RIEMANNIAN SYMMETRIC SPACE

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CONTENTS

Part 1. Riemannian symmetric space

1. GEOMETRIC VIEWPOINTS

1.A. Basic definitions and properties.

1.A.1. *Riemannian symmetric space.*

Definition 1.1 (Riemannian symmetric space)**.** *A Riemannian manifold (M, g) is called a Riemannian symmetric space if for each $p \in M$ there exists an isometry $\varphi : M \to M$, which is called a symmetry at p, such that $\varphi(p) = p$ and $(\mathrm{d}\varphi)_p = -\mathrm{id}$.*

Remark 1.2. Note that Theorem A.1, that is rigidity property of isometry, implies if symmetry at point p exists, then it's unique.

Proposition 1.3. *The following statements are equivalent.*

(1) *(M, g) is a Riemannian symmetric space.*
(2) *For each $p \in M$, there exists an isometry $\varphi : M \to M$ such that $\varphi^2 = \mathrm{id}$ and p is an isolated fixed point of φ.*

Proof. From (1) to (2). Let φ be a symmetry at $p \in M$. Since $(\mathrm{d}\varphi^2)_p = (\mathrm{d}\varphi)_p \circ (\mathrm{d}\varphi)_p = \mathrm{id}$ and $\varphi^2(p) = p$, one has $\varphi^2 = \mathrm{id}$ by Theorem A.1. If p is not an isolated fixed point, then there exists a sequence $\{p_i\}_{i=1}^{\infty}$ converging to p such that $\varphi(p_i) = p_i$. For $0 < \delta < \mathrm{inj}(p)$, there exists sufficiently large k such that $p_k \in B(p, \delta)$, and we denote $v = \exp_p^{-1}(p_k)$. Since φ is an isometry, one has $\varphi(\exp_p(tv))$ and $\exp_p(tv)$ are two geodesics connecting p and p_k, and thus

$$\varphi(\exp_p(tv)) = \exp_p(tv)$$

by uniqueness. In particular, one has $v = (\mathrm{d}\varphi)_p v$, which is a contradiction.

From (2) to (1). From $\varphi^2 = \mathrm{id}$ we have $(\mathrm{d}\varphi)_p^2 = \mathrm{id}$, so only possible eigenvalues of $(\mathrm{d}\varphi)_p$ are ± 1. Now it suffices to show all eigenvalues of $(\mathrm{d}\varphi)_p$ are -1. Otherwise if it has an eigenvalue 1, there exists some non-zero $v \in T_pM$ such that $(\mathrm{d}\varphi)_p v = v$. Since φ is an isometry, one has $\varphi(\exp_p(tv))$ and $\exp_p(tv)$ are geodesics with the same direction at p. Thus

$$\varphi(\exp_p(tv)) = \exp_p(tv)$$

for $0 < t < \mathrm{inj}(p)$. In particular, p is not an isolated fixed point, which is a contradiction. □

Proposition 1.4. *The fundamental group of a Riemannian symmetric space is abelian.*

Corollary 1.5. *A surface of genus $g \geq 2$ does not admit a Riemannian metric with respect to which it is a symmetric space.*

1.A.2. *Locally Riemannian symmetric space.*

Definition 1.6 (locally Riemannian symmetric space)**.** *A Riemannian manifold (M, g) is called a locally Riemannian symmetric space if each $p \in M$ has a neighborhood U such that there exists an isometry $\varphi : U \to U$ such that $\varphi(p) = p$ and $(\mathrm{d}\varphi)_p = -\mathrm{id}$.*

Theorem 1.7. *Let (M, g) be a complete Riemannian manifold. The following statements are equivalent.*

(1) (M, g) *is a locally Riemannian symmetric space.*
(2) $\nabla R = 0$.

Proof. From (1) to (2). If φ is the symmetry at point $p \in M$, then it's an isometry such that $(\mathrm{d}\varphi)_p = -\mathrm{id}$, and thus for $u, v, w, z \in T_pM$, one has

$$\begin{aligned} -\nabla_u R(v,w)z &= (\mathrm{d}\varphi)_p\,(\nabla_u R(v,w)z) \\ &= \nabla_{(\mathrm{d}\varphi)_p u}((\mathrm{d}\varphi)_p)v, (\mathrm{d}\varphi)_p w)(\mathrm{d}\varphi)_p z \\ &= \nabla_u R(v,w)z \end{aligned}$$

This shows $(\nabla R)_p = 0$, and thus $\nabla R = 0$ since p is arbitrary.

From (2) to (1). For arbitrary $p \in M$, it suffices to show

$$\varphi = \exp_p \circ \Phi_0 \circ \exp_p^{-1} : B(p,\delta) \to B(p,\delta)$$

is an isometry, where $0 < \delta < \mathrm{inj}(p)$ and $\Phi_0 = -\mathrm{id} : T_pM \to T_pM$. For $v \in T_pM$ with $|v| < \delta$ and $\gamma(t) = \exp_p(tv), \widetilde{\gamma}(t) = \exp_p(t\Phi_0(v))$, if we define

$$\Phi_t = P_{0,t;\widetilde{\gamma}} \circ \Phi_0 \circ P_{t,0;\gamma}$$

then direct computation shows

$$\begin{aligned} \Phi_t^* R_{\widetilde{\gamma}(t)} &= (P_{t,0;\gamma})^* \circ \Phi_0^* \circ (P_{0,t;\widetilde{\gamma}})^* R_{\widetilde{\gamma}(t)} \\ &\overset{(a)}{=} (P_{t,0;\gamma})^* \circ \Phi_0^* R_{\widetilde{\gamma}(0)} \\ &\overset{(b)}{=} (P_{t,0;\gamma})^* R_{\gamma(0)} \\ &\overset{(c)}{=} R_{\gamma(t)} \end{aligned}$$

where

(a) and (c) holds from Proposition A.5.
(b) holds from $\widetilde{\gamma}(0) = \gamma(0)$ and R is a $(0,4)$-tensor.

Then by Theorem A.2, that is Cartan-Ambrose-Hicks's theorem, φ is an isometry, which completes the proof. □

1.B. **Transvection.**

Definition 1.8 (transvection)**.** *Let* (M, g) *be a Riemannian symmetric space and* γ *a geodesic. The transvection along* γ *is defined as*

$$T_t = s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}$$

where s_p *is the symmetry at point* p*.*

Proposition 1.9. *Let* (M, g) *be a Riemannian symmetric space and* T_t *be the transvection along geodesic* γ*. Then*

(1) For any $a, t \in \mathbb{R}$, $s_{\gamma(a)}(\gamma(t)) = \gamma(2a - t)$*.*
(2) T_t *translates the geodesic* γ*, that is* $T_t(\gamma(s)) = \gamma(t + s)$*.*
(3) $(\mathrm{d}T_t)_{\gamma(s)} : T_{\gamma(s)}M \to T_{\gamma(t+s)}M$ *is the parallel transport* $P_{s,t+s;\gamma}$*.*
(4) T_t *is one-parameter subgroup of* $\mathrm{Iso}(M, g)$*.*

Proof. For (1). It follows from the uniqueness of geodesics with given initial value.

For (2). By (1) one has

$$\begin{aligned} T_t(\gamma(s)) &= s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}(\gamma(s)) \\ &= s_{\gamma(\frac{t}{2})}(\gamma(-s)) \\ &= \gamma(t+s) \end{aligned}$$

For (3). Let X be a parallel vector field along γ. By uniqueness of parallel vector fields with given initial data, we have $(\mathrm{d}s_{\gamma(0)})_{\gamma(s)} X_{\gamma(s)} = -X_{\gamma(-s)}$ for all s, since $(\mathrm{d}s_{\gamma(0)})_{\gamma(0)} X_{\gamma(0)} = -X_{\gamma(0)}$. Thus

$$\begin{aligned} (\mathrm{d}T_t)_{\gamma(s)} X_{\gamma(s)} &= (\mathrm{d}s_{\gamma(\frac{t}{2})})_{\gamma(-s)}(-X_{\gamma(-s)}) \\ &= X_{\gamma(t+s)} \end{aligned}$$

This shows $(\mathrm{d}T_t)_{\gamma(s)} = P_{s,t+s;\gamma}$.

For (4). In order to show $T_{t+s} = T_t \circ T_s$, it suffices to check they're same at some point, so do their derivatives, since isometry can be determined by these two values. Note that

$$\begin{aligned} T_{t+s}(\gamma(0)) &= \gamma(t+s) \\ &= T_t \circ T_s(\gamma(0)) \\ (\mathrm{d}T_{t+s})_{\gamma(0)} &= P_{0,t+s;\gamma} \\ &= P_{s,t+s;\gamma} \circ P_{0,s;\gamma} \\ &= (\mathrm{d}T_t)_{\gamma(s)} \circ (\mathrm{d}T_s)_{\gamma(0)} \\ &= (\mathrm{d}(T_t \circ T_s))_{\gamma(0)} \end{aligned}$$

This completes the proof. $\square$

1.C. **Symmetric space, locally symmetric space and homogeneous space.** In this section, we will show any complete locally Riemannian symmetric space is a quotient of Riemannian symmetric space (Corollary 1.11), and any Riemannian symmetric space is a Riemannian homogeneous space (Corollary 1.15).

