RIEMANNIAN GEOMETRY

BOWEN LIU

Contents

0. Preface

0.1. About this lecture.

0.2. To readers.

This note is divided into several parts:

1. In the **First** part, we firstly introduce connections on a vector bundle E in different viewpoints. Holding a connection on E, one can construct connection on its dual bundle E^*, tensor product $E \otimes E^*$ and so on. When E is chosen to be tangent bundle equipped with a Riemannian metric, there is a unique connection which is compatible with metric and torsion-free, which is called Levi-Civita connection.

   A section of tensor products of tangent bundle with its dual bundle is called a tensor, and tensor computation is a powerful tool of Riemannian geometry so we collect some basic properties and operations about tensor in section 2.

   However, tensor computation may be quite complicated in general. To give a neat local computation for tensor, we introduce geodesic in section 3 in order to introduce normal coordinate. By the way we also introduce Hopf-Rinow's theorem about completeness.
2. The **Second** part is about curvature. We introduce curvature using two different views: curvature form and curvature tensor and prove Bianchi identities in these two views. We also introduce Ricci identity for tensor, which is a crucial step in Bochner's technique. In the end we introduce some other important curvatures such as sectional curvature, Ricci curvature and scalar curvature.
3. The **Third** part is about Bochner's technique, which is one of the most important technique in modern Riemannian geometry. Holding this technical, we can see how does bounded Ricci curvature appear as an obstruction to the existence of Killing fields and harmonic 1-forms. Aside these, we also introduce Hodge theory, which allows us to use harmonic 1-forms to represent elements in the first homology group, then Bochner's technique gives a kind of vanishing theorem.
4. The goal of **Fourth part** is to solve the following question: "Given two points p, q, what' the length-minimizing curve connecting p, q in a Riemannian manifold?". To answer this, we consider the arc-length functional, and
   - (a) First variation formula implies geodesics are critical points of arc-length functional.
   - (b) Second variation formula implies if a geodesic contains no interior conjugate points, then it's locally minimum of arc-length functional.

   Along the way we develop the tools of index form and Jacobi fields, which are also quite important in the following parts.
5. The **Fifth part** generalizes geodesic and Hessian of smooth function to some extend. In this part we define what is second fundamental form,

and when a smooth map between Riemannian manifold is harmonic map. Finally we consider its variation and Bochner's formula.

6. The **Sixth part** introduces how does curvature condition controls the topology of the whole manifold. We mainly consider the following three cases:
   (a) A Riemannian manifold M with non-positive sectional curvature is $K(\pi_1(M), 1)$, that is M is covered by $\mathbb{R}^n$. A fact in topology says if a finite dimensional CW complex is a $K(G, 1)$ space for some group G, we must have G is torsion-free. Here Cartan's torsion-free theorem gives a neat proof of this fact via deck transformations and some basic facts about Lie group action. Furthermore, $\pi_1(M)$ deserves many other interesting properties:
      I Preissmann's theorem says if M is compact with negative sectional curvature, then any non-trivial abelian subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$ and $\pi_1(M)$ itself is not abelian.
      II Byers' theorem says more: if M is compact with negative sectional curvature, then any non-trivial solvable subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$.
   (b) A Riemannian manifold with curvature lower bounded is also quite interesting.
      I Myers' theorem says a Riemannian manifold with positive Ricci curvature is compact, and with finite fundamental group. However, it's meaningless to consider what will happen if Ricci curvature is upper bounded, since every Riemannian n-manifold admits a complete metric with $\mathrm{Ric} < 0$ if $n \geq 3$.
      II Synge's theorem says a little about fundamental group of Riemannian manifold M with positive sectional curvature and even dimension: If it's orientable, then it's simply-connected, otherwise $\pi_1(M) = \mathbb{Z}_2$.
   (c) Finally, a celebrated theorem of Hopf implies every Riemannian manifold with constant sectional curvature is covered by three basic models, which are called space forms.
7. The **Seventh part** is also about curvature, but it shows how to use comparison in curvatures to obtain comparison in other objects, such as length, metrics, volume and Hessian or Laplacian operators. A philosophy is that the "larger" curvature is, the "smaller" other thing is. It also gives us some rigidity theorem, an interesting result is that Cheng's theorem.
   (a) If (M, g) be a Riemannian n-manifold with $\mathrm{Ric}(g) \geq (n-1)kg$ for some constant $k > 0$, then Myers's theorem implies $\mathrm{diam}(M) \leq \pi/\sqrt{k}$. If $\mathrm{diam}(M) = \pi/\sqrt{k}$, then Cheng's theorem says (M, g) is isometric to $\mathbb{S}^n(1/\sqrt{k})$ with standard metric.
8. The **Final part** is about Riemannian symmetric space.

0.3. **Some notations and conventions.**

0.3.1. *Conventions.*

1. We always use Einstein summation.
2. When we say M is a smooth manifold, we assume it's a real smooth manifold and it's connected.
3. When we consider vector bundles, we assume it's a real vector bundle.

0.3.2. *Notations about smooth manifolds.*

1. For a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$, we use $\frac{\partial f}{\partial x^i}$ to denote its partial derivative with respect to x^i, where x^i are coordinates of $\mathbb{R}^n$.
2. For a smooth manifold M, we use TM, T^*M to denote its tangent space and cotangent space respectively, and we also use Ω^k_M to denote the bundle of k-forms, that is $\bigwedge^k T^*M$.
3. We always use X, Y, Z to denote vector fields, ω to denote 1-forms and φ, ψ to denote k-forms.
4. Given a vector bundle $E \to M$ over a smooth manifold M, we use $C^\infty(M, E)$ to denote the set of all smooth sections of E.

0.3.3. *Notations about Riemannian manifolds.*

1. We use (M, g) to denote a Riemannian manifold, where M is a smooth manifold, and g is its Riemannian metric. If there is no ambiguity, we will omit g.
2. For a Riemannian metric g, we sometimes use $\langle -, - \rangle_g$ to denote it, or directly $\langle -, - \rangle$ if there is no ambiguity.

Part 1. Basic settings

1. Connections

Connection is a very basic conception in realm of geometry of vector bundles, and there are too many definitions of it which seem to be different. This part is divided into four parts:

1. In the first section, we will introduce one approach to connection in two different ways, the first one is often used in complex geometry and the second is given in [Car92].
2. In the second section, we will give another characterization of connection using parallel transport, and we will see all these approaches are same in fact.
3. In the third section, we will put more restrictions on our connection, such as compatibility with metric and torsion-free.
4. In the fourth section, we will construct many new connections from a given connection, which play an important role in our later discuss.

1.1. Two different viewpoints to connection.

1.1.1. *First viewpoint.* When I first learn Riemannian geometry or complex geometry, I'm quite confused about why we need connection, and why we define it like this? In fact, given a vector bundle $\pi : E \to M$, connections on E are arised to take "derivative" of a section $s : M \to E$ in a given direction.

It's quite natural to ask such a question, since when we learn calculus, we already know how to take derivative of a smooth function $f\colon M \to \mathbb{R}^m$ to obtain a 1-form, that is a section of T^*M. In another point of view, any smooth function $f\colon M \to \mathbb{R}^m$ can be regarded as a section of trivial vector bundle $M \times \mathbb{R}^m$, as follows

$$x \mapsto (x, f(x))$$

and we can also regard its derivative $\mathrm{d}f$ as a section of $T^*M \otimes (M \times \mathbb{R}^m)$. So taking derivative can be seen as the following operator:

$$\nabla\colon C^\infty(M, M \times \mathbb{R}^m) \to C^\infty(M, T^*M \otimes (M \times \mathbb{R}^m))$$

In general, we can define a connection as follows:

Definition 1.1.1 (connection)**.** A connection ∇ on a vector bundle E on a smooth manifold M is a linear operator

$$\nabla\colon C^\infty(M, E) \to C^\infty(M, T^*M \otimes E)$$

satisfying Leibniz rule $\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s$, where $s \in C^\infty(M, E)$.

Remark 1.1.1 (local form)*.* Suppose $\{e_\alpha\}$ is a local frame of E, then any section s of E locally can be written as $s^\alpha e_\alpha$, then Leibniz rule implies

$$\nabla(s^\alpha e_\alpha) = \mathrm{d}s^\alpha e_\alpha + s^\alpha \nabla e_\alpha$$

that is to say ∇ is determined by

$$\nabla e_\alpha = \omega_\alpha^\beta e_\beta$$

where ω_α^β are 1-forms, which forms a 1-form valued matrix ω. Note that ω depends on the choice of local frame $\{e_\alpha\}$. Suppose there is another local frame $\{\widetilde{e}_\alpha\}$, with transition functions $\widetilde{e}_\alpha = g_\alpha^\beta e_\beta$, then direct computation shows

$$\begin{aligned}
\nabla \widetilde{e}_\alpha &= \nabla(g_\alpha^\beta e_\beta) \\
&= g_\alpha^\beta \nabla e_\beta + \mathrm{d}g_\alpha^\beta \otimes e_\beta \\
&= g_\alpha^\beta \omega_\beta^\gamma \otimes e_\gamma + \mathrm{d}g_\alpha^\beta \otimes e_\beta \\
\nabla \widetilde{e}_\alpha &= \widetilde{\omega}_\alpha^\gamma \otimes g_\gamma^\beta e_\beta
\end{aligned}$$

In matrix notation, if we regard $\{e_\alpha\}$ as a row vector e, then

$$eg\widetilde{\omega} = e(\mathrm{d}g + \omega g)$$

which implies $\widetilde{\omega} = g^{-1}\omega g + g^{-1}\mathrm{d}g$.

1.1.2. *Second viewpoint.* The following is the definition given in [Car92].

Definition 1.1.2 (connection)**.** A connection ∇ on a vector bundle E on a smooth manifold M is a mapping

$$\begin{aligned}
\nabla \colon C^\infty(M, TM) \times C^\infty(M, E) &\to C^\infty(M, E) \\
(X, s) &\mapsto \nabla_X s
\end{aligned}$$

satisfying the following properties:

1. $\nabla_{fX+gY} s = f\nabla_X s + g\nabla_Y s$
2. $\nabla_X(s + s') = \nabla_X s + \nabla_X s'$
3. $\nabla_X(fs) = f\nabla_X s + X(f)s$

where $X, Y \in C^\infty(M, TM), f, g \in C^\infty(M)$ and $s, s' \in C^\infty(M, E)$.

Remark 1.1.2 (local form). If we write a vector field X and a section s of E locally as $X = X^i \frac{\partial}{\partial x^i}$ and $e = s^\alpha e_\alpha$. Then

$$\begin{aligned}
\nabla_X s &= \nabla_{X^i \frac{\partial}{\partial x^i}} s^\alpha e_\alpha \\
&= X^i \nabla_{\frac{\partial}{\partial x^i}} s^\alpha e_\alpha \\
&= X^i s^\alpha \nabla_{\frac{\partial}{\partial x^i}} e_\alpha + X^i \frac{\partial s^\alpha}{\partial x^i} e_\alpha \\
&= X^i s^\alpha \nabla_{\frac{\partial}{\partial x^i}} e_\alpha + X(s^\alpha) e_\alpha
\end{aligned}$$

If we write $\nabla_{\frac{\partial}{\partial x^i}} s_\alpha = \Gamma_{i\alpha}^\beta e_\beta$, then

$$\nabla_X s = (X^i s^\alpha \Gamma_{i\alpha}^\beta + X(s^\beta)) e_\beta$$

So as we can see, $\Gamma_{i\alpha}^\beta$, which is sometimes called Christoffel symbol, completely determines our connection ∇.

Remark 1.1.3 (The equivalence between two definitions). Locally a connection in Definition 1.1.1 is a 1-form valued matrix ω, and write it as $\omega_\alpha^\beta = \Gamma_{j\alpha}^\beta \mathrm{d}x^j$. Then

$$\begin{aligned}\nabla e_\alpha &= \omega_\alpha^\beta e_\beta \\ &= \Gamma_{i\alpha}^\beta \mathrm{d}x^i e_\beta\end{aligned}$$

So if want to define $\nabla_{\frac{\partial}{\partial x^i}} e_\alpha$, ∇e_α need to "eat" a vector field, and luckily $\mathrm{d}x^j$ can eat one, so we can define it as follows

$$\begin{aligned}\nabla_{\frac{\partial}{\partial x^i}} e_\alpha :=& \Gamma_{j\alpha}^\beta \mathrm{d}x^j (\frac{\partial}{\partial x^i}) e_\beta \\ =& \Gamma_{i\alpha}^\beta e_\beta\end{aligned}$$

From this we can see these two definitions are same.

Remark 1.1.4 (connection and covariant derivative). Some authors may also use terminology "covariant derivative", here we make a clearification: Here we give two definitions of connection ∇ on a vector bundle E. Given a section s of E and a vector field X, we call $\nabla_X s$ the covariant derivative of s with respect to X. In fact, you can see connection and covariant derivative the same thing, just different terminology.

1.2. **Compatibility and torsion-free.**

1.2.1. *Compatibility with metric.* Now consider a vector bundle E with a metric g, which can be locally written as $g_{\alpha\beta} e^\alpha \otimes e^\beta$. So if there is a connection ∇ on E, it's natural to ask it to be compatible with our metric.

Definition 1.2.1 (compatibility)**.** A connection ∇ on vector bundle E is compatible with metric g, if for any two section s, t of E, we have

$$\mathrm{d}g(s,t) = g(\nabla s, t) + g(s, \nabla t)$$

Remark 1.2.1 (local form). Locally we can compute it as

$$\begin{aligned}\mathrm{d}g_{\alpha\beta} &= \mathrm{d}g(e_\alpha, e_\beta) \\ &= g(\nabla e_\alpha, e_\beta) + g(e_\alpha, \nabla e_\beta) \\ &= \omega_\alpha^\gamma g_{\gamma\beta} + g_{\alpha\gamma}\omega_\beta^\gamma\end{aligned}$$

So in matrix notation we have[1]

$$\mathrm{d}g = \omega g + g\omega^t$$

In particular we have

$$\frac{\partial}{\partial x^i} g_{\alpha\beta} = \Gamma_{i\alpha}^\gamma g_{\gamma\beta} + \Gamma_{i\beta}^\gamma g_{\alpha\gamma}$$

for all i, α, β.

[1]Here we need to pay more attention, although as a number $g_{\alpha\gamma}\omega_\beta^\gamma = \omega_\beta^\gamma g_{\alpha\gamma}$, we can not write this matrix notation as $\mathrm{d}g = \omega g + \omega g^t$, since $\omega_\beta^\gamma g_{\gamma\alpha}$ is (β,α)-entry of ωg^t, but $\mathrm{d}g_{\alpha\beta}$ and $g_{\alpha\gamma}\omega_\beta^\gamma$ are (α,β)-entries of $g\omega^t$.

1.2.2. *Torsion-free.* Now let's choose our vector bundle E to be tangent bundle of a Riemannian manifold (M, g).

Definition 1.2.2 (torsion-free)**.** A connection ∇ of TM is torsion-free if

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

where X, Y are vector fields.

Remark 1.2.2 (local form). If we choose $X = \frac{\partial}{\partial x^i}, Y = \frac{\partial}{\partial x^j}$, then we have

$$\begin{aligned}\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} &= (\Gamma^k_{ij} - \Gamma^k_{ji}) \frac{\partial}{\partial x^k} \\ &= 0\end{aligned}$$

which is equivalent to say Γ^k_{ij} is symmetric in i and j.

1.3. **Levi-Civita connection.** There are infinitely many connections on tangent bundle of a Riemannian manifold, but an interesting thing is that there is only one of them which is both compatible with Riemannian metric and torsion-free.

It suffices to see a connection which is compatible with metric and torsion-free is completely determined, in other words, Γ^k_{ij} is completely determined. Note that compatibility implies

$$\begin{aligned}Xg(Y, Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Yg(Z, X) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ Zg(X, Y) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y)\end{aligned}$$

where X, Y and Z are vector fields. Adding first two equations, substract the third and use torsion-free condition, we will see

$$Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) = g([X, Z], Y) + g([Y, Z], X) + g([X, Y], Z) + 2g(Z, \nabla_Y X)$$

thus

$$g(Z, \nabla_Y X) = \frac{1}{2}(Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g([X, Z], Y) - g([Y, Z], X) - g([X, Y], Z))$$

which implies $\nabla_X Y$ is uniquely determined. Above formula is also called Koszul formula.

Remark 1.3.1 (local form). Firstly, compatibility implies

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma^l_{ki} g_{lj} + \Gamma^l_{kj} g_{il}$$

By permuting i, j, k we obtain the following two equations

$$\begin{aligned}\frac{\partial g_{jk}}{\partial x^i} &= \Gamma^l_{ij} g_{lk} + \Gamma^l_{ik} g_{jl} \\ \frac{\partial g_{ki}}{\partial x^j} &= \Gamma^l_{jk} g_{li} + \Gamma^l_{ji} g_{kl}\end{aligned}$$

By the symmetry of Γ_{ij}^l in i,j and symmetry of g_{ij}, we have
$$2\Gamma_{ij}^l g_{lk} = \frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}$$
If we use (g^{ij}) to denote the inverse matrix of (g_{ij}), then we have
$$\Gamma_{ij}^l = \frac{1}{2} g^{kl} (\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k})$$
which implies Christoffel symbol is completely determined by Riemannian metric and its partial derivatives.

2. Induced connections

2.1. **Transpose.** In this section we collect some notations about transpose we will use later.

2.1.1. *Transpose via duality.* Let V be a vector space with dual vector space V^*. Suppose $\{e_\alpha\}$ is a basis of V with dual basis. For $B = B_\alpha^\beta e^\alpha \otimes e_\beta \in V^* \otimes V$, there is a natural way to define its transpose B^T via the following pairing

$$(Be_\alpha, e^\beta) = (e_\alpha, B^T e^\beta)$$

where (-,-) is dual pairing. This shows

$$B_\alpha^\beta = (B^T)_\alpha^\beta$$

which implies

$$B^T = (B^T)_\alpha^\beta e_\beta \otimes e^\alpha = B_\alpha^\beta e_\beta \otimes e^\alpha \in V \otimes V^*$$

2.1.2. *Transpose via metric.* Let S, Q be two Euclidean vector spaces, that is a vector space equipped with an inner product. Suppose $\{e_\alpha\}$ is an orthonormal basis of S while $\{t_\beta\}$ is an orthonormal basis of Q. For $B = B_\alpha^\beta e^\alpha \otimes t_\beta \in S^* \otimes Q$, its transpose $B^T \in Q^* \otimes S$, given by

$$\langle Be_\alpha, t_\beta \rangle_Q = \langle e_\alpha, B^T t_\beta \rangle_S$$

where $\langle$-,-$\rangle$ are inner products on S, Q. This shows

$$B_\alpha^\beta = (B^T)_\beta^\alpha$$

which implies

$$B^T = (B^T)_\beta^\alpha t^\beta \otimes e_\alpha = B_\alpha^\beta t^\beta \otimes e_\alpha$$

Remark 2.1.1. The reason for Einstein summation fails is that here we choose orthonormal basis of S and Q, thus there are some kronecker symbols we omit.

2.2. **Induced connections on vector bundle.** Given a vector bundle E over a smooth manifold M, you can construct many new vector bundles by algebraic method, such as considering its dual bundle E^*, tensor product $E \otimes E$ and so on. Now let's see if we already have a connection on E, how to construct some new connections on new vector bundles.

2.2.1. *Induced connection on dual bundle.* Let E be a vector bundle equipped with connection ∇^E, induced connection on dual bundle E^* is defined as follows

$$\mathrm{d}(s, t) = (\nabla^{E^*} s, t) + (s, \nabla^E t)$$

where s, t are sections of E^* and E respectively, and (-,-) denotes the dual pairing. Suppose $\{e_\alpha\}$ is a local frame of E with dual frame $\{e^\alpha\}$, then

$$0 = ((\omega^*)_\gamma^\alpha e^\gamma, e_\beta) + (e^\alpha, \omega_\beta^\gamma e_\gamma)$$

where ω and ω^* are connection 1-forms of ∇^E and ∇^{E^*} respectively. This shows

$$(\omega^*)^\alpha_\beta + \omega^\alpha_\beta = 0$$

This shows

$$\omega^* = (\omega^*)^\alpha_\beta e_\alpha \otimes e^\beta = -\omega^\alpha_\beta e_\alpha \otimes e^\beta$$

that is $\omega^* = -\omega^T$.

Remark 2.2.1 (another viewpoint of torsion-free)*.* Let ∇ be a connection on TM, given by Christoffel symbol Γ^k_{ij}, then induced connection on T^*M is given by

$$\nabla \mathrm{d}x^k = -\Gamma^k_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$$

Given a section s of T^*M, there is a natural 2-form obtained from taking exterior derivative $\mathrm{d}s$, and note that $\bigwedge^2 T^*M$ is just the skew-symmetrization of $T^*M \otimes T^*M$, so it's natural to require the skew-symmetrization of ∇s is $\mathrm{d}s$. Locally that's to say skew-symmetrization of $\nabla \mathrm{d}x^k = 0$, that is,

$$-\Gamma^k_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j = 0$$

This shows torsion-free if and only if the skew-symmetrization of ∇s is $\mathrm{d}s$.

2.2.2. *Induced connection on tensor product.* Let E, F be two vector bundles equipped with connection ∇^E, ∇^F respectively, induced connection on $E \oplus F$ is given by

$$\nabla^{E\otimes F}(s \otimes t) := \nabla^E s \otimes t + s \otimes \nabla^F t$$

where s, t are sections of E, F respectively.

2.2.3. *Induced connection on wedge product.* Let E be a vector bundle equipped with connection ∇, there is an induced connection on $\bigwedge^2 E$, since it's a subbundle of $\bigotimes^2 E$. To be explicit

$$\begin{aligned}\nabla^{\wedge^2 E}(s \wedge t) :=& \nabla^{\otimes^2 E}(s \otimes t - t \otimes s) \\ =& \nabla s \otimes t + s \otimes \nabla t - \nabla t \otimes s - t \otimes \nabla s \\ =& \nabla s \wedge t + s \wedge \nabla t\end{aligned}$$

where s, t are sections of E.

Remark 2.2.2*.* In general case, there is an induced connection on $\otimes^k E$, given by

$$\nabla^{\otimes^k E}(s_1 \otimes \cdots \otimes s_k) = \sum_{i=1}^k s_1 \otimes \cdots \otimes \nabla s_i \otimes \cdots \otimes s_k$$

where $s_1, \ldots, s_k$ are sections of E. Its restriction on $\bigwedge^k E$ gives a connection on $\bigwedge^k E$, that is,

$$\nabla^{\otimes^k E}(s_1 \wedge \cdots \wedge s_k) = \sum_{i=1}^k s_1 \wedge \cdots \wedge \nabla s_i \wedge \cdots \wedge s_k$$

2.2.4. *Induced connection on endomorphism bundle.* Let E be a vector bundle equipped with connection ∇^E, there is an induced connection ∇ on $\operatorname{End} E$, since we have $\operatorname{End} E \cong E \otimes E^*$. Suppose $\{e_\alpha\}$ is a local frame of E with dual frame $\{e^\alpha\}$, for section s of $E \otimes E^*$ locally written as $s = s^\alpha_\beta e_\alpha \otimes e^\beta$, direct computation shows

$$\begin{aligned}\nabla^{E\otimes E^*}(s^\alpha_\beta e_\alpha \otimes e^\beta) &= \mathrm{d}s^\alpha_\beta e_\alpha \otimes e^\beta + s^\alpha_\beta(\nabla^E e_\alpha \otimes e^\beta + e_\alpha \otimes \nabla^{E^*} e^\beta) \\ &= \mathrm{d}s^\alpha_\beta e_\alpha \otimes e^\beta + s^\alpha_\beta \omega^\gamma_\alpha e_\gamma \otimes e^\beta - s^\alpha_\beta \omega^\beta_\gamma e_\alpha \otimes e^\gamma \\ &= (\mathrm{d}s^\alpha_\beta + s^\alpha_\beta \omega^\gamma_\alpha - \omega^\beta_\gamma s^\alpha_\beta) e_\alpha \otimes e^\beta\end{aligned}$$

Thus in matrix notation we have

$$\nabla s = \mathrm{d}s + s\omega - \omega s$$

Remark 2.2.3. There is another way to construct a connection on $E \otimes E^*$: For any section s of $E \otimes E^*$, we have a function $s(e^\alpha, e_\beta)$, the induced connection on $E \otimes E^*$ is defined as

$$\mathrm{d}s(e^\alpha, e_\beta) = \nabla^{E\otimes E^*} s(e^\alpha, e_\beta) + s(\nabla^{E^*} e^\alpha, e_\beta) + s(e^\alpha, \nabla^E e_\beta)$$

Locally if we write $s = s^\alpha_\beta e_\alpha \otimes e^\beta$, then

$$\begin{aligned}\mathrm{d}(s^\alpha_\beta) &= (\nabla s)^\alpha_\beta + s(-\omega^\alpha_\gamma e^\gamma, e_\beta) + s(e^\alpha, \omega^\gamma_\beta e_\gamma) \\ &= (\nabla s)^\alpha_\beta - s^\gamma_\beta \omega^\alpha_\gamma + \omega^\gamma_\beta s^\alpha_\gamma\end{aligned}$$

which implies connections obtained from these two ways are same. In fact, we will use this way to define induced connections of arbitrary tensor.

2.3. **Induced connections on tensor.** Let M be a smooth manifold.

Definition 2.3.1 (tensor)**.** A section of $\bigotimes^s TM \otimes \bigotimes^r T^*M$ is called a (s, r)-tensor.

Example 2.3.1. A smooth function f is a $(0, 0)$-tensor.

Example 2.3.2. A vector field X is a $(1, 0)$-tensor.

Example 2.3.3. A 1-form ω is a $(0, 1)$-tensor.

Example 2.3.4. The Riemannian metric g is a $(0, 2)$-tensor.

Definition 2.3.2 (connection on tensor)**.** For a (s, r)-tensor T, ∇T is a $(s, r + 1)$-tensor, which is defined by

$$\begin{aligned}\nabla T(\mathrm{d}x^{j_1}, \ldots, \mathrm{d}x^{j_s}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_r}}) :=& \frac{\partial}{\partial x^i} T(\mathrm{d}x^{j_1}, \ldots, \mathrm{d}x^{j_s}, \frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_r}}) \\ &- \sum_{l=1}^{s} T(\mathrm{d}x^{j_1}, \ldots, \nabla_{\frac{\partial}{\partial x^i}} \mathrm{d}x^{j_l}, \ldots, \mathrm{d}x^{j_s}, \frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_r}}) \\ &- \sum_{m=1}^{r} T(\mathrm{d}x^{j_1}, \ldots, \mathrm{d}x^{j_s}, \frac{\partial}{\partial x^{i_1}}, \ldots, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^{i_m}}, \ldots, \frac{\partial}{\partial x^{i_r}})\end{aligned}$$

Definition 2.3.3 (covariant derivative of tensor)**.** For a (s,r)-tensor T, the covariant derivative of T with respect to vector field X, which is a (s,r)-tensor, is defined as

$$\nabla_X T := \nabla T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},X,\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}})$$

Remark 2.3.1 (local form)*.* If we write a (s,r)-tensor T locally as

$$T^{j_1\dots j_s}_{i_1\dots i_r}\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_s}}\otimes\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_r}$$

and $(s,r+1)$-tensor ∇T locally as

$$\nabla_i T^{j_1\dots j_s}_{i_1\dots i_r}\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_s}}\otimes\mathrm{d}x^{i}\otimes\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_r}$$

Then by definition we have

$$\nabla_i T^{j_1\dots j_s}_{i_1\dots i_r}=\frac{\partial T^{j_1\dots j_s}_{i_1\dots i_r}}{\partial x^i}+\sum_{l=1}^s\Gamma^{j_l}_{iq}T^{j_1\dots j_{l-1}qj_{l+1}\dots j_s}_{i_1\dots i_r}-\sum_{m=1}^r\Gamma^q_{ii_m}T^{j_1\dots j_s}_{i_1\dots i_{m-1}qi_{m+1}\dots i_r}$$

Example 2.3.5. Consider $(0,0)$-tensor f, that is a smooth function. Then ∇f is a $(0,1)$-tensor, given by

$$\nabla f=\nabla_i f\mathrm{d}x^i$$

by definition $\nabla_i f=\frac{\partial f}{\partial x^i}$, it coincides with our usual notations.

Notation 2.3.1. For a smooth function $f\colon M\to\mathbb{R}$, the following notations are same in Riemannian geometry:

1. $\frac{\partial f}{\partial x^i}$.
2. $\partial_i f$.
3. $\nabla_i f$.

Any of them denotes the partial derivatives of f with respect to $\frac{\partial}{\partial x^i}$.

Inductively, we can define $\nabla^2 T$ to be $\nabla(\nabla T)$, which is a $(s,r+2)$-tensor, and locally write it as

$$\nabla^2 T=\nabla^2_{k,i}T^{j_1\dots j_s}_{i_1\dots i_r}\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_s}}\otimes\mathrm{d}x^k\otimes\mathrm{d}x^{i}\otimes\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_r}$$

Now there is a natural question: $\nabla^2_{k,i}T$ is a (s,r)-tensor, and $\nabla_k\nabla_i T$ is also a (s,r)-tensor, does they agree? Unfortunately, it's false in general.

Example 2.3.6. For $(0,0)$-tensor f, by definition we have $\nabla^2 f$ is $\nabla(\nabla_i f\mathrm{d}x^i)$, which is called the Hessian of f, denoted by $\operatorname{Hess} f$. More explicitly

$$\begin{aligned}\operatorname{Hess} f&=\nabla(\nabla_i f\mathrm{d}x^i)\\&=\frac{\partial\nabla_i f}{\partial x^k}\mathrm{d}x^k\otimes\mathrm{d}x^i-\nabla_i f\Gamma^i_{kj}\mathrm{d}x^k\otimes\mathrm{d}x^j\\&=(\frac{\partial^2 f}{\partial x^k\partial x^i}-\Gamma^j_{ki}\frac{\partial f}{\partial x^j})\mathrm{d}x^k\otimes\mathrm{d}x^i\end{aligned}$$

that is $\nabla^2_{k,i}f = \frac{\partial^2 f}{\partial x^k \partial x^i} - \Gamma^j_{ki}\frac{\partial f}{\partial x^j}$. But $\nabla_k\nabla_i f = \frac{\partial^2 f}{\partial x^k \partial x^i}$, so $\nabla^2_{k,i} \neq \nabla_k\nabla_i f$ in general.

Proposition 2.3.1.

$$\nabla^2_{k,i} T^{j_1\dots j_s}_{i_1\dots i_r} = \nabla_k\nabla_i T^{j_1\dots j_s}_{i_1\dots i_r} - \Gamma^j_{ki}\nabla_j T^{j_1\dots j_s}_{i_1\dots i_r}$$

Proof. Direct computation shows

$$\begin{aligned}
\nabla^2_{k,i} T^{j_1\dots j_s}_{i_1\dots i_r} =& \nabla^2 T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}}) \\
=& \nabla_{\frac{\partial}{\partial x^k}}\nabla T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}}) \\
=& \underbrace{\frac{\partial}{\partial x^k}\nabla T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}})}_{\text{part I}} \\
& \underbrace{-\nabla T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}})}_{\text{part II}} \\
& \underbrace{-\sum_{l=1}^{s}\nabla T(\mathrm{d}x^{j_1},\dots,\nabla_{\frac{\partial}{\partial x^k}}\mathrm{d}x^{j_l},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_r}})}_{\text{part III}} \\
& \underbrace{-\sum_{m=1}^{r}\nabla T(\mathrm{d}x^{j_1},\dots,\mathrm{d}x^{j_s},\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^{i_1}},\dots,\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^{i_m}},\dots,\frac{\partial}{\partial x^{i_r}})}_{\text{part IV}}
\end{aligned}$$

Note that

1. Part I+III+IV is $\nabla_k\nabla_i T^{j_1\dots j_s}_{i_1\dots i_r}$.
2. Part II is $\Gamma^j_{ki}\nabla_j T^{j_1\dots j_s}_{i_1\dots i_r}$.

□

Remark 2.3.2 (another viewpoint of compatibility). Note that we can regard our Riemannian metric g as a $(0,2)$-tensor. Recall our definition for compatibility is for any two vector fields X, Y we have

$$\mathrm{d}g(X,Y) = g(\nabla X, Y) + g(X, \nabla Y)$$

Or more explicit for vector field Z, we have

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

However, by definition of ∇g we have

$$\nabla_Z g(X,Y) = Zg(X,Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)$$

which shows that compatibility is equivalent to $\nabla g = 0$.

2.4. **Type change of tensor.** In general, for a (s, r)-tensor, we can change its type into any type of $(s-k, r+k)$ for all k such that $s-k \geq 0, r+k \geq 0$, since TM is canonically isomorphic to T^*M, which is called music isomorphism. More explicitly, for any vector field X, it gives a 1-form by

$$X^\flat \colon Y \mapsto g(X, Y)$$

where Y is a vector field. Locally we have

$$\begin{aligned} g(\frac{\partial}{\partial x^i}, Y) &= g(\frac{\partial}{\partial x^i}, \mathrm{d}x^j(Y)\frac{\partial}{\partial x^j}) \\ &= \mathrm{d}x^j(Y) g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \\ &= g_{ij}\mathrm{d}x^j(Y) \end{aligned}$$

that is $(\frac{\partial}{\partial x^i})^\flat = g_{ij}\mathrm{d}x^j$ of T^*M. Similarly for any 1-form ω, it can be regarded as a section of $TM = T^{**}M$ via

$$\omega^\sharp \colon \beta \mapsto g(\omega, \beta)$$

and locally we have

$$\begin{aligned} g(\mathrm{d}x^j, \beta) &= g(\mathrm{d}x^j, \beta(\frac{\partial}{\partial x^i})\mathrm{d}x^i) \\ &= \beta(\frac{\partial}{\partial x^i}) g^{ij} \end{aligned}$$

that is $(\mathrm{d}x^j)^\sharp = g^{ij}\frac{\partial}{\partial x^i}$. In a summary, we have the so-called music isomorphism locally looks like

$$\begin{aligned} \flat : TM &\to T^*M & \sharp : T^*M &\to TM \\ \frac{\partial}{\partial x^i} &\mapsto g_{ij}\mathrm{d}x^j & \mathrm{d}x^j &\mapsto g^{ij}\frac{\partial}{\partial x^i} \end{aligned}$$

Example 2.4.1. For a smooth function f, ∇f is a $(0, 1)$-tensor, locally written as

$$\nabla f = \frac{\partial f}{\partial x^i}\mathrm{d}x^i$$

Then we can change its type into $(1, 0)$, that is

$$\nabla f = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}$$

More generally, for a 1-form ω, locally looks like $\omega_i \mathrm{d}x^i$, then we can change it into a $(1, 0)$-tensor, and it locally looks like

$$\omega^\sharp = g^{ij}\omega_i\frac{\partial}{\partial x^j}$$

Example 2.4.2 (Induced metric on T^*M)**.** Recall that a Riemannian metric g is a $(0, 2)$-tensor, locally written as

$$g = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$$

Then we can change its type into $(2,0)$, that is

$$g_{ij}g^{ik}g^{jl}\frac{\partial}{\partial x^k}\otimes\frac{\partial}{\partial x^l}=\delta_j^k g^{jl}\frac{\partial}{\partial x^k}\otimes\frac{\partial}{\partial x^l}=g^{kl}\frac{\partial}{\partial x^k}\otimes\frac{\partial}{\partial x^l}$$

that is a metric on T^*M.

2.5. **Induced metric on tensor.** If g is a Riemannian metric, then its $(2,0)$-type is a metric on T^*M. Now we can induce a metric on $T^*M\otimes T^*M$ as follows: Take two $(0,2)$-tensors T,S and write them locally as $T=T_{ij}\mathrm{d}x^i\otimes\mathrm{d}x^j, S=S_{kl}\mathrm{d}x^k\otimes\mathrm{d}x^l$, then

$$\begin{aligned}g(T,S)&=T_{ij}S_{kl}g(\mathrm{d}x^i\otimes\mathrm{d}x^j,\mathrm{d}x^k\otimes\mathrm{d}x^l)\\&:=T_{ij}S_{kl}g^{ik}g^{jl}\end{aligned}$$

Remark 2.5.1. In general we also have induced metric on $\bigotimes^k T^*M$, and on Ω^k_M, which will be used later in Hodge theory.

Proposition 2.5.1. If connection ∇ on vector bundle T^*M is compatible with metric g on it, then induced connection on $T^*M\otimes T^*M$ is compatible with induced metric g on it.

Proof. It suffices to check

$$\frac{\partial}{\partial x^m}g(\mathrm{d}x^i\otimes\mathrm{d}x^j,\mathrm{d}x^k\otimes\mathrm{d}x^l)=g(\nabla_{\frac{\partial}{\partial x^m}}\mathrm{d}x^i\otimes\mathrm{d}x^j,\mathrm{d}x^k\otimes\mathrm{d}x^l)+g(\mathrm{d}x^i\otimes\mathrm{d}x^j,\nabla_{\frac{\partial}{\partial x^m}}\mathrm{d}x^k\otimes\mathrm{d}x^l)$$

By compatibility of ∇ and g, we have

$$\frac{\partial g^{ij}}{\partial x^k}=-\Gamma^i_{kl}g^{lj}-\Gamma^j_{kl}g^{il}$$

Thus direct computation shows

$$\begin{aligned}\frac{\partial}{\partial x^m}g(\mathrm{d}x^i\otimes\mathrm{d}x^j,\mathrm{d}x^k\otimes\mathrm{d}x^l)&=-(\Gamma^i_{mn}g^{nk}+\Gamma^k_{mn}g^{in})g^{jl}-g^{ik}(\Gamma^j_{mn}g^{nl}+\Gamma^l_{mn}g^{jn})\\g(\nabla_{\frac{\partial}{\partial x^m}}\mathrm{d}x^i\otimes\mathrm{d}x^j,\mathrm{d}x^k\otimes\mathrm{d}x^l)&=-\Gamma^i_{mn}g^{nk}g^{jl}-\Gamma^j_{mn}g^{ik}g^{nl}\\g(\mathrm{d}x^i\otimes\mathrm{d}x^j,\nabla_{\frac{\partial}{\partial x^m}}\mathrm{d}x^k\otimes\mathrm{d}x^l)&=-\Gamma^k_{mn}g^{in}g^{jl}-\Gamma^l_{mn}g^{ik}g^{jn}\end{aligned}$$

This yields the desired result. □

2.6. **Trace of tensor.** Let's see a simple example: For a $(1,1)$-tensor T, we can define its "trace", since there is a natural isomorphism between $TM\otimes T^*M$ and $\mathrm{End}(TM)$, thus we can take its trace in the sense of matrix. To be explicit, if we locally write T as $T=T^i_j\frac{\partial}{\partial x^i}\otimes\mathrm{d}x^j$, then trace of T, denoted by $\mathrm{tr}_g T$, is defined as T^i_i.

If T is not in $(1,1)$-type, then we change it into $(1,1)$-type and then take trace:

1. If $T=T_{ij}\mathrm{d}x^i\otimes\mathrm{d}x^j$, then $T=g^{ik}T_{ij}\frac{\partial}{\partial x^k}\otimes\mathrm{d}x^j$, thus $\mathrm{tr}_g T=g^{ij}T_{ij}$.
2. If $T=T^{ij}\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j}$, then $T=g_{kj}T^{ij}\frac{\partial}{\partial x^i}\otimes\mathrm{d}x^k$, thus $\mathrm{tr}_g T=g_{ij}T^{ij}$.

In general, if a tensor of type (r,s) with $r+s=2n$, we can change its type into (n,n) and take trace n times to obtain a number. Later we will see we obtain Ricci curvature by taking trace of curvature, and we obtain scalar curvature by taking trace of Ricci curvature.

Remark 2.6.1 (scalar Laplacian). For a smooth function $f\colon M\to\mathbb{R}$, $\nabla^2 f$ is a $(0,2)$-form, locally looks like

$$\nabla^2_{i,j} f \mathrm{d}x^i\otimes \mathrm{d}x^j$$

Then its trace looks like

$$\operatorname{tr}_g \nabla^2 f = g^{ij}\nabla^2_{i,j} f$$

That's called scalar Laplacian of f, denoted by Δf.

Remark 2.6.2. If g is induced metric on $(0,2)$-tensor, then for any $(0,2)$-tensor T, we have

$$\begin{aligned} g(g,T) &= g(g_{ij}\mathrm{d}x^i\otimes\mathrm{d}x^j, T_{kl}\mathrm{d}x^k\otimes\mathrm{d}x^l)\\ &= g_{ij}T_{kl}g^{ik}g^{jl}\\ &= \delta^k_j g^{jl}T_{kl}\\ &= g^{kl}T_{kl}\\ &= \operatorname{tr}_g T \end{aligned}$$

Proposition 2.6.1 (magic formula)**.** For a $(0,2)$-tensor T, we have

$$X(\operatorname{tr}_g T) = g(g,\nabla_X T)$$

Proof. From above remark we can see $\operatorname{tr}_g T = g(g,T)$, then ∇ is compatible with metric completes the proof. □

Remark 2.6.3 (local form). Locally we have

$$\nabla_i(g^{jk}T_{jk}) = g^{jk}(\nabla_i T_{jk})$$

that is, g^{jk} can "pass through" taking covariant derivative, which is called "magic formula".

3. Geodesic and normal coordinate

In this section we always assume (M, g) is a Riemannian manifold equipped with Levi-Civita connection ∇.

3.1. Geodesic.

Definition 3.1.1 (geodesic)**.** A smooth curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$ is called a geodesic, if for each local coordinate $\{x^i\}$, it satisfies

$$\frac{\mathrm{d}^2\gamma^k}{\mathrm{d}t} + \frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\Gamma^k_{ij} \circ \gamma = 0$$

where $\gamma^i = x^i \circ \gamma$.

Remark 3.1.1. In Section 10, we will give a definition of geodesic via pullback connection.

Theorem 3.1.1. For any $p \in M, v \in T_pM$, there exists $\varepsilon > 0$ and a geodesic $\gamma\colon (-\varepsilon, \varepsilon) \to M$ such that

$$\gamma(0) = p$$
$$\gamma'(0) = v$$

Moreover, any two such geodesics agree on their common domain.

Proof. Follows from standard result in ODEs' theory. □

Remark 3.1.2. Note that standard result in ODEs' theory only guarantees the short time existence of geodesic. If we use I to denote the maximal interval such that γ can be defined on it, then in general $I \subsetneq \mathbb{R}$.

Notation 3.1.1. For $v \in T_pM$, γ_v denotes the unique geodesic such that $\gamma(0) = p$ and $\gamma'(0) = v$.

Lemma 3.1.1. For each $p \in M, v \in T_pM$ and $c, t \in \mathbb{R}$, then

$$\gamma_{cv}(t) = \gamma_v(ct)$$

whenever either side is defined.

Proof. It's clear by uniqueness. □

Definition 3.1.2. For any $p \in M$, V_p is a subspace of T_pM defined by

$$V_p := \{v \in T_pM \mid \gamma_v(1) \text{ is defined}\}$$

Remark 3.1.3. From Lemma 3.1.1, $v \in V_p$ if $|v| < \varepsilon$ for sufficiently small $\varepsilon > 0$.

Definition 3.1.3 (exponential map)**.** For $p \in M$, the exponential map at point p is the map

$$\begin{aligned} \exp_p\colon V_p &\to M \\ v &\mapsto \gamma_v(1) \end{aligned}$$

Theorem 3.1.2. The exponential map $\exp_p$ maps a neighborhood $0 \in T_pM$ diffeomorphically onto a neighborhood of $p \in M$.

Proof. Note that

$$(\mathrm{d}\exp_p)_0\colon T_0(T_pM)\to T_pM$$

and we can identify $T_0(T_pM)$ with T_pM via $0+tv\mapsto v$, since T_pM is just a vector space. In this viewpoint $(\mathrm{d}\exp_p)_0$ then becomes a map from T_pM onto itself. To see what we need, it suffices to check $(\mathrm{d}\exp_p)_0$ is identity map. For all $v\in T_pM$,

$$\begin{aligned}(\mathrm{d}\exp_p)_0(v) &\overset{(1)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\exp_p(0+tv)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\gamma_{tv}(1)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\gamma_v(t)\\ &= \gamma_v'(0)\\ &= v\end{aligned}$$

where (1) holds from our identification $T_0(T_pM)\cong T_pM$ and definition of differential. □

Let $\{\frac{\partial}{\partial x^1}\big|_p,\dots,\frac{\partial}{\partial x^n}\big|_p\}$ be an orthonormal basis of T_pM with respect to Riemannian metric g. Consider the following linear isomorphism

$$\begin{aligned}\Phi\colon T_pM&\to\mathbb{R}^n\\ v^i\frac{\partial}{\partial x^i}\Big|_p&\mapsto(v^1,\dots,v^n)\end{aligned}$$

Then Theorem 3.1.2 implies there exists a neighborhood U of p which is mapped by $\Phi\circ\exp_p^{-1}$ diffeomorphically onto a neighborhood of $0\in\mathbb{R}^n$. Thus $(\Phi\circ\exp_p^{-1},U,p)$ gives a local coordinates of M with center p, which is called normal coordinate.

Theorem 3.1.3. In normal coordinate we have

$$\begin{aligned}g_{ij}(0)&=\delta_{ij}\\ \Gamma_{ij}^k(0)&=0\end{aligned}$$

Remark 3.1.4. Recall for a smooth function $f\colon T_pM\to\mathbb{R}$ and (φ,U,p) is a local coordinate centered at p, the expression of f in this local coordinate is $f\circ(\mathrm{d}\varphi^{-1})_p\colon T_pU\to\mathbb{R}$.

Proof. Let e_i denotes $(0, \dots, \underbrace{1}_{i-\mathrm{th}}, \dots, 0) \in \mathbb{R}^n$, then

$$\begin{aligned}
g_{ij}(0) &\overset{(1)}{=} \langle \mathrm{d}(\exp_p \circ \Phi^{-1})_0 e_i, \mathrm{d}(\exp_p \circ \Phi^{-1})_0 e_j \rangle_p \\
&\overset{(2)}{=} \langle (\mathrm{d}\exp_p)_0 \left.\frac{\partial}{\partial x^i}\right|_p, (\mathrm{d}\exp_p)_0 \left.\frac{\partial}{\partial x^j}\right|_p \rangle_p \\
&\overset{(3)}{=} \langle \left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p \rangle_p \\
&\overset{(4)}{=} \delta_{ij}
\end{aligned}$$

where

(1) holds from Remark 3.1.4.
(2) holds from Φ is a linear map, thus $\mathrm{d}\Phi^{-1} = \Phi^{-1}$.
(3) holds from Theorem 3.1.2.
(4) holds from our choice of $\{\frac{\partial}{\partial x^i}\big|_p\}$.

To see local expression of Christoffel symbol: For arbitrary $v = (v^1, \dots, v^n) \in \mathbb{R}^n$, consider geodesic $\gamma(t) = \exp_p(t\Phi^{-1}(v))$ with $\gamma(0) = p$ and $\gamma'(t) = \Phi^{-1}(v)$. In normal coordinate γ looks like $\gamma(t) = (tv^1, \dots, tv^n)$, thus geodesic equation simplifies to

$$\Gamma_{ij}^k(tv)v^i v^j = 0$$

Evaluating this expression at $t = 0$ shows $\Gamma_{ij}^k(0)v^i v^j = 0$ for arbitrary index k. Now take $v = \frac{1}{2}(e_i + e_j)$ to conclude $\Gamma_{ij}^k(0) = 0$ for all i, j, k. □

Corollary 3.1.1. In normal coordinate, the Taylor expansion of g_{ij} around zero is

$$g_{ij}(x) = \delta_{ij} + O(|x|^2)$$

Proof. Note that

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ki}^l(0)g_{lj}(0) + \Gamma_{kj}^l(0)g_{il}(0) = 0$$

□

3.2. **Arts of computation.** Tensor comutation is one of the hallmarks of Riemannian geometry, but sometimes there is a way to avoid some complicated computations if you don't want to do it. In this section we collect some useful tools which provide a neat to way to compute.

If we want to prove an identity of tensors, it suffices to check it pointwisely, since zero tensor is independent of the choice of coordinates. Thus the normal coordinate is a good tool to use, since by Theorem 3.1.3, one has

$$x^i(p) = 0$$
$$g_{ij}(p) = \delta_{ij}$$
$$\Gamma_{ij}^k(p) = 0$$

Example 3.2.1. For a (s, r)-tensor T, from Proposition 2.3.1 one has

$$\nabla^2_{k,i} T^{j_1 \dots j_s}_{i_1 \dots i_r} = \nabla_k \nabla_i T^{j_1 \dots j_s}_{i_1 \dots i_r}$$

under normal coordinate. In particular, Hessian of a smooth function f can be written as $\nabla_k \nabla_i f \mathrm{d}x^k \otimes \mathrm{d}x^i$ locally, which is relatively easier to compute.

Lemma 3.2.1. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Given an arbitrary local frame $\{\frac{\partial}{\partial x^i}\}$ of TM with dual basis $\{\mathrm{d}x^i\}$, then

$$\mathrm{d} = \mathrm{d}x^i \wedge \nabla_{\frac{\partial}{\partial x^i}}$$

Proof. Firstly note that exterior derivative is independent of the choice of coordinates, and direct computation also shows $\mathrm{d}x^i \wedge \nabla_{\frac{\partial}{\partial x^i}}$ is independent of the choice of coordinates. Now it suffices to check $\mathrm{d} = \mathrm{d}x^i \wedge \nabla_{\frac{\partial}{\partial x^i}}$ in normal coordinate, that is clear, since for arbitrary k-form ω, without lose of generality we may write it as $f \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^k$, then

$$\begin{aligned} \mathrm{d}x^i \wedge \nabla_{\frac{\partial}{\partial x^i}} \omega &= \mathrm{d}x^i \wedge \nabla_{\frac{\partial}{\partial x^i}} (f \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^k) \\ &= \mathrm{d}x^i \wedge \frac{\partial f}{\partial x^i} \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^k \\ &= \frac{\partial f}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^k \\ &= \mathrm{d}\omega \end{aligned}$$

□

3.3. **Hopf-Rinow's theorem.**

In this section we will figure out when does exponential map is defined on the whole T_pM.

Definition 3.3.1 (geodesically complete)**.** A Riemannian manifold M is geodesically complete if for all $p \in M$, the exponential map $\exp_p$ is defined on the whole T_pM.

At this stage it's convenient to introduce a distance function on a Riemannian manifold M which is not necessarily geodesic complete as follows: For $p, q \in M$, consider all the piecewise smooth curves joining p and q. Since M is connected, such curves always exist (cover a continuous curve joining p and q by a finite number of coordinates neighborhood and replace each piece contained in a coordinate neighborhood by a smooth one).

Definition 3.3.2 (distance)**.** Let (M, g) be a Riemannian manifold, $p, q \in M$, the distance between p and q is defined by the infimum of the lengths of all piecewise smooth curves joining p and q, denoted by $\mathrm{dist}(p, q)$.

Proposition 3.3.1. The topology induced by distance function on M coincides with the original topology on M.

Proof. See Proposition 2.6 in Page146 of [Car92]. □

Theorem 3.3.1 (Hopf-Rinow)**.** Let (M, g) be a Riemannian manifold and $p \in M$. The following statements are equivalent:

1. M is geodesically complete.
2. The closed and bounded sets of M are compact.
3. M is complete as a topological space.

In addition, any of statements above implies that for any $p, q \in M$, there exists a geodesic joining p and q with length $\mathrm{dist}(p, q)$.

Proof. See Theorem 2.8 in Page146 of [Car92]. □

Remark 3.3.1. Note that (2) is equivalent to (3) is a basic fact in general topology.

Definition 3.3.3 (complete)**.** A Riemannian manifold is called complete, if it's geodesically complete, or it's complete as a topological space.

Corollary 3.3.1. If M is compact, then it's complete.

Part 2. Curvature

4. Riemannian Curvature

4.1. Curvature form. Let (M, g) be a Riemannian manifold with connection ∇ of a vector bundle E over M. Now we're going to extend connection to something called exterior derivative[2] defined on sections of vector bundle valued k-forms as follows

$$\begin{aligned}\mathrm{d}^\nabla\colon C^\infty(M, \Omega^k_M\otimes E) &\to C^\infty(M, \Omega^{k+1}_M\otimes E)\\ \omega\otimes e &\mapsto \mathrm{d}\omega\otimes e + (-1)^k\omega\wedge\nabla e\end{aligned}$$

Suppose $\{e_\alpha\}$ is a local frame of E, then

$$\begin{aligned}(\mathrm{d}^\nabla)^2(s^\alpha e_\alpha) &= \mathrm{d}^\nabla(\mathrm{d}s^\alpha\otimes e_\alpha + s^\alpha\omega_\alpha^\beta\otimes e_\beta)\\ &= -\mathrm{d}s^\alpha\wedge\omega_\alpha^\beta\otimes e_\beta + \mathrm{d}(s^\alpha\omega_\alpha^\beta)\otimes e_\beta - s^\alpha\omega_\alpha^\beta\wedge\omega_\beta^\gamma\otimes e_\gamma\\ &= s^\alpha(\mathrm{d}\omega_\alpha^\beta - \omega_\alpha^\gamma\wedge\omega_\gamma^\beta)\otimes e_\beta\\ (\mathrm{d}^\nabla)^2(e_\alpha) &= \mathrm{d}^\nabla(\omega_\alpha^\beta\otimes e_\beta)\\ &= \mathrm{d}\omega_\alpha^\beta\otimes e_\beta - \omega_\alpha^\beta\wedge\nabla e_\beta\\ &= \mathrm{d}\omega_\alpha^\beta\otimes e_\beta - \omega_\alpha^\beta\wedge\omega_\beta^\gamma\otimes e_\gamma\\ &= (\mathrm{d}\omega_\alpha^\beta - \omega_\alpha^\gamma\wedge\omega_\gamma^\beta)\otimes e_\beta\end{aligned}$$

This shows smooth functions commutes with $(\mathrm{d}^\nabla)^2$. In fact it's a tensor property, which implies

1. $(\mathrm{d}^\nabla)^2(e_\alpha)$ completely determines $(\mathrm{d}^\nabla)^2$ locally, thus we can say $(\mathrm{d}^\nabla)^2$ locally looks like $\mathrm{d}\omega - \omega\wedge\omega$.
2. $(\mathrm{d}^\nabla)^2$ is a global section of $\Omega^2_M\otimes\operatorname{End}E$, that is it's compatible with change of basis. Indeed, for two local frame $e, \widetilde{e}$ such that $\widetilde{e} = ge$, we will see

$$\begin{aligned}g(\mathrm{d}^\nabla)^2 e &= (\mathrm{d}^\nabla)^2 ge\\ &= (\mathrm{d}^\nabla)^2\widetilde{e}\\ &= (\mathrm{d}\widetilde{\omega} - \widetilde{\omega}\wedge\widetilde{\omega})\widetilde{e}\\ &= (\mathrm{d}\widetilde{\omega} - \widetilde{\omega}\wedge\widetilde{\omega})ge\end{aligned}$$

   which implies

$$g^{-1}(\mathrm{d}\widetilde{\omega} - \widetilde{\omega}\wedge\widetilde{\omega})g = \mathrm{d}\omega - \omega\wedge\omega$$

In general case, for $s\in C^\infty(M, \Omega^k_M\otimes E)$, locally written as $s = s^\alpha e_\alpha$, where s^α is a k-form, direct computation still yields

$$(\mathrm{d}^\nabla)^2(s^\alpha e_\alpha) = s^\alpha\wedge(\mathrm{d}^\nabla)^2(e_\alpha)$$

[2] Just like exterior derivative learnt in calculus.

Definition 4.1.1 (curvature form)**.** Let E be a vector bundle over M equipped with connection ∇, there exists a section $\Theta \in C^\infty(M, \Omega^2_M \otimes \operatorname{End} E)$, called curvature form, such that

$$(\mathrm{d}^\nabla)^2 s = \Theta \wedge s$$

for all $s \in C^\infty(X, \Omega^k_M \otimes E)$.

Remark 4.1.1 (local form). If we write $\Theta = \Theta^\beta_\alpha e_\beta \otimes e^\alpha$ under local frames, where

$$\Theta^\beta_\alpha = \Omega^\beta_{ij\alpha} \mathrm{d}x^i \wedge \mathrm{d}x^j$$

Then $\Theta = \mathrm{d}\omega - \omega \wedge \omega$ shows

$$\begin{aligned}
\Theta^\beta_{ij\alpha} \mathrm{d}x^i \wedge \mathrm{d}x^j &= \Theta^\beta_\alpha \\
&= \mathrm{d}\omega^\beta_\alpha - \omega^\gamma_\alpha \wedge \omega^\beta_\gamma \\
&= \mathrm{d}(\Gamma^\beta_{i\alpha} \mathrm{d}x^i) - (\Gamma^\gamma_{i\alpha} \mathrm{d}x^i) \wedge (\Gamma^\beta_{j\gamma} \mathrm{d}x^j) \\
&= (-\partial_j \Gamma^\beta_{i\alpha} - \Gamma^\gamma_{i\alpha} \Gamma^\beta_{j\gamma}) \mathrm{d}x^i \wedge \mathrm{d}x^j
\end{aligned}$$

that is $\Theta^\beta_{ij\alpha} = -(\partial_j \Gamma^\beta_{i\alpha} + \Gamma^\gamma_{i\alpha} \Gamma^\beta_{j\gamma})$.

Remark 4.1.2*.* In physicists' language, a connection is a "field", the curvature is the "strength" of the field, and choosing a local frame is called "fixing the gauge". The reason for these names comes from H. Weyl's work, rewriting Maxwell's equations.

4.2. **Curvature tensor.** Let M be a smooth manifold, E a vector bundle over M equipped with connection ∇, in [Car92], the curvature of a connection ∇ is defined as follows:

$$\begin{aligned}
R \colon C^\infty(M, TM) \times C^\infty(M, TM) \times E &\to E \\
(X, Y, s) &\mapsto R(X, Y)s
\end{aligned}$$

where $R(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$.

Proposition 4.2.1. The curvature R has tensorial property.

Remark 4.2.1 (local form). Suppose $\{e_\alpha\}$ is a local frame of E, then

$$R = R^\beta_{ij\alpha} \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes e^\alpha \otimes e_\beta$$

To see $R^\beta_{ij\alpha}$, it suffices to compute

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} e_\alpha &= \nabla_{\frac{\partial}{\partial x^i}} (\Gamma^\beta_{j\alpha} e_\beta) \\
&= \partial_i \Gamma^\beta_{j\alpha} e_\beta + \Gamma^\beta_{j\alpha} \Gamma^\gamma_{i\beta} e_\gamma \\
&= (\partial_i \Gamma^\beta_{j\alpha} + \Gamma^\gamma_{j\alpha} \Gamma^\beta_{i\gamma}) e_\beta
\end{aligned}$$

Thus

$$R^\beta_{ij\alpha} e_\beta = (\partial_i \Gamma^\beta_{j\alpha} - \partial_j \Gamma^\beta_{i\alpha} + \Gamma^\gamma_{j\alpha} \Gamma^\beta_{i\gamma} - \Gamma^\gamma_{i\alpha} \Gamma^\beta_{j\gamma}) e_\beta$$

or in other words,

$$R_\alpha^\beta = (\partial_i\Gamma_{j\alpha}^\beta - \partial_j\Gamma_{i\alpha}^\beta + \Gamma_{j\alpha}^\gamma\Gamma_{i\gamma}^\beta - \Gamma_{i\alpha}^\gamma\Gamma_{j\gamma}^\beta)\mathrm{d}x^i\otimes\mathrm{d}x^j$$

Recall that our curvature form Ω is a section of $\Omega_M^2\otimes\operatorname{End}E$, and you can regard it as a section of $T^*M\otimes T^*M\otimes\operatorname{End}E$, that is

$$\begin{aligned}\Theta_\alpha^\beta &= (-\partial_j\Gamma_{i\alpha}^\beta - \Gamma_{i\alpha}^\gamma\Gamma_{j\gamma}^\beta)\mathrm{d}x^i\wedge\mathrm{d}x^j\\ &= (\partial_i\Gamma_{j\alpha}^\beta - \partial_j\Gamma_{i\alpha}^\beta + \Gamma_{j\alpha}^\gamma\Gamma_{i\gamma}^\beta - \Gamma_{i\alpha}^\gamma\Gamma_{j\gamma}^\beta)\mathrm{d}x^i\otimes\mathrm{d}x^j\end{aligned}$$

So if you regard curvature form as a tensor, then it's exactly curvature tensor we defined here.

If we take E to be tangent bundle, then curvature tensor R is a $(1,3)$-tensor, locally looks like

$$R_{ijk}^r\mathrm{d}x^i\otimes\mathrm{d}x^j\otimes\mathrm{d}x^k\otimes\frac{\partial}{\partial x^r}$$

However, we always use its $(0,4)$ type, that is

$$R_{ijkl} = g_{rl}R_{ijk}^r$$

Now let's give a more explicit expression about R_{ijkl}. Direct computation shows

$$\begin{aligned}R_{ijkl} &= R(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l})\\ &= \langle\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\rangle\\ &= \partial_i\langle\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\rangle - \langle\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k},\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^l}\rangle - (\partial_j\langle\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\rangle - \langle\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k},\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^l}\rangle)\\ &= \underbrace{\partial_i\langle\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\rangle - \partial_j\langle\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\rangle}_{\text{part I}} + \underbrace{\langle\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k},\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^l}\rangle - \langle\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k},\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^l}\rangle}_{\text{part II}}\end{aligned}$$

For part II, we have

$$g_{rs}(\Gamma_{ik}^r\Gamma_{jl}^s - \Gamma_{jk}^r\Gamma_{il}^s)$$

For part I, note that

$$\begin{aligned}\partial_i(\Gamma_{jk}^r g_{rl}) &= \partial_i(\frac{1}{2}g^{rs}(\partial_j g_{ks} + \partial_k g_{js} - \partial_s g_{jk})g_{rl})\\ &= \partial_i(\frac{1}{2}\delta_l^s(\partial_j g_{ks} + \partial_k g_{js} - \partial_s g_{jk}))\\ &= \frac{1}{2}\partial_i(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk})\end{aligned}$$

Thus we have part I is

$$\partial_i(\Gamma_{jk}^r g_{rl}) - \partial_j(\Gamma_{ik}^r g_{rl}) = \frac{1}{2}(\partial_i\partial_k g_{jl} + \partial_j\partial_l g_{ik} - \partial_i\partial_l g_{jk} - \partial_j\partial_k g_{il})$$

So we have an explicit expression for R_{ijkl}

$$R_{ijkl} = \frac{1}{2}(\partial_i\partial_k g_{jl} + \partial_j\partial_l g_{ik} - \partial_i\partial_l g_{jk} - \partial_j\partial_k g_{il}) + g_{rs}(\Gamma^r_{ik}\Gamma^s_{jl} - \Gamma^r_{jk}\Gamma^s_{il})$$

From this expression, we can see in general curvature depends on second order partial derivatives of metric. Furthermore, there are some (skew) symmetries of R_{ijkl} listed as follows:

1. $R_{ijkl} = -R_{jikl}$.
2. $R_{ijkl} = -R_{ijlk}$.
3. $R_{ijkl} = R_{klij}$.

