.

Conversely, if we have a locally free sheaf, how can we get a holomorphic vector bundle? □

1.3. **Tangent bundle.**

Definition 1.3.1 (tangent bundle)**.** *X is a differential manifold,* $\dim_{\mathbb{R}} X = n$*, and $\{U_\alpha, \phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n\}$ is a atlas of X. The* (*real*) *tangent bundle $T_{X,\mathbb{R}}$ is defined through transition functions*

$$\begin{aligned} g_{\alpha\beta} : U_\alpha \cap U_\beta &\stackrel{\text{diff}}{\to} \mathrm{GL}(n, \mathbb{R}) \\ x &\mapsto J_{\phi_\alpha \circ \phi_\beta^{-1}}(\phi_\beta(x)) \end{aligned}$$

Then $T_{X,\mathbb{C}} = T_{X,\mathbb{R}} \otimes \mathbb{C}$ is a complex vector bundle, called the complexified tangent vector bundle.

Remark 1.3.2. The following statement may be a little bit boring, I write it down just to make myself more clear and to get familiar with two definition of vector bundle.

The tangent bundle $T_{X,\mathbb{R}}$ can be defined as the set

$$T_{X,\mathbb{R}} = \coprod_{x \in X} T_{X,x}$$

and note that there is a natural projection $\pi : T_{X,\mathbb{R}} \to X$, sending $v \in T_{X,x}$ to $x \in X$. Now we want to give a chart on $T_{X,\mathbb{R}}$ to make it into a differential manifold. Let $\{(U_i, \phi_i = (x_i^1, \dots, x_i^n)\}$ be a chart of X, then we can define a chart on X by considering $\{(\pi^{-1}(U_i), \widetilde{\phi}_i)\}$, where $\widetilde{\phi}_i$ is defined through

$$\widetilde{\phi}_i(v) = (\phi_i(\pi(v)), (\mathrm{d}x_i^1)_{\pi(v)}(v), \dots, (\mathrm{d}x_i^n)_{\pi(v)}(v)) \subset \mathbb{R}^n \times \mathbb{R}^n$$

note that such $\widetilde{\varphi}_i$ is bijective. And it's easy to equip $T_{X,\mathbb{R}}$ with a topology such that $\widetilde{\varphi}_i$ is diffeomorphism.

Now I need to calculate transition function to confirm myself as follows: For two charts $(U, \phi = (x_1, \dots, x_n)), (V, \psi = (y_1, \dots, y_n))$, then calculate

$$\widetilde{\psi} \circ \widetilde{\phi}^{-1} : \phi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$$

Note that

$$\widetilde{\phi}^{-1}(r_1, \cdots, r_n, u_1, \cdots, u_n) = \sum_i u_i \frac{\partial}{\partial x_i}|_{\phi^{-1}(r_1, \cdots, r_n)} \in T_{\phi^{-1}(r_1, \cdots, r_n)} M$$

But

$$\mathrm{d}y_j (\sum_i u_i \frac{\partial}{\partial x_i}) = \sum_i u_i (\frac{\partial}{\partial x_i}(y_j)) = \sum_i \frac{\partial y_i}{\partial x_j} u_i$$

Thus transition functions are

$$\begin{aligned}\widetilde{\psi} \circ \widetilde{\phi}^{-1}(r_1, \cdots, r_n, u_1, \cdots, u_n) &= (\psi \circ \phi^{-1}(r), (\sum_i \frac{\partial y_1}{\partial x_i}(r) u_i, \dots, \sum_i \frac{\partial y_n}{\partial x_i}(r) u_i)) \\ &= (\psi \circ \phi^{-1}(r), (\frac{\partial y_j}{\partial x_i}(r))(\begin{matrix} u_1 \\ \vdots \\ u_n \end{matrix}))\end{aligned}$$

So transition function $g_{\alpha\beta}$ are exactly Jacobian of $\varphi_\alpha \circ \varphi_\beta^{-1}$.

Definition 1.3.3 (holomorphic tangent bundle)**.** *X is a complex manifold, $\dim_{\mathbb{C}} X = n$, and $\{U_\alpha, \phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^n\}$ is a atlas of X. The holomorphic tangent bundle T_X is defined through transition functions*

$$\begin{aligned} g_{\alpha\beta} : U_\alpha \cap U_\beta &\to \mathrm{GL}(n, \mathbb{C}) \\ z &\mapsto J^{holo}_{\phi_\alpha \circ \phi_\beta^{-1}}(\phi_\beta(x)) \end{aligned}$$

where J^{holo} is holomorphic Jacobian.

Remark 1.3.4. Since $\phi_\alpha \circ \phi_\beta^{-1} : V_\beta \to V_\alpha$ is a holomorphic functions, then holomorphic Jacobian means the matrix

$$(\frac{\partial(\phi_\alpha \circ \phi_\beta^{-1})^j}{\partial z_i})_{1 \le i,j \le n}$$

Remark 1.3.5. Clearly, $T_X \neq T_{X,\mathbb{C}}$, even they don't have the same rank! For example, if X is a n-dimensional complex manifold, then

$$\dim T_X = n \neq 2n = \dim T_{X,\mathbb{C}}$$

Later we will see the relationship between them.

Remark 1.3.6 (sheaf viewpoint)**.** $\mathcal{O}_X$ is the sheaf of holomorphic function, then define the stalk at x is

$$\mathcal{O}_{X,x} := \varinjlim_{x \in U \subset X} \mathcal{O}_X(U)$$

The elements of $\mathcal{O}_{X,x}$ are called germs.

For a tangent vector, we can take derivation in this direction, so

$$\text{tangent vector} \longrightarrow \text{derivation } D : \mathcal{O}_{X,x} \to \mathbb{C}$$

where a derivation satisfies

1. $\mathbb{C}$-linear
2. Leibniz rule $D(fg) = D(f)g + fD(g)$

In fact, the above correspondence is $1-1$, i.e., $T_{X,x} \cong$ space of derivations of $\mathcal{O}_{X,x}$.

Definition 1.3.7 (cotangent bundle/(anti)canonical bundle)**.** $\Omega_X = T_X^*$ *is called holomorphic cotangent bundle;* $K_X = \det \Omega_X$ *is called canonical bundle;* $K_X^* = \det T_X$ *is called the anticanonical bundle.*

Exercise 1.3.8. We calculate tangent bundle of $\mathbb{P}^n$ through the following exact sequence called Euler sequence[4].

$$0 \to \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \xrightarrow{\psi} T_{\mathbb{P}^n} \to 0$$

Let's clearify what does the map look like in a geometry viewpoint:

Let $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ denote the canonical projection from $\mathbb{C}^n\backslash\{0\} \to \mathbb{P}^n$. We know that basis of tangent vector at $z \in \mathbb{C}^{n+1}\backslash\{0\}$ is $\{\frac{\partial}{\partial z_0}, \dots, \frac{\partial}{\partial z_n}\}$, but these are not tangent vector for $\mathbb{P}^n$. Since function $f([z])$ defined on $\mathbb{P}^n$ satisfies $f([z]) = f([\lambda z]), \forall \lambda \in \mathbb{C}\backslash\{0\}$, regard it as a function defined on $\mathbb{C}^n\backslash\{0\}$ and use tangent vector $\frac{\partial}{\partial z_i}$ to act on both sides of this equation, we have

$$\frac{\partial f}{\partial z_i} = \lambda \frac{\partial f}{\partial z_i}$$

A contradiction. However, $z_i \frac{\partial}{\partial z_i}$ will descend to tangent vector at $[z] \in \mathbb{P}^n$.

Recall $z_0, \dots, z_n$ form basis of $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and span $(\mathcal{O}_{\mathbb{P}^n}(1))_{[z]}$, so can define

$$\psi : (\mathcal{O}_{\mathbb{P}^n}(1))_{[z]}^{\oplus n+1} \to T_{\mathbb{P}^n,[z]}$$
$$(0, \dots, \underbrace{z_i}_{j-th}, \dots, 0) \mapsto z_i \frac{\partial}{\partial z_j}$$

But $\sum_{i=0}^n z_i \frac{\partial}{\partial z_i}$ tangent to the fibers of π, so descends to zero at $[z] \in \mathbb{P}^n$, so we can define ϕ as

$$\phi : \mathcal{O}_{\mathbb{P}^n,[z]} \to (\mathcal{O}_{\mathbb{P}^n}(1))_{[z]}^{\oplus n+1}$$
$$1 \to (z_0, \dots, z_n)$$

In fact, for a homogenous polynomial f with degree d, we have the famous relation

$$\sum_{i=0}^n x_i \frac{\partial f}{\partial x_i} = df$$

[4]Refer to pages 408-409 of Griffiths-Harris for more details.

which is discovered by Euler, and that's why this sequence is called Euler sequence.

Exercise 1.3.9. For Grassmannian manifold $G(r,n)$, we have

$$0 \to E \to G(r,n) \otimes \mathbb{C}^n \to Q \to 0$$

Show that

$$T_{G(r,n)} \cong \mathrm{Hom}(E,Q)$$

Exercise 1.3.10. Let $\pi : L \to X$ is a holomorphic line bundle, given $s \in \Gamma(X,L)$, suppose that $D = \{x \in X \mid s(x) = 0\}$ is a smooth submanifold of codimensional 1. Show that the following sequence is exact:

$$0 \to T_D \to T_X|_D \to L|_D \to 0$$

then we can get

$$K_X^*|_D \cong K_D^* \otimes L|_D$$

which is called adjunchon formula.

