RIEMANNIAN SYMMETRIC SPACE

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CONTENTS

1. GEOMETRIC VIEWPOINTS

1.A. **Basic definitions and properties.**

1.A.1. *Riemannian symmetric space.*

Definition 1.1 (Riemannian symmetric space)**.** *A Riemannian manifold (M, g) is called a Riemannian symmetric space if for each $p \in M$ there exists an isometry $\varphi : M \to M$, which is called a symmetry at p, such that $\varphi(p) = p$ and $(\mathrm{d}\varphi)_p = -\mathrm{id}$.*

Remark 1.2. Note that Theorem A.1, that is rigidity property of isometry, implies if symmetry at point p exists, then it's unique.

Example 1.3. Let g_{can} be the Euclidean metric on $\mathbb{R}^n$. For each $p \in \mathbb{R}^n$, the reflection

$$\varphi(x) = 2p - x$$

is a symmetric at point p. Thus $(\mathbb{R}^n, g_{\mathrm{can}})$ is a Riemannian symmetric space.

Example 1.4. Let g_{can} be the metric of S^n induced from $(\mathbb{R}^{n+1}, g_{\mathrm{can}})$. For each $p \in S^n$, the reflection

$$\varphi(x) = 2\langle x, p\rangle p - x$$

is a symmetric at point p. Thus (S^n, g_{can}) is a Riemannian symmetric space.

Proposition 1.5. *The following statements are equivalent.*

(1) (M, g) is a Riemannian symmetric space.
(2) For each $p \in M$, there exists an isometry $\varphi : M \to M$ such that $\varphi^2 = \mathrm{id}$ and p is an isolated fixed point of φ.

Proof. From (1) to (2). Let φ be a symmetry at $p \in M$. Since $(\mathrm{d}\varphi^2)_p = (\mathrm{d}\varphi)_p \circ (\mathrm{d}\varphi)_p = \mathrm{id}$ and $\varphi^2(p) = p$, one has $\varphi^2 = \mathrm{id}$ by Theorem A.1. If p is not an isolated fixed point, then there exists a sequence $\{p_i\}_{i=1}^{\infty}$ converging to p such that $\varphi(p_i) = p_i$. For $0 < \delta < \mathrm{inj}(p)$, there exists sufficiently large k such that $p_k \in B(p, \delta)$, and we denote $v = \exp_p^{-1}(p_k)$. Since φ is an isometry, one has $\varphi(\exp_p(tv))$ and $\exp_p(tv)$ are two geodesics connecting p and p_k, and thus

$$\varphi(\exp_p(tv)) = \exp_p(tv)$$

by uniqueness. In particular, one has $v = (\mathrm{d}\varphi)_p v$, which is a contradiction.

From (2) to (1). From $\varphi^2 = \mathrm{id}$ we have $(\mathrm{d}\varphi)_p^2 = \mathrm{id}$, so only possible eigenvalues of $(\mathrm{d}\varphi)_p$ are ± 1. Now it suffices to show all eigenvalues of $(\mathrm{d}\varphi)_p$ are -1. Otherwise if it has an eigenvalue 1, there exists some non-zero $v \in T_pM$ such that $(\mathrm{d}\varphi)_p v = v$. Since φ is an isometry, one has $\varphi(\exp_p(tv))$ and $\exp_p(tv)$ are geodesics with the same direction at p. Thus

$$\varphi(\exp_p(tv)) = \exp_p(tv)$$

for $0 < t < \mathrm{inj}(p)$. In particular, p is not an isolated fixed point, which is a contradiction. □

Proposition 1.6. *The fundamental group of a Riemannian symmetric space is abelian.*

Corollary 1.7. *A surface of genus $g \geq 2$ does not admit a Riemannian metric with respect to which it is a symmetric space.*

1.A.2. *Locally Riemannian symmetric space.*

Definition 1.8 (locally Riemannian symmetric space). *A Riemannian manifold (M, g) is called a locally Riemannian symmetric space if each $p \in M$ has a neighborhood U such that there exists an isometry $\varphi : U \to U$ such that $\varphi(p) = p$ and $(\mathrm{d}\varphi)_p = -\mathrm{id}$.*

Theorem 1.9. *Let (M, g) be a complete Riemannian manifold. The following statements are equivalent.*
(1) (M, g) is a locally Riemannian symmetric space.
(2) $\nabla R = 0$.

Proof. From (1) to (2). If φ is the symmetry at point $p \in M$, then it's an isometry such that $(\mathrm{d}\varphi)_p = -\mathrm{id}$, and thus for $u, v, w, z \in T_pM$, one has

$$\begin{aligned} -\nabla_u R(v, w)z &= (\mathrm{d}\varphi)_p \left(\nabla_u R(v, w)z\right) \\ &= \nabla_{(\mathrm{d}\varphi)_p u}((\mathrm{d}\varphi)_p)v, (\mathrm{d}\varphi)_p w)(\mathrm{d}\varphi)_p z \\ &= \nabla_u R(v, w)z \end{aligned}$$

This shows $(\nabla R)_p = 0$, and thus $\nabla R = 0$ since p is arbitrary.

From (2) to (1). For arbitrary $p \in M$, it suffices to show

$$\varphi = \exp_p \circ \Phi_0 \circ \exp_p^{-1} : B(p, \delta) \to B(p, \delta)$$

is an isometry, where $0 < \delta < \mathrm{inj}(p)$ and $\Phi_0 = -\mathrm{id} : T_pM \to T_pM$. For $v \in T_pM$ with $|v| < \delta$ and $\gamma(t) = \exp_p(tv), \widetilde{\gamma}(t) = \exp_p(t\Phi_0(v))$, if we define

$$\Phi_t = P_{0,t;\widetilde{\gamma}} \circ \Phi_0 \circ P_{t,0;\gamma}$$

then direct computation shows

$$\begin{aligned} \Phi_t^* R_{\widetilde{\gamma}(t)} &= (P_{t,0;\gamma})^* \circ \Phi_0^* \circ (P_{0,t;\widetilde{\gamma}})^* R_{\widetilde{\gamma}(t)} \\ &\overset{(a)}{=} (P_{t,0;\gamma})^* \circ \Phi_0^* R_{\widetilde{\gamma}(0)} \\ &\overset{(b)}{=} (P_{t,0;\gamma})^* R_{\gamma(0)} \\ &\overset{(c)}{=} R_{\gamma(t)} \end{aligned}$$

where

(a) and (c) holds from Proposition A.5.
(b) holds from $\widetilde{\gamma}(0) = \gamma(0)$ and R is a $(0, 4)$-tensor.

Then by Theorem A.2, that is Cartan-Ambrose-Hicks's theorem, φ is an isometry, which completes the proof. □

1.B. **Symmetric space, locally symmetric space and homogeneous space.** In this section, we will show any complete locally Riemannian symmetric space is a quotient of Riemannian symmetric space (Corollary 1.11), and any Riemannian symmetric space is a Riemannian homogeneous space (Corollary 1.16).

1.B.1. *Riemannian symmetric space and locally Riemannian symmetric space.*

Theorem 1.10. *Let (M, g) be a complete, simply-connected locally Riemannian symmetric space. Then (M, g) is a Riemannian symmetric space.*

Proof. For $p \in M$ and $0 < \delta < \operatorname{inj}(p)$, suppose $\varphi : B(p,\delta) \to B(p,\delta)$ is an isometry such that $\varphi(p) = p$ and $(\mathrm{d}\varphi)_p = -\mathrm{id}$. For arbitrary $q \in M$, we use $\Omega_{p,q}$ to denote all curves γ with $\gamma(0) = p, \gamma(1) = q$, and for $c \in \Omega_{p,q}$ we choose[1] a covering $\{B(p_i,\delta_i)\}_{i=0}^k$ of c such that

(1) $0 < \delta_i < \operatorname{inj}(p_i)$.
(2) $B(p_0,\delta_0) = B(p,\delta)$ and $p_k = q$.
(3) $p_{i+1} \in B(p_i,\delta_i)$.