4.3. **Curvature of induced connections.**

4.3.1. *Curvature form of induced connection on dual bundle.* Let E be a vector bundle over a smooth manifold M, equipped with connection ∇ with curvature Θ, and Θ^* is curvature form of induced connection on E^*. Suppose $\{e_\alpha\}$ is a local frame of E with dual frame $\{e^\alpha\}$, then

$$\begin{aligned}(\Theta^*)^\alpha_\beta &= \mathrm{d}(\omega^*)^\alpha_\beta - (\omega^*)^\beta_\gamma \wedge (\omega^*)^\gamma_\alpha \\ &= -\mathrm{d}\omega^\alpha_\beta - \omega^\beta_\gamma \wedge \omega^\gamma_\alpha \\ &= -\mathrm{d}\omega^\alpha_\beta + \omega^\gamma_\alpha \wedge \omega^\beta_\gamma \\ &= -\Theta^\alpha_\beta\end{aligned}$$

Thus

$$\Theta^* = (\Theta^*)^\alpha_\beta e_\alpha \otimes e^\beta = -\Theta^\alpha_\beta e_\alpha \otimes e^\beta$$

that is $\Theta^* = -\Theta^T$.

4.3.2. *Curvature form of induced connection on tensor product.* Let E be a vector bundles over a smooth manifold M, equipped with connection ∇ with curvature Θ. Suppose $\{e_\alpha\}$ is a local frame of E, here we define

$$\begin{aligned}\otimes\colon C^\infty(M, \Omega^p_M \otimes E) \times C^\infty(M, \Omega^q_M \otimes E) &\to C^\infty(M, \Omega^{p+q}_M \otimes E \otimes E) \\ (s^\alpha e_\alpha, t^\beta e_\beta) &\mapsto s^\alpha \wedge t^\beta e_\alpha \otimes e_\beta\end{aligned}$$

Then for $s \in C^\infty(M, \Omega^p_M \otimes E), t \in C^\infty(M, \Omega^q_M \otimes E)$, direct computation shows

$$\begin{aligned}\nabla^{E\otimes E}(s \otimes t) =& \nabla^{E\otimes E}(s^\alpha \wedge t^\beta e_\alpha \otimes e_\beta) \\ =& \mathrm{d}(s^\alpha \wedge t^\beta) e_\alpha \otimes e_\beta + (-1)^{p+q} s^\alpha \wedge t^\beta (\nabla e_\alpha \otimes e_\beta + e_\alpha \otimes \nabla e_\beta) \\ =& (\mathrm{d}s^\alpha \wedge t^\beta + (-1)^p s^\alpha \wedge \mathrm{d}t^\beta) e_\alpha \otimes e_\beta + (-1)^{p+q} s^\alpha \wedge t^\beta (\nabla e_\alpha \otimes e_\beta + e_\alpha \otimes \nabla e_\beta) \\ =& (\mathrm{d}s^\alpha \wedge t^\beta e_\alpha \otimes e_\beta + (-1)^{p+q} s^\alpha \wedge t^\beta \nabla e_\alpha \otimes e_\beta) \\ &+ (-1)^p s^\alpha \wedge \mathrm{d}t^\beta e_\alpha \otimes e_\beta + (-1)^{p+q} s^\alpha \wedge t^\beta e_\alpha \otimes \nabla e_\beta \\ =& (\mathrm{d}s^\alpha \wedge t^\beta e_\alpha \otimes e_\beta + (-1)^p s^\alpha \wedge \nabla e_\alpha \otimes t) + (-1)^p \{s \otimes \mathrm{d}t^\beta e_\beta + (-1)^q s \otimes (t^\beta \wedge \nabla e_\beta)\} \\ =& \nabla s \otimes t + (-1)^p s \otimes \nabla t\end{aligned}$$

Then curvature of $\nabla^{E\otimes E}$ can be computed as

$$\begin{aligned}(\mathrm{d}^{\nabla^{E\otimes E}})^2(s\otimes t) &= \mathrm{d}^{\nabla^{E\otimes E}}(\nabla s\otimes t+s\otimes\nabla t)\\ &\overset{(1)}{=} \Theta s\otimes t-\nabla s\otimes\nabla t+\nabla s\otimes\nabla t+s\otimes\Theta t\\ &= \Theta s\otimes t+s\otimes\Theta t\end{aligned}$$

where (1) holds from above computation. This shows

$$\Theta^{E\otimes E}=\Theta\otimes\mathrm{id}+\mathrm{id}\otimes\Theta$$

In general case, by induction one can show

$$\Theta^{\otimes^k E}=\sum_{i=1}^k \mathrm{id}\otimes\cdots\otimes\underbrace{\Theta}_{i\text{-th}}\otimes\cdots\otimes\mathrm{id}$$

4.3.3. *Curvature form of induced connection on wedge product.* Let E be a vector bundles over a smooth manifold M, equipped with connection ∇ with curvature Θ. Suppose $\{e_\alpha\}$ is a local frame of E, here we define

$$\begin{aligned}\wedge\colon C^\infty(M,\Omega_M^p\otimes E)\times C^\infty(M,\Omega_M^q\otimes E)&\to C^\infty(M,\Omega_M^{p+q}\otimes\wedge^2 E)\\ (s^\alpha e_\alpha,t^\beta e_\beta)&\mapsto s^\alpha\wedge t^\beta e_\alpha\wedge e_\beta\end{aligned}$$

By definition, for $s\in C^\infty(M,\Omega_M^p\otimes E), t\in C^\infty(M,\Omega_M^q\otimes E)$ one has

$$\begin{aligned}s\wedge t&=s^\alpha\wedge t^\beta e_\alpha\wedge e_\beta\\ &=s^\alpha\wedge t^\beta(e_\alpha\otimes e_\beta-e_\beta\otimes e_\alpha)\\ &=s\otimes t-(-1)^{pq}t\otimes s\end{aligned}$$

Thus

$$\begin{aligned}\nabla^{\wedge^2 E}(s\wedge t):=&\nabla^{E\otimes E}(s\otimes t-(-1)^{pq}t\otimes s)\\ =&\nabla s\otimes t+(-1)^p s\otimes\nabla t-(-1)^{pq}\{\nabla t\otimes s+(-1)^q t\otimes\nabla s\}\\ =&\nabla s\otimes t-(-1)^{(p+1)q}s\otimes t+(-1)^p\{s\otimes\nabla t-(-1)^{p(q+1)}\nabla t\otimes s\}\\ =&\nabla s\wedge t+(-1)^p s\wedge\nabla t\end{aligned}$$

Remark 4.3.1. In general case, by induction one can show

$$\nabla^{\wedge^k E}(s_1\wedge\cdots\wedge s_k)=\sum_{i=1}^k(-1)^{p_1+\cdots+p_{i-1}}s_1\wedge\cdots\wedge\nabla s_i\wedge\cdots\wedge s_k$$

where $s_i\in C^\infty(M,\Omega_M^{p_i}\otimes E)$. Now let's show curvature of $\nabla^{\wedge^k E}$

4.3.4. *Curvature form of induced connection on determinant.* Let E be a vector bundle of rank r over a smooth manifold M, equipped with connection ∇ with curvature Θ, there is an induced connection on canonical line bundle $\det E$, since $\det E=\bigwedge^r E$. In this case $\Theta^{\det E}\in C^\infty(X,\Omega_M^2)$, now we're going to show $\Theta^{\det E} = \mathrm{tr}_E \Theta$. By tensorial of curvature form, it suffices to check on local frame $\{e_\alpha\}$, and

$$
\begin{aligned}
\Theta^{\det E}(e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_r}) &= \sum_{i=1}^{n} e_{\alpha_1} \wedge \cdots \wedge \Theta^E e_{\alpha_i} \wedge \cdots \wedge e_{\alpha_r} \\
&= \sum_{i=1}^{n} e_{\alpha_1} \wedge \cdots \wedge (R_{ij\alpha_i}^{\beta} \mathrm{d}x^i \otimes e_\beta) \wedge \cdots \wedge e_{\alpha_r} \\
&= R_{ij\alpha}^{\alpha} \mathrm{d}x^i \otimes e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_r} \\
&= \mathrm{tr}_E \Theta \otimes e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_r}
\end{aligned}
$$

This yields the desired result.

4.4. **Taylor expansion of metric.**

Proposition 4.4.1. In normal coordinate, the Taylor expansions of g_{ij} is

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{iklj}(0) x^k x^l + O(|x|^3)$$

Proof. The compatibility of metric and connection implies

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ki}^m g_{mj} + \Gamma_{kj}^m g_{mi}$$

Take differential with respect to x^l, evaluate at $x = 0$ and use the fact that Christoffel symbol vanishes at $x = 0$, one has

$$\frac{\partial^2 g_{ij}}{\partial x^l \partial x^k}(0) = \frac{\partial \Gamma_{ki}^m}{\partial x^l}(0) g_{mj}(0) + \frac{\partial \Gamma_{kj}^m}{\partial x^l}(0) g_{mi}(0)$$

Now we claim

$$\frac{\partial \Gamma_{ij}^k}{\partial x^l}(0) + \frac{\partial \Gamma_{li}^k}{\partial x^j}(0) + \frac{\partial \Gamma_{jl}^k}{\partial x^i}(0) = 0$$

Indeed, in normal coordinate we have

$$0 = \Gamma_{ij}^k(tx) x^i x^j$$

Take derivative with respect to x^l and evaluate at $t = 0$, we have

$$0 = \frac{\partial \Gamma_{ij}^k}{\partial x^l}(0) x^i x^j x^l$$

which implies

$$\sum_{\sigma \in S_3} \frac{\partial \Gamma_{\sigma(i)\sigma(j)}^k}{\partial x^{\sigma(l)}}(0) = 0$$

Then the claim holds from the symmetry of Christoffel symbol in term i, j. From $R^l_{ijk}(0) = \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j}$ we have

$$\begin{aligned} R_{ijkl}(0) &= (\frac{\partial \Gamma^m_{jk}}{\partial x^i}(0) - \frac{\partial \Gamma^m_{ik}}{\partial x^j}(0))g_{ml}(0) \\ &= -(\frac{\partial \Gamma^m_{ij}}{\partial x^k}(0) + \frac{\partial \Gamma^m_{ki}}{\partial x^j}(0) + \frac{\partial \Gamma^m_{ik}}{\partial x^j}(0))g_{ml}(0) \\ &= -(\frac{\partial \Gamma^m_{ij}}{\partial x^k}(0) + 2\frac{\partial \Gamma^m_{ki}}{\partial x^j}(0))g_{ml}(0) \end{aligned}$$

Thus we have

$$\begin{aligned} 2R_{ikjl}(0)x^k x^l =& - (R_{iklj}(0) + R_{jlki}(0))x^k x^l \\ =&(\frac{\partial \Gamma^m_{ik}}{\partial x^l}(0) + 2\frac{\partial \Gamma^m_{il}}{\partial x^k}(0))g_{mj}(0)x^k x^l \\ &+ (\frac{\partial \Gamma^m_{jl}}{\partial x^k}(0) + 2\frac{\partial \Gamma^m_{jk}}{\partial x^l}(0))g_{mi}(0)x^k x^l \\ =&3\frac{\partial g_{ij}}{\partial x^k \partial x^l}(0)x^k x^l \end{aligned}$$

Thus we get for the second term in the Taylor expansion

$$\begin{aligned} \frac{1}{2}\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}(0)x^k x^l &= \frac{1}{3}R_{ikjl}(0)x^k x^l \\ &= -\frac{1}{3}R_{iklj}(0)x^k x^l \end{aligned}$$

that is

$$g_{ij} = \delta_{ij} - \frac{1}{3}R_{iklj}(0)x^k x^l + O(|x|^3)$$

□

Corollary 4.4.1. In normal coordinate we have
1. $g^{ij} = \delta_{ij} + \frac{1}{3}R_{iklj}(0)x^k x^l + O(|x|^3)$
2. $\det(g_{ij}) = 1 - \frac{1}{3}R_{kl}(0)x^k x^l + O(|x|^3)$
3. $\sqrt{\det(g_{ij})} = 1 - \frac{1}{6}R_{kl}(0)x^k x^l + O(|x|^3)$

Proof. For (1). Note that g^{ij} gives a Riemannian metric on T^*M, and Levi-Civita connection ∇ on T^*M with respect to g^{ij} is exactly the induced connection from the one on TM. Note that

$$\nabla \mathrm{d}x^k = -\Gamma^k_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$$

where Γ^k_{ij} is the Christoffel symbol for Levi-Civita connection on TM, we have curvature form in this case differs a sign since

$$R^l_{ijk}(0) = \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j}$$

Thus all computations are same as proof above, but result differs a sign in curvature.

For (2). By Jacobi's formula, we have

$$\frac{\partial \det(g_{ij})}{\partial x^k} = \det(g_{ij}) g^{ij} \frac{\partial g_{ij}}{\partial x^k}$$

Thus $\frac{\partial \det(g_{ij})}{\partial x^k}(0) = 0$, since first-order partial derivatives of g_{ij} vanishes. Furthermore, since first-order partial derivatives of g^{ij} also vanishes, we have

$$\begin{aligned}
\frac{1}{2}\frac{\partial^2 \det(g_{ij})}{\partial x^l \partial x^k} &= \det(g_{ij}) g^{ij} \frac{1}{2} \frac{\partial^2 g_{ij}}{\partial x^l \partial x^k} \\
&= \det(g_{ij}) g^{ij} (-\frac{1}{3} R_{iklj} x^k x^l) \\
&= -\frac{1}{3}\det(g_{ij}) R_{kl} x^k x^l
\end{aligned}$$

which implies

$$\det(g_{ij}) = 1 - \frac{1}{3} R_{kl}(0) x^k x^l + O(|x|^3)$$

For (3). It follows from (2) directly. □

4.5. **Ricci identity for tensor.**

Theorem 4.5.1 (Ricci identity)**.** Let (M,g) be a Riemannian manifold and T a (s,r)-tensor, locally written as $T^{j_1\cdots j_s}_{i_1\dots i_r}\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_s}}\otimes \mathrm{d}x^{i_1}\otimes\cdots\otimes \mathrm{d}x^{i_r}$. Then

$$\nabla^2_{k,i} T^{j_1\cdots j_s}_{i_1\dots i_r} - \nabla^2_{i,k} T^{j_1\cdots j_s}_{i_1\dots i_r} = \sum_{l=1}^{s} R^{j_l}_{kiq} T^{j_1\dots j_{l-1} q j_{l+1}\cdots j_s}_{i_1\dots i_r} - \sum_{m=1}^{r} R^q_{kii_m} T^{j_1\cdots j_s}_{i_1\dots i_{m-1} q i_{m+1}\dots i_r}$$

Proof. Without lose of generality, we may choose normal coordinate, by Proposition 2.3.1 and Remark 2.3.1, one has

$$\begin{aligned}
\nabla^2_{k,i} T^{j_1\cdots j_s}_{i_1\dots i_r} &= \nabla_k \nabla_i T^{j_1\cdots j_s}_{i_1\dots i_r} \\
&= \nabla_k \Big(\frac{\partial T^{j_1\cdots j_s}_{i_1\dots i_r}}{\partial x^i} + \sum_{l=1}^{s} \Gamma^{j_l}_{iq} T^{j_1\dots j_{l-1} q j_{l+1}\cdots j_s}_{i_1\dots i_r} - \sum_{m=1}^{r} \Gamma^q_{ii_m} T^{j_1\cdots j_s}_{i_1\dots i_{m-1} q i_{m+1}\dots i_r}\Big) \\
&= \frac{\partial^2 T^{j_1\cdots j_s}_{i_1\dots i_r}}{\partial x^k \partial x^i} + \sum_{l=1}^{s} \frac{\partial \Gamma^{j_l}_{iq}}{\partial x^k} T^{j_1\dots j_{l-1} q j_{l+1}\cdots j_s}_{i_1\dots i_r} - \sum_{m=1}^{r} \frac{\partial \Gamma^q_{ii_m}}{\partial x^k} T^{j_1\cdots j_s}_{i_1\dots i_{m-1} q i_{m+1}\dots i_r}
\end{aligned}$$

This completes the proof, since in normal coordinate one has

$$R^l_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j}$$

□

5. Bianchi identities

There are two famous Bianchi identities in Riemannian geometry, in [Car92] they are stated as follows

1. First Bianchi identity: $R(X,Y,Z,W)+R(Y,Z,X,W)+R(Z,X,Y,W) = 0$.
2. Second Bianchi identity: $\nabla_X R(Y,Z,W,R)+\nabla_Y R(Z,X,W,R)+\nabla_Z R(X,Y,W,R) = 0$.

5.1. First Bianchi.

Locally we have first Bianchi identity as

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0$$

In order to compute we use $(1,3)$ type as follows

$$R^r_{ijk} + R^r_{jki} + R^r_{kij} = 0$$

since we have

$$R^r_{ijk} = \underbrace{\partial_i \Gamma^r_{jk} - \partial_j \Gamma^r_{ik}}_{\text{part I}} + \underbrace{\Gamma^s_{jk}\Gamma^r_{is} - \Gamma^s_{ik}\Gamma^r_{js}}_{\text{part II}}$$

1. For the first part, if we permuting i,j,k, we have
$$\partial_i\Gamma^r_{jk} - \partial_j\Gamma^r_{ik} + \partial_j\Gamma^r_{ki} - \partial_k\Gamma^r_{ji} + \partial_k\Gamma^r_{ij} - \partial_i\Gamma^r_{kj} = 0$$
since $\Gamma^r_{ij} = \Gamma^r_{ji}$ by torsion-free.
2. For the second part, if we permuting i,j,k, we have
$$\Gamma^s_{jk}\Gamma^r_{is} - \Gamma^s_{ik}\Gamma^r_{js} + \Gamma^s_{ki}\Gamma^r_{js} - \Gamma^s_{ji}\Gamma^r_{ks} + \Gamma^s_{ij}\Gamma^r_{ks} - \Gamma^s_{kj}\Gamma^r_{is} = 0$$
by the same reason.

Thus we obtain first Bianchi identity, which is just a consequence of torsion-free.

Remark 5.1.1. If we consider connection on arbitrary vector bundle E, there is no first Bianchi identity, since e_α is just a section of E, not a section of TM, so $R(e_\alpha, \text{-})$ or $R(\text{-}, e_\alpha)$ is nonsense.

5.2. Second Bianchi.

In fact, we can write second Bianchi identity for arbitrary vector bundle E as follows

$$\nabla_X R(Y,Z,s,t) + \nabla_Y R(Z,X,s,t) + \nabla_Z R(X,Y,s,t) = 0$$

where $s,t \in C^\infty(M,E)$ and $X,Y,Z \in C^\infty(M,TM)$. That is to say

$$\nabla_i R_{jk\alpha\beta} + \nabla_j R_{ki\alpha\beta} + \nabla_k R_{ij\alpha\beta} = 0$$

holds for arbitrary indices.

5.2.1. *The fisrt approach.* To prove it, for convenience we consider normal coordinate. Then

$$\nabla_{\frac{\partial}{\partial x^i}} g(\nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^l}-\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^l},\frac{\partial}{\partial x^m}) = g(\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^l}-\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^l},\frac{\partial}{\partial x^m})$$

By permuting i,j,k we have

$$\begin{aligned}
&\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^l}-\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^l}\\
&+\nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^l}-\nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^l}\\
&+\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^l}-\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^l}\\
&=R(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x^j})\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^l}+R(\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k})\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x^l}+R(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x_i})\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^l}\\
&=0
\end{aligned}$$

This completes the computation of second Bianchi identity.

5.2.2. *The second approach.* From another aproach, recall that our curvature form Θ is a section of $\Omega^2_M\otimes\operatorname{End}E$, which can be written as $\Theta^\alpha_\beta e_\alpha\otimes e^\beta$ locally. According to Section 2.2.4, $\nabla\Theta$ can be written as

$$\nabla\Theta=\mathrm{d}\Theta+\Theta\wedge\omega-\omega\wedge\Theta$$

However, $\nabla\Theta=0$, since

$$\begin{aligned}
\nabla\Theta&=\mathrm{d}\Theta+\Theta\wedge\omega-\omega\wedge\Theta\\
&=\mathrm{d}(\mathrm{d}\omega-\omega\wedge\omega)+(\mathrm{d}\omega-\omega\wedge\omega)\wedge\omega-\omega\wedge(\mathrm{d}\omega-\omega\wedge\omega)\\
&=\mathrm{d}^2\omega-\mathrm{d}\omega\wedge\omega+\omega\wedge\mathrm{d}\omega+\mathrm{d}\omega\wedge\omega-\omega\wedge\omega\wedge\omega-\omega\wedge\mathrm{d}\omega+\omega\wedge\omega\wedge\omega\\
&=0
\end{aligned}$$

If we back to local form, we have

$$\mathrm{d}\Theta^\beta_\alpha+\Theta^\gamma_\alpha\wedge\omega^\beta_\gamma-\omega^\gamma_\alpha\wedge\Theta^\beta_\gamma=0$$

More explicitly, if we write $\Theta^\beta_\alpha=\Omega^\beta_{ij\alpha}\mathrm{d}x^i\wedge\mathrm{d}x^j$, we obtain

$$(\partial_k\Theta^\beta_{ij\alpha}+\Theta^\gamma_{ij\alpha}\Gamma^\beta_{k\gamma}-\Gamma^\gamma_{k\alpha}\Theta^\beta_{ij\gamma})\mathrm{d}x^k\wedge\mathrm{d}x^i\wedge\mathrm{d}x^j=0$$

In other words

$$\begin{aligned}
&\partial_k\Theta^\beta_{ij\alpha}+\Theta^\gamma_{ij\alpha}\Gamma^\beta_{k\gamma}-\Gamma^\gamma_{k\alpha}\Theta^\beta_{ij\gamma}\\
&+\partial_i\Theta^\beta_{jk\alpha}+\Theta^\gamma_{jk\alpha}\Gamma^\beta_{i\gamma}-\Gamma^\gamma_{i\alpha}\Theta^\beta_{jk\gamma}\\
&+\partial_j\Theta^\beta_{ki\alpha}+\Theta^\gamma_{ki\alpha}\Gamma^\beta_{j\gamma}-\Gamma^\gamma_{j\alpha}\Theta^\beta_{ki\gamma}=0
\end{aligned}$$

Note that $2\Theta^\beta_{ij\alpha}=R^\beta_{ij\alpha}$, and

$$\nabla_kR^\beta_{ij\alpha}=\partial_kR^\beta_{ij\alpha}+\Gamma^\beta_{k\gamma}R^\gamma_{ij\alpha}-\Gamma^\gamma_{k\alpha}R^\beta_{ij\gamma}$$

So $\nabla\Theta=0$ locally looks like

$$\nabla_kR^\beta_{ij\alpha}+\nabla_iR^\beta_{jk\alpha}+\nabla_jR^\beta_{ki\alpha}=0$$

This shows two Bianchi identities are same.

6. Other curvatures

6.1. **Sectional curvature.** Closely related to Riemannian curvature is sectional curvature that we're going to define, which is used to characterize a two dimensional subspace of tangent space.

Fix $p \in M$ and let x, y are two linearly independent tangent vectors in T_pM, then sectional curvature for these two vectors are defined as

$$K_p(x, y) = \frac{R(x, y, y, x)}{g(x, x)g(y, y) - g(x, y)^2}$$

In order to show it's a invariant defined for a two dimensional subspace, we need to check if $\mathrm{span}_{\mathbb{R}}\{x, y\} = \mathrm{span}_{\mathbb{R}}\{z, w\}$, then

$$K_p(x, y) = K_p(z, w)$$

Indeed, if we write

$$\begin{cases} z = ax + by \\ w = cx + dy \end{cases}$$

Then by symmetry and skew symmetry properties of R we have

$$\begin{aligned} R(z, w, w, z) &= R(ax + by, cx + dy, cx + dy, ax + by) \\ &= R(ax, dy, dy, ax) + R(ax, dy, cx, by) + R(by, cx, dy, ax) + R(by, cx, cx, by) \\ &= a^2d^2R(x, y, y, x) - abcdR(x, y, y, x) - abcdR(x, y, y, x) + b^2c^2R(x, y, y, x) \\ &= (ad - bc)^2R(x, y, y, x) \end{aligned}$$

And by the same computations we have

$$g(z, z)g(w, w) - g(z, w)^2 = (ad - bc)^2\{g(x, x)g(y, y) - g(x, y)^2\}$$

Thus

$$K_p(x, y) = K_p(z, w)$$

So the following definition is well-defined:

Definition 6.1.1 (sectional curvature)**.** The sectional curvature $K_p(\sigma)$ for two dimensional subspace $\sigma \subseteq T_pM$ is defined as

$$K_p(\sigma) := K_p(x, y)$$

where $\{x, y\}$ is a basis of σ.

Definition 6.1.2 (isotropic)**.** A Riemannian manifold (M, g) is called isotropic, if for each point $p \in M$, the sectional curvature $K_p(\sigma)$ is independent of σ.

Definition 6.1.3 (constant sectional curvature)**.** A Riemannian manifold (M, g) has constant sectional curvature, if $K_p(\sigma)$ is constant for arbitrary $\sigma \subset T_pM, p \in M$.

Remark 6.1.1. By definition, we can see if a Riemannian manifold has constant sectional curvature, then it must be isotropic. Conversely, if the dimension of a Riemannian manifold ≥ 3, then isotropic is equivalent to constant sectional curvature, see Corollary 7.1.1.

Lemma 6.1.1.

$$-6R(X,Y,Z,W) = \frac{\partial^2}{\partial s \partial t}\Big|_{s=t=0}\{R(X+sZ,Y+tW,Y+tW,X+sZ) - R(X+sW,Y+tZ,Y+tZ,X+sW)\}$$

where X,Y,Z,W are vector fields.

Proof. It suffices to compute coefficients of st of $R(X+sZ,Y+tW,Y+tW,X+sZ)$ and exchange Z with W to obtain coefficients of st of $R(X+sW,Y+tZ,Y+tZ,X+sW)$.

It's easy to see coefficients of st of $R(X+sZ,Y+tW,Y+tW,X+sZ)$ is

$$R(Z,W,Y,X)+R(Z,Y,W,X)+R(X,W,Y,Z)+R(X,Y,W,Z)$$

So coefficients of st of $R(X+sZ,Y+tW,Y+tW,X+sZ)$ is

$$R(W,Z,Y,X)+R(W,Y,Z,X)+R(X,Z,Y,W)+R(X,Y,Z,W)$$

Thus the right hand of our desired identity is

$$-4R(X,Y,Z,W)-(R(Y,Z,W,X)+R(W,Y,Z,X))-(R(W,X,Y,Z)+R(W,Y,Z,X))$$

By first Bianchi identity we have

$$\begin{aligned} R(Y,Z,W,X)+R(W,Y,Z,X) &= R(Y,Z,W,X)+R(Z,X,W,Y) \\ &= R(X,Y,Z,W) \\ R(W,X,Y,Z)+R(W,Y,Z,X) &= R(Y,Z,W,X)+R(Z,X,W,Y) \\ &= R(X,Y,Z,W) \end{aligned}$$

This completes the proof. □

Notation 6.1.1. For convenience, we use $R_0(X,Y,Z,W)$ to denote

$$R_0(X,Y,Z,W) = g(X,W)g(Y,Z) - g(X,Z)g(Y,W)$$

where X,Y,Z,W are vector fields. Then we can write sectional curvature as

$$K_p(\sigma) = \frac{R(x,y,y,x)}{R_0(x,y,y,x)}$$

where $\sigma \subset T_pM$ is spanned by x,y.

Proposition 6.1.1. A Riemannian manifold has constant sectional curvature K_p at point $p \in M$ if and only if $R = K_pR_0$, where K_p is a constant(may depend on p), R is curvature tensor.

Proof. If $R = K_pR_0$, then for a arbitrary x,y, we have

$$K_p(x,y) = \frac{R(x,y,y,x)}{R_0(x,y,y,x)} = K_p$$

Conversely, if $K(\sigma)$ is constant at point $p \in M$, that is for arbitrary x,y we have

$$\frac{R(x,y,y,x)}{R_0(x,y,y,x)} = K_p$$

If we denote

$$F(s,t) = R(x+sz, y+tw, y+tw, x+sz) - R(x+sw, y+tz, y+tz, x+sw)$$

$$F_0(s,t) = R_0(x+sz, y+tw, y+tw, x+sz) - R_0(x+sw, y+tz, y+tz, x+sw)$$

we still have $F(s,t) = K_p F_0(s,t)$. By Lemma 6.1.1, we have

$$R(x,y,z,w) = -\frac{1}{6}\left.\frac{\partial^2}{\partial s \partial t}\right|_{s=t=0} F(s,t)$$

and it's easy to see

$$R_0(x,y,z,w) = -\frac{1}{6}\left.\frac{\partial^2}{\partial s \partial t}\right|_{s=t=0} F_0(s,t)$$

This completes the proof. □

Corollary 6.1.1. A Riemannian manifold is isotropic if and only if $R = KR_0$, where K is a smooth function.

Corollary 6.1.2. A Riemannian manifold has constant sectional curvature K if and only if $R = KR_0$, where K is a constant.

Remark 6.1.2. An important corollary is that curvature tensor of Riemannian manifold with constant sectional curvature K is quite simple, since

$$R_{ijkl} = K(g_{il}g_{jk} - g_{ik}g_{jl})$$

that is, curvature is completely determined by metric, not second order in general.

Remark 6.1.3. Suppose the dimension of Riemannian manifold (M,g) is 2, and $\{e_1, e_2\}$ is a basis of T_pM. Then

$$K_p = K_p(e_1, e_2) = \frac{R(e_1, e_2, e_2, e_1)}{|e_1|^2|e_2|^2 - |g(e_1,e_2)|^2}$$

is exactly Gauss curvature we learnt in theory of surface.

6.2. **Ricci curvature and scalar curvature.**

Definition 6.2.1 (Ricci curvature)**.** For a Riemannian manifold (M,g), the Ricci curvature is a $(0,2)$-tensor, which is defined as

$$\mathrm{Ric}(X,Y) := \mathrm{tr}_g(Z \mapsto R(Z,X)Y)$$

where X, Y are vector fields.

Remark 6.2.1 (local form). The trace of above endomorphism is exactly R^i_{ijk}, and it can be written as

$$g^{il}R_{ijkl}$$

In other words, Ricci curvature tensor is the contracted tensor of curvature with respect to the first and fourth index.

Definition 6.2.2 (Ricci curvature in one direction)**.** For a point $p \in M$, and $x \in T_pM$, Ricci curvature in the direction x is defined as

$$\mathrm{Ric}_p(x) := \mathrm{Ric}(x,x)$$

Remark 6.2.2. For $x \in T_pM$, if we write it as $x = x^i e_i$ with respect to basis $\{e_i\}$ of T_pM, then

$$\operatorname{Ric}_p(x) = R_{jk} x^j x^k$$

Definition 6.2.3 (scalar curvature)**.** For a Riemannian manifold (M, g), the scalar curvature S at $p \in M$ is defined as $\operatorname{tr}_g \operatorname{Ric}$.

Remark 6.2.3 (local form). Locally we have

$$S = g^{jk} R_{jk}$$

Proposition 6.2.1 (contracted Bianchi identity)**.**

$$g^{jk} \nabla_k R_{ij} = \frac{1}{2} \nabla_i S$$

where R_{ij} is Ricci curvature and S is scalar curvature.

Proof. Direct computation shows

$$\begin{aligned} g^{jk} \nabla_k R_{ij} &= g^{jk} \nabla_k g^{pq} R_{pijq} \\ &= g^{jk} g^{pq} \nabla_k R_{pijq} \\ &= g^{jk} g^{pq} (-\nabla_p R_{ikjq} - \nabla_i R_{kpjq}) \\ &= -g^{pq} \nabla_p R_{iq} + \nabla_i S \\ &= -g^{jk} \nabla_k R_{ij} + \nabla_i S \end{aligned}$$

This completes the proof. □

Lemma 6.2.1.

$$S(p) = \sum_{i=1}^{n} \operatorname{Ric}_p(e_i)$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM.

Proposition 6.2.2. The scalar curvature S at $p \in M$ is given by

$$S(p) = \frac{1}{\alpha_n} \int_{\mathbb{S}^{n-1}} \operatorname{Ric}_p(x) \mathrm{d}\mathbb{S}^{n-1}$$

where α_n is the volume of n-dimension unit ball in $\mathbb{R}^{n+1}$.

Proof. Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ in T_pM and write $x = x^i e_i$, then

$$\begin{aligned} \operatorname{Ric}_p(x) &= \operatorname{Ric}_p(x^i e_i) \\ &= (x^i)^2 \operatorname{Ric}_p(e_i) \end{aligned}$$

Since $|x| = 1$, then the vector $\mu = (x^1, \dots, x^n)$ is a unit normal vector on $\mathbb{S}^{n-1}$. Denoting $V = (x^1 \operatorname{Ric}_p(e_1), \dots, x^n \operatorname{Ric}_p(e_n))$, then

$$\begin{aligned}
\frac{1}{\alpha_n} \int_{\mathbb{S}^{n-1}} (x^i)^2 \operatorname{Ric}_p(e_i) \mathrm{d}\mathbb{S}^{n-1} &= \frac{1}{\alpha_n} \int_{\mathbb{S}^{n-1}} \langle V, \mu \rangle \mathrm{d}\mathbb{S}^{n-1} \\
&\overset{(1)}{=} \frac{1}{\alpha_n} \int_{B^n} \operatorname{div} V \mathrm{d}B^n \\
&= \operatorname{div} V \\
&\overset{(2)}{=} \sum_{i=1}^{n} \operatorname{Ric}_p(e_i) \\
&= S(p)
\end{aligned}$$

where

(1) holds from Stokes theorem.
(2) holds from Lemma 6.2.1.

□

Remark 6.2.4. From the proof it's clear to see

$$S(p) = \int_{\mathbb{S}^{n-1}(t)} \operatorname{Ric}_p(x) \mathrm{d}\mathbb{S}^{n-1}$$

Theorem 6.2.1. Let (M, g) be a Riemannian manifold, then for all $p \in M$ and r sufficiently small, the volume of $B(p, r)$ is

$$\operatorname{Vol}(B(p, r)) = \alpha_n r^n \left(1 - \frac{S(p)}{6(n+2)} r^2 + O(r^3)\right)$$

where α_n is the volume of n-dimension unit ball in $\mathbb{R}^{n+1}$.

Proof. Direct computation shows

$$\begin{aligned}
\operatorname{Vol}(B(p, r)) &= \int_0^r \int_{\mathbb{S}^{n-1}(t)} \sqrt{\det g} \mathrm{d}S \mathrm{d}t \\
&\overset{(1)}{=} \int_0^r \int_{\mathbb{S}^{n-1}(t)} \left(1 - \frac{1}{6} \operatorname{Ric}_p(x) + O(|x|^3)\right) \mathrm{d}S \mathrm{d}t \\
&\overset{(2)}{=} \alpha_n r^n - \frac{\alpha_n}{6} \int_0^r S(p) t^{n+1} \mathrm{d}t + O(r^{n+3}) \\
&= \alpha_n r^n - \frac{\alpha_n S(p) r^{n+2}}{6(n+2)} + O(r^{n+3}) \\
&= \alpha_n r^n \left(1 - \frac{S(p)}{6(n+2)} r^2 + O(r^3)\right)
\end{aligned}$$

where

(1) holds from Corollary 4.4.1.
(2) holds from Proposition 6.2.2.

□

7. Basic models

7.1. Einstein manifold.

Definition 7.1.1 (Einstein manifold)**.** A Riemannian manifold (M,g) is called Einstein manifold, if its Ricci curvature satisfies $R_{ij}=\lambda g_{ij}$ for some $\lambda\in\mathbb{R}$.

Lemma 7.1.1 (Schur's lemma)**.** Let (M,g) be a Riemannian manifold with $\dim M\geq 3$, suppose $R_{ij}=fg_{ij}$, where f is a smooth function, then (M,g) is an Einstein manifold.

Proof. If $R_{ij}=fg_{ij}$, then contracted Bianchi identity shows

$$\begin{aligned}\frac{n}{2}\nabla_i f&=g^{jk}\nabla_k f g_{ij}\\&=\nabla_i f\end{aligned}$$

for arbitrary i, which implies f is constant, since $n\geq 3$. □

Corollary 7.1.1. For a Riemannian manifold (M,g) with $\dim M\geq 3$, it is isotropic if and only if it has constant sectional curvature.

Proof. By Remark 6.1.2, it suffices to show if M is isotropic then it has constant sectional curvature. If M is isotropic, then there exists a smooth function K such that

$$R_{ijkl}=K(g_{il}g_{jk}-g_{ik}g_{jl})$$

Consider its Ricci curvature, that is

$$R_{jk}=(n-1)Kg_{jk}$$

Then Schur's lemma implies $(n-1)K$ is constant, that is K is constant. □

Proposition 7.1.1. Let (M,g) be an Einstein 3-manifold, then (M,g) has constant sectional curvature.

Proof. For arbitrary point $p\in M$, without lose of generality we consider normal coordinate, that is $g_{ij}=\delta_{ij}$. Then

$$R_{11}=g^{ij}R_{i11j}=R_{2112}+R_{3113}=\lambda$$

Similarly we have

$$\begin{aligned}R_{1221}+R_{3223}&=\lambda\\R_{1331}+R_{2332}&=\lambda\end{aligned}$$

Thus we can conclude

$$R_{1221}=R_{1331}=R_{2332}=\frac{\lambda}{2}$$

that is

$$R_{ijkl}=\frac{\lambda}{2}(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl})$$

This shows (M,g) has constant sectional curvature $\lambda/2$. □

Remark 7.1.1. In fact, it's a special case of Ricci curvature controls curvature. For a n-dimensional Riemannian manifold, it's easy to see R_{jk} has $n(n+1)/2$ independent components. But for R_{ijkl}, this counting problem becomes a little bit complicated, it has

$$\frac{n^2(n^2-1)}{12}$$

independent components. Indeed, since R_{ijkl} is skew symmetric in ij and kl, this means that these pair of indices can take

$$m = \binom{n}{2} = \frac{n(n-1)}{2}$$

R_{ijkl} is also symmetric when you swap ij with kl, this means there would be

$$\frac{m(m+1)}{2} = \frac{n^4 - 2n^3 + 3n^2 - 2n}{8}$$

choices. However, these are not independent, since there is first Bianchi identity

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0$$

, and it provides

$$\binom{n}{4} = \frac{n^4 - 6n^3 + 11n^2 - 6n}{24}$$

relations between these components, thus the number of independent components of R_{ijkl} is

$$\frac{n^4 - 2n^3 + 3n^2 - 2n}{8} - \frac{n^4 - 6n^3 + 11n^2 - 6n}{24} = \frac{n^4 - n^2}{12} = \frac{n^2(n^2-1)}{12}$$

Therefore curvature is fully determined by the Ricci curvature if and only if

$$\frac{n^2(n^2-1)}{12} \leq \frac{n(n+1)}{2}$$

or in other words, $n \leq 3$.

7.2. **Sphere.**

Example 7.2.1 (sphere)**.** Let $\mathbb{S}^n(R)$ denote n-dimensional sphere with radius R. There is a natural inclusion $f\colon \mathbb{S}^n(R) \hookrightarrow (\mathbb{R}^{n+1}, g_{\mathrm{can}})$, and we can use f to pullback g_{can} to obtain a metric on $\mathbb{S}^n(R)$, denoted by g. Given a local chart (x^i, U), we can write

$$f(x^1, \ldots, x^n) = (x^1, \ldots, x^n, \sqrt{R^2 - \sum_{i=1}^{n}(x^i)^2})$$

For any $\frac{\partial}{\partial x^i}$, we have

$$\begin{aligned}\mathrm{d}f(\frac{\partial}{\partial x^i}) &= \frac{\partial f^j}{\partial x^i}\frac{\partial}{\partial x^j}\\ &= \frac{\partial}{\partial x^i} - \frac{x^i}{\sqrt{K^2-\sum_{i=1}^n (x^i)^2}}\frac{\partial}{\partial x^{n+1}}\end{aligned}$$

Thus for any two $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}$ we have

$$\begin{aligned}g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) &= g_{\mathrm{can}}(\mathrm{d}f(\frac{\partial}{\partial x^i}), \mathrm{d}f(\frac{\partial}{\partial x^j}))\\ &= g_{\mathrm{can}}(\frac{\partial}{\partial x^i} - \frac{x^i}{\sqrt{R^2-\sum_{i=1}^n (x^i)^2}}\frac{\partial}{\partial x^{n+1}}, \frac{\partial}{\partial x^j} - \frac{x^j}{\sqrt{R^2-\sum_{i=1}^n (x^i)^2}}\frac{\partial}{\partial x^{n+1}})\\ &= \delta_{ij} + \frac{x^i x^j}{R^2-\sum_{i=1}^n (x^i)^2}\end{aligned}$$

which implies

$$g_{ij} = \delta_{ij} + \frac{x^i x^j}{T^2}$$

where $T^2 = R^2 - \sum (x^i)^2$. Thus we have

$$\begin{aligned}g^{ij} &= \delta^{ij} - \frac{x^i x^j}{R^2}\\ \frac{\partial g_{ij}}{\partial x^k} &= \frac{\delta_{ki}x^j + \delta_{kj}x^i}{T^2} + \frac{2x^i x^j x^k}{T^4}\end{aligned}$$

So Christoffel symbol can be computed as

$$\begin{aligned}\Gamma^k_{ij} &= \frac{1}{2}g^{kl}(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l})\\ &= \sum_l \frac{1}{2}(\delta^{kl} - \frac{x^k x^l}{R^2})(\frac{\delta_{ij}x^l + \delta_{il}x^j}{T^2} + \frac{2x^i x^j x^l}{T^4} + \frac{\delta_{ji}x^l + \delta_{jl}x^i}{T^2} + \frac{2x^i x^j x^l}{T^4} - \frac{\delta_{li}x^j + \delta_{kj}x^i}{T^2} - \frac{2x^i x^j x^l}{T^4})\\ &= \sum_l \frac{x^l}{T^2}(\delta_{ij} + \frac{x^i x^j}{T^2})(\delta^{kl} - \frac{x^k x^l}{R^2})\\ &= \frac{g_{ij}}{T^2}x^k(1 - \frac{\sum_{l=1}^n (x^l)^2}{R^2})\\ &= \frac{x^k}{R^2}g_{ij}\end{aligned}$$

Direct computation shows curvature can be written as[3]

$$\begin{aligned}R_{ijkl} &= \frac{1}{2}(\partial_i\partial_k g_{jl} + \partial_j\partial_l g_{ik} - \partial_i\partial_l g_{jk} - \partial_j\partial_k g_{il}) + g_{rs}(\Gamma^r_{ik}\Gamma^s_{jl} - \Gamma^r_{jk}\Gamma^s_{il})\\ &= \frac{1}{R^2}(g_{il}g_{jk} - g_{ik}g_{jl})\end{aligned}$$

[3]Here I omit a huge computation, and I suggest you compute it by yourself. Maybe first it's quite tough for you to do this first time, but you should try.

So Ricci curvature and scalar curvature can be computed as follows

$$\begin{aligned}
R_{jk} &= g^{il} R_{ijkl} \\
&= \frac{1}{R^2} g^{il}(g_{il}g_{jk} - g_{ik}g_{jl}) \\
&= \frac{1}{R^2}(n g_{jk} - \delta^l_k g_{jl}) \\
&= \frac{n-1}{R^2} g_{jk} \\
S &= g^{jk} R_{jk} \\
&= \frac{n(n-1)}{R^2}
\end{aligned}$$

7.3. **Hyperbolic space.**

Example 7.3.1 (hyperbolic upper plane)**.** Let $\mathbb{H}^n(R) = \{(x^1, \dots, x^{n-1}, y) \in \mathbb{R}^n \mid y > 0\}$ with metric

$$g = R^2 \frac{\delta_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j + \mathrm{d}y \otimes \mathrm{d}y}{y^2}$$

Example 7.3.2 (Poincaré disk)**.** Let $\mathbb{B}^n(R) = \{x \in \mathbb{R}^n \mid |x| < R\}$ with metric

$$g = 4R^4 \frac{\delta_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j}{(R^2 - |x|^2)^2}$$

7.4. **Lie group.**

7.4.1. *Invariant metrics.*

Definition 7.4.1 (left-invariant metric)**.** A Riemannian metric $\langle \text{-}, \text{-} \rangle$ on a Lie group G is called left-invariant if

$$\langle (\mathrm{d}L_g)X, (\mathrm{d}L_g)Y \rangle = \langle X, Y \rangle$$

holds for arbitrary $g \in G$ and vector fields X, Y.

Remark 7.4.1. Similarly we can define a right-invariant metric, and a Riemannian metric which is both left-invariant and right-invariant is called bi-invariant metric.

Proposition 7.4.1. Any compact Lie group G admits a bi-invariant metric.

Proposition 7.4.2. There is a bijective correspondence between left-invariant metrics on a Lie group G and inner products on the Lie algebra $\mathfrak{g}$ of G.

Proof. Given an inner product $\langle \text{-}, \text{-} \rangle_e$ on Lie algebra $\mathfrak{g}$, then we have an inner product on G defined as follows

$$\langle X_g, Y_g \rangle := \langle (\mathrm{d}L_{g^{-1}})X_g, (\mathrm{d}L_{g^{-1}})Y_g \rangle_e$$

where X, Y are two vector fields on G. It's left-invariant, since

$$\begin{aligned}
\langle (\mathrm{d}L_h)X_g, (\mathrm{d}L_h)Y_g \rangle &= \langle (\mathrm{d}L_{(hg)^{-1}}) \circ (\mathrm{d}L_h)X_g, (\mathrm{d}L_{(hg)^{-1}}) \circ (\mathrm{d}L_h)Y_g \rangle_e \\
&= \langle (\mathrm{d}L_{g^{-1}})X_g, (\mathrm{d}L_{g^{-1}})Y_g \rangle_e
\end{aligned}$$

Conversely, if we have a left-invariant inner product $\langle -,- \rangle$ on G, then it's clear we have an inner product on $\mathfrak{g}$, by just considering its value at identity. Furthermore, these two constructions are inverse to each other, this completes the proof. $\square$

Proposition 7.4.3. There is a bijective correspondence between bi-invariant metrics on a Lie group G, and Ad-invariant inner products on the Lie algebra $\mathfrak{g}$ of G.

Proof. Given a Ad-invariant inner product $\langle -,- \rangle_e$ on the Lie algebra $\mathfrak{g}$, by Proposition 7.4.2, there is a left-invariant metric $\langle -,- \rangle$ on G, it suffices to check it's also right-invariant:

$$\begin{aligned}\langle (\mathrm{d}L_h)X_g, (\mathrm{d}L_h)Y_g \rangle &= \langle (\mathrm{d}L_{(hg)^{-1}}) \circ (\mathrm{d}L_h)X_g, (\mathrm{d}L_{(hg)^{-1}}) \circ (\mathrm{d}R_h)Y_g \rangle_e \\ &= \langle \mathrm{Ad}(h^{-1})(\mathrm{d}L_{g^{-1}})X_g, \mathrm{Ad}(h^{-1})(\mathrm{d}L_{g^{-1}})Y_g \rangle_e \\ &= \langle (\mathrm{d}L_{g^{-1}})X_g, (\mathrm{d}L_{g^{-1}})Y_g \rangle_e \\ &= \langle X_g, Y_g \rangle\end{aligned}$$

Conversely, if we start with a bi-invariant metric, then it's restriction to the Lie algebra is a Ad-invariant, since $\mathrm{Ad}(g)$ is exactly the differential of $L_g \circ R_{g^{-1}}$. $\square$

Lemma 7.4.1. Let G be a Lie group equipped with left-invariant metric $\langle -,- \rangle$, and ∇ the Levi-Civita connection with respect to it. Then for all left-invariant vector fields X, Y, Z,

$$\langle X, \nabla_Y Y \rangle = \langle Y, [X, Y] \rangle$$

Proof. Recall that

$$\langle X, \nabla_Y Z \rangle = \frac{1}{2}(Y\langle Z, X \rangle + Z\langle X, Y \rangle - X\langle Y, Z \rangle - \langle [Y, X], Z \rangle - \langle [Z, X], Y \rangle - \langle [Y, Z], X \rangle)$$

But $Y\langle Z, X \rangle = Z\langle X, Y \rangle = X\langle Y, Z \rangle = 0$ since both metric and X, Y, Z are left-invariant, that is

$$\langle X, \nabla_Y Z \rangle = \frac{1}{2}\{\langle Z, [X, Y] \rangle + \langle Y, [X, Z] \rangle + \langle X, [Z, Y] \rangle\}$$

Now set $Y = Z$ to conclude. $\square$

Proposition 7.4.4. Let G be a Lie group equipped with bi-invariant metric $\langle -,- \rangle$, and ∇ the Levi-Civita connection with respect to it. Then for all left-invariant vector fields X, Y, Z,

$$\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle$$

Proof. Let y_t be the flow of Y, then

$$[X, Y] = \lim_{t \to 0} \frac{1}{t}((\mathrm{d}y_t)(X) - X)$$

On the other hand, since Y is left-invariant, that is $L_g \circ y_t = y_t \circ L_g$, giving

$$y_t(g) = y_t(L_g(e)) = L_g y_t(e) = g y_t(e) = R_{y_t(e)}(g)$$

Thus $\mathrm{d}y_t = \mathrm{d}R_{y_t(e)}$ and

$$[X,Y] = \lim_{t\to 0}\frac{1}{t}((\mathrm{d}R_{y_t(e)})(X) - X)$$

Note that the metric is bi-invariant, thus

$$\begin{aligned}\langle X, Z\rangle &= \langle(\mathrm{d}R_{y_t(e)})\circ(\mathrm{d}L_{y_t^{-1}(e)})X, (\mathrm{d}R_{y_t(e)})\circ(\mathrm{d}L_{y_t^{-1}(e)})Z\rangle\\ &= \langle(\mathrm{d}R_{y_t(e)})X, (\mathrm{d}R_{y_t(e)})Z\rangle\end{aligned}$$

Differentiating the expression above with respect to t and setting $t = 0$ we conclude

$$0 = \langle[X,Y], Z\rangle + \langle X, [Z,Y]\rangle$$

This completes the proof. □

7.4.2. *Levi-Civita connection of bi-invariant metric.*

Theorem 7.4.1. Let G be a Lie group equipped with bi-invariant metric $\langle \text{-}, \text{-}\rangle$, and ∇ the Levi-Civita connection with respect to it. Then for every left-invariant vector field X on G, then $\nabla_X X = 0$.

Proof. From Lemma 7.4.1, we have

$$\langle Y, \nabla_X X\rangle = \langle X, [Y,X]\rangle$$

From Proposition 7.4.4, we have

$$\langle X, [Y,X]\rangle = \langle [X,Y], X\rangle = -\langle X, [Y,X]\rangle$$

that is $\langle Y, \nabla_X X\rangle = 0$ for arbitrary vector field Y, which implies $\nabla_X X = 0$. □

Corollary 7.4.1. The assumptions are as above. If X, Y are left-invariant vector fields, then $\nabla_X Y = \frac{1}{2}[X,Y]$.

Proof. Note that

$$\begin{aligned}0 &= \nabla_{X+Y}(X+Y)\\ &= \nabla_X Y + \nabla_Y X + \nabla_X X + \nabla_Y Y\\ &= \nabla_X Y + \nabla_Y X\\ &= 2\nabla_X Y - [X,Y]\end{aligned}$$

Division by two finally yields

$$\nabla_X Y = \frac{1}{2}[X,Y]$$

□

Corollary 7.4.2. The assumptions are as above. If X, Y, Z are left-invariant vector fields, then $R(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$.

Proof. Directly from $\nabla_X Y = \frac{1}{2}[X,Y]$ and Jacobi's identity. □

Corollary 7.4.3. The assumptions are as above. If X, Y are left-invariant vector fields which are orthogonal, and σ is the plane generated by X and Y. Then

$$K(\sigma) = \frac{1}{4}\|[X, Y]\|^2$$

Proof.

$$K(\sigma) = -\frac{1}{4}\langle[[X, Y], Y], X\rangle = -\frac{1}{4}\langle[X, Y], [Y, X]\rangle = \frac{1}{4}\|[X, Y]\|^2$$

$\square$

Remark 7.4.2. Therefore, sectional curvature of a Lie group with bi-invariant metric is non-negative. Furthermore, if the center of Lie algebra $\mathfrak{g}$ is trivial, then the sectional curvature is positive.

Theorem 7.4.2. Let G be a Lie group equipped with bi-invariant metric, the geodesics on G are precisely the integral curves of left-invariant vector fields.

Proof. Let $X \in \mathfrak{g}$ be a left-invariant vector field, and $\gamma\colon \mathbb{R} \to G$ its integral curve. Then

$$\begin{aligned} \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma_*(\frac{\mathrm{d}}{\mathrm{d}t}) &= \nabla_{\gamma_*(\frac{\mathrm{d}}{\mathrm{d}t})} \gamma_*(\frac{\mathrm{d}}{\mathrm{d}t}) \\ &= \nabla_X X \\ &= 0 \end{aligned}$$

which implies integral curves of left-invariant vector fields are geodesics. Furthermore, since geodesics are unique, we have geodesics are precisely integral curves of left-invariant vector fields. $\square$

Corollary 7.4.4. The exponential map for the Lie group coincides with the exponential map of the Levi-Civita connection with respect to bi-invariant metric.

Part 3. Bochner's technique

8. Hodge theory on Riemannian manifold

For convenience, in this section we assume (M, g) is a compact oriented Riemannian n-manifold, since we need to consider integral.

8.1. Inner product on Ω_M^k.

Before we talk about Hodge theory on (M, g), let's recall some basic facts about differential k-forms. For a k-form φ, locally it can be written as

$$\varphi = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \varphi_{i_1 \ldots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

where $\varphi_{i_1 \ldots i_k} := \varphi(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}})$, is skew-symmetric. If we don't want indices are arranged in order, we can write

$$\varphi = \frac{1}{k!} \varphi_{i_1 \ldots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

where the summation runs over arbitrary different k indices. It's clear this two expressions are same, since both $\varphi_{i_1 \ldots i_k}$ and $\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$ are skew-symmetric.

Notation 8.1.1. $\varphi_I \mathrm{d}x^I$ denotes $\varphi_{i_1 \ldots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$.

Recall that we already have a induced metric g on $\bigotimes^k T^*M$, and Ω_M^k is a subbundle of $\bigotimes^k T^*M$. Thus we can define a metric on Ω_M^k as follows

Definition 8.1.1. Let φ, ψ be two k-forms, define

$$\langle \varphi, \psi \rangle := \frac{1}{k!} g(\varphi, \psi)$$

where g is induced metric on $\bigotimes^k T^*M$.

Lemma 8.1.1. For $\varphi = \varphi_I \mathrm{d}x^I, \psi = \psi_J \mathrm{d}x^J$, then

$$\langle \varphi, \psi \rangle = \varphi_I \psi_J g^{IJ}$$

where

$$g^{IJ} = \frac{1}{k!} g(\mathrm{d}x^I, \mathrm{d}x^J) = \det \begin{pmatrix} g^{i_1 j_1} & \cdots & g^{i_1 j_k} \\ \cdots & \cdots & \cdots \\ g^{i_k j_1} & \cdots & g^{i_k j_k} \end{pmatrix}$$

Proof. It suffices to show

$$g(\mathrm{d}x^I, \mathrm{d}x^J) = k! \det \begin{pmatrix} g^{i_1 j_1} & \cdots & g^{i_1 j_k} \\ \cdots & \cdots & \cdots \\ g^{i_k j_1} & \cdots & g^{i_k j_k} \end{pmatrix}$$

By definition one has

$$\mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} = \sum_{\sigma \in S_k} (-1)^{|\sigma|} e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(k)}}$$

Then

$$\begin{aligned}
g(\mathrm{d}x^I, \mathrm{d}x^J) &= \sum_{\sigma,\tau}(-1)^{|\sigma|}(-1)^{|\tau|} g(\mathrm{d}x^{i_{\sigma(1)}} \otimes \cdots \otimes \mathrm{d}x^{i_{\sigma(k)}}, \mathrm{d}x^{j_{\tau(1)}} \otimes \cdots \otimes \mathrm{d}x^{j_{\tau(k)}}) \\
&= \sum_{\sigma,\tau}(-1)^{|\sigma|}(-1)^{|\tau|} g^{i_{\sigma(1)}j_{\tau(1)}} \dots g^{i_{\sigma(k)}j_{\tau(k)}} \\
&= \sum_{\sigma,\tau}(-1)^{|\sigma\tau^{-1}|} g^{i_{\sigma\tau^{-1}(1)}j_1} \dots g^{i_{\sigma\tau^{-1}(k)}j_k} \\
&= \sum_{\sigma}\sum_{\rho}(-1)^{|\rho|} g^{i_{\rho(1)}j_1} \dots g^{i_{\rho(k)}j_k} \\
&= \sum_{\sigma} \det(g^{i_pj_q}) \\
&= k! \det(g^{i_pj_q})
\end{aligned}$$

□

Remark 8.1.1. Note that here we don't assume φ_I, ψ_I is skew-symmetric, they can be arbitrary functions.

Corollary 8.1.1. For two k-forms φ, ψ, locally written as

$$\begin{aligned}
\varphi &= \sum_{1\leq i_1<\dots<i_k\leq n} \varphi_{i_1\dots i_k} \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_k} \\
\psi &= \sum_{1\leq j_1<\dots<j_k\leq n} \varphi_{j_1\dots j_k} \mathrm{d}x^{j_1} \wedge \dots \wedge \mathrm{d}x^{j_k}
\end{aligned}$$

with φ_I, ψ_J is skew-symmetric, then

$$\langle \varphi, \psi \rangle = \sum_{\substack{1\leq i_1<\dots i_k\leq n \\ 1\leq j_1<\dots j_k\leq n}} \varphi_{i_1\dots i_k}\psi_{j_1\dots j_k} \det \begin{pmatrix} g^{i_1j_1} & \dots & g^{i_1j_k} \\ \dots & \dots & \dots \\ g^{i_kj_1} & \dots & g^{i_kj_k} \end{pmatrix}$$

Example 8.1.1. Let φ, ψ be two 2-forms, locally written as

$$\begin{aligned}
\varphi &= \varphi_{i_1i_2} \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \\
\psi &= \psi_{j_1j_2} \mathrm{d}x^{j_1} \wedge \mathrm{d}x^{j_2}
\end{aligned}$$

where $i_1 < i_2, j_1 < j_2$. Then

$$\begin{aligned}
\langle \varphi, \psi \rangle &= \frac{1}{2}\varphi_{i_1i_2}\psi_{j_1j_2} g(\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2}, \mathrm{d}x^{j_1} \wedge \mathrm{d}x^{j_2}) \\
&= \frac{1}{2}\varphi_{i_1i_2}\psi_{j_1j_2} g(\mathrm{d}x^{i_1} \otimes \mathrm{d}x^{i_2} - \mathrm{d}x^{i_2} \otimes \mathrm{d}x^{i_1}, \mathrm{d}x^{j_1} \otimes \mathrm{d}x^{j_2} - \mathrm{d}x^{j_2} \otimes \mathrm{d}x^{j_1}) \\
&= \frac{1}{2}\varphi_{i_1i_2}\psi_{j_1j_2} (g^{i_1j_1}g^{i_2j_2} - g^{i_1j_2}g^{i_2j_1} - g^{i_2j_1}g^{i_1j_2} + g^{i_2j_2}g^{i_1j_1}) \\
&= \varphi_{i_1i_2}\psi_{j_1j_2} (g^{i_1j_1}g^{i_2j_2} - g^{i_1j_2}g^{i_2j_1}) \\
&= \varphi_{i_1i_2}\psi_{j_1j_2} \det \begin{pmatrix} g^{i_1j_1} & g^{i_1j_2} \\ g^{i_2j_1} & g^{i_2j_2} \end{pmatrix}
\end{aligned}$$

Definition 8.1.2 (volume form)**.** The volume form vol is a n-form such that $\langle \mathrm{vol}, \mathrm{vol}\rangle = 1$.

Remark 8.1.2 (local form). Locally the volume form is given by $\sqrt{\det g}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$.

Definition 8.1.3 (inner product on Ω^k_M)**.** For two k-forms φ, ψ, their inner product is defined as

$$(\varphi, \psi) := \int_M \langle \varphi, \psi\rangle \,\mathrm{vol}$$

Definition 8.1.4 (formal adjoint)**.** For a k-form φ and a $(k+1)$-form ψ, the formal adjoint of d is an operator $\mathrm{d}^* : C^\infty(M, \Omega^{k+1}_M) \to C^\infty(M, \Omega^k_M)$ such that

$$(\mathrm{d}\varphi, \psi) = (\varphi, \mathrm{d}^*\psi)$$

Remark 8.1.3. There is no gurantee for existence, but later we will see such d^* do exists, and give an explicit formula.

Definition 8.1.5 (Laplace-Beltrami operator)**.** The Laplace-Beltrami operator $\Delta_g : C^\infty(M, \Omega^k_M) \to C^\infty(M, \Omega^k_M)$ is defined as

$$\Delta_g = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$$

Definition 8.1.6 (harmonic)**.** A k-form α is called harmonic, if $\Delta_g \alpha = 0$.

Notation 8.1.2. The space of all harmonic forms is denoted by $\mathcal{H}^k(M)$.

Lemma 8.1.2. A k-form α is harmonic if and only if $\mathrm{d}\alpha = 0$ and $\mathrm{d}^*\alpha = 0$.

Proof. Note that

$$\begin{aligned}(\alpha, \Delta\alpha) &= (\alpha, \mathrm{d}\mathrm{d}^*\alpha) + (\alpha, \mathrm{d}^*\mathrm{d}\alpha)\\ &= \|\mathrm{d}^*\alpha\|^2 + \|\mathrm{d}\alpha\|^2\end{aligned}$$

□

8.2. **Hodge star operator.** Although we have defined an inner product on Ω^k_M, it's still quite difficult to compute it. However, inner product on Ω^k_M is independent of the choice of local frame, so we can use normal coordinate to give a local frame, and define Hodge star operator on it, which will gives us an effective method to compute.

