RIEMANNIAN SYMMETRIC SPACE

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Contents

Part 1. Riemannian symmetric space

1. GEOMETRIC VIEWPOINTS

1.A. Basic definitions and properties.

1.A.1. *Riemannian symmetric space.*

Definition 1.1 (Riemannian symmetric space)**.** *A Riemannian manifold (M, g) is called a Riemannian symmetric space if for each $p \in M$ there exists an isometry $\varphi : M \to M$, which is called a symmetry at p, such that $\varphi(p) = p$ and $(\mathrm{d}\varphi)_p = -\mathrm{id}$.*

Remark 1.2. Theorem B.8 implies if symmetry at point p exists, then it's unique.

Proposition 1.3. *The following statements are equivalent:*

(1) (M, g) is a Riemannian symmetric space.
(2) For each $p \in M$, there exists an isometry $\varphi : M \to M$ such that $\varphi^2 = \mathrm{id}$ and p is an isolated fixed point of φ.

Proof. From (1) to (2). Let φ be a symmetry at $p \in M$. Since $(\mathrm{d}\varphi^2)_p = (\mathrm{d}\varphi)_p \circ (\mathrm{d}\varphi)_p = \mathrm{id}$ and $\varphi^2(p) = p$, one has $\varphi^2 = \mathrm{id}$ by Theorem B.8. If p is not an isolated fixed point, then there exists a sequence $\{p_i\}_{i=1}^{\infty}$ converging to p such that $\varphi(p_i) = p_i$. For $0 < \delta < \mathrm{inj}(p)$, there exists sufficiently large k such that $p_k \in B(p, \delta)$, and we denote $v = \exp_p^{-1}(p_k)$. Since φ is an isometry, one has $\varphi(\exp_p(tv))$ and $\exp_p(tv)$ are two geodesics connecting p and p_k, and thus

$$\varphi(\exp_p(tv)) = \exp_p(tv)$$

by uniqueness. In particular, one has $v = (\mathrm{d}\varphi)_p v$, which is a contradiction.

From (2) to (1). From $\varphi^2 = \mathrm{id}$ we have $(\mathrm{d}\varphi)_p^2 = \mathrm{id}$, so only possible eigenvalues of $(\mathrm{d}\varphi)_p$ are ± 1. Now it suffices to show all eigenvalues of $(\mathrm{d}\varphi)_p$ are -1. Otherwise if it has an eigenvalue 1, there exists some non-zero $v \in T_pM$ such that $(\mathrm{d}\varphi)_p v = v$. Since φ is an isometry, one has $\varphi(\exp_p(tv))$ and $\exp_p(tv)$ are geodesics with the same direction at p. Thus

$$\varphi(\exp_p(tv)) = \exp_p(tv)$$

for $0 < t < \mathrm{inj}(p)$. In particular, p is not an isolated fixed point, which is a contradiction. $\square$

Proposition 1.4. *The fundamental group of a Riemannian symmetric space is abelian.*

Corollary 1.5. *A surface of genus $g \geq 2$ does not admit a Riemannian metric with respect to which it is a symmetric space.*

1.A.2. *Locally Riemannian symmetric space.*

Definition 1.6 (locally Riemannian symmetric space)**.** *A Riemannian manifold (M, g) is called a locally Riemannian symmetric space if each $p \in M$ has a neighborhood U such that there exists an isometry $\varphi : U \to U$ such that $\varphi(p) = p$ and $(\mathrm{d}\varphi)_p = -\mathrm{id}$.*

Theorem 1.7. *Let (M, g) be a Riemannian manifold. Then the following statements are equivalent:*

(1) (M, g) is a locally Riemannian symmetric space.

(2) $\nabla R = 0$.

Proof. From (1) to (2). If φ is the symmetry at point $p \in M$, then it's an isometry such that $(\mathrm{d}\varphi)_p = -\mathrm{id}$, and thus for $u, v, w, z \in T_pM$, one has

$$\begin{aligned} -\nabla_u R(v,w)z &= (\mathrm{d}\varphi)_p\left(\nabla_u R(v,w)z\right) \\ &= \nabla_{(\mathrm{d}\varphi)_p u}((\mathrm{d}\varphi)_p)v, (\mathrm{d}\varphi)_p w)(\mathrm{d}\varphi)_p z \\ &= \nabla_u R(v,w)z \end{aligned}$$

This shows $(\nabla R)_p = 0$, and thus $\nabla R = 0$ since p is arbitrary.

From (2) to (1). For arbitrary $p \in M$, it suffices to show

$$\varphi = \exp_p \circ \Phi_0 \circ \exp_p^{-1} : B(p,\delta) \to B(p,\delta)$$

is an isometry, where $0 < \delta < \mathrm{inj}(p)$ and $\Phi_0 = -\mathrm{id} : T_pM \to T_pM$. For $v \in T_pM$ with $|v| < \delta$ and $\gamma(t) = \exp_p(tv), \widetilde{\gamma}(t) = \exp_p(t\Phi_0(v))$, if we define

$$\Phi_t = P_{0,t;\widetilde{\gamma}} \circ \Phi_0 \circ P_{t,0;\gamma}$$

then direct computation shows

$$\begin{aligned} \Phi_t^* R_{\widetilde{\gamma}(t)} &= (P_{t,0;\gamma})^* \circ \Phi_0^* \circ (P_{0,t;\widetilde{\gamma}})^* R_{\widetilde{\gamma}(t)} \\ &\overset{(a)}{=} (P_{t,0;\gamma})^* \circ \Phi_0^* R_{\widetilde{\gamma}(0)} \\ &\overset{(b)}{=} (P_{t,0;\gamma})^* R_{\gamma(0)} \\ &\overset{(c)}{=} R_{\gamma(t)} \end{aligned}$$

where

(a) and (c) holds from Proposition B.12.
(b) holds from $\widetilde{\gamma}(0) = \gamma(0)$ and R is a $(0,4)$-tensor.

Then by Theorem B.9, that is Cartan-Ambrose-Hicks's theorem, φ is an isometry, which completes the proof. □

Remark 1.8. The proof for locally Riemannian symmetric space has parallel curvature tensor can be applied to other situations. For example, one can easy show if a p-form ω is invariant under isometries, that is $\varphi^*\omega = \omega$ for arbitrary isometry, then $\mathrm{d}\omega = 0$, and in Section 7 we will use this idea to show any almost Hermitian symmetric space is Kähler.

1.B. **Transvection.**

Definition 1.9 (transvection). *Let (M, g) be a Riemannian symmetric space and γ be a geodesic. The transvection along γ is defined as*

$$T_t = s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)},$$

where s_p is the symmetry at point p.

Proposition 1.10. *Let (M, g) be a Riemannian symmetric space and T_t be the transvection along geodesic γ. Then*

(1) For any $a, t \in \mathbb{R}$, $s_{\gamma(a)}(\gamma(t)) = \gamma(2a - t)$.
(2) T_t translates the geodesic γ, that is $T_t(\gamma(s)) = \gamma(t + s)$.

(3) $(\mathrm{d}T_t)_{\gamma(s)} : T_{\gamma(s)}M \to T_{\gamma(t+s)}M$ *is the parallel transport* $P_{s,t+s;\gamma}$.
(4) T_t *is one-parameter subgroup of* $\mathrm{Iso}(M,g)$.

Proof. For (1). It follows from the uniqueness of geodesics with given initial value.

For (2). By (1) one has

$$\begin{aligned} T_t(\gamma(s)) &= s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}(\gamma(s)) \\ &= s_{\gamma(\frac{t}{2})}(\gamma(-s)) \\ &= \gamma(t+s). \end{aligned}$$

For (3). Let X be a parallel vector field along γ. By uniqueness of parallel vector fields with given initial data, we have $(\mathrm{d}s_{\gamma(0)})_{\gamma(s)}X_{\gamma(s)} = -X_{\gamma(-s)}$ for all s, since $(\mathrm{d}s_{\gamma(0)})_{\gamma(0)}X_{\gamma(0)} = -X_{\gamma(0)}$. Thus

$$\begin{aligned} (\mathrm{d}T_t)_{\gamma(s)}X_{\gamma(s)} &= (\mathrm{d}s_{\gamma(\frac{t}{2})})_{\gamma(-s)}(-X_{\gamma(-s)}) \\ &= X_{\gamma(t+s)}. \end{aligned}$$

This shows $(\mathrm{d}T_t)_{\gamma(s)} = P_{s,t+s;\gamma}$.

For (4). In order to show $T_{t+s} = T_t \circ T_s$, it suffices to check they're same at some point, so do their derivatives, since isometry can be determined by these two values. Note that

$$\begin{aligned} T_{t+s}(\gamma(0)) &= \gamma(t+s) \\ &= T_t \circ T_s(\gamma(0)), \\ (\mathrm{d}T_{t+s})_{\gamma(0)} &= P_{0,t+s;\gamma} \\ &= P_{s,t+s;\gamma} \circ P_{0,s;\gamma} \\ &= (\mathrm{d}T_t)_{\gamma(s)} \circ (\mathrm{d}T_s)_{\gamma(0)} \\ &= (\mathrm{d}(T_t \circ T_s))_{\gamma(0)}. \end{aligned}$$

This completes the proof. □

1.C. **Symmetric space, locally symmetric space and homogeneous space.** In this section, we will show any complete locally Riemannian symmetric space is a quotient of Riemannian symmetric space (Corollary 1.12), and any Riemannian symmetric space is a Riemannian homogeneous space (Corollary 1.16).

1.C.1. *Riemannian symmetric space and locally Riemannian symmetric space.*

Theorem 1.11. *Let* (M,g) *be a complete, simply-connected locally Riemannian symmetric space. Then* (M,g) *is a Riemannian symmetric space.*

Proof. For $p \in M$ and $0 < \delta < \mathrm{inj}(p)$, suppose $\varphi : B(p,\delta) \to B(p,\delta)$ is an isometry such that $\varphi(p) = p$ and $(\mathrm{d}\varphi)_p = -\mathrm{id}$. For arbitrary $q \in M$, we use $\Omega_{p,q}$ to denote all curves γ with $\gamma(0) = p, \gamma(1) = q$, and for $c \in \Omega_{p,q}$ we choose[1] a covering $\{B(p_i,\delta_i)\}_{i=0}^k$ of c such that

(1) $0 < \delta_i < \mathrm{inj}(p_i)$.
(2) $B(p_0,\delta_0) = B(p,\delta)$ and $p_k = q$.
(3) $p_{i+1} \in B(p_i,\delta_i)$.

[1]Since injective radius is a continuous function, it has a positive minimum on curve c, so such covering exists.

If we set $\varphi = \varphi_0$, then we can define isometries $\varphi_i : B(p_i, \delta_i) \to M$ such that $\varphi_i(p_i) = \varphi_{i-1}(p_i)$ and $(\mathrm{d}\varphi_i)_{p_i} = (\mathrm{d}\varphi_{i-1})_{p_i}$ by using Cartan-Ambrose-Hicks's theorem successively, and by Theorem B.8 one has φ_i and φ_{i+1} coincide on $B(p_i, \delta_i) \cap B(p_{i+1}, \delta_i)$. The covering together with isometries we construct is denoted by $\mathcal{A} = \{B(p_i, \delta_i), \varphi_i\}_{i=0}^k$. For arbitrary $x \in [0,1]$, if $c(x) \in B(p_m, \delta_m)$, we may define

$$\begin{aligned} \varphi_{\mathcal{A}}(c(x)) &:= \varphi_m(c(x)), \\ (\mathrm{d}\varphi_{\mathcal{A}})_{c(x)} &:= (\mathrm{d}\varphi_m)_{c(x)}. \end{aligned}$$

In particular, $\varphi_{\mathcal{A}}(q) := \varphi_k(q)$. If $\mathcal{B} = \{\widetilde{B}(\widetilde{p}_i, \widetilde{\delta}_i), \widetilde{\varphi}_i\}_{i=0}^l$ is another covering of c, let's show $\varphi_{\mathcal{A}}(q) = \varphi_{\mathcal{B}}(q)$. Consider

$$I = \{x \in [0,1] \mid \varphi_{\mathcal{A}}(c(x)) = \varphi_{\mathcal{B}}(c(x)), (\mathrm{d}\varphi_{\mathcal{A}})_{c(x)} = (\mathrm{d}\varphi_{\mathcal{B}})_{c(x)}\}.$$

It's clear $I \neq \varnothing$, since $0 \in I$. Now it suffices to show it's both open and closed to conclude $1 \in I$.

