ALGEBRAIC CURVES

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Contents

0. Preface

0.1. **Notations.**

(1) X, Y always denote Riemann surfaces.
(2) C always denotes the algebraic plane curve.
(3) $\Phi, \Psi \colon X \to Y$ always denote the holomorphic map between Riemann surfaces.
(4) f, g sometimes denote functions (smooth,holomorphic,meromorphic), sometimes denote polynomials, and sometimes denote the convergent power series.
(5) F, G always denote polynomials, and most of time they denote homogenous polynomials given by polynomials f, g.
(6) f_x always denote the partial derivative of f with respect to variable x.

0.2. **Motivations.**

0.2.1. *Meromorphic functions.* Let $U \subseteq \mathbb{C}$ be an open subset with coordinate $\{z\}$. In complex analysis we learnt that a meromorphic function f is a function that is holomorphic on all of U except for a set of isolated points, which are poles of the function. In other words, a meromorphic function can be regarded as a function $f\colon U \to \mathbb{C} \cup \{\infty\}$.

Topologically speaking, $\mathbb{C} \cup \{\infty\}$ is S^2, and in fact there is a complex manifold structure on it. More precisely, we can glue two pieces of complex plane via $w = 1/z$ to obtain a complex manifold called Riemann sphere

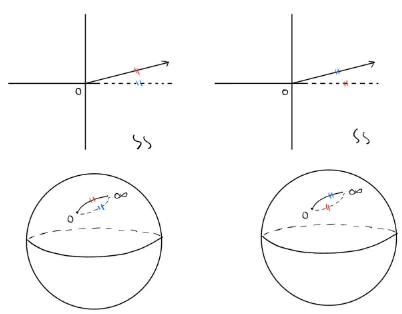
$$\mathbb{P}^1 = \mathbb{C} \cup_{\mathbb{C}^*} \mathbb{C},$$

and topologically $\mathbb{P}^1$ is exactly $\mathbb{C} \cup \{\infty\}$. By using this viewpoint, meromorphic function on U is exactly the same thing as holomorphic map from U to the Riemann sphere, and thus it gives us a lovely way to study meromorphic functions by using theories of holomorphic maps between Riemann surfaces, such as the number (counted with multiplicity) of zeros is equal to the number (counted with multiplicity) of poles.

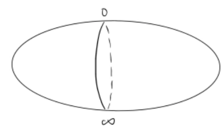
0.2.2. *Multivalueness of holomorphic functions.* For complex number $z = \rho e^{\sqrt{-1}\theta}$, where $\rho \in [0, \infty)$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, one has

$$(\sqrt{\rho} e^{\sqrt{-1}\theta/2})^2 = (\sqrt{\rho} e^{\sqrt{-1}\theta/2+\pi})^2 = z.$$

This shows there are two candidates for $\sqrt{z}$, and this phenomenon is called multivalueness of holomorphic function. If we define square root as $\sqrt{z} = \sqrt{\rho} e^{\sqrt{-1}\theta/2}$, then it's only well-defined on $\mathbb{C} \setminus [0, \infty)$, since it will "jump" when passing through the two sides of $[0, \infty)$, and $\mathbb{C} \setminus [0, \infty)$ is called a single value component of $\sqrt{z}$.



The ideal to solve this phenomenon is that, when passing the segment $[0, \infty)$, $\sqrt{z}$ should come into another single value component. In other words, if we want to make square root $\sqrt{z}$ defined on the whole complex plane, it should be no longer a function from $\mathbb{C}$ to $\mathbb{C}$, but a function from $\mathbb{C}$ to an object we obtained from gluing two single value components together. This construction also gives the Riemann sphere.

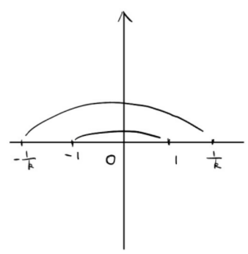


Similarly, $f(z) = \sqrt{1-z^2}$ is well-defined on $\mathbb{C}\setminus[-1,1]$, and it gives a well-defined function from $\mathbb{C}$ to something obtained by gluing two copies of $\mathbb{C}\setminus[-1,1]$, which is also the Riemann sphere.

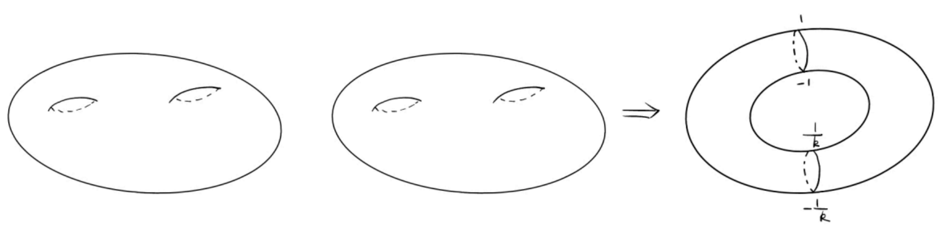
Now let's consider a more complicated example. For

$$f(z) = \sqrt{(1-z^2)(1-k^2z^2)},$$

where $k \neq \pm 1$, it gives a well-defined function on $\mathbb{C}$ minus two line segments connecting $-1, 1$ and $-1/k, 1/k$.



If we want to obtain a function defined on $\mathbb{C}$, we should glue two copies of above single value components. This gives a new Riemann surface called complex torus.



0.2.3. *Abelian integrals.*

Example 0.2.1 (arc-length of ellipse)**.** For ellipse given by $(x/a)^2+(y/b)^2 = 1$, by using parameterization

$$x = a\cos\theta$$
$$y = b\sin\theta,$$

it's easy to see arc-length is given by

$$\begin{aligned}\int_{\theta_0}^{\theta_1}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\mathrm{d}\theta &= a\int_{\theta_0}^{\theta_1}\sqrt{1-k^2\sin^2\theta}\mathrm{d}\theta \\ &\overset{z=\sin\theta}{=} \int_{z_0}^{z_1}\frac{\sqrt{1-k^2z^2}}{\sqrt{1-z^2}}\mathrm{d}z \\ &= \int_{z_0}^{z_1}\frac{1-k^2z^2}{\sqrt{(1-k^2z^2)(1-z^2)}}\mathrm{d}z,\end{aligned}$$

where $k=\sqrt{1-b^2/a^2}$. For $k=0$, since $\arcsin z$ is a primitive function of $1/\sqrt{1-z^2}$, one has

$$\int_{z_0}^{z_1}\frac{1}{\sqrt{1-z^2}}\mathrm{d}z=\arcsin z_1-\arcsin z_0.$$

The classical theory of "addition formula" gives

$$\sin(\alpha+\beta)=\sin\alpha\sqrt{1-\sin^2\beta}+\sqrt{1-\sin^2\alpha}\sin\beta.$$

In terms of integration

$$\int_0^{z_1}\frac{1}{\sqrt{1-t^2}}\mathrm{d}t+\int_0^{z_2}\frac{1}{\sqrt{1-t^2}}\mathrm{d}t=\int_0^{z_1\sqrt{1-z_2^2}+z_2\sqrt{1-z_1^2}}\frac{1}{\sqrt{1-t^2}}\mathrm{d}t.$$

For analogue of above case, if we define ellipse sine sn as

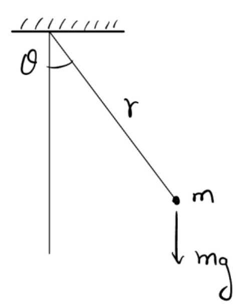
$$\int_0^{\arcsin z}\frac{1}{\sqrt{1-k^2\sin^2 t}}\mathrm{d}t=\mathrm{sn}^{-1}(z),$$

one can also show it satisfies some addition formula

$$\mathrm{sn}(\alpha+\beta)=\frac{\mathrm{sn}\,\alpha\sqrt{(1-\mathrm{sn}^2\beta)(1-k^2\,\mathrm{sn}^2\beta)}+\mathrm{sn}\,\beta\sqrt{(1-\mathrm{sn}^2\alpha)(1-k^2\,\mathrm{sn}^2\alpha)}}{1-k^2\,\mathrm{sn}^2\alpha\,\mathrm{sn}^2\beta}.$$

However, the ellipse sine cannot be expressed as an elementary function, and this is closely related to the fact that $y^2=(1-z^2)(1-k^2z^2)$ is not a Riemann sphere.

Example 0.2.2 (simple pendulum)**.** Suppose there is an object with mass m is released at $\theta=\alpha$ with zero initial velocity, and the length of pendulum is r.



The conservation of energy gives the following equation

$$\frac{1}{2}mr^2(\frac{\mathrm{d}\theta}{\mathrm{d}t})^2 = mgr\cos\theta - mgr\cos\alpha.$$

In other words,

$$(\frac{\mathrm{d}\theta}{\mathrm{d}t})^2 = 2\frac{g}{r}(\cos\theta - \cos\alpha) = 4\frac{g}{r}(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}). \tag{0.1}$$

An approximation with θ sufficiently small, one has

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \sqrt{\frac{g}{r}(\alpha^2 - \theta^2)}.$$

This shows

$$t = \int_0^\theta \sqrt{\frac{r}{g}}\frac{1}{\sqrt{\alpha^2 - s^2}}\mathrm{d}s.$$

Thus the period of the simple pendulum is given by

$$T = 4\int_0^\alpha \sqrt{\frac{r}{g}}\frac{1}{\sqrt{\alpha^2 - s^2}}\mathrm{d}s = 2\pi\sqrt{\frac{r}{g}}.$$

However, if we don't use the approximation, and use substitution

$$\sin\varphi = \frac{\sin\frac{\theta}{2}}{\sin\frac{\alpha}{2}}$$

in (0.1), one has

$$(\frac{\mathrm{d}\varphi}{\mathrm{d}t})^2 = \frac{g}{r}(1 - \sin^2\frac{\alpha}{2}\sin^2\varphi).$$

Then

$$t = \sqrt{\frac{r}{g}}\int_0^\varphi \frac{1}{\sqrt{1 - k^2\sin^2 s}}\mathrm{d}s,$$

where $k = \sin\dfrac{\alpha}{2}$, and thus explicit formula for the period of simple pendulum is

$$T = 4\sqrt{\frac{r}{g}}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2\sin^2 s}}\mathrm{d}s.$$

This is exactly ellipse integral.

Remark 0.2.1 (general case). Let f be a polynomial of two variables and $y = \Phi(X)$ be a solution of equation $f(x, y) = 0$. Then

$$\int R(x, f(x)) = 0$$

can be expressed as elementary function if and only if $\deg f = 0, 1, 2$, and in fact $\deg f$ is closely related to the topology of algebraic curves.

1. Riemann surface and algebraic curves

1.1. Riemann surface.

1.1.1. *Definitions.*

Definition 1.1.1 (complex atlas)**.** Let X be a topological space. A complex atlas on X consists of the following data:
(1) $\{U_i\}_{i\in I}$ is an open covering of X.
(2) For each $i \in I$, there exists a homeomorphism $\varphi_i\colon U_i \to \varphi_i(U_i) \subseteq \mathbb{C}$.
(3) For $i, j \in I$, if $U_i \cap U_j \neq \varnothing$, then the transition function

$$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1}\colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

is holomorphic.

Remark 1.1.1. If $\{U_i, \varphi_i\}$ is a complex atlas on a topological space, then all transition functions φ_{ij} are not only holomorphic, but biholomorphic with inverse φ_{ji}.

Definition 1.1.2 (complex structure)**.** Two complex atlas $\mathscr{A}, \mathscr{B}$ are equivalent if $\mathscr{A} \cup \mathscr{B}$ is also a complex atlas, and a complex structure is an equivalent class of atlas on X.

Definition 1.1.3 (Riemann surface)**.** A Riemann surface is a connected, second countable, Hausdorff topological space X together with a complex structure on X.

Remark 1.1.2. A Riemann surface X is a complex manifold with $\dim_{\mathbb{C}} X = 1$, and it's called a surface since $\dim_{\mathbb{R}} X = 2$.

1.1.2. *Examples.*

Example 1.1.1 (Riemann sphere)**.** Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$ be 2-sphere and $\{U_1 = S^2 \setminus (0, 0, 1), U_2 = S^2 \setminus (0, 0, -1)\}$ be an open covering of S^2. Consider

$$\begin{aligned} \varphi_1\colon U_1 &\to \mathbb{C} \\ (x_1, x_2, x_3) &\mapsto \frac{x_1}{1-x_3} + \sqrt{-1}\frac{x_2}{1-x_3}, \end{aligned}$$

and

$$\begin{aligned} \varphi_2\colon U_1 &\to \mathbb{C} \\ (x_1, x_2, x_3) &\mapsto \frac{x_1}{1+x_3} - \sqrt{-1}\frac{x_2}{1+x_3}. \end{aligned}$$

A direct computation shows that

$$\left(\frac{x_1}{1-x_3} + \sqrt{-1}\frac{x_2}{1-x_3}\right)\left(\frac{x_1}{1+x_3} - \sqrt{-1}\frac{x_2}{1+x_3}\right) = \frac{x_1^2 + x_2^2}{1 - x_3^2} = 1,$$

and thus the transition function $\varphi_2 \circ \varphi_1^{-1}(z) = 1/z$. This shows $\{U_1, U_2\}$ is a complex atlas of S^2. It's clear as a topological space S^2 is connected, second countable and Hausdorff, and thus S^2 is a Riemann surface, called Riemann sphere.

Remark 1.1.3. There is another construction of Riemann sphere, given by gluing two complex planes together on $\mathbb{C}^*$, and the gluing data on $\mathbb{C}^*$ is given by $z \sim 1/w$. One thing to mention is that it's not clear object constructed in this way is Hausdorff. For example, if we glue two complex planes together on $\mathbb{C}^*$ by using gluing data $z \sim w$, then the object obtained is not Hausdorff.

