A quick review of topology

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2023/06/28

In this talk we give a quick review of topology which we will use frequently, and the main topics are listed as follows:

- Homotopy and fundamental group.
- Covering spaces.
- Continuous group action.

Definition (homotopy)

Let X and Y be topological spaces and $f, g\colon X \to Y$ be continuous maps. A homotopy from f to g is a continuous map $F\colon X \times I \to Y$ such that for all $x \in X$, one has

$$F(x,0) = f(x)$$
$$F(x,1) = g(x)$$

If there exists a homotopy from f to g, then we say f and g are homotopic, and write $f \simeq g$.

Definition (stationary homotopy)

Let X and Y be topological spaces and $A \subseteq X$ an arbitrary subset. A homotopy F between continuous maps $f, g \colon X \to Y$ is said to be stationary on A if

$$F(x, t) = f(x)$$

for all $x \in A$ and $t \in I$. If there exists such a homotopy, then we say f and g are homotopic relative to A.

Remark.

If f and g are homotopic relative to A, then f must agree with g on A.

Definition (path homotopy)

Let X be a topological space and γ_1, γ_2 be two paths in X. They are said to be path homotopic if they are homotopic relative on $\{0,1\}$, and write $\gamma_1 \simeq \gamma_2$.

Definition (loop homotopy)

Let X be a topological space and γ_1, γ_2 be two loops in X. They're called loop homotopic if they are homotopic relative on $\{0\}$, and write $\gamma_1 \simeq \gamma_2$.

Remark.

For convenience, if γ_1, γ_2 are paths (or loops), then when we say γ_1 is homotopic to γ_2, we mean γ_1 is path (or loop) homotopic to γ_2.

Definition (free loop homotopy)

Let X be a topological space and γ_1, γ_2 be two loops in X. They are said to be freely loop homotopic if they're homotopic through loops (but not necessarily preserving the base point), that is, there exists a homotopy $F(x, t)\colon [0,1] \times [0,1] \to X$ such that

$$F(t, 0) = \gamma_1(t)$$
$$F(t, 1) = \gamma_2(t)$$
$$F(0, s) = F(1, s) \text{ holds for all } s \in [0,1]$$

Lemma

Let X be a topological space. For any $p, q \in X$, path homotopy is an equivalence relation on the set of all paths in X from p to q. For any path γ in X, the path homotopy class is denoted by $[\gamma]$.

Proof.

For path $\gamma\colon I \to X$, γ is homotopic to itself by $F(s,t) = \gamma(s)$. If γ_1 is homotopic to γ_2 by F, then γ_2 is homotopic to γ_1 by $G(s,t) = F(s, 1-t)$. Finally, suppose γ_1 is homotopic to γ_2 by F, γ_2 is homotopic to γ_3 by G. Then consider

$$H = \begin{cases} F(s, 2t) & 0 \leq t \leq \frac{1}{2} \\ G(s, 2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

which is a homotopy from γ_1 to γ_3. This shows path homotopy is an equivalence relation. □

Definition (reparametrization)

A reparametrization of a path $f\colon I \to X$ is of the form $f \circ \varphi$ for some continuous map $\varphi\colon I \to I$ fixing 0 and 1.

Lemma

Any reparametrization of a path f is homotopic to f.

Proof.

Suppose $f \circ \varphi$ is a reparametrization of f, and let $F\colon I \times I \to I$ denote the straight-line homotopy from the identity map to φ. Then $f \circ F$ is a path homotopy from f to $f \circ \varphi$. $\square$

Definition (product of path)

Let X be a topological space and f, g be paths. f and g are composable if $f(1) = g(0)$. If f and g are composable, their product $f \cdot g \colon I \to X$ is defined by

$$f \cdot g(s) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

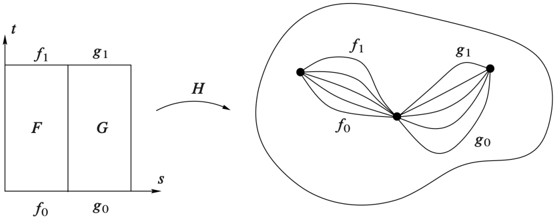
Lemma

Let X be a topological space and f_0, f_1, g_0, g_1 be paths in X such that f_0, g_0 are composable and f_1, g_1 are composable. If $f_0 \simeq g_0$, $f_1 \simeq g_1$, then $f_0 \cdot g_0 \simeq f_1 \cdot g_1$.

Proof.

Suppose the homotopy from f_0 to f_1 is given by F and the homotopy from g_0 to g_1 is given by G. Then the required homotopy H from $f_0 \cdot g_0$ to $f_1 \cdot g_1$ is given by

$$H(s,t) = \begin{cases} F(2s,t) & 0 \leq s \leq \frac{1}{2}, 0 \leq t \leq 1 \\ G(2s-1,t) & \frac{1}{2} \leq s \leq 1, 0 \leq t \leq 1 \end{cases}$$



Lemma

Let X be a topological space and f, g be paths in X such that $f \simeq g$. If $\overline{f}$ is the path obtained by reversing f, that is $\overline{f}(s) := f(1-s)$, then $\overline{f} \simeq \overline{g}$.

Proof.

Suppose f is homotopic to g by homotopy F. Then $G(s,t) := F(1-s,t)$ is a homotopy from $\overline{f}$ to $\overline{g}$ since

$$G(s,0) = F(1-s,0) = f(1-s) = \overline{f}(s)$$
$$G(s,1) = F(1-s,1) = g(1-s) = \overline{g}(s)$$

$\square$

Theorem

Let X be a topological space and $[f], [g], [h]$ be homotopy classes of loops based at $p \in X$.

1. $[c_p] \cdot [f] = [f] \cdot [c_p] = [f]$, *where c_p is constant loop based at p.*
2. $[f] \cdot [\overline{f}] = [c_p]$ *and* $[\overline{f}] \cdot [f] = [c_p]$.
3. $[f] \cdot ([g] \cdot [h]) = ([f] \cdot [g]) \cdot [h]$.

Proof.