(a) It's open: For $x \in I$, we assume $c(x) \in B(p_m, \delta_m) \cap \widetilde{B}(\widetilde{p}_n, \widetilde{\delta}_n)$, that is

$$\begin{aligned} \varphi_m(c(x)) &= \widetilde{\varphi}_n(c(x)), \\ (\mathrm{d}\varphi_m)_{c(x)} &= (\mathrm{d}\widetilde{\varphi}_n)_{c(x)}. \end{aligned}$$

Then one has

$$\begin{aligned} \varphi_m \circ \exp_{c(x)}(v) &= \exp_{\varphi_m(c(x))} \circ (\mathrm{d}\varphi_m)_{c(x)}(v) \\ &= \exp_{\widetilde{\varphi}_n(c(x))} \circ (\mathrm{d}\widetilde{\varphi}_n)_{c(x)}(v) \\ &= \widetilde{\varphi}_n \circ \exp_{c(x)}(v). \end{aligned}$$

Since $\exp_{c(x)}$ maps onto a neighborhood of $c(x)$, it follows that some neighborhood of x also lies in I, and thus I is open.

(b) It's closed: Let $\{x_i\}_{i=1}^\infty \subseteq I$ be a sequence converging to x. Without lose of generality we may assume $\{x_i\}_{i=1}^\infty \subseteq B(p_m, \delta_m) \cap \widetilde{B}(\widetilde{p}_n, \widetilde{\delta}_n)$, then one has

$$\begin{aligned} \varphi_m(c(x_i)) &= \widetilde{\varphi}_n(c(x_i)), \\ (\mathrm{d}\varphi_m)_{c(x_i)} &= (\mathrm{d}\widetilde{\varphi}_n)_{c(x_i)}. \end{aligned}$$

By taking limit we obtain the desired results.

Since $\varphi_{\mathcal{A}}(q)$ is independent of the choice of coverings, we use $\varphi(q)$ to denote it for convenience, and as a consequence we obtain the following map

$$\begin{aligned} F : \Omega_{p,q} &\to M \\ c &\mapsto \varphi(q). \end{aligned}$$

Note that $F(c)$ is locally constant, and thus it's independent of the choice of homotopy classes of c. Since M is simply-connected, one has $F : \Omega_{p,q} \to M$ is constant, so we obtain a local isometry $\varphi : M \to M$ which extends $\varphi : B(p, \delta) \to B(p, \delta)$. By Proposition B.10 φ is a Riemannian covering map since M is complete, and thus φ is a diffeomorphism since M is simply-connected, which implies φ is an isometry. $\square$

Corollary 1.12. *Let (M, g) be a complete locally Riemannian symmetric space. Then it's isometric to $(\widetilde{M}/\Gamma, \widetilde{g})$ where $(\widetilde{M}, \widetilde{g})$ is a Riemannian symmetric space and $\Gamma \cong \pi_1(M)$ is a discrete Lie group acting on $\widetilde{M}$ freely, properly and isometrically.*

Proof. Let $(\widetilde{M}, \widetilde{g})$ be the universal covering of (M, g) with pullback metric. Then $(\widetilde{M}, \widetilde{g})$ is a simply-connected Riemannian manifold with parallel curvature tensor. Moreover, by Proposition B.13 it's complete, hence it is symmetric. □

1.C.2. *Riemannian symmetric space and Riemannian homogeneous space.*

Definition 1.13 (Riemannian homogeneous space)**.** *A Riemannian manifold* (M, g) *is called a Riemannian homogeneous space, if* $\mathrm{Iso}(M, g)$ *acts on* M *transitively.*

Proposition 1.14. *Let* (M, g) *be a Riemannian homogeneous space. If there exists a symmetry at some point* $p \in M$*, then* (M, g) *is a Riemannian symmetric space.*

Proof. Let φ be a symmetry at $p \in M$. For arbitrary $q \in M$, there exists an isometry $\psi : M \to M$ such that $\psi(p) = q$ since (M, g) is a Riemannian homogeneous space. Then

$$\varphi_q := \psi \circ \varphi \circ \psi^{-1}$$

is the desired symmetry at q. □

Theorem 1.15. *Let* (M, g) *be a Riemannian symmetric space. Then*

(1) (M, g) *is complete.*

(2) the identity component of isometry group acts transitively on M*.*

Proof. For (1). For arbitrary geodesic $\gamma : [0, 1] \to M$ with $\gamma(0) = p, \gamma'(0) = v$, the curve $\beta(t) = \varphi(\gamma(t)) : [0, 1] \to M$ is also a geodesic with $\beta(0) = p$ and $\beta'(0) = -v$. Now we obtain a smooth extension $\gamma' : [0, 2] \to M$ of γ, given by

$$\gamma'(t) = \begin{cases} \beta(1-t), & t \in [0, 1] \\ \gamma(t-1), & t \in [1, 2]. \end{cases}$$

Repeat above process to extend γ to a geodesic defined on $\mathbb{R}$, which shows completeness.

For (2). For $p, q \in M$, let γ be a geodesic connecting p, q. Then the transvection along γ gives an isometry which maps p to q. Since the transvection lies in the identity component of isometry group, one has the identity component of isometry group acts transitively on M. □

Corollary 1.16. *The Riemannian symmetric space* (M, g) *is a Riemannian homogeneous space.*

2. Algebraic viewpoints

2.A. **Riemannian symmetric space as a Lie group quotient.**

Definition 2.1 (involution)**.** *An automorphism σ of a Lie group G is called an involution if $\sigma^2 = \mathrm{id}_G$.*

Definition 2.2 (Cartan decomposition)**.** *Let G be a Lie group and σ be an involution of G. The eigen-decomposition of $\mathfrak{g}$ given by $(\mathrm{d}\sigma)_e$ is called Cartan decomposition, that is,*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m},$$

where

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid (\mathrm{d}\sigma)_e(X) = X\},$$
$$\mathfrak{m} = \{X \in \mathfrak{g} \mid (\mathrm{d}\sigma)_e(X) = -X\}.$$

Proposition 2.3. *Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the Cartan decomposition given by σ. Then*

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}.$$

Proof. Since σ is a Lie group homomorphism, $(\mathrm{d}\sigma)_e$ gives a Lie algebra homomorphism, and thus

$$(\mathrm{d}\sigma)_e([X, Y]) = [(\mathrm{d}\sigma)_e(X), (\mathrm{d}\sigma)_e(Y)],$$

where $X, Y \in \mathfrak{g}$. □

Lemma 2.4. *Let G be a Lie group and $K \subseteq G$ be a closed subgroup. A left invariant metric on G which is also right invariant under K gives a left-invariant metric on G/K.*

Theorem 2.5. *Let (M, g) be a Riemannian symmetric space and $G = (\mathrm{Iso}(M, g))_0$. For $p \in M$, K denotes the isotropic group of G_p.*

(1) The mapping $\sigma : G \to G$, given by $\sigma(g) = s_p g s_p$ is an involution automorphism of G.
(2) If G^σ is the set of fixed points of σ in G, then $(G^\sigma)_0 \subseteq K \subseteq G^\sigma$.
(3) If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the Cartan decomposition given by σ, then $\mathfrak{k}$ is the Lie algebra of K, and thus $\mathfrak{m} \cong T_pM$ as vector spaces.
(4) There is a left invariant metric on G/K such that G/K with this metric is isometric to (M, g).

Proof. For (1). It's clear σ preserves G, and it's an involution since for arbitrary $g \in G$, one has $\sigma^2(g) = \sigma(s_p g s_p) = s_p^2 g s_p^2 = g$.

For (2). It follows from the following two steps:

(a) To show $K \subseteq G^\sigma$. For any $k \in K$, in order to show $k = s_p k s_p$, it suffices to show they and their differentials agree at some point by Theorem B.8, since both of them are isometries, and p is exactly the point we desired.

(b) To see $(G^\sigma)_0 \subseteq K$. Suppose $\exp(tX) \subseteq (G^\sigma)_0$ is a one-parameter subgroup. Since $\sigma(\exp(tX)) = \exp(tX)$, one has

$$\exp(tX)(p) = s_p \exp(tX) s_p(p) = s_p \exp(tX)(p).$$

But p is an isolated fixed point of s_p, which implies $\exp(tX)(p) = p$ for all t. This shows the one-parameter subgroup lies in K. Since exponential map of Lie group is a diffeomorphism in a small neighborhood of identity element e and $(G^\sigma)_0$ can be generated by a neighborhood of e, which implies the whole $(G^\sigma)_0 \subseteq K$.

For (3). Note that $(G^\sigma)_0\subseteq K\subseteq G^\sigma$, it suffices to show $\mathfrak{k}\cong\operatorname{Lie}G^\sigma$. For $X\in\mathfrak{k}$, we claim $\gamma_2(t)=\sigma(\exp(tX))\colon\mathbb{R}\to G$ is a one-parameter subgroup. Indeed, note that

$$\begin{aligned}\gamma_2(t)\cdot\gamma_2(s)&=s_p\exp(tX)s_p\cdot s_p\exp(sX)s_p\\&=\sigma(\exp(tX+sX))\\&=\gamma_2(t+s).\end{aligned}$$

Moreover, $\gamma_2(t)=\sigma(\exp(tX))$ and $\gamma_1(t)=\exp(tX)$ are two one-parameter subgroups of G such that $\gamma_1(0)=\gamma_2(0)$ and $\gamma_2'(0)=(\mathrm{d}\sigma)_e(X)=X=\gamma_1'(0)$. Then $\gamma_1(t)=\gamma_2(t)$, and thus $\exp(tX)\in G^\sigma$ for all $t\in\mathbb{R}$. This shows $\mathfrak{k}\subseteq\operatorname{Lie}G^\sigma$, and the converse inclusion is clear, so one has $\mathfrak{k}=\operatorname{Lie}G^\sigma$.

For (4). Let $\pi\colon G\to M$ be the natural projection given by $\pi(g)=gp$. Then for $k\in K$ and $X\in\mathfrak{g}$ one has

$$\begin{aligned}(\mathrm{d}\pi)_e(\operatorname{Ad}(k)X)&=(\mathrm{d}\pi)_e\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}k\exp(tX)k^{-1}\right)\\&=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\pi(k\exp(tX)k^{-1})\\&=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}k\exp(tX)k^{-1}\cdot p\\&=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}k\exp(tX)\cdot p\\&=(\mathrm{d}L_k)_p(\mathrm{d}\pi)_e(X).\end{aligned}$$

By using the equivalent isomorphism $(\mathrm{d}\pi)_e|_{\mathfrak{m}}\colon\mathfrak{m}\to T_pM$, one has an $\operatorname{Ad}(K)$-invariant metric on $\mathfrak{m}$, and then we can extend it to an $\operatorname{Ad}(K)$-invariant metric on $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$ by choosing[2] arbitrary $\operatorname{Ad}(K)$-invariant metric on $\mathfrak{k}$ such that $\mathfrak{m}\perp\mathfrak{k}$. This shows one has a left-invariant metric on G which is also right invariant with respect to K, and by Lemma 2.4 it gives a left-invariant metric on G/K. Now it suffices to show G/K with this metric is isometric to (M,g). For any $gK\in G/K$, consider the following communicative diagram

$$\begin{tikzcd}\mathfrak{m}=T_{eK}G/K \arrow[r, "(\mathrm{d}\pi)_e|_{\mathfrak{m}}"] \arrow[d, "\mathrm{d}L_g"'] & T_pM \arrow[d, "\mathrm{d}L_g"]\\ T_{gK}G/K \arrow[r] & T_{gp}M\end{tikzcd}$$

Since both $(\mathrm{d}\pi)_e|_{\mathfrak{m}}$ and $(\mathrm{d}L_g)$ are linear isometries, one has $T_{gK}G/K$ is isometric to $T_{gp}M$, and thus G/K with this metric is isometric to (M,g). □

2.B. **Riemannian symmetric pair.** In Theorem 2.5 one can see that if (M,g) is a symmetric space, then it gives a pair of Lie groups (G,K) with an involution σ on G such that

$$(G^\sigma)_0\subseteq K\subseteq G^\sigma.$$

Moreover, there exists a left-invariant metric on G/K such that G/K with this metric is isometric to (M,g). This motivates us a useful way to construct Riemannian symmetric

[2]Such metric exists since K is compact.

spaces from a pair of Lie groups with certain properties, and such a pair is called a Riemannian symmetric pair.