Example 1.1.2 (complex projective line)**.** The complex projective line $\mathbb{P}^1 = \mathbb{C}^2 \setminus (0,0)/\sim$, where $(x,y) \sim (z,w)$ if and only if $(\lambda x, \lambda y) = (z,w)$ for some $\lambda \in \mathbb{C}^*$, and the equivalent class for (x,y) is denoted by $[x,y]$, called the homogenous coordinate. The quotient topology on $\mathbb{P}^1$ which makes it second countable, Hausdorff and compact. Consider

$$U_0 = \{[z,w] \mid z \neq 0\} \xrightarrow{\varphi_0} \mathbb{C}$$

where φ_0 is defined as $\varphi_1([z,w]) = z/w$. Similarly consider

$$U_1 = \{[z,w] \mid w \neq 0\} \xrightarrow{\varphi_1} \mathbb{C}$$

where φ_1 is defined as $\varphi_1([z,w]) = w/z$. For $z \in \varphi_1(U_0 \cap U_1)$, one has

$$z \xrightarrow{\varphi_1^{-1}} [z:1] = [1:\frac{1}{z}] \xrightarrow{\varphi_0} \frac{1}{z}.$$

This shows the transition function $\varphi_{01}(z) = 1/z$, which is holomorphic, and thus $\{(U_0,\varphi_0),(U_1,\varphi_1)\}$ gives a complex atlas on $\mathbb{P}^1$.

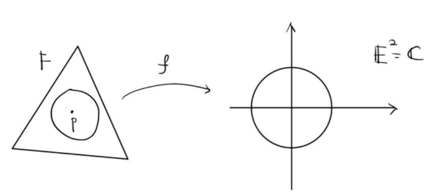
Remark 1.1.4 (complex projective space). In general, the complex projective space $\mathbb{P}^n$ is defined by $\mathbb{P}^n = \mathbb{C}^{n+1} \setminus (0,0)/\sim$, where $(x_0,x_1,\dots,x_n) \sim (y_0,y_1,\dots,y_n)$ if and only if there exists $\lambda \in \mathbb{C}^*$ such that $y_i = \lambda x_i$ holds for all $i = 0,1,\dots,n$, and the equivalent class $[x_0 : x_1 : \cdots : x_n]$ is call the homogenous coordinate of $\mathbb{P}^n$. There is a canonical affine open covering $\{(U_i,\varphi_i)\}$ of $\mathbb{P}^n$ defined by

$$U_i = \{[x_0 : x_1 : \cdots : x_n] \mid x_i \neq 0\} \xrightarrow{\varphi_i} \mathbb{C}^n,$$

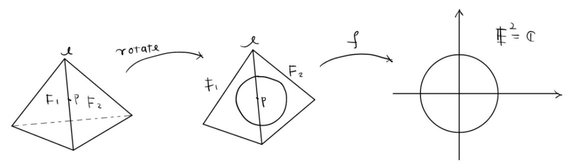
where $\varphi_i([x_0 : x_1 : \cdots : x_n]) = (x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i)$, and it makes $\mathbb{P}^n$ to be a complex n-manifold.

Example 1.1.3. Let P be a convex polyhedra in Euclidean 3-dimensional space $\mathbb{E}^3$. Topologically P is S^2, and let's construct a complex atlas on it.

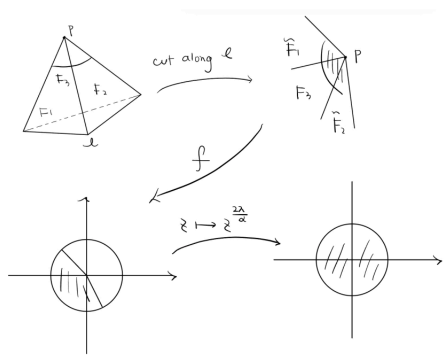
(1) Suppose p is the interior point of some face F. Since F can be isometrically embedded into $\mathbb{E}^2$, we choose an orientation-preserving, isometric embedding f which maps an open neighborhood U of p into $\mathbb{E}^2 = \mathbb{C}$.



(2) Suppose p is the interior point of some edge $l = F_1 \cap F_2$. Firstly we rotate F_2 along l to the plane of F_1, and then choose an orientation-preserving, isometric embedding f which maps an open neighborhood U of p into $\mathbb{E}^2 = \mathbb{C}$.



(3) Suppose p is an vertex which is the intersection of three faces F_1, F_2 and F_3. Firstly we cut it along some edge $l = F_1 \cap F_2$, and then rotate F_1, F_2 to the plane of F_3. Then we use some orientation-preserving, isometric embedding f to map it into $\mathbb{E}^2$, and finally composite it with $z \mapsto z^{2\pi/\alpha}$.



Exercise 1.1.1. Prove that above constructions give a complex atlas on convex polyhedra.

Remark 1.1.5. All of above three examples give complex structure on topological sphere S^2, and we will see all of them are the "same" after we define the isomorphism between Riemann surfaces. In fact, there is only one complex structure on S^2.

Example 1.1.4 (complex torus)**.** For non-zero $w_1, w_2 \in \mathbb{C}$ such that w_1, w_2 are $\mathbb{R}$-linearly independent, $L = \mathbb{Z}\,w_1 + \mathbb{Z}\,w_2$ is a discrete subgroup of $(\mathbb{C}, +)$. Then $T = \mathbb{C}/L$ equipped with quotient topology is a connected, Hausdorff and second countable topological space. Let $\pi\colon \mathbb{C} \to T$ be the natural projection. For $p \in T$, suppose z_0 is an inverse image of p. For $\varepsilon \in \mathbb{R}_{>0}$ such that

$$B_{2\varepsilon}(0) \cap L = \{0\},$$

one has $B_\varepsilon(z_0) \xrightarrow{\pi} \pi(B_\varepsilon(z_0)) \subseteq T$ is injective, and thus $\pi^{-1}\colon \pi(B_\epsilon(z_0)) \to B_\epsilon(z_0) \subseteq \mathbb{C}$ is a homeomorphism. Then $\{\pi(B_\varepsilon(\pi^{-1}(p))\}_{p\in T}$ gives an open covering of T, and together with π^{-1} it gives a complex atlas of T.

Remark 1.1.6. It's clear complex structure constructed above depends on the choice of w_1, w_2, but it's not obvious to see whether w_1, w_2 and w_1', w_2' give the same complex structure or not. Moreover, all complex structure on torus are arisen in this way.

1.1.3. *Morphisms.*

Definition 1.1.4 (holomorphic map)**.** Let X, Y be two Riemann surfaces and $\Phi\colon X \to Y$ be a continous map. For $p \in X$, F is called holomorphic at x, if there exists a chart (U, φ) of x, and a chart (V, ψ) of $\Phi(x)$, such that

$$\psi \circ \Phi \circ \varphi^{-1}\colon \varphi\left(U \cap \Phi^{-1}(V)\right) \to \psi\left(V \cap \Phi(U)\right)$$

is holomorphic at $\varphi(x)$. Moreover, Φ is called holomorphic in an open subset $W \subseteq X$, if Φ is holomorphic at any point in W.

Exercise 1.1.2. Show that the definition of holomorphic map is independent of the choice of charts.

Definition 1.1.5 (isomorphism)**.** Let $\Phi\colon X \to Y$ be a holomorphic map between Riemann surfaces. Φ is called an isomorphism if it's bijective and holomorphic.

Proposition 1.1.1. Let $\Phi\colon X \to Y$ be a holomorphic map between Riemann surfaces. Φ is an isomorphism if and only if Φ has an two-side inverse Ψ, and Ψ is holomorphic.

Proposition 1.1.2. The complex projective space is isomorphic to Riemann sphere.

Theorem 1.1.1 (open map theorem)**.** Let $\Phi\colon X \to Y$ be a non-constant holomorphic map between Riemann surfaces. Then Φ is open.

Corollary 1.1.1. Let $\Phi\colon X \to Y$ be a non-constant holomorphic map between Riemann surfaces and X is compact. Then $\Phi(X) = Y$, and thus Y is compact.

Proof. By open map theorem, $\Phi(X)$ is an open subset of Y, and $\Phi(X)$ is compact in Y, since continous function maps compact set to compact set. Then $\Phi(X)$ is both open and closed in Y, and thus $\Phi(X) = Y$ since Y is assumed to be connected. □

Theorem 1.1.2. Let $\Phi\colon X \to Y$ be a non-constant holomorphic map between Riemann surfaces. Then for any $p \in Y$, $\Phi^{-1}(y)$ is a discrete set. In particular, if X is compact, then $\Phi^{-1}(y)$ is a non-empty finite set.

1.1.4. *Meromorphic functions.*

Definition 1.1.6 (singularity)**.** Let X be a Riemann surface and f be a holomorphic function defined on $U \setminus \{x\}$ where $U \subseteq X$ is an open subset. The point x is called a removbale singularity/pole/essential singularity, if there exists a chart (U, φ) of x, such that $f \circ \varphi^{-1} \colon \varphi(U) \to \mathbb{C}$ has $\varphi(x)$ as a removbale singularity/pole/essential singularity.

Remark 1.1.7.

(1) If $|f(x)|$ is bounded in a punctured neighborhood of x, then x is a removable singularity, and we can cancel the singularity by defining $f(x) = \lim_{y \to x} f(y)$.
(2) If $\lim_{y \to x} |f(y)| = \infty$, then x is a pole.
(3) If $\lim_{y \to x} |f(y)|$ doesn't exist, then p is a essential singularity.

Definition 1.1.7 (meromorphic function)**.** Let X be a Riemann surface and f be a holomorphic function defined on $U \setminus \{x\}$ where $U \subseteq X$ is an open subset.

(1) f is called a meromorphic function at x if x is either a removbale singularity or a pole, or f is holomorphic at x.
(2) f is called a meromorphic function on W, if it's meromorphic at any point $x \in W$.

Remark 1.1.8. If f, g are meromorphic on W, then $f \pm g, fg$ are meromorphic on W. If in addition, $g \not\equiv 0$, then f/g is also meromorphic on W. In other words, the set of meromorphic functions on W forms a field, which is called meromorphic function field.

Example 1.1.5. Consider f, g are two polynomials in variable z with $g \not\equiv 0$, then f/g is a meromorphic function on Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$. In fact, all meromorphic functions on S^2 are in this form.

Theorem 1.1.3 (discreteness of singularities and zeros)**.** Let X be a Riemann surface and $W \subseteq X$ be an open subset. If f is a meromorphic function on W, then set of singularities and zeros of f is discrete, unless $f \equiv 0$.

Corollary 1.1.2. Let X be a compact Riemann surface.

(1) If f is a non-zero meromorphic function, then f has finitely many poles and zeros on X.
(2) If f, g are two meromorphic functions on an open subset $W \subseteq X$, and f agrees with g on a set with limit point in W, then $f \equiv g$.

1.2. **Algebraic curves.**

1.2.1. *Affine plane curves.* Let $V \subseteq \mathbb{C}$ be a connected open subset and g be a holomorphic function defined on V. The graph X of g, as a subset of $\mathbb{C}^2$ is defined by

$$\{(z, g(z)) \mid z \in V\}.$$

Given X the subspace topology, and let $\pi\colon X \to V$ be the projection to the first factor. Note that π is a homeomorphism, whose inverse sends the point $z \in V$ to the ordered pair $(z, g(z))$. This makes X a Riemann surface.

A generalization of the graph of holomorphic function is that we consider "Riemann surface" which is locally a graph, but perhaps not globally. The tools we use is implicit function theorem in fact.

Theorem 1.2.1 (The implicit function theorem)**.** Let $f(z,w)\colon \mathbb{C}^2 \to \mathbb{C}$ be holomorphic function of two variables and $X = \{(z,w) \in \mathbb{C}^2 \mid f(z,w) = 0\}$ be its zero loucs. Let $p = (z_0, w_0)$ be a point of X and $\partial f/\partial z(p) \neq 0$. Then there exists a function $g(w)$ defined and holomorphic in a neighborhood of w_0 such that, near p, X is equal to the graph $z = g(w)$.

