# TOPCIS IN COMPLEX ALGEBRAIC GEOMETRY

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## Contents

# 0. Preface

0.1. **Introduction.** In this lecture, the object we're most interested in is the complex variety.

**Definition 0.1** (complex variety)**.** A complex algebraic variety or simply a complex variety is a quasi-projective[1] variety over $\mathbb{C}$.

**Definition 0.2** (non-singular)**.** A complex variety $X$ is non-singular if the sheaf of Kähler differentials $\Omega_{X/\mathbb{C}}$ is locally free.

Given any non-singular projective complex variety $X$, one can show that $X \subseteq \mathbb{CP}^n$ is a submanifold by using inverse function theorem. Conversely, Chow showed that

**Theorem 0.1** (Chow)**.** Any compact complex submanifold[2] of complex projective space must be a complex variety.

Chow's theorem implies that there is a deep connection between complex manifolds and complex varieties, and thus techniques from complex differential geometry may be used to solve some questions in algebraic geometry, such as corollaries of Calabi-Yau theorem. On the other hand, motivated by Chow's theorem, it's natural to ask whether a compact complex manifold can be (holomorphically) embedded into complex projective space or not.

**Theorem 0.2** (Riemann)**.** Any compact Riemann surface can be embedded into $\mathbb{CP}^N$.

**Theorem 0.3** (Kodaira)**.** A compact complex manifold with a positive holomorphic line can be embedded into $\mathbb{CP}^N$.

*Remark* 0.1. In fact, Riemann's result can be obtained from Kodaira's embedding. Given a Hermitian holomorphic line bundle $(L, h)$, its Chern curvature $\sqrt{-1}\Theta_h$ represents the first Chern class $c_1(L)$, and $\partial\overline{\partial}$-lemma shows that any real $(1,1)$-form which represents $c_1(L)$ can be realized as the Chern curvature of some Hermitian metric $h$. Thus if we want to see whether a holomorphic line bundle is positive or not, it suffice to compute its first Chern class, and there always exists holomorphic line with positive first Chern class[3].

The Kähler manifold is an important object in the complex differential geometry, which lies in the intersection of Riemannian manifold, complex

[1]The set $X \subseteq \mathbb{CP}^n$ is a projective variety if it's the zero-locus of some (finite) family of homogeneous polynomials, that generate a prime ideal, and it's called quasi-projective if it's an open subset of a projective variety.

[2]In fact, "submanifold" can be replaced by analytic subvariety, that is, we allow some singularities.

[3]For holomorphic line bundle $L$ over Riemann surface, the "positivity" of first Chern class is determined by its degree, that is, holomorphic line bundle with positive degree has positive first Chern class.

manifold and symplectic manifold, and has many elegant properties. One of the most profound results is the Hodge decomposition.

**Theorem 0.4** (Hodge)**.** Let $(X,\omega)$ be a compact Kähler manifold. Then there is a decomposition

$$H^n(X)\cong\bigoplus_{p+q=n}H^{p,q}(X),$$

where $H^{p,q}(X)$ is the Dolbeault cohomology of $X$.

*Remark* 0.2. The Hodge decomposition is independent of the choice of Kähler form $\omega$, but for the proof, we need to use theory of harmonic forms and Kähler identities.

The Hodge decomposition has lots of consequences in algebraic geometry. Let $X$ be a non-singular projective complex variety. The algebraic de Rham complex is defined by

$$\Omega^\bullet_{X/\mathbb{C}}\colon\ \mathscr{O}_X\xrightarrow{\mathrm{d}}\Omega_{X/\mathbb{C}}\xrightarrow{\mathrm{d}}\dots\xrightarrow{\mathrm{d}}\Omega^n_{X/\mathbb{C}},$$

where $n=\dim X$, and the algebraic de Rham cohomology is defined by the hypercohomology of above complex as follows

$$H^k_{alg}(X)=\mathbb{H}^k(\Omega^\bullet_{X/\mathbb{C}}),$$

where $k\in\mathbb{Z}_{\geq 0}$. Note that there is a natural filteration on algebraic de Rham complex

$$\Omega^\bullet_{X/\mathbb{C}}=F^0\Omega^\bullet_{X/\mathbb{C}}\supseteq F^1\Omega^\bullet_{X/\mathbb{C}}\supseteq\cdots\supseteq F^n\Omega^\bullet_{X/\mathbb{C}}=\{0\},$$

where

$$F^p\Omega^\bullet_{X/\mathbb{C}}\colon 0\to\cdots\to 0\to\Omega^p_{X/\mathbb{C}}\to\cdots\to\Omega^n_{X/\mathbb{C}}.$$

This filteration gives the Hodge to de Rham spectral sequence.

**Theorem 0.5** ($E_1$-degeneration)**.** Let $X$ be a non-singular projective complex variety. The Hodge to de Rham spectral sequence

$$E_1^{p,q}=H^q(X,\Omega^p_{X/\mathbb{C}})\Longrightarrow H^{p+q}_{alg}(X)$$

degenerates at $E_1$-page, and

$$\dim_{\mathbb{C}}H^p(X,\Omega^q_{X/\mathbb{C}})=\dim_{\mathbb{C}}H^q(X,\Omega^p_{X/\mathbb{C}}).$$

*Remark* 0.3. The inequality

$$\dim H^k_{alg}(X)\leq\sum_{p+q=k}H^q(X,\Omega^p_{X/\mathbb{C}})$$

always holds, and the equality holds if and only if the Hodge to de Rham spectral sequence degenerates at $E_1$-page.

There are several important developments in Kähler geometry after Hodge and Kodaira, such as the solution to Calabi conjecture given by Shing-Tung Yau, and the connection between stable vector bundles and Hermitian-Yang-Mills metrics proved by Uhlenbeck-Yau.

**Theorem 0.6** (Calabi-Yau)**.** Let $(X, \omega)$ be a compact Kähler manifold and $\chi$ be a real $(1,1)$-form that represents the first Chern class. Then there exists a unique $\omega_h \in [\omega]$ such that $\mathrm{Ric}(\omega_h) = \chi$.

**Corollary 0.1.** There exists a unique Ricci-flat Kähler metric on compact Kähler manifold with vanishing first Chern class.

Now let's introduce some algebraic consequence of Calabi-Yau theorem.

**Theorem 0.7.** Let $X$ be a non-singular projective complex variety with ample canonical bundle $K_X$. Then

$$(-1)^n \left( c_1^n(X) - \frac{2(n+1)}{n} c_1^{n-2}(X) c_2(X) \right) \leq 0.$$

Moreover, the equality holds if and only if $X$ is a locally symmetric variety of ball type.

**Corollary 0.2.** If $X$ is a locally symmetric variety of ball type, then $X^\sigma$ is again a locally symmetric variety of ball type for any $\sigma \in \mathrm{Aut}(\mathbb{C})$.

**Theorem 0.8.** Let $X$ be a non-singular projective complex variety with $c_1(X) = 0$. Then for any ample line bundle $L$ on $X$,

$$c_2(X) \cdot L^{n-2} \geq 0.$$

Moreover, the equality holds if and only if $X$ is an abelian variety.

To state Uhlenbeck-Yau's theorem, we need the following preparations.

**Definition 0.3** (slope)**.** Let $(X, \omega)$ be a compact Kähler and $E$ be a holomorphic vector bundle. The slope of $E$ with respect to $\omega$ is defined by

$$\mu_\omega(E) = \frac{\deg_\omega(E)}{\operatorname{rk} E},$$

where

$$\deg_\omega(E) = \int_X c_1(E) \cdot \omega^{n-1}.$$

**Definition 0.4** (stability)**.** Let $(X, \omega)$ be a compact Kähler and $E$ be a holomorphic vector bundle.

(1) $E$ is $\mu_\omega$-stable if for all subbundle $F \subseteq E$, one has

$$\mu_\omega(F) < \mu_\omega(E).$$

(2) $E$ is $\mu_\omega$-semistable if for all subbundle $F \subseteq E$, one has

$$\mu_\omega(F) \leq \mu_\omega(E).$$

**Definition 0.5** (Hermitian-Yang-Mills metric)**.** Let $(X, \omega)$ be a compact Kähler and $E$ be a holomorphic vector bundle. A Hermitian metric $h$ on $E$ is called Hermitian-Yang-Mills if

$$\wedge_\omega \Theta_h = \lambda \,\mathrm{id}_E,$$

where $\lambda \in \mathbb{R}$.

**Theorem 0.9** (Uhlenbeck-Yau)**.** Let $(X, \omega)$ be a compact Kähler and $E$ be a $\mu_\omega$-stable holomorphic bundle. Then there exists a unique Hermitian-Yang-Mills metric on $E$.

It also has lots of algebraic consequences.

**Theorem 0.10.** Let $X$ be a non-singular projective complex variety and $H$ be a line bundle. Let $E$ be a $\mu_H$-semistable vector bundle. Then

$$\left(c_1^2(E) - \frac{\mathrm{rk}(E)+1}{\mathrm{rk}(E)} c_2(E)\right) \cdot H^{n-2} \geq 0$$

**Theorem 0.11.** Let $X$ be a non-singular projective complex variety and $H$ be a line bundle. Let $E$ be a $\mu_H$-semistable vector bundle. Then

$$H^p(X, \Omega_X^q \otimes E \otimes H) = 0$$

for all $p + q > \dim X$.

In particular, above vanishing theorem generalizes the classical Kodaira vanishing theorem, which is a consequence of Hodge theory.

## 0.2. **Outlines.**

### 0.2.1. *Part I: Hodge theory.*

(1) Existence of harmonic forms.
(2) Kähler condition and Hodge package.
(3) Kodaira's vanishing theorem.
(4) Cartier descent theorem.
(5) De Rham decomposition theorem of Deligne-Illusie's theorem.
(6) Hodge symmetry.

### 0.2.2. *Part II: Non-abelian Hodge theory.*

(1) Existence of Hermitian-Yang-Mills metrics.
(2) Higgs bundle and the variant.
(3) Non-abelian Hodge theory.
(4) Ogus-Vologodsky theorem.
(5) Higgs-de Rham flow.

**Part 1. Hodge theory**

## 1. Existence of harmonic forms

Let $(M, g)$ be an oriented compact Riemannian manifold. Then there is a $L^2$-metric on $\mathcal{A}^n(M) := \Gamma(M, \Omega^n_M)$ defined by

$$\begin{aligned} (\text{-},\text{-})_{L^2} \colon \mathcal{A}^n(M) \times \mathcal{A}^n(M) &\to \mathbb{C} \\ (\omega, \tau) &\mapsto \int_X \langle \omega, \tau \rangle_g \operatorname{vol}_g. \end{aligned}$$

The vector space of harmonic forms is defined by

$$\mathcal{H}^n(M) := \{\omega \in \mathcal{A}^n(M) \mid \Delta_g(\omega) = 0\}.$$

**Theorem 1.1** (Hodge)**.** Let $(M, g)$ be an oriented compact Riemannian manifold. Then $\mathcal{H}^n(M) \cong H^n_{dR}(M)$.

*Remark* 1.1. The Hodge theorem gives a split of following exact sequence.

$$\begin{tikzcd} 0 \arrow[r] & B^n(M) \arrow[r] & Z^n(M) \arrow[r] & H^n_{dR}(M) \arrow[r] & 0, \\ & & \mathcal{H}^n(M) \arrow[u, hook] \arrow[ur, "\cong"] & & \end{tikzcd}$$

and one of advantages of above split is that we can regard elements of de Rham cohomology as a closed forms with certain properties, not an equivalent class.

### 1.1. Elliptic operators.

Let $E, F$ be smooth complex vector bundles over a smooth manifold $M$ of dimension $d$.

#### 1.1.1. *Differential operators.*

**Definition 1.1.1** (differential operator)**.** A differential operator of order $k$ is a $\mathbb{C}$-linear map

$$P \colon \Gamma(M, E) \to \Gamma(M, F)$$

such that on every local trivialization $U$ of $E, F$ with local coordinate $\{x^1, \ldots, x^d\}$, the map $P$ is given by a matrix $(p_{ij})$, where

$$p_{ij} = \sum_{|I| \leq k} P_{I,ij} \frac{\partial}{\partial x^I},$$

and $P_{I,ij} \in \mathcal{A}^0(U)$.