1.C.1. *Riemannian symmetric space and locally Riemannian symmetric space.*

Theorem 1.10. *Let (M, g) be a complete, simply-connected locally Riemannian symmetric space. Then (M, g) is a Riemannian symmetric space.*

Proof. For $p \in M$ and $0 < \delta < \mathrm{inj}(p)$, suppose $\varphi : B(p, \delta) \to B(p, \delta)$ is an isometry such that $\varphi(p) = p$ and $(\mathrm{d}\varphi)_p = -\mathrm{id}$. For arbitrary $q \in M$, we use $\Omega_{p,q}$ to denote all curves γ with $\gamma(0) = p, \gamma(1) = q$, and for $c \in \Omega_{p,q}$ we choose[1] a covering $\{B(p_i, \delta_i)\}_{i=0}^{k}$ of c such that

(1) $0 < \delta_i < \mathrm{inj}(p_i)$.
(2) $B(p_0, \delta_0) = B(p, \delta)$ and $p_k = q$.
(3) $p_{i+1} \in B(p_i, \delta_i)$.

If we set $\varphi = \varphi_0$, then we can define isometries $\varphi_i : B(p_i, \delta_i) \to M$ such that $\varphi_i(p_i) = \varphi_{i-1}(p_i)$ and $(\mathrm{d}\varphi_i)_{p_i} = (\mathrm{d}\varphi_{i-1})_{p_i}$ by using Cartan-Ambrose-Hicks's theorem successively, and by Theorem A.1 one has φ_i and φ_{i+1} coincide on $B(p_i, \delta_i) \cap B(p_{i+1}, \delta_i)$. The covering

[1] Since injective radius is a continuous function, it has a positive minimum on curve c, so such covering exists.

together with isometries we construct is denoted by $\mathcal{A} = \{B(p_i, \delta_i), \varphi_i\}_{i=0}^{k}$. For arbitrary $x \in [0,1]$, if $c(x) \in B(p_m, \delta_m)$, we may define

$$\begin{aligned}\varphi_{\mathcal{A}}(c(x)) &:= \varphi_m(c(x)) \\ (\mathrm{d}\varphi_{\mathcal{A}})_{c(x)} &:= (\mathrm{d}\varphi_m)_{c(x)}\end{aligned}$$

In particular, $\varphi_{\mathcal{A}}(q) := \varphi_k(q)$. If $\mathcal{B} = \{\widetilde{B}(\widetilde{p}_i, \widetilde{\delta}_i), \widetilde{\varphi}_i\}_{i=0}^{l}$ is another covering of c, let's show $\varphi_{\mathcal{A}}(q) = \varphi_{\mathcal{B}}(q)$. Consider

$$I = \{x \in [0,1] \mid \varphi_{\mathcal{A}}(c(x)) = \varphi_{\mathcal{B}}(c(x)), (\mathrm{d}\varphi_{\mathcal{A}})_{c(x)} = (\mathrm{d}\varphi_{\mathcal{B}})_{c(x)}\}$$

It's clear $I \neq \varnothing$, since $0 \in I$. Now it suffices to show it's both open and closed to conclude $1 \in I$.

(a) It's open: For $x \in I$, we assume $c(x) \in B(p_m, \delta_m) \cap \widetilde{B}(\widetilde{p}_n, \widetilde{\delta}_n)$, that is

$$\begin{aligned}\varphi_m(c(x)) &= \widetilde{\varphi}_n(c(x)) \\ (\mathrm{d}\varphi_m)_{c(x)} &= (\mathrm{d}\widetilde{\varphi}_n)_{c(x)}\end{aligned}$$

Then one has

$$\begin{aligned}\varphi_m \circ \exp_{c(x)}(v) &= \exp_{\varphi_m(c(x))} \circ (\mathrm{d}\varphi_m)_{c(x)}(v) \\ &= \exp_{\widetilde{\varphi}_n(c(x))} \circ (\mathrm{d}\widetilde{\varphi}_n)_{c(x)}(v) \\ &= \widetilde{\varphi}_n \circ \exp_{c(x)}(v)\end{aligned}$$

Since $\exp_{c(x)}$ maps onto a neighborhood of $c(x)$, it follows that some neighborhood of x also lies in I, and thus I is open.

(b) It's closed: Let $\{x_i\}_{i=1}^{\infty} \subseteq I$ be a sequence converging to x. Without lose of generality we may assume $\{x_i\}_{i=1}^{\infty} \subseteq B(p_m, \delta_m) \cap \widetilde{B}(\widetilde{p}_n, \widetilde{\delta}_n)$, then one has

$$\begin{aligned}\varphi_m(c(x_i)) &= \widetilde{\varphi}_n(c(x_i)) \\ (\mathrm{d}\varphi_m)_{c(x_i)} &= (\mathrm{d}\widetilde{\varphi}_n)_{c(x_i)}\end{aligned}$$

By taking limit we obtain the desired results.

Since $\varphi_{\mathcal{A}}(q)$ is independent of the choice of coverings, we use $\varphi(q)$ to denote it for convenience, and as a consequence we obtain the following map

$$\begin{aligned}F : \Omega_{p,q} &\to M \\ c &\mapsto \varphi(q)\end{aligned}$$

Note that $F(c)$ is locally constant, and thus it's independent of the choice of homotopy classes of c. Since M is simply-connected, one has $F : \Omega_{p,q} \to M$ is constant, so we obtain a local isometry $\varphi : M \to M$ which extends $\varphi : B(p,\delta) \to B(p,\delta)$. By Proposition A.3 φ is a Riemannian covering map since M is complete, and thus φ is a diffeomorphism since M is simply-connected, which implies φ is an isometry. □

Corollary 1.11. *Let (M, g) be a complete locally Riemannian symmetric space. Then it's isometric to $(\widetilde{M}/\Gamma, \widetilde{g})$ where $(\widetilde{M}, \widetilde{g})$ is a Riemannian symmetric space and $\Gamma \cong \pi_1(M)$ is a discrete Lie group acting on $\widetilde{M}$ freely, properly and isometrically.*

Proof. Let $(\widetilde{M}, \widetilde{g})$ be the universal covering of (M, g) with pullback metric. Then $(\widetilde{M}, \widetilde{g})$ is a simply-connected Riemannian manifold with parallel curvature tensor. Furthermore, by Proposition A.6 it's complete, hence it is symmetric. □

1.C.2. *Riemannian symmetric space and Riemannian homogeneous space.*

Definition 1.12 (Riemannian homogeneous space)**.** *A Riemannian manifold* (M, g) *is called a Riemannian homogeneous space, if* $\mathrm{Iso}(M, g)$ *acts on* M *transitively.*

Proposition 1.13. *Let* (M, g) *be a Riemannian homogeneous space. If there exists a symmetry at some point* $p \in M$*, then* (M, g) *is a Riemannian symmetric space.*

Proof. Let φ be a symmetry at $p \in M$. For arbitrary $q \in M$, there exists an isometry $\psi : M \to M$ such that $\psi(p) = q$ since (M, g) is a Riemannian homogeneous space. Then

$$\varphi_q := \psi \circ \varphi \circ \psi^{-1}$$

is the desired symmetry at q. □

Theorem 1.14. *Let* (M, g) *be a Riemannian symmetric space. Then*

(1) (M, g) *is complete.*

(2) the identity component of isometry group acts transitively on M*.*

Proof. For (1). For arbitrary geodesic $\gamma : [0, 1] \to M$ with $\gamma(0) = p, \gamma'(0) = v$, the curve $\beta(t) = \varphi(\gamma(t)) : [0, 1] \to M$ is also a geodesic with $\beta(0) = p$ and $\beta'(0) = -v$. Now we obtain a smooth extension $\gamma' : [0, 2] \to M$ of γ, given by

$$\gamma'(t) = \begin{cases} \beta(1 - t), & t \in [0, 1] \\ \gamma(t - 1), & t \in [1, 2] \end{cases}$$

Repeat above process to extend γ to a geodesic defined on $\mathbb{R}$, this shows completeness.

