RIEMANN SURFACE

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ABSTRACT. It's a lecture note I typed for "Riemann surface" taught by Xiaobo Liu, in spring 2022. This note mainly follows the blackboard-writing of Prof. Liu. I also add some details and my understandings in it.

Attention: There may be a considerable number of mistakes in this note, and that's all my fault. If you have any advice for this note, please email me. Here is my email: bowenl@sdu.edu.cn.

CONTENTS

1. Riemann surface

1.1. **Definitions and Examples.**

Definition 1.1.1 (almost complex structure)**.** *If X is a surface, a (almost) complex structure is a smooth map $J : TX \to TX$, such that for any $p \in X$, $J_p : T_pX \to T_pX$ is a linear map with $J_p^2 = -\operatorname{id}$.*

Remark 1.1.2. If X admits a complex structure, then X is orientable.

Example 1.1.3. Assume X has a Riemann metric, and X is orientable. For any $v \in T_pX$, define $J(v)$ to be the tangent vector obtained by rotating v by $\pi/2$ counterclockwise.

Corollary 1.1.4. *Any orientable surface admits a complex structure.*

Example 1.1.5. If $X = \mathbb{C}$, then $T_qX \cong \mathbb{C}, \forall q \in X$, choose $v \in T_qX$, define $J(v) = iv$, then J is a complex structure on X.

Definition 1.1.6 (complex chart)**.** *Assume X is a topological space. A complex chart on X is an open subset $U \subset X$ together with a homeomorphism $\varphi : U \to V \subset \mathbb{C}$, where V is an open subset. If $p \in U$, and $\varphi(p) = 0$, then (U, φ) is called a chart centered at p. For $q \in U$, $z = \varphi(q)$ is called a local coordinate of q.*

Definition 1.1.7 (compatible chart)**.** *If $(U_1, \varphi_1), (U_2, \varphi_2)$ are two charts on X, we say they're compatible if $U_1 \cap U_2 = \varnothing$ or*

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is holomorphic.

Definition 1.1.8 (atlas)**.** *An atlas is a collection of compatible charts $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$, such that $\bigcup_{\alpha \in I} U_\alpha = X$. Two atlas $\mathscr{A}, \mathscr{B}$ are equivalent if every chart in $\mathscr{A}$ and every chart in $\mathscr{B}$ is compatible.*

Definition 1.1.9 (complex structure)**.** *A complex structure on X is an equivalent class of atlas on X.*

Remark 1.1.10. Given an atlas $\mathscr{A}$ on X, we can use charts in $\mathscr{A}$ to define a almost complex structure $J : TX \to TX$ such that $J^2 = -\operatorname{id}$. However, the converse may not hold, that is not every almost complex structure will define a complex structure on X, it still needs some integrable condition.

Definition 1.1.11 (Riemann surface)**.** *A Riemann surface is a second countable, connected, Hausdorff topological space X together with a complex chart on X.*

Example 1.1.12. Every open subset of $\mathbb{C}$ is a Riemann surface.

Example 1.1.13 (complex sphere)**.** $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, consider

$$U_1 = S^2 \backslash \{(0, 0, 1)\} \xrightarrow{\varphi_1} \mathbb{C}$$

where φ_1 is defined as $\varphi_1(x, y, z) = \frac{x}{1-z} + i\frac{y}{1-z} = w$. Similarly consider

$$U_2 = S^2 \backslash \{(0, 0, -1)\} \xrightarrow{\varphi_2} \mathbb{C}$$

where φ_2 is defined as $\varphi_2(x, y, z) = \frac{x}{1+z} - i\frac{y}{1+z} = w'$. Note that $ww' = \frac{x^2+y^2}{1-z^2} = 1$. And it's easy to see the transition function is $T(w) = \frac{1}{w}$. So $\{U_1, U_2\}$ is an atlas of S^2.

Example 1.1.14 (complex projective space)**.** $\mathbb{CP}^1 = \{\text{complex 1-dimensional subspaces of } \mathbb{C}^2\}$, is called a 1-dimensional projective space. Given a point $(0, 0) \neq (z, w) \in \mathbb{C}^2$, exists a unique point $[z, w] \in \mathbb{CP}^1$, called the homogenous coordinate of $\mathbb{CP}^1$. Consider

$$U_1 = \{[z, w] \mid z \neq 0\} \xrightarrow{\varphi_1} \mathbb{C}$$

where φ_1 is defined as $\varphi_1([z, w]) = z/w$. Similarly consider

$$U_2 = \{[z, w] \mid w \neq 0\} \xrightarrow{\varphi_2} \mathbb{C}$$

where φ_2 is defined as $\varphi_2([z, w]) = w/z$. It's easy to check $\{U_1, U_2\}$ is a atlas of $\mathbb{CP}^1$.

In fact, $\mathbb{CP}^1$ is a Riemann surface which is isomorphic to S^2.

Example 1.1.15 (complex torus)**.** Given two nonzero $w_1, w_2 \in \mathbb{C}$, with $w_1 \neq aw_2$ for any $a \in \mathbb{C}$. Define lattice:

$$L = \mathbb{Z}w_1 + \mathbb{Z}w_2$$

In fact, L is a subgroup of $\mathbb{C}$ with respect to operation "+".

Then $T = \mathbb{C}/L$ is a Riemann surface called complex torus. Consider the projection $\pi : \mathbb{C} \to T$. For $p \in T$, find one of its inverse image of π, denoted by z_0. Choose $\varepsilon \in \mathbb{R}^+$ small enough such that

$$B_{2\varepsilon} \cap L = \{0\}$$

Consider

$$B_\varepsilon(z_0) \xrightarrow{\pi} \pi(B_\varepsilon(z_0)) \subset T$$

and the condition on ε implies $\pi|_{B_\varepsilon}$ is injective. So let $\{\pi(B_\varepsilon(z_0))\}$ be a open cover of T, and π^{-1} is the parametrization, this is an atlas of T.

Remark 1.1.16. The complex structure of complex torus depends on w_1, w_2. In fact, all complex structure of complex torus forms a Riemann surface which is the same as $\mathbb{C}$. [1]

[1] The space consists of all complex structure of a Riemann surface is called the moduli space of it.

1.2. **Holomorphic function and Properties.**

Definition 1.2.1 (holomorphic function)**.** *If X is a Riemann surface, $W \subset X$ is a open subset. The function $f : W \to \mathbb{C}$ is a complex valued function on W. f is called holomorphic at $p \in W$, if there exists a chart (U, φ) of p such that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ is holomorphic at $\varphi(p)$. f is called holomorphic on W, if it is holomorphic at any $p \in W$.*

Theorem 1.2.2 (Maximum modulus theorem)**.** *For a Riemann surface X, $W \subset X$ is an open subset, and f is a holomorphic function on W. If there exists a point $p \in W$, such that $|f(p)| \geq |f(x)|$ for all $x \in W$, then f must be a constant.*

Proof. Clear. □

Corollary 1.2.3. *If X is a compact Riemann surface, then any global holomorphic funtion f must be constant.*

So, it's boring to consider holomorphic funtions on a compact Riemann surface. In order to get something interesting, we need to consider meromorphic functions.

Definition 1.2.4 (singularity)**.** *If X is a Riemann surface, let f be a holomorphic function defined on $U \backslash \{p\}$ where $U \subset X$ is an open subset. p is called a removbale singularity/pole/essential singularity, if there exists a chart (U, φ) of p, such that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ has $\varphi(p)$ as a removbale singularity/pole/essential singularity.*

Remark 1.2.5. We have the following criterions:

1. If $|f(x)|$ is bounded in a punctured neighborhood of p, then p is a removable singularity. And we can cancel the singularity by defining $f(p) = \lim_{x\to p} f(x)$.
2. If $\lim_{x\to p} |f(x)| = \infty$, then p is a pole.
3. If $\lim_{x\to p} |f(x)|$ doesn't exist, then p is a essential singularity.

Definition 1.2.6 (meromorphic function)**.** *f is called a meromorphic function at p if p is either a removbale singularity or a pole, or f is holomorphic at p; f is called a meromorphic function on W, if it's meromorphic at any point $p \in W$.*

Remark 1.2.7. If f, g are meromorphic on W, then $f \pm g, fg$ are also meromorphic on W. If in addition, $g \not\equiv 0$, then f/g is also meromorphic on W. In other words, the set of meromorphic functions one W forms a field, which is called meromorphic function field.

Example 1.2.8. Consider f, g are two polynomials in variable z with $g \not\equiv 0$, then f/g is a meromorphic function on $S^2 = \mathbb{C} \cup \{\infty\}$. In fact, all meromorphic functions on S^2 are in this form.

Theorem 1.2.9 (discreteness of singularities and zeros)**.** *Let X be a Riemann surface and $W \subset X$ is an open subset, f is a meromorphic function on W, then set of singularities and zeros of f is discrete, unless $f \equiv 0$.*

Corollary 1.2.10. *If X is compact, $f \not\equiv 0$, then f has finitely many poles and zeros on X. As a consequence, if f, g are two meromorphic functions on an open subset $W \subset X$, and f agrees with g on a set with limit point in W, then $f \equiv g$.*

Definition 1.2.11 (holomorphic map)**.** *Let X, Y be two Riemann surfaces, $F : X \to Y$. For a point $p \in X$, f is called holomorphic at p, if there exists a chart (U, φ) of p, and a chart (V, ψ) of $F(p)$, such that*

$$\psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V \cap F(U))$$

is holomorphic at $\varphi(p)$; F is called holomorphic in W, if F is holomorphic at any point in W.

Remark 1.2.12. $\psi \circ F \circ \varphi^{-1}$ is called the local representation or local form of F at p.

Example 1.2.13. Any meromorphic function on X can be seen as a holomorphic map from X to S^2; Conversely, we can construct a meromorphic function from a holomorphic map from X to S^2.

Definition 1.2.14 (biholomorphic)**.** *Two Riemann surfaces are called biholomorphic or isomorphic to each other, if there are two holomorphic map $F : X \to Y, G : Y \to X$, such that $F \circ G = G \circ F = \mathrm{id}$.*

Example 1.2.15. S^2 is biholomorphic to $\mathbb{RP}^2$.

Theorem 1.2.16 (Open mapping theorem)**.** *$F : X \to Y$ is a non-constant holomorphic map, then F is an open map.*

Corollary 1.2.17. *If X is compact, and Y is connected, $F : X \to Y$ is a non-constant holomorphic map, then Y is compact and $F(X) = Y$.*

Proof. By open mapping theorem, $F(X)$ is an open subset of Y, and $F(X)$ is compact in Y, since continous function maps compact set to compact set. Then $F(X)$ is both open and closed in Y, then $F(X) = Y$. $\square$

1.3. **Ramification covering.**

Theorem 1.3.1. *$F : X \to Y$ is a non-constant holomorphic function on Riemann surfaces, then for any $p \in Y$, $F^{-1}(y)$ is a discrete set. Furthemore, if X is compact, then $F^{-1}(y)$ only contains finite many points.*

So we wonder what's exact number of $F^{-1}(y)$, the local normal form tells you answer.

Theorem 1.3.2 (Local normal form)**.** *$F : X \to Y$ is a non-constant holomorphic function on X, then there is a local representation of F at $p \in X$, such that*

$$\psi \circ F \circ \varphi^{-1}(z) = z^k, \quad \forall z \in \varphi(U \cap F^{-1}(V))$$

k is called the multiplicity[2] *of F at p, denoted by $\mathrm{mult}_p(F)$. In fact, k is independent of the choice of charts.*

[2]Sometimes this number is also called ramification of F at p.

Proof. Fix a chart (U_2, φ_2) of $F(p)$, choose an arbitary local chart (U, ψ) of p such that $F(U) \subset U_2$, denote $\varphi_2 \circ F \circ \psi^{-1} = T$, then $T(0) = 0$. Consider the Taylor expansion of T at $w = 0$ has

$$T(w) = \sum_{k=m}^{\infty} a_k w^k, \quad a_m \neq 0$$

So $T(w) = w^m S(w)$, where $S(w)$ is a holomorphic function with $S(0) \neq 0$, then there exists a holomorphic function $R(w)$ such that $R^m(w) = S(w)$.

Then $T(w) = (wR(w))^m = (\eta(w))^m$, so $\eta(0) = 0, \eta'(0) = R(0) \neq 0$, so η is invertible near $w = 0$ by inverse funtion theorem. So there exists another chart of $p \in U_1 \subset U$, with

$$U \supset U_1 \xrightarrow{\psi} V \xrightarrow{\eta} V_1 \subset \mathbb{C}$$

then we can define a local chart $(U_1, \varphi_1 = \eta \circ \psi)$, and check

$$\varphi_2 \circ F \circ \varphi_1^{-1}(z) = \varphi_2 \circ F \circ \psi^{-1} \circ \eta^{-1}(z) = T(w) = (\eta(w))^m = z^m$$

What's more, we can see from the local normal form that for any $q \in Y, q \neq F(p)$ and q lies in a small neighborhood of p such that $F^{-1}(q)$ lies in a small neighborhood of p, then $F^{-1}(q)$ consists of exactly k points. So the ramification index is independent of the charts we choose. □

Definition 1.3.3 (ramification points)**.** *p is called a ramification point of a holomorphic map $F : X \to Y$, if $\mathrm{mult}_p(F) > 1$, such $F(p)$ is called a ramification value.*

Lemma 1.3.4. *p is a ramification point of a holomorphic map $F : X \to Y$ if $T'(w) = 0$, for any local representation of F.*

Corollary 1.3.5. *The set of ramification points of a holomorphic map is a discrete set.*

Theorem 1.3.6. *Assume X, Y are complex Riemann surface, $F : X \to Y$ is non-constant holomorphic function, for $q \in Y$, let*

$$d_q(F) = \sum_{p \in F^{-1}(q)} \mathrm{mult}_p(F)$$

then $d_q(F)$ is independent of $q \in Y$, and denoted by $\deg(F)$.

Proof. Consider $F : \mathbb{D} \to \mathbb{D}$, defined by $z \mapsto z^m$, it's easy to check $d_q(F) = m$, for all $q \in \mathbb{D}$.

For general case, for $q \in Y$, let $F^{-1}(q) = \{p_1, \ldots, p_k\} \subset X$. Fix a chart (U_2, φ_2) centered at $q \in Y$, for any $i = 1, \ldots, k$, we can find local chart $(U_{1,q}, \psi_i$ centered at $p_i \in X$, such that

$$\varphi_2 \circ F \circ \psi_i^{-1}(z) = z^{m_i}, \quad z \in \psi_i(U_{1,i})$$

where $m_i = \mathrm{mult}_{p_i}(F)$. Choose $q \in W \subset U_2$ such that $F^{-1}(W) \subset \bigcup_{i=1}^{k} U_{1,i}$, then for any $q \in W$

$$d_q(F) = \sum_{i=1}^{k} m_i$$

which can be seen from trivial case we discuss firstly. Then $d_q(F)$ is a locally constant function, then $d_q(F)$ must be global constant, since Y is connected. $\square$

Corollary 1.3.7. *X is a compact Riemann surface, and f is a meromorphic function on X, then the number (counted with multiplicity) of zeros is equal to the number (counted with multiplicity) of poles.*

Proof. Note that meromorphic function f on X is equivalent to the holomorphic map F from X to S^2. Then the number of zeros is the multiplicity of F at zero and the number of poles is the multiplicity of F at ∞. $\square$

1.4. **Hurwitz Formula.** Now let us forget the complex structure of Riemann surface, and recall some facts about topological invariants.

Let X be a compact oriented surface, we can say the genus of X is the number of "holes" which X has, informally. We can use genus to classify all oriented compact surfaces: any two surfaces which have the same genus are diffeomorphic to each other.

We can also define Euler characterisitic of X, as

$$\chi(X) := \sum_i (-1)^i \dim H_i(X)$$

And there is a connection between genus of X and $\chi(X)$,

$$\chi(X) = 2 - 2\,\mathrm{genus}(X)$$

so we can also use $\chi(X)$ to classify oriented compact surface.

Theorem 1.4.1 (Hurwitz Formula)**.** *Let X, Y be two compact Riemann surfaces, and $F : X \to Y$ be a non-constant holomorphic map, then*

$$2\,\mathrm{genus}(Y) - 2 = \deg(F)(2\,\mathrm{genus}(X) - 2) + \sum_{p \in X} (\mathrm{mult}_p(F) - 1)$$

Note that the set of ramification points is finite, then $\sum_{p \in X} (\mathrm{mult}_p(F) - 1)$ is a finite sum, and denoted by $B(F)$.

Proof. Choose a triangulation of Y such that its vertex are exactly ramification values of F. Let v denote the number of vertices of Δ, c and t denote the number of edges and triangles of Δ, where Δ denotes a triangulation of Y. We can get a triangulation Δ' of X, by pulling back Δ through F, and use v', c' and t' to denote the same thing in Δ'.

Then we have the following obvious relations

$$t' = td, \quad e' = ed$$

where $d = \deg(F)$. The relation between v and v' is a little bit complicated, consider $q \in Y$, then

$$|F^{-1}(q)| = \sum_{p \in F^{-1}(q)} 1 = d + \sum_{p \in F^{-1}(q)} (1 - \mathrm{mult}_p(F))$$

then

$$\begin{aligned} v' &= \sum_{\text{vertex } q \text{ of } \Delta} |F^{-1}(q)| \\ &= \sum_{\text{vertex } q \text{ of } \Delta} (d + \sum_{p \in F^{-1}(q)} (1 - \mathrm{mult}_p(F))) \\ &= vd + \sum_{\text{vertex } q \text{ of } \Delta} \sum_{p \in F^{-1}(q)} (1 - \mathrm{mult}_p(F)) \\ &= vd + \sum_{p \in X} (1 - \mathrm{mult}_p(F)) \end{aligned}$$

Then by the relation between Euler characterisitic and triangulation, we get the desired conclusion. □

Definition 1.4.2 (ramified holomorphic map)**.** *A holomorphic map F is called ramified if $B(F) > 0$, this is equivalent to F has at least one ramification point; A holomorphic map F is called unramified if $B(F) = 0$, this is equivalent to F is a covering map.*

Corollary 1.4.3. *Let X, Y be two compact Riemann surfaces, and $F : X \to Y$ is a non-constant holomorphic map, then consider*
1. *If $Y = S^2$, and $\deg(F) > 1$, then F must be ramified.*
2. *If $\mathrm{genus}(X) = \mathrm{genus}(Y) = 1$, then F must be unramified.*
3. *$\mathrm{genus}(X) \geq \mathrm{genus}(Y)$.*
4. *If $\mathrm{genus}(X) = \mathrm{genus}(Y) > 1$, then F must be an isomorphism.*

Proof. All of them are simple applications of Hurwitz Formula.

1. By Hurwitz Formula we have
$$B(F) = 2(\deg(F) - 1) + 2\,\mathrm{genus}(X) > 0$$
2. By Hurwitz Formula we have
$$0 = 0 + B(F)$$
3. If $\mathrm{genus}(Y) = 0$, it's trivial. Otherwise, we have
$$2\,\mathrm{genus}(X) - 2 \geq 2\,\mathrm{genus}(Y) - 2 + B(F)$$

since $\deg F \geq 1$.

4. By Hurwitz Formula we have
$$(1 - \deg(F))(2\,\mathrm{genus}(X) - 2) = B(F)$$

Then $\deg(F) = 1$, since $\deg(F) \geq 1$, $2\,\mathrm{genus}(X) - 2 > 0$ and $B(F) \geq 0$. □

Remark 1.4.4. From above corollary, we can see that genus, as a topological invariants, controls geometric properties heavily.

1.5. **Automorphism groups of lower genus surface.**

1.5.1. *Automorphism group of Riemann sphere.* Firstly we determine what does the holomorphic maps $f : S^2 \to S^2$ look like

Proposition 1.5.1. *Let $f : S^2 \to S^2$ be a holomorphic map. Then f is a rational function, i.e.*

$$f(z) = \frac{p(z)}{q(z)}$$

where $p(z), q(z) \in \mathbb{C}[z]$, and $q(z) \neq 0$.

Proof. Consider f as a meromorphic from S^2 to $\mathbb{C}$. Since the Riemann sphere is compact, f can have only finitely many poles, for otherwise a sequence of poles would cluster somewhere, giving a non-isolated singularity. Especially, f has only finitely many poles in the plane. Let the poles occur at the plane z_1 through z_n with multiplicities e_1 through e_n. Define a polynomial

$$q(z) = \prod_{i=1}^{n} (z - z_i)^{e_j}$$

Then the function

$$p(z) = f(z)q(z)$$

has removbale singularities at the poles of f in $\mathbb{C}$, i.e. it is entire. So p has a power series representation on all of $\mathbb{C}$. Also, p is meromorphic at ∞, since both f and q are. This forces p to be a polynomial. This completes the proof. □

Corollary 1.1. *The biholomorphic maps on S^2 take the form*

$$f(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}$$

Proof. If the numerator or denominator of f were to have degree greater than 1 then by the local normal form, f would not be bijective. □

Furthermore, we assume that f is expressed in the lowest term, i.e. the numerator is not a scalar multiple of denominator. This discussion narrows our considerations to functions of the form

$$f(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}, ad - bc \neq 0$$

Then there is a surjective map

$$\mathrm{GL}_2(\mathbb{C}) \longrightarrow \mathrm{Aut}(S^2), \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto f(z) = \frac{az + b}{cz + d}$$

And after an direct check we will see it's a group homomorphism. But this homomorphism is clearly not injective, since all nonzero scalar multiples of a given matrix are taken to the same automorphism. The kernel of this homomorphism is

$$\mathbb{C}^{\times} I=\left\{\lambda\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]: \lambda \in \mathbb{C}, \lambda \neq 0\right\}$$

And by the first isomorphism theorem we have

$$\mathrm{GL}_2(\mathbb{C}) / \mathbb{C}^{\times} \mathrm{I}_2 \xrightarrow{\sim} \operatorname{Aut}(S^2)$$

Furthermore, we have

$$\mathrm{GL}_2(\mathbb{C}) / \mathbb{C}^{\times} \mathrm{I}_2 \cong \mathrm{PSL}_2(\mathbb{C})$$

And we have its complex dimension is 3, as a complex manifold.

1.5.2. *Automorphism group of complex torus.* Consider a lattice $L=\mathbb{Z} w_1+\mathbb{Z} w_2$, let X denote the complex torus $X=\mathbb{C} / L$, a Riemann surface with genus 1. Moreover, there is a group structure on X, induced by $(\mathbb{C},+)$ through natural projection $\pi: \mathbb{C} \rightarrow X$, defined as follows

$$[z_1]+[z_2]:=[z_1+z_2]$$

So, inversion map

$$[z] \mapsto[-z]$$

gives an automorphism.

For $a \in \mathbb{C}$,we can define a transformation

$$T_a: X \rightarrow X, \quad[z] \mapsto[z+a]$$

which is also an automorphism.

So, as we can see, there are too many automorphism on X, let $\operatorname{Aut}(X)$ denote all automorphisms on X, which forms a group which can reflect the symmetry of X.

Obviously, we have the following inclusion

$$\operatorname{Aut}(X) \supset\{T_{[a]} \mid [a] \in X\} \cup\{\text{inversion}\}$$

In fact, we will see later that $\operatorname{Aut}(X)$ is a complex manifold with $\dim_{\mathbb{C}} \operatorname{Aut}(X)=1$, but for now, we can only conclude that $\dim_{\mathbb{C}} \operatorname{Aut}(X) \geq 1$.

Before we come to see what is the automorphism group of X, we consider a more general case, holomorphic map between complex torus.

Assume L, M are two different lattices in $\mathbb{C}$, $X=\mathbb{C} / L, Y=\mathbb{C} / M$ are two complex torus.

Let $F: X \rightarrow Y$ be a non-constant holomorphic map, after composing some translation T_a, we can assume that $F([0])=[0]$. Since $\operatorname{genus}(X)=\operatorname{genus}(Y)$, then by Hurwitz formula F must be a covering map.

Let $\pi_X: \mathbb{C} \rightarrow X, \pi_Y: \mathbb{C}$ are natural projection. In fact, they're universal covering map.

Consider

$$\mathbb{C} \xrightarrow{\pi_X} X \xrightarrow{F} Y$$

then $F \circ \pi_X$ is also a universal covering of Y. By uniqueness of universal covering, there exists a holomorphic map[3] $G : \mathbb{C} \to \mathbb{C}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{G} & \mathbb{C} \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow{F} & Y \end{array}$$

Since $F([0]) = [0]$, then $G(0) \in M$. After composing with a translation in $\mathbb{C}$ with respect to $-G(0)$, we can assume $G(0) = 0$.

For any $z \in \mathbb{C}, l \in L$, consider the difference $w(z,l) := G(z+l) - G(z)$. First note that $w(z,l)$ is a holomorphic with respect to z. What's more, $w(z,l)$ must lie in M. So $w(z,l)$ must be a constant with respect to z, since M is discrete. So

$$\frac{\partial}{\partial z} w(z,l) = G'(z+l) - G'(z) = 0$$

That is, $G'(z)$ is periodic with respect to L, so $|G'(z)|$ is bounded. By Liouville's theorem, we have $G'(z)$ is constant.

So G must have the form $G(z) = \gamma z, \gamma \in \mathbb{C}$, since we assume $G(0) = 0$. Since $G(L) \subset G(M)$, we have

$$\gamma L \subset M$$

Since $G(z) = \gamma z$ is a group homomorphism, then F is also a group homomorphism between X and Y.

Clearly, F is an isomorphism if and only if $\gamma L = M$.