In particular, let $X = \mathbb{P}^n$ and $L = \mathcal{O}_{\mathbb{P}^n}$, then Exercise 2.1.44 tells us that $D \subset \mathbb{P}^n$ is a smooth hypersurface defined by a homogenous polynomial with degree d. Then we have

$$K_D^* \cong \mathcal{O}_{\mathbb{P}^n}(n+1-d)$$

and we call it

$$\begin{cases} \text{Fano}, & d < n+1 \\ \text{Calabi-Yau}, & d = n+1 \\ \text{General type}, & d > n+1 \end{cases}$$

1.4. **Almost complex structure and integrable theorem.** Now let us talk about some linear algebra:

Consider a $2n$-dimensional real vector space V, a complex structure on V is a $\mathbb{R}$-linear transformation $J : V \to V$ such that $J^2 = -\mathrm{id}$. We can regard V as a complex vector space, by

$$(a+bi)v = av + bJ(v), \quad a,b \in \mathbb{R}$$

If we consider $V \otimes \mathbb{C}$, then J can extend to $V \otimes \mathbb{C}$, by $J(v \otimes \alpha) = J(v) \otimes \alpha$, then we can decompose $V \otimes \mathbb{C}$ into

$$V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$$

such that $\overline{V^{1,0}} = V^{0,1}$, where conjugate means $\overline{v \otimes \alpha} = v \otimes \overline{\alpha}$.

Such definition may be a little abstract for someone, let's calculate in a concrete example to show how it works.

Example 1.4.1. Let $V = \mathbb{C}^n \cong \mathbb{R}^{2n}$, stand coordinate in $\mathbb{C}^n$ is $(z_1, \dots, z_n)$, and in $\mathbb{R}^{2n}$ we always write $(x_1, y_1, \dots, x_n, y_n)$. Note that complex structure on $\mathbb{C}^n$ maps $z_i = x_i + \sqrt{-1}y_i$ to $\sqrt{-1}(x_i + \sqrt{-1}y_i) = -y_i + \sqrt{-1}x_i$.

So we can define $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by $(x_1, y_1, \dots, x_n, y_n) \mapsto (-y_1, x_1, \dots, -y_n, x_n)$. If we complexify $\mathbb{R}^{2n}$ into a complex vector space $\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$, with a basis $\{e_1, f_1, \dots, e_n, f_n\}$, J is defined by $e_j \mapsto f_j, f_j \mapsto -e_j$, then eigenspace decomposition is as follows

$$V^{1,0} = \{\frac{1}{2}(e_j - if_j)\}, \quad V^{0,1} = \{\frac{1}{2}(e_j + if_j)\}$$

such that conjugation $\overline{e_j - if_j} = e_j + if_j$

Definition 1.4.2 (almost complex structure)**.** *X is a differential manifold of $\dim_{\mathbb{R}} X = 2n$. An almost complex structure on X is a complex structure on $T_{X,\mathbb{R}}$ i.e., an isomorphism of differential vector bundles $J : T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$ such that $J^2 = -\operatorname{id}$.*

It's natural to ask, if X is a complex manifold, and forget its complex structure and just regard it as a differential manifold, can we give a natural almost complex structure on it? That's the following example:

Example 1.4.3. X is a complex manifold, and $T_{X,\mathbb{R}}$ is its (real) tangent bundle if we just see X as a differential manifold. Locally we have

$$T_{X,\mathbb{R}}|_U \cong U \times \mathbb{C}^n$$

for open subset U in X, where we regard $\mathbb{C}^n$ as a $2n$ dimension real vector space. But there is a natural almost complex structure on it arising from multiplying i. So we get $J : T_{X,\mathbb{R}}|_U \to T_{X,\mathbb{R}}|_U$. If we want to get a global one, it suffices to glue them together. So we consider

For two charts $(U_1, \varphi_1) = (z_1 = x_1 + iy_1, , \dots, z_n = x_n + iy_n)), (U_2, \varphi_2) = (g_1 = u_1 + iv_1, \dots, g_n = u_n + iv_n))$ with $U_1 \cap U_2 \neq \varnothing$, there are two ways to define J on $U_1 \cap U_2$

$$\begin{tikzcd}
U_1 \cap U_2 \times \mathbb{C}^n \arrow[ddd, "\times i"] & & & U_1 \cap U_2 \times \mathbb{C}^n \arrow[ddd, leftrightarrow, "\times i"] \\
& T_X|_{U_1 \cap U_2} \arrow[ul, "\widetilde{\varphi}_1"] \arrow[d, "J"] & T_X|_{U_1 \cap U_2} \arrow[l] \arrow[ur, "\widetilde{\varphi}_2"] \arrow[d, "J"] & \\
& T_X|_{U_1 \cap U_2} \arrow[dl, "\widetilde{\varphi}_1"] & T_X|_{U_1 \cap U_2} \arrow[l] \arrow[dr, "\widetilde{\varphi}_2"] & \\
U_1 \cap U_2 \times \mathbb{C}^n & & & U_1 \cap U_2 \times \mathbb{C}^n
\end{tikzcd}$$

We need to check transition function commutes with J, calculated in a local chart as follows: For a 2×2 part, Jacobian of $\varphi_2 \circ \varphi_1^{-1}$ is

$$\begin{pmatrix} \frac{\partial u_j}{\partial x_k} & \frac{\partial u_j}{\partial y_k} \\ \frac{\partial v_j}{\partial x_k} & \frac{\partial v_j}{\partial y_k} \end{pmatrix} \overset{\text{C-R}}{=} \begin{pmatrix} \frac{\partial v_j}{\partial y_k} & \frac{\partial u_j}{\partial y_k} \\ -\frac{\partial u_j}{\partial y_k} & \frac{\partial v_j}{\partial y_k} \end{pmatrix}$$

and J is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

So they commute with each other.

So complex structure gives a almost complex structure (naturally), but the question is: Does every complex structure on a complex manifold can be induced from a almost complex structure on a even-dimensional differential manifold? Unfortunately, it's false in general, but we have the following theorem.

Theorem 1.4.4 (Newlander-Nirenberg)**.** *Let (X, J) be a complex manifold, J is induced by a almost complex structure on X is equivalent to*

$$[T_X^{1,0}, T_X^{1,0}] \subset T_X^{1,0}$$

which is called an integrable condition.

1.5. **Operator ∂ and $\overline{\partial}$.** In this section, we will discuss the relationship between $T_X, T_{X,\mathbb{R}}, T_{X,\mathbb{C}}$ and so on, for a complex manifold X.[5]

First, we have $T_X \hookrightarrow T_{X,\mathbb{C}}$ as complex vector bundle[6], with image $T_{X,\mathbb{C}}^{1,0}$. In fact, if we consider locally, take $(z_1, \ldots, z_n) \in V \subset \mathbb{C}^n$, $z_j = x_j + iy_j$ as a coordinate, then

$$T_X \ni \frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}) \in T_{X,\mathbb{C}}^{1,0}$$

moreover, we can define the conjugation as

$$\frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}) \in T_{X,\mathbb{C}}^{0,1}$$

If we consider its dual space, we get the differential forms

$$\Omega_{X,\mathbb{C}}^1 = \Omega_{X,\mathbb{R}}^1 \otimes \mathbb{C} = \Omega_X^{1,0} \oplus \Omega_X^{0,1}$$

and take wedge product k times, then we get

$$\Omega_{X,\mathbb{C}}^k = \bigwedge^k \Omega_{X,\mathbb{C}}^1 = \bigoplus_{p+q=k} \Omega_X^{p,q}, \quad \text{where } \Omega_X^{p,q} = \bigwedge^p \Omega_X^{1,0} \otimes \bigwedge^q \Omega_X^{0,1}$$

[5] T_X is the tangent bundle of complex manifold X, and $T_{X,\mathbb{R}}$ is the underlying real tangent bundle of T_X, and $T_{X,\mathbb{C}} = T_{X,\mathbb{R}} \otimes \mathbb{C}$.

[6] However, not as a holomorphic vector bundle.

Remark 1.5.1. We look it locally, the dual of $\frac{\partial}{\partial z_j}$ is

$$\mathrm{d}z_j = \mathrm{d}x_j + i\mathrm{d}y_j \in \Omega_X^{1,0}$$

and the dual of $\frac{\partial}{\partial \overline{z_j}}$ is

$$\mathrm{d}\overline{z_j} = \mathrm{d}x_j - i\mathrm{d}y_j \in \Omega_X^{0,1}$$

For any $\alpha \in C^\infty(X, \Omega^1_{X,\mathbb{C}})$, locally we have

$$\alpha = \sum \alpha_j \mathrm{d}x_j + \sum \beta_j \mathrm{d}y_j$$

then we can decompose it into

$$\alpha = \sum \frac{1}{2}(\alpha_j - i\beta_j)\mathrm{d}z_j + \sum \frac{1}{2}(\alpha_j + i\beta_j)\mathrm{d}\overline{z_j}$$

where the first part lies in $\Omega_X^{1,0}$ and the later part lies in $\Omega_X^{0,1}$.

Definition 1.5.2 (differential k-form)**.** *A k-form α of type (p,q) is a differential section of $\Omega_X^{p,q}$, that is*

$$\alpha \in C^\infty(X, \Omega_X^{p,q}) \subset C^\infty(X, \Omega^k_{X,\mathbb{C}})$$

Example 1.5.3. For $\mathbb{C}^n \cong \mathbb{R}^{2n}$, we have[7]

$$w = \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}x_n = (\frac{i}{2})^n \mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\overline{z_n}$$

1.6. **Exterior differential.** Recall: Let X be a differential manifold, with real dimension n, we have exterior differential

$$\mathrm{d} : C^\infty(X, \Omega^k_{X,\mathbb{R}}) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{R}})$$

such that

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \mathrm{d}\beta$$

Exterior differential has an important property:

$$\mathrm{d}^2 = 0$$

So we can consider such chain complex

$$0 \to C^\infty(X, \Omega^0_{X,\mathbb{R}}) \to C^\infty(X, \Omega^1_{X,\mathbb{R}}) \to C^\infty(X, \Omega^2_{X,\mathbb{R}}) \to \cdots \to C^\infty(X, \Omega^n_{X,\mathbb{R}}) \to 0$$

with de Rham cohomology group

$$H^k(X, \mathbb{R}) := Z^k(X, \mathbb{R}) / B^k(X, \mathbb{R})$$

The following theorem implies that the de Rham cohomology is just a topological data.

Theorem 1.6.1 (comparision theorem)**.** $H^k(X, \mathbb{R})$ *computes the singular cohomology of X with real coefficient.*

[7]By induction on n

Theorem 1.6.2 (Poincaré lemma). *Let $X = B(x_0, r_0) \subset \mathbb{R}^n$ is a open ball, then $H^k(X, \mathbb{R}) = 0, \forall k > 0$.*

Remark 1.6.3. Poincaré lemma implies that for small enough open set, the cohomology groups are trivial, so only for global differential forms, de Rham cohomology tells interesting information.