For (2). For arbitrary $p, q \in M$, let γ be a geodesic connecting p, q. Then the transvection along γ gives an isometry which maps p to q. Since the transvection lies in the identity component of isometry group, one has the identity component of isometry group acts transitively on M. □

Corollary 1.15. *The Riemannian symmetric space* (M, g) *is a Riemannian homogeneous space.*

2. Lie group viewpoints

2.A. **Riemannian symmetric space as a Lie group quotient.**

Definition 2.1 (involution)**.** *Let G be a Lie group. An automorphism σ of G is called an involution if $\sigma^2 = \mathrm{id}_G$.*

Definition 2.2 (Cartan decomposition)**.** *Let G be a Lie group and σ be an involution of G. The eigen-decomposition of $\mathfrak{g}$ given by $(\mathrm{d}\sigma)_e$ is called Cartan decomposition, that is,*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

where

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid (\mathrm{d}\sigma)_e(X) = X\}$$
$$\mathfrak{m} = \{X \in \mathfrak{g} \mid (\mathrm{d}\sigma)_e(X) = -X\}$$

Proposition 2.3. *Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be a Cartan decomposition given by σ. Then*

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$$

Proof. It follows from

$$(\mathrm{d}\sigma)_e([X, Y]) = [(\mathrm{d}\sigma)_e(X), (\mathrm{d}\sigma)_e(Y)]$$

where $X, Y \in \mathfrak{g}$. □

Theorem 2.4. *Let (M, g) be a Riemannian symmetric space and G be the identity component of $\mathrm{Iso}(M, g)$. For $p \in M$, K denotes the isotropic group of G_p.*

(1) The mapping $\sigma : G \to G$, given by $\sigma(g) = s_p g s_p$ is an involution automorphism of G.
(2) If G^σ is the set of fixed points of σ in G, and $(G^\sigma)_0$ is the identity component of G^σ, then $(G^\sigma)_0 \subseteq K \subseteq G^\sigma$.
(3) If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the Cartan decomposition given by σ, then $\mathfrak{k}$ is the Lie algebra of K.
(4) There is a left invariant metric on G which is also right invariant under K, such that G/K with the induced metric is isometric to (M, g).

Proof. For (1). σ is an involution since for arbitrary $g \in G$, one has $\sigma^2(g) = \sigma(s_p g s_p) = s_p^2 g s_p^2 = g$ since $s_p^2 = \mathrm{id}$.

For (2). It follows from the following two steps:

(a) To show $K \subseteq G^\sigma$. For any $k \in K$, in order to show $k = s_p k s_p$, it suffices to show they and their differentials agree at some point by Theorem A.1, since both of them are isometries, and p is exactly the point we desired.
(b) To see $(G^\sigma)_0 \subseteq K$. Suppose $\exp(tX) \subseteq (G^\sigma)_0$ is a one-parameter subgroup. Since $\sigma(\exp(tX)) = \exp(tX)$, one has

$$\exp(tX)(p) = s_p \exp(tX) s_p(p) = s_p \exp(tX)(p)$$

But p is an isolated fixed point of s_p, which implies $\exp(tX)(p) = p$ for all t. This shows the one-parameter subgroup lies in K. Since exponential map of Lie group is a diffeomorphism in a small neighborhood of identity element e and $(G^\sigma)_0$ can be generated by a neighborhood of e, which implies the whole $(G^\sigma)_0 \subseteq K$.

For (3). Note that $(G^\sigma)_0 \subseteq K \subseteq G^\sigma$, it suffices to show $\mathfrak{k} \cong \operatorname{Lie} G^\sigma$. For $X \in \mathfrak{k}$, we claim $\gamma_2(t) = \sigma(\exp(tX)) \colon \mathbb{R} \to G$ is a one-parameter subgroup. Indeed, note that

$$\begin{aligned}\gamma_2(t) \cdot \gamma_2(s) &= s_p \exp(tX) s_p \cdot s_p \exp(sX) s_p \\ &= \sigma(\exp(tX + sX)) \\ &= \gamma_2(t+s)\end{aligned}$$

Furthermore, $\gamma_2(t) = \sigma(\exp(tX))$ and $\gamma_1(t) = \exp(tX)$ are two one-parameter subgroups of G such that $\gamma_1(0) = \gamma_2(0)$ and $\gamma_2'(0) = (\mathrm{d}\sigma)_e(X) = X = \gamma_1'(0)$. Then $\gamma_1(t) = \gamma_2(t)$, and thus $\exp(tX) \in G^\sigma$ for all $t \in \mathbb{R}$. This shows $\mathfrak{k} \subseteq \operatorname{Lie} G^\sigma$, and the converse inclusion is clear, so one has $\mathfrak{k} = \operatorname{Lie} G^\sigma$.

For (4). Let $\pi \colon G \to M$ be the natural projection given by $\pi(g) = gp$. Then for $k \in K$ and $X \in \mathfrak{g}$ one has

$$\begin{aligned}(\mathrm{d}\pi)_e(\mathrm{Ad}_k X) &= (\mathrm{d}\pi)_e\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} k\exp(tX)k^{-1}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi(k\exp(tX)k^{-1}) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} k\exp(tX)k^{-1} \cdot p \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} k\exp(tX) \cdot p \\ &= (\mathrm{d}L_k)_p(\mathrm{d}\pi)_e(X)\end{aligned}$$

By using the equivalent isomorphism $(\mathrm{d}\pi)_e|_{\mathfrak{m}} \colon \mathfrak{m} \to T_pM$, one has an $\mathrm{Ad}(K)$-invariant metric on $\mathfrak{m}$, and then we can extend it to an $\mathrm{Ad}(K)$-invariant metric on $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ by choosing[2] arbitrary $\mathrm{Ad}(K)$-invariant metric on $\mathfrak{k}$ such that $\mathfrak{m} \perp \mathfrak{k}$. This shows one has a left-invariant metric on G which is also right invariant with respect to K. Now it suffices to show G/K with the induced metric is isometric to (M, g). For any $gK \in G/K$, consider the following communicative diagram

$$\begin{array}{ccc} \mathfrak{m} = T_{eK}G/K & \xrightarrow{(\mathrm{d}\pi)_e|_{\mathfrak{m}}} & T_pM \\ {\scriptstyle \mathrm{d}L_g}\downarrow & & \downarrow{\scriptstyle \mathrm{d}L_g} \\ T_{gK}G/K & \longrightarrow & T_{gp}M \end{array}$$

Since both $(\mathrm{d}\pi)_e|_{\mathfrak{m}}$ and $(\mathrm{d}L_g)$ are linear isometries, one has $T_{gK}G/K$ is isometric to $T_{gp}M$, and thus G/K with induced metric is isometric to (M, g). □

2.B. **Riemannian symmetric pair.** In Theorem 2.4 one can see that if (M, g) is a symmetric space, then it gives a pair of Lie groups (G, K) with an involution σ on G such that

$$(G^\sigma)_0 \subseteq K \subseteq G^\sigma$$

Then there exists a left-invariant metric on G/K such that G/K with this metric is isometric to (M, g). This motivates us an effective way to construct Riemannian

[2]Such metric exists since K is compact.

symmetric spaces from a pair of Lie groups with certain properties, and such a pair is called a Riemannian symmetric pair.

Definition 2.5 (Riemannian symmetric pair)**.** *Let G be a connected Lie group and $K \subseteq G$ be a compact subgroup. The pair (G, K) is called a Riemannian symmetric pair if there exists an involution $\sigma : G \to G$ with $(G^\sigma)_0 \subseteq K \subseteq G^\sigma$.*

Proposition 2.6. *Let (G, K) be a symmetric pair given by σ. Then there is an isomorphism as Lie algebras*

$$\mathfrak{k} \cong \operatorname{Lie} K$$

and an isomorphism as vector spaces

$$\mathfrak{m} \cong T_{eK}G/K$$

Proof. $\mathfrak{k} \cong \operatorname{Lie} K$ follows from the same as proof of (3) in Theorem 2.4, and $\mathfrak{m} \cong T_{eK}G/K$ is an immediate consequence. □

Corollary 2.7. *Let $\widetilde{\sigma} : G/K \to G/K$ be the automorphism of G/K induced σ. Then $(\mathrm{d}\widetilde{\sigma})_{eK} = -\mathrm{id}_{G/K}$.*

Proof. Since $K \subseteq G^\sigma$, one has $\sigma : K \to K$, and thus $\widetilde{\sigma} : G/K \to G/K$ is well-defined. By construction one has $(\mathrm{d}\widetilde{\sigma})_{eK} = (\mathrm{d}\sigma)_e|_{\mathfrak{m}}$. Then $(\mathrm{d}\widetilde{\sigma})_{eK} = -\mathrm{id}_{G/K}$ since $\mathfrak{m} = \{X \in \mathfrak{g} \mid (\mathrm{d}\sigma)_e X = -X\}$. □

Theorem 2.8. *Let (G, K) be a Riemannian symmetric pair given by σ. Then there exists a left-invariant metric on G which is also right invariant on K such that the induced metric on G/K making it to be a Riemannian symmetric space.*

Proof. For convenience we use M to denote G/K. Note that a left-invariant metric on G which is also right invariant on K is equivalent to a metric on $\mathfrak{g}$ which is $\mathrm{Ad}(K)$-invariant. Since K is compact, it admits a $\mathrm{Ad}(K)$-invariant metric, and it can be extended to a $\mathrm{Ad}(K)$-invariant metric on $\mathfrak{g}$ as what we have done in the proof of (4) in Theorem 2.4. Furthermore, by Corollary 2.7 one has $(\mathrm{d}\widetilde{\sigma})_{eK} = -\mathrm{id}_M$.