For (1). Let us show that $c_p \cdot f \simeq f$, and the other case is similar. Define $H\colon I \times I \to X$ by

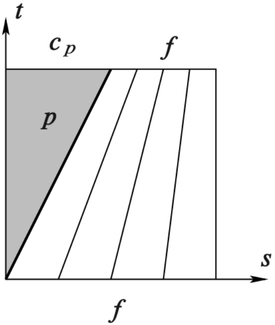
$$H(s,t) = \begin{cases} p & t \geq 2s \\ f\left(\frac{2s-t}{2-t}\right) & t \leq 2s \end{cases}$$

Continuation.

For (2). It suffices to show that $f \cdot \overline{f} \simeq c_p$, since the reverse path of $\overline{f}$ is f, the other relation follows by interchanging the roles of f and $\overline{f}$. Define

$$H(s,t) = \begin{cases} f(2s) & 0 \leq s \leq \frac{t}{2} \\ f(t) & \frac{t}{2} \leq s \leq 1 - \frac{t}{2} \\ f(2-2s) & 1 - \frac{t}{2} \leq s \leq 1 \end{cases}$$

It is easy to check that H is a homotopy from c_p to $f \cdot \overline{f}$.
For (3). It suffices to show $(f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$. The first path follows f and then g at quadruple speed for $s \in [0, \frac{1}{2}]$, and then follows h at double speed for $s \in [\frac{1}{2}, 1]$, while the second follows f at double speed and then g and h at quadruple speed. The two paths are therefore reparametrizations of each other and thus homotopic by Lemma 8. $\square$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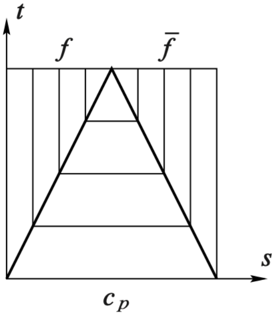
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Definition (fundamental group)

Let X be a topological space. The fundamental group of X based at p, denoted by $\pi_1(X,p)$, is the set of path homotopy classes of loops based at p equipped with composition as its group structure.

Theorem (base point change)

Let X be a topological space, $p,q \in X$ and g is any path from p to q. The map

$$\begin{aligned}\Phi_g\colon \pi_1(X,p) &\to \pi_1(X,q)\\ [f] &\mapsto [\overline{g}]\cdot[f]\cdot[g]\end{aligned}$$

is a group isomorphism with inverse $\Phi_{\overline{g}}$.

Proof.

It suffices to show Θ_g is a group homomorphism, since it's clear $\Phi_g \circ \Phi_{\overline{g}} = \Phi_{\overline{g}} \circ \Phi_g = \mathrm{id}$. For $[\gamma_1], [\gamma_2] \in \pi_1(X, p)$, one has

$$\begin{aligned}
\Phi_g[\gamma_1] \cdot \Phi[\gamma_2] &= [\overline{g}] \cdot [\gamma_1] \cdot [g] \cdot [\overline{g}] \cdot [\gamma_2] \cdot [g] \\
&= [\overline{g}] \cdot [\gamma_1] \cdot [c_p] \cdot [\gamma_2] \cdot [g] \\
&= [\overline{g}] \cdot [\gamma_1] \cdot [\gamma_2] \cdot [g] \\
&= \Phi_g([\gamma_1] \cdot [\gamma_2])
\end{aligned}$$

□

Corollary

If X is a path-connected topological space, then its fundamental is independent of the choice of base point, and denoted by $\pi_1(X)$ for convenience.

Theorem

The fundamental group of a topological manifold is countable.

In this section we assume all spaces are connected and locally path connected topological spaces, and all maps are continuous. We are including these hypotheses[1] since most of the interesting results (such as lifting criterion) require them, and most of the interesting topological space (such as connected topological manifold) satisfy them.

[1] In fact, it's almost the strongest connected hypotheses, since if a topological space is connected and locally path-connected, then it's also path connected.

Definition (covering space)

A covering space of X is a map $\pi\colon \widetilde{X} \to X$ such that there exists a discrete space D and for each $x \in X$ an open neighborhood $U \subseteq X$, such that $\pi^{-1}(U) = \coprod_{d\in D} V_d$ and $\pi|_{V_d}\colon V_d \to U$ is a homeomorphism for each $d \in D$.

1. Such a U is called evenly covered by $\{V_d\}$.
2. The open sets $\{V_d\}$ are called sheets.
3. For each $x \in X$, the discrete subset $\pi^{-1}(x)$ is called the fiber of x.
4. The degree of the covering is the cardinality of the space D.

Definition (isomorphism between covering spaces)

Let $\pi_1\colon \widetilde{X}_1 \to X$ and $\pi_2\colon \widetilde{X}_2 \to X$ be two covering spaces. An isomorphism between covering spaces is a homeomorphism $f\colon \widetilde{X}_1 \to \widetilde{X}_2$ such that $\pi_1 = \pi_2 \circ f$.

Definition (proper)

Let $f\colon X \to Y$ be a continuous map between topological spaces. f is called proper if preimage of any compact set in Y is a compact subset in X.

Lemma

Let $p\colon X \to Y$ be a proper map between topological spaces and Y be locally compact and Hausdorff. Then p is a closed map.

Proof.

Let C be a closed subset of X. It suffices to show $Y \setminus p(C)$ is open. Let $y \in Y \setminus p(C)$. Then y has an compact neighborhood V since Y is locally compact and $p^{-1}(V)$ is compact. Let $E = C \cap p^{-1}(V)$. Then E is a compact and hence so is $p(E)$. Then $p(E)$ is closed since compact set in Hausdorff space is closed. Let $U = V \setminus p(E)$. Then U is an open neighborhood of y and disjoint from $p(C)$.

Corollary

Let $p\colon X \to Y$ be a proper map between topological spaces and Y be locally compact and Hausdorff. If $y \in Y$ and V is an open neighborhood of $p^{-1}(y)$, then there exists an open neighborhood U of y with $p^{-1}(U) \subseteq V$.

Proof.

Since V is open, one has $X \setminus V$ is closed, and thus $A := p(X \setminus V)$ is also closed with $y \notin A$ since p is a closed map by Lemma 20. Thus $U := Y \setminus A$ is an open neighborhood of y such that $p^{-1}(U) \subseteq V$. □

Theorem

Let $p\colon X \to Y$ be a proper local homeomorphism between topological spaces and Y be locally compact and Hausdorff. Then p is a covering map.