If we set $\varphi = \varphi_0$, then we can define isometries $\varphi_i : B(p_i,\delta_i) \to M$ such that $\varphi_i(p_i) = \varphi_{i-1}(p_i)$ and $(\mathrm{d}\varphi_i)_{p_i} = (\mathrm{d}\varphi_{i-1})_{p_i}$ by using Cartan-Ambrose-Hicks's theorem successively, and by Theorem A.1 one has φ_i and φ_{i+1} coincide on $B(p_i,\delta_i) \cap B(p_{i+1},\delta_i)$. The covering together with isometries we construct is denoted by $\mathcal{A} = \{B(p_i,\delta_i),\varphi_i\}_{i=0}^k$. For arbitrary $x \in [0,1]$, if $c(x) \in B(p_m,\delta_m)$, we may define

$$\begin{aligned}\varphi_{\mathcal{A}}(c(x)) &:= \varphi_m(c(x))\\ (\mathrm{d}\varphi_{\mathcal{A}})_{c(x)} &:= (\mathrm{d}\varphi_m)_{c(x)}\end{aligned}$$

In particular, $\varphi_{\mathcal{A}}(q) := \varphi_k(q)$. If $\mathcal{B} = \{\widetilde{B}(\widetilde{p}_i,\widetilde{\delta}_i),\widetilde{\varphi}_i\}_{i=0}^l$ is another covering of c, let's show $\varphi_{\mathcal{A}}(q) = \varphi_{\mathcal{B}}(q)$. Consider

$$I = \{x \in [0,1] \mid \varphi_{\mathcal{A}}(c(x)) = \varphi_{\mathcal{B}}(c(x)), (\mathrm{d}\varphi_{\mathcal{A}})_{c(x)} = (\mathrm{d}\varphi_{\mathcal{B}})_{c(x)}\}$$

It's clear $I \neq \varnothing$, since $0 \in I$. Now it suffices to show it's both open and closed to conclude $1 \in I$.

(a) It's open: For $x \in I$, we assume $c(x) \in B(p_m,\delta_m) \cap \widetilde{B}(\widetilde{p}_n,\widetilde{\delta}_n)$, that is

$$\begin{aligned}\varphi_m(c(x)) &= \widetilde{\varphi}_n(c(x))\\ (\mathrm{d}\varphi_m)_{c(x)} &= (\mathrm{d}\widetilde{\varphi}_n)_{c(x)}\end{aligned}$$

Then one has

$$\begin{aligned}\varphi_m \circ \exp_{c(x)}(v) &= \exp_{\varphi_m(c(x))} \circ (\mathrm{d}\varphi_m)_{c(x)}(v)\\ &= \exp_{\widetilde{\varphi}_n(c(x))} \circ (\mathrm{d}\widetilde{\varphi}_n)_{c(x)}(v)\\ &= \widetilde{\varphi}_n \circ \exp_{c(x)}(v)\end{aligned}$$

Since $\exp_{c(x)}$ maps onto a neighborhood of $c(x)$, it follows that some neighborhood of x also lies in I, and thus I is open.

(b) It's closed: Let $\{x_i\}_{i=1}^\infty \subseteq I$ be a sequence converging to x. Without lose of generality we may assume $\{x_i\}_{i=1}^\infty \subseteq B(p_m,\delta_m) \cap \widetilde{B}(\widetilde{p}_n,\widetilde{\delta}_n)$, then one has

$$\begin{aligned}\varphi_m(c(x_i)) &= \widetilde{\varphi}_n(c(x_i))\\ (\mathrm{d}\varphi_m)_{c(x_i)} &= (\mathrm{d}\widetilde{\varphi}_n)_{c(x_i)}\end{aligned}$$

By taking limit we obtain the desired results.

Since $\varphi_{\mathcal{A}}(q)$ is independent of the choice of coverings, we use $\varphi(q)$ to denote it for convenience, and as a consequence we obtain the following map

$$\begin{aligned}F : \Omega_{p,q} &\to M\\ c &\mapsto \varphi(q)\end{aligned}$$

[1]Since injective radius is a continuous function, it has a positive minimum on curve c, so such covering exists.

Note that $F(c)$ is locally constant, and thus it's independent of the choice of homotopy classes of c. Since M is simply-connected, one has $F: \Omega_{p,q} \to M$ is constant, so we obtain a local isometry $\varphi: M \to M$ which extends $\varphi: B(p,\delta) \to B(p,\delta)$. By Proposition A.3 φ is a Riemannian covering map since M is complete, and thus φ is a diffeomorphism since M is simply-connected, which implies φ is an isometry. □

Corollary 1.11. *Let (M,g) be a complete locally Riemannian symmetric space. Then it's isometric to $(\widetilde{M}/\Gamma, \widetilde{g})$ where $(\widetilde{M}, \widetilde{g})$ is a Riemannian symmetric space and Γ is a discrete Lie group acting on $\widetilde{M}$ freely, properly and isometrically.*

Proof. Let $(\widetilde{M}, \widetilde{g})$ be the universal covering of (M,g) with pullback metric. Then $(\widetilde{M}, \widetilde{g})$ is a simply-connected Riemannian manifold with parallel curvature tensor. Furthermore, by Proposition A.6 it's complete, hence it is symmetric. □

As a consequence, above argument about analytic continuation can be used to give a proof of Hopf's theorem.

Theorem 1.12 (Hopf)**.** *Let (M,g) be a complete, simply-connected Riemannian manifold with constant sectional curvature K. Then (M,g) is isometric to*

$$(\widetilde{M}, g_{can}) = \begin{cases} (\mathbb{S}^n(\frac{1}{\sqrt{K}}), g_{can}) & K > 0 \\ (\mathbb{R}^n, g_{can}) & K = 0 \\ (\mathbb{H}^n(\frac{1}{\sqrt{-K}}), g_{can}) & K < 0 \end{cases}$$

Proof. For $p \in M, \widetilde{p} \in \widetilde{M}$ and $\delta < \min\{\mathrm{inj}(p), \mathrm{inj}(\widetilde{p})\}$. By Cartan-Ambrose-Hicks's theorem, there exists an isometry $\varphi: B(p,\delta) \to B(\widetilde{p},\delta)$ such that $\varphi(p) = \widetilde{p}$ and $(\mathrm{d}\varphi)_p$ equals to a given linear isometry, since both (M,g) and $(\widetilde{M}, \widetilde{g})$ have constant sectional curvature K. By the same argument in proof of Theorem 1.10, there is an isometry $\varphi: (M,g) \to (\widetilde{M}, \widetilde{g})$ which extends $\varphi: B(p,\delta) \to B(\widetilde{p},\delta)$. In particular, this completes the proof. □

1.B.2. *Riemannian symmetric space and Riemannian homogeneous space.*

Definition 1.13 (Riemannian homogeneous space)**.** *A Riemannian manifold (M,g) is called a Riemannian homogeneous space, if $\mathrm{Iso}(M,g)$ acts on M transitively.*

Proposition 1.14. *Let (M,g) be a Riemannian homogeneous space. If there exists a symmetry at some point $p \in M$, then (M,g) is a Riemannian symmetric space.*

Proof. Let φ be a symmetry at $p \in M$. For arbitrary $q \in M$, there exists an isometry $\psi: M \to M$ such that $\psi(p) = q$ since (M,g) is a Riemannian homogeneous space. Then

$$\varphi_q := \psi \circ \varphi \circ \psi^{-1}$$

is the desired symmetry at q. □

Theorem 1.15. *Let (M,g) be a Riemannian symmetric space. Then*

(1) (M,g) is complete.