Method one. If we write $z = a + \sqrt{-1}b, w = c + \sqrt{-1}d$ and $f(z,w) = u + \sqrt{-1}v$, then u, v are smooth functions of a, b, c, d. Moreover, the Cauchy-Riemann equations give

$$\frac{\partial f}{\partial z} = \frac{\partial u}{\partial a} + \sqrt{-1}\frac{\partial v}{\partial a} = \frac{\partial v}{\partial b} - \sqrt{-1}\frac{\partial u}{\partial b} = A + \sqrt{-1}B.$$

Then

$$\frac{\partial(u,v)}{\partial(a,b)} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

and $\det \dfrac{\partial(u,v)}{\partial(a,b)} = A^2 + B^2 \neq 0$ if and only if $A + \sqrt{-1}B \neq 0$. Then the classical implicit function theorem implies the zero loucs

$$\begin{cases} u = 0 \\ v = 0 \end{cases}$$

is locally given by

$$\begin{cases} a = a(c,d) \\ b = b(c,d). \end{cases}$$

In other words, $z = g(w)$. Now it suffices to compute $\partial g/\partial \overline{w}$ to show g is holomorphic. Again by Cauchy-Riemann equations

$$\frac{\partial f}{\partial w} = \frac{\partial u}{\partial c} + \sqrt{-1}\frac{\partial v}{\partial c} = \frac{\partial v}{\partial d} - \sqrt{-1}\frac{\partial u}{\partial d} = C + \sqrt{-1}D.$$

Then by chain rule one has

$$\begin{aligned} \frac{\partial(a,b)}{\partial(c,d)} &= \left(\frac{\partial(u,v)}{\partial(a,b)}\right)^{-1} \frac{\partial(u,v)}{\partial(c,d)} \\ &= \begin{pmatrix} A & B \\ -B & A \end{pmatrix}^{-1} \begin{pmatrix} C & D \\ -D & C \end{pmatrix} \\ &= \frac{1}{A^2+B^2} \begin{pmatrix} AC+BD & AD-BC \\ BC-AD & BD+AC \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial g}{\partial \overline{w}} &= \frac{1}{2}\left(\frac{\partial}{\partial c} + \sqrt{-1}\frac{\partial}{\partial d}\right)\left(a + \sqrt{-1}b\right) \\ &= \frac{1}{2}\left(\frac{\partial a}{\partial c} + \sqrt{-1}\frac{\partial b}{\partial c} + \sqrt{-1}\frac{\partial a}{\partial d} - \frac{\partial b}{\partial d}\right) \\ &= 0\end{aligned}$$

□

Method two. Firstly let's recall some basic facts in complex analysis: For a holomorphic function f defined on U, the integral

$$\frac{1}{2\pi\sqrt{-1}}\oint_{\partial U}\frac{f'(z)}{f(z)}\mathrm{d}z$$

counts the number of zeros of $f(z)$ in U with multiplicity, and the integral

$$\frac{1}{2\pi\sqrt{-1}}\oint_{\partial U} z\frac{f'(z)}{f(z)}\mathrm{d}z$$

is the summation of zeros of $f(z)$ in U. Now let's prove the implicit function theorem by using above observation. Fix $w = w_0$, the holomorphic function $f(z, w_0)$ has a zero at $z = z_0$, and we may choose an open neighborhood U of z_0 such that z_0 is the only zero of $f(z, w_0)$ in U since holomorphic function has discrete zeros. Consider the integral

$$\frac{1}{2\pi\sqrt{-1}}\oint_{\partial U}\frac{f_z(z, w)}{f(z, w)}\mathrm{d}z = N(w),$$

which is well-defined on sufficiently small neighborhood D_{w_0} of w_0. It gives a continous, integer-valued function with $N(w_0) = 1$. This shows $N(w) = 1$ for all $w \in D_{w_0}$, and thus $f(z, w)$ has only one zero for every $w \in D_{w_0}$. Moreover, this zero point z is given by

$$\frac{1}{2\pi\sqrt{-1}}\oint_{\partial U} z\frac{f_z(z, w)}{f(z, w)}\mathrm{d}z = g(w),$$

which is holomorphic with respect to w. □

Definition 1.2.1 (affine plane curve)**.** An affine plane curve is the loucs of zeros in $\mathbb{C}^2$ of a (non-trivial) polynomial $p(z, w)$.

Definition 1.2.2 (non-singular)**.**

(1) A polynomial $f(x, y)$ is non-singular at root x if either $\partial f/\partial x$ or $\partial f/\partial y$ is not zero at x, otherwise it's called singular.
(2) The affine plane curve X defined by $f(x, y)$ is non-singular is non-singular at $p \in X$ if f is non-singular at x.
(3) The curve X is non-singular if it's non-singular at each of its points.

Example 1.2.1. The affine plane curve $C \subseteq \mathbb{C}^2$ defined by $x^2 + y^2 - 1$ is non-singular.

Given a non-singular affine plane curve C, by the implicit function theorem, one has C is locally a graph, and thus it gives a complex structure of C. To be precise, suppose C is defined by the non-singular polynomial $f(x, w)$. Let $p = (x_0, y_0) \in C$ with $\partial f/\partial x(p) \neq 0$, then there exists a holomorphic function $g(x)$ such that in an open neighborhood U of p, C is the graph $w = g(x)$. Thus the projection $\pi\colon U \to \mathbb{C}$, which maps $(x, y) \to x$ is a homeomorphism from U to its image, which is an open subset in $\mathbb{C}$. This gives a complex chart of C.

A straightforward computation shows that complex charts given as above are compatible with each other, and thus it gives a complex structure on C. Moreover, C is second countable and Hausdorff, as a subspace of $\mathbb{C}^2$. The only thing we need to check is C is connected. However, if p is an arbitary non-singular polynomial, the affine plane curve defined by p may not be connected. For example, consider

$$p(z, w) = (z + w)(z + w - 1).$$

Then the affine plane curve defined by above non-singular polynomial is the union of two complex planes which do not meet. Later in Section 3.2.3 we will show that the plane curve defined by an irreducible polynomial must be connected. Thus we have the following theorem.

Theorem 1.2.2. A non-singular affine plane curve defined by an irreducible polynomial is a Riemann surface.

1.2.2. *Projective plane curve.*

Definition 1.2.3 (projective plane curve)**.** Let F be a homogenous polynomial in $\mathbb{C}[x, y, z]$. A projective plane curve C defined by F is the zero loucs of F, that is,

$$C = \{[x : y : z] \in \mathbb{P}^2 \mid F(x, y, z) = 0\}.$$

Remark 1.2.1 (relations between affine plane curve and projective plane curve)*.* Given a projective plane curve C given by homogenous polynomial F. Consider

$$\begin{aligned} \varphi_0\colon U_0 = \mathbb{C}^2 &\to \mathbb{P}^2 \\ (y, z) &\mapsto [1 : y : z] \end{aligned}$$

Then $\varphi_0^{-1}(U_0 \cap C) = \{(y, z) \in \mathbb{C}^2 \mid F(1, y, z) = 0\}$ is an affine plane curve, and similarly there are other affine plane curves given by $\varphi_0^{-1}(U_1 \cap C)$ and $\varphi_0^{-1}(U_2 \cap C)$. Conversely, given an affine plane curve C defined by $f \in \mathbb{C}[y, z]$. Consider the homogenous polynomial $F(x, y, z)$ defined by

$$F(x, y, z) = x^d f(\frac{y}{x}, \frac{z}{x})$$

where $d = \deg f$. Then P defines a projective plane curve such that the affine plane curve it gives on affine chart U_0 is exactly C.

Definition 1.2.4 (non-singular)**.** A projective plane curve C is non-singular if the affine plane curves $\varphi_i^{-1}(U_i\cap C)$ are non-singular for $i=0,1,2$, where $\varphi_i\colon U_i\to\mathbb{P}^2$ are standard affine covering of $\mathbb{P}^2$.

Proposition 1.2.1. A projective plane curve $C=\{[x:y:z]:F(x,y,z)=0\}\subseteq\mathbb{P}^2$ is non-singular if and only if

$$F=\frac{\partial F}{\partial x}=\frac{\partial F}{\partial y}=\frac{\partial F}{\partial y}=0$$

has no solution in $\mathbb{P}^2$.

Proof. Since F is a homogenous polynomial, it satisfies the Euler's formula

$$dF=x\frac{\partial F}{\partial x}+y\frac{\partial F}{\partial y}+z\frac{\partial F}{\partial z},$$

where $d=\deg F$. Now let's start our proof as follows:

(1) Suppose $F=\partial F/\partial x=\partial F/\partial y=\partial F/\partial z=0$ has a solution (a,b,c) with $a\neq 0$. Then

$$\begin{aligned}\frac{\partial F}{\partial y}(1,\frac{b}{a},\frac{c}{a})&=\frac{1}{a^{d-1}}\frac{\partial F}{\partial y}(a,b,c)=0\\ \frac{\partial F}{\partial z}(1,\frac{b}{a},\frac{c}{a})&=\frac{1}{a^{d-1}}\frac{\partial F}{\partial z}(a,b,c)=0\\ F(1,\frac{b}{a},\frac{c}{a})&=\frac{1}{a^{d}}F(a,b,c)=0.\end{aligned}$$

This shows the affine plane curve $\varphi_0^{-1}(U_0\cap C)$ is singular, and thus C is singular.

(2) Conversely, if the projective plane curve defined by F is singular, without lose of generality we may assume $X_0:=\varphi_0^{-1}(U_0\cap C)$ is singular. Then there exists a solution $(b,c)\in\mathbb{C}^2$ such that

$$F(1,b,c)=\frac{\partial F}{\partial y}(1,b,c)=\frac{\partial F}{\partial z}(1,b,c)=0.$$

By Euler's formula one has

$$\frac{\partial F}{\partial x}(1,b,c)=dF(1,b,c)-b\frac{\partial F}{\partial y}-c\frac{\partial F}{\partial z}=0.$$

As a consequence, $(1,a,b)$ is a solution of $F=\partial F/\partial x=\partial F/\partial y=\partial F/\partial z=0$.

□

Theorem 1.2.3. Any non-singular projective plane curve C is a compact Riemann surface.

Proof. Later we will show that a non-singular homogenous polynomial must be irreducible (See Proposition 3.2.1). Then the three affine charts of C are non-singular affine plane curve defined by irreducible polynomials, and thus Riemann surfaces by Theorem 1.2.2. A straightforward computation shows that the complex structures on each affine charts are compatible, and thus C is a Riemann surface. Moreover, it's compact since $\mathbb{P}^2$ is compact and C is a closed subset of $\mathbb{P}^2$. □

Remark 1.2.2. One way to understand projective plane curve is to regard it as a compactifications of affine plane curve.

Example 1.2.2 (Fermat curve)**.** $x^d + y^d = z^d$ gives a non-singular projective plane curve.

Example 1.2.3. The polynomial $f(x,y) = y^2 - (1-x^2)(1-k^2x^2), k \neq 0, \pm 1$ gives a non-singular affine plane curve C. Now we consider the compactification of C. Let $F(x,y,z)$ be the homogenous polynomial given by $f(x,y)$, that is,

$$F(x,y,z) = z^2y^2 - (z^2 - x^2)(z^2 - k^2x^2).$$

$F(x,y,z)$ gives a projective plane curve, and the affine plane curve it gives on the affine chart U_2 is exactly C, so it suffices to see the affine plane curves it gives on the other affine charts.

(1) The affine plane curve it gives on the affine chart U_1 is defined by

$$f(x,1,z) = z^2 - (z^2 - x^2)(z^2 - k^2x^2).$$

In this case there is a new point $[0:1:0]$, which is singular.

(2) The affine plane curve it gives on the affine chart U_0 is defined by

$$f(1,y,z) = z^2y^2 - (z^2 - 1)(z^2 - k^2).$$

But in this case, there is no more new point since there is no solution satisfying $z = 0$.

In a summary, the compactification of the affine plane curve C adds a singular point, and later we will see how to handle with singularities by resolutions.

1.2.3. *Quadratic.* A homogenous polynomial F of degree 2 can be written as

$$F = (x,y,z)A\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where $A \in M_{3\times 3}(\mathbb{C})$ is a symmetric matrix. In this section we will see the projective plane curve C defined by F is determined by the rank of A.

Proposition 1.2.2. If $\operatorname{rk} A = 3$, then F is non-singular, and C is isomorphic to $\mathbb{P}^1$.

Method one. If $\operatorname{rk} A = 3$, then there exists $P \in \mathrm{GL}(3,\mathbb{C})$ such that

$$P^TAP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

This shows that afer a suitable change of coordinate, we may assume the projective plane curve C defined by F is $\{[x : y : z] \mid x^2+y^2-z^2 = 0\} \subseteq \mathbb{P}^2$. The following map gives an isomorphism between C and $\mathbb{P}^1$.

$$\begin{aligned}\Phi\colon \mathbb{P}^1 &\to C\\ [1:t] &\mapsto [1-t^2 : 2t : 1+t^2].\end{aligned}$$

□

Method two. Consider the following holomorphic embedding

$$\begin{aligned}\Phi\colon \mathbb{P}^1 &\to \mathbb{P}^2\\ [t_0:t_1] &\mapsto [t_0^2 : t_0t_1 : t_1^2].\end{aligned}$$

Note that the image of F is a projective plane curve defined by the equation $xz = y^2$. On the other hand, after a suitable change of coordinate we may also assume C is defined by this equation since there also exists $P \in \mathrm{GL}(3,\mathbb{C})$ such that

$$P^TAP = \begin{pmatrix} 0 & 0 & \frac{1}{2}\\ 0 & 1 & 0\\ \frac{1}{2} & 0 & 0\end{pmatrix}.$$

□

Proposition 1.2.3. If $\operatorname{rk} A = 2$, then C is isomorphic to the union of two $\mathbb{P}^1$.

Proof. If $\operatorname{rk} A = 2$, then there exists $P \in \mathrm{GL}(3,\mathbb{C})$ such that

$$P^TAP = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

This shows the projective plane curve C is defined by $x^2 + y^2 = (x + \sqrt{-1}y)(x-\sqrt{-1}y)$, which is the union of two $\mathbb{P}^1$ which intersects at $[0:0:1]$. In particular, it's singular. □

Proposition 1.2.4. If $\operatorname{rk} A = 1$, then C is isomorphic to a double line.

Proof. If $\operatorname{rk} A = 1$, then there exists $P \in \mathrm{GL}(3,\mathbb{C})$ such that

$$P^TAP = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

This shows the projective plane curve C is defined by $x^2 = 0$, which is a singular projective plane curve called double line. □

2. Ramification

Topologically speaking a Riemann surface is an orientable 2-dimensional real manifold without boundary. In particular, the topology of a compact Riemann surface can be classified by its genus. So there is a natural question: Given a non-singular projective plane curve C defined by the homogenous polynomial $F(x, y, z) = y^2z - x(x - z)(x - \lambda z), \lambda \neq 0, 1$, topologically C is a closed orientable surface, is there any way to compute its genus?