Now Let X be a complex manifold, with complex dimension n, then

$$\mathrm{d} : C^\infty(X, \Omega^k_{X,\mathbb{C}}) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{C}})$$

Example 1.6.4. For $\alpha \in C^\infty(X, \Omega^0_{X,\mathbb{C}})$, then

$$\mathrm{d}\alpha \in C^\infty(X, \Omega^1_{X,\mathbb{C}}) = C^\infty(X, \Omega^{1,0}_X) \oplus C^\infty(X, \Omega^{0,1}_X)$$

Locally, we have

$$\begin{aligned}
\mathrm{d}\alpha &= \sum \frac{\partial \alpha}{\partial x_j}\mathrm{d}x_j + \sum \frac{\partial \alpha}{\partial y_j}\mathrm{d}y_j \\
&= \sum \frac{1}{2}(\frac{\partial \alpha}{\partial x_j} - i\frac{\partial \alpha}{\partial y_j})\mathrm{d}z_j + \sum \frac{1}{2}(\frac{\partial \alpha}{\partial x_j} + i\frac{\partial \alpha}{\partial y_j})\mathrm{d}\overline{z_j} \\
&= \sum \frac{\partial \alpha}{\partial z_j}\mathrm{d}z_j + \sum \frac{\partial \alpha}{\partial \overline{z_j}}\mathrm{d}\overline{z_j}
\end{aligned}$$

More generally, for $\alpha \in C^\infty(X, \Omega^{p,q}_X)$, then locally

$$\alpha = \sum_{|J|=p,|K|=q} \alpha_{J,K}\mathrm{d}z_J \wedge \mathrm{d}\overline{z_K}$$

then

$$\mathrm{d}\alpha = \sum_{|J|=p,|K|=q} \frac{\partial \alpha_{J,K}}{\partial z_l}\mathrm{d}z_l \wedge \mathrm{d}z_J \wedge \mathrm{d}\overline{z_K} + \sum_{|J|=p,|K|=q} \frac{\partial \alpha_{J,K}}{\partial \overline{z_l}}\mathrm{d}\overline{z_l} \wedge z_J \wedge \overline{z_K}$$

Definition 1.6.5 (partial operator). *For $\alpha \in C^\infty(X, \Omega^{p,q}_X)$, we can define partial operator and its conjugation $\partial\alpha, \overline{\partial}\alpha$ as follows*

$$\mathrm{d}\alpha = \partial\alpha + \overline{\partial}\alpha$$

where $\partial\alpha \in C^\infty(X, \Omega^{p+1,q}_X), \overline{\partial}\alpha \in C^\infty(X, \Omega^{p,q+1}_X)$

More generally, if $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{C}})$, write $\alpha = \sum \alpha^{p,q}$, then

$$\partial\alpha = \sum_{p,q} \partial\alpha^{p,q}, \quad \overline{\partial}\alpha = \sum_{p,q} \overline{\partial}\alpha^{p,q}$$

Remark 1.6.6. We have the following relations

1. Leibniz rule

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{\deg \alpha}\alpha \wedge \partial\beta$$

2. [8]

$$\partial^2 = \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0$$

[8]Hint: consider $\mathrm{d}^2 = (\partial + \overline{\partial})^2 = 0$

So we can do the same thing for ∂ by consider the following chain complex

$$0 \to C^\infty(X, \Omega_X^{p,0}) \xrightarrow{\overline{\partial}} C^\infty(X, \Omega_X^{p,1}) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} C^\infty(X, \Omega_X^{p,n}) \to 0$$

Definition 1.6.7 (Dolbeault cohomology)**.**

$$H^{p,q} := Z^{p,q}(X)/B^{p,q}(X) = H^q_{\overline{\partial}}(C^\infty(X, \Omega_X^{p,*}))$$

Key question: Since we have $C^\infty(X, \Omega^k_{X,\mathbb{C}}) = \bigoplus_{p+q=k} C^\infty(X, \Omega_X^{p,q})$, could we have the following decomposition?

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

Example 1.6.8. What is $H^{p,0}(X)$? Since $B^{p,0} = 0$, then

$$H^{p,0}(X) = Z^{p,0}(X) = \{\alpha \in C^\infty(X, \Omega_X^{p,0}) \mid \overline{\partial}\alpha = 0\}$$

Locally $\alpha = \sum_{|J|=p} \alpha_J \mathrm{d}z_J$, then

$$\overline{\partial}\alpha = \sum_{|J|=p} \frac{\partial \alpha_J}{\partial \overline{z_k}} \mathrm{d}\overline{z_k} \wedge \mathrm{d}z_J = 0 \implies \frac{\partial \alpha_J}{\partial \overline{z_k}} = 0$$

That is, α_J is holomorphic function. Since $\Omega_X^{p,0} \cong \Omega_X^p$ as complex vector bundle, we have $H^{p,0}(X) = \Gamma(X, \Omega_X^p)$. [9]

Example 1.6.9. For a holomorphic map $f : X \to Y$ between complex manifold, then

$$f^* : C^\infty(Y, \Omega^k_{Y,\mathbb{C}}) \to C^\infty(X, \Omega^k_{X,\mathbb{C}})$$

Then[10]

$$f^* : C^\infty(Y, \Omega^{p,q}_{Y,\mathbb{C}}) \to C^\infty(X, \Omega^{p,q}_{X,\mathbb{C}})$$

and

$$f^* : H^{p,q}(Y) \to H^{p,q}(X)$$

so Dolbeault cohomology is a contravariant functor.

Example 1.6.10. Dolbeault cohomology of a holomorphic vector bundle[11] $E \to X$, define

$$\overline{\partial_E} : C^\infty(X, \Omega_X^{0,q} \otimes E) \to C^\infty(X, \Omega_X^{0,q+1} \otimes E)$$

satisfies $\overline{\partial_E}^2 = 0$, so we can construct a chain complex and define its cohomology, denoted by

$$H^q(X, E) = H^q_{\overline{\partial_E}}(C^\infty(X, \Omega_X^{0,*} \otimes E))$$

and we can calculate

$$H^0(X, E) = \Gamma(X, E)$$

[9] This implies that Dolbeault cohomology computes useful information indeed.

[10] Check this, we need back to definition, a holomorphic map induces a tangent map $T_f : T_{X,\mathbb{C}} \to f^* T_{Y,\mathbb{C}}$,and consider its dual we get cotangent map $\Omega_f : f^*\Omega_{Y,\mathbb{C}} \to \Omega_{X,\mathbb{C}}$

[11] In previous, $E = \Omega_X^p$

Theorem 1.6.11 (Dolbeault lemma)**.** *Let $X = D(z_0, r_0) \subset \mathbb{C}^n$ be a polydisk, then*

$$H^{p,q}(X) = 0, \quad \forall p \geq 0, q > 0$$

1.7. **Čech cohomology.** Let X be a topological space, and $\mathcal{U} = (U_i)_{i \in I}$ be an open covering, such that I is countable and an ordered set. For all $i_0, \dots, i_p \in I$ write

$$U_{i_0 \dots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$$

Let $\mathscr{F}$ be a sheaf of abelian group, define a chain complex $C^*(\mathcal{U}, \mathscr{F})$ as

$$0 \to C^0(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta} \dots$$

where

$$C^p = \prod_{i_0 < \cdots < i_p} \mathscr{F}(U_{i_0 \dots i_p})$$

and δ is defined as

$$(\delta\alpha)_{i_0 \dots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \dots \widehat{i_k} \dots i_{p+1}} |_{U_{i_0 \cdots i_{p+1}}}$$

Exercise 1.7.1 (Once and only once exercise in your whole life)**.** Check that $\delta \circ \delta = 0$

So we can define Čech cohomology as

$$\check{H}^q(\mathcal{U}, \mathscr{F}) := H^q_\delta(C^*(\mathcal{U}, \mathscr{F}))$$

Example 1.7.2. We consider

$$\check{H}^0(\mathcal{U}, \mathscr{F}) = \{\alpha \in C^0(\mathcal{U}, \mathscr{F}) \mid \delta\alpha = 0\}$$

then if $\alpha = \prod_{i_0} \alpha_{i_0}$, then $\delta\alpha = 0$ implies

$$\alpha_i|_{U_i \cap U_j} = \alpha_j|_{U_i \cap U_j}$$

then we have

$$\check{H}^0(\mathcal{U}, \mathscr{F}) = \mathscr{F}(X)$$

However, we want out definition is independent of open cover, so

Definition 1.7.3. *We define Čech cohomology as*

$$\check{H}^q(X, \mathscr{F}) = \varinjlim_{\mathcal{U}} \check{H}^q(\mathcal{U}, \mathscr{F})$$

Remark 1.7.4. In other words, $\alpha = \alpha' \in \check{H}^q(X, \mathscr{F})$ is equivalent to there exists a common refinement $\mathcal{U}''$ such that

$$\alpha = \alpha' \in \check{H}^q(\mathcal{U}'', \mathscr{F})$$

Why we want to introduce Čech cohomology here? In fact, it provides a method to compute de Rham cohomology and Dolbeault cohomology we defined before. However, we will use a cohomology theory to unify all cohomology theory later.

Recall that if X is a complex manifold, $E \to X$ is a holomorphic vector bundle. Then we can define a sheaf of holomorphic sections, defined by

$$\mathcal{O}_X(E) : U \mapsto \Gamma(U, E|_U)$$

then we get a Čech cohomology of this sheaf

$$\check{H}^q(X, \mathcal{O}_X(E)) = \varinjlim_{\mathcal{U}} \check{H}^q(\mathcal{U}, \mathcal{O}_X(E))$$

Theorem 1.7.5 (comparision)**.** *We have the following isomorphism*

$$\check{H}^q(X, \mathcal{O}_X(E)) \cong H^q(X, E) = H^q_{\overline{\partial}_E}(C^\infty(X, \Omega^{0,*}_X \otimes E))$$

In particular, let $E = \Omega^p_X$, we have

$$\check{H}^q(X, \mathcal{O}_X(\Omega^p_X)) \cong H^{p,q}(X) = H^q_{\overline{\partial}}(C^\infty(X, \Omega^{p,*}_X))$$

Remark 1.7.6. In fact, Theorem 2.7.5 uses the sequence of sheaves

$$0 \to E \to \Omega^{0,0}_X \otimes E \xrightarrow{\overline{\partial_E}} \Omega^{0,1}_E \otimes E \xrightarrow{\overline{\partial_E}} \cdots \xrightarrow{\overline{\partial_E}} \Omega^{0,n}_X \otimes E \to 0$$

And Dolbeault lemma implies the above sequence is exact. That's what behind the comparision theorem, and tells us why Dolbeault cohomology is about holomorphic information, since here we use $\mathcal{O}_X(\Omega^p_X)$, sheaf of holomorphic sections.