Now it suffices to show for any $gK \in M$, $(\mathrm{d}\widetilde{\sigma})_{gK} : T_{gK}M \to T_{\sigma(g)K}M$ is an isometry. Note that $\widetilde{\sigma}(ghK) = \sigma(g)\sigma(h)K = \sigma(g)\widetilde{\sigma}(hK)$ holds for all $h \in G$. This shows $\widetilde{\sigma} \circ L_g = L_{\sigma(g)} \circ \widetilde{\sigma}$, where $L_g : M \to M$ is given by $L_g(hK) = ghK$. By taking differential one has the following communicative diagram

$$\begin{tikzcd} T_{eK}M \arrow[r, "(\mathrm{d}\widetilde{\sigma})_{eK}"] \arrow[d, "(\mathrm{d}L_g)_{eK}"'] & T_{eK}M \arrow[d, "(\mathrm{d}L_{\sigma(g)})_{eK}"] \\ T_{gK}M \arrow[r, "(\mathrm{d}\widetilde{\sigma})_{gK}"] & T_{\sigma(g)K}M \end{tikzcd}$$

Since $(\mathrm{d}L_g)_{eK}, (\mathrm{d}L_{\sigma(g)})_{eK}, (\mathrm{d}\widetilde{\sigma})_{eK}$ are isometries, one has $(\mathrm{d}\widetilde{\sigma})_{gK}$ is also an isometry as desired. □

2.C. **Examples of Riemannian symmetric space.**

Example 2.9. $G = \mathrm{SL}(n, \mathbb{R})$ together with $K = \mathrm{SO}(n)$ gives a Riemannian symmetric pair, where σ is defined by

$$\begin{aligned} \sigma : \mathrm{SL}(n, \mathbb{R}) &\to \mathrm{SL}(n, \mathbb{R}) \\ g &\mapsto (g^{-1})^T. \end{aligned}$$

Indeed, note that

$$(\mathrm{SL}(n,\mathbb{R}))^\sigma = \mathrm{SO}(n).$$

Thus $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ is a Riemannian symmetric space, and it can be viewed as a generalization of hyperbolic plane $\mathbb{H}^2$, since $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2) \cong \mathbb{H}^2$.

Example 2.10. $G = \mathrm{SO}(n+1)$ together with $K = \mathrm{SO}(n)$ gives a Riemannian symmetric pair, where σ is defined by

$$\begin{aligned} \sigma :\ \mathrm{SO}(n+1) &\to \mathrm{SO}(n+1) \\ a &\mapsto I_{1,n} a I_{1,n}^{-1}, \end{aligned}$$

where $I_{1,n} = diag\{-1, 1, \dots, 1\}$. Indeed, a direct computation shows

$$\mathrm{SO}(n+1)^\sigma = \{a \in \mathrm{SO}(n+1) \mid I_{1,n} a = a I_{1,n}\} = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & b \end{pmatrix} \in \mathrm{SO}(n+1) \mid b \in \mathrm{O}(n) \right\},$$

which implies $(\mathrm{SO}(n+1)^\sigma)_0 = \mathrm{SO}(n) \subsetneq \mathrm{SO}(n+1)$. Thus $S^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$ is a Riemannian symmetric space.

Example 2.11 (compact Grassmannian)**.** Consider the Grassmannian of oriented k-planes in $\mathbb{R}^{k+l}$, denoted by $M = \widehat{Gr}_k(\mathbb{R}^{k+l})$. It's clear that $\mathrm{SO}(k+l)$ acts on M transitively with isotropy group $\mathrm{SO}(k) \times \mathrm{SO}(l)$, and thus $M \cong \mathrm{SO}(k+l)/\mathrm{SO}(k) \times \mathrm{SO}(l)$. Consider the involution

$$\begin{aligned} \sigma :\ \mathrm{SO}(k+l) &\to \mathrm{SO}(k+l) \\ a &\mapsto I_{k,l} a I_{k,l}^{-1}, \end{aligned}$$

where $I_{k,l} = diag\{\underbrace{-1, \dots, -1}_{k \text{ times}}, \underbrace{1, \dots, 1}_{l \text{ times}}\}$. A direct computation shows

$$\mathrm{SO}(k+l)^\sigma = S(\mathrm{O}(k) \times \mathrm{O}(l))$$

Then $(\mathrm{SO}(k+l)^\sigma)_0 = \mathrm{SO}(k) \times \mathrm{SO}(l) \subsetneq \mathrm{SO}(k+l)^\sigma$, and thus M is a Riemannian symmetric space, called compact Grassmannian. In particular, $S^n = \widehat{Gr}_1(\mathbb{R}^{n+1})$.

Example 2.12 (hyperbolic Grassmannian)**.** In $\mathbb{R}^{k,l}$ with $k \geq 2, l \geq 1$, let's consider the following quadratic form

$$v^t I_{k,l} w = v^t \begin{pmatrix} I_k & 0 \\ 0 & -I_l \end{pmatrix} w = \sum_{i=1}^{k} v_i w_i - \sum_{j=k+1}^{k+l} v_j w_j$$

The group of linear transformation X that preserves this quadratic form is denoted by $\mathrm{O}(k,l)$, that is

$$X I_{k,l} X^t = I_{k,l}$$

and $\mathrm{SO}(k,l)$ are those with positive determinant. Now consider set consisting of those oriented k-dimensional subspaces of $\mathbb{R}^{k,l}$ on which quadratic form $I_{k,l}$ are positive definite. This space is called the hyperbolic Grassmannian $M = \widehat{Gr}_k(\mathbb{R}^{k,l})$, which is also an open subset of $\widehat{Gr}_k(\mathbb{R}^{k+l})$. It's clear $G = \mathrm{SO}(k,l)$ acting transitively on M with isotropy group $G_p = \mathrm{SO}(k) \times \mathrm{SO}(l)$. As in Example 2.11 one can also construct an involution σ to show $\widehat{Gr}_k(\mathbb{R}^{k,l})$ is a Riemannian symmetric space.

Example 2.13. Let K be a connected compact Lie group and $G = K \times K$. Then (G, K) is a Riemannian symmetric pair given by σ, where $\sigma :\ G \to G$ is given by $(x, y) \mapsto (y, x)$,

since
$$G^\sigma = \{(a, a) \mid a \in K\} \cong K$$
Then any compact Lie group is a Riemannian symmetric space.

3. Curvature of Riemannian symmetric space

3.A. **Formulas.**

Proposition 3.1. *Let (M, g) be a Riemannian symmetric space and $G = \mathrm{Iso}(M, g)$ with Lie algebra $\mathfrak{g}$. For any $p \in M$, one has Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{k}$ is Lie algebra of isotropy group G_p and $\mathfrak{m} \cong T_pM$. Then for any $X \in \mathfrak{k}$, one has*

$$B(X, X) \leq 0$$

where B is the Killing form of $\mathfrak{g}$. Furthermore, the identity holds if and only if $X = 0$.