Proof.

For $y \in Y$, one has $\{y\}$ is a compact set since Y is locally compact and Hausdorff, and hence so is $p^{-1}(y)$ since p is proper. On the other hand, $p^{-1}(y)$ is a discrete set since p is a local homeomorphism. Then $p^{-1}(y)$ is a finite set, and we denote it by $\{x_1, \dots, x_n\}$. Since p is a local diffeomorphism, for each $i = 1, \dots, n$, there exists an open neighborhood W_i of x_i and an open neighborhood U_i of x such that $p|_{W_i}$ is a homeomorphism. Without lose of generality we may assume W_i are pairwise disjoint. Now $W_1 \cup \dots \cup W_n$ is an open neighborhood of $p^{-1}(y)$. Thus by Corollary 21 there exists an open neighborhood $U \subseteq U_1 \cap \dots \cap U_n$ of y with $p^{-1}(U) \subseteq W_1 \cup \dots \cup W_n$. If we let $V_i = W_i \cap p^{-1}(U)$, then the V_i are disjoint open sets with

$$p^{-1}(U) = V_1 \cup \dots \cup V_n$$

and all the mappings $p|_{V_i}$ are homeomorphisms.

Theorem (unique lifting property)

Let $\pi\colon \widetilde{X} \to X$ be a covering space and a map $f\colon Y \to X$. If two lifts $\widetilde{f}_1, \widetilde{f}_2\colon Y \to \widetilde{X}$ of f agree at one point of Y, then $\widetilde{f}_1$ and $\widetilde{f}_2$ agree on all of Y.

Proof.

Let A be the set consisting of points of Y where $\widetilde{f}_1$ and $\widetilde{f}_2$ agree. If $\widetilde{f}_1$ agrees with $\widetilde{f}_2$ at some point of Y, then A is not empty, and we may assume $A \neq Y$, otherwise there is nothing to prove. For $y \notin A$, let $\widetilde{U}_1$ and $\widetilde{U}_2$ be the sheets containing $\widetilde{f}_1(y)$ and $\widetilde{f}_2(y)$ respectively. By continuity of $\widetilde{f}_1$ and $\widetilde{f}_2$, there exists a neighborhood N of y mapped into $\widetilde{U}_1$ by $\widetilde{f}_1$ and mapped into $\widetilde{U}_2$ by $\widetilde{f}_2$. Since $\widetilde{f}_1(y) \neq \widetilde{f}_2(y)$, then $\widetilde{U}_1 \cap \widetilde{U}_2 = \varnothing$. This shows $\widetilde{f}_2 \neq \widetilde{f}_2$ throughout the neighborhood N, and thus $Y \backslash A$ is open, that is A is closed. To see A is open, for $y \in A$ one has $\widetilde{f}_1(y) = \widetilde{f}_2(y)$, and thus $\widetilde{U}_1 = \widetilde{U}_2$. Since $\pi|_{\widetilde{U}_1}$ is a diffeomorphism, one has $\widetilde{f}_1 = \pi^{-1} \circ f = \widetilde{f}_2$ on $\widetilde{U}_i$. This shows the set A is open, and thus $A = Y$ since Y is connected. □

Theorem (homotopy lifting property)

Let $\pi\colon \widetilde{X} \to X$ be a covering space and $F\colon Y \times I \to X$ be a homotopy. If there exists a map $\widetilde{F}\colon Y \times \{0\} \to \widetilde{X}$ which lifts $F|_{Y\times\{0\}}$, then there exists a unique homotopy $\widetilde{F}\colon Y \times I \to \widetilde{X}$ which lifts F and restricting to the given $\widetilde{F}$ on $Y \times \{0\}$. Furthermore, if F is stationary on A, so is $\widetilde{F}$.

Proof.

Firstly, let's construct a lift $\widetilde{F}\colon N \times I \to \widetilde{X}$ for some neighborhood N in Y of a given point $y_0 \in Y$. Since F is continuous, every point $(y_0, t) \in Y \times I$ has a product neighborhood $N_t \times (a_t, b_t)$ such that $F(N_t \times (a_t, b_t))$ is contained in an evenly covered neighborhood of $F(y_0, t)$. By compactness of $\{y_0\} \times I$, finitely many such products $N_t \times (a_t, b_t)$ cover $\{y_0\} \times I$.

Continuation.

This implies that we can choose a single neighborhood N of y_0 and a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ of I such that for each i, one has $F(N \times [t_i, t_{i+1}])$ is contained in an evenly covered neighborhood U_i. Suppose $\widetilde{F}$ has been constructed on $N \times [0, t_i]$, starting with the given $\widetilde{F}$ on $N \times \{0\}$. Since U_i is evenly covered, there is an open set $\widetilde{U}_i$ of $\widetilde{X}$ projecting homeomorphically onto U_i by π and containing the point $\widetilde{F}(y_0, t_i)$. After replacing N by a smaller neighborhood of y_0 we may assume that $\widetilde{F}(N \times \{t_i\})$ is contained in $\widetilde{U}_i$. Now we can define $\widetilde{F}$ on $N \times [t_i, t_{i+1}]$ to be the composition of F with the homeomorphism $\pi^{-1} \colon U_i \to \widetilde{U}_i$ since $F(N \times [t_i, t_{i+1}]) \subseteq U_i$, After a finite number of steps we eventually get a lift $\widetilde{F} \colon N \times I \to \widetilde{X}$ for some neighborhood N of y_0.

Continuation.