Similarly, in de Rham cohomology, we have the same story. There is a sequence of sheaves

$$0 \to \underline{\mathbb{C}} \xrightarrow{i} \Omega^0_{X,\mathbb{C}} \xrightarrow{\mathrm{d}} \Omega^1_{X,\mathbb{C}} \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n_{X,\mathbb{C}} \to 0$$

where $\underline{\mathbb{C}}$ is the sheaf of locally constant functions, i.e.

$$\underline{\mathbb{C}} : U \mapsto \{\text{locally constant functions } f : U \to \mathbb{C}\}$$

Then Poincaré lemma implies the above sequence is also exact. Parallel to Theorem 2.7.5, we will get

$$\check{H}^q(X, \underline{\mathbb{C}}) \cong H^k(X, \mathbb{C}) = H^k_{dR}(C^\infty(X, \Omega^*_{X,\mathbb{C}}))$$

This also explain why de Rham cohomology is just a topological information, since here we just use $\underline{\mathbb{C}}$, a pure topological information.

Theorem 1.7.7 (Leray)**.** *Let $\mathcal{U}$ be a covering such that for all $i_0 \dots i_k \in I$, and for all $q > 0$, we have*

$$H^q(U_{i_0 \dots i_k}, E|_{U_{i_0 \dots i_k}}) = 0$$

then $\mathcal{U}$ is called acyclic for E. Then

$$\check{H}^q(\mathcal{U}, \mathcal{O}_X(E)) \cong H^q(X, E)$$

Remark 1.7.8. This provides us a practical way to compute Čech cohomology.

Example 1.7.9. Consider $\mathcal{O}_X^\times \subset \mathcal{O}_X$, the sheaf of invertible holomorphic functions.

Then we have

$$\check{H}^1(X, \mathcal{O}_X^\times) \cong \mathrm{Pic}(X)$$

2. Geometry of vector bundles

2.1. Connections.

Definition 2.1.1 (connection)**.** *X is a differential manifold, and $\pi : E \to X$ is a complex vector bundle. A connection on E is a $\mathbb{C}$-linear operator*

$$D : C^\infty(X, E) \to C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes E)$$

satisfying the Leibiz rule

$$D(f\sigma) = \mathrm{d}f \otimes \sigma + fD(\sigma)$$

for $f \in C^\infty(X)$ and $\sigma \in C^\infty(X, E)$.

Remark 2.1.2. In fact, if we ask D to satisfy the Leibniz rule, it induces

$$D : C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes E) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{C}} \otimes E)$$

for any k, by setting

$$D(\varphi \otimes \sigma) = \mathrm{d}\varphi \otimes \sigma + (-1)^{\deg \varphi} \varphi \wedge (D\sigma)$$

for $\varphi \in C^\infty(X, \Omega^k_{X,\mathbb{C}})$ and $\sigma \in C^\infty(X, E)$.

Remark 2.1.3. Let's see what's going on in local pointview.

Locally around $x \in U \subset X$, then $\pi^{-1}(U) \cong U \times \mathbb{C}^r$, there is a basis $\{e_1, \dots, e_n\}$ for $\mathbb{C}^r$. For $\sigma \in C^\infty(U, E|_U)$, we have

$$\sigma = \sum_{j=1}^r s_j e_j, \quad s_j \in C^\infty(U)$$

By Leibniz rule, we have

$$D\sigma = \sum_{j=1}^r (\mathrm{d}s_j \otimes e_j + s_j De_j)$$

where $De_j \in C^\infty(U, \Omega^1_{U,\mathbb{C}} \otimes E)$. So we can write more explictly as

$$De_j = \sum_{i=1}^r a_{ij} \otimes e_i, \quad a_{ij} \in C^\infty(U, \Omega^1_{U,\mathbb{C}})$$

So we have

$$D\sigma = \sum_{j=1}^r (\mathrm{d}s_j \otimes e_j + \sum_{i=1}^r a_{ij} s_j \otimes e_i)$$

We can rewrite the above formula in frame of $\{e_1, \dots, e_r\}$ as

$$D\sigma = Ds = \mathrm{d}s + As$$

where

$$\sigma = s = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}, \quad A = (a_{ij}) \in C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes \mathrm{End}(E|_U))$$

Here we chose a local trivialization of the vector bundle E, so we may wonder what will happen if we change our choice.

If $x \in U' \subset X$ is another trivialization, so $\pi^{-1}(U') \cong U' \times \mathbb{C}^r$, and $\{e'_1, \dots, e'_r\}$ is another local frame. Then

$$D\sigma = \begin{cases} Ds = \mathrm{d}s + As \\ Ds' = \mathrm{d}s' + A's' \end{cases}$$

so we wonder the relationship between A and A'. Transition functions between U and U' are

$$g : U \cap U' \to \mathrm{GL}(r, \mathbb{C})$$

so we have $s = gs'$ and $Ds = gDs'$. We compute as follows

$$\begin{aligned} \mathrm{d}s &= \mathrm{d}(gs') = (\mathrm{d}g)s' + g(\mathrm{d}s') = g(g^{-1}(\mathrm{d}g)s' + \mathrm{d}s') \\ \mathrm{d}s + As &= g(g^{-1}(\mathrm{d}g)s' + \mathrm{d}s' + g^{-1}Ags') \\ &= g(\mathrm{d}s' + (g^{-1}\mathrm{d}g + g^{-1}Ag)s') \end{aligned}$$

Since we have

$$\mathrm{d}s + As = gDs' = g(\mathrm{d}s' + A's')$$

So we have

$$A' = g^{-1}\mathrm{d}g + g^{-1}Ag$$

You may feel quite uncomfortable since A' does not conjugate to A under the change of the frame, but if we apply D twice, something interesting may happen.

Before that, we compute what does $D : C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes E) \to C^\infty(X, \Omega^2_{X,\mathbb{C}} \otimes E)$ look like locally:

Take $\sigma \in C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes E)$, locally we can write as

$$\sigma = \sum_{j=1}^r s_j \otimes e_j, \quad s_j \in \Omega^1_{X,\mathbb{C}}$$

then

$$
\begin{aligned}
D\sigma &= D(\sum_{j=1}^{r} s_j \otimes e_j) \\
&= \sum_{j=1}^{r} \mathrm{d}s_j \otimes e_j - s_j \wedge De_j \\
&= \sum_{j=1}^{r} \mathrm{d}s_j \otimes e_j - s_j \wedge \sum_{i=1}^{r} a_{ij} \otimes e_j \\
&= \sum_{j=1}^{r} \mathrm{d}s_j + \sum_{i=1}^{r} a_{ij} \wedge s_j \otimes e_j \\
&= \mathrm{d}s + A \wedge s
\end{aligned}
$$

So we know that $D : C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes E) \to C^\infty(X, \Omega^2_{X,\mathbb{C}} \otimes E)$ still looks like[12]

$$D\sigma = Ds = \mathrm{d}s + A \wedge s$$

Here we can see clearly what does $A \wedge s$ look like. Furthermore, we can see that $A \wedge A$ isn't trivial, unless in the case of line bundle.

So we can compute as follows.

$$
\begin{aligned}
D^2\sigma = D(\mathrm{d}s + As) &= \mathrm{d}(\mathrm{d}s + As) + A \wedge (\mathrm{d}s + As) \\
&= \mathrm{d}^2 s + \mathrm{d}(As) + A \wedge \mathrm{d}s + A \wedge As \\
&= \mathrm{d}^2 s + (\mathrm{d}A)s - A \wedge \mathrm{d}s + A \wedge \mathrm{d}s + A \wedge As \\
&= (\mathrm{d}A + A \wedge A)s
\end{aligned}
$$

And we check what will happen if we choose another trivialization

$$
\begin{aligned}
D^2\sigma = D^2 s = (\mathrm{d}A + A \wedge A)s &= (\mathrm{d}A + A \wedge A)gs' \\
&= gD^2 s' = g(\mathrm{d}A' + A' \wedge A')s'
\end{aligned}
$$

so we have

$$\mathrm{d}A' + A' \wedge A' = g^{-1}(\mathrm{d}A + A \wedge A)g$$

that is, $\mathrm{d}A + A \wedge A$ behaves "well" under the change of frame, object with such property we always call it a "tensor"[13].

From discussion above, we can give the following definition

Definition 2.1.4 (curvature)**.** *There exists a global section $H_D \in C^\infty(X, \Omega^2_{X,\mathbb{C}} \otimes \mathrm{End}(E))$ such that*

$$D^2\sigma = H_D \wedge \sigma, \quad \forall \sigma \in C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes E)$$

such H_D is called the curvature tensor of connection D.

[12]In general, D always looks like $\mathrm{d} + A\wedge$ locally, A can be seen as a kind of structure coefficient.

[13]But what is a "tensor"? Here I quote a motto said by Leonard Susskind, a well-known physicist. I'm quite impressed when I first heard it in my childhood. "Tensor is something which behaves like a tensor."

Definition 2.1.5 (Hermitian metric)**.** *X is a differential manifold, and $\pi : E \to X$ is a complex vector bundle. A Hermitian metric h on E is a hermitian inner product on each fiber E_x, such that for all open subset $U \subset X$, and $\xi, \eta \in C^\infty(U, E|_U)$, we have*

$$\begin{aligned}\langle \xi, \eta \rangle : U &\to \mathbb{C} \\ x &\mapsto \langle \xi(x), \eta(x) \rangle\end{aligned}$$

is a C^∞-function.