8.2.1. *Baby case.* Recall that for a $\mathbb{F}$-vector space V with inner product $\langle$-,-$\rangle$, and $\{e_1, \ldots, e_n\}$ is a basis of V. For any $0 \le k \le n$, there is a natural basis of $\bigwedge^k V$, consisting of $\{e_I := e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \le i_1 < \cdots < i_k \le n\}$.

Here are two special cases:

1. For $k = 0$, we regard $\bigwedge^0 V^k$ as base field $\mathbb{F}$, and $e_I = 1$.
2. For $k = n$, we use vol to denote basis $e_1 \wedge \cdots \wedge e_n$.

With respect to this basis, we can write down the induced metric on $\bigwedge^k V$ as

$$\langle e_{i_1} \wedge \cdots \wedge e_{i_k}, e_{j_1} \wedge \cdots \wedge e_{j_k} \rangle = \det \begin{pmatrix} \langle e_{i_1}, e_{j_1} \rangle & \dots & \langle e_{i_1}, e_{j_k} \rangle \\ \vdots & & \vdots \\ \langle e_{i_k}, e_{j_1} \rangle & \dots & \langle e_{i_k}, e_{j_k} \rangle \end{pmatrix}$$

It's clear if $\{e_1, \dots, e_n\}$ is an orthonormal basis of V, then $\{e_I\}$ is an orthonormal basis of $\bigwedge^k V$. From now on, we assume $\{e_I\}$ is an orthonormal basis of $\bigwedge^k V$.

Definition 8.2.1 (Hodge star)**.** Hodge star operator is defined as

$$\begin{aligned} \star : \bigwedge^k V &\to \bigwedge^{n-k} V \\ e_I &\mapsto \operatorname{sign}(I, I^c) e_{I^c} \end{aligned}$$

where I^c is $[n] - I = \{i'_1, \dots, i'_{n-k}\}$ and $\operatorname{sign}(I, I^c)$ is the sign of the permutation $(i_1, \dots, i_k, i'_1, \dots, i'_{n-k})$.

Example 8.2.1. It's clear $\star 1 = \mathrm{vol}$ and $\star \mathrm{vol} = 1$.

Proposition 8.2.1.

$$\star^2 = (-1)^{k(n-k)} \operatorname{id}$$

holds on $\bigwedge^k V$.

Proof. It suffices to check on basis e_I as follows

$$\begin{aligned} \star^2 e_I &= \star(\operatorname{sign}(I, I^c) e_{I^c}) \\ &= \operatorname{sign}(I, I^c) \operatorname{sign}(I^c, I) e_I \\ &= (-1)^{k(n-k)} e_I \end{aligned}$$

□

Proposition 8.2.2. For $u \in \bigwedge^k V, v \in \bigwedge^{n-k} V$, we have

$$\star(u \wedge v) = (-1)^{k(n-k)} \langle u, \star v \rangle$$

Proof. It suffices to check on basis $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}, e_J = e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}$. Furthermore, it's clear $e_I \wedge e_J = 0$, if $J \neq I^c$, so we may assume $J = I^c$.

$$\begin{aligned} \star(e_I \wedge e_{I^c}) &= \star(\operatorname{sign}(I, I^c) \mathrm{vol}) \\ &= \operatorname{sign}(I, I^c) \\ \langle e_I, \star e_{I^c} \rangle &= \langle e_I, \operatorname{sign}(I, I^c) e_I \rangle \\ &= \operatorname{sign}(I, I^c) \langle e_I, e_I \rangle \\ &= \operatorname{sign}(I, I^c) \end{aligned}$$

□

Corollary 8.2.1. For $u, v \in \bigwedge^k V$, we have

1. $u \wedge \star v = v \wedge \star u = \langle u, v \rangle \mathrm{vol}$.

2. $\langle \star u, \star v\rangle = \langle u, v\rangle$.

Proof. For (1).

$$\star(u \wedge \star v) = (-1)^{k(n-k)}\langle u, \star^2 v\rangle = \langle u, v\rangle$$

which implies $u \wedge \star v = \langle u, v\rangle \operatorname{vol}$. Since $\langle u, v\rangle = \langle v, u\rangle$, we obtain $u \wedge \star v = v \wedge \star u$.

For (2).

$$\begin{aligned}\langle \star u, \star v\rangle &= (-1)^{k(n-k)} \star (\star u \wedge v)\\ &= (-1)^{2k(n-k)} \star (v \wedge \star u)\\ &= (-1)^{3k(n-k)}\langle v \wedge \star^2 u\rangle\\ &= (-1)^{4k(n-k)}\langle v, u\rangle\\ &= \langle u, v\rangle\end{aligned}$$

□

Remark 8.2.1. Here are two remarks about this corollary:

(1) gives us a method to compute inner product, that's why we define Hodge star, some authors also use this property to denote Hodge star operator.
(2) implies that Hodge star operator is an isometry between $\bigwedge^k V$ and $\bigwedge^{n-k} V$.

Corollary 8.2.2 (almost self-adjoint)**.** For $u \in \bigwedge^k V, v \in \bigwedge^{n-k} V$, we have

$$\langle u, \star v\rangle = (-1)^{k(n-k)}\langle \star u, v\rangle$$

Proof.

$$\langle u, \star v\rangle = \langle \star u, \star^2 v\rangle = (-1)^{k(n-k)}\langle \star u, v\rangle$$

□

Remark 8.2.2. This corollary implies the adjoint operator of $\star$ is $(-1)^{k(n-k)}\star$, so here I call it almost self-adjoint.

8.2.2. *General case.* Now we're going to define Hodge star operator on M, that's an operator from $C^\infty(M, \Omega^k_M)$ to $C^\infty(M, \Omega^{n-k}_M)$. If we define it pointwise, then everything reduces to the baby case. For each point $p \in M$, consider the local frame $\{\frac{\partial}{\partial x^1}, \ldots \frac{\partial}{\partial x^n}\}$ of TM given by normal coordinate, with dual frame $\{\mathrm{d}x^1, \ldots, \mathrm{d}x^n\}$, then

$$\star(\mathrm{d}x^I) := \operatorname{sign}(I, I^c)\mathrm{d}x^{I^c}$$

Theorem 8.2.1.

1. $\star 1 = \operatorname{vol}, \star \operatorname{vol} = 1$.
2. $\star^2 = (-1)^{k(n-k)}$ on k-forms.
3. If u is a k-form and v a $(n-k)$-form, then

$$\begin{aligned}\star(u \wedge v) &= (-1)^{k(n-k)}\langle u, \star v\rangle\\ \langle u, \star v\rangle &= (-1)^{k(n-k)}\langle \star u, v\rangle\end{aligned}$$

4. For any two k-forms u, v, then

$$\begin{aligned} u \wedge \star v = v \wedge \star u &= \langle u, v\rangle \operatorname{vol} = \langle v, u\rangle \operatorname{vol} \\ \langle \star u, \star v\rangle &= \langle u, v\rangle \end{aligned}$$

5. $\mathrm{d}^* = (-1)^{nk+n+1} \star \mathrm{d}\star$ on k-forms.

Proof. It suffices to check (5), other cases we have already solved in the case of linear algebra. Take any $(k-1)$-form α and k-form β, we need to show

$$(\mathrm{d}\alpha, \beta) = (\alpha, \mathrm{d}^*\beta)$$

that is to show

$$\int_M \mathrm{d}\alpha \wedge \star\beta = \int_M \alpha \wedge \star \mathrm{d}^*\beta$$

By Stokes theorem and Leibniz rule we have

$$0 = \int_M \mathrm{d}(\alpha \wedge \star\beta) = \int_M \mathrm{d}\alpha \wedge \star\beta + (-1)^{k-1}\int_M \alpha \wedge \mathrm{d}\star\beta$$

Since $\star^2 = (-1)^{(n-k+1)(k-1)}$ on $(n-k+1)$-forms, then

$$(-1)^{k-1}\int_M \alpha \wedge \mathrm{d}\star\beta = (-1)^{k-1+(n-k+1)(k-1)}\int_M \alpha \wedge \star^2 \mathrm{d}\star\beta$$

Therefore

$$\begin{aligned} (\mathrm{d}\alpha, \beta) &= \int_M \mathrm{d}\alpha \wedge \star\beta \\ &= (-1)^{k+(n-k+1)(k-1)}\int_M \alpha \wedge \star\star\, \mathrm{d}\star\beta \\ &= (-1)^{nk+k+1}\int_M \alpha \wedge \star(\star \mathrm{d}\star\beta) \end{aligned}$$

which implies

$$\mathrm{d}^*\beta = (-1)^{nk+k+1} \star \mathrm{d}\star\beta$$

□

Remark 8.2.3. (4) allows us to give a new expression for inner product (φ, ψ), where φ, ψ are two k-forms, that is

$$(\varphi, \psi) := \int_M \langle\varphi, \psi\rangle \operatorname{vol} = \int_M \varphi \wedge \star\psi$$

Some authors use this expression to define Hodge star operator, and prove its properties shown in Theorem 8.2.1.

8.3. **Computations of adjoint operator.**

Lemma 8.3.1 (Jacobi's formula)**.** For a function $(a_{ij}(t))$ valued in $\mathrm{GL}(n, \mathbb{R})$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\det(a_{ij}(t)) = \det(a_{ij}(t))a^{ij}(t)\frac{\mathrm{d}a_{ij}(t)}{\mathrm{d}t}$$

where $(a^{ij}(t))$ is the inverse matrix of $(a_{ij}(t))$.

Lemma 8.3.2. Let (M, g) be a Riemannian manifold, for any two vector fields X, Y, one has

$$\nabla_Y \circ \iota_X = \iota_X \circ \nabla_Y + \iota_{\nabla_Y X}$$

Proof. Let ω be a $(k+1)$-form, then for vector fields $Y_1, \dots, Y_k$, direct computation shows

$$\begin{aligned}
\nabla_Y \circ \iota_X \omega(Y_1, \dots, Y_k) =& \nabla_Y \omega(X, Y_1, \dots, Y_k) \\
=& Y\omega(X, Y_1, \dots, Y_k) - \omega(\nabla_X Y, Y_1, \dots, Y_k) \\
& - \sum_{i=1}^{k} \omega(X, Y_1, \dots, Y_{i-1}, \nabla_X Y_i, Y_{i+1}, \dots, Y_k) \\
& = (\iota_X \circ \nabla_Y \omega + \iota_{\nabla_X Y} \omega)(Y_1, \dots, Y_k)
\end{aligned}$$

□

Lemma 8.3.3. Let (M, g) be a compact Riemannian manifold, then

$$\langle \mathrm{d}x^i \wedge \alpha, \beta \rangle = \langle \alpha, g^{ij} \iota_{\frac{\partial}{\partial x^j}} \beta \rangle$$

where $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ is a local frame of TM, and α, β are forms with appropriate degrees.

Proof. It suffices to check with respect to normal coordinate. If we locally write $\alpha = \alpha_I \mathrm{d}x^I$ and $\beta = \beta_J \mathrm{d}x^J$, it suffices to check the case there is no $\mathrm{d}x^i$ in $\mathrm{d}x^I$ and there is $\mathrm{d}x^i$ in $\mathrm{d}x^J$, since other cases are trivial. Suppose $|I| = k$ and $|J| = k+1$, and $\mathrm{d}x^i$ in the m-th position of $\mathrm{d}x^J$, then

$$\langle \alpha, \iota_{\frac{\partial}{\partial x^i}} \beta \rangle = (-1)^{m+2} \alpha_I \beta_J \det G$$

where G is a $k \times k$ matrix. By definition if we write

$$\langle \mathrm{d}x^i \wedge \alpha, \beta \rangle = \alpha_I \beta_J \det G'$$

where G' is a $(k+1) \times (k+1)$ matrix. It's clear $\det G' = (-1)^{m+2} \det G$ by expansion of $\det G'$ by the first row. □

Lemma 8.3.4. Let (M, g) be a Riemannian manifold with volume form vol, then

$$\begin{aligned}
\mathcal{L}_X \mathrm{vol} &= (\frac{\partial X^i}{\partial x^i} + \frac{1}{2} X^i g^{pq} \frac{\partial g_{pq}}{\partial x^i}) \mathrm{vol} \\
&= (\frac{\partial X^i}{\partial x^i} + \Gamma^j_{ij} X^i) \mathrm{vol}
\end{aligned}$$

Proof. Cartan's magic formula shows that

$$\mathcal{L}_X = \iota_X \circ \mathrm{d} + \mathrm{d} \circ \iota_X$$

So

$$\begin{aligned}
\mathcal{L}_X \operatorname{vol} &= (\iota_X \circ \mathrm{d} + \mathrm{d} \circ \iota_X) \operatorname{vol} \\
&= \mathrm{d} \circ \iota_X \operatorname{vol} \\
&= \mathrm{d}\{(-1)^{i-1} X^i \sqrt{\det g} \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n\} \\
&= \frac{1}{\sqrt{\det g}} \frac{\partial (X^i \sqrt{\det g})}{\partial x^i} \operatorname{vol} \\
&= \frac{1}{\sqrt{\det g}} (\frac{\partial X^i}{\partial x^i} \sqrt{\det g} + X^i \frac{\partial \sqrt{\det g}}{\partial x^i}) \operatorname{vol} \\
&= (\frac{\partial X^i}{\partial x^i} + X^i \frac{\partial \log \sqrt{\det g}}{\partial x^i}) \operatorname{vol} \\
&= (\frac{\partial X^i}{\partial x^i} + \frac{1}{2} X^i \frac{\partial \log \det g}{\partial x^i}) \operatorname{vol}
\end{aligned}$$

Then the following Jacobi's formula shows the first equality

$$\frac{\partial \log \det g}{\partial x^i} = \frac{1}{\det g} \frac{\partial \det g}{\partial x^i} = g^{pq} \frac{\partial g_{pq}}{\partial x^i}$$

and the second equality holds from the formula of Christoffel in terms of metric, that is $g^{jk}(\Gamma^l_{ij} g_{lk} + \Gamma^l_{ik} g_{jl}) = 2\Gamma^j_{ij}$. □

Proposition 8.3.1. Let (M, g) be a compact Riemannian manifold equipped with Levi-Civita connection ∇, then

$$\mathrm{d}^* = -g^{ij} \iota_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}}$$

where $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ is a local frame of TM.

Proof. Direct computation shows

$$\begin{aligned}
0 &= \int_M \mathrm{d}(\alpha \wedge \star\beta) \\
&= \int_M \mathcal{L}_{\frac{\partial}{\partial x^i}} (\mathrm{d}x^i \wedge \alpha \wedge \star\beta) \\
&= \int_M \mathcal{L}_{\frac{\partial}{\partial x^i}} (\langle \mathrm{d}x^i \wedge \alpha, \beta \rangle \operatorname{vol}) \\
&\overset{(1)}{=} \int_M \mathcal{L}_{\frac{\partial}{\partial x^i}} (\langle \alpha, g^{ij} \iota_{\frac{\partial}{\partial x^j}} \beta \rangle \operatorname{vol}) \\
&\overset{(2)}{=} \int_M (\langle \nabla_i \alpha, g^{ij} \iota_{\frac{\partial}{\partial x^j}} \beta \rangle + \langle \alpha, \nabla_i (g^{ij} \iota_{\frac{\partial}{\partial x^j}} \beta) \rangle + \frac{1}{2} g^{pq} \frac{\partial g_{pq}}{\partial x^i} \langle \alpha, g^{ij} \iota_{\frac{\partial}{\partial x^j}} \beta \rangle) \operatorname{vol} \\
&= \int_M (\langle \mathrm{d}x^i \wedge \nabla_i \alpha, \beta \rangle + \langle \alpha, \frac{\partial g^{ij}}{\partial x^i} \iota_{\frac{\partial}{\partial x^j}} \beta \rangle + \langle \alpha, g^{ij} \nabla_i (\iota_{\frac{\partial}{\partial x^j}} \beta) \rangle + \frac{1}{2} g^{pq} \frac{\partial g_{pq}}{\partial x^i} \langle \alpha, g^{ij} \iota_{\frac{\partial}{\partial x^j}} \beta \rangle) \operatorname{vol} \\
&\overset{(3)}{=} \int_M (\langle \mathrm{d}x^i \wedge \nabla_i \alpha, \beta \rangle + \langle \alpha, \frac{\partial g^{ij}}{\partial x^i} \iota_{\frac{\partial}{\partial x^j}} \beta \rangle + \langle \alpha, g^{ij} \iota_{\frac{\partial}{\partial x^j}} (\nabla_i \beta) \rangle + \langle \alpha, g^{il} \Gamma^j_{il} \iota_{\frac{\partial}{\partial x^j}} \rangle \\
&\quad + \frac{1}{2} g^{pq} \frac{\partial g_{pq}}{\partial x^i} \langle \alpha, g^{ij} \iota_{\frac{\partial}{\partial x^j}} \beta \rangle) \operatorname{vol}
\end{aligned}$$

where

(1) holds from Lemma 8.3.3.
(2) holds from Lemma 8.3.4.
(3) holds from Lemma 8.3.2.

It's easy to see

$$\frac{\partial g^{ij}}{\partial x^i}+g^{il}\Gamma_{il}^j+\frac{1}{2}g^{ij}g^{pq}\frac{g^{pq}}{\partial x^i}=0$$

since

$$\Gamma_{ij}^l=\frac{1}{2}g^{kl}(\frac{\partial g_{kj}}{\partial x^i}+\frac{\partial g_{ik}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k})$$

□

In the following examples, we always compute with respect to normal coordinate.

Example 8.3.1. For a 1-form ω locally written as $\omega_i \mathrm{d}x^i$, then

$$\mathrm{d}^*\omega=-\sum_{i=1}^n\frac{\partial\omega_i}{\partial x^i}$$

Example 8.3.2. For a smooth function f, then

$$\begin{aligned}\Delta_g f&=(\mathrm{d}\mathrm{d}^*+\mathrm{d}^*\mathrm{d})f\\&=\mathrm{d}^*\mathrm{d}f\\&=\mathrm{d}^*(\frac{\partial f}{\partial x^i}\mathrm{d}x^i)\\&=-\sum_{i=1}^n\frac{\partial^2 f}{\partial x^i\partial x^i}\end{aligned}$$

So as you can see, Laplace-Beltrami operator differs a sign with scalar Laplacian.

Example 8.3.3. For a n-form ω written as $f\,\mathrm{vol}$, where f is a smooth function, then

$$\begin{aligned}\mathrm{d}^*\omega&=(-1)^n\star\mathrm{d}\star(f\,\mathrm{vol})\\&=(-1)^n\star\mathrm{d}f\\&=(-1)^n\star(\frac{\partial f}{\partial x^i}\mathrm{d}x^i)\\&=\sum_{i=1}^n(-1)^{n+i-1}\frac{\partial f}{\partial x^i}\mathrm{d}x^1\wedge\cdots\wedge\widehat{\mathrm{d}x^i}\wedge\cdots\wedge\mathrm{d}x^n\end{aligned}$$

8.4. **Divergence.**

Definition 8.4.1 (divergence)**.** For any vector field X, its divergence $\operatorname{div} X$ is defined as $\operatorname{tr}\nabla X$.

Remark 8.4.1 (local form). If we locally write X as $X^i\frac{\partial}{\partial x^i}$, then

$$\nabla X = \nabla_i X^j \mathrm{d}x^i \otimes \frac{\partial}{\partial x^j}$$

Then

$$\mathrm{div}\, X = \nabla_i X^i$$

Proposition 8.4.1.

$$\mathrm{div}\, X \,\mathrm{vol} = \mathcal{L}_X \,\mathrm{vol}$$

Proof. Cartan's magic formula shows that

$$\mathcal{L}_X = \iota_X \circ \mathrm{d} + \mathrm{d} \circ \iota_X$$

So

$$\begin{aligned}
\mathcal{L}_X \,\mathrm{vol} &= (\iota_X \circ \mathrm{d} + \mathrm{d} \circ \iota_X)\,\mathrm{vol} \\
&= \mathrm{d} \circ \iota_X \,\mathrm{vol} \\
&= \mathrm{d}((-1)^{i-1} X^i \sqrt{\det g}\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n) \\
&= \frac{1}{\sqrt{\det g}} \frac{\partial (X^i \sqrt{\det g})}{\partial x^i} \,\mathrm{vol} \\
&= \frac{1}{\sqrt{\det g}} (\frac{\partial X^i}{\partial x^i} \sqrt{\det g} + X^i \frac{\partial \sqrt{\det g}}{\partial x^i}) \,\mathrm{vol} \\
&= (\frac{\partial X^i}{\partial x^i} + X^i \frac{\partial \log \sqrt{\det g}}{\partial x^i}) \,\mathrm{vol} \\
&= (\frac{\partial X^i}{\partial x^i} + \frac{1}{2} X^i \frac{\partial \log \det g}{\partial x^i}) \,\mathrm{vol}
\end{aligned}$$

Note that Jacobi's formula says

$$\frac{\partial \log \det g}{\partial x^i} = \frac{1}{\det g} \frac{\partial \det g}{\partial x^i} = g^{jk} \frac{\partial g_{jk}}{\partial x^i} = g^{jk}(\Gamma^l_{ij} g_{lk} + \Gamma^l_{ik} g_{jl}) = 2\Gamma^j_{ij}$$

Thus

$$\begin{aligned}
\mathcal{L}_X \,\mathrm{vol} &= (\frac{\partial X^i}{\partial x^i} + \frac{1}{2} X^i \frac{\partial \log \det g}{\partial x^i}) \,\mathrm{vol} \\
&= (\frac{\partial X^i}{\partial x^i} + \Gamma^j_{ij} X^i) \,\mathrm{vol} \\
&= (\frac{\partial X^i}{\partial x^i} + \Gamma^i_{ij} X^j) \,\mathrm{vol} \\
&= \nabla_i X^i \,\mathrm{vol}
\end{aligned}$$

□

Remark 8.4.2. From the proof, we can say there is the following formula for divergence of a vector field X written as $X^i \frac{\partial}{\partial x^i}$, one has

$$\mathrm{div}\, X = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} (\sqrt{\det g} X^i)$$

Corollary 8.4.1 (divergence theorem)**.**

$$\int_M \operatorname{div} X \operatorname{vol} = 0$$

Proposition 8.1. Let f be a smooth function on M, then for any smooth function φ, one has

$$-\int_M \langle \nabla\varphi, \nabla f\rangle \operatorname{vol} = \int_M \Delta\varphi \cdot f \operatorname{vol}$$

Proof. Direct computation shows

$$\begin{aligned}
\operatorname{div}(f\nabla\varphi) &= \nabla_k (f\nabla\varphi)^k \\
&= \frac{\partial (f\nabla\varphi)^k}{\partial x^k} + \Gamma^k_{ks}(f\nabla\varphi)^s \\
&= \frac{\partial (f g^{ik}\frac{\partial\varphi}{\partial x^i})}{\partial x^k} + \Gamma^k_{ks} f g^{is}\frac{\partial\varphi}{\partial x^i} \\
&= \underbrace{g^{ik}\frac{\partial f}{\partial x^k}\frac{\partial\varphi}{\partial x^i}}_{\text{part I}} + \underbrace{f(\frac{\partial g^{ik}}{\partial x^k}\frac{\partial\varphi}{\partial x^i} + g^{ik}\frac{\partial^2\varphi}{\partial x^k\partial x^i} + g^{is}\Gamma^k_{ks}\frac{\partial\varphi}{\partial x^i})}_{\text{part II}}
\end{aligned}$$

We have the following observations:

1. Part I equals

$$\begin{aligned}
g^{ik}\frac{\partial f}{\partial x^k}\frac{\partial\varphi}{\partial x^i} &= g_{lj} g^{lk}\frac{\partial f}{\partial x^k} g^{ji}\frac{\partial\varphi}{\partial x^i} \\
&= \langle g^{lk}\frac{\partial f}{\partial x^k}\frac{\partial}{\partial x^l}, g^{ji}\frac{\partial\varphi}{\partial x^i}\frac{\partial}{\partial x^j}\rangle \\
&= \langle \nabla f, \nabla\varphi\rangle
\end{aligned}$$

2. Note

$$\begin{aligned}
\frac{\partial g^{ik}}{\partial x^k} + g^{is}\Gamma^k_{ks}\frac{\partial\varphi}{\partial x^i} &= -g^{is}g^{kt}\frac{\partial g_{st}}{\partial x^k} + \frac{1}{2}g^{is}g^{kt}(\frac{\partial g_{kt}}{\partial x^s} + \frac{\partial g_{st}}{\partial x^k} - \frac{\partial g_{ks}}{\partial x^t}) \\
&= -\frac{1}{2}g^{is}g^{kt}(\frac{\partial g_{ks}}{\partial x^t} + \frac{\partial g_{st}}{\partial x^k} - \frac{\partial g_{kt}}{\partial x^s}) \\
&= -g^{kt}\Gamma^i_{kt}
\end{aligned}$$

   where $\frac{\partial g^{ik}}{\partial x^k} = -g^{is}g^{kt}\frac{\partial g_{st}}{\partial x^k}$ holds from the fact $g^{ik}g_{kt} = \delta^i_t$, then take partial derivative with respect to x^k to conclude.

3. From (2) and local expression of Δ, it's clear part II equals $f\Delta\varphi$.

Thus we have

$$\operatorname{div}(f\nabla\varphi) = \langle\nabla\varphi, \nabla f\rangle + f\Delta\varphi$$

Then divergence theorem completes the proof. □

Proposition 8.4.2. Let ω be a 1-form, then

$$\mathrm{d}^*\omega = -\operatorname{div}(\omega^\sharp)$$

Proof. It suffices to check with respect to normal coordinate:

1. Remark 8.4.2 or direct computation shows

$$\operatorname{div}\omega^\sharp = \sum_{i=1}^n \frac{\partial \omega_i}{\partial x^i}$$

2. Example 8.3.1 implies

$$\mathrm{d}^*\omega = -\sum_{i=1}^n \frac{\partial \omega_i}{\partial x^i}$$

This completes the proof. $\square$

8.5. **Conformal Laplacian.** For a smooth function u, according to Proposition 8.4.2 and Remark 8.4.2, we can write $\Delta_g u$ as follows

$$\begin{aligned}\Delta_g u &= \mathrm{d}^*\mathrm{d}u \\ &= -\operatorname{div}(g^{ij}\frac{\partial u}{\partial x^i}\frac{\partial}{\partial x^j}) \\ &= -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^j}(\sqrt{\det g}g^{ij}\frac{\partial u}{\partial x^i})\end{aligned}$$

Thus Laplace-Beltrami Δ_g with respect to g is

$$\Delta_g = -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^j}(\sqrt{\det g}g^{ij}\frac{\partial}{\partial x^i})$$

So if we consider conformal transformation $\widetilde{g} = e^{2f}g$ for some smooth function f, we have

$$\begin{aligned}\widetilde{g}_{ij} &= e^{2f}g_{ij} \\ \widetilde{g}^{ij} &= e^{-2f}g^{ij} \\ \det\widetilde{g} &= e^{2nf}\det g \\ \sqrt{\widetilde{g}} &= e^{nf}\sqrt{\det g}\end{aligned}$$

Thus

$$\begin{aligned}\Delta_{\widetilde{g}} &= -\frac{1}{e^{nf}\sqrt{\det g}}\frac{\partial}{\partial x^j}(e^{nf}\sqrt{\det g}e^{-2f}g^{ij}\frac{\partial}{\partial x^i}) \\ &= -\frac{e^{-nf}}{\sqrt{\det g}}\frac{\partial}{\partial x^j}(e^{(n-2)f}\sqrt{\det g}g^{ij}\frac{\partial}{\partial x^i}) \\ &= -\frac{e^{-2f}}{\sqrt{\det g}}\frac{\partial}{\partial x^j}(\sqrt{\det g}g^{ij}\frac{\partial}{\partial x^i}) - \frac{(n-2)e^{-2f}}{\sqrt{\det g}}\frac{\partial f}{\partial x^j}\sqrt{\det g}g^{ij}\frac{\partial}{\partial x^i} \\ &= -e^{-2f}\Delta_g - (n-2)e^{-2f}g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^i}\end{aligned}$$

So we have

$$\Delta_{\widetilde{g}} = -e^{-2f}\Delta_g$$

when $n = 2$. It's a kind of conformal invariance. However this fails in higher dimension. Let's consider the following so-called conformal Laplacian when $n > 3$

$$\begin{aligned} L : C^\infty(M) &\to C^\infty(M) \\ u &\mapsto -\frac{4(n-1)}{n-2}\Delta_g u + Su \end{aligned}$$

where S is scalar curvature. Let's show

$$\widetilde{L}u = e^{-\frac{n+2}{2}f} L(e^{\frac{n-2}{2}f}u)$$

where $\widetilde{L}$ is the conformal Laplacian after conformal transformation. Divide computations into several parts:

(1)

$$\begin{aligned} \nabla^2(e^{\frac{n-2}{2}f}u) =& \nabla(\frac{n-2}{2}e^{\frac{n-2}{2}f}\frac{\partial f}{\partial x^i}u\mathrm{d}x^i + e^{\frac{n-2}{2}f}\frac{\partial u}{\partial x^i}\mathrm{d}x^i) \\ =& e^{\frac{n-2}{2}f}\nabla^2 u + \frac{n-2}{2}e^{\frac{n-2}{2}f}\frac{\partial f}{\partial x^j}\frac{\partial u}{\partial x^i}\mathrm{d}x^i \otimes \mathrm{d}x^j \\ &+ (\frac{(n-2)^2}{4}e^{\frac{n-2}{2}f}u\frac{\partial f}{\partial x^j}\frac{\partial f}{\partial x^i} + \frac{n-2}{2}e^{\frac{n-2}{2}f}\frac{\partial f}{\partial x^i}\frac{\partial u}{\partial x^j})\mathrm{d}x^i \otimes \mathrm{d}x^j + \frac{n-2}{2}e^{\frac{n-2}{2}f}u\nabla^2 f \end{aligned}$$

(2)

$$\begin{aligned} \Delta_g(e^{\frac{n-2}{2}f}u) =& \operatorname{tr}_g \nabla^2(e^{\frac{n-2}{2}f}u) \\ =& e^{\frac{n-2}{2}f}\Delta_g u + \frac{n-2}{2}e^{\frac{n-2}{2}f}g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial u}{\partial x^i} \\ &+ g^{ij}(\frac{(n-2)^2}{4}e^{\frac{n-2}{2}f}u\frac{\partial f}{\partial x^j}\frac{\partial f}{\partial x^i} + \frac{n-2}{2}e^{\frac{n-2}{2}f}\frac{\partial f}{\partial x^i}\frac{\partial u}{\partial x^j}) + \frac{n-2}{2}e^{\frac{n-2}{2}f}u\Delta_g f \end{aligned}$$

(3)

$$\begin{aligned} e^{-\frac{n+2}{2}f}L(e^{\frac{n-2}{2}f}u) =& -\frac{4(n-1)}{n-2}e^{-2f}\Delta_g u - 4(n-1)e^{-2f}g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial u}{\partial x^i} \\ &- g^{ij}(n-2)(n-1)e^{-2f}u\frac{\partial f}{\partial x^j}\frac{\partial f}{\partial x^i} - 2(n-1)e^{-2f}u\Delta_g f + e^{-2f}Su \\ =& -\frac{4(n-1)}{n-2}e^{-2f}\Delta_g u - 4(n-1)e^{-2f}g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial u}{\partial x^i} \\ &- (n-2)(n-1)e^{-2f}u|\mathrm{d}f|^2 - 2(n-1)e^{-2f}u\Delta_g f + e^{-2f}Su \end{aligned}$$

(4)

$$-\frac{4(n-1)}{n-2}\Delta_{\widetilde{g}}u = -\frac{4(n-1)}{n-2}e^{-2f}\Delta_g u - 4(n-1)e^{-2f}g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial u}{\partial x^i}$$

(5) Note that

$$\widetilde{S} = e^{-2f}S - 2(n-1)e^{-2f}\Delta_g f - (n-2)(n-1)e^{-2f}|\mathrm{d}f|^2$$

This completes the computation. In particular, in (2) if we take $u=1$ we have

$$-\frac{4(n-1)}{n-2}\Delta_g(e^{\frac{n-2}{2}f}) = -(n-2)(n-1)e^{\frac{n-2}{2}f}|\mathrm{d}f|^2 - 2(n-1)e^{\frac{n-2}{2}f}\Delta_g f$$

Thus we have

$$\widetilde{S} = e^{-\frac{n+2}{2}f}(-\frac{4(n-1)}{n-2}\Delta_g e^{\frac{n-2}{2}f} + Se^{\frac{n-2}{2}f}) = e^{-\frac{n+2}{2}f}L(e^{\frac{n-2}{2}f})$$

So if we put $e^{2f} = \varphi^{\frac{4}{n-2}}$, we have

$$\widetilde{S} = \varphi^{-\frac{n+2}{n-2}}L\varphi$$

So it's clear g is conformal to $\widetilde{g}$ with constant scalar curvature λ if and only if φ is a smooth positive solution to the Yamabe equation

$$L\varphi = \lambda\varphi^{\frac{n+2}{n-2}}$$

8.6. **Hodge theorem and corollaries.**

Theorem 8.6.1 (Hodge theorem)**.** Consider the Laplace operator $\Delta_g : C^\infty(M,\Omega_M^k)\to C^\infty(M,\Omega_M^k)$, then

1. $\dim_{\mathbb{R}}\mathcal{H}^k(M)<\infty$.
2. There is an orthogonal decomposition

$$C^\infty(M,\Omega_M^k) = \mathcal{H}^k(M)\perp \operatorname{im}\Delta_g$$

Proof. See Appendix C. □

Corollary 8.6.1. More explicitly, we have the following orthogonal decomposition

$$C^\infty(M,\Omega_M^k) = \mathcal{H}^k(M)\oplus \mathrm{d}(C^\infty(M,\Omega_M^{k-1}))\oplus \mathrm{d}^*(C^\infty(M,\Omega_M^{k+1}))$$

Proof. It suffices to check $\mathrm{d}(C^\infty(M,\Omega_M^{k-1}))$ is orthogonal to $\mathrm{d}^*(C^\infty(M,\Omega_M^{k+1}))$. Take $\mathrm{d}\alpha$ and $\mathrm{d}^*\beta$, where α is a $k-1$-form and β is a $k+1$-form. Then

$$(\mathrm{d}\alpha,\mathrm{d}^*\beta) = (\mathrm{d}^2\alpha,\beta) = 0$$

□

Corollary 8.6.2.

$$\ker\mathrm{d} = \mathcal{H}^k(M)\oplus\mathrm{d}(C^\infty(M,\Omega_M^{k-1}))$$
$$\ker\mathrm{d}^* = \mathcal{H}^k(M)\oplus\mathrm{d}^*(C^\infty(M,\Omega_M^{k+1}))$$

Proof. Clear from above corollary. □

Corollary 8.6.3. The natural map $\mathcal{H}^k(M)\to H^k(M.\mathbb{R})$ is an isomorphism. In other words, every element in $H^k(M.\mathbb{R})$ is represented by a unique harmonic form.

Proof. Clear from above corollary. □

Corollary 8.6.4. $\star:\mathcal{H}^k(M)\to\mathcal{H}^{n-k}(M)$ is an isomorphism.

Proof. It suffices to show $*$ maps harmonic forms to harmonic forms, since we already have $\star$ maps k-forms to k-forms By Lemma 8.1.2, we just need to show $\mathrm{d}\star\alpha = \mathrm{d}^*\star\alpha = 0$ for a harmonic form α. Directly compute as follows

$$\mathrm{d}\star\alpha = (-1)^{\bullet_1}\star\star\mathrm{d}\star\alpha = (-1)^{\bullet_2}\star\mathrm{d}^\star\alpha = 0$$
$$\mathrm{d}^*\star\alpha = (-1)^{\bullet_3}\star\mathrm{d}\star\star\alpha = (-1)^{\bullet_4}\star\mathrm{d}\alpha = 0$$

Here we use $\bullet, \bullet'$ to denote the power of (-1), since it's not neccessary for us to know what exactly it is. □

Remark 8.6.1. In fact, above corollary follows from the following identity

$$\Delta_g\star = \star\Delta_g$$

which can be directly checked. In other words, Hodge star commutes with Laplacian Δ. Here gives a method of computation: From what we have done in the proof, we will see

$$\star\mathrm{d}^*\mathrm{d} = (-1)^{\bullet_2}\mathrm{d}\star\mathrm{d} = (-1)^{\bullet_2+\bullet_4}\mathrm{d}\mathrm{d}^*\star$$
$$\star\mathrm{d}\mathrm{d}^* = (-1)^{\bullet_4}\mathrm{d}^*\star\mathrm{d}^* = (-1)^{\bullet_2+\bullet_4}\mathrm{d}^*\mathrm{d}\star$$

So all we need to do is to figure out the precise number of $\bullet_2, \bullet_4$ and show that $\bullet_2 + \bullet_4$ is even.

Corollary 8.6.5 (Poincaré duality)**.** $H^k(M.\mathbb{R}) \cong H^{n-k}(M.\mathbb{R})$.

Proof. Clear from Corollary 8.6.3 and Corollary 8.6.4. □

9. Bochner's technique

9.1. **Bochner formula.** Let (M,g) be a Riemannian manifold with Levi-Civita connection ∇. Recall in Example 2.3.6 and Remark 2.6.1 we have already seen Hessian of a smooth function f and its scalar Laplacian locally as follows:

$$\begin{aligned}\operatorname{Hess} f &= \nabla^2_{i,j} f \mathrm{d}x^i \otimes \mathrm{d}x^j \\ \Delta f &= g^{ij}\nabla^2_{i,j} f\end{aligned}$$

where

$$\nabla^2_{i,j} f = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}$$

We also see in Example 8.3.2 that scalar Laplacian differs a sign with Laplace-Beltrami operator.

Remark 9.1.1. Unless otherwise specified, we use Δ to denote scalar Laplacian and Δ_g to denote Laplace-Beltrami operator.

Theorem 9.1.1. Let $f\colon (M,g)\to\mathbb{R}$ be a smooth function. Then

1. $p\in M$ is a local minimum(maximum), then $\nabla f(p)=0$.
2. $p\in M$ is a local minimum, then

$$\begin{cases}\operatorname{Hess} f(p)\geq 0\\ \Delta f(p)\geq 0\end{cases}$$

3. $p\in M$ is a local maximum, then

$$\begin{cases}\operatorname{Hess} f(p)\leq 0\\ \Delta f(p)\leq 0\end{cases}$$

Proposition 9.1.1 (Bochner formula)**.** Let $f\colon (M,g)\to\mathbb{R}$ be a smooth function, then

$$\frac{1}{2}\Delta|\nabla f|^2 = |\operatorname{Hess} f|^2 + \operatorname{Ric}(\nabla f,\nabla f) + g(\nabla\Delta f,\nabla f)$$

Proof. For a tensor, its norm is independent of which type it is, so we write $\nabla f = g^{ij}\nabla_i f\frac{\partial}{\partial x^j}$, then

$$\begin{aligned}|\nabla f|^2 &= g(\nabla f,\nabla f)\\ &= g(g^{ij}\nabla_i f\frac{\partial}{\partial x^j}, g^{kl}\nabla_k f\frac{\partial}{\partial x^l})\\ &= g^{ij}g^{kl}g_{jl}\nabla_i f\nabla_k f\\ &= g^{ij}\nabla_i f\nabla_j f\end{aligned}$$

In the following computation we may use normal coordinate, then in this case

$$
\begin{aligned}
\frac{1}{2}\Delta|\nabla f|^2 &\overset{(1)}{=} \frac{1}{2}g^{kl}\nabla_k\nabla_l(g^{ij}\nabla_i f\nabla_j f) \\
&\overset{(2)}{=} \frac{1}{2}g^{kl}g^{ij}\nabla_k\nabla_l(\nabla_i f\nabla_j f) \\
&\overset{(3)}{=} g^{kl}g^{ij}\nabla_l\nabla_i f\cdot\nabla_k\nabla_j f + g^{kl}g^{ij}\nabla_k\nabla_l\nabla_i f\cdot\nabla_j f \\
&= |\operatorname{Hess} f|^2 + g^{kl}g^{ij}\nabla_k\nabla_l\nabla_i f\cdot\nabla_j f
\end{aligned}
$$

where

(1) holds from in normal coordinate $\Delta f = g^{ij}\nabla_i\nabla_j f$.
(2) holds from Proposition 2.6.1, that is magic formula.
(3) holds from the following direct computation

$$
\begin{aligned}
\nabla_k\nabla_l(\nabla_i f\nabla_j f) =&\nabla_k(\nabla_l\nabla_i f\cdot\nabla_j f + \nabla_i f\cdot\nabla_l\nabla_j f) \\
=&\nabla_k\nabla_l\nabla_i f\cdot\nabla_j f + \nabla_l\nabla_i f\cdot\nabla_k\nabla_j f \\
&+ \nabla_k\nabla_i f\cdot\nabla_l\nabla_j f + \nabla_i f\cdot\nabla_k\nabla_l\nabla_j f \\
=&2\nabla_l\nabla_i f\cdot\nabla_k\nabla_j f + 2\nabla_k\nabla_l\nabla_i f\cdot\nabla_j f
\end{aligned}
$$

Then the following computation completes the proof:

$$
\begin{aligned}
g^{kl}g^{ij}\nabla_k\nabla_l\nabla_i f\cdot\nabla_j f &\overset{(4)}{=} g^{kl}g^{ij}\nabla_k\nabla_i\nabla_l f\cdot\nabla_j f \\
&\overset{(5)}{=} g^{kl}g^{ij}(\nabla_i\nabla_k\nabla_l f - R^s_{kil}\nabla_s f)\cdot\nabla_j f \\
&= g^{ij}\nabla_i(g^{kl}\nabla_k\nabla_l f)\cdot\nabla_j f + g^{ij}R^s_i\nabla_s f\cdot\nabla_j f \\
&= g^{ij}\nabla_i(\Delta f)\cdot\nabla_j f + \operatorname{Ric}(\nabla f,\nabla f) \\
&= g(\nabla\Delta f,\nabla f) + \operatorname{Ric}(\nabla f,\nabla f)
\end{aligned}
$$

where

(4) holds from symmetry of Hessian.
(5) holds from Theorem 4.5.1, that is Ricci identity.

□

9.2. **Obstruction to the existence of Killing fields.**

Definition 9.2.1 (Killing field)**.** A vector field X on a Riemannian manifold (M,g) is called a Killing field, if $\mathcal{L}_X g = 0$.

Remark 9.2.1. Since vector field can generate local flows, then X is a Killing field if and only if local flows generated by X acts on M as isometries.

Theorem 9.2.1. The followings are equivalent:

1. X is a Killing field.
2. For any two vector fields Y, Z, we have

$$
\langle\nabla_Y X, Z\rangle + \langle Y, \nabla_Z X\rangle = 0
$$

Proof. To see (1) is equivalent to (2). Just note that

$$\begin{aligned}\mathcal{L}_X\langle Y,Z\rangle &= X\langle Y,Z\rangle - \langle \mathcal{L}_X Y, Z\rangle - \langle Y, \mathcal{L}_X Z\rangle \\ &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle - \langle [X,Y], Z\rangle - \langle Y, [X,Z]\rangle \\ &= \langle \nabla_Y X, Z\rangle + \langle Y, \nabla_Z X\rangle\end{aligned}$$

□

Remark 9.2.2. For (2) locally we have

$$g_{kj}\nabla_i X^j = -g_{ij}\nabla_k X^j$$

Thus X is a Killing vector if and only if ∇X is a skew-symmetric $(1,1)$-tensor, that is $\nabla_i X^j$ is skew-symmetric in i, j.

Corollary 9.2.1. If X is a Killing field, then for arbitrary vector field Y we have

$$\langle \nabla_Y X, Y\rangle = 0$$

Proof. Set $Y = Z$ in $\langle \nabla_Y X, Z\rangle + \langle Y, \nabla_Z X\rangle = 0$. □

Corollary 9.2.2. If X is parallel, that is $\nabla X = 0$, then X is Killing.

Proof. A zero matrix must be skew-symmetric. □

Corollary 9.2.3. If X is Killing, then $\operatorname{div} X = \nabla_i X^i = 0$.

Proof. The trace of a skew-symmetric matrix is zero. □

Lemma 9.2.1 (Bochner formula for Killing field)**.** Let X be a Killing field, and $f = \frac{1}{2}|X|^2$. Then

1. $\nabla f = -\nabla_X X$.
2. $\operatorname{Hess} f(Y,Y) = \langle \nabla_Y X, \nabla_Y X\rangle - R(Y,X,X,Y)$ holds for any vector field Y.
3. $\Delta f = |\nabla X|^2 - \operatorname{Ric}(X,X)$.

Proof. For (1). By direct computation we have

$$\begin{aligned}\nabla f &= \langle \nabla X, X\rangle \\ &= \langle \nabla_k X^i \mathrm{d}x^k \otimes \frac{\partial}{\partial x^i}, X^j\frac{\partial}{\partial x^j}\rangle \\ &= g_{ij}X^j\nabla_k X^i \mathrm{d}x^k \\ &\overset{\mathrm{I}}{=} -g_{ik}X^j\nabla_j X^i \mathrm{d}x^k \\ \nabla_X X &= X^j\nabla_j X^i\frac{\partial}{\partial x^i} \\ &= g_{ik}X^j\nabla_j X^i\mathrm{d}x^k\end{aligned}$$

where I holds from skew-symmetry of ∇X.

For (2). By direct computation we have

$$\begin{aligned}
\operatorname{Hess} f(Y,Y) &= \frac{1}{2}Y^iY^j\nabla_i\nabla_j(g_{kl}X^kX^l)\\
&= Y^iY^jg_{kl}(\nabla_iX^k\nabla_jX^l+\nabla_i\nabla_jX^k\cdot X^l)\\
&= \langle\nabla_YX,\nabla_YX\rangle+Y^iY^jg_{kl}\nabla_i\nabla_jX^k\cdot X^l
\end{aligned}$$

and

$$\begin{aligned}
Y^iY^jg_{kl}\nabla_i\nabla_jX^k\cdot X^l &= -Y^iY^jg_{kj}\nabla_i\nabla_lX^k\cdot X^l\\
&\overset{\mathrm{II}}{=} -Y^iY^jg_{kj}X^l(\nabla_l\nabla_iX^k+R_{ilm}^kX^m)\\
&= -Y^iY^jX^lX^mR_{ilmj}\\
&= -R(Y,X,X,Y)
\end{aligned}$$

where (II) holds from $g_{kj}X^l\nabla_l\nabla_iX^k=0$, since this expression is skew symmetric in i,j.

(3) holds from (2) directly. □

Theorem 9.2.2 (Bochner)**.** Let (M,g) be a compact, oriented Riemannian manifold,

1. If $\operatorname{Ric}(g)\leq 0$, then every Killing field is parallel.
2. If $\operatorname{Ric}(g)\leq 0$ and $\operatorname{Ric}(g)<0$ at some point, then there is no non-trivial Killing field.

Proof. For (1). Let X be a Killing field and set $f=\frac{1}{2}|X|^2$, then

$$\begin{aligned}
0 &= \int_M \Delta f\operatorname{vol}\\
&= \int_M (|\nabla X|^2-\operatorname{Ric}(X,X))\operatorname{vol}\\
&\geq \int_M |\nabla X|^2\operatorname{vol}\\
&\geq 0
\end{aligned}$$

Thus $|\nabla X|\equiv 0$, that is X is parallel.

For (2). From proof of (1) one can see if $\operatorname{Ric}(g)\leq 0$ and X is a Killing field, then

$$\int_M \operatorname{Ric}(X,X)=0$$

which implies $\operatorname{Ric}(X,X)\equiv 0$. So if $\operatorname{Ric}(g)<0$ at some point $p\in M$, then $X_p=0$, thus $X\equiv 0$, since it's parallel. □

9.3. **Obstruction to the existence of harmonic 1-forms.** To some extend, Killing field is dual to harmonic 1-form. Let's explain this in more detail.

Lemma 9.3.1. For a harmonic 1-form α, locally written as $\alpha_i \mathrm{d}x^i$, we have

$$\nabla_i \alpha_j = \nabla_j \alpha_i$$
$$g^{ij} \nabla_j \alpha_i = 0$$

Proof. Recall α is harmonic if and only if

$$\mathrm{d}\alpha = 0$$
$$\mathrm{d}^* \alpha = 0$$

It's clear

$$\mathrm{d}(\alpha_j \mathrm{d}x^j) = \nabla_i \alpha_j \mathrm{d}x^i \wedge \mathrm{d}x^j = 0$$

implies $\nabla_i \alpha_j = \nabla_j \alpha_i$. Similarly explicit expression for d^* implies the second identity. □

Remark 9.3.1. Recall Killing field implies $g_{ij} \nabla_k X^j$ is skew-symmetric in i, k, we can see both Killing field and harmonic 1-form implies some (skew)symmetries.

Lemma 9.3.2. If α is a harmonic 1-form, then

$$\frac{1}{2} \Delta |\alpha|^2 = |\nabla \alpha|^2 + \mathrm{Ric}(\alpha^\sharp, \alpha^\sharp)$$

Proof. Routine computation as follows:

$$\begin{aligned}
\frac{1}{2} \Delta |\alpha|_g^2 &= \frac{1}{2} g^{kl} \nabla_k \nabla_l (g^{ij} \alpha_i \alpha_j) \\
&= |\nabla \alpha|^2 + g^{kl} g^{ij} \nabla_k \nabla_l \alpha_i \cdot \alpha_j \\
&= |\nabla \alpha|^2 + g^{kl} g^{ij} \nabla_k \nabla_i \alpha_l \cdot \alpha_j \\
&= |\nabla \alpha|^2 + g^{kl} g^{ij} (\nabla_i \nabla_k \alpha_l - R_{kil}^s \alpha_s) \alpha_j \\
&= |\Delta \alpha|^2 - g^{kl} g^{ij} R_{kil}^s \alpha_s \cdot \alpha_j \\
&= |\Delta \alpha|^2 + \mathrm{Ric}(\alpha^\sharp, \alpha^\sharp)
\end{aligned}$$

□

Theorem 9.3.1 (Bochner)**.** Let (M, g) be a compact, oriented Riemannian manifold,

1. If $\mathrm{Ric}(g) \geq 0$, then every harmonic 1-form is parallel.
2. If $\mathrm{Ric}(g) \geq 0$ and $\mathrm{Ric}(g) > 0$ at some point, then there is no non-trivial harmonic 1-form.

Proof. The same as before. □

Corollary 9.3.1. Let (M, g) be a compact, oriented Riemannian manifold with $\mathrm{Ric}(g) \geq 0$ and $\mathrm{Ric}(g) > 0$ at some point, then $b_1(M) = 0$.

Proof. It's clear from above theorem and Corollary 8.6.3. □

Remark 9.3.2. It's a kind of vanishing theorem. In geometry, "positivity" may cause "vanishing", that's a philosophy.

Corollary 9.3.2. Let (M, g) be a compact Riemannian n-manifold with $\mathrm{Ric}(g) \geq 0$, then $b_1(M) \leq n$. Moreover, if $b_1(M) = n$ if and only if (M, g) is isometric to a flat torus.

Proof. By Corollary 8.6.3 we have $b_1(M) = \dim \mathcal{H}^1(M)$. Now if $\mathrm{Ric}(g) \geq 0$, then any harmonic 1-form is parallel, thus linear map $\mathcal{H}^1(M) \to T_pM$ that evaluates ω at point p is injective. In particular, $\dim \mathcal{H}^1 \leq n$.

If the equality holds, we obviously have n linearly independent parallel fields $E_i, i = 1, \ldots, n$. This shows M is flat. Thus the universal covering of (M, g) is $(\mathbb{R}^n, g_{\mathrm{can}})$ with $\Gamma = \pi_1(M)$ acting by isometries. Now pull E_i back to $\widetilde{E}_i$ to $\mathbb{R}^n$, these vector fields are again parallel and are therefore constant vector field. This means we can see them as usual Cartesian coordinate vector field $\frac{\partial}{\partial x^i}$. In addition, they are invariant under the action of Γ. Thus Γ consists of translations, which implies Γ is finitely generated, abelian and torsion-free, thus $\Gamma = \mathbb{Z}^q$ for some q. We must have $q = n$, otherwise $\mathbb{R}^n / \mathbb{Z}^q$ is not compact. $\square$

Part 4. Minimal length curve problem

10. Pullback connection

10.1. Pullback and pushforward.

Definition 10.1.1 (pullback vector bundle)**.** Let $f\colon M\to N$ be a smooth map between manifolds, E a vector bundle over N. The pullback vector bundle f^*E over M is defined by the set

$$\widehat{E}=f^*E:=\{(p,v)\in M\times E\mid f(p)=\pi(v)\}$$

endowed with subspace topology.

Remark 10.1.1 (local form)*.* A local frame of $\widehat{E}$ can be written as

$$\widehat{e}_\alpha(x):=f^*e_\alpha(x)=e_\alpha\circ f(x)$$

where $x\in M$ and $\{e_\alpha\}$ is a local frame of E.

Let $f\colon M\to N$ be a smooth map between manifolds, and $\mathrm{d}f\colon TM\to TN$ its differential. There is another viewpoint to see it, consider

$$\begin{aligned}\mathrm{d}f\colon TM&\to f^*TN\subset M\times TN\\ X_p&\mapsto (p,\mathrm{d}f_p(X_p))\end{aligned}$$

that is one can regard $\mathrm{d}f$ as a section of $T^*M\otimes f^*TN$.

Definition 10.1.2 (pushforward)**.** For vector field X over M, the pushforward of X is defined as $f_*X=\mathrm{d}f\circ X\in C^\infty(M,f^*TN)$.

Remark 10.1.2*.* In a word, pushforward of a vector field is no longer a vector field, but a section of pullback bundle.

Remark 10.1.3 (local form)*.* Let $\{x^i\}$ and $\{y^m\}$ be local coordinates of M,N respectively, one has

$$f_*(\frac{\partial}{\partial x^i})=\frac{\partial f^m}{\partial x^i}f^*(\frac{\partial}{\partial y^m})$$

10.2. Pullback connection.

Let $f\colon M\to N$ be a smooth map between manifolds, E a vector bundle over N with connection ∇. Now we want to give a connection $\widehat{\nabla}$ on pullback bundle $\widehat{E}$ induced by ∇. Let $\{e_\alpha\}$ be a local frame of E, then $\{\widehat{e}_\alpha:=f^*e_\alpha\}$ is a local frame of $\widehat{E}$. Now we define

$$\widehat{\nabla}(\widehat{e}_\alpha):=f^*(\nabla e_\alpha)$$

and

$$\widehat{\nabla}(f\widehat{s}):=\mathrm{d}f\otimes\widehat{s}+f\widehat{\nabla}\widehat{s}\tag{10.1}$$

where $\widehat{s}=f^*s$, and s is a section of E. Suppose ∇ is given by Christoffel symbol $\Gamma^\beta_{m\alpha}$, then by definition

$$\widehat{\nabla}(\widehat{e}_\alpha)=f^*(\Gamma^\beta_{m\alpha}\mathrm{d}y^m\otimes e_\beta)=\frac{\partial f^m}{\partial x^i}\Gamma^\beta_{m\alpha}(f)\cdot\mathrm{d}x^i\otimes\widehat{e}_\beta\tag{10.2}$$

Then here comes a natural question, we need to check our definition is independent of the choice of local frame, that is to check if $\{\widehat{e}'_\beta\}$ is another local frame with $\widehat{e}_\alpha = g_\alpha^\beta \widehat{e}'_\beta$, then

$$\widehat{\nabla}(\widehat{e}_\alpha) = \widehat{\nabla}(g_\alpha^\beta \widehat{e}'_\beta)$$

It's straightforward computation, since the left hand is computed by (10.2), and right hand can be computed by (10.1). Thus we obtain an linear operator

$$\widehat{\nabla}\colon C^\infty(M,\widehat{E}) \to C^\infty(M, T^*M \otimes \widehat{E})$$

Now it remains to show it's an affine connection, that is to check for any smooth function $f \in C^\infty(M)$ and $s \in C^\infty(M,\widehat{E})$, one has

$$\widehat{\nabla}(fs) = \mathrm{d}f \otimes s + f\widehat{\nabla}s$$

Note that it doesn't follow from (10.1), since here locally if we write $s = s^\alpha \widehat{e}_\alpha$, and by definition we only obtain

$$\widehat{\nabla}(fs) = \widehat{\nabla}(fs^\alpha \widehat{e}_\alpha) = \mathrm{d}(fs^\alpha) \otimes \widehat{e}_\alpha + fs^\alpha \widehat{\nabla}\widehat{e}_\alpha$$

and it's neccessary to do a stepforward computation to show above equation equals to the following one

$$\mathrm{d}f \otimes s^\alpha e_\alpha + f\widehat{\nabla}(s^\alpha e_\alpha)$$

Definition 10.2.1 (pullback metric)**.** Let g be a metric on E, the pullback metric on f^*E is $\widehat{g} = f^*g$.

Remark 10.2.1 (local form). On local frames one has

$$\begin{aligned} \widehat{g}_{\alpha\beta}\widehat{e}^\alpha \otimes \widehat{e}^\beta :=& f^*(g_{\alpha\beta} e^\alpha \otimes e^\beta) \\ =& g_{\alpha\beta}(f) \cdot \widehat{e}^\alpha \otimes \widehat{e}^\beta \end{aligned}$$

that is $\widehat{g}_{\alpha\beta} = g_{\alpha\beta}(f)$.

Proposition 10.2.1. If connection ∇ is compatible with metric g, then pullback connection $\widehat{\nabla}$ is compatible with $\widehat{g}$, that is for any vector field X of M and section s,t of $\widehat{E}$, we have

$$X\widehat{g}(s,t) = \widehat{g}(\widehat{\nabla}_X s, t) + \widehat{g}(s, \widehat{\nabla}_X t)$$

Proof. It suffices to check on local frames, consider $X = \frac{\partial}{\partial x^i}, s = \widehat{e}_\alpha, t = \widehat{e}_\beta$, then

$$\begin{aligned} \frac{\partial}{\partial x^i}\widehat{g}_{\alpha\beta} &= \frac{\partial}{\partial x^i} g_{\alpha\beta}(f) \\ &= \frac{\partial f^m}{\partial x^i}\frac{\partial}{\partial y^m} g_{\alpha\beta}(f) \\ &\overset{(1)}{=} \frac{\partial f^m}{\partial x^i}(\Gamma_{m\alpha}^\gamma(f) \cdot g_{\gamma\beta}(f) + \Gamma_{m\beta}^\gamma(f) \cdot g_{\alpha\gamma}(f)) \end{aligned}$$

$$\widehat{g}(\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{e}_\alpha, \widehat{e}_\beta) = \Gamma^\gamma_{m\alpha}(f)\cdot\frac{\partial f^m}{\partial x^i}\cdot g_{\gamma\beta}(f)$$
$$\widehat{g}(\widehat{e}_\alpha, \widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{e}_\beta) = \Gamma^\gamma_{m\beta}(f)\cdot\frac{\partial f^m}{\partial x^i}\cdot g_{\alpha\gamma}(f)$$

where (1) holds from ∇ is compatible with g. □

10.3. **Pullback curvature.**

Definition 10.3.1 (pullback curvature)**.** Let $f\colon M\to N$ be a smooth map between manifolds, E a vector bundle over N with connection ∇. The curvature tensor $\widehat{R}$ of pullback connection $\widehat{\nabla}$ on vector bundle $\widehat{E}\to M$ is given by

$$\widehat{R}(X,Y,s,t) = \widehat{g}(\widehat{\nabla}_X\widehat{\nabla}_Y s - \widehat{\nabla}_Y\widehat{\nabla}_X s, t)$$

where X,Y are vector fields on M and s,t are sections of $\widehat{E}$.

Remark 10.3.1 (local form)*.*

$$\widehat{R}_{ij\alpha\beta} = R_{mn\alpha\beta}\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j}$$

where $R_{mn\alpha\beta}$ is curvature of ∇.

Proof.

$$\begin{aligned}
\widehat{R}_{ij\alpha\beta} &= \widehat{R}(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j},\widehat{e}_\alpha,\widehat{e}_\beta)\\
&= \widehat{g}(\widehat{R}(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})\widehat{e}_\alpha,\widehat{e}_\beta)\\
&= \widehat{g}(\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{\nabla}_{\frac{\partial}{\partial x^j}}\widehat{e}_\alpha - \widehat{\nabla}_{\frac{\partial}{\partial x^j}}\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{e}_\alpha,\widehat{e}_\beta)
\end{aligned}$$

The first term can be computed as follows

$$\begin{aligned}
\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{\nabla}_{\frac{\partial}{\partial x^j}}\widehat{e}_\alpha &= \widehat{\nabla}_{\frac{\partial}{\partial x^i}}(\Gamma^\gamma_{m\alpha}(f)\cdot\frac{\partial f^m}{\partial x^j}\widehat{e}_\gamma)\\
&= \frac{\partial}{\partial x^i}(\Gamma^\gamma_{m\alpha}(f)\cdot\frac{\partial f^m}{\partial x^j})\widehat{e}_\gamma + \Gamma^\gamma_{m\alpha}(f)\cdot\frac{\partial f^m}{\partial x^j}\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{e}_\gamma\\
&= (\frac{\partial\Gamma^\gamma_{m\alpha}}{\partial y^n}\frac{\partial f^n}{\partial x^i}\frac{\partial f^m}{\partial x^j} + \Gamma^\gamma_{m\alpha}(f)\cdot\frac{\partial^2 f^m}{\partial x^i\partial x^j})\widehat{e}_\gamma + \frac{\partial f^m}{\partial x^j}\frac{\partial f^n}{\partial x^i}\Gamma^\gamma_{m\alpha}\Gamma^\delta_{n\gamma}\widehat{e}_\delta\\
&= \frac{\partial f^m}{\partial x^j}\frac{\partial f^n}{\partial x^i}(\frac{\partial\Gamma^\gamma_{m\alpha}}{\partial y^n} + \Gamma^\delta_{m\alpha}\Gamma^\gamma_{n\delta})\widehat{e}_\gamma + \Gamma^\gamma_{m\alpha}\frac{\partial^2 f^m}{\partial x^i\partial x^j}\widehat{e}_\gamma
\end{aligned}$$

Thus

$$\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{\nabla}_{\frac{\partial}{\partial x^j}}\widehat{e}_\alpha - \widehat{\nabla}_{\frac{\partial}{\partial x^j}}\widehat{\nabla}_{\frac{\partial}{\partial x^i}}\widehat{e}_\alpha = \frac{\partial f^m}{\partial x^j}\frac{\partial f^n}{\partial x^i}R^\gamma_{mn\alpha}\widehat{e}_\gamma$$

that is

$$\begin{aligned}
\widehat{R}_{ij\alpha\beta} &= \widehat{g}(\frac{\partial f^m}{\partial x^j}\frac{\partial f^n}{\partial x^i}R^\gamma_{mn\alpha}\widehat{e}_\gamma,\widehat{e}_\beta)\\
&= \frac{\partial f^m}{\partial x^j}\frac{\partial f^n}{\partial x^i}R^\gamma_{mn\alpha}g_{\gamma\beta}\\
&= \frac{\partial f^m}{\partial x^j}\frac{\partial f^n}{\partial x^i}R_{mn\alpha\beta}
\end{aligned}$$

□

10.4. **Parallel transport.** Let (M,g) be a Riemannian manifold with Levi-Civita connection ∇, $\gamma\colon I\to M$ a smooth curve, E a vector bundle over M and γ^*E endowed with pullback connection $\widehat{\nabla}$.