We summarize as follows:

Theorem 1.5.2. *Any holomorphic map $F : \mathbb{C}/L \to \mathbb{C}/M$ is induced by a linear map*

$$G : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \gamma z + a, \quad \gamma, a \in \mathbb{C}$$

such that $\gamma L \subseteq M$. Moreover, F is a biholomorphic map if and only if $\gamma L = M$, for some $\gamma \in \mathbb{C}$.

Now consider non-constant biholomorphic map $F : X \to X$, where $X = \mathbb{C}/L$. After composing some translation, we may assume $F([0]) = [0]$. Then F is induced by a map $z \mapsto \gamma z$ such that $\gamma L = L$.

Note that this condition is a quite strong for γ. We list some facts as follows

1. $|\gamma| = 1$, otherwise the shortest length of non-zero element in L and γL will be different.
2. There exists integers $m \geq 1$ such that

   $$\gamma^m = 1$$

   otherwise L contains infty many points in a circle, a contradiction to the discreteness.

[3] Clearly, G is not unique.

Note that $\gamma = \pm 1$ is allowed, $\gamma = 1$ is equivalent to F is identity and $\gamma = -1$ is equivalent to the inversion. Assume $\gamma \notin \mathbb{R}$, choose $w \in L\backslash\{0\}$, such that

$$|w| \leq |v|, \quad \text{for all } v \in L\backslash\{0\}$$

We claim that:

Lemma 1.5.3. *$L = \mathbb{Z}w + \mathbb{Z}\gamma w$, for w we choose above.*

Proof. Let $L' = \mathbb{Z}w + \mathbb{Z}\gamma w \subset L$. If $L' \neq L$, we will find an elment $v \in L\backslash L'$. Adding an element in L' if necessary, we may assume v lies in the parallelogram spanned by w and γw. Then

$$|v - w| + |v - \gamma w| < |w| + |\gamma w| = 2|v|$$

So either $|v - w|$ or $|v - \gamma w|$ is less than $|v|$, a contradiction. □

Since $\gamma L = L$, then $\gamma^2 w \in L = \mathbb{Z}w + \mathbb{Z}\gamma w$, so

$$\gamma^2 w = mw + n\gamma w, \quad m, n \in \mathbb{Z}$$

After canceling w we have the quadratic equation that γ must satisfy

$$\gamma^2 = m + n\gamma$$

so we have

$$\gamma = \frac{1}{2}(n \pm \sqrt{n^2 + 4m})$$

Since $\gamma \notin \mathbb{R}$, we have $n^2 + 4m < 0$. And

$$|\gamma|^2 = \frac{1}{4}(n^2 - (n^2 + 4m)) = -m$$

so we must have $m = -1$. So $n^2 < 4$ implies $n = \pm 1, 0$. Then all possible γ are listed as follows

$$\gamma = \begin{cases} \pm i, & n = 0 \\ \frac{1}{2}(\pm 1 \pm \sqrt{3}i), & n = \pm 1 \end{cases}$$

When $n = 0$, L is called a square lattice. When $\gamma = \pm 1$, $L = \mathbb{Z}w + \mathbb{Z}w \cdot e^{\frac{\pi}{3}i}$, is called a hexagonal lattice.

We summarize as follows

Theorem 1.5.4. *If we define* $\mathrm{Aut}_0(X) = \{$*automorphism* $F : X \to X \mid F([0]) = [0]\}$, *then*

$$\mathrm{Aut}_0(X) = \begin{cases} \mathbb{Z}_4, & L \textit{ is a square lattice} \\ \mathbb{Z}_6, & L \textit{ is a hexagonal lattice} \\ \mathbb{Z}_2, & \textit{otherwise} \end{cases}$$

So we have

$$\mathrm{Aut}(X) = \mathrm{Aut}_0(X) \ltimes \{T_{[a]} \mid [a] \in X\}$$

In particular, we have

$$\dim_{\mathbb{C}} \mathrm{Aut}(X) = 1$$

Remark 1.5.5. As we can see, the three cases above are not isomorphic to each other, since Riemann surfaces which are isomorphic to each other have the same automorphism group. This is the first example we meet, surfaces with the same topological structure but different complex structures.

It's worth mentioning that automorphism groups of higher genus are very small.

Theorem 1.5.6. *For genus ≥ 2, the automorphism groups are finite.*

1.6. **Moduli space of complex torus.** Since the above results show some different complex structures on a topological torus, we want to ask: How many different complex structures are there on a topological torus? And in general, how many different complex structures are there on a given Riemann surfaces? That leads to the conception of Moduli space.

For any lattice $M = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, if we let $\gamma = \frac{1}{\omega_1}$, then

$$L = \gamma M = \mathbb{Z} + \mathbb{Z}\frac{\omega_2}{\omega_1}$$

So it suffices to consider the complex torus of form $X_\tau = \mathbb{C}/L_\tau$, where

$$L_\tau = \mathbb{Z} + \mathbb{Z}\tau$$

Since $L_{-\tau} = L_\tau$, so we can assume that $\operatorname{Im}\tau > 0$.

Let

$$\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$$

Given $\tau, \tau' \in \mathbb{H}$, we want to ask when $X_{\tau'}$ and $X_{\tau'}$ give the same complex structure on a topological torus. It is equivalent to that there exists $\gamma \in \mathbb{C}$, such that

$$\gamma L_\tau = L_{\tau'}$$

i.e.

$$\mathbb{Z}\gamma + \mathbb{Z}\gamma\tau = \mathbb{Z} + \mathbb{Z}\tau'$$

So there exists $a, b, c, d \in \mathbb{Z}$, such that

$$\begin{cases} \gamma = c + d\tau' \\ \gamma\tau = a + b\tau' \end{cases}$$

moreover,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible since it's a base change and its inverse matrix must have integral entries, so its determinant must be ± 1.

So, it's the famous Möbius transformation

$$\tau = \frac{a + b\tau'}{c + d\tau'}$$

Since we require both γ and γ have positive imaginary part, we compute as follows

$$\tau = \frac{(a\tau' + b)(c\bar{\tau}' + b)}{|c\tau' + d|^2} \implies \operatorname{Im}\tau = \frac{ad - bc}{|c\tau' + d|^2}\operatorname{Im}\tau'$$

So we need $A \in \mathrm{SL}_2(\mathbb{Z})$.

We summarize as follows

Theorem 1.6.1. *$X_\tau \cong X_{\tau'}$ if and only if there exists $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that*

$$\tau = \frac{a + b\tau'}{c + d\tau'}$$

For any $A \in \mathrm{SL}_2(\mathbb{Z})$, it induces a map from $\mathbb{H}$ to itself, defined by

$$\tau \mapsto \frac{a + b\tau}{c + d\tau}$$

In fact, it's an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. Furthermore, A and $-A$ gives the same action. So the above theorem can be rephrased as follows

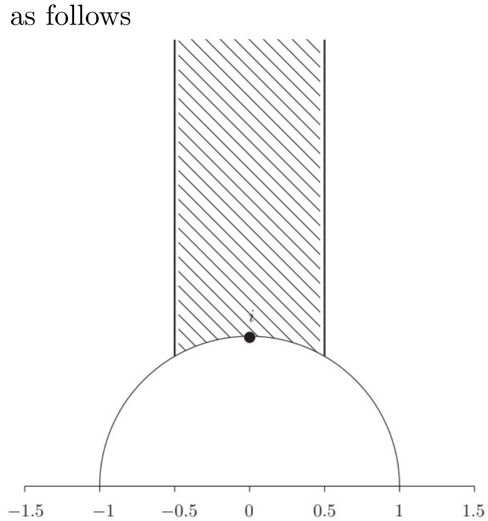
Theorem 1.6.2. *The set of isomorphism classes of complex structure on complex torus is $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z}) = \mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$,where $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm \mathrm{I}_2\}$.*

Remark 1.6.3. As we have shown, all complex structures on a complex torus $\mathbb{C}/L$ are $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$. In fact, it contains all possible complex structure on surface with genus 1[4], called the moduli space of surface with genus 1, denoted by $\mathscr{M}(1)$.

So we wonder what's the fundamental domain[5] of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{H}$. We will show that it is

$$D = \{\tau \in \mathbb{C} \mid |\tau| \geq 1, -\frac{1}{2} \leq \operatorname{Re}\tau \leq \frac{1}{2}\}$$

and can be drawn as follows



Theorem 1.6.4. *D is the fundamental domain of $\mathrm{PSL}_2(\mathbb{Z})$ action on $\mathbb{H}$.*

[4]We will show this later, using Abel's theorem.

[5]Fundamental domain is usually defined as a set of representatives for the orbits. However, definition we give here is sometimes called a fundamental domain with boundary.

Definition 1.6.5. *Consider the following two matrices*

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Theorem 1.6.6. $\mathrm{SL}_2(\mathbb{Z})$ *is generated by* S *and* T*.*

Remark 1.6.7. Before proving the theorem, let's see what's the action of S and T on $\mathbb{H}$

$$S : \tau = re^{i\theta} \mapsto -\frac{1}{\tau} = \frac{1}{\tau} e^{i(\pi-\theta)}$$

So S preserves the upper semicircle, and $S(i) = i$.

$$T : \tau \mapsto \tau + 1$$

So T is just a translation by 1.

Proof. Proof of Theorem1.6.6 and Theorem1.6.7

Let Γ be the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by S and T. We need to show $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

Step one: For any $z \in \mathbb{H}$, there exists $A \in \Gamma$ such that $A(z) \in D$. Fix $z \in \mathbb{H}$, from the relation between $\operatorname{Im} A(z)$ and $\operatorname{Im} z$

$$\operatorname{Im}(A(z)) = \frac{1}{|cz+d|^2} \operatorname{Im}(z)$$

we have that $\{\operatorname{Im} A(z) \mid A \in \mathrm{SL}_2(\mathbb{Z})\}$ is a bounded set. Since $\mathrm{SL}_2(\mathbb{Z})$ is discrete, there exists $w \in \{A(z) \mid A \in \Gamma\}$ such that

$$\operatorname{Im} w \geq \operatorname{Im} A(z), \quad \forall A \in \Gamma$$

Since the transition by T doesn't change the imaginary part of w, so we may assume w such that

$$-\frac{1}{2} \leq \operatorname{Re} w < \frac{1}{2}$$

We claim $w \in D$ to finish step one. It suffices to show $|w| \geq 1$. If not, write $w = re^{i\theta}, r < 1, 0 < \theta < \pi$. Then $S(w) = \frac{1}{r} e^{i(\pi-\theta)}$, so we have

$$\operatorname{Im} S(w) = \frac{1}{r} \sin(\pi - \theta) > r \sin(\pi - \theta) = \operatorname{Im} w$$

a contradiction to the choice of w.

Step two: Assume $z, w \in D$, and there exists $A \in \mathrm{SL}_2(\mathbb{Z})$ such that $w = A(z)$, then

1. $A \in \Gamma$;
2. if $z \neq w$, then z and w lies in the boundary of D;
3. if $z = w \in D \backslash \partial D$, then $A = \pm \mathrm{I}_2$.

We may assume $\operatorname{Im} w \geq \operatorname{Im} z$, and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \geq 0$, then we have

$$w = \frac{az+b}{cz+d}$$

and the requirement on imaginary part implies that

$$|cz+d|\leq 1$$

Since $z\in D$, then $\operatorname{Im} z\geq \frac{\sqrt{3}}{2}$. Then

$$1\geq |cz+d|\geq \operatorname{Im}(cz+d)=c\operatorname{Im} z\geq c\frac{\sqrt{3}}{2}$$

then c must be 0 or 1.

If $c=0$, then $\det A=ad=1$, then $a=d=\pm 1$. Replacing A by $-A$, we may assume $A=\begin{pmatrix} 1 & b\\ 0 & 1\end{pmatrix}$, then $w=A(z)=z+b\in W$, then $b=0,\pm 1$. If $b=0$, then $A=\mathrm{I}_2$. We will see later it's the only case that $z=w\in D\backslash\partial D$. If $b=\pm 1$, then $A=T$ or T^{-1}, then $A\in\Gamma$. And

$$|\operatorname{Re} z|=|\operatorname{Re} w|=\frac{1}{2}$$

implies $z=w\in\partial D$.

If $c=1$, then

$$1\geq |cz+d|=|z+d|=\sqrt{(\operatorname{Re} z+d)^2+(\operatorname{Im} z)^2}\geq\sqrt{(\operatorname{Re} z+d)^2+\frac{4}{3}}$$

Since $|\operatorname{Re} z|\leq\frac{1}{2}$, then $d=0,\pm 1$. If $d=0$, then $A=\begin{pmatrix} a & -1\\ 1 & 0\end{pmatrix}=T^aS\in\Gamma$. And

$$1\geq |cz+d|=|z|$$

then $z\in\partial D$, since $z\in D$. Then $w=A(z)\in\partial D$. If $d=1$, then $1\geq |cz+d|=|z+1|$, then $z=\rho=-\frac{1}{2}+\frac{\sqrt{3}}{2}i\in\partial D$. Then $A=\begin{pmatrix} a & a-1\\ 1 & 1\end{pmatrix}$, and

$$A(z)=\frac{az+a-1}{z+1}=a-\frac{1}{z+1}=a-\frac{1}{2}+\frac{\sqrt{3}}{2}i\in D$$

then $a=0,1$, so $A(z)\in\partial D$. The case $d=-1$ is similar to $d=1$.

Step three: $\Gamma=\mathrm{SL}_2(\mathbb{Z})$. For $z\in D\backslash\partial D$. For any $B\in\mathrm{SL}_2(\mathbb{Z})$. By step one, there exists $A\in\Gamma$ such that $AB(z_0)=A(B(z_0))\in D$, then by step two, we have $AB(z_0)=z_0$, and $AB=\pm\mathrm{I}_2$, i.e. $B=\pm A^{-1}\in\Gamma$. So we have $\Gamma=\mathrm{SL}_2(\mathbb{Z})$. □

Remark 1.6.8. Topologically we have

$$\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})\cong S^2\backslash\{\mathrm{pt}\}$$

and we have

$$\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})\cong\mathbb{C}$$

as Riemann surface.

2. Differential forms

2.1. **Definitions.** Recall what we've learnt in complex analysis. Consider $\{z, \overline{z}\}$ as a coordinate on $\mathbb{C}$, smooth 1-forms on $\mathbb{C}$ have the form

$$f(z, \overline{z})\mathrm{d}z + g(z, \overline{z})\mathrm{d}\overline{z}$$

Where f, g are smooth functions.

Let $z = T(w)$ be a holomorphic change of coordinate, then

$$\frac{\partial z}{\partial \overline{w}} = \frac{\partial \overline{z}}{\partial w} = 0, \quad \frac{\partial \overline{z}}{\partial \overline{w}} = \overline{\frac{\partial z}{\partial w}} = \overline{T'(w)}$$

then we have

$$\begin{aligned} \mathrm{d}z &= \frac{\partial z}{\partial w}\mathrm{d}w + \frac{\partial z}{\partial \overline{w}}\mathrm{d}\overline{w} = T'(w)\mathrm{d}w \\ \mathrm{d}\overline{z} &= \overline{T'(w)}\mathrm{d}\overline{w} \end{aligned}$$

A form $f\mathrm{d}z$ is called a $(1,0)$-form, and a form $g\mathrm{d}\overline{z}$ is called a $(0,1)$-form, and these concepts are invariant under the change of holomorphic change of coordinate, so we define them on Riemann surfaces.

Let's see deeper why it is independent of the choice of the charts. Since we have $T_p\mathbb{C} \cong \mathbb{C}$, and we identify

$$\frac{\partial}{\partial x} = 1, \quad \frac{\partial}{\partial y} = i$$

then we have J as

$$\begin{aligned} J(\frac{\partial}{\partial x}) &= \frac{\partial}{\partial y} \\ J(\frac{\partial}{\partial y}) &= -\frac{\partial}{\partial x} \end{aligned}$$

this induces linear map

$$J : T_p^*\mathbb{C} \to T_p^*\mathbb{C}$$

given by

$$\langle J(\theta), v\rangle = \langle \theta, J(v)\rangle$$

where $\theta \in T_p^*\mathbb{C}, v \in T_p\mathbb{C}$.

If we want to see what is $J(\mathrm{d}x)$, then

$$\begin{aligned} \langle J(\mathrm{d}x), \frac{\partial}{\partial x}\rangle &= \langle \mathrm{d}x, J(\frac{\partial}{\partial x})\rangle = 0 \\ \langle J(\mathrm{d}x), \frac{\partial}{\partial y}\rangle &= \langle \mathrm{d}x, J(\frac{\partial}{\partial y})\rangle = -1 \end{aligned}$$

so we have $J(\mathrm{d}x) = -\mathrm{d}y$, similarly we have $J(\mathrm{d}y) = \mathrm{d}x$. So as we can see, there is no eigenvector of J in $T_p^*\mathbb{C}$, but if we consider $T_p^*\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, and

$$\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$$

then we have

$$J(\mathrm{d}z) = J(\mathrm{d}x) + iJ(\mathrm{d}y) = -\mathrm{d}y + i\mathrm{d}x = iJ(\mathrm{d}z)$$

So, our $(1,0)$-form defined above just the eigenvectors of J with respect to the eigenvalue i, and $(0,1)$-form defined above just the eigenvectors of J with respect to the eigenvalue $-i$.

So $(1,0)$-form and $(0,1)$-form are independent of the choice of charts, since J is independent.

And what's more, we have the dual of $\mathrm{d}z$ and $\mathrm{d}\overline{z}$.

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) \in T_p\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$$
$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}) \in T_p\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$$

and J acts on them as follows

$$J(\frac{\partial}{\partial z}) = i\frac{\partial}{\partial z}$$
$$J(\frac{\partial}{\partial \overline{z}}) = -i\frac{\partial}{\partial \overline{z}}$$

For a complex function f, we have $f = u + iv$, where u and v are real-valued function, then

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}) + \frac{1}{2}i(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$$

Then we have $\dfrac{\partial f}{\partial \overline{z}} = 0$ is equivalent to the Cauthy-Riemann equations.

Now let's consider what will happen on a Riemann surface X.

Definition 2.1 (differential form)**.** *A differential 1-form θ on X assigns to any local chart $U \xrightarrow{\varphi} V$ a form $f\mathrm{d}z + g\mathrm{d}\overline{z}$, and compatible with the charts.*

Remark 2.2. Compatiblity means if $U' \xrightarrow{\varphi'} V'$ is another local chart, and θ is represented in this chart by

$$s\mathrm{d}w + t\mathrm{d}\overline{w}$$

and let $w = T(z) = \varphi' \circ \varphi^{-1}(z)$, then we have

$$s(T(z), \overline{T(z)})T'(z)\mathrm{d}z + t(T(z), \overline{T(z)})\overline{T'(z)}\mathrm{d}\overline{z} = f\mathrm{d}z + g\mathrm{d}\overline{z}$$

Remark 2.3. Similarly, we can define what is a 2-form on X. That is, a 2-form η on X assigns each local chart a form

$$f\mathrm{d}z \wedge \mathrm{d}\overline{z}$$

and compatible with the charts, i.e. If there is another local chart, and η is represented by

$$g\mathrm{d}w \wedge \mathrm{d}\overline{w}$$

and T is the transition function between two charts, then

$$f\mathrm{d}z\wedge\mathrm{d}\overline{z} = g(T(w), \overline{T(w)})T'(w)\overline{T'(w)}\mathrm{d}z\wedge\mathrm{d}\overline{z} = g(T(w), \overline{T(w)})|T'(z)|^2\mathrm{d}z\wedge\mathrm{d}\overline{z}$$

Since we have the differential form on a Riemann surface, then we define what is a $(1,0)$-form or a $(0,1)$-form, as what we have done.

Definition 2.4 ($(1,0)$ or $(0,1)$-form)**.** *A differential form θ on a Riemann surface is called a $(1,0)$-form, if it can be represented as $f\mathrm{d}z$ locally. Similarly we can define what is a $(0,1)$-form.*

Definition 2.5. *A holomorphic 1-form θ is a differential 1-form which can be locally represented as $f(z)\mathrm{d}z$, with f is holomorphic; A meromorphic 1-form θ is a differential 1-form which can be locally represented as $f(z)\mathrm{d}z$, with f is meromorphic.*

If f is a function, we can define

$$\mathrm{d}f = \frac{\partial f}{\partial z}\mathrm{d}z + \frac{\partial f}{\partial \overline{z}}\mathrm{d}\overline{z}$$

so we define

$$\partial f := \frac{\partial f}{\partial z}\mathrm{d}z$$
$$\overline{\partial} f := \frac{\partial f}{\partial \overline{z}}\mathrm{d}\overline{z}$$

For a 1-form θ, locally given by

$$\theta = f\mathrm{d}z + g\mathrm{d}\overline{z}$$

we have

$$\mathrm{d}\theta = \mathrm{d}f \wedge \mathrm{d}z + \mathrm{d}g \wedge \mathrm{d}\overline{z} = \frac{\partial f}{\partial \overline{z}}\mathrm{d}\overline{z} \wedge \mathrm{d}z + \frac{\partial g}{\partial z}\mathrm{d}z \wedge \mathrm{d}\overline{z} = (\frac{\partial g}{\partial z} - \frac{\partial f}{\partial \overline{z}})\mathrm{d}z \wedge \mathrm{d}\overline{z}$$

so we define

$$\partial\theta := \partial g \wedge \mathrm{d}\overline{z}$$
$$\overline{\partial}\theta := \overline{\partial} f \wedge \mathrm{d}z$$

Theorem 2.6. *For the exterior differential defined above, we have*
1. $\mathrm{d}^2 = \partial^2 = \overline{\partial}^2 = 0$.
2. $\partial\overline{\partial} = -\overline{\partial}\partial$.
3. *A $(1,0)$-form θ is holomorphic is equivalent to $\overline{\partial}\theta = 0$, and is also equivalent to $\mathrm{d}\theta = 0$.*
4. $\mathrm{d}, \partial, \overline{\partial}$ *satisfy the Leibniz rule.*

Remark 2.7. The third property implies that a $(1,0)$-form is a holomorphic form is equivalent to it's a closed form.

If X and Y are two Riemann surface, and $F : X \to Y$ is a holomorphic map, then we can pullback differential forms on Y to those on X, defined as follows.

Let (U_1, φ_1) be a local chart of X and (U_2, φ_2) be a local chart of Y, such that $F(U_1) \subseteq U_2$, and let $w = T(z) = \varphi_2 \circ F \circ \varphi_1^{-1}(z)$.

Then we define pullback F^*

$$F^*(f\mathrm{d}w + g\mathrm{d}\overline{w}) = f(T(z), \overline{T(z)})T'(z)\mathrm{d}z + g(T(z), \overline{T(z)})\overline{T'(z)}\mathrm{d}\overline{z}$$
$$F^*(f\mathrm{d}w \wedge \mathrm{d}\overline{w}) = f(T(z), \overline{T(z)})|T'(z)|^2\mathrm{d}z \wedge \mathrm{d}\overline{z}$$

Furthermore, it's easily to check F^* commutes with $\mathrm{d}, \partial, \overline{\partial}$.

If we have a differential form, then we can integral it. Let θ be a 1-form on X, and γ be a piecewise smooth curve on X, write $\gamma = \gamma_1 \cup \cdots \cup \gamma_n$, each γ_i lies in a local chart (U_i, φ_i).

Then we can define

$$\int_\gamma \theta = \sum_{i=1}^n \int_{\gamma_i} \theta = \sum_{i=1}^n \int_{a_i}^{b_i} \{f(z_i, \overline{z_i}) z_i'(t) + g(z_i, \overline{z_i}) \overline{z_i'(t)}\} \mathrm{d}t$$

if θ is locally given by

$$f(z_i, \overline{z_i}) \mathrm{d}z_i + g(z_i, \overline{z_i}) \mathrm{d}\overline{z_i}$$

and z_i is $\varphi_i \circ \gamma_i : [a_i, b_i] \to \varphi(U_i)$.

Similarly we can integral an 2-form on a reigon D on X. If η is a 2-form and D is a region on X. Write $D = D_1 \cup \cdots \cup D_n$ such that each D_i lies in a local chart (U_i, φ_i).

Note that

$$\mathrm{d}z_i \wedge \mathrm{d}\overline{z_i} = (\mathrm{d}x_i + i\mathrm{d}y_i) \wedge (\mathrm{d}x_i - i\mathrm{d}y_i) = -2i \mathrm{d}x_i \wedge \mathrm{d}y_i$$

If η is given locally by

$$f(z_i, \overline{z_i}) \mathrm{d}z_i \wedge \mathrm{d}\overline{z_i}$$

then we can define

$$\int_D \eta = \sum_{j=1}^n \int_{D_j} \eta = \sum_{j=1}^n \int_{\varphi_j(D_j)} (-2i) f(x_j + iy_j, x_j - iy_j) \mathrm{d}x_j \wedge \mathrm{d}y_j$$

And we have a famous theorem

Theorem 2.8 (Stokes)**.** *If D is a compact reigon and ∂D is piecewise smooth, then*

$$\int_D \mathrm{d}\theta = \int_{\partial D} \theta$$

where θ is a smooth 1-form.

2.2. **Order of meremorphic function.** Let X be a meromorphic function on a Riemann surface X, for $p \in X$, we choose a local coordinate z centered at p.

We can define the Laurend series of f at p by consider the Laurent series of $f \circ \varphi^{-1}(z)$ as

$$f(z) = \sum_{n=m}^{\infty} c_n z^n, \quad c_m \neq 0$$

So we define the order of f at p is m, denoted by $\mathrm{ord}_p(f)$.