**Notation 1.1.1.** The set of all differential operators from $E$ to $F$ of order $k$ is denoted by $\operatorname{Diff}_k(E, F)$, and the set of all differential operators from $E$ to $E$ of order $k$ is denoted by $\operatorname{Diff}_k(E)$ for convenience.

**Example 1.1.1.** Let $M$ be a smooth manifold of dimension 1 and $E = F = M \times \mathbb{C}$ be the trivial bundle of rank one. For convenience, we assume the trivialization of $M$ is given by two charts as follows

$$x\colon U \to \mathbb{R}$$
$$y\colon V \to \mathbb{R}\,.$$

Let $P$ be a differential operator of order $k$ locally given by $\partial_x^k$ on the trivialization $U$. By chain rule and Leibniz's rule one has

$$\partial_y^k \equiv \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^k \partial_x^k \quad \text{(mod lower order terms)}.$$

*Remark* 1.1.1. More generally, let $P \in \mathrm{Diff}_k(E,F)$ be a differential operator of order $k$, which is locally given by the matrix $(p_{ij})$. If we define

$$P_{ij}^k = \sum_{|I|=k} P_{I,ij} \frac{\partial}{\partial x^I}.$$

Then by chain rule and Leibniz's rule one can see the coefficients of matrix $P^k$ are transformed like the sections of the $\mathrm{Sym}^k TM$ and similarly by a change of trivialization of the bundles $E$ and $F$, the matrix transforms like a section of $\mathrm{Hom}(E,F)$. It gives a section of $\mathrm{Sym}^k TM \otimes \mathrm{Hom}(E,F)$, which is called the **symbol** of $P$, and denoted by $\sigma_P$.

**Definition 1.1.2.** A differential operator $P \in \mathrm{Diff}_k(E,F)$ is said to be **elliptic** if for all $x \in M$ and $0 \neq \omega \in \Omega_{M,x}$, the homomorphism

$$\sigma_P(\omega)\colon E_x \to F_x$$

is an isomorphism.

1.1.2. *Adjoint operator.* Let $(M,g)$ be an oriented compact Riemannian manifold and $P \in \mathrm{Diff}_k(E,F)$ be a differential operator from smooth vector bundles $E$ to $F$. A **formal adjoint** of $P$, denoted by $P^*$, is a differential operator from $F$ to $E$ such that

$$(\alpha, P\beta)_{L^2} = (P^*\alpha, \beta)_{L^2}$$

holds for all $\alpha \in \Gamma(M,F)$ and $\beta \in \Gamma(M,E)$. The symbol of $P^*$ equals to the adjoint of the symbol of $P$, that is,

$$\sigma_{P^*,\omega} = (\sigma_{P,\omega})^*.$$

In particular, if $E$ and $F$ are of equal rank, then $P$ is elliptic if and only if its adjoint is elliptic.

**Definition 1.1.3.** Let $P \in \mathrm{Diff}_k(E)$ be a differential operator. Then $P$ is called **self-adjoint** if $P = P^*$.

**Example 1.1.2.** The Laplacian operator $\Delta_{\mathrm{d}} = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$ is a self-adjoint elliptic operator of order 2.

1.1.3. *Fundamental decomposition theorem for self-adjoint elliptic operator.* Let $(X, g)$ be a Riemannian manifold and $E$ be a Hermitian vector bundle on $(X, g)$.

**Theorem 1.1.1.** Let $L \in \mathrm{Diff}_k(E)$ be self-adjoint and elliptic. Then there exist linear mappings $H_L, G_L \colon \Gamma(X, E) \to \Gamma(X, E)$ such that

(1) $H_L(\Gamma(X, E)) = \mathcal{H}_L$, and $\dim \mathcal{H}_L < \infty$, where
$$\mathcal{H}_L = \{\alpha \in \Gamma(X, E) \mid L\alpha = 0\}.$$
(2) $L \circ G_L + H_L = G_L \circ L + H_L = \mathrm{id}_E$.
(3) The following decomposition is orthogonal with respect to $L^2$-norm
$$\begin{aligned}\Gamma(X, E) &= \mathcal{H}_L \oplus G_L \circ P(\Gamma(X, E)) \\ &= \mathcal{H}_L \oplus L \circ G_L(\Gamma(X, E)).\end{aligned}$$

*Proof.* See Theorem 4.12 in [Wel80]. □

*Remark* 1.1.2. The above theorem was first proved by Hodge for the case $E = \Omega^p_M$ and $L$ is the Laplacian operator with respect to the Riemannian metric on $(X, g)$. In this case, $\mathcal{H}_L$ is exactly the vector spaces of harmonic forms

1.1.4. *Elliptic complex and generalized Laplacian operator.* In this section, we study a generalization of elliptic operators to be called elliptic complexes. Suppose there is a sequence of differential operators,

$$\Gamma(M, E_0) \xrightarrow{L_0} \Gamma(M, E_1) \xrightarrow{L_1} \Gamma(M, E_2) \xrightarrow{L_2} \dots \xrightarrow{L_{N-1}} \Gamma(M, E_N), \tag{1.1}$$

where $L_i$ is a differential operator of order $k$ such that $L_i \circ L_{i-1} = 0$ and $E_0, E_1, \dots, E_N$ are a sequence of smooth complex vector bundles on smooth manifold $M$.

**Definition 1.1.4.** The complex (1.1) is called a **elliptic complex** if for all $x \in X$ and $0 \neq \omega \in \Omega_{M,x}$, the sequence

$$(E_0)_x \xrightarrow{\sigma_{L_0}(\omega)} (E_1)_x \xrightarrow{\sigma_{L_1}(\omega)} (E_2)_x \xrightarrow{\sigma_{L_2}(\omega)} \dots \xrightarrow{\sigma_{L_{N-1}}(\omega)} (E_{N-1})_x$$

is exact.

**Example 1.1.3.**
$$\mathcal{A}^0(M) \xrightarrow{\mathrm{d}} \mathcal{A}^1(M) \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \mathcal{A}^n(M)$$
is an elliptic complex.

**Definition 1.1.5.** The **Laplacian operator** of the complex (1.1) is defined by
$$\Delta_i = L_i^* \circ L_i + L_{i-1} \circ L_{i-1}^*,$$
which is a differential operator from $E_i$ to $E_i$.

**Lemma 1.1.** Let $U, V, W$ be finite-dimensional vector spaces. If we have a diagram of finite-dimensional vector spaces

$$\begin{tikzcd} U \arrow[r, "A"] \arrow[d, dash] & V \arrow[r, "B"] \arrow[d, dash] & W \arrow[d, dash] \\ U & V \arrow[l, "A^*"] & W, \arrow[l, "B^*"] \end{tikzcd}$$

which is exact at $V$. Then
(1) $V = \operatorname{im} A \oplus \operatorname{im} B^*$.
(2) $AA^*$ is injective on $\operatorname{im} A$ and is zero on $\operatorname{im} B^*$.
(3) $BB^*$ is injective on $\operatorname{im} B^*$ and is zero on $\operatorname{im} A$.
(4) $AA^* + BB^* \colon V \to V$ is an isomorphism.

**Corollary 1.1.1.** If the complex (1.1) is elliptic, then $\Delta_i$ is elliptic for all $i$.

**Theorem 1.1.2.** Let $(E, L)$ be an elliptic complex equipped with an inner product, where $E = \bigoplus_{i=0}^N E_i$ and $L = \bigoplus_{i=0}^N L_i$.
(1) The following decomposition is orthogonal
$$\Gamma(X, E) = \mathcal{H}(E) \oplus LL^*G(\Gamma(X, E)) \oplus L^*LG(\Gamma(X, E)).$$
(2) The following commutation relations are valid
$$\begin{aligned} HG = GH = H\Delta = \Delta H = 0 \\ LH = HL = L^*H = HL^* = 0 \\ L\Delta = \Delta L, L^*G = GL^* \\ LG = GL, L^*G = GL^*. \end{aligned}$$
(3) The dimension of $\mathcal{H}(E)$ is finite, and there is a canonical isomorphism
$$\mathcal{H}(E_i) \cong H^i(E)$$
for each $i$.

*Proof.* See Theorem 5.2 in [Wel80]. □

1.2. **Heat equation.** Let $(M, g)$ be a compact Riemannian manifold and $\omega(t) \colon \mathbb{R}_{\geq 0} \to \mathcal{A}^n(M)$. The heat equation is given by
$$\begin{cases} (\frac{\partial}{\partial t} + \Delta)(\omega(t)) = 0, \\ \omega(0) = \omega_0. \end{cases} \tag{1.2}$$
Now let's explain why the heat equation can be used to prove Hodge theorem. The idea is to use heat equation to flow what we have to something we desired, and this philosophy is frequently used in solving other problems, such as Ricci flow, Kähler-Einstein flow and so on.

Suppose $\omega(t)$ is a solution defined on $\mathbb{R}$ for heat equation (1.2). Roughly speaking we desire
$$\lim_{t\to\infty} \frac{\partial}{\partial t}\omega(t) = 0,$$
and thus $\omega_\infty := \lim_{t\to\infty} \omega(t)$ is expected to be a harmonic form. On the other hand, we desire the flow doesn't change the cohomology class, that is,

$[\omega_0] = [\omega(t)]$. If so, for $\alpha \in H^n_{dR}(M)$, we pick an arbitary representative $\omega_0$ and consider the heat equation with initial value $\omega_0$. If there exists a unique solution $\omega(t)$ defined on $\mathbb{R}$, then $\omega_\infty$ is the unique harmonic representative of $\alpha$. This proves the Hodge theorem.

Before we come into deep theories about partial differential equations, let's consider a baby example.

**Example 1.2.1.** Let $M = S^1$ equipped with the metric induced from $\mathbb{R}^2$. Then the heat equation is

$$(\partial_t - \partial_\theta^2) f(t,\theta) = 0,$$

where $\theta$ is the coordinate on $S^1$. Let $f_0(\theta)$ be the initial value with Fourier expansion

$$f_0(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{\sqrt{-1}n\theta},$$

and

$$f(t,\theta) = \sum_{n=\infty} a_n(t) e^{\sqrt{-1}n\theta}.$$

Then the heat equation is given by

$$(\partial_t - (\partial_\theta)^2)(f(t,\theta)) = \sum_{n=\infty}^{\infty} (a_n'(t) + a_n(t)n^2) e^{\sqrt{-1}n\theta}.$$

This shows

$$a_n(t) = a_n e^{-n^2 t}.$$

Thus the solution is given by

$$\lim_{t\to\infty} f(t,\theta) = a_0.$$

1.2.1. *Existence and uniqueness of the heat equation.*

**Theorem 1.2.1.** For any $\omega_0 \in \mathcal{A}^n(M)$, there exists a unique smooth map

$$\omega(t)\colon \mathbb{R}_{\geq 0} \to \mathcal{A}^n(M)$$

such that

$$\begin{cases} (\partial_t + \Delta_g)(\omega(t)) = 0 \\ \omega(0) = 0 \end{cases}$$

**Definition 1.2.1.** For $t \in [0,\infty)$, let $H_t\colon \mathcal{A}^n(M) \to \mathcal{A}^n(M)$ be the **heat-equation-solver operator**, which takes a form $\omega_0$ to its solution under the heat flow (1.2) at time $t$.

**Lemma 1.2.1.**

(1) $H_0 = \mathrm{id}$.
(2) $H_{t+s} = H_s \circ H_t = H_t \circ H_s$.
(3) The operator $H_t$ commutes with both $\Delta$ and d.

**Proposition 1.2.1.** Let $W_0(\mathcal{A}^n(M))$ be the $L^2$-complete of $\mathcal{A}^n(M)$. Then the operator $H_t$ extends to compact self-adjoint operator

$$H_t\colon W_0(\mathcal{A}^n(M)) \to W_0(\mathcal{A}^n(M)).$$

*Proof.* □

**Theorem 1.2.2** (spectral decomposition)**.** Let $T\colon H \to H$ be a compact self-adjoint operator on countable dimensional Hilbert space. Then there is an orthogonal eigenvalue decomposition

$$H = \bigoplus_{k=0}^{\infty} \langle v_n \rangle$$

such that

(1) each eigenspace $\langle v_n \rangle$ is finite dimensional.
(2) if $Tv_k = \gamma_n v_k$, then $\lim_{k\to\infty} \gamma_k = 0$.