Definition 10.4.1 (parallel)**.** Let s be a section of γ^*E, it's called parallel along γ, if $\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}s=0$.

From local form we can see $\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}s=0$ is a system of ODEs locally, which can always be solved uniquely in a sufficiently short interval if we given a initial value, that's how we define parallel transport.

Definition 10.4.2 (parallel transport)**.** For $t_0,t\in I$, parallel transport $P^{\gamma}_{t_0,t}$ is an isomorphism[4] between vector spaces defined by

$$\begin{aligned} P^{\gamma}_{t_0,t}: E_{\gamma(t_0)} &\to E_{\gamma(t)} \\ s_0 &\mapsto s(t) \end{aligned}$$

where s is the unique parallel section along γ satisfying $s(t_0)=s_0$.

Definition 10.4.3 (parallel orthonormal frame)**.** Suppose $\{e_\alpha\}$ is an orthonormal basis of $E_{\gamma(t_0)}$, then there is a local frame $\{e_\alpha(t)\}$ of $E_{\gamma(t)}$ along γ obtained by parallel transport, such that $e_\alpha(0)=e_\alpha$.

Remark 10.4.1. In fact, connection and parallel transport are the same things in different viewpoint.

Proposition 10.4.1. A connection ∇ is compatible with metric if and only if for arbitrary curve $\gamma\colon I\to M$ and two parallel sections s_1,s_2 along γ we have $g(s_1,s_2)$ is constant.

Proof. It's clear if ∇ is compatible with metric g, then and two sections s,t are parallel along γ, we have

$$\mathrm{d}g(s_1,s_2)=g(\nabla s_1,s_2)+g(s_1,\nabla s_2)=0$$

which implies $g(s,t)$ is constant. Conversely, suppose $\{e_\alpha(t)\}$ is a parallel orthonormal frame with respect to g along γ and write

$$s_1(t)=s_1^\alpha(t)e_\alpha,\quad s_2(t)=s_2^\alpha(t)e_\alpha(t)$$

Then we have

$$\begin{aligned} g(\nabla s_1,s_2)+g(s_1,\nabla s_2) &= \sum_\alpha \frac{\mathrm{d}s_1^\alpha}{\mathrm{d}t}s_2^\alpha+s_1^\alpha\frac{\mathrm{d}s_2^\alpha}{\mathrm{d}t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\sum_\alpha s_1^\alpha s_2^\alpha) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}g(s_1,s_2) \end{aligned}$$

□

[4]Its inverse is P^{γ}_{t,t_0}.

10.5. **Second fundamental form.** In this section, we fix the following notations:

1. Let $(M,g),(N,g')$ be Riemannian manifolds with Levi-Civita connection ∇ and ∇' respectively.
2. $f\colon M\to N$ is a smooth map between manifolds.
3. Γ_{ij}^k is used to denote Christoffel symbol of ∇ and Γ_{mn}^l is used to denote Christoffel symbol of ∇'.
4. $\widehat{\nabla}$ is the connection on f^*TN induced by ∇'.

Definition 10.5.1 (second fundamental form)**.** The second fundamental form $B\in C^\infty(M,T^*M\otimes T^*M\otimes f^*TN)$ of f is defined as

$$B(X,Y):=\widehat{\nabla}_X(f_*Y)-f_*(\nabla_XY)\in C^\infty(M,f^*TN)$$

where $X,Y\in C^\infty(M,TM)$.

Remark 10.5.1 (local form). Suppose that $X=\frac{\partial}{\partial x^i},Y=\frac{\partial}{\partial x^j}$, then one has

$$f_*(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j})=\Gamma_{ij}^kf_*(\frac{\partial}{\partial x^k})=\Gamma_{ij}^k\frac{\partial f^m}{\partial x^k}f^*(\frac{\partial}{\partial y^m})$$

And

$$\begin{aligned}\widehat{\nabla}_{\frac{\partial}{\partial x^i}}(\frac{\partial f^m}{\partial x^j}f^*(\frac{\partial}{\partial y^m}))&=\frac{\partial^2 f^m}{\partial x^i\partial x^j}f^*(\frac{\partial}{\partial y^m})+\frac{\partial f^m}{\partial x^j}\widehat{\nabla}_{\frac{\partial}{\partial x^i}}f^*(\frac{\partial}{\partial y^m})\\&=(\frac{\partial^2 f^l}{\partial x^i\partial x^j}+\frac{\partial f^n}{\partial x^i}\frac{\partial f^m}{\partial x^j}\Gamma_{nm}^l\circ f)f^*(\frac{\partial}{\partial y^l})\end{aligned}$$

Therefore

$$\begin{aligned}B_{ij}:&=B(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})\\&=(\frac{\partial^2 f^l}{\partial x^i\partial x^j}+\frac{\partial f^n}{\partial x^i}\frac{\partial f^m}{\partial x^j}\Gamma_{mn}^l\circ f-\Gamma_{ij}^k\frac{\partial f^l}{\partial x^k})f^*(\frac{\partial}{\partial y^l})\end{aligned}$$

Proposition 10.5.1. The second fundamental form B is symmetric.

Proof. It's clear from local expression. □

Corollary 10.5.1. For $X,Y\in C^\infty(M,TM)$, one has

$$\widehat{\nabla}_X(f_*Y)-\widehat{\nabla}_Y(f_*X)=f_*(\nabla_XY-\nabla_YX)=f_*([X,Y])$$

Proof. The first equality holds from the symmetric of second fundamental form and the second equality holds since ∇ is torsion-free. □

Remark 10.5.2. This can be viewed as a kind of torsion-free property of pullback connection.

Example 10.5.1 (geodesic)**.** Consider a smooth curve $\gamma\colon I\to M$, it can be regarded as $\gamma\colon(I,g,\nabla)\to(M,g^M,\nabla^M)$, where metric and connection on interval I are standard. Thus second fundamental form in this case is

$$B=(\frac{\mathrm{d}^2\gamma^k}{\mathrm{d}t^2}+\frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\Gamma_{ij}^k\circ\gamma)\mathrm{d}t\otimes\mathrm{d}t\otimes\gamma^*(\frac{\partial}{\partial x^k})$$

since standard metrics on I has vanishing Christoffel symbol. In this viewpoint, a smooth curve is a geodesic, if it has vanishing second fundamental form as smooth maps between Riemannian manifolds.

Example 10.5.2 (Hessian)**.** Consider smooth function f, it can be regarded as $f\colon (M, g^M, \nabla^M) \to (\mathbb{R}, g, \nabla)$, where metric and connection on $\mathbb{R}$ are standard. Thus second fundamental form in this case is

$$B = (\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k})\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes f^*(\frac{\partial}{\partial y})$$

since standard metrics on R has vanishing Christoffel symbol. Recall Hessian of f is

$$\operatorname{Hess} f = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k}\mathrm{d}x^i \otimes \mathrm{d}x^j$$

So second fundamental form generalizes Hessian of smooth function.

Remark 10.5.3. Since Hessian of a smooth function is $\nabla(\nabla f)$, where $\nabla f \in C^\infty(M, T^*M)$. This motivates us to express our second fundamental form B as $\widetilde{\nabla}\mathrm{d}f$, where $\mathrm{d}f \in C^\infty(M, T^*M \otimes f^*TN)$ and $\widetilde{\nabla}$ is the connection on $T^*M \otimes f^*TN$ induced by ∇^M together with pullback connection on f^*TN. Indeed, note that locally we have

$$\mathrm{d}f = \frac{\partial f^m}{\partial x^i}\mathrm{d}x^i \otimes \frac{\partial}{\partial y^m}$$

Then

$$\begin{aligned}
\widetilde{\nabla}\mathrm{d}f &= \widetilde{\nabla}(\frac{\partial f^m}{\partial x^i}\mathrm{d}x^i \otimes \frac{\partial}{\partial y^m}) \\
&= \frac{\partial^2 f^m}{\partial x^j \partial x^i}\mathrm{d}x^j \otimes \mathrm{d}x^i \otimes \frac{\partial}{\partial y^m} - \frac{\partial f^m}{\partial x^i}\Gamma_{jk}^i \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \frac{\partial}{\partial y^m} + \frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j}\Gamma_{mn}^l \mathrm{d}x^i \otimes \mathrm{d}x^j \frac{\partial}{\partial y^l} \\
&= (\frac{\partial^2 f^l}{\partial x^i \partial x^j} - \frac{\partial f^l}{\partial x^k}\Gamma_{ij}^k + \frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j}\Gamma_{mn}^l)\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \frac{\partial}{\partial y^l} \\
&= B
\end{aligned}$$

as desired.

11. Variation formulas

In this section and Section 12, we fix the following notations:

1. $I = [a, b] \subset \mathbb{R}$ is a closed interval.
2. (M, g) is a Riemannian manifold equipped with Levi-Civita connection ∇.
3. For two different points $p, q \in M$, the space of piecewise smooth curves from p to q is denoted as $\mathcal{L}_{p,q}$.
4. For $\gamma \in \mathcal{L}_{p,q}$, $\gamma'(t)$ denotes $\gamma_*(\frac{\mathrm{d}}{\mathrm{d}t})$, which is a piecewise smooth vector field along γ.
5. The arc-length functional and energy functional on $\mathcal{L}_{p,q}$ are defined as follows
$$L(\gamma) = \int_a^b |\gamma'(t)|_{\widehat{g}} \mathrm{d}t$$
$$E(\gamma) = \frac{1}{2} \int_a^b |\gamma'(t)|^2_{\widehat{g}} \mathrm{d}t$$
where $\widehat{g}$ is pullback metric on $\gamma^* TM$.

11.1. **First variation formula.**

Definition 11.1.1 (variation)**.** Given $\gamma \in \mathcal{L}_{p,q}$, a variation (fixing endpoints) of γ is a map
$$\alpha \colon [a, b] \times (-\varepsilon, \varepsilon) \to M$$
such that

1. $\alpha(\text{-}, s) \in \mathcal{L}_{p,q}$ for any $s \in (-\varepsilon, \varepsilon)$.
2. There is a subdivision $a = t_0 < t_1 < \cdots < t_k = b$ of I such that α is smooth on each strip $(t_{i-1}, t_i] \times (-\varepsilon, \varepsilon)$ for $i = 1, \ldots, k$.
3. $\alpha(t, 0) = \gamma(t)$ for any $t \in [a, b]$.

Remark 11.1.1. In general, we can consider variations of γ without fixing endpoints, but in this section we only consider variations fixing endpoints.

Notation 11.1.1. For pullback bundle $\alpha^* TM$, $\overline{\nabla}$ and $\overline{g}$ denote connection and metric pulled back from the ones on TM respectively. By definition the restriction of $\overline{\nabla}$ on $\gamma^* TM$ is exactly $\widehat{\nabla}$, and the restriction of $\overline{g}$ on $\gamma^* TM$ is $\widehat{g}$.

Definition 11.1.2 (variation vector field)**.** For a variation α of $\gamma \in \mathcal{L}_{p,q}$, $\alpha_*(\frac{\partial}{\partial s})\big|_{s=0}$ is called variation vector field of variation α, which is a piecewise smooth vector field along γ.

Remark 11.1.2. Note that for a variation
$$\begin{cases} \alpha(a, s) = p \\ \alpha(b, s) = q \end{cases}$$

holds for any $s \in (-\varepsilon, \varepsilon)$. Thus we have

$$\begin{cases} \alpha_*(\frac{\partial}{\partial s})(a,s) = 0 \\ \alpha_*(\frac{\partial}{\partial s})(b,s) = 0 \end{cases}$$

holds for any $s \in (-\varepsilon, \varepsilon)$. In particular it holds for $s = 0$. In other words, variation vector field vanishes at endpoints.

Lemma 11.1.1. Let $\gamma \in \mathcal{L}_{p,q}$ and X a piecewise smooth vector field along γ which vanishes at endpoints. Then there exists a variation α of γ such that the variation vector field is exactly X, that is

$$\alpha_*(\frac{\partial}{\partial s})\Big|_{s=0} = X$$

Proof. See Proposition 2.2 in Page193 of [Car92]. □

Theorem 11.1.1 (first variation formula of smooth version)**.** Let $\gamma \colon I \to (M, g)$ be a unit-speed smooth curve, α a variation of γ with variation vector fields V. Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} L(\alpha(\text{-},s)) \overset{(1)}{=} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} E(\alpha(\text{-},s)) &\overset{(2)}{=} \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma'(t)\rangle \mathrm{d}t \\ &\overset{(3)}{=} -\int_a^b \langle V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma'(t)\rangle \mathrm{d}t \end{aligned}$$

Proof. Note that

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} L(\alpha(\text{-},s)) = \int_a^b \frac{1}{2|\gamma'(t)|} \frac{\partial}{\partial s}\Big|_{s=0} |\alpha_*(\frac{\partial}{\partial t})|^2 \mathrm{d}t = \frac{1}{|\gamma'(t)|} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} E(\alpha(\text{-},s))$$

since γ is unit-speed, This show equation marked by (1). Since $\gamma(t)$ is smooth, integration by parts shows

$$0 = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\langle V, \gamma'(t)\rangle \mathrm{d}t = \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma'(t)\rangle \mathrm{d}t + \int_a^b \langle V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma'(t)\rangle \mathrm{d}t$$

This shows equation marked by (3). For equation marked by (2), direct computation shows

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} E(\alpha(\text{-},s)) &= \frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{2}\int_a^b |\alpha_*(\frac{\partial}{\partial t})|^2 \mathrm{d}t \\ &= \frac{1}{2}\int_a^b \frac{\partial}{\partial s}|\alpha_*(\frac{\partial}{\partial t})|^2 \mathrm{d}t \\ &= \frac{1}{2}\int_a^b 2\langle \overline{\nabla}_{\frac{\partial}{\partial s}} \alpha_*(\frac{\partial}{\partial t}), \alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}} \mathrm{d}t \\ &\overset{(4)}{=} \int_a^b \langle \overline{\nabla}_{\frac{\partial}{\partial t}} \alpha_*(\frac{\partial}{\partial s}), \alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}} \mathrm{d}t \end{aligned}$$

The hallmark of above computation is the equality marked by (4), which can be seen from Corollary 10.5.1. Thus

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} E(\alpha(\text{-}, s)) = \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma'(t)\rangle \mathrm{d}t$$

since $\alpha_*(\frac{\partial}{\partial s})\big|_{s=0} = V$ and $\alpha_*(\frac{\partial}{\partial t})\big|_{s=0} = \gamma'(t)$. □

Corollary 11.1.1 (first variation formula of piecewise smooth version)**.** Let $\gamma\colon I \to (M, g)$ be a unit-speed piecewise smooth curve with breakpoints $a = t_0 < t_1 < \cdots < t_k = b$, α a variation of γ with the variation vector field V. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} L(\alpha(\text{-}, s)) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} E(\alpha(\text{-}, s)) &= \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma'(t)\rangle \mathrm{d}t \\ &= -\int_a^b \langle V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma'(t)\rangle \mathrm{d}t - \sum_{i=1}^{k-1} \langle V_{t_i}, \Delta_{t_i}\gamma'\rangle\end{aligned}$$

where $\Delta_t\gamma' = \gamma'(t_+) - \gamma'(t_-)$.

Proof. Note that $\gamma(t)$ is smooth on each $(t_{i-1}, t_i]$, then by Theorem 11.1.1 one has

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} L(\alpha(\text{-}, s)) = \int_{t_{i-1}}^{t_i} \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma'(t)\rangle \mathrm{d}t$$

and integration by parts shows

$$\langle V, \gamma'(t)\rangle\big|_{t_{i-1}}^{t_i} = \int_{t_{i-1}}^{t_i} \frac{\mathrm{d}}{\mathrm{d}t}\langle V, \gamma'(t)\rangle \mathrm{d}t = \int_{t_{i-1}}^{t_i} \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V, \gamma'(t)\rangle \mathrm{d}t + \int_{t_{i-1}}^{t_i} \langle V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma'(t)\rangle \mathrm{d}t$$

Then we add these equations together to obtain desired equation. □

Remark 11.1.3. Suppose α is a variation of γ without fixing endpoints with variation vector field V, from the proof of equality marked by (3), it's clear to see its first variation formula is

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{s=0} L(\alpha(\text{-}, s)) = \langle V(b), \gamma'(b)\rangle - \langle V(a), \gamma'(a)\rangle - \int_a^b \langle V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma'(t)\rangle \mathrm{d}t - \sum_{i=1}^{k-1} \langle V_{t_i}, \Delta_{t_i}\gamma'\rangle$$

Corollary 11.1.2. Given $\gamma \in \mathcal{L}_{p,q}$. The followings are equivalent:

1. γ is a critical point of energy functional $E\colon \mathcal{L}_{p,q} \to \mathbb{R}$.
2. γ has constant speed $|\gamma'(t)| = c > 0$ and γ is a critical point of arc-length functional $L\colon \mathcal{L}_{p,q} \to \mathbb{R}$.
3. γ is a geodesic. In particular, it's smooth.

Proof. From (3) to (2): Firstly a geodesic must have constant speed c, and $c > 0$ since p, q are distinct points. It's also a critical point of L since first variation formula implies

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} L(\alpha(\text{-}, s)) = -\int_a^b \langle V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma'(t)\rangle \mathrm{d}t = 0$$

Note that there is only one term in first variation formula, since geodesic is a smooth curve.

From (2) to (1): It's clear, since from above proof we have already seen for constant speed curve, the first variation of arc-length functional and energy functional only differs a scalar.

From (1) to (3): In order to show $\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\gamma'(t) = 0$ and it's smooth, it suffices to choose appropriate variation vector fields to conclude. □

11.2. **Second variation formula.** We already know a geodesic γ is a critical point for energy functional or arc-length functional, so it suffices to compute second variation of geodesics to determine whether it's local minimum or not. To see this, we need to consider the following 2-dimensional variation

$$\alpha\colon [a,b]\times(-\varepsilon_1,\varepsilon_1)\times(-\varepsilon_2,\varepsilon_2)$$

such that

1. $\alpha(t,0,0)=\gamma(t)$
2. $\alpha(\text{-},s_1,s_2)$ is a smooth curve connecting p and q.

11.2.1. *Second variation formula for energy.*

Theorem 11.2.1 (second variation formula for energy)**.** Let $\gamma\colon[a,b]\to(M,g)$ be a smooth curve. If α is a 2-dimensional variation of γ with variation fields V,W. Then

$$\begin{aligned}\frac{\partial^2}{\partial s_1\partial s_2}\Big|_{s_1=s_2=0} E(\alpha(\text{-},s_1,s_2)) &= \int_a^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}W\rangle \mathrm{d}t\\ &- \int_a^b R(V,\gamma',\gamma',W)\mathrm{d}t - \int_a^b \langle \overline{\nabla}_{\frac{\partial}{\partial s_1}}\ \alpha_*(\frac{\partial}{\partial s_2})\Big|_{s_1=s_2=0}, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\gamma'(t)\rangle\mathrm{d}t\end{aligned}$$

Proof. By first variation formula we have

$$\frac{\partial}{\partial s_2}E(\alpha(\text{-},s_1,s_2)) = -\int_a^b \langle \alpha_*(\frac{\partial}{\partial s_2}), \overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t$$

Thus

$$\frac{\partial^2}{\partial s_1\partial s_2}E(\alpha(\text{-},s_1,s_2)) = \underbrace{-\int_a^b \langle \overline{\nabla}_{\frac{\partial}{\partial s_1}}\alpha_*(\frac{\partial}{\partial s_2}), \overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t}_{\text{part I}} \underbrace{-\int_a^b \langle \alpha_*(\frac{\partial}{\partial s_2}), \overline{\nabla}_{\frac{\partial}{\partial s_1}}\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t}_{\text{part II}}$$

For part II, we have

$$\overline{\nabla}_{\frac{\partial}{\partial s_1}}\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial t}) = R(\alpha_*(\frac{\partial}{\partial s_1}),\alpha_*(\frac{\partial}{\partial t}))\alpha_*(\frac{\partial}{\partial t}) + \overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial s_1}}\alpha_*(\frac{\partial}{\partial t})$$

Thus we can write part II as

$$-\int_a^b \langle \alpha_*(\frac{\partial}{\partial s_2}), R(\frac{\partial}{\partial s_1},\frac{\partial}{\partial t})\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t \underbrace{-\int_a^b \langle \alpha_*(\frac{\partial}{\partial s_2}), \overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial s_1}}\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t}_{\text{part III}}$$

For part III, we have

$$\begin{aligned}-\int_a^b\langle\alpha_*(\frac{\partial}{\partial s_2}),\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial s_1}}\alpha_*(\frac{\partial}{\partial t})\rangle_{\overline{g}}\mathrm{d}t&=-\int_a^b\langle\alpha_*(\frac{\partial}{\partial s_2}),\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial s_1})\rangle_{\overline{g}}\mathrm{d}t\\&=\int_a^b\langle\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial s_2}),\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial s_1})\rangle_{\overline{g}}\mathrm{d}t\end{aligned}$$

Now let's evaluate at $s_1=s_2=0$, then we have

1. Part I

$$-\int_a^b\langle\overline{\nabla}_{\frac{\partial}{\partial s_1}}\ \alpha_*(\frac{\partial}{\partial s_2})\Big|_{s_1=s_2=0},\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\gamma'(t)\rangle\mathrm{d}t$$

2. Part II

$$\int_a^b\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V,\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}W\rangle\mathrm{d}t-\int_a^b R(V,\gamma',\gamma',W)\mathrm{d}t$$

This completes the proof. $\square$

Corollary 11.2.1. Let $\gamma\colon[a,b]\to(M,g)$ be a geodesic, then

$$\frac{\partial^2}{\partial s_1\partial s_2}\Big|_{s_1=s_2=0}E(\alpha(\text{-},s_1,s_2))=\int_a^b\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V,\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}W\rangle\mathrm{d}t-\int_a^b R(V,\gamma',\gamma',W)\mathrm{d}t$$

11.2.2. *Second variation formula for arc-length.*

Theorem 11.2.2 (second variation formula for arc-length)**.** Let $\gamma\colon[a,b]\to(M,g)$ be a unit-speed curve. If α is a 2-dimensional variation of γ with variation fields V,W. Then

$$\begin{aligned}\frac{\partial^2}{\partial s_1\partial s_2}\Big|_{s_1=s_2=0}L(\alpha(\text{-},s_1,s_2))&=\int_a^b\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V,\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}W\rangle\mathrm{d}t-\int_a^b R(V,\gamma',\gamma',W)\mathrm{d}t\\&\quad-\int_a^b\langle\overline{\nabla}_{\frac{\partial}{\partial s_1}}\ \alpha_*(\frac{\partial}{\partial s_2})\Big|_{s_1=s_2=0},\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\gamma'(t)\rangle\mathrm{d}t\\&\quad-\int_a^b\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V,\gamma'\rangle\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}W,\gamma'\rangle\mathrm{d}t\end{aligned}$$

Corollary 11.2.2. Let $\gamma\colon[a,b]\to(M,g)$ be a unit-speed geodesic. If α is a 2-dimensional variation of γ with variation fields V,W. Then

$$\begin{aligned}\frac{\partial^2}{\partial s_1\partial s_2}\Big|_{s_1=s_2=0}L(\alpha(\text{-},s_1,s_2))&=\int_a^b\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V,\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}W\rangle\mathrm{d}t-\int_a^b R(V,\gamma',\gamma',W)\mathrm{d}t\\&\quad-\int_a^b\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V,\gamma'\rangle\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}W,\gamma'\rangle\mathrm{d}t\\&=\int_a^b\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V^\perp,\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}W^\perp\rangle\mathrm{d}t-\int_a^b R(V^\perp,\gamma',\gamma',W^\perp)\mathrm{d}t\end{aligned}$$

where

$$V^\perp=V-\langle V,\gamma'\rangle\gamma',\quad W^\perp=W-\langle W,\gamma'\rangle\gamma'$$

Proof. It suffices to check the second equality. Direct computation shows:

$$\begin{aligned}\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V,\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}W\rangle &= \langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}(V^{\perp}+\langle V,\gamma'\rangle\gamma'),\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}(W^{\perp}+\langle W,\gamma'\rangle\gamma')\rangle \\ &= \langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V^{\perp},\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}W^{\perp}\rangle+\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V,\gamma'\rangle\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}W,\gamma'\rangle\end{aligned}$$

Thus

$$\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V,\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}W\rangle-\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V,\gamma'\rangle\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}W,\gamma'\rangle=\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V^{\perp},\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}W^{\perp}\rangle$$

since

$$\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}(\langle V,\gamma'\rangle\gamma')=\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V,\gamma'\rangle\gamma'$$

and it's clear

$$R(V,\gamma',\gamma',W)=R(V^{\perp},\gamma',\gamma',W^{\perp})$$

□

So if we want to show a geodesic γ is a (locally) minimal geodesic, it suffices to show for any 2-dimensional variation α with variation vector fields, we have

$$\frac{\partial^2}{\partial s_1\partial s_2}\bigg|_{s_1=s_2=0}L(\alpha(\text{-},s_1,s_2))\geq 0$$

This motivate us to consider the following bilinear form defined on the space of variation vector fields:

Definition 11.2.1 (index form)**.** Let $\gamma\colon[a,b]\to(M,g)$ be a unit-speed geodesic. The index form I_γ is defined as

$$I_\gamma(V,W)=\int_a^b\langle\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{dt}}}V,\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{dt}}}W\rangle\mathrm{d}t-\int_a^b R(V,\gamma',\gamma',W)\mathrm{d}t$$

where V,W are vector fields along γ.

Remark 11.2.1. By Corollary 11.2.2, a geodesic γ is locally minimal if and only if index form defined on the space of normal[5] variation fields are semipositive-definite.

In the following section, we will study when the index form defined on the normal variation vector fields along γ is positive-definite, semipositive-definite or not.

[5]A vector field V along γ is called normal, if V is perpendicular to γ'.

12. Jacobi fields I: as the null space

12.1. First properties.

Definition 12.1.1 (Jacobi field)**.** A vector field J along geodesic γ is called a Jacobi field, if it satisfies

$$\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}J + R(J,\gamma')\gamma' = 0$$

Notation 12.1.1. For convenience,

$$J' = \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}J$$
$$J'' = \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}J$$

Remark 12.1.1 (local form). Suppose $\{e_1,\dots,e_n\}$ is a parallel orthonormal frame along γ, and $J(t) = J^i(t)e_i(t)$, the condition for Jacobi fields becomes

$$\frac{\mathrm{d}^2 J^k}{\mathrm{d}t^2} + \langle J^j R(e_j,\gamma')\gamma', e_k\rangle = 0$$

Thus by standard results in ODEs, a Jacobi field J is completely determined by its initial conditions

$$J(0), J'(0) \in T_{\gamma(0)}M$$

Consequently, the set of Jacobi fields is a vector space with dimension $2n$.

Example 12.1.1. There is always a trivial Jacobi field along geodesic $\gamma\colon [a,b]\to M$, that is $J(t) = (at+b)\gamma'(t)$.

Proposition 12.1.1. Let $J(t)$ be a Jacobi field along geodesic γ.

1. If $J(t)\not\equiv 0$, then the set consisting of zeros of $J(t)$ is discrete.
2. There exist constants λ,μ such that
$$J(t) = J^{\perp}(t) + (\lambda t+\mu)\gamma'(t)$$
where $\langle J^{\perp}(t),\gamma'\rangle \equiv 0$.
3. $J(t)\perp\gamma'$ if and only if there exist $t_1\neq t_2$ such that
$$\langle J(t_1),\gamma'(t_1)\rangle = \langle J(t_2),\gamma'(t_2)\rangle = 0$$
4. If there exists t_0 such that
$$\langle J(t_0),\gamma'(t_0)\rangle = \langle J'(t_0),\gamma'(t_0)\rangle = 0$$
then $J(t)\perp\gamma'(t)$.

Proof. For (3). Note that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle J(t),\gamma'(t)\rangle = \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}J,\gamma'\rangle = \langle R(J,\gamma')\gamma',\gamma'\rangle = 0$$

Thus $\langle J(t),\gamma'(t)\rangle = \lambda t+\mu$. Note that $\langle J_i(t_1),\gamma'(t_1)\rangle = \langle J_i(t_2),\gamma'(t_2)\rangle = 0$, which implies $\langle J(t),\gamma'(t)\rangle\equiv 0$.

□

Proposition 12.1.2. Let $\gamma\colon [a,b]\to(M,g)$ be a geodesic and $\alpha\colon [a,b]\times(-\varepsilon,\varepsilon)\to(M,g)$ a variation of γ consisting of geodesics, then

$$J=\alpha_*(\frac{\partial}{\partial s})\Big|_{s=0}$$

is a Jacobi field.

Proof. Direct computation shows

$$\begin{aligned}\overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial s}) &\overset{(1)}{=} \overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\nabla}_{\frac{\partial}{\partial s}}\alpha_*(\frac{\partial}{\partial t})\\ &= R(\frac{\partial}{\partial t},\frac{\partial}{\partial s})\alpha_*(\frac{\partial}{\partial t})+\overline{\nabla}_{\frac{\partial}{\partial s}}\overline{\nabla}_{\frac{\partial}{\partial t}}\alpha_*(\frac{\partial}{\partial t})\\ &\overset{(2)}{=} R(\frac{\partial}{\partial t},\frac{\partial}{\partial s})\alpha_*(\frac{\partial}{\partial t})\end{aligned}$$

where

(1) holds from Corollary 10.5.1.
(2) holds from α is a variation consisting of geodesics.

Set $s=0$ one has

$$\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}J=R(\gamma',J)\gamma'=-R(J,\gamma')\gamma'$$

which implies J is a Jacobi field. □

Remark 12.1.2. In fact, all Jacobi fields can be obtained by above construction.

Corollary 12.1.1. Let $\gamma\colon[0,1]\to M$ be a geodesic with $\gamma(0)=p,\gamma'(0)=v$, where $v\in T_pM$. Consider the following variation of $\gamma(t)$ consisting of geodesics

$$\alpha(t,s)=\exp_p(t(v+sw))$$

where $w\in T_pM$, then $J(t)=\alpha_*(\frac{\partial}{\partial s})\big|_{s=0}$ is a Jacobi field along γ such that

$$\begin{aligned}J(0)&=0\\ J(1)&=w\\ J'(0)&=w\end{aligned}$$

Proof. In normal coordinate (x^i,U,p), variation $\alpha(t,s)$ can be written explicitly as

$$\alpha(t,s)=(t(v^1+sw^1),\dots,t(v^n+sw^n))$$

where $v=(v^1,\dots,v^n),w=(w^1,\dots,w^n)$. Thus Jacobi field J is given by the formula

$$J(t)=tw^i\frac{\partial}{\partial x^i}\Big|_{\gamma(t)}$$

It's clear $J(t)$ satisfies $J(0)=0,J(1)=w,J'(0)=w$. □

Corollary 12.1.2. Let (M, g) be a Riemannian manifold, (x^i, U, p) a normal coordinate centered at $p \in M$. For each $q \in U \setminus \{p\}$, and $w \in T_qM$, there exists a Jacobi field J along a radial geodesic such that $J(0) = 0$ and $J(1) = w$.

12.2. **Conjugate points.**

Definition 12.2.1 (conjugate points)**.** Let $p \neq q$ be two endpoints of a geodesic γ. p and q are called conjugate along γ if there exists a non-zero Jacobi field J along γ which vanishes at endpoints.

Notation 12.2.1. The conjugate set of p, denoted by $\mathrm{conj}(p)$ is defined as

$$\mathrm{conj}(p) := \{q \in M \mid p \text{ and } q \text{ are conjugate along some geodesic.}\}$$

Remark 12.2.1. There are at most $n-1$ linearly independent Jacobi fields along γ such that $J(a) = J(b) = 0$. Indeed, by Remark 12.1.1, there are at most n linearly independent Jacobi fields such that $J(a) = 0$. However, trivial Jacobi field $J(t) = (t-a)\gamma'(t)$ never vanishes at $t = b$.

Theorem 12.2.1. Let (M, g) be a Riemannian manifold, $p \in M$ and $v \in V_p \subset T_pM$. Let $\gamma_v : [0, 1] \to M$ be the geodesic $\gamma_v(t) = \exp_p(tv)$ and $q = \gamma_v(1)$. Then $(\mathrm{d}\exp_p)_v$ is not injective if and only if q is conjugate to p along γ_v.

Proof. For any $w \in T_pM$, consider Jacobi field given by

$$J(t) = (\mathrm{d}\exp_p)_{tv}(tw)$$

So if $w \neq 0$ lies in the kernel of $(\mathrm{d}\exp_p)_v$, then $J(0) = J(1) = 0$, that is p is conjugate to q. Conversely, if p and q are conjugate along γ, then there exists a Jacobi field J such that $J(0) = J(1) = 0$, then it's clear

$$J(t) = (\mathrm{d}\exp_p)_{tv}(tw)$$

where $0 \neq w = J'(0) \in T_pM$. Thus

$$(\mathrm{d}\exp_p)_v(w) = J(1) = 0$$

which implies $(\mathrm{d}\exp_p)_v$ is not injective. $\square$

Corollary 12.2.1. Let (M, g) be a complete Riemannian manifold, $p \in M$. If the conjugate locus $\mathrm{conj}(p) = \varnothing$, then $\exp_p \colon T_pM \to M$ is a local diffeomorphism.

Proof. Since M is complete, then $\exp_p \colon T_pM \to M$ is surjective. Furthermorem, since the conjugate locus $\mathrm{conj}(p) = \varnothing$, so for arbitrary $v \in T_pM$, we have $(\mathrm{d}\exp_p)_v$ is non-degenerated, which implies $\exp_p$ is a local diffeomorphism at $v \in T_pM$. $\square$

Example 12.2.1. For $p \in \mathbb{S}^n$, we have $\mathrm{conj}(p) = \{-p\}$.

Example 12.2.2. For $p \in \mathbb{S}^1 \times \mathbb{R}$, we have $\mathrm{conj}(p) = \varnothing$.

12.3. **Locally minimal geodesic.**

Lemma 12.3.1. Let $\gamma\colon [a,b]\to (M,g)$ be a unit-speed geodesic with no conjugate points, there exist Jacobi fields $J_2,\dots,J_n$ along γ such that

1. $J_i(a)=0, i\geq 2$ and $\{\gamma'(b),J_2(b),\dots,J_n(b)\}$ is an orthonormal basis of $T_{\gamma(b)}M$.
2. $\langle J_i(t),\gamma'(t)\rangle\equiv 0$ for any $t\in[a,b]$.
3. $\{\gamma'(t),J_2(t),\dots,J_n(t)\}$ are linearly independent for $t\in(a,b]$.

Proof. Suppose $\{\gamma'(b),e_2,\dots,e_n\}$ is an orthonormal basis of $T_{\gamma(b)}M$, since there is no conjugate points along γ, there exists a unique Jacobi field J_i such that

$$J_i(a)=0, J_i(b)=e_i$$

for each $i=2,\dots,n$. Now it suffices to show Jacobi fields $J_i(t)$ satisfy properties (2) and (3).

For (2). Note that $\langle J_i(a),\gamma'(a)\rangle=\langle J_i(b),\gamma'(b)\rangle=0$, then by (3) of Proposition 12.1.1 one $\langle J_i(t),\gamma'(t)\rangle\equiv 0$.

For (3). Suppose there exists $c\in(a,b]$ and $\lambda_i\in\mathbb{R}$ such that

$$\sum_{i=2}^{n}\lambda_i J_i(c)=0$$

which implies

$$W(t)=\sum_{i=2}^{n}\lambda_i J_i(t)\equiv 0$$

on $(a,c]$ since there is no conjugate points. By (1) of Proposition 12.1.1 one has $W(t)\equiv 0$ on $(a,b]$, thus we have $\lambda_i=0, i=2,\dots,n$, since $\{\gamma'(b),J_2(b),\dots,J_n(b)\}$ is linearly independent. $\square$

Theorem 12.3.1. Let $\gamma\colon[a,b]\to(M,g)$ be a unit-speed geodesic, then

1. If γ has no conjugate points, then index form I_γ is **positive-definite** on vector space consisting of normal variation fields.
2. If γ only has conjugate points as endpoints, then index form is **semipositive-definite** on vector space consisting of normal variation fields. Furthermore, Jacobi field is null space.
3. If γ has an interior conjugate point, then index form is **not positive-definite** on vector space consisting of normal variation fields.

Proof. For (1). Suppose $\gamma(t)$ has no conjugate points, choose Jacobi fields $\{J_1(t)=\gamma'(t),J_2(t),\dots,J_n(t)\}$ in Lemma 12.3.1. Then for any normal variation vector V along γ we write it as

$$V=\sum_{i=2}^{n}V^i(t)J_i(t)$$

Then it's clear $V^i(b)=0$ since $V(b)=0$ and $\{J_2(b),\dots,J_n(b)\}$ is linearly independent. Direct computation shows

$$
\begin{aligned}
I_\gamma(V,V)=\sum_{i,j=2}^n&\underbrace{\int_a^b V^iV^j\langle J_i',J_j'\rangle+\frac{\mathrm{d}V^i}{\mathrm{d}t}V^j\langle J_i,J_j'\rangle+V^i\frac{\mathrm{d}V^j}{\mathrm{d}t}\langle J_i',J_j\rangle\mathrm{d}t}_{\text{Part I}}\\
&+\underbrace{\int_a^b\left\{\frac{\mathrm{d}V^i}{\mathrm{d}t}\frac{\mathrm{d}V^j}{\mathrm{d}t}\langle J_i,J_j\rangle-V^iV^jR(J_i,\gamma',\gamma',J_j)\right\}\mathrm{d}t}_{\text{Part II}}
\end{aligned}
$$

Note that

$$\langle J_i',J_j\rangle=\langle J_i,J_j'\rangle$$

Then Part I is

$$\int_a^b\left\{(V^iV^j\langle J_i',J_j\rangle)'-V^iV^j\langle J_i'',J_j\rangle\right\}\mathrm{d}t$$

Thus

$$
\begin{aligned}
I_\gamma(V,V)&=\sum_{i,j=2}^n V^iV^j\langle J_i',J_j\rangle\big|_a^b+\sum_{i,j=2}^n\int_a^b\frac{\mathrm{d}V^i}{\mathrm{d}t}\frac{\mathrm{d}V^j}{\mathrm{d}t}\langle J_i,J_j\rangle\mathrm{d}t\\
&=\sum_{i,j=2}^n\int_a^b\frac{\mathrm{d}V^i}{\mathrm{d}t}\frac{\mathrm{d}V^j}{\mathrm{d}t}\langle J_i,J_j\rangle\mathrm{d}t\\
&\geq 0
\end{aligned}
$$

Furthermore, $I_\gamma(V,V)=0$ if and only if $\sum_{i=2}^n\frac{\mathrm{d}V^i}{\mathrm{d}t}J_i(t)=0$ if and only if $\frac{\mathrm{d}V^i}{\mathrm{d}t}(t)=0, t\in[a,b]$, thus $V^i(t)\equiv 0$, that is $V=0$.

For (2). For any $c\in(a,b)$, consider geodesic $\gamma^c\colon[a,c]\to(M,g)$. By (1) it's clear I_{γ^c} is positive-definite on the vector space consisting of normal variation fields along γ^c, then a standard approximation argument shows I_γ is semipositive-definite. To see its null space: It's clear a normal variation Jacobi field V satisfies $I_\gamma(V,V)=0$. Conversely, if a normal variation field V satisfies $I_\gamma(V,V)=0$, then by a variation argument we have for arbitrary W we have

$$I_\gamma(V,W)=0$$

Take appropriate W to see V satisfies the equation for Jacobi fields.

For (3). If $\gamma(a)$ is conjugate to $\gamma(c)$ for some $c\in(a,b)$, then there exists a non-zero normal Jacobi field J_1 along $\gamma([a,c])$ such that $J_1(a)=J_1(c)=0$. Consider

$$J=\begin{cases}J_1(t) & t\in[a,c]\\ 0 & t\in[c,b]\end{cases}$$

It's easy to see $I_\gamma(J,J)=0$. Note that here our J may not be smooth. Let W be a smooth normal variation vector field along γ such that $W(c)=$

$-\lim_{t\to c^-}\nabla_{\frac{\mathrm{d}}{\mathrm{d}t}}J_1$. It' clear $W(c)\neq 0$. Consider $J_\varepsilon = J+\varepsilon W$ and so[6]

$$I_\gamma(J_\varepsilon, J_\varepsilon) = 2\varepsilon I_\gamma(J,W)+\varepsilon^2 I_\gamma(W,W)$$

And integration by parts we have

$$I_\gamma(J,W) = \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}J_1, W\rangle\Big|_a^c = -W(c)^2 < 0$$

So for sufficiently small ε we have $I_\gamma(J_\varepsilon, J_\varepsilon) < 0$, and by approximation argument we can show there exists a smooth normal variation field such that $I_\gamma(V,V) < 0$. □

Corollary 12.3.1. Let $\gamma\colon [a,b]\to (M,g)$ be a unit-speed geodesic with no conjugate points, and V,W are normal vector fields satisfying $V(a) = W(a), V(b) = W(b)$. If V is a Jacobi field, then $I_\gamma(V,V)\leq I_\gamma(W,W)$, and the equality holds if and only if $V=W$.

Proof. Since V,W agree at end points, then $V-W$ is a normal variation field, thus we have

$$0\leq I_\gamma(V-W,V-W) = I_\gamma(V,V)+I_\gamma(W,W)-2I_\gamma(V,W)$$

Since V is a Jacobi field, then integration by parts shows

$$I_\gamma(V,V) = \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V, V\rangle\Big|_a^b = \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}V, W\rangle\Big|_a^b = I_\gamma(V,W)$$

Hence we get $I_\gamma(V,V)\leq I_\gamma(W,W)$, and the equality holds if and only if $V=W$. □

Remark 12.3.1. From second variation formula, we can conclude that a geodesic γ is a **locally minimal geodesic** if and only if it has no interior conjugate points. However, it may not be **globally minimal geodesic**. Indeed, consider $M=\mathbb{S}^1\times\mathbb{R}$, it's clear there is no conjugate points for any geodesic on M, thus for geodesic $\gamma\colon [a,b]\to M$ starting at $(x,y)\in M$, it's locally minimal, but if there exists $c\in(a,b)$ such that $\gamma(c)\in\{-x\}\times\mathbb{R}$, then γ is not globally minimal.

[6]Note that here our J and J_ε may not be smooth, so keep in mind here we already extend our index form I_γ to the one defined on piecewise smooth vector field.

13. Cut locus and injective radius

13.1. Cut locus.

Definition 13.1.1 (cut time/point/locus)**.** Let (M,g) be a complete Riemannian manifold, $p\in M$ and $v\in T_pM$.

1. The cut time of (p,v) is defined as
$$t_{\mathrm{cut}}(p,v)=\sup\{c>0\mid \gamma_v|_{[0,c]}\text{ is a minimal geodesic}\}$$
2. Suppose $t_{\mathrm{cut}}(p,v)<\infty$, the cut point of p along γ along γ_v is $\gamma_v(t_{\mathrm{cut}}(p,v))\in M$.
3. The cut locus of p, denoted by $\mathrm{cut}(p)$ is the set
$$\mathrm{cut}(p)=\{q\in M\mid \exists v\in T_pM\text{ such that }q\text{ is a cut point of }p\text{ along }\gamma_v.\}$$

Remark 13.1.1.

1. It's possibly for $t_{\mathrm{cut}}(p,v)$ to be $+\infty$. For example, just let M be Euclidean space with standard metric.
2. The cut point (if it exists) occurs at or before the first conjugate point along every geodesic.
3. It's clear that $t_{\mathrm{cut}}(p,v)$ depends on the $|v|$, but $\gamma_v(t_{\mathrm{cut}}(p,v))$ is independent of $|v|$. So when we consider cut points of p along some geodesic γ, we always assume γ is unit-speed.

Theorem 13.1.1. Let (M,g) be a complete Riemannian manifold, $p\in M, v\in T_pM$ with unit length. Let $c=t_{\mathrm{cut}}(p,v)\in(0,\infty]$, then

1. If $0<b<c$ and b is finite, then $\gamma_v|_{[0,b]}$ has no conjugate point and it is the unique minimal unit-speed geodesic between endpoints.
2. If $c<\infty$, then $\gamma_v|_{[0,c]}$ is a minimal geodesic.
3. In the case of (2), one or both of the following holds:
   (a) $\gamma_v(c)$ is conjugate to p along γ_v.
   (b) There are two or more different unit-speed minimal geodesic connecting p and $\gamma_v(c)$.

Proof. For (1). It's clear $\gamma_v|_{[0,b]}$ has no conjugate point and it's minimal. To see it's unique, suppose $\sigma\colon[0,b]\to M$ is another minimal unit-speed geodesic. Note that $\gamma_v'(b)\neq\sigma(b)$, otherwise by uniqueness we will have $\gamma_v(t)=\sigma(t)$ in $t\in[0,b]$. Now take $b'\in(b,c)$, and consider a new unit-speed curve
$$\widetilde{\gamma}(t)=\begin{cases}\sigma(t), & t\in[0,b]\\ \gamma_v(t), & t\in(b,b']\end{cases}$$
Then $\widetilde{\gamma}$ has length b', so it's also a minimal curve from p to $\gamma_v(b')$, since $\mathrm{dist}(p,\gamma_v(b'))=b'$. However, it's not smooth at $t=b$, contradicting to the fact that minimal curve are smooth geodesics.

For (2). By definition of cut time, there exists a sequence b_i increasing to c such that $\gamma_v|_{[0,b_i]}$ is minimal. By continuity of distance function, one has
$$\mathrm{dist}(p,\gamma_v(c))=\lim_{i\to\infty}\mathrm{dist}(p,\gamma_v(b_i))=\lim_{i\to\infty}b_i=c$$

which implies γ_v is minimal on $[0, c]$.

For (3). Assume $\gamma_v(c)$ is not conjugate to p along γ_v, we shall prove the existence of another unit-speed minimal geodesic from p to $\gamma_v(c)$.

Firstly we choose a sequence $\{b_i\}$ descending to c. Note that $\gamma_v : [0, b_i] \to M$ is not a minimal geodesic, thus there exists a unit-speed minimal geodesic $\gamma_i : [0, a_i] \to M$ connecting p and $\gamma_v(b_i)$. In particular we have

1. $\gamma_i(a_i) = \gamma_v(b_i)$.
2. $a_i < b_i$.

If we denote $\omega_i = \gamma_i'(0) \in T_pM$, by compactness of unit sphere on T_pM and the fact $\{a_i\}$ is bounded, we can find a subsequence of $\{\gamma_i\}$ such that ω_i converging to some $w \in T_pM$ with $|w| = 1$, and $\lim_{i\to\infty} a_i = a$. For convenience we still denote this subsequence by $\{\gamma_i\}$.

On one hand $\gamma_i(a_i) = \exp_p(a_i w_i)$ converges to $\exp_p(aw)$. On the other hand, $\gamma_i(a_i) = \gamma_v(b_i)$ implies $\exp_p(cv) = \gamma_v(c) = \exp_p(aw)$. Furthermore,

$$c = \operatorname{dist}(p, \gamma_v(c)) = \lim_{i\to\infty} \operatorname{dist}(p, \gamma_v(b_i)) = \lim_{i\to\infty} \operatorname{dist}(p, \gamma_i(a_i)) = \lim_{i\to\infty} a_i = a$$

So it suffices to check $v \neq w$.

By assumption we have $\gamma_v(c)$ is not conjugate to p, thus cv is not a critical point of $\exp_p$, that is $\exp_p$ is injective in $B_\varepsilon(cv)$, where $\varepsilon > 0$ is sufficiently small. On one hand we have $a_i w_i \neq b_i v$ since $a_i < b_i$. On the other hand we have

$$\exp_p(b_i v) = \gamma_v(b_i) = \gamma_i(a_i) = \exp_p(a_i w_i)$$

Thus injectivity implies $a_i w_i \notin B_\varepsilon(cv)$ for sufficiently large i, since in this case $b_i v \in B_\varepsilon(cv)$. Taking limits we have

$$aw \neq cv$$

that is $w \neq v$. □

Remark 13.1.2. From the proof of (1) one can deduce: If $\gamma\colon [0, b] \to M$ is a minimal geodesic connecting $\gamma(0)$ and $\gamma(b)$, then it's the unique minimal geodesic connecting any two points strictly between $\gamma(0)$ and $\gamma(b)$.

Corollary 13.1.1. Let (M, g) be a complete Riemannian manifold with $p, q \in M$.

1. If $q \in \operatorname{cut}(p)$, then $p \in \operatorname{cut}(q)$.
2. If $q \notin \operatorname{cut}(p)$, then there exists a unique minimal geodesic connecting p and q.

Proof. For (1). If q is cut point of p along geodesic γ, then γ is a minimal geodesic connecting p and q, and by Theorem 13.1.1, there are two cases:

(a) q is conjugate to p along γ.
(b) There are two more different unit-speed geodesic connecting p and q.

Note that we already have γ^{-1} is a minimal geodesic connecting q and p, so if we want to show $p \in \operatorname{cut}(q)$, it suffices to show γ^{-1} is no longer minimal after p.

(a) It's clear in the first case γ^{-1} is no longer minimal after p, since if q is conjugate to p, then p is also conjugate to q.
(b) In the second case, if γ^{-1} is still minimal after p, then by Remark 13.1.2, we will know γ^{-1} is the unique minimal geodesic connecting q and p, contradicting to the second case.

For (2). If there exist two or more minimal geodesic connecting p and q, then for any minimal geodesic γ connecting p and q, it's no longer minimal after q by Remark 13.1.2, a contradiction to $q \notin \mathrm{cut}(p)$. □

Example 13.1.1.

1. $M = \mathbb{S}^n$, then $\mathrm{cut}(p) = \mathrm{conj}(p) = \{-p\}$. In this case both $(a),(b)$ hold in Theorem 13.1.1.
2. $M = \mathbb{S}^1 \times \mathbb{R}$, then $\mathrm{cut}(p) = \{-p\} \times \mathbb{R}$. In this case (a) fails and (b) holds in Theorem 13.1.1.

Definition 13.1.2 (tangent cut locus and injectivity domain)**.** Let (M, g) be a complete Riemannian manifold, given $p \in M$, we define

1. the tangent cut locus
$$\mathrm{TCL}(p) := \{v \in T_pM : |v| = t_{\mathrm{cut}}(p, v/|v|)\}$$
2. the injectivity domain
$$\mathrm{ID}(p) := \{v \in T_pM : |v| < t_{\mathrm{cut}}(p, v/|v|)\}$$

It's clear that $\mathrm{TCL}(p) = \partial\,\mathrm{ID}(p)$ and $\mathrm{cut}(p) = \exp_p(\mathrm{TCL}(p))$. Furthermore, we have the following properties.

Proposition 13.1.1. Let (M, g) be a complete Riemannian manifold and $p \in M$, then

1. The cut locus of p is a closed subset of M of measure zero.
2. The restriction of $\exp_p$ to $\mathrm{ID}(p)$ is a diffeomorphism onto $M \setminus \mathrm{cut}(p)$.

Proof. See Theorem 10.34 of Page311 of [Lee18]. □

13.2. **Injective radius.**

Definition 13.2.1 (injective radius)**.** Let (M, g) be a Riemannian manifold, $p \in M$. The injective radius of p is defined as

$$\mathrm{inj}(p) := \sup\{\rho > 0 : \exp_p \text{ is defined on } B(0,\rho) \subset T_pM \text{ and injective}\}$$

The injectivity radius of M is

$$\mathrm{inj}(M) := \inf_{p \in M} \mathrm{inj}(p)$$

Theorem 13.2.1. Let (M, g) be a complete Riemannian manifold, then

$$\mathrm{inj}(p) = \begin{cases} \mathrm{dist}(p, \mathrm{cut}(p)) & \mathrm{cut}(p) \neq \varnothing \\ \infty & \mathrm{cut}(p) = \varnothing \end{cases}$$

Proof. See Proposition 10.36 in Page312 of [Lee18]. □

Proposition 13.2.1. Let (M, g) be a complete Riemannian manifold and $p \in M$. Suppose there exists some point $q \in \mathrm{cut}(p)$ such that $\mathrm{dist}(p, q) = \mathrm{dist}(p, \mathrm{cut}(p))$, then

1. Either q is a conjugate point of p along some minimizing geodesic from p to q, or there are exactly two minimizing geodesics from p to q, say $\gamma_1, \gamma_2 : [0, b] \to M$, such that $\gamma_1'(b) = -\gamma_2'(b)$.
2. If in addition that $\mathrm{inj}(p) = \mathrm{inj}(M)$, and q is not conjugate to p along any minimizing geodesic, then there is a closed unit-speed geodesic $\gamma\colon [0, 2b] \to M$ such that $\gamma(0) = \gamma(2b) = p$ and $\gamma(b) = q$ where $b = \mathrm{dist}(p, q)$.

Proof. For (1). Suppose q is not conjugate to p along any minimizing geodesic, then by Theorem 13.1.1 there are at least two unit-speed minimal geodesics $\gamma_1(t), \gamma_2(t)$ such that $\gamma_1(b) = \gamma_2(b) = q$. Suppose $\gamma_1'(b) \neq -\gamma_2'(b)$, then there exists a unit vector $v \in T_qM$ such that

$$\langle v, \gamma_1'(b)\rangle < 0, \quad \langle v, \gamma_2'(b)\rangle < 0$$

Since q is not conjugate to p along γ_1, there exists a neighborhood U_1 of $b\gamma_1'(0)$ in T_pM such that $\exp_p|_{U_1}$ is diffeomorphism. Now choose a sufficiently small s and let

$$\xi_1(s) = (\exp_p|_{U_1})^{-1}\exp_q(sv)$$

Consider the following variation of γ_1 consisting of geodesics:

$$\alpha_1(t, s) = \exp(\frac{t}{b}\xi_1(s))$$

It's clear $\alpha_1(t, 0) = \gamma_1(t)$, since $\xi_1(0) = (\exp_p|_{U_1})^{-1}\exp_q(0) = (\exp_p|_{U_1})^{-1}(q) = b\gamma_1'(0)$. Then by Remark 11.1.3, that is the first variation formula of general variation, one has

$$\frac{\mathrm{d}L(\gamma_s)}{\mathrm{d}s}\Big|_{s=0} = \langle v, \gamma_1'(b)\rangle < 0$$

which implies for sufficiently small s we have $L(\alpha_1(t, s)) < L(\gamma_1(t))$. For γ_2 we can do the same construction and the same argument implies for sufficiently small s we have $L(\alpha_2(t, s)) < L(\gamma_2(t))$. Thus for each sufficiently small s we have two geodesics $\alpha_1(t, s), \alpha_2(t, s)$ from p to $\exp_q(sX_q)$. However,

$$\mathrm{d}(p, \exp_q(sv)) \leq L(\alpha_1(t, s)) < L(\gamma_1(t)) = \mathrm{dist}(p, q) = \mathrm{inj}(p) \tag{13.1}$$

A contradiction to the definition of injective radius. So any two different minimizing geodesics γ_1, γ_2 from p to q satisfy $\gamma_1'(b) = -\gamma_2'(b)$, which implies there are exactly two minimizing geodesics from p to q.

For (2). By (1) we know that there exists exactly two geodesics γ_1, γ_2 such that $\gamma_1(b) = \gamma_2(b) = q$ with $\gamma_1'(b) = \gamma_2'(b)$. Consider the loop $\gamma = \gamma_1 \circ \gamma_2^{-1}$, then it's a unit-speed geodesic such that $\gamma(0) = \gamma(2b) = p, \gamma(b) = q$, where $b = \mathrm{dist}(p, q)$, since we have already shown $\gamma_1'(b) = -\gamma_2'(b)$. To show γ is a closed geodesic, it suffices to show $\gamma'(2b) = \gamma'(0)$, that is equivalent to show $(\gamma_1^{-1})'(b) = (\gamma_2^{-1})'(b)$. Note that in the proof of (1), condition of $\mathrm{dist}(p, q) = \mathrm{dist}(p, \mathrm{cut}(p)) = \mathrm{inj}(p)$ is used in inequality (13.1), and in fact we only need $\operatorname{dist}(p,q) \leq \operatorname{inj}(p)$, strict equality is not neccessary. So if $\operatorname{inj}(p) = \operatorname{inj}(M)$, thus

$$\operatorname{dist}(q,p) = \operatorname{dist}(p,q) = \operatorname{inj}(p) \leq \operatorname{inj}(q)$$

Then (1) implies $(\gamma_1^{-1})'(b) = (\gamma_2^{-1})'(b)$. $\square$

Part 5. Harmonic maps

14. Harmonic map

In this section we fix a smooth map $f\colon (M,g)\to(N,h)$ between Riemannian manifolds with second fundamental form B.

14.1. Harmonic map and totally geodesic.

Definition 14.1.1 (scalar Laplacian)**.** The scalar Laplacian of f is defined as

$$\Delta f\colon = \operatorname{tr}_g B \in C^\infty(M, f^*TN)$$

Definition 14.1.2 (harmonic map)**.** f is called a harmonic map if its scalar Laplacian $\Delta f = 0$.

Definition 14.1.3 (totally geodesic)**.** f is called totally geodesic, if its second fundamental form $B = 0$.

Example 14.1.1. For a geodesic $\gamma\colon [a,b]\to(M,g)$, if we endow $[a,b]$ with standard metric, then γ is totally geodesic, thus it's harmonic.

Example 14.1.2. For a smooth function $f\colon (M,g)\to\mathbb{R}$, if we endow $\mathbb{R}$ with standard metric, then f is a harmonic map if and only if it's a harmonic function.

Lemma 14.1.1. Let $\gamma\colon [a,b]\to M$ be a smooth curve and $\widetilde{\gamma} = f\circ\gamma$. If we use $\widehat{\nabla}$ and $\widetilde{\nabla}$ to denote the induced connection on γ^*TM and $\widetilde{\gamma}^*TN$ respectively, then

$$\widetilde{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\widetilde{\gamma}_*(\frac{\mathrm{d}}{\mathrm{d}t}) = f_*(\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\gamma_*(\frac{\mathrm{d}}{\mathrm{d}t})) + \gamma^* B$$

Proof. Directly compute

$$\begin{aligned}
\widetilde{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\widetilde{\gamma}_*(\frac{\mathrm{d}}{\mathrm{d}t}) &= (\frac{\mathrm{d}^2\widetilde{\gamma}^l}{\mathrm{d}t^2} + \Gamma^l_{mn}(\widetilde{\gamma})\frac{\mathrm{d}\widetilde{\gamma}^m}{\mathrm{d}t}\frac{\mathrm{d}\widetilde{\gamma}^n}{\mathrm{d}t})\frac{\partial}{\partial y^l} \\
&= \{\frac{\partial f^l}{\partial x^k}\frac{\mathrm{d}^2\gamma^k}{\mathrm{d}t^2} + (\frac{\partial^2 f^l}{\partial x^i\partial x^j} + \Gamma^l_{mn}\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j})\frac{\partial\gamma^i}{\mathrm{d}t}\frac{\partial\gamma^j}{\mathrm{d}t}\}\frac{\partial}{\partial y^l} \\
&= \{\frac{\partial f^l}{\partial x^k}(\frac{\mathrm{d}^2\gamma^k}{\mathrm{d}t^2} + \Gamma^k_{ij}\frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}) + (\frac{\partial^2 f^l}{\partial x^i\partial x^j} + \frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j}\Gamma^l_{mn} - \Gamma^k_{ij}\frac{\partial f^l}{\partial x^k})\frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\}\frac{\partial}{\partial y^l} \\
&= f_*(\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}\gamma_*(\frac{\mathrm{d}}{\mathrm{d}t})) + \gamma^* B
\end{aligned}$$

□

Theorem 14.1.1. The following statements are equivalent:

1. f is totally geodesic.
2. f maps geodesics in M to geodesics in N.

Proof. It's clear from above lemma. □

14.2. **First variation of smooth map.**

Definition 14.2.1 (energy functional)**.** The energy density of smooth function $f\colon (M,g)\to(N,h)$ is
$$e(f)=|\mathrm{d}f|^2$$
The energy functional of f is
$$E(f)=\frac{1}{2}\int_M e(f)\,\mathrm{vol}$$

Remark 14.2.1 (local form)*.* Locally energy density can be written as
$$\begin{aligned}\langle \mathrm{d}f,\mathrm{d}f\rangle&=\langle\frac{\partial f^m}{\partial x^i}\mathrm{d}x^i\otimes\frac{\partial}{\partial y^m},\frac{\partial f^n}{\partial x^j}\mathrm{d}x^j\otimes\frac{\partial}{\partial y^n}\rangle\\&=g^{ij}h_{mn}(f)\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j}\end{aligned}$$

Theorem 14.2.1. The Euler-Lagrange equation of $E(f)$ is
$$\widehat{\nabla}^*\mathrm{d}f=0$$
where $\widehat{\nabla}^*$ is the formal adjoint operator of $\widehat{\nabla}$.

Proof. We fix the following notations in the proof:
1. Consider a smooth variation $f\colon M\times\mathbb{R}\to N$ of f, we also write $f_t(\text{-})=F(\text{-},t)$ for convenience.
2. Set $\overline{M}=M\times\mathbb{R}$ and there is a natural metric $\overline{g}=g\times g_{\mathbb{R}}$ on $\overline{M}$.
3. The pullback F^*TN bundle is denoted by W, and induced connection on W is denoted by ∇^W.
4. Fix $t\in\mathbb{R}$, $f_t:M\to N$, then $\mathrm{d}f_t$ is a section of $T^*M\otimes f_t^*TN$, and we can regard it as a section of $T^*\overline{M}\otimes W$.