Definition 2.6 (Riemannian symmetric pair)**.** *Let G be a connected Lie group and $K \subseteq G$ be a closed subgroup. The pair (G, K) is called a symmetric pair if there exists an involution $\sigma : G \to G$ with $(G^\sigma)_0 \subseteq K \subseteq G^\sigma$. If, in addition, the group $\mathrm{Ad}(K) \subseteq \mathrm{GL}(\mathfrak{g})$ is compact, then (G, K) is said to be a Riemannian symmetric pair.*

Remark 2.7. The first condition of above definition means K is compact up to the center of G since the kernel of Ad is the center of G. By Theorem 2.5 every Riemannian symmetric space gives a Riemannian symmetric pair.

Definition 2.8 (associated)**.** *If (M, g) is a Riemannian symmetric space, $G = (\mathrm{Iso}(M, g))_0$ and K is the isotropy group G_p of some point $p \in M$, then (G, K) is a Riemannian symmetric pair. In this case (G, K) is called the Riemannian symmetric pair associated to (M, g).*

Proposition 2.9. *Let (G, K) be a symmetric pair given by σ. Then there is an isomorphism as Lie algebras*

$$\mathfrak{k} \cong \mathrm{Lie}\, K,$$

and an isomorphism as vector spaces

$$\mathfrak{m} \cong T_{eK}G/K$$

Proof. It's the same as proof of (3) in Theorem 2.5. □

Corollary 2.10. *Let $\widetilde{\sigma} : G/K \to G/K$ be the automorphism given by $\widetilde{\sigma}(gK) = \sigma(g)K$. Then $(\mathrm{d}\widetilde{\sigma})_{eK} = -\mathrm{id}_{G/K}$.*

Proof. $\widetilde{\sigma}$ is well-defined since $K \subseteq G^\sigma$, and by construction one has $(\mathrm{d}\widetilde{\sigma})_{eK} = (\mathrm{d}\sigma)_e|_{\mathfrak{m}}$. Then $(\mathrm{d}\widetilde{\sigma})_{eK} = -\mathrm{id}_{G/K}$ since $\mathfrak{m} = \{X \in \mathfrak{g} \mid (\mathrm{d}\sigma)_e X = -X\}$. □

Theorem 2.11. *Let (G, K) be a Riemannian symmetric pair given by σ. Then there exists a left-invariant metric on $M = G/K$ making it to be a Riemannian symmetric space.*

Proof. Since $\mathrm{Ad}(K) \subseteq \mathrm{GL}(\mathfrak{g})$ is a compact subgroup, by averaging trick there exists an inner product on $\mathfrak{g}$ which is also $\mathrm{Ad}(K)$-invariant, and thus it gives a left-invariant metric on M by Lemma 2.4. Moreover, by Corollary 2.10 one has $(\mathrm{d}\widetilde{\sigma})_{eK} = -\mathrm{id}_M$.

Now it suffices to show for any $gK \in M$, $(\mathrm{d}\widetilde{\sigma})_{gK} : T_{gK}M \to T_{\sigma(g)K}M$ is an isometry. Note that $\widetilde{\sigma}(ghK) = \sigma(g)\sigma(h)K = \sigma(g)\widetilde{\sigma}(hK)$ holds for all $h \in G$. This shows $\widetilde{\sigma} \circ L_g = L_{\sigma(g)} \circ \widetilde{\sigma}$, where $L_g : M \to M$ is given by $L_g(hK) = ghK$. By taking differential one has the following communicative diagram

$$\begin{tikzcd} T_{eK}M \arrow[r, "(\mathrm{d}\widetilde{\sigma})_{eK}"] \arrow[d, "(\mathrm{d}L_g)_{eK}"'] & T_{eK}M \arrow[d, "(\mathrm{d}L_{\sigma(g)})_{eK}"] \\ T_{gK}M \arrow[r, "(\mathrm{d}\widetilde{\sigma})_{gK}"] & T_{\sigma(g)K}M \end{tikzcd}$$

Since $(\mathrm{d}L_g)_{eK}, (\mathrm{d}L_{\sigma(g)})_{eK}, (\mathrm{d}\widetilde{\sigma})_{eK}$ are isometries, one has $(\mathrm{d}\widetilde{\sigma})_{gK}$ is also an isometry as desired. □

Remark 2.12. In Theorem 3.4 we will see the curvature tensor of G/K is independent of the choice of the left-invariant metric on it, so here we only care about existence, which is guaranteed by $\mathrm{Ad}(K)$ is compact.

2.C. **Examples of Riemannian symmetric pair.**

Example 2.13. $G = \mathrm{SL}(n,\mathbb{R})$ together with $K = \mathrm{SO}(n)$ gives a Riemannian symmetric pair, where σ is defined by

$$\begin{aligned} \sigma\colon\ \mathrm{SL}(n,\mathbb{R}) &\to \mathrm{SL}(n,\mathbb{R}) \\ g &\mapsto (g^{-1})^T. \end{aligned}$$

Indeed, note that

$$(\mathrm{SL}(n,\mathbb{R}))^\sigma = \mathrm{SO}(n).$$

Thus $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ is a Riemannian symmetric space, and it can be viewed as a generalization of hyperbolic plane $\mathbb{H}^2$, since $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2) \cong \mathbb{H}^2$.

Example 2.14. $G = \mathrm{SO}(n+1)$ together with $K = \mathrm{SO}(n)$ gives a Riemannian symmetric pair, where σ is defined by

$$\begin{aligned} \sigma\colon\ \mathrm{SO}(n+1) &\to \mathrm{SO}(n+1) \\ a &\mapsto I_{1,n} a I_{1,n}^{-1}, \end{aligned}$$

where $I_{1,n} = diag\{-1, 1, \dots, 1\}$. Indeed, a direct computation shows

$$\mathrm{SO}(n+1)^\sigma = \{a \in \mathrm{SO}(n+1) \mid I_{1,n}a = aI_{1,n}\} = \left\{\begin{pmatrix} \pm 1 & 0 \\ 0 & b \end{pmatrix} \in \mathrm{SO}(n+1) \mid b \in \mathrm{O}(n)\right\},$$

which implies $(\mathrm{SO}(n+1)^\sigma)_0 = \mathrm{SO}(n) \subseteq \mathrm{SO}(n+1)$. Thus $S^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$ is a Riemannian symmetric space.

Example 2.15 (compact Grassmannian)**.** Consider the Grassmannian of oriented k-planes in $\mathbb{R}^{k+l}$, denoted by $M = \widehat{Gr}_k(\mathbb{R}^{k+l})$. It's clear that $\mathrm{SO}(k+l)$ acts on M transitively with isotropy group $\mathrm{SO}(k) \times \mathrm{SO}(l)$, and thus $M \cong \mathrm{SO}(k+l)/\mathrm{SO}(k) \times \mathrm{SO}(l)$. Consider the involution

$$\begin{aligned} \sigma\colon\ \mathrm{SO}(k+l) &\to \mathrm{SO}(k+l) \\ a &\mapsto I_{k,l} a I_{k,l}^{-1}, \end{aligned}$$

where $I_{k,l} = diag\{\underbrace{-1, \dots, -1}_{k \text{ times}}, \underbrace{1, \dots, 1}_{l \text{ times}}\}$. A direct computation shows

$$\mathrm{SO}(k+l)^\sigma = S(\mathrm{O}(k) \times \mathrm{O}(l)).$$

Then $(\mathrm{SO}(k+l)^\sigma)_0 = \mathrm{SO}(k)\times\mathrm{SO}(l) \subseteq \mathrm{SO}(k+l)^\sigma$, and thus M is a Riemannian symmetric space, called compact Grassmannian. In particular, $S^n = \widehat{Gr}_1(\mathbb{R}^{n+1})$.

Example 2.16 (hyperbolic Grassmannian)**.** In $\mathbb{R}^{k,l}$ with $k \geq 2, l \geq 1$, let's consider the following quadratic form

$$v^t I_{k,l} w = v^t \begin{pmatrix} I_k & 0 \\ 0 & -I_l \end{pmatrix} w = \sum_{i=1}^{k} v_i w_i - \sum_{j=k+1}^{k+l} v_j w_j.$$

The group of linear transformation X that preserves this quadratic form is denoted by $\mathrm{O}(k,l)$, that is

$$X I_{k,l} X^t = I_{k,l},$$

and $\mathrm{SO}(k,l)$ are those with positive determinant. Now consider set consisting of those oriented k-dimensional subspaces of $\mathbb{R}^{k,l}$ on which quadratic form $I_{k,l}$ are positive definite. This space is called the hyperbolic Grassmannian $M = \widehat{Gr}_k(\mathbb{R}^{k,l})$, which is also an open subset of $\widehat{Gr}_k(\mathbb{R}^{k+l})$. It's clear $G = \mathrm{SO}(k,l)$ acting transitively on M with isotropy group $G_p = \mathrm{SO}(k) \times \mathrm{SO}(l)$. As in Example 2.15 one can also construct an involution σ to show $\widehat{Gr}_k(\mathbb{R}^{k,l})$ is a Riemannian symmetric space.

Example 2.17. Suppose K is a compact connected Lie group. Then $(K \times K, \Delta K)$ is a Riemannian symmetric pair given by σ, where $\sigma : K \times K \to K \times K$ is given by $(x, y) \mapsto (y, x)$, since

$$(K \times K)^{\sigma} = \{(a, a) \mid a \in K\} = \Delta K.$$

Then any compact Lie group is a Riemannian symmetric space.

3. Curvature of Riemannian symmetric space

3.A. **Formulas.** Let (M,g) be a Riemannian symmetric space with isometry group G and isotropy group G_p. On one hand, there is a Cartan decomposition of Lie algebra $\mathfrak{g}$ given by involution $\sigma : G \to G$, that is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m},$$

where $\mathfrak{m} \cong T_pM$ as vector spaces, and $\mathfrak{k}$ is the Lie algebra of isotropy group G_p. On the other hand, by Corollary B.6 there is another decomposition of $\mathfrak{g}$ given by

$$\mathfrak{g} = \mathfrak{k}' \oplus \mathfrak{m}',$$

where

$$\begin{aligned}\mathfrak{k}' &= \{X \in \mathfrak{g} \mid X_p = 0\},\\ \mathfrak{m}' &= \{X \in \mathfrak{g} \mid (\nabla X)_p = 0\}.\end{aligned}$$

In fact, for any complete Riemannian manifold, the following proposition shows $\mathfrak{k} \cong \mathfrak{k}'$, and thus above two Cartan decompositions are exactly the same.