*Proof.* □

**Proposition 1.2.2.** There is an orthogonal decomposition $W_0(\mathcal{A}^n(M)) = \bigoplus_{k=0}^{\infty} \langle \omega_k \rangle$ such that

$$\Delta(\omega_k) = \lambda_k \omega_k$$
$$H_t(\omega_k) = e^{-\lambda_k t} \omega_k$$

with $\lambda_k \geq 0$ with finite multiplicity.

*Proof.* By spectral decomposition theorem (Theorem 1.2.2), for each $H_t$ with $t > 0$, there is an orthogonal decomposition

$$W_0(\mathcal{A}^n(M)) = \bigoplus_{k=0}^{\infty} \langle \omega_n(t) \rangle$$

with $H_t(\omega_k(t)) = \gamma_k(t)\omega_k(t)$. On the other hand, since $H_t \circ H_s = H_s \circ H_t$ for all $t, s$, one has $\{H_t\}_{t>0}$ are simultaneously diagonalizable. As a result, $\{\omega_k(t)\}_{t>0}$ is independent on $t$.

In the following let's prove the eigenvalue of $H_t$ are strictly positive for $t > 0$.

(1) Note that
$$\gamma_k(t) = \langle H_t(\omega_k), \omega_k \rangle_{L^2} = \langle H_{\frac{t}{2}}(\omega_k), H_{\frac{t}{2}}(\omega_k) \rangle_{L^2} = \|H_{\frac{t}{2}}(\omega_k)\|_{L^2}^2 \geq 0$$
This shows $\gamma_k(t) \geq 0$.
(2) Now let's prove $\gamma_n(t) \neq 0$. Suppose $H_t(\omega) = 0$ for some $t$ and $\omega$. Then
$$0 = \langle H_t(\omega), \omega \rangle_{L^2} = \langle H_{\frac{t}{2}}(\omega), H_{\frac{t}{2}}(\omega) \rangle_{L^2} = \|H_{\frac{t}{2}}(\omega)\|_{L^2}^2$$
implies that $H_{\frac{t}{2}}(\omega) = 0$. Repeating above process one has $H_{\frac{t}{2^m}}(\omega) = 0$ for all $m \in \mathbb{N}$. Then
$$0 = \lim_{m\to\infty} H_{\frac{t}{2^m}}(\omega) = \lim_{t\to 0} H_t(\omega) = \omega.$$
Thus $H_t$ is injective for each $t > 0$.

□

Now let's use heat flow to give a proof of Hodge theorem.

*Proof of Theorem 1.1.* □

## 2. Kähler identities and Hodge package

Let $X$ be a compact complex $n$-manifold. Before Hodge, we only know a little about $\bigoplus_{k=0}^{2n} H^n(X, \mathbb{Q})$. One thing is Poincaré duality, that is,

$$H^k(X, \mathbb{Q}) \otimes H^{2n-k}(X, \mathbb{Q}) \xrightarrow{\cup} H^{2n}(X, \mathbb{Q}) \cong \mathbb{Q}$$

is a perfect pairing. But after Hodge, $\bigoplus_{k=0}^{2n} H^k(X, \mathbb{Q})$ turns out to be a rich and rigid object.

**Theorem 2.1** (Kähler identities)**.** Let $(X, \omega)$ be a compact Kähler $n$-manifold. The Kähler identities are given by

$$\begin{aligned} [\overline{\partial}^*, L] &= \sqrt{-1}\partial \\ [\partial^*, L] &= -\sqrt{-1} \cdot \overline{\partial} \\ [\Lambda, \overline{\partial}] &= -\sqrt{-1}\partial^* \\ [\Lambda, \partial] &= \sqrt{-1} \cdot \overline{\partial}^*, \end{aligned}$$

where $L$ is the Lefschetz operator and $\Lambda$ is its dual operator.

## 3. Kodaira vanishing

**Theorem 3.1** (Kodaira-Akizuki-Nakano vanishing)**.** Let $X$ be a non-singular projective complex variety with dimension $d$ and $L$ be an ample line bundle on $X$. Then

$$H^q(X, \Omega_X^p \otimes L) = 0$$

for all $p + q > d$.

**Theorem 3.2** (Kodaira vanishing)**.** Let $X$ be a non-singular projective complex variety and $L$ be an ample line bundle on $X$. Then

$$H^q(X, K_X \otimes L) = 0$$

for all $q > 0$.

In birational geometry, the following vanishing theorem is also extremely useful.

**Theorem 3.3** (Kawamata-Viehweg vanishing)**.** Let $X$ be a non-singular projective complex variety and $D = \sum_i a_i D_i$ be an effective $\mathbb{Q}$-divisor, where $a_i \in [0,1) \cap \frac{1}{N}\mathbb{Z}$ for some integer $N \geq 1$. For the line bundle $L$ such that $L^{\otimes N} \otimes \mathcal{O}_X(-ND)$ is ample[4], it holds that

$$H^q(X, K_X \otimes L) = 0$$

for all $q > 0$.

3.1. **Differential geometry method.** Let $(X, \omega)$ be a compact Kähler manifold and $(E, h)$ be a Hermitian holomorphic vector bundle over $X$ equipped with Chern connection $\nabla_h$. For the following operators

$$\begin{aligned}
\partial_E &\colon \mathcal{A}^{p,q}(X, E) \to \mathcal{A}^{p+1,q}(X, E) \\
\overline{\partial}_E &\colon \mathcal{A}^{p,q}(X, E) \to \mathcal{A}^{p,q+1}(X, E) \\
L &\colon \mathcal{A}^{p,q}(X, E) \to \mathcal{A}^{p+1,q+1}(X, E) \\
\Lambda &\colon \mathcal{A}^{p,q}(X, E) \to \mathcal{A}^{p-1,q-1}(X, E),
\end{aligned}$$

there are also Kähler identities

$$\begin{aligned}
[\overline{\partial}_E^*, L] &= \sqrt{-1}\partial_E \\
[\partial_E^*, L] &= -\sqrt{-1} \cdot \overline{\partial}_E \\
[\Lambda, \overline{\partial}_E] &= -\sqrt{-1}\partial_E^* \\
[\Lambda, \partial_E] &= \sqrt{-1} \cdot \overline{\partial}_E^*,
\end{aligned}$$

and

$$[L, \Lambda] = (p + q - n)\,\mathrm{id}$$

holds on $E$-valued $(p, q)$-forms.

[4] In fact, nef and big.

**Proposition 3.1.1** (Bochner-Kodaira-Nakano identity)**.** Let $(X, \omega)$ be a compact Kähler manifold and $(E, h)$ be a Hermitian holomorphic vector bundle. Then

$$\Delta_{\overline{\partial}_E} = [\sqrt{-1}\Theta_h, \Lambda] + \Delta_{\partial_E}.$$

*Proof.* Direct computation shows

$$\begin{aligned}\Delta_{\overline{\partial}_E} &= [\overline{\partial}_E, \overline{\partial}_E^*] \\ &= -\sqrt{-1}[\overline{\partial}_E, [\Lambda, \partial_E]] \\ &= -\sqrt{-1}[\Lambda, [\partial_E, \overline{\partial}_E]] - \sqrt{-1}[\partial_E, [\overline{\partial}_E, \Lambda]] \\ &= -\sqrt{-1}[\Lambda, \Theta_h] - \sqrt{-1}[\partial_E, \sqrt{-1}\partial_E^*] \\ &= [\sqrt{-1}\Theta_h, \Lambda] + \Delta_{\partial_E}.\end{aligned}$$

□

**Corollary 3.1.1** (Bochner-Kodaira-Nakano inequality)**.** Let $(X, \omega)$ be a compact Kähler manifold and $(E, h)$ a Hermitian holomorphic vector bundle. Then for $\alpha \in \mathcal{A}^{p,q}(X, E)$, one has

$$([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) \leq (\Delta_{\overline{\partial}_E}\alpha, \alpha)$$

In particular, if $\alpha$ is $\Delta_{\overline{\partial}_E}$-harmonic, then $([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) \leq 0$.

*Proof.* Direct computation shows

$$\begin{aligned}(\Delta_{\overline{\partial}_E}\alpha, \alpha) - ([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) &= (\Delta_{\partial_E}\alpha, \alpha) \\ &= \|\partial_E\alpha\|^2 + \|\partial_E^*\alpha\|^2 \geq 0.\end{aligned}$$

□

**Theorem 3.4** (Kodaira-Akizuki-Nakano vanishing)**.** Let $X$ be a compact $n$-manifold, $(L, h)$ a positive Hermitian holomorphic line bundle. Then

$$H^{p,q}(X, L) = 0$$

for $p + q > n$.

*Proof.* Let $X$ be endowed with the Kähler metric $\omega$ given by Chern curvature of $L$. Then there is an isomorphism $H^{p,q}(X, L) \cong \mathcal{H}^{p,q}(X, L)$. For $\alpha \in \mathcal{H}^{p,q}(X, L)$, by Corollary 3.1.1 one has

$$[\sqrt{-1}\Theta_h, \Lambda]\alpha \leq 0.$$

On the other hand,

$$([\sqrt{-1}\Theta_h, \Lambda]\alpha, \alpha) = 2\pi(p + q - n)\|\alpha\|^2 \geq 0.$$

Thus if $p + q > n$, one has $\alpha = 0$. This completes the proof. □

**Corollary 3.1.2** (Kodaira vanishing)**.** Let $X$ be a compact $n$-manifold and $(L, h)$ be a positive holomorphic line bundle over $X$. Then

$$H^q(X, K_X \otimes L) = 0$$

for $q > 0$.

3.2. **Algebraic geometry method.** An algebraic geometry proof of Kodaira's vanishing theorem uses the cyclic cover tricks and logarithmic differential forms. Here we give a brief introduction about these techniques, and a good reference is [EV92].

For convenience, $X$ always denotes a non-singular projective complex variety of dimension $d$ unless otherwise stated.

3.2.1. *Logarithmic differential forms.* Let $D$ be a reduced normal crossing divisor[5] in $X$. The sheaf of differential $k$-forms with logarithmic singularities along $D$, denoted by $\Omega^k_X(\log D)$, is the subsheaf of $\Omega^k_X(*D)$[6] defined by the following condition:

- If $\alpha$ is a meromorphic differential forms on $U$, holomorphic on $U\setminus D\cap U$, then $\alpha\in\Omega^k_X(\log D)|_U$ if $\alpha$ admits a pole of order at most 1 along $D$, and the same holds for $\mathrm{d}\alpha$.

**Lemma 3.1.** Let $\{z_1,\dots,z_n\}$ be a local coordinate on an open subset $U$ of $X$, in which $D\cap U$ is defined by the equation $z_1\dots z_r=0$. For convenience we denote

$$\delta_j=\begin{cases}\mathrm{d}z_j/z_j & j\le r\\ \mathrm{d}z_j & j>r,\end{cases}$$

and for $I=\{j_1,\dots,j_k\}\subseteq\{1,\dots,n\}$ with $j_1<\dots<j_s$, we denote

$$\delta_I=\delta_{j_1}\wedge\dots\wedge\delta_{j_k}.$$

Then $\Omega^k_X(\log D)|_U$ is a sheaf of free $\mathcal{O}_U$-modules with basis $\{\delta_I\}_{|I|=k}$.

*Proof.* See Proposition 2.2 in [EV92]. □

**Corollary 3.2.1.**

(1) $\Omega^k_X(\log D)=\bigwedge^k\Omega^1_X(\log D)$.
(2) The sheaves $\Omega^k_X(\log D)$ are sheaves of locally free $\mathcal{O}_X$-modules.

Notations for local frames of logarithmic differential forms as Lemma 3.1 will be used along the way. For example, one can defined the following several maps by using local frames.

(1) The first one is

$$\alpha\colon\Omega^1_X(\log D)\to\bigoplus_{j=1}^r\mathcal{O}_{D_j}$$

which is locally defined by $\sum_{j=1}^n a_j\delta_j\mapsto\bigoplus_{j=1}^r a_j|_{D_j}$.