Next we show the uniqueness part in the special case that Y is a point, since in this case we can omit Y from the notation. Suppose $\widetilde{F}$ and $\widetilde{F}'$ are two lifts of $F\colon I \to X$ such that $\widetilde{F}(0) = \widetilde{F}'(0)$. As before, choose a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ of I so that for each i, one has $F([t_i, t_{i+1}])$ is contained in some evenly covered neighborhood U_i. Assume inductively that $\widetilde{F} = \widetilde{F}'$ on $[0, t_i]$. Since $[t_i, t_{i+1}]$ is connected, so is $\widetilde{F}([t_i, t_{i+1}])$, which must therefore lie in a single one of the disjoint open sets $\widetilde{U}_i$ projecting homeomorphically to U_i. Similarly, $\widetilde{F}'([t_i, t_{i+1}])$ lies in a single $\widetilde{U}_i$, in fact in the same one that contains $\widetilde{F}([t_i, t_{i+1}])$ since $\widetilde{F}'(t_i) = \widetilde{F}(t_i)$. Because π is injective on $\widetilde{U}_i$ and $\pi \circ \widetilde{F} = \pi \circ \widetilde{F}'$, it follows that $\widetilde{F} = \widetilde{F}'$ on $[t_i, t_{i+1}]$, and the induction step is finished.

Continuation.

The last step in the proof of is to observe that since the $\widetilde{F}$ constructed above on sets of the form $N \times I$ are unique when restricted to each segment $\{y\} \times I$, they must agree whenever two such sets $N \times I$ overlap, which gives well-defined $\widetilde{F}$ on $Y \times I$. □

Corollary (path lifting property)

Let $\pi\colon \widetilde{X} \to X$ be a covering space. Suppose $\gamma\colon I \to X$ is any path, and $\widetilde{x} \in \widetilde{X}$ is any point in the fiber of $\pi^{-1}(\gamma(0))$. Then there exists a unique lift $\widetilde{\gamma}\colon I \to \widetilde{X}$ of γ such that $\widetilde{\gamma}(0) = \widetilde{x}$.

Corollary (monodromy theorem)

Let $\pi\colon \widetilde{X} \to X$ be a covering space. Suppose γ_1 and γ_2 are paths in X with the same initial point and the same terminal point, and $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ are their lifts with the same initial point. Then $\widetilde{\gamma}_1$ is homotopic to $\widetilde{\gamma}_2$.

Corollary

Let $\pi\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering space. Then

1. *The map $\pi_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective.*
2. *$\pi_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ consists of the homotopy class of loops in X whose lifts to $\widetilde{X}$ are still loops.*
3. *The index of $\pi_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ in $\pi_1(X, x_0)$ is the degree of covering. In particular, the degree of universal covering equals $|\pi_1(X, x_0)|$.*

Proof.

For (1). An element of $\ker \pi_*$ is represented by a loop $\widetilde{\gamma}_0\colon I \to \widetilde{X}$ with a homotopy F of $\gamma_0 = \pi \circ \widetilde{\gamma}_0$ to the trivial loop γ_1. By Theorem 24 there is a lifted homotopy of loops $\widetilde{F}$ starting with $\widetilde{\gamma}_0$ and ending with a constant loop. Hence $[\widetilde{\gamma}_0] = 0$ in $\pi_1(\widetilde{X}, \widetilde{x}_0)$ and π_* is injective.

Continuation.

For (2). The loops at x_0 lifting to loops at $\widetilde{x}_0$ certainly represent elements of the image of $\pi_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$. Conversely, a loop representing an element of the image of π_* is homotopic to a loop having such a lift, so by Theorem 24, the loop itself must have such a lift.

For (3). For a loop γ in X based at x_0, let $\widetilde{\gamma}$ be its lift to $\widetilde{X}$ starting at $\widetilde{x}_0$. A product $h \cdot \gamma$ with $[h] \in H = \pi_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ has the lift $\widetilde{h} \cdot \widetilde{\gamma}$ ending at the same point as $\widetilde{\gamma}$ since $\widetilde{h}$ is a loop. Thus we may define a function Φ from cosets $H[\gamma]$ to $\pi^{-1}(x_0)$ by sending $H[\gamma]$ to $\widetilde{\gamma}(1)$. The path-connectedness of $\widetilde{X}$ implies that Φ is surjective since $\widetilde{x}_0$ can be joined to any point in $\pi^{-1}(x_0)$ by a path $\widetilde{\gamma}$ projecting to a loop γ at x_0. To see that Φ is injective, observe that $\Phi(H[\gamma_1]) = \Phi(H[\gamma_2])$ implies that $\gamma_1 \cdot \overline{\gamma}_2$ lifts to a loop in $\widetilde{X}$ based at $\widetilde{x}_0$, so $[\gamma_1][\gamma_2]^{-1} \in H$ and hence $H[\gamma_1] = H[\gamma_2]$. Thus the index of H is the same as $|\pi^{-1}(x_0)|$. □

Theorem (lifting criterion)

Let $\pi\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering space and $f\colon (Y, y_0) \to (X, x_0)$ be a map. A lift $\widetilde{f}\colon (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ of f exists if and only if $f_*(\pi_1(Y, y_0)) \subseteq \pi_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.

Proof.

The only if statement is obvious since $f_* = \pi_* \circ f_*$. Conversely, let $y \in Y$ and let γ be a path in Y from y_0 to y. By Corollary 25, the path $f\gamma$ in X starting at x_0 has a unique lift $\widetilde{f\gamma}$ starting at $\widetilde{x}_0$, and we define $\widetilde{f}(y) = \widetilde{f\gamma}(1)$.

To see it's well-defined, let γ' be another path from y_0 to y. Then $(f\gamma') \cdot (\overline{f\gamma})$ is a loop h_0 at x_0 with $[h_0] \in f_*(\pi_1(Y, y_0)) \subseteq \pi_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. This means there is a homotopy H of h_0 to a loop h_1 that lifts to a loop $\widetilde{h}_1$ in $\widetilde{X}$ based at $\widetilde{x}_0$.

Continuation.

Apply Theorem 24 to H to get a lifting $\widetilde{H}$. Since $\widetilde{h}_1$ is a loop at $\widetilde{x}_0$, so is $\widetilde{h}_0$. By Theorem 23, that is uniqueness of lifted paths, the first half of $\widetilde{h}_0$ is $\widetilde{f\gamma'}$ and the second half is $\widetilde{f\gamma}$ traversed backwards, with the common midpoint $\widetilde{f\gamma}(1) = \widetilde{f\gamma'}(1)$. This shows $\widetilde{f}$ is well-defined.