(2) for any isometry $\varphi: M \to M$ with $(\mathrm{d}\varphi)_p = -\mathrm{id}$ and $\varphi(p) = p$, if $v \in T_pM$, then

$$\varphi(\exp_p(v)) = \exp_p(-v)$$

(3) the isometry group $\mathrm{Iso}(M, g)$ *acts transitively on* M.

Proof. For (1). For arbitrary geodesic $\gamma : [0,1] \to M$ with $\gamma(0) = p, \gamma'(0) = v$, the curve $\beta(t) = \varphi(\gamma(t)) : [0,1] \to M$ is also a geodesic with $\beta(0) = p$ and $\beta'(0) = -v$. Now we obtain a smooth extension $\gamma' : [0,2] \to M$ of γ, given by

$$\gamma'(t) = \begin{cases} \beta(1-t), & t \in [0,1] \\ \gamma(t-1), & t \in [1,2] \end{cases}$$

Repeat above process to extend γ to a geodesic defined on $\mathbb{R}$, this shows completeness.

For (2). Note that $\varphi(\exp_p(tv)$ and $\exp_p(-tv)$ are geodesics starting at p with the same direction since φ is an isometry, and thus $\varphi(\exp_p(tv)) = \exp_p(-tv)$. Furthermore, since (M, g) is complete, one has $\varphi(\exp_p(tv)$ and $\exp_p(-tv)$ are defined on $\mathbb{R}$. In particular, one has $\varphi(\exp_p(v)) = \exp_p(-v)$ by considering $t = 1$.

For (3). Let $\gamma : [0,1] \to M$ be a geodesic connecting $p, q \in M$, and $\varphi_m : M \to M$ is the symmetry at $m = \gamma(\frac{1}{2})$. If we consider $\beta(t) = \varphi_m(\gamma(\frac{1}{2} - t))$, then $\beta(0) = m, \beta'(0) = \gamma'(\frac{1}{2})$, which implies $\beta(t) = \gamma(\frac{1}{2} + t)$. Therefore $q = \gamma(1) = \beta(\frac{1}{2}) = \varphi_m(\gamma(0)) = \varphi_m(p)$. $\square$

Corollary 1.16. *The Riemannian symmetric space* (M, g) *is a Riemannian homogeneous space.*

2.A. **Review of Killing fields.**

2.A.1. *Basic properties.*

Proposition 2.1. *Let (M, g) be a Riemannian manifold and X be a Killing field.*
(1) If γ is a geodesic, then $J(t) = X(\gamma(t))$ is a Jacobi field.
(2) For any two vector fields Y, Z,

$$\nabla_Y \nabla_Z X - \nabla_{\nabla_Y Z} X + R(X, Y)Z = 0$$

Proof. For (1). Suppose φ_s is the flow generated by X. Then we obtain a variation $\alpha(s,t) = \varphi_s(\gamma(t))$ consisting of geodesics, and thus

$$X(\gamma(t)) = \frac{\partial \varphi_s(\gamma(t))}{\partial s}\Big|_{s=0}$$

is a Jacobi field.

For (2). It's an equation of tensors, so we check it pointwisely and use normal coordinate $\{x^i\}$ centered at p. Furthermore, we assume $X = X^i \frac{\partial}{\partial x^i}, Y = \frac{\partial}{\partial x^j}, Z = \frac{\partial}{\partial x^k}$. Then

$$\begin{aligned}\nabla_Y \nabla_Z X - \nabla_{\nabla_Y Z} X + R(X, Y)Z &= \nabla_j \nabla_k X + X^i R^l_{ijk} \frac{\partial}{\partial x^l} \\ &= \Big(\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma^l_{ki}}{\partial x^j} + X^i R^l_{ijk}\Big)\frac{\partial}{\partial x^l} \\ &= \Big(\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma^l_{jk}}{\partial x^i}\Big)\frac{\partial}{\partial x^l}\end{aligned}$$

since $R^l_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \Gamma^s_{jk}\Gamma^l_{is} - \Gamma^s_{ik}\Gamma^l_{js}$. Now it suffices to show $\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma^l_{jk}}{\partial x^i} \equiv 0$. In order to show this, for arbitrary $p \in M$, consider a geodesic γ starting at p and consider Jacobi field $J(t) = X(\gamma(t))$. Direct computation shows

$$\begin{aligned}J'(t) &= \Big(\frac{\partial X^i}{\partial x^k}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t} + X^i \Gamma^l_{ki}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}\Big)\frac{\partial}{\partial x^l}\Big|_{\gamma(t)} \\ J''(0) &= \Big(\frac{\partial^2 X^l}{\partial x^j \partial x^k}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t} + X^i \frac{\partial \Gamma^l_{ki}}{\partial x^j}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}\Big)\frac{\partial}{\partial x^l}\Big|_p \\ &= \Big(\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma^l_{ki}}{\partial x^j}\Big)\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}\frac{\partial}{\partial x^l}\Big|_p \\ &= \Big(\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma^l_{jk}}{\partial x^i} + X^i \frac{\partial \Gamma^l_{ki}}{\partial x^j} - X^i \frac{\partial \Gamma^l_{jk}}{\partial x^i}\Big)\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}\frac{\partial}{\partial x^l}\Big|_p \\ &= \Big(\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma^l_{jk}}{\partial x^i}\Big)\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}\frac{\partial}{\partial x^l}\Big|_p - R(X, \gamma')\gamma'\end{aligned}$$

which implies

$$\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma^l_{jk}}{\partial x^i} = 0$$

holds at point p, and since p is arbitrary, this completes the proof. □

Corollary 2.2. *Let (M,g) be a complete Riemannian manifold and $p \in M$. Then a Killing field X is determined by the values X_p and $(\nabla X)_p$ for arbitrary $p \in M$.*

Proof. The equation $\mathcal{L}_X g \equiv 0$ is linear in X, so the space of Killing fields is a vector space. Therefore, it suffices to show if $X_p = 0$ and $(\nabla X)_p = 0$, then $X \equiv 0$. For arbitrary $q \in M$, let $\gamma : [0,1] \to M$ be a geodesic connecting p and q with $\gamma'(0) = v$. Since $J(t) = X(\gamma(t))$ is a Jacobi field, and a direct computation shows

$$(\nabla_v X)_p = J'(0)$$

Thus $J(t) \equiv 0$, since Jacobi field is determined by two initial values. In particular, $X_q = J(1) = 0$, and since q is arbitrary, one has $X \equiv 0$. □

Corollary 2.3. *The dimension of vector space consisting of Killing fields $\leq n(n+1)/2$.*

Proof. Note that ∇X is skew-symmetric and the dimension of skew-symmetric matrices is $n(n-1)/2$. Thus the dimension of vector space consisting of Killing fields $\leq n + n(n-1)/2 = n(n+1)/2$. □

2.A.2. *Killing field as the Lie algebra of isometry group.*

Lemma 2.4. *Killing field on a complete Riemannian manifold (M,g) is complete.*

Proof. For a Killing field X, we need to show the flow $\varphi_t : M \to M$ generated by X is defined for $t \in \mathbb{R}$. Otherwise, we assume φ_t is defined on (a,b). Note that for each $p \in M$, curve $\varphi_t(p)$ is a curve defined on (a,b) having finite constant speed, since φ_t is isometry. Then we have $\varphi_t(p)$ can be extended to the one defined on $\mathbb{R}$, since M is complete. □

Theorem 2.5. *Let (M,g) be a complete Riemannian manifold and $\mathfrak{g}$ the space of Killing fields. Then $\mathfrak{g}$ is isomorphic to the Lie algebra of $G = \mathrm{Iso}(M,g)$.*

Proof. It's clear $\mathfrak{g}$ is a Lie algebra since $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$. Now let's see it's isomorphic to Lie algebra consisting of Killing field as Lie algebra.