Example 2.1.6. Locally, for $x \in U \subset X$, we have $\pi^{-1}(U) \cong U \times \mathbb{C}^r$, and $\{e_1, \ldots, e_r\}$ is a local frame. Then our Hermitian metric is just a matrix

$$H = (h_{\lambda\mu})$$

where $h_{\lambda\mu} \in C^\infty(U)$, defined by

$$h_{\lambda\mu}(x) = \langle e_\lambda(x), e_\mu(x) \rangle$$

In our local frame, two sections $\xi, \eta \in C^\infty(U, E|_U)$ can be write as

$$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}$$

We have

$$h(\xi, \eta) = (\xi_1, \ldots, \xi_n) H \begin{pmatrix} \overline{\eta_1} \\ \vdots \\ \overline{\eta_n} \end{pmatrix} = \xi^t H \overline{\eta}$$

And take another $x \in U' \subset X$, $\pi^{-1}(U') \cong U' \times \mathbb{C}^r$, with $\{e_1', \ldots, e_r'\}$, with g is the transition function, we have

$$H' = g^t H \overline{g}$$

Proposition 2.1.7. *Every complex vector bundle admits a Hermitian metric*

Proof. Use partition of unity. □

Now for a complex vector bundle over a differential manifold, we have two structures on it, connection and Hermitian metric, so it's natural to require them exist in a harmony.

For a Hermitian metric h, it induces a pairing $\{\cdot, \cdot\}$

$$C^\infty(X, \Omega^p_{X,\mathbb{C}} \otimes E) \times C^\infty(X, \Omega^q_{X,\mathbb{C}} \otimes E) \to C^\infty(X, \Omega^{p+q}_{X,\mathbb{C}})$$

Locally, consider $x \in U \subset X$, $\pi^{-1}(U) \cong U \times \mathbb{C}^r$, and $\{e_1, \ldots, e_r\}$ is a local frame. Then $\sigma \in C^\infty(X, \Omega^p_{X,\mathbb{C}} \otimes E)$ and $\eta \in C^\infty(X, \Omega^q_{X,\mathbb{C}} \otimes E)$ are in form

$$\sigma = \sum_{j=1}^r s_j \otimes e_j, \quad \tau = \sum_{j=1}^r t_j \otimes e_j$$

where s_j are p-forms and t_j are q-forms, then the pairing is locally look like

$$\sum_{i,j=1}^{r} s_i \wedge t_j h_{ij} = s^t H \bar{t}$$

Use this pairing, we can define when a connection is called Hermitian.

Definition 2.1.8 (Hermitian connection)**.** *(E, h) is a Hermitian vector bundle on X. A connection D on E is Hermitian if for all $\sigma \in C^\infty(X, \Omega^p_{X,\mathbb{C}} \otimes E), \eta \in C^\infty(X, \Omega^q_{X,\mathbb{C}} \otimes E)$,*

$$\mathrm{d}\{\sigma, \tau\} = \{D\sigma, \tau\} + (-1)^{\deg \sigma}\{\sigma, D\tau\}$$

Since we know that a connection locally looks like $D = \mathrm{d} + A\wedge$. Then let's compute in a local frame to show what condition A needs to satisfy for a Hermitian connection.

Take $\sigma = s = (s_1, \ldots, s_n)^T, \tau = t = (t_1, \ldots, t_n)^T$. WLOG, we assume $\{e_1, \ldots, e_r\}$ is a orthonormal basis, i.e. H is identity matrix, then

$$\{\sigma, \tau\} = s^t \bar{t}$$

then we compute

$$\begin{aligned}
\mathrm{d}\{\sigma, \tau\} &= (\mathrm{d}s)^t \wedge \bar{t} + (-1)^{\deg \sigma} s^t \wedge \mathrm{d}\bar{t} \\
\{D\sigma, \tau\} &= (\mathrm{d}s + A \wedge s)^t = (\mathrm{d}s)^t \wedge \bar{t} + (-1)^{\deg \sigma} s^t \wedge A^t \wedge \bar{t} \\
\{\sigma, D\tau\} &= s^t \wedge \overline{\mathrm{d}t + A \wedge t} = s^t \wedge \mathrm{d}\bar{t} + s^t \wedge \overline{A} \wedge \bar{t}
\end{aligned}$$

then

$$\mathrm{d}\{\sigma, \tau\} - \{D\sigma, \tau\} - \{\sigma, D\tau\} = (-1)^{\deg \sigma} s^t \wedge (A^t + \overline{A}^t) \wedge \bar{t}$$

So D is a Hermitian connection if and only if $A^t + \overline{A} = 0$.

Let's make it more beautiful. We define D^{adj}, adjoint connection, locally given by $-\overline{A}^t$ with respect to $H = I$, then we always have

$$\mathrm{d}\{\sigma, \tau\} = \{D\sigma, \tau\} + (-1)^{\deg \sigma}\{\sigma, D^{adj}\tau\}$$

Take $\frac{1}{2}(D + D^{adj})$, which is also a connection, locally looks like

$$\frac{1}{2}(A - \overline{A}^t)$$

is a Hermitian connection. So it's easy to get a Hermitian connection, just average A with its adjoint.

Proposition 2.1.9. *Every Hermitian vector bundle admits a Hermitian connection.*

Proof. Use partition of unity to show the existence of connection, and take average. □

2.2. **Connections and metrics on holomorphic vector bundles.** In section, let's see when the base space is a complex manifold, and the vector bundle is holomorphic, what will happen?

Recall that for a complex manifold X, we have

$$\Omega^1_{X,\mathbb{C}} = \Omega^{1,0}_X \oplus \Omega^{0,1}_X$$

Consider $E \to X$ is a complex vector bundle, and D is a connection, then we can decompose $D = D^{1,0} + D^{0,1}$ as

$$\begin{tikzcd}
 & & C^\infty(X, \Omega^{1,0}_X \otimes E) \\
C^\infty(X,E) \arrow[r, "D"] & C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes E) \arrow[ur] \arrow[dr] & \\
 & & C^\infty(X, \Omega^{0,1}_X \otimes E)
\end{tikzcd}$$

Locally, we have $D = \mathrm{d} + A$, then

$$D^{1,0} = \partial + A^{1,0}, \quad D^{0,1} = \overline{\partial} + A^{0,1}$$

both $D^{1,0}$ and $D^{0,1}$ satisfy Leibniz rule.

Now consider X is a complex manifold, and $E \to X$ is a holomorphic vector bundle. Recall that we already have

$$\overline{\partial}_E : C^\infty(X,E) \to C^\infty(X, \Omega^{0,1}_X \otimes E)$$

We want to compare $D^{0,1}_E$ and $\overline{\partial}_E$

Theorem 2.2.1 (Chern connection)**.** *X is a complex manifold, (E,h) is a Hermitian holomorphic vector bundle, then there exists a unique Hermitian connection D_E such that $D^{0,1}_E = \overline{\partial}_E$. D_E is called the Chern connection of (E,h).*

Proof. Uniqueness: locally $x \in U \subset X$, $\{e_1, \dots, e_r\}$ is holomorphic local frame. And smooth section $\sigma = s = (s_1, \dots, s_n)^t$. Then

$$D_E\sigma = \mathrm{d}s + As$$

$$D^{0,1}_E \sigma = \overline{\partial} s + A^{0,1}s = \overline{\partial}_E \sigma$$

If s is a holomorphic section, then $\overline{\partial}_E \sigma = \overline{\partial} s = 0$, implies $A^{0,1} = 0$.

Since we have

$$\mathrm{d}H = A^t H + H\overline{A}$$

then

$$\overline{\partial} H = H\overline{A}$$

So A is uniquely determined by

$$A = \overline{H^{-1}}\partial\overline{H}$$

Existence: It suffices to prove we can glue A together to get a global connection, i.e. compatible with holomorphic change of frames.

Consider another holomorphic local chart $x \in U' \subset X$, with frame $\{e'_1, \ldots, e'_r\}$. And the metric with respect to this new frame is H', we have

$$H' = g^t H \overline{g}$$

Then

$$\begin{aligned} A' = \overline{H'^{-1}} \partial \overline{H'} &= g^{-1} \overline{H^{-1} (g^t)^{-1}} \partial (\overline{g^t} \overline{H} g) \\ &= g^{-1} \overline{H^{-1} (g^t)^{-1}} ((\partial \overline{g^t}) + \overline{g^t} (\partial \overline{H}) g + \overline{g^t} \overline{H} \partial g) \\ &= g^{-1} \overline{H^{-1}} (\partial \overline{H} g) + g^{-1} \mathrm{d} g \\ &= g^{-1} \mathrm{d} g + g^{-1} A g \end{aligned}$$

As we desire. □

Corollary 2.2.2. *If X is a complex manifold, (E,h) is a Hermitian holomorphic vector bundle. D_E is Chern connection on it, and H_E is Chern curvature. A is the matrix of D_E with respect to holomorphic local frame, then*

1. *A is of type $(1,0)$, with $\partial A = -A \wedge A$*
2. *locally $\overline{\partial} A$ of type $(1,1)$*
3. *$\overline{\partial} H_E = 0$*

Proof. Locally we have $A = \overline{H}^{-1} \partial \overline{H}$, so it's of type $(1,0)$, and we compute

$$\begin{aligned} \partial A = \partial (\overline{H}^{-1} \partial \overline{H}) &= \partial \overline{H}^{-1} \wedge \partial \overline{H} \\ &= (-\overline{H}^{-1} \partial H \overline{H}^{-1}) \wedge \partial \overline{H} \\ &= -(\overline{H}^{-1} \partial \overline{H}) \wedge (\overline{H}^{-1} \partial \overline{H}) \\ &= -A \wedge A \end{aligned}$$

Chern curvature locally looks like

$$H_E = \mathrm{d} A + A \wedge A = \mathrm{d} A - \partial A = \overline{\partial} A$$

, which is of type $(1,1)$. And clearly $\overline{\partial} H_E = 0$. □

Exercise 2.2.3. (E,h) is a Hermitian holomorphic vector bundle, and $S \hookrightarrow E$ is a holomorphic subbundle. $S^\perp$ is defined by $(S^\perp)_x = (S_x)^\perp$ with respect to h. We have $E = S \oplus S^\perp$ as complex vector bundle.[14] We have a projection

$$P_s : C^\infty (X, \Omega^k_{X,\mathbb{C}} \otimes E) \to C^\infty (X, \Omega^k_{X,\mathbb{C}} \otimes S)$$

Show that $D_S = P_s \circ D_E$.

