YANG-MILLS EQUATIONS ON RIEMANN SURFACES AND MODULI SPACES

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Abstract. This article is mainly divided into three parts. In the first part, as a preparation for further theories, I studied the geometry of vector bundle and principal bundle, such as covariant derivative or connection and curvature on them. I introduced the motivations of studying covariant derivative and what roles does connection of principal bundle play. In order to get a better understanding of it, I studied two different views of connection. I mainly followed "Differential Geometry" written by Taubes [2], and refer [3], [4] for some supplements.

In the second part, I reviewed some basic definitions in geometry, such as volume form or Hodge star operator, since they're crucial in defining Yang-Mills functional. Then I gave an introduction about Yang-Mills functional and its variations.

In the third part, I mainly followed [3], which give a neat introduction about Yang-Mills equations' applications on the stability of holomorphic vector bundles.

Key words: principle bundle, connection, curvature, Yang-Mills equations, stability of holomorphic vector bundles.

Contents

1. Principle Bundles and Connections

1.1. **Principle bundles.**

Definition 1.1.1 (principle G-bundle)**.** *A principle G-bundle is a smooth manifold P, with the following data:*

1. *A Lie group G acting freely and transitively on P on the right:*
$$\begin{aligned} R_g : P \times G &\to P \\ (p, g) &\mapsto pg \end{aligned}$$
2. *A surjective map $\pi : P \to M$ that is G-invariant, that is, $\pi(pg) = \pi(p), \forall p \in P, g \in G$.*
3. *For every point $x \in M$, there exists a neighborhood U of x and a diffeomorphism $\varphi : P|_U := \pi^{-1}(U) \to U \times G$ that is G-equivariant, and the following diagram commutes*
$$\begin{tikzcd} P|_U \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & \arrow[d, phantom, "\circlearrowleft"] & U \times G \arrow[dl] \\ & U & \end{tikzcd}$$
where $U \times G \to U$ is projection onto the first factor.

Remark 1.1.2. If we consider two local trivialization $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ with non-empty intersection, then
$$\varphi_\alpha \circ \varphi_\beta^{-1} : \pi^{-1}(U_\alpha \cap U_\beta) \to U_\alpha \cap U_\beta \times G$$
then for any $x \in U_\alpha \cap U_\beta$, we will get $\varphi_\alpha \circ \varphi_\beta^{-1}(x) \in \mathrm{Aut}(G)$. Furthermore, since local trivialization is G-equivariant, then[1]
$$\varphi_\alpha \circ \varphi_\beta^{-1}(x) \in \mathrm{Aut}_G(G) := \{f : G \to G \mid f(xg) = f(x)g, \forall x, g \in G, f \text{ is a diffeomorphism.}\}$$
and we have $\mathrm{Aut}_G(G) \cong G$. Indeed, $G \subseteq \mathrm{Aut}_G(G)$ automatically holds, since for any $g \in G$, we can define a map $x \mapsto gx$. Conversely, note that such f is completely determined by its value at any point. If we let g to denote $f(e)$, where $e \in G$ is the identity. Then $f(x) = f(ex) = gx$ for all $x \in G$, that is, f is just $x \mapsto gx$.

So not similar to case of vector bundles, the gluing data of principal bundles are given by G-equivariant maps
$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$$
satisfying cocycle conditions. Conversely, this determines the principal bundles uniquely as case of vector bundles.

Many principal bundles arise from vector bundles, such as frame bundles we will discuss immediately.

[1]Here may be a little misleading, $\mathrm{Aut}(G)$ is always reserved for group automorphisms, but here f is not a group automorphism, but just a diffeomorphism.

Example 1.1.3 (frame bundle)**.** Let $E \to M$ be a real vector bundle of rank k. For $x \in M$, let $\mathcal{B}_x$ denote the set of all bases of the fiber E_x, i.e. the set of linear isomorphisms $\mathbb{R}^k \to E_x$. This has a natural right action of $\mathrm{GL}_k(\mathbb{R})$ by precomposition. Then let

$$\mathcal{B}_{\mathrm{GL}_k(\mathbb{R})}(E) := \coprod_{x \in M} \mathcal{B}_x$$

Using local trivialization of the vector bundle E, we equip $\mathcal{B}_{\mathrm{GL}_k(\mathbb{R})}(E)$ with the structure of a smooth manifold such that $\pi : \mathcal{B}_{\mathrm{GL}_k(\mathbb{R})}(E) \to M$ taking $\mathcal{B}_x$ to x. This gives $\mathcal{B}_{\mathrm{GL}_k(\mathbb{R})}(E)$ the structure of $\mathrm{GL}_k(\mathbb{R})$-bundle, called the frame bundle of E.

Example 1.1.4 (orthonormal frame bundle)**.** Let $E \to M$ be a rank k vector bundle equipped with a fiber metric, i.e. a smoothly varying inner product on the fibers E_x. Then the orthonormal frame bundle of E, denoted $\mathcal{B}_O(E)$, is the principal O_k-bundle where the fiber over $x \in M$ is the linear isometries $\mathbb{R}^k \to E_x$, where we use the standard inner product on $\mathbb{R}^k$ and the fiber metric restricted to E_x.

Remark 1.1.5. A near identical story holds for complex vector bundles: from any complex vector bundle we get a principal $\mathrm{GL}_k(\mathbb{C})$-bundle of frames, and if we fix a Herimitian fiber metric, we get a principle U_k-bundle of orthogonal frames, which is quite important in future.