(1) Given a Killing field X, by Lemma 2.4, one deduces that the flow $\varphi : \mathbb{R} \times M \to M$ generated by X is a one parameter subgroup $\gamma : \mathbb{R} \to G$, and $\gamma'(0) \in T_e G$.
(2) Given $v \in T_e G$, consider the one-parameter subgroup $\gamma(t) = \exp(tv) : \mathbb{R} \to G$ which gives a flow by
$$\begin{aligned} \varphi : \mathbb{R} \times M &\to M \\ (t,p) &\mapsto \exp(tv) \cdot p \end{aligned}$$
Then the vector field X generated by this flow is a Killing field.

This gives a one to one correspondence between Killing fields and Lie algebra of G, and it's a Lie algebra isomorphism. □

Corollary 2.6 (Cartan decomposition)**.** *Let (M,g) be a complete Riemannian manifold and $G = \mathrm{Iso}(M,g)$ with Lie algebra $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ of G has a decomposition as vector spaces*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

where

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid X_p = 0\}$$
$$\mathfrak{m} = \{X \in \mathfrak{g} \mid (\nabla X)_p = 0\}$$

and they satisfy

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$$

Proof. The decomposition follows from Corollary 2.2 and Theorem 2.5, and it's easy to see

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$$

For arbitrary $X \in \mathfrak{k}, Y \in \mathfrak{m}$ and $v \in T_pM$, one has

$$\begin{aligned}\nabla_v[X, Y] &= \nabla_v \nabla_X Y - \nabla_v \nabla_Y X \\ &= -R(Y, v)X + \nabla_{\nabla_v X} Y + R(X, v)Y - \nabla_{\nabla_v Y} X \\ &= 0\end{aligned}$$

since $X_p = 0$ and $(\nabla Y)_p = 0$. This shows $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$. □

2.B. **Riemannian symmetric space as a quotient.**

Definition 2.7 (involution)**.** *Let G be a Lie group. An automorphism σ of G is called an involution if $\sigma^2 = \mathrm{id}_G$.*

Definition 2.8 (Cartan decomposition)**.** *Let G be a Lie group and σ be an involution of G. The eigen-decomposition of $\mathfrak{g}$ given by $(\mathrm{d}\sigma)_e$ is called Cartan decomposition, that is,*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

where

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid (\mathrm{d}\sigma)_e(X) = X\}$$
$$\mathfrak{m} = \{X \in \mathfrak{g} \mid (\mathrm{d}\sigma)_e(X) = -X\}$$

Proposition 2.9. *Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be a Cartan decomposition given by σ. Then*

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$$

Proof. It follows from

$$(\mathrm{d}\sigma)_e([X, Y]) = [(\mathrm{d}\sigma)_e(X), (\mathrm{d}\sigma)_e(Y)]$$

where $X, Y \in \mathfrak{g}$. □

Theorem 2.10. *Let (M, g) be a Riemannian symmetric space and G be the identity component of $\mathrm{Iso}(M, g)$. For $p \in M$, K denotes the isotropic group of G_p.*

(1) The mapping $\sigma : G \to G$, given by $\sigma(g) = s_p g s_p$ is an involution automorphism of G.
(2) If G^σ is the set of fixed points of σ in G, and $(G^\sigma)_0$ is the identity component of G^σ, then $(G^\sigma)_0 \subseteq K \subseteq G^\sigma$.
(3) If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the Cartan decomposition given by σ, then $\mathfrak{k}$ is the Lie algebra of K.
(4) There is a left invariant metric on G which is also right invariant under K, such that G/K with the induced metric is isometric to (M, g).

Proof. For (1). σ is an involution since for arbitrary $g \in G$, one has $\sigma^2(g) = \sigma(s_p g s_p) = s_p^2 g s_p^2 = g$ since $s_p^2 = \mathrm{id}$.

For (2). It follows from the following two steps:

(a) To show $K \subseteq G^\sigma$. For any $k \in K$, in order to show $k = s_p k s_p$, it suffices to show they and their differentials agree at some point by Theorem A.1, since both of them are isometries, and p is exactly the point we desired.

(b) To see $(G^\sigma)_0 \subseteq K$. Suppose $\exp(tX) \subseteq (G^\sigma)_0$ is a one-parameter subgroup. Since $\sigma(\exp(tX)) = \exp(tX)$, one has

$$\exp(tX)(p) = s_p \exp(tX) s_p(p) = s_p \exp(tX)(p)$$

But p is an isolated fixed point of s_p, which implies $\exp(tX)(p) = p$ for all t. This shows the one-parameter subgroup lies in K. Since exponential map of Lie group is a diffeomorphism in a small neighborhood of identity element e and $(G^\sigma)_0$ can be generated by a neighborhood of e, which implies the whole $(G^\sigma)_0 \subseteq K$.

For (3). Note that $(G^\sigma)_0 \subseteq K \subseteq G^\sigma$, it suffices to show $\mathfrak{k} \cong \operatorname{Lie} G^\sigma$. For $X \in \mathfrak{k}$, we claim $\gamma_2(t) = \sigma(\exp(tX)) : \mathbb{R} \to G$ is a one-parameter subgroup. Indeed, note that

$$\begin{aligned}\gamma_2(t) \cdot \gamma_2(s) &= s_p \exp(tX) s_p \cdot s_p \exp(sX) s_p \\ &= \sigma(\exp(tX + sX)) \\ &= \gamma_2(t+s)\end{aligned}$$

Furthermore, $\gamma_2(t) = \sigma(\exp(tX))$ and $\gamma_1(t) = \exp(tX)$ are two one-parameter subgroups of G such that $\gamma_1(0) = \gamma_2(0)$ and $\gamma_2'(0) = (\mathrm{d}\sigma)_e(X) = X = \gamma_1'(0)$. Then $\gamma_1(t) = \gamma_2(t)$, and thus $\exp(tX) \in G^\sigma$ for all $t \in \mathbb{R}$. This shows $\mathfrak{k} \subseteq \operatorname{Lie} G^\sigma$, and the converse inclusion is clear, so one has $\mathfrak{k} = \operatorname{Lie} G^\sigma$.