Proof. Since a Killing field is determined by X_p and $(\nabla X)_p$, one has elements in $\mathfrak{k}$ is determined by $(\nabla X)_p$, and note that ∇X is a skew-symmetric matrice, so

$$\mathfrak{k} \cong \{(\nabla X) \in \mathfrak{so}(T_pM) \mid X \in \mathfrak{k}\}$$

By using this identification, there is a natural metric on $\mathfrak{k}$ given by

$$\langle S_1, S_2 \rangle = -\mathrm{tr}(S_1 S_2)$$

Then one has metric on $\mathfrak{g}$ since there is a metric on $\mathfrak{m}$ obtained from $\mathfrak{m} \cong T_pM$. For any $S \in \mathfrak{k}$, we claim with respect to this metric, $\mathrm{ad}_S : \mathfrak{g} \to \mathfrak{g}$ is skew-symmetric. Indeed, for $X_1, X_2 \in \mathfrak{k}$, one has

$$\begin{aligned}
\langle \mathrm{ad}_S X_1, X_2 \rangle &= -\mathrm{tr}(\mathrm{ad}_S X_1 X_2) \\
&= -\mathrm{tr}((SX_1 - X_1 S)X_2) \\
&= \mathrm{tr}(X_1(SX_2 - X_2 S)) \\
&= -\langle X_1, \mathrm{ad}_S X_2 \rangle
\end{aligned}$$

For $Y_1, Y_2 \in \mathfrak{m}$, since $S_p = 0$ and $(\nabla S)_p$ is skew-symmetric, one has

$$\begin{aligned}
\langle \mathrm{ad}_S Y_1, Y_2 \rangle &= \langle \nabla_S Y_1 - \nabla_{Y_1} S, Y_2 \rangle \\
&= -\langle \nabla_{Y_1} S, Y_2 \rangle \\
&= \langle \nabla_{Y_2} S, Y_1 \rangle \\
&= -\langle Y_1, \nabla_S Y_2 - \nabla_{Y_2} S \rangle \\
&= -\langle Y_1, \mathrm{ad}_S Y_2 \rangle
\end{aligned}$$

Then one has

$$B(S, S) = \mathrm{tr}(\mathrm{ad}_S \circ \mathrm{ad}_S) = \sum \langle \mathrm{ad}_S \circ \mathrm{ad}_S(e_i), e_i \rangle = -\sum \langle \mathrm{ad}_S(e_i), \mathrm{ad}_S(e_i) \rangle \leq 0$$

Furthermore, if $B(S, S) = 0$, then $\mathrm{ad}_S = 0$ and for any $X \in \mathfrak{g}$, one has

$$0 = \mathrm{ad}_S(X) = [S, X] = \nabla_S X - \nabla_X S = -\nabla_X S$$

since $S_p = 0$. This implies $(\nabla S)_p = 0$, and thus $S = 0$. $\square$

Theorem 3.2. *Let (M, g) be a Riemannian symmetric space and $G = \mathrm{Iso}(M, g)$. For any $p \in M$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ with $\mathfrak{m} \cong T_pM$.*

(1) For any $X, Y, Z \in \mathfrak{m}$, there holds

$$R(X, Y)Z = -[Z, [Y, X]]$$

$$\mathrm{Ric}(Y, Z) = -\frac{1}{2} B(Y, Z)$$

(2) If $\mathrm{Ric}(g) = \lambda g$*, then for* $X, Y \in \mathfrak{m}$*, one has*

$$2\lambda R(X, Y, Y, X) = -B([X, Y], [X, Y])$$

Proof. For (1). For any $X, Y, Z \in \mathfrak{m}$, direct computation shows

$$\begin{aligned} R(X, Y)Z &\overset{(a)}{=} R(X, Z)Y - R(Y, Z)X \\ &\overset{(b)}{=} \nabla_Z \nabla_Y X - \nabla_{\nabla_Z Y} X - \nabla_Z \nabla_X Y + \nabla_{\nabla_Z X} Y \\ &\overset{(c)}{=} -\nabla_Z [X, Y] \\ &\overset{(d)}{=} -[Z[X, Y]] \end{aligned}$$

where

(a) holds from the first Bianchi identity.
(b) holds from (2) of Proposition C.1.
(c) holds from $X, Y \in \mathfrak{m}$, and thus $(\nabla X)_p = (\nabla Y)_p = 0$.
(d) holds from

$$\nabla_Z [X, Y] - \nabla_{[X,Y]} Z = [Z, [X, Y]]$$

and $(\nabla Z)_p = 0$.

To see Ricci curvature, note that for $Y \in \mathfrak{m}$

$$\mathrm{ad}_Y : \mathfrak{k} \to \mathfrak{m}, \quad \mathrm{ad}_Y : \mathfrak{m} \to \mathfrak{k}$$

Thus $\mathrm{ad}_Z \circ \mathrm{ad}_Y$ preserves the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ if $Y, Z \in \mathfrak{m}$. Then

$$\begin{aligned} \mathrm{tr}(\mathrm{ad}_Z \circ \mathrm{ad}_Y |_{\mathfrak{m}}) &= \mathrm{tr}(\mathrm{ad}_Z |_{\mathfrak{k}} \circ \mathrm{ad}_Y |_{\mathfrak{m}}) \\ &= \mathrm{tr}(\mathrm{ad}_Y |_{\mathfrak{m}} \circ \mathrm{ad}_Z |_{\mathfrak{k}}) \\ &= \mathrm{tr}(\mathrm{ad}_Y \circ \mathrm{ad}_Z |_{\mathfrak{k}}) \end{aligned}$$

Hence we obtain

$$B(Y, Y) = \mathrm{tr}(\mathrm{ad}_Y \circ \mathrm{ad}\, Y|_{\mathfrak{k}}) + \mathrm{tr}(\mathrm{ad}_Y \circ \mathrm{ad}\, Y|_{\mathfrak{m}}) = 2\mathrm{tr}(\mathrm{ad}_Y \circ \mathrm{ad}_Y |_{\mathfrak{m}})$$

Since Ricci tensor is trace of curvature tensor, and thus

$$\mathrm{Ric}(Y, Y) = -\mathrm{tr}(\mathrm{ad}_Y \circ \mathrm{ad}_Y |_{\mathfrak{m}}) = -\frac{1}{2} B(Y, Y)$$

Then by using Polarization identity, one has $\mathrm{Ric}(Y, Z) = -B(Y, Z)/2$.

For (2). If $\mathrm{Ric}(g) = \lambda g$, then

$$\begin{aligned} 2\lambda g(R(X, Y)Y, X) &= -2\lambda g(\mathrm{ad}_Y \circ \mathrm{ad}_Y X, X) \\ &= -2\mathrm{Ric}(\mathrm{ad}_Y \circ \mathrm{ad}_Y X, X) \\ &= B(\mathrm{ad}_Y \circ \mathrm{ad}_Y X, X) \\ &= -B(\mathrm{ad}_Y X, \mathrm{ad}_Y X) \\ &= -B([X, Y], [X, Y]) \end{aligned}$$

□

Corollary 3.3. *Let* (M, g) *be a Riemannian symmetric space which is an Einstein manifold with Einstein constant* λ*. Then*

(1) If $\lambda > 0$, then (M, g) has non-negative sectional curvature.
(2) If $\lambda < 0$, then (M, g) has non-positive sectional curvature.
(3) If $\lambda = 0$, then (M, g) is flat.

Proof. By Theorem 3.2 one has

$$2\lambda R(X,Y,Y,X) = -B([X,Y],[X,Y]) \geq 0,$$

since $[X,Y] \in [\mathfrak{m},\mathfrak{m}] \subseteq \mathfrak{k}$ and B is negative definite on $\mathfrak{k}$. This shows (1) and (2). If $\lambda = 0$, one has $B([X,Y],[X,Y]) \equiv 0$ for arbitrary X, Y. Then by Proposition 3.1 one has $[X,Y] \equiv 0$ for arbitrary X, Y, and thus (M, g) is flat. □

3.B. **Computations of curvature.**

Example 3.4. In Example 2.9 we have already shown that $M = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ is a Riemannian symmetric space. Consider its Cartan decomposition

$$\mathfrak{sl}(n) = \mathfrak{so}(n) \oplus \mathfrak{m},$$

where $\mathfrak{m}$ consists of symmetric matrices and $\mathfrak{m} \cong T_pM$ for $p \in M$. On $\mathfrak{m}$ we can put the usual Euclidean metric, that is for $X, Y \in \mathfrak{m}$, we define

$$\langle X, Y\rangle = \mathrm{tr}(XY^T) = \mathrm{tr}(XY) = \frac{1}{2n}B(X,Y),$$

where B is the Killing form of $\mathfrak{sl}(n)$. By Theorem 3.2 the corresponding metric on M has the curvature formulas

$$\mathrm{Ric}(g) = -\frac{B}{2} = -ng$$
$$R(X,Y,Y,X) = \frac{B([X,Y],[X,Y])}{2n} \leq 0$$

Hence it has non-positive sectional curvatures. One can also show its sectional curvature is non-positive by computing curvature tensor as follows

$$\begin{aligned}
R(X,Y,Z,W) &= \mathrm{tr}([Z,[X,Y]]W) \\
&= \mathrm{tr}(Z[X,Y]W - [X,Y]ZW) \\
&= \mathrm{tr}(WZ[X,Y] - [X,Y]ZW) \\
&= \mathrm{tr}([X,Y][Z,W]) \\
&= -\mathrm{tr}([X,Y][Z,W]^T) \\
&= -\langle [X,Y],[Z,W]\rangle
\end{aligned}$$

Example 3.5 (compact Grassmannian)**.** In Example 2.11 we have already shown that $M = \widehat{Gr}_k(\mathbb{R}^{k+l})$ is a Riemannian symmetric space with Cartan decomposition

$$\mathfrak{so}(k+l) = \mathfrak{so}(k) \oplus \mathfrak{so}(l) \oplus \mathfrak{m},$$

where $\mathfrak{m} \cong T_pM$ for $p \in M$. Note that one has the block decomposition of matrices in $\mathfrak{so}(k+l)$ as follows

$$\mathfrak{so}(k+l) = \left\{\begin{pmatrix} X_1 & B \\ -B^T & X_2 \end{pmatrix} \mid X_1 \in \mathfrak{so}(k), X_2 \in \mathfrak{so}(l), B \in M_{k\times l}(\mathbb{R})\right\}.$$