Lemma 2.9. $\mathrm{ord}_p(f)$ *is independent of the choice of local coordinate.*

Proof. Clearly f corresponds to a holomorphic map $F : X \to S^2$. If p is a zero point of f, then $\mathrm{ord}_p(f) = \mathrm{mult}_p(F)$; and if p is a pole of f, then $\mathrm{ord}_p(f) = -\mathrm{mult}_p(f)$. □

Let θ be a meromorphic 1-form on X, in local coordinate z centered at p, we can write

$$\theta = f(z)\mathrm{d}z$$

so we can define $\mathrm{ord}_p(\theta) = \mathrm{ord}_p(f)$, and clearly it's independent of the choice of local coordinate.

However, the order of f lose some information given by the coefficient of its Laurent series. We want to keep track coefficient which are invariant under the change of local coordinate. Luckily, there exists such a coefficient, that is c_{-1}.

Definition 2.10 (residue)**.** *We define the residue of a meromorphic* 1*-form* θ *by* $\mathrm{Res}_p(\theta) = c_{-1}$

Lemma 2.11. $\mathrm{Res}_p(\theta)$ *is independent of local coordinate.*

This follows from the following lemma.

Lemma 2.12. *Let* D *be any compact region in* X *with* $p \in D \backslash \partial D$*,* ∂D *is piecewise smooth, and* θ *can not have pole in* $D\backslash\{p\}$*, then*

$$\mathrm{Res}_p \theta = \frac{1}{2\pi i}\int_{\partial D} \theta$$

Proof. Choose $D' \subset D$ such that $p \in D'\backslash\partial D'$, $\partial D'$ is smooth, and D' is contained in a local chart with local coordinate z centered at p. In this local chart, we can write θ as

$$\theta = (\sum_{n-m}^{\infty} c_n)\mathrm{d}z$$

Consider

$$\int_{\partial D}\theta - \int_{\partial D'}\theta \overset{\text{Stokes}}{=} \int_{D\backslash D'} \mathrm{d}\theta = 0$$

The last equality holds since θ is holomorphic in $D\backslash D'$. So our origin integral becomes more easy to compute, since D' is very good. We have

$$\int_{\partial D}\theta = \int_{\partial D'}\theta = \int_{\varphi(\partial D')} (\sum_{n=m}^{\infty} c_n z^n)\mathrm{d}z = 2\pi i c_{-1} = 2\pi i \,\mathrm{Res}_p(\theta)$$

$\square$

Theorem 2.13 (residue theorem)**.** *Let* X *be a compact Riemann surface, and* θ *is a meromorphic* 1*-form on* X*, then*

$$\sum_{p\in X} \mathrm{Res}_p(\theta) = 0$$

Proof. Since X is compact, then θ can only have finite poles, denoted by $p_1, \ldots, p_k$. And for each $1 \leq j \leq k$, we can choose a neighborhood D_j of p_j which plays the role of D' in Lemma 2.12. Then

$$\sum_{p\in X}\mathrm{Res}(\theta) = \sum_{j=1}^{k}\mathrm{Res}_{p_j}(\theta) = \frac{1}{2\pi i}\sum_{j=1}^{k}\int_{\partial D_j}\theta = \frac{1}{2\pi i}\int_{D\backslash\bigcup_{j=1}^{k} D_j}\mathrm{d}\theta = 0$$

□

2.3. **Divisors.** Given function $D : X \to \mathbb{Z}$, we define its support $\operatorname{supp}(D) = \{x \in X \mid D(x) \neq 0\}$.

Definition 2.14 (divisors). *A divisor on X is a function $D : X \to \mathbb{Z}$ such that $\overline{\operatorname{supp}(D)}$ is discrete.*

Remark 2.15. Usually, we write a divisor D as a formal sum

$$D = \sum_{p \in X} D(p) \cdot p$$

In particular, if X is compact, then the above formal sum is a finite sum.

We use $\operatorname{Div}(X)$ to denote the set of all divisors on X. In fact, $\operatorname{Div}(X)$ is an abelian group.

Definition 2.16 (degree). *If X is compact, we can define the degree of a divisor D as*

$$\deg(D) = \sum_{p \in X} D(p)$$

Remark 2.17. So degree defines a map $\deg : \operatorname{Div}(X) \to \mathbb{Z}$. In fact, it's a group homomorphism. So it's natural to ask what's the kernel of this homomorphism

$$\operatorname{Div}_0(X) := \operatorname{Ker} \deg = \{D \in \operatorname{Div}(X) \mid \deg D = 0\}$$

is a normal subgroup of $\operatorname{Div}(X)$.

Now let's how to construct a divisor.

Example 2.18 (principal divisor). If $f \not\equiv 0$ is a meromorphic function on X, define

$$\operatorname{div}(f) := \sum_{p \in X} \operatorname{ord}_p(f) \cdot p$$

called a principal divisor on X. And use $\operatorname{PDiv}(X)$ to denote the set of all principal divisors on X.

Lemma 2.19. *we have the following properties of principal divisor*

1. $\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$
2. $\operatorname{div}(f/g) = \operatorname{div}(f) - \operatorname{div}(g)$
3. $\operatorname{div}(1/f) = -\operatorname{div}(f)$

Proof. Clear. □

Corollary 2.20. $\operatorname{PDiv}(X)$ *is a subgroup of* $\operatorname{Div}(X)$.

Lemma 2.21. *If X is compact, $f \not\equiv 0$ is a meromorphic function on X, then*

$$\deg(\operatorname{div}(f)) = 0$$

Proof. Let $F : X \to S^2$ be the holomorphic map induced by f. Use the relation between multiplicity of F and order of f. We have

$$\deg(\mathrm{div}(f)) = \sum_{p\in X} \mathrm{ord}_p(f) = \sum_{\substack{p\in X \\ p \text{ is zero of } F}} \mathrm{mult}_p(F) - \sum_{\substack{p\in X \\ p \text{ is pole of } F}} \mathrm{mult}_p(F) = 0$$

□

Corollary 2.22. *We have*

$$\mathrm{PDiv}(X) \subset \mathrm{Div}_0(X) \subset \mathrm{Div}(X)$$

Example 2.23. Let $f \not\equiv 0$ be a meromorphic function on X, we can define

$$\mathrm{div}_0(f) := \sum_{\substack{p\in X \\ \mathrm{ord}_p(f)>0}} \mathrm{ord}_p(f) \cdot p$$

which is named divisor of zeros. And similarly we can define

$$\mathrm{div}_\infty(f) := -\sum_{\substack{p\in X \\ \mathrm{ord}_p(f)<0}} \mathrm{ord}_p(f) \cdot p$$

which is named divisor of poles. Clearly we have

$$\mathrm{div}(f) = \mathrm{div}_0(f) - \mathrm{div}_\infty(f)$$

Remark 2.24. Since meromorphic 1-form also have the conception of order, so what we have done can be tranlated to meromorphic 1-form θ. So we have $\mathrm{div}(\theta)$, and call it a canonical divisor, later we will see why it's called canonical.

Since we have seen that the degree of a principal divisor is zero, so it's natural to ask the degree of a canonical divisor. However, it may be not zero.

Example 2.25. Let $X = S^2 = \mathbb{C} \cup \{\infty\}$, and consider $\theta = \mathrm{d}z$, where z is the coordinate of $\mathbb{C}$. Clearly $\mathrm{d}z$ is a meromorphic 1-form.

If $p \in \mathbb{C}$, then $\mathrm{ord}_p(\theta) = 0$, otherwise $p = \infty$, then consider $w = 1/z$, which is a local coordinate of ∞ centered at ∞. In this new coordinate, we have

$$\theta = -\frac{1}{w^2}\mathrm{d}w$$

so we have $\mathrm{ord}_\infty(\theta) = -2$. So in this quite simple example, we have

$$\deg(\mathrm{div}(\theta)) = -2 \neq 0$$

Example 2.26. Let $X = S^2 = \mathbb{C} \cup \{\infty\}$, and consider

$$f(z) = c\prod_{j=1}^{n}(z - \lambda_i)^{a_j}, \quad c \neq 0, a_i \in \mathbb{Z}, \lambda_j \neq \lambda_j \in \mathbb{C}$$

and let $\theta = f(z)\mathrm{d}z$, is a meromorphic 1-form. So we have

$$\mathrm{ord}_{\lambda_j}(\theta) = a_j, \quad \forall j = 1, 2, \dots, n$$

And for $p = \infty$, and consider $w = 1/z$, so we have

$$\theta = c\prod_{j=1}^{n}(\frac{1}{w} - \lambda_j)^{a_j}(-\frac{1}{w^2})\mathrm{d}w$$

so

$$\mathrm{ord}_\infty(\theta) = -2 - \sum_{j=1}^{n} a_j$$

Surprisingly we have

$$\deg(\mathrm{div}(\theta)) = \sum_{j=1}^{n} a_j - 2 - \sum_{j=1}^{n} a_j = -2$$

In fact, it's not an coincidence!

Lemma 2.27. *If f is a meromorphic function, and θ is a meromorphic 1-form, then $f\theta$ is also a meromorphic 1-form, and*

$$\mathrm{div}(f\theta) = \mathrm{div}(f) + \mathrm{div}(\theta)$$

Proof. Clear. □

Remark 2.28. Above lemma implies that

$$\mathrm{PDiv}(X) + \mathrm{KDiv}(X) \subset \mathrm{KDiv}(X)$$

where $\mathrm{KDiv}(X)$ is the set of all canonical divisors of X. In particular, $\mathrm{KDiv}(X)$ is not a subgroup of $\mathrm{Div}(X)$.

Conversely, we have

Lemma 2.29. *If θ_1, θ_2 are meromorphic 1-form, then there exists a meromorphic function f such that*

$$\theta_1 = f\theta_2$$

Proof. Locally we have

$$\theta_1 = f_1\mathrm{d}z, \quad \theta_2 = f_2\mathrm{d}z$$

then locally we can define f as f_1/f_2, is a meromorphic function. We need to check it's independent of the choice of local charts. Indeed, things come from the change of charts cancel with each other, since one of them is on the denominator and the other one is on the numerator. □

Corollary 2.30. *The difference of any two canonical divisors is a principal divisor.*

Definition 2.31 (linearly equivalent)**.** *Let $D_1, D_2 \in \mathrm{Div}(X)$ are called linearly equivalent, if $D_1 - D_2$ is a principal divisor, denoted by $D_1 \sim D_2$.*

Example 2.32. Any two canonical divisors are linearly equivalent.

Example 2.33. $\mathrm{div}_0(f)$ is linearly equivalent to $\mathrm{div}_\infty(f)$.

If X is compact, then we can compute the degree of a divisor on it, but the degree of principal divisor is zero, then we have:

Proposition 2.34. *If X is compact, and $D_1 \sim D_2$, then $\deg(D_1) = \deg(D_2)$. In particular, canonical divisors have the same degree.*

So it's natural to ask what is the degree of a canonical divisor?

Lemma 2.35. *If $F : X \to Y$ is a holomorphic map between Riemann surfaces X and Y, and θ is a meromorphic 1-form on Y. For any $p \in X$,*

$$\mathrm{ord}_p(F^*(\theta)) + 1 = (\mathrm{ord}_{F(p)}(\theta) + 1) \cdot \mathrm{mult}_p(F)$$

Proof. Choose local coordinate w centered at p and local coordinate z at $F(p)$ good enough, such that F is given by

$$z = w^n$$

where $n = \mathrm{mult}_p(F)$. Let $k = \mathrm{ord}_{F(p)}(\theta)$, then in local coordinate z, θ is given by

$$\theta = (\sum_{j=k}^{\infty} c_j z^j)\mathrm{d}z, \quad c_k \neq 0$$

so we have

$$\begin{aligned} F^*(\theta) &= (c_k(w^n)^k + \text{higher order terms})nw^{n-1}\mathrm{d}w \\ &= (nc_k w^{n(k+1)-1} + \text{higher order terms})\mathrm{d}w \end{aligned}$$

that is

$$\mathrm{ord}_p(F^*(\theta)) + 1 = (\mathrm{ord}_{F(p)}(\theta) + 1) \cdot \mathrm{mult}_p(F)$$

□

To compute the degree of a canonical divisor, We need the following fact:

Proposition 2.36. *Any compact Riemann surface has a non-constant meromorphic function.*[6]

Proof. See Farkas-Kra: Compact Riemann surface. □

Theorem 2.37. *Let X be a compact Riemann surface with genus g. The degree of any canonical divisor on X is $2g - 2 = -\chi(X)$.*

Proof. Let f be a be a meromorphic function on X, and let $F : X \to S^2$ be the holomorphic map it corresponds to. Let $d = \deg(F)$. Consider canonical divisor $\theta = z\mathrm{d}$ on S^2, and Example 2.25 tells us $\deg(\mathrm{div}(\theta)) = -2$. Then

[6]It's a quite untrivial fact, and in higher dimensions, this proposition fails.

pull it back to X, we have $F^(\theta)$ is a meromorphic 1-form on X, and Lemma 2.35 tell us

$$\begin{aligned}\deg(\operatorname{div}(F^*(\theta))) &= \sum_{p\in X}\operatorname{ord}_p(F^*(\theta)) = \sum_{p\in X}\{(\operatorname{ord}_{F(p)}(\theta)+1)\cdot\operatorname{mult}_p(F)-1\} \\ &= \sum_{p\notin F^{-1}(\infty)}(\operatorname{mult}_p(F)-1) + \sum_{p\in F^{-1}(\infty)}(-\operatorname{mult}_p(F)-1) \\ &= \sum_{p\in X}(\operatorname{mult}_p(F)-1) - 2\sum_{p\in F^{-1}(\infty)}\operatorname{mult}_p(F) \\ &= 2g-2+2d-2\sum_{p\in F^{-1}(\infty)}\operatorname{mult}_p(F) \\ &= 2g-2\end{aligned}$$

The forth equality we used Hurwitz formula. And for the last one, we used $\sum_{p\in F^{-1}(\infty)}\operatorname{mult}_p(F)=d$. □

Since we can pullback a meromorphic 1-form, so we consider how to pullback a divisor.

Let $F: X\to Y$ be a non-constant holomorphic map. For $q\in Y$, we can regard it as a divisor. So we consider how to pullback such a special divisor.

We define

$$F^*(q) := \sum_{p\in F^{-1}(q)}\operatorname{mult}_p(F)\cdot p$$

After this, we can define how to pullback a general divisor as follows: For any $D\in\operatorname{Div}(X)$, we write

$$D=\sum_{q\in Y}n_q\cdot q$$

then

$$F^*(D)=\sum_{q\in Y}n_qF^*(q)$$

and we can compute the degree of it, for an example

$$\deg(F^*(q)) = \sum_{p\in F^{-1}(q)}\operatorname{mult}_p(F) = \deg(F)$$

so we have

$$\deg(F^*(D)) = \sum_{q\in Y}n_q\deg(F^*(q)) = \deg(F)\deg(D)$$

since deg is a group homomorphism. What a beautiful result!

Lemma 2.38. *For pullback, we have the following properties:*
1. $F^*:\operatorname{Div}(Y)\to\operatorname{Div}(X)$ *is a group homomorphism.*
2. $F^*(\operatorname{PDiv}(Y))\subset\operatorname{PDiv}(X)$.

Proof. The first statement is clear. For the second, let $f \not\equiv 0$ be a meromorphic 1-form on Y. Then

$$F^*(\operatorname{div}(f)) = \operatorname{div}(f \circ F)$$

Indeed, for any $p \in X$, we have

$$F^*(\operatorname{div}(f))(p) = \operatorname{mult}_p(F)\operatorname{div}(f)(F(p)) = \operatorname{mult}_p(F)\operatorname{ord}_{F(p)}(f) = \operatorname{ord}_p(f \circ F) = \operatorname{div}(f \circ F)(p)$$

□

Corollary 2.39. *If $D_1 \sim D_2$ on Y, then $F^*(D_1) \sim F^*(D_2)$ on X.*

Definition 2.40 (ramification divisor)**.** *For a holomorphic map $F : X \to Y$, we can define a divisor R_F as*

$$R_F := \sum_{p \in X} (\operatorname{mult}_p - 1) \cdot p$$

called ramification divisor.

Remark 2.41. This divisor is well-defined since we already know the set of ramification points is discrete. And for a compact Riemann surface, the degree of it is

$$\begin{aligned} \deg(R_F) &= \sum_{p \in X} (\operatorname{mult}_p(F) - 1) \\ &= \text{total branch number of } F \end{aligned}$$

Recall Hurwitz formula, it tells

$$2\operatorname{genus}(X) - 2 = (2\operatorname{genus}(Y) - 2)\deg(F) + \sum_{p \in X} (\operatorname{mult}_p(F) - 1)$$

Let θ be any non-zero meromorphic 1-form on Y, then

$$\deg(\operatorname{div}(\theta)) = 2\operatorname{genus}(Y) - 2$$

and

$$\deg(\operatorname{div}(F^*\theta)) = 2\operatorname{genus}(X) - 2$$

so we can rephrased Hurwitz formula as follows

$$\begin{aligned} \deg(\operatorname{div}(F^*(\theta))) &= \deg(\operatorname{div}(\theta))\deg(F) + \deg(R_F) \\ &= \deg(F^*(\operatorname{div}(\theta))) + \deg(R_F) \end{aligned}$$

So the order of pullback and take divisor of a meromorphic 1-fom really matters, when F is ramified. However, for a function, such thing won't happen, since

$$F^*(\operatorname{div}(f)) = \operatorname{div}(f \circ F) = \operatorname{div}(F^*(f))$$

Corollary 2.42. *If $R_F \neq 0$, then the pullback of a canonical divisor may not be canonical*[7]*, i.e.*

$$\operatorname{div}(F^*(\theta)) \neq F^*(\operatorname{div}(\theta))$$

[7] In fact, we have $\operatorname{div}(F^*(\theta)) = F^*(\operatorname{div}(\theta)) + R_F$

We already know that the degree of a principal divisor is zero, so we wonder if the degree of a divisor is zero, will it be a principal divisor? Unfortunately, this conjecture fails for general cases, but for complex sphere, it is true.

Theorem 2.43. *Let $D \in \mathrm{Div}(S^2)$, then*

$$D \in \mathrm{PDiv}(S^2) \Longleftrightarrow \deg(D) = 0$$

Proof. Take a divisor with zero degree, write it as

$$D = \sum_{j=1}^{n} n_j \cdot \lambda_j + n_\infty \cdot \infty, \quad \lambda_j \in \mathbb{C}$$

If $\deg(D) = 0$, then $n_\infty = -\sum_{j=1}^{n} \lambda_j$. Let $f = \prod_{j=1}^{n}(z - \lambda_j)^{n_j}$, then $\mathrm{div}(f) = D$. □

Remark 2.44. In fact, the converse of above theorem still holds.

Corollary 2.45. *Two divisors D_1, D_2 on S^2 are linearly equivalent if and only if* $\deg(D_1) = \deg(D_2)$.

Corollary 2.46. *Any two points on S^2 are linearly equivalent as divisors.*

Since pullback preserves linearly equivalence. Let f be a meromorphic function on X, and $F : X \to S^2$ is the holomorphic map which corresponds to f. For any two points $p, q \in S^2$, then $F^*(p) \sim F^*(q)$ on X as divisors.

In particular, we recover a fact we already know $\mathrm{div}_0(f) = F^*(0) \sim F^*(\infty) = \mathrm{div}_\infty(f)$.

Now we will give a partial order on divisors.

Definition 2.47 (effective divisors)**.** *For $D \in \mathrm{Div}(X)$, we say $D \geq 0$, if $D(p) \geq 0$ for all $p \in X$, and call it effective divisors*[8]*. Similarly we define $D > 0$ if $D \geq 0$ and $D \neq 0$.*

Remark 2.48. For any divisor D, we can write it as a difference of two effective divisors. There are two many ways, one of the easiest is to write

$$D = \sum_{\substack{p \in X \\ D(p) \geq 0}} D(p) \cdot p - \sum_{\substack{p \in X \\ D(q) < 0}} (-D(p)) \cdot p$$

Definition 2.49 (partial order of divisors)**.** *For two divisors D_1, D_2, we say $D_1 \geq D_2$ if $D_1 - D_2 \geq 0$.*

2.4. **Spaces of $L(D)$.** From now on, we only consider compact Riemann surface X. Let $\mathcal{M}(X)$ denote the set of all meromorphic functions on X.

Given $D \in \mathrm{Div}(X)$, we can define such a set

$$L(D) := \{f \in \mathcal{M}(X) \mid \mathrm{div}(f) + D \geq 0\}$$

Convention: if $f \equiv 0$, we define $\mathrm{ord}_p(f) = \infty$. So this convention allow us to have $\mathrm{div}(0) \in L(D)$. We make such convention in order to make $L(D)$ to be a complex vector space.

[8]Some authors also call it integral divisors

Remark 2.50. To some extent, $L(D)$ consists of meromorphic functions with poles not too bad, since $\mathrm{ord}_p(f) \geq -D(p)$. If $D(p) = -n < 0$, then p must be a zero of f with order $\geq n$; If $D(p) = n > 0$, then p may be a pole, but its order at least won't be larger than n.

Example 2.51. Consider $L(0)$, by definition, $\mathrm{ord}_p(f) \geq 0$, i.e. f can't have a pole. So f is a holomorphic function. Since X is a compact Riemann surface, then

$$D(0) \cong \mathbb{C}$$

Lemma 2.52. *If $D_1 \leq D_2$ are two divisors, and $f \in L(D_1)$, then $f \in L(D_2)$.*

Proof. Clear. □

Lemma 2.53. *If $\deg(D) < 0$, then $L(D) = \{0\}$.*

Proof. If $f \in L(D)$ and $f \not\equiv 0$, then by definition, we have $\mathrm{div}(f) + D \geq 0$. Take degree we have

$$0 = \deg(\mathrm{div}(f)) \geq -\deg(D) > 0$$

A contradiction. □

Definition 2.54 (complete linear system)**.** *$|D|$ is called the complete linear system of D, where*

$$|D| = \{E \in \mathrm{Div}(X) \mid E \geq 0, E \sim D\}$$

Remark 2.55. Clearly, if $D_1 \sim D_2$, then $|D_1| = |D_2|$. And $\deg(D) < 0$, then $|D| = \varnothing$. Indeed, if $E \in |D|$, then $\deg(E) = \deg(D) < 0$, contradicts to $E \geq 0$.

Let's see the relation between complete linear system $|D|$ and $L(D)$. For $f \in L(D)\backslash\{0\}$, we can define

$$S(f) := \mathrm{div}(f) + D$$

by definition $S(f) \geq 0$, and is linearly equivalent to D, so $S(f) \in |D|$. But S is not injective, since $S(f) = S(\lambda f), \forall \lambda \in \mathbb{C}\backslash\{0\}$. In order to make S to be an injective map, we can consider the projectivization of $L(D)$.

Recall that if we have a complex vector space W, with dimension n. Then we define its projectivization as

$$\begin{aligned} \mathbb{P}(W) :&= W\backslash\{0\}/v \sim \lambda v, \quad \forall v \in W, \lambda \in \mathbb{C}\backslash\{0\} \\ &= \text{Set of all complex 1-dimensional vector subspaces of } W. \end{aligned}$$

$\mathbb{P}(W)$ is called the projectivization of W.

In fact, $\mathbb{P}(W) \cong \mathbb{CP}^{n-1}$, is a $(n-1)$-dimensional complex manifold and it is compact.

Then we descend S to projectivization of $L(D)$, that is

$$S : \mathbb{P}(L(D)) \to |D|$$

and it's injective. Furthemore, it's bijective. Indeed, for injectivity, take $f_1, f_2 \in L(D)\backslash\{0\}$ with $S(f_1) = S(f_2)$, then $\operatorname{div}(f_1/f_2) = 0$, that is f_1/f_2 is a holomorphic function, that is f_1/f_2 is constant. So f_1, f_2 are same in $\mathbb{P}(L(D))$, that's injective. For surjectivity, take any $E \in |D|$, then $E = D + \operatorname{div}(f)$, for some meromorphic function f. Since $E \geq 0$, we have $f \in L(D)$. Then we have $S(f) = E$. Summarize as

Lemma 2.56.

$$\begin{aligned} S : \mathbb{P}(L(D)) &\to |D| \\ [f] &\mapsto \operatorname{div}(f) + D \end{aligned}$$

is bijective.

Corollary 2.57. $\dim L(D) \geq 1$ *is equivalent to* $|D| \neq \varnothing$*.*

Proof. Clear, since $\dim L(D) \geq 1$ is equivalent to $\mathbb{P}(L(D)) \neq \varnothing$. □

Lemma 2.58. *If* $D_1 \sim D_2$ *are two divisors, then* $L(D_1) \cong L(D_2)$ *as vector spaces.*

Proof. Since $D_1 \sim D_2$, then there exists a meromorphic function h such that $D_1 = D_2 + \operatorname{div}(h)$. For any $f \in L(D_1)$, then

$$\operatorname{div}(fh) = \operatorname{div}(f) + \operatorname{div}(h) \geq -D_1 + D_1 - D_2 = -D_2$$

so we define such a linear map

$$\begin{aligned} \mu_h : L(D_1) &\to L(D_2) \\ f &\mapsto fh \end{aligned}$$

and $\mu_{h^{-1}} : L(D_2) \to L(D_1)$ is its inverse, so we have $L(D_1) \cong L(D_2)$. □

Corollary 2.59. *If* $D \in \operatorname{PDiv}(X)$*, then* $L(D) \cong L(0) \cong \mathbb{C}$*.*

Similarly, if we let $\mathcal{M}^{(1)}(X)$ to be the set of all meromorphic 1-forms on X. We can define

$$L^{(1)}(D) = \{\omega \in \mathcal{M}^{(1)}(X) \mid \operatorname{div}(\omega) + D \geq 0\}$$

Example 2.60. Consider $L^{(1)}(0)$, similarly we have that it's set of all holomorphic 1-forms, and sometimes is denoted by $\Omega^1(X)$. Not like holomorphic form, there may be many holomorphic 1-forms on X. So $L^{(1)}(0)$ is a quite non-trivial space.