For (4). Let $\pi : G \to M$ be the natural projection given by $\pi(g) = gp$. Then for $k \in K$ and $X \in \mathfrak{g}$ one has

$$\begin{aligned}(\mathrm{d}\pi)_e(\mathrm{Ad}_k X) &= (\mathrm{d}\pi)_e\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} k\exp(tX)k^{-1}\right) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \pi(k\exp(tX)k^{-1}) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} k\exp(tX)k^{-1} \cdot p \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} k\exp(tX) \cdot p \\ &= (\mathrm{d}L_k)_p(\mathrm{d}\pi)_e(X)\end{aligned}$$

By using the equivalent isomorphism $(\mathrm{d}\pi)_e|_{\mathfrak{m}} : \mathfrak{m} \to T_pM$, one has an $\mathrm{Ad}(K)$-invariant metric on $\mathfrak{m}$, and then we can extend it to an $\mathrm{Ad}(K)$-invariant metric on $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ by choosing[2] arbitrary $\mathrm{Ad}(K)$-invariant metric on $\mathfrak{k}$ such that $\mathfrak{m} \perp \mathfrak{k}$. This shows one has a left-invariant metric on G which is also right invariant with respect to K. Now it suffices to show G/K with the induced metric is isometric to (M, g). For any $gK \in G/K$, consider the following communicative diagram

[2]Such metric exists since K is compact.

$$\begin{tikzcd}
\mathfrak{m} = T_{eK}G/K \arrow[r, "(\mathrm{d}\pi)_e|_{\mathfrak{m}}"] \arrow[d, "\mathrm{d}L_g"'] & T_pM \arrow[d, "\mathrm{d}L_g"] \\
T_{gK}G/K \arrow[r] & T_{gp}M
\end{tikzcd}$$

Since both $(\mathrm{d}\pi)_e|_{\mathfrak{m}}$ and $(\mathrm{d}L_g)$ are linear isometries, one has $T_{gK}G/K$ is isometric to $T_{gp}M$, and thus G/K with induced metric is isometric to (M,g). $\square$

2.C. **Riemannian symmetric pair.** In Theorem 2.10 one can see that if (M,g) is a symmetric space, then it gives a pair of Lie groups (G,K) with an involution σ on G such that

$$(G^\sigma)_0 \subseteq K \subseteq G^\sigma$$

Then there exists a left-invariant metric on G/K such that G/K with this metric is isometric to (M,g). This motivates us an effective way to construct Riemannian symmetric spaces from a pair of Lie groups with certain properties, and such a pair is called a Riemannian symmetric pair. Unless otherwise specified, we assume G is a connected Lie group with Lie algebra $\mathfrak{g}$.

Definition 2.11 (Riemannian symmetric pair)**.** *Let K be a compact subgroup of G. The pair (G,K) is called a Riemannian symmetric pair if there exists an involution $\sigma: G \to G$ with $(G^\sigma)_0 \subseteq K \subseteq G^\sigma$.*

Example 2.12. $G = \mathrm{SO}(n+1)$ and $K = \mathrm{SO}(n)$ is a Riemannian symmetric pair given by

$$\begin{aligned} \sigma:\ \mathrm{SO}(n+1) &\to \mathrm{SO}(n+1) \\ a &\mapsto sas^{-1} \end{aligned}$$

where $s = diag\{-1,1,\dots,1\}$. Indeed,

$$\mathrm{SO}(n+1)^\sigma = \{a \in \mathrm{SO}(n+1) \mid sa = as\} = \left\{\begin{pmatrix} \pm 1 & 0 \\ 0 & b \end{pmatrix} \mid b \in \mathrm{O}(n)\right\}$$

which implies $(\mathrm{SO}(n+1)^\sigma)_0 = \mathrm{SO}(n) \subseteq \mathrm{SO}(n+1)$.

Example 2.13. $G = \mathrm{SL}(n,\mathbb{R})$ and $K = \mathrm{SO}(n)$ is a Riemannian symmetric pair given by

$$\begin{aligned} \sigma:\ \mathrm{SL}(n,\mathbb{R}) &\to \mathrm{SL}(n,\mathbb{R}) \\ g &\mapsto (g^{-1})^T \end{aligned}$$

Indeed,

$$(\mathrm{SL}(n,\mathbb{R}))^\sigma = \mathrm{SO}(n)$$

Example 2.14. Let K be a compact Lie group and $G = K \times K$. Then (G,K) is a Riemannian symmetric pair given by σ, where $\sigma: G \to G$ is given by $(x,y) \mapsto (y,x)$, since

$$G^\sigma = \{(a,a) \mid a \in K\} \cong K$$

Proposition 2.15. *Let (G,K) be a symmetric pair given by σ. Then there is an isomorphism as Lie algebras*

$$\mathfrak{k} \cong \mathrm{Lie}\,K$$

and an isomorphism as vector spaces

$$\mathfrak{m} \cong T_{eK}G/K$$

Proof. $\mathfrak{k} \cong \operatorname{Lie} K$ follows from the same as proof of (3) in Theorem 2.10, and $\mathfrak{m} \cong T_{eK}G/K$ is an immediate consequence. $\square$

Corollary 2.16. *Let $\widetilde{\sigma}: G/K \to G/K$ be the automorphism of G/K induced σ. Then $(\mathrm{d}\widetilde{\sigma})_{eK} = -\mathrm{id}_{G/K}$.*

Proof. Since $K \subseteq G^{\sigma}$, one has $\sigma: K \to K$, and thus $\widetilde{\sigma}: G/K \to G/K$ is well-defined. By construction one has $(\mathrm{d}\widetilde{\sigma})_{eK} = (\mathrm{d}\sigma)_e|_{\mathfrak{m}}$. Then $(\mathrm{d}\widetilde{\sigma})_{eK} = -\mathrm{id}_{G/K}$ since $\mathfrak{m} = \{X \in \mathfrak{g} \mid (\mathrm{d}\sigma)_e X = -X\}$. $\square$

Theorem 2.17. *Let (G, K) be a Riemannian symmetric pair given by σ. Then there exists a left-invariant metric on G which is also right invariant on K such that the induced metric on G/K making it to be a Riemannian symmetric space.*

Proof. For convenience we use M to denote G/K. Note that a left-invariant metric on G which is also right invariant on K is equivalent to a metric on $\mathfrak{g}$ which is $\operatorname{Ad}(K)$-invariant. Since K is compact, it admits a $\operatorname{Ad}(K)$-invariant metric, and it can be extended to a $\operatorname{Ad}(K)$-invariant metric on $\mathfrak{g}$ as what we have done in the proof of (4) in Theorem 2.10. Furthermore, by Corollary 2.16 one has $(\mathrm{d}\widetilde{\sigma})_{eK} = -\mathrm{id}_M$.

Now it suffices to show for any $gK \in M$, $(\mathrm{d}\widetilde{\sigma})_{gK}: T_{gK}M \to T_{\sigma(g)K}M$ is an isometry. Note that $\widetilde{\sigma}(ghK) = \sigma(g)\sigma(h)K = \sigma(g)\widetilde{\sigma}(hK)$ holds for all $h \in G$. This shows $\widetilde{\sigma} \circ L_g = L_{\sigma(g)} \circ \widetilde{\sigma}$, where $L_g: M \to M$ is given by $L_g(hK) = ghK$. By taking differential one has the following communicative diagram

$$\begin{tikzcd}
T_{eK}M \arrow[r, "(\mathrm{d}\widetilde{\sigma})_{eK}"] \arrow[d, "(\mathrm{d}L_g)_{eK}"'] & T_{eK}M \arrow[d, "(\mathrm{d}L_{\sigma(g)})_{eK}"] \\
T_{gK}M \arrow[r, "(\mathrm{d}\widetilde{\sigma})_{gK}"] & T_{\sigma(g)K}M
\end{tikzcd}$$

Since $(\mathrm{d}L_g)_{eK}, (\mathrm{d}L_{\sigma(g)})_{eK}, (\mathrm{d}\widetilde{\sigma})_{eK}$ are isometries, one has $(\mathrm{d}\widetilde{\sigma})_{gK}$ is also an isometry as desired. $\square$

Example 2.18. S^n is a Riemannian symmetric space, since $S^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$ and $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ is a Riemannian symmetric pair.

Example 2.19. $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ is a Riemannian symmetric space, and it can be viewed as a generalization of hyperbolic plane $\mathbb{H}^2$, since $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2) \cong \mathbb{H}^2$.

Example 2.20. Any compact Lie group K is a Riemannian symmetric space, since $(K \times K, K)$ is a Riemannian symmetric pair.