Then one has $\mathfrak{m} \cong \left\{ \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} \mid B \in M_{k\times l}(\mathbb{R}) \right\}$. If we put the usual Euclidean metric on $\mathfrak{m}$, that is

$$\begin{aligned}\left\langle \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} \right\rangle &= \operatorname{tr}\left(\begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}^T \right) \\ &= -\operatorname{tr}\left(\begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \right) \\ &= -\frac{1}{k+l-2} B\left(\begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} \right),\end{aligned}$$

where B is the Killing form of $\mathfrak{so}(n)$. Then the corresponding metric on M has the curvature formulas

$$\begin{aligned}\operatorname{Ric}(g) &= -\frac{B}{2} = \frac{k+l-2}{2} g, \\ R(X,Y,Y,X) &= -\frac{B([X,Y],[X,Y])}{k+l-2} \geq 0,\end{aligned}$$

where $X, Y \in \mathfrak{m}$. This shows the compact Grassmannian has the non-negative sectional curvature.

Example 3.6 (hyperbolic Grassmannian)**.** In Example 2.12 we have already shown that $M = \widehat{Gr}_k(\mathbb{R}^{k,l})$ is a Riemannian symmetric space with Cartan decomposition

$$\mathfrak{so}(k,l) = \mathfrak{so}(k) \oplus \mathfrak{so}(l) \oplus \mathfrak{m},$$

where $\mathfrak{m} \cong T_p M$ for $p \in M$. Note that one has the block decomposition of matrices in $\mathfrak{so}(k,l)$ as follows

$$\mathfrak{so}(k,l) = \left\{ \begin{pmatrix} X_1 & B \\ B^T & X_2 \end{pmatrix} \mid X_1 \in \mathfrak{so}(k), X_2 \in \mathfrak{so}(l), B \in M_{k\times l}(\mathbb{R}) \right\}.$$

Then one has $\mathfrak{m} \cong \left\{ \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \mid B \in M_{k\times l}(\mathbb{R}) \right\}$. If we put the usual Euclidean metric on $\mathfrak{m}$, then

$$\left\langle \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \right\rangle = \frac{1}{k+l-2} B\left(\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \right),$$

where B is the Killing form of $\mathfrak{so}(k,l)$. Then the corresponding metric on M has the curvature formulas

$$\begin{aligned}\operatorname{Ric}(g) &= -\frac{B}{2} = -\frac{k+l-2}{2} g, \\ R(X,Y,Y,X) &= \frac{B([X,Y],[X,Y])}{k+l-2} \leq 0,\end{aligned}$$

where $X, Y \in \mathfrak{m}$. This shows the hyperbolic Grassmannian has non-positive sectional curvature.

Example 3.7. In Example 2.13 one has a connected compact Lie group $G \cong G \times G/G^{\Delta}$ is a Riemannian symmetric space with Cartan decomposition $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}^{\Delta} \oplus \mathfrak{g}^{\perp}$, where

$$\begin{aligned} \mathfrak{g}^{\Delta} &= \{(X, X) \mid X \in \mathfrak{g}\}, \\ \mathfrak{g}^{\perp} &= \{(X, -X) \mid X \in \mathfrak{g}\}. \end{aligned}$$

Then one has $\mathfrak{m} \cong \mathfrak{g}^{\perp}$, and thus curvature tensor can be computed as follows

$$\begin{aligned} R(X, Y)Z &= R((X, -X), (Y, -Y))(Z, -Z) \\ &= [(Z, -Z), [(X, -X), (Y, -Y)]] \\ &= ([Z, [X, Y]], -[Z, [X, Y]]) \end{aligned}$$

Hence, we arrive at that the formula

$$R(X, Y)Z = [Z, [X, Y]].$$

Remark 3.8. If one computes the curvature tensor in the standard way using bi-invariant metric, then the formula has a factor $1/4$ on it.

4. Classifications

4.A. **Holonomy group.**

Definition 4.1 (holonomy). *Let (M, g) be a Riemannian manifold and γ be a piecewise smooth loop centered at $p \in M$. Then the parallel along γ gives an isometry on T_pM, and the set of all such isometries forms a group called holonomy group, denoted by* $\mathrm{Hol}_p(M, g)$*.*

Remark 4.2. Note that if q is another base point, and γ is a path from p to q, then $\mathrm{Hol}_q = P_\gamma \mathrm{Hol}_p P_\gamma^{-1}$, and thus they are isomorphic, so for convenience we just denote it by Hol.

Theorem 4.3. *Let (M, g) be a Riemannian manifold. Then*
(1) Hol *is a Lie group and its identity component* Hol^0 *is compact.*
(2) Hol^0 *is given by parallel transport along null homotopic loops. As a consequence, if M is simply-connected, then* $\mathrm{Hol} = \mathrm{Hol}^0$.

Proposition 4.4. *Let (M, g) be a Riemannian symmetric space with $G = \mathrm{Iso}(M, g)$ and $K = G_p$ for some $p \in M$. Then $\mathrm{Hol}_p \subseteq K$.*

Proof. Note that holonomy group is the group of parallel transports along all piecewise smooth loops centered at p, and such a loop γ be written as a limit of geodesic polygons γ_i. The parallel transport along any edge of the polygon is given by applying a transvection along that edge, and so the parallel transport along the full polygon is a composition of isometries which sends p back to itself, hence it is an element of the isotropy group K. Since K is compact, the sequence of parallel transports along geodesic polygons approximating the given loop has a convergent subsequence, and thus $\mathrm{Hol}_p \subseteq K$. □

Definition 4.5 (decomposability). *A Riemannian manifold (M, g) is called decomposable if M is a product of $N_1 \times N_2$ and the Riemannian metric is a product metric. Otherwise (M, g) is called indecomposable.*

Theorem 4.6 (de Rham). *Let M be a simply-connected Riemannian manifold, $p \in M$ and Hol_p the holonomy group. Let $T_pM = V_0 \oplus V_1 \oplus \cdots \oplus V_k$ be decomposition into Hol_p irreducible subspace with $V_0 = \{v \in T_pM \mid hv = v \text{ for all } h \in \mathrm{Hol}_p\}$. Then M is a Riemannian product $M = M_0 \times \cdots \times M_k$, where M_0 is isometric to flat $\mathbb{R}^n$. If $p = (p_0, p_1, \dots, p_k)$, then $T_{p_i}M_i \cong V_i$ and M_i is indecomposable if $i \geq 1$. Furthermore, the decomposition is unique up to order and $\mathrm{Hol}_p \cong \mathrm{Hol}_{p_1} \times \cdots \times \mathrm{Hol}_{p_k}$.*

4.B. **Irreducibility.**

Definition 4.7 (isotropy irreducible). *Let (M, g) be a Riemannian symmetric space with $G = \mathrm{Iso}(M, g)$ and $K = G_p$ for some $p \in M$. If the identity component of K acts irreducibly on T_pM, then M is called irreducible. Otherwise M is called reducible.*

Lemma 4.8. *Let B_1, B_2 be two symmetric bilinear forms on a vector space V such that B_1 is positive definite. If a group K acts irreducibly on V such that B_1 and B_2 are invariant under K, then $B_2 = \lambda B_1$ for some constant λ.*

Proof. Since B_1 is positive definite, there exists an endomorphism $L : V \to V$ such that

$$B_2(u, v) = B_1(Lu, v)$$

where $u, v \in V$. Since B_1, B_2 are invariant under K, one has for any $k \in K$

$$B_1(kLu, v) = B_1(Lu, k^{-1}v) = B_2(u, k^{-1}v) = B_2(ku, v) = B_1(Lku, v)$$

holds for arbitrary $u, v \in V$, which implies $Lk = kL$ for all $k \in K$. Moreover, the symmetry of B_1, B_2 implies

$$B_1(Lu, v) = B_2(u, v) = B_2(v, u) = B_1(Lv, u) = B_1(u, Lv)$$

Hence L is symmetric with respect to B_1, and thus the eigenvalues of L are real. If $E \subseteq V$ is an eigenspace with eigenvalue λ, the fact $kL = Lk$ implies E is invariant under K. Since K acts irreducibly on V, one has $E = V$, that is $L = \lambda I$, which implies $B_2 = \lambda B_1$. □

Theorem 4.9. *The irreducible Riemannian symmetric space is Einstein, and the metric is unique determined up to a scalar.*

Proof. Since isometries preserves the metric and curvature, and Ricci tensor is also a symmetric bilinear form, by Lemma 4.8 there exists smooth function λ such that

$$\mathrm{Ric}(g) = \lambda g$$

Note that Riemannian curvature of Riemannian symmetric space is parallel, so is Ricci curvature. Thus we have λ is a constant. □

Theorem 4.10. *Let (M, g) be a simply-connected Riemannian symmetric space. Then (M, g) is isometric to*

$$(M_1, g_1) \times \cdots \times (M_k, g_k)$$

where (M_i, g_i) are irreducible Riemannian symmetric space for $i = 1, \dots, k$.