Holding above notations, we have
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(f_t)&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_M|\mathrm{d}f_t|^2\,\mathrm{vol}\\&=\int_M\langle\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial t}}\mathrm{d}f_t,\mathrm{d}f_t\rangle\,\mathrm{vol}\end{aligned}$$
Here we claim
$$\langle\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial t}}\mathrm{d}f_t,\mathrm{d}f_t\rangle\overset{1}{=}\langle\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial t}}\mathrm{d}F,\mathrm{d}f_t\rangle\overset{2}{=}\langle\nabla^W F_*(\frac{\partial}{\partial t}),\mathrm{d}f_t\rangle$$

1. For equation marked 1: Note that
$$\begin{aligned}\mathrm{d}F-\mathrm{d}f_t&=\frac{\partial F^m}{\partial x^i}\mathrm{d}x^i\otimes\frac{\partial}{\partial y^m}+\frac{\partial F^m}{\partial t}\mathrm{d}t\otimes\frac{\partial}{\partial y^m}-\frac{\partial f_t^m}{\partial x^i}\mathrm{d}x^i\otimes\frac{\partial}{\partial y^m}\\&=\frac{\partial F^m}{\partial t}\mathrm{d}t\otimes\frac{\partial}{\partial y^m}\end{aligned}$$

since $\frac{\partial F^m}{\partial x^i} = \frac{\partial f_t^m}{\partial x^i}$. So we have

$$\begin{aligned}\nabla^{T^*\overline{M}\otimes W}(\mathrm{d}F - \mathrm{d}f_t) &= \frac{\partial^2 F^l}{\partial t^2}\mathrm{d}t\otimes\mathrm{d}t\otimes\frac{\partial}{\partial y^l} + \frac{\partial F^m}{\partial t}\mathrm{d}t\otimes(\frac{\partial F^n}{\partial t}\Gamma^l_{mn}\mathrm{d}t\otimes\frac{\partial}{\partial y^l} + \frac{\partial F^n}{\partial x^i}\Gamma^l_{mn}\mathrm{d}x^i\otimes\frac{\partial}{\partial y^l})\\ &= (\frac{\partial^2 F^l}{\partial t^2} + \frac{\partial F^m}{\partial t}\frac{\partial F^n}{\partial t}\Gamma^l_{mn})\mathrm{d}t\otimes\mathrm{d}t\otimes\frac{\partial}{\partial y^l} + \frac{\partial F^m}{\partial t}\frac{\partial F^n}{\partial x^i}\Gamma^l_{mn}\mathrm{d}x^i\otimes\mathrm{d}t\otimes\frac{\partial}{\partial y^l}\end{aligned}$$

Thus we have

$$\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial t}}(\mathrm{d}F - \mathrm{d}f_t) = (\frac{\partial^2 F^l}{\partial t^2} + \frac{\partial F^m}{\partial t}\frac{\partial F^n}{\partial t}\Gamma^l_{mn})\mathrm{d}t\otimes\frac{\partial}{\partial y^l}$$

From above expression it's clear

$$\langle\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial t}}(\mathrm{d}F - \mathrm{d}f_t), \mathrm{d}f_t\rangle = 0$$

since there is no $\mathrm{d}t$ in $\mathrm{d}f_t$, which implies equation marked 1 holds.

2. For equation marked 2: For arbitrary $X \in C^\infty(M, TM) \subset C^\infty(\overline{M}, T^*\overline{M})$, since second fundamental form is symmetric, thus

$$\begin{aligned}(\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial t}}\mathrm{d}F)(X) &= (\nabla^{T^*\overline{M}\otimes W}_X\mathrm{d}F)(\frac{\partial}{\partial t})\\ &= \nabla^W_X F_*(\frac{\partial}{\partial t}) - F_*(\nabla^{\overline{M}}_X\frac{\partial t}{\partial t})\\ &= \nabla^W_X F_*(\frac{\partial}{\partial t})\end{aligned}$$

Now let v be an arbitrary variation vector field, that is

$$v = F_*(\frac{\partial}{\partial t})\Big|_{t=0} \in C^\infty(M, f^*TN)$$

Hence when $t = 0$ we have

$$(\nabla^W F_*(\frac{\partial}{\partial t}))\Big|_{t=0} = \widehat{\nabla}v$$

where $\widehat{\nabla}$ is the induced connection on f^*TN. So we have first variation formula

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} E(f_t) &= \int_M \langle\widehat{\nabla}v, \mathrm{d}f\rangle\,\mathrm{vol}\\ &= \int_M \langle v, \widehat{\nabla}^*\mathrm{d}f\rangle\,\mathrm{vol} = 0\end{aligned}$$

where $\widehat{\nabla}^*$ is the formal adjoint operator of $\widehat{\nabla}$. since v is arbitrary, we deduce $\widehat{\nabla}^*\mathrm{d}f = 0$. □

14.3. **Second variation formula of harmonic map.** Consider the following variation map of f

$$f\colon M\times(-\varepsilon_1,\varepsilon_1)\times(-\varepsilon_2,\varepsilon_2)\to N$$

with variation fields

$$v=F_*(\frac{\partial}{\partial t})\Big|_{s=t=0}\in C^\infty(M,f^*TN)$$
$$w=F_*(\frac{\partial}{\partial s})\Big|_{s=t=0}\in C^\infty(M,f^*TN)$$

For convenience we denote $F(\text{-},s,t)=f_{s,t}(\text{-})$.

Theorem 14.3.1 (second variation formula)**.** If $f\colon(M,g)\to(N,h)$ is a harmonic map, then the second variation of the harmonic map f along v,w is

$$\frac{\partial^2}{\partial s\partial t}\Big|_{s=t=0}E(f_{s,t})=\int_M\langle\widehat{\nabla}v,\widehat{\nabla}w\rangle\,\mathrm{vol}-\int_M g^{ij}R_{pmnq}v^pw^q\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j}\,\mathrm{vol}$$

Proof. In this proof, we still use the notations in proof of first variation formula. By first variation formula, we have

$$\frac{\partial}{\partial t}E(f_{s,t})=\int_M\langle\nabla^WF_*(\frac{\partial}{\partial t}),\mathrm{d}f_{s,t}\rangle\,\mathrm{vol}$$

So

$$\begin{aligned}\frac{\partial^2}{\partial s\partial t}E(f_{s,t})=&\underbrace{\int_M\langle\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial s}}\nabla^WF_*(\frac{\partial}{\partial t}),\mathrm{d}f_{s,t})\rangle\,\mathrm{vol}}_{\text{part I}}\\&+\underbrace{\int_M\langle\nabla^WF_*(\frac{\partial}{\partial t}),\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial s}}\mathrm{d}f_{s,t}\rangle\,\mathrm{vol}}_{\text{part II}}\end{aligned}$$

Note that

$$\begin{aligned}\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial s}}\mathrm{d}f_{s,t}&=\nabla^{T^*\overline{M}\otimes W}_{\frac{\partial}{\partial s}}(\frac{\partial F^m}{\partial x^i}\mathrm{d}x^i\otimes\frac{\partial}{\partial y^m})\\&=\frac{\partial^2F^m}{\partial s\partial x^i}\mathrm{d}x^i\otimes\frac{\partial}{\partial y^m}+\frac{\partial F^m}{\partial x^i}\frac{\partial F^n}{\partial s}\Gamma^l_{mn}\mathrm{d}x^i\otimes\frac{\partial}{\partial y^l}\\&=(\frac{\partial^2F^l}{\partial s\partial x^i}+\frac{\partial F^m}{\partial x^i}\frac{\partial F^n}{\partial s}\Gamma^l_{mn})\mathrm{d}x^i\otimes\frac{\partial}{\partial y^l}\\\widehat{\nabla}w&=\widehat{\nabla}_{\frac{\partial}{\partial x^i}}(\frac{\partial F^n}{\partial s}\Big|_{t=s=0})\mathrm{d}x^i\otimes\frac{\partial}{\partial y^n}+\frac{\partial F^m}{\partial s}\frac{\partial F^n}{\partial x^i}\Big|_{t=s=0}\Gamma^l_{mn}\mathrm{d}x^i\otimes\frac{\partial}{\partial y^l}\\&=(\frac{\partial^2F^l}{\partial s\partial x^i}+\frac{\partial F^m}{\partial x^i}\frac{\partial F^n}{\partial s}\Big|_{t=s=0}\Gamma^l_{mn})\mathrm{d}x^i\otimes\frac{\partial}{\partial y^l}\end{aligned}$$

which implies setting $t=s=0$ we have part II is

$$\int_M\langle\widehat{\nabla}v,\widehat{\nabla}w\rangle\,\mathrm{vol}$$

For part I, take arbitrary $X \in C^\infty(M, TM) \subset C^\infty(\overline{M}, T^*\overline{M})$, we have
Hence we obtain

$$\nabla^{T*\overline{M}\otimes W}_{\frac{\partial}{\partial s}} \nabla^W F_*(\frac{\partial}{\partial t})(X) = (\nabla^{T*\overline{M}\otimes W} \nabla^W F_*(\frac{\partial}{\partial t})(X))(\frac{\partial}{\partial s}, X)$$

Setting $t = s = 0$ we have
Hence

$$\frac{\partial^2}{\partial s \partial t}\Big|_{t=s=0} E(f_{s,t}) = \int_M \langle \widehat{\nabla}(\nabla^W_{\frac{\partial}{\partial s}} F_*(\frac{\partial}{\partial t})\Big|_{s=t=0}), \mathrm{d}f\rangle \operatorname{vol}$$
$$+ \int_M g^{ij} R_{pmqn} v^p w^q \frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j} \operatorname{vol} + \int_M \langle \widehat{\nabla} w, \widehat{\nabla} v\rangle \operatorname{vol}$$

If f is harmonic, that is $\widehat{\nabla}^*\mathrm{d}f = 0$, we obtain the desired formula. □

14.4. **Bochner formula for harmonic map.** Recall that for a smooth function $f\colon (M, g) \to \mathbb{R}$,

$$\frac{1}{2}\Delta|\mathrm{d}f|^2 = |\operatorname{Hess} f|^2 + \operatorname{Ric}(\nabla f, \nabla f) + g(\nabla\Delta f, \nabla f)$$

In this section we generalizes this formula to smooth map $f\colon (M, g) \to (N, h)$ between Riemannian manifolds, to get similar Bochner's theorem we have proven before.

Theorem 14.4.1. Let $f\colon (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds, then

$$\frac{1}{2}\Delta|\mathrm{d}f|^2 = |\widetilde{\nabla}\mathrm{d}f|^2 + \langle\widehat{\nabla}(\mathrm{d}f), \mathrm{d}f\rangle + g^{ik}g^{jl}R_{ij}\frac{\partial f^m}{\partial x^k}\frac{\partial f^n}{\partial x^l}h_{mn} - g^{kl}g^{ij}R_{mnpq}\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j}\frac{\partial f^p}{\partial x^k}\frac{\partial f^q}{\partial x^l}$$

Theorem 14.4.2. Let $f\colon (M, g) \to (N, h)$ be a harmonic map between Riemannian manifolds. If

1. M is compact with positive Ricci curvature.
2. N has non-positive sectional curvature.

Then f is constant.

Proof. Suppose $|\mathrm{d}f|^2$ attains its maximum at some point $p \in M$, we have

$$\Delta|\mathrm{d}f|^2(p) \leq 0$$

On the other hand,

$$\frac{1}{2}\Delta|\mathrm{d}f|^2 \geq g^{ik}g^{jl}R_{ij}\frac{\partial f^m}{\partial x^k}\frac{\partial f^n}{\partial x^l}h_{mn} - g^{kl}g^{ij}R_{mnpq}\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^j}\frac{\partial f^p}{\partial x^k}\frac{\partial f^q}{\partial x^l}$$

since $|\widetilde{\nabla}\mathrm{d}f|^2 + \langle\widehat{\nabla}(\mathrm{d}f), \mathrm{d}f\rangle \geq 0$.

Without lose of generality, we may assume $g_{ij}(p) = \delta_{ij}, h_{mn}(f(p)) = \delta_{mn}$ by choosing normal coordinates. Then

$$\frac{1}{2}\Delta|\mathrm{d}f|^2 \geq \sum_{i,j,m} R_{ij}\frac{\partial f^m}{\partial x^i}\frac{\partial f^m}{\partial x^j} - \sum_{i,j} R_{mnpq}\frac{\partial f^m}{\partial x^i}\frac{\partial f^n}{\partial x^i}\frac{\partial f^p}{\partial x^j}\frac{\partial f^q}{\partial x^j} \geq 0$$

since R_{ij} is positive, which implies $\mathrm{d}f \equiv 0$, thus f is constant since we always assume M is connected. □

Corollary 14.4.1. Let (M, g) be a compact Riemannian manifold with non-negative Ricci curvature, (N, h) a Riemannian manifold with non-positive sectional curvature, and $f\colon (M, g) \to (N, h)$ a harmonic map, then

1. f is totally geodesic.
2. If $\mathrm{Ric}(g)$ is strictly positive at some point, then f is constant.
3. If sectional curvature of h is negative, then either f is constant or its image is a closed geodesic.

Part 6. Topology of Riemannian manifold

15. Topology of non-positive sectional curvature manifold

15.1. Cartan-Hadamard manifold.

Definition 15.1.1 (Cartan-Hadamard manifold)**.** A simply-connected, complete Riemannian manifold with non-positive sectional curvature is called Cartan-Hadamard manifold.

15.1.1. *Expansion property of exponential map of Cartan-Hadamard manifold.* In this section we explore some properties of Cartan-Hadamard manifold using Jacobi fields.

Proposition 15.1.1. Let $p \in M$ and $\gamma\colon [0,1] \to M$ be a geodesic such that $\gamma(0) = p, \gamma'(0) = v$. Then for any $w \in T_pM$ with $|w| = 1$, let $J(t)$ be the Jacobi field along γ given by

$$J(t) = (\mathrm{d}\exp_p)_{tv}(tw)$$

Then we have the following Taylor expansions about $t = 0$

$$\begin{aligned}|J(t)|^2 &= t^2 - \frac{1}{3}R(J', \gamma', \gamma', J')(0)t^4 + O(t^4)\\ |J(t)| &= t - \frac{1}{6}R(J', \gamma', \gamma', J')(0)t^3 + O(t^3)\end{aligned}$$

Proof. For (1). Since $J(0) = 0, J'(0) = w$, the first three coefficients are given as

$$\begin{aligned}\langle J, J\rangle(0) &= 0\\ \langle J, J\rangle'(0) &= 2\langle J, J'\rangle(0) = 0\\ \langle J, J\rangle''(0) &= 2\langle J', J'\rangle(0) + 2\langle J'', J\rangle(0) = 2\\ \langle J, J\rangle'''(0) &= 6\langle J', J''\rangle(0) + 2\langle J''', J\rangle(0) = 0\\ &= 6\langle J', R(J, \gamma')\gamma'\rangle(0) = 0\\ \langle J, J\rangle''''(0) &= 8\langle J', J'''\rangle(0) + 6\langle J'', J''\rangle(0) + 2\langle J'''', J\rangle(0)\\ &= 8\langle J', J'''\rangle(0) + 6\langle R(J, \gamma')\gamma', R(J, \gamma')\gamma'\rangle(0)\\ &= 8\langle J', J'''\rangle(0)\end{aligned}$$

So we need to compute J'''. For arbitrary vector field W along γ, direct computation shows

$$\begin{aligned}\langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} R(J, \gamma')\gamma', W\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\langle R(J, \gamma')\gamma', W\rangle - \langle R(J, \gamma')\gamma, W'\rangle\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\langle R(W, \gamma')\gamma', J\rangle - \langle R(J, \gamma')\gamma, W'\rangle\\ &= \langle R(W, \gamma')\gamma', J'\rangle - \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} R(W, \gamma')\gamma', J\rangle - \langle R(J, \gamma')\gamma, W'\rangle\\ &= \langle R(J', \gamma')\gamma', W\rangle - \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} R(W, \gamma')\gamma', J\rangle - \langle R(J, \gamma')\gamma, W'\rangle\end{aligned}$$

Setting $t=0$ we obtain

$$\langle J', J'''\rangle(0) = -\langle J'(0), \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} R(J,\gamma')\gamma'\Big|_{t=0}\rangle = -R(J',\gamma',\gamma',J')(0)$$

So we have

$$|J(t)|^2 = t^2 - \frac{1}{3}R(J',\gamma',\gamma',J')(0)t^4 + O(t^4)$$

For (2). It follows directly from (1). □

Theorem 15.1.1. Let (M,g) be a simply-connected complete Riemannian manifold. The followings are equivalent:

1. M is Cartan-Hadamard manifold.
2. For any $p\in M$ and $v,w\in T_pM$, we have
$$|(\mathrm{d}\exp_p)_v w| \geq |w|$$
3. For any $p\in M, T>0$ and $v,w\in T_pM$, we have
$$|v-w| \leq \frac{\mathrm{dist}(\exp_p(tv), \exp_p(tw))}{t}$$
holds for arbitrary $t>0$.

Proof. From (1) to (2). For all $p\in M$ and $v,w\in T_pM$, consider geodesic $\exp_p(tv)$ and Jacobi field

$$J(t) = (\mathrm{d}\exp_p)_{tv}(tw)$$

along it. If M has non-positive sectional curvature, direct computation shows

$$|J(t)|'' = \frac{|J^2||J'|^2 - \langle J, J'\rangle^2}{|J|^3} - \frac{R(J,\gamma',\gamma',J)}{|J|} \geq 0$$

for all $t>0$. Thus consider

$$f(t) = |J(t)| - t|w|$$

It's clear $f''(t)\geq 0$ and $f'(0)=0$, thus $f(t)\geq 0$ for all $t>0$ since $f(0)=0$. In particular, set $t=1$ we have

$$|(\mathrm{d}\exp_p)_v(w)| - |w| \geq 0$$

From (2) to (1). If M has sectional curvature $K(\sigma)>0$ at $p\in M$, where σ is the plane spanned by v,w with $|v|=|w|=1$. Then consider geodesic $\exp_p(tv)$ and Jacobi field

$$J(t) = (\mathrm{d}\exp_p)_{tv}(tw)$$

along it. Then by Proposition 15.1.1 we have $|J(t)|''<0$ for sufficiently small t. If we set $f(t)=|J(t)|-t|w|$, then we can see $f(0)=0, f'(0)=0$ and $f''(0)<0$ for sufficiently small t. In particular, we have

$$|(\mathrm{d}\exp_p)_{\varepsilon v}(\varepsilon w)| - |\varepsilon w| = f(\varepsilon) < 0$$

where $\varepsilon>0$ is sufficiently small. This leads to a contradiction.

From (2) to (3). For arbitrary $t > 0$. Let $\gamma(s) : [0, 1] \to M$ be a geodesic connecting $\exp_p(tv), \exp_p(tw)$ and choose a curve $v(s) \in T_pM$ such that

$$\exp_p(v(s)) = \gamma(s)$$

for all $s \in [0, 1]$. Hence $v(0) = tv, v(1) = tw$. Then

$$\begin{aligned}
\operatorname{dist}(\exp_p(tv), \exp_p(tw)) &= \int_0^1 |\gamma'(s)|\mathrm{d}s \\
&= \int_0^1 |(\mathrm{d}\exp_p)_{v(s)}(v'(s))|\mathrm{d}s \\
&\geq |\int_0^1 v'(s)\mathrm{d}s| \\
&= t|v - w|
\end{aligned}$$

This shows

$$|v - w| \leq \frac{\operatorname{dist}(\exp_p(tv), \exp_p(tw))}{t}$$

holds for arbitrary $t > 0$.

From (3) to (2). Note that

$$\begin{aligned}
|(\mathrm{d}\exp_p)_v(w)| &= \lim_{t\to 0} \frac{\operatorname{dist}(\exp_p(v + tw), \exp_p(v))}{t} \\
&= \lim_{t\to 0} \frac{\operatorname{dist}(\exp_p(tv' + tw), \exp_p(tv'))}{t} \\
&\geq |v' + w - v'| \\
&= |w|
\end{aligned}$$

□

Corollary 15.1.1. Let (M, g) be a Cartan-Hadamard manifold with $a, b, c \in M$. Such points determine a unique geodesic triangle T with vertices a, b, c. Let α, β, γ be the angles of the vertices a, b, c respectively, and let A, B, C be the lengths of the side opposite the vertices a, b, c respectively. Then

1. $A^2 + B^2 - 2AB\cos\gamma \leq C^2 (< C^2, \text{if } K < 0)$.
2. $\alpha + \beta + \gamma \leq \pi (< \pi, \text{if } K < 0)$

Proof. See Lemma 3.1 in Page259 of [Car92]. □

So you find that the exponential map of simply-connected complete Riemannian manifold with non-positive sectional curvature has a property of "expansion".

15.1.2. *Complete Riemannian manifold with non-positive sectional curvature is $K(G, 1)$.*

Lemma 15.1.1. If (M, g) is a complete Riemannian manifold with sectional curvature $K \leq 0$, then for any $p \in M$, the conjugate locus $\operatorname{conj}(p) = \varnothing$. In particular, $\exp_p : T_pM \to M$ is a local diffeomorphism.

Proof. Suppose q is conjugate to p along $\gamma\colon [0,1]\to M$, and without lose of generality we may assume there is no conjugate point for $t\in(0,1)$. Let J be a Jacobi field along γ with $J(0)=J(1)=0$, then

$$\begin{aligned}(\frac{1}{2}|J|^2)' &= (g(J',J))'\\ &= g(J'',J)+g(J',J')\\ &= -R(J,\gamma',\gamma',J)+|J'|^2\\ &\geq |J'|^2\end{aligned}$$

Since $J'(0)\neq 0$, we have

$$\begin{aligned}g(J',J)(t) &\geq \int_0^t |J'|^2+g(J'(0),J(0))\\ &= \int_0^t |J'|^2\\ &> 0\end{aligned}$$

which implies $(\frac{1}{2}|J|^2)'=g(J',J)>0$, a contradiction to $J(1)=0$. □

Theorem 15.1.2 (Cartan-Hadamard)**.** If (M,g) is a complete Riemannian manifold with sectional curvature $K\leq 0$, then $\exp_p\colon T_pM\to M$ is a covering map.

Proof. Lemma 15.1.1, together with Proposition B.2.4 and Theorem 15.1.1 completes the proof. □

Corollary 15.1.2. Cartan-Hadamard manifold is diffeomorphic to $\mathbb{R}^n$.

Corollary 15.1.3. If (M,g) is a complete Riemannian manifold with $K\leq 0$, then $\pi_k(M)=0, k\geq 2$, that is M is $K(\pi_1(M),1)$.

Remark 15.1.1. Thoery in topology says if a finite dimension CW complex is a $K(G,1)$ space, then its fundamental group is torsion-free. So if M is a complete Riemannian manifold with $K\leq 0$, we have $\pi(M)$ is torsion-free. We will prove this fact later by tools of Riemannian manifold, called Cartan's torsion-free theorem.

Corollary 15.1.4. If M and N are two compact Riemannian manifold and one of them is simply-connected, then $M\times N$ has no metric with non-positive sectional curvature.

Proof. If both of M and N are simply-connected, and $M\times N$ admits a metric with non-positive sectional curvature, then it's diffeomorphic to $\mathbb{R}^n$ for some positive integer n, a contradiction to compactness.

So suppose M is simply-connected and N is not simply-connected with universal covering $\widetilde{N}$, then there is a universal covering

$$\pi: M\times\widetilde{N}\to M\times N$$

If $M \times N$ admits a Riemannian metric g with non-positive sectional curvature, then $\pi^* g$ is a complete metric of non-positive sectional curvature on $M \times \widetilde{N}$, so we have $M \times \widetilde{N}$ is diffeomorphic to $\mathbb{R}^n$ for some n. M is orientable since it's simply-connected, thus $H^m(M) = \mathbb{Z}$, where $m = \dim M$, thus by Künneth formula $H^m(M \times \widetilde{N}) \neq 0$, a contradiction to Poincaré lemma. $\square$

Remark 15.1.2. The condition simply-connected is crucial, for example $S^1 \times S^1$.

15.2. **Cartan's torsion-free theorem.**

Lemma 15.2.1. Let (M, g) be a Cartan-Hadamard manifold, $p \in M$ and $v \in T_pM$. For all $q \in M$ we have

$$2 \operatorname{dist}(p, q)^2 + \operatorname{dist}(p_0, p)^2 + \operatorname{dist}(p_1, p)^2 \leq \operatorname{dist}(p_0, q)^2 + \operatorname{dist}(p_1, q)^2$$

where $p_0 = \exp_p(-v), p_1 = \exp_p(v)$.

Proof. Since $\exp_p \colon T_pM \to M$ is a diffeomorphism, there exists $w \in T_pM$ such that $q = \exp_p(w)$ with $\operatorname{dist}(p, q) = |w|$. So we have

$$\begin{aligned}
\operatorname{dist}(p_0, q) &= \operatorname{dist}(\exp_p(-v), \exp_p(w)) \geq |w + v| \\
\operatorname{dist}(p_1, q) &= |w - v| \\
\operatorname{dist}(p, q)^2 &= |w|^2 \\
&= \frac{|w + v|^2 + |w - v|^2}{2} - |v|^2 \\
&\leq \frac{\operatorname{dist}(p_0, q)^2 + \operatorname{dist}(p_1, q)^2}{2} - \frac{\operatorname{dist}(p_0, p)^2 + \operatorname{dist}(p_1, p)^2}{2}
\end{aligned}$$

$\square$

Lemma 15.2.2 (Serre)**.** Let (M, g) be a Cartan-Hadamard manifold, $p \in M$ and $B(p, r)$ the closed ball of radius r. If $\Omega \subset M$ is non-empty bounded set and define

$$r_\Omega = \inf\{r > 0 \mid \Omega \subset B(p, r), p \in M\}$$

There exists a unique $p_\Omega \in M$ such that $\Omega \subset B(p_\Omega, r_\Omega)$.

Proof. Existence: Choose a sequence $r_i > r_\Omega$ and $p_i \in M$ such that

$$\Omega \subset B(p_i, r_i), \lim r_i = r_\Omega$$

Fix arbitrary $q \in \Omega$, one has $\operatorname{dist}(q, p_i) \leq r_i$ for each i, thus $\{p_i\}$ is bounded since we can choose $\{r_i\}$ is bounded, which has a convergent subsequence since M is complete. The limit of this convergent subsequence is p_Ω.

Uniqueness: Let $p_0, p_1 \in M$ such that

$$\Omega \subset B(p_0, r_\Omega) \cap B(p_1, r_\Omega)$$

Since $\exp_{p_0}$ is a diffeomorphism, there exists unique v_0 such that $p_1 = \exp_{p_0} v_0$. Set $p = \exp_{p_0}(v_0/2)$, for all $q \in \Omega$ we have

$$\begin{aligned}\operatorname{dist}(p,q)^2 &\leq \frac{\operatorname{dist}(p_0,q)^2 + \operatorname{dist}(p_1,q)^2}{2} - \frac{\operatorname{dist}(p_0,p_1)^2}{4} \\ &\leq r_\Omega^2 - \frac{\operatorname{dist}(p_0,p_1)^2}{4}\end{aligned}$$

By definition of r_Ω, we have $\operatorname{dist}(p_0,p_1) = 0$, hence $p_0 = p_1$. □

Theorem 15.2.1 (Cartan's fixed-point theorem)**.** Let (M,g) be a Cartan-Hadamard manifold and G a compact Lie group acting isometrically on M, then G has a fixed-point.

Proof. Let $p \in M$, consider its orbit

$$\Omega = \{gp \mid g \in G\}$$

it's a bounded since M is compact. Note

$$\Omega = g\Omega \subset B(gp_\Omega, r_\Omega)$$

Then by uniqueness of p_Ω, we have p_Ω is a fixed-point of G. □

Corollary 15.2.1. If (M,g) is a complete Riemannian manifold with $K \leq 0$, then $\pi_1(M)$ is torsion-free.

Proof. Let $(\widetilde{M}, \widetilde{g})$ be the universal covering of M with pullback metric. Then $(\widetilde{M}, \widetilde{g})$ is a Cartan-Hadamard manifold, and M is isometric to a Riemannian quotient $\widetilde{M}/\Gamma$, where Γ is a subgroup of $\operatorname{Iso}(\widetilde{M}, \widetilde{g})$, which is isomorphic to $\pi_1(M)$.

Now it suffices to show Γ has no torsion element. If there exists a torsion element φ, consider the finite group G generated by φ, it's a 0-dimension Lie group with discrete topology. By Cartan's fixed-point theorem there exists a fixed-point of G, which implies φ is identity, since Γ acts on $\widetilde{M}$ freely. □

15.3. **Preissmann's Theorem.**

Definition 15.3.1 (axis)**.** Let (M,g) be a complete Riemannian manifold, $\varphi\colon M \to M$ is an isometry. A non-trivial geodesic $\gamma\colon \mathbb{R} \to M$ is called an axis of φ if $\varphi \circ \gamma$ is a non-trivial translation of γ, that is there exists $c \neq 0$ such that

$$\varphi(\gamma(t)) = \gamma(t+c)$$

Definition 15.3.2 (axial)**.** An isometry with no fixed points that has an axis is said to be axial.

Lemma 15.3.1. $\varphi\colon (M,g) \to (M,g)$ is an isometry of complete Riemannian manifold, if $\delta_\varphi(p) = \operatorname{dist}(p,\varphi(p))$ has a positive minimum, then φ has a axis.

Proof. Suppose δ_φ attains its minimum at some $p \in M$ and $\gamma(t) : [0,1] \to M$ is a minimum geodesic connecting p and $\varphi(p)$, then $\varphi \circ \gamma \colon [0,1] \to M$ is also a minimum geodesic connecting $\varphi(p)$ and $\varphi^2(p)$, since φ is an isometry. We claim these two geodesics form an angle π at point $\varphi(p)$ and thus fit together an extension of γ to $[0,2]$. Indeed, for any $t \in [0,1]$,

$$\begin{aligned}
\delta_\varphi(p) &= \operatorname{dist}(p, \varphi(p)) \\
&\leq \delta_\varphi(\gamma(t)) \\
&= \operatorname{dist}(\gamma(t), \varphi \circ \gamma(t)) \\
&\leq \operatorname{dist}(\gamma(t), \gamma(1)) + \operatorname{dist}(\gamma(1), \varphi \circ \gamma(t)) \\
&= \operatorname{dist}(\gamma(t), \gamma(1)) + \operatorname{dist}(\varphi \circ \gamma(0), \varphi \circ \gamma(t)) \\
&= \operatorname{dist}(\gamma(t), \gamma(1)) + \operatorname{dist}(\gamma(0), \gamma(t)) \\
&= \delta_\varphi(p)
\end{aligned}$$

Thus we have $(\varphi \circ \gamma)(t) = \gamma(1+t)$ for $0 \leq t \leq 1$. Repeating this argument to obtain a geodesic $\gamma \colon \mathbb{R} \to M$ with period 1, and it's an axis for φ. □

Lemma 15.3.2. Let (M, g) be a compact Riemannian manifold and $\varphi \colon \widetilde{M} \to \widetilde{M}$ a non-trivial deck transformation, where $\widetilde{M}$ is the universal covering of M. Then

1. δ_φ has a positive minimum and $\delta_\varphi \geq 2 \operatorname{inj}(M)$. In particular, φ has an axis $\gamma \colon \mathbb{R} \to \widetilde{M}$.
2. $\pi \circ \gamma$ is a closed geodesic in M whose length is minimal in the homotopy class $[\pi \circ \gamma]$.

Lemma 15.3.3. Let (M, g) be a Cartan-Hadamard manifold with $K < 0$, if isometry $\varphi \colon M \to M$ has an axis, then it's unique up to reparametrization.

Proof. Suppose $\gamma_1, \gamma_2 : \mathbb{R} \to M$ are two axes of φ, without lose of generality we may assume

$$\begin{aligned}
\varphi(\gamma_1(t)) &= \gamma_1(t+1) \\
\varphi(\gamma_2(t)) &= \gamma_2(t+1)
\end{aligned}$$

Suppose γ_1, γ_2 do not intersect, then points $A = \gamma_1(0), B = \gamma_1(1) = \varphi(A), C = \gamma_2(0)$ and $D = \gamma_2(1) = \varphi(C)$ are all distinct. Let γ be a geodesic from A to C, then $\varphi \circ \gamma$ is the geodesic from B to D. Furthermore, the geodesic quadrilateral $ABCD$ has angle sum 2π, since φ preserves angles. However, according to Lemma 15.1.1, triangle $\triangle ABC$ and $\triangle BCD$ have angle sum strictly less than π, and

$$\begin{aligned}
\angle ACD &\leq \angle ACB + \angle BCD \\
\angle ABD &\leq \angle ABC + \angle CBD
\end{aligned}$$

thus the angle sum of $ABCD$ is strictly less than 2π, a contradiction. Hence γ_1 and γ_2 must intersect at some point $p = \gamma_1(t_1) = \gamma_2(t_2)$, then

$$\begin{aligned}
\varphi(p) &= \varphi(\gamma_1(t_1)) = \gamma_1(t_1 + 1) \\
&= \varphi(\gamma_2(t_2)) = \gamma_2(t_2 + 1)
\end{aligned}$$

is another intersection point. Since (M, g) is a Cartan-Hadamard manifold, any two points are joined by a unique geodesic, thus γ_1 is a reparametrization of γ_2. □

Lemma 15.3.4. If H is a additive subgroup of $\mathbb{R}$, then either H is dense in $\mathbb{R}$ or $H \cong \mathbb{Z}$.

Proof. Let H be an additive subgroup of $\mathbb{R}$, it's clear $H \cap \mathbb{R}_{>0} \neq \varnothing$, consider

$$b := \inf\{h \in H \cap \mathbb{R}_{>0}\}$$

1. If $b > 0$: Let $h \in H$ and $k \in \mathbb{Z}$ such that

   $$kb \leq |h| < (k+1)b$$

   then we have $|h| - kb \in H$, and $0 \leq |h| - kb < (k+1)b - kb = b$. By the choice of b, we have $|h| - kb$, which implies $h = \pm kb$. In this case $H = b\mathbb{Z}$.
2. If $b = 0$: For arbitrary $r \in \mathbb{R}_{\geq 0}$ and $\varepsilon > 0$, there exists $h \in H \cap (0, \varepsilon]$ since $b = 0$ and $k \in \mathbb{N}$ such that

   $$kh \leq r \leq (k+1)h$$

   Thus

   $$0 \leq r - kh \leq (k+1)h - kh = h \leq \varepsilon$$

   which implies $|r - kh| \leq \varepsilon$, that is H is dense in $\mathbb{R}_{\geq 0}$. For the same argument you can show H is also dense in $\mathbb{R}_{\leq 0}$.

□

Theorem 15.3.1 (Preissmann)**.** If (M, g) is a compact Riemannian manifold with negative sectional curvature, then any non-trivial abelian subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$.

Proof. Let $(\widetilde{M}, \widetilde{g})$ be the universal covering of M equipped with pullback metric, then it's a Cartan-Hadamard manifold with negative sectional curvature. Now it suffices to show every non-trivial abelian subgroup H of group consisting of deck transformations is isomorphic to $\mathbb{Z}$. Let φ be a non-trivial deck transformation in H, then Lemma 15.3.2, φ has an axis $\gamma\colon \mathbb{R} \to \widetilde{M}$, that is there exists $c \neq 0$ such that

$$\varphi \circ \gamma(t) = \gamma(t + c)$$

for all $t \in \mathbb{R}$. If ψ is another non-trivial element of H, then for any $t \in \mathbb{R}$ we have

$$\varphi \circ \psi(\gamma(t)) = \psi \circ \varphi(\gamma(t)) = \psi \circ \gamma(t + c)$$

which implies $\psi \circ \gamma$ is also an axis of φ. So by Lemma 15.3.3 we have $\psi \circ \gamma$ is a reparametrization of γ. Furthermore, $\psi \circ \gamma$ and γ have the same speed since ψ is an isometry, thus there are two cases:

1. $\psi \circ \gamma(t) = \gamma(t + a)$.
2. $\psi \circ \gamma(t) = \gamma(-t + a)$

(2) can't happen, otherwise $\psi \circ \gamma(\frac{a}{2}) = \gamma(\frac{a}{2})$, contradicts to deck transformation acts on $\widetilde{M}$ freely. Consider

$$\begin{aligned} f\colon H &\to \mathbb{R} \\ \psi &\mapsto a \end{aligned}$$

where a is determined $\psi \circ \gamma(t) = \gamma(t+a)$. It's easy to see F is a group homomorphism with trivial kernel thus $F(H)$ is an additive subgroup of $\mathbb{R}$. Consider

$$b := \inf\{h \in F(H) \cap \mathbb{R}_{>0}\}$$

By Lemma 15.3.4, it suffices to show $b > 0$. If $b = 0$, then there exist $a \in (0, \mathrm{inj}(M))$ and $\psi \in H$ such that $a = F(\psi)$, that is

$$\psi \circ \gamma(t) = \gamma(t+a)$$

Since $\pi \circ \psi = \pi$, we have $\pi \circ \gamma(t) = \pi \circ \gamma(t+a)$. Set $t = 0$ one has

$$\pi \circ \gamma(a) = \pi \circ \gamma(0)$$

A contradiction to $0 < a < \mathrm{inj}(M)$ since $\pi \circ \gamma$ is a geodesic. $\square$

Corollary 15.3.1. Suppose M and N are compact smooth manifolds, then $M \times N$ doesn't admit a Riemannian metric with negative sectional curvature.

Proof. If $M \times N$ admits a Riemannian metric with negative sectional curvature, Cartan's torsion-free theorem implies $\pi_1(M \times N)$ is torsion-free, thus for arbitrary $\alpha \in \pi_1(M), \beta \in \pi_1(N)$, unless either M or N is simply-connected, $\pi_1(M \times N)$ will contain an abelian subgroup $\mathbb{Z} \times \mathbb{Z}$ generated by α, β, which contradicts to Preissmann's theorem.

So we may assume M is simply-connected, then consider the universal covering $M \times \widetilde{N}$ of $M \times N$, Cartan-Hadamard's theorem implies it's diffeomorphic to $\mathbb{R}^n$ for $n \in \mathbb{Z}_{>0}$, but M is orientable since it's simply-connected, so $H^m(M) = \mathbb{Z}$ where $m = \dim M$. So by Künneth formula $H^n(M \times \widetilde{N}) \neq 0$, a contradiction to Poincaré lemma. $\square$

Lemma 15.3.5. Let (M, g) be a complete Riemannian manifold with non-positive sectional curvature and $\widetilde{M}$ is its universal covering. If $\gamma\colon \mathbb{R} \to \widetilde{M}$ is a common axis for all deck transformations, then M is not compact.

Theorem 15.3.2 (Preissmann)**.** If (M, g) is a compact Riemannian manifold with negative sectional curvature, then $\pi_1(M)$ is not abelian.

Proof. Suppose $\pi_1(M)$ is abelian, then let γ be the axis of some deck transformation, then it's the axis of all deck transformations since $\pi_1(M)$ is abelian, which implies M is non-compact, a contradiction. $\square$

15.4. **Other facts.**

Theorem 15.4.1 (Byers)**.** If (M, g) is a compact Riemannian manifold with negative sectional curvature, then any non-trivial solvable subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$.

Theorem 15.4.2 (Yau)**.** Let (M, g) be a compact Riemannian manifold with non-positive sectional curvature. If $\pi_1(M)$ is solvable, then M is flat.

Theorem 15.4.3 (Farrell-Jones)**.** Let $(M_i, g_i), i = 1, 2$ be two compact Riemannian manifolds with non-positive sectional curvature. If $\pi_1(M_1) = \pi_1(M_2)$, then M_1 and M_2 are homeomorphic.

16. Topology of positive curvature manifold

16.1. **Myers' theorem.**

Theorem 16.1.1 (Myers)**.** Let (M, g) be a complete Riemannian n-manifold with $\mathrm{Ric}(g) \geq \frac{n-1}{R^2} g$, then
1. $\mathrm{diam}(M) \leq \pi R$.
2. M is compact.

Proof. For (1). If $\mathrm{diam}(M) > \pi R$, then there exists $b > \pi R$ and a (locally) minimal geodesic $\gamma\colon [0, b] \to M$ of unit-speed, since M is complete. Choose a parallel orthonormal basis $\{e_1(t), e_2(t), \ldots, e_n(t)\}$ along γ with $e_1(t) = \gamma'(t)$, and for each $i = 2, \ldots, n$

$$V_i(t) = \sin(\frac{\pi t}{b}) e_i(t)$$

It's clear $V_i(0) = V_i(b) = 0$ for $2 \leq i \leq n$. Let $\alpha\colon (-\varepsilon, \varepsilon) \to M$ be a variation of γ with variation field $V(t) = \sum_{i=2}^n V_i(t)$, then by second variation formula we have

$$\begin{aligned}
\frac{\mathrm{d}^2 L(\alpha(t,s))}{\mathrm{d}s^2}\Big|_{s=0} &= \sum_{i=2}^n \int_0^b \langle \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V_i, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} V_i \rangle \mathrm{d}t - \sum_{i=2}^n \int_0^b R(V_i, \gamma', \gamma', V_i) \mathrm{d}t \\
&= \sum_{i=2}^n \int_0^b (\frac{\pi}{b})^2 \cos^2(\frac{\pi t}{l}) \mathrm{d}t - \sum_{i=2}^n \int_0^b \sin^2(\frac{\pi t}{l}) R(e_i, e_1, e_1, e_i) \mathrm{d}t \\
&\leq (n-1)(\frac{\pi}{b})^2 \int_0^l \cos^2(\frac{\pi t}{b}) \mathrm{d}t - \frac{(n-1)}{R^2} \int_0^b \sin^2(\frac{\pi t}{b}) \mathrm{d}t \\
&< 0
\end{aligned}$$

A contradiction to γ is minimal.

(2) follows from (1). □

Corollary 16.1.1. Let M be a complete Riemannian manifold with positive Ricci curvature, then the universal covering of M is compact. In particular, the fundamental group $\pi_1(M)$ is finite.

Proof. Endow the universal covering $\widetilde{M}$ with pullback metric, thus $\widetilde{M}$ is a complete Riemannian manifold with positive Ricci curvature, thus $\widetilde{M}$ is compact, which implies $\pi : \widetilde{M} \to M$ is a finite covering, thus $\pi_1(M)$ is finite, since $|\pi_1(M)|$ equals the number of sheets of covering. □

Remark 16.1.1. The estimate for the diameter given by Myers's theorem can't be improved. Indeed, the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ has constant sectional curvature $K = 1$ and $\mathrm{diam}(\mathbb{S}^n) = \pi$.

A surprising theorem is that this example is unique in the following sense: Let (M, g) be a complete Riemannian n-manifold, $\mathrm{Ric}(g) \geq \frac{n-1}{R^2} g$ and $\mathrm{diam}(M) = \pi R$, then M is isometric to sphere $\mathbb{S}^n(R)$ with standard metric, that's Cheng's theorem, we will see it in Theorem 20.4.2.

16.2. **Synge's theorem.**

Lemma 16.2.1. Let A be an orthogonal linear transformation of $\mathbb{R}^{n-1}$ and suppose $\det A = (-1)^n$. Then 1 is an eigenvalue of A.

Proof. We consider the following cases:

1. If n is even, then $\det(\lambda I - A)$ is a polynomial of odd degree, therefore A has at least a real eigenvalue, and it must be ± 1 since A is orthogonal. Furthermore, since $\det A = 1$ and the product of complex eigenvalue is positive, there is at least a real eigenvalue which equals 1.
2. If n is odd, then $\det A = -1$. Because the product of complex eigenvalue is positive, there are at least two real eigenvalues, and one of them is 1.

$\square$

Theorem 16.2.1 (Synge)**.** Let (M, g) be a compact Riemannian manifold with positive sectional curvature, then

1. If $\dim M$ is even and orientable, then M is simply-connected.
2. If $\dim M$ is odd, then M is orientable.

Proof. Let $(\widetilde{M}, \widetilde{g})$ be the universal covering of M equipped with pullback metric.

1. If $\dim M$ is even, equip $\widetilde{M}$ the pullback orientation.
2. If $\dim M$ is odd, equip $\widetilde{M}$ arbitrary orientation.

Suppose the conclusions are not correct, thus $\pi_1(M)$ is non-trivial. Choose a non-trivial deck transformation $f\colon \widetilde{M} \to \widetilde{M}$ such that

1. If $\dim M$ is even, F is orientation preserving.
2. If $\dim M$ is odd, F is orientation reserving.

By Lemma, there exists an axis $\widetilde{\gamma} : \mathbb{R} \to \widetilde{M}$ for F and $\gamma = \pi \circ \widetilde{\gamma}$ is a closed geodesic in M that minimizes the length in $[\gamma]$,

$$F(\widetilde{\gamma}(t)) = \widetilde{\gamma}(t+1)$$

$\square$

Corollary 16.2.1. Let (M, g) be a compact Riemannian manifold with even dimension, if M is non-orientable, then $\pi_1(M) = \mathbb{Z}_2$.

Proof. If M is non-orientable, it has a double-sheeted orientable covering manifold $\widetilde{M}$, Synge's theorem implies $\widetilde{M}$ is simply-connected, thus it's the universal covering of M, which implies $\pi_1(M) = \mathbb{Z}_2$. $\square$

Example 16.1. $\mathbb{RP}^n \times \mathbb{RP}^n$ admits no Riemannian metric with positive sectional curvature, since its fundamental group is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Conjecture 16.1 (Hopf conjecture)**.** Does $S^2 \times S^2$ admit a Riemannian metric with positive sectional curvature?

16.3. **Other facts.**

Theorem 16.3.1. Let (M, g) be a compact, simply-connected Riemannian n-manifold.

1. (Hamilton) If $n = 3$, $\mathrm{Ric}(g) > 0$, then M is diffeomorphism to S^3.
2. (Hamilton) If $n = 4$ with curvature operator > 0, then M is diffeomorphism to S^4.
3. (Böhm-Wilking) If curvature operator > 0, then M is diffeomorphism to S^n.

Theorem 16.3.2 (soul theorem)**.** Let (M, g) be a complete, non-compact Riemannian n-manifold,

1. If M has non-negative sectional curvature, then there exists a compact totally geodesic submanifold $S \subseteq M$ (called a soul of M) such that M is diffeomorphic to the normal bundle of S in M.
2. If M has positive sectional curvature, then its soul is a point and M is diffeomorphic to $\mathbb{R}^n$.

Theorem 16.3.3 (differentiable sphere theorem)**.** Let (M, g) be a compact, simply-connected Riemannian n-manifold with $n \geq 4$. If sectional curvature satisfies $\frac{1}{4} < K \leq 1$, then M is diffeomorphism to S^n.

17. Topology of constant sectional curvature manifold

17.1. Cartan-Ambrose-Hicks theorem.

Theorem 17.1.1 (Cartan-Ambrose-Hicks)**.** Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be two Riemannian manifold $p \in M, \widetilde{p} \in \widetilde{M}$ and $\Phi_0 : T_pM \to T_{\widetilde{p}}\widetilde{M}$ some fixed linear isometry. Suppose $0 < \delta < \min\{\mathrm{inj}_p(M), \mathrm{inj}_{\widetilde{p}}(\widetilde{M})\}$. Then the followings are equivalent:

1. There exists an isometry $\varphi\colon B(p,\delta) \to B(\widetilde{p},\delta)$ such that $\varphi(p) = \widetilde{p}$ and $(\mathrm{d}\varphi)_p = \Phi_0$.
2. For $v \in T_pM, |v| < \delta, \gamma(t) = \exp_p(tv), \widetilde{\gamma}(t) = \exp_{\widetilde{p}}(t\Phi_0 v)$, if
$$\Phi_t = P^{\widetilde{\gamma}}_{0,t} \circ \Phi_0 \circ P^{\gamma}_{t,0} : T_{\gamma(t)}M \to T_{\widetilde{\gamma}(t)}\widetilde{M}$$
then Φ_t preserves curvature, that is
$$R(x, y, z, w) = \widetilde{R}(\Phi_t x, \Phi_t y, \Phi_t z, \Phi_t w)$$
where $x, y, z, w \in T_{\gamma(t)}M$.

Proof. From (1) to (2). If we can show $\Phi_t = (\mathrm{d}\varphi)_{\gamma(t)}$, then it's clear that Φ_t preserves curvature, since φ is an isometry. By definition of Φ_t, it suffices to show the following diagram commutes

$$\begin{array}{ccc} T_pM & \xrightarrow{(\mathrm{d}\varphi)_p} & T_{\widetilde{p}}\widetilde{M} \\ \downarrow{\scriptstyle P^{\gamma}_{0,t}} & & \downarrow{\scriptstyle P^{\widetilde{\gamma}}_{0,t}} \\ T_{\gamma(t)}M & \xrightarrow{(\mathrm{d}\varphi)_{\gamma(t)}} & T_{\widetilde{\gamma}(t)}\widetilde{M} \end{array}$$

since $(\mathrm{d}\varphi)_p = \Phi_0$. Note that $\varphi(\gamma(t)) = \widetilde{\gamma}(t)$ since both of them are geodesics, and they and their derivatives agree at $t = 0$. So it's tautological that
$$P^{\varphi\circ\gamma}_{0,t} \circ (\mathrm{d}\varphi)_p(v) = (\mathrm{d}\varphi)_{\gamma(t)} \circ P^{\gamma}_{0,t}(v)$$
where $v = \gamma'(0)$, since
$$\begin{aligned} P^{\gamma}_{0,t}(v) &= \gamma'(t) \\ (\mathrm{d}\varphi)_{\gamma(t)}(\gamma'(t)) &= (\varphi\circ\gamma)'(t) = P^{\varphi\circ\gamma}_{0,t} \circ (\mathrm{d}\varphi)_p(v) \end{aligned}$$
Now consider $w \in T_pM$ which is not parallel to $v = \gamma'(0)$. Since both $(\mathrm{d}\varphi)_{\gamma(t)}$ and parallel transport preserve angles, so $P^{\varphi\circ\gamma}_{0,t} \circ (\mathrm{d}\varphi)_p(w)$ and $(\mathrm{d}\varphi)_{\gamma(t)} \circ P^{\gamma}_{0,t}(w)$ has the same angle with $(\mathrm{d}\varphi)_{\gamma(t)}(\gamma'(t))$, and the they have the same length, so they're equal.

From (2) to (1). Define
$$\varphi = \exp_{\widetilde{p}} \circ \Phi_0 \circ \exp_p^{-1}$$
It suffices to show for any $q \in B(p, \delta)$,
$$(\mathrm{d}\varphi)_q : T_qM \to T_{\varphi(q)}\widetilde{M}$$

is a linear isometry. For any $w \in T_qM$, by Corollary 12.1.2, there exists a geodesic $\gamma\colon [0,1] \to M$ with $\gamma(0) = p, \gamma(1) = q$ and a Jacobi field J such that $J(0) = 0, J(1) = w$ along γ. Now we claim:

1. **Claim 1**: $\widetilde{J}(t) = \Phi_t(J(t))$ is a Jacobi field.
2. **Claim 2**: $\widetilde{J}(1) = (\mathrm{d}\varphi)_q(J(1))$.

From claim 2 we have

$$|(\mathrm{d}\varphi)_q(w)| = |\widetilde{J}(1)| = |J(1)| = |w|$$

since Φ_t preserves length. This completes the proof. Now let's give proofs of these two claims.

1. **Proof of Claim 1**: Given an orthonormal $\{e_1(0) = \frac{\gamma'(0)}{|\gamma'(0)|}, e_2(0), \dots, e_n(0)\}$ of T_pM and use parallel transport to obtain a parallel frame along γ. With respect to this frame we can write $J(t) = J^i(t)e_i(t)$, then $\widetilde{J}(t) = J^i(t)\widetilde{e}_i(t)$, where $\widetilde{e}_i(t) = \Phi_t(e_i(t))$. Furthermore, $\widetilde{e}_i(t)$ is also a parallel frame by definition of Φ_t. Then $\widetilde{J}(t)$ is a Jacobi field, since
$$\begin{aligned}&\frac{\mathrm{d}^2 J^j}{\mathrm{d}t^2} + J^i(t)|\widetilde{\gamma}(t)|^2\widetilde{R}(\widetilde{e}_i(t), \widetilde{e}_1(t), \widetilde{e}_1(t), \widetilde{e}_j(t))\\ =&\frac{\mathrm{d}^2 J^j}{\mathrm{d}t^2} + J^i(t)|\gamma(t)|^2 R(e_i(t), e_1(t), e_1(t), e_j(t))\\ =&0\end{aligned}$$
holds for arbitrary j, where we use the fact Φ_t preserves the length and curvature, and $J(t)$ is a Jacobi field.
2. **Proof of Claim 2**: Since $\widetilde{J}(t) = \Phi_t(J(t))$, then $\widetilde{J}'(0) = \Phi_0 J'(0)$. On the other hand, by Corollary one has
$$\begin{aligned}J(t) &= (\mathrm{d}\exp_p)_{t\gamma'(0)}(tJ'(0))\\ \widetilde{J}(t) &= (\mathrm{d}\exp_{\widetilde{p}})_{t\widetilde{\gamma}'(0)}(t\widetilde{J}'(0))\end{aligned}$$
Therefore
$$\begin{aligned}\widetilde{J}(1) &= (\mathrm{d}\exp_{\widetilde{p}})_{\widetilde{\gamma}'(0)} \circ \Phi_0(J'(0))\\ &= (\mathrm{d}\exp_{\widetilde{p}})_{\widetilde{\gamma}'(0)} \circ \Phi_0 \circ (\mathrm{d}\exp_p)^{-1}_{\gamma'(0)}(J(1))\end{aligned}$$
which completes the proof of claim 2.

□

Theorem 17.1.2. Let (M, g) be a Riemannian manifold. Suppose φ and ψ are two local isometries from (M, g) to $(\widetilde{M}, \widetilde{g})$. If there exists $p \in M$ such that
$$\begin{aligned}\varphi(p) &= \psi(p)\\ (\mathrm{d}\varphi)_p &= (\mathrm{d}\psi)_p\end{aligned}$$
Then $\varphi = \psi$.

Proof. Suppose $\varphi|_U, \psi|_U$ is diffeomorphism and U is a normal coordinate, then

$$f := (\varphi^{-1} \circ \psi)|_U : U \to U$$

satisfies $f(p) = p, (\mathrm{d}f)_p = \mathrm{id}$. Given $q \in V$, there exists unique $v \in T_pM$ such that $\exp_p(v) = q$, then

$$f(q) = \exp_p \circ \mathrm{id} \circ \exp_p^{-1}(q) = q$$

which implies φ agrees with ψ in U. Consider the following set

$$D = \{p \in M \mid \psi(p) = \varphi(p)\}$$

Above argument shows it's open, and it's clearly closed, since it's zero set of smooth function. Then $D = M$, since we always assume our manifold M is connected. This completes the proof. □

17.2. **Hopf's theorem.**

Theorem 17.2.1 (Hopf)**.** Let (M, g) be a simply-connected complete Riemannian manifold with constant sectional curvature K, then (M, g) is isometric to $(\widetilde{M}, g_{\mathrm{can}})$, where

$$(\widetilde{M}, g_{\mathrm{can}}) = \begin{cases} \mathbb{S}^n(\frac{1}{\sqrt{K}}), & K > 0 \\ \mathbb{R}^n, & K = 0 \\ \mathbb{H}^n(\frac{1}{\sqrt{-K}}), & K < 0 \end{cases}$$

Proof. Let M be a simply-connected complete Riemannian manifold with constant sectional curvature K.

1. If $K \leq 0$, let $\widetilde{M} = \mathbb{R}^n$ or $\mathbb{H}^n(\frac{1}{\sqrt{-K}})$. Fix $p \in M, \widetilde{p} \in \widetilde{M}$ and a linear isometry $\Phi_0 : T_{\widetilde{p}}\widetilde{M} \to T_pM$, Cartan-Ambrose-Hicks's theorem implies

$$\varphi = \exp_p \circ \Phi_0 \circ \exp_{\widetilde{p}}^{-1} : \widetilde{M} \to M$$

is a local isometry. Furthermore, Cartan-Hadamard's theorem implies φ is a diffeomorphism, since $M, \widetilde{M}$ are simply-connected with non-positive sectional curvature. This completes the proof of this part.

2. If $K > 0$, let $\widetilde{M} = \mathbb{S}^n(\frac{1}{\sqrt{K}})$. Fix $p \in M, \widetilde{p} \in \widetilde{M}$ and a linear isometry $\Phi_0 : T_{\widetilde{p}}\widetilde{M} \to T_pM$. Consider the following smooth map

$$\varphi_1 = \exp_p \circ \Phi_0 \circ \exp_{\widetilde{p}}^{-1} : \widetilde{M} \setminus \{-\widetilde{p}\} \to M$$

it's well-defined since the only cut point of $\widetilde{p}$ is its antipodal point $-\widetilde{p}$. Then Cartan-Ambrose-Hicks's theorem implies φ_1 is a local isometry. Choose $\widetilde{q} \in \widetilde{M} \setminus \{\widetilde{p}, -\widetilde{p}\}$, $q = \varphi_1(\widetilde{q})$ and $\Psi_0 = (\mathrm{d}\varphi_1)_{\widetilde{q}} : T_{\widetilde{q}}\widetilde{M} \to T_qM$, then the same argument shows

$$\varphi_2 = \exp_q \circ \Phi_0 \circ \exp_{\widetilde{q}}^{-1} : \widetilde{M} \setminus \{-q\} \to M$$

is a well-defined local isometry defined on $\widetilde{M}\setminus\{-\widetilde{q}\}$. Note that

$$\varphi_2(\widetilde{q}) = q = \varphi_1(\widetilde{q})$$
$$(\mathrm{d}\varphi_2)_{\widetilde{q}} = \Psi_0 = (\mathrm{d}\varphi_1)_{\widetilde{q}}$$

So by Theorem 17.1.2, we have the φ_1 agrees with φ_2 on $\widetilde{M}\setminus\{-\widetilde{p},-\widetilde{q}\}$. Thus

$$\varphi(x) = \begin{cases}\varphi_1(x), & x\in\widetilde{M}\setminus\{-\widetilde{p}\}\\ \varphi_2(x), & x\in\widetilde{M}\setminus\{-\widetilde{q}\}\end{cases}$$

is a well-defined local isometry from $\widetilde{M}\to M$. In particular, φ is a local diffeomorphism, then by Proposition B.2.1 we have φ is a diffeomorphism, since $\mathbb{S}^n$ is compact and simply-connected, thus φ is an isometry.

□

Corollary 17.2.1. Let (M,g) be a Riemannian manifold with constant sectional curvature K, then (M,g) is isometric to $\widetilde{M}/\Gamma$, where $\Gamma\subset\mathrm{Iso}(\widetilde{M},\widetilde{g})$ and is isomorphic to $\pi_1(M)$ and

$$(\widetilde{M},\widetilde{g}) = \begin{cases}(\mathbb{S}^n(\frac{1}{\sqrt{K}}), g_{\mathrm{can}}) & K>0\\ (\mathbb{R}^n, g_{\mathrm{can}}) & K=0\\ (\mathbb{H}^n(\frac{1}{\sqrt{-K}}), g_{\mathrm{can}}) & K<0\end{cases}$$

Proof. Let $(\widetilde{M},\widetilde{g})$ be the universal covering of M with pullback metric, then M is isometric to $\widetilde{M}/\Gamma$, where $\Gamma\subset\mathrm{Iso}(\widetilde{M},\widetilde{g})$, which is isomorphic to $\pi_1(M)$. Furthermore, Hopf's theorem implies what does $(\widetilde{M},\widetilde{g})$ look like. □

Definition 17.2.1 (space form)**.** A complete, simply-connected Riemannian n-manifold with constant sectional curvature k is called space form, and denoted by $S(n,k)$.

Example 17.2.1. Let (M,g) be a complete Riemannian manifold with constant sectional curvature $K=1$. If $\dim M=2n$, then (M,g) is isometric to the sphere $(\mathbb{S}^{2n},g_{\mathrm{can}})$ or the real projective space $(\mathbb{RP}^{2n},g_{\mathrm{can}})$.

Proof. Note that Hopf's theorem implies (M,g) is isometric to $(\mathbb{S}^{2n}/\Gamma,g_{\mathrm{can}})$, where Γ is isomorphic to $\pi_1(M)$, and Synge's theorem implies if $\dim M$ is even and $K>0$, then $\pi_1(M)=\{e\}$ or $\pi_1(M)=\mathbb{Z}_2$.

1. If $\pi_1(M)=\{e\}$, then (M,g) is isometric to $(\mathbb{S}^{2n},g_{\mathrm{can}})$.
2. If $\pi_1(M)=\{e,\varphi\}$, to show (M,g) is isometric to $(\mathbb{RP}^{2n},g_{\mathrm{can}})$, it suffices to show φ is antipodal map. Note that only possible eigenvalues of φ is ±1, and if 1 is an eigenvalue of φ, then it exists a fixed point, which implies $\varphi=e$, since $\pi_1(M)$ acts on $\mathbb{S}^{2n}$ freely.

□

Remark 17.2.1. In general, we have no ideal about what does $\pi_1(M)$ look like.

Part 7. Comparison theorems

18. Jacobi fields II: a useful tool

18.1. Gauss lemma. Let (M,g) be a Riemannian manifold and (x^i,U,p) a normal coordinate centered at $p\in M$.

Definition 18.1.1 (radial distance function)**.** The radial distance function r defined on U is given by

$$r(q):=\sqrt{\sum_{i=1}^n (q^i)^2}$$

where $q=(q^1,\dots,q^n)$ in normal coordinate (x^i,U,p).

Definition 18.1.2 (radial vector field)**.** The radial vector field in $U\setminus\{p\}$ is defined as

$$\partial_r=\frac{x^i}{r}\frac{\partial}{\partial x^i}$$

Proposition 18.1.1. The geodesic starting at p with unit-speed is the integral curve of radial vector field ∂_r over $U\setminus\{p\}$.

Proof. Let $\gamma\colon I\to U$ be a geodesic with $\gamma(0)=p,\gamma'(0)=v$, where $|v|=1$, by definition we need to show

$$\gamma'(b)=\partial_r|_{\gamma(b)}$$

where I is an open interval and $b\in I$. In normal coordinate γ looks like $\gamma(t)=(tv^1,\dots,tv^n)$. If we denote $\gamma(b)=q=(q^1,\dots,q^n)$, then it's clear $v^i=q^i/b$, and $r(q)=b$, since $|v|=1$. Then

$$\gamma'(b)=v^i\frac{\partial}{\partial x^i}\Big|_q=\frac{q^i}{b}\frac{\partial}{\partial x^i}\Big|_q=\frac{q^i}{r(q)}\frac{\partial}{\partial x^i}\Big|_q=\partial_r|_q$$

□

Lemma 18.1.1. Let $f\colon M\to\mathbb{R}$ be a smooth function and X a vector field over M, if

1. $Xf=|X|^2$.
2. X is perpendicular to the level set of f.

then $X=\nabla f$.