To see $\widetilde{f}$ is continuous, let $U \subseteq X$ be an open neighborhood of $f(y)$ having a lift $\widetilde{U} \subseteq \widetilde{X}$ containing $\widetilde{f}(y)$ such that $\pi\colon \widetilde{U} \to U$ is a homeomorphism. Choose a path-connected open neighborhood V of y with $f(V) \subseteq V$. For paths from y_0 to points $y' \in V$, we can take a fixed path γ from y_0 to y followed by paths η in V from y to points y'. Then the paths $(f\gamma) \cdot (f\eta)$ in X have lifts $(\widetilde{f\gamma}) \cdot (\widetilde{f\eta})$ where $\widetilde{f\eta} = \pi^{-1} f\eta$. Thus $\widetilde{f}(V) \subseteq \widetilde{U}$ and $\widetilde{f}|_V = \pi^{-1} f$, so $\widetilde{f}$ is continuous at y. $\square$

Theorem

Suppose M is a topological manifold, E is a Hausdorff space and $\pi\colon E \to M$ is a local homeomorphism with the path lifting property. Then π is a covering space.

1. Overview
2. Homotopy and fundamental group
3. Covering space
   - Proper maps
   - Lifting theorems
   - The classification of the covering spaces
   - The structure of the deck transformation group
   - Covering of topological manifold
4. Continuous group action

Definition (universal covering)

A simply-connected covering space of X is called universal covering.

Definition (semilocally simply-connected)

A topological space X is called semilocally simply-connected if each $x \in X$ has a neighborhood U such that the inclusion induced map $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.

Theorem

If X is a semilocally simply-connected topological space, then X has a universal covering $\widetilde{X}$.

Theorem

Let X be a semilocally simply-connected topological space. Then there is a bijection between the set of basepoint-preserving isomorphism classes of covering spaces $\pi\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ and the the set of subgroups of $\pi_1(X, x_0)$ obtained by associating the subgroup $\pi_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ to the covering space $(\widetilde{X}, \widetilde{x}_0)$. If basepoints are ignored, this correspondence gives a bijection between isomorphism classes of covering spaces $\pi\colon \widetilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$.

Corollary

Let X be a semilocally simply-connected topological space. Then the universal covering of X is unique up to isomorphism.

Definition (deck transformation)

Let $\pi\colon \widetilde{X} \to X$ be a covering space. The deck transformation group is following set

$$\mathrm{Aut}_\pi(\widetilde{X}) = \{f\colon \widetilde{X} \to \widetilde{X} \text{ is homeomorphism} \mid \pi \circ f = \pi\}$$

equipped with composition as group operation.

Definition (normal)

A covering $\pi\colon \widetilde{X} \to X$ is called normal, if any deck transformation acts transitively on each fiber of $x \in X$.

Lemma

Let $\pi\colon \widetilde{X} \to X$ be a covering space. The deck transformation group $\mathrm{Aut}_\pi(\widetilde{X})$ acts on $\widetilde{X}$ freely.

Proof.

Suppose $f\colon \widetilde{X} \to \widetilde{X}$ is a deck transformation admitting a fixed point. Since $\pi \circ f = p$, we may regard f as a lift of π, and identity map of $\widetilde{X}$ is another lift of π. By Theorem 23, that is unique lifting property, one has f is exactly identity map since it agrees with identity map at fixed point. □

Lemma

Let $\pi\colon \widetilde{X} \to X$ be a normal covering. Then $\widetilde{X}/\operatorname{Aut}_\pi(\widetilde{X})$ is homeomorphic to X.

Proof.

Let $\Phi\colon \widetilde{X}/\operatorname{Aut}_\pi(\widetilde{X}) \to X$ be the map sending the orbit $\mathcal{O}_{\widetilde{x}}$ to $\pi(\widetilde{x})$, where $\widetilde{x} \in \widetilde{X}$. It's clear Φ is well-defined bijection since $\operatorname{Aut}_\pi(\widetilde{X})$ acts on $\widetilde{X}$ fiberwise transitive, and the following diagram commutes

$$\begin{tikzcd}
\widetilde{X} \arrow[d, "p"'] \arrow[dr, "\pi"] & \\
\widetilde{X}/\operatorname{Aut}_\pi(\widetilde{X}) \arrow[r, "\Phi"'] & X
\end{tikzcd}$$

This diagram shows Φ is both continuous and open, since p is the quotient map and π is continuous and open, which shows $\widetilde{X}/\operatorname{Aut}_\pi(\widetilde{X})$ is homeomorphic to X.

Theorem

Let $\pi\colon(\widetilde{X},\widetilde{x}_0)\to(X,x_0)$ *be a covering space and* $H=\pi_*(\pi_1(\widetilde{X},\widetilde{x}_0))\subseteq\pi_1(X,x_0)$. *Then*

1. π *is a normal covering if and only if* H *is a normal subgroup of* $\pi_1(X,x_0)$.
2. $\mathrm{Aut}_\pi(\widetilde{X})$ *is isomorphic to the quotient* $N(H)/H$, *where* $N(H)$ *is the normalizer of* H *in* $\pi_1(X,x_0)$. *In particular, if* $\pi\colon\widetilde{X}\to X$ *is the universal covering, then* $\mathrm{Aut}_\pi(\widetilde{X})\cong\pi_1(X,x_0)$.

Lemma

Let X be a topological space admitting a countable open covering $\{U_i\}$ such that each set U_i is second countable in the subspace topology. Then X is second countable.

Proof.

Let $\mathcal{B}_\alpha$ be a countable base for U_α. Its members are by definition open in U_α, and as all U_α are open in X, these sets are also open in X. So $\mathcal{B} = \bigcup_\alpha \mathcal{B}_\alpha$ is a countable family of open sets in X. Suppose that $x \in X$ and V is open in X with $x \in V$. Then $x \in U_\beta$ for some index β. Now apply the definition of a base to see that for some $B \in \mathcal{B}_\beta$ we have $x \in B \subseteq V \cap U_\beta$. This $B \in \mathcal{B}$ and $x \in B \subseteq V$. This shows that $\mathcal{B}$ is a countable base for X. $\square$

Theorem

Suppose M is a topological n-manifold and let $\pi\colon \widetilde{M} \to M$ be a covering map. Then $\widetilde{M}$ is a topological n-manifold.