Proposition 3.1. *Let (M,g) be a complete Riemannian manifold with isometry group G and isotropy group G_p. Then the Lie algebra of G_p is*

$$\{X \in \mathfrak{g} \mid X_p = 0\}.$$

Proof. Let $X \in \mathfrak{g}$ with $X_p = 0$ and $\varphi_t : M \to M$ be the flow of X. If we denote $\gamma_p(t) = \varphi_t(p)$, then it suffices to show $\gamma_p(t) \equiv p$. For any smooth function $f : M \to \mathbb{R}$, one has

$$\begin{aligned}\gamma_p'(s)f &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} f\circ\gamma_p(t)\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f\circ\gamma_p(s+t)\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f\circ\varphi_s)(\gamma_p(t))\\ &= X_p(f\circ\varphi_s)\\ &= 0\end{aligned}$$

□

Proposition 3.2. *Let (M,g) be a Riemannian symmetric space and $G = \mathrm{Iso}(M,g)$ with Lie algebra $\mathfrak{g}$. For any $p \in M$, one has Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then for any $S \in \mathfrak{k}$, one has*

$$B(S,S) \leq 0,$$

where B is the Killing form of $\mathfrak{g}$. Moreover, the identity holds if and only if $S = 0$.

Proof. Since a Killing field is determined by X_p and $(\nabla X)_p$, one has elements in $\mathfrak{k}$ are determined by $(\nabla X)_p$, and note that ∇X is a skew-symmetric matrice, so

$$\mathfrak{k} \cong \{(\nabla X)_p \in \mathfrak{so}(T_pM) \mid X \in \mathfrak{k}\}.$$

By using this identification, there is a natural inner product on $\mathfrak{k}$ given by

$$\langle S_1, S_2\rangle = \mathrm{tr}(S_1 S_2^T) = -\mathrm{tr}(S_1 S_2).$$

By adding inner product on $\mathfrak{m}$ obtained from $\mathfrak{m} \cong T_pM$ and the one on $\mathfrak{k}$ constructed as above together, one can construct an inner product on $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is orthogonal. For any $S \in \mathfrak{k}$, we claim with respect to this metric, $\mathrm{ad}(S) : \mathfrak{g} \to \mathfrak{g}$ is skew-symmetric. Indeed, for $X_1, X_2 \in \mathfrak{k}$, one has

$$\begin{aligned}\langle \mathrm{ad}(S)X_1, X_2\rangle &= -\mathrm{tr}((\mathrm{ad}(S)X_1)X_2)\\ &= -\mathrm{tr}((SX_1 - X_1S)X_2)\\ &= \mathrm{tr}(X_1(SX_2 - X_2S))\\ &= -\langle X_1, \mathrm{ad}(S)X_2\rangle.\end{aligned}$$

For $Y_1, Y_2 \in \mathfrak{m}$, since $S_p = 0$ and $(\nabla S)_p$ is skew-symmetric, one has

$$\begin{aligned}\langle \mathrm{ad}(S)Y_1, Y_2\rangle &= \langle \nabla_S Y_1 - \nabla_{Y_1} S, Y_2\rangle\\ &= -\langle \nabla_{Y_1} S, Y_2\rangle\\ &= \langle \nabla_{Y_2} S, Y_1\rangle\\ &= -\langle Y_1, \nabla_S Y_2 - \nabla_{Y_2} S\rangle\\ &= -\langle Y_1, \mathrm{ad}(S)Y_2\rangle.\end{aligned}$$

If $X \in \mathfrak{k}$ and $Y \in \mathfrak{m}$, since $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$, one has

$$\begin{aligned}\langle \mathrm{ad}(S)X, Y\rangle &= 0,\\ \langle X, \mathrm{ad}(S)Y\rangle &= 0.\end{aligned}$$

Similarly one has

$$\begin{aligned}\langle \mathrm{ad}(S)Y, X\rangle &= 0,\\ \langle Y, \mathrm{ad}(S)X\rangle &= 0.\end{aligned}$$

This completes the proof of our claim. Then one has

$$B(S, S) = \mathrm{tr}(\mathrm{ad}(S)\circ\mathrm{ad}(S)) = \sum_i \langle \mathrm{ad}(S)\circ\mathrm{ad}(S)e_i, e_i\rangle = -\sum_i \langle \mathrm{ad}(S)e_i, \mathrm{ad}(S)e_i\rangle \le 0.$$

Moreover, if $B(S, S) = 0$, then $\mathrm{ad}(S) = 0$ and for any $X \in \mathfrak{g}$, one has

$$0 = \mathrm{ad}(S)X = \nabla_S X - \nabla_X S = -\nabla_X S,$$

since $S_p = 0$. This implies $(\nabla S)_p = 0$, and thus $S = 0$. □

Remark 3.3. For $S \in \mathfrak{k}$, the most important part of the proof of $B(S, S) = 0$ if and only if $S = 0$ is $\mathrm{ad}(S) = 0$ if and only if $S = 0$. In other words, $\mathfrak{k} \cap \mathfrak{z} = \{0\}$, where $\mathfrak{z}$ is the center of Lie algebra $\mathfrak{g}$.

Theorem 3.4. *Let (M, g) be a Riemannian symmetric space and $G = \mathrm{Iso}(M, g)$. For any $p \in M$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ with $\mathfrak{m} \cong T_pM$.*

(1) For any $X, Y, Z \in \mathfrak{m}$, there holds

$$\begin{aligned}R(X, Y)Z &= -[Z, [Y, X]],\\ \mathrm{Ric}(Y, Z) &= -\frac{1}{2}B(Y, Z).\end{aligned}$$

(2) If $\mathrm{Ric}(g) = \lambda g$, then for $X, Y \in \mathfrak{m}$, one has

$$2\lambda R(X, Y, Y, X) = -B([X, Y], [X, Y]).$$

Proof. For (1). For any $X, Y, Z \in \mathfrak{m}$, direct computation shows

$$\begin{aligned} R(X,Y)Z &\overset{(a)}{=} R(X,Z)Y - R(Y,Z)X \\ &\overset{(b)}{=} \nabla_Z \nabla_Y X - \nabla_{\nabla_Z Y} X - \nabla_Z \nabla_X Y + \nabla_{\nabla_Z X} Y \\ &\overset{(c)}{=} -\nabla_Z [X,Y] \\ &\overset{(d)}{=} -[Z[X,Y]], \end{aligned}$$

where

(a) holds from the first Bianchi identity.
(b) holds from (2) of Proposition B.1.
(c) holds from $X, Y \in \mathfrak{m}$, and thus $(\nabla X)_p = (\nabla Y)_p = 0$.
(d) holds from

$$\nabla_Z [X,Y] - \nabla_{[X,Y]} Z = [Z,[X,Y]],$$

and $(\nabla Z)_p = 0$.

To see Ricci curvature, note that for $Y \in \mathfrak{m}$,

$$\operatorname{ad}(Y) : \mathfrak{k} \to \mathfrak{m}, \quad \operatorname{ad}(Y) : \mathfrak{m} \to \mathfrak{k}.$$

Thus if $Y, Z \in \mathfrak{m}$, one has $\operatorname{ad}(Z) \circ \operatorname{ad}(Y)$ preserves the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then

$$\begin{aligned} \operatorname{tr}(\operatorname{ad}(Z) \circ \operatorname{ad}(Y)|_{\mathfrak{m}}) &= \operatorname{tr}(\operatorname{ad}(Z)|_{\mathfrak{k}} \circ \operatorname{ad}(Y)|_{\mathfrak{m}}) \\ &= \operatorname{tr}(\operatorname{ad}(Y)|_{\mathfrak{m}} \circ \operatorname{ad}(Z)|_{\mathfrak{k}}) \\ &= \operatorname{tr}(\operatorname{ad}(Y) \circ \operatorname{ad}(Z)|_{\mathfrak{k}}). \end{aligned}$$

Hence we obtain

$$B(Y,Y) = \operatorname{tr}(\operatorname{ad}(Y) \circ \operatorname{ad} Y|_{\mathfrak{k}}) + \operatorname{tr}(\operatorname{ad}(Y) \circ \operatorname{ad} Y|_{\mathfrak{m}}) = 2\operatorname{tr}(\operatorname{ad}(Y) \circ \operatorname{ad}(Y)|_{\mathfrak{m}}).$$

Since Ricci tensor is trace of curvature tensor, and thus

$$\operatorname{Ric}(Y,Y) = -\operatorname{tr}(\operatorname{ad}(Y) \circ \operatorname{ad}(Y)|_{\mathfrak{m}}) = -\frac{1}{2} B(Y,Y).$$

Then by using polarization identity, one has $\operatorname{Ric}(Y,Z) = -B(Y,Z)/2$.

For (2). If $\operatorname{Ric}(g) = \lambda g$, then

$$\begin{aligned} 2\lambda g(R(X,Y)Y,X) &= -2\lambda g(\operatorname{ad}(Y) \circ \operatorname{ad}(Y)X, X) \\ &= -2\operatorname{Ric}(\operatorname{ad}(Y) \circ \operatorname{ad}(Y)X, X) \\ &= B(\operatorname{ad}(Y) \circ \operatorname{ad}(Y)X, X) \\ &= -B(\operatorname{ad}(Y)X, \operatorname{ad}(Y)X) \\ &= -B([X,Y],[X,Y]). \end{aligned}$$

□

Corollary 3.5. *Let (M,g) be a Riemannian symmetric space which is an Einstein manifold with Einstein constant λ. Then*

(1) If $\lambda > 0$, then (M,g) has non-negative sectional curvature.
(2) If $\lambda < 0$, then (M,g) has non-positive sectional curvature.
(3) If $\lambda = 0$, then (M,g) is flat.

Proof. By Theorem 3.4 one has

$$2\lambda R(X,Y,Y,X) = -B([X,Y],[X,Y]) \geq 0,$$

since $[X,Y] \in [\mathfrak{m},\mathfrak{m}] \subseteq \mathfrak{k}$ and B is negative definite on $\mathfrak{k}$. This shows (1) and (2). If $\lambda = 0$, one has $B([X,Y],[X,Y]) \equiv 0$ for arbitrary X,Y. Then by Proposition 3.2 one has $[X,Y] \equiv 0$ for arbitrary X,Y, and thus (M,g) is flat. □

3.B. **Computations.**

Example 3.6. In Example 2.13 we have already shown that $M = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ is a Riemannian symmetric space. Consider its Cartan decomposition

$$\mathfrak{sl}(n) = \mathfrak{so}(n) \oplus \mathfrak{m},$$

where $\mathfrak{m}$ consists of symmetric matrices and $\mathfrak{m} \cong T_pM$ for $p \in M$. On $\mathfrak{m}$ we can put the usual Euclidean metric, that is for $X,Y \in \mathfrak{m}$, we define

$$\langle X,Y\rangle = \mathrm{tr}(XY^T) = \mathrm{tr}(XY) = \frac{1}{2n}B(X,Y),$$

where B is the Killing form of $\mathfrak{sl}(n)$. By Theorem 3.4 the corresponding metric on M has the curvature formulas

$$\begin{aligned}\mathrm{Ric}(g) &= -\frac{B}{2} = -ng,\\ R(X,Y,Y,X) &= \frac{B([X,Y],[X,Y])}{2n} \leq 0.\end{aligned}$$

Hence it has non-positive sectional curvatures. One can also show its sectional curvature is non-positive by computing curvature tensor as follows

$$\begin{aligned}R(X,Y,Z,W) &= \mathrm{tr}([Z,[X,Y]]W)\\ &= \mathrm{tr}(Z[X,Y]W - [X,Y]ZW)\\ &= \mathrm{tr}(WZ[X,Y] - [X,Y]ZW)\\ &= \mathrm{tr}([X,Y][Z,W])\\ &= -\mathrm{tr}([X,Y][Z,W]^T)\\ &= -\langle [X,Y],[Z,W]\rangle.\end{aligned}$$

Example 3.7 (compact Grassmannian)**.** In Example 2.15 we have already shown that $M = \widehat{Gr}_k(\mathbb{R}^{k+l})$ is a Riemannian symmetric space with Cartan decomposition

$$\mathfrak{so}(k+l) = \mathfrak{so}(k) \oplus \mathfrak{so}(l) \oplus \mathfrak{m},$$

where $\mathfrak{m} \cong T_pM$ for $p \in M$. Note that one has the block decomposition of matrices in $\mathfrak{so}(k+l)$ as follows

$$\mathfrak{so}(k+l) = \left\{ \begin{pmatrix} X_1 & B \\ -B^T & X_2 \end{pmatrix} \mid X_1 \in \mathfrak{so}(k), X_2 \in \mathfrak{so}(l), B \in M_{k\times l}(\mathbb{R}) \right\}.$$

Then one has $\mathfrak{m} \cong \left\{ \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} \mid B \in M_{k\times l}(\mathbb{R}) \right\}$. If we put the usual Euclidean metric on $\mathfrak{m}$, that is

$$\begin{aligned}
\left\langle \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} \right\rangle &= \operatorname{tr}\left(\begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}^T \right) \\
&= -\operatorname{tr}\left(\begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \right) \\
&= -\frac{1}{k+l-2} B\left(\begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} \right),
\end{aligned}$$

where B is the Killing form of $\mathfrak{so}(n)$. Then the corresponding metric on M has the curvature formulas

$$\begin{aligned}
\operatorname{Ric}(g) &= -\frac{B}{2} = \frac{k+l-2}{2} g, \\
R(X,Y,Y,X) &= -\frac{B([X,Y],[X,Y])}{k+l-2} \geq 0,
\end{aligned}$$

where $X, Y \in \mathfrak{m}$. This shows the compact Grassmannian has the non-negative sectional curvature.