Theorem 18.1.1 (Gauss lemma)**.** The radial vector field ∂_r is perpendicular to the level set of radial distance function r on $U\setminus\{p\}$.

Proof. For any $q\in U\setminus\{p\}$ written as $q=(q^1,\dots,q^n)$ in normal coordinate with $b=r(q)$. Given $w\in T_qM$ which is tangent to the level set of r, we need to show $\langle\partial_r|_q,w\rangle=0$. By definition there exists a curve $c(s)\colon(-\varepsilon,\varepsilon)\to M$ such that $c(0)=q,c'(0)=w$ with $\sum_{i=1}^n(c^i(s))^2=b$, where c^i is the coordinates of c in normal coordinate. Taking derivative with respect to s one has

$$\sum_{i=1}^{n} 2c^i(s)(c^i(s))' = 0$$

In particular one has $\sum_{i=1}^{n} c^i(0)(c^i)'(0) = 0$. By Corollary 12.1.2 there is a Jacobi field $J(t)$ along geodesic connecting p, q such that $J(0) = 0, J'(0) = w$ and $J(1) = w$. Note that the metric at T_pM is standard metric, thus $\langle J'(0), \gamma'(0)\rangle = 0$, and by construction $\langle J(0), \gamma'(0)\rangle = 0$, then by (4) of Proposition 12.1.1 one has $\langle J(t), \gamma'(t)\rangle \equiv 0$. In particular, one has $\langle J(1), \gamma'(1)\rangle$, which complete the proof since $\gamma'(t)$ is integral curve of ∂_r. □

Corollary 18.1.1.

1. $|\partial_r|^2 = 1$.
2. $g^{ij}\frac{\partial r}{\partial x^i}\frac{\partial}{\partial x^j} = \nabla r = \partial_r$.

Proof. For (1). It's clear, since we have already shown geodesic with unit-speed is integral curve of ∂_r.

For (2). By Lemma 18.1.1 and Theorem 18.1.1, it suffices to show $Xr = |\partial_r|^2$, which can be seen from

$$Xr = \frac{x^i}{r}\frac{\partial r}{\partial x^i} = \sum_{i=1}^{n}\frac{(x^i)^2}{r^2} = 1 = |\partial_r|^2$$

□

Corollary 18.1.2. The following identities hold in (x^i, U, p):

1. $g_{ij}x^j = x^i$.
2. $g_{im} = \delta_{im} - \frac{\partial g_{ij}}{\partial x^m}x^j$.
3. $\frac{\partial g_{ij}}{\partial x^m}x^j = \frac{\partial g_{mj}}{\partial x^i}x^j$.
4. $\frac{\partial g_{ij}}{\partial x^m}x^j x^i = \frac{\partial g_{mj}}{\partial x^i}x^j x^i = 0$.
5. $\Gamma_{ij}^k x^i x^j = 0$.

Proof. For (1). On one hand by Corollary 18.1.1 we have $\partial_r = \nabla r = g^{ij}\frac{x^i}{r}\frac{\partial}{\partial x^j}$. On the other hand by definition of ∂_r we have $\partial_r = \frac{x^j}{r}\frac{\partial}{\partial x^j}$, which implies

$$g^{ij}x^i = x^j$$

This shows (1).

For (2). Take partial derivatives of (1) with respect to x^m, we have

$$\frac{\partial g_{ij}}{\partial x^m}x^j + g_{ij}\delta_{jm} = \delta_{im}$$

This shows (2).

For (3). It follows from (2), since g_{im}, δ_{im} are symmetric in i, m.

For (4). It follows from (1) and (2), since

$$\frac{\partial g_{ij}}{\partial x^m}x^j x^i \overset{(2)}{=} (\delta_{im} - g_{im})x^i = x^m - g_{im}x^i \overset{(1)}{=} 0$$
$$\frac{\partial g_{mj}}{\partial x^i}x^j x^i \overset{(2)}{=} (\delta_{mi} - g_{mi})x^i = x^m - g_{im}x^i \overset{(1)}{=} 0$$

For (5). It follows from (4) and

$$\Gamma_{ij}^k = \frac{1}{2}g^{mk}(\frac{\partial g_{mj}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m})$$

□

Corollary 18.1.3.

$$\nabla_{\partial_r}\partial_r = 0$$

holds in $U \setminus \{p\}$.

Proof. Direct computation shows

$$\begin{aligned}\nabla_{\partial_r}\partial_r &= \frac{x^k}{r}\nabla_{\frac{\partial}{\partial x^k}}(g^{ij}\frac{x^i}{r}\frac{\partial}{\partial x^j}) \\ &= g^{ij}\frac{x^k}{r}\{\underbrace{(\frac{\delta_{ki}}{r} - \frac{x^k x^i}{r^3})\frac{\partial}{\partial x^j}}_{\text{part I}} + \underbrace{\frac{x^i}{r}\Gamma_{kj}^m\frac{\partial}{\partial x^m}}_{\text{part II}}\}\end{aligned}$$

By (1) and (5) of Corollary 18.1.2 one has

$$g^{ij}\frac{x^k x^i}{r}\Gamma_{kj}^m = \frac{1}{r}\Gamma_{kj}^m x^k x^j = 0$$

which implies part II is zero. For part I, we have

$$\frac{1}{r^2}(g^{ij}x^k\delta_{ki} - \frac{(x^k)^2}{r^2}g^{ij}x^i) = \frac{1}{r^2}(g^{ij}x^i - g^{ij}x^i) = 0$$

□

18.2. Jacobi fields on constant sectional curvature manifold.

Proposition 18.2.1. Let (M, g) be a Riemannian manifold with constant sectional curvature k and $\gamma\colon [0, b] \to M$ a unit-speed geodesic. Then the normal Jacobi field with $J(0) = 0$ is of the form

$$J(t) = m\,\mathrm{sn}_k(t)E(t)$$

where

1. The constant m is determined by $J'(0) = mE(0)$.
2. $$\mathrm{sn}_k(t) = \begin{cases} t, & k = 0 \\ \frac{\sin(\sqrt{k}t)}{\sqrt{k}}, & k > 0 \\ \frac{\sinh(\sqrt{-k}t)}{\sqrt{-k}}, & k < 0 \end{cases}$$
3. $E(t)$ is a normal parallel vector field along γ with $|E(t)| = 1$

Proof. Since (M, g) has constant sectional curvature k, thus $R_{ijkl} = k(g_{il}g_{jk} - g_{ik}g_{jl})$, so for any normal vector field J along γ we have

$$\begin{aligned} R(J, \gamma', \gamma', W) &= k(\langle J, W\rangle\langle\gamma', \gamma'\rangle - \langle J, \gamma'\rangle\langle\gamma', W\rangle) \\ &= k\langle J, W\rangle \end{aligned}$$

which implies

$$R(J, \gamma')\gamma' = kJ$$

since γ is unit-speed and J is normal. Thus equation for Jacobi field can be written as

$$0 = J'' + kJ$$

Assume $J = u(t)E(t)$, then

$$(u''(t) + ku(t))E(t) = 0$$

So if we want to find normal Jacobi fields J, it suffices to solve

$$\begin{cases} u''(t) + ku = 0 \\ u(0) = 0 \end{cases}$$

and it's clear $\mathrm{sn}_k(t)$ is the solution of this ODE. $\square$

18.3. **Polar decomposition of metric with constant sectional curvature.** Let $\pi : \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1}$ given by $\pi(x) = x/|x|$. We can use π to pullback canonical metric on $\mathbb{S}^{n-1}$, and still use $g_{\mathbb{S}^{n-1}}$ to denote it.

Lemma 18.3.1. Let $\overline{g}$ be the Euclidean metric on $\mathbb{R}^n \setminus \{0\}$, then

$$\overline{g} = \mathrm{d}r \otimes \mathrm{d}r + r^2 g_{\mathbb{S}^{n-1}}$$

where $r(x) = |x|$.

Theorem 18.3.1 (polar decomposition)**.** Let (x^i, U, p) be a normal coordinate centered at $p \in S(n, k)$, then in U the metric g can be written as

$$g = \mathrm{d}r \otimes \mathrm{d}r + \mathrm{sn}_k^2(r) g_{\mathbb{S}^{n-1}}$$

where r is radial distance function.

Proof. We use g_c to denote metric $\mathrm{d}r \otimes \mathrm{d}r + \mathrm{sn}_k^2(r) g_{\mathbb{S}^{n-1}}$ and $\overline{g}$ to denote standard metric on Euclidean space. By Corollary 18.1.1, we have

$$g(\partial_r, \partial_r) = 1 = g_c(\partial_r, \partial_r)$$

So it remains to show for each $q \in U \setminus \{p\}$ and $w_1, w_2 \in T_qM$ such that $g(w_i, \partial_r|_q) = 0, i = 1, 2$, we have

$$g(w_1, w_2) = g_c(w_1, w_2)$$

By polarization it suffices to show that $g(w, w) = g_c(w, w)$ for every such vector w.

Suppose $\mathrm{dist}(p, q) = b$, on one hand we have

$$|w|^2_{g_c} \overset{(1)}{=} \mathrm{sn}_k^2(b)|w|^2_{g_{\mathbb{S}^{n-1}}} \overset{(2)}{=} \frac{\mathrm{sn}_k(b)}{b^2}|w|^2_{\overline{g}}$$

where

(1) holds from definition of g_c.
(2) holds from polar decomposition of standard metric of Euclidean space, that is Lemma 18.3.1.

On the other hand, let $\gamma\colon [0,b]\to U$ be a unit-speed geodesic connecting p,q, and we can write it with respect to normal coordinate U as

$$\gamma(t) = (\frac{tq^1}{b},\dots,\frac{tq^n}{b})$$

where $q=(q^1,\dots,q^n)$ in normal coordinate U. Let J be a Jacobi field such that $J(0)=0, J(b)=w$, then we have

$$|w|_g^2 = |J(b)|_g^2 \overset{(3)}{=} \mathrm{sn}_k^2(b)|J'(0)|_g^2 \overset{(4)}{=} \mathrm{sn}_k^2(b)|J'(0)|_{\overline{g}}^2$$

where

(3) holds from the fact Jacobi field on constant sectional curvature space is of form $J(t)=|J'(0)|\,\mathrm{sn}_k(t)E(t)$.
(4) holds from the metric on T_pM is standard metric in normal coordinate.

Furthermore, suppose $J'(0)=a$, then we can write it as $J(t)=\alpha_*(\frac{\partial}{\partial s})\big|_{s=0}$, where

$$\alpha(s,t) = \exp_p(t(\gamma'(0)+sJ'(0)))$$

In normal coordinate we can write $\alpha(s,t)$ explicitly as

$$\alpha(s,t) = (\frac{tq^1}{b}+tsa^1,\dots,\frac{tq^n}{b}+tsa^n)$$

thus $J(t) = ta^i\frac{\partial}{\partial x^i}\big|_{\gamma(t)}$. We can conclude $a^i=\frac{w^i}{b}$ by setting $t=b$, in particular we have $J'(0)=\frac{w^i}{b}\frac{\partial}{\partial x^i}\Big|_p$. Then

$$\mathrm{sn}_k^2(b)|J'(0)|_{\overline{g}}^2 = \mathrm{sn}_k^2(b)\frac{|w|_{\overline{g}}^2}{b^2} = |w|_{g_c}^2$$

□

Remark 18.3.1. Note that there are four points we need in the proof of above theorem, and the **key point** is (3), that is Jacobi field of $S(n,k)$ has the form of

$$J(t) = m\,\mathrm{sn}_k(t)E(t)$$

So this motivate us that if on a normal neighborhood of some point, the Jacobi field has the above form, then metric g can be written as

$$g = \mathrm{d}r\otimes\mathrm{d}r + \mathrm{sn}_k(r)^2 g_{\mathbb{S}^{n-1}}$$

in U. In particular, it has constant sectional curvature k.

18.4. **A criterion for constant sectional curvature space.** Recall that for a smooth function $f\colon M \to \mathbb{R}$, $\operatorname{Hess} f$ is a $(0,2)$-tensor, we use $\mathscr{H}_f$ to denote its $(1,1)$-type, that is

$$g(\mathscr{H}_f(X), Y) = \operatorname{Hess} f(X, Y)$$

where X, Y are two vector fields.

In particular, if r is the radial distance function on a normal coordinate, then Hessian r is a $(2,0)$-tensor, that is $\nabla^2 r$, then we have

$$\mathscr{H}_r = \nabla \partial_r$$

since $(1,0)$-type of ∇r is ∂_r.

Proposition 18.4.1. Let (M, g) be a complete Riemannian manifold, (x^i, U, p) a normal coordinate centered at p and r the radial distance function on U. If $\gamma\colon [0,b] \to M$ is unit-speed geodesic with $\gamma(0) = p, \gamma'(0) = v \in T_pM$, and J is a normal Jacobi field along γ with $J(0) = 0$. Then for all $t \in (0, b]$

$$\begin{aligned}\mathscr{H}_r(J(t)) &= J'(t)\\ \mathscr{H}_r(\gamma'(t)) &= 0\end{aligned}$$

Proof. Here we only prove the first identity, the second can be computed in the same method. Let $J'(0) = w$, then $J(t) = tw^i \frac{\partial}{\partial x^i}\big|_{\gamma(t)}$,

$$\begin{aligned}
J'(t) &= \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}}(tw^i \frac{\partial}{\partial x^i}\Big|_{\gamma(t)})\\
&= w^i \frac{\partial}{\partial x^i}\Big|_{\gamma(t)} + tw^i \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \frac{\partial}{\partial x^i}\Big|_{\gamma(t)}\\
&= w^i \frac{\partial}{\partial x^i}\Big|_{\gamma(t)} + tw^i \Gamma_{ij}^k(\gamma(t)) \frac{\mathrm{d}\gamma^j}{\mathrm{d}t} \frac{\partial}{\partial x^k}\Big|_{\gamma(t)}\\
&= (w^k + tw^i v^j \Gamma_{ij}^k(\gamma(t))) \frac{\partial}{\partial x^k}\Big|_{\gamma(t)}\\
\mathscr{H}_r(J(t)) &= \nabla_{J(t)} \partial_r\\
&= \nabla_{tw^i \frac{\partial}{\partial x^i}\big|_{\gamma(t)}} (\frac{x^j}{r} \frac{\partial}{\partial x^j})\\
&= tw^i \nabla_{\frac{\partial}{\partial x^i}\big|_{\gamma(t)}} (\frac{x^j}{r} \frac{\partial}{\partial x^j})\\
&= tw^i \frac{x^j}{r} \Gamma_{ij}^k(\gamma(t)) \frac{\partial}{\partial x^k}\Big|_{\gamma(t)} + \sum_{i=1}^n tw^i (\frac{\delta_{ij}}{r} - \frac{x^i x^j}{r^3}) \frac{\partial}{\partial x^j}\Big|_{\gamma(t)}
\end{aligned}$$

However, we have the following observations:

1. $r(\gamma(t)) = t$.
2. $x^i = tv^i$.
3. $\sum_{i=1}^n a^i v^i = 0$

where the last equality holds since J is a normal vector field, then

$$0 = \langle J(t), \gamma'(t)\rangle = \langle J(0), \gamma'(0)\rangle + \langle J'(0), \gamma'(0)\rangle t$$

implies $\langle J'(0), \gamma'(0)\rangle = \sum_{i=1}^n a^i v^i = 0$. □

Corollary 18.4.1. With the same assumption as above proposition, for any vector field W along γ with $W(0) = 0$,

$$\begin{aligned}
\operatorname{Hess} r(J(s), W(s)) &\overset{(1)}{=} g(\mathscr{H}_r(J(s), W(s))) \\
&\overset{(2)}{=} g(J'(t), W(s)) \\
&\overset{(3)}{=} \int_0^s \langle J'(t), W(t)\rangle' \mathrm{d}t \\
&\overset{(4)}{=} \int_0^s \langle J'(t), W'(t)\rangle - R(J, \gamma', \gamma', W) \mathrm{d}t
\end{aligned}$$

Proof. It's clear, since

(1) holds from definition of $\mathscr{H}_r$.
(2) holds from $\mathscr{H}_r(J(t)) = J'(t)$.
(3) holds from $W(0) = 0$.
(4) holds from J is a Jacobi field.

□

Corollary 18.4.2. Let $p \in U \subset S(n, k)$, where U is a normal neighborhood of p, then the following holds in $U \setminus \{p\}$

$$\mathscr{H}_r = \frac{\operatorname{sn}_k'(r)}{\operatorname{sn}_k(r)} \pi_r$$

where r is the radial distance function on U, and for each $q \in U \setminus \{p\}$, $\pi_r : T_qM \to T_qM$ is the orthogonal projection onto the orthogonal complement of $\partial_r|_q$.

Proof. For $p \in U \setminus \{q\}$, it's clear

$$\mathscr{H}_r(\partial_r|_q) = 0 = \frac{\operatorname{sn}_k'(r)}{\operatorname{sn}_k(r)} \pi_r(\partial_r|_q)$$

For $w \in T_qM$ such that $g(w, \partial_r|_q)$, choose a unit-speed geodesic $\gamma \colon [0, b] \to M$ connecting p and q and $J(t)$ is the Jacobi field such that $J(0) = 0, J(b) = w$. Then we must have

$$J(t) = m \operatorname{sn}_k(t) E(t)$$

where $E(t)$ is a normal parallel vector field along γ with $|E(t)| = 1$. Then

$$\begin{aligned}
m \operatorname{sn}_k'(t) E(t) &= J'(t) \\
&= \mathscr{H}_r(J(t)) \\
&= \mathscr{H}_r(m \operatorname{sn}_k(t) E(t)) \\
&= m \operatorname{sn}_k(t) \mathscr{H}_r(E(t))
\end{aligned}$$

Setting $t = b$ and dividing by $\mathrm{sn}_k(b)$ one has

$$\mathscr{H}_r(E(b)) = \frac{\mathrm{sn}_k'(b)}{\mathrm{sn}_k(b)} E(b)$$

Note that $w = m\,\mathrm{sn}_k(b)E(b)$, this completes the proof. □

Furthermore, the converse of above corollary still holds:

Proposition 18.4.2. Let (M, g) be a Riemannian manifold and U a normal neighborhood of $p \in M$, r radial distance function. If

$$\mathscr{H}_r = \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)} \pi_r$$

holds in $U \setminus \{p\}$, then (M, g) has constant sectional curvature k in U.

Proof. Let $\gamma\colon [0, b] \to U$ be a unit-speed geodesic $r(0) = p$, J is a normal Jacobi vector field along γ with $J(0) = 0$, then $\mathscr{H}_r(J) = J'$ implies

$$J'(t) = \frac{\mathrm{sn}_k'(t)}{\mathrm{sn}_k(t)} J(t)$$

holds for $t \in (0, b]$, that is

$$\left(\frac{J(t)}{\mathrm{sn}_k(t)}\right)' = 0$$

holds for $t \in (0, b]$. So we can write every normal Jacobi fields as $J(t) = m\,\mathrm{sn}_k(t)E(t)$, where E is normal a parallel vector field with $|E| = 1$ and $t \in [0, b]$. Thus by Remark 18.3.1, g has constant sectional curvature k in U. □

Remark 18.4.1. For convenience, we record the exact formulas for the quotient $\frac{\mathrm{sn}_k'}{\mathrm{sn}_k}$ as follows

$$\frac{\mathrm{sn}_k'(t)}{\mathrm{sn}_k(t)} = \begin{cases} \frac{1}{t}, & k = 0 \\ \frac{1}{\sqrt{k}} \cot \frac{t}{\sqrt{k}}, & k > 0 \\ \frac{1}{\sqrt{k}} \coth \frac{t}{\sqrt{k}}, & k < 0 \end{cases}$$

and we can draw the graph as follows.

19. Comparison theorems based on sectional curvature

In this section, we will see the following philosophy: "The larger curvature is, the smaller the distance is."

19.1. Rauch comparison.

Rauch comparison. Rauch comparison theorem is one of the most important comparison theorems, which gives bounds on the sizes of Jacobi fields based on sectional curvature bounds. Recall that Jacobi field is a quite useful tool, based on the following observations:

1. Corollary 12.1.2 implies that in a normal neighborhood of p, every vector field can be represented as the value of Jacobi field that vanishes at p.
2. The zeros of Jacobi fields corresponds to conjugate points, beyond which geodesics can't be minimal.

Theorem 19.1.1 (Rauch comparison)**.** Let (M,g) and $(\widetilde{M},\widetilde{g})$ be two Riemannian manifold with $\dim M \leq \dim \widetilde{M}$. Suppose $\gamma\colon[0,b]\to M$ and $\widetilde{\gamma}:[0,b]\to\widetilde{M}$ are two unit-speed geodesics such that

1. For all $t\in[0,b]$, and any planes $\Sigma\subset T_{\gamma(t)}M, \gamma'(t)\in\Sigma, \widetilde{\Sigma}\subset T_{\widetilde{\gamma}(t)}\widetilde{M}, \widetilde{\gamma}'(t)\in\widetilde{\Sigma}$, we have $K_{\gamma(t)}(\Sigma)\leq K_{\widetilde{\gamma}(t)}(\widetilde{\Sigma})$.
2. $\widetilde{\gamma}(0)$ has no conjugate points along $\widetilde{\gamma}|_{[0,b]}$.

Then for any Jacobi fields $J(t)$ and $\widetilde{J}(t)$ with

1. $$\begin{cases} J(0)=c\gamma'(0)\\ \widetilde{J}(0)=c\widetilde{\gamma}'(0)\end{cases}$$
2. $|J'(0)|=|\widetilde{J}'(0)|$.
3. $\langle J'(0),\gamma'(0)\rangle=\langle\widetilde{J}'(0),\widetilde{\gamma}'(0)\rangle$.

we have $|J(t)|\geq|\widetilde{J}(t)|$ for all $t\in[0,b]$.

Proof. Firstly we consider the following simple case:

1. $J(0)=\widetilde{J}(0)=0$.
2. $|J'(0)|=|\widetilde{J}'(0)|$.
3. $\langle J'(0),\gamma'(0)\rangle=\langle\widetilde{J}'(0),\widetilde{\gamma}'(0)\rangle=0$.

Since $\widetilde{\gamma}(0)$ has no conjugate points along $\widetilde{\gamma}|_{[0,b]}$, then $\frac{|J(t)|^2}{|\widetilde{J}(t)|^2}$ is well-defined for all $t\in(0,b]$, and standard calculus implies

$$\lim_{t\to0}\frac{|J|^2}{|\widetilde{J}^2|}=\lim_{t\to0}\frac{\langle J'(t),J(t)\rangle}{\langle\widetilde{J}'(t),\widetilde{J}(t)\rangle}=\lim_{t\to0}\frac{|J'|^2}{|\widetilde{J}'|^2}=1$$

So it suffices to show in $(0,b]$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\frac{|J|^2}{|\widetilde{J}|^2})\geq0$$

Direct computation shows above inequality is equivalent to:

$$\frac{\langle J'(t), J(t)\rangle}{|J(t)|^2} \geq \frac{\langle \widetilde{J}'(t), \widetilde{J}(t)\rangle}{|\widetilde{J}(t)|^2}$$

holds for arbitrary $t \in (0, b]$. For arbitrary $s \in (0, b]$, we can define the following Jacobi fields by scaling $J(t)$:

$$W_s(t) = \frac{J(t)}{|J(s)|}, \quad \widetilde{W}_s(t) = \frac{\widetilde{J}(t)}{|\widetilde{J}(s)|}$$

Then

$$\frac{\langle J'(s), J(s)\rangle}{|J(s)|^2} = \langle W_s'(s), W_s(s)\rangle$$

So it suffices to show

$$\langle W_s'(s), W_s(s)\rangle \geq \langle \widetilde{W}_s'(s), \widetilde{W}_s(s)\rangle$$

holds for arbitrary $s \in (0, b]$. Direct computation shows:

$$\begin{aligned}
\langle W_s'(s), W_s(s)\rangle &= \int_0^s (\langle W_s(t), W_s(t)\rangle)'\mathrm{d}t \\
&= \int_0^s \langle W_s'(t), W_s'(t)\rangle \mathrm{d}t + \int_0^s \langle W_s''(t), W_s(t)\rangle \mathrm{d}t \\
&= \int_0^s \langle W_s'(t), W_s'(t)\rangle \mathrm{d}t - \int_0^s R(W_s(t), \gamma'(t), \gamma'(t), W(t))\mathrm{d}t
\end{aligned}$$

Choose a parallel orthonormal frame $\{e_1(t), \ldots, e_n(t)\}$ with $e_1(t) = \gamma'(t), e_2(t) = W_s(t)$. With respect to this frame we write

$$W_s(t) = \lambda^i(t)e_i(t)$$

Similarly we choose a parallel orthogonal frame $\{\widetilde{e}_1(t), \ldots, \widetilde{e}_n(t)\}$ and construct the following vector field

$$\widetilde{V}(t) = \lambda^i(t)\widetilde{e}_i(t)$$

Then it's clear we have

$$\int_0^s \langle W_s'(t), W_s'(t)\rangle \mathrm{d}t = \int_0^s \langle \widetilde{V}'(t), \widetilde{V}'(t)\rangle \mathrm{d}t$$

and our curvature condition implies

$$\int_0^s R(W_s(t), \gamma'(t), \gamma'(t), W_s(t))\mathrm{d}t \leq \int_0^s \widetilde{R}(\widetilde{V}(t), \gamma'(t), \gamma'(t), \widetilde{V}(t))\mathrm{d}t$$

Thus we have

$$\begin{aligned}
\langle W_s'(s), W_s(s)\rangle &\leq \int_0^s \langle \widetilde{V}'(t), \widetilde{V}'(t)\rangle \mathrm{d}t - \int_0^s R(\widetilde{V}(t), \gamma'(t), \gamma'(t), \widetilde{V}(t))\mathrm{d}t \\
&= \widetilde{I}(\widetilde{V}, \widetilde{V})
\end{aligned}$$

where $\widetilde{I}$ is index form on $\widetilde{M}$. According to Corollary 12.3.1, we have

$$\widetilde{I}(\widetilde{V}, \widetilde{V}) \geq \widetilde{I}(\widetilde{W}_s, \widetilde{W}_s)$$

since $\widetilde{W}_s$ is a Jacobi field. This shows the desired result.

For general case, we consider the following decomposition

$$J(t) = J_1(t) + \langle J(t), \gamma'(t)\rangle\gamma'(t)$$
$$\widetilde{J}(t) = \widetilde{J}_1(t) + \langle \widetilde{J}(t), \widetilde{\gamma}'(t)\rangle\widetilde{\gamma}'(t)$$

Then it's clear $J_1(t)$ and $\widetilde{J}_1(t)$ satisfy requirement of our simple case, that is for $t \in [0,1]$ we have

$$|J_1(t)| \geq |\widetilde{J}_1(t)|$$

Furthermore,

$$\langle J(t), \gamma'(t)\rangle = \langle \widetilde{J}(t), \widetilde{\gamma}'(t)\rangle$$

always holds, since

$$\begin{aligned}\langle J(t), \gamma'(t)\rangle &= \langle J(0), \gamma'(0)\rangle + \langle J'(0), \gamma'(0)\rangle t \\ &\overset{(1)}{=} \langle \widetilde{J}(0), \widetilde{\gamma}'(0)\rangle + \langle \widetilde{J}'(0), \widetilde{\gamma}'(0)\rangle t \\ &= \langle \widetilde{J}(t), \widetilde{\gamma}'(t)\rangle\end{aligned}$$

where (1) holds from our assumption. □

Corollary 19.1.1. Let (M, g) be a Riemannian manifold, U a normal neighborhood of $p \in M$, $\gamma\colon [0, b] \to U$ a unit-speed geodesic with $\gamma(0) = p$ and J a Jacobi field along γ with $J(0) = 0$.

1. If the sectional curvature $K \leq k$ in U, then $|J(t)| \geq \mathrm{sn}_k(t)|J'(0)|$, for all $t \in [0, b_0]$, where
$$b_0 = \begin{cases} b, & k \leq 0 \\ \min\{b, \pi R\}, & k = \frac{1}{R^2} > 0 \end{cases}$$
2. If the sectional curvature $K \geq k$ in U, then
$$|J(t)| \leq \mathrm{sn}_k(t)|J'(0)|$$
for all $t \in [0, b]$.

Proof. Apply Rauch comparison between M and space form $\widetilde{M} = S(n, k)$ to conclude. However, in order to avoid geodesic $\widetilde{\gamma}$ of $\widetilde{M}$ from having conjugate points, we need to let $b_0 < \min\{b, \pi R\}$, when $k = \frac{1}{R^2} > 0$. □

Remark 19.1.1. In particular, from above corollary, we immediately have the following corollary when $K \leq k$:

1. If $k \leq 0$, we have already known M has no conjugate point along any geodesic.
2. If $k = \frac{1}{R^2} > 0$, then there is no conjugate point along any geodesic with length $< \pi R$. Or in other words, the distance between two consecutive conjugate points is $\geq \pi R$.

Corollary 19.1.2 (metric comparison)**.** Let (M, g) be a Riemannian n-manifold, U a normal neighborhood of $p \in M$. For all $k \in \mathbb{R}$, we use g_k to denote the metric $\mathrm{d}r \otimes \mathrm{d}r + \mathrm{sn}_k(r) g_{\mathbb{S}^{n-1}}$ in $U \setminus \{p\}$.

1. If $K \leq k$ holds for all $q \in U \setminus \{p\}$, then for $w \in T_qM$ we have
$$g(w,w) \geq g_k(w,w)$$
holds in $U_0 \setminus \{p\}$, where
$$U_0 = \begin{cases} U, & k \leq 0 \\ U \cap B(p, \pi R), & k = \frac{1}{R^2} > 0 \end{cases}$$
2. If $K \geq k$ holds for all $q \in U \setminus \{p\}$, then for $w \in T_qM$ we have
$$g(w,w) \leq g_k(w,w)$$
holds in $U \setminus \{p\}$.

Proof. If $w = \partial_r|_q$, it's clear
$$g(\partial_r|_q, \partial_r|_q) = 1 = g_k(\partial_r|_q, \partial_r|_q)$$
by Gauss lemma, then it suffices to check for $w \in T_qM$ such that $g(w, \partial_r|_q) = 0$, we have
$$g(w,w) \geq g_k(w,w)$$
Let $\gamma\colon [0,b] \to M$ be a unit-speed geodesic connecting p and q, and J a Jacobi field such that $J(0) = 0, J(b) = w$. In normal coordinate $J(t)$ can be written as $ta^i \frac{\partial}{\partial x^i}\big|_{\gamma(t)}$ for some a^i.

Since (x^i, U, p) is both normal coordinate for metric g and g_k, thus γ is also a radial geodesic for g_c, and $J(t)$ is also a Jacobi field with respect to g_c along γ. Thus we have
$$g(w,w) = |J(b)|_g^2$$
$$g_k(w,w) = |J(b)|_{g_k}^2$$
Then by Corollary 19.1.1, this completes the proof. □

Remark 19.1.2. The **ideal** of this proof and the proof of Theorem 18.3.1 is almost the same, that is via Corollary 12.1.2 to construct a Jacobi field valued a given vector, and then one can use Rauch comparison to compare length of given vectors.

Corollary 19.1.3. Let (M,g) and $(\widetilde{M}, \widetilde{g})$ be two Riemannian manifolds with $K \leq \widetilde{K}$. Fix $p \in M, \widetilde{p} \in \widetilde{M}$, linear isometry $\Phi_0 : T_pM \to T_{\widetilde{p}}\widetilde{M}$ and $0 \leq \delta < \min(\mathrm{inj}(p), \mathrm{inj}(\widetilde{p}))$. Then for any smooth curve $\gamma\colon [0,1] \to \exp_p(B(0,\delta))$ and $\widetilde{\gamma}(t) = \exp_{\widetilde{p}} \circ \Phi_0 \circ \exp_p^{-1}(\gamma(t))$, we have
$$L(\gamma) \geq L(\widetilde{\gamma})$$

Proof. Let $c(s) = \exp_p^{-1} \circ \gamma(s)$ and $\widetilde{c}(s) = \exp_{\widetilde{p}}^{-1} \circ \widetilde{\gamma}(s)$, then $\widetilde{c}(s) = \Phi_0(c(s))$. Consider the following variations
$$\alpha(t,s) = \exp_p(tc(s))$$
$$\widetilde{\alpha}(t,s) = \exp_{\widetilde{p}}(t\widetilde{c}(s))$$

and Jacobi fields

$$J_s(t)=\alpha_*(\frac{\partial}{\partial s})(t,s)$$
$$\widetilde{J}_s(t)=\widetilde{\alpha}_*(\frac{\partial}{\partial s})(t,s)$$

A crucial observation is for arbitrary $s_0\in[0,1]$, we have

$$J_{s_0}(1)=\gamma'(s_0)$$
$$\widetilde{J}_{s_0}(1)=\widetilde{\gamma}'(s_0)$$

So it suffices to prove $|J_{s_0}(1)|\geq|\widetilde{J}_{s_0}(1)|$ holds for arbitrary $s_0\in[0,1]$, that is we need to use Rauch comparison to Jacobi fields $J_{s_0}(t),\widetilde{J}_{s_0}(t)$ along γ_{s_0} and $\widetilde{\gamma}_{s_0}$, where $\gamma_{s_0}(t)=\alpha(t,s_0)$ and $\widetilde{\gamma}_{s_0}(t)=\widetilde{\alpha}(t,s_0)$. Check requirements as follows:

1. $J_{s_0}(0)=\widetilde{J}_{s_0}(0)=0$.
2. $J'_{s_0}(0)=c'(s_0),\widetilde{J}'_{s_0}(0)=\widetilde{c}'(s_0)$, and $\widetilde{c}(s_0)=\Phi_0(c(s_0))$ implies $|J'_{s_0}(0)|=|\widetilde{J}'_{s_0}(0)|$, since Φ_0 is linear isometry.
3. $\langle\widetilde{J}'_{s_0}(0),\widetilde{\gamma}'_{s_0}(0)\rangle=\langle\Phi_0(c'(s_0)),\Phi_0(c(s_0))\rangle=\langle c'(s_0),c(s_0)\rangle=\langle J'_{s_0}(0),\gamma'_{s_0}(0)\rangle$.

□

Corollary 19.1.4. Let (M,g) be a Riemannian n-manifold, $0<k_1\leq K\leq k_2$. Let γ be any geodesic in M and b the distance along γ between two consective conjugate points, then

$$\frac{\pi}{\sqrt{k_2}}\leq b\leq\frac{\pi}{\sqrt{k_1}}$$

Proof. Without lose of generality, we assume $\gamma\colon[0,b]\to M$ is a unit-speed geodesic with $\gamma(0)=p,\gamma(b)=q$ and p,q are two consective conjugate points along γ.

1. By Remark 19.1.1, we have already seen $b\geq\frac{\pi}{\sqrt{k_2}}$.
2. Apply Rauch comparison to (M,g) and $(\mathbb{S}^n(\frac{\pi}{\sqrt{k_1}}),g_{\mathrm{can}})$, we have
$$|J(t)|\leq|\widetilde{J}(t)|$$
for $t\in[0,b]$, where $J(t),\widetilde{J}(t)$ are defined the same as before. Suppose $b>\frac{\pi}{\sqrt{k_1}}$, then take $t=\frac{\pi}{\sqrt{k_1}}$, we have
$$0<|J(t)|\leq|\widetilde{J}(t)|=0$$
A contradiction.

□

Theorem 19.1.2. Let (M,g) be a compact Riemannian manifold with sectional curvature $K\leq k,k>0$. If we define

$$l(M):=\inf\{L(\gamma)\mid\gamma\text{ is a closed geodesic in }M\}$$

Then either $\operatorname{inj}(M)\geq\frac{\pi}{\sqrt{k}}$ or $\operatorname{inj}(M)=\frac{l(M)}{2}$.

Proof. By compactness of M, there exists $p, q \in M, q \in \mathrm{cut}(p)$ such that $\mathrm{dist}(p,q) = \mathrm{inj}(M) = \mathrm{inj}(p)$. Let $\gamma\colon [0,b] \to M$ be a minimal geodesic connecting p and q, that is $b = \mathrm{dist}(p,q) = \mathrm{inj}(M)$. Then

1. If p and q are conjugate along γ, then by Corollary 19.1.4 we have $\mathrm{inj}(M) = b \geq \frac{\pi}{\sqrt{k}}$.
2. If p and q are not conjugate along γ, then by Proposition 13.2.1 there exists a unit-speed closed geodesic $\gamma\colon [0,2b] \to M$ with $\gamma(0) = p, \gamma(b) = q$, where $b = \mathrm{dist}(p,q) = \mathrm{inj}(M)$. On one hand by definition of $l(M,g)$ one has $2b \geq l(M)$. On the other hand, $l(M) \geq 2b$, since $\mathrm{dist}(p,q) = q$. Thus in this case $\mathrm{inj}(M) = \frac{l(M)}{2}$.

□

19.2. **Hessian comparison.**

Theorem 19.2.1 (Hessian comparison)**.** Let (M,g) and $(\widetilde{M},\widetilde{g})$ be two Riemannian manifolds with the same dimension, $U \subset M, \widetilde{U} \subset \widetilde{M}$ normal neighborhoods around $p \in M$ and $\widetilde{p} \in \widetilde{M}$ respectively. Suppose

$$\gamma\colon [0,b] \to U, \gamma(0) = p, \gamma(b) = q$$
$$\widetilde{\gamma} : [0,b] \to \widetilde{U}, \widetilde{\gamma}(0) = \widetilde{p}, \widetilde{\gamma}(b) = \widetilde{q}$$

are two unit-speed geodesics such that

For all $t \in [0,b]$, and any planes $\Sigma \subset T_{\gamma(t)}M, \gamma'(t) \in \Sigma, \widetilde{\Sigma} \subset T_{\widetilde{\gamma}(t)}\widetilde{M}, \widetilde{\gamma}'(t) \in \widetilde{\Sigma}$, we have $K_{\gamma(t)}(\Sigma) \leq K_{\widetilde{\gamma}(t)}(\widetilde{\Sigma})$.

Then for any $v \in T_qM, \widetilde{v} \in T_{\widetilde{q}}\widetilde{M}$ with unit length and $v \perp \gamma'(b), \widetilde{v} \perp \widetilde{\gamma}'(b)$, we have

1. $\mathrm{Hess}\, r(v,v) \geq \mathrm{Hess}\, \widetilde{r}(\widetilde{v},\widetilde{v})$.
2. $\Delta r(\gamma(t)) \geq \widetilde{\Delta}\widetilde{r}(\widetilde{\gamma}(t))$ for all $t \in (0,b]$.
3. Moreover, the equality holds if and only if $K_\Sigma(\gamma(t)) = \widetilde{K}_{\widetilde{\Sigma}}(\widetilde{\gamma}(t))$.

Proof. For (1). Let $\{e_1(t), \ldots, e_n(t)\}$ be a parallel orthonormal basis along γ such that $e_n(t) = \gamma'(t)$ and $\{\widetilde{e}_1(t), \ldots, \widetilde{e}_n(t)\}$ a parallel orthonormal basis along $\widetilde{\gamma}$ suc that $\widetilde{e}_n(t) = \widetilde{\gamma}'(t)$. Without lose of generality we may assume $\langle v, e_i(b)\rangle_g = \langle \widetilde{v}, \widetilde{e}_i(b)\rangle_{\widetilde{g}}$ for $i = 1, \ldots, n-1$, it's just a trick of linear algebra.

Use Corollary 12.1.2 to construct Jacobi fields

$$\begin{cases} J(0) = 0, J(b) = v \\ \widetilde{J}(0) = 0, \widetilde{J}(b) = \widetilde{v} \end{cases}$$

With respect to $\{\widetilde{e}_i(t)\}$ we can write $\widetilde{J}(t)$ as $\widetilde{J}(t) = \lambda^i(t)\widetilde{e}_i(t)$, and construct $V(t) = \lambda^i(t)e_i(t)$. Then

$$\begin{aligned}
\operatorname{Hess} r(v,v) &= \operatorname{Hess} r(J(b), J(b)) \\
&\overset{\mathrm{I}}{=} \int_0^b \langle J'(t), J'(t)\rangle - R(J,\gamma',\gamma',J)\mathrm{d}t \\
&\overset{\mathrm{II}}{\geq} \int_0^b \langle V'(t), V'(t)\rangle - R(V,\gamma',\gamma',V)\mathrm{d}t \\
&\overset{\mathrm{III}}{\geq} \int_0^b \langle \widetilde{J}'(t), \widetilde{J}'(t)\rangle - \widetilde{R}(\widetilde{J},\widetilde{\gamma},\widetilde{\gamma},\widetilde{J})\mathrm{d}t \\
&= \operatorname{Hess} \widetilde{r}(\widetilde{J}(b), \widetilde{J}(b)) \\
&= \operatorname{Hess} \widetilde{r}(\widetilde{v}, \widetilde{v})
\end{aligned}$$

where

I holds from Corollary 18.4.1.
II holds from Corollary 12.3.1.
III holds from our assumption on curvature and the choice of V.

For (2) and (3). They directly follow from (1) and proof of (1).

□

Corollary 19.2.1 (Hessian and Laplacian comparison)**.** Let (M,g) be a Riemannian n-manifold and U a normal neighborhood of $p \in M$.

1. If sectional curvature $K \leq k$ in $U \setminus \{p\}$, then

$$\mathscr{H}_r \geq \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\pi_r, \quad \Delta r \geq (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}$$

holds in $U_0 \setminus \{p\}$, where

$$U_0 = \begin{cases} U, & k \leq 0 \\ U \cap B(p, \pi R), & k = \frac{1}{R^2} > 0 \end{cases}$$

2. If sectional curvature $K \geq k$ in $U \setminus \{p\}$, then

$$\mathscr{H}_r \leq \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\pi_r, \quad \Delta r \leq (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}$$

holds in $U \setminus \{p\}$.
3. Moreover, if equality holds, g has constant sectional curvature k in U_0 or U.

Proof. For (1). Apply Hessian comparison to (M,g) and space form $S(n,k)$, then we directly have

$$\operatorname{Hess} r(v,v) \geq \operatorname{Hess} \widetilde{r}(\widetilde{v}, \widetilde{v})$$

for any $v \in T_qM, \widetilde{v} \in T_qS(n,k)$ with unit length and $v \perp \gamma'(b), \widetilde{v} \perp \widetilde{\gamma}'(b)$, where

$$\gamma\colon [0,b] \to U, \gamma(0) = p, \gamma(b) = q$$
$$\widetilde{\gamma} : [0,b] \to \widetilde{U}, \widetilde{\gamma}(0) = \widetilde{p}, \widetilde{\gamma}(b) = \widetilde{q}$$

are two unit-speed geodesics, and $U, \widetilde{U}$ are normal neighborhoods of $p, \widetilde{p}$ respectively. However, we must be careful here, since if sectional curvature of M is ≤ 0, then b can be infinite, and in this case if $k > 0$, the diameter of $\widetilde{U}$ is $< \frac{\pi}{\sqrt{k}}$. Thus we only have

$$\operatorname{Hess} r(v,v) \geq \operatorname{Hess} \widetilde{r}(\widetilde{v}, \widetilde{v})$$

for $0 < b < \frac{\pi}{\sqrt{k}}$ if $k > 0$, and there is no restriction for b if $k \leq 0$. Thus by taking different geodesics and different Jacobi fields, we can show this holds for arbitrary $v \in T_qM, \widetilde{v} \in T_qS(n,k)$, where $q \in U_0 \setminus \{p\}$, that is we have

$$\mathscr{H}_r \geq \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)} \pi_r$$

holds in $U_0 \setminus \{p\}$. By taking trace we obtain $\Delta r \geq (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}$ holds in $U_0 \setminus \{p\}$, since π_r is a projection onto a subspace with codimension 1.

For (2), the same as (1).

For (3), if

$$\mathscr{H}_r = \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)} \pi_r$$

holds in $U \setminus \{p\}$, then it's directly from Proposition 18.4.2. If

$$\Delta r \geq (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}$$

holds in $U \setminus \{p\}$, that is the trace of $\mathscr{H}_r - \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\pi_r$ vanishes identically in $U \setminus \{p\}$, then $\mathscr{H}_r - \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\pi_r$ vanishes identically, since it's semipositive-definite. $\square$

20. Comparison theorems based on Ricci curvature

20.1. Local Laplacian comparison.

Theorem 20.1.1 (local Laplacian comparison)**.** Let (M,g) be a Riemannian n-manifold and U a normal coordinate of $p\in M$. If there exists $k\in\mathbb{R}$ such that $\operatorname{Ric}(g)\geq(n-1)kg$, then

$$\Delta r\leq(n-1)\frac{\operatorname{sn}_k'(r)}{\operatorname{sn}_k(r)}$$

holds in $U_0\setminus\{p\}$, where

$$U_0=\begin{cases}U, & k\leq 0\\ U\cap B(p,\pi R), & k=\frac{1}{R^2}>0\end{cases}$$

Moreover, if equality holds, then g has constant sectional curvature in U_0.

20.1.1. *Proof via Jacobi fields.*

Proof of Theorem 20.1.1 via Jacobi fields. For arbitrary $q\in U_0\setminus\{p\}$, choose a unit-speed geodesic $\gamma\colon[0,b]\to M$ with $\gamma(0)=p,\gamma(b)=q$, and $\{e_1(t),\dots,e_n(t)\}$ is a parallel orthonormal frame along γ with $e_n(t)=\gamma'(t)$. Then by definition $\Delta r=\sum_{i=1}^n\operatorname{Hess}r(e_i,e_i)$.

By Corollary 12.1.2 one can construct Jacobi fields $J_i(t), i=1,\dots,n$ such that $J_i(0)=0, J_i(b)=e_i(b)$, then we have

$$\Delta r=\sum_{i=1}^{n-1}\operatorname{Hess}r(J_i(b),J_i(b))\overset{(1)}{=}\sum_{i=1}^{n-1}I(J_i,J_i)$$

where (1) holds from Corollary 18.4.1. Now let $\widetilde{M}$ be the space form $S(n,k)$ and $\widetilde{U}$ a normal coordinate of $\widetilde{p}\in\widetilde{M}$. Repeat the same process as above we have

$$(n-1)\frac{\operatorname{sn}_k'(r)}{\operatorname{sn}_k(r)}=\widetilde{\Delta}\widetilde{r}=\sum_{i=1}^{n-1}\widetilde{I}(\widetilde{J}_i,\widetilde{J}_i)$$

If we denote $V_i(t)=f(t)e_i(t)$, routine computation shows

$$\begin{aligned}\Delta r&=\sum_{i=1}^{n-1}I(J_i,J_i)\\&\leq\sum_{i=1}^{n-1}I(V_i,V_i)\\&=\sum_{i=1}^{n-1}\int_0^b\langle V_i'(t),V_i'(t)\rangle-R(V_i,\gamma',\gamma',V_i)\mathrm{d}t\end{aligned}$$

Until now, all computations are the same as what we have done in Hessian comparison based on sectional curvature. A crucial observation is that

$\widetilde{J}_i(t) = f(t)\widetilde{e}_i(t)$, and the **key point** is that $f(t)$ is independent of i, then

$$
\begin{aligned}
\Delta r &\overset{(2)}{=} \sum_{i=1}^{n-1} \int_0^b \langle V_i'(t), V_i'(t)\rangle - f^2(t) R(e_i, e_n, e_n, e_i) \mathrm{d}t \\
&= \sum_{i=1}^{n-1} \int_0^b \langle V_i'(t), V_i'(t)\rangle - \int_0^b f^2(t) \operatorname{Ric}(e_n, e_n) \mathrm{d}t \\
&\leq \sum_{i=1}^{n-1} \int_0^b \langle \widetilde{J}_i(t), \widetilde{J}_i(t)\rangle - \int_0^b (n-1) k f^2(t) \mathrm{d}t \\
&= \sum_{i=1}^{n-1} \widetilde{I}(\widetilde{J}_i, \widetilde{J}_i) \\
&= (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}
\end{aligned}
$$

the key point is used in equality marked by (2), and others are routines. □

20.1.2. *Proof via Bochner's technique.*

Lemma 20.1.1. Let (M, g) be a Riemannian manifold, (x^i, U, p) a normal coordinate centered at p, then

$$\Delta r = \partial_r \log(r^{n-1}\sqrt{\det g})$$

in $U \setminus \{p\}$. Moreover, along any unit-speed geodesic $\gamma\colon [0, b] \to U$ with $\gamma(0) = p$, if we define $f(t) := \Delta r(\gamma(t))$, then

$$f(t) = \frac{n-1}{t} + O(1)$$

Proof. Direct computation shows

$$
\begin{aligned}
\Delta r &= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}(g^{ij}\sqrt{\det g}\frac{\partial r}{\partial x^j}) \\
&= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}(g^{ij}\sqrt{\det g}\frac{x^j}{r}) \\
&= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}(\frac{x^i}{r}\sqrt{\det g}) \\
&= \frac{\partial}{\partial x^i}(\frac{x^i}{r}) + \frac{1}{\sqrt{\det g}}\frac{x^i}{r}\frac{\partial}{\partial x^i}(\sqrt{\det g}) \\
&= \frac{n-1}{r} + \frac{1}{\sqrt{\det g}}\partial_r(\sqrt{\det g}) \\
&= \partial_r \log(r^{n-1}\sqrt{\det g})
\end{aligned}
$$

Moreover, for unit-speed geodesic $\gamma\colon [0, b] \to U$, we have

$$f(t) = \frac{n-1}{r(\gamma(t))} + \partial_r(\log\sqrt{\det g})\Big|_{\gamma(t)}$$

Then note that
1. $r(\gamma(t)) = t$, since γ is unit-speed geodesic.
2. Jacobi's formula implies

$$\partial_r(\log\sqrt{\det g})\Big|_{\gamma(t)} = \frac{1}{2}g^{ij}\frac{\partial g_{ij}}{\partial x^k}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t} = O(1)$$

we obtain the desired results. □

Lemma 20.1.2 (Riccati comparison theorem)**.** If $f\colon (0,b)\to\mathbb{R}$ is a smooth function satisfying
1. $f(t) = \frac{1}{t} + O(1)$.
2. $f' + f^2 + k \leq 0$.

Then

$$f(t) \leq \frac{\mathrm{sn}_k'(t)}{\mathrm{sn}_k(t)}$$

for all $t\in(0,b)$, where $k>0, b\leq\frac{\pi}{\sqrt{k}}$.

Proof. Consider $f_k(t) = \frac{\mathrm{sn}_k'(t)}{\mathrm{sn}_k(t)}$, it's a smooth function defined on $(0,b)$ satisfying
1. $f_k(t) = \frac{1}{t} + O(1)$
2. $f_k' + f_k^2 + k = 0$

Choose a smooth function $f\colon (0,b)\to\mathbb{R}$ satisfying
1. $F(t) = 2\log t + O(1)$.
2. $F'(t) = f + f_k$

Then

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(e^F(f-f_k)) = e^F(f^2 - f_k^2 + f' - f_k') \leq 0\\ &\lim_{t\to 0} e^F(f-f_k) = 0\end{aligned}$$

Then we have $f(t)\leq f_k(t)$ holds for all $t\in(0,b)$. □

Lemma 20.1.3.

$$|\operatorname{Hess} r|^2 \geq \frac{(\Delta r)^2}{n-1}$$

Proof. Let $\{e_1,\dots,e_n\}$ be an orthonormal frame with $e_1 = \partial_r$. Then

$$\begin{aligned}|\operatorname{Hess} r|^2 &= \sum_{i,j=1}^n (\langle\nabla_{e_i}\partial_r, e_j\rangle)^2\\ &= \sum_{i,j=2}^n (\langle\nabla_{e_i}\partial_r, e_j\rangle)^2\\ &\geq \frac{1}{n-1}\sum_{i=2}^n(\langle\nabla_{e_i}\partial_r, e_i\rangle)^2\\ &= \frac{1}{n-1}(\Delta r)^2\end{aligned}$$

The inequality

$$|A|^2 \geq \frac{1}{k}|\operatorname{tr}(A)|^2$$

for a $k \times k$ matrix A is a direct consequence of the Cauchy-Schwarz inequality. □

Proof of Theorem 20.1.1 via Bochner's technique. Recall Bochner's technique says

$$\frac{1}{2}\Delta|\nabla f|^2 = |\operatorname{Hess} f|^2 + \operatorname{Ric}(\nabla f, \nabla f) + g(\nabla\Delta f, \nabla f)$$

Set $f = r$ we have

$$\begin{aligned} 0 &= |\operatorname{Hess} r|^2 + \operatorname{Ric}(\nabla r, \nabla r) + g(\nabla\Delta r, \nabla r) \\ &\overset{(1)}{\geq} |\operatorname{Hess} r|^2 + \partial_r(\Delta r) + (n-1)k \\ &\overset{(2)}{\geq} (\frac{\Delta r}{n-1})^2 + \partial_r(\frac{\Delta r}{n-1}) + k \end{aligned}$$

where

(1) holds from $\partial_r = \nabla_r$ and lower bounded of Ricci.
(2) holds from Lemma 20.1.3 and divided by $n-1$.

Thanks to Lemma 20.1.1, we can apply Riccati comparison to $f(r) = \frac{\Delta r}{n-1}$, then we have

$$\frac{\Delta r}{n-1} \leq \frac{\operatorname{sn}_k'(r)}{\operatorname{sn}_k(r)}$$

This shows desired comparison.

Furthermore, if equality holds

$$\frac{\Delta r}{n-1} = \frac{\operatorname{sn}_k'(r)}{\operatorname{sn}_k(r)}$$

then direct computation shows

$$(\frac{\Delta r}{n-1})^2 + \partial_r(\frac{\Delta r}{n-1}) + k = 0$$

which implies inequalities in (1) and (2) are in fact equalities. In particular one has

$$|\operatorname{Hess} r|^2 = \frac{(\Delta r)^2}{n-1}$$

that is inequality in Cauchy-Schwarz inequality holds, which implies

$$\mathscr{H}_r = \frac{\operatorname{sn}_k'(r)}{\operatorname{sn}_k(r)}\pi_r$$

Then g has constant sectional curvature k in U_0 by Proposition 18.4.2. □

20.2. Maximal principle.

Proposition 20.2.1. Let (M, g) be a Riemannian manifold and f, h be two smooth functions on M. If there is a point p such that $f(p) = h(p)$ and $f(x) \geq h(x)$ for all x near p, then

$$\nabla f(p) = \nabla h(p), \quad \text{Hess } f|_p \geq \text{ Hess } h|_p, \quad \Delta f(p) \geq \Delta h(p).$$

Proof. Firstly let's consider the case $(M, g) \subset (\mathbb{R}^n, g_{\text{can}})$, it's a simple calculus since we can use Taylor expansion. To be explicit, for all x near p, we have

$$f(x) = f(p) + \nabla f(p)^T(x-p) + \frac{1}{2}(x-p)^T \operatorname{Hess} f|_p(x-p) + O(|x|^3)$$

where ∇f is a n column vector and $\operatorname{Hess} f$ is a $n \times n$ matrix in this case. Similarly we have

$$h(x) = h(p) + \nabla h(p)^T(x-p) + \frac{1}{2}(x-p)^T \operatorname{Hess} h|_p(x-p) + O(|x|^3)$$

Then consider

$$f(x)-h(x) = (\nabla f - \nabla h)(p)^T(x-p) + \frac{1}{2}(x-p)^T \operatorname{Hess}(f-h)|_p(x-p) + O(|x|^3)$$

Since $f(x) - h(x) \geq 0$ for all x near p, then we must have

$$\nabla f(p) = \nabla h(p)$$
$$\operatorname{Hess} f|_p \geq \operatorname{Hess} h|_p$$

By taking trace we have

$$\Delta f(p) \geq \Delta h(p)$$

For general case, take $\gamma\colon (-\varepsilon, \varepsilon) \to M$ to be a geodesic with $\gamma(0) = p$, then use previous case on $f \circ \gamma, h \circ \gamma$ to obtain

$$\nabla_{\gamma'(0)} f(p) = \nabla_{\gamma'(0)} h(p)$$
$$\operatorname{Hess} f_p(\gamma'(0), \gamma'(0)) \geq \operatorname{Hess} h_p(\gamma'(0), \gamma'(0))$$

Then it's clear this proposition holds if we let $v = \gamma'(0)$ run over all $v \in T_pM$. □

Definition 20.2.1 (barrier sense)**.** Let (M, g) be a Riemannian manifold and $f \in C(M)$. Suppose f_q is a C^2 function defined in a neighborhood of U of $q \in M$.

1. f_q is called a lower barrier function of f at q if
$$f_q(q) = f(q), \quad f_q(x) \leq f(x), \quad x \in U.$$
2.
$$\Delta f(q) \geq c$$
in the barrier sense if for all $\varepsilon > 0$, there exists a lower barrier function $f_{q,\varepsilon}$ of f at q such that
$$\Delta f_{q,\varepsilon}(q) \geq c - \varepsilon$$

3.
$$\Delta f(q) \leq c$$
in the barrier sense if for all $\varepsilon > 0$, there exists a upper barrier function $f_{q,\varepsilon}$ of f at q such that
$$\Delta f_{q,\varepsilon}(q) \leq c + \varepsilon$$

Definition 20.2.2 (distribution sense)**.** Let (M, g) be an orientable Riemannian manifold and $f \in C(M)$.
$$\Delta f \leq h$$
in distribution sense, if
$$\int_M f \Delta\varphi \leq \int_M h\varphi$$
holds for all $\varphi \geq 0 \in C_c^\infty(M)$

Theorem 20.2.1 (maximal principle)**.** Let (M, g) be a Riemannian manifold and $f \in C(M)$.

1. If $\Delta f \geq 0$ in the barrier sense or distribution sense, then if f has a local(global) maximum, then it's local(global) constant.
2. If $\Delta f \leq 0$ in the barrier sense or distribution sense, then if f has a local(global) minimal, then it's local(global) constant.
3. $\Delta f = 0$ implies $f \in C^\infty(M)$.

Proof. See Theorem 66 in Page280 of [Pet06]. □

20.3. **Global Laplacian comparison.**

20.3.1. *In the barrier sense.*

Proposition 20.3.1. Let (M, g) be a complete Riemannian manifold and $p, q \in M$. Let $\gamma : [0, b] \to M$ be a unit-speed minimal geodesic with $\gamma(0) = p$ and $\gamma(b) = q$. For any small $\varepsilon > 0$,
$$r_\varepsilon(x) = \varepsilon + \mathrm{dist}(\gamma(\varepsilon), x)$$
where $x \in M$. Then

1. $q \notin \mathrm{cut}(\gamma(\varepsilon))$ and in particular, r_ε is smooth at q.
2. r_ε is an upper barrier function of $r(x) = \mathrm{dist}(p, x)$ at point q.

Proof. For (1). If $q \in \mathrm{cut}(\gamma(\varepsilon))$, by Corollary 13.1.1, one has $\gamma(\varepsilon) \in \mathrm{cut}(q)$, a contradiction to γ. is a minimal geodesic connecting p and q.

For (2). Firstly note that $\gamma(b) = q$, then
$$r(q) = \mathrm{dist}(p, q) = \mathrm{dist}(\gamma(0), \gamma(b)) \overset{\mathrm{I}}{=} \mathrm{dist}(\gamma(0), \gamma(\varepsilon)) + \mathrm{dist}(\gamma(\varepsilon), \gamma(b)) \overset{\mathrm{II}}{=} r_\varepsilon(q)$$
where

I holds since γ is a minimal geodesic.
II holds since γ is unit-speed minimal geodesic, then $\mathrm{dist}(\gamma(0), \gamma(\varepsilon)) = \varepsilon$.

By triangle inequality, one has

$$r(q') = \operatorname{dist}(p, q') \leq \varepsilon + \operatorname{dist}(\gamma(\varepsilon), q) = r_\varepsilon(q')$$

for all q' near q. Combining these two facts together we have r_ε is an upper barrier function of r. □

Theorem 20.3.1 (global Laplacian comparison)**.** Let (M, g) be a complete Riemannian manifold with

$$\operatorname{Ric}(g) \geq (n-1)kg$$

Then for $q \in M$

$$\Delta r(q) \leq (n-1)\frac{\operatorname{sn}_k'(r(q))}{\operatorname{sn}_k(r(q))}$$

in the barrier sense.

Proof. We consider the following three cases:

1. If $q \in M \setminus \{p\} \cup \operatorname{cut}(p)$, it's exactly smooth case we have proven.
2. If $q = p$, it's clear, since the right hand is infinite.
3. For arbitrary $q \in \operatorname{cut}(p)$, there exists a unit-speed $\gamma\colon [0, b] \to M$ with $\gamma(0) = p, \gamma(b) = q$. Then for each $\gamma > 0$, define

$$\gamma_\varepsilon(x) = \varepsilon + \operatorname{dist}(\gamma(\varepsilon), x)$$

   Then by Proposition 20.3.1 we have $\gamma_\varepsilon(x)$ is an upper barrier of $r(x)$ and γ_ε is smooth at q. Thus we have

$$\begin{aligned}\Delta\gamma_\varepsilon(q) &= \Delta \operatorname{dist}(\gamma(\varepsilon), q)\\ &\leq (n-1)\frac{\operatorname{sn}_k'(\gamma_\varepsilon(q)-\varepsilon)}{\operatorname{sn}_k(\gamma_\varepsilon(q)-\varepsilon)}\\ &= (n-1)\frac{\operatorname{sn}_k'(\gamma(q)-\varepsilon)}{\operatorname{sn}_k(\gamma(q)-\varepsilon)}\end{aligned}$$

   which descends to $(n-1)\frac{\operatorname{sn}_k'(\gamma(q))}{\operatorname{sn}_k(\gamma(q))}$ as $\varepsilon \to 0$ by monotonicity. This completes the proof.

□

20.3.2. *In the distribution sense.*

Proposition 20.1. Let (M, g) be an orientable Riemannian manifold and $f\colon M \to \mathbb{R}$ a Lipschitz function. Then for any $\varphi \in C_0^\infty(M, \mathbb{R})$, one has

$$-\int_M \langle \nabla\varphi, \nabla f\rangle d\mathrm{vol}_g = \int_M \Delta\varphi \cdot f d\mathrm{vol}_g.$$

Theorem 20.3.2 (global Laplacian comparison II)**.** Let (M, g) be a complete Riemannian manifold with

$$\operatorname{Ric}(g) \geq (n-1)kg$$

Then for $x \in M$

$$\Delta r(x) \leq (n-1)\frac{\mathrm{sn}_k'(r(x))}{\mathrm{sn}_k(r(x))}$$

in the distribution sense.