Proof.

Since π is a local diffeomorphism and M is locally Euclidean, one has $\widetilde{M}$ is also locally Euclidean. Now let's show $\widetilde{M}$ is Hausdorff, let $\widetilde{x}_1, \widetilde{x}_2$ be two distinct points in $\widetilde{M}$. If $\pi(\widetilde{x}_1) = \pi(\widetilde{x}_2)$ and $U \subseteq M$ is an evenly covered open subset containing $\pi(\widetilde{x}_1)$, then the component of $\pi^{-1}(U)$ containing $\widetilde{x}_1$ and $\widetilde{x}_2$ are disjoint open subsets of $\widetilde{M}$ that separate $\widetilde{x}_1$ and $\widetilde{x}_2$. If $\pi(\widetilde{x}_1) \neq \pi(\widetilde{x}_2)$, there are disjoint open subsets $U_1, U_2 \subseteq M$ containing $\pi(\widetilde{x}_1)$ and $\pi(\widetilde{x}_2)$ since M is Hausdorff, and then $\pi^{-1}(U_1)$ and $\pi^{-1}(U_2)$ are disjoint open subsets of $\widetilde{M}$ containing $\widetilde{x}_1$ and $\widetilde{x}_2$, and thus $\widetilde{M}$ is Hausdorff.

Continuation.

To see $\widetilde{M}$ is second countable, we will show first that each fiber of π is countable. Given $x \in M$ and an arbitrary point $\widetilde{x}$ in $\pi^{-1}(x)$, we will construct a surjective map $\beta\colon \pi_1(M, x) \to \pi^{-1}(x)$, and since by 16, one has each fiber is countable. Let $[\gamma] \in \pi_1(M, x)$ be the homotopy class of an arbitrary loop $\gamma\colon I \to M$ based at x. The path lifting property guarantees that there is a lift $\widetilde{\gamma}\colon I \to \widetilde{M}$ starting at $\widetilde{x}$, and Corollary 26 implies the endpoint $\widetilde{\gamma}(1)$ depends only on the homotopy class of γ, so it makes sense to define $\beta([\gamma]) = \widetilde{\gamma}(1)$. To see β is surjective, it suffices to note that for any $\widetilde{y} \in \pi^{-1}(x)$, there is a path $\widetilde{\gamma}$ in $\widetilde{M}$ from $\widetilde{x}$ to $\widetilde{y}$, and then $\gamma = \pi \circ \widetilde{\gamma}$ is a loop in M such that $\widetilde{y} = \beta([\gamma])$.

Continuation.

The collection of all evenly covered open subsets is an open covering of M, and therefore has a countable subcover $\{U_i\}$. For any given i, each component of $\pi^{-1}(U_i)$ contains exactly one point in each fiber over U_i, so $\pi^{-1}(U_i)$ has countably many components. The collection of all components of all sets of the form $\pi^{-1}(U_i)$ is a countable open covering of $\widetilde{M}$. Since each such component is second countable, by Lemma 40 one has $\widetilde{M}$ is also second countable. $\square$

Definition (group action)

Let G be a group and S be a set. A left G-action on S is a function

$$\theta\colon G \times S \to S$$

satisfying the following two axioms:

1. $\theta(e, s) = s$, where $e \in G$ is the identity element.
2. $\theta(g_1, \theta(g_2, s)) = \theta(g_1 g_2, s)$, where $g_1, g_2 \in G$.

For convenience we denote $\theta(g, s) = gs$ for $g \in G, s \in S$.

Definition (G-set)

Let G be a group. A set S endowed with a left (or right) G-action is called a left (or right) G-set.

Definition

Let G be a group and S be a left G-set.

1. For $g \in G$, if $gs = s$ for some $s \in S$ implies $g = e$, then the group action is called free.
2. For $g \in G$, if $gs = s$ for all $s \in S$ implies $g = e$, then the group action is called effective.
3. If for arbitrary $s_1, s_2 \in S$, there exists $g \in G$ such that $gs_1 = s_2$, then the group action is called transitive.

Definition (isotropy group)

Let G be a group and S be a right G-set. For any $s \in G$, the isotropy group of s, denoted by G_s, is the set of all elements of G that fix s, that is

$$G_s = \{g \in G \mid gs = s\}$$

Definition (act by homeomorphisms)

Let Γ be a group and X be a topological space. The group Γ is calld acting X by homeomorphisms, if Γ acts on X, and for every $g \in \Gamma$, the map $x \mapsto gx$ is a homeomorphism.

Definition (topological group)

A group is called a topological group, if it's a topological space such that the multiplication and the inversion are continuous.

Definition (continuous action)

Let X be a topological space and G a topological group. A continuous G-action on X is given by the following data:

1. G acts on X by homeomorphisms.
2. The map $G \times X \to X$ given by $(g, x) \mapsto gx$ is continuous.

Lemma

Let X be a topological space and Γ a group acting on X by homeomorphisms. Then the quotient map $\pi\colon X \to X/\Gamma$ is an open map.

Proof.

For any $g \in \Gamma$ and any subset $U \subseteq X$, the set $gU \subseteq X$ is defined as

$$gU = \{gx \mid x \in U\}$$

If $U \subseteq X$ is open, then $\pi^{-1}(\pi(U))$ is the union of all sets of the form gU as g ranges over G. Since $p \mapsto gp$ is a homeomorphism, each set is open, and therefore $\pi^{-1}(\pi(U))$ is open in X. Since π is a quotient map, this implies $\pi(U)$ is open in X/Γ, and therefore π is an open map. □

Definition (proper)

Let X be a topological space and G a topological group. A continuous G-action on X is called proper if the continuous map

$$\begin{aligned} \Theta\colon G \times X &\to X \times X \\ (g, x) &\mapsto (gx, x) \end{aligned}$$

is proper, that is, the preimage of a compact set is compact.

Lemma

Let X, Y be topological spaces and $\pi\colon X \to Y$ be an open quotient map. Then Y is Hausdorff if and only if the set $\mathcal{R} = \{(x_1, x_2) \mid \pi(x_1) = \pi(x_2)\}$ is closed in $X \times X$.