Lemma 2.61. *If* $D_1 \sim D_2$*, we have* $L^{(1)}(D_1) \cong L^{(1)}(D_2)$

Proof. Similar to Lemma 2.58. □

Theorem 2.62. *Let* K *be a canonical divisor on* X*, then for any* $D \in \operatorname{Div}(X)$*, we have*

$$L^{(1)}(D) \cong L(K + D)$$

Proof. By definition, there exists a meromorphic 1-form ω such that $K = \operatorname{div}(\omega)$. For any $f \in L(K+D)$, we have

$$\operatorname{div}(f\omega) = \operatorname{div}(f) + \operatorname{div}(\omega) \geq -(K+D) + K = -D$$

so we have $f\omega \in L^{(1)}(D)$. So we have such a linear map

$$\begin{aligned} \mu_\omega : L(K+D) &\to L^{(1)}(D) \\ f &\mapsto f\omega \end{aligned}$$

clearly μ_ω is injective. Now we need to show it's also surjective. For any $\theta \in L^{(1)}(D)$, then there exists meromorphic function f such that $\theta = f\omega$, it suffices to show $f \in L(K+D)$. Directly compute

$$-D \leq \operatorname{div}(\theta) = \operatorname{div}(f) + \operatorname{div}(\omega) = \operatorname{div}(f) + K \implies \operatorname{div}(f) + (D+K) \geq 0$$

as desired. $\square$

Note that both $\mathcal{M}(X)$ and $\mathcal{M}^{(1)}(X)$ are infinity-dimensional vector spaces. For such spaces, it is always difficult to study. but we may wonder whether $L(D)$ and $L^{(1)}(D)$ are finite-dimensional vector spaces or not, since we have already put some restrictions on it, that is we don't allow such meromorphic functions have too bad poles.

In fact, they're really finite-dimensional, and we can give a relatively nice upper bound of its dimension.

Lemma 2.63. *For any $D \in \operatorname{Div}(X)$, and $p \in X$, then $L(D-p) \subset L(D)$. Furthermore, either $L(D-p) = L(D)$ or $L(D-p)$ has codimension 1 in $L(D)$ holds.*

Proof. Let $n = -D(p)$, and choose a local coordinate z centered at p. For any $f \in L(D)$, the Laurent series of f at p must have the following form

$$cz^n + \text{higher order terms}$$

Define $\alpha : L(D) \to \mathbb{C}$, defined by $f \mapsto c$. If $\alpha \not\equiv 0$, it's a surjective linear map clearly. Claim that $\ker(\alpha) = D(L-p)$. Indeed, if $f \in \ker(\alpha)$, then $\operatorname{ord}_p(f) \geq n+1$, so $\operatorname{ord}_p(f) + D(p) - 1 \geq 0$, that is $f \in L(D-p)$. The converse is similar.

If $\alpha \equiv 0$, then $L(D-p) = L(D)$, otherwise codimension of $L(D-p)$ in $L(D)$ is 1, since $\dim \mathbb{C} = 1$. $\square$

Theorem 2.64. *For any $D \in \operatorname{Div}(X)$, write $D = P - N$ such that $P, N \geq 0$ and $\operatorname{supp}(P) \cap \operatorname{supp}(N) = \varnothing$. Then*

$$\dim L(D) \leq 1 + \deg(P)$$

Proof. Induction on $\deg(P)$. If $\deg(P) = 0$, then $P = 0$, so we have $L(P) \cong \mathbb{C}$. Since $D \leq P$, then $\dim L(D) \leq \dim L(P) = 1 = 1 + \deg(P)$. Assume theorem holds for $\deg(P) = k-1$, and let D be a divisor such that $D = P - N$ with $\deg(P) = k$. Since $\operatorname{supp}(P) \neq \varnothing$, choose $q \in \operatorname{supp}(P)$, then $D - q =$

$(P-q)-N$, then $\operatorname{supp}(D(P-q)) \cap \operatorname{supp}(N) = \varnothing$ and $\deg(D-q) = k-1$, so by induction, we have

$$\dim L(D-q) \leq 1 + \deg(P-q) = 1 + k - 1 = k$$

and by Lemma 2.63, we have

$$\dim L(D) \leq \dim L(D-q) + 1 \leq k + 1 = \deg(P) + 1$$

□

Corollary 2.65. *For any $D \in \operatorname{Div}(X)$, we have $L(D)$ and $L^{(1)}(D)$ are finite-dimensional vector spaces.*

So we wonder how to compute $\dim L(D)$ or $\dim L^{(1)}(D)$, that's what Riemann-Roch theorem will tell us later.

2.5. **Riemann-Roch theorem.** For any point $p \in X$, fix a local coordinate z_p centered at p. We can define

Definition 2.66 (Laurent tail divisor)**.** *A Laurent tail divisor is a formal finite sum*

$$\sum_{p \in X} r_p(z_p) \cdot p$$

where $r_p(z_p)$ is a Laurent polynomial[9] *in z_p.*

Let $T(X)$ be the set of all Laurent tail divisors on X. For any $D \in \operatorname{Div}(X)$, define

$$T[D](X) = \{\sum_p r_p(z_p) \in T(X) \mid \text{highest term of } r_p(z_p) \text{ has degree less than } -D(p) \text{ for all } p \in X\}$$

Consider such divisor map

$$\begin{aligned} \alpha_D : \mathcal{M}(X) &\to T[D](X) \\ f &\mapsto \sum_{p \in X} r_p(z_p) p \end{aligned}$$

where $r_p(z_p)$ is obtained from the Laurent series of f in z_p by cutting off all terms with degree $\geq -D(p)$.

In fact, α_D is a group homomorphism, and the kernel of it is $L(D)$.

Lemma 2.67. $\ker \alpha_D = L(D)$.

Proof. Let $\alpha_D(f) = \sum_p r_p(z_p) p$, then

$$\begin{aligned} f \in L(D) &\iff \operatorname{div}(f) \geq -D \\ &\iff \operatorname{ord}_p(f) \geq -D(p) \\ &\iff r_p(z_p) = 0 \\ &\iff \alpha_D(f) = 0 \end{aligned}$$

□

[9]A Laurent polynomial is $\sum_{n=k}^{m} c_n z^n$, where k may be negative.

So it's natural to ask what's the image of α_D, and that's Mittag-Leffler problem: Given $Z \in T[D](X)$, can we find $f \in \mathcal{M}(X)$ such that $\alpha_D(f) = Z$? In other words, does $Z \in \operatorname{im} \alpha_D$?

We define

$$H^1(D) := \operatorname{coker} \alpha_D = T[D](X) / \operatorname{im} \alpha_D$$

the size of this space measures the failure of solving Mittag-Leffler problem. Use this notation, we have the following exact sequence

$$0 \to L(D) \longrightarrow \mathcal{M}(X) \xrightarrow{\alpha_D} T[D](X) \longrightarrow H^1(D) \to 0$$

It induces short exact sequence

$$0 \to \mathcal{M}(X)/L(D) \xrightarrow{\alpha_D} T[D](X) \to H^1(D) \to 0$$

Given two divisors D_1, D_2 with $D_1 \leq D_2$, define truncation map

$$t = t_{D_2}^{D_1} : T[D_1](X) \to T[D_2](X)$$
$$\sum_p r_p(z_p)p \mapsto \sum_p \widetilde{r_p}(z_p)p$$

where $\widetilde{r_p}(z_p)$ is obtained from $r_p(z_p)$ by cutting off all terms with degree $\geq -D_2(p)$.

Since $D_1 \leq D_2$, then $L(D_1) \subset L(D_2)$, so there exists a canonical map $\Phi : \mathcal{M}(X)/L(D_1) \to \mathcal{M}(X)/L(D_2)$. Then there exists a canonical map $\Psi : H^1(D_1) \to H^1(D_2)$ such that the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{M}(X)/L(D_1) \arrow[r] \arrow[d, "\Phi"] & T[D_1](X) \arrow[r] \arrow[d, "t_{D_2}^{D_1}"] & H^1(D_1) \arrow[r] \arrow[d, "\Psi"] & 0 \\ 0 \arrow[r] & \mathcal{M}(X)/L(D_2) \arrow[r] & T[D_2](X) \arrow[r] & H^1(D_2) \arrow[r] & 0 \end{tikzcd}$$

By snake lemma, we have

$$0 \to \ker \Phi \to \ker t_{D_2}^{D_1} \to \ker \Psi \to \operatorname{coker} \Phi \to \operatorname{coker} t_{D_2}^{D_1} \to \operatorname{coker} \Psi \to 0$$

But clearly we have Φ and $t_{D_2}^{D_1}$ are surjective, that is $\operatorname{coker} \Phi = \operatorname{coker} t_{D_2}^{D_1} = 0$. So we have Ψ is also surjective.

Furthemore, we have the following short exact sequence

$$0 \to \ker \Phi \to \ker t_{D_2}^{D_1} \to \ker \Psi \to 0$$

For Φ, we have

$$\dim \ker \Phi = \dim L(D_2) - \dim L(D_1)$$

and for $t_{D_2}^{D_1}$, we have

$$\ker t_{D_2}^{D_1} = \{\sum_p r_p(z_p) \in T(X) \mid r_p(z_p) = \sum_{k=-D_2(p)}^{-D_1(p)-1} c_n z_p^k\}$$

then we have

$$\dim \ker t_{D_2}^{D_1} = \sum_{p \in X} (-D_1(p) - 1 - (-D_2(p) - 1)) = -\deg(D_1) + \deg(D_2)$$

If we define $H^1(D_1/D_2) := \ker\Psi$, by the property of short exact sequence, we have

$$\begin{aligned}\dim H^1(D_1/D_2) &= \dim\ker t^{D_1}_{D_2} - \dim\Phi\\ &= -\deg(D_1) + \deg(D_2) - \dim L(D_2) + \dim L(D_1)\\ &= (\dim L(D_1) - \deg(D_1)) - (\dim L(D_2) - \deg(D_2))\end{aligned}$$

In fact, $H^1(D)$ is finite-dimensional[10], if we admit this fact, we have

$$\dim H^1(D_1/D_2) = \dim H^1(D_1) - H^1(D_2)$$

Summarize, if $D_1 \leq D_2$, we have

$$\dim L(D_1) - \deg(D_1) - \dim H^1(D_1) = \dim L(D_2) - \deg(D_2) - \dim H^1(D_2)$$

However, we can drop the condition $D_1 \leq D_2$, since for any two divisors D_1, D_2, we can find a divisor D such that $D_1 \leq D, D_2 \leq D$.

In particular, if we let $D_2 = 0$, then we have the first form of Riemann-Roch theorem.

Theorem 2.68 (Riemann-Roch)**.** *For any divisor D, we have*

$$\dim L(D) - \dim H^1(D) = \deg(D) + 1 - \dim H^1(0)$$

However, it's still difficult to compute $H^1(D)$. We will see later Serre duality tells us how to compute it. Serre duality wants to construct a map

$$L^{(1)}(-D) \to H^1(D)^*$$

and prove that it is an isomorphism. Then we will get the dimension of $H^1(D)$.

If we already have Serre duality, then

$$\dim H^1(D)^* = \dim H^1(D) = \dim L^{(1)}(-D) = \dim L(K-D)$$

where K is an canonical divisor. And let $D = K$, then

$$\dim L(K) - 1 = \deg(K) + 1 - \dim L(K) \implies \dim L(K) = \mathrm{genus}(X)$$

So we get the second form of Riemman-Roch.

Theorem 2.69 (Riemann-Roch)**.** *For any divisor, we have*

$$\dim L(D) - \dim L(K-D) = \deg(D) + 1 - \mathrm{genus}(X)$$

Remark 2.70. Note that $L^{(1)}(0) = L(K)$, and $L^{(1)}(0)$ is the set of all holomorphic forms on X, sometimes is denoted by $\Omega(X)$. So we have

$$\dim\Omega(X) = \dim L^{(1)}(0) = \mathrm{genus}(X)$$

A amazing result, since $\Omega(X)$ is defined by an analytic information, but it is in fact a topological information.

Corollary 2.71 (Riemann inequality)**.** $\dim L(D) \geq \deg(D) + 1 - \mathrm{genus}(X)$.

[10]See Miranda for a proof

In fact, Riemann found this inequality and his student Roch made it into an equality. However, in many cases, Riemann inequality is an equality.

Lemma 2.72. *If* $\deg(D) \geq 2\,\mathrm{genus}(X) - 1$*, then the Riemann inequality is an equality.*

Proof. Recall that if $\deg(D) < 0$, then $\dim L(D) = 0$. And note that

$$\deg(K - D) = \deg(K) - \deg(D) = 2\,\mathrm{genus}(X) - 2 - \deg(D)$$

□

Lemma 2.73. *If there exists* $p \in X$ *such that* $\dim L(p) > 1$*, then* X *must be a Riemann sphere.*

Proof. If $\dim L(p) > 1$ for some $p \in X$, there exists a non-constant function $f \in L(p)$. We use F to denote the holomorphic map $F : X \to S^2$ which corresponds to the meromorphic function f. Consider the degree of F, the only possible pole of f is p, since $f \in L(p)$. And p must be a pole of f, since f is non-constant. So $\mathrm{ord}_p(f) = 1$, that is $F^{-1}(\infty) = \{p\}$, and $\deg F = 1$. So F is an isomorphism. □

Corollary 2.74. *Any Riemann surface* X *with genus zero is isomorphic to* S^2*.*

Proof. For any $p \in X$, $\deg(p) = 1 > 2g - 1$, then $\dim L(p) = \deg(p) + 1 - 0 = 2 > 1$. So by Lemma 2.73 X must be a Riemann sphere. □

Corollary 2.75. *Any two complex structures on a topological sphere are same.*

2.6. **Serre Duality.** For $\omega \in L^{(1)}(-D) \subset \mathcal{M}^{(1)}(X)$, we need to define linear map from $H^1(D)$ to $\mathbb{C}$. We first define a residue map as follows

$$\begin{aligned} \mathrm{Res}_\omega : T[D](X) &\to \mathbb{C} \\ \sum_p r_p(z_p)p &\mapsto \sum_p \mathrm{Res}_p(r_p(z_p)\omega) \end{aligned}$$

Now let's see whether this residue map can descend to $H^1(D)$.

Lemma 2.76. *For* $f \in \mathcal{M}(X)$*, we have* $\mathrm{Res}_\omega(\alpha_D(f)) = 0$*.*

Proof. Write Laurent series of f at p as

$$\sum_k a_k z_p^k$$

and ω can be rephrased near p as

$$(\sum_{n=D(p)}^{\infty} c_n z_p^n)\mathrm{d}z_p$$

sum begins from $D(p)$ since $\omega \in L^{(1)}(-D)$. Then

$$\begin{aligned}\mathrm{Res}_p(f\omega) &= \text{coefficient of } z_p^{-1} \text{ in } (\sum_k a_k z_p^k)((\sum_{n=D(p)}^{\infty} c_n z_p^n)\mathrm{d}z_p) \\ &= \sum_{n=D(p)}^{\infty} a_{-n-1}c_n\end{aligned}$$

So only a_k with $k < -D(p)$ can contribute to $\mathrm{Res}_p(f\omega)$. By definition of α_D, we have

$$\mathrm{Res}_p(f\omega) = \mathrm{Res}_p(r_p(z_p)w)$$

where $\alpha_D(f) = \sum_p r_p(z_p)p$. By residue theorem, we have

$$\mathrm{Res}_p(\alpha_D(f)) = \sum_p \mathrm{Res}_p(f\omega) = 0$$

□

So we have a map

$$\mathrm{Res}_\omega : H^1(D) \to \mathbb{C}$$

that is, $\mathrm{Res}_\omega \in H^1(D)^*$. In other words, we have

$$\begin{aligned}\mathrm{Res} : L^{(1)}(-D) &\to H^1(D)^* \\ \omega &\mapsto \mathrm{Res}_\omega\end{aligned}$$

Theorem 2.77 (Serre duality)**.** Res *is an isomorphism.*

Proof. Injectivity. For any $0 \neq \omega \in L^{(1)}(-D)$, we fix $p \in X$ and let $k = \mathrm{ord}_p(\omega) \geq D(p)$. Let

$$Z = \frac{1}{z_p^{k+1}}p \in T[D](X)$$

Near p, we write ω as

$$(\sum_{n=k}^{\infty} c_k z_p^k)\mathrm{d}z_p, \quad c_k \neq 0$$

then

$$\mathrm{Res}_\omega(Z) = c_k \neq 0$$

So $\mathrm{Res}_\omega \neq 0$, that's injectivity.

Surjectivity. It's a long way to prove it, let's make some preparations. For $f \in \mathcal{M}(X), D \in \mathrm{Div}(X)$, we define multiplicative map

$$\begin{aligned}\mu_f = \mu_f^D : T[D](X) &\to T[D - \mathrm{div}(f)](X) \\ \sum_p r_p p &\mapsto \text{suitable truncation of } \sum_p (fr_p)p\end{aligned}$$

Exercise 2.78. If $f \not\equiv 0$, we have μ_f is an isomorphism with inverse $\mu_{\frac{1}{f}}$.

Exercise 2.79. For $f, g \in \mathcal{M}(X), D \in \mathrm{Div}(X)$, we have

$$\mu_f(\alpha_D(g)) = \alpha_{D-\mathrm{div}(f)}(fg)$$

that is

$$\begin{tikzcd}
\mathcal{M}(X) \arrow[r, "f"] \arrow[d, "\alpha_D"] & \mathcal{M}(X) \arrow[d, "\alpha_D - \mathrm{div}(f)"] \\
T[D](X) \arrow[r, "\mu_f"] & T[D](X)
\end{tikzcd}$$

Deduce that

$$\mu_f(\mathrm{im}\,\alpha_D) \subset \mathrm{im}(\alpha_{D-\mathrm{div}(f)})$$

Remark 2.80. For any $\varphi \in H^1(D)^*$, we have

$$T[D](X) \xrightarrow{\text{projection}} H^1(D) \xrightarrow{\varphi} \mathbb{C}$$

we use $\widetilde{\varphi}$ to denote φ compose with projection, $\widetilde{\varphi}$ satisfies

$$\widetilde{\varphi}|_{\mathrm{im}\,\alpha_D} = 0$$

Clearly we can identify such $\widetilde{\varphi} : T[D](X) \to \mathbb{C}$ with $\varphi : H^1(D) \to \mathbb{C}$.

Consider

$$T[D + \mathrm{div}(f)](X) \xrightarrow{\mu_f} T[D](X) \xrightarrow{\widetilde{\varphi}} \mathbb{C}$$

Exercise 2.79 implies that

$$\widetilde{\varphi} \circ \mu_f|_{\mathrm{im}(\alpha_{D+\mathrm{div}(f)})} = 0$$

so by Remark 2.80 $\widetilde{\varphi} \circ \mu_f$ induces a map $H^1(D + \mathrm{div}(f)) \to \mathbb{C}$.

Lemma 2.81. *For any $A \in \mathrm{Div}(X)$, and two non-zero $\varphi_1, \varphi_2 \in H^1(A)^*$, there exists $B \in \mathrm{Div}(X), B > 0$, and non-zero functions $f_1, f_2 \in L(B)$ such that*

$$\widetilde{\varphi}_1 \circ t_A^{A-B-\mathrm{div}(f_1)} \circ \mu_{f_1} = \widetilde{\varphi_2} \circ t_A^{A-B-\mathrm{div}(f_2)} \circ \mu_{f_2}$$

i.e. the following diagram commutes

$$\begin{tikzcd}
& T[A-B-\mathrm{div}(f_1)](X) \arrow[r, "t"] & T[A](X) \arrow[dr, "\widetilde{\varphi}_1"] & \\
T[A-B](X) \arrow[ur, "\mu_{f_1}"] \arrow[dr, "\mu_{f_2}"'] & & & \mathbb{C} \\
& T[A-B-\mathrm{div}(f_2)](X) \arrow[r, "t"] & T[B](X) \arrow[ur, "\widetilde{\varphi}_2"'] &
\end{tikzcd}$$

Proof. Note that for any $g \in \mathcal{M}(X)$, we have

$$t_A^{A-B-\mathrm{div}(f_i)} \circ \mu_{f_i}(\alpha_{A-B}) = t_A^{A-B-\mathrm{div}(f_i)} \alpha_{A-B-\mathrm{div}(f_i)}(f_i g) = \alpha_A(f_i g) \in \mathrm{im}(\alpha_A)$$

Suppose this lemma fails, then for any divisor $B > 0$, the map

$$L(B) \times L(B) \to H^1(A-B)^*$$
$$(f_1, f_2) \mapsto \widetilde{\varphi}_1 \circ t_A^{A-B-\mathrm{div}(f_1)} \circ \mu_{f_1} - \widetilde{\varphi_2} \circ t_A^{A-B-\mathrm{div}(f_2)} \circ \mu_{f_2}$$

is injective. So $2\dim L(B) \leq \dim H^1(A-B)$, by the Riemann-Roch theorem in the first form, we have

$$\begin{aligned} \dim H^1(A-B) &= \dim L(A-B) - \deg(A-B) - 1 + \dim H^1(0) \\ &\leq \dim L(A) - \deg(A) - 1 + \dim H^1(0) + \deg(B) \\ &:= a + \deg(B) \end{aligned}$$

where a is constant. And

$$\dim L(B) = \dim H^1(B) + \deg(B) - 1 + \dim H^1(0) \geq \deg(B) + 1 - \dim H^1(0) := \deg(B) + b$$

where b is constant. So

$$a + \deg(B) \geq \dim H^1(A-B) \geq 2\dim L(B) \geq 2b + 2\deg(B)$$

This inequality can not hold for sufficiently large $\deg(B)$, a contradiction. □

Lemma 2.82. *For $D_1, D_2 \in \mathrm{Div}(X), D_1 \leq D_2$, and $\omega \in L^{(1)}(-D_1)$. If $\mathrm{Res}_\omega : T[D_1](X) \to \mathbb{C}$ satisfies*

$$\mathrm{Res}_\omega |_{\ker t_{D_2}^{D_1}} = 0$$

then $\omega \in L^{(1)}(-D_2)$, and

$$\begin{tikzcd} T[D_1](X) \arrow[rr, "t_{D_2}^{D_1}"] \arrow[dr, "\mathrm{Res}_\omega"'] & & T[D_2](X) \arrow[dl, "\mathrm{Res}_\omega"] \\ & \mathbb{C} & \end{tikzcd}$$

Proof. Assume $\omega \notin L^{(1)}(-D_2)$, then there exists $p \in X$ such that

$$D_1(p) \leq k = \mathrm{ord}_p(\omega) < D_2(p)$$

Let $Z = z_p^{-k-1}p \in T[D_1](X)$, then $t_{D_2}^{D_1}(Z) = 0$, but $\omega = (\sum_{n=k}^{\infty} c_n z_p^n)\mathrm{d}z_p$

$$\mathrm{Res}_\omega(Z) = c_k \neq 0$$

A contradiction, so we have $\omega \in L^{(1)}(-D_2)$. For any $Z = \sum_p r_p(z)p \in T[D_1](X)$, $\mathrm{Res}_\omega(Z)$ only depends on terms in r_p with order $< -D_2(p) \leq -D_1(p)$, this proves that the diagram commutes. □

Now we give the proof of the surjectivity of Res: For any $0 \neq \varphi \in H^1(D)^*$, and let ω be any meromorphic 1-form on X, $K = \mathrm{div}(\omega)$ is a canonical divisor. Choose $A \in \mathrm{Div}(X)$ such that $A \leq D$ and $A \leq K$, so we have $\omega \in L^{(1)}(-A)$.

$$0 \neq \mathrm{Res}_\omega : T[A](X) \to \mathbb{C}$$

which induces an element $\mathrm{Res}_\omega \in H^1(A)^*$. Since $A \le D$, we have

$$T[A](X) \xrightarrow{t_D^A} T[D](X) \xrightarrow{\widetilde{\varphi}} \mathbb{C}$$

and use φ_A to denote the composition of $\widetilde{\varphi}$ and t_D^A. Clearly $\varphi_A \neq 0$. By Lemma 2.81, there exists a divisor $0 < B$ and non-zero functions $f_1, f_2 \in L(B)$ such that

$$\varphi_A \circ t_A^{A-B-\mathrm{div}(f_1)} \circ \mu_{f_1} = \mathrm{Res}_\omega \circ t_A^{A-B-\mathrm{div}(f_2)} \circ \mu_{f_2}$$

For RHS, we have

$$\begin{tikzcd} T[A-B](X) \arrow[r, "\mu_{f_2}"] & T[A-B-\mathrm{div}(f_2)] \arrow[r, "t_A^{A-B-\mathrm{div}(f_2)}"] & T[A](X) \arrow[d, "\mathrm{Res}_\omega"] \\ & & \mathbb{C} \end{tikzcd}$$

And note that

$$\mathrm{div}(\omega) \ge A \ge A - B - \mathrm{div}(f_2)$$
$$\mathrm{div}(f_2\omega) \ge A - B$$

we can add two more arrows in above diagram and this diagram commutes

$$\begin{tikzcd} T[A-B](X) \arrow[r, "\mu_{f_2}"] \arrow[rrd, "\mathrm{Res}_{f_2\omega}"'] & T[A-B-\mathrm{div}(f_2)] \arrow[r, "t_A^{A-B-\mathrm{div}(f_2)}"] \arrow[rd, "\mathrm{Res}_\omega"] & T[A](X) \arrow[d, "\mathrm{Res}_\omega"] \\ & & \mathbb{C} \end{tikzcd}$$

So we have

$$\varphi_A \circ t^{A-B-\mathrm{div}(f_1)} \circ \mu_{f_1} = \mathrm{Res}_{f_2\omega}$$

composing $\mu_{f_1}^{-1}$, we have

$$\varphi_A \circ t_A^{A-B-\mathrm{div}(f_1)} = \mathrm{Res}_{f_2\omega} = \mathrm{Res}_{\frac{f_2}{f_1}\omega}$$

Let $\widetilde{\omega} = \dfrac{f_2}{f_1}\omega$, then

$$T[A-B-\mathrm{div}(f_1)](X) \overset{t_A^{A-B-\mathrm{div}(f_1)}}{\longrightarrow} T[A](X) \xrightarrow{\varphi_A} \mathbb{C}$$

implies that

$$\mathrm{Res}_{\widetilde{\omega}}\big|_{\ker t_A^{A-B-\mathrm{div}(f_1)}} = 0$$

So by Lemma 2.82, we have $\widetilde{\omega} \in L^{(1)}(-A)$, thus $\mathrm{Res}_{\widetilde{\omega}} = \varphi_A$, by same argument, we have $\mathrm{Res}_{\widetilde{\omega}}\big|_{\ker t_D^A} = 0$. Again by Lemma 2.82, we have $\widetilde{\omega} \in L^{(1)}(-D)$ such that $\mathrm{Res}_{\widetilde{\omega}} = \widetilde{\varphi}$, this completes the proof. $\square$

3. Abel theorem

3.1. **Some facts about topology.** Recall that the first homology group of X is denoted by $H_1(X,\mathbb{Z})$, and we have

$$H_1(X,\mathbb{Z}) = \pi_1(X)/[\pi_1(X),\pi_1(X)]$$

So every loop α defines an element $[\alpha]\in H_1(X,\mathbb{Z})$. If $\alpha_1\cup(-\alpha_2)=\partial\sum$, where $\sum\subset X$ is a surface with boundary, then $[\alpha_1]=[\alpha_2]$.