Consider the following map

$$\begin{aligned} \Phi\colon C \setminus [0:1:0] &\to \mathbb{P}^1 \\ [x:y:z] &\mapsto [x:z]. \end{aligned}$$

It's clear that Φ is well-defined holomorphic map. If we desire to extend F to a holomorphic map $\widetilde{\Phi}$ defined on C, we need to consider the behavior of C around $[0:1:0]$. On affine chart $U_1 = \{[x:1:z] \mid x, z \in \mathbb{C}\}$, it gives an affine plane curve defined by

$$f(x, z) = z - x(x - z)(x - \lambda z).$$

A direct computation shows that

$$\frac{\partial f}{\partial z}\bigg|_{(0,0)} = 1, \quad \frac{\partial f}{\partial x}\bigg|_{(0,0)} = 0.$$

Then by implicit function theorem, C is given by $[x:1:z(x)]$ locally around $[0:1:0]$, and

$$z'(0) = -\frac{\partial p}{\partial x}\bigg|_{(0,0)} / \frac{\partial p}{\partial z}\bigg|_{(0,0)} = 0/1 = 0.$$

Thus $x = 0$ is a removable singularity of $z(x)/x$, so it's reasonable to define $\widetilde{\Phi}([0:1:0]) = [1:0]$ to give an extension of F since for $x \neq 0$,

$$\Phi([x:1:z(x)]) = [x:z(x)] = [1:\frac{z(x)}{x}].$$

There are four special points for $\widetilde{\Phi}\colon C \to \mathbb{P}^1$, listed as follows

$$\begin{aligned} [0:1:0] &\mapsto [1:0] \\ [0:0:1] &\mapsto [0:1] \\ [z:0:1] &\mapsto [z:1] \\ [\lambda z:0:1] &\mapsto [\lambda z:1]. \end{aligned}$$

These points are called ramification points/values of $\widetilde{\Phi}$, and besides these points, $\widetilde{\Phi}$ is a double covering. Such a holomorphic map is called a ramification covering, and in this section we will show that all holomorphic maps between Riemann surfaces are ramification coverings. Moreover, we introduce the Riemann-Hurwitz formula, which gives a method to compute the genus of the ramification covering of a given space.

2.1. Ramification covering.

Theorem 2.1.1 (local normal form)**.** Let $\Phi\colon X \to Y$ be a non-constant holomorphic map. Then there are local coordinates (U, φ) and (V, ψ) of p and $\Phi(p)$ respectively, such that

$$\psi \circ \Phi \circ \varphi^{-1}(z) = z^k$$

holds for all $z \in \varphi(U \cap \Phi^{-1}(V))$.

Proof. Firstly we fix a local coordinate (V, ψ) of $\Phi(p)$, and choose a local coordinate (U_1, φ_1) of p such that $\Phi(U) \subset V$. If we denote $\psi \circ \Phi \circ \varphi_1^{-1} = T$, then $T(0) = 0$. Suppose the Taylor expansion of T at $w = 0$ is

$$T(w) = \sum_{k=m}^{\infty} a_k w^k, \quad a_m \neq 0.$$

Then $T(w) = w^m S(w)$, where $S(w)$ is a holomorphic function with $S(0) \neq 0$, and thus there exists a holomorphic function $R(w)$ such that $R^m(w) = S(w)$.

Then $T(w) = (wR(w))^m = (\eta(w))^m$, where $\eta(0) = 0, \eta'(0) = R(0) \neq 0$. By inverse function theorem, there exists a sufficiently small neighborhood $U \subseteq U_1$ of p such that η is invertible in $\varphi_1(U)$, and thus this gives a new local coordinate of p as

$$U_1 \supseteq U \xrightarrow{\varphi_1} \varphi_1(U) \xrightarrow{\eta} \eta \circ \varphi_1(U) \subset \mathbb{C}.$$

If we define $\varphi = \eta \circ \varphi_1$, then with respect to (U, φ) and (V, ψ), the local representation of Φ is given by

$$\psi \circ \Phi \circ \varphi^{-1}(z) = \psi \circ \Phi \circ \varphi_1^{-1} \circ \eta^{-1}(z) = T(\eta^{-1}(z)) = z^m.$$

□

Definition 2.1.1 (multiplicity)**.** Let $\Phi\colon X \to Y$ be a holomorphic map between Riemann surfaces. If its local normal form at point $p \in X$ is given by $z \mapsto z^k$, then k is called the multiplicity[1] of F at p, denoted by $\mathrm{mult}_p(\Phi)$.

Definition 2.1.2 (ramification point and ramification value)**.** Let $\Phi\colon X \to Y$ be a holomorphic map between Riemann surfaces. A point $p \in X$ is called a ramification point if $\mathrm{mult}_p(\Phi) > 1$, and the image of ramification point is called a ramification value.

Lemma 2.1.1. Let $\Phi\colon X \to Y$ be a non-constant holomorphic map between Riemann surfaces. A point $p \in X$ is a ramification point if there exists some local representation of Φ, denoted by T, such that $T'(0) = 0$.

Corollary 2.1.1. The set of ramification points of a holomorphic map is a discrete set.

[1]Sometimes this number is also called ramification of F at p.

Theorem 2.1.2. Let $\Phi\colon X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces and define

$$d_q(\Phi) = \sum_{p\in\Phi^{-1}(q)} \mathrm{mult}_p(\Phi).$$

Then $d_q(\Phi)$ is independent of $q \in Y$, which is called the degree of F, and denoted by $\deg(\Phi)$.

Proof. Suppose $X = Y = \mathbb{D}$ are unit disks and $F\colon \mathbb{D} \to \mathbb{D}$ is a holomorphic map defined by $z \mapsto z^m$. Then it's easy to show $d_q(\Phi) = m$, for all $q \in \mathbb{D}$, since for $q = 0$, there is only one preimage of multiplicity m and for $q \neq 0$, there are m preimages of multiplicity 1.

Let's consider the general case. For $q \in Y$, since X is compact, $\Phi^{-1}(q)$ only consists of finitely many points, denoted by $\{p_1, \dots, p_k\}$. Fix a local coordinate (V, ψ) centered at $q \in Y$, for any $i = 1, \dots, k$, there is a local coordinate (U_i, φ_i) centered at $p_i \in X$ such that

$$\psi \circ \Phi \circ \varphi_i^{-1}(z) = z^{m_i}, \quad z \in \varphi_i(U_i),$$

where $m_i = \mathrm{mult}_{p_i}(\Phi)$. If we choose another neighborhood $q \in W \subseteq V$ such that $\Phi^{-1}(W) \subseteq \bigcup_{i=1}^k U_i$, then for any $q \in W$, from the trivial case discussed before one has

$$d_q(\Phi) = \sum_{i=1}^k m_i.$$

This shows $d_q(\Phi)$ is a locally constant function, and thus $d_q(\Phi)$ is constant since Y is connected. □

Corollary 2.1.2. A holomorphic map between compact Riemann surfaces is an isomorphism if and only if it has degree one.

Corollary 2.1.3. X is a compact Riemann surface, and f is a meromorphic function on X, then the number (counted with multiplicity) of zeros is equal to the number (counted with multiplicity) of poles.

Proof. Note that meromorphic function f on X is equivalent to the holomorphic map Φ from X to S^2. Then the number of zeros is the multiplicity of Φ at zero and the number of poles is the multiplicity of Φ at ∞. □

2.2. **Riemann-Hurwitz formula.** In this section we talk about Riemann-Hurwitz formula, which computes the genus from a given ramification covering. Before that we recall some basic facts in topology. Let X be a compact oriented surface, the Euler number of X can be defined by the triangulation of X as follows: Suppose a triangulation of X is given with v vertices, e edges and t tirangles. Then the Euler characterisitic of X is defined by $v - e + t$. On the other hand, the Euler number can also be defined as

$$\chi(X) := \sum_i (-1)^i \dim H_i(X; \mathbb{R}),$$

where $H_i(X;\mathbb{R})$ is the i-th singular holology of X. The genus of X is defined by

$$\chi(X) = 2 - 2\,\mathrm{genus}(X).$$

Theorem 2.2.1 (Riemann-Hurwitz formula)**.** Let $\Phi\colon X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces. Then

$$\chi(X) = \deg(\Phi)\chi(Y) - \sum_{p\in X}(\mathrm{mult}_p(\Phi) - 1)$$

Proof. Choose a triangulation Δ of Y such that its vertex are exactly ramification values of F. Let v, e, t denote the number of vertices, edges and triangles of Δ respectively. Suppose Δ' is the triangulation of X obtained by pulling back Δ through F, and use v', e' and t' to denote the number of vertices, edges and triangles of Δ' respectively.

It's clear we have the following relations

$$t' = td, \quad e' = ed$$

where $d = \deg(\Phi)$. For $q \in Y$, note that

$$|\Phi^{-1}(q)| = \sum_{p\in\Phi^{-1}(q)} 1 = d + \sum_{p\in\Phi^{-1}(q)} (1 - \mathrm{mult}_p(\Phi)).$$

Then the relation between v and v' is given by

$$\begin{aligned} v' &= \sum_{\text{vertex } q \text{ of } \Delta} |\Phi^{-1}(q)| \\ &= \sum_{\text{vertex } q \text{ of } \Delta} \left(d + \sum_{p\in\Phi^{-1}(q)} (1 - \mathrm{mult}_p(\Phi)) \right) \\ &= vd + \sum_{\text{vertex } q \text{ of } \Delta} \left(\sum_{p\in\Phi^{-1}(q)} (1 - \mathrm{mult}_p(\Phi)) \right) \\ &= vd + \sum_{p\in X} (1 - \mathrm{mult}_p(\Phi)). \end{aligned}$$

Thus by the relation between Euler number and triangulation, we obtain the desired conclusion. □

Remark 2.2.1. Since the set of ramification points is finite, then $\sum_{p\in X}(\mathrm{mult}_p(\Phi) - 1)$ is a finite number, and for convenience we denote it by $B(\Phi)$. It describes how many ramification points of F are there on X.

Definition 2.2.1 (ramified holomorphic map)**.** A holomorphic map Φ is called ramified if $B(\Phi) > 0$.

Definition 2.2.2 (unramified holomorphic map)**.** A holomorphic map Φ is called unramified if $B(\Phi) = 0$.

Remark 2.2.2. A unramified holomorphic map is a covering map, and thus ramified holomorphic map is sometimes called ramified covering map.

Corollary 2.2.1. Let $\Phi\colon X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces. Then

(1) If Y is Riemann sphere and $\deg(\Phi) > 1$, then F must be ramified.
(2) If $\mathrm{genus}(X) = \mathrm{genus}(Y) = 1$, then F must be unramified.
(3) If $\mathrm{genus}(X) = \mathrm{genus}(Y) > 1$, then F must be an isomorphism.

Proof.

(1) Since Riemann sphere has genus zero, one has
$$B(\Phi) = 2(\deg(\Phi) - 1) + 2\,\mathrm{genus}(X) > 0.$$
(2) By Riemann-Hurwitz formula we have
$$0 = 0 + B(\Phi).$$
(3) By Riemann-Hurwitz formula we have
$$(1 - \deg(\Phi))(2\,\mathrm{genus}(X) - 2) = B(\Phi).$$
Then $\deg(\Phi) = 1$, since $\deg(\Phi) \geq 1$, $2\,\mathrm{genus}(X) - 2 > 0$ and $B(\Phi) \geq 0$.

□

2.2.1. *Genus of projective plane curve.* Now we're going to use Riemann-Hurwitz formula to compute the genus of projective plane curves. Firstly consider the example at the beginning of this section, that is, the non-singular projective plane curve C is defined by homogenous polynomial
$$F(x, y, z) = y^2z - x(x - z)(x - \lambda z),$$
where $\lambda \neq 0, 1$. The ramification covering $\widetilde{\Phi}\colon C \to \mathbb{P}^1$ has degree 2, and the ramification values are $[1 : 0], [0 : 1], [z : 1], [\lambda z : 1]$. Then by Riemann-Hurwitz formula one has
$$\chi(C) = 2 \times 2 - 4$$
This shows the genus of C is 1.

Example 2.2.1 (Fermat curve)**.** Let C be the projective plane curve defined by the homogenous polynomial $F(x, y, z) = x^d + y^d - z^d$. A direct computation shows C is non-singular, and thus it gives a Riemann surface. Consider the holomorphic map
$$\Phi\colon C \to \mathbb{P}^1$$
$$[x : y : z] \mapsto [x : z].$$
Note that
$$y^d = z^d - x^d = (x - \alpha_1 z) \dots (x - \alpha_d z),$$
where $\alpha_1, \dots, \alpha_d \in \mathbb{C}$ are different d-th unit roots. Then Φ is a ramification covering of degree d, and has d ramification values. Then by Riemann-Hurwitz formula,
$$\chi(C) = 2 \times d - d(d - 1).$$

This shows the genus of C is $(d-1)(d-2)/2$.

Remark 2.2.3. In general, for a non-singular projective plane curve C defined by a homogenous polynomial of degree d, the genus of C is $(d-1)(d-2)/2$, and this is called Plücker's formula or genus-degree formula. Moreover, if C is singular, then

$$\frac{(d-1)(d-2)}{2} - \delta,$$

where $\delta > 0$ is related to the type of singularities of C.

3. Bezout theorem

3.1. **Statement and proof.** Let C, C' be a non-singular projective plane curves defined by a homogenous polynomial $F, G \in \mathbb{C}[x,y,z]$ respectively with $G \nmid F$. In this section we will show how to count the number of the intersections of C and C'.

Definition 3.1.1 (multiplicity)**.** The multiplicity of intersection $p \in C \cap C'$ is the order of zero of G at p on some affine chart on C.

Remark 3.1.1. Note that the change of affine charts does not change the vanishing order of a polynomial. This shows the multiplicity of an intersection is well-defined. For convenience, the multiplicity of an intersection $p \in C \cap C'$ is denoted by $\mathrm{mult}_p(C, C')$, and it's a standard exercise to show $\mathrm{mult}_p(C, C') = \mathrm{mult}_p(C', C)$.