Lemma

Let X be a topological space and G a topological group acting on X continuously. If the action is also proper, then the orbit space is Hausdorff.

Proof.

Let $\Theta\colon G \times X \to X \times X$ be the proper map $\Theta(g, x) = (gx, x)$ and $\pi\colon X \to X/G$ be the quotient map. Define the orbit relation $\mathcal{O} \subseteq X \times X$ by

$$\mathcal{O} = \Theta(G \times X) = \{(gx, x) \mid x \in X, g \in G\}$$

Since proper continuous map is closed, it follows that $\mathcal{O}$ is closed in $X \times X$, and since π is open by Lemma 49, one has X/G is Hausdorff by Lemma 51. $\square$

Theorem

Let M be a topological manifold and G a topological group acting on M continuously. The following statements are equivalent.

1. *The action is proper.*
2. *If $\{p_i\}$ is a sequence in M and $\{g_i\}$ is a sequence in G such that both $\{p_i\}$ and $\{g_ip_i\}$ converge, then a subsequence of $\{g_i\}$ converges.*
3. *For every compact subset $K \subseteq M$, the set $G_K = \{g \in G \mid gK \cap K \neq \varnothing\}$ is compact.*

Proof.

Along the proof, let $\Theta\colon G \times M \to M \times M$ denote the map $(g, p) \mapsto (gp, p)$. For (1) to (2). Suppose Θ is proper, and $\{p_i\}$, $\{g_i\}$ are sequences satisfying the hypotheses of (2). Let U and V be precompact[2] neighborhoods of $p = \lim_i p_i$ and $q = \lim_i g_i p_i$. The assumption implies $\Theta(g_i, p_i)$ all lie in compact set $\overline{U} \times \overline{V}$ when i is sufficiently large, so there exists a subsequence of $\{(g_i, p_i)\}$ converges in $G \times M$ since Θ is proper. In particular, this means that a subsequence of $\{g_i\}$ converges in G.

For (2) to (3). Let K be a compact subset of M, and suppose $\{g_i\}$ is any sequence in G_K. This means for each i, there exists $p_i \in g_i K \cap K$, which is to say that $p_i \in K$ and $g_i^{-1} p_i \in K$. By passing to a subsequence twice, we may assume both $\{p_i\}$ and $\{g_i^{-1} p_i\}$ converge, and the assumption implies there exists a convergent subsequence of $\{g_i\}$. Since each sequence of G_K has a convergent subsequence, G_K is compact.

Continuation.

For (3) to (1). Suppose $L \subseteq M \times M$ is compact, and let $K = \pi_1(L) \cup \pi_2(L)$, where $\pi_1, \pi_2 \colon M \times M \to M$ are the projections onto the first and second factors, respectively. Then

$$\Theta^{-1}(L) \subseteq \Theta^{-1}(K \times K) = \{(g, p) \mid gp \in K, p \in K\} \subseteq G_K \times K$$

By assumption $G_K \times K$ is compact, and thus $\Theta^{-1}(L)$ is compact since it's a closed subset of a compact subset, which implies the action is proper. $\square$

Corollary

Let M be a topological manifold and G a compact topological group. Then every continuous G-action on M is proper.

Definition (properly discontinuous)

Let Γ be a group acting on a topological space X by homeomorphisms. The action is called properly discontinuous, if every point $x \in X$ has a neighborhood U such that for each $g \in G$, $gU \cap U = \varnothing$ unless $g = e$.

Lemma

Suppose Γ be a group acting properly discontinuous on a topological space X. Then every subgroup of Γ still acts properly discontinuous on X.

Lemma

Let $\pi\colon \widetilde{X} \to X$ be a covering space. Then $\mathrm{Aut}_\pi(\widetilde{X})$ acts on $\widetilde{X}$ properly discontinuous.

Proof.

Let $\widetilde{U} \subseteq \widetilde{X}$ project homeomorphically to $U \subseteq X$. For $g \in \mathrm{Aut}_\pi(\widetilde{X})$, if $g(\widetilde{U}) \cap \widetilde{U} \neq \varnothing$, then $g\widetilde{x}_1 = \widetilde{x}_2$ for some $\widetilde{x}_1, \widetilde{x}_2 \in U$. Since $\widetilde{x}_1$ and $\widetilde{x}_2$ lie in the same set $\pi^{-1}(x)$, which intersects $\widetilde{U}$ in only one point, we must have $\widetilde{x}_1 = \widetilde{x}_2 = \widetilde{x}$. Then $\widetilde{x}$ is a fixed point of g, which implies $g = e$ by Lemma 37. □

Theorem (covering space quotient theorem)

Let E be a topological space and Γ be a group acting on E by homeomorphisms effectively . Then the quotient map $\pi\colon E \to E/\Gamma$ is a covering map if and only if Γ acts on E properly discontinuous. In this case, π is a normal covering and $\mathrm{Aut}_{\pi}(E) = \Gamma$.

Proof.

Firstly, assume π is a covering map. Then the action of each $g \in \Gamma$ is an automorphism of the covering since it's a homeomorphism satisfying $\pi(ge) = \pi(e)$ for all $g \in \Gamma, e \in E$, so we can identify Γ with a subgroup of $\mathrm{Aut}_{\pi}(E)$. Then Γ acts on E properly discontinuous by Lemma 56 and Lemma 57.

Continuation.

Conversely, suppose the action is properly discontinuous. To show π is a covering map, suppose $x \in E/\Gamma$ is arbitrary. Choose $e \in \pi^{-1}(x)$, and let U be a neighborhood of e such that for each $g \in \Gamma$, $gU \cap U = \varnothing$ unless $g = 1$. Since E is locally path-connected, by passing to the component of U containing e, we may assume U is path-connected. Let $V = \pi(U)$, which is a path-connected neighborhood of x. Now $\pi^{-1}(V)$ is equal to the union of the disjoint connected open subsets gU for $g \in \Gamma$, so to show π is a covering space it remains to show π is a homeomorphism from each such set onto V. For each $g \in \Gamma$, the restriction map $g\colon U \to gU$ is a homeomorphism, and the diagram

$$\begin{tikzcd} U \arrow[rr, "g"] \arrow[dr, "\pi"'] & & gU \arrow[dl, "\pi"] \\ & V & \end{tikzcd}$$

Continuation.