[5] A divisor $D=\sum_{j=1}^r D_j$ is called a reduced normal crossing divisor, if locally there exists coordinate $\{z_1,\dots,z_n\}$ on $X$ such that $D$ is defined by the equation $z_1\dots z_r=0$ for an integer $r$ which naturally depends on the considered open set.

[6] $\Omega^k_X(*D)$ is the sheaf of meromorphic forms on $X$, holomorphic on $X\setminus D$.

(2) For $k \geq 1$, one has
$$\beta_1 \colon \Omega_X^k(\log D) \to \Omega_{D_1}^{k-1}(\log(D - D_1)|_{D_1})$$
which is given by: For local section
$$\varphi = \varphi_1 + \varphi_2 \wedge \frac{\mathrm{d}z_1}{z_1},$$
where $\varphi_1$ lies in the span of the $\delta_I$ with $1 \notin I$ and $\varphi_2 = \sum_{1\in I} a_I \delta_{I\setminus\{1\}}$, we define
$$\beta_1(\varphi) = \sum a_I \delta_{I\setminus\{1\}}|_{D_1}.$$

(3) Finally the natural restriction gives
$$\gamma_1 \colon \Omega_X^k(\log(D - D_1)) \to \Omega_{D_1}^k(\log(D - D_1)|_{D_1}).$$
Note that $\{z_1 \cdot \delta_I \mid 1 \in I\} \cup \{\delta_I \mid 1 \notin I\}$ gives a local frame of $\Omega_X^k(\log(D - D_1))$. Then $\gamma_1$ can be described as
$$\gamma_1(\sum_{1\in I} z_1 a_I \delta_I + \sum_{1\notin I} a_I \delta_I) = \sum_{1\notin I} a_I \delta_I|_{D_1}.$$

*Remark* 3.2.1. Similarly, $\beta_i$ and $\gamma_i$ are the corresponding map for the $i$-th component $D_i$.

**Proposition 3.2.1.** The following sequences of sheaves are exact.

(1)
$$0 \to \Omega_X^1 \to \Omega_X^1(\log D) \xrightarrow{\alpha} \bigoplus_{j=1}^{r} \mathcal{O}_{D_j} \to 0$$

(2)
$$0 \to \Omega_X^k(\log(D - D_1)) \to \Omega_X^k(\log D) \to \Omega_{D_1}^{k-1}(\log(D - D_1)|_{D_1}) \xrightarrow{\beta_1} 0$$

(3)
$$0 \to \Omega_X^k(\log D)(-D_1) \to \Omega_X^k(\log(D - D_1)) \to \Omega_{D_1}^k(\log(D - D_1)|_{D_1}) \xrightarrow{\gamma_1} 0$$

*Proof.* It follows from the definition of $\alpha, \beta_1$ and $\gamma_1$. □

**Definition 3.2.1** (logarithmic connection)**.** Let $\mathcal{E}$ be a locally free coherent sheaf on $X$. A logarithmic connection is a $\mathbb{C}$-linear map $\nabla \colon \mathcal{E} \to \Omega_X^1(\log D) \otimes \mathcal{E}$ satisfying the Lebniz rule, that is
$$\nabla(f \cdot e) = f \cdot \nabla e + \mathrm{d}f \otimes e.$$
One defines
$$\nabla \colon \Omega_X^k(\log D) \otimes \mathcal{E} \to \Omega_X^{k+1}(\log D) \otimes \mathcal{E}$$
by the rule
$$\nabla(\omega \otimes e) = \mathrm{d}\omega \otimes e + (-1)^k \omega \wedge \nabla e.$$

**Definition 3.2.2** (flat logarithmic connection)**.** A logarithmic connection $\nabla$ is called flat[7] if it's curvature is zero, that is, $\nabla^2 = 0$.

[7]Sometimes is also called integrable.

**Definition 3.2.3** (residue)**.** For a flat logarithmic connection
$$\nabla\colon \mathcal{E} \to \Omega^1_X(\log D) \otimes \mathcal{E},$$
the residue map along $D_1$ is defined to be the composed map
$$\mathrm{Res}_{D_1}(\nabla)\colon \mathcal{E} \xrightarrow{\nabla} \Omega^1_X(\log D) \otimes \mathcal{E} \xrightarrow{\beta_1 \otimes \mathrm{id}} \mathcal{O}_{D_1} \otimes \mathcal{E}.$$

*Remark* 3.2.2.

3.2.2. $\mathbb{Q}$-*divisors.*

**Definition 3.2.4** ($\mathbb{Q}$-divisor)**.** Let $\mathrm{Div}(X)$ be the group of divisors on $X$ and $\mathrm{Div}_{\mathbb{Q}}(X) = \mathrm{Div}(X) \otimes_Z \mathbb{Q}$. A $\mathbb{Q}$-divisor $\Delta \in \mathrm{Div}_{\mathbb{Q}}(X)$ is a sum
$$\Delta = \sum_{j=1}^{r} \alpha_j D_j$$
where $D_j$ are irreducible divisors and $\alpha_j \in \mathbb{Q}$.

**Definition 3.2.5** (integeral part)**.** For a $\mathbb{Q}$-divisor $\Delta = \sum_{j=1}^{r} \alpha_j D_j$, its integeral part
$$[\Delta] := \sum_{j=1}^{r} [a_j] D_j$$
where $[\alpha]$ denotes the integeral part of $\alpha$.

**Definition 3.2.6.** Let $\mathcal{L}$ be an invertible sheaf, $D = \sum_{j=1}^{r} \alpha_j D_j$ be an effective divisor and $N$ be a positive natural number such that $\mathcal{L}^N = \mathcal{O}_X(D)$. Then we will write
$$\mathcal{L}^{(i,D)} = \mathcal{L}^i \otimes \mathcal{O}_X(-[\frac{i}{N} D])$$
and
$$\mathcal{A} = \bigoplus_{i=0}^{N-1} \mathcal{L}^{(i,D)^{-1}}$$

*Remark* 3.2.3. If there is no ambiguous, then we write $\mathcal{L}^{(i)}$ for convenience.

**Theorem 3.5.** Let $\mathcal{L}$ be an invertible sheaf, $D = \sum_{j=1}^{r} \alpha_j D_j$ be an effective divisor and $N$ be a positive natural number such that $\mathcal{L}^N = \mathcal{O}_X(D)$. Then for $i = 0, 1, \ldots, N-1$, the sheaf $\mathcal{L}^{(i)^{-1}}$ has a flat logarithmic connection
$$\nabla^{(i)}\colon \mathcal{L}^{(i)^{-1}} \to \Omega^1_X(\log D^{(i)}) \otimes \mathcal{L}^{(i)^{-1}}$$
where $D^{(i)} = \sum_{j=1, \frac{i\alpha_j}{N} \notin \mathbb{Z}}^{r} D_j$, and $\nabla^{(i)}$ satisfies

(1) The residue of $\nabla^{(i)}$ along $D_j$ is given by multiplication with
$$(i \cdot \alpha_j - N \cdot [\frac{i \cdot \alpha_j}{N}]) \cdot N^{-1}$$

(2)
(3)

3.2.3. *Cyclic coverings.* Let $\mathcal{L}$ be an invertible sheaf, $D = \sum_{j=1}^{r} \alpha_j D_j$ be an effective divisor and $N$ be a positive natural number such that $\mathcal{L}^N = \mathcal{O}_X(D)$. Let $s \in H^0(X, \mathcal{L}^N)$ be a section whose zero divisor is $D$. Then the dual of $s\colon \mathcal{O}_X \to \mathcal{L}^N$ gives a $\mathcal{O}_X$-algebra structure on

$$\mathcal{A}' = \bigoplus_{i=0}^{N-1} \mathcal{L}^{-i}$$

Let $Y' = \operatorname{Spec} \mathcal{A}' \to X$ be the spectrum of the $\mathcal{O}_X$-algebra $\mathcal{A}'$ and $\pi\colon Y \to X$ be the finite morphism obtained by normalizing $Y' \to X$. Then $Y$ is called cyclic covering obtained by taking $n$-th root out of $D$.

The inclusion

$$\mathcal{L}^{-i} \to \mathcal{L}^{(i)^{-1}} = \mathcal{L}^{-i} \otimes \mathcal{O}_X([\frac{i}{N} D])$$

gives a morphism of $\mathcal{O}_X$-module $\phi\colon \mathcal{A}' \to \mathcal{A}$.

**Lemma 3.2.** $\mathcal{A}$ has a structure of $\mathcal{O}_X$-algebra such that $\phi$ is a homomorphism of algebras.

Now let $\xi_N$ be a fixed primitive $N$-th root of unit and $G = \langle \sigma \rangle$ be the cyclic group of order $N$. Then $G$ acts on $\mathcal{A}$ by $\mathcal{O}_X$-algebra homomorphisms by $\sigma(l) = \xi_N^i \cdot l$, where $l$ is a local section of $\mathcal{L}^{(i)^{-1}}$, and it's clear the invariants under this $G$-action are $\mathcal{A}^G = \mathcal{O}_X$.

**Proposition 3.2.2.** The cyclic group $G$ acts on $Y$ and on $\pi_* \mathcal{O}_Y$. One has $Y/G = X$ and the decomposition

$$\pi_* \mathcal{O}_Y = \bigoplus_{i=0}^{N-1} \mathcal{L}^{(i)^{-1}}$$

is the decomposition in eigenspaces.

**Proposition 3.2.3.** $\operatorname{Spec} \mathcal{A} \to X$ is finite and $\operatorname{Spec} \mathcal{A}$ is normal.

3.2.4. *Cyclic cover.* Now suppose $X$ is a non-singular projective complex variety with dimension $d$ and $L$ is an ample line bundle. Take $N \gg 1$ large sufficiently such that there exists $s \in \Gamma(X, L^{\otimes N})$, and the divisor $D = \operatorname{div}(s)$ is non-singular. Then it gives a $N$-cyclic cover $\pi\colon Y \to X$.

**Proposition 3.2.4.** Notations as above. Then
(1) $\pi^* \Omega_X^n(\log D) = \Omega_Y^n(\log \pi^* D)$.
(2) For each $1 \leq i \leq N-1$, $\mathrm{d}\colon \mathcal{O}_Y \to \Omega_Y$ induces a flat log connection

$$\nabla^i\colon L^{-i} \to L^{-i} \otimes \Omega_X^1(\log D)$$

such that

$$\bigoplus_{i=1}^{N-1} \nabla^i = \pi_* \mathrm{d}\colon \pi_* \mathcal{O}_Y \to \pi_*(\Omega_Y^1(\log \pi^* D))$$

is an eigen-decomposition with respect to the action of cyclic group $\mathbb{Z}/N\mathbb{Z}$.

(3) $\pi_*\Omega_Y^n = \Omega_X^n \oplus \bigoplus_{i=1}^{N-1} \Omega_X^n(\log D) \otimes L^{-i}$.

**Proposition 3.2.5.** The Hodge to de Rham spectral sequence associated to the log de Rham complex for $1 \leq i \leq N-1$

$$L^{-i} \xrightarrow{\nabla^i} L^{-i} \otimes \Omega_X^1(\log D) \xrightarrow{\nabla^i} L^{-i} \otimes \Omega_X^2(\log D) \to \dots$$

degenerates at $E_1$-page.

3.2.5. *An Algebraic proof of Kodaira's vanishing theorem.* In this section we present an algebraic proof of Kodaira's vanishing theorem by Serre vanishing, $E_1$-degeneration and techniques introduced above.

*Proof.* Consider the de Rham complex

$$\mathcal{O}_Y \xrightarrow{\mathrm{d}} \Omega_Y^1 \xrightarrow{\mathrm{d}} \Omega_Y^2 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega_Y^d.$$

The $E_1$-degeneration implies

$$E_1^{p,q} = H^q(Y, \Omega_Y^p) \xrightarrow{\mathrm{d}=0} H^q(Y, \Omega_Y^{p+1}) = E_1^{p+1,q}.$$

Then

$$\pi_*\mathcal{O}_Y \xrightarrow{\pi_*\mathrm{d}} \pi_*\Omega_Y^1 \xrightarrow{\pi_*\mathrm{d}} \pi_*\Omega_Y^2 \xrightarrow{\pi_*\mathrm{d}} \cdots \xrightarrow{\pi_*\mathrm{d}} \pi_*\Omega_Y^d$$

also has $E_1$-degeneration since $\pi$ is a finite morphism.