Formally we write the sum

$$(C, C') = \sum_{p \in C \cap C'} \mathrm{mult}_p(C, C') \cdot p,$$

and call it the intersection divisor[2]. The degree of the intersection divisor is defined by

$$\deg(C, C') := \sum_{p \in C \cap C'} \mathrm{mult}_p(C, C').$$

It's called the intersection number of C and C'

Theorem 3.1.1 (Bezout theorem)**.** Let C, C' be two non-singular projective plane curves defined by homogenous polynomials F, G with $\deg F = e, \deg G = d$ and $G \nmid F$. Then the intersection number

$$\deg(F, G) = ed.$$

Proof. Let L be a linear homogenous polynomial such that $L \nmid P$ and H be the projective line defined by L. Consider the holomorphic map

$$\Phi \colon C \to \mathbb{P}^1$$
$$[x : y : z] \mapsto [L^d : G].$$

Since C is compact, by Corollary 1.1.1 one has Φ is surjective.

(1) Suppose Φ is a non-constant holomorphic map. Note that the order of zeros of Φ equals $\deg(C, H^d)$, and the order of poles of Φ equals to $\deg(C, C')$. Then

$$\deg(C, H^d) = \deg(C, C').$$

since both order of zeros and order of poles are degree of Φ. By definition one has

$$\deg(C, H^d) = d \deg(C, H).$$

[2]Later in Section we will introduce divisor and its degree systematically.

Now it suffices to show a projective plane curve defined by a homogenous polynomial with degree e intersects a projective line e times, which is straightforward.

(2) If Φ is a constant holomorphic map, then there exists a constant $\lambda \in \mathbb{C}^*$ such that $G = \lambda L^d$. Again one has
$$\deg(C, L^d) = \deg(C, \lambda H^d) = \deg(C, C'),$$
since $\lambda \neq 0$.

□

3.2. **Applications.**

3.2.1. *Plücker formula.* In this section we will prove Plücker formula as a consequence of Bezout theorem, but before that we prove a technique lemma.

Lemma 3.2.1. Let C be a projective plane curve of degree d. Then there exists an affine coordinate $[x : y : 1] \subseteq \mathbb{P}^2$ such that C is given by the following equation
$$f(x, y) = y^d + a_1(x)y^{d-1} + \cdots + a_d(x) = 0,$$
where $a_i(x) \in \mathbb{C}[x]$ with $\deg a_i(x) \leq i$, or $a_j(x) = 0$.

Proof. Let $[z : w : 1]$ be an arbitary affine coordinate of $\mathbb{P}^2$ and C is defined by
$$p'(z, w) = 0, \quad \deg p' = d.$$
If p' is not of the necessary form, then consider the following coordinate transformation
$$\begin{aligned} z &= x + \lambda y \\ w &= y. \end{aligned}$$
Consider the coefficient $b(\lambda)$ of the term involving y^n in $f'(x + \lambda y, y)$. It's clear $b(\lambda)$ is a non-zero polynomial in λ, and hence can equal 0 for only a finite number of values of λ. Then we choose λ such that $b(\lambda) \neq 0$, and for such a chosen λ, we consider
$$f(x, y) = \frac{1}{b(\lambda)} f'(x + \lambda y, y).$$
Then in affine coordinate $[x : y : 1]$, the equation of C is
$$f(x, y) = 0,$$
which satisfies our desire. □

Corollary 3.2.1 (Plücker formula)**.** Let $C \subseteq \mathbb{P}^2$ be a non-singular projective plane curve of degree d. Then the genus of C is $(d-1)(d-2)/2$.

Proof. By Lemma 3.2.1, without lose of generality we may assume C is defined by the non-singular homogenous polynomial P with
$$F(x, y, z) = y^d - a_1(x, z)y^{d-1} - \cdots - a_d(x, z).$$

Then consider the following holomorphic map

$$\begin{aligned}\Phi\colon C&\to\mathbb{P}^1\\ [x:y:z]&\mapsto[x:z],\end{aligned}$$

which is a ramification covering in fact. Now by Riemann-Hurwitz formula it suffices to compute the ramification data of Φ. On affine charts $U_2=\{[x:y:1]\}\subseteq\mathbb{P}^2$, C is defined by

$$f(x,y)=y^d-a_1(x,1)y^{d-1}-\cdots-a_d(x,1)=0.$$

(1) If $f_y(x_0,y_0)\neq 0$, then by implicit function theorem, around the point $[x_0:y_0:1]$, the affine plane curve $C\cap U_2$ is given by $[x:y(x):1]$, and thus Φ is a local diffeomorphism at this point.
(2) If $f_y(x_0,y_0)=0$, then $f_x(x_0,y_0)\neq 0$ since f is non-singular. By implicit function theorem again, around $[x_0:y_0:1]$, there exists a local coordinate $y\mapsto[x(y):y:1]$, and F is given by $y\mapsto x(y)$. By chain rule one has

$$x'(y)=-(f_x(x(y),y))^{-1}f_y(x(y),y).$$

This shows the order of zero of $x'(y)$ equals to the order of zero of $f_y(x(y),y)$.

This shows $B(\Phi)=\sum_{p\in C}(\mathrm{mult}_p(\Phi)-1)$ is exactly the intersection number of F and F_y, and since both F and F_y are non-singular homogenous polynomial, by Bezout theorem one has

$$B(\Phi)=d(d-1).$$

By Rmann-Hurwitz formula, the genus of C is $(d-1)(d-2)/2$. □

3.2.2. *Non-singular homogenous polynomial is irreducible.* Another application of Bezout theorem is that any non-singular homogenous polynomial is irreducible.

Proposition 3.2.1. Let F be a non-singular homogenous polynomial. Then F is irreducible.

Proof. On contrary we suppose $F=F_1F_2$. By chain rule of derivative it's easy to see both F_1 and F_2 are non-singular. Then by Bezout theorem, F_1 and F_2 have at least a common zero, which contradicts to P is non-singular, since P is singular at the common zero of F_1 and F_2, which can be shown by chain rule of derivatives again. □

3.2.3. *Connectedness of irreducible plane curve.* In this section, we will prove the connectedness of plane curves as we mentioned before. In fact, we will prove the following stronger theorem.

Theorem 3.2.1. Let F be an irreducible homogenous polynomial and C be the projective plane curve defined by F. Then the set of singularities S is finite, and $C\setminus S$ is connected.

Before starting the proof, we prepare some basic facts we will use.

Lemma 3.2.2. If R is a UFD and

$$\begin{aligned} f &= a_0x^m + a_1x^{m-1} + \cdots + a_m, \\ g &= b_0x^n + b_1x^{n-1} + \cdots + b_n \end{aligned}$$

are polynomials in $R[x]$ with $a_0 \neq 0, b_0 \neq 0$. Then f, g has a non-trivial common divisor if and only if there exists $F, G \in R[x]$ with $\deg F < m, \deg G < n$ such that

$$f \cdot G = F \cdot g.$$

Proof. On one hand, if f, g has a non-trivial common divisor h, then

$$\begin{aligned} f &= h \cdot F \\ g &= h \cdot G. \end{aligned}$$

This shows $f \cdot G = F \cdot g$, where $\deg F < \deg f \leq m$ and $\deg G < \deg g \leq n$.

On the other hand, if $f \cdot G = F \cdot g$ with $\deg F < m$ and $\deg G < n$, then all factors of f cannot be all factors of F since $\deg f > \deg F$. Hence there exists a non-trivial divisor of f which is also a divisor of g since $R[x]$ is UFD by Gauss lemma. $\square$

Suppose

$$\begin{aligned} F(x) &= A_0x^{m-1} + \cdots + A_{m-1} \\ G(x) &= B_0x^{n-1} + \cdots + B_{n-1}. \end{aligned}$$

Then $f \cdot G = F \cdot g$ if and only if

$$\begin{cases} a_0B_0 = b_0A_0 \\ a_1B_0 + a_0B_1 = b_1A_0 + b_0A_1 \\ \vdots \\ a_mB_{m-1} = b_nA_{m-1}. \end{cases} \tag{3.1}$$

Thus $f \cdot G = F \cdot g$ has non-zero solutions F, G if and only if (3.1) has a non-zero solution $(A_0, \ldots A_{m-1}, B_0, \ldots, B_{n-1})$. Then by basic theory of systems of linear equations, (3.1) has a non-zero solution if and only if the following determinant equals to zero.

$$\det \begin{pmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 & b_1 & b_0 & \cdots & 0 \\ a_2 & a_1 & \ldots & 0 & b_2 & b_1 & \ldots & 0 \\ \vdots & \vdots & \ldots & a_0 & \vdots & \vdots & \ldots & b_0 \\ a_m & a_{m-1} & \cdots & \vdots & b_n & b_{n-1} & \cdots & \vdots \\ 0 & a_m & \ldots & \vdots & 0 & b_n & \ldots & \vdots \\ \vdots & \vdots & \ldots & a_{m-1} & \vdots & \vdots & \ldots & b_{n-1} \\ 0 & 0 & \cdots & a_m & 0 & 0 & \cdots & b_n \end{pmatrix} \tag{3.2}$$

Definition 3.2.1 (resultant)**.** If R is a ring and

$$f = a_0x^m + a_1x^{m-1} + \cdots + a_m,$$
$$g = b_0x^n + b_1x^{n-1} + \cdots + b_n$$

are polynomials in $R[x]$. The resultant of f, g is defined as the determinant in (3.2), and denoted by $\mathscr{R}(f, g)$.

Theorem 3.2.2. If R is a UFD and

$$f = a_0x^m + a_1x^{m-1} + \cdots + a_m,$$
$$g = b_0x^n + b_1x^{n-1} + \cdots + b_n$$

are polynomials in $R[x]$ with $a_0 \neq 0$, then

(1) f, g have a non-trivial common divisor if and only if $\mathscr{R}(f, g) = 0$;
(2) there exists polynomial $\alpha, \beta \in R[x]$, with $\deg \alpha < n, \deg \beta < m$ such that

$$\alpha(x)f(x) + \beta(x)g(x) = \mathscr{R}(f, g).$$

Definition 3.2.2 (discriminant)**.** Let R be a ring and $f \in R[x]$. The discriminant of p is defined by $\mathscr{D}(f) := \mathscr{R}(f, f')$, where f' is the formal derivative of f.

Corollary 3.2.2. Let R be a UFD and $f \in R[x]$. Then f has a multiple root if and only if $\mathscr{D}(f) = 0$.

Now let's start the proof of Theorem 3.2.1.

Proof. Firstly let's shows F has only finitely many singularities. By Lemma 3.2.1, without lose of generality we may assume C is defined by

$$f(x, y) = y^d + a_1(x)y^{d-1} + \cdots + a_d(x)$$

on some affine chart. If we regard $f(x, y)$ and $f_y(x, y)$ as elements in $\mathbb{C}[x][y]$, then $R(f, f_y) \in \mathbb{C}[x]$, which is a non-zero polynomial since $f(x, y)$ is irreducible. By Theorem 3.2.2 there exists $\alpha, \beta \in \mathbb{C}[x, y]$ such that

$$\alpha(x, y)f(x, y) + \beta(x, y)f_y(x, y) = \mathscr{R}(f, f_y)(x).$$

If point (x_0, y_0) such that $f(x_0, y_0) = f_y(x_0, y_0) = 0$, then

$$\mathscr{R}(f, f_y)(x_0) = 0.$$

This shows $f(x, y) = f_y(x, y) = 0$ has finitely many solutions, and thus C only has finitely many singularities on this affine chart. On the other hand, by Bezout theorem one has the projective plane curve defined by $z = 0$ only intersects with C finitely many times. Then C has only finitely many singularities.

To prove $C^* = C \setminus S$ is connected, it suffices to show C^* is connected on the affine chart $U_2 = \{[x : y : 1]\}$ since

$$C^* \cap U_2 \subseteq C^* \subseteq C = \overline{C^* \cap U_2},$$

and a basic fact in point set topology says that if a set is connected, so is its closure. For convenience, in the following proof we still use C to denote the affine plane curve $C \cap U_2$. Now consider the ramification covering

$$\begin{aligned} \Phi\colon C &\to \mathbb{C} \\ (x, y) &\mapsto x. \end{aligned}$$

If we define $B = \{x_0 \in \mathbb{C} \mid \mathscr{R}(f, f_y)(x_0) = 0\} \subseteq \mathbb{C}^1$, then the argument in the proof of Corollary 3.2.1 can be used here to show $\Phi\colon C \setminus \Phi^{-1}(B) \to \mathbb{C} \setminus B$ is a local diffeomorphism. Thus $\Phi\colon C \setminus \Phi^{-1}(B) \to \mathbb{C} \setminus B$ is a covering map on each component of $C \setminus \Phi^{-1}(B)$ since Φ is a proper.

For each point $x_0 \notin B$, the fiber $\Phi^{-1}(x_0)$ are exactly the d distinct solutions of $y^d + a_1(x_0)y^{d-1} + \cdots + a_d(x_0) = 0$ has d distinct solutions, denoted by $\{y_1(x_0), \ldots, y_d(x_0)\}$. By the basic theory of covering space, there is an action of the fundamental group $\pi_1(\mathbb{C} \setminus B, x_0)$ on the fiber $\Phi^{-1}(x_0)$. To be precise, given $[\gamma] \in \pi_1(\mathbb{C} \setminus B, x_0)$, we choose arbitary representive $\gamma \in [\gamma]$ and consider its lift $\widetilde{\gamma}$, which is independent of the choice of γ. If $y_i(x_0)$ and $y_j(x_0)$ are endpoints of $\widetilde{\gamma}$, then $[\gamma] \cdot y_i(x_0) = y_j(x_0)$. Thus it's clear to see the number of connected components of $C \setminus \Phi^{-1}(B)$ equals to the number of orbits of $\Phi^{-1}(x_0)$ under the $\pi_1(\mathbb{C} \setminus B, x_0)$-action.