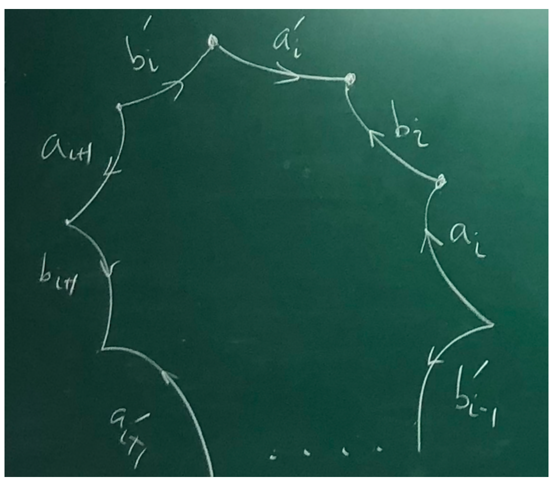
If ω is a smooth closed 1-form on X and $[\alpha_1]=[\alpha_2]$, then Stokes theorem implies that

$$\int_{\alpha_1}\omega - \int_{\alpha_2}\omega = \int_{\sum}\mathrm{d}\omega = 0$$

thus

$$\int_{\alpha_1}\omega = \int_{\alpha_2}\omega$$

If X is a Riemann surface with genus g, then topologically X can be obtained from a polygon P_g with $4g$ edges in the following way



As shown above, all $4g$ vertices of P_g are glued to be one point in X, so a_i, a_i' give loops in X, that is, in $H_1(X,\mathbb{Z})$

$$[a_i]=[a_i'],\quad i=1,\dots,g$$

In general, we have

$$H_1(X,\mathbb{Z}) = \bigoplus_{i=1}^{g}\mathbb{Z}[a_i]\oplus\mathbb{Z}[b_i]\cong\mathbb{Z}^{2g}$$

3.2. **Abel-Jacobi map.** Let $\Omega^1(X)$ be the space of all holomorphic 1-forms on X, and Riemann-Roch theorem tells us that $\Omega^1(X)=L^{(1)}(0)$, with dimension g.

For any $[c] \in H_1(X) := H_1(X, \mathbb{Z})$, we define the following linear map

$$\int_{[c]} : \Omega^1(X) \to \mathbb{C}$$
$$\omega \mapsto \int_c \omega$$

Stokes theorem implies it's well-defined. So we have $\int_{[c]} \in \Omega^1(X)^*$, we call it a period of X. Let Λ to denote the set of all periods of X. Clearly Λ is a subgroup of $\Omega^1(X)^*$, and call it the period group of X

Definition 3.1 (Jacobian)**.** *The Jacobian of X is defined as*

$$\operatorname{Jac}(X) := \Omega^1(X)^*/\Lambda$$

Example 3.2. If genus $g = 0$, then $\operatorname{Jac}(S^1) = \{0\}$, since in this case $\Omega^1(S^1)$ is zero dimensional.

Example 3.3. If genus $g = 1$, and $X = \mathbb{C}/L$, where L is a lattice. We have

$$\Omega^1(X) = \mathbb{C} \cdot \mathrm{d}z \cong \mathbb{C}$$

where $\mathrm{d}z$ is a holomorphic 1-form on $\mathbb{C}$, and L preserves it, so it descends to X. So we have $\Omega^1(X)^* \cong \mathbb{C}$. We can check $\operatorname{Jac}(X) = X$

We will see later Riemann bilinear relation tells us that Λ is a lattice in $\Omega^1(X)^* \cong \mathbb{C}^g \cong \mathbb{R}^{2g}$, here we adimit this fact first. So the quotient $\operatorname{Jac}(X)$ is a compact complex group. More explictly, a g-dimensional complex torus.

In order to connect X and its Jacobian, we need to define Abel map. Fix a base point $p_0 \in X$. For any $p \in X$, choose a path γ_p from p_0 to p. Define

$$\int_{\gamma_p} : \Omega^1(X) \to \mathbb{C}$$
$$\omega \mapsto \int_{\gamma_p} \omega$$

Clearly $\int_{\gamma_p} \in \Omega^1(X)^*$, but it depends on the choice of γ_p. If we take another path γ_p', then

$$\int_{\gamma_p} - \int_{\gamma_p'} = \int_{\gamma \cup (-\gamma_p')} \in \Lambda$$

that is,

$$\int_{\gamma_p} \equiv \int_{\gamma_p'} \pmod{\Lambda}$$

Definition 3.4 (Abel-Jacobi map)**.** *We define Abel-Jacobi map A as follows*

$$A : X \to \Omega^1(X)^*/\Lambda$$
$$p \to \int_{\gamma_p}$$

And we can extent our definition of Abel-Jacobi map to group of divisors, using the group structure of Jacobian.

$$\begin{aligned} A : \mathrm{Div}(X) &\to \mathrm{Jac}(X) \\ \sum_p n_p p &\mapsto \sum_p n_p A(p) \end{aligned}$$

Clearly this map is a group homomorphism, depending on p_0. But if we restrict our definition on $\mathrm{Div}_0(X)$, and denoted it by A_0, then

Lemma 3.5. *$A_0 : \mathrm{Div}_0(X) \to \mathrm{Jac}(X)$ is independent of the choice of p_0.*

Proof. Let p_0' be another base point, and use A' to denote the Abel-Jacobi map corresponding to p_0' and get A_0'. We want to show $A_0 = A_0'$.

Take any path α from p_0 to p_0', then

$$\begin{aligned} A(p) - A'(p) &= \int_{\gamma_p} - \int_{\gamma_p'} \\ &= \int_{\gamma_p \cup (-\gamma_p') \cup (-\alpha)} + \int_\alpha \\ &\equiv \int_\alpha \pmod{\Lambda} \end{aligned}$$

Given any $D \in \mathrm{Div}_0(X)$, then

$$\begin{aligned} A_0(D) - A_0'(D) &= \sum_p n_p (A(p) - A'(p)) \\ &\equiv \sum_p n_p \int_\alpha \pmod{\Lambda} \\ &\equiv \int_\alpha \sum_p n_p \pmod{\Lambda} \\ &\equiv 0 \pmod{\Lambda} \end{aligned}$$

This completes the proof. □

The Abel theorem says what's the kernel of A_0.

Theorem 3.6 (Abel theorem)**.** $\ker A_0 = \mathrm{PDiv}(X)$.

Using Abel theorem, we can show that all Riemann surfaces with genus 1 are constructed as $\mathbb{C}/L$, where L is a lattice. Roughly we prove Abel-Jacobi map is an isomorphism in the case of genus one, and Jacobian of genus one is exactly $\mathbb{C}/L$.

In general, we have Abel-Jacobi map is injective.

Lemma 3.7. *If $\mathrm{genus}(X) \geq 1$, then $A : X \to \mathrm{Jac}(X)$ is injective.*

Proof. If not, $p \neq p' \in X$ such that $A(p) = A(p')$. Then consider divisor $D = p - p'$ with degree 0. We have

$$A_0(D) = A(p) - A(p') = 0 \in \mathrm{Jac}(X)$$

Then Abel theorem says $D \in \ker A_0 = \mathrm{PDiv}(X)$. So there exists a meromorphic function f such that $D = \mathrm{div}(f)$. Let $F : X \to S^2$ be the holomorphic map corresponding to f. Then $F^{-1}(\infty) = p'$, and the multiplicity of p' is 1. Then degree of F is exactly 1. So F is an isomorphism. A contradiction to the fact genus$(X) \geq 1$. □

Theorem 3.8. *Assume X is a compact Riemann surface with genus 1, then $X \cong \mathbb{C}/L$, where L is a lattice.*

Proof. Since genus$(X) = 1$, then $\dim_{\mathbb{C}} \Omega^1(X) = 1$, so we have $\Omega^1(X)^* \cong \mathbb{C}$. Then $\mathrm{Jac}(X) \cong \mathbb{C}/L$ for some $L \subset \mathbb{C}$.

Consider Abel-Jacobi map $A : X \to \mathrm{Jac}(X)$, an injective holomorphic map. As a consequence we have $\mathrm{Jac}(X)$ is a compact Riemann surface, since X is. So L must be a lattice and the fact that A is injective implies that we have that $\deg(A) = 1$. Then A is an isomorphism from $X \to \mathbb{C}/L$. □

3.3. **Proof of Abel theorem: Part I.** Let X, Y be compact Riemann surfaces. Consider a non-constant holomorphic map $F : X \to Y$ with degree $m \geq 1$. Let $B(F)$ denote the set of all ramification value of F in Y, that is, $y \in Y$ is a ramification value, if there is a ramification point in the set of $\{F^{-1}(y)\}$.

First consider $q \in Y \backslash B(F)$, then $F^{-1}(q) = \{p_1, \ldots, p_m\}$ with $p_i \neq p_j$. Choose a neighborhood U of q such that $U \cap B(F) = \varnothing$. Then $F^{-1}(U) = \bigcup_{i=1}^m V_i$, where $V_i \cap V_j = \varnothing$ and $p_i \in V_i$. Furthermore, $F|_{V_i} : V_i \to U$ is an isomorphism.

Given any function f and 1-form θ on X. Then define

$$\mathrm{Tr}(f)|_U = \sum_{i=1}^m f \circ F_i^{-1}$$
$$\mathrm{Tr}(\theta)|_U = \sum_{i=1}^m (F_i^{-1})^*(\theta)$$

Here Tr means trace. However, now we only define near a non ramification value.

Theorem 3.9. *If f and θ are meromorphic, then $\mathrm{Tr}(f)$ and $\mathrm{Tr}(\theta)$ can be extended to globally defined meromorphic function and meromorphic 1-forms on Y. Furthermore, if f and θ are holomorphic, then $\mathrm{Tr}(f)$ and $\mathrm{Tr}(\theta)$ are holomorphic.*

Proof. Consider the case $q \in B(F)$ and $F^{-1}(q) = \{p\}$. Then $\mathrm{mult}_p F = m$. By local normal form, we can choose local coordinate w centered at p and local coordinate z centered at q such that locally F is given by $z = w^m$.

Let f be a meromorphic function on X, in local coordinate,

$$f(w) = \sum_n c_n w^n$$

Let $\xi = e^{\frac{2\pi i}{m}}$ be a m-th unit root. For any $z \neq 0$, choose w such that $w^m = z$. Then

$$F^{-1}(z) = \{w\xi^j \mid j = 0, 1, \dots, m-1\}$$

Since $z \neq 0$, we have

$$\begin{aligned}
\mathrm{Tr}(f)(z) &= \sum_{j=0}^{m-1} f(w\xi^j) \\
&= \sum_{j=0}^{m-1} \sum_n c_n (w\xi^j)^n \\
&= \sum_n c_n (\sum_{j=0}^{m-1} \xi^{jn}) w^n
\end{aligned}$$

By directly commuting, we have

$$(\xi^n - 1) \sum_{j=0}^{m-1} \xi^{jn} = \xi^{mn} - 1 = 0$$

So

$$\sum_{j=0}^{m-1} \xi^{jn} = \begin{cases} 0, & \xi^n \neq 1 \\ m, & \xi^n = 1 \end{cases}$$

And $\xi^n = 1$ is equivalent to $n = km$ for some $k \in \mathbb{Z}$. So we have

$$\begin{aligned}
\mathrm{Tr}(f)(z) &= \sum_k c_{mk} m w^{mk} \\
&= \sum_k m c_{mk} (w^m)^k \\
&= \sum_k m c_{mk} z^k
\end{aligned}$$

is a meromorphic function in a neighborhood of $z = 0$. Furthermore, if f is holomorphic at $w = 0$, then $k \geq 0$, so we have $\mathrm{Tr}(f)$ is holomorphic.

Now let's see the case of 1-form. Near p, locally we have

$$\theta = (\sum_n c_n w^n) \mathrm{d}w$$

Since $z = w^m$, then $\mathrm{d}z = m w^{m-1} \mathrm{d}w$. Then

$$\theta = (\sum_n c_n w^n) \frac{1}{m w^{m-1}} \mathrm{d}z$$

At $z \neq 0$, we have

$$\begin{aligned}
\mathrm{Tr}(\theta) &= \sum_{j=0}^{m-1} \sum_n \frac{c_n}{m} (w\xi^j)^{n-m+1} \mathrm{d}z \\
&= \sum_n \frac{c_n}{m} (\sum_{j=0}^{m-1} \xi^{j(n-m+1)}) w^{n-m+1} \mathrm{d}z \\
&= \sum_k c_{mk+m-1} w^{mk} \mathrm{d}z \\
&= \sum_k c_{mk+m-1} z^k \mathrm{d}z
\end{aligned}$$

So $\mathrm{Tr}(\theta)$ defines a meromorphic 1-form near $z = 0$. If θ is holomorphic, then $\mathrm{Tr}(\theta)$ is holomorphic.

Furthermore, we can see that the residue of $\mathrm{Tr}(\theta)$ equals to the residue of θ. Since if $k = -1$, we have

$$c_{mk+m-1} = c_{-m+m-1} = c_{-1}$$

In general case, $q \in B(F)$ and $F^{-1}(q) = \{p_1, \ldots, p_n\}$. We still choose $W \ni q$ such that $F^{-1}(W) = V_1 \cup \cdots \cup V_n, p_i \in V_i, V_i \cap V_j \neq \varnothing$. We define

$$\mathrm{Tr}(f) := \sum_{i=1}^{n} \mathrm{Tr}(f|_{V_i})$$

$$\mathrm{Tr}(\theta) := \sum_{i=1}^{n} \mathrm{Tr}(\theta|_{V_i})$$

□

Lemma 3.10. *If θ is a meromorphic 1-form on X, $q \in Y$. Then*

$$\mathrm{Res}_q(\mathrm{Tr}(\theta)) = \sum_{p \in F^{-1}(q)} \mathrm{Res}_p(\theta)$$

Proof. Clear. □

Let γ be a piecewise smooth curve in Y such that $F^{-1}(\gamma)$ doesn't contain poles of θ. Then there is no poles of $\mathrm{Tr}(\theta)$ on γ. Thus we can integrate $\mathrm{Tr}(\theta)$ on γ. That is

$$\int_\gamma \mathrm{Tr}(\theta)$$

is well-defined. Away from ramification points, γ can be lifted to exactly $m = \deg(F)$ non-intersecting curves in X. Taking closures of these curves, we obtain curves $\gamma_1, \ldots, \gamma_m \subset X$. And $F^{-1}(\gamma) = \gamma_1 \cup \cdots \cup \gamma_m$. Define $F^*(\gamma) = \gamma_1 + \cdots + \gamma_m$.

Lemma 3.11.

$$\int_\gamma \operatorname{Tr}(\theta) = \int_{F^*(\gamma)} \theta := \sum_{i=1}^m \int_{\gamma_i} \theta$$

Proof. Removing finitely many points doesn't affect the integration. So we only need to prove this lemma for the case where γ doesn't have ramification point.

We can find a small neighborhood of γ, denoted by U, which doesn't have ramification points and

$$F^{-1}(U) = V_1 \cup \cdots \cup V_m$$

such that $V_i \cap V_j \neq \varnothing, \gamma_i \subset V_i, F|_{V_i}$ is an isomorphism. Then

$$\begin{aligned}
\int_{F^*(\gamma)} \theta &= \sum_{i=1}^m \int_{\gamma_i} \theta \\
&= \sum_{i=1}^m \int_{F(\gamma_i)} (F_i^{-1})^* \theta \\
&= \int_\gamma \sum_{i=1}^m (F_i^{-1})^* \theta \\
&= \int_\gamma \operatorname{Tr}(\theta)
\end{aligned}$$

□

With above tools, we can prove one direction of Abel theorem. Recall that Abel-Jacobi map

$$A_0 : \operatorname{Div}_0(X) \to \operatorname{Jac}(X)$$

And Abel theorem says that $\ker A_0 = \operatorname{PDiv}(X)$. Now let show $\operatorname{PDiv}(X) \subseteq \ker A_0$.

For any $D \in \operatorname{PDiv}(X)$, there exists a meromorphic function f such that $\operatorname{div}(f) = D$. Let F be the holomorphic map corresponding to f with degree d.

Choose a curve γ in S^1 from ∞ to 0, and it contains no ramification values of F except 0 and ∞. Then $F^*(\gamma) = \gamma_1 + \cdots + \gamma_d, \gamma_i \in X$, each γ_i is a curve from a pole q_i of f to a zero p_i of f. Then $D = \sum_{i=1}^d (p_i - q_i)$.

Fixe $x \in X$, and use α_i, β_i to denote the curve from x to p_i and q_i. Then

$$A_0(D) = \sum_{i=1}^d (\int_{\alpha_i} - \int_{\beta_i})$$

Let $\eta = \alpha_i - \gamma_i - \beta_i$. Then

$$A_0(D) = \sum_{i=1}^{d}(\int_{\eta} + \int_{\gamma_i}) \pmod{\Lambda}$$
$$= \sum_{i=1}^{d}\int_{\gamma_i} \pmod{\Lambda}$$

For any $\theta \in \Omega^1(X)$, we have

$$\begin{aligned} A_0(D)(\theta) &= \sum_{i=1}^{d}\int_{\gamma_i}\theta \\ &= \int_{F^*(\gamma)}\theta \\ &= \int_{\gamma}\mathrm{Tr}(\theta) \\ &= 0 \end{aligned}$$

Thus $A_0(D) = 0$, as desired.

3.4. **Proof of Abel theorem: Part II.** Recall that X is obtained from gluing a $4g$-polygon V. And the homology group $H_1(X) = \mathrm{span}_{\mathbb{Z}}\{[a_i], [b_i] \mid i = 1, \ldots, g\}$. For any closed 1-form ω on X. We define

$$A_i(\omega) = \int_{a_i}\omega, \quad i = 1, \ldots, g$$

and call them a-periods of ω. Similarly we can define b-periods of ω as

$$B_i(\omega) = \int_{b_i}\omega, \quad i = 1, \ldots, g$$

We can also consider ω as a closed 1-form defined in a neighborhood of polygon V. Fix a base point $x \in V$, define

$$f_\omega(p) = \int_x^p \omega$$

where integration along any path from x to p inside V. Since ω is closed, this integration is independent of the choice of path. Thus f_ω is a well-defined function on a neighborhood of V, and $\mathrm{d}f_\omega = \omega$.

However, it's worth to mention that f_ω is not well-defined on X! You may think f_ω as a multi-valued function on X, since different points in V are glued to the same point on X.

Lemma 3.12. *Let ω, θ be closed 1-form on X. Then*

$$\int_X \omega \wedge \theta = \int_{\partial V} f_\omega\theta = \sum_{i=1}^{g} A_i(\omega)B_i(\theta) - A_i(\theta)B_i(\omega)$$

Proof. For any $p \in a_i$, we use $p' \in a_i'$ to denote the point glued to p. Let α_p be a curve from p to p'. Then

$$\begin{aligned} f_\omega(p) - f_\omega(p') &= \int_x^p \omega - \int_x^{p'} \omega \\ &= -\int_{\alpha_p} \omega \\ &= -\int_{b_i} \omega \\ &= -B_i(\omega) \end{aligned}$$

Similarly we can take $p \in b_i$ and $p' \in b_i'$, and we can see

$$f_\omega(p) - f_\omega(p') = A_i(\omega)$$

Now for any smooth 1-form θ, define in a neighborhood of $\bigcup_{i=1}^q (a_i \cup b_i)$ in X. Then

$$\begin{aligned} \int_{\partial V} f_\omega \theta &= \sum_{i=1}^g (\int_{a_i} + \int_{b_i} - \int_{a_i'} - \int_{b_i'}) f_\omega \theta \\ &= \sum_{i=1}^g \int_{p \in a_i} (f_\omega(p) - f_\omega(p'))\theta + \int_{q \in b_i} (f_\omega(q) - f_\omega(q'))\theta \\ &= \sum_{i=1}^g -B_i(\omega)A_i(\theta) + A_i(\omega)B_i(\theta) \end{aligned}$$

As desired. □

Remark 3.13. This formula also holds if θ is a meromorphic 1-form on X with no poles along a_i and b_i.

Now let's see some applications of this lemma. First we have

Lemma 3.14. *Let ω be a holomorphic 1-form on X which is not identically zero, then*

$$\operatorname{Im} \sum_{i=1}^g A_i(\omega) B_i(\omega) < 0$$

Proof. In each local coordinate z, ω can be written as $\omega = f(z)\mathrm{d}z$ for some holomorphic function $f(z)$, so $\overline{\omega} = \overline{f(z)}\mathrm{d}z$. Then

$$\begin{aligned} \omega \wedge \overline{\omega} &= |f(z)|^2 \mathrm{d}z \wedge \mathrm{d}\overline{z} \\ &= -2i|f(z)|^2 \mathrm{d}x \wedge \mathrm{d}y \end{aligned}$$

so $i \int_X \omega \wedge \overline{\omega} > 0$, since $|f(z)|^2 \geq 0$ and not identically zero. By previous lemma, we have

$$\mathbb{R} \ni i \sum_{j=1}^g \{A_j(\omega)B_j(\overline{\omega}) - A_j(\overline{\omega})B_j(\omega)\} = i \int_X \omega \wedge \overline{\omega} > 0$$

Since $\int_\gamma \overline{\omega} = \overline{\int_\gamma \omega}$, then

$$A_j(\overline{\omega}) = \overline{A_j(\omega)}, \quad B_j(\overline{\omega}) = \overline{B_j(\omega)}$$

Thus

$$\operatorname{Im}\sum_{i=1}^{g} A_i(\omega)B_i(\overline{\omega}) = \frac{1}{2}\operatorname{Im}\sum_{i=1}^{g}\{A_i(\omega)B_i(\overline{\omega}) - A_i(\overline{\omega})B_i(\omega)\} < 0$$

□

Corollary 3.15. *Let $\omega \in \Omega^1(X)$. If $A_i(\omega) = 0$ for all $i = 1, \dots, g$, then $\omega = 0$. If $B_i(\omega) = 0$ for all $i = 1, \dots, g$, then $\omega = 0$.*

Proof. Assume $A_i(\omega) = 0$ for all $i = 1, \dots, g$. If $\omega \neq 0$, then by previous lemma, we have

$$\operatorname{Im}\sum_{i=1}^{g} A_i(\omega)B_i(\overline{\omega}) < 0$$

A contradiction, so we have $\omega = 0$. The proof still holds for the case of $B_i(\omega) = 0, i = 1, \dots, g$. □

Recall $\dim \Omega^1(X) = \dim L^{(1)}(0) = g$. Fix a basis $\{\omega_1, \dots, \omega_g\}$ of $\Omega^1(X)$.

Definition 3.16 (period matrices)**.** *Define two matrices A, B as*

$$A = (A_i(\omega_j))_{g\times g}, \quad B = (B_i(\omega_j))_{g\times g}$$

Then A, B are called period matrices of X.

Remark 3.17. A, B depends on the choice of basis $\{\omega_1, \dots, \omega_g\}$ and generators $\{a_i, b_i\}$ of $H_1(X, \mathbb{Z})$.