Proof. For fixed $p \in M$, the domain $\Sigma(p)$ of injective radius $\mathrm{inj}(p)$ is a star-shaped open subset of T_pM and $M = \exp_p(\Sigma(p)) \cup \mathrm{cut}(p)$. The boundary of $\Sigma(p)$ is locally a graph of continuous function and so there exists a family of star-shaped domains $\{U_j\}$ with smooth boundaries such that

$$U_j \subset U_{j+1} \subset \cdots \subset \Sigma(p), \quad \Sigma(p) = \bigcup U_j$$

If we set $\Omega = \exp_p(\Sigma(p))$, then $\Omega = \bigcup \Omega_j$, where $\Omega_j = \exp_p(U_j)$. Since each U_j is star-shaped, by Gauss lemma, on each boundary $\partial\Omega_j$, one has $\frac{\partial r}{\partial v} = g(\nabla r, v) \geq 0$ where v is the outer normal vector on $\partial\Omega_j$.

Therefore for each $\varphi \in C_c^\infty(M)$ with $\varphi \geq 0$, one has

$$\begin{aligned}
\int_M r\Delta\varphi \,\mathrm{vol} &\overset{(1)}{=} -\int_M \langle \nabla r, \nabla\varphi\rangle \,\mathrm{vol} \\
&\overset{(2)}{=} -\lim_j \int_{\Omega_j\backslash\{p\}} \langle \nabla r, \nabla\varphi\rangle \\
&\overset{(3)}{=} \lim_j \Big(\int_{\Omega_j\backslash\{p\}} \Delta r\varphi \,\mathrm{vol} - \int_{\partial\Omega_j} \varphi\frac{\partial r}{\partial v}\Big) \\
&\overset{(4)}{\leq} \lim_j \int_{\Omega_j\backslash\{p\}} \Delta r\varphi \,\mathrm{vol} \\
&\overset{(5)}{\leq} \lim_j \int_{\Omega_j\backslash\{p\}} (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\varphi \,\mathrm{vol} \\
&\overset{(6)}{=} \int_{\Omega\backslash\{p\}} (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k'(r)} \,\mathrm{vol} \\
&\overset{(7)}{=} \int_M (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\varphi \,\mathrm{vol}
\end{aligned}$$

where

(1) holds from the fact r is Lipschitz and Proposition 20.1.
(2) and (6) holds from dominated convergence theorem.
(3) holds from Stokes theorem.
(4) holds from $\varphi \geq 0$ and $\frac{\partial r}{\partial v} \geq 0$.
(5) holds from Local Laplacian comparison theorem, that is Theorem 20.1.1.
(7) holds from the fact $\mathrm{cut}(p)$ is zero-measure.

□

20.4. **Volume comparison.**

Lemma 20.4.1. Let (M, g) be a complete, connected Riemannian manifold and $p \in M$. For any $\delta \in \mathbb{R}^+$

$$\exp_p(B(0,\delta) \cap \Sigma(p)) \subset B(p,\delta) \subset \exp_p(B(0,\delta) \cap \Sigma(p)) \cup \mathrm{cut}(p)$$

In particular, under the map $\Phi\colon \mathbb{R}^+ \times \mathbb{S}^{n-1} \to T_pM \setminus \{0\}$ given by $\Phi(\rho, \omega) = \rho\omega$

$$\begin{aligned}
\mathrm{Vol}(B(p,\delta)) &= \mathrm{Vol}(\exp_p(B(0,\delta)) \cap \Sigma(p)) \\
&= \int_{B(0,\delta)\cap\Sigma(p)} \exp_p^* \mathrm{vol} \\
&= \int_{B(0,\delta)} \chi_{\Sigma(p)} \exp_p^* \mathrm{vol} \\
&= \int_{\mathbb{S}^{n-1}} \int_0^\delta \chi_{\Sigma(p)} \sqrt{\det g} \circ \Phi(\rho,\omega) \rho^{n-1} \mathrm{d}\rho \, \mathrm{vol}_{\mathbb{S}^{n-1}}
\end{aligned}$$

Corollary 20.4.1. Let $p \in S(n, k)$

1. If $k \leq 0$, then for any $\delta \in \mathbb{R}^+$

$$\mathrm{Vol}(B(p,\delta)) = \int_{\mathbb{S}^{n-1}} \int_0^\delta \mathrm{sn}_k^{n-1}(\rho) \mathrm{d}\rho \, \mathrm{vol}_{\mathbb{S}^{n-1}}$$

2. If $k = \frac{1}{R^2} \geq 0$, then for any $\delta \in \mathbb{R}^+$

$$\mathrm{Vol}(B(p,\delta)) = \int_{\mathbb{S}^{n-1}} \int_0^\delta \chi_{B(0,\pi R)} \mathrm{sn}_k^{n-1}(\rho) \mathrm{d}\rho \, \mathrm{vol}_{\mathbb{S}^{n-1}}$$

Lemma 20.4.2. Let (M, g) be a Riemannian manifold, and (x^i, U, p) be a geodesic ball chart of radius b around $p \in M$.

1. If $K \leq k$, then for each fixed $\omega \in \mathbb{S}^{n-1}$ the volume density ratio

$$\lambda(\rho,\omega) = \frac{\rho^{n-1}\sqrt{\det g} \circ \Phi(\rho,\omega)}{\mathrm{sn}_k^{n-1}(\rho)}$$

   is non-decreasing in $\rho \in (0, b_0)$ where

$$b_0 = \begin{cases} b, & k \leq 0 \\ \min\{b, \pi R\}, & k = \frac{1}{R^2} \end{cases}$$

   Moreover, $\lim_{\rho\to 0} \lambda(\rho,\omega) = 1$.
2. If $K \geq k$ or $\mathrm{Ric}(g) \geq (n-1)kg$, then for each fixed $\omega \in \mathbb{S}^{n-1}$ the volume density ratio $\lambda(\rho,\omega)$ is non-increasing in $\rho \in (0, b)$ and $\lim_{\rho\to 0} \lambda(\rho,\omega) = 1$.

Proof. By Corollary 19.2.1 and Lemma 20.1.1

$$\partial_r \log(r^{n-1}\sqrt{\det g}) = \Delta r \geq (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)} = \partial_r \log(\mathrm{sn}_k^{n-1}(r))$$

Hence $\log\left(\frac{r^{n-1}\sqrt{\det g}}{\mathrm{sn}_k^{n-1}(r)}\right)$ is a non-decreasing function of r along each radial geodesic γ, that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\log\left(\frac{r^{n-1}\sqrt{\det g}}{\mathrm{sn}_k^{n-1}(r)}\right)\circ\gamma(t)\right)\geq 0$$

Hence, $f(r)=\frac{r^{n-1}\sqrt{\det g}}{\mathrm{sn}_k^{n-1}(r)}$ is a non-decreasing function of r along each radial geodesic γ. It is easy to see that $r\circ\Phi=\rho$ (the exponential map is used in normal coordinate). Hence,

$$\lambda(\rho,\omega)=f\circ\Phi(\rho,\omega)$$

is nondecreasing in ρ for any fixed $\omega\in\mathbb{S}^{n-1}$. It is obvious that

$$\lim_{\rho\to 0}\sqrt{\det g}=\lim_{\rho\to 0}\frac{\rho^{n-1}}{\mathrm{sn}_k^{n-1}(\rho)}=1.$$

The proof of (2) is similar. □

Lemma 20.4.3. Let $f\colon[0,+\infty)\to[0,+\infty), g:[0,+\infty)\to(0,+\infty)$ be two integrable functions. If

$$\lambda(t)=\frac{f(t)}{g(t)}:[0,+\infty)\to[0,+\infty)$$

is non-increasing, then

$$F(t)=\frac{\int_0^t f(\tau)\mathrm{d}\tau}{\int_0^t g(\tau)\mathrm{d}\tau}:[0,+\infty)\to[0,+\infty)$$

is non-increasing. Moreover, if there exists $0<t_1<t_2$ such that

$$F(t_1)=F(t_2),$$

then $\lambda(t)\equiv\lambda(t_1)$ for almost all $t\in[0,t_2]$.

Proof. We can assume $f(t)>0$ for all $t\in[0,+\infty)$, otherwise we replace it by $f(t)+\varepsilon g(t)$ for some $\varepsilon>0$. Given $0<t_1<t_2$, we need to show

$$\int_0^{t_1}f(\tau)d\tau\int_0^{t_2}g(\tau)d\tau-\int_0^{t_2}f(\tau)d\tau\int_0^{t_1}g(\tau)d\tau\geq 0.$$

Indeed,

$$\begin{aligned}
&\int_0^{t_1} f(\tau)d\tau \int_0^{t_2} g(\tau)d\tau - \int_0^{t_2} f(\tau)d\tau \int_0^{t_1} g(\tau)d\tau \\
=&\int_0^{t_1} f(\tau)d\tau \int_0^{t_2} g(\tau)d\tau - \int_0^{t_1} f(\tau)d\tau \int_0^{t_1} g(\tau)d\tau - \int_{t_1}^{t_2} f(\tau)d\tau \int_0^{t_1} g(\tau)d\tau \\
=&\int_0^{t_1} f(\tau)d\tau \int_{t_1}^{t_2} g(\tau)d\tau - \int_{t_1}^{t_2} f(\tau)d\tau \int_0^{t_1} g(\tau)d\tau \\
\overset{(1)}{\geq}&\int_0^{t_1} \frac{f(t_1)}{g(t_1)} g(\tau)d\tau \int_{t_1}^{t_2} \frac{g(t_1)}{f(t_1)} f(\tau)d\tau - \int_{t_1}^{t_2} f(\tau)d\tau \int_0^{t_1} g(\tau)d\tau \\
=&0
\end{aligned}$$

where (1) holds from $\lambda(t)$ is non-increasing. It is clear that if $F(t_1) = F(t_2)$, then the inequality marked by (1) is an equality, which implies for almost all $t \in [0, t_2], \lambda(t) \equiv \lambda(t_1)$. $\square$

Remark 20.4.1. For any $0 \leq \delta_1 < \delta_2 \leq \delta_3 < \delta_4$, we can slightly adapt above proof to show

$$\frac{\int_{\delta_3}^{\delta_4} f(\tau)\mathrm{d}\tau}{\int_{\delta_3}^{\delta_4} g(\tau)\mathrm{d}\tau} \leq \frac{\int_{\delta_1}^{\delta_2} f(\tau)\mathrm{d}\tau}{\int_{\delta_1}^{\delta_2} g(\tau)\mathrm{d}\tau}$$

Indeed, just note that

$$\begin{aligned}
&\int_{\delta_3}^{\delta_4} f(\tau)\mathrm{d}\tau \int_{\delta_2}^{\delta_1} g(\tau)\mathrm{d}\tau - \int_{\delta_1}^{\delta_2} f(\tau)\mathrm{d}\tau \int_{\delta_3}^{\delta_4} g(\tau)\mathrm{d}\tau \\
\leq&\int_{\delta_3}^{\delta_4} \frac{f(\delta_3)}{g(\delta_3)} g(\tau)\mathrm{d}\tau \int_{\delta_2}^{\delta_1} g(\tau)\mathrm{d}\tau - \int_{\delta_1}^{\delta_2} \frac{f(\delta_2)}{g(\delta_2)} g(\tau)\mathrm{d}\tau \int_{\delta_3}^{\delta_4} g(\tau)\mathrm{d}\tau \\
=&\left(\frac{f(\delta_3)}{g(\delta_3)} - \frac{f(\delta_2)}{g(\delta_2)}\right) \int_{\delta_2}^{\delta_1} g(\tau)\mathrm{d}\tau \int_{\delta_3}^{\delta_4} g(\tau)\mathrm{d}\tau \\
\leq&0
\end{aligned}$$

Theorem 20.4.1 (Bishop-Gromov)**.** Let (M, g) be a complete Riemannian manifold and $p \in M$. Let $B(p, \delta)$ be the metric ball centered at p with radius δ and g_k be the metric with constant sectional curvature k on $B(p, \delta) \setminus \{p\}$.

1. Suppose $K \leq k$, then the volume ratio $\frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))}$ is non-decreasing for any $0 < \delta \leq \delta_0$ where $\delta_0 = \mathrm{inj}(p)$ if $k \leq 0$, and $\delta_0 = \min\{\mathrm{inj}(p), \pi/\sqrt{k}\}$ if $k > 0$. Moreover,

   $$\lim_{\delta \to 0} \frac{\mathrm{Vol}_g(B(p, \delta))}{\mathrm{Vol}_{g_k}(B(p, \delta))} = 1.$$

   In particular,

   $$\mathrm{Vol}_g(B(p, \delta)) \geq \mathrm{Vol}_{g_k}(B(p, \delta)),$$

2. If $K \geq k$ or $\mathrm{Ric}(g) \geq (n-1)kg$, then the volume ratio $\frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))}$ is non-increasing for $\delta \in \mathbb{R}^+$. Moreover,
$$\lim_{\delta\to 0}\frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))} = 1$$
In particular,
$$\mathrm{Vol}_g(B(p,\delta)) \leq \mathrm{Vol}_{g_k}(B(p,\delta)),$$
3. Furthermore, if there exists $\delta_1 < \delta_2$ such that
$$\frac{\mathrm{Vol}_g(B(p,\delta_1))}{\mathrm{Vol}_{g_k}(B(p,\delta_1))} = \frac{\mathrm{Vol}_g(B(p,\delta_2))}{\mathrm{Vol}_{g_k}(B(p,\delta_2))}$$
then $\mathrm{Vol}_g(B(p,\delta)) = \mathrm{Vol}_{g_k}(B(p,\delta))$ for any $\delta \in [0,\delta_2]$ and g has constant sectional curvature k on $B(p,\delta_2)$.

Proof. For (1). By the assumption, we know the metric ball $B(p,\delta)$ is actually a geodesic ball. We have the expression
$$\begin{aligned}\frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))} &\overset{\mathrm{I}}{=} \frac{\int_{\mathbb{S}^{n-1}}\int_0^\delta \rho^{n-1}\sqrt{\det g}\circ\Phi(\rho,\omega)\mathrm{d}\rho\mathrm{d}\,\mathrm{Vol}_{\mathbb{S}^{n-1}}}{\int_{\mathbb{S}^{n-1}}\int_0^\delta \mathrm{sn}_k^{n-1}(\rho)\mathrm{d}\rho\mathrm{d}\,\mathrm{Vol}_{\mathbb{S}^{n-1}}}\\ &\overset{\mathrm{II}}{=} \frac{1}{\mathrm{Vol}(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}}\left(\frac{\int_0^\delta \rho^{n-1}\sqrt{\det g}\circ\Phi(\rho,\omega)\mathrm{d}\rho}{\int_0^\delta \mathrm{sn}_k^{n-1}(\rho)\mathrm{d}\rho}\right)\mathrm{d}\,\mathrm{Vol}_{\mathbb{S}^{n-1}}.\end{aligned}$$
where

I holds from Lemma 20.4.1.

II holds from Fubini's theorem.

By Lemma 20.4.2, one has $\lambda(\rho,\omega) = \frac{\rho^{n-1}\sqrt{\det g}\circ\Phi(\rho,\omega)}{\mathrm{sn}_k^{n-1}(\rho)}$ is non-decreasing in ρ, then by Lemma 20.4.3 we have $\frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))}$ is non-decreasing in ρ. On ther other hand,
$$\lim_{\delta\to 0}\frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))} = 1$$
Hence, for any $0<\delta\leq\delta_0, \mathrm{Vol}_g(B(p,\delta)) \geq \mathrm{Vol}_{g_k}(B(p,\delta))$

For (2). Let's divide into the following two cases:

(a) If $k\leq 0$, for any $\delta \in \mathbb{R}^+$, we get
$$\begin{aligned}\frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))} &= \frac{\int_{\mathbb{S}^{n-1}}\int_0^\delta \chi_{\Sigma(p)}\rho^{n-1}\sqrt{\det g}\circ\Phi(\rho,\omega)\mathrm{d}\rho\mathrm{d}\,\mathrm{Vol}_{\mathbb{S}^{n-1}}}{\int_{\mathbb{S}^{n-1}}\int_0^\delta \mathrm{sn}_k^{n-1}(\rho)\mathrm{d}\rho\mathrm{d}\,\mathrm{Vol}_{\mathbb{S}^{n-1}}}\\ &= \frac{1}{\mathrm{Vol}(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}}\left(\frac{\int_0^\delta \chi_{\Sigma(p)}\rho^{n-1}\sqrt{\det g}\circ\Phi(\rho,\omega)\mathrm{d}\rho}{\int_0^\delta \mathrm{sn}_k^{n-1}(\rho)\mathrm{d}\rho}\right)\mathrm{d}\,\mathrm{Vol}_{\mathbb{S}^{n-1}}.\end{aligned}$$
where these two equalities hold from the same reasons. So in this case we consider
$$\widetilde{\lambda}(\rho,\omega) := \chi_{\Sigma(p)}\lambda(\rho,\omega)$$

It's clear $\widetilde{\lambda}$ is also non-increasing in ρ, since $\chi_{\Sigma(p)}$ is just a cut-off function, then the same argument implies for arbitrary $\delta \in \mathbb{R}^+$, one has $\mathrm{Vol}_g(B(p,\delta)) \leq \mathrm{Vol}_{g_k}(B(p,\delta))$.

(b) If $k = \frac{1}{R^2} > 0$, for any $\delta \in \mathbb{R}^+$, we get

$$\begin{aligned}\frac{\mathrm{Vol}_g(B(p,\delta))}{\mathrm{Vol}_{g_k}(B(p,\delta))} &= \frac{\int_{\mathbb{S}^{n-1}} \int_0^\delta \chi_{\Sigma(p)} \rho^{n-1}\sqrt{\det g} \circ \Phi(\rho,\omega) \mathrm{d}\rho \mathrm{d}\,\mathrm{Vol}_{\mathbb{S}^{n-1}}}{\int_{\mathbb{S}^{n-1}} \int_0^\delta \chi_{B(0,\pi R)} \mathrm{sn}_k^{n-1}(\rho) \mathrm{d}\rho \mathrm{d}\,\mathrm{Vol}_{\mathbb{S}^{n-1}}} \\ &= \frac{1}{\mathrm{Vol}(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} \Big(\frac{\int_0^\delta \chi_{\Sigma(p)} \rho^{n-1}\sqrt{\det g} \circ \Phi(\rho,\omega)\mathrm{d}\rho}{\int_0^\delta \chi_{B(0,\pi R)} \mathrm{sn}_k^{n-1}(\rho)\mathrm{d}\rho}\Big) \mathrm{d}\,\mathrm{Vol}_{\mathbb{S}^{n-1}}.\end{aligned}$$

So in this case we consider[7]

$$\widetilde{\lambda}(\rho,\omega) := \frac{\chi_{\Sigma(p)}}{\chi_{B(0,\pi R)}} \lambda(\rho,\omega)$$

Then the same argument shows the result.

For (3).

$\square$

Corollary 20.4.2. Let (M,g) be a complete Riemannian n-manifold with $\mathrm{Ric}(g) \geq 0$. Then the volume growth of (M,g) satisfies

$$\mathrm{Vol}_g(B(p,r)) \leq c_n r^n$$

where c_n is a constant > 0 depending only on n.

Proof. Consider $k = 0$ and use Theorem 20.4.1, one has

$$\mathrm{Vol}_g(B(p,r)) \leq \mathrm{Vol}_{g_0}(B(p,r)) = \frac{\mathrm{Vol}_{g_1}(\mathbb{S}^{n-1}) r^n}{n}$$

where $\mathbb{S}^{n-1}$ is the unit sphere. Thus we just set $c_n = \mathrm{Vol}_{g_1}(\mathbb{S}^{n-1})/n$ to conclude. $\square$

Corollary 20.4.3. Let (M,g) be a complete Riemannian n-manifold with $\mathrm{Ric}(g) \geq 0$. If

$$\lim_{r\to\infty} \frac{\mathrm{Vol}_g(B(p,r))}{r^n} \geq \frac{\mathrm{Vol}_{g_1}(\mathbb{S}^{n-1})}{n}$$

where $\mathbb{S}^{n-1}$ is unit sphere, then (M,g) is isometric to $(\mathbb{R}^n, g_{\mathrm{can}})$.

[7] Be careful, our notation here is a little bit ambiguous, since it's nonsense if $\chi_{B(0,\pi R)} = 0$. However, Myers' theorem implies $\mathrm{diam}(M,g) \leq \pi R$, hence $\Sigma(p) \subset B(0,\pi R)$, so here the explicit means of $\frac{\chi_{\Sigma(p)}}{\chi_{B(0,\pi R)}}$ is as follows

$$\frac{\chi_{\Sigma(p)}}{\chi_{B(0,\pi R)}} = \begin{cases} 1, & \delta \in \Sigma(p) \\ 0, & \text{otherwise} \end{cases}$$

Proof. Note that $\mathrm{Vol}_{g_0}(B(p,r)) = \frac{\mathrm{Vol}_{g_1}(\mathbb{S}^{n-1})r^n}{n}$, then our assumption is equivalent to say

$$\lim_{r\to\infty} \frac{\mathrm{Vol}_g(B(p,r))}{\mathrm{Vol}_{g_0}(B(p,r))} = 1$$

However, by Theorem 20.4.1 we know volume ratio $\frac{\mathrm{Vol}_g(B(p,r))}{\mathrm{Vol}_{g_0}(B(p,r))}$ is non-increasing, with

$$\lim_{r\to 0} \frac{\mathrm{Vol}_g(B(p,r))}{\mathrm{Vol}_{g_0}(B(p,r))} = 1$$

which implies $\frac{\mathrm{Vol}_g(B(p,r))}{\mathrm{Vol}_{g_0}(B(p,r))} = 1$ holds for arbitrary $r > 0$. By rigidity of volume comparison, we conclude g has constant sectional curvature 0 on $B(p,r)$ for arbitrary $r > 0$. Since $\overline{B(p,\infty)} = M$, we deduce (M,g) has constant sectional curvature 0.

Thanks to Hopf's theorem, now it suffices to show M is simply-connected, suppose $\pi : \mathbb{R}^n \to M$ is the universal covering, one deduces that

$$|\pi_1(M)| = \frac{\mathrm{Vol}_{g_0}(\mathbb{R}^n)}{\mathrm{Vol}_g(M)} = 1$$

which implies M is simply-connected. $\square$

Corollary 20.4.4. Let (M,g) be a complete Riemannian n-manifold with $\mathrm{Ric}(g) \geq (n-1)kg$ for some constant $k > 0$. Then

$$\mathrm{Vol}_g(M) \leq \mathrm{Vol}_{g_k}(\mathbb{S}^n(\frac{1}{\sqrt{k}}))$$

If the equality holds, then (M,g) is isometric to $\mathbb{S}^n(1/\sqrt{k})$ with standard metric.

Proof. Let $k = 1/R^2$, then Myers's theorem implies $\mathrm{diam}(M,g) \leq \pi R$, thus compact. Hence, for any $p \in M$ one has $\Sigma(p) \subset B(0,\pi R)$. Therefore

$$\mathrm{Vol}_g(B(p,\pi R)) = \mathrm{Vol}_g(M)$$

where $B(p,\pi R)$ is a metric ball in M. On the other hand, it is obvious that

$$\mathrm{Vol}_{g_k}(B(p,\pi R)) = \mathrm{Vol}_{g_k}(\mathbb{S}^n(R))$$

Hence by Theorem 20.4.1, one has

$$\mathrm{Vol}_g(M) \leq \mathrm{Vol}_{g_k}(\mathbb{S}^n(R))$$

Furthermore, if the equality holds, g has constant sectional curvature on $B(p,\pi R)$. Then use the argument in Corollary 20.4.3 completes the proof. $\square$

Corollary 20.4.5. Let (M,g) be a complete Riemannian manifold and $p \in M$. Let $B(p,\delta)$ be the metric ball centered at p with radius δ and

g_k be the metric with constant sectional curvature k on $B(p,\delta)\setminus\{p\}$. If $\operatorname{Ric}(g)\geq (n-1)kg$, then for any $0\leq \delta_1<\delta_2\leq\delta_3<\delta_4$

$$\frac{\operatorname{Vol}_g(B(p,\delta_4))-\operatorname{Vol}_g(B(p,\delta_3))}{\operatorname{Vol}_g(B(p,\delta_2))-\operatorname{Vol}_g(B(p,\delta_1))}\leq\frac{\operatorname{Vol}_{g_k}(B(p,\delta_4))-\operatorname{Vol}_{g_k}(B(p,\delta_3))}{\operatorname{Vol}_{g_k}(B(p,\delta_2))-\operatorname{Vol}_{g_k}(B(p,\delta_1))}$$

Proof. Just note that volume density ratio is non-decreasing, then by Remark 20.4.1, one has

$$\frac{\operatorname{Vol}_g(B(p,\delta_4))-\operatorname{Vol}_g(B(p,\delta_3))}{\operatorname{Vol}_{g_k}(B(p,\delta_4))-\operatorname{Vol}_{g_k}(B(p,\delta_3))}\leq\frac{\operatorname{Vol}_g(B(p,\delta_2))-\operatorname{Vol}_g(B(p,\delta_1))}{\operatorname{Vol}_{g_k}(B(p,\delta_2))-\operatorname{Vol}_{g_k}(B(p,\delta_1))}$$

This gives desired result. □

Theorem 20.4.2 (Cheng)**.** Let (M,g) be a complete Riemannian n-manifold with $\operatorname{Ric}(g)\geq(n-1)kg$ for some constant $k>0$. If $\operatorname{diam}(M)=\pi/\sqrt{k}$, then (M,g) is isometric to $\mathbb{S}^n(1/\sqrt{k})$ with standard metric.

Proof. Let $k=1/R^2$. Since M is complete, there exist points $p,q\in M$ and $\operatorname{dist}(p,q)=\pi R$, thus for any $\delta\in(0,\pi R)$

$$B(p,\delta)\cap B(q,\pi R-\delta)=\varnothing$$

Then

$$\begin{aligned}\operatorname{Vol}_g(M)&\overset{(1)}{\geq}\operatorname{Vol}_g(B(p,\delta))+\operatorname{Vol}_g(B(q,\pi R-\delta))\\&\overset{(2)}{\geq}\operatorname{Vol}_{g_k}(B(p,\delta))\frac{\operatorname{Vol}_g(B(p,\pi R))}{\operatorname{Vol}_{g_k}(B(p,\pi R))}+\operatorname{Vol}_{g_k}(B(q,\pi R-\delta))\frac{\operatorname{Vol}_g(B(q,\pi R))}{\operatorname{Vol}_{g_k}(B(q,\pi R))}\\&\overset{(3)}{=}\operatorname{Vol}_g(M)\end{aligned}$$

where

(1) holds from $B(p,\delta)\cap B(q,\pi R-\delta)=\varnothing$.
(2) holds from Theorem 20.4.1.
(3) holds since for any $x,y\in M$, $\operatorname{Vol}_g(B(x,\pi R))=\operatorname{Vol}_g(M)$ and

$$\begin{aligned}&\operatorname{Vol}_{g_k}(B(x,\pi R))=\operatorname{Vol}_{g_k}(\mathbb{S}^n(R))\\&\operatorname{Vol}_{g_k}(B(x,\delta))+\operatorname{Vol}_{g_k}(B(y,\pi R-\delta))=\operatorname{Vol}_{g_k}(\mathbb{S}^n(R))\end{aligned}$$

Hence, for any $0<\delta<\pi R$.

$$\frac{\operatorname{Vol}_g(B(p,\delta))}{\operatorname{Vol}_{g_k}(B(p,\delta))}=\frac{\operatorname{Vol}_g(B(p,\pi R))}{\operatorname{Vol}_{g_k}(B(p,\pi R))}=\frac{\operatorname{Vol}_g(M)}{\operatorname{Vol}_{g_k}(\mathbb{S}^n(R))}.$$

Let $\delta\to 0$, and we deduce $\operatorname{Vol}_g(M)=\operatorname{Vol}_{g_k}(\mathbb{S}^n(R))$. By Proposition 20.4.4, (M,g) is isometric to $\mathbb{S}^n(R)$ with standard metric. □

Theorem 20.4.3 (Bishop-Yau)**.** Let (M,g) be a complete non-compact Riemannian n-manifold with $\operatorname{Ric}(g)\geq 0$. Then the volume growth of (M,g) satisfies

$$c_n\operatorname{Vol}_g(B(p,1))\cdot r\leq\operatorname{Vol}_g(B(p,r))$$

for $r\geq 1$, where c_n is a positive constant depending only on n.

Proof. Let $x \in \partial B(p, 1+r)$, then

$$B(p,1) \subset B(x, 2+r) \setminus B(x,r), \quad B(x,r) \subset B(p, 1+2r)$$

By Corollary 20.4.5, one has

$$\begin{aligned}
\mathrm{Vol}_g(B(p,1)) &\leq \mathrm{Vol}_g(B(x,2+r)) - \mathrm{Vol}_g(B(x,r)) \\
&\leq \mathrm{Vol}_g(B(x,r)) \cdot \frac{\mathrm{Vol}(B(x,2+r)) - \mathrm{Vol}(B(x,r))}{\mathrm{Vol}(B(x,r))} \\
&\leq \mathrm{Vol}_g(B(p,1+2r)) \cdot \frac{(2+r)^n - r^n}{r^n} \\
&\leq \mathrm{Vol}_g(B(p,1+2r)) \cdot \frac{1}{r} c_n
\end{aligned}$$

where $r \geq 1$. By changing variable, we obtain the lower bound. □

Proposition 20.4.1. Let (M, g) be a Cartan-Hadamard manifold with $\mathrm{Ric}(g) \leq -kg$ for some $k > 0$. Then for any $p \in M$

$$\mathrm{Vol}_g(B(p,r)) \geq c_n e^{\sqrt{k} r}$$

where c_n is a positive constant depending only on n.

Proposition 20.4.2 (Cheeger-Colding)**.** For each integer $n \geq 2$, there exists a real number $\delta(n) \in (0,1)$ with the following property: if (M, g) is a compact Riemannian manifold of dimension n with $\mathrm{Ric}(g) \geq (n-1)g$ and

$$\mathrm{Vol}(M, g) \geq (1 - \delta(n)) \, \mathrm{Vol}(\mathbb{S}^n)$$

then M is diffeomorphic to $\mathbb{S}^n$.

21. Splitting theorem

21.1. Geodesic rays.

Definition 21.1.1 (geodesic ray)**.** A geodesic ray is a unit-speed geodesic $\gamma\colon [0,\infty) \to M$ such that for any $s, t \geq 0$,

$$\mathrm{dist}(\gamma(s), \gamma(t)) = |s-t|$$

Lemma 21.1.1. Let (M, g) be a complete Riemannian manifold. then the following statements are equivalent:

1. M is non-compact.
2. For any $p \in M$, there exists a geodesic ray $\gamma\colon [0,\infty) \to M$ starting from p.

Proof. From (1) to (2) If M is non-compact, for any $p \in M$, there is a sequence of points $\{p_i\}$ such that $\mathrm{dist}(p, p_i) = i$. Let $\gamma_i(t) = \exp_p(tv_i)$ be a unit-speed minimal geodesic connecting p and p_i, that is $\gamma_i(0) = p$ and $\gamma_i(i) = p_i$. By possibly passing to a subsequence, we may assume $v_i \to v \in T_pM$. Then

$$\gamma(t) = \exp_p(tv), \quad t \in [0, +\infty)$$

is a unit-speed geodesic ray. Indeed, for any $s, t \geq 0$, and for any $k > \max\{s, t\}$, one has

$$\mathrm{dist}(\gamma_k(s), \gamma_k(t)) = |s-t|.$$

By continuity of exponential map $\exp_p$, one obtains

$$\mathrm{dist}(\gamma(s), \gamma(t)) = \lim_{k \to +\infty} \mathrm{dist}(\gamma_k(s), \gamma_k(t)) = |s-t|$$

Hence γ is a geodesic ray.

From (2) to (1). It's trivial. $\square$

21.2. Buseman function.

Definition 21.2.1. Let (M, g) be a complete Riemannian manifold, $p \in M$ and $\gamma\colon [0,\infty) \to M$ be a geodesic ray starting from p. For any $t \geq 0$, $b^t_\gamma : M \to \mathbb{R}$ as

$$b^t_\gamma(x) := \mathrm{dist}(x, \gamma(t)) - t$$

Proposition 21.2.1. Let (M, g) be a complete non-compact Riemannian manifold, $p \in M$ and γ be a geodesic ray starting from p. The function $b^t_\gamma(x) : M \to \mathbb{R}$ has the following properties:

1. For any fixed $x \in M, b^t_\gamma(x)$ is non-increasing in t.
2. For any $x \in M$ and $t \geq 0, |b^t_\gamma(x)| \leq \mathrm{dist}(x, \gamma(0))$.
3. For any $x, y \in M$ and $t \geq 0, |b^t_\gamma(x) - b^t_\gamma(y)| \leq \mathrm{dist}(x, y)$.

Proof. For (1). Note that for $t > s > 0$, one has

$$\begin{aligned} b^t_\gamma(x) - b^s_\gamma(t) &= \mathrm{dist}(x, \gamma(t)) - \mathrm{dist}(x, \gamma(s)) + s - t \\ &\leq \mathrm{dist}(\gamma(t), \gamma(s)) + s - t \\ &= |t - s| + s - t \\ &= 0 \end{aligned}$$

For (2),(3). Directly from triangle inequality. □

Definition 21.2.2 (Buseman function)**.** The Buseman function with respect to the geodesic ray is defined as

$$b_\gamma := \lim_{t\to\infty} b^t_\gamma(x)$$

Example 21.2.1 (Buseman function on hyperbolic plane)**.** Note that geodesics on $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ are

1. Semicircles centered on $\mathbb{R}$.
2. Straight lines perpendicular to $\mathbb{R}$.

Given $x \in \mathbb{H}$, in order to compute Buseman function

$$b_\gamma(x) = \lim_{t\to\infty} \mathrm{dist}(x, \gamma(t)) - \mathrm{dist}(\gamma(0), \gamma(t))$$

It suffices to solve the following calculus: Fix $z_1, z_2 \in \mathbb{H}$ and $\alpha \in \partial\mathbb{H} = \mathbb{R}\cup\{\infty\}$, solve

$$\lim_{q\to\alpha} \mathrm{dist}(q, z_1) - \mathrm{dist}(q, z_2) =? \tag{21.1}$$

then we can set $q = \gamma(t), \alpha = \gamma(\infty), z_1 = x, z_2 = \gamma(0)$ to conclude. Let's divide into several steps:

Step one: For arbitrary $r > s > 0$, the distance between ri, si in $\mathbb{H}$ is $\ln \frac{r}{s}$, where i is imaginary number. Indeed, since metric on this line is exactly $\frac{\mathrm{d}y\otimes\mathrm{d}y}{y^2}$.

Step two: In hyperbolic planes, it's possible to use isometry to translate any two points to the positive imaginary axis. To be explicit, consider the Möbius transformation V mapping Poincaré disk $\mathbb{D}$ to $\mathbb{H}$ with inverse V^{-1}, given by

$$\begin{aligned} z = V(w) &= \frac{-iw + 1}{w - i} \\ w = V^{-1}(z) &= \frac{iz + 1}{z + i} \end{aligned}$$

Now for arbitrary $z_1, z_2 \in \mathbb{H}$, firstly use V^{-1} to send z_1, z_2 to $w_1, w_2 \in \mathbb{D}$ respectively, then let $S(w) = e^{i\theta}\frac{w-w_1}{1-\overline{w_1}w}$ be transformation in $\mathbb{D}$ that send w_1 to 0, with θ chosen carefully so that w_2 get sent to the positive imaginary axis, that is, w_2 get sent to the point ki, where $k = |S(w_2)|$. Finally apply V to this situation, 0 gets sent to i and ki get sents to $\frac{1+k}{1-k}i$.

Step three: Combine step one and two, one can conclude that for arbitrary $z_1, z_2 \in \mathbb{H}$, the distance between them are

$$\mathrm{dist}(z_1, z_2) = \ln \frac{1+k}{1-k}$$

If we express k in terms of z_1, z_2, one has

$$\mathrm{dist}(z_1, z_2) = \ln \frac{|z_1 + z_2| + |z_1 - z_2|}{|z_1 + z_2| - |z_1 - z_2|}$$

Step four: Consider a special case of (21.1), that is we assume $z_1 = ri, z_2 = i$, where $\ln r = \mathrm{dist}(z_1, z_2)$. Now we choose a sequence $q_n = u_n + iv_n$ such that $u_n \to \alpha$ and $v_n \to v$, where $v = 0$ as $n \to \infty$. Then

$$\begin{aligned}
\lim_{q\to\alpha}(\mathrm{dist}(q, ri) - \mathrm{dist}(q, i)) &= \lim_{n\to\infty}(\ln(\frac{|q_n + ri| + |q_n - ri|}{|q_n + ri| - |q_n - ri|}) - \ln(\frac{|q_n + i| + |q_n - i|}{|q_n + i| - |q_n - i|})) \\
&= \lim_{n\to\infty}(\ln(\frac{|q_n + ri| + |q_n - ri|}{|q_n + i| + |q_n - i|}) + \ln(\frac{|q_n + i| - |q_n - i|}{|q_n + ri| - |q_n - ri|})) \\
&= \lim_{n\to\infty} \ln \frac{\sqrt{u_n^2 + (v_n + r)^2} + \sqrt{u_n^2 + (v_n - r)^2}}{\sqrt{u_n^2 + (v_n + 1)^2} + \sqrt{u_n^2 + (v_n - 1)^2}} \\
&\quad + \lim_{n\to\infty} \ln \frac{\sqrt{u_n^2 + (v_n + 1)^2} - \sqrt{u_n^2 + (v_n - 1)^2}}{\sqrt{u_n^2 + (v_n + r)^2} - \sqrt{u_n^2 + (v_n - r)^2}} \\
&= \lim_{v_n\to 0} \ln \underbrace{\frac{\sqrt{\alpha^2 + (v_n + r)^2} + \sqrt{\alpha^2 + (v_n - r)^2}}{\sqrt{\alpha^2 + (v_n + 1)^2} + \sqrt{\alpha^2 + (v_n - 1)^2}}}_{\text{part I}} \\
&\quad + \lim_{v_n\to 0} \ln \underbrace{\frac{\sqrt{\alpha^2 + (v_n + 1)^2} - \sqrt{\alpha^2 + (v_n - 1)^2}}{\sqrt{\alpha^2 + (v_n + r)^2} - \sqrt{\alpha^2 + (v_n - r)^2}}}_{\text{part II}}
\end{aligned}$$

It's clear Part I is $\frac{\sqrt{\alpha^2+r^2}}{\sqrt{\alpha^2+1}}$, and apply L'Hospital's rule to Part II one has

$$\lim_{v_n\to 0} \frac{\sqrt{\alpha^2 + (v_n + 1)^2} - \sqrt{\alpha^2 + (v_n - 1)^2}}{\sqrt{\alpha^2 + (v_n + r)^2} - \sqrt{\alpha^2 + (v_n - r)^2}} = \lim_{v_n\to 0} \frac{\frac{v_n+1}{\sqrt{\alpha^2+(v_n+1)^2}} - \frac{v_n-1}{\sqrt{\alpha^2+(v_n-1)^2}}}{\frac{v_n+r}{\sqrt{\alpha^2+(v_n+r)^2}} - \frac{v_n-r}{\sqrt{\alpha^2+(v_n-r)^2}}} = \frac{\sqrt{\alpha^2 + r^2}}{r\sqrt{\alpha^2+1}}$$

which implies

$$\lim_{q\to\alpha} \mathrm{dist}(q, ri) - \mathrm{dist}(q, i) = \ln \frac{\alpha^2 + r^2}{\alpha^2 + 1} - \ln r$$

Step five: In order to solve general case of (21.1), we can use processes in step two to translate z_1, z_2 to the positive imaginary axis. However, α is also translated into a new point α', that is

$$\alpha' = V \circ S \circ V^{-1}(\alpha)$$

where V, V^{-1} and S are defined in step two. Thus from step four one has

$$\lim_{q\to\alpha}\operatorname{dist}(q,z_1)-\operatorname{dist}(q,z_2)=\ln\frac{(\alpha')^2+r^2}{(\alpha')^2+1}-\ln r$$

where $\ln r=\operatorname{dist}(z_1,z_2)$.

Proposition 21.2.2. Let (M,g) be a complete non-compact Riemannian manifold, $p\in M$ and γ be a geodesic ray starting from p. The Busemann function $b_\gamma\colon M\to\mathbb{R}$ is Lipschitz continuous with $\operatorname{Lip}(b_\gamma)\leq 1$

Proof. It follows from Arezla-Ascoli lemma. □

Proposition 21.2.3. Let (M,g) be a complete non-compact Riemannian manifold, and γ be a geodesic ray starting from $p\in M$. If $\operatorname{Ric}(g)\geq 0$, then

$$\Delta b_\gamma\leq 0$$

in the sense of distribution.

Proof. For any non-negative smooth function $\varphi\in C_0^\infty(M)$, one has

$$\begin{aligned}\int_M\Delta\varphi b_\gamma^t\operatorname{vol}&=\int_M\Delta\varphi(\operatorname{dist}(x,\gamma(t))-t)\operatorname{vol}\\&\overset{(1)}{=}\int_M\Delta\varphi\operatorname{dist}(x,\gamma(t))\operatorname{vol}\\&\overset{(2)}{\leq}\int_M\frac{(n-1)\varphi}{\operatorname{dist}(x,\gamma(t))}\operatorname{vol}\end{aligned}$$

where
(1) holds from Stokes' theorem.
(2) holds from Theorem 20.3.2.

Then Lebesgue's dominated convergence implies

$$\int_M\Delta\varphi b_\gamma\operatorname{vol}\leq 0$$

□

Definition 21.2.3 (geodesic line)**.** A geodesic line is a unit-speed geodesic $\gamma\colon(-\infty,\infty)\to M$ such that for any $s,t\in\mathbb{R}$,

$$\operatorname{dist}(\gamma(s),\gamma(t))=|s-t|$$

Lemma 21.2.1. Let (M,g) be a connected, non-compact Riemannian manifold. If M contains a compact subset K such that $M\setminus K$ has at least two unbounded components[8], then there is a geodesic line passing through K.

Proof. Since $M\setminus K$ has at least two unbounded components, there are two unbounded sequences of points $\{p_i\}$ and $\{q_i\}$ such that any curve from p_i to q_i passes through K. Let $\gamma_i:[-a_i,b_i]\to M$ be minimal geodesics connecting p_i and q_i with $\gamma_i(-a_i)=p_i$, $\gamma_i(b_i)=q_i$ and $\gamma_i(0)\in K$. Hence, $a_i\to+\infty$

[8]Some authors use "ends" to call such unbounded components.

and $b_i \to +\infty$. By possibly passing to subsequences, $\{\gamma_i\}$ converges to a geodesic line $\gamma_\infty : (-\infty, +\infty) \to M$. □

Proposition 21.2.4. Let (M, g) be a complete non-compact Riemannian manifold with $\mathrm{Ric}(g) \geq 0$. If (M, g) contains a geodesic line γ, then $b_{\gamma_\pm} : M \to \mathbb{R}$ are smooth harmonic functions with

$$|\nabla b_{\gamma_\pm}| = 1, \quad \mathrm{Hess}\, b_{\gamma_\pm} = 0$$

where $\gamma_\pm(t) = \gamma(\pm t) : [0, +\infty) \to M$.

Proof. Let $b(x) = b_{\gamma_+}(x) + b_{\gamma_-}(x)$. By the triangle inequality

$$\begin{aligned} b(x) &= \lim_{s \to +\infty} \mathrm{dist}(x, \gamma_+(s)) + \mathrm{dist}(x, \gamma_-(s)) - 2s \\ &= \lim_{s \to +\infty} \mathrm{dist}(x, \gamma(s)) + \mathrm{dist}(x, \gamma(-s)) - 2s \\ &\geq 0 \end{aligned}$$

By Proposition 21.2.3, $\Delta b \leq 0$ in the sense of distributions. On the other hand,

$$b(\gamma(t)) = \lim_{s \to +\infty} \mathrm{dist}(\gamma(t), \gamma(s)) + \mathrm{dist}(\gamma(t), \gamma(-s)) - 2s = 0$$

Hence the subharmonic function b attains its absolute minimum, by Theorem 20.2.1, $b \equiv 0$, that is $b_{\gamma_+} = -b_{\gamma_-}$. Hence $\Delta b_{\gamma_+} = \Delta b_{\gamma_-} = 0$, and by Wely's lemma one has $b_{\gamma_\pm}$ are smooth.

Bochner's formula says

$$\frac{1}{2}\Delta|\nabla f|^2 = |\,\mathrm{Hess}\, f|^2 + \mathrm{Ric}(\nabla f, \nabla f) + g(\nabla \Delta f, \nabla f)$$

Let $f = b_{\gamma_+}$, then

$$\frac{1}{2}\Delta|\nabla b_{\gamma_+}|^2 \geq |\,\mathrm{Hess}\, b_{\gamma_+}|^2 \geq 0$$

since b_{γ_+} is harmonic and $\mathrm{Ric}(g) \geq 0$, thus $|\nabla b_{\gamma_+}|^2$ is superharmonic. On the other hand, by Proposition 21.2.2, $\mathrm{Lip}(b_{\gamma_+}) \leq 1$, and so $|\nabla b_{\gamma_+}| \leq 1$. Note that

$$b_{\gamma_+}(\gamma_+(t)) = \lim_{s \to +\infty} \mathrm{dist}(\gamma_+(t), \gamma_+(s)) - s = \lim_{s \to +\infty} |t - s| - s = -t$$

For any $x = \gamma_+(t_0)$

$$|\nabla b_{\gamma_+}|(x) \overset{(1)}{=} |\nabla b_{\gamma_+}||\gamma_+'(t_0)| \overset{(2)}{\geq} |\langle \nabla b_{\gamma_+}(x), \gamma_+'(t_0)\rangle| = 1$$

where

(1) holds from the trivial fact γ_+ is unit-speed.
(2) holds from Cauchy-Schwarz inequality.

Hence, the superharmonic function $|\nabla b_{\gamma_+}|^2$ attains its absolute maximum in M, hence $|\nabla b_{\gamma_+}|^2 \equiv 1$ on M. Again by the Bochner formula, one has Hess $b_{\gamma_+} = 0$. The same argument holds for b_{γ_-}, this completes the proof. □

Lemma 21.2.2. Let (M, g) be a complete Riemannian manifold, and V a smooth vector field with $|V|_g \leq C$ for some constant C. Then V is a complete vector field.

Proof. We need to show the integral curve of V is globally defined, that is defined on $\mathbb{R}$. Suppose $\gamma\colon (a, b) \to M$ is an integral curve of M and $b < \infty$. For arbitrary $t, s \in (a, b)$, we have

$$\gamma(t) = \gamma(s) + \int_s^t V(\gamma(\tau))\mathrm{d}\tau$$

By using the boundedness of V, we can conclude that

$$|\gamma(t) - \gamma(s)| \leq C|t - s|$$

which implies $\gamma(t)$ is uniformly continuous on (a, b), thus it's possible to extend γ to $(a, b]$ since $b < \infty$, a contradiction. $\square$

Proposition 21.2.5. Let (M, g) be a complete Riemannian manifold. Suppose $f \in C^\infty(M, \mathbb{R})$ satisfies

$$|\nabla f| = 1 \quad \text{and} \quad \operatorname{Hess} f = 0.$$

Let Σ denote $f^{-1}(0)$, with induced metric $h := g|_\Sigma$.

1. (Σ, h) is a totally geodesic submanifold of (M, g).
2. The map

$$f\colon (\mathbb{R} \times \Sigma, g_{\mathbb{R}} \oplus h) \to (M, g), \quad F(t, p) = \exp_p(t\nabla_p f)$$

   is an isometry.

Proof. For (1). Recall that (Σ, h) is a totally geodesic submanifold of (M, g) if the second fundamental form of Σ vanishes, and facts in basic differential geometry says the second fundamental form of a hyperplane Σ with induced metric is given by

$$\mathbf{II}(v, w) := \langle \nabla_v n, w \rangle$$

where n is the normal vector of Σ. In this case, if we consider $\Sigma = f^{-1}(0)$, then the normal vector of Σ is exactly ∇f, and thus

$$\mathbf{II}(v, w) := \langle \nabla_v \nabla f, w \rangle$$

Then $\operatorname{Hess} f = \nabla^2 f = 0$ implies the second fundamental form of Σ vanishes, that is Σ is a totally geodesic submanifold of (M, g).

For (2). For a fixed p, let $X = \nabla f$, and consider $\gamma(t) = \exp_p(tX_p)$. Since $\nabla X = 0$, we have $E(t) = X(\gamma(t))$ and $\gamma'(t)$ are two parallel vector fields along γ with the same initial value. Hence

$$\gamma'(t) = X(\gamma(t))$$

that is γ is exactly the integral curve of X. Furthermore, since $|X| = 1$, by Lemma 21.2.2 one has γ is globally defined, and one can deduce F is a global flow of X, thus it's a diffeomorphism.

Now it remains to prove that F is an isometry. For $v \in T_p\Sigma$, let J be the Jacobi field along γ with $J(0) = 0$ and $J'(0) = v$. By the radial curvature equation

$$R(\text{-}, \nabla f, \nabla f, \text{-}) = \mathrm{Hess}(\frac{1}{2}|\nabla f|^2)(\text{-},\text{-}) - (\nabla_{\nabla f} \mathrm{Hess}\, f)(\text{-},\text{-}) - \mathrm{Hess}\, f(\nabla_{\text{-}} \nabla f, \text{-})$$

one has $R(\text{-}, \nabla f, \nabla f, \text{-}) = 0$, thus Jacobi equation

$$J''(t) + R(J, \gamma')\gamma' = 0$$

reduces to $J''(t) = 0$. It implies that $J'(t)$ is a parallel vector field and in particular, $|J'(t)| \equiv |J'(0)| = |v|$. By uniqueness of Jacobi fields, we deduce

$$J(t) = tJ'(t)$$

Then F is an isometry holds as follows:

(*a*) It is easy to see that $(\mathrm{d}F)_{(1,p)}v = J(1)$, thus $|(\mathrm{d}F)_{(1,p)}v| = |J(1)| = |J'(1)| = |v|$.

(*b*) $|(\mathrm{d}F)_{(0,p)}\partial_t| = |\nabla f| = 1 = |\partial_t|$.

□

21.3. **Splitting theorem and its corollaries.**

Theorem 21.3.1 (splitting theorem)**.** Let (M, g) be a complete Riemannian n-manifold with $\mathrm{Ric}(g) \geq 0$. If there is a geodesic line in M, then (M, g) is isometric to $(\mathbb{R} \times N, g_{\mathbb{R}} \oplus g_N)$, where $\mathrm{Ric}(g_N) \geq 0$.

Proof. Directly from Proposition 21.2.4 and Proposition 21.2.5. □

Corollary 21.3.1. Let (M, g) be a complete Riemannian n-manifold with $\mathrm{Ric}(g) \geq 0$

1. (M, g) is isometric to $(\mathbb{R}^k \times N, g_{\mathbb{R}^k} \oplus g_N)$, where N does not contain a geodesic line and $\mathrm{Ric}(g_N) \geq 0$.
2. The isometry group splits
$$\mathrm{Iso}(M, g) \cong \mathrm{Iso}(\mathbb{R}^k) \times \mathrm{Iso}(N, g_N)$$

Theorem 21.3.2 (structure theorem for manifold with $\mathrm{Ric} \geq 0$)**.** Let (M, g) be a compact Riemannian manifold with $\mathrm{Ric}(g) \geq 0$, and $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ is its universal covering with the pullback metric.

1. There exists some integer $k \geq 0$ and a compact Riemannian manifold (N, g_N) with $\mathrm{Ric}(g_N) \geq 0$ such that $(\widetilde{M}, \widetilde{g})$ is isometric to $(\mathbb{R}^k \times N, g_{\mathrm{can}} \oplus g_N)$.
2. The isometry group splits
$$\mathrm{Iso}(\tilde{M}, \widetilde{g}) \cong \mathrm{Iso}(\mathbb{R}^k) \times \mathrm{Iso}(N, g_N)$$

Proof. For (1). Suppose to the contrary that N is non-compact, then fix a point $x_0 \in N$, there exists a geodesic ray $\gamma\colon [0, \infty) \to N$ starting from x_0. Since M is compact, there exists a compact subset $\widetilde{K} \subset \widetilde{M}$ such that

$$\mathrm{Aut}_\pi(\widetilde{M})\widetilde{K} = \widetilde{M}$$

□

Corollary 21.3.2. $\mathbb{S}^n \times \mathbb{S}^1$ doesn't admit any Ricci flat metrics when $n = 2, 3$.

Proof. If $\mathbb{S}^n \times \mathbb{S}^1$ admits a Ricci flat metric, after splitting its universal covering we obtain a Ricci flat metric on $\mathbb{S}^n$. However, $\mathbb{S}^n$ doesn't admit such a metric when $n = 2, 3$. Indeed, since any Einstein manifold with dimension 2 or 3 has constant sectional curvature, thus if $\mathbb{S}^n, n = 2, 3$ admit a Ricci flat metric, then it has constant sectional curvature 0, and it's also simply-connected, so Hopf's theorem implies it's diffeomorphic to $\mathbb{R}^n$, a contradiction. □

Remark 21.3.1. It's clear $\mathbb{S}^1 \times \mathbb{S}^1$ admits a Ricci flat metric, and when $n \geq 4$, we don't know whether $\mathbb{S}^n$ admit a Ricci flat metric or not.

Corollary 21.3.3. Let (M, g) be a compact Riemannian manifold with $\mathrm{Ric}(g) \geq 0$, and $(\widetilde{M}, \widetilde{g})$ is its universal covering equipped with pullback metric.

1. If $\widetilde{M}$ is contractible, then $(\widetilde{M}, \widetilde{g})$ is isometric to $(\mathbb{R}^n, g_{\mathrm{can}})$ and (M, g) is flat.
2. If $(\widetilde{M}, \widetilde{g})$ doesn't contain a geodesic line, then $\pi_1(M)$ is finite and $b_1(M) = 0$.

Proof. For (1). If $\widetilde{M} \cong N \times \mathbb{R}^k$ is contractible, we must have N is just a point, since it's compact,

For (2). If $\widetilde{M}$ doesn't contain a geodesic line, then $\widetilde{M}$ is compact, which implies $|\pi_1(M)|$ is finite. Furthermore, since there is a natural Hurwicz surjective

$$h : \pi_1(M) \to H_1(M.\mathbb{Z})$$

thus $H_1(M.\mathbb{Z})$ can't have free part, otherwise h can't be surjective, since there is no surjective map from a finite group to an infinite one. So we have $b_1(M) = 0$. □

Corollary 21.3.4. Let (M, g) be a compact Riemannian manifold with $\mathrm{Ric}(g) \geq 0$. If there exists a point $p \in M$ such that $\mathrm{Ric}(g) > 0$ on T_pM, then $\pi_1(M)$ is finite and $b_1(M) = 0$.

Proof. Since $\mathrm{Ric}(g) > 0$ on the whole tangent space T_pM, the universal covering $(\widetilde{M}, \widetilde{g})$ can't split into a product $(\mathbb{R}^k \times N, g_{\mathrm{can}} \oplus g_N)$, since metric on $\widetilde{M}$ is pullback metric, and g_{can} on $\mathbb{R}^k$ has vanishing Ricci curvature. Thus $\widetilde{M}$ is compact, consequently we have $|\pi_1(M)|$ is finite and $b_1(M) = 0$. □

Remark 21.3.2. We have already seen this in Bochner's technique.

Part 8. Riemannian symmetric space

22. Two viewpoints to Riemannian symmetric space

22.1. A geometric viewpoint.

Definition 22.1.1 (Riemannian symmetric space)**.** A Riemannian manifold (M, g) is called a Riemannian symmetric space if for each $p \in M$ there exists an isometry $\varphi\colon M \to M$, which is called symmetry at p, such that $\varphi(p) = p$ and $(\mathrm{d}\varphi)_p = -\operatorname{id}$.

Remark 22.1.1. Note that Theorem 17.1.2, that is rigidity property of isometry, implies if symmetry at point p exists, it's unique.

Definition 22.1.2 (locally Riemannian symmetric space)**.** A Riemannian manifold (M, g) is called a locally Riemannian symmetric space if each $p \in M$ has a neighborhood U such that there exists an isometry $\varphi\colon U \to U$ such that $\varphi(p) = p$ and $(\mathrm{d}\varphi)_p = -\operatorname{id}$.

Lemma 22.1.1. The following statements are equivalent:
1. (M, g) is a Riemannian symmetric space.
2. For each $p \in M$, there exists an isometry $\varphi\colon M \to M$ such that $\varphi^2 = \operatorname{id}$ and p is an isolated fixed point of φ.

Proof. From (1) to (2). Let φ be symmetry at $p \in M$, note that $(\mathrm{d}\varphi^2)_p = (\mathrm{d}\varphi)_p \circ (\mathrm{d}\varphi)_p = \operatorname{id}$ and $\varphi^2(p) = p$, then by Theorem 17.1.2 one has $\varphi^2 = \operatorname{id}$. If p is not an isolated fixed point, then there exists a sequence $\{p_i\}_{i=1}^{\infty}$ converging to p such that $\varphi(p_i) = p_i$. Consider $0 < \delta < \operatorname{inj}(M), q \in B(0, \delta)$ and $v = \exp_p^{-1}(q)$. Note that $\varphi(\exp_p(tv))$ and $\exp_p(tv)$ are two geodesics connecting p and q, since φ is an isometry, thus

$$\varphi(\exp_p(tv)) = \exp_p(tv)$$

by uniqueness. In particular, one has $v = (\mathrm{d}\varphi)_p v$, which is a contradiction.

From (2) to (1). From $\varphi^2 = \operatorname{id}$ we have $(\mathrm{d}\varphi)_p^2 = \operatorname{id}$, so only possibly eigenvalues of $(\mathrm{d}\varphi)_p$ are ± 1. Now it suffices to show all eigenvalues of $(\mathrm{d}\varphi)_p$ are -1. Otherwise if it has an eigenvalue 1, there exists some non-zero $v \in T_pM$ such that $(\mathrm{d}\varphi)_p v = v$. Then

$$\varphi(\exp_p(tv)) = \exp_p(tv)$$

are the same geodesics for sufficiently small t, since both $\varphi(\exp_p(tv))$ and $\exp_p(tv)$ are geodesics and they have the same starting point and direction. In particular, p is not an isolated fixed point, which is a contradiction. □

22.2. Riemannian homogeneous space.

In appendix A.3, we review what is homogeneous space. Now let's consider its analogy in Riemannian geometry.

Definition 22.2.1 (Riemannian G-homogeneous space)**.** If Lie group G acts smoothly and transitively as isometries on a Riemannian manifold (M, g), then (M, g) is called a Riemannian G-homogeneous space.

Theorem 22.2.1 (Myers-Steenrod)**.** Let (M, g) be a Riemannian manifold and $G = \mathrm{Iso}(M, g)$. Then

1. G is a Lie group with respect to compact-open topology.
2. For each $p \in M$, the isotropy group G_p is compact.
3. If M is compact, then G is compact.

Proof. See [MS39]. □

Definition 22.2.2 (Riemannian homogeneous space)**.** A Riemannian manifold (M, g) is called a Riemannian homogeneous space, if $\mathrm{Iso}(M, g)$ acts on M transitively.

Lemma 22.2.1. Let (M, g) be a Riemannian homogeneous space. If at some point $p \in M$ there exists a symmetry at p, then (M, g) is a Riemannian symmetric space.

Proof. Suppose φ is a symmetry at $p \in M$. Since (M, g) is a Riemannian homogeneous space, then for any $q \in M$, there exists an isometry $\psi\colon M \to M$ such that $\psi(p) = q$, then

$$\varphi_q := \psi \circ \varphi \circ \psi^{-1}$$

is a symmetry at q. □

22.3. **The relations between symmetric, locally symmetric and homogeneous space.** In this section, we will show any Riemannian symmetric space is a Riemannian homogeneous space (in Corollary 22.3.2), and any locally Riemannian symmetric space is a quotient of Riemannian symmetric space (in Theorem 23.2.3).

22.3.1. *Riemannian symmetric space and locally Riemannian symmetric space.* Firstly, let's give another characterization of locally Riemannian symmetric space via curvature, which is based on the following lemma.

Lemma 22.3.1. Let (M, g) be a Riemannian manifold, $\gamma\colon I \to M$ a smooth curve and $P^{\gamma}_{s,t}\colon T_{\gamma(s)}M \to T_{\gamma(t)}M$ is the parallel transport along γ. For any $s \in I$ with $v = \gamma'(s)$, one has

$$\nabla_v R = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=s} (P^{\gamma}_{s,t})^* R_{\gamma(t)}$$

In particular, if $\nabla R = 0$ then

$$(P^{\gamma}_{s,t})^* R_{\gamma(t)} = R_{\gamma(s)}$$

holds for arbitrary $t, s \in I$.

Proof. Let $v_1 = v$ and choose $v_2, \dots, v_5 \in T_{\gamma(s)}M$. For each $1 \leq i \leq 5$, we define vector fields along $\gamma(t)$ via $X_i(t) = P^{\gamma}_{s,t}(v_i)$. Hence

$$\begin{aligned}
\nabla R(v_1, \dots, v_5) &= \lim_{t \to s} \nabla R(X_1, \dots, X_5) \\
&= \lim_{t \to s} X_1 R(X_2, \dots, X_5) - \sum_{i=2}^{5} \widehat{R}(X_2, \dots, \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} X_i, \dots, X_5) \\
&\overset{(1)}{=} \lim_{t \to s} X_1 R(X_2, \dots, X_5) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=s} R_{\gamma(t)}(X_2(t), \dots, X_5(t)) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=s} (P^{\gamma}_{s,t})^* R_{\gamma(t)}(v_2, \dots, v_5)
\end{aligned}$$

where (1) holds from $X_i(t)$ are parallel vector fields. $\square$

Theorem 22.3.1. Let (M, g) be a complete Riemannian manifold, the following statements are equivalent:

1. (M, g) is a locally Riemannian symmetric space.
2. $\nabla R = 0$.