2.D. **Transvection.** Let (M, g) be a Riemannian manifold and $\mathfrak{g}$ be the Lie algebra of isometry group. Recall in Corollary 2.6 we have the following decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

In this section we will give more explicit descriptions for this decomposition in case of Riemannian symmetric space.

Theorem 2.21. *Let (M, g) be a complete Riemannian manifold with isometry group G. For any $p \in M$, the Lie algebra of the isotropy subgroup G_p is isomorphic to*

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid X_p = 0\}$$

where $\mathfrak{g}$ is the Lie algebra of G.

Proof. Let $X \in \mathfrak{g}$ with $X_p = 0$ and $\varphi_t : M \to M$ be the flow of X. It suffices to show $\varphi_t(p) = p$ for all $t \in \mathbb{R}$. If we use $\gamma_p(t)$ to denote $\varphi_t(p)$, then for any smooth function $f : M \to \mathbb{R}$ and $s \in \mathbb{R}$, one has

$$\begin{aligned}
\gamma_p'(s)f &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=s} f \circ \gamma_p(t) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f \circ \gamma_p(t+s) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f \circ \varphi_s \circ \varphi_t(p) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (f \circ \varphi_s)(\gamma_p(t)) \\
&= \gamma_p'(0)(f \circ \varphi_s) \\
&= X_p(f \circ \varphi_s) \\
&= 0
\end{aligned}$$

Hence $\gamma_p'(s) = 0$ for all $s \in \mathbb{R}$, and thus $\gamma_p(s)$ is constant, which implies $\gamma_p(s) = \gamma_p(0) = p$. □

In order to describe $\mathfrak{m}$, we need to introduce transvection.

Definition 2.22 (transvection)**.** *Let (M, g) be a Riemannian symmetric space and γ a geodesic. The transvection along γ is defined as*

$$T_t = s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}$$

where s_p is the symmetry at point p.

Proposition 2.23. *Let (M, g) be a Riemannian symmetric space, γ a geodesic and T_t the transvection along γ. Then*

(1) For any $a, t \in \mathbb{R}$, $s_{\gamma(a)}(\gamma(t)) = \gamma(2a - t)$.
(2) T_t translates the geodesic γ, that is $T_t(\gamma(s)) = \gamma(t + s)$.
(3) $(\mathrm{d}T_t)_{\gamma(s)} : T_{\gamma(s)}M \to T_{\gamma(t+s)}M$ is the parallel transport $P_{s,t+s;\gamma}$.
(4) T_t is one-parameter subgroup of $\mathrm{Iso}(M, g)$*.*

Proof. For (1). It follows from the uniqueness of geodesics with given initial value.

For (2). By (1) one has

$$\begin{aligned}
T_t(\gamma(s)) &= s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}(\gamma(s)) \\
&= s_{\gamma(\frac{t}{2})}(\gamma(-s)) \\
&= \gamma(t + s)
\end{aligned}$$

For (3). Let X be a parallel vector field along γ. By uniqueness of parallel vector fields with given initial data, we have $(\mathrm{d}s_{\gamma(0)})_{\gamma(s)} X_{\gamma(s)} = -X_{\gamma(-s)}$ for all s, since $(\mathrm{d}s_{\gamma(0)})_{\gamma(0)} X_{\gamma(0)} =$

$-X_{\gamma(0)}$. Thus

$$\begin{aligned}(\mathrm{d}T_t)_{\gamma(s)}X_{\gamma(s)} &= (\mathrm{d}s_{\gamma(\frac{t}{2})})_{\gamma(-s)}(-X_{\gamma(-s)})\\ &= X_{\gamma(t+s)}\end{aligned}$$

This shows $(\mathrm{d}T_t)_{\gamma(s)} = P_{s,t+s;\gamma}$.

For (4). In order to show $T_{t+s} = T_t \circ T_s$, it suffices to check they're same at some point, so do their derivatives, since isometry can be determined by these two values. Note that

$$\begin{aligned}T_{t+s}(\gamma(0)) &= \gamma(t+s)\\ &= T_t \circ T_s(\gamma(0))\\ (\mathrm{d}T_{t+s})_{\gamma(0)} &= P_{0,t+s;\gamma}\\ &= P_{s,t+s;\gamma} \circ P_{0,s;\gamma}\\ &= (\mathrm{d}T_t)_{\gamma(s)} \circ (\mathrm{d}T_s)_{\gamma(0)}\\ &= (\mathrm{d}(T_t \circ T_s))_{\gamma(0)}\end{aligned}$$

This completes the proof. □

Definition 2.24 (infinitesimal transvection)**.** *Let (M, g) be a Riemannian symmetric space. For any point $p \in M$ and any $v \in T_pM$, the infinitesimal generator X of transvections T_t along γ_v is given by*

$$X_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T_t(p)$$

This Killing field X is called an infinitesimal transvection.

Theorem 2.25. *Let (M, g) be a Riemannian symmetric space and X an infinitesimal transvection of transvection T_t along geodesic $\gamma = \exp_p(tv)$. Then*

$$X_p = v, \quad (\nabla X)_p = 0$$

Proof. It's clear $X_p = v$. For any $w \in T_pM$, let c be a curve in M with $c(0) = p$ and $c'(0) = w$. Then

$$\begin{aligned}\nabla_w X &= \left.\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}s}} X(c(s))\right|_{s=0}\\ &= \left.\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}s}} \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} T_t(c(s))\right|_{t=s=0}\\ &= \left.\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}s}} T_t(c(s))\right|_{t=s=0}\\ &= \left.\widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \big((\mathrm{d}T_t)_p(w)\big)\right|_{t=0}\\ &= 0\end{aligned}$$

□

Corollary 2.26. *The space of infinitesimal transvection is exactly $\mathfrak{m}$, and there is an isomorphism between $\mathfrak{m} \cong T_pM$ given by $X \mapsto X_p$.*

3. Curvature of Riemannian symmetric space

Proposition 3.1. *Let (M,g) be a Riemannian symmetric space and $G = \mathrm{Iso}(M,g)$ with Lie algebra $\mathfrak{g}$. For any $p \in M$, one has Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{k}$ is Lie algebra of isotropy group G_p and $\mathfrak{m} \cong T_pM$. Then for any $X \in \mathfrak{m}$, one has*

$$B(X,X) \leq 0$$

where B is the Killing form of $\mathfrak{g}$. Furthermore, the identity holds if and only if $X = 0$.

Proof. Since a Killing field is determined by X_p and $(\nabla X)_p$, one has elements in $\mathfrak{k}$ is determined by $(\nabla X)_p$, and note that ∇X is a skew-symmetric matrice, so

$$\mathfrak{k} \cong \{(\nabla X) \in \mathfrak{so}(T_pM) \mid X \in \mathfrak{k}\}$$

By using this identification, there is a natural metric on $\mathfrak{k}$ given by

$$\langle S_1, S_2\rangle = -\mathrm{tr}(S_1S_2)$$

Then one has metric on $\mathfrak{g}$ since there is a metric on $\mathfrak{m}$ obtained from $\mathfrak{m} \cong T_pM$. For any $S \in \mathfrak{k}$, we claim with respect to this metric, $\mathrm{ad}_S : \mathfrak{g} \to \mathfrak{g}$ is skew-symmetric. Indeed, for $X_1, X_2 \in \mathfrak{k}$, one has

$$\begin{aligned}
\langle \mathrm{ad}_S X_1, X_2\rangle &= -\mathrm{tr}(\mathrm{ad}_S X_1 X_2)\\
&= -\mathrm{tr}((SX_1 - X_1S)X_2)\\
&= \mathrm{tr}(X_1(SX_2 - X_2S))\\
&= -\langle X_1, \mathrm{ad}_S X_2\rangle
\end{aligned}$$