Example 3.8 (hyperbolic Grassmannian)**.** In Example 2.16 we have already shown that $M = \widehat{Gr}_k(\mathbb{R}^{k,l})$ is a Riemannian symmetric space with Cartan decomposition

$$\mathfrak{so}(k,l) = \mathfrak{so}(k) \oplus \mathfrak{so}(l) \oplus \mathfrak{m},$$

where $\mathfrak{m} \cong T_pM$ for $p \in M$. Note that one has the block decomposition of matrices in $\mathfrak{so}(k,l)$ as follows

$$\mathfrak{so}(k,l) = \left\{ \begin{pmatrix} X_1 & B \\ B^T & X_2 \end{pmatrix} \mid X_1 \in \mathfrak{so}(k), X_2 \in \mathfrak{so}(l), B \in M_{k\times l}(\mathbb{R}) \right\}.$$

Then one has $\mathfrak{m} \cong \left\{ \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \mid B \in M_{k\times l}(\mathbb{R}) \right\}$. If we put the usual Euclidean metric on $\mathfrak{m}$, then

$$\left\langle \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \right\rangle = \frac{1}{k+l-2} B\left(\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \right),$$

where B is the Killing form of $\mathfrak{so}(k,l)$. Then the corresponding metric on M has the curvature formulas

$$\begin{aligned}
\operatorname{Ric}(g) &= -\frac{B}{2} = -\frac{k+l-2}{2} g, \\
R(X,Y,Y,X) &= \frac{B([X,Y],[X,Y])}{k+l-2} \leq 0,
\end{aligned}$$

where $X, Y \in \mathfrak{m}$. This shows the hyperbolic Grassmannian has non-positive sectional curvature.

Remark 3.9. Later we will see compact Grassmannian and hyperbolic Grassmannian are dual to each other in Example 6.5.

Example 3.10. In Example 2.17 one has a compact connected Lie group $G \cong G \times G/G^{\Delta}$ is a Riemannian symmetric space with Cartan decomposition $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}^{\Delta} \oplus \mathfrak{g}^{\perp}$, where

$$\mathfrak{g}^{\Delta} = \{(X, X) \mid X \in \mathfrak{g}\},$$
$$\mathfrak{g}^{\perp} = \{(X, -X) \mid X \in \mathfrak{g}\}.$$

Then one has $\mathfrak{m} \cong \mathfrak{g}^{\perp}$, and thus curvature tensor can be computed as follows

$$\begin{aligned} R(X,Y)Z &= R((X,-X),(Y,-Y))(Z,-Z) \\ &= [(Z,-Z),[(X,-X),(Y,-Y)]] \\ &= ([Z,[X,Y]], -[Z,[X,Y]]). \end{aligned}$$

Hence, we arrive at that the formula

$$R(X,Y)Z = [Z,[X,Y]].$$

Remark 3.11. If one computes the curvature tensor in the standard way using bi-invariant metric, then the formula has a factor $1/4$ on it.

Part 2. Classifications

4. DECOMPOSITIONS

So far, we have seen that any Riemannian symmetric space (M, g) gives a Riemannian symmetric pair (G, K) with involution σ, and any Riemannian symmetric pair gives a pair $(\mathfrak{g}, s)$ of Lie algebra $\mathfrak{g}$ and involution s of $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. In this section, we will study such pairs of Lie algebras and prove decomposition theorems, which will give decomposition theorems for symmetric spaces.

4.A. **Orthogonal symmetric Lie algebra.**

4.A.1. *Basic definitions.*

Definition 4.1 (orthogonal symmetric Lie algebra)**.** *An orthogonal symmetric Lie algebra is a pair $(\mathfrak{g}, s)$ consisting of a real Lie algebra $\mathfrak{g}$ and an involution $s \neq \mathrm{id}$ of $\mathfrak{g}$ such that $\mathfrak{k}$ is a compactly embedded subalgebra*[3]*, where $\mathfrak{k}$ is given by Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$.*

Remark 4.2. For an orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$, the term "orthogonal" is motivated by the fact that Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an orthogonal direct sum with respect to the Killing form of $\mathfrak{g}$.

Example 4.3. Let (G, K) be a Riemannian symmetric pair given by involution σ. Then it gives an orthogonal symmetric pair $(\mathfrak{g}, s)$, where $\mathfrak{g} = \operatorname{Lie} G$ and $s = (\mathrm{d}\sigma)_e$.

Definition 4.4 (isomorphism)**.** *Two orthogonal symmetric Lie algebra $(\mathfrak{g}_1, s_1), (\mathfrak{g}_2, s_2)$ are called isomorphic to each other, if there exists a Lie algebra isomorphism $\rho\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $s_2 \circ \rho = \rho \circ s_1$.*

Definition 4.5 (effective)**.** *Let $(\mathfrak{g}, s)$ be an orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. It's called effective if $\mathfrak{k} \cap \mathfrak{z} = 0$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$.*

Remark 4.6. Let's try to explain the motivation we define "effective" like this: Let (G, K) be a Riemannian symmetric pair associated to a Riemannian symmetric space (M, g). Note that G acts on M effectively, and thus we claim K contains no non-zero subgroup of G. Otherwise if N is a normal subgroup of G contained in K, it suffices to show for any $n \in N$, it fixes every point of M since G acts on M effectively. For any $q \in M$, suppose $q = gp$ for some $g \in G$ and hence

$$nq = ngp = g(g^{-1}ng)p = gp = q.$$

In particular, $Z(G) \cap K = \{e\}$, and thus $\mathfrak{k} \cap \mathfrak{z} = 0$. As a consequence, if (G, K) is a Riemannian symmetric pair associated to a Riemannian symmetric space and $(\mathfrak{g}, s)$ is the orthogonal symmetric Lie algebra given by (G, K), then it's effective.

Proposition 4.7. *Let $(\mathfrak{g}, s)$ be an effective orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then the Killing form of $\mathfrak{g}$ is negative definite on $\mathfrak{k}$.*

Proof. Let B be the Killing form of $\mathfrak{g}$ and $K \subseteq \mathrm{GL}(\mathfrak{g})$ be the compact Lie group such that $\operatorname{Lie} K = \mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$. Without lose of generality we may assume $K \leq \mathrm{SO}(n)$, and thus $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$

[3]See Definition A.7

consisting of skew-symmetric matrices. Hence for $S \in \mathfrak{k}$,

$$B(S,S) = \operatorname{tr}(\operatorname{ad}(S)\circ\operatorname{ad}(S)) = \sum_i \langle \operatorname{ad}(S)\circ\operatorname{ad}(S)e_i, e_i\rangle = -\sum_i \langle \operatorname{ad}(S)e_i, \operatorname{ad}(S)e_i\rangle \le 0,$$

and the equality holds if and only if $S \in \mathfrak{z} \cap \mathfrak{k} = 0$. $\square$

4.A.2. *Relations between Riemannian symmetric space, Riemannian symmetric pair and orthogonal symmetric Lie algebra.* Untill now, we have encountered three categories listed as follows:

(1) Riemannian symmetric space.
(2) Riemannian symmetric pair.
(3) Orthogonal symmetric Lie algebra.

Theorem 4.8. *Every orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ gives a Riemannian symmetric pair (G, K) with G/K simply-connected.*

Proof. By Theorem A.1 there exists a unique connected and simply-connected Lie group $\widetilde{G}$ such that $\operatorname{Lie}\widetilde{G} = \mathfrak{g}$ and there also exists a unique connected Lie subgroup $\widetilde{K} \subseteq \widetilde{G}$ with Lie algebra $\mathfrak{k}$ by Theorem A.2. Moreover, by Theorem A.3 there exists a unique involution $\sigma: \widetilde{G} \to \widetilde{G}$ such that $(d\sigma)_e = s$, and $\widetilde{K}$ is the identity component of $\widetilde{G}_\sigma$. Then $(\widetilde{G}, \widetilde{K})$ is the Riemannian symmetric pair given by σ. To see $M = \widetilde{G}/\widetilde{K}$ is a simply-connected Riemannian symmetric space, we consider the exact sequence

$$0 \to \widetilde{K} \to \widetilde{G} \to M \to 0,$$

which gives a long exact sequence of homotopy groups as

$$\cdots \to \pi_1(\widetilde{G}) \to \pi_1(M) \to \pi_0(\widetilde{K}) \to \cdots$$

Since $\widetilde{K}$ is connected and $\widetilde{G}$ is simply-connected, M is simply-connected as desired. $\square$

All in all, relations we have known is shown in the following diagram

$$\begin{tikzcd}
(1) \arrow[r, "\text{Theorem 2.5}"'] & (2) \arrow[l, bend right, "\text{Theorem 2.11}"'] \arrow[r, "\text{Example 4.3}"'] & (3) \arrow[l, bend right, "\text{Theorem 4.8}"']
\end{tikzcd}$$

Remark 4.9. Given a Riemannian symmetric space (M, g), there is a Riemannian symmetric pair (G, K), and thus we obtain an orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$. If we use $\widetilde{M} = \widetilde{G}/\widetilde{K}$ to denote the Riemannian symmetric space given by $(\mathfrak{g}, s)$, a natural question is what's the relationship between M and $\widetilde{M}$? Since $\widetilde{G}$ is simply-connected and has the same Lie algebra as G, there exists a covering map $p: \widetilde{G} \to G$. Moreover, since $\widetilde{K}$ and K also have the same Lie algebra, $p(\widetilde{K}) = K$ and thus p induces a covering map $\overline{p}: \widetilde{M} \to M$, which gives an isomorphism $T_{\widetilde{p}}\widetilde{M} \cong \mathfrak{m} \cong T_pM$. If we endow $\widetilde{M}$ with Riemannian metric obtained from $\overline{p}$, then $\overline{p}$ is a local isometry, and thus $\overline{p}$ is a Riemannian covering since $\widetilde{M}$ is complete. In particular, if M is simply-connected, then it's isometric to $\widetilde{M}$.

4.B. **Decomposition into pieces of different types.**

4.B.1. *Types.*

Definition 4.10 (types). *Let $(\mathfrak{g}, s)$ be an effective orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and Killing form B. Then $(\mathfrak{g}, s)$ is of*

(1) of compact type if $B|_{\mathfrak{m}} < 0$;
(2) of non-compact type if $B|_{\mathfrak{m}} > 0$;
(3) of Euclidean type if $B|_{\mathfrak{m}} = 0$;
(4) of semisimple type if $\mathfrak{g}$ is semisimple, or equivalently, B is non-degenerate.