Proof. We can decompose T_pM into irreducible subspaces V_i under the action of K_0. Since $\mathrm{Hol}_p = \mathrm{Hol}_p^0 \subseteq K_0$, these subspaces can be further decomposed into irreducible ones under Hol_p. Applying Theorem 4.6, M has a corresponding decomposition as a Riemannian product. By collecting factors whose tangent spaces lie in V_i, we get a decomposition $M_1 \times \cdots \times M_k$ with $M_1 \cong \mathbb{R}^n$ flat (if a flat factor exists) and $T_{p_i}M_i \cong V_i$. If s_p is the symmetry at $p = (p_1, \dots, p_k)$, then the uniqueness of the decomposition also implies that $s_p = (s_{p_1}, \dots, s_{p_k})$ since, due to $\mathrm{d}(s_p)_p = -\mathrm{id}$, s_p cannot permute factors in the decomposition. Thus each factor M_i is a symmetric space which is irreducible by construction. □

Corollary 4.11. *A simply-connected symmetric space with simple isometry group is irreducible.*

5. Type and duality

5.A. **Compact, non-compact and Euclidean types.**

Definition 5.1 (types). *Let (G, K) be a Riemannian symmetric pair with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and B be the Killing form of $\mathfrak{g}$. The pair is called*

(1) of compact type if $B|_{\mathfrak{m}} < 0$;
(2) of non-compact type if $B|_{\mathfrak{m}} > 0$;
(3) of Euclidean type if $B|_{\mathfrak{m}} = 0$.

Theorem 5.2. *Let (G, K) be a Riemannian symmetric pair.*

(1) If (G, K) is irreducible, then it's either of compact type, non-compact type or Euclidean type.
(2) If $M = G/K$ is simply-connected, then M is isometric to a Riemannian product $M = M_0 \times M_1 \times M_2$ with M_0 of Euclidean type, M_1 of compact type and M_2 of non-compact type.
(3) If (G, K) is of compact type, then G is semisimple, and both G and M is compact.
(4) If (G, K) is of non-compact type, then G is semisimple, and both G and M are non-compact.
(5) (G, K) is of Euclidean type if and only if $[\mathfrak{m}, \mathfrak{m}] = 0$. Furthermore, if G/K is simply-connected, then it's isometric to $\mathbb{R}^n$.

Proposition 5.3. *Let (G, K) be a Riemannian symmetric pair.*

(1) If (G, K) is of compact type, then it has non-negative sectional curvature.
(2) If (G, K) is of non-compact type, then it has non-positive sectional curvatures.
(3) If (G, K) is of Euclidean type, then it's flat.

5.B. **Duality.** In this section we discuss the important concept of duality. Let (G, K) be a symmetric pair with G/K is simply-connected. Since G is connected, K is connected as well. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the Cartan decomposition of $\mathfrak{g}$. Then we can consider $\mathfrak{g}$ as a real subalgebra of $\mathfrak{g} \otimes \mathbb{C}$ and define a new real Lie algebra $\mathfrak{g}^* \subseteq \mathfrak{g} \otimes \mathbb{C}$ by $\mathfrak{g}^* = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{m}$. Now let G^* be the simply-connected Lie group with Lie algebra $\mathfrak{g}^*$ and K^* be the connected subgroup with Lie algebra $\mathfrak{k} \subseteq \mathfrak{g}^*$. The Riemannian symmetric pair (G^*, K^*) is called dual of (G, K).

Theorem 5.4. *Let (G, K) be a symmetric pair with dual symmetric space pair (G^*, K^*).*

(1) If (G, K) is of compact type, then (G^*, K^*) is of non-compact type, and vice versa.
(2) If (G, K) is of Euclidean type, then (G^*, K^*) is of Euclidean type.
(3) (G, K) is irreducible if and only if (G^*, K^*) is irreducible.

Example 5.5. $\mathrm{SU}(n)/\mathrm{SO}(n)$ is the dual of $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$.

Part 2. **Hermitian symmetric space**

Part 3. **Appendix**

Appendix A. Basic facts in Riemannian geometry

Theorem A.1. *Let $\varphi, \psi : (M, g_M) \to (N, g_N)$ be two local isometries between Riemannian manifolds, and M is connected. If there exists $p \in M$ such that*

$$\varphi(p) = \psi(p)$$
$$(\mathrm{d}\varphi)_p = (\mathrm{d}\psi)_p$$

then $\varphi = \psi$.

Theorem A.2 (Cartan-Ambrose-Hicks)**.** *Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be two Riemannian manifolds, and $\Phi_0 : T_pM \to T_{\widetilde{p}}\widetilde{M}$ is a linear isometry, where $p \in M, \widetilde{p} \in \widetilde{M}$. For $0 < \delta < \min\{\mathrm{inj}_p(M), \mathrm{inj}_{\widetilde{p}}(\widetilde{M})\}$, The following statements are equivalent.*

(1) There exists an isometry $\varphi : B(p, \delta) \to B(\widetilde{p}, \delta)$ such that $\varphi(p) = \widetilde{p}$ and $(\mathrm{d}\varphi)_p = \Phi_0$.

(2) For $v \in T_pM, |v| < \delta, \gamma(t) = \exp_p(tv), \widetilde{\gamma}(t) = \exp_{\widetilde{p}}(t\Phi_0(v))$, if we define

$$\Phi_t = P_{0,t;\widetilde{\gamma}} \circ \Phi_0 \circ P_{t,0;\gamma} : T_{\gamma(t)}M \to T_{\widetilde{\gamma}(t)}\widetilde{M}$$

then Φ_t preserves curvature, that is $(\Phi_t)^*R = R$.

Proposition A.3. *Let $(M, g_M), (N, g_N)$ be complete Riemannian manifolds and $f : M \to N$ be a local diffeomorphism such that for all $p \in M$ and for all $v \in T_pM$, one has $|(\mathrm{d}f)_p v| \geq |v|$. Then f is a Riemannian covering map.*

Theorem A.4 (Myers-Steenrod)**.** *Let (M, g) be a Riemannian manifold and $G = \mathrm{Iso}(M, g)$. Then*

(1) G is a Lie group with respect to compact-open topology.

(2) for each $p \in M$, the isotropy group G_p is compact.

(3) G is compact if M is compact.

Proposition A.5. *Let (M, g) be a Riemannian manifold, $\gamma : I \to M$ a smooth curve and $P_{s,t;\gamma} : T_{\gamma(s)}M \to T_{\gamma(t)}M$ is the parallel transport along γ. For any $s \in I$ with $v = \gamma'(s)$, one has*

$$\nabla_v R = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} (P_{s,t;\gamma})^* R_{\gamma(t)}$$

In particular, if $\nabla R = 0$ then

$$(P_{s,t;\gamma})^* R_{\gamma(t)} = R_{\gamma(s)}$$

holds for arbitrary $t, s \in I$.

Proposition A.6. *If $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ is a Riemannian covering, then M is complete if and only if $\widetilde{M}$ is.*

Appendix B. Hopf theorem

The argument about analytic continuation in Theorem 1.10 can be used to give a proof of Hopf's theorem.

Theorem B.1 (Hopf). *Let (M, g) be a complete, simply-connected Riemannian manifold with constant sectional curvature K. Then (M, g) is isometric to*

$$(\widetilde{M}, g_{can}) = \begin{cases} (\mathbb{S}^n(\frac{1}{\sqrt{K}}), g_{can}) & K > 0 \\ (\mathbb{R}^n, g_{can}) & K = 0 \\ (\mathbb{H}^n(\frac{1}{\sqrt{-K}}), g_{can}) & K < 0 \end{cases}$$

Proof. For $p \in M, \widetilde{p} \in \widetilde{M}$ and $\delta < \min\{\mathrm{inj}(p), \mathrm{inj}(\widetilde{p})\}$. By Cartan-Ambrose-Hicks's theorem, there exists an isometry $\varphi\colon B(p, \delta) \to B(\widetilde{p}, \delta)$ such that $\varphi(p) = \widetilde{p}$ and $(\mathrm{d}\varphi)_p$ equals to a given linear isometry, since both (M, g) and $(\widetilde{M}, \widetilde{g})$ have constant sectional curvature K. By the same argument in proof of Theorem 1.10, there is an isometry $\varphi\colon (M, g) \to (\widetilde{M}, \widetilde{g})$ which extends $\varphi\colon B(p, \delta) \to B(\widetilde{p}, \delta)$. In particular, this completes the proof. $\square$

APPENDIX C. KILLING FIELDS

C.A. **Basic properties.**

Proposition C.1. *Let (M,g) be a Riemannian manifold and X be a Killing field.*

(1) If γ is a geodesic, then $J(t)=X(\gamma(t))$ is a Jacobi field.