For $Y_1, Y_2 \in \mathfrak{m}$, since $S_p = 0$ and $(\nabla S)_p$ is skew-symmetric, one has

$$\begin{aligned}
\langle \mathrm{ad}_S Y_1, Y_2\rangle &= \langle \nabla_S Y_1 - \nabla_{Y_1} S, Y_2\rangle\\
&= -\langle \nabla_{Y_1} S, Y_2\rangle\\
&= \langle \nabla_{Y_2} S, Y_1\rangle\\
&= -\langle Y_1, \nabla_S Y_2 - \nabla_{Y_2} S\rangle\\
&= -\langle Y_1, \mathrm{ad}_S Y_2\rangle
\end{aligned}$$

Then one has

$$B(S,S) = \mathrm{tr}(\mathrm{ad}_S \circ \mathrm{ad}_S) = \sum \langle \mathrm{ad}_S \circ \mathrm{ad}_S(e_i), e_i\rangle = -\sum \langle \mathrm{ad}_S(e_i), \mathrm{ad}_S(e_i)\rangle \leq 0$$

Furthermore, if $B(S,S) = 0$, then $\mathrm{ad}_S = 0$ and for any $X \in \mathfrak{g}$, one has

$$0 = \mathrm{ad}_S(X) = [S,X] = \nabla_S X - \nabla_X S = -\nabla_X S$$

since $S_p = 0$. This implies $(\nabla S)_p = 0$, and thus $S = 0$. $\square$

Theorem 3.2. *Let (M,g) be a Riemannian symmetric space and $G = \mathrm{Iso}(M,g)$. For any $p \in M$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ with $\mathfrak{m} \cong T_pM$.*

(1) For any $X, Y, Z \in \mathfrak{m}$, there holds

$$R(X,Y)Z = -[Z,[Y,X]]$$

$$\mathrm{Ric}(Y,Z) = -\frac{1}{2}B(Y,Z)$$

(2) If $\mathrm{Ric}(g) = \lambda g$*, then for* $X, Y \in \mathfrak{m}$*, one has*

$$2\lambda R(X, Y, Y, X) = -B([X, Y], [X, Y])$$

Proof. For (1). For any $X, Y, Z \in \mathfrak{m}$, direct computation shows

$$\begin{aligned} R(X, Y)Z &\overset{(a)}{=} R(X, Z)Y - R(Y, Z)X \\ &\overset{(b)}{=} \nabla_Z \nabla_Y X - \nabla_{\nabla_Z Y} X - \nabla_Z \nabla_X Y + \nabla_{\nabla_Z X} Y \\ &\overset{(c)}{=} -\nabla_Z [X, Y] \\ &\overset{(d)}{=} -[Z[X, Y]] \end{aligned}$$

where

(a) holds from the first Bianchi identity.
(b) holds from (2) of Proposition 2.1.
(c) holds from $X, Y \in \mathfrak{m}$, and thus $(\nabla X)_p = (\nabla Y)_p = 0$.
(d) holds from

$$\nabla_Z [X, Y] - \nabla_{[X,Y]} Z = [Z, [X, Y]]$$

and $(\nabla Z)_p = 0$.

Too see Ricci curvature, note that for $Y \in \mathfrak{m}$

$$\mathrm{ad}_Y : \mathfrak{k} \to \mathfrak{m}, \quad \mathrm{ad} : Y : \mathfrak{m} \to \mathfrak{k}$$

Thus $\mathrm{ad}_Z \circ \mathrm{ad}_Y$ preserves the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ if $Y, Z \in \mathfrak{m}$. Then

$$\begin{aligned} \mathrm{tr}(\mathrm{ad}_Z \circ \mathrm{ad}_Y |_{\mathfrak{m}}) &= \mathrm{tr}(\mathrm{ad}_Z |_{\mathfrak{k}} \circ \mathrm{ad}_Y |_{\mathfrak{m}}) \\ &= \mathrm{tr}(\mathrm{ad}_Y |_{\mathfrak{m}} \circ \mathrm{ad}_Z |_{\mathfrak{k}}) \\ &= \mathrm{tr}(\mathrm{ad}_Y \circ \mathrm{ad}_Z |_{\mathfrak{k}}) \end{aligned}$$

Hence we obtain

$$B(Y, Y) = \mathrm{tr}(\mathrm{ad}_Y \circ \mathrm{ad}\, Y|_{\mathfrak{k}}) + \mathrm{tr}(\mathrm{ad}_Y \circ \mathrm{ad}\, Y|_{\mathfrak{m}}) = 2\mathrm{tr}(\mathrm{ad}_Y \circ \mathrm{ad}_Y |_{\mathfrak{m}})$$

Since Ricci tensor is trace of curvature tensor, and thus

$$\mathrm{Ric}(Y, Y) = -\mathrm{tr}(\mathrm{ad}_Y \circ \mathrm{ad}_Y |_{\mathfrak{m}}) = -\frac{1}{2} B(Y, Y)$$

Then by using Polarization identity, one has $\mathrm{Ric}(Y, Z) = -B(Y, Z)/2$.

For (2). If $\mathrm{Ric}(g) = \lambda g$, then

$$\begin{aligned} 2\lambda g(R(X, Y)Y, X) &= -2\lambda g(\mathrm{ad}_Y \circ \mathrm{ad}_Y X, X) \\ &= -2\mathrm{Ric}(\mathrm{ad}_Y \circ \mathrm{ad}_Y X, X) \\ &= B(\mathrm{ad}_Y \circ \mathrm{ad}_Y X, X) \\ &= -B(\mathrm{ad}_Y X, \mathrm{ad}_Y X) \\ &= -B([X, Y], [X, Y]) \end{aligned}$$

□

Corollary 3.3. *Let* (M, g) *be a Riemannian symmetric space which is an Einstein manifold with Einstein constant* λ*. Then*

(1) If $\lambda > 0$, then (M, g) has non-negative sectional curvature.
(2) If $\lambda < 0$, then (M, g) has non-positive sectional curvature.
(3) If $\lambda = 0$, then (M, g) is flat.

Proof. By Theorem 3.2 one has

$$2\lambda R(X, Y, Y, X) = -B([X, Y], [X, Y]) \geq 0$$

since $[X, Y] \in [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{m}$ and B is negative definite on $\mathfrak{m}$. This shows (1) and (2). If $\lambda = 0$, one has $B([X, Y], [X, Y]) \equiv 0$ for arbitrary X, Y. Then by Proposition 3.1 one has $[X, Y] \equiv 0$ for arbitrary X, Y, and thus (M, g) is flat. □

4. Classifications and examples

4.A. **Irreducible space.**

Definition 4.1 (isotropy irreducible)**.** *Let (M, g) be a Riemannian symmetric space with $G = \mathrm{Iso}(M, g)$ and $K = G_p$ for some $p \in M$. If the identity component K_0 acts irreducibly on T_pM, then M is called irreducible. Otherwise M is called reducible.*

Lemma 4.2. *Let B_1, B_2 be two symmetric bilinear forms on a vector space V such that B_1 is positive definite. If a group K acts irreducibly on V such that B_1 and B_2 are invariant under K, then $B_2 = \lambda B_1$ for some constant λ.*