Suppose $\{y_1(x_0), \ldots, y_l(x_0)\}$ is an orbit of $\pi_1(\mathbb{C} \setminus B, x_0)$-action. Then for any $x \notin B$, we choose a path $\gamma\colon [0, 1] \to \mathbb{C} \setminus B$ connecting x_0 and x. Then γ has l different liftings ending at points $y_1(x), \ldots, y_l(x)$, which can be extended as holomorphic functions defined on an open neighborhood of x. If we define

$$\begin{aligned} \sigma_1(x) &= \sum_i y_i(x) \\ \sigma_2(x) &= \sum_{i<j} y_i(x) y_j(x) \\ &\vdots \\ \sigma_l(x) &= y_1(x) \ldots y_d(x), \end{aligned}$$

then $\sigma_i(x)$ does not depend on the choice of paths connecting x_0 and x, and thus $\sigma_i(x)$ are holomorphic functions defined over $\mathbb{C} \setminus B$. By Rouché's theorem, one can see these $\sigma_i(x)$ has polynomial growth, that is, there exists constants C and N such that

$$|\sigma_i(x)| < C|x|^N$$

holds for all $i = 1, \ldots, l$. Then by Riemann extension theorem one has $\sigma_i(x)$ are defined on $\mathbb{C}$, and they are polynomials of x in fact. Note that

$$(y - y_1(x)) \ldots (y - y_l(x)) \mid f(x, y).$$

Then

$$g(x, y) = y^d - \sigma_1(x)y^{d-1} + \sigma_2(x)y^{d-2} + \cdots + (-1)^l \sigma_l(x) \in \mathbb{C}[x, y]$$

also divides $f(x, y)$. But since $f(x, y)$ is irreducible, one has $f = g$, and thus the $\pi_1(\mathbb{C} \setminus B, x_0)$-action is transitive as desired. $\square$

4. Differential forms

4.1. Differential forms, differential operators and integrations.

4.1.1. *Differential forms.* Firstly let's consider the differential forms defined on an open subset $U \subseteq \mathbb{C}$. Suppose $\{z\}$ is the coordinate on $\mathbb{C}$. Then

(1) A smooth 1-form is of the form $f\mathrm{d}z + g\mathrm{d}\overline{z}$, where f, g are smooth functions, and the set of all smooth 1-forms defined on U is denoted by $\mathcal{A}^1(U)$.
(2) A smooth 1-form is a $(1,0)$-form, if it's of the form $f\mathrm{d}z$, where f is a smooth function, and the set of all $(1,0)$-form defined on U is denoted by $\mathcal{A}^{1,0}(U)$.
(3) A smooth 1-form is a $(0,1)$-form, if it's of the form $f\mathrm{d}\overline{z}$, where f is a smooth function, and the set of all $(0,1)$-form defined on U is denoted by $\mathcal{A}^{0,1}(U)$.
(4) A smooth 1-form is a holomorphic 1-form, if it's of the form $f\mathrm{d}z$, where f is a holomorphic function, and the set of all holomorphic 1-form defined on U is denoted by $\Omega^1(U)$.
(5) A smooth 2-form is of the form $f\mathrm{d}z \wedge \mathrm{d}\overline{z}$, where f is a smooth function, and the set of all smooth 2-forms defined on U is denoted by $\mathcal{A}^2(U)$.
(6) A holomorphic 2-form is of the form $f\mathrm{d}z \wedge \mathrm{d}\overline{z}$, where f is a holomorphic function, and the set of all holomorphic 2-forms defined on U is denoted by $\Omega^2(U)$.

Remark 4.1.1. It's clear $\mathcal{A}^1(U) = \mathcal{A}^{1,0}(U) \oplus \mathcal{A}^{0,1}(U)$.

If we want to define differential forms on Riemann surfaces, a natural idea is to define them on each coordinate chart, and glue them together in a suitable way, so we need to know what will happen under the holomorphic change of coordinate.

Suppose $\Phi\colon U \to V$ is a holomorphic function between open subsets $U, V \subseteq \mathbb{C}$ and $\theta = f\mathrm{d}w + g\mathrm{d}\overline{w}$ is a smooth 1-form on V. Then the pullback of θ is defined by

$$\Phi^*(\theta) = f(\Phi(z))\Phi'(z)\mathrm{d}z + g(F(z))\overline{\Phi'(z)}\mathrm{d}\overline{z}.$$

Similarly, if $\theta = f\mathrm{d}w \wedge \mathrm{d}\overline{w}$ is a smooth 2-form, then the pullback is defined by

$$\Phi^*(\theta) = f(\Phi(z))|\Phi'(z)|^2\mathrm{d}z \wedge \mathrm{d}\overline{z}.$$

In fact, pullback is a contravariant functor.

Definition 4.1.1 (differential k-form)**.** A differential k-form θ on a Riemann surface X assigns to any local coordinate $\varphi\colon U \to V$ a k-form α, and assignments are compatible[3] with the charts.

[3] This means if $U' \xrightarrow{\varphi'} V'$ is another local coordinate assigned with k-form β, then

$$\Phi^*(\beta) = \alpha,$$

where $\Phi = \varphi' \circ \varphi^{-1}(z)$.

Definition 4.1.2 ($(1,0)$-form and $(0,1)$-form)**.** A differential 1-form θ on a Riemann surface X is called

(1) a $(1,0)$-form, if it can be represented as $f\mathrm{d}z$ locally, where f is a smooth function;
(2) a $(0,1)$-form, if it can be represented as $f\mathrm{d}\overline{z}$ locally, where f is a smooth function.

Definition 4.1.3 (holomorphic 1-form)**.** A holomorphic 1-form θ on a Riemann surface X is a differential $(1,0)$-form which can be locally represented as $f(z)\mathrm{d}z$, where f is a holomorphic function.

4.1.2. *Differential operators.* Given a smooth function f defined on an open subset $U\subseteq\mathbb{C}$, one has

$$\mathrm{d}f=\frac{\partial f}{\partial z}\mathrm{d}z+\frac{\partial f}{\partial\overline{z}}\mathrm{d}\overline{z}.$$

Then we can define the operators ∂ and $\overline{\partial}$ on smooth functions as follows

$$\partial f:=\frac{\partial f}{\partial z}\mathrm{d}z$$
$$\overline{\partial}f:=\frac{\partial f}{\partial\overline{z}}\mathrm{d}\overline{z}.$$

Similarly for a smooth 1-form $\theta=f\mathrm{d}z+g\mathrm{d}\overline{z}$. Then

$$\mathrm{d}\theta=\frac{\partial f}{\partial\overline{z}}\mathrm{d}\overline{z}\wedge\mathrm{d}z+\frac{\partial g}{\partial z}\mathrm{d}z\wedge\mathrm{d}\overline{z}=(\frac{\partial g}{\partial z}-\frac{\partial f}{\partial\overline{z}})\mathrm{d}z\wedge\mathrm{d}\overline{z}.$$

Then we can define the operators ∂ and $\overline{\partial}$ on smooth 1-form $\theta=f\mathrm{d}z+g\mathrm{d}\overline{z}$ as follows

$$\partial\theta:=\partial g\wedge\mathrm{d}\overline{z}$$
$$\overline{\partial}\theta:=\overline{\partial}f\wedge\mathrm{d}z.$$

Thus we have constructed differential operators d,∂ and $\overline{\partial}$ on open subset $U\subseteq\mathbb{C}$, and above constructions can also be paralled to the Riemann surface X.

Theorem 4.1.1.

(1) $\mathrm{d}=\partial+\overline{\partial}$.
(2) $\mathrm{d}^2=\partial^2=\overline{\partial}^2=0$.
(3) $\partial\overline{\partial}=-\overline{\partial}\partial$.
(4) A $(1,0)$-form θ is holomorphic if and only if $\mathrm{d}\theta=\overline{\partial}\theta=0$.
(5) d,∂ and $\overline{\partial}$ satisfy the Leibniz rule, and commute with pullback.

4.1.3. *Integrations of differential forms.* Let θ be a smooth 1-form on a Riemann surface X and γ be a piecewise smooth curve on X. Suppose the curve γ is divided into $\gamma=\gamma_1\cup\cdots\cup\gamma_n$, such that $\gamma_i\colon[a_i,b_i]\to U_i$, where (U_i,φ_i) is a local coordinate. If θ is given by $f_i\mathrm{d}z_i+g_i\mathrm{d}\overline{z}_i$ in the local chart (U_i,φ_i), then the integration of θ along γ is defined by

$$\int_\gamma\theta=\sum_{i=1}^n\int_{\gamma_i}\theta:=\sum_{i=1}^n\int_{a_i}^{b_i}\{f\cdot z_i'(t)+g\cdot\overline{z_i'(t)}\}\mathrm{d}t.$$

Similarly, if η is a 2-form and D is a region on X, we also divide D into $D = D_1 \cup \cdots \cup D_n$ such that each D_i lies in some local chart (U_i, φ_i). If we write $z_i = x_i + \sqrt{-1}y_i$, then

$$\mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i = (\mathrm{d}x_i + \sqrt{-1}\mathrm{d}y_i) \wedge (\mathrm{d}x_i - \sqrt{-1}\mathrm{d}y_i) = -2\sqrt{-1}\mathrm{d}x_i \wedge \mathrm{d}y_i.$$

Thus if η is given locally by

$$f\mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i,$$

then the integration is defined by

$$\int_D \eta = \sum_{i=1}^n \int_{D_i} \eta := \sum_{i=1}^n \int_{\varphi_i(D_i)} -2\sqrt{-1}f\mathrm{d}x_i \wedge \mathrm{d}y_i.$$

Theorem 4.1.2 (Stokes)**.** Let X be a Riemann surface and θ be a smooth 1-form. If D is a compact reigon with piecewise smooth boundary ∂D, then

$$\int_D \mathrm{d}\theta = \int_{\partial D} \theta.$$

4.2. **Holomorphic 1-form and meromorphic 1-form.**

4.2.1. *Holomorphic 1-form.*

Example 4.2.1. Consider the non-singular affine plane curve C defined by $f(x, y) = y^2 - x(x-1)(x-\lambda) = 0$. Then $\mathrm{d}x/y$ is a holomorphic 1-form on C.

(1) For point (x, y) with $y \neq 0$, $\mathrm{d}x/y$ is a well-defined holomorphic 1-form.
(2) For point (x, y) with $y = 0$, since C is non-singular, at this point one has $f_x \neq 0$. Note that $f(x, y) = 0$ holds on C, and thus one has $f_x\mathrm{d}x + f_y\mathrm{d}y = 0$ holds on C, which implies

$$\frac{\mathrm{d}x}{2y} = -\frac{\mathrm{d}y}{f_x}.$$

This shows $\mathrm{d}x/y$ is always a well-defined holomorphic 1-form on C.

More generally, arguments shown in above example can be used to prove the following proposition.

Proposition 4.2.1. Let C be a non-singular affine plane curve defined by $f(x, y) = 0$. Then

$$\omega = \frac{\mathrm{d}x}{f_y} = \frac{\mathrm{d}y}{f_x}$$

is a holomorphic 1-form on C.

Proposition 4.2.2. Let C be a non-singular projective plane curve defined by $F(x, y, z) = 0$ with $\deg F \geq 3$. Then the holomorphic 1-form

$$\omega = \frac{\mathrm{d}x}{F_y(x, y, 1)}$$

on the affine piece $\{z = 1\}$ extends to a holomorphic 1-form on C.

Proof. Firstly we extend the holomorphic 1-form as follows

$$\omega = \frac{\mathrm{d}(x/z)}{F_y(x/z, y/z, z/z)}.$$

Then on the affine piece defined by $x = 1$, one has

$$\omega = -\frac{z^{d-3}\mathrm{d}z}{F_y(1, y, z)} = \frac{z^{d-3}\mathrm{d}z}{F_z(1, y, z)}.$$

Thus if $d \geq 3$, the extension of ω is a holomorphic 1-form defined on C. $\square$

Remark 4.2.1. More generally, if $g(x, y) \in \mathbb{C}[x, y]$ is a polynomial, then by the same argument one can show that the holomorphic 1-form

$$\omega = \frac{g(x, y)\mathrm{d}x}{F_y(x, y, 1)}$$

defined on affine piece also extends to a holomorphic 1-form on C if $\deg g \leq d - 3$. Note that the dimension of vector space consisting of homogenous polynomial with degree $d-3$ in three variables are $\binom{d-1}{2} = (d-1)(d-2)/2$. In fact, any holomorphic 1-form on C is of this form, and dimension of vector space consisting of all holomorphic 1-forms is genus. This gives an another viewpoint to genus formula.

4.2.2. *Meromorphic* 1*-forms.*

Definition 4.2.1 (meromorphic 1-form)**.** A meromorphic 1-form θ on a Riemann surface X is a smooth $(1, 0)$-form which can be locally represented as $f(z)\mathrm{d}z$, where f is a meromorphic function.

Recall that given a meromorphic meromorphic function f on a Riemann surface X, for $p \in X$, we can chosoe a local coordinate z centered at p, and consider the Laurend series of $f \circ \varphi^{-1}(z)$ as

$$f(z) = \sum_{n=m}^{\infty} c_n z^n, \quad c_m \neq 0.$$

The order of f at p is defined by m and denoted by $\mathrm{ord}_p(f)$.

Lemma 4.2.1. $\mathrm{ord}_p(f)$ is independent of the choice of local coordinate.

Proof. A meromorphic function f on a Riemann surface X corresponds to a holomorphic map $\Phi\colon X \to S^2$. If p is a zero point of f, then $\mathrm{ord}_p(f) = \mathrm{mult}_p(\Phi)$, and if p is a pole of f, then $\mathrm{ord}_p(f) = -\mathrm{mult}_p(\Phi)$. $\square$

Let θ be a meromorphic 1-form on Riemann surface X, in local coordinate z centered at p, we can write

$$\theta = f(z)\mathrm{d}z$$

so we can define $\mathrm{ord}_p(\theta) = \mathrm{ord}_p(f)$, and clearly it's independent of the choice of local coordinate.