Thus it suffices to show $\pi|_U \colon U \to V$ is a homeomorphism. It's surjective, continuous and open, and it's injective since $\pi(e) = \pi(e')$ for $e, e' \in U$ implies $e' = ge$ for some $g \in \Gamma$, so $e = e'$ by the choice of U. This shows π is a covering map.
To prove the final statement of the theorem, suppose the action is a covering space action. As noted above, each map $e \mapsto ge$ is a covering automorphism, so $\Gamma \subseteq \mathrm{Aut}_\pi(E)$. By construction, Γ acts transitively on each fiber, so $\mathrm{Aut}_\pi(E)$ does too, and thus π is a normal covering. If φ is any covering automorphism, choose $e \in E$ and let $e' = \varphi(e)$. Then there is some $g \in \Gamma$ such that $ge = e'$. Since φ and $x \mapsto gx$ are deck transformation that agree at a point, so they are equal. Thus $\Gamma = \mathrm{Aut}_\pi(E)$. □

Lemma

Suppose G is a discrete topological group acting continuously and freely on a topological manifold M. The action is proper if and only if the following conditions both hold.

1. *G acts on M properly discontinuous.*
2. *If $p, p' \in M$ are not in the same orbit, then there exist a neighborhood V of p and V' of p' such that $gV \cap V' = \varnothing$ for all $g \in G$.*

Proof.

Firstly, suppose that the action is free and proper and let $\pi \colon M \to M/G$ denote the quotient map. By Lemma 52, the orbit space M/G is Hausdorff. If $p, p' \in M$ are not in the same orbit, we can choose disjoint neighborhoods W of $\pi(p)$ and W' of $\pi(p')$.

Firstly, suppose that the action is free and proper and let $\pi\colon M \to M/G$ denote the quotient map. By Lemma 52, the orbit space M/G is Hausdorff. If $p, p' \in M$ are not in the same orbit, we can choose disjoint neighborhoods W of $\pi(p)$ and W' of $\pi(p')$, and then $V = \pi^{-1}(W)$ and $V' = \pi^{-1}(W')$ satisfy the conclusion of condition (2). To show G acts on M properly discontinuous, we need to show for each $p \in M$, there exists an open neighborhood U of p such that $gU \cap U = \varnothing$ unless $g = e$. Let V be a precompact neighborhood of p. By Theorem 53, the set $G_{\overline{V}}$ is a compact subset of G, and hence finite because G is discrete, so we write $G_{\overline{V}} = \{e, g_1, \ldots, g_m\}$. Shrinking V if necessary, we may assume that $g_i^{-1}p \notin \overline{V}$ for $i = 1, \ldots, m$. Consider open subset

$$U = V \setminus (g_1\overline{V} \cup \cdots \cup g_m\overline{V})$$

It's clear $gU \cap U = \varnothing$ unless $g = e$.

Continuation.

Then $V=\pi^{-1}(W)$ and $V'=\pi^{-1}(W')$ satisfy the conclusion of condition (2).

To show G acts on M properly discontinuous, we need to show for each $p\in M$, there exists an open neighborhood U of p such that $gU\cap U=\varnothing$ unless $g=e$. Let V be a precompact neighborhood of p. By Theorem 53, the set $G_{\overline{V}}$ is a compact subset of G, and hence finite because G is discrete, so we write $G_{\overline{V}}=\{e,g_1,\dots,g_m\}$. Shrinking V if necessary, we may assume that $g_i^{-1}p\notin\overline{V}$ for $i=1,\dots,m$. Consider open subset

$$U=V\setminus(g_1\overline{V}\cup\cdots\cup g_m\overline{V})$$

It's clear $gU\cap U=\varnothing$ unless $g=e$.

Continuation.

Conversely, assume that (1) and (2) hold. Suppose $\{g_i\}$ is a sequence in G and $\{p_i\}$ is a sequence in M such that $p_i \to p$ and $g_i p_i \to p'$. If p and p' are in different orbits, there exist neighborhoods V of p and V' of p' as in (2), but for large enough i, we have $p_i \in V$ and $g_i p_i \in V'$, which contradicts the fact that $g_i V \cap V' = \varnothing$. This shows p and p' are in the same orbit, so there exists $g \in G$ such that $gp = p'$. This implies $g^{-1} g_i p_i \to p$. Since G acts on M properly discontinuous, there exists an open neighborhood U such that $gU \cap U = \varnothing$ unless $g = e$. For large enough i, one has p_i and $g^{-1} g_i p_i$ are both in U, and by the choice of U one has $g^{-1} g_i = e$. So $g_i = g$ when i is large enough, which certainly converges. By (2) of Theorem 53, the action is proper. $\square$

Theorem

Let M be a topological manifold and $\pi\colon \widetilde{M} \to M$ be a covering space. If $\mathrm{Aut}_\pi(\widetilde{M})$ is equipped with the discrete topology, then it acts on $\widetilde{M}$ continuously, freely and properly.

Proof.

By Lemma 37 one has $\mathrm{Aut}_\pi(\widetilde{M})$ acts on $\widetilde{M}$ freely and the action is also continuously since $\mathrm{Aut}_\pi(\widetilde{M})$ is equipped with discrete topology. To see the action is properly, it suffices to show the action satisfies the two conditions in Theorem 53.

Continuation.

(a) By Lemma 57, one already has $\mathrm{Aut}_\pi(\widetilde{M})$ acts on $\widetilde{M}$ properly discontinuous.

(b) Since $\pi\colon \widetilde{M} \to M$ is a normal covering, one has the orbit space is homeomorphic to M by Lemma 38 and thus orbit space is Hausdorff. If $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{M}$ are in different orbits, we can choose disjoint neighborhoods W of $\pi(\widetilde{x}_1)$ and W' of $\pi(\widetilde{x}_2)$ since orbit space is Hausdorff, and it follows that $V = \pi^{-1}(W)$ and $V' = \pi^{-1}(W')$ satisfy the second condition.

$\square$

Thanks!