[14]If we have a short exact sequence of holomorphic vector bundle

$$0 \to S \to E \to Q \to 0$$

this exact sequence generally won's split. That's why we perfer short exact sequence rather than direct sum in algebraic geometry.

2.3. **Case of Line bundle.** In this case we will consider a special case, i.e. line bundle, to find some interesting things.

Recall that if X is a differential manifold, and $\pi : X \to L$ is a complex line bundle, D is a connection on L. Since

$$H_D \in C^\infty(X, \Omega^2_{X,\mathbb{C}} \otimes \mathrm{End}(L))$$

But for line bundle, $\mathrm{End}(L) \cong L^* \otimes L \cong X \times \mathbb{C}$ is just trivial bundle. So in fact, $H_D \in C^\infty(X, \Omega^2_{X,\mathbb{C}})$, that is, curvature of connection is exactly a 2-form, without coefficient.

Furthermore, in a local pointview, D is represented by 1-form A, then $H_D = \mathrm{d}A + A \wedge A = \mathrm{d}A$, and for a line bundle, we clearly have $A \wedge A = 0$, since A is just a 1×1 matrix, and forms are skew symmetric.

Then $\mathrm{d}H_D = 0$, i.e. H_D is a closed form, so we get

$$[H_D] \in H^2(X, \mathbb{C})$$

de Rham cohomology.

Now it's natural to ask what's the relationship between closed form from different connections. A surprising result is that they exactly lie in the same cohomology class.

If we consider another connection $\widetilde{D}$, and $\widetilde{A}$, let's compare H_D with $H_{\widetilde{D}}$. For all $\sigma \in C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes L)$, locally $\sigma = s$ with respect to $\{e_1, \ldots, e_r\}$.

Then

$$\begin{aligned} D(\sigma) - \widetilde{D}(\sigma) &= (\mathrm{d}s + A \wedge s) - (\mathrm{d}s - \widetilde{A} \wedge s) \\ &= (A - \widetilde{A}) \wedge s \\ &= B \wedge \sigma \end{aligned}$$

where $B = (A - \widetilde{A}) \in C^\infty(X, \Omega^1_{X,\mathbb{C}})$.

Then

$$H_D - H_{\widetilde{D}} = \mathrm{d}B$$

So different connections give the same cohomology class in $H^2(X, \mathbb{C})$. What a beautiful result!

Definition 2.3.1 (First Chern class)**.** *Let $\pi : E \to X$ be a complex line bundle, D is any connection. Then define*

$$c_1(L) := [\frac{i}{2\pi} H_D] \in H^2(X, \mathbb{C})$$

, which is called the first Chern class of line bundle.

Remark 2.3.2. The first Chern class is a property of line bundle itself, and independent of connections on it. In other words, it's a topological information.

Let's explain why we need coefficient here.

Lemma 2.3.3. *(E, h) is a Hermitian line bundle, and D is a Hermitian connection, then*

$$\frac{i}{2\pi} H_D \in C^\infty(X, \Omega^2_{X,\mathbb{R}})$$

hence $c_1(L) \in H^2(X, \mathbb{R})$.

Proof. Locally we have $\overline{A} = -A$

$$\overline{\frac{i}{2\pi} H_D} = -\frac{i}{2\pi}\overline{H_D} = -\frac{i}{2\pi}\overline{\mathrm{d}A} = -\frac{i}{2\pi}\mathrm{d}\overline{A} = \frac{i}{2\pi}\mathrm{d}A = \frac{i}{2\pi} H_D$$

□

Remark 2.3.4. However, why we need 2π here? Later we will see in fact $c_1(L) \in H^2(X, \mathbb{Z})$.

Exercise 2.3.5. Let $E \to X$ be a complex vector bundle of rank r, define

$$c_1(E) := c_1(\det E)$$

If $L \to X$ is a complex line bundle, Show that

$$c_1(E \otimes L) = c_1(E) + rc_1(L)$$

Now, Let's combine all we have together, to see what will happen. Let X be a complex manifold and L be a holomorphic line bundle, with a Hermitian metric. D_L is the Chern connection of L, and its Chern curvature is H_L.

By Corollary 3.2.2 and Lemma 3.3.3, we have

$$\frac{i}{2\pi} H_L \in C^\infty(X, \Omega^2_{X,\mathbb{R}}) \cap C^\infty(X, \Omega^{1,1}_X)$$

such that

$$\mathrm{d}(\frac{i}{2\pi} H_L) = \overline{\partial}(\frac{i}{2\pi} H_L) = 0$$

that is

$$[\frac{i}{2\pi} H_L] \in H^2(X, \mathbb{R}), \quad [\frac{i}{2\pi} H_L] \in H^{1,1}(X)$$

Exercise 2.3.6. Show that $[\frac{i}{2\pi} H_L] \in H^{1,1}(X)$ is independent of h.

Example 2.3.7. Locally we have $x \in U \subset X$, with $\pi^{-1}(U) \cong U \times \mathbb{C}$, $\{e_1\}$ is the local frame. Then Hermitian metric is

$$H(z) = \langle e_1(z), e_1(z) \rangle = \|e_1(z)\|_h^2$$

Write $\varphi(z) = -\log H(z)$, a function $U \to \mathbb{R}$. Then

$$A = \overline{H^{-1}} \partial \overline{H} = e^{\varphi(z)} \partial e^{-\varphi(z)} = -\partial\varphi(z)$$

then

$$H_L = \overline{A} = -\overline{\partial}\partial\varphi(z) = \partial\overline{\partial}\varphi(z)$$

then we have

$$\frac{i}{2\pi}H_L = \frac{i}{2\pi}\partial\overline{\partial}\varphi(z) = \frac{i}{2\pi}\partial\overline{\partial}(-\log H(z))$$
$$= \frac{1}{2\pi i}\partial\overline{\partial}\log\|e_1(z)\|_h^2$$

Summarize as follows

Proposition 2.3.8. *X is a complex manifold, (L, h) is a Hermitian holomorphic line bundle. Then $c_1(L)$ is represented by a real $(1,1)$-form, given locally by*

$$\frac{i}{2\pi}H_L = \frac{1}{2\pi i}\partial\overline{\partial}\log\|e_1(z)\|_h^2$$

Before we go into deeper, let's discuss some facts about linear algebra we will need.

Let V be a n-dimensional complex vector space, and use $V_{\mathbb{R}}$ to denote the underlying real vector space, with real dimension $2n$. And J acts on $V_{\mathbb{R}}$ as $\times i$. Then

$$V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$$

Consider its dual space $W_{\mathbb{R}} = V_{\mathbb{R}}^*$, and $W_{\mathbb{C}} = W_{\mathbb{R}} \otimes \mathbb{C} = W^{1,0} \oplus W^{0,1}$. Then $W^{1,1} = W^{1,0} \otimes W^{0,1} = W^{1,1} \subset \bigwedge^2 W_{\mathbb{C}}$.

For a Hermitian form $h : V \times V \to \mathbb{C}$, we have the following magic correspondence.

Lemma 2.3.9. *We have the following canonical correspondence*

$$\{\textit{Hermitian forms on } V\} \longleftrightarrow \{\textit{real } (1,1)\textit{-form on } V_{\mathbb{R}}\}$$

Proof. Given a Hermitian form h, then $h \mapsto \omega = -\operatorname{Im}(h)$; Conversely, given such a form ω, we define

$$h = \omega(\cdot, J\cdot) - i\omega(\cdot,\cdot)$$

Now let's check it: In one direction, write $h = \operatorname{Re} h + i \operatorname{Im} h$, then

$$\operatorname{Re} h(u,v) + i\operatorname{Im} h(u,v) = h(u,v) = \overline{h(v,u)} = \operatorname{Re} h(v,u) - i\operatorname{Im} h(v,u)$$

So $\operatorname{Im} h$ is skew symmetric, that is, $\omega = -\operatorname{Im} h$ is an alternating real form.

Now we need to show ω is a $(1,1)$-form. Since $W^{1,1} = (V^{1,1})^*$, where $\bigwedge^2 V_{\mathbb{C}} = V^{2,0} \oplus V^{1,1} \oplus V^{0,2}$. So we have $\omega \in W^{1,1}$ is equivalent to $\omega(V^{1,0}, V^{1,0}) = \omega(V^{0,1}, V^{0,1}) = 0$

Recall that $V^{1,0}$ is spanned by $u - iJ(u)$, then

$$\omega(u - iJ(u), v - iJ(v)) = \omega(u,v) - \omega(J(u), J(v)) - i(\omega(u, J(v)) + \omega(J(u), v))$$

Since

$$h(J(u), J(v)) = ih(u, J(v)) = i \times (-i)h(u,v) = h(u,v)$$

then

$$\omega(u,v) = \omega(J(u), J(v))$$

Similarly, we can check the last two terms cancel with each other.

On the other hand, if ω is a real $(1,1)$-form, then

$$\begin{aligned}\overline{h(u,v)} := \overline{\omega(u,J(v)) - i\omega(u,v)} &= \omega(u,J(v)) + i\omega(u,v) \\ &= -\omega(J(u),v) - i\omega(v,u) \\ &= -\omega(J^2(u),J(v)) - i\omega(v,u) \\ &= \omega(u,J(v)) - i\omega(v,u) \\ &= h(v,u)\end{aligned}$$

□

Remark 2.3.10. Though the correspondence above is canonical, we can choose a basis to see what's going on:[15]

If we choose a basis $z_1,\ldots,z_n$ of V. Then Hermitian forms on V can be write as $h = \sum_{j,k} h_{jk} z_j^* \otimes \overline{z_k}^*$, where z_k^* is the dual basis of z_k.