By Proposition 3.2.5 one has

$$E_1^{0,n} = H^n(X, L^{-1}) \xrightarrow{\nabla_1} H^n(X, L^{-1} \otimes \Omega_X(\log D)) = E_1^{1,n}$$

is a zero map. Hence

$$\begin{tikzcd}
H^n(X, L^{-1}) \arrow[rr, "0"] \arrow[dr, "\nabla_1"'] & & H^n(D, L^{-1}|_D) \\
& H^n(X, L^{-1} \otimes \Omega_X(\log D)) \arrow[ur, "\mathrm{Res}"'] &
\end{tikzcd}$$

On the other hand,

$$L^{-1} \xrightarrow{\nabla_1} L^{-1} \otimes \Omega_X(\log D) \xrightarrow{\mathrm{Res}} L^{-1}|_D$$

is just the restriction map up to the factor $1/N$. Thus the long exact sequence of

$$0 \to L^{-1} \otimes \mathcal{O}_X(-D) \to L^{-1} \to L^{-1}|_D \to 0$$

reads

$$\cdots \to H^n(X, L^{-1} \otimes \mathcal{O}_X(-D)) \to H^n(X, L^{-1}) \xrightarrow{0} H^n(D, L^{-1}|_D) \to \cdots.$$

Note that $L^{-1} \otimes \mathcal{O}_X(-D) = L^{-N-1}$ and by Serre vanishing theorem

$$H^n(X, L^{-N-1}) = 0$$

for $n < \dim X$ and $N \gg 1$. Therefore

$$H^n(X, L^{-1}) = 0$$

for $n < \dim X$ as desired. $\square$

## 4. Cartier descent theorem

In this section we assume $k$ is an algebraically closed field with positive characteristic $p$, $F_k\colon k \to k$ is the Frobenius map and $X$ is a non-singular variety over $k$. Let $X'$ be the base change of $X$ given by the Frobenius map, that is, there is the following commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[d] & X \arrow[d] \\ k \arrow[r, "F_k"] & k. \end{tikzcd}$$

*Remark* 4.1.

On the other hand, the Frobenius map $F_k\colon k \to k$ induces a homomorphism $F_X\colon X \to X$, called the absolute Frobenius map, which also satisfies the universal property of of fiber product. Then there exists a morphism $F_{X/k}\colon X \to X'$ such that the following diagram commutes, which is called relative Frobenius map.

$$\begin{tikzcd} X \arrow[dr, dotted] \arrow[drr, "F_X"] \arrow[ddr, "\alpha"'] & & \\ & X' \arrow[r, "\pi"'] \arrow[d, "\alpha'"] & X \arrow[d, "\alpha"] \\ & k \arrow[r, "F_k"] & k. \end{tikzcd}$$

*Remark* 4.2.

**Definition 4.1** ($k$-connection)**.** A $k$-connection on $X/k$ is a pair $(\mathcal{E}, \nabla)$, which consists of the following data:

(1) $\mathcal{E}$ is a (quasi)-coherent $\mathcal{O}_X$-module.
(2) $\nabla\colon \mathcal{E} \to \mathcal{E} \otimes \Omega_{X/k}$ is $k$-linear such that

$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s$$

*Remark* 4.3. Given a $k$-connection $(\mathcal{E}, \nabla)$ on $X/k$, it can be regarded as a "quasi-representation" as follows

$$\nabla\colon T_{X/k} \to \mathrm{End}_k(\mathcal{E}).$$

On $T_{X/k}$, there is a Lie bracket

$$[\text{-},\text{-}]\colon T_{X/k} \times T_{X/k} \to T_{X/k},$$

which is given by $[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$. The curvature of $k$-connection $(\mathcal{E}, \nabla)$ is given by

$$\begin{aligned} \Theta_\nabla\colon \bigwedge^2 T_{X/k} &\to \mathrm{End}_k(\mathcal{E}) \\ D_1 \wedge D_2 &\mapsto [D_1, D_2] - \nabla_{[D_1, D_2]}. \end{aligned}$$

This measures the failure of $\nabla$ to be a Lie algebra representation. Moreover, one can show that $\Theta_\nabla\colon \bigwedge^2 T_{X/k} \to \mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$.

In the case of positive characteristic, there is so called $p$-curvature which doesn't appear in the zero characteristic case, since the $p$-th power map $D \mapsto D^p := \underbrace{D \circ \cdots \circ D}_{p \text{ times}}$ gives a map between $T_{X/k} \to T_{X/k}$.

**Definition 4.2** ($p$-curvature)**.** The $p$-curvature of a $k$-connection $(\mathcal{E}, \nabla)$ over $X/k$ is defined by

$$\begin{aligned} \Psi_\nabla \colon T_{X/k} &\mapsto \operatorname{End}_k(\mathcal{E}) \\ D &\mapsto (\nabla_D)^p - \nabla_{D^p}. \end{aligned}$$

**Proposition 4.1.** For any $D \in T_{X/k}$, one has $\Psi_\nabla(D)$ is $\mathcal{O}_X$-linear. In other words,

$$\Psi_\nabla \colon T_{X/k} \to \operatorname{End}_{\mathcal{O}_X}(\mathcal{E}).$$

*Proof.* For any $f \in \mathcal{O}_X$ and $s \in \mathcal{E}$, one has

$$(\nabla_D)^p(fs) = \sum_{i=0}^{p} \binom{p}{i} D^i(f)(\nabla_D)^{p-i}(s) = D^p(f)s + f\nabla_D^p(s).$$

On the other hand, it's clear

$$\nabla_{D^p}(fs) = \nabla^p(f)s + f\nabla_{D^p}(s).$$

Thus it follows

$$\Psi_\nabla(D)(fs) = f((\nabla_D)^p - \nabla_{D^p})(s) = f\Psi_\nabla(D)(s).$$

$\square$

**Proposition 4.2.** Let $(\mathcal{E}, \nabla)$ be a $k$-connection over $X/k$. If the curvature $\Theta_\nabla$ vanishes, then

(1) $\Psi_\nabla \colon T_{X/k} \to \operatorname{End}_{\mathcal{O}_X}(\mathcal{E})$ is additive.
(2) $\Psi_\nabla$ is $F_X$-linear, that is,

$$\Psi_\nabla(fD) = f^p\Psi_\nabla(D).$$

(3) $\Psi_\nabla$ is integrable, that is, $\Psi_\nabla \wedge \Psi_\nabla = 0$.

It's a highly non-trivial fact, whose proof relies on the following algebra result.

**Lemma 4.1.** Let $R$ be an associated ring with positive characteristic $p$. For $a, b \in R$,

(1) $(a+b)^p = a^p + b^p + \sum_{i=1}^{p-1} s_i(a,b)$, where

$$(\operatorname{ad}(ta+b))^p(a) = \sum_{i=0}^{p-1} is_i(a,b)t^i.$$

(2) If $\{a^{(n)}\}_{n\geq 1}$ are mutually commutative, then

$$(ab)^p = a^pb^p + a(a^{p-1})^{(p-1)}b,$$

where

$$a^{(n)} := (\operatorname{ad} b)^n(a).$$

*Proof.* See [Kat70]. □

Now let's begin the proof of Proposition 4.2.

*Proof of Proposition 4.2.* For (1). For arbitary $D_1, D_2 \in T_{X/k}$, by using (1) of Lemma 4.1 one has

$$\begin{aligned} (\nabla_{D_1+D_2})^p &= (\nabla_{D_1} + \nabla_{D_2})^p \\ &= (\nabla_{D_1})^p + (\nabla_{D_2})^p + \sum_i s_i(D_1, D_2) \\ \nabla_{(D_1+D_2)^p} &= \nabla_{(\nabla_1^p + \nabla_2^p + \sum_i s_i(D_1,D_2))} \\ &= \nabla_{D_1^p} + \nabla_{D_2^p} + \sum_i \nabla_{s_i(D_1,D_2)}. \end{aligned}$$

Then

$$\Psi_\nabla(D_1 + D_2) = (\nabla_{D_1+D_2})^p - \nabla_{(D_1+D_2)^p} = \Psi_\nabla(D_1) + \Psi_\nabla(D_2).$$

For (2). For arbitary $f \in \mathcal{O}_X$ and $D \in T_{X/k}$, by using (2) of Lemma 4.1, one has

$$\begin{aligned} (fD)^p &= f^p D^p + f(\mathrm{ad}(D))^{p-1}(f^{p-1})D \\ &= f^p D^p + f(D^{p-1}(f^{p-1}))D, \end{aligned}$$

since $\mathrm{ad}(D)(f^{p-1}) = D \circ f^{p-1} - f^{p-1}D = D(f^{p-1})$. Thus

$$\nabla_{(fD)^p} = f^p \nabla_{D^p} + f(D^{p-1}(f^{p-1}))\nabla_D.$$

Applying (2) of Lemma 4.1 again, one has

$$\begin{aligned} (\nabla_{fD})^p = (f\nabla_D)^p &= f^p(\nabla_D)^p + f(\mathrm{ad}(\nabla_D))^{p-1}(f^{p-1})\nabla_D \\ &= f^p(\nabla_D)^p + f(D^{p-1}(f^{p-1}))\nabla_D. \end{aligned}$$

This completes the proof of (2).

For (3) □

*Remark* 4.4. In other words,

$$\Psi_D \colon \mathcal{E} \to \mathcal{E} \otimes F^*_{X/k}\Omega_X$$

is a $\mathcal{O}_X$-linear morphism.

Holding notations as above, now we can state the main theorem of this section, which is a very basic theorem in geometry over field $k$ with characteristic $p$.

**Theorem 4.1** (Cartier)**.** There is a natural equivalent of categories between category of (quasi)-coherent $\mathcal{O}_X$-module and the category of flat $k$-connections $(\mathcal{E}, \nabla)$ with vanishing $p$-curvatures. More explicitly, the correspondence is given by

(1) For (quasi)-coherent $\mathcal{O}_X$-module $\mathcal{E}$, the flat $k$-connection $(F^*\mathcal{E}, \nabla_{\mathrm{can}})$ is given by

$$\nabla_{\mathrm{can}}(e \otimes f) = \mathrm{d}f \otimes e.$$

(2) For flat $k$-conenction $(\mathcal{E}, \nabla)$ with vanishing $p$-curvature, the corresponding (quasi)-coherent $\mathcal{O}_X$-module is the $\mathcal{O}_X$-submodule $\mathcal{E}^{\nabla=0} \subseteq \mathcal{E}$.

## 5. De Rham decomposition theorem of Deligne-Illusie

5.1. **Introduction.** In this section, unless otherwise specified, $k$ always denotes an algebraically closed field with positive characteristic $p$. Let $X$ be a non-singular variety over $k$ and $F\colon X \to X'$ denote the relative Frobenius map. Then

$$F_*\Omega^\bullet_{X/k}\colon F_*\mathcal{O}_X \to F_*\Omega^1_{X/k} \to F_*\Omega^2_{X/k} \to \dots$$

is a finite complex of coherent $\mathcal{O}_{X'}$-module with $\mathcal{O}_{X'}$-linear differential.

**Theorem 5.1** (Deligne-Illusie)**.** Let $X/k$ be a non-singular variety such that $X/k$ is $W_2(k)$-liftable and $\dim_k X < p$. Then there is a quasi-isomorphism

$$(F_*\Omega^\bullet_{X/k}, F_*\mathrm{d}) \cong \bigoplus_{i=0}^{d} \Omega^i_{X/k}[-i].$$

*Remark* 5.1.

(1) The condition of $W_2(k)$-liftable cannot be removed, and the first counterexample is given by Michel Raynaud by showing Kodaira's vanishing theorem fails in positive characteristic in [Ray78].
(2) The statement still holds for $\dim_k X = p$, but in [Pet23] the author shows that it fails when $\dim_k X > p$ .

5.1.1. *Witt vectors of length two.* The Witt ring $W_2(k)$ can be interpreted as the set $k \times k$, where the multiplication and addition for $a = (a_0, a_1)$ and $b = (b_0, b_1)$ are defined by

$$ab = (a_0a_1, b_0a_1^p + b_1a_0^p),$$

and

$$a + b = (a_0 + b_0, a_1 + b_1 - \sum_{i=1}^{p-1} p^{-1}\binom{p}{t} a_0^i b_0^{p-i}).$$

*Remark* 5.2. In fact, the operations on $W_2(k)$ makes the ghost polynomial $\Phi(a_0, a_1) = a_0^p + pa_1$ a ring homomorphism.