Proof. Since B_1 is positive definite, there exists an endomorphism $L : V \to V$ such that

$$B_2(u, v) = B_1(Lu, v)$$

where $u, v \in V$. Since B_1, B_2 are invariant under K, one has for any $k \in K$

$$B_1(kLu, v) = B_1(Lu, k^{-1}v) = B_2(u, k^{-1}v) = B_2(ku, v) = B_1(Lku, v)$$

holds for arbitrary $u, v \in V$, which implies $Lk = kL$ for all $k \in K$. Moreover, the symmetry of B_1, B_2 implies

$$B_1(Lu, v) = B_2(u, v) = B_2(v, u) = B_1(Lv, u) = B_1(u, Lv)$$

Hence L is symmetric with respect to B_1, and thus the eigenvalues of L are real. If $E \subseteq V$ is an eigenspace with eigenvalue λ, the fact $kL = Lk$ implies E is invariant under K. Since K acts irreducibly on V, one has $E = V$, that is $L = \lambda I$, which implies $B_2 = \lambda B_1$. □

Theorem 4.3. *The irreducible Riemannian symmetric space is Einstein, and the metric is unique determined up to a scalar.*

Proof. Since isometries preserves the metric and curvature, and Ricci tensor is also a symmetric bilinear form, by Lemma 4.2 there exists smooth function λ such that

$$\mathrm{Ric}(g) = \lambda g$$

Note that Riemannian curvature of Riemannian symmetric space is parallel, so is Ricci curvature. Thus we have λ is a constant. □

4.B. **Classification of Riemannian symmetric space.**

Theorem 4.4. *Let (M, g) be a simply-connected Riemannian symmetric space. Then (M, g) is isometric to*

$$(M_1, g_1) \times \cdots \times (M_k, g_k)$$

where (M_i, g_i) are irreducible Riemannian symmetric space for $i = 1, \dots, k$.

4.C. **Examples of Riemannian symmetric space.**

4.C.1. *Matrix groups as symmetric spaces.*

Example 4.5 (hyperbolic Grassmannian)**.** In $\mathbb{R}^{k,l}$ with $k \geq 2, l \geq 1$, consider the following quadratic form

$$v^t I_{k,l} w = v^t \begin{pmatrix} I_k & 0 \\ 0 & -I_l \end{pmatrix} w = \sum_{i=1}^{k} v_i w_i - \sum_{j=k+1}^{k+l} v_j w_j$$

The group of linear transformation X that preserves this quadratic form is denoted by $\mathrm{O}(k,l)$, that is

$$XI_{k,l}X^t = I_{k,l}$$

and $\mathrm{SO}(k,l)$ are those with positive determinant. The Lie algebra $\mathfrak{so}(k,l)$ of $\mathrm{SO}(k,l)$ is

$$\mathfrak{so}(k,l) = \{X = \begin{pmatrix} X_1 & B \\ B^t & X_2 \end{pmatrix} \in \mathfrak{gl}(k+l,\mathbb{R}) \mid X_1 \in \mathfrak{so}(k), X_2 \in \mathfrak{so}(l), B \in M_{k\times l}\}$$

Now consider set consisting of those oriented k-dimensional subspaces of $\mathbb{R}^{k,l}$ on which quadratic form $I_{k,l}$ are positive definite. This gives a manifold which is called the hyperbolic Grassmannian $M = \widehat{Gr}(k,\mathbb{R}^{k,l})$. It's clear $G = \mathrm{O}(k,l)$ acting transitively on M with isotropy group $G_p = \mathrm{SO}(k) \times \mathrm{O}(l)$. Then we have the decomposition of Lie algebra $\mathfrak{g}$ of G as follows

$$\mathfrak{so}(k,l) \cong \mathfrak{so}(k) \oplus \mathfrak{so}(l) \oplus \mathfrak{m}$$

If we give the following metric on $\mathfrak{m} \cong T_pM$

$$\langle X,Y\rangle = \mathrm{tr}(XY) = \frac{1}{k+l-2}B(X,Y)$$

where B is the Killing form of $\mathfrak{so}(k,l)$. Then the corresponding metric on M has the curvature formulas

$$\mathrm{Ric}(g) = -\frac{B}{2} = -\frac{k+l-2}{2}g$$

$$R(X,Y,Y,X) = \frac{B([X,Y],[X,Y])}{k+l-2} \leq 0$$

Hence the hyperbolic Grassmannian has non-positive curvatures.

Example 4.6. Let $G = \mathrm{SL}(n,\mathbb{R}), K = \mathrm{SO}(n)$ with Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$. Consider $M = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$, one has

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

If we give the following metric on $\mathfrak{m} \cong T_pM$

$$\langle X,Y\rangle = \mathrm{tr}(XY) = \frac{1}{2n}B(X,Y)$$

where B is the Killing form of $\mathfrak{so}(k,l)$. Then the corresponding metric on M has the curvature formulas

$$\mathrm{Ric}(g) = -\frac{B}{2} = -ng$$

$$R(X,Y,Y,X) = \frac{B([X,Y],[X,Y])}{2n} \leq 0$$

Hence it has non-positive curvatures.

Appendix A. Appendix

Theorem A.1. *Let $\varphi, \psi : (M, g_M) \to (N, g_N)$ be two local isometries between Riemannian manifolds, and M is connected. If there exists $p \in M$ such that*

$$\varphi(p) = \psi(p)$$
$$(\mathrm{d}\varphi)_p = (\mathrm{d}\psi)_p$$

then $\varphi = \psi$.

Theorem A.2 (Cartan-Ambrose-Hicks). *Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be two Riemannian manifolds, and $\Phi_0 : T_pM \to T_{\widetilde{p}}\widetilde{M}$ is a linear isometry, where $p \in M, \widetilde{p} \in \widetilde{M}$. For* $0 < \delta < \min\{\mathrm{inj}_p(M), \mathrm{inj}_{\widetilde{p}}(\widetilde{M})\}$, *The following statements are equivalent.*

(1) There exists an isometry $\varphi : B(p, \delta) \to B(\widetilde{p}, \delta)$ such that $\varphi(p) = \widetilde{p}$ and $(\mathrm{d}\varphi)_p = \Phi_0$.
(2) For $v \in T_pM, |v| < \delta, \gamma(t) = \exp_p(tv), \widetilde{\gamma}(t) = \exp_{\widetilde{p}}(t\Phi_0(v))$, if we define

$$\Phi_t = P_{0,t;\widetilde{\gamma}} \circ \Phi_0 \circ P_{t,0;\gamma} : T_{\gamma(t)}M \to T_{\widetilde{\gamma}(t)}\widetilde{M}$$

then Φ_t preserves curvature, that is $(\Phi_t)^*R = R$.

Proposition A.3. *Let $(M, g_M), (N, g_N)$ be complete Riemannian manifolds and $f : M \to N$ be a local diffeomorphism such that for all $p \in M$ and for all $v \in T_pM$, one has $|(\mathrm{d}f)_p v| \geq |v|$. Then f is a Riemannian covering map.*

Theorem A.4 (Myers-Steenrod). *Let (M, g) be a Riemannian manifold and* $G = \mathrm{Iso}(M, g)$. *Then*

(1) G is a Lie group with respect to compact-open topology.
(2) for each $p \in M$, the isotropy group G_p is compact.
(3) G is compact if M is compact.

Proposition A.5. *Let (M, g) be a Riemannian manifold, $\gamma : I \to M$ a smooth curve and $P_{s,t;\gamma} : T_{\gamma(s)}M \to T_{\gamma(t)}M$ is the parallel transport along γ. For any $s \in I$ with $v = \gamma'(s)$, one has*

$$\nabla_v R = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} (P_{s,t;\gamma})^* R_{\gamma(t)}$$

In particular, if $\nabla R = 0$ then

$$(P_{s,t;\gamma})^* R_{\gamma(t)} = R_{\gamma(s)}$$

holds for arbitrary $t, s \in I$.

Proposition A.6. *If $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ is a Riemannian covering, then M is complete if and only if $\widetilde{M}$ is.*

References

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