Then the corresponding real $(1,1)$-form is

$$\omega = \frac{i}{2}\sum_{j,k} h_{jk} z_j^* \wedge \overline{z_k}^*$$

Definition 2.3.11 (positive form)**.** *For a real $(1,1)$-form ω, it is called positive, if the corresponding Hermitian form h is positive definite.*

Definition 2.3.12 (positive line bundle)**.** *X is a complex manifold, L is a holomorphic line bundle. L is called positive if it admits a Hermitian metric h, such that*

$$\frac{i}{2\pi} H_L$$

corresponds to a Hermitian metric on the holomorphic tangent bundle T_X.

Remark 2.3.13. For any $x \in X$, then

$$\left(\frac{i}{2\pi} H_L\right)_x \in (\Omega^2_{X,\mathbb{R}} \cap \Omega^{1,1}_X)_x$$

is a real $(1,1)$-form on $(T_{X,\mathbb{R}})_x$. Then by Lemma 3.3.9, we know that there is a one to one correspondence with Hermitian form on $T_{X,x}$. So globally we have that $\frac{i}{2\pi} H_L$ will correspond to a Hermitian metric on T_X.

Locally, we have

$$\frac{i}{2\pi} H_L = \frac{i}{2\pi}\partial\overline{\partial}\varphi(z) = \frac{i}{2\pi}\sum_{j,k}\frac{\partial^2\varphi}{\partial_{z_j}\partial_{\overline{z_k}}} \mathrm{d}z_j \wedge \mathrm{d}\overline{z_k}$$

Then L is positive is equivalent to the Hermitian metric

$$\left(\frac{\partial\varphi^2}{\partial z_j \partial_{\overline{z_k}}}\right)$$

is everywhere positive definite.

[15]There may be a mistake, I will fix it later.

Exercise 2.3.14. L is positive if and only if $L^{\otimes m}$ is positive for some $m \in \mathbb{N}$.

Exercise 2.3.15. L is positive, and M is any holomorphic line bundle, then there exists $N_0 \in \mathbb{N}$ such that $M \otimes L^{\otimes N}$ positive for $N \geq N_0$.

Facts, If X is a complex complex manifold, then positive is equivalent to ample.

2.4. **Lefschetz $(1,1)$-theorem.** Now we know that given a Hermitian holomorphic line bundle (L, h), then consider its Chern curvature we will get a real $(1,1)$-form. So we may wonder the converse of this statement. Is there any real $(1,1)$-form comes from such a Hermitian holomorphic line bundle? That's main theorem for this section.

Theorem 2.4.1 (Lefschetz $(1,1)$-theorem)**.** *X is a complex manifold, $\omega \in C^\infty(X, \Omega^2_{X,\mathbb{R}}) \cap C^\infty(X, \Omega^{1,1}_X)$, a real $(1,1)$-form, such that $\mathrm{d}\omega = 0$. And*

$$[\omega] \in \mathrm{Im}(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R}))$$

Then there exists a Hermitian holomorphic line bundle (L, h) such that

$$\frac{i}{2\pi} H_L = \omega$$

Before proving this theorem, let's elaborate what does the following map mean

$$H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$$

since in de Rham cohomology, it's meaningless to say cohomology with $\mathbb{Z}$ coefficient. Here we use comparision $H^2(X, \mathbb{R}) \cong \check{H}^2(X, \underline{\mathbb{R}})$, where $\underline{\mathbb{R}}$ is the sheaf of local constant $\mathbb{R}$-functions, and prove in an explict method, since later we will use it.

In sketch, the philosophy of this method is that we descend the degree of differential forms, but the price is we need to consider functions defined on intersections of many open subsets.

X is a differential manifold, and $Z^1 \subset \Omega^1_{X,\mathbb{R}}$, sheaf of closed 1-form. Then we have the following exact sequence of sheaves.

$$0 \to \underline{\mathbb{R}} \to C^\infty \xrightarrow{d} Z^1 \to 0$$

Locally constant functions are clearly C^∞ functions, such that d acts on them is zero, so the exactness for the first two is trivial. But for the last one, it is equivalent to that a closed form locally must be an exact form, that's Poincaré lemma.

Similarly, define $Z^2 \subset \Omega^2_{X,\mathbb{R}}$, sheaf of closed 2-forms. then

$$0 \to Z^1 \to \Omega^1_{X,\mathbb{R}} \xrightarrow{\mathrm{d}} Z^2 \to 0$$

This sequence is exact for the same reason.

By the definition of de Rham cohomology, we have

$$H^2(X, \mathbb{R}) = \frac{C^\infty(X, Z^2)}{\mathrm{d}C^\infty(X, \Omega^1_{X,\mathbb{R}})}$$

In order to avoid the limit in the definition of Čech cohomology, we take open covering $\mathcal{U} = \{U_\alpha\}$ good enough, such that

$$\mathrm{d} : C^\infty(U_\alpha, \Omega^1_{U_\alpha,\mathbb{R}}) \to C^\infty(U_\alpha, Z^2)$$

is surjective for any α. And

$$\mathrm{d} : C^\infty(U_\alpha \cap U_\beta) \to C^\infty(U_\alpha \cap U_\beta, Z^1)$$

is surjective for any α, β.

If ω is a closed real 2-form, i.e. $[\omega] \in H^2(X, \mathbb{R})$. For any α, choose $A_\alpha \in C^\infty(U_\alpha, \Omega^1_{U_\alpha,\mathbb{R}})$ such that

$$\omega|_{U_\alpha} = \mathrm{d}A_\alpha$$

then

$$\prod_{\alpha,\beta}(A_\alpha - A_\beta)$$

is a Čech 1-cocchain in $C^1(\mathcal{U}, Z^1)$, it's d closed since $\mathrm{d}(A_\alpha - A_\beta)|_{U_\alpha \cap U_\beta} = \omega - \omega = 0$.

For any α, β, choose $f_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta)$, such that

$$(A_\alpha - A_\beta)_{\alpha\beta} = \mathrm{d}f_{\alpha\beta}$$

then

$$f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}|_{U_\alpha \cap U_\beta \cap U_\gamma}$$

is closed, hence locally constant,

$$\breve{\omega} = \prod_{\alpha,\beta,\gamma}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta})$$

is a Čech 2-cocycle, in $C^2(\mathcal{U}, \underline{\mathbb{R}})$. We have $\delta\breve{\omega} = 0$, and $[\omega]$ corresponds to $[\breve{\omega}]$, that's the explict construction for comparision theorem in dimension 2. In fact, the general case is proved in the same method.

And we also the some lemmas in multiply complex analysis.

Lemma 2.4.2. *Locally on a polydisk $D \subset \mathbb{C}^n$, and $\omega \in C^\infty(D, \Omega^2_{D,\mathbb{R}} \cap C^\infty(D, \Omega^{1,1}_D)$ is a d closed real $(1,1)$-form. Then there exists a smooth function $\varphi : D \to \mathbb{R}$ such that*

$$\omega = i\partial\overline{\partial}\varphi$$

Proof. Poincaré lemma implies that $\omega = \mathrm{d}A = \mathrm{d}(A^{1,0} + A^{0,1}) = (\partial + \overline{\partial})(A^{1,0} + A^{0,1})$, and since A is real, then $\overline{A^{1,0}} = A^{0,1}$.

Since ω is a $(1,1)$-form, then

$$\begin{cases} \partial A^{1,0} = 0 \\ \overline{\partial} A^{0,1} = 0 \\ \omega = \overline{\partial} A^{1,0} + \partial A^{0,1} \end{cases}$$

Dolbeault lemma implies that $A^{0,1} = \overline{\partial} f$, so $A^{1,0} = \partial \overline{f}$, so we have

$$\begin{aligned}\omega &= \overline{\partial}\partial\overline{f} + \partial\overline{\partial} f \\ &= \partial\overline{\partial}(f - \overline{f}) \\ &= i\partial\overline{\partial}\varphi\end{aligned}$$

□

Lemma 2.4.3. *Locally on $U \subset \mathbb{C}^n$, a simply connected open subset, and a smooth function $\varphi : U \to \mathbb{R}$, such that $\partial\overline{\partial}\varphi = 0$[16]. Then there exists a holomorphic functions $f : U \to \mathbb{C}$, such that $\varphi = \mathrm{Re}(f)$.*

Now let's prove Lefschetz $(1,1)$-theorem

Proof. Let's first see how does the above two lemmas play a role in our proof. We will choose a good enough open cover $\mathcal{U} = \{U_\alpha\}$ of open polydisk such that for all α, β, we have $U_\alpha \cap U_\beta$ is simply connected.

Since ω is a d closed real $(1,1)$-form, Lemma 3.4.2 implies that there exists smooth function $\varphi_\alpha : U_\alpha \to \mathbb{R}$ such that

$$\omega|_{U_\alpha} = \frac{i}{2\pi}\partial\overline{\partial}\varphi_\alpha$$

On any two intersection $U_\alpha \cap U_\beta$, we have $\partial\overline{\partial}(\varphi_\alpha - \varphi_\beta) = 0$, then Lemma 3.4.3 implies that there exists a holomorphic function $f_{\alpha\beta}$, such that

$$(\varphi_\alpha - \varphi_\beta)|_{U_\alpha \cap U_\beta} = 2\,\mathrm{Re}(f_{\alpha\beta}) = f_{\alpha\beta} + \overline{f_{\alpha\beta}}$$

Consider $\prod f_{\alpha\beta} \in C^1(\mathcal{U}, \mathcal{O}_X)$, then

$$(\delta f)_{\alpha\beta\gamma} = (f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta})|_{U_\alpha \cap U_\beta \cap U_\gamma}$$

Note that $2\,\mathrm{Re}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta})_{\alpha\beta\gamma} = 0$, so it must be a locally constant imaginary number, i.e. it lies in $2\pi i\underline{\mathbb{R}}(U_\alpha \cap U_\beta \cap U_\gamma)$.

Consider real form[17]

$$A_\alpha = \frac{i}{4\pi}(\overline{\partial}\varphi_\alpha - \partial\varphi_\alpha)$$

and by directly computing, we can note that $\omega|_{U_\alpha} = \mathrm{d}A_\alpha$, and that's why we define A_α in this method.