Lemma 3.18. *Both A and B are invertible.*

Proof. Assume A is not invertible, then there exists $c = (c_1, \dots, c_g)^T \in \mathbb{C}^g, c \neq 0$ such that $Ac = 0$. Let $\omega = \sum_{j=1}^{g} c_j\omega_j \in \Omega^1(X)$. Then

$$A_i(\omega) = \sum_{j=1}^{g} c_j A_i(\omega_j) = 0, \quad \text{for all } i = 1, \dots, g$$

By above corollary, we have $\omega = 0$, a contradiction to the fact $\{\omega_1, \dots, \omega_g\}$ is a basis, so A is invertible. The proof still holds for the case of B. □

Lemma 3.19 (First Riemann bilinear relation)**.** *A^TB is a symmetric matrix.*

Proof. For any $1 \leq j, k \leq g$, clearly $\omega_i \wedge \omega_j = 0$, since both of them are $(1,0)$-form. So

$$0 = \int_X \omega_j \wedge \omega_k = \sum_{i=1}^{g}\{A_i(\omega_j)B_i(\omega_k) - A_i(\omega_k)B_i(\omega_k)\}$$

And this is exactly (j,k)-th entry of $A^TB - B^TA$, thus $A^TB = B^TA$, as desired. □

Lemma 3.20 (Second Riemann bilinear relation)**.** *$i(A^T\overline{B} - B^T\overline{A})$ is a positive definite Hermitian matrix.*

Proof. We have proven that for any $\omega \in \Omega^1(X)$,

$$i(\sum_{j=1}^{g}\{A_j(\omega)B_j(\overline{\omega}) - A_j(\overline{\omega})B_j(\omega)\}) > 0$$

For any $0 \neq c = (c_1, \dots, c_g)^T \in \mathbb{C}^g$, applying above equation to $\omega = \sum_{j=1}^{g} c_j\omega_j$, we have

$$\begin{aligned} 0 &< i\sum_{j=1}^{g}\sum_{k,l}^{g} c_k\overline{c_l}\{A_j(\omega)B_j(\overline{\omega}) - A_j(\overline{\omega})B_j(\omega)\} \\ &= ic^T(A^T\overline{B} - B^T\overline{A})\overline{c} \end{aligned}$$

This completes the proof. □

Remark 3.21. Note if we choose another basis $\{\omega_1', \dots, \omega_g'\}$ of $\Omega^1(X)$, there exists an invertible matrix $M = (m_{ij})$ such that

$$\omega_i = \sum_{j=1}^{g} m_{ij}\omega_j'$$

Let A', B' be the period matrices with respect to $\{\omega_1', \dots, \omega_g'\}$. Then

$$A_i(\omega_j) = \sum_k m_{jk}A_i(\omega_k'), \quad \text{for all } i, j$$

Thus

$$A = A'M^T$$

Similarly we have $B = B'M^T$. Since period matrices A, B are always invertible, we can choose a basis $\{\omega_1, \dots, \omega_g\}$ such that $A = I$, that is

$$\int_{a_i} \omega_j = \delta_{ij}, \quad \text{for all } i, j = 1, \dots, g$$

Such basis is called normalized basis, in this case, b-period matrix B is called normalized b-period matrix.

First Riemann relation is equivalent to B is symmetric, and second Riemann bilinear relation is equivalent to $\mathrm{Im}(B)$ is positive definite.

Lemma 3.22. *The $2g$ rows of any period matrices of A and B are linear independent over $\mathbb{R}$.*

Proof. If suffices to prove for any $\alpha, \beta \in \mathbb{R}^n$, then

$$\alpha^T A + \beta^T B = 0 \implies \alpha = \beta = 0$$

Since under a change of basis of $\Omega^1(X)$, A and B will be multiplied by the same invertible matrix from the right. So it suffices to show for the case $A = I$, that is

$$0 = \alpha^T + \beta^T B = 0$$

so we have

$$\beta^T \operatorname{Im}(B) = 0$$

But $\operatorname{Im}(B)$ is positive definite, then $\beta = 0$, so is α. $\square$

Corollary 3.23. *The period group Λ is a lattice in $\Omega^1(X)^*$ of rank $2g$.*

Remark 3.24. Two Riemann bilinear relations and previous lemma will be needed to study Riemann theta function.

Definition 3.25 (linear system)**.** *A linear system Q is a subspace of some complete linear system $|D|$, where D is a divisor.*

Definition 3.26 (base point)**.** *Given a linear system Q on X, a point $p \in X$ is called base point if $E \geq p$ for all $E \in Q$.*

Lemma 3.27. *Assume $Q \subset |D|$ for some $D \in \operatorname{Div}(X)$, $V \subset L(D)$ is a subspace corresponds to Q. Then $p \in X$ is a base point of Q if and only if*

$$V \subset L(D - p)$$

In particular, p is a base point of $|D|$ is equivalent to

$$\dim L(D) = \dim L(D - p)$$

Proof. By definition, $Q = \{\operatorname{div}(f) + D \mid f \in V\}$, p is a base point of Q is equivalent to

$$\operatorname{ord}_p(f) + D(p) \geq 1, \quad \forall f \in V$$

In other words,

$$\operatorname{ord}_p(f) \geq -D(p) + 1, \quad \forall f \in V$$

and the right hand is exactly values of divisor $D - p$ at p. So it's equivalent to $f \in L(D - p)$. This completes the proof. $\square$

Lemma 3.28. *Assume the genus of X $\operatorname{genus}(X) \geq 1$. Let K be a canonical divisor on X. Then the complete linear system $|K|$ is base point free.*

Proof. For any $p \in X$, we have proven that if $\dim L(p) \geq 1$, then $X \cong S^1$. Since $g \geq 1$, then $\dim L(p) = 1$ for all $p \in X$. By Riemann-Roch theorem, we have

$$\dim L(p) - \dim L(K - p) = \deg(p) + 1 - g$$

Thus $\dim L(K - p) = g - 1 < g = \dim L(K)$, for all $p \in X$. This completes the proof. $\square$

Theorem 3.29. *For any compact Riemann surface X, given finite set of distinct point $\{p_i\}$ on X and a corresponding set of complex numbers $\{\gamma_i\}$ with $\sum_i \gamma_i = 0$, then there exists a meromorphic 1-form ω on X such that the poles of ω are exactly $\{p_i\}$, all thoes poles are simpy poles with residue $\{\gamma_i\}$.*

Proof. If $g = 0$, then $X = \mathbb{C} \cup \{\infty\}$, we can construct as follows

$$\omega = \sum_i \frac{\gamma_i}{z - p_i} \mathrm{d}z$$

Now assume $g \geq 1$, let's see a lemma firstly. Note that this lemma has no requirement on genus.

Lemma 3.30. *If Q is a linear system without base point, for any finite set of points $\{p_1, \dots, p_n\}$, there exists a divisor $E \in Q$ such that $p_i \notin \operatorname{supp}(E)$ for all $i = 1, \dots, n$.*

Proof. Assume $Q \subset |D|$ for some divisor D, $V \subset L(D)$ is the space corresponding to Q. Since p_i is not base point of Q, then $V \not\subset L(D - p_i)$ for all i. So $V \setminus \bigcup_{i=1}^n L(D - p_i)$ is non-empty. Choose $f \in V \setminus \bigcup_{i=1}^n L(D - p_i)$. Then $\operatorname{ord}_{p_i}(f) = -D(p_i)$ for all i. Let $E = \operatorname{div}(f) + D \in Q$, we have $E(p_i) = 0$ and $p_i \notin \operatorname{supp}(E)$ for all i. This completes the proof of claim. □

Since $g \geq 1$, then complete linear system of canonical divisor K is base point free. So we may choose a canonical divisor $K \geq 0$, such that $p_i \notin \operatorname{supp}(K)$ for all i. Let ω_0 be the meromorphic 1-form corresponding to K, since $K \geq 0$, then ω is holomorphic. We want to find $f \in \mathcal{M}(X)$ such that $\omega = f\omega_0$, which satisfies our requirements. Choose local coordinate z_i centered at p_i. In this coordinate, ω_0 can be written as

$$\omega_0 = (c_i + z_i g_i(z_i)) \mathrm{d}z_i$$

where g_i is a holomorphic function. Since $p_i \notin \operatorname{supp}(\omega_0)$, then $c_i \neq 0$. Consider Laurent tail divisor $Z = \sum_i \frac{\gamma_i}{c_i} z_i^{-1} \cdot p_i$. Since

$$-K(p_i) = 0 > -1, \quad \text{for all } i$$

Then $Z \in T[K](X)$. Let $\alpha_K : \mathcal{M}(X) \to T[K](X)$ be the divisor map and $H^1(K) = \operatorname{coker}(\alpha_K)$. Let $\pi : T[K](X) \to H^1(K)$ be the projection. $Z \in \operatorname{Im}(\alpha_K)$ if and only if $\pi(Z) = 0$, and if and only if $\theta(\pi(Z)) = 0, \forall \theta \in H^1(K)^*$. By Serre duality, the residue map

$$\operatorname{Res} : L^{(1)}(K) \to H^1(K)^*$$

is an isomorphism.

Note that $\omega_0 \in L^{(1)}(-K)$ since $\operatorname{div}(\omega_0) = K$, and $\dim L^{(1)}(-K) = \dim L(0) = 1$, then $L^{(1)}(-K) = \{a\omega_0 \mid a \in \mathbb{C}\}$. So Res. Indeed,

$$\begin{aligned}
\operatorname{Res}_{\omega_0}(Z) &= \sum_i \operatorname{Res}_{z_i=0}\{\frac{\gamma_i}{c_i} z_i^{-1}(c_i + z_i g_i(z_i))\} \mathrm{d}z_i \\
&= \sum_i \gamma_i \\
&= 0
\end{aligned}$$

So there exists $f \in \mathcal{M}(X)$ such that $\alpha_K(f) = Z$. Let $\omega = f\omega_0$. If $q \neq p_i$, then $\alpha_K(f)(q) = 0$, so

$$\operatorname{ord}_{p_i}(f) \geq -K(q)$$

So

$$\mathrm{ord}_q(\omega) = \mathrm{ord}_q(f) + \mathrm{ord}_q(\omega_0) \geq 0$$

□

Lemma 3.31. *Let $D \in \mathrm{Div}_0(X)$ such that $A_0(D) = 0 \in \mathrm{Jac}(X)$ where A_0 is the Abel-Jacobi map. Then there exists a meromorphic 1-form ω on X such that*

1. $\mathrm{supp}(D) =$ *set of poles of ω and ω only has simple poles;*
2. $\mathrm{Res}_p(\omega) = D(p)$*;*
3. *periods of ω are integral multiples of $2\pi i$.*

Proof. Since $\sum_{p\in X} D(p) = 0$, then by Theorem 3.29, there exists a meromorphic 1-form θ on X satisfying 1 and 2. Let $\{\omega_1, \dots, \omega_n\}$ be a basis of $\Omega^1(X)$. Let $\omega = \theta - \sum_{i=1}^{g} c_i\omega_i$ with $c_i \in \mathbb{C}$. Then ω still satisfies 1 and 2. The difficultly is to find suitable c_i such that ω satisfies 3.

Choose closed paths a_i, b_i which generate $H_1(X, \mathbb{Z})$ such that $\mathrm{supp}(D) \subset X \backslash \bigcup_i (a_i \cup b_i)$. For $i = 1, \dots, g$, define

$$\rho_k = \frac{1}{2\pi i} \sum_{i=1}^{g} \{A_i(\omega_k)B_i(\theta) - A_i(\theta)B_i(\omega_k)\}$$

By Lemma 3.12 we have

$$\begin{aligned}\rho_k &= \frac{1}{2\pi i} \int_{\partial V} f_{\omega_k}\theta \\ &= \sum_{p\in V} \mathrm{Res}_p(f_{\omega_k}\theta) \\ &= \sum_{p\in X} \mathrm{Res}_p(f_{\omega_k}\theta) \\ &= \sum_{p\in \mathrm{supp}(D)} f_{\omega_k}(p)D(p)\end{aligned}$$

the last equality holds since f_{ω_k} is holomorphic and θ satisfies 1 and 2. Thus

$$\rho_k = \sum_{p} D(p) \int_{p_0}^{p} \omega_k$$

where p_0 is a fixed base point in interior of $\mathcal{P}$.

Consider the identification

$$\begin{aligned}\Omega^1(X)^* &\xrightarrow{\Phi} \mathbb{C}^g \\ \alpha &\mapsto (\alpha(\omega_1), \dots, \alpha(\omega_g))\end{aligned}$$

and $\Lambda = \mathrm{span}_{\mathbb{Z}}\{\Phi(\int_{a_i}), \Phi(\int_{b_i})\}$, and note that

$$\begin{aligned}\Phi(a_i) &= (A_i(\omega_1), \dots, A_i(\omega_g)) \\ \Phi(b_i) &= (B_i(\omega_1), \dots, B_i(\omega_g))\end{aligned}$$

Thus Φ induces isomorphism

$$\Phi : \mathrm{Jac}(X) \to \mathbb{C}^g/\Lambda$$

a complex g-dimensional torus. By the definition of Abel-Jacobi map

$$(\rho_1, \ldots, \rho_g) \equiv \Phi(A_0(D)) \pmod{\Lambda}$$

If $A_0(D) = 0$, then $(\rho_1, \ldots, \rho_g) \in \Lambda$, so there exists $m_j, n_j \in \mathbb{Z}$ such that

$$(\rho_1, \ldots, \rho_g) = \sum_{i=1}^{g} m_j(A_j(\omega_1), \ldots, A_j(\omega_g)) - \sum_{i=1}^{g} n_j(B_j(\omega_1), \ldots, B_j(\omega_g))$$

By definition of ρ_k, we have

$$\rho_k = \frac{1}{2\pi i} \sum_{i=1}^{g} \{A_i(\omega_k)B_i(\theta) - A_i(\theta)B_i(\omega_k)\}$$

we must have

$$\sum_{j=1}^{g} (B_j(\theta) - 2\pi i m_j)A_j(\omega_k) = \sum_{j=1}^{g} (A_j(\theta) - 2\pi i n_j)B_j(\omega_k), \quad 1 \leq k \leq g$$

Let $\widetilde{b}_j = B_j(\theta) - 2\pi i m_j, \widetilde{a}_j = A_j(\theta) - 2\pi i n_j$. Then above equations can be expressed as

$$A^T b = B^T a$$

where $a = (\widetilde{a}_1, \ldots, \widetilde{a}_g)^T, b = (\widetilde{b}_1, \ldots, \widetilde{b}_g)^T$, and A, B are period matrices.

Consider linear transformations

$$\mathbb{C}^g \xrightarrow{\alpha} \mathbb{C}^{2g} \xrightarrow{\beta} \mathbb{C}^g$$

where

$$\alpha = \begin{pmatrix} A \\ B \end{pmatrix}, \quad \beta = (B^T, -A^T)$$

Since A, B are invertible, then α is injective and β is surjective. And second Riemann bilinear relation tells us $\beta \circ \alpha(v) = (B^T A - A^T B)v = 0$. So we have

$$\operatorname{Im} \alpha \subset \ker \beta$$

and the injectivity of α and surjectivity of β tells us $\operatorname{Im} \alpha$ and $\ker \beta$ have the same dimension, so the following sequence is exact.

$$0 \to \mathbb{C}^g \xrightarrow{\alpha} \mathbb{C}^{2g} \xrightarrow{\beta} \mathbb{C}^g \to 0$$

Since $\beta\begin{pmatrix} a \\ b \end{pmatrix} = 0$. Thus there exists c such that $\alpha(c) = \begin{pmatrix} a \\ b \end{pmatrix}$. In other words, $a = Ac, b = Bc$. Let $\omega = \theta - \sum_{j=1}^{g} c_j\omega_j$. Then periods of ω is

$$\begin{aligned}
A_k(\omega) &= A_k(\theta) - \sum_j c_j A_k(\omega_j) \\
&= A_k(\theta) - (A_k(\theta) - 2\pi i n_k) \\
&= 2\pi i n_k \\
B_k(\omega) &= B_k(\theta) - \sum_j c_j B_k(\omega_j) \\
&= B_k(\theta) - (B_k(\theta) - 2\pi i m_k) \\
&= 2\pi i m_k
\end{aligned}$$

As desired. □

Now we are ready to prove the other direction of Abel theorem.

Proof. Assume $D \in \mathrm{Div}_0(X)$ such that $A_0(D) = 0 \in \mathrm{Jac}(X)$. Let ω be a meromorphic 1-form on X satisfying three conditions in previous lemma.

Fix a base point $p_0 \in X$ which is not a pole of Ω. Define

$$f(p) := \exp(\int_{p_0}^{p} \omega), \quad \forall p \in X$$

where the integral is along any path from p_0 to p which doesn't pass poles of ω. Since period of ω are integral multiples of $2\pi i$ and residue of ω are integers. So $f(p)$ doesn't depend on the choice of path in the integral $\int_{p_0}^{p} \omega$. In other words, f is well-defined for p which is not a pole of ω, and f is holomorphic and non-zero at such points.

Since $\mathrm{supp}(D) =$ poles of ω, f is holomorphic on $X \setminus \mathrm{supp}(D)$. For $p \in \mathrm{supp}(D)$ and $n = D(p)$. Choose a local coordinate z centered at p. Since $\mathrm{Res}_p(\omega) = n$ and $\mathrm{ord}_p(\omega) = 1$, then near p

$$\omega = (nz^{-1} + g(z))\mathrm{d}z$$

where g is holomorphic. Thus near p we have

$$f(z) = \exp(\int_{p_0}^{p} \omega) = \exp(n \log z + h(z)) = z^n e^{h(z)}$$

Thus f is meromorphic and $\mathrm{ord}_p(f) = n = D(p)$, so $D = \mathrm{div}(f) \in \mathrm{PDiv}(X)$. This completes the proof of Abel theorem. □

3.5. **Jacobi inversion theorem.** Abel theorem tells us what does the kernel of $A_0 : \mathrm{Div}_0(X) \to \mathrm{Jac}(X)$ look like. Jacobi inversion theorem tells us Abel-Jacobi map $A : \mathrm{Div}(X) \to \mathrm{Jac}(X)$ is surjective.

Lemma 3.32. *Given* $D \in \mathrm{Div}(X), D \geq 0, \deg(D) = g = \mathrm{genus}(X) \geq 1$. *Then there exists divisor* $D' \geq 0$ *close to* D *such that* $\dim L(K - D') = 0$ *and* $D' = \sum_{i=1}^{g} p_i'$ *with* $p_i' \neq p_j'$ *if* $i \neq j$.

Remark 3.33. If $D = \sum_i p_i$, U_i is a neighborhood of p_i, we say $D' = \sum_i p_i'$ is closed to D if $p_i' \in U_i$.

Remark 3.34. By Riemann-Roch, if $\deg(D) = g$, then

$$\dim L(K - D) = 0 \Longleftrightarrow \dim L(D) = 1$$

Proof. Assume $D = \sum_{i=1}^g p_i$, where p_i may repeat. Let $D_j = \sum_{i=1}^j p_i, D_0 = 0$. Then $D_g = D$. The lemma holds from following claim: For $j \geq 1$, we can find divisor D_j' such that $D_j' = D_{j-1}' + p_j', p_j' \notin \operatorname{supp}(D_{j-1}')$ and p_j' is arbitary close to p_j, and $\dim L(K - D_j') = g - j$. $\square$

Theorem 3.35 (Jacobi inversion theorem)**.** *For $\xi \in \mathrm{Jac}(X)$, there exists $D \in \mathrm{Div}(X)$ such that $D \geq 0, \deg(D) = g$ and $A(D) = \xi$, where A is the Abel-Jacobi map.*

Proof. If $g = 0$, the case is trivial. Assume $g \geq 1$, by Lemma 3.32, there exists $p_1, \ldots, p_g \in X$ such that $p_i \neq p_j$ if $i \neq j$ and $\dim L(K - D_0) = 0$, where $D_0 = p_1 + \cdots + p_g$. Let $\xi_0 = A(D_0) = \sum_{i=1}^g A(p_i) \in \mathrm{Jac}(X)$. Let $\{\omega_1, \ldots, \omega_g\}$ be a basis of $\Omega^1(X)$, then

$$\mathrm{Jac}(X) \cong \mathbb{C}^g/\Lambda$$

where $\Lambda = \operatorname{span}_{\mathbb{Z}}\{(\int_{a_i}\omega_1, \ldots, \int_{a_i}\omega_g), (\int_{b_i}\omega_1, \ldots, \int_{b_i}\omega_g) \mid i = 1, \ldots, g\}$. Under this identification, Abel-Jacobi map can be written as

$$A(p) = (\int_{p_0}^p \omega_1, \ldots, \int_{p_0} \omega_g) \in \mathbb{C}^g/\Lambda$$

where p_0 is a fixed base point in X, and the integral is along any path from p_0 to p.

For each point $p_i, i = 1, \ldots, g$, we choose a local coordinate z_i centered at p_i and defined over an open neighborhood U_i of p_i. Over U_i, ω_j can be written as

$$\omega_j = f_{ji}(z_i)\mathrm{d}z_i$$

where f_{ji} are holomorphic function over U_i.

For any $q_i \in U_i, i = 1, \ldots, g$, we have

$$\begin{aligned} A(q_1 + \cdots + q_g) - \xi_0 &= \sum_{i=1}^g \{A(q_i) - A(p_i)\} \\ &= \sum_{i=1}^g (\int_{p_i}^{q_i} \omega_1, \ldots, \int_{p_i}^{q_i} \omega_g) \in \mathbb{C}^g/\Lambda \end{aligned}$$

Hence we can consider $\int_{p_i}^{q_i} \omega_j$ as an integral $\int_0^{z_i} f_{ji}(z)\mathrm{d}z_j$, where z_i is the coordinate of q_i.

Thus $A(q_1 + \cdots + q_g) - \xi_0$ can be considered as a map

$$(z_1, \ldots, z_g) \mapsto (\varphi_1, \ldots, \varphi_g) = \varphi$$

defined over an open neighborhood of $(0,\dots,0)\in\mathbb{C}^g$, where

$$\varphi_j=\sum_{i=1}^{g}\int_0^{z_i} f_{ji}(z_i)\mathrm{d}z_i,\quad j=1,\dots,g$$

Note that

$$\frac{\partial\varphi_j}{\partial z_k}=f_{jk}(z_k)$$

If we consider its value at $(0,\dots,0)$, we have

$$\frac{\partial\varphi_j}{\partial z_k}(0)=f_{jk}(0)=\omega_j(p_k)/\mathrm{d}z_k$$

Thus Jacobian of the map φ at $(0,\dots,0)$ is

$$J=\begin{pmatrix}\omega_1(p_1) & \dots & \omega_1(p_g)\\ \vdots & & \vdots\\ \omega_g(p_1) & \dots & \omega_g(p_g)\end{pmatrix}/\mathrm{d}z_1\dots\mathrm{d}z_g$$

If $\det J=0$, then there exists $(c_1,\dots,c_g)\neq(0,\dots,0)$ such that

$$\omega=\sum_{j=1}^{g}c_j\omega_j$$

is zero at all $p_1,\dots,p_g$. Thus $0\neq\omega\in L^{(1)}(-D_0)$. So

$$\dim L(K-D_0)=\dim L^{(1)}(-D_0)\geq 1$$

A contradiction to the choice of D_0. Thus J is invertible, so φ is a homeomorphism in an open neighborhood of $(0,\dots,0)$. This is equivalent to say $A-\xi_0$ defines a homeomorphism from an open neighborhood $U\subset U_1\times\cdots\times U_g$ of $(p_1,\dots,p_g)$ to an open neighborhood V of the zero element in $\mathrm{Jac}(X)$.

For any $\xi\in\mathrm{Jac}(X)$, we can choose integer N large enough such that $\xi/N\in V$. So there exists $(q_1,\dots,q_g)\in U_1\times\cdots\times U_g$ such that

$$A(q_1+\cdots+q_g)-\xi_0=\frac{\xi}{N}$$

In other words

$$N\{A(q_1+\dots,q_g)-\xi_0\}=\xi$$

Let $E=-gp_0+\sum_{i=1}^{g}(Np_i-Ng_i)\in\mathrm{Div}(X)$. By Riemann-Roch,

$$\dim L(-E)=\dim L(K-E)+\deg(-E)+1-g\geq 1$$

So there exists a non-zero $f\in L(-E)$ such that

$$D=\mathrm{div}(f)-E\geq 0$$

And $\deg(D)=\deg(-E)=g$. By Abel theorem we have $A(\mathrm{div}(f))=0$. So

$$\begin{aligned}A(D)&=A(-E)\\&=gA(p_0)+N\{A(q_1+\cdots+q_g)-A(p_1+\cdots+p_g)\}\\&=N\{A(q_1+\cdots+q_g)-\xi_0\} \qquad\qquad =\xi\end{aligned}$$

This completes the proof. □

4. Automorphism group for Riemann surface with genus ≥ 2

4.1. **Hyperelliptic Riemann surface.**

Theorem 4.1. *If $T \in \mathrm{Aut}(X)$ and T is not identity. Then T has at most $2g+2$ fixed points.*

Proof. Let $\mathrm{Fix}(T)$ be the set of fixed points of T. For $p \notin \mathrm{Fix}(T)$, by Riemann-Roch we have

$$\dim L((g+1)p) \geq \deg((g+1)p) + 1 - g = 2$$

so there exists a non-constant $f \in L((g+1)p)$ such that

$$\mathrm{div}_\infty(f) = rp, \quad 1 \leq r \leq p+1$$

Let $h = f - f \circ T \in \mathcal{M}(X)$, then

$$\mathrm{div}_\infty(h) = rp + rq, \quad q = T^{-1}(p)$$

so

$$\deg(\mathrm{div}_0(h)) = \deg(\mathrm{div}_\infty(h)) = 2r \leq 2g+2$$

Since each fixed point of T is a zero of h, then

$$|\mathrm{Fix}(T)| \leq \deg(\mathrm{div}_0(h)) \leq 2g+2$$

□

Definition 4.2 (hyperelliptic)**.** *A compact Riemann surface X is called hyperelliptic if there exists a holomorphic map $F: X \to S^2$ such that* $\deg(F) = 2$.