Definition 4.11 (types).

(1) A Riemannian symmetric pair is of one of above types if its corresponding orthogonal symmetric Lie algebra is.
(2) A Riemannian symmetric space is of one of above types if its corresponding Riemannian symmetric pair is.

Proposition 4.12. *Let $(\mathfrak{g}, s)$ be an effective orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. It's of Euclidean type if and only if $[\mathfrak{m}, \mathfrak{m}] = 0$.*

Proof. If $(\mathfrak{g}, s)$ is of Euclidean type, then $B(\mathfrak{k}, \mathfrak{m}) = 0$ and $B|_{\mathfrak{k}} < 0$ implies $\mathfrak{m}$ is the kernel of Killing form B, and thus $\mathfrak{m}$ is an ideal. Then

$$[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{m} \cap \mathfrak{k} = 0.$$

Conversely, if $[\mathfrak{m}, \mathfrak{m}] = 0$, then by definition of Killing form it's clear $B|_{\mathfrak{m}} = 0$. □

Proposition 4.13. *Let (G, K) be a Riemannian symmetric pair and $M = G/K$.*

(1) If (G, K) is of compact type, then M has non-negative sectional curvature.
(2) If (G, K) is of non-compact type, then M has non-positive sectional curvatures.
(3) If (G, K) is of Euclidean type, then M is flat[4]. In particular, if M is simply-connected, then it's isometric to $\mathbb{R}^n$.

Proof. If (G, K) is of compact type, we may assume $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{m}$ is given by $-B|_{\mathfrak{m}}$, and thus by 3.4 one has

$$\mathrm{Ric} = -\frac{1}{2}B.$$

This shows M is Einstein with Einstein constant $1/2$, and thus by Corollary 3.5 one has M has non-negative sectional curvature. Similarly one can show if (G, K) is of non-compact type, then M has non-positive sectional curvatures, and (G, K) is of Euclidean type, then M is flat. □

4.B.2. *Decomposition of effective orthogonal symmetric Lie algebra.*

Theorem 4.14. *Let $(\mathfrak{g}, s)$ be an effective orthogonal symmetric Lie algebra and B be the Killing form of $\mathfrak{g}$. Then there exists ideals $\mathfrak{g}_0$, $\mathfrak{g}_-$ and $\mathfrak{g}_+$ with the following properties:*

(1) $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_- \oplus \mathfrak{g}_+$.
(2) $\mathfrak{g}_0$, $\mathfrak{g}_-$ and $\mathfrak{g}_+$ are invariant under s and orthogonal with respect to Killing form B of $\mathfrak{g}$.

[4] A Riemannian manifold is called flat, if its sectional curvatures are zero.

(3) *Let s_0, s_-, s_+ be the restrictions of s to $\mathfrak{g}_0, \mathfrak{g}_-$ and $\mathfrak{g}_+$. The pairs $(\mathfrak{g}_0, s_0), (\mathfrak{g}_-, s_-)$ and $(\mathfrak{g}_+, s_+)$ are effective orthogonal symmetric Lie algebras of the Euclidean type, compact type and non-compact type, respectively.*

Proof. See Theorem 1.1 in Chapter V of [Hel78]. $\square$

4.B.3. *Decomposition of Riemannian symmetric space.*

Theorem 4.15. *Let (M, g) be a simply-connected symmetric space. Then $M = M_0 \times M_+ \times M_-$ is the Riemannian product of symmetric space of Euclidean, non-compact and compact types respectively.*

Proof. Let (G, K) with involution σ be the Riemannian symmetric pair given by (M, g) and $(\mathfrak{g}, s)$ be the corresponding effective orthogonal symmetric Lie algebra. Let $p : \widetilde{G} \to G$ be the universal covering and $\widetilde{K}$ be the identity component of $p^{-1}(K)$. Then it induces a covering map of $\overline{p} : \widetilde{G}/\widetilde{K} \to G/K$ by $g\widetilde{K} \to \varphi g\widetilde{K}$. Since M is simply-connected, $M = \widetilde{G}/\widetilde{K}$.

By Theorem 4.14, we obtain a decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_- \oplus \mathfrak{g}_+$. By Theorem A.1, there exists simply-connected Lie groups G_0, G_- and G_+ with Lie algebras $\mathfrak{g}_0, \mathfrak{g}_-$ and $\mathfrak{g}_+$. Then it gives a decomposition $\widetilde{G} = G_0 \times G_- \times G_+$. If $\widetilde{K} = K_0 \times K_- \times K_+$ is the corresponding decomposition, then the spaces $M_0 = G_0/K_0, M_- = G_-/K_-$ and $M_+ = G_+/K_+$ gives the desired decomposition. $\square$

5. Irreducibility

5.A. **Irreducible orthogonal symmetric Lie algebra.**

Definition 5.1 (irreducible)**.** *Suppose $(\mathfrak{g}, s)$ is an orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then $(\mathfrak{g}, s)$ is called irreducible if*

(1) $\mathfrak{g}$ is semisimple and $\mathfrak{k}$ contains no ideal of $\mathfrak{g}$;
(2) the Lie algebra $\mathrm{ad}(\mathfrak{k})$ acts irreducibly on $\mathfrak{m}$.

Remark 5.2. Any irreducible orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$ is effective, since $\mathfrak{z} \cap \mathfrak{k}$ is an ideal in $\mathfrak{k}$, and thus vanishes.

Definition 5.3 (irreducible)**.**

(1) A Riemannian symmetric pair is called irreducible if its corresponding orthogonal symmetric Lie algebra is.
(2) A Riemannian symmetric space is called irreducible if its corresponding Riemannian symmetric pair is.

Lemma 5.4 (Schur lemma)**.** *Let B_1, B_2 be two symmetric bilinear forms on a vector space V such that B_1 is positive definite. If a group K acts irreducibly on V such that B_1 and B_2 are invariant under K, then $B_2 = \lambda B_1$ for some constant λ.*

Proof. Since B_1 is positive definite, there exists an endomorphism $L : V \to V$ such that

$$B_2(u, v) = B_1(Lu, v),$$

where $u, v \in V$. Since B_1, B_2 are invariant under K, one has for any $k \in K$

$$B_1(kLu, v) = B_1(Lu, k^{-1}v) = B_2(u, k^{-1}v) = B_2(ku, v) = B_1(Lku, v),$$

holds for arbitrary $u, v \in V$, which implies $Lk = kL$ for all $k \in K$. On the other hand, the symmetry of B_1, B_2 implies

$$B_1(Lu, v) = B_2(u, v) = B_2(v, u) = B_1(Lv, u) = B_1(u, Lv).$$

Hence L is symmetric with respect to B_1, and thus the eigenvalues of L are real. If $0 \neq E \subseteq V$ is an eigenspace with eigenvalue λ, the fact $kL = Lk$ implies E is invariant under K. Since K acts irreducibly on V, one has $E = V$, that is $L = \lambda I$, which implies $B_2 = \lambda B_1$. □

Proposition 5.5. *Let (G, K) be an irreducible Riemannian symmetric pair given by σ. Then there is up to scaling a unique left-invariant metric on $M = G/K$.*

Proof. It suffices to show there is up to scaling a unique $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{m}$. Since (G, K) is an irreducible Riemannian symmetric pair, then K acts on $\mathfrak{m}$ irreducibly by adjoint representation, and thus by Lemma 5.4 any two $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{m}$ are scalar multiples of each other. In particular, $-B|_{\mathfrak{m}}$ and $B|_{\mathfrak{m}}$ give such an inner product in compact and non-compact cases respectively. □

5.B. **Decomposition into irreducible pieces.**

Theorem 5.6. *Let $(\mathfrak{g}, s)$ be an effective orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ such that $\mathfrak{g}$ is semisimple and $\mathfrak{k}$ does not contain an ideal of $\mathfrak{g}$. Then there are ideals $(\mathfrak{g}_i)_{i \in I}$ of $\mathfrak{g}$ such that*

(1) $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$.
(2) *The ideals* $\mathfrak{g}_i$ *are mutually orthogonal with respect to Killing form* B *of* $\mathfrak{g}$*, and they are invariant under* s*.*
(3) *Denoting by* s_i *the restriction if* s *to* $\mathfrak{g}_i$*, each* $(\mathfrak{g}_i, s_i)$ *is an irreducible orthogonal symmetric Lie algebra.*

Proof. See Proposition 5.2 in Chapter VIII of [Hel78]. □

As Theorem 4.15, this decomposition of effective orthogonal symmetric Lie algebra gives a decomposition of Riemannian symmetric space as follows.

Theorem 5.7. *Let* (M, g) *be a simply-connected Riemannian symmetric space. Then* M *is a product*

$$(M, g) \cong (M_0, g_0) \times (M_1, g_1) \times \cdots \times (M_n, g_n),$$

where (M_0, g_0) *is a Riemannian symmetric space of Euclidean type and for* $i \geq 1$*, the factors* (M_i, g_i) *are irreducible Riemannian symmetric spaces.*

Proof. See Proposition 5.5 in Chapter VIII of [Hel78]. □

5.C. **Classifications of irreducible orthogonal symmetric Lie algebra.**

Theorem 5.8. *Let* $(\mathfrak{g}, s)$ *be an effective semisimple orthogonal symmetric Lie algebra. Then it must be isomorphic to one of the following four cases:*

CI $\mathfrak{g}$ *is a compact simple Lie algebra and* s *is an involution of* $\mathfrak{g}$*;*
CII $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$*, where* $\mathfrak{g}_1, \mathfrak{g}_2$ *are compact simple Lie algebra and* $s(X, Y) = (Y, X)$*;*
NI $\mathfrak{g}$ *is a non-compact simple Lie algebra such that its complexification* $\mathfrak{g}_{\mathbb{C}}$ *is a complex simple Lie algebra, and* s *is an involution of* $\mathfrak{g}$*.*
NII $\mathfrak{g}$ *is a non-compact simple Lie algebra such that its complexification* $\mathfrak{g}_{\mathbb{C}}$ *is not a complex simple Lie algebra, and* s *is an involution of* $\mathfrak{g}$*.*

Theorem 5.9. *Let* $(\mathfrak{g}, s)$ *be an effective semisimple orthogonal symmetric Lie algebra of non-compact type. Then Riemannian symmetric space arisen from* $(\mathfrak{g}, s)$ *is unique up to isometry, the center of* $(\mathrm{Iso}(M, g))_0 = \{e\}$ *and* M *is simply-connected.*

6. DUALITY

Let $\mathfrak{g}$ be a real Lie algebra. Then its complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ is a complex Lie algebra, with Lie bracket

$$[X_1 + \sqrt{-1}Y_1, X_2 + \sqrt{-1}Y_2] := [X_1, X_2] - [Y_1, Y_2] + \sqrt{-1}\,([Y_1, X_2] + [X_1, Y_2])$$

Definition 6.1 (real form)**.** *Let $\mathfrak{h}$ be a complex Lie algebra. A real form of $\mathfrak{h}$ is a real Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}_{\mathbb{C}}$ is isomorphic to $\mathfrak{h}$ as complex Lie algebras.*

Remark 6.2. It's clear a real Lie algebra is a real form of its complexification but in general there are many pairwise non-isomorphic real forms.