Similar to what we have done in the proof of comparision theorem, we want to consider $A_\alpha - A_\beta$ on the intersection $U_\alpha \cap U_\beta$. So we compute the difference of each term of A_α and A_β as follows

$$\begin{aligned}\partial(\varphi_\beta - \varphi_\alpha) &= \partial(f_{\alpha\beta} + \overline{f_{\alpha\beta}}) \\ &= \partial f_{\alpha\beta} \\ &= \mathrm{d} f_{\alpha\beta}\end{aligned}$$

[16] Such φ is called pluriharmonic

[17] Here we need to consider some queer coefficients, in order to get a beautiful result. In fact, we need to use $e^{2\pi i} = 1$, a god given formula.

Similarly we have

$$\overline{\partial}(\varphi_\beta - \varphi_\alpha) = \mathrm{d}\overline{f_{\alpha\beta}}$$

then

$$(A_\beta - A_\alpha)_{\alpha\beta} = \frac{i}{4\pi}\mathrm{d}(\overline{f_{\alpha\beta}} - f_{\alpha\beta}) = \frac{1}{2\pi}\mathrm{d}(\mathrm{Im}(f_{\alpha\beta}))$$

Via $H^2(X,\mathbb{R}) \cong \check{H}^2(X,\mathbb{R})$, $[\omega]$ corresponds to $[\check{\omega}]$, above process is just what we have done in the proof of comparision theorem, so we have

$$\begin{aligned}\check{\omega} &= \prod(\frac{1}{2\pi}\,\mathrm{Im}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}))_{\alpha\beta\gamma} \\ &= \prod(\frac{1}{2\pi i}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}))_{\alpha\beta\gamma}\end{aligned}$$

Hypothesis tells that $[\check{\omega}]$ is an image of $[\prod n_{\alpha\beta\gamma}] \in \check{H}^2(X,\underline{\mathbb{Z}})$. However, it doesn't mean that $f_{\alpha\beta}$ are exactly integers, but not too bad, we just need some correction terms, that is

$$\prod(\frac{1}{2\pi i}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}))_{\alpha\beta\gamma} = \prod n_{\alpha\beta\gamma} + \delta(\prod c_{\alpha\beta})$$

where $\prod(c_{\alpha\beta})$ is real 1-cochain.

So we set $f'_{\alpha\beta} = f_{\alpha\beta} - 2\pi i c_{\alpha\beta}$. Then

$$\frac{1}{2\pi i}(f'_{\beta\gamma} - f'_{\alpha\gamma} + f'_{\alpha\beta})_{\alpha\beta\gamma} = 2\pi i n_{\alpha\beta\gamma} \in 2\pi i\underline{\mathbb{Z}}(U_\alpha \cap U_\beta \cap U_\gamma)$$

Note that $e^{2\pi i} = 1$, Then consider $g_{\alpha\beta} = \exp(-f'_{\alpha\beta})$, a holomorphic from $U_\alpha \cap U_\beta$ to $\mathbb{C}^*$, it satisfies the cocycle condition

$$g_{\beta\gamma}g_{\alpha\gamma}^{-1}g_{\alpha\beta} = 1$$

so we get a holomorphic line bundle L.

Remark 2.4.4. It's important to keep in mind vector bundles are encoded in their gluing data, and you can regard it as an element in $\check{H}^1$. So if we want to get a holomorphic line bundle, we need to determine its transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ satisfying the cocycle conditions, that is, $\prod g_{\alpha\beta} \in C^1(\mathcal{U},\mathcal{O}_X^*)$ such that $\delta(\prod g_{\alpha\beta}) = 1$, i.e. $[\prod g_{\alpha\beta}] \in \check{H}^1(X,\mathcal{O}_X^*) = \mathrm{Pic}(X)$. That's what Example 2.7.9 already tells us.

Now we need to give a Hermitian metric on this holomorphic line bundle H, and calculate its curvature to complete the proof.

Note that

$$(\varphi_\alpha - \varphi_\beta)_{U_\alpha\cap U_\beta} = 2\,\mathrm{Re}(f_{\alpha\beta}) = 2\,\mathrm{Re}(f_{\alpha\beta})' \quad = -\log|g_{\alpha\beta}|^2$$

then we get a Hermitian metric

$$H_\alpha = -\exp(-\varphi_\alpha), \quad \text{on } U_\alpha$$

Indeed, since $H_\beta = |g_{\alpha\beta}|^2 H_\alpha = g_{\alpha\beta}^T H_\alpha \overline{g_{\alpha\beta}}$.

Finally,

$$\frac{i}{2\pi}H_L = \frac{i}{2\pi}\partial\overline{\partial}\varphi_\alpha = \omega$$

This completes the proof. □

Remark 2.4.5. Now if we already have a holomorphic line bundle L, determined by its transition functions $g_{\alpha\beta}$. We can try to reverse what we have done above. That is, take its logarithm, and divide it by $2\pi i$, and consider its alternating sum and get an element in $\check{H}^2(X, \mathbb{R})$, and that's exact $-c_1$, where c_1 is the first Chern class.

However, we can rephrase it as a basical operation in homological algebra, consider the exponential sequence

$$0 \to \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1$$

Taking cohomology we will get a boundary map ∂

$$\mathrm{Pic}(X) \cong \check{H}^1(X, \mathcal{O}_X^*) \xrightarrow{\partial} \check{H}^2(X, \underline{\mathbb{Z}}) \to \check{H}^2(X, \underline{\mathbb{R}})$$

that's just what we have done, so this boundary map sometimes is denoted by $-c_1$.

2.5. **Hypersurface and Divisors.** This section we will briefly introduce some definitions and theorems about hypersurfaces and divisors wihout proofs.

Definition 2.5.1 (hypersurface)**.** *X is a complex manifold, a hypersurface of X is a closed subset $D \subset X$ such that for all $x \in D$, there exists an open subset $U \subset X$ containing x and a nonzero holomorphic function $f : U \to \mathbb{C}$ such that*

$$D \cap U = \{x \in U \mid f(x) = 0\}$$

Remark 2.5.2. $x \in D$ is called smooth, if we can choose $f : D \to \mathbb{C}$ is a submersion. The set of all smooth point is denoted by D_{sm}; If $D_{sm} = D$, then $D_{sm} = D$. And note that we do not assume D is connected.

Exercise 2.5.3. Let $D \subset X$ be a smooth hypersurface, then there exists an open covering $\{U_\alpha : f_\alpha \to \mathbb{C}\}$ where f_α is holomorphic submersion, such that

$$D \cap U_\alpha = \{x \in U_\alpha \mid f_\alpha(x) = 0\}$$

Then $g_{\alpha\beta} = f_\alpha / f_\beta : U_\alpha \cap U_\beta \to \mathbb{C}^*$ is a holomorphic function. Then we get a holomorphic line bundle $\mathcal{O}_X(D)$ on X. In particular, if we take $X = \mathbb{P}^n, D = \mathbb{P}^{n-1}$, then $\mathcal{O}_X(D) = \mathcal{O}_{\mathbb{P}^n}(1)$.

In fact, we can drop the assumption of "smoothness" by some technical method.

Lemma 2.5.4. *If $D \subset X$ is a hypersurface, then $D_{sm} \subset D$ is open and dense.*

Proposition 2.1. *If $D \subset X$ is a hypersurface, then there exists a holomorphic line bundle $\mathcal{O}_X(D)$ on X with global section σ, such that $D = \{x \in X \mid \sigma(x) = 0\}$*

Proof. Sketch. Set $Z = D\backslash D_{sm}$. Then $D_{sm} \subset X\backslash Z$ is a smooth hypersurface. Then we get transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \backslash Z \to \mathbb{C}^*$, so we get a holomorphic line bundle over $X\backslash Z$. Then Lemma 3.5.4 tells us Z contains no hypersurface of X, and Hartgos theorem tells us we can extend this holomorphic function $g_{\alpha\beta}$ to $U_\alpha \cap U_\beta$. So we define a holomorphic line bundle. □

Definition 2.5.5 (irreducible hypersurface)**.** *A hypersurface $D \subset X$ is called irreducible, if it can not be written as union of two hypersurface.*

Definition 2.5.6 (divisor)**.** *X is a complex manifold, a divisor on X is a finite formal sum*

$$D = \sum_i a_i D_i$$

where $a_i \in \mathbb{Z}$ and D_i is a irreducible hypersurface. And formally we can define

$$\mathcal{O}_X(D) := \bigotimes_i \mathcal{O}_X(D_i)^{\otimes a_i}$$

Remark 2.5.7. We have seen that from divisors we can get a holomorphic line bundle. So it's natural to guess all holomorphic line bundle are arised in this form. However, it's false.

Example 2.5.8. There exists a complex torus with no hypersurface, but there is non trivial holomorphic line bundle on it.

But

Theorem 2.5.9. *If X is a projective manifold, then for any holomorphic line bundle L, there exists a divisor D ($D = D_1 - D_2$, and D_1, D_2 are hypersurface in X), such that*

$$L \cong \mathcal{O}_X(D)$$

Let X be a compact complex manifold of dimension n, and $D \subset X$ is a smooth hypersurface, we can define

$$D : C^\infty(X, \Omega^{2n-2}_{X,\mathbb{R}}) \to \mathbb{R}$$
$$\omega \mapsto \int_D \omega|_D$$

and if ω is a exact form, then $\int_D \omega|_D = 0$ by Stokes.

So we get a $[D] : H^{2n-2}(X, \mathbb{R}) \to \mathbb{R}$, then Poincaré duality tells us we get $[D] \in H^2(X, \mathbb{R})$.

Surprisingly,

Theorem 2.5.10 (Lelong-Poincaré)**.** *X is a compact complex manifold, $D \subset X$ is a smooth hypersurface, then*

$$[D] = c_1(\mathcal{O}_X(D)) \in H^2(X, \mathbb{R})$$

Remark 2.5.11. We can also drop smoothness condition. We need to make sense of $[D] \in H^2(X, \mathbb{R})$, i.e. we need to check the following integral make sense:

$$\int_{D_{sm}} \omega|_{D_{sm}}$$

It's not trivial, since D_{sm} is just an open subset, and integral over an open subset may be quite bad.

Part 2. **Hodge theory**

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