Proof. From (1) to (2). Suppose (M, g) is a locally Riemannian symmetric space. For arbitrary $p \in M$, there exists a neighborhood U of p and an isometry $\varphi \colon U \to U$ such that $\varphi(p) = p$ and $(\mathrm{d}\varphi)_p = -\mathrm{id}$. Since φ is an isometry, so it preserves curvature, thus for any $v_i \in T_pM, i = 1, 2, \dots, 5$, one has

$$\begin{aligned}
\varphi^*(\nabla R)(v_1, \dots, v_5) &= \nabla R((\mathrm{d}\varphi)_p v_1, \dots, (\mathrm{d}\varphi)_p v_5) \\
&= -\nabla R(v_1, \dots, v_5)
\end{aligned}$$

which implies $\nabla R = 0$ at point p, and since p is arbitrary, thus $\nabla R = 0$.

From (2) to (1). Suppose $\nabla R = 0$. For arbitrary $p \in M$, let $\Phi_0 = -\mathrm{id} : T_pM \to T_pM$ and $0 < \delta < \mathrm{inj}(p)$. Then

$$\varphi = \exp_p \circ \Phi_0 \circ \exp_p^{-1} \colon B(p, \delta) \to B(p, \delta)$$

is an isometry with $\varphi(p) = p$ and $(\mathrm{d}\varphi)_p = \Phi_0$. Indeed, if $v \in T_pM$ with $|v| < \delta$ and $\gamma(t) = \exp_p(tv), \widetilde{\gamma}(t) = \exp_p(\Phi_0 v)$, consider

$$\Phi_t = P^{\widetilde{\gamma}}_{0,t} \circ \Phi_0 \circ P^{\gamma}_{t,0}$$

By Lemma 22.3.1, one has

$$\begin{aligned}
\Phi_t^* R &= (P^{\gamma}_{t,0})^* \circ \Phi_0^* \circ (P^{\widetilde{\gamma}}_{0,t})^* R_{\widetilde{\gamma}(t)} \\
&= R_{\gamma(t)}
\end{aligned}$$

Then by Cartan-Ambrose-Hicks's theorem, φ is the desired isometry. $\square$

Theorem 22.3.2. Let (M, g) be a complete, simply-connected locally Riemannian symmetric space, then (M, g) is a Riemannian symmetric space.

Corollary 22.3.1. Let (M, g) be a complete locally Riemannian symmetric space, then it's isometric to $\widetilde{M}/\Gamma$, where $\widetilde{M}$ is a Riemannian symmetric space and $\Gamma \subseteq \mathrm{Iso}(\widetilde{M}, \widetilde{g})$.

22.3.2. *Riemannian symmetric space and Riemannian homogeneous space.*

Theorem 22.3.3. Let (M, g) be a Riemannian symmetric space, then
1. (M, g) is complete.
2. For any isometry $\varphi \colon M \to M$ with $(\mathrm{d}\varphi)_p = -\mathrm{id}$ and $\varphi(p) = p$, if $v \in T_pM$, then
$$\varphi(\exp_p(v)) = \exp_p(-v)$$
3. The isometry group $\mathrm{Iso}(M, g)$ acts transitively on M.

Proof. For (1). For arbitrary geodesic $\gamma \colon [0, 1] \to M$ with $\gamma(0) = p, \gamma'(0) = v$. the curve $\beta(t) = \varphi(\gamma(t)) \colon [0, 1] \to M$ is also a geodesic with $\beta(0) = p$ and $\beta'(0) = -v$. Now we obtain a smooth extension $\gamma' \colon [0, 2] \to M$ of γ, given by
$$\gamma'(t) = \begin{cases} \beta(1 - t), & t \in [0, 1] \\ \gamma(t - 1), & t \in [1, 2] \end{cases}$$
Repeat above process to extend γ to a geodesic defined on $\mathbb{R}$, this shows completeness.

For (2). Just consider geodesics $\varphi(\exp_p(tv)) = \exp_p(-tv)$.

For (3). Let p, q be any two points in M and $\gamma \colon [0, 1] \to M$ be a geodesic with $\gamma(0) = p, \gamma(1) = q$. Let $m = \gamma(\frac{1}{2})$ and $\varphi_m : M \to M$ the symmetry at m. Consider $\beta(t) = \varphi_m(\gamma(\frac{1}{2} - t))$, then $\beta(0) = m, \beta'(0) = \gamma'(\frac{1}{2})$, which implies $\beta(t) = \gamma(\frac{1}{2} + t)$. Therefore $q = \gamma(1) = \beta(\frac{1}{2}) = \varphi_m(\gamma(0)) = \varphi_m(p)$. ∎

Corollary 22.3.2. The Riemannian symmetric space (M, g) is a Riemannian homogeneous space.

22.4. **An algebraic viewpoint: Riemannian symmetric pair.** In this section, we assume G is a connected Lie group with Lie algebra $\mathfrak{g}$.

Definition 22.4.1 (involution)**.** An automorphism σ of G is called an involution, if $\sigma^2 = \mathrm{id}_G$.

Definition 22.4.2 (symmetric pair)**.** Let K be a closed subgroup of G, the pair (G, K) is called a symmetric pair if there exists an involution $\sigma \colon G \to G$ with $(G^\sigma)_0 \subseteq K \subseteq G^\sigma$, where G^σ is the set of fixed points of σ in G, and $(G^\sigma)_0$ is the identity component of G^σ.

Example 22.4.1. $G = \mathrm{SO}(n + 1)$ and $K = \mathrm{SO}(n)$ is a symmetric pair. If we consider involution
$$\begin{aligned} \sigma \colon \mathrm{SO}(n + 1) &\to \mathrm{SO}(n + 1) \\ a &\mapsto sas^{-1} \end{aligned}$$

where $s = \operatorname{diag}\{-1, 1, \dots, 1\}$, then

$$\mathrm{SO}(n+1)^\sigma = \{a \in \mathrm{SO}(n+1) \mid sa = as\} = \{\begin{pmatrix} \pm 1 & 0 \\ 0 & b \end{pmatrix} \mid b \in \mathrm{O}(n)\}$$

Thus $(\mathrm{SO}(n+1)^\sigma)_0 = \mathrm{SO}(n) \subset \mathrm{SO}(n+1)$.

Example 22.4.2. The pair $(G \times G, G)$ is a symmetric pair. If we consider involution

$$\begin{aligned} \sigma \colon G \times G &\to G \times G \\ (a, b) &\mapsto (b, a) \end{aligned}$$

Then

$$(G \times G)^\sigma = \{(a, a) \mid a \in G\} \cong G$$

Let (G, K) be a symmetric pair given by involution σ, then its differential $\mathrm{d}\sigma$ gives an isomorphism of $\mathfrak{g}$, which is also an involution. Consider the eigenspace decomposition of $\mathfrak{g}$ with respect to $(\mathrm{d}\sigma)_e$

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

where

$$\begin{aligned} \mathfrak{k} &= \{X \in \mathfrak{g} \mid (\mathrm{d}\sigma)_e(X) = X\} \\ \mathfrak{m} &= \{X \in \mathfrak{g} \mid (\mathrm{d}\sigma)_e(X) = -X\} \end{aligned}$$

This decomposition is called Cartan decomposition of $\mathfrak{g}$.

Proposition 22.4.1. $\mathfrak{k} = \operatorname{Lie} K$.

Proof. It follows from

$$\operatorname{Lie} K = \operatorname{Lie} K_0 = \operatorname{Lie}(G^\sigma)_0 = \{X \in \mathfrak{g} \mid (\mathrm{d}\sigma)_e(X) = X\} = \mathfrak{k}$$

□

Corollary 22.4.1. $\mathfrak{m} \cong \operatorname{Lie} G/K$.

Corollary 22.4.2. Let $\widetilde{\sigma} \colon G/K \to G/K$ be the automorphism of G/K arisen from σ, then $(\mathrm{d}\widetilde{\sigma})_{eK} = -\operatorname{id}_{G/K}$.

Proposition 22.4.2.

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$$

Proof. It follows from

$$(\mathrm{d}\sigma)_e([X, Y]) = [(\mathrm{d}\sigma)_e(X), (\mathrm{d}\sigma)_e(Y)]$$

where $X, Y \in \mathfrak{g}$. □

Definition 22.4.3 (Riemannian symmetric pair)**.** A symmetric pair (G, K) is called a Riemannian symmetric pair, if there exists a bi-invariant metric on G/K.

Example 22.4.3. $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ is a Riemannian symmetric pair.

Example 22.4.4. If G is a compact Lie group, then $(G \times G, G)$ is a Riemannian symmetric pair.

Theorem 22.4.1. Let (G, K) be a symmetric pair given by σ, if it's also a Riemannian symmetric pair, then $\widetilde{\sigma}\colon G/K \to G/K$ arisen from σ is an isometry.

Corollary 22.4.3. If (G, K) is a Riemannian symmetric pair, then G/K is a Riemannian symmetric space.

Proof. Suppose (G, K) is a symmetric pair given by σ, then $\widetilde{\sigma}\colon G/K \to G/K$ arisen from σ gives a symmetry at eK, and thus by Lemma 22.2.1 one has G/K is a Riemannian symmetric space, since G/K is a homogeneous space. $\square$

23. Riemannian symmetric space to Riemannian symmetric pair

23.1. Killing field as Lie algebra of isometry group.

Proposition 23.1.1. Let (M,g) be a Riemannian manifold and X a Killing field.

1. If γ is a geodesic, then $J(t)=X(\gamma(t))$ is a Jacobi field.
2. For any two vector fields Y,Z,

$$\nabla_Y\nabla_Z X-\nabla_{\nabla_Y Z}X+R(X,Y)Z=0$$

Proof. For (1). Suppose φ_s is the flow generated by X, then we obtain a variation $\alpha(s,t)=\varphi_s(\gamma(t))$ consisting of geodesics, thus

$$X(\gamma(t))=\left.\frac{\partial\varphi_s(\gamma(t))}{\partial s}\right|_{s=0}$$

is a Jacobi field.

For (2). It's an equation of tensors, thus we check it pointwisely and use normal coordinate $\{x^i\}$ centered at p, and we assume $X=X^i\frac{\partial}{\partial x^i}, Y=\frac{\partial}{\partial x^j}, Z=\frac{\partial}{\partial x^k}$. Then

$$\begin{aligned}\nabla_Y\nabla_Z X-\nabla_{\nabla_Y Z}X+R(X,Y)Z&=\nabla_j\nabla_k X+X^iR_{ijk}^l\frac{\partial}{\partial x^l}\\&=(\frac{\partial^2X^l}{\partial x^j\partial x^k}+X^i\frac{\partial\Gamma_{ki}^l}{\partial x^j}+X^iR_{ijk}^l)\frac{\partial}{\partial x^l}\\&=(\frac{\partial^2X^l}{\partial x^j\partial x^k}+X^i\frac{\partial\Gamma_{jk}^l}{\partial x^i})\frac{\partial}{\partial x^l}\end{aligned}$$

since $R_{ijk}^l=\frac{\partial\Gamma_{jk}^l}{\partial x^i}-\frac{\partial\Gamma_{ik}^l}{\partial x^j}+\Gamma_{jk}^s\Gamma_{is}^l-\Gamma_{ik}^s\Gamma_{js}^l$. Now it suffices to show $\frac{\partial^2X^l}{\partial x^j\partial x^k}+X^i\frac{\partial\Gamma_{jk}^l}{\partial x^i}\equiv0$. In order to show this, for arbitrary $p\in M$, consider a geodesic γ starting at p and consider Jacobi field $J(t)=X(\gamma(t))$. Direct computation shows

$$\begin{aligned}J'(t)&=\left.(\frac{\partial X^i}{\partial x^k}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}+X^i\Gamma_{ki}^l\frac{\mathrm{d}\gamma^k}{\mathrm{d}t})\frac{\partial}{\partial x^l}\right|_{\gamma(t)}\\J''(0)&=\left.(\frac{\partial^2X^l}{\partial x^j\partial x^k}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}+X^i\frac{\partial\Gamma_{ki}^l}{\partial x^j}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t})\frac{\partial}{\partial x^l}\right|_p\\&=\left.(\frac{\partial^2X^l}{\partial x^j\partial x^k}+X^i\frac{\partial\Gamma_{ki}^l}{\partial x^j})\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}\frac{\partial}{\partial x^l}\right|_p\\&=\left.(\frac{\partial^2X^l}{\partial x^j\partial x^k}+X^i\frac{\partial\Gamma_{jk}^l}{\partial x^i}+X^i\frac{\partial\Gamma_{ki}^l}{\partial x^j}-X^i\frac{\partial\Gamma_{jk}^l}{\partial x^i})\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}\frac{\partial}{\partial x^l}\right|_p\\&=\left.(\frac{\partial^2X^l}{\partial x^j\partial x^k}+X^i\frac{\partial\Gamma_{jk}^l}{\partial x^i})\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}\frac{\partial}{\partial x^l}\right|_p-R(X,\gamma')\gamma'\end{aligned}$$

which implies

$$\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma^l_{jk}}{\partial x^i} = 0$$

holds at point p. Since p is arbitrary, this completes the proof. □

Corollary 23.1.1. Let (M, g) be a complete Riemannian manifold and $p \in M$, then a Killing field X is determined by the values X_p and $(\nabla X)_p$ for arbitrary $p \in M$.

Proof. It suffices to show if $X_p = 0$ and $(\nabla X)_p = 0$, then $X \equiv 0$. For arbitrary $q \in M$, let $\gamma\colon [0,1] \to M$ be a geodesic connecting p and q with $\gamma'(0) = v$. Since $J(t) = X(\gamma(t))$ is a Jacobi field, a direct computation shows

$$(\nabla_v X)_p = J'(0)$$

thus $J(t) \equiv 0$, since Jacobi field is determined by two initial values. In particular, $X_q = J(1) = 0$, and since q is arbitrary, one has $X \equiv 0$. □

Corollary 23.1.2. The dimension of vector field consisting of Killing field $\leq \frac{n(n+1)}{2}$.

Proof. Note that ∇X is skew-symmetric, and the dimension of skew-symmetric matrices is $\frac{n(n-1)}{2}$, thus the dimension of vector field consisting of Killing field

$$\leq n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

□

Lemma 23.1.1. Killing field on a complete Riemannian manifold (M, g) is complete.

Proof. Let X be a Killing field, we need to show the flow $\varphi_t : M \to M$ generated by X is defined for $t \in \mathbb{R}$. Otherwise, we assume φ_t is defined on (a, b). Note that for each $p \in M$, curve $\varphi_t(p)$ is a curve defined on (a, b) having finite constant speed, since φ_t is isometry. Then we have $\varphi_t(p)$ can be extended to the one defined on $\mathbb{R}$, since M is complete. □

Theorem 23.1.1. Let (M, g) be a complete Riemannian manifold and $\mathfrak{g}$ the space of Killing fields, then $\mathfrak{g}$ is isomorphic to the Lie algebra of $G = \mathrm{Iso}(M, g)$.

Proof. Since $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$, we know $\mathfrak{g}$ is a Lie algebra. Now let's construct correspondence:

1. Given a Killing field X, by Lemma 23.1.1, one deduces that the flow $\varphi\colon \mathbb{R} \times M \to M$ generated by X is a one parameter subgroup $\gamma\colon \mathbb{R} \to G$, and $\gamma'(0) \in T_eG$.
2. Given $v \in T_eG$, consider the one-parameter subgroup $\gamma(t) = \exp(tv) : \mathbb{R} \to G$ which gives a flow by

$$\begin{aligned} \varphi\colon \mathbb{R} \times M &\to M \\ (t, p) &\mapsto \exp(tv) \cdot p \end{aligned}$$

Then the vector field X generated by this flow is a Killing field.

This gives an one to one correspondence between Killing fields and Lie algebra of G, and it's a Lie algebra isomorphism in fact. □

23.2. **Cartan decomposition of Killing fields.** Together with Corollary 23.1.1 and Theorem 23.1.1, we can decompose Lie algebra $\mathfrak{g}$ of isometry group G as direct sum of following vector spaces

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

where

$$\begin{aligned}\mathfrak{k} &= \{X \in \mathfrak{g} \mid X_p = 0\}\\ \mathfrak{m} &= \{X \in \mathfrak{g} \mid (\nabla X)_p = 0\}\end{aligned}$$

A direct computation shows

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$$

For arbitrary $X \in \mathfrak{k}, Y \in \mathfrak{m}$ and $v \in T_pM$, one has

$$\begin{aligned}\nabla_v[X, Y] &= \nabla_v\nabla_X Y - \nabla_v\nabla_Y X\\ &= -R(Y, v)X + \nabla_{\nabla_v X} Y + R(X, v)Y - \nabla_{\nabla_v Y} X\\ &= 0\end{aligned}$$

since $X_p = 0$ and $(\nabla Y)_p = 0$. This shows $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, that is we recover similar relations in Proposition 22.4.2.

Theorem 23.2.1. Let (M, g) be a Riemannian symmetric space and G the isometry group. For any $p \in M$, the Lie algebra of the isotropy subgroup G_p is isomorphic to

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid X_p = 0\}$$

where $\mathfrak{g}$ is the Lie algebra of G.

Proof. Let $X \in \mathfrak{g}$ with $X_p = 0$, and $\varphi_t : M \to M$ the flow of X. It suffices to show $\varphi_t(p) = p$ for all $t \in \mathbb{R}$. We use $\gamma_p(t)$ to denote $\varphi_t(p)$, then for any smooth function $f\colon M \to \mathbb{R}$ and $s \in \mathbb{R}$, we have

$$\begin{aligned}\gamma_p'(s)f &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} f \circ \gamma_p(t)\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f \circ \gamma_p(t+s)\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f \circ \varphi_s \circ \varphi_t(p)\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ \varphi_s)(\gamma_p(t))\\ &= \gamma_p'(0)(f \circ \varphi_s)\\ &= X_p(f \circ \varphi_s)\\ &= 0\end{aligned}$$

Hence $\gamma_p'(s) = 0$ for all $s \in \mathbb{R}$, thus $\gamma_p(s)$ is constant, which implies $\gamma_p(s) = \gamma_p(0) = p$. $\square$

In order to describe $\mathfrak{m}$, we need to introduce transvection.

Definition 23.2.1 (transvection)**.** Let (M, g) be a Riemannian symmetric space and γ a geodesic. The transvection along γ is defined as

$$T_t = s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}$$

where s_p is the symmetry at point p.

Proposition 23.2.1. Let (M, g) be a Riemannian symmetric space, γ a geodesic and T_t the transvection along γ. Then

1. For any $a, t \in \mathbb{R}$, $s_{\gamma(a)}(\gamma(t)) = \gamma(2a - t)$.
2. T_t translates the geodesic γ, that is $T_t(\gamma(s)) = \gamma(t + s)$.
3. $(\mathrm{d}T_t)_{\gamma(s)} : T_{\gamma(s)}M \to T_{\gamma(t+s)}M$ is the parallel transport $P^{\gamma}_{s,t+s}$.
4. T_t is one-parameter subgroup of $\mathrm{Iso}(M, g)$.

Proof. For (1). It follows from the uniqueness of geodesics with given initial value.

For (2). Note that

$$\begin{aligned} T_t(\gamma(s)) &= s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}(\gamma(s)) \\ &= s_{\gamma(\frac{t}{2})}(\gamma(-s)) \\ &= \gamma(t + s) \end{aligned}$$

For (3). Let X be a parallel vector field along γ. By uniqueness of parallel vector fields with given initial data, we have $(\mathrm{d}s_{\gamma(0)})_{\gamma(s)} X_{\gamma(s)} = -X_{\gamma(-s)}$ for all s, since $(\mathrm{d}s_{\gamma(0)})_{\gamma(0)} X_{\gamma(0)} = -X_{\gamma(0)}$. Thus

$$\begin{aligned} (\mathrm{d}T_t)_{\gamma(s)} X_{\gamma(s)} &= (\mathrm{d}s_{\gamma(\frac{t}{2})})_{\gamma(-s)}(-X_{\gamma(-s)}) \\ &= X_{\gamma(t+s)} \end{aligned}$$

This shows $(\mathrm{d}T_t)_{\gamma(s)} = P^{\gamma}_{s,t+s}$.

For (4). In order to show $T_{t+s} = T_t \circ T_s$, it suffices to check they're same at some point, so do their derivatives, since isometry can be determined by these two values. Note that

$$\begin{aligned} T_{t+s}(\gamma(0)) &= \gamma(t + s) \\ &= T_t \circ T_s(\gamma(0)) \\ (\mathrm{d}T_{t+s})_{\gamma(0)} &= P^{\gamma}_{0,t+s} \\ &= P^{\gamma}_{s,t+s} \circ P^{\gamma}_{0,s} \\ &= (\mathrm{d}T_t)_{\gamma(s)} \circ (\mathrm{d}T_s)_{\gamma(0)} \\ &= (\mathrm{d}(T_t \circ T_s))_{\gamma(0)} \end{aligned}$$

This completes the proof. $\square$

Definition 23.2.2 (infinitesimal transvection)**.** Let (M,g) be a Riemannian symmetric space. For any point $p\in M$ and any $v\in T_pM$, the infinitesimal generator X of transvections T_t along γ_v is given by

$$X_q=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}T_t(q).$$

This Killing field X is called an infinitesimal transvection.

Theorem 23.2.2. Let (M,g) be a Riemannian symmetric space and X an infinitesimal transvection of transvection T_t along geodesic $\gamma=\exp_p(tv)$. Then

$$X_p=v,\quad (\nabla X)_p=0$$

Corollary 23.2.1. The space of infinitesimal transvection is exactly $\mathfrak{m}$, and there is an isomorphism between $\mathfrak{m}\cong T_pM$ given by $X\mapsto X_p$.

Theorem 23.2.3. Let (M,g) be a Riemannian symmetric space, G is identity component of isometry group, and K is the isotropy group G_p for some $p\in M$.

1. The mapping
$$\begin{aligned}\sigma\colon G&\to G\\ g&\mapsto s_pgs_p\end{aligned}$$
is an involutive automorphism of G, where s_p is symmetry at p.
2. $$(G^\sigma)_0\subseteq K\subseteq G^\sigma$$
3. We have
   (a) $\mathfrak{k}=\{X\in\mathfrak{g}:(\mathrm{d}\sigma)_eX=X\}$ as Lie algebra, where $\mathfrak{k}$ is the Lie algebra of K.
   (b) $\mathfrak{m}\cong\{X\in\mathfrak{g}:(\mathrm{d}\sigma)_eX=-X\}$ as vector space.
4. There is a left invariant metric on G which is also right-invariant under K, such that G/K with the induced metric is isometric to (M,g).

Proof. For (1). It's clear, since $s_p^2=\mathrm{id}$.

For (2). For any $k\in K$, if we want to show isometries k and $\sigma(k)=s_pks_p$ are same, it suffices to check they and their differentials agree at some point by Theorem 17.1.2. Now just consider point p to conclude $K\subset G^\sigma$. To see $(G^\sigma)_0\subset K$, let $\exp(tX)\subset(G^\sigma)_0$ be a one-parameter subgroup. Since $\sigma(\exp(tX))=\exp(tX)$, then acting them on p yields

$$s_p\exp(tX)s_p(p)=s_p\exp(tX)(p)=\exp(tX)(p)$$

But p is an isolated fixed point of s_p, thus $\exp(tX)(p)=p$ for all t, this shows the one-parameter subgroup lies in K. Since exponential map of Lie group is a diffeomorphism in a small neighborhood of identity element e and $(G^\sigma)_0$ can be generated by a neighborhood of e, which implies the whole $(G^\sigma)_0\subset K$.

For (3). Let $E_1 = \{X \in \mathfrak{g} : (\mathrm{d}\sigma)_e X = X$. If $X \in E_1$, then $\gamma_2(t) = \sigma(\exp(tX)) : \mathbb{R} \to G$ is a one-parameter subgroup. Indeed,

$$\begin{aligned}\gamma_2(t)\gamma_2(s) &= s_p \exp(tX) s_p^2 \exp(sX) s_p \\ &= s_p \exp((t+s)X) s_p \\ &= \gamma_2(t+s)\end{aligned}$$

Furthermore, $\gamma_2(t)$ and $\gamma_1(t) = \exp(tX)$ are the same one-parameter subgroup, since $\gamma_2'(0) = (\mathrm{d}\sigma)_e(X) = X = \gamma_1'(0)$, which implies $\exp(tX) \in G^\sigma$ for all t. In particular one has $X \in \mathrm{Lie}(G^\sigma)$ and thus $E_1 \subseteq \mathrm{Lie}(G^\sigma)$. Then $E_1 = \mathrm{Lie}(G^\sigma)$ since converse inclusion is clear. By (2) we have $\mathfrak{k} = \mathrm{Lie}(G^\sigma)$, this shows $\mathfrak{k} = E_1$. Let $E_{-1} = \{X \in \mathfrak{g} : (\mathrm{d}\sigma)_e X = -X\}$, then $\mathfrak{g} = E_1 \oplus E_{-1}$. On the other hand, since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and $\mathfrak{k} \cong E_1$, one has $\mathfrak{m} \cong E_{-1}$.

For (4). L $\square$

24. Curvature of Riemannian symmetric space

24.1. Curvature of Riemannian symmetric space.

Proposition 24.1.1. Let (M, g) be a Riemannian symmetric space and $G = \mathrm{Iso}(M, g)$. For any $p \in M$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{k}$ is Lie algebra of isotropy group G_p and $\mathfrak{m} \cong T_pM$. For any $X \in \mathfrak{m}$, one has

$$B_{\mathfrak{g}}(X, X) \leq 0$$

and the identity holds if and only if $X = 0$.

Theorem 24.1.1. Let (M, g) be a Riemannian symmetric space and $G = \mathrm{Iso}(M, g)$. For any $p \in M$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ with $\mathfrak{m} \cong T_pM$.

1. For any $X, Y, Z \in \mathfrak{m}$, there holds

$$\begin{aligned} R(X, Y)Z &= -[Z, [Y, X]] \\ \mathrm{Ric}(Y, Z) &= -\frac{1}{2}B(Y, Z) \end{aligned}$$

2. If $\mathrm{Ric}(g) = \lambda g$, then for $X, Y \in \mathfrak{m}$, one has

$$2\lambda R(X, Y, Y, X) = -B([X, Y], [X, Y])$$

Proof. For (1). For any $X, Y, Z \in \mathfrak{m}$, direct computation shows

$$\begin{aligned} R(X, Y)Z &\overset{\mathrm{I}}{=} R(X, Z)Y - R(Y, Z)X \\ &\overset{\mathrm{II}}{=} \nabla_Z\nabla_Y X - \nabla_{\nabla_Z Y}X - \nabla_Z\nabla_X Y + \nabla_{\nabla_Z X}Y \\ &\overset{\mathrm{III}}{=} -\nabla_Z[X, Y] \\ &\overset{\mathrm{IV}}{=} -[Z[X, Y]] \end{aligned}$$

where

I holds from the first Bianchi identity.
II holds from (2) of Proposition 23.1.1.
III holds from $X, Y \in \mathfrak{m}$, thus $(\nabla X)_p = (\nabla Y)_p = 0$.
IV holds from

$$\nabla_Z[X, Y] - \nabla_{[X,Y]}Z = [Z, [X, Y]]$$

and $(\nabla Z)_p = 0$.

Too see Ricci curvature, note that for $Y \in \mathfrak{m}$

$$\mathrm{ad}_Y : \mathfrak{k} \to \mathfrak{m}, \quad \mathrm{ad}_Y : \mathfrak{m} \to \mathfrak{k}$$

Thus $\mathrm{ad}_Z \circ \mathrm{ad}_Y$ preserves the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ if $Y, Z \in \mathfrak{m}$. Then

$$\begin{aligned} \mathrm{tr}(\mathrm{ad}_Z \circ \mathrm{ad}_Y|_{\mathfrak{m}}) &= \mathrm{tr}(\mathrm{ad}_Z|_{\mathfrak{k}} \circ \mathrm{ad}_Y|_{\mathfrak{m}}) \\ &= \mathrm{tr}(\mathrm{ad}_Y|_{\mathfrak{m}} \circ \mathrm{ad}_Z|_{\mathfrak{k}}) \\ &= \mathrm{tr}(\mathrm{ad}_Y \circ \mathrm{ad}_Z|_{\mathfrak{k}}) \end{aligned}$$

Hence we obtain

$$B(Y, Y) = \mathrm{tr}(\mathrm{ad}_Y \circ \mathrm{ad}\,Y|_{\mathfrak{k}}) + \mathrm{tr}(\mathrm{ad}_Y \circ \mathrm{ad}\,Y|_{\mathfrak{l}}) = 2\,\mathrm{tr}(\mathrm{ad}_Y \circ \mathrm{ad}\,Y|_{\mathfrak{m}})$$

Since Ricci tensor is trace of curvature tensor, thus

$$\mathrm{Ric}(Y,Y) = -\operatorname{tr}(\mathrm{ad}_Y \circ \mathrm{ad}_Y |_{\mathfrak{m}}) = -\frac{1}{2}B(Y,Y)$$

By using symmetry for $Y+Z$, one has $\mathrm{Ric}(Y,Z) = -\frac{1}{2}B(Y,Z)$.

For (2). If $\mathrm{Ric}(g) = \lambda g$, then

$$\begin{aligned}
2\lambda g(R(X,Y)Y,X) &= -2\lambda g(\mathrm{ad}_Y \circ \mathrm{ad}_Y X, X) \\
&= -2\,\mathrm{Ric}(\mathrm{ad}_Y \circ \mathrm{ad}_Y X, X) \\
&= B(\mathrm{ad}_Y \circ \mathrm{ad}_Y X, X) \\
&= -B(\mathrm{ad}_Y X, \mathrm{ad}_Y X) \\
&= -B([X,Y],[X,Y])
\end{aligned}$$

□

Corollary 24.1.1. Let (M,g) be a Riemannian symmetric space which is an Einstein manifold with Einstein constant λ. Then

1. If $\lambda > 0$, then (M,g) has non-negative sectional curvature.
2. If $\lambda < 0$, then (M,g) has non-positive sectional curvature.
3. If $\lambda = 0$, then (M,g) is flat.

Proof. By above theorem one has

$$2\lambda R(X,Y,Y,X) = -B([X,Y],[X,Y]) \geq 0$$

since $[X,Y] \in [\mathfrak{m},\mathfrak{m}] \in \mathfrak{m}$ and B is negative-definite on $\mathfrak{m}$. □

24.2. **Irreducible space.**

Definition 24.2.1 (isotropy irreducible)**.** Let (M,g) be a Riemannian symmetric space with $G = \mathrm{Iso}(M,g)$ and $K = G_p$ for some $p \in M$. If the identity component K_0 acts irreducibly on T_pM, then M is called irreducible. Otherwise M is called reducible.

Lemma 24.2.1. Let B_1, B_2 be two symmetric bilinear forms on a vector space V such that B_1 is positive-definite. If a group K acts irreducibly on V such that B_1 and B_2 are invariant under K, then $B_2 = \lambda B_1$ for some constant λ.

Proof. Since B_1 is positive-definite, then there exists an endomorphism $L : V \to V$ such that

$$B_2(u,v) = B_1(Lu,v)$$

where $u,v \in V$. Since B_1, B_2 are invariant under K, then for any $k \in K$

$$B_1(kLu,v) = B_1(Lu,k^{-1}v) = B_2(u,k^{-1}v) = B_2(ku,v) = B_1(Lku,v)$$

holds for arbitrary $u,v \in V$, which implies $Lk = kL$ for all $k \in K$. Moreover, the symmetry of B_1, B_2 implies

$$B_1(Lu,v) = B_2(u,v) = B_2(v,u) = B_1(Lv,u) = B_1(u,Lv)$$

Hence L is symmetric with respect to B_1, thus the eigenvalues of L are real. If $E \subset V$ is an eigenspace with eigenvalue λ, the fact $kL = Lk$ implies E is invariant under K. Since K acts irreducibly on V, thus $E = V$, that is $L = \lambda I$, which implies $B_2 = \lambda B_1$. □

Theorem 24.2.1. An irreducible Riemannian symmetric space is Einstein. Moreover, the metric is unique determined up to a multiple.

Proof. Since isometries preserves the metric and curvature, and Ricci tensor is also a symmetric bilinear form, thus there exists smooth λ such that

$$\mathrm{Ric}(g) = \lambda g$$

Since Ric is parallel, which relys on $\nabla R = 0$, then we have λ is a constant. □

Definition 24.2.2. Let (M, g) be an irreducible Riemannian symmetric manifold.

1. If the Ricci curvature is positive, then M is called of compact type.
2. If the Ricci curvature is negative, then M is called of non-compact type.
3. If the Ricci curvature is zero, then M is called of Euclidean type.

Remark 24.2.1. We have the following observations:

1. By Myer's theorem, if M is of compact type, then it's compact.
2. If M is of non-compact type, then it's non-compact, otherwise by Bochner's technique there is no non-trivial Killing vector field.
3. If M is of Euclidean type, then it is flat and so it is covered by $\mathbb{R}^n$.

25. Examples of Riemannian symmetric space

25.1. **Compact Lie group as Riemannian symmetric space.**

Theorem 25.1.1. Let G be a compact Lie group and $\mathfrak{g}$ be its Lie algebra.
1. G equipped with bi-invariant g is a Riemannian symmetric space.
2. Every left-invariant vector field is a Killing field.
3. For any $X \in \mathfrak{g}$, $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is a skew-symmetric with respect to g.
4. For any $X \in \mathfrak{g}$, $B_{\mathfrak{g}}(X, X) \leq 0$ and the identity holds if and only if X lies in center of $\mathfrak{g}$.
5. (G, g) has non-negative sectional curvature.
6. If $\mathfrak{g}$ has no center, then B induces a bi-invariant metric g_B with $\mathrm{Ric}(g_B) = \frac{1}{2} g_B$.

Proof. For (1). Consider Lie group $G \times G$ and its compact subgroup G, we claim the involution σ on $G \times G$, given by $(g, h) \mapsto (h, g)$ makes pair $(G \times G, G)$ a Riemannian symmetric pair. Indeed,

$$G \cong G^{\sigma} = G^{\Delta} := \{(g, g) \in G \times G \mid g \in G\}$$

By Theorem 23.2.3, one has $G \times G/G^{\Delta}$ with induced metric is a Riemannian symmetric space. Note that the following diffeomorphism

$$\begin{aligned} G \times G/G^{\Delta} &\to G \\ (a, b)G^{\Delta} &\mapsto ab^{-1} \end{aligned}$$

is an isometry. This shows G is a Riemannian symmetric space.

For (2). Since the flows of a left-invariant vector fields are left translations which are isometries, thus every left-invariant vector field is Killing.

For (3).

For (4). $B_g(X, X) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_X) \leq 0$, since

For (5) and (6). They follow from Theorem 24.1.1. □

Remark 25.1.1.

25.2. **Examples.**

Example 25.2.1 (hyperbolic Grassmannian)**.** In $\mathbb{R}^{k,l}$ with $k \geq 2, l \geq 1$, consider the following quadratic form

$$v^t I_{k,l} w = v^t \begin{pmatrix} I_k & 0 \\ 0 & -I_l \end{pmatrix} w = \sum_{i=1}^{k} v_i w_i - \sum_{j=k+1}^{k+l} v_j w_j$$

The group of linear transformation X that preserves this quadratic form is denoted by $O(k, l)$, that is

$$X I_{k,l} X^t = I_{k,l}$$

If $k, l > 0$, $O(k, l)$ is not compact, but it contains a compact subgroup $O(k) \times O(l)$

Part 9. Appendix

Appendix A. Review of smooth manifolds

In this section we give a quick review of facts in differential geometry we may use.

A.1. Lie group.

Definition A.1.1 (Lie group). A Lie group G is a smooth manifold which is also endowed with a group structure such that the multiplication map and the inverse map are smooth.

Since the multiplication map is smooth, then for any $g \in G$, there are two smooth maps L_g, R_g, defined by

$$
\begin{aligned}
L_g(h) &= gh \\
R_g(h) &= hg
\end{aligned}
$$

Furthermore, they're also diffeomorphisms with inverse $L_{g^{-1}}, R_{g^{-1}}$, since inverse maps are also smooth.

Definition A.1.2 (invariant vector field). A vector field X on a Lie group G is called left-invariant, if

$$(\mathrm{d}L_g)X = X$$

for arbitrary $g \in G$.

Remark A.1.1. It's clear there is the following isomorphism

$$
\begin{aligned}
\{\text{left-invaraint vector fields}\} &\to T_eG \\
X &\mapsto X_e
\end{aligned}
$$

where X_e is its value in T_eG. Furthermore, since Lie bracket of two left-invariant vector fields is still left-invariant, we can equip T_eG a Lie bracket.

Definition A.1.3 (Lie algebra). The tangent space T_eG of a Lie group G equipped with Lie bracket is called Lie algebra of G, denoted by $\mathfrak{g}$.

Definition A.1.4 (adjoint representation). The adjoint representation is defined as follows

$$
\begin{aligned}
\mathrm{Ad} : G &\to \mathrm{GL}(\mathfrak{g}) \\
g &\mapsto \mathrm{d}R_{g^{-1}} \circ \mathrm{d}L_g
\end{aligned}
$$

Definition A.1.5 (integral curve). Let X be a vector field of G and $g \in G$, then an integral curve of X through the point p is a smooth curve $\gamma\colon I \subseteq \mathbb{R} \to G$ such that

$$
\begin{aligned}
\gamma(0) &= g \\
\gamma'(t) &= X(\gamma(t))
\end{aligned}
$$

Definition A.1.6 (complete vector field). A vector field X is called complete, if its integral curve is defined for all $t \in \mathbb{R}$.

Proposition A.1.1. Every left-invariant vector field on a Lie group G is complete.

Proof. Let X be a left-invariant vector field, γ the unique integral curve for X such that $\gamma(0) = e$, defined on $(-\varepsilon, \varepsilon)$. Then $\gamma_g := L_g\gamma$ is an integral curve for X such that $\gamma_g(0) = g$. Indeed,

$$\begin{aligned}\gamma_g'(t) &= \mathrm{d}(L_g)_{\gamma(t)}(\gamma'(t))\\ &= \mathrm{d}(L_g)_{\gamma(t)}(X(\gamma(t)))\\ &= X(L_g\gamma(t))\\ &= X(\gamma_g(t))\end{aligned}$$

In particular, for $t_0 \in (-\varepsilon, \varepsilon)$, the curve $t \mapsto \gamma(t_0)\gamma(t)$ is an integral curve for X starting at $\gamma(t_0)$. By uniqueness, this curve coincides with $\gamma(t_0 + t)$ for all $t \in (-\varepsilon, \varepsilon) \cap (-\varepsilon - t_0, \varepsilon - t_0)$. Define

$$\widetilde{\gamma}(t) = \begin{cases}\gamma(t), & t \in (-\varepsilon, \varepsilon)\\ \gamma(t_0)\gamma(t), & t \in (-\varepsilon - t_0, \varepsilon - t_0)\end{cases}$$

Repeat above operations to get our desired extension. □

Remark A.1.2. From this proof we can see integral curve of left-invariant vector fields through identity e is just a Lie group homomorphism $\gamma\colon \mathbb{R} \to G$, such homomorphism is called a one parameter subgroup.

A.2. **Killing form.**

Definition A.2.1 (Killing form)**.** Let $\mathfrak{g}$ be a Lie algebra.

1. For any $X \in \mathfrak{g}$, the adjoint linear map $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is defined as $\mathrm{ad}_X Y = [X, Y]$.
2. The Killing form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is a bilinear symmetric form defined as

$$B(X, Y) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$$

Lemma A.2.1. Let B be the Killing form on Lie algebra $\mathfrak{g}$ of Lie group G. Then for any $g \in G$ and $X, Y, Z \in \mathfrak{g}$, then

1. $B(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) = B(X, Y)$.
2. $B(\mathrm{ad}_Z X, Y) = -B(X, \mathrm{ad}_Z Y)$.

Remark A.2.1. Recall the following facts about Lie algebra:

1. For $g \in G$, Ad_g is the differential at identity element of inner automorphism $x \mapsto gxg^{-1}$ of G, and it's a Lie algebra homomorphism.
2. For $X, Y \in \mathfrak{g}$, $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \mathrm{Ad}_{\exp(tX)} Y = \mathrm{ad}_X Y$

Proof. For (1). For any $X, Y \in \mathfrak{g}$, one has

$$\begin{aligned}[\mathrm{Ad}_g X, Y] &= [\mathrm{Ad}_g X, \mathrm{Ad}_g \circ \mathrm{Ad}_{g^{-1}}(Y)]\\ &= \mathrm{Ad}_g([X, \mathrm{Ad}_{g^{-1}} Y])\\ &= \mathrm{Ad}_g \circ \mathrm{ad}_X \circ (\mathrm{Ad}_g)^{-1}(Y)\end{aligned}$$

If we use σ to denote Ad_g, then $\mathrm{ad}_{\sigma(X)} = \sigma \circ \mathrm{ad}_X \circ \sigma^{-1}$. Hence,

$$B(\sigma(X), \sigma(Y)) = \mathrm{tr}(\mathrm{ad}_{\sigma(X)} \circ \mathrm{ad}_{\sigma(Y)}) = \mathrm{tr}(\sigma \circ \mathrm{ad}_X \circ \mathrm{ad}_Y \circ \sigma^{-1}) = B(X, Y)$$

For (2). For $Z \in \mathfrak{g}$, from (1) one has

$$B(\mathrm{Ad}_{\exp(tZ)} X, \mathrm{Ad}_{\exp(tZ)} Y) = B(X, Y)$$

By taking derivative at point $t = 0$, one has

$$B(\mathrm{ad}_Z X, Y) + B(X, \mathrm{ad}_Z Y) = 0$$

□

Proposition A.2.1. Let $\mathfrak{g}$ be a Lie algebra with Killing form B, $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then Killing form on $\mathfrak{h}$ is exactly the restriction of B on it.

In the following computations we always use ϕ to denote $\mathrm{ad}_X \circ \mathrm{ad}_Y$.

Example A.2.1. Killing form $B(X, Y)$ on $\mathfrak{gl}(n)$ is $2n\,\mathrm{tr}(XY) - 2\,\mathrm{tr}(X)\,\mathrm{tr}(Y)$.

Proof. There is a canonical basis of $\mathfrak{gl}(n)$, that is $\{E_{ij}\}$, where E_{ij} is the matrix such that

$$(E_{ij})_{kl} = \begin{cases} 1, & (k, l) = (i, j) \\ 0, & \text{otherwise} \end{cases}$$

We compute the trace of ϕ in terms of this basis. A direct computation shows

$$\phi(E_{ij}) = \sum_{k=1}^{n} (XY)_{jk} E_{ik} + (XY)_{ki} E_{kj} - \sum_{k,l=1}^{n} (X_{ki} Y_{jl} + Y_{ki} X_{jl}) E_{kl}$$

which implies the trace of ϕ is

$$\sum_{i,j=1}^{n} (XY)_{jj} + (XY)_{ii} - X_{ii} Y_{jj} - Y_{ii} X_{jj} = 2n\,\mathrm{tr}(XY) - 2\,\mathrm{tr}(X)\,\mathrm{tr}(Y)$$

□

Example A.2.2. Killing form $B(X, Y)$ on $\mathfrak{sl}(n)$ is $2n\,\mathrm{tr}(XY)$.

Proof. Note that $\mathfrak{sl}(n)$ is an ideal of $\mathfrak{gl}(n)$, which implies the restriction of Killing form on $\mathfrak{gl}(n)$ to $\mathfrak{sl}(n)$ is exactly the one on $\mathfrak{sl}(n)$. Thus by Example A.2.1 one has Killing form on $\mathfrak{sl}(n)$ is $2n\,\mathrm{tr}(XY)$, since $\mathfrak{sl}(n)$ consisting of matrices with vanishing trace. □

Example A.2.3. Killing form $B(X, Y)$ on $\mathfrak{so}(n)$ is $(n-2)\,\mathrm{tr}(XY)$.

Proof. There is a natural basis of $\mathfrak{so}(n)$, that is $\{E_{ij} - E_{ji}\}_{i<j}$. If we denote

$$\phi(E_{ij}) = a_{ij,ij} E_{ij} + a_{ij,ji} E_{ji} + \dots$$

The computation in Example A.2.1 shows

$$\begin{aligned} a_{ij,ij} &= (XY)_{jj} + (XY)_{ii} - X_{ii} Y_{jj} - Y_{ii} X_{jj} \\ a_{ij,ji} &= \delta_{ij}((XY)_{jj} + (XY)_{ii}) - X_{ji} Y_{ji} - Y_{ji} X_{ji} \end{aligned}$$

Note that

$$\phi(E_{ij} - E_{ji}) = (a_{ij,ij} - a_{ij,ji})(E_{ij} - E_{ji}) + \dots$$

Thus the Killing form on $\mathfrak{so}(n)$ is

$$\begin{aligned}
B(X,Y) &= \sum_{i<j}((XY)_{jj} + (XY)_{ii} - X_{ii}Y_{jj} - X_{jj}Y_{ii} + X_{ji}Y_{ji} + Y_{ji}X_{ji}) \\
&= \frac{1}{2}\sum_{i\neq j}((XY)_{jj} + (XY)_{ii} - X_{ii}Y_{jj} - X_{jj}Y_{ii} + X_{ji}Y_{ji} + Y_{ji}X_{ji}) \\
&= (n-1)\operatorname{tr}(XY) + \frac{1}{2}\sum_{i\neq j}(-X_{ii}Y_{jj} - X_{jj}Y_{ii} + X_{ji}Y_{ji} + Y_{ji}X_{ji}) \\
&\overset{(1)}{=} (n-1)\operatorname{tr}(XY) - \operatorname{tr}(X)\operatorname{tr}(Y) - \frac{1}{2}\sum_{i\neq j}(X_{ji}Y_{ij} - Y_{ji}X_{ij}) \\
&\overset{(2)}{=} (n-1)\operatorname{tr}(XY) - \frac{1}{2}(\operatorname{tr}(XY) + \operatorname{tr}(YX)) \\
&= (n-2)\operatorname{tr}(XY)
\end{aligned}$$

where

(1) holds from X, Y are skew-symmetric.
(2) holds from skew-symmetry matrix has vanishing trace.

□

A.3. **Homogeneous space.** One can refer to Page120 of [War10] for more details.

Definition A.3.1 (smooth Lie group action)**.** A Lie group G acts on a smooth manifold M smoothly, if the following conditions are satisfied:

1. Every $g \in G$ induces a diffeomorphism of M, denoted by $x \to gx$, where $x \in M$.
2. The map $G \times M \to M$ given by $(g,x) \mapsto gx$ is smooth.
3. For $g_1, g_2 \in M$ and $x \in M, (g_1g_2)x = g_1(g_2x)$.

Definition A.3.2 (G-homogeneous space)**.** A smooth manifold M endowed with a transitive smooth G-action is called a homogeneous G-space, where G is a Lie group.

Definition A.3.3 (isotropy group)**.** If Lie group G acts on smooth manifold M smoothly, the isotropy group at $p \in M$ is defined as

$$G_p = \{g \in G \mid gp = p\}$$

Theorem A.3.1. Let M be a G-homogeneous space and $p \in M$. Then the isotropy group G_p is a closed subgroup of G and the map

$$\begin{aligned} G/G_p &\to M \\ gG_p &\mapsto gp \end{aligned}$$

is an G-equivariant diffeomorphism.

Here are some tools which can be used to construct homogeneous manifolds. In fact, the most interesting examples of homogeneous space comes from this construction.

Theorem A.3.2. Let G be a Lie group and H be a closed subgroup of G. Then

1. The left coset space G/H is a topological manifold of dimension $\dim G - \dim H$.
2. G/H admits a smooth structure, such that the quotient map $\pi : G \to G/H$ is a smooth submersion.
3. The left action
$$\begin{aligned} G \times G/H &\to G/H \\ (g_1, g_2H) &\mapsto (g_1g_2)H \end{aligned}$$
turns G/H into a G-homogeneous space.

Appendix B. Covering spaces

B.1. **The topological covering.** In this section we mainly follows [Hat02], and we always assume X is a path connected topological space.

Definition B.1.1 (covering space)**.** A (topological) covering of X is a continuous map $\pi : \widetilde{X} \to X$ such that there exists a discrete space D and for each $x \in X$ an open neighborhood $U \subset X$, such that $\pi^{-1}(U) = \coprod_{d\in D} V_d$ and $\pi|_{V_d} : V_d \to U$ is a homeomorphism for each $d \in D$. Furthermore,

1. The open sets V_d are called sheets.
2. For each $x \in X$, the discrete subset $x \in X$ is called the fiber of x.
3. The degree of the covering is the cardinality of the space D.

Proposition B.1.1 (homotopy lifting property)**.** Given a covering space $\pi : \widetilde{X} \to X$, a homotopy $f_t : Y \to X$ and a map $\widetilde{f_0} : Y \to \widetilde{X}$ lifting f_0, then there exists a unique homotopy $\widetilde{f_t} : Y \to \widetilde{X}$ lifts f_t.

Remark B.1.1. Note that above statement says if there is a lift of f_0, then there is a unique homotopy which lifts f_t. However, what's the existence and uniqueness of such lifts? Here are two results:

1. Existence: If Y is path connected and locally path connected[9], then a lift $\widetilde{f} : Y \to \widetilde{X}$ of f exists if and only if $f_*(\pi_1(Y)) \subset \pi_*(\pi_1(\widetilde{X}))$.
2. Uniqueness: If two lifts $\widetilde{f}, \widetilde{g} : Y \to X$ of f agree at one point of Y and Y is connected, then $\widetilde{f}$ and $\widetilde{g}$ agree on all Y.

Remark B.1.2. Here are two special cases:

1. Taking Y to be a point gives the path lifting property.
2. Taking Y to be I, we see that every homotopy f_t of a path f_0 in X lifts to a homotopy $\widetilde{f_t}$ of each lift $\widetilde{f_0}$ of f_0.

Corollary B.1.1. The map $\pi_* : \pi_1(\widetilde{X}) \to \pi_1(X)$ induced by $\pi : \widetilde{X} \to X$ is injective. Furthermore,

1. $\pi_*(\pi_1(\widetilde{X}))$ consists of the homotopy class of loops in X whose lifts to $\widetilde{X}$ are still loops.
2. The index of $\pi_*(\pi_1(\widetilde{X}))$ in $\pi_1(X)$ is the degree of covering.

Definition B.1.2 (universal covering)**.** A simply-connected covering space of X is called universal covering.

Corollary B.1.2. The degree of universal covering equals $|\pi_1(X)|$.

Definition B.1.3 (deck transformation)**.** Let $\pi : \widetilde{X} \to X$ be a covering, the deck transformation group of this covering is defined as

$$\mathrm{Aut}_\pi(\widetilde{X}) = \{f\colon \widetilde{X} \to \widetilde{X} \text{ is homeomorphism} \mid \pi \circ F = \pi\}$$

[9]Note that a path connected space may not be locally path connected. For example, the topologist's sine curve.

Definition B.1.4. A covering $\pi : \widetilde{X} \to X$ is called normal, if deck transformation is transitive on each fiber of $x \in X$.

Corollary B.1.3. If $\pi : \widetilde{X} \to X$ is a normal covering, then X is homeomorphic to $\widetilde{X}/\pi_1(X)$.

Proposition B.1.2. Let $\pi : \widetilde{X} \to X$ be a path-connected covering space of the path-connected, locally path-connected space X, and let H be $\pi_*(\pi_1(\widetilde{X}))$. Then

1. This covering space is normal if and only if H is a normal subgroup of $\pi_1(X)$.
2. The group of deck transformation is isomorphic to the quotient $N(H)/H$, where $N(H)$ is the normalizer of H.
3. In particular, the group of deck transformation is isomorphic to $\pi_1(X)$, if $\widetilde{X}$ is universal covering.

Corollary B.1.4. The universal covering is a normal covering. In particular, X is homeomorphic to $\widetilde{X}/\pi_1(X)$.

The group of deck transformation is a special case of the general notation of groups acting on spaces.

Definition B.1.5 (group acting on space)**.** Given a group G and a topological space X, then an action of G on X is a homomorphism ρ from G to the group $\mathrm{Homeo}(X)$ consisting of all homeomorphisms from X to itself.

Remark B.1.3. Thus to each $g \in G$ is associated a homeomorphism $\rho(g) : X \to X$, which for notational simplicity we write simply as $g : X \to X$.

Definition B.1.6 (properly discontinuous)**.** An action is called properly discontinuous if each $x \in X$ has a neighborhood U such that all images $g(U)$ for varying $g \in G$ are disjoint, that is, $g_1(U) \cap g_2(U) \neq \varnothing$ implies $g_1 = g_2$.

Remark B.1.4. If an action is properly discontinuous, then it's free. Indeed, if G acts on X properly discontinuous and there exists $g \in G$ such that $gx = x$ for all $x \in X$, then for arbitrary neighborhood U of x, one has $g(U) \cap U \neq \varnothing$, thus $g = e$.

Proposition B.1.3. For a covering $\pi : \widetilde{X} \to X$, the group of deck transformation $\mathrm{Aut}_\pi(\widetilde{X})$ acts on $\widetilde{X}$ properly discontinuous.

In a summary, we have the group of deck transformations acts on covering space

1. homeomorphically.
2. transitively.
3. properly discontinuous. In particular, freely.

B.2. **Riemannian covering.**

Definition B.2.1 (smooth covering)**.** If $\widetilde{M}, M$ are smooth manifolds, then a smooth map $\pi : \widetilde{M} \to M$ is called a smooth covering, if

1. π is a topological covering.
2. π is a local diffeomorphism.

Proposition B.2.1. Let $\widetilde{M}, M$ be smooth manifolds and $f\colon \widetilde{M} \to M$ a proper map which is a local diffeomorphism, then f is a covering.

Definition B.2.2 (Riemannian covering)**.** If $(\widetilde{M}, \widetilde{g}), (M, g)$ are Riemannian manifolds, then $\pi : \widetilde{M} \to M$ is called a Riemannian covering, if

1. π is a smooth covering.
2. π is a local isometry.

We always consider the following case:

Example B.2.1. Let (M, g) be a Riemannian manifold with smooth covering $\pi : \widetilde{M} \to M$, then we can equip $\widetilde{M}$ with pullback metric $\widetilde{g} = \pi^* g$, since we can use local diffeomorphism to pullback metric, then $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ is a Riemannian covering.

Proposition B.2.2. Let $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ be a Riemannian universal covering with deck transformation $\Gamma \subset \mathrm{Iso}(\widetilde{M}, \widetilde{g})$, then

1. M is isometric to $\widetilde{M}/\Gamma$.
2. Γ acts on $\widetilde{M}$ isometrically, transitively and properly discontinuous.

Proposition B.2.3. If $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ is a Riemannian covering, then M is complete if and only if $\widetilde{M}$ is.

Proposition B.2.4. Let M be a complete Riemannian manifold and $f\colon M \to N$ a local diffeomorphism onto a Riemannian manifold N which has the following property: For all $p \in M$ and for all $v \in T_pM$, we have $|(\mathrm{d}f)_p v| \geq |v|$. Then f is a covering map.

Proof. See Lemma 3.3 in Page150 of [Car92]. $\square$

Appendix C. Hodge theorem

In this section, we mainly follow the Chapter 6 of [War10].

C.1. **Introduction and proof of Hodge theorem.** We shall use Δ^* to denote the adjoint of Laplace-Beltrami operator on Ω^k_M. This operator is precisely Δ itself, since Laplace-Beltrami operator is self-adjoint, and we usually make no distinction between Δ and Δ^*. However, this distinction will be important for the form of the following definition.

An important question is to find a neccessary and sufficient condition for there to exist a solution ω of equation $\Delta\omega = \alpha$, where α is a given k-form. Suppose ω is a solution, then

$$(\Delta\omega, \varphi) = (\alpha, \varphi)$$

holds for all k-forms φ. Equivalently we have

$$(\omega, \Delta^*\varphi) = (\alpha, \varphi)$$

holds for all k-forms φ. In this viewpoint, we can regard a solution of $\Delta\omega = \alpha$ as a centain type of linear functional on $C^\infty(M, \Omega^k_M)$, namely solution ω determines a bounded linear functional l on $C^\infty(M, \Omega^k_M)$ by

$$l(\varphi) = (\omega, \varphi), \quad \varphi \in C^\infty(M, \Omega^k_M)$$

such that

$$l(\Delta^*\varphi) = (\alpha, \varphi)$$

holds for all k-forms φ.

Definition C.1.1 (weak solution)**.** A linear functional l on $C^\infty(M, \Omega^k_M)$ is called a weak solution of $\Delta\omega = \alpha$, if

$$l(\Delta^*\varphi) = (\alpha, \varphi)$$

holds for all k-forms φ.

We have seen that each ordinary solution of $\Delta\omega = \alpha$ determines a weak solution of it, it turns out that the major effort of this section will be to prove a regularity theorem which says that the converse of this is true, that is each weak solution determines an ordinary solution. The main step is to show if l is a weak solution of $\Delta\omega = \alpha$, then there exists a smooth form ω such that

$$l(\varphi) = (\alpha, \varphi), \quad \varphi \in C^\infty(M, \Omega^k_M)$$

Then ω is an ordinary solution follows from

$$(\Delta, \varphi) = (\omega, \Delta^*\varphi) = l(\Delta^*\varphi) = (\alpha, \varphi)$$

holds for all k-forms φ, which implies $\Delta\omega = \alpha$.

The key theorems we will prove are listed as follows:

Theorem C.1.1 (regularity theorem)**.** Let $\alpha \in C^\infty(M, \Omega^k_M)$, and l be a weak solution of $\Delta\omega = \alpha$, then there exists $\omega \in C^\infty(M, \Omega^k_M)$ such that

$$l(\varphi) = (\omega, \varphi)$$

holds for every k-forms φ. In particular, $\Delta\omega = \alpha$.

Theorem C.1.2. Let $\{\alpha_n\}$ be a sequence of smooth k-forms on M such that $\|\alpha_n\| \leq c$ and $\|\Delta\alpha_n\| \leq c$ for all n and for some constant $c > 0$. Then a subsequence of $\{\alpha_n\}$ is a Cauchy sequence in $C^\infty(M, \Omega^k_M)$.

Corollary C.1.1. There exists a constant $c > 0$ such that

$$\|\psi\| \leq c\|\Delta\psi\|$$

holds for all $\psi \in (\mathcal{H}^k)^\perp$

Proof. Suppose the contraty, then there exists a sequence $\psi_j \in (\mathcal{H}^k)^\perp$ with $\|\psi_j\| = 1$ and $\|\Delta\psi_j\| \to 0$. By Theorem C.1.2, there exists a subsection of $\{\psi_j\}$ which for convenience we can assume to be $\{\psi_j\}$ itself, is Cauchy. Thus for each $\varphi \in C^\infty(M, \Omega^k_M)$, $\lim_{j\to\infty}(\psi_j, \varphi)$ exists. Consider the linear functional l on $C^\infty(M, \Omega^k_M)$ defined by

$$l(\varphi) := \lim_{j\to\infty}(\psi_j, \varphi), \quad \varphi \in C^\infty(M, \Omega^k_M)$$

It's clear l is bounded, and

$$l(\Delta\varphi) = \lim_{j\to\infty}(\psi, \Delta\varphi) = \lim_{j\to\infty}(\Delta\psi_j, \varphi) = 0$$

holds for all $\varphi \in C^\infty(M, \Omega^k_M)$, which implies l is a weak solution of $\Delta\psi = 0$. By Theorem C.1.1, there exists a k-form ψ such that $l(\varphi) = (\psi, \varphi)$, where $\varphi \in C^\infty(M, \Omega^k_M)$. Consequently $\psi_j \to \psi$, and $\psi \in (\mathcal{H}^k)^\perp$ with $\|\psi\| = 1$. However, Theorem C.1.1 implies $\psi \in \mathcal{H}^k$, a contradiction. □

Holding above results, we can prove Hodge theorem.

Theorem C.1.3 (Hodge theorem)**.** Consider the Laplace operator $\Delta : C^\infty(M, \Omega^k_M) \to C^\infty(M, \Omega^k_M)$, then

1. $\dim_{\mathbb{R}} \mathcal{H}^k < \infty$.
2. There is an orthogonal direct sum decomposition

$$C^\infty(M, \Omega^k_M) = \mathcal{H}^k \oplus \operatorname{im}\Delta$$

Proof. For (1). If $\mathcal{H}^k$ is not finite dimensional, then there exists an infinite orthonormal sequence. By Theorem C.1.2, this orthonormal sequence contains a Cauchy sequence, which is impossible. Thus $\mathcal{H}^k$ is finite dimensional.

For (2). Note that we naturally have the following orthogonal decomposition

$$C^\infty(M, \Omega^k_M) = (\mathcal{H}^k)^\perp \oplus \mathcal{H}^k$$

The theorem will be proved by showing that $(\mathcal{H}^k)^\perp = \operatorname{im}\Delta$. We use $\mathcal{H}$ to denote the projection from $C^\infty(M, \Omega^k_M)$ to $\mathcal{H}^k$, that is $\mathcal{H}(\alpha)$ is the harmonic part of α.

It's easy to see $\operatorname{im}\Delta \subset (\mathcal{H}^k)^\perp$, since for all $\omega \in C^\infty(M, \Omega_M^k)$ and $\alpha \in \mathcal{H}^k$, we have

$$(\Delta\omega, \alpha) = (\omega, \Delta\alpha) = 0$$

To see converse, for $\alpha \in (\mathcal{H}^k)^\perp$, we define a linear functional l on $\operatorname{im}\Delta$ by setting

$$l(\Delta\varphi) := (\alpha, \varphi)$$

for all $\varphi \in C^\infty(M, \Omega_M^k)$.

1. l is well-defined, since if $\Delta\varphi_1 = \Delta\varphi_2$, then $\varphi_1 - \varphi_2 \in \mathcal{H}^k$, then $(\alpha, \varphi_1 - \varphi_2) = 0$.
2. l is bounded. Indeed, for $\varphi \in C^\infty(M, \Omega_M^k)$, let $\psi = \varphi - \mathcal{H}(\varphi)$. Then

$$\begin{aligned} |l(\Delta\varphi)| &= |l(\Delta\psi)| \\ &= |(\alpha, \psi)| \\ &\leq \|\alpha\|\|\psi\| \\ &\overset{*}{\leq} c\|\alpha\|\|\Delta\psi\| \\ &= c\|\alpha\|\|\Delta\varphi\| \end{aligned}$$

where $*$ holds from Corollary C.1.1.

By Hahn-Banach theorem, l extends to a bounded linear functional on $C^\infty(M, \Omega_M^k)$, thus l is a weak solution of $\Delta\omega = \alpha$. By Theorem C.1.1, there exists a k-form ω such that $\Delta\omega = \alpha$. Hence

$$(\mathcal{H}^k)^\perp = \operatorname{im}\Delta$$

This completes the proof of Hodge theorem. □

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Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, P.R. China,

Email address: liubw22@mails.tsinghua.edu.cn