**Example 5.1.** If $k = \mathbb{Z}/p\,\mathbb{Z}$, then $W_2(k) = \mathbb{Z}/p^2\,\mathbb{Z}$.

**Proposition 5.1.** $pW_2(k) = \{(0, a) \mid a \in k\}$ is a maximal ideal of $W_2(k)$, and the following sequence is exact

$$0 \to pW_2(k) \to W_2(k) \to k \to 0$$

**Proposition 5.2.** The ring homomorphism $F_{W_2(k)}\colon W_2(k) \to W_2(k)$ given by $(a_0, a_1) \mapsto (a_0^p, a_1^p)$ reduces to the Frobenius map $F_k$ on $k$ modulo $p$.

**Definition 5.1** ($W_2(k)$-liftable)**.** Let $X/k$ be a non-singular variety over $k$. If there exists a flat morphism $\widetilde{X} \to W_2(k)$ such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "\cong"] \arrow[dr] & \widetilde{X} \times_{\operatorname{Spec} W_2(k)} \operatorname{Spec} k \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] \\
& \operatorname{Spec} k \arrow[r] & \operatorname{Spec} W_2(k),
\end{tikzcd}$$

then $X/k$ is $W_2(k)$-liftable.

*Remark* 5.3. Not every non-singular variety $X/k$ is $W_2(k)$-liftable, and there is an obstruction in $\operatorname{ob}(\alpha) \in H^2(X, T_{X/k})$, such that $\operatorname{ob}(\alpha) = 0$ if and only if $X/k$ is $W_2(k)$-liftable.

5.1.2. *Cartier isomorphism.* Before the proof of Deligne-Illusie, there are many evidences, such as Cartier descent theorem.

**Theorem 5.2.** Let $X/k$ be a non-singular variety and $F\colon X \to X'$ be the relative Frobenius map. Then there is a unique isomorphism of graded $\mathcal{O}_{X'}$-algebra

$$C^{-1}\colon \bigoplus_{i=0}^{d} \Omega^i_{X'/k} \to \bigoplus_{i=0}^{d} \mathscr{H}^i(F_*\Omega^\bullet_{X/k}),$$

which is determined by

(1) On degree zero, $C^{-1}\colon \mathcal{O}_{X'} \to \mathscr{H}^0(F_*\Omega^\bullet_{X/k}) = \ker\{F_*\mathrm{d}\colon F_*\mathcal{O}_X \to F_*\Omega_{X/k}\}$ is the morphism $\mathcal{O}_{X'} \xrightarrow{F_*} F_*\mathcal{O}_X$.

(2) On degree one, there is the following commutative diagram

$$\begin{tikzcd}
C^{-1}\colon \Omega^1_{X'/k} \arrow[r] \arrow[dr] & \mathscr{H}^1 = Z^1/B^1 \\
& Z^1 \arrow[u]
\end{tikzcd}$$

such that $C^{-1}(\mathrm{d}x') = x^{p-1}\mathrm{d}x \pmod{B^1}$, where $x \in \mathcal{O}_X$ and $x' =$

*Proof.* Construct a derivation $\mathcal{O}_{X'} \to H^1(F_*\Omega^\bullet_{X/k})$ by sending $x$ to $x^{p-1}\mathrm{d}x$. Then it factor throught $\Omega^1_{X'}$. To show

$$(x+y)^{p-1}\mathrm{d}(x+y) = x^{p-1}\mathrm{d}x + y^{p-1}\mathrm{d}y \pmod{B^1}$$
$$(xy)^{p-1}\mathrm{d}(xy) = (x^{p-1}\mathrm{d}x)y + (y^{p-1}\mathrm{d}y)x$$

Suppose $\dim X = 2$.

$\mathcal{O}_{X'}\{t_1^{i_1}t_2^{i_2} \mid 0 \leq i_1 \leq p-1, 0 \leq i_2 \leq p-1\} \to \mathcal{O}_{X'}\{t_1^{i_1}t_2^{i_2} \mid \dots\} \otimes \mathrm{d}t_1 \oplus \mathcal{O}_{X'}\{t_1^{i_1}t_2^{i_2} \mid \dots\} \otimes \mathrm{d}t_2 \to \mathcal{O}_{X'}\{t_1^{i_1}t_2^{i_2}\}$

In general it's difficult to compute, but here is a trick by using Kunneth formula. Firstly we may write $\mathcal{O}_{X'} \otimes_k k$, secondly let's consider

$$k[t_1^i \mid 0 \leq i \leq p-1] \xrightarrow{\mathrm{d}_1} k[t_1^i \mid 0 \leq i \leq p-1] \otimes \mathrm{d}t_1$$
$$k[t_2^i \mid 0 \leq i \leq p-1] \xrightarrow{\mathrm{d}_2} k[t_2^i \mid 0 \leq i \leq p-1] \otimes \mathrm{d}t_2$$

Then consider the tensor product of above two complexes $\square$

5.1.3. *Deformation theory.* $f\colon X \to Y$ is locally finite presented if

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \\
\operatorname{Spec} A \arrow[u, hook] \arrow[rr] \arrow[d, hook] & & \operatorname{Spec} B \arrow[u, hook] \\
\operatorname{Spec} B[t_1, \dots, t_n] \arrow[urr] & &
\end{tikzcd}$$

$I = \ker(B[t_1, \dots, t_n] \to A)$ is finitely generated.

**Definition 5.2.** A morphism $f\colon X \to Y$ is smooth/étale/unramified if for every first order thickening

$$\begin{tikzcd}
T_0 \arrow[r, "g_0"] \arrow[d, hook] & X \arrow[d, "f"] \\
T \arrow[r] \arrow[ur, dashed, "g"] & Y
\end{tikzcd}$$

there exists local lifting/ there exists a unique local lifting(global lifting)/at most one local lifting.

**Theorem 5.3.** Let $f\colon X \to Y$ is smooth. Then

(1) $\Omega_{X/Y}$ is locally free $\mathcal{O}_X$-module of rank $r$.

(2) Suppose $X \xrightarrow{f} Y \xrightarrow{g} S$. Then
$$0 \to g^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$
is exact, locally split

(3)

**Corollary 5.1.** $f\colon X \to Y$ is smooth. Then $f$ is locally of the following type

$$\begin{tikzcd}
X \arrow[r, "etale"] \arrow[d, "f"'] & \mathbb{A}^n_Y = A^n \times_{\mathbb{Z}} Y \arrow[dl, "pr"] \\
Y &
\end{tikzcd}$$

**Corollary 5.2.** If $f$ is smooth and $g$ is also smooth, then for $x \in Z$, there exists $x \in U$, $\{s_1, \dots, s_r\} \subseteq I(U)$ such that $\{(s_1)_x, \dots, (s_r)_x\}$ generates $I_x$ and $\{(\mathrm{d}s_1)(x), \dots, \mathrm{d}s_r)(x)\}$ linearly independent in $\Omega_{X/Y}(x)$.

$$\begin{tikzcd}
Z \arrow[r, hook, "i"] \arrow[d, "g"'] & X \arrow[dl, "f"] \\
Y &
\end{tikzcd}$$

**Theorem 5.4.** Let $f\colon X \to Y$ is smooth. Then

(1) there exists $\mathrm{ob}(g_0) \in \mathrm{Ext}^1(g_0^*\Omega_{X/Y}, I)$, such that $\mathrm{ob}(g_0) = 0$ if and only if there exists a global lifting $g\colon T \to X$.

(2) Assume $\mathrm{ob}(g_0) = 0$. Then the set of all liftings $g$ is an affine space under $\mathrm{Hom}(g_0^*\Omega_{X/Y}, I)$, called torsor.

$$\begin{tikzcd}
T_0 \arrow[r, "g_0"] \arrow[d] & X \arrow[d, "f"] \\
T \arrow[r, "g_2"'] \arrow[d] & Y \\
x \in U_x \arrow[uur, bend left, "g_1"] \arrow[uur, bend right] &
\end{tikzcd}$$

Then we claim

$$(g_1^* - g_2^*)(ab) = (g_1^* - g_2^*)(a)b + a(g_1^* - g_2^*)(b)$$

Then

$$(g_1^* - g_2^*)(ab) = g_1^*(ab) - g_2^*(ab)$$

5.2. **Explicit quasi-isomorphism.** In this section we give the proof of Deligne-Illusie's decomposition theorem.

*Proof of Theorem 5.1.* Firstly let's consider the case that the relative Frobenius map $F\colon X \to X'$ lifts over $W_2(k)$. In other words, there exists a morphism $\widetilde{F}\colon \widetilde{X} \to \widetilde{X}'$ such that the following diagram commutes

$$\begin{tikzcd}
 & X' \arrow[rr] \arrow[dd] & & \widetilde{X}' \arrow[dd] \\
X \arrow[ur, "F"] \arrow[rr] \arrow[dr] & & \widetilde{X} \arrow[ur, dashed, "\widetilde{F}"] \arrow[dr] & \\
 & \operatorname{Spec} k \arrow[rr] & & \operatorname{Spec} W_2(k),
\end{tikzcd}$$

where $\widetilde{X}'$ is the base change of $\widetilde{X}$ given by the Frobenius map of $W_2(k)$. For any $x \in \mathcal{O}_{\widetilde{X}'}$, one has

$$\widetilde{F}^*(x) = x^p + pa$$

since $\widetilde{F}$ is a lifting of $F$ and $F(x) = x^p$.

Then

$$\begin{aligned}
\mathrm{d}\widetilde{F}(\mathrm{d}x) &= \mathrm{d}(\widetilde{F}^*x) \\
&= \mathrm{d}(x^p + pa) \\
&= p(x^{p-1}\mathrm{d}x + \mathrm{d}a).
\end{aligned}$$

Therefore $\mathrm{d}\widetilde{F}(\widetilde{F}^*\Omega_{\widetilde{X}'/W_2(k)}) \subseteq p\Omega_{\widetilde{X}/W_2(k)}$, and thus one has

$$0 \to pW_2 \to W_2 \to k \to 0$$

$$0 \to p\mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_X \to 0$$

$$\begin{tikzcd}
\Omega_{X'/k} \arrow[r] \arrow[dr] & F_*\Omega_{X/k} \arrow[d] \\
& Z^1F_*\Omega_{X/k}/B^1F_*\Omega_{X/k}
\end{tikzcd}$$

Take any open affine covering $\mathcal{U} = \{U_i\}$ of $X$ such that there exists

$$\widetilde{F}_{U_i}\colon \widetilde{U}_i \to \widetilde{U}_i \hookrightarrow \widetilde{X}'$$

$$\begin{tikzcd}
\Omega_{X'} \arrow[r] \arrow[dr] & F_*\Omega_{U_{ij}/k} \arrow[d, "F_*d"] \\
\mathcal{O}_{X'} \arrow[u, "d"] \arrow[r, "\frac{1}{[p]}(\widetilde{F}^*_{U_j} - \widetilde{F}^*_{U_i})"'] & F_*\mathcal{O}_{U_{ij}}
\end{tikzcd}$$

□

### 5.3. **Applications of de Rham decomposition.**

5.3.1. *$E_1$-degeneration.*

**Theorem 5.5.** Let $X/k$ be a non-singular proper variety such that $X/k$ is $W_2(k)$-liftable and $\dim X < p$. Then the Hodge to de Rham spectral sequence degenerates at $E_1$.

*Proof.* Note that one has

$$\dim_k H^n_{dR}(X/k) \leq \sum_{i+j=n} H^j(X, \Omega^i_{X/k}) < \infty.$$

Since the absolute Frobenius $F_X\colon X \to X$ is an identity topologically, one has

$$\mathbb{H}^n(X, \Omega^\bullet_{X/k}) = \mathbb{H}^n(X, (F_X)_*\Omega^\bullet_{X/k}) = \mathbb{H}^n(X, \bigoplus_{i=0}^{d} \Omega^i_{X/k}[-i]) = \bigoplus_{i=0}^{d} H^{n-i}(X, \Omega^i_{X/k})$$

□

5.3.2. *Kodaira-Akizuki-Nakano theorem.*

**Theorem 5.6.** Let $X/k$ be a non-singular projective variety such that $X/k$ is $W_2(k)$-liftable and $\dim_k X < p$. Then for any ample line bundle $L$ on $X$, one has

$$H^j(X, \Omega^i_{X/k} \otimes L) = 0$$

for all $i + j > d$.