Now suppose $(\mathfrak{g}, s)$ is an orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then there are following bracketing relations:
(1) $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$.
(2) $[\mathfrak{k}, \sqrt{-1}\mathfrak{m}] = \sqrt{-1}[\mathfrak{k}, \mathfrak{m}] \subseteq \sqrt{-1}\mathfrak{m}$.
(3) $[\sqrt{-1}\mathfrak{m}, \sqrt{-1}\mathfrak{m}] = -[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$.
In particular, $\mathfrak{g}^* := \mathfrak{k} \oplus \sqrt{-1}\mathfrak{m}$ is a real Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let $s_{\mathbb{C}}$ be the $\mathbb{C}$-linear extension of s to $\mathfrak{g}_{\mathbb{C}}$ and s^* be the restriction of $s_{\mathbb{C}}$ to $\mathfrak{g}^*$. Then $(\mathfrak{g}^*, s^*)$ is also an orthogonal symmetric Lie algebra, which is defined to be the dual of $(\mathfrak{g}, s)$.

Theorem 6.3. *Let $(\mathfrak{g}, s)$ be an orthogonal symmetric Lie algebra with dual $(\mathfrak{g}^*, s^*)$.*
(1) If $(\mathfrak{g}, s)$ is of compact type, then $(\mathfrak{g}^*, s^*)$ is of non-compact type, and vice versa.
(2) If $(\mathfrak{g}, s)$ is of Euclidean type, then $(\mathfrak{g}^*, s^*)$ is of Euclidean type.
(3) $(\mathfrak{g}, s)$ is irreducible if and only if $(\mathfrak{g}^*, s^*)$ is irreducible.

Proof. For (1) and (2). It suffices to establish a relation between the respective Killing forms. Note that there is an isomorphism of vector spaces $\Psi : \mathfrak{g} \to \mathfrak{g}^*$ given by $X + Y \mapsto X + \sqrt{-1}Y$. For $Z_1, Z_2 \in \mathfrak{m}$, a direct computation shows

$$\begin{aligned}
\mathrm{ad}_{\mathfrak{g}^*}(\sqrt{-1}Z_1)\mathrm{ad}_{\mathfrak{g}^*}(\sqrt{-1}Z_2)(X + \sqrt{-1}Y) &= \left[\sqrt{-1}Z_1, [\sqrt{-1}Z_2, X + \sqrt{-1}Y]\right] \\
&= -\,[Z_1, [Z_2, X]] - \sqrt{-1}\,[Z_1, [Z_2, Y]] \\
&= -\Psi([Z_1, [Z_2, X + Y]]) \\
&= -\Psi(\mathrm{ad}_{\mathfrak{g}}(Z_1)\mathrm{ad}_{\mathfrak{g}}(Z_2)(X + Y)).
\end{aligned}$$

Therefore $B_{\mathfrak{g}^*}(\sqrt{-1}Z_1, \sqrt{-1}Z_2) = -B_{\mathfrak{g}}(Z_1, Z_2)$. As a consequence, $B_{\mathfrak{g}}|_{\mathfrak{m}} > 0$ if and only if $B_{\mathfrak{g}^*}|_{\sqrt{-1}\mathfrak{m}} < 0$ and vice versa.

For (3). Note that $\mathfrak{g}$ is semisimple if and only if its Killing form is non-degenerate, so $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g}^*$ is, and thus $(\mathfrak{g}, s)$ is irreducible if and only if $(\mathfrak{g}^*, s^*)$ is irreducible. □

6..1. *Examples of duality.*

Example 6.4. Consider the orthogonal symmetric Lie algebra $(\mathfrak{sl}(n, \mathbb{R}), s)$, where $s : X \mapsto -X^T$. Its Cartan decomposition is given by

$$\mathfrak{k} = \{X \in \mathfrak{sl}(n, \mathbb{R}) \mid X^T + X = 0\},$$
$$\mathfrak{m} = \{X \in \mathfrak{sl}(n, \mathbb{R}) \mid X^T = X\}.$$

Then $\mathfrak{sl}(n,\mathbb{R})_{\mathbb{C}} = \mathfrak{sl}(n,\mathbb{C})$ and

$$\begin{aligned}\mathfrak{k} + \sqrt{-1}\mathfrak{m} &= \{Z \in \mathfrak{sl}(n,\mathbb{C}) \mid Z = X + \sqrt{-1}Y, X^T + X = 0, Y^T = Y\}\\ &= \{Z \in \mathfrak{sl}(n,\mathbb{C}) \mid Z + \overline{Z}^T = 0\}\\ &= \mathfrak{su}(n).\end{aligned}$$

As a consequence, the Riemannian symmetric space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ and $\mathrm{SU}(n)/\mathrm{SO}(n)$ are dual to each other. For $n = 2$, one has $\mathbb{H}^2$ is dual to S^2, since $\mathrm{SU}(2)$ is the universal covering of $\mathrm{SO}(3)$.

Example 6.5. Consider the orthogonal symmetric Lie algebra $(\mathfrak{so}(n), s)$, where s is given by

$$\begin{aligned} s: \mathfrak{so}(n) &\to \mathfrak{so}(n)\\ X &\mapsto I_{k,l} X I_{k,l}\end{aligned}$$

where $k + l = n$. Its Cartan decomposition is given by

$$\mathfrak{so}(n) = \mathfrak{so}(k) \oplus \mathfrak{so}(l) \oplus \mathfrak{m},$$

where

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} \mid B \in M_{k\times l}(\mathbb{R}) \right\}.$$

Then

$$\mathfrak{g}^* = \left\{ \begin{pmatrix} X_1 & \sqrt{-1}B \\ -\sqrt{-1}B^T & X_2 \end{pmatrix} \mid X_1 \in \mathfrak{so}(k), X_2 \in \mathfrak{so}(l), B \in M_{k\times l}(\mathbb{R}) \right\}.$$

It's easy to verify the mapping

$$\begin{pmatrix} X_1 & \sqrt{-1}B \\ -\sqrt{-1}B^T & X_2 \end{pmatrix} = \begin{pmatrix} X_1 & B \\ B^T & X_2 \end{pmatrix}$$

is a Lie algebra isomorphism of $\mathfrak{g}^*$ to $\mathfrak{so}(p,q)$. This shows compact Grassmannian and hyperbolic Grassmannian are dual to each other.

Part 3. Hermitian symmetric space

7. Hermitian symmetric space

Definition 7.1 (Hermitian symmetric space)**.** *Let (M, g) be a Riemannian symmetric space. (M, g) is said to be a Hermitian symmetric manifold if (M, g) is a Hermitian manifold and the symmetric at each point is a holomorphic isometry.*

Lemma 7.2. *Any almost Hermitian structure on a Riemannian symmetric space (M, g) is integrable, and any Hermitian symmetric space is Kähler.*

Proof. Suppose φ is the symmetry at point $p \in M$ and J is an almost Hermitian structure of (M, g). Since φ is a holomorphic isometry one has $(\mathrm{d}\varphi)_p \circ J = J \circ (\mathrm{d}\varphi)_p$, and thus

$$\begin{aligned}
-N_J(X, Y) &= (\mathrm{d}\varphi)_p N_J(X, Y) \\
&= (\mathrm{d}\varphi)_p \left([JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]\right) \\
&= [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] \\
&= N_J(X, Y).
\end{aligned}$$

This shows $N_J = 0$ at point p, and since p is arbitrary one has $N_J \equiv 0$, which implies J is integrable. By the same argument one can show $\nabla J = 0$, and thus (M, g) is Kähler. □

Proposition 7.3. *Let (G, K) be a symmetric pair with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. If $J : \mathfrak{m} \to \mathfrak{m}$ satisfies*

(1) J is orthogonal and $J^2 = -\mathrm{id}$.

(2) $J \circ \mathrm{Ad}(k) = \mathrm{Ad}(k) \circ J$ for all $k \in K$.

Then $M = G/K$ is a Hermitian symmetric space, and thus Kähler.

Corollary 7.4. *Let (G, K) be a symmetric pair. Then*

(1) (G, K) is Hermitian symmetric if and only if its dual is Hermitian symmetric.

(2) If (G, K) is irreducible and Hermitian symmetric, then it's Kähler-Einstein.

Proposition 7.5. *Let (G, K) be an irreducible symmetric pair.*

(1) If (G, K) is of compact type, then it's Hermitian symmetric if and only if $H^2(M, \mathbb{R}) \neq 0$.

(2) (G, K) is Hermitian symmetric if and only if K is not semisimple.

(3) The complex structure J is unique up to a sign.

Proof. For (1). It's clear if (G, K) is Hermitian symmetric, then $H^2(M, \mathbb{R}) \neq 0$ since its Kähler form lies in it; Conversely, for $0 \neq \omega \in H^2(M, \mathbb{R})$, we may construct a new 2-form $\widetilde{\omega}$ by

$$\widetilde{\omega}_p := \int_G \omega_{gp} \mathrm{d}g.$$

It's clear $\widetilde{\omega}$ is invariant under isometries. □

8. Bounded symmetric domains

8.A. **The Bergman metrics.**

8.B. **Classical bounded symmetric domains.**

8.C. **Curvatures of classical bounded symmetric domains.**

Part 4. **Appendix**

Appendix A. Lie group and Lie algebra

A.A. **Fundamental theorems.**

Theorem A.1. *Every finite-dimensional (real) Lie algebra is the Lie algebra of some simply-connected Lie group.*

Theorem A.2. *If G is a Lie group and $\mathfrak{h} \subseteq \operatorname{Lie} G$ is a Lie subalgebra, then there exists a unique connected Lie subgroup $H \subseteq G$ with $\operatorname{Lie} H = \mathfrak{h}$.*

Theorem A.3. *If $\Phi : \operatorname{Lie} G \to \operatorname{Lie} H$ is a Lie group homomorphism and G is simply-connected, then there exists a unique Lie group homomorphism $\varphi : G \to H$ such that $\Phi = (\mathrm{d}\varphi)_e$.*

Lemma A.4. *Suppose G, H are connected Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$ and $\varphi : G \to H$ is a Lie group homomorphism. If $(\mathrm{d}\varphi)_e : \mathfrak{g} \to \mathfrak{h}$ is bijective, then φ is a covering map.*

Corollary A.5. *If $\widetilde{G}, G$ are connected Lie groups having isomorphic Lie algebra and $\widetilde{G}$ is simply-connected, then $\widetilde{G}$ is the universal covering of G.*

Corollary A.6. *If connected and simply-connected Lie groups G, H have isomorphic Lie algebra, then G and H are isomorphic.*

A.B. **Adjoint action.**

Definition A.7 (compactly embedded)**.** *Let $\mathfrak{g}$ be a Lie algebra. A subalgebra $\mathfrak{k} \leq \mathfrak{g}$ is compactly embedded if $\operatorname{ad}(\mathfrak{k})$ is the Lie algebra of a compact subgroup of $\mathrm{GL}(\mathfrak{g})$.*

A.C. **Semisimple Lie algebras.**

Definition A.8 (semisimple)**.** *A Lie algebra $\mathfrak{g}$ is called semisimple if the Killing form B of $\mathfrak{g}$ is non-degenerate.*

Definition A.9 (simple)**.** *A Lie algebra $\mathfrak{g}$ is called simple if it's semisimple and has no ideals except $\{0\}$ and $\mathfrak{g}$.*

Definition A.10. *A Lie group is called semisimple (simple) if its Lie algebra is semisimple (simple).*

Proposition A.11. *A semisimple Lie algebra has center $\{0\}$.*

Proposition A.12. *A semisimple Lie algebra $\mathfrak{g}$ is the direct sum*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \mathfrak{g}_r.$$

where $\mathfrak{g}_i$ $(1 \leq i \leq r)$ are all the simple ideals in $\mathfrak{g}$. Every ideal $\mathfrak{a}$ of $\mathfrak{g}$ is the direct sum of certain $\mathfrak{g}_i$.

Proposition A.13.

(1) Let $\mathfrak{g}$ be a semisimple Lie algebra. Then $\mathfrak{g}$ is compact if and only if the Killing form of $\mathfrak{g}$ is negative definite.
(2) Every compact Lie algebra $\mathfrak{g}$ is a direct sum $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and the ideal $[\mathfrak{g}, \mathfrak{g}]$ is semisimple and compact.