Example 4.2.2. Let $X = S^2 = \mathbb{C} \cup \{\infty\}$ and $\{(\mathbb{C}, z), (\mathbb{C}, w)\}$ be an atlas of S^2, where the transition is given by $w = 1/z$. Consider 1-form θ which locally looks like $\mathrm{d}z$ on $(\mathbb{C}, z)$. Using holomorphic change of coordinate, we have θ looks like

$$\theta = \frac{-1}{z^2}\mathrm{d}z$$

on $(\mathbb{C}, w)$. Clearly θ is a meromorphic 1-form, and

$$\mathrm{ord}_p(\theta) = \begin{cases} 0, & p \in S^2 \setminus \{\infty\} \\ -2, & p = \infty \end{cases}$$

Then

$$\sum_{p \in S^2} \mathrm{ord}_p(\theta) = -2.$$

Example 4.2.3. Let $\{(\mathbb{C}, z), (\mathbb{C}, w)\}$ be an atlas of $\mathbb{C}_\infty$ with transition function $z = 1/z$. Suppose meromorphic 1-form θ is locally given by a rational function $r(z)$ on $(\mathbb{C}, z)$, where

$$r(z) = c\prod_{j=1}^{n}(z - \lambda_i)^{a_j},$$

where $c \neq 0, a_i \in \mathbb{Z}, \lambda_j \neq \lambda_j \in \mathbb{C}$. Thus in the coordinate chart $(\mathbb{C}, w)$, one has

$$\theta = c\prod_{j=1}^{n}(\frac{1}{w} - \lambda_j)^{a_j}(-\frac{1}{w^2})\mathrm{d}w$$

This shows

$$\mathrm{ord}_p(\theta) = \begin{cases} a_j, & p = \lambda_j \\ -2 - \sum_{j=1}^{n} a_j, & p = \infty. \end{cases}$$

Then

$$\sum_{p \in S^2} \mathrm{ord}_p(\theta) = -2.$$

Remark 4.2.2. This shows for a meromorphic 1-form θ on Riemann sphere S^2, one always has

$$\sum_{p \in S^2} \mathrm{ord}_p(\theta) = 2\,\mathrm{genus}(S^2) - 2.$$

Later we will see it's not a coincidence.

4.3. **Residue theorem.** Let θ be a meromorphic 1-form on a Riemann surface X. Suppose θ is locally given by $f\mathrm{d}z$, where f is a meromorphic function. The order of f lose too many information given by the coefficient of its Laurent series and we want to keep track coefficients which are invariant under the holomorphic change of local coordinate. Luckily, there exists such an invariant, that is -1-th coefficient of Laurent series c_{-1}.

Definition 4.3.1 (residue)**.** The residue of a meromorphic 1-form θ is defined by $\mathrm{Res}_p(\theta) = c_{-1}$.

The following lemma shows that the residue is independent of the choice of local coordinate, and gives a formula to compute it.

Lemma 4.3.1. Let D be any compact region in Riemann surface X such that $p \in D\backslash\partial D$, ∂D is piecewise smooth, and θ cannot have poles in $D\backslash\{p\}$. Then

$$\mathrm{Res}_p(\theta) = \frac{1}{2\pi\sqrt{-1}}\int_{\partial D}\theta.$$

Proof. Choose $D' \subseteq D$ such that $p \in D' \setminus \partial D'$, $\partial D'$ is smooth, and D' is contained in a local chart with local coordinate z centered at p. In this local chart, we can write θ as

$$\theta = (\sum_{n=m}^{\infty} c_n z^n)\mathrm{d}z.$$

Then

$$\int_{\partial D}\theta - \int_{\partial D'}\theta = \int_{D\backslash D'}\mathrm{d}\theta = 0,$$

where the last equality holds since θ is holomorphic in $D \setminus D'$. As a consequence,

$$\int_{\partial D}\theta = \int_{\partial D'}\theta = \int_{\varphi(\partial D')}(\sum_{n=m}^{\infty} c_n z^n)\mathrm{d}z = 2\pi\sqrt{-1}c_{-1} = 2\pi\sqrt{-1}\,\mathrm{Res}_p(\theta).$$

□

Theorem 4.3.1 (residue theorem)**.** Let X be a compact Riemann surface and θ be a meromorphic 1-form on X. Then

$$\sum_{p\in X}\mathrm{Res}_p(\theta) = 0$$

Proof. Since X is compact, there are only finitely many poles of θ, denoted by $\{p_1, \ldots, p_k\}$. For each $1 \leq j \leq k$, we can choose a neighborhood D_j of p_j which plays the role of D' in Lemma 4.3.1. Then

$$\sum_{p\in X}\mathrm{Res}(\theta) = \sum_{j=1}^{k}\mathrm{Res}_{p_j}(\theta) = \frac{1}{2\pi\sqrt{-1}}\sum_{j=1}^{k}\int_{\partial D_j}\theta = \frac{1}{2\pi\sqrt{-1}}\int_{D\backslash\bigcup_{j=1}^{k} D_j}\mathrm{d}\theta = 0.$$

□

Corollary 4.3.1. Let X be a compact Riemann surface and f be a meromorphic function on X. Then

$$\sum_{p\in X}\mathrm{ord}_p(f) = 0$$

Proof. It suffices to note that

$$\operatorname{ord}_p(f) = \operatorname{Res}_p(\frac{\mathrm{d}f}{f}).$$

□

4.4. **Poincaré-Hopf theorem.** Let M be a real closed 2-manifold and σ be a smooth 1-form with isolated zeros. Suppose σ is locally given by $\sigma = u\mathrm{d}x + v\mathrm{d}y$ on an open neighborhood U of zero p such that $U \setminus \{p\}$ contains no zero of σ. Then the index of σ at p, denoted by $\operatorname{Ind}_p(\sigma)$, is defined by the degree of the following map

$$\begin{aligned}\Phi\colon S^1(\epsilon) &\to S^1\\ (x,y) &\mapsto \frac{(u,v)}{\sqrt{u^2+v^2}},\end{aligned}$$

where $S^1(\epsilon)$ is the sphere of radius ϵ contained in U. The Poincaré-Hopf theorem[4] says that

$$\sum_{i=1}^k \operatorname{Ind}_{p_i}(\sigma) = \chi(M),$$

where $\{p_1,\dots,p_k\}$ are all zeros of σ. Moreover, Poincaré-Hopf theorem still holds if σ is smooth except finitely many singularities, by adding the index of these singularities. In this section we will use Poincaré-Hopf theorem to show that the phenomenon we have seen in Example 4.2.2 and Example 4.2.3 are not coincidences.

Theorem 4.4.1. Let X be a compact Riemann surface and θ be a meromorphic 1-form on X. Then

$$\sum_{p\in X} \operatorname{ord}_p(\theta) = -\chi(X) = 2g-2.$$

Proof. Consider the 1-form $\sigma = \operatorname{Re}(\theta)$, which is a smooth 1-form besides the poles of θ, and the zeros of σ are exactly the one of θ. For any zero or pole $p \in X$ of θ, without lose of generality we may assume θ is of the form $z^m\mathrm{d}z$ locally. Then

$$\sigma = r^m\left(\cos(m\theta)\mathrm{d}x - \sin(m\theta)\mathrm{d}y\right)$$

where $r = |z|$. Thus the index at point p is

$$\begin{aligned}\operatorname{Ind}_p(\sigma) &= \frac{1}{2\pi}\int_0^{2\pi} \cos(m\theta)\mathrm{d}\sin(-m\theta) + \sin(-m\theta)\mathrm{d}\cos(m\theta)\\ &= \frac{1}{2\pi}\int_0^{2\pi} -m\left(\sin(-m\theta)^2 + \cos(-m\theta)^2\right)\mathrm{d}\theta\\ &= -\operatorname{ord}_p(\theta).\end{aligned}$$

[4]See page35 of [Mil65].

Thus by Poincaré-Hopf theorem one has

$$\sum_{p\in X}\operatorname{ord}_p(\theta)=-\chi(X).$$

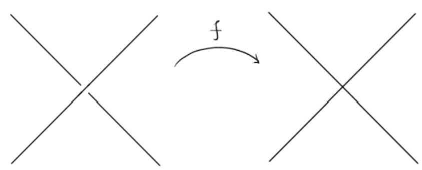
$\square$

Remark 4.4.1. In fact, above theorem is equivalent to the Riemann-Hurwitz formula, which is left an exercise in homework.

5. Normalization: Analytic viewpoint

In this section we will deal with singularities of algebraic curves. Roughly speaking, after resoluting all singularities of a curve C, we should obtain a Riemann surface, which is "isomorphic" to C besides these singularities. Before the formal definitions, let's see some examples of singularities we have already seen.

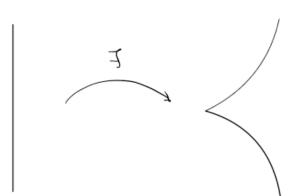
Example 5.1. The affine plane curve C defined by $x^2-y^2=0$ has a singular point $(0,0)$. Geometrically speaking, there are two projective line intersect at the point $(0,0)$, which cause the singularity. Thus one way to solve the singularity is to "split" these two lines.



Formally speaking, we should consider the disjoint union of two copy of $\mathbb{C}$, which is mapped to C as follows

$$\begin{aligned} f\colon \mathbb{C}\coprod\mathbb{C} &\to C \\ \{t_1\},\{t_2\} &\mapsto (t_1,t_1),(t_2,-t_2). \end{aligned}$$

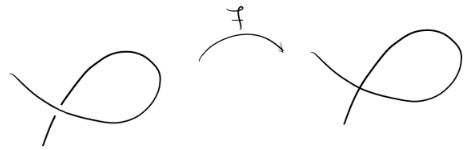
Example 5.2. The affine plane curve C defined by $x^2-y^3=0$ has a singular point $(0,0)$. To solve this singularity, geometrically thinking we should pull this curve "straightly", which can be seen as



Formally speaking, we should consider the parameterization

$$\begin{aligned} f\colon \mathbb{C} &\to C \\ t &\mapsto (t^3,t^2). \end{aligned}$$

Example 5.3. The affine plane curve defined by $y^2-x^2(x-1)=0$ has a singular point $(0,0)$. From the following picture we can see that if we want to solve the singularity, we should also "split" the two part which intersect at $(0,0)$, as what we have done in the Example 5.1.



However, there is no global parameterization as in Example 5.1. Note that

$$y = \pm\sqrt{x^2(x-1)} = \pm x\sqrt{x-1}.$$

Then for $D_1 = D_2 = \{|x| < 1\}$, we should consider

$$\begin{aligned} f\colon D_1 \coprod D_2 &\to C \\ \{t_1\}, \{t_2\} &\mapsto (t_1, t_1\sqrt{t_1 - 1}), (t_2, -t_2\sqrt{t_2 - 1}). \end{aligned}$$

5.1. **Weierstrass preparation theorem.** Denote

$$\mathbb{C}\{x\} = \{\sum_{k=1}^{\infty} a_k x^k \mid \text{convergent series with positive convergence radius.}\}$$

$$\mathbb{C}\{x, y\} = \{\sum_{k=1}^{\infty} a_{kl} x^k y^l \mid \text{convergent series with positive convergence radius.}\}$$

They are called germs of holomorphic functions.

Definition 5.1.1 (Weierstrass polynomial)**.** An element $f(x, y) \in \mathbb{C}\{x, y\}$ is called a Weierstrass polynomial if $f(x, y) = y^d + a_1(x)y^{d-1} + \cdots + a_d(x) \in \mathbb{C}\{x, y\}$, where $a_i(x) \in \mathbb{C}\{x\}$ and $a_i(0) = 0$.

Theorem 5.1.1 (Weierstrass preparation theorem)**.** If $f \in \mathbb{C}\{x, y\}$ such that $f(0, y)$ is not identically zero, then there exist unique $u \in \mathbb{C}\{x, y\}^*$ and a unique Weierstrass polynomial w such that $f = uw$.

Proof. Firstly we may assume $f(0, 0) = 0$, otherwise $f \in \mathbb{C}\{x, y\}^*$ and there is nothing to prove. If so, then $f(0, y)$ has an isolated zero at $y = 0$, that is, there exists $\epsilon > 0$ such that

$$\{f(0, y) = 0\} \cap \{|y| \leq \epsilon\} = \{y = 0\}.$$

By continuity we may choose $\rho > 0$ sufficiently small such that $f(x, y) \neq 0$ on $\{|x| < \rho, |y| = \epsilon\}$. Then the number of zeros of $f(x, y)$ in $|y| \leq \epsilon$ for a fixed x is computed by

$$n(x) = \frac{1}{2\pi\sqrt{-1}} \int_{|y|=\epsilon} \frac{f_y(x, y)}{f(x, y)} \mathrm{d}y,$$

which is an integer-valued holomorphic function, and thus $n(x) \equiv m$ is a constant.

For all $|x| < \rho$, suppose $y_1(x), \ldots, y_m(x)$ are zeros of $f(x,y)$ contained in $\{|y| \leq \epsilon\}$. Then we claim that $w(x,y) = (y - y_1(x)) \cdots (y - y_n(x))$ is a Weierstrass polynomial. Indeed, note that

$$\sigma_k(x) := \sum_{i=1}^m y_i^k(x) = \frac{1}{2\pi\sqrt{-1}} \int_{|y|=\epsilon} y^k \frac{f_y(x,y)}{f(x,y)} \mathrm{d}y$$

are holomorphic, and thus if we write

$$w(x,y) = y^m + a_1(x) y^{m-1} + \cdots + a_m(x),$$

then $a_i(x)$ are polynomials of $\sigma_1(x), \ldots, \sigma_m(x)$. This shows $a_i(x) \in \mathbb{C}\{x\}$, and $a_i(0) = 0$ for all i since $y_i(0) = 0$ for all i.