(2) For any two vector fields Y,Z,

$$\nabla_Y\nabla_Z X-\nabla_{\nabla_Y Z}X+R(X,Y)Z=0$$

Proof. For (1). Suppose φ_s is the flow generated by X. Then we obtain a variation $\alpha(s,t)=\varphi_s(\gamma(t))$ consisting of geodesics, and thus

$$X(\gamma(t))=\frac{\partial\varphi_s(\gamma(t))}{\partial s}\Big|_{s=0}$$

is a Jacobi field.

For (2). It's an equation of tensors, so we check it pointwisely and use normal coordinate $\{x^i\}$ centered at p. Furthermore, we assume $X=X^i\frac{\partial}{\partial x^i},Y=\frac{\partial}{\partial x^j},Z=\frac{\partial}{\partial x^k}$. Then

$$\begin{aligned}\nabla_Y\nabla_Z X-\nabla_{\nabla_Y Z}X+R(X,Y)Z&=\nabla_j\nabla_k X+X^iR^l_{ijk}\frac{\partial}{\partial x^l}\\&=(\frac{\partial^2X^l}{\partial x^j\partial x^k}+X^i\frac{\partial\Gamma^l_{ki}}{\partial x^j}+X^iR^l_{ijk})\frac{\partial}{\partial x^l}\\&=(\frac{\partial^2X^l}{\partial x^j\partial x^k}+X^i\frac{\partial\Gamma^l_{jk}}{\partial x^i})\frac{\partial}{\partial x^l}\end{aligned}$$

since $R^l_{ijk}=\frac{\partial\Gamma^l_{jk}}{\partial x^i}-\frac{\partial\Gamma^l_{ik}}{\partial x^j}+\Gamma^s_{jk}\Gamma^l_{is}-\Gamma^s_{ik}\Gamma^l_{js}$. Now it suffices to show $\frac{\partial^2X^l}{\partial x^j\partial x^k}+X^i\frac{\partial\Gamma^l_{jk}}{\partial x^i}\equiv 0$. In order to show this, for arbitrary $p\in M$, consider a geodesic γ starting at p and consider Jacobi field $J(t)=X(\gamma(t))$. Direct computation shows

$$\begin{aligned}J'(t)&=(\frac{\partial X^i}{\partial x^k}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}+X^i\Gamma^l_{ki}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t})\frac{\partial}{\partial x^l}\Big|_{\gamma(t)}\\J''(0)&=(\frac{\partial^2X^l}{\partial x^j\partial x^k}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}+X^i\frac{\partial\Gamma^l_{ki}}{\partial x^j}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t})\frac{\partial}{\partial x^l}\Big|_p\\&=(\frac{\partial^2X^l}{\partial x^j\partial x^k}+X^i\frac{\partial\Gamma^l_{ki}}{\partial x^j})\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}\frac{\partial}{\partial x^l}\Big|_p\\&=(\frac{\partial^2X^l}{\partial x^j\partial x^k}+X^i\frac{\partial\Gamma^l_{jk}}{\partial x^i}+X^i\frac{\partial\Gamma^l_{ki}}{\partial x^j}-X^i\frac{\partial\Gamma^l_{jk}}{\partial x^i})\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}\frac{\partial}{\partial x^l}\Big|_p\\&=(\frac{\partial^2X^l}{\partial x^j\partial x^k}+X^i\frac{\partial\Gamma^l_{jk}}{\partial x^i})\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}\frac{\partial}{\partial x^l}\Big|_p-R(X,\gamma')\gamma'\end{aligned}$$

which implies

$$\frac{\partial^2X^l}{\partial x^j\partial x^k}+X^i\frac{\partial\Gamma^l_{jk}}{\partial x^i}=0$$

holds at point p, and since p is arbitrary, this completes the proof. □

Corollary C.2. *Let (M, g) be a complete Riemannian manifold and $p \in M$. Then a Killing field X is determined by the values X_p and $(\nabla X)_p$ for arbitrary $p \in M$.*

Proof. The equation $\mathcal{L}_X g \equiv 0$ is linear in X, so the space of Killing fields is a vector space. Therefore, it suffices to show if $X_p = 0$ and $(\nabla X)_p = 0$, then $X \equiv 0$. For arbitrary $q \in M$, let $\gamma : [0, 1] \to M$ be a geodesic connecting p and q with $\gamma'(0) = v$. Since $J(t) = X(\gamma(t))$ is a Jacobi field, and a direct computation shows

$$(\nabla_v X)_p = J'(0)$$

Thus $J(t) \equiv 0$, since Jacobi field is determined by two initial values. In particular, $X_q = J(1) = 0$, and since q is arbitrary, one has $X \equiv 0$. □

Corollary C.3. *The dimension of vector space consisting of Killing fields $\leq n(n+1)/2$.*

Proof. Note that ∇X is skew-symmetric and the dimension of skew-symmetric matrices is $n(n-1)/2$. Thus the dimension of vector space consisting of Killing fields $\leq n + n(n-1)/2 = n(n+1)/2$. □

C.B. **Killing field as the Lie algebra of isometry group.**

Lemma C.4. *Killing field on a complete Riemannian manifold (M, g) is complete.*

Proof. For a Killing field X, we need to show the flow $\varphi_t : M \to M$ generated by X is defined for $t \in \mathbb{R}$. Otherwise, we assume φ_t is defined on (a, b). Note that for each $p \in M$, curve $\varphi_t(p)$ is a curve defined on (a, b) having finite constant speed, since φ_t is isometry. Then we have $\varphi_t(p)$ can be extended to the one defined on $\mathbb{R}$, since M is complete. □

Theorem C.5. *Let (M, g) be a complete Riemannian manifold and $\mathfrak{g}$ the space of Killing fields. Then $\mathfrak{g}$ is isomorphic to the Lie algebra of $G = \mathrm{Iso}(M, g)$.*

Proof. It's clear $\mathfrak{g}$ is a Lie algebra since $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$. Now let's see it's isomorphic to Lie algebra consisting of Killing field as Lie algebra.

(1) Given a Killing field X, by Lemma C.4, one deduces that the flow $\varphi : \mathbb{R} \times M \to M$ generated by X is a one parameter subgroup $\gamma : \mathbb{R} \to G$, and $\gamma'(0) \in T_e G$.
(2) Given $v \in T_e G$, consider the one-parameter subgroup $\gamma(t) = \exp(tv) : \mathbb{R} \to G$ which gives a flow by
$$\begin{aligned} \varphi : \mathbb{R} \times M &\to M \\ (t, p) &\mapsto \exp(tv) \cdot p \end{aligned}$$
Then the vector field X generated by this flow is a Killing field.

This gives a one to one correspondence between Killing fields and Lie algebra of G, and it's a Lie algebra isomorphism. □

Corollary C.6 (Cartan decomposition)**.** *Let (M, g) be a complete Riemannian manifold and $G = \mathrm{Iso}(M, g)$ with Lie algebra $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ of G has a decomposition as vector spaces*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

where

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid X_p = 0\}$$
$$\mathfrak{m} = \{X \in \mathfrak{g} \mid (\nabla X)_p = 0\}$$

and they satisfy

$$[\mathfrak{k},\mathfrak{k}]\subseteq\mathfrak{k},\quad [\mathfrak{m},\mathfrak{m}]\subseteq\mathfrak{k},\quad [\mathfrak{k},\mathfrak{m}]\subseteq\mathfrak{m}$$

Proof. The decomposition follows from Corollary C.2 and Theorem C.5, and it's easy to see

$$[\mathfrak{k},\mathfrak{k}]\subseteq\mathfrak{k},\quad [\mathfrak{m},\mathfrak{m}]\subseteq\mathfrak{k}$$

For arbitrary $X\in\mathfrak{k}, Y\in\mathfrak{m}$ and $v\in T_pM$, one has

$$\begin{aligned}\nabla_v[X,Y] &= \nabla_v\nabla_X Y-\nabla_v\nabla_Y X\\ &= -R(Y,v)X+\nabla_{\nabla_v X}Y+R(X,v)Y-\nabla_{\nabla_v Y}X\\ &= 0\end{aligned}$$

since $X_p=0$ and $(\nabla Y)_p=0$. This shows $[\mathfrak{k},\mathfrak{m}]\subseteq\mathfrak{m}$. □

References

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