*Proof given by M.Raynaud.* Note that $F^*L^{-1} = (L^{-1})^p$, then

$$L^{-p} \xrightarrow{\nabla_{\text{can}}} L^{-p} \otimes \Omega_{X/k} \xrightarrow{\nabla_{\text{can}}} \dots.$$

Then we project it

$$F_*L^{-p} \xrightarrow{F_*\nabla_{\text{can}}} F_*(L^{-p} \otimes \Omega^p_{X/k}) \to \dots.$$

By projection formula one has

$$F_*L^{-p} = L^{-1} \otimes F_*\mathcal{O}_X, \quad F_*(L^{-p} \otimes \Omega^p_{X/k}) = L^{-1} \otimes F_*\Omega_{X/k}$$

One can find that $F_*\nabla_{\mathrm{can}} = \mathrm{id} \otimes F_*\mathrm{d}$, and thus above complex is $(F_*\Omega_{X/k}, F_*\mathrm{d}) \otimes L^{-1}$. By de Rham decomposition one has

$$\bigoplus_{i=0}^{d} \Omega^i_{X/k}[-i] \otimes L^{-1}$$

Then

$$\begin{aligned}
\dim \mathbb{H}^n(\bigoplus \Omega^i_{X/k}[-i] \otimes L^{-1}) &= \dim \mathbb{H}^n(X, F_*(L^{-p} \otimes \Omega_{X/k})) \\
&= \dim \mathbb{H}^n(X, L^{-p} \otimes \Omega_{X/k}) \\
&\leq \sum_{i+j=n} \dim H^j(X, L^{-p} \otimes \Omega^i_{X/k}) \\
&\leq \sum_{i+j=n} \dim H^j(X, L^{-Np} \otimes \Omega^i_{X/k})
\end{aligned}$$

Then by Serre vanishing one has

$$H^j(X, \Omega^i_{X/k} \otimes L^{-1}) = 0$$

for all $i + j < d$. $\square$

5.4. **From characteristic $p$ to characteristic $0$.**

**Lemma 5.1.** Let $\{A_i\}_{i \in I}$ be a direct system with direct limit $A$.

(1) $E$ is a $A$-module of finite presented, then there exists $i_0 \in I$ and $E_{i_0}$ is a $A_{i_0}$-module of finite presented such that

$$E_{i_0} \otimes_{A_{i_0}} A \cong E.$$

(2) Let $f\colon X \to S = \operatorname{Spec} A$ is a finite presented morphism (l.f.p., qcqs). Then there exists $i_0 \in I$ and $f_{i_0}\colon X_{i_0} \to \operatorname{Spec} S_{i_0} = \operatorname{Spec} A_{i_0}$ such that

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & X_{i_0} \arrow[d, "f_{i_0}"] \\
S \arrow[r] & S_{i_0}
\end{tikzcd}$$

(3) Moreover, if $f$ is smooth/proper/projective, then there exists $i_0 \in I$, and $f_{i_0}\colon X_{i_0} \to S_{i_0}$ such that $f_{i_0}$ is smooth/proper/projective.

**Theorem 5.7.**

# Part 2. Non-abelian Hodge theory

## 6. Non-abelian Hodge theory

The correspondence between Higgs bundles and local systems can be viewed as a Hodge theorem for non-abelian cohomology.

*Carlos T. Simpson*

6.1. **Introduction.** Let $X$ be a compact Kähler manifold. The Hodge theory says that the following cohomology groups are isomorphic

$$\underbrace{H^n(X,\mathbb{Z})\otimes\mathbb{C}}_{\text{topological aspect}} \cong \underbrace{H^n_{dR}(X)}_{\text{smooth aspect}} \cong \underbrace{\bigoplus_{p+q=n} H^{p,q}(X)}_{\text{holomorphic aspect}},$$

where

(1) $H^n(X,\mathbb{Z})$ is the $n$-th singular cohomology;
(2) $H^n_{dR}(X)$ is the $n$-th de Rham cohomology;
(3) $H^{p,q}(X)=H^q(X,\Omega^p_X)$ is the $(p,q)$-th Dolbeault cohomology.

In other words, topological, smooth and holomorphic aspects are related to each other closely by the classical Hodge theory.

However, many people thought about "analogue" for cohomology group with coefficients in a non-abelian group for a long time. One of the ideals is to note that

$$H^1(X,\mathbb{C})=\mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{C})=\mathrm{Hom}(\pi_1(X),\mathbb{C}),$$

since the singular homology $H_1(X,\mathbb{Z})$ is the abelianization of the fundamental group $\pi_1(X)$ by Hurwitz theorem. This interesting observation motivates us, instead of considering cohomology groups with coefficients in a non-abelian group $G$, we may consider $\pi_1(X)$-representations, that is, group homomorphisms from $\pi_1(X)$ to $G$.

In particular, we're interested in the case $G=\mathrm{GL}(n,\mathbb{C})$, since by Riemann-Hilbert correspondence, the following three objects are same:

(1) $\pi_1(X)$-representations (up to conjugacy).
(2) rank $n$ local systems (up to isomorphism).
(3) smooth flat vector bundles of rank $n$ on $X$ (up to isomorphism).

The first two things are living in the topological world, while the third one stands for the smooth category. Thus, parallel to the classical Hodge theory, the missing piece is some object living in the holomorphic world, and that's Higgs bundle we're going to define.

**Definition 6.1.1** (Higgs bundle)**.** A Higgs bundle over a compact Kähler manifold $X$ is a pair $(E,\theta)$, where $E\to X$ is a holomorphic vector bundle and $\theta\colon E \to E \otimes \Omega^1$ is an $\mathrm{End}(E)$-valued holomorphic 1-form such that $\theta \wedge \theta = 0$, called Higgs field.

Before we introduce more definitions, let's see a baby example.

**Example 6.1.1.** Let's consider $n = 1$. Note that

$$\begin{aligned}
\mathrm{Hom}(\pi_1(X), \mathbb{C}^*) &= \mathrm{Hom}(H_1(X,\mathbb{Z}), \mathbb{C}^*) \\
&\cong \frac{\mathrm{Hom}(H_1(X,\mathbb{Z}), \mathbb{C})}{\mathrm{Hom}(H_1(X,\mathbb{Z}), \mathbb{Z})} \\
&\cong \frac{H^0(X, \Omega_X) \oplus H^1(X, \mathcal{O}_X)}{H^1(X,\mathbb{Z})} \\
&\cong \mathrm{Pic}^0(X) \oplus H^0(X, \Omega_X).
\end{aligned}$$

Thus there is a bijection between the following sets

$$\{\rho\colon \pi_1(X) \to \mathbb{C}^*\} \longleftrightarrow \{(L,\theta) \mid L \in \mathrm{Pic}^0(X), \theta \in H^0(X, \Omega_X)\}.$$

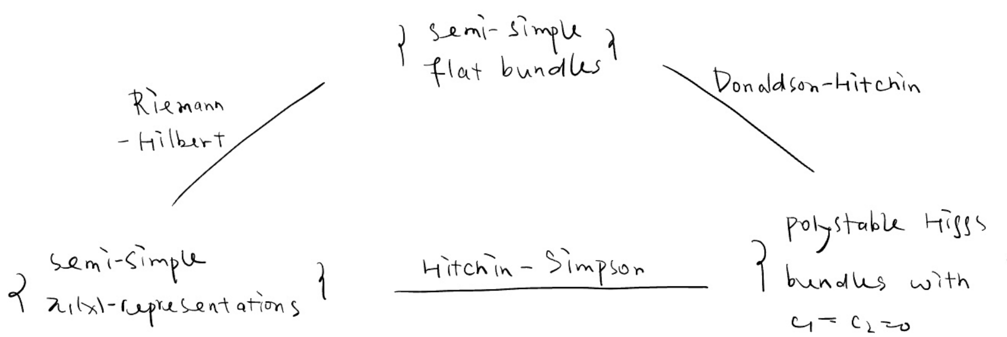
This is exactly the non-abelian Hodge correspondence.

For higher dimension cases, we need to consider polystable Higgs bundles and semi-simple representation to obtain the desired correspondence.

**Definition 6.1.2** (stability)**.** Let $(E,\theta)$ be a Higgs bundle on a compact Kähler manifold $(X,\omega)$. It's called

(1) stable if for every Higgs subbundle[8] $F$, one has $\mu_\omega(F) < \mu_\omega(E)$.
(2) semi-stable if for every Higgs subbundle $F$, one has $\mu_\omega(F) \leq \mu_\omega(E)$.
(3) polystable if it's direct sum of stable Higgs bundles, all having the same slope.

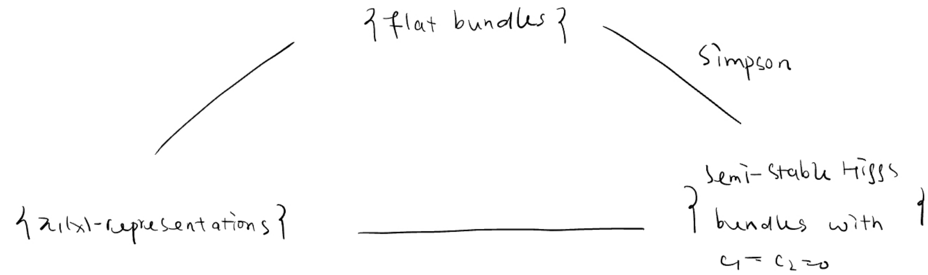
Now we can state the non-abelian Hodge correspondence by drawing the following picture.



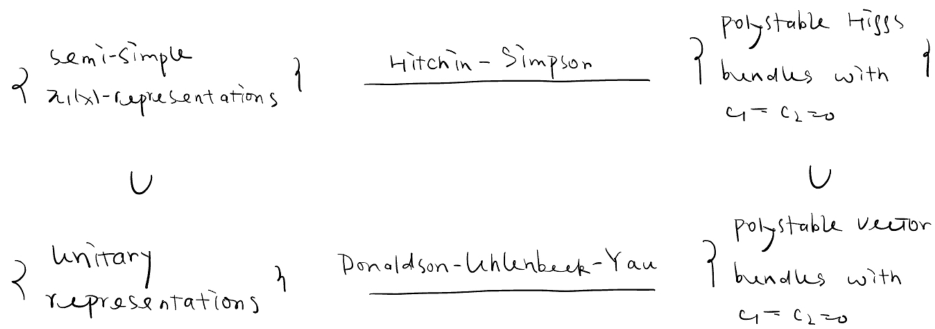

*Remark* 6.1.1.

(1) If $X$ is projective, then above triangle extends to

[8] A subbundle $F \subseteq E$ is a Higgs subbundle, if $\theta|_F$ is a holomorphic 1-form valued in $F$.




(2) The Donaldson-Uhlenbeck-Yau correspondence[9] gives the correspondence between unitary representations[10] and polystable vector bundles as follows




6.2. **Harmonic bundle.** In this section, $X$ always denotes a compact complex manifold, $V$ denotes a (smooth) complex vector bundle on $X$, and $E$ denotes a holomorphic vector bundle on $X$.

6.2.1. *Smooth Higgs bundle.*

**Definition 6.2.1** (Higgs field)**.** A Higgs field on $V$ is a first order differential operator $D'' \colon V \to V \otimes \mathcal{A}^1_X$ satisfying the $\overline{\partial}$-Lebnize rule, that is,

$$D''(fs) = \overline{\partial} fs + f D''(s)$$

and integerality $D'' \wedge D'' = 0$.

**Definition 6.2.2** (smooth Higgs bundle)**.** A smooth vector bundle $V$ equipped with a Higgs field $D''$ is called a smooth Higgs bundle.

[9]In particular, when $X$ is a Riemann surface, this correspondence is due to Narasimhan–Seshadri.

[10]A $\pi_1(X)$-representation is said to be unitary, if it factors as $\pi_1(X) \to \mathrm{U}(n) \hookrightarrow \mathrm{GL}(n, \mathbb{C})$.