For convenience we denote $D = \{|x| < \rho, |y| \leq \epsilon\}$. By definition $u(x,y) = f(x,y)/w(x,y)$ is well-defined in $D \setminus \{w = 0\}$. For fixed $|x| < \rho$, by construction $w(x,y)$ and $f(x,y)$ have the same zeros in y. Therefore $u(x,y) \neq 0$ on D and $u(x,y)$ is holomorphic in variable y for each x. Now for given y_0 with $|y_0| < \epsilon$, one has

$$u(x, y_0) = \frac{1}{2\pi\sqrt{-1}} \int_{|y|=\epsilon} \frac{u(x,y)}{y - y_0} \mathrm{d}y.$$

This shows $u(x,y)$ is holomorphic in variables x and y, and thus one has $u(x,y)$ is holomorphic. Moreover, since u has no zeros, it has a non-zero constant term $u(0,0)$, and thus $u \in \mathbb{C}\{x,y\}^*$.

Finally let's see the uniqueness. If $f = u'w'$ in D, then

$$w' = y^d + c_1(x) y^{d-1} + \cdots + c_d(x) = (y - y_1(x)) \ldots (y - y_m(x)) = w.$$

This shows $w = w'$ and thus $u = u'$. $\square$

Corollary 5.1.1. $\mathbb{C}\{x,y\}$ is UFD.

Proof. Firstly note that $\mathbb{C}\{x\}$ is UFD, since for $f \in \mathbb{C}\{x\}$, one has

$$f = x^\mu g,$$

where $g \in \mathbb{C}\{x\}$ is a unit. Then by Gauss lemma one has $\mathbb{C}\{x\}[y]$ is UFD. Now for $f \in \mathbb{C}\{x,y\}$, suppose $f = x^\mu g$ with $g(0,y) \neq 0$. Since μ is unique, it suffices to show the unique factorization for g. By Weierstrass preparation theorem there is a decomposition

$$g(x,y) = uw.$$

Since Weierstrass polynomial w belongs to $\mathbb{C}\{x\}[y]$ which is UFD, there is a unique decomposition

$$w = w_1^{p_1} \ldots w_k^{p_k},$$

where $w_i \in \mathbb{C}\{x\}[y]$ is monic irreducible. Now we need to show each w_i is irreducible in $\mathbb{C}\{x,y\}$. If not, suppose $w_i = a_i b_i$ in $\mathbb{C}\{x,y\}$. Then $w_i(0,y) \not\equiv 0$ implies both $a_i(0,y) \not\equiv 0$ and $b_i(0,y) \not\equiv 0$, and again by Weierstrass preparation theorem one has

$$a_i = u_i' w_i'$$
$$b_i = u_i'' w_i'',$$

where $w_i', w_i'' \in \mathbb{C}\{x\}[y]$. Since the decomposition in Weierstrass preparation theorem is unique, one has $u_i'u_i'' = 1$ and $w_i = w_i'w_i''$ in $\mathbb{C}\{x\}[y]$, a contradiction. Thus we obtain a decomposition of g into

$$g = uw_1^{p_1} \dots w_k^{p_k},$$

where u is a unit in $\mathbb{C}\{x, y\}$ and w_i are irreducible in $\mathbb{C}\{x, y\}$.

Now let's prove the uniqueness of decomposition of g. Suppose g is decomposed as

$$g = uw_1^{p_1} \dots w_k^{p_k} = v\widetilde{w}_1^{q_1} \dots \widetilde{w}_l^{q_l}.$$

Since $g(0, y) \not\equiv 0$, then again by Weierstrass preparation theorem we may decomposition these w_i and $\widetilde{w}_j$ into

$$\begin{aligned} w_i &= u_i w_i' \\ \widetilde{w}_j &= v_j \widetilde{w}_j'. \end{aligned}$$

By the uniqueness of the decomposition in Weierstrass preparation and the factorization in $\mathbb{C}\{x\}[y]$, one has $\{w_i\}$ and $\{\widetilde{w}_j\}$ are the same up to ordering, and $l = k$. This completes the proof. □

Remark 5.1.1. Although $y^2 - x^2(x-1)$ is irreducible in $\mathbb{C}[x, y]$, it's reducible in $\mathbb{C}\{x\}[y]$, that is,

$$(y - x\sqrt{x-1})(y + x\sqrt{x-1}).$$

In the following section we will see such local decomposition gives the local resolution of singularities.

5.2. **Resolution of singularities.** Let C be an irreducible projective plane curve with singularities $\operatorname{sing}(C)$. A normalization of C is a compact Riemann surface $\widetilde{C}$ together with a continous map $f\colon \widetilde{C} \to C$ such that f is surjective and

$$f\colon \widetilde{C} \setminus \Phi^{-1}(\operatorname{sing}(C)) \to C \setminus \operatorname{sing}(C)$$

is an isomorphism. In this section we will use unique factorization of $\mathbb{C}\{x, y\}$ to construct the normalization of C. Firstly let's give a rough ideal about what we're going to do.

Suppose p is a singularity of C, and without lose of generality we may assume $p = [1 : 0 : 0]$ by after a suitable $\mathrm{PGL}(3, \mathbb{C})$ transformation. Moreover, we may put the affine equation of C in the following form

$$f(x, y) = y^n + a_1(x)y^{n-1} + \dots + a_n(x) = 0,$$

where $a_i(x) \in \mathbb{C}[x]$. Since $f(x, y)$ is irreducible in $\mathbb{C}[x, y]$, an observation[5] is that $f(x, y)$ has no multiple divisors in $\mathbb{C}\{x\}[y]$.

The idea is to find sufficiently small $D = \{|x| < \rho, |y| < \epsilon\}$ such that $C \cap D$ decomposed into several pieces $C_1 \cup \dots \cup D^l$, where each C_i is homeomorphic to a disk and the union attaches them only at their centers. In fact, suppose $f = uw_1 \dots w_l$ in $\mathbb{C}\{x, y\}$, where $w_1, \dots, w_l$ are irreducible

[5]Since $f(x, y)$ has no multiple divisors in $\mathbb{C}[x, y]$, it's clear $R(f, f_y) \neq 0$ in $\mathbb{C}[x]$, and thus $R(f, f_y) \neq 0$ in $\mathbb{C}\{x\}$, which implies $f(x, y)$ has no multiple divisors in $\mathbb{C}\{x\}[y]$.

Weierstrass polynomials and $u \in \mathbb{C}\{x, y\}^*$. Then each C_i is the zero loucs $\{w_i = 0\}$. If we can construct homeomorphism φ_i from disk Δ_i to C_i for each i and repeat this procedure for all singularities, then we may construct the normalization $\widetilde{C}$ by adding these C_i to $C \setminus \mathrm{sing}(C)$ in a suitable way.

In the following sections we will explain above procedures in detail. The construction of homeomorphism for each singularity is called local resolution and adding these C_i to $C \setminus \mathrm{sing}(C)$ is called the global resolution.

5.2.1. *Local resolution of singularities.*

5.2.2. *Global resolution of singularities.*

5.3. **Bezout theorem for singular curve.**

The curve $C = \{f(x, y) = 0\}$, where f has no multiple divisors.

Example 5.3.1. Consider the curve $y^2 = x^2(1 - x)$. The tangent cone at $(0, 0)$ is $y^2 = x^2$, that is, $y = \pm x$. Thus the multiplicity of the singularity $(0, 0)$ is 2.

If $\{f_d = 0\}$ is union of d distinct lines, call $(0, 0)$ is an ordinary d-tuple point.

Proposition 5.3.1. If $f(x, y)$ has ordinary d-tuple point at $(0, 0)$, then

$$f(x, y) = f_1 \dots f_d$$

in $\mathbb{C}\{x, y\}$, where f_i is irreducible, distinct in $\mathbb{C}\{x, y\}$.

Example 5.3.2. Consider the curve $y^2 - y^3 + x^3 = 0$. The tangent cone at $(0, 0)$ is $y = 0$, double point. Let $y = xw$, then

$$w^2x^2 - x^3w^3 + x^3 = 0$$

thus $x = 0$ implies $w = 0$.

$$\frac{\partial g}{\partial x} = 1 \neq 0, \quad \frac{\partial g}{\partial x} = 0$$

Then $x = x(w)$, $x(0) = 0$ and

$$x'(w) = \frac{-\frac{\partial g}{\partial w}}{\frac{\partial g}{\partial x}} = 0$$

but $x''(w) \neq 0$. Then

$$x(w) = w^2(c + \dots), \quad y(w) = w^3(c + \dots)$$

Let $\widetilde{w} = w\sqrt{c + \dots}$, then

$$x(\widetilde{w}) = \widetilde{w}^2 \quad y(\widetilde{w})$$

Example 5.3.3. Consider the curve $y^2 - y^3 - x^4 = 0$. $y = xw$, then

$$g(x, w) = w^2 - w^3x - x^2 = 0$$

Then $x = 0$ implies $w = 0$. then

$$g_x|_{(0,0)} = 0, \quad g_w|_{(0,0)} = 0$$

Then $g(x, w)$ has ordinary double point at $(0, 0)$. suppose

$$w = xt,$$

then $w_1 = xt_1(x)$, $w_2 = xt_2$, then

$$y_1 = xw = x^2t_1, \quad y_2 = x^2t_2$$

Then $y^2 - y^3 - x^4$ is reducible $(y - y_1(x))(y - y_2(x))$.

5.4. **Genus formula for singular case.** Suppose the projective plane curve C defined by f has only ordinary m-tuple points, $\mathrm{sing}(C) = \{p_1, \dots, p_k\}$, and C has ordinary m_i-tuple point at p_i, $\deg f = d$. Then

$$\widetilde{C} \to C$$

is the normalization, then

$$\mathrm{genus}(\widetilde{C}) = \binom{d-1}{2} - \sum_{i=1}^{k} \binom{m_i}{2} \geq 0$$

Proof. Then

$$d(d-1) = \deg R_{\widetilde{F}} - \sum_{i=1}^{k} m_i(m_i - 1).$$

This shows

$$\deg R_{\widetilde{F}} = d(d-1) - \sum_{i=1}^{k} m_i(m_i - 1).$$

By Riemann-Hurwitz formula one has

$$2 - 2g = 2d - \deg R_{\widetilde{F}}$$

$\square$

Terminology:

$$f(x, y) = f_d(x, y) + f_{d+1}(x, y) + \cdots +$$

defines a singularity with multiplicity d. If $\{f_d(x, y)\}$ is a union of d distinc lines, then call the singularity ordinary d multiple point.

Choose a point $p \notin C$ and not lies in the tangent cone of C for ant singularity. Then apply Riemann-Hurwitz theorem to $\widetilde{F}$. If $q \in C \setminus \mathrm{sing}(C)$, one has

$$\mathrm{mult}_{\pi^{-1}(q)} \widetilde{F} - 1 = \mathrm{mult}_q F - 1 = (f, f_y)_p$$

Suppose q is a ordinary d point. $q_i \in \pi^{-1}(q)$, and then $\mathrm{mult}_{q_i} \widetilde{F} = 1$, since C is locally defined by irreducible linear function. This shows

$$R_{\widetilde{F}} = \sum_{q \in C \setminus \mathrm{sing}(C)} (f, f_y)_q \cdot q$$

Finally it suffices to figure out the intersection number in singular case

Note that $\pi(t) = (t, y_1(t))$.

$$f(x, y) = (y - a_1 x + o(x^2)) \dots (y - a_n x + o(x^2)) = (y - y_1(x)) \dots (y - y_n(x)).$$

Then

$$(f, f_y)_{(0,0)} = \sum_{q_i} \mathrm{ord}_{q_i} \pi^*(f_y) = \sum_{q_i} \mathrm{ord}_{t=0} \prod_{j \neq i} (y_i(t) - y_j(t)) = \sum_{q_i} m-1 = m(m-1)$$

For generally for a cusp

$$\deg(R_{\widetilde{F}} - \sum_{q\in cusp} \pi^{-1}(q)) + \sum_{q\in cusp} 3 = \sum_{q\in C} (f, f_y)_q = d(d-1)$$

5.5. **Poincaré-Hopf.** Note that

$$y = \frac{\mathrm{d}x}{f_y(x,y,1)} = -\frac{\mathrm{d}y}{f_x(x,y,1)}$$

Suppose $f(x,y)$ has singularity $(0,0)$ with multiplicity d.

$$\eta = \frac{\mathrm{d}x}{f_y(x,y)}$$

assume $x=0$ is not in the tangent cone of f at $(0,0)$, that is, $x \nmid f_m(x,y)$. Consider

$$g(x,w) = \frac{f(x,xw)}{x^m}$$

Then

$$x^m g_w(x,w) = f_y(x,y)|_{y=xw} \cdot x = x f_y(x,xw)$$

This shows

$$x^{m-1} g_w(x,w) = f_y(x,xw)$$

Thus

$$\eta = \frac{\mathrm{d}x}{x^{m-1} g_w(x,xw)} = x^{-(m-1)} \frac{\mathrm{d}x}{g_w(x,w)}$$

By induction,

Example 5.5.1. For $y^n = x^m$, with $gcd(m,n) = 1$. Then $\delta_{(0,0)} = (m-1)(n-1)/2$. If $m < n$, then $m_0 = m$, and $y = xw$ implies

$$w^{n-m} = x^m$$

This shows

$$\delta(n,m) = \delta(n-m,m) + \binom{m}{2}$$

6. Normalization: Algebraic viewpoint

Let $f(x,y) \in \mathbb{C}[x,y]$ be an irreducible polynomial and $A = \mathbb{C}[x,y]/(f(x,y))$ be the integral domain. Suppose $\widetilde{A}$ is the integral closure of A in its fractional field $K(A)$. Then $\widetilde{A}$ is a finitely generated $\mathbb{C}$-algebra with

$$\widetilde{A} = \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_k).$$

Then $\widetilde{C} = \{f_1 = 0, \ldots, f_k = 0\} \subseteq \mathbb{C}^n$ is non-singular, and thus has a structure of Riemann surface. Then

$$\widetilde{C} \to C$$

is the normalization. In this case, the δ-invariance

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