B.A. **Killing fields.**

B.A.1. *Basic properties.*

Proposition B.1. *Let (M, g) be a Riemannian manifold and X be a Killing field.*
(1) If γ is a geodesic, then $J(t) = X(\gamma(t))$ is a Jacobi field.
(2) For any two vector fields Y, Z,

$$\nabla_Y \nabla_Z X - \nabla_{\nabla_Y Z} X + R(X, Y)Z = 0$$

Proof. For (1). Suppose φ_s is the flow generated by X. Then we obtain a variation $\alpha(s,t) = \varphi_s(\gamma(t))$ consisting of geodesics, and thus

$$X(\gamma(t)) = \left.\frac{\partial \varphi_s(\gamma(t))}{\partial s}\right|_{s=0}$$

is a Jacobi field.

For (2). It's an equation of tensors, so we check it pointwisely and use normal coordinate $\{x^i\}$ centered at p. Moreover, we assume $X = X^i \frac{\partial}{\partial x^i}, Y = \frac{\partial}{\partial x^j}, Z = \frac{\partial}{\partial x^k}$. Then

$$\begin{aligned}
\nabla_Y \nabla_Z X - \nabla_{\nabla_Y Z} X + R(X, Y)Z &= \nabla_j \nabla_k X + X^i R^l_{ijk} \frac{\partial}{\partial x^l} \\
&= \Big(\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma^l_{ki}}{\partial x^j} + X^i R^l_{ijk}\Big)\frac{\partial}{\partial x^l} \\
&= \Big(\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma^l_{jk}}{\partial x^i}\Big)\frac{\partial}{\partial x^l}
\end{aligned}$$

since $R^l_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \Gamma^s_{jk}\Gamma^l_{is} - \Gamma^s_{ik}\Gamma^l_{js}$. Now it suffices to show $\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma^l_{jk}}{\partial x^i} \equiv 0$. In order to show this, for arbitrary $p \in M$, consider a geodesic γ starting at p and consider Jacobi field $J(t) = X(\gamma(t))$. Direct computation shows

$$\begin{aligned}
J'(t) &= \left.\Big(\frac{\partial X^i}{\partial x^k}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t} + X^i \Gamma^l_{ki}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}\Big)\frac{\partial}{\partial x^l}\right|_{\gamma(t)} \\
J''(0) &= \left.\Big(\frac{\partial^2 X^l}{\partial x^j \partial x^k}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t} + X^i \frac{\partial \Gamma^l_{ki}}{\partial x^j}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}\Big)\frac{\partial}{\partial x^l}\right|_p \\
&= \left.\Big(\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma^l_{ki}}{\partial x^j}\Big)\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}\frac{\partial}{\partial x^l}\right|_p \\
&= \left.\Big(\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma^l_{jk}}{\partial x^i} + X^i \frac{\partial \Gamma^l_{ki}}{\partial x^j} - X^i \frac{\partial \Gamma^l_{jk}}{\partial x^i}\Big)\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}\frac{\partial}{\partial x^l}\right|_p \\
&= \left.\Big(\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma^l_{jk}}{\partial x^i}\Big)\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}\frac{\partial}{\partial x^l}\right|_p - R(X, \gamma')\gamma'
\end{aligned}$$

which implies

$$\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma^l_{jk}}{\partial x^i} = 0$$

holds at point p, and since p is arbitrary, this completes the proof. $\square$

Corollary B.2. *Let (M,g) be a complete Riemannian manifold and $p \in M$. Then a Killing field X is determined by the values X_p and $(\nabla X)_p$ for arbitrary $p \in M$.*

Proof. The equation $\mathcal{L}_X g \equiv 0$ is linear in X, so the space of Killing fields is a vector space. Therefore, it suffices to show if $X_p = 0$ and $(\nabla X)_p = 0$, then $X \equiv 0$. For arbitrary $q \in M$, let $\gamma : [0,1] \to M$ be a geodesic connecting p and q with $\gamma'(0) = v$. Since $J(t) = X(\gamma(t))$ is a Jacobi field, and a direct computation shows

$$(\nabla_v X)_p = J'(0)$$

Thus $J(t) \equiv 0$, since Jacobi field is determined by two initial values. In particular, $X_q = J(1) = 0$, and since q is arbitrary, one has $X \equiv 0$. $\square$

Corollary B.3. *The dimension of vector space consisting of Killing fields $\leq n(n+1)/2$.*

Proof. Note that ∇X is skew-symmetric and the dimension of skew-symmetric matrices is $n(n-1)/2$. Thus the dimension of vector space consisting of Killing fields $\leq n + n(n-1)/2 = n(n+1)/2$. $\square$

B.A.2. *Killing field as the Lie algebra of isometry group.*

Lemma B.4. *Killing field on a complete Riemannian manifold (M,g) is complete.*

Proof. For a Killing field X, we need to show the flow $\varphi_t : M \to M$ generated by X is defined for $t \in \mathbb{R}$. Otherwise, we assume φ_t is defined on (a,b). Note that for each $p \in M$, curve $\varphi_t(p)$ is a curve defined on (a,b) having finite constant speed, since φ_t is isometry. Then we have $\varphi_t(p)$ can be extended to the one defined on $\mathbb{R}$, since M is complete. $\square$

Theorem B.5. *Let (M,g) be a complete Riemannian manifold and $\mathfrak{g}$ the space of Killing fields. Then $\mathfrak{g}$ is isomorphic to the Lie algebra of $G = \mathrm{Iso}(M,g)$.*

Proof. It's clear $\mathfrak{g}$ is a Lie algebra since $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$. Now let's see it's isomorphic to Lie algebra consisting of Killing field as Lie algebra.

(1) Given a Killing field X, by Lemma B.4, one deduces that the flow $\varphi : \mathbb{R} \times M \to M$ generated by X is a one parameter subgroup $\gamma : \mathbb{R} \to G$, and $\gamma'(0) \in T_e G$.
(2) Given $v \in T_e G$, consider the one-parameter subgroup $\gamma(t) = \exp(tv) : \mathbb{R} \to G$ which gives a flow by
$$\begin{aligned} \varphi : \mathbb{R} \times M &\to M \\ (t,p) &\mapsto \exp(tv) \cdot p \end{aligned}$$
Then the vector field X generated by this flow is a Killing field.

This gives a one to one correspondence between Killing fields and Lie algebra of G, and it's a Lie algebra isomorphism. $\square$

Corollary B.6 (Cartan decomposition)**.** *Let (M,g) be a complete Riemannian manifold and $G = \mathrm{Iso}(M,g)$ with Lie algebra $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ of G has a decomposition as vector spaces*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

where

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid X_p = 0\}$$
$$\mathfrak{m} = \{X \in \mathfrak{g} \mid (\nabla X)_p = 0\}$$

and they satisfy

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$$

Proof. The decomposition follows from Corollary B.2 and Theorem B.5, and it's easy to see

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$$

For arbitrary $X \in \mathfrak{k}, Y \in \mathfrak{m}$ and $v \in T_pM$, one has

$$\begin{aligned} \nabla_v[X, Y] &= \nabla_v \nabla_X Y - \nabla_v \nabla_Y X \\ &= -R(Y, v)X + \nabla_{\nabla_v X} Y + R(X, v)Y - \nabla_{\nabla_v Y} X \\ &= 0 \end{aligned}$$

since $X_p = 0$ and $(\nabla Y)_p = 0$. This shows $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$. □

B.B. **Hopf theorem.** The argument about analytic continuation in Theorem 1.11 can be used to give a proof of Hopf's theorem.

Theorem B.7 (Hopf)**.** *Let (M, g) be a complete, simply-connected Riemannian manifold with constant sectional curvature K. Then (M, g) is isometric to*

$$(\widetilde{M}, g_{can}) = \begin{cases} (\mathbb{S}^n(\frac{1}{\sqrt{K}}), g_{can}) & K > 0 \\ (\mathbb{R}^n, g_{can}) & K = 0 \\ (\mathbb{H}^n(\frac{1}{\sqrt{-K}}), g_{can}) & K < 0 \end{cases}$$

Proof. For $p \in M, \widetilde{p} \in \widetilde{M}$ and $\delta < \min\{\mathrm{inj}(p), \mathrm{inj}(\widetilde{p})\}$. By Cartan-Ambrose-Hicks's theorem, there exists an isometry $\varphi: B(p, \delta) \to B(\widetilde{p}, \delta)$ such that $\varphi(p) = \widetilde{p}$ and $(\mathrm{d}\varphi)_p$ equals to a given linear isometry, since both (M, g) and $(\widetilde{M}, \widetilde{g})$ have constant sectional curvature K. By the same argument in proof of Theorem 1.11, there is an isometry $\varphi: (M, g) \to (\widetilde{M}, \widetilde{g})$ which extends $\varphi: B(p, \delta) \to B(\widetilde{p}, \delta)$. In particular, this completes the proof. □

B.C. **Other basic facts.**

Theorem B.8. *Let $\varphi, \psi: (M, g_M) \to (N, g_N)$ be two local isometries between Riemannian manifolds, and M is connected. If there exists $p \in M$ such that*

$$\varphi(p) = \psi(p)$$
$$(\mathrm{d}\varphi)_p = (\mathrm{d}\psi)_p$$

then $\varphi = \psi$.

Theorem B.9 (Cartan-Ambrose-Hicks)**.** *Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be two Riemannian manifolds and $\Phi_0: T_pM \to T_{\widetilde{p}}\widetilde{M}$ be a linear isometry, where $p \in M, \widetilde{p} \in \widetilde{M}$. For $0 < \delta < \min\{\mathrm{inj}_p(M), \mathrm{inj}_{\widetilde{p}}(\widetilde{M})\}$, The following statements are equivalent.*

(1) There exists an isometry $\varphi: B(p, \delta) \to B(\widetilde{p}, \delta)$ such that $\varphi(p) = \widetilde{p}$ and $(\mathrm{d}\varphi)_p = \Phi_0$.

(2) *For* $v \in T_pM, |v| < \delta, \gamma(t) = \exp_p(tv), \widetilde{\gamma}(t) = \exp_{\widetilde{p}}(t\Phi_0(v))$*, if we define*

$$\Phi_t = P_{0,t;\widetilde{\gamma}} \circ \Phi_0 \circ P_{t,0;\gamma} : T_{\gamma(t)}M \to T_{\widetilde{\gamma}(t)}\widetilde{M}$$

then Φ_t *preserves curvature, that is* $(\Phi_t)^*R = R$.

Proposition B.10. *Let* $(M, g_M), (N, g_N)$ *be complete Riemannian manifolds and* $f : M \to N$ *be a local diffeomorphism such that for all* $p \in M$ *and for all* $v \in T_pM$*, one has* $|(\mathrm{d}f)_p v| \geq |v|$*. Then* f *is a Riemannian covering map.*

Theorem B.11 (Myers-Steenrod)**.** *Let* (M, g) *be a Riemannian manifold and* $G = \mathrm{Iso}(M, g)$*. Then*

(1) *G is a Lie group with respect to compact-open topology.*
(2) *for each* $p \in M$*, the isotropy group* G_p *is compact.*
(3) *G is compact if M is compact.*

Proposition B.12. *Let* (M, g) *be a Riemannian manifold,* $\gamma : I \to M$ *a smooth curve and* $P_{s,t;\gamma} : T_{\gamma(s)}M \to T_{\gamma(t)}M$ *is the parallel transport along* γ*. For any* $s \in I$ *with* $v = \gamma'(s)$ *and tensor* T*, one has*

$$\nabla_v T = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} (P_{s,t;\gamma})^* T_{\gamma(t)}$$

In particular, if $\nabla T = 0$ *then*

$$(P_{s,t;\gamma})^* T_{\gamma(t)} = T_{\gamma(s)}$$

holds for arbitrary $t, s \in I$*.*

Proposition B.13. *If* $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ *is a Riemannian covering, then M is complete if and only if* $\widetilde{M}$ *is.*

REFERENCES

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