*Remark* 6.2.1. Given a Higgs field $D''$ on $V$, we may intepret $D''$ in a different way. Since $V \otimes \mathcal{A}^1_X = V \otimes (\mathcal{A}^{1,0} \oplus \mathcal{A}^{0,1})$, we can write

$$D'' = \theta + \overline{\partial},$$

where $\theta\colon V \to V \otimes \mathcal{A}^{1,0}$ and $\overline{\partial}\colon V \to V \otimes \mathcal{A}^{0,1}$. Then

$$\begin{aligned} D'' \wedge D'' &= (\theta + \overline{\partial}) \wedge (\theta + \overline{\partial}) \\ &= \theta \wedge \theta + (\theta \wedge \overline{\partial} + \overline{\partial} \wedge \theta) + \overline{\partial} \wedge \overline{\partial} \end{aligned}$$

Note that for any $s \in V$, one has

$$(\overline{\partial} \wedge \theta)(a) = \overline{\partial}(\theta a) = \overline{\partial}(\theta)a - \theta \wedge \overline{\partial}a,$$

that is, $\overline{\partial} \wedge \theta + \theta \wedge \overline{\partial} = \overline{\partial}(\theta)$. As a consequence, $D'' \wedge D'' = 0$ is equivalent to the following equations

$$\begin{cases} \theta \wedge \theta = 0 \\ \overline{\partial}(\theta) = 0 \\ \overline{\partial} \wedge \overline{\partial} = 0. \end{cases}$$

The operator $\overline{\partial}$ can be used to define a holomorphic structure on $V$, that is, $E := V^{\overline{\partial}=0} = \{a \in V \mid \overline{\partial}a = 0\}$ is a holomorphic vector bundle. As $\overline{\partial}$ also satisfies the $\overline{\partial}$-Leibniz rule, $\theta = D'' - \overline{\partial}$ is a zero order differential operator. The condition $\overline{\partial}(\theta) = 0$ means that

$$\theta \in H^0(X, \mathrm{End}(E) \otimes \Omega^1_X).$$

In other words, given a smooth Higgs bundle $(V, D'' = \theta + \overline{\partial})$, it gives a Higgs bundle $(V^{\overline{\partial}=0}, \theta)$.

**Lemma 6.2.1.** The following map is a bijection.

$$\begin{aligned} \{\text{Higgs bundle}\} &\to \{\text{smooth Higgs bundle}\} \\ (E, \theta) &\mapsto (E \otimes \mathcal{A}^0_X, D'' = \theta + \overline{\partial}) \\ (V^{\overline{\partial}=0}, \theta) &\leftarrow\!\shortmid (V, D''). \end{aligned}$$

6.2.2. *From flat bundle to Higgs bundle.*

**Definition 6.2.3** (flat connection)**.** A flat connection on $V$ is a first order differential operator $D\colon V \to V \otimes \mathcal{A}^1_X$ such that $D \wedge D = 0$, and $(V, D)$ is called a flat bundle.

Now we try to use flat connection $D$ to construct a Higgs field on $V$. Firstly we pick any Hermitian metric $h$ on $V$ and let $D = \mathrm{d}' + \mathrm{d}''$ be the type decomposition. Set $\delta'$ and $\delta''$ to be the unique operator of type $(1,0)$ and $(0,1)$ such that $\delta' + \mathrm{d}''$ and $\delta'' + \mathrm{d}'$ preseves the metric. Then one has the following four operators

| | (1,0)-type | (0,1)-type |
|---|---|---|
| 1-st order | $\partial = (\mathrm{d}' + \delta')/2$ | $\overline{\partial} = (\mathrm{d}'' + \delta'')/2$ |
| 0-th order | $\theta = (\mathrm{d}' - \delta')/2$ | $\overline{\theta} = (\mathrm{d}'' - \delta'')/2$ |

**Definition 6.2.4** (quasi-Higgs field)**.** A quasi-Higgs field $D''_h$ is defined to be

$$D''_h := \overline{\partial} + \theta = \frac{1}{2}(\mathrm{d}' + \mathrm{d}'' - \delta' + \delta''),$$

and it's a Higgs field if and only if $D''_h \wedge D''_h = 0$.

**Definition 6.2.5** (harmonic metric on flat bundle)**.** A Hermitian metric $h$ on a flat bundle is said to be harmonic, if the pseudo-curvature $G_h = D''_h \wedge D''_h$ vanishes.

*Remark* 6.2.2. In other words, the harmonicity is so-defined such that we obtain a Higgs field out of a flat connection.

6.2.3. *From Higgs bundle to flat bundle.* Given a Higgs bundle $(E, \theta)$ equipped with a Hermitian metric $h$, we set $\partial_h$ to be the Chern connection of Hermitian metric $h$, that is, $\partial_h + \overline{\partial}_E$ preseves $h$, and $\overline{\theta}_h$ is the adjoint of $\theta$ with respect to $h$. Thus we have the following four operators

| | (1,0)-type | (0,1)-type |
|---|---|---|
| 1-st order | $\partial_h$ | $\overline{\partial}_E$ |
| 0-th order | $\theta$ | $\overline{\theta}_h$ |

Now we define

$$D'_h = \partial_h + \overline{\theta}_h$$
$$D'' = \overline{\partial}_E + \theta,$$

and set $D_h = D'_h + D''$, which is called a quasi-flat connection.

**Definition 6.2.6** (harmonic metric on Higgs bundle)**.** A Hermitian metric $h$ on Higgs bundle $(E, \theta)$ is called harmonic if the quasi-flat connection $D_h$ is flat.

**Lemma 6.2.2.** The following map is a bijection.

$$\{\text{smooth flat bundle with harmonic metric}\} \to \{\text{smooth Higgs bundle with harmonic metric}\}$$
$$(V, D, h) \mapsto (V, D''_h, h)$$
$$(V, D_h, h) \leftarrow\!\shortmid (V, D'', h).$$

6.2.4. *Harmonic bundle.*

**Definition 6.2.7** (harmonic bundle)**.** A harmonic bundle on $X$ is a flat bundle together with a harmonic metric, or a Higgs bundle together with a harmonic metric.

*Remark* 6.2.3. A choice of a harmonic metric is not a part of defining data for a harmonic bundle.

**Lemma 6.2.3** (Kähler identities)**.** Suppose $(X, \omega)$ is a Kähler manifold. Then for any Hermitian metric $h$ on Higgs bundle $(E, \theta)$,

$$\begin{aligned}(D'_h)^* &= \sqrt{-1}\left[\Lambda_\omega, D''\right]\\ (D'')^* &= -\sqrt{-1}\left[\Lambda_\omega, D'_h\right]\end{aligned}$$

*Remark* 6.2.4. Consider the trivial Higgs bundle $(E, 0)$, that is, a holomorphic vector bundle $E$ equipped with trivial Higgs field $\theta = 0$. Then above lemma recovers Kälher identities shown in Theorem 2.1.

Analogously, for any Hermitian metric $h$ on smooth flat bundle $(V, D)$, one has the following identities.

**Lemma 6.2.4.** Suppose $(X, \omega)$ is a Kähler manifold. Then for any Hermitian metric $h$ on Higgs bundle $(E, \theta)$,

$$\begin{aligned}(D^c_h)^* &= -\sqrt{-1}\left[\Lambda_\omega, D\right]\\ D^* &= \sqrt{-1}\left[\Lambda_\omega, D^c_h\right],\end{aligned}$$

where $D^c_h = D'_h - D''_h$.

*Remark* 6.2.5. For flat bundle $(V, D) = (\mathcal{A}^0, \mathrm{d})$, $D^c_h = \overline{\partial} - \partial$, and thus

$$\begin{aligned}(\mathrm{d}^c)^* &= -\sqrt{-1}\left[\Lambda_\omega, \mathrm{d}\right]\\ \mathrm{d}^* &= \sqrt{-1}\left[\Lambda_\omega, \mathrm{d}^c\right].\end{aligned}$$

**Lemma 6.2.5** (Bianchi identity)**.**

(1) Let $(E, \theta, h)$ be a harmonic bundle. Then

$$\begin{aligned}F_h &= (D_h)^2 = D'_h D'' + D'' D'_h\\ D'_h F_h &= D'' F_h = 0\end{aligned}$$

(2) Let $(V, D, h)$ be a harmonic bundle. Then

$$\begin{aligned}G_h &= (D''_h)^2 = \frac{D D^c_h + D^c_h D}{4}\\ D G_h &= D^c_h G_h = 0\end{aligned}$$

**Lemma 6.2.6.**

(1) Let $(E, \theta)$ be a Higgs bundle admitting a Hermitian-Yang-Mills metric $h$. If $c_1(E)[\omega]^{n-1} = c_2(E)[\omega]^{n-2} = 0$, then $F_h = 0$.
(2) Let $(V, D)$ be a flat bundle admitting a metric satisfying $\Lambda G_h = 0$. Then $G_h = 0$.

*Remark* 6.2.6. Results in non-linear analysis shows that

(1) A Higgs bundle $(E, \theta)$ has a Hermitian-Yang-Mills metric if and only if it's polystable.
(2) A flat bundle $(V, D)$ admits a metric $h$ satisfying $\Lambda G_h = 0$ if and only if it's semi-simple.

## 6.3. Complex variation of Hodge structure.

**Definition 6.3.1** ($\mathbb{C}$-VHS)**.** A (polarized) complex variation of Hodge structure ($\mathbb{C}$-VHS) consists of the following data:

(1) A smooth vector bundle $V$ with a decomposition $V = \bigoplus_{p+q=n} V^{p,q}$.

(2) A flat connection $D$ on $V$ satisfies

$$D\colon V^{p,q} \to \mathcal{A}^{0,1}(V^{p+1,q-1}) \oplus \mathcal{A}^{1,0}(V^{p,q}) \oplus \mathcal{A}^{0,1}(V^{p,q}) \oplus \mathcal{A}^{1,0}(V^{p-1,q+1}),$$

which is called Griffiths transversality.

(3) A parallel Hermitian form $Q$ which makes the decomposition in (1) is orthogonal and which on $V^{p,q}$ is positive definite if $p$ is even and negative definite if $p$ is odd.

## 7. Higgs-de Rham flow abd their applications

### 7.1. **Higgs-de Rham flow.**

**Definition 7.1.1.** Let $(V, \nabla)$ be a flat module. A filteration of coherent $\mathcal{O}_X$ -module on $V$ of the form

$$V = Fil^0 \supseteq Fil^1 \supseteq \cdots \supseteq Fil^l \supseteq \dots$$

is said to be

(1) of length $\leq l$ if $Fil^{l+1} = \{0\}$.
(2) saturated if $V$ is torsion free and $Fil^i \subseteq V$ is saturated.
(3) Griffiths transverse if $\nabla(Fil^i) \subseteq Fil^{i-1} \otimes \Omega_{X/k}$
(4) when $V$ is locally free, it's called a Hodge module, if it's Griffiths transverse and $Fil^i \subseteq V$ is a subbundle for each $i$

**Definition 7.1.2.** A de Rham module is a triple $(V, \nabla, Fil)$, where

(1) $(V, \nabla)$ is a flat module.
(2) $Fil$ is a Griffiths transverse filteration.

**Definition 7.1.3** (Higgs-de Rham flow)**.** A Higgs-de Rham flow over $X$ is a diagram of the form

$$\begin{tikzcd}
 & (V_0, \nabla_0) \arrow[dr, "Gr_{Fil_0}"] & & \dots \\
(E_0, \theta_0) \arrow[ur, "C^{-1}"] & & (E_1, \theta_1) \arrow[ur] &
\end{tikzcd}$$

(1) $(E_0, \theta_0) \in HIG_{\leq p-1}(X)$.
(2) $Fil_i$ is a Griffiths transverse filteration on $(V_i, \nabla_i)$ of length $\leq p - 1$.

**Example 7.1.1** (arithmetic)**.** A perodic Higgs-de Rham flow is of the form

From now on, we assume $(X, L)$ is a smooth projective variety.

**Example 7.1.2** (geometry)**.** A semi-stable Higgs-de Rham flow over $X$ if $(E_i, \theta_i)$ is $L$-semistable.

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