Lemma 4.3. *There exists an automorphism on hyperelliptic X with genus g which has $2g+2$ fixed points.*

Proof. By Hurwitz formula, if genus of X is g, then

$$2g - 2 = \deg(F)(2 \times 0 - 2) + B(F)$$

where $B(F) = \sum_{p \in X}\{\mathrm{mult}_p(F) - 1\}$ is the total branch number. So $B(F) = 2g-2$. In other words, F has exactly $2g+2$ ramification points $x_1, \ldots, x_{2g+2} \in X$, and $2g+2$ ramification values $b_i = F(x_i) \in S^2$.

For any $z \in S^2 \backslash \{b_1, \ldots, b_{2g+2}\}$. F^{-1} has exactly 2 points. Define $T: X \to X$ by $T(x_i) = x_i, i = 1, \ldots, 2g+2$ and $T(p) = q$ if $F(p) = F(q)$ and $p \neq q$. This is an involution on X, called the hyperelliptic involution of X. And $\mathrm{Fix}(T) = \{x_1, \ldots, x_{2g+2}\}$. So the bound in Theorem 4.1 is sharp. □

Lemma 4.4. *A Riemann surface X is hyperelliptic if and only if there exists a divisor $D \in \mathrm{Div}(X)$ such that $D \geq 0, \deg(D) = 2$ and $\dim L(D) \geq 2$.*

Proof. If X is hyperelliptic, then there exists a holomorphic map $F : X \to S^2$ with degree 2. F defines a non-constant meromorphic function $f \in \mathcal{M}(X)$. Let $D = \mathrm{Div}_\infty(f) \geq 0$, then $\deg(D) = \deg(F) = 2$. Moreover,

$$\mathrm{div}(f) = \mathrm{div}_0(f) - \mathrm{div}_\infty(f) \geq -D$$

So $f \in L(D)$, then $\dim L(D) \geq 2$.

Conversely, given $D \geq 0, \deg(D) = 2, \dim L(D) \geq 2$. There exists a non-constant $f \in L(D)$ which gives a holomorphic map $F : X \to S^2$. Then

$$1 \leq \deg(F) = \deg(\mathrm{div}_\infty(f)) \leq \deg(D) = 2$$

So $\deg(F) = 1$ or 2. If $X \cong S^2$, then $\deg(F) = 2$ and X is hyperelliptic. If $X \cong S^2$, consider $z \mapsto z^2$, so X is also hyperelliptic. ☐

Theorem 4.5. *If* $\mathrm{genus}(X) \leq 2$*, then X must be hyperelliptic.*

Proof. Let $D \in \mathrm{Div}(X)$ with $D \geq 0, \deg(D) = 2$. By Riemann-Roch theorem we have

$$\dim L(D) - \dim L(K - D) = \deg(D) + 1 - g$$

So $\dim L(D) \geq 2$ if $g \leq 1$. By Lemma 4.4, we have X is hyperelliptic.

When $g = 2$, there exists a non-zero holomorphic 1-form ω since $\dim L^{(1)}(0) = g \geq 2$. Let $K = \mathrm{div}(\omega) \geq 0$ since ω is holomorphic. Let $K = \mathrm{div}(\omega)$, then $\deg(K) = 2g - 2 = 2$. By Riemann-Roch

$$\dim L(K) = 2$$

So X is also hyperelliptic. ☐

Here is some facts we omit proofs

Theorem 4.6. *There exists hyperelliptic Riemann surface of any genus.*

Theorem 4.7. *If X is hyperelliptic with $g \geq 2$, then hyperelliptic involution is the unique involution on X which has $2g + 2$ fixed points.*

4.2. **Gap theory.**

4.2.1. *Noether gap theorem.* In this section we assume genus g of X is ≥ 1, given $p_1, p_2, \dots \in X$. Define sequence of divisors $D_0 = 0, \dots, D_n = \sum_{i=1}^n p_i$. Since $D_{n-1} \leq D_n$, then

$$\dim L(D_{n-1}) \leq \dim L(D_n), \quad \forall n \geq 0$$

So it's natural to ask such a question: Does there exist $f \in L(D_n)$ but $f \notin L(D_{n-1})$? We call n a noether gap if the answer to this question is no.

Theorem 4.8. *There are precisely g integers n_k with*

$$1 = n_1 < \cdots < n_g < 2g$$

such that n_k is a noether gap.

Proof. Note that n is a noether gap is equivalent to $\dim L(D_n) = \dim L(D_{n-1})$. We have proved before that if there exists $p \in X$ such that $\dim L(p) > 1$, then X is isomorphic to S^2. Since $g \geq 1$, so X can not be isomorphic to S^2, then $\dim L(D_1) = 1 = \dim L(D_0)$, so $n_1 = 1$ is a gap.

Applying Riemann-Roch to D_{n-1} and D_n, we have

$$\begin{aligned}
\dim L(D_{n-1}) - \dim L(K - D_{n-1}) &= \deg(D_{n-1}) + 1 - g \\
\dim L(D_n) - \dim L(K - D_n) &= \deg(D_n) + 1 - g
\end{aligned}$$

So we have

$$\dim L(D_n) - \dim L(D_{n-1}) = 1 + \dim L(K - D_n) - \dim L(K - D_{n-1}), \quad \forall n \geq 1$$

Adding equations for each n together, we have

$$\begin{aligned}
\dim L(D_n) - \dim L(D_0) &= \sum_{i=1}^{n} \{\dim L(D_i) - \dim L(D_{i-1})\} \\
&= n + \sum_{i=1}^{n} \{\dim L(K - D_i) - \dim L(K - D_{i-1})\} \\
&= n + \dim L(K - D_n) - \dim L(K - D_0) \\
&= n + \dim L(K - D_n) - g \\
&\leq n
\end{aligned}$$

So the number of non-gaps[11] is $\leq n$, since for each non-gaps, $\dim L(D_i) - \dim L(D_{i-1})$ counts one.

For $n > 2g - 2$, $\deg(K - D_n) = \deg(K) - \deg(D_n) = 2g - 2 - n < 0$, thus $\dim L(K - D_n) = 0$. So we have

$$\dim L(D_n) - \dim L(D_0) = n - g$$

that is, the number of non-gaps between 1 and n is equal to $n - g$ when $n > 2g - 2$. In particular, take $n = 2g - 1$, we have exactly $2g - 1 - g = g - 1$ non-gaps between 1 and $2g - 1$. In other words, there are exactly g gaps between 1 and $2g - 1$. □

Remark 4.9. From above proof, we can see that if $n > 2g - 2$, $\dim L(D_n) = n - g + 1$, so every $n \geq 2g$ is a non-gap. As a summary, there is only g gaps, lying in $[1, 2g) \cap \mathbb{Z}$.

Theorem 4.10 (Weierstrass gap theorem). *If $g = \mathrm{genus}(X) \geq 1, p \in X$, then there are exactly g integers n_k with*

$$1 = n_1 < n_2 < \cdots < n_g < 2g$$

such that does not exist $f \in \mathcal{M}(X)$ which is holomorphic in $X \backslash \{p\}$ and has a hole of order n_k at p, such n_k is called a Weierstrass gap at p.

[11] n is a non-gap if n is not a gap.

Proof. $f \in L(np)$ means f is holomorphic over $X\backslash\{p\}$ and has a pole at p of order $\leq n$. So $f \in L(np)$ but $f \notin L((n-1)p)$ means f has a pole at p of order n and has no other poles.

So a Weierstrass gap is just a noether gap for the special case $D_n = np$. This completes the proof. □

Remark 4.11. If m and n are two Weierstrass non-gap at p, so there exists f, h holomorphic over $X\backslash\{p\}$ and $\mathrm{ord}_p(f) = -m$ and $\mathrm{ord}_p(h) = -n$. Then fh is again holomorphic over $X\backslash\{p\}$ and $\mathrm{ord}_p(fh) = -(m+n)$. So $m+n$ is again a non-gap. In other words, the set of non-gaps at p is a semi-group.

Lemma 4.12. *Let α_k be the first g non-gaps at p with $1 < \alpha_1 < \alpha_2 < \cdots < \alpha_g = 2g$. Then for any $0 < j < g$, we have*

$$\alpha_j + \alpha_{g-j} \geq 2g$$

Proof. Assume there exists $0 < j < g$ such that

$$\alpha_j + \alpha_{g-j} < 2g$$

Then for all $1 \leq k \leq j$, then $\alpha_k + \alpha_{g-j} < 2g$ is also a non-gap $> \alpha_{g-j}$. So the number of non-gaps between 1 and $2g$ is more than

$$\underbrace{(g-j)}_{\text{for } \alpha_1,\ldots,\alpha_{g-j}} + \underbrace{j}_{\text{for non-gaps } < 2g \text{ and } > \alpha_{g-j}} + \underbrace{1}_{\text{for } \alpha_g = 2g} = g+1$$

Contradicts to the fact that there are exactly g non-gaps between 1 and $2g$. □

Lemma 4.13. *If $\alpha_1 = 2$, then $\alpha_j = 2j$ for all $j = 1, \ldots, g$.*

Proof. Since the set of non-gaps is a semi-group, then $\alpha_1 + \cdots + \alpha_j = 2j$ are non-gaps for all $j \geq 1$. In particular, $\{2j \mid j = 1, \ldots, g\}$ are g non-gaps $\leq 2g$. □

Lemma 4.14. *For $g \geq 2$, and if $\alpha_1 > 2$, then there exists $0 < j < g$ such that*

$$\alpha_j + \alpha_{g-j} > 2g$$

Proof. If $g = 2$, then $\alpha_2 = 4$, and $2 < \alpha_1 < \alpha_2 = 4$, so $\alpha_1 = 3$. Take $j = 1$, we have $\alpha_j + \alpha_{g-j} = 2\alpha_1 = 6 > 2g$. If $g = 3$, then $\alpha_1 \geq 3, \alpha_2 \geq \alpha_1 + 1 = 4$. Take $j = 1$, then

$$\alpha_j + \alpha_{g-j} = \alpha_1 + \alpha_2 \geq 7 > 2g$$

Now assume $g \geq 4$, assume the lemma fails, then by Lemma 5.5 we have

$$\alpha_j + \alpha_{g-j} = 2g, \quad \forall 0 < j < g$$

Let r be the greates integer $\leq \frac{2g}{\alpha_1}$. Then $k\alpha_1$ is a non-gap $\leq 2g$ for all integers $1 \leq k \leq r$.

If $\alpha_1 > 2$ and $g > 4$, then $r \leq \frac{2g}{\alpha_1} < g - 1 < g$, so there exists a non-gap $\leq 2g$ which is not a multiple of α_1. Let α be the smallest such non-gap.

There exists integer $1 \leq s \leq r$ such that

$$s\alpha_1 < \alpha < (s+1)\alpha_1$$

So we have non-gaps

$$\alpha_1, \alpha_2 = 2\alpha_1, \dots, \alpha_s = s\alpha_1, \alpha_{s+1} = \alpha, \dots$$

and by assumption

$$\alpha_{g-1} = 2g - \alpha_1, \dots, \alpha_{g-s} = 2g - s\alpha_1, \alpha_{g-s-1} = 2g - \alpha$$

Let $n = \alpha_1 + \alpha_{g-s-1} = \alpha_1 + 2g - \alpha = 2g - (\alpha - \alpha_1)$, which is a non-gap. Since $0 < \alpha - \alpha_1 < s\alpha_1$, we have $\alpha_g = 2g > n > 2g - s\alpha_1 = \alpha_{g-s}$. Then there exists non-gap not equal to any of α_j for $g - s \leq j \leq g$. A contradiction to the choice of order of α_i. $\square$

Corollary 4.15. $\sum_{j=1}^{g-1} \alpha_j \geq g(g-1)$ *and the equality holds if and only if* $\alpha_1 = 2$.

Proof. If $g-1$ is even, then

$$\sum_{j=1}^{g-1} \alpha_j = \sum_{j=1}^{\frac{g-1}{2}} (\alpha_j + \alpha_{g-j}) \geq \frac{g-1}{2} \times 2g = g(g-1)$$

If $g-1$ is odd, then

$$\sum_{j=1}^{g-1} \alpha_j = \{\sum_{j=1}^{\frac{g-2}{2}} (\alpha_j + \alpha_{g-j})\} + \alpha_{\frac{g}{2}} \geq \frac{g-2}{2} \times 2g + g = g(g-1)$$

If $\alpha_1 = 2$, then clearly the inequality is an equality. If $\alpha_1 > 2$, then $\alpha_j + \alpha_{g-j} > 2g$, so the above inequality must be a strict inequality. $\square$

Remark 4.16 (forms version)**.** Note that $n \geq 1$ is a gap at p is equivalent to $\dim L(np) = \dim L((n-1)p)$, By Riemann-Roch, we have

$$\dim L(np) - \dim L((n-1)p) = 1 + \dim L(K - np) - \dim L(K - (n-1)p)$$

So $n \geq 1$ is a gap at p is equivalent to

$$\dim L(K - (n-1)p) - \dim L(K - np) = 1$$

But we have isomorphism

$$L^{(1)}(D) = L(K + D)$$

So we can rephrase what is a gap in term of differential forms, that is: $n \geq 1$ is a gap at p is equivalent to there exists $\omega \in L^{(1)}(-(n-1)p)$ but $\omega \notin L^{(1)}(-np)$. In other words, there exists a holomorphic 1-form ω which has a zero at p of order $n-1$.

Lemma 4.17. *n is a gap at p if and only if there exists holomorphic 1-form ω with a zero of order $n-1$ at p.*

Since $n = 1$ is always a gap, we have:

Corollary 4.18. *For all $p \in X$, there exists holomorphic 1-form such that $\omega(p) \neq 0$.*

4.2.2. *Gap numbers for linear system.* Recall that when we say a linear system Q, we means that it corresponds to a subspace $\mathbb{P}(V)$ of projective space $\mathbb{P}(L(D))$ which is isomorphic to $|D|$ by S, where $V \subset L(D)$. We define the dimension of a linear system as its dimension as a complex manifold, that is $\dim Q = \dim V - 1$.

Definition 4.19. *Let Q be a linear system, then define the degree of Q as*

$$\deg(Q) := \deg(E), \quad E \in Q$$

Remark 4.20. It's well-defined, since the degree of principal divisor is zero, and the difference of any two $E, E' \in Q$ is a principal divisor.

Notation 4.21. A linear system Q is g_d^n, if

$$\dim(Q) = n, \quad \deg(Q) = d$$

For any $p \in X$, we define

$$V(-np) = V \cap L(D - np) = \{f \in V \mid \mathrm{ord}_p(f) \geq -D(p) + n\}$$

Note that $V(-np) \subseteq V(-mp)$ if $n \geq m$, since $L(D - np) \subseteq L(D - mp)$ if $n \geq m$. And $V(-np) = \{0\}$ if n is sufficiently large, since if $\deg(D-np) < 0$, then $L(D - np) = \{0\}$.

Moreover, $\dim V(-(n-1)p) - \dim V(-np)$ equals to zero or one. So we can use this to define a gap number for linear system.

Definition 4.22 (gap number for linear system)**.** *An integer $n \geq 1$ is a gap number for Q at p if $V(-np) \subsetneq V(-(n-1)np)$.*

Notation 4.23. We use $G_p(Q)$ to denote the set of all gap numbers of Q at p.

Lemma 4.24. *Let Q be a non-empty linear system on X with $\dim(Q) = r$ and $\deg(Q) = d$. Fix $p \in X$. Then*

1. *$G_p(Q)$ is a finite set with $|G_p(Q)| = 1 + r$;*
2. *$G_p(Q) \subset \{1, \ldots, 1 + d\}$;*
3. *p is a base point of Q is equivalent to $1 \notin G_p(Q)$;*
4. *$1 + d \in G_p(Q)$ is equivalent to $dp \in Q$;*
5. *If $|Q| = |0|$, the complete linear system of zero divisor, then $G_p(Q) = \{1\}$ for all $p \in X$.*

Proof. (1) Let $V \subset L(D)$ be the subspace corresponding to Q. Then $\dim V = \dim Q + 1 = r + 1$ and $\deg(D) = d$. For each gap number n at p,

$$\dim V(-np) = \dim V(-(n-1)p) - 1$$

since $V(0 \cdot p) = V \cap L(D - 0) = V$, and $\dim V = r + 1$. Furthermore, we will see in the proof of (2), $V(-np) = \{0\}$ when n is sufficiently large. So there are exactly $r+1$ gap numbers, since each gap number reduces the dimension by 1.

(2) For $n \geq d+1$, then

$$\deg(D - np) = \deg(D) - n < 0$$

So $L(D - np) = \{0\}$, and $V(-np) = V \cap L(D - np) = \{0\}$ for all $n \geq d+1$, so any number $\geq d+2$ is not a gap number.

(3) If $1 \in G_p(Q)$, then $V = V(0 \cdot p) \neq V(-p) = V \cap L(D - p)$, so there exists $f \in V \subset L(D)$ but $f \notin L(D - p)$, So $\mathrm{ord}_p(f) = -D(p)$. Let $E = S(f) = \mathrm{div}(f) + D \in Q$, then

$$E(p) = \mathrm{ord}_p(f) + D(p) = 0$$

So E is not $\geq p$, by definition of base point, if p is a base of Q, then for any divisor $E \in D$, we have $E \geq p$. So p is not a base point of Q.

Conversely, if $1 \notin G_p(Q)$, we claim that p is a base point of Q. Indeed, if p is not a base point, then there exists $E \in Q$ such that $E(p) < 1$, but $E \geq 0$, then $E(p) = 0$. Since $Q = S(V)$, then there exists $f \in V$ such that

$$E = S(f) = \mathrm{div}(f) + D$$

Note that $0 = E(p) = \mathrm{ord}_f(f) + D(p)$, then

$$\mathrm{ord}_p(f) = -D(p) < -D(p) + 1$$

that is $f \notin L(D - p)$. So $V \neq V(-p) = V \cap L(D - p)$, a contradiction to $1 \notin G_p(Q)$.

(4) $1 + d \in G_p(Q)$ if and only if $V \cap L(D - (d+1)p) \neq V \cap L(D - dp)$, so it's equivalent to that there exists $f \in V$ and $\mathrm{ord}_p(f) = -D(p) + d$. And it's equivalent to there exists $E \in Q$ such that $E(p) = d$, if we take $E = S(f)$. Note that $E \geq 0$ and $\deg(E) = \deg(D) = d$, it's equivalent to $E = dp$.

(5) If $D = 0$, then we have $d = 0, r = 0$, then together (1) and (2) to get desired result. □

Remark 4.25. In particular, if we take D to be the canonical divisor K, and consider its gap number. n is a gap number for $|K|$ at p if and only if

$$L(K - np) \subsetneq L(K - (n-1)p) \Longleftrightarrow L^{(1)}(-np) \subsetneq L^{(1)}(-(n-1)p)$$

By Remark 5.9, we know a gap number for $|K|$ at p is a Weierstrass gap at p, where K is a canonical divisor.

Definition 4.26 (inflection point)**.** *$p \in X$ is an inflection point of a linear system Q if $G_p(Q) \neq \{1, \dots, r+1\}$ where $r = \dim Q$.*

Definition 4.27 (Weierstrass point)**.** *Inflection points for $|K|$ are called Weierstrass points.*

Assume $r = \dim Q = \dim V - 1$, Let

$$G_p(Q) = \{n_1, \dots, n_{r+1}\}$$

with $n_1 < n_2 < \cdots < n_{r+1}$. For each i, choose $f_i \in V(-(n_i-1)p) \backslash V(-n_i p) \neq \varnothing$, that is $f_i \in L(D - (n_i - 1)p) \backslash L(D - n_i p)$. In other words, $\mathrm{ord}_p(f_i) =$

$n_i - 1 - D(p)$. Then $\{f_1, \dots, f_{r+1}\}$ are linearly independent, so they form a basis of V. We call such a basis an inflection basis of V with respect to p.

Definition 4.28 (Wronskian)**.** *Given holomorphic functions $g_1, \dots, g_{r+1}$, the Wronskian of $g_1, \dots, g_{r+1}$ is the function $W[g_1, \dots, g_{r+1}]$ defined by*

$$W[g_1, \dots, g_{r+1}] = \det \begin{pmatrix} g_1(z) & g_1'(z) & \dots & g_1^{(r)}(z) \\ g_2(z) & g_2'(z) & \dots & g_2^{(r)}(z) \\ \vdots & \vdots & \dots & \vdots \\ g_{r+1}(z) & g_{r+1}'(z) & \dots & g_{r+1}^{(r)}(z) \end{pmatrix}$$

Remark 4.29. However, if our g_i are not holomorphic functions, but all of them lie in $L(D)$, then $\mathrm{ord}_p(g_i) \geq -D(p)$. Let z be a local coordinate on X centered at p. we $z^{D(p)}g_i(z)$ is holomorphic in a small neighborhood near $z = 0$, we can use these holomorphic functions to define Wronskian of these special meromorphic functions.

Lemma 4.30. *p is an inflection point for Q if and only if for any basis $\{h_1, \dots, h_{r+1}\}$ of V, the Wronskian $W[z^{D(p)}h_1, \dots, z^{D(p)}h_{r+1}]$ is zero at $z = 0$ where z is a local coordinate centered at p.*

Proof. Let $\{f_1, \dots, f_{r+1}\}$ be an inflection basis of V with respect to p. Then $\mathrm{ord}_p(f_i) = n_i - 1 - D(p)$ with $n_i \in G_p(Q)$ and $n_1 < \cdots < n_{r+1}$. For all p, we have $n_i \geq i, i = 1, \dots, r+1$. So $\mathrm{ord}_p(z^{D(p)}f_i) = n_i - 1$ is strictly increasing as i increases. So $W[z^{D(p)}f_1, \dots, z^{D(p)}f_{r+1}](0)$ is the determinant of an upper triangular matrix which is invertible if and only if $n_i = i$ for all $i = 1, \dots, r+1$. By definition, p is not an inflection point of Q if and only if $n_i = i$ for all i, and that's equivalent to Wronskian $W[z^{D(p)}f_1, \dots, z^{D(p)}f_{r+1}](0)$ is not zero.

For an arbitary basis $\{h_1, \dots, h_{r+1}\}$ of V, there exists $(r+1) \times (r+1)$ invertible matrix A with constant entries such that

$$\begin{bmatrix} h_1 \\ \vdots \\ h_{r+1} \end{bmatrix} = A \begin{bmatrix} f_1 \\ \vdots \\ f_{r+1} \end{bmatrix}$$

Since A is constant, then

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k} z^{D(p)} \begin{bmatrix} h_1 \\ \vdots \\ h_{r+1} \end{bmatrix} = A \frac{\mathrm{d}^k}{\mathrm{d}z^k} (z^{D(p)} \begin{bmatrix} f_1 \\ \vdots \\ f_{r+1} \end{bmatrix})$$

Thus

$$W[z^{D(p)}h_1, \dots, z^{D(p)}h_{r+1}] = \det A \cdot W[z^{D(p)}f_1, \dots, z^{D(p)}f_{r+1}]$$

and $\det A \neq 0$. This completes the proof. □

Corollary 4.31. *For any fixed linear system Q on X, there are only finite number of inflection points of Q.*

Proof. Since X is compact, so we only need to show the set of inflection points of Q is discrete. Assume Q is the linear system corresponding to a subspace $V \subset L(D)$ for some $D \in \mathrm{Div}(X)$. Fix a basis $\{h_1, \dots, h_{r+1}\}$ of V.

Fix $p \in X$ and a local coordinate z centered at p. Let $d = D(p)$. By previous lemma we have that p is an inflection point of Q if and only if

$$W[z^d h_1, \dots, z^d h_{r+1}](0) = 0$$

Since support of D is discrete, there is a neighborhood U of p such that $D(q) = 0$ for all $q \in U \backslash \{0\}$. We can choose U small enough such that the domain of local coordinate z covers U.

Since the k-th column of the matrix in defining Wronskian $W[z^d h_1, \dots, z^d h_{r+1}]$ is given by

$$\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} z^d \begin{bmatrix} h_1 \\ \vdots \\ h_{r+1} \end{bmatrix} = \sum_{n=0}^{k-1} \binom{k-1}{n} (\frac{\mathrm{d}^n}{\mathrm{d}z^n} z^d) \frac{\mathrm{d}^{k-1-n}}{\mathrm{d}z^{k-1-n}} \begin{bmatrix} h_1 \\ \vdots \\ h_{r+1} \end{bmatrix}$$

At points where $z \neq 0$,

In particular, for all $q \in U \backslash \{p\}$, q is an inflection point of Q if and only if $W[h_1, \dots, h_{r+1}]$ is 0 at q, thus if and only if $W[z^d h_1, \dots, z^d h_{r+1}]$ is 0 at q.

So we have proved that for all $q \in U$, q is an inflection point of Q if and only if $W[z^d h_1, \dots, z^d h_{r+1}]$ is 0 at q.

Note that $W[z^d h_1, \dots, z^d h_{r+1}]$ is holomorphic and is not identically equal to 0 since $\{h_1, \dots, h_{r+1}\}$ are linearly independent. So there exists a neighborhood V of p such that $W[z^d h_1, \dots, z^d h_{r+1}]$ is not 0 at every point $q \in V \backslash \{p\}$. This shows that inflection points are isolated. $\square$

Lemma 4.32. *If $g_1(z), \dots, g_l(z)$ are holomorphic and $W[g_1, \dots, g_l]$ is identically equal to zero in a neighborhood of $z = 0$, then $\{g_1, \dots, g_l\}$ is linearly dependent.*

Proof. For any $n_1, \dots, n_l \in \mathbb{Z}_{\geq 0}$, let

$$D(n_1, \dots, n_l) := \det \begin{pmatrix} g_1^{(n_1)}(z) & g_1^{(n_2)}(z) & \dots & g_1^{(n_l)}(z) \\ g_2^{(n_1)}(z) & g_2^{(n_2)}(z) & \dots & g_2^{(n_l)}(z) \\ \vdots & \vdots & \dots & \vdots \\ g_l^{(n_1)}(z) & g_l^{(n_2)}(z) & \dots & g_l^{(n_l)}(z) \end{pmatrix}$$

Then by product formula,

$$\frac{\mathrm{d}}{\mathrm{d}z} D(n_1, \dots, n_l) = \sum_{i=1}^{l} D(n_1, \dots, n_{i-1}, n_i + 1, n_{i+1}, \dots, n_l)$$

Note that $W[g_1, \dots, g_l] = D(0, 1, \dots, l-1)$, we can show by induction on k that

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k} W[g_1, \dots, g_l] = \sum_{\substack{n_1, \dots, n_l \geq 0 \\ n_1 + \dots + n_l = k}} a_{n_1, \dots, n_l} D(0 + n_1, \dots, l - 1 + n_l) \tag{4.1}$$

where $a_{n_1,\dots,n_l}$ are positive integers and $0+n_1 < 1+n_2 < \cdots < l-1+n_l$.

For $n \geq 0$, let

$$v_n = \begin{bmatrix} g_1^{(n)}(0) \\ \vdots \\ g_l^{(n)}(0) \end{bmatrix} \in \mathbb{C}^l$$

We claim that if $W[g_1,\dots,g_l] = 0$ in a neighborhood of $z = 0$, then $\dim_{\mathbb{C}} \operatorname{span}_{\mathbb{C}}\{v_n \mid n \geq 0\} \leq l-1$.

If the claim is false, then there exists positive integers $n_1 < \cdots < n_l$ such that $\{v_{n_1},\dots,v_{n_l}\}$ is linear independent, and for each $i = 1,\dots,l$, n_i is the smallest integer such that $v_{n_i} \notin \operatorname{span}_{\mathbb{C}}\{v_{n_1},\dots,v_{n_{i-1}}\}$.

Note that $D(m_1,\dots,m_l)|_{z=0} = \det\{v_{m_1},\dots,v_{m_l}\}$. For $m_1 < \cdots < m_l$, if $j \in \{1,\dots,l\}$ such that $m_j < n_j$, then

$$D(m_1,\dots,m_l)|_{z=0} = 0$$

since $v_{m_1},\dots,v_{m_j} \in \operatorname{span}_{\mathbb{C}}\{v_{n_1},\dots,v_{n_{j-1}}\}$, and $\dim \operatorname{span}_{\mathbb{C}}\{v_{n_1},\dots,v_{n_{j-1}}\} = j-1$. In particular, if $m_1 < \cdots < m_l$ and $\sum_{i=1}^l m_i = \sum_{i=1}^l n_i$, then

$$D(m_1,\dots,m_l)|_{z=0} = 0, \quad \text{if } (m_1,\dots,m_l) \neq (n_1,\dots,n_l)$$

Since $W[g_1,\dots,g_l] \equiv 0$ in a neighborhood of $z = 0$, then

$$\left.\frac{\mathrm{d}^k}{\mathrm{d}z^k}\right|_{t=0} W[g_1,\dots,g_l] = 0, \quad \text{for all } k \geq 0$$

In particular, for $k = \sum_{i=1}^l (n_i - i + 1)$, equation 5.1 implies $D(n_1,\dots,n_l)|_{z=0} = 0$, contradicts to the fact that $\{v_1,\dots,v_n\}$ is linearly independent. So the claim is proved.

Since $\dim \operatorname{span}\{v_n \mid n \geq 0\} \leq l-1$, there exists $(c_1,\dots,c_l) \in \mathbb{C}^l \backslash \{0\}$ such that

$$\sum_{i=1}^n c_i g_i^{(n)}(0) = (c_1,\dots,c_n)v_n = 0, \quad \text{for all } n \geq 0$$

Let $g(z) = \sum_{i=1}^n c_i g_i(z)$, then $g^{(n)}(0)$ for all $n \geq 0$, then $g(z) \equiv 0$, since g is holomorphic. So $\{g_1,\dots,g_l\}$ is linearly dependent. $\square$

4.2.3. *Number of inflection points.* Since the number of inflection points is finite, it's natural to ask what's the exactly number? To answer this question, we need the concept of higher order differentials.

Definition 4.33 (n-fold differential)**.** *A meromorphic n-fold differential on an open set $V \subset \mathbb{C}$ is an expression of the form $\mu = f(z)(\mathrm{d}z)^n$, where f is a meromorphic function on V. We say μ a n-fold differential in coordinate z.*

Remark 4.34. This is not a n-form or in other words, an exterior product of n copies of $\mathrm{d}z$, since it's meaningless to talk about this on Riemann surface, a 1-dimensional complex manifold. This is just a formal symbol.

Given two local coordinate V_1, V_2 and their transition function T as follows

$$\mathbb{C} \supset V_1 \xleftarrow{T} V_2\mathbb{C}$$

Let $\mu_1 = f(z)(\mathrm{d}z)^n$ be a n-fold differential on V_1, $\mu_2 = g(w)(\mathrm{d}w)^n$ a n-fold differential on V_2.

Definition 4.35 (compatible)**.** *We say μ_1, μ_2 are compatible if*

$$g(w) = f(T(w))(T'(w))^n$$

Definition 4.36 (n-fold differential)**.** *Let X be a Riemann surface, a meromorphic n-fold differential on X is a collection of meromorphic n-fold differential $\{\mu_\alpha\}$, one for each coordinate chart, such that if two charts intersects, the corresponding differential μ_α, μ_β are compatible.*

Remark 4.37. A 1-fold differential is just a 1-form. If $\omega_1, \dots, \omega_n$ are meromorphic 1-forms, define $\mu = \omega_1 \dots \omega_n$ in the following way: For any local coordinate z, locally we can write $\omega_i = f_i(z)\mathrm{d}z$, and μ is given locally by

$$\mu = f_1(z) \dots f_n(z)(\mathrm{d}z)^n$$

Then μ is a meromorphic n-fold differential. In particular, if ω is a 1-form, then ω^n is an n-fold differential.

Lemma 4.38. *Let $f_1, \dots, f_l \in \mathcal{M}(X)$, then we can define a meromorphic $\frac{l(l-1)}{2}$-fold differential $W(f_1, \dots, f_l)$ in the following way: For any local coordinate z,*

$$W(f_1, \dots, f_l) = W[f_1(z), \dots, f_l(z)](\mathrm{d}z)^{\frac{l(l-1)}{2}}$$

Proof. It's obvious $W[f_1(z), \dots, f_l(z)]$ is a meromorphic function of z. So it suffices to show what will happen under the change of coordinate $z = T(w)$. Let

$$V(z) = \begin{bmatrix} f_1(z) \\ \vdots \\ f_l(z) \end{bmatrix}, \quad V^{(k)}(z) = \frac{\mathrm{d}^k}{\mathrm{d}z^k} V(z)$$

Then $W[f_1(z), \dots, f_l(z)] = \det[V(z), \dots, V^{(l-1)}(z)]$ □

Definition 4.39 (order)**.** *Let μ be a meromorphic n-fold differential on X. For $p \in X$, define the order of μ at p by*

$$\mathrm{ord}_p(\mu) = \mathrm{ord}_p(f(z))$$

if $\mu = f(z)(\mathrm{d}z)^n$ in a local coordinate z at p.

Definition 4.40 (divisor)**.** *Let μ be a meromorphic n-fold differential on X, then divisor of μ is defined by*

$$\mathrm{div}(\mu) = \sum_{p \in X} \mathrm{ord}_p(\mu) \cdot p$$

Definition 4.41. *For $D \in \mathrm{Div}(X)$, define*

$$L^{(n)}(D) = \{\textit{meromorphic } n\textit{-fold differential } \mu \textit{ such that } \mathrm{div}(\mu) + D \geq 0\}$$

Lemma 4.42. *Let ω be a meromorphic 1-form on X and $K = \operatorname{div}(\omega)$ be a canonical divisor. For each $D \in \operatorname{Div}(X)$, the multiplicative map*

$$\begin{aligned} F : L(D + nK) &\to L^{(n)}(D) \\ f &\mapsto f\omega^n \end{aligned}$$

is an isomorphism of vector space.

Proof. For any $p \in X$, let z be a local coordinate centered at p. Locally we can write $\omega = g(z)\mathrm{d}z$, then $f\omega^n = f(z)g^n(z)(\mathrm{d}z)^n$. So we can compute the order

$$\begin{aligned} \operatorname{ord}_p(f\omega^n) &= \operatorname{ord}_p(f(z)g^n(z)) \\ &= \operatorname{ord}_p(f) + n \operatorname{ord}_p(g) \\ &= \operatorname{ord}_p(f) + nK(p) \end{aligned}$$

Thus we have

$$\operatorname{div}(f\omega^n) = \operatorname{div}(f) + nK$$

If $f \in L(D + nK)$, then

$$\operatorname{div}(f\omega^n) \geq (-D - nK) + nK = -D$$

So $F(f) = f\omega^n \in L^{(n)}(D)$. It's clear that F is an injective linear map. We need to show that F is also a surjective map. For any $\mu \in L^{(n)}(D)$, if z is a local coordinate, then we can write $\mu = h(z)(\mathrm{d}z)^n$ and $\omega = g(z)\mathrm{d}z$. Let

$$f(z) = \frac{h(z)}{g(z)^n}$$

If $\widetilde{z} = T(w)$ is another local coordinate, then

$$\mu = \widetilde{h}(\widetilde{z})(\mathrm{d}\widetilde{z})^n, \quad \omega = \widetilde{g}(\widetilde{z})\mathrm{d}\widetilde{z}$$

then $h(z) = \widetilde{h}(T(z))(T'(z))^n$ and $g(z) = \widetilde{g}(T(z))T'(z)$, so

$$\frac{h(z)}{g(z)^n} = \frac{\widetilde{h}(T(z))}{\widetilde{g}(T(z))^n} = \frac{\widetilde{h}(\widetilde{z})}{\widetilde{g}(\widetilde{z})}$$

So the definition of f doesn't depend on the choice of local coordinate and defines a meromorphic function on X. It's easy to check $f \in L(D + nK)$ and $F(f) = \mu$. This completes the proof. □

Lemma 4.43. *If $f_1, \dots, f_l \in L(D)$ for some divisor D, then $W(f_1, \dots, f_n) \in L^{(n)}(lD)$, where $n = \frac{l(l-1)}{2}$.*

Let Q be a linear system corresponding to subspace $V \subset L(D)$ for some divisor D. Let $r = \dim Q$, then $\dim V = r + 1$, choose a basis $\{f_1, \dots, f_{r+1}\}$ of V.

Let $W(Q) = W(f_1, \dots, f_{r+1})$ be the n-fold differential defined by Wronskian, where $n = \frac{r(r+1)}{2}$. If we choose another basis of V, then $W(Q)$ will be changed by multiplying with a non-zero constant, so $\operatorname{div}(W(Q))$ doesn't depend on the choice of basis of V.

Lemma 4.44. *For degree of $W(Q)$, we have*

$$\deg(\operatorname{div}(W(Q))) = \sum_p \operatorname{ord}_p(W(Q)) = r(r+1)(g-1)$$

where $r = \dim Q$ and g is the genus of X.

Lemma 4.45. *Let $G_p(Q) = \{n_1, \dots, n_{r+1}\}$ be the set of gap numbers of Q at p with $n_1 < n_2 < \cdots < n_{r+1}$, $\{f_1, \dots, f_{r+1}\}$ a basis of V, $d = D(p)$. Let z be a local coordinate centered at p. Then*

$$\operatorname{ord}_p(W[z^d f_1, \dots, z^d f_{r+1}]) = \sum_{i=1}^{r+1} (n_i - i)$$

Definition 4.46 (inflection weight)**.** *The inflection weight of Q at p is defined to be*

$$w_p(Q) = \sum_{i=1}^{r+1} (n_i - i)$$

where n_i are gap numbers of Q at p.

Remark 4.47. Clearly, p is not an inflection point of Q if and only if $w_p(Q) = 0$. So on a compact Riemann surface, there is only finitely many p such that $w_p(Q) \neq 0$.

Theorem 4.48 (Inflection weight formula)**.** *For a compact Riemann surface with genus g, let Q be a linear system on X with $\dim Q = r, \deg(Q) = d$. Then*

$$\sum_{p \in X} w_p(Q) = (r+1)(d + rg - r)$$

Recall Weierstrass gaps are gaps number for the complete linear system $|K|$ of canonical divisor K. $p \in X$ is a Weierstrass point if p is an inflection point for $|K|$, in other words, $G_p(|K|) \neq \{1, 2, \dots, g\}$.

Note $\dim |K| = \dim L(K) - 1 = g - 1, \deg |K| = \deg(K) = 2g - 2$.

Theorem 4.49. *If X is a compact Riemann surface with genus $g \geq 1$, then there are $g^3 - g$ Weierstrass points on X, each counted according to their weights.*

Proof. By inflection weight formula, $r = g - 1, d = 2g - 2$. So

$$\begin{aligned}
\sum_{p \in X} w_p(|K|) &= g(2g - 2 + (g-1)g - (g-1)) \\
&= g(g-1)(2 + g - 1) \\
&= g(g-1)(g+1) \\
&= g^3 - g
\end{aligned}$$

$\square$

Corollary 4.50. *Let W be the number of Weierstrass points on X without counting weights, then*

$$2g+2 \leq W \leq g^3 - g$$

Proof. Clear $W \leq g^3 - g$. For the other hand, we need to estimate the upper bound of $w_p(|K|)$ for each p.

Claim: For each $p \in X$, $w_p(|K|) \leq \frac{1}{2}g(g-1)$ with equality holds if and only if 2 is not a gap number. In this case $G_p(|K|) = \{1, 3, 5, \ldots, 2g-1\}$. In fact, let $\alpha_1, \ldots, \alpha_{g-1}$ be the first non-gaps at p, we have proved before $\sum_{i=1}^{g-1} \alpha_i \geq g(g-1)$ with equality holds if and only if $\alpha_1 = 2$. So

$$\begin{aligned} w_p(|K|) &= \sum_{n=1}^{2g-1} n - \sum_{i=1}^{g-1} \alpha_i - \sum_{i=1}^{g} i \\ &=\leq \frac{1}{2}(2g-1)2g - g(g-1) - \frac{1}{2}g(g+1) \\ &= \frac{g(g-1)}{2} \end{aligned}$$

with equality holds if and only if 2 is not a gap number. So

$$W \geq (\sum_{p \in X} w_p(|K|))/\frac{1}{2}g(g-1) = (g^3-g)/\frac{1}{2}g(g-1) = 2g+2$$

□

Lemma 4.51. *p is a Weierstrass point if and only if $\dim L(gp) \geq 2$, and this is equivalent to that there exists a non-constant meromorphic function f such that f is holomorphic over $X \backslash \{p\}$ and has a pole at p of order less than g, where g is the genus of X.*

Proof. p is a Weierstrass point if and only if there exists $1 \leq n \leq g$ which is not a gap number, that is there exists $1 \leq n \leq g$ such that

$$\dim L(np) > \dim L((n-1)p) \geq \dim L(0) = 1$$

This is equivalent to $\dim L(gp) \geq 2$, since $\dim L(gp) \geq \dim L(np)$ if $g \geq n$. □

Lemma 4.52. *If X is a hyperelliptic with genus $g \geq 2$ and $F : X \to S^2$ is a holomorphic map with $\deg(F) = 2$. Then Weierstrass points on X are precisely branch points of F. Therefore number of Weierstrass points on X is $2g+2$.*

Proof. Let f be meromorphic function corresponding to $F : X \to S^2$, $p \in X$ a branch point of F. If $F(p) = \infty$, then p is a pole of f of order 2 and f is holomorphic in $X \backslash \{p\}$. So $\dim L(gp) \geq \dim L(2p) \geq 2$ since $g \geq 2$, so p is a Weierstrass point; If $F(p) \neq \infty$, let

$$h = \frac{1}{f - f(p)}$$

then h is holomorphic in $X\backslash\{p\}$, and p is a pole of order 2 of h. By the same reason p is a Weierstrass point.

For a branch p, whether it's a pole or not, 2 is not a gap number at p. Indeed, if 2 is a gap number at p, then

$$\dim L(p) = \dim L(2p) > 1$$

By Lemma 2.73 we have X must be a Riemann sphere, a contradiction.

So gap numbers at p are exactly $G_p(|K|) = \{1, 3, \dots, 2g-1\}$, so weight

$$w_p(|K|) = \sum_{i=1}^{g}\{(2i-1)-1\} = \frac{1}{2}g(g-1)$$

Since there are $2g+2$ branch point of F, then

$$\sum_{p \text{ branch point}} w_p(|K|) = (2g+2)\cdot\frac{1}{2}g(g-1) = g^3 - g = \sum_{p\in X} w_p(|K|)$$

So every Weierstrass point must be a branch point. This completes the proof. $\square$

Lemma 4.53. *Let X be hyperelliptic with genus $g \geq 2$. If F, G are two holomorphic map $X \to S^2$ with degree 2, then F and G have the same branch points and there exists $a, b, c, d \in \mathbb{C}$ such that*

$$G = \frac{a+bF}{c+dF}, \quad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$$

Proof. Branch points of F and G are precisely Weierstrass points of X, so they must coincide. Let f, g be the meromorphic functions on X corresponding to F and G respectively.

Fix a Weierstrass point $p \in X$, we claim $\mathrm{div}_\infty(f) \sim 2p \sim \mathrm{div}_\infty(g)$. Indeed, recall $F^*: \mathrm{Div}(S^2) \to \mathrm{Div}(X)$ preserves linearly equivalent relation. Let $q = F(p)$, then $q \sim \infty$ since any two points on S^2 are linearly equivalent. So

$$\mathrm{div}_\infty(f) = F^*(\infty) \sim F^*(q) = 2p$$

Similarly we have $\mathrm{div}_\infty(g) \sim 2p$. Let $\mathrm{div}_\infty(f) = p_1 + q_1, \mathrm{div}(g) = p_2 + q_2$. Then $f \in L(p_1 + q_1)$ since $\mathrm{div}(f) \geq -\mathrm{div}_\infty(f)$. Similarly we have $g \in L(p_2 + q_2)$.

Since 1 is a gap number for any point, then

$$\dim L(p_1) = \dim L(0) = 1$$
$$\dim L(p_1 + q_1) \leq \dim L(p_1) + 1 \leq 2$$

So we have $\dim L(p_1 + q_1) = 2$, since $f \in L(p_1 + q_1)$ is a non-constant function. Furthemore, $\{1, f\}$ is a basis of $L(p_1 + q_1)$. Similarly $\{1, g\}$ is a basis of $L(p_2 + q_2)$.

Since $p_1 + q_1 \sim p_2 + q_2$, then there exists a meromorphic function h such that $\mathrm{div}(h) = (p_1 + q_1) - (p_2 + q_2)$. And $L(p_1 + q_1) \to L(p_2 + q_2)$ defined by multiplication by h is an isomorphism. So $\{h, hf\}$ is also a basis of $L(p_2 + q_2)$. So there exists $a, b, c, d \in \mathbb{C}$ such that

$$\begin{cases} 1 = ch + dhf \\ g = ah + bhf \end{cases}, \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$$

So we have

$$g = \frac{a + bf}{c + df} \Longleftrightarrow G = \frac{a + bF}{c + dF}$$

□

Corollary 4.54. *Hyperelliptic involution is unique.*

Lemma 4.55. *Let X be a compact Riemann surface with genus $g \geq 2$. Then X is hyperelliptic if and only if X has exactly $2g + 2$ Weierstrass points.*

Proof. It suffices to show X has exactly $2g + 2$ Weierstrass points, then X is hyperelliptic. For any $p \in X$, the weight

$$w_p(|K|) \leq \frac{1}{2} g(g-1)$$

with equality holds if and only if 2 is not a gap number at p. If X has exactly $2g + 2$ Weierstrass points, then

$$g^3 - g = \sum_{p \in X} w_p(|K|) \leq (2g + 2) \cdot \frac{1}{2} g(g-1) = g^3 - g$$

So every Weierstrass point p must have weight $\frac{1}{2} g(g-1)$ and 2 is not a gap number. By definition of Weierstrass gap number, there exists non-constant meromorphic function f such that f is holomorphic in $X \backslash \{p\}$ and has a pole of order 2 at p. The corresponding holomorphic map $F : X \to S^2$ has degree 2, so X is hyperelliptic. This completes the proof. □

Remark 4.56. If X is not hyperelliptic, then number of Weierstrass points of X will $\geq 2g + 6$. You can find a proof in Miranda, page 245.

4.3. **Fixed point of automorphism on hyperelliptic Riemann surface.** Recall we have proved before, for any $T \in \operatorname{Aut}(X)$, T is not identity, then $|\operatorname{Fix}(T)| \leq 2g + 2$. If X is hyperelliptic, $F : X \to S^2$ is holomorphic map with degree 2. Then there exists an involution $J : X \to X$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[rr, "J"] \arrow[dr, "F"'] & & X \arrow[dl, "F"] \\ & S^2 & \end{tikzcd}$$

such that

$$\begin{aligned} \operatorname{Fix}(J) &= \text{set of branch points of } F \\ &= \text{set of Weierstrass points of } X \end{aligned}$$

J is called the hyperelliptic involution of X, clearly $|\operatorname{Fix}(J)| = 2g + 2$.

Lemma 4.57. *If X is hyperelliptic with genus $g \geq 2$, $T \in \operatorname{Aut}(X)$. If $T \notin \{identity, J\}$, then $|\operatorname{Fix}(T)| \leq 4$.*

Proof. Given $F : X \to S^2$ is a holomorphic map with degree 2. Let $G = F \circ T$, then clearly $\deg G = 2$. By Lemma 5.46, there exists

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})/\{\pm \mathrm{I}_2\} = \mathrm{PSL}(2, \mathbb{C})$$

such that

$$G = \frac{a + bF}{c + dF}$$

Note that A gives a automorphism on S^2, so $F \circ T = G = A \circ F$, that is $F(T(p)) = A(F(p))$ for all $p \in X$. So if $p \in \operatorname{Fix}(T)$, then $F(p) \in \operatorname{Fix}(A)$.

Here is a fact which will be proved later: if $T \notin \{\text{Identity}, J\}$, then $A \neq \pm \mathrm{I}_2$. And the following Exercise tells us the upper bound of fixed points of Möbius transformation which is not an identity.

Exercise 4.58. Any automorphism of S^2 which is not an identity won't have more than 2 fixed points.

So $|\operatorname{Fix}(A)| \leq 2$. Note that $\operatorname{Fix}(F) \subset F^{-1}(\operatorname{Fix}(A))$, so we have $|\operatorname{Fix}(T)| \leq 4$ since $\deg(F) = 2$. $\square$

If $A = \pm \mathrm{I}_2$, then $G = F$ and T must be an involution which fixes all branch points of F, so the fact we mentioned in the proof follows from following theorem.

Theorem 4.59. *Let X be a compact Riemann surface with genus $g \geq 2$.* Identity $\neq T \in \operatorname{Aut}(X)$ *such that $T^2 = \mathrm{Id}$ and $|\operatorname{Fix}(T)| \geq 2g + 2$. Then X is hyperelliptic and T is the hyperelliptic involution.*

Proof. We have proved before $|\operatorname{Fix}(T)| \leq 2g + 2$, then $|\operatorname{Fix}(T)| = 2g + 2$. Let $H = \{\text{Identity}, T\}$ be the subgroup of $\operatorname{Aut}(X)$ generated by T. So $H \cong \mathbb{Z}/2\mathbb{Z}$ since T is an involution. Let $M = X/H$, endowed with quotient topology, $\pi : X \to M$ projection.

Fix any $p \in X$, if $T(p) \neq p$, then there exists a local coordinate defined on a neighborhood U of p such that $T(U) \cap U = \varnothing$, giving a local coordinate in a neighborhood of $\pi(p)$.

Here we give a fact which will be proved later: if $T(p) = p$, there exists local coordinate chart centered at p

$$X \supset U \xrightarrow{\varphi} D_r = \{z \in \mathbb{C} \mid |z| < r\}$$

such that $T(U) = U$ and $T(z) = -z$. Consider the following communicative diagram

$$\begin{tikzcd} X \supset U \arrow[r, "\varphi"] \arrow[d, "\pi"] & D_r \arrow[d, "z^2"] \\ M \supset \pi(U) \arrow[r, "\psi"] & D_r \end{tikzcd}$$

defines a local coordinate of $\pi(U)$. So $(\pi(U), \psi)$ gives a local coordinate in a neighborhood of $\pi(p)$. So M is a compact Riemann surface, and $\pi : X \to M$ is a holomorphic map with degree 2. By Hurwitz formula

$$2\,\mathrm{genus}(X) - 2 = 2(2\,\mathrm{genus}(M) - 2) + 2g + 2$$

So $\mathrm{genus}(M) = 0$ and $M \cong S^2$. Thus X is hyperelliptic and T is hyperelliptic involution. $\square$

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