Definition 1.1.6 (trivial principal bundles)**.** *We say that the bundle P is trivial, if there exists a diffeomorphism $\varphi : P \to M \times G, \varphi(p) = (\pi(p), g(p))$ such that $g(ph) = g(p)h, \forall h \in G$.*

Definition 1.1.7 (section)**.** *A section is a smooth map $s : M \to P$ such that $\pi \circ s = \mathrm{id}$.*

Remark 1.1.8. In other words, a section is a smooth assignment to each point in the base of a point in the fiber over it. However, sections are quite rare.

Proposition 1.1.9. *A principal bundle admits a section if and only if it is trivial.*[2]

Proof. If $s : M \to P$ is a smooth section, we define

$$\begin{aligned} \varphi : P &\to M \times G \\ p &\mapsto (\pi(p), g(p)) \end{aligned}$$

where $g(p) \in G$ is such that $p = s(\pi(p))g(p)$, it always exsits since the right action of G is transitive on each fiber and it is unique since the action is free on each fiber.

Clearly, it's G-equivariant, since

$$\varphi(ph) = (\pi(ph), g(ph)) = (\pi(p), g(p)h)$$

[2] This is in sharp contrast with vector bundles, which always admit sections.

and the last equality holds since

$$ph = s(\pi(ph))g(ph) = s(\pi(p))g(ph) = pg^{-1}(p)g(ph) \implies h = g^{-1}(p)g(ph)$$

And it's easy to see φ is a bijection, with inverse map

$$\begin{aligned}\varphi^{-1} : M \times G &\to P \\ (p, g) &\mapsto s(p)g\end{aligned}$$

The smoothness of the section and of the G-action on P imply smoothness.

□

However, since P is locally trivial, local sections do exist. In fact, there are local sections $s_\alpha : U_\alpha \to \pi^{-1}U_\alpha$ canonically associated to the trivialization, defined so that for every $m \in U_\alpha, \psi_\alpha(s_\alpha(m)) = (m, e)$, where $e \in G$ is the identity element. In other words, $g_\alpha \circ s_\alpha : U_\alpha \to G$ is the constant function sending every point to the identity.

1.2. **Associated vector bundles.** We have seen that for a vector bundle, we can construct its frame principal bundle or orthogonal frames bundle. As explained here, this construction has an inverse of sorts.

To set the stage, suppose that G is a Lie group and $\pi : P \to M$ is a principal G bundle. The construction of a vector bundle from P requires an additional input, that is a representation ρ of the group G into $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$, and we use V to denote representation space.

Definition 1.2.1 (associated vector bundle)**.** *Let $P \to M$ be a principal G-bundle, and $\rho : G \to \mathrm{GL}(V)$ be a representation of G. The associated vector bundle, denoted by $P \times_G V$ is the space*

$$P \times_G V := (P \times V)/G$$

where the G-action on $P \times_G V$ is the diagonal action, i.e. $(p, v) \cdot g := (pg, \rho(g^{-1})v)$.

Remark 1.2.2. More generally, V need not to be a vector space, it might be a manifold admitting a G action.

The following proposition shows that why associated fiber bundle indeed gives us a fiber bundle with model fiber.

Proposition 1.2.3. *$P \times_G V$ is a vector bundle over M with model fiber V.*

Proof. Consider the map taking an equivalence class $[p, f]$ to $\pi(p)$. To see the local structure, since we already have the local structure of principal bundle P, i.e. for any $x \in M$, there exists open $U \ni x$ and $\varphi : \pi^{-1}(U) \to U \times G$. Use $\psi : P \to G$ to denote the composition of the first φ and then projection from $U \times G \to G$. Now we define the local trivialization of $P \times_G V$ as

$$\begin{aligned}\varphi^V : (P \times_G V)|_U &\to U \times V \\ (p, v) &\mapsto (\pi(p), \rho(\psi(p))v)\end{aligned}$$

First note that this is well-defined, since

$$(pg, \rho(g^{-1})v) \mapsto (\pi(pg), \rho(\psi(pg))\rho(g^{-1})v) = (\pi(p), \rho(\psi(p)gg^{-1})v) = (\pi(p), \rho(\psi(p))v)$$

And this map is one to one, and invertible, its inverse sends $(x, v) \to U \times V$ to the equivalence class of $(\varphi^{-1}(x, 1), v)$. □

Remark 1.2.4. Though we've find the local trivialization of $P \times_G V$, it's also necessary to see what does the transition functions look like.

Let U, U' be intersecting open sets, and φ, φ' be local trivialization of principal bundles, with transition functions

$$\begin{aligned} \varphi' \circ \varphi^{-1} : U \cap U' \times G &\to U \cap U' \times G \\ (x, g) &\mapsto (x, g_{UU'}(x)(g)) \end{aligned}$$

then we can compute the transition functions of associated vector bundles as follows

$$\begin{aligned} \varphi'^V \circ (\varphi^V)^{-1} : U \cap U' \times V &\to U \cap U' \times V \\ (x, v) &\mapsto (x, \rho(g_{UU'}(x))v) \end{aligned}$$

Example 1.2.5. If $E \to M$ is a given vector bundle, with fiber $V = \mathbb{R}^n$, then we will have its frame principal G bundle P with $G = \mathrm{GL}(n, \mathbb{R})$. Let ρ be the defining representation of G on V, i.e. the representation $\rho : G \to \mathrm{GL}(n, \mathbb{R})$ is identity map. Then we claim that $P \times_G V$ is canonically isomorphic to E, since from Remark 1.2.3 we can directly see that they have the same transition functions.

Remark 1.2.6. The above example shows that we can recover a vector bundle from its frame bundle using associated vector bundle. But we need to use defining representation, so that what "of sorts" means, i.e. principal bundles indeed encode more information.

Example 1.2.7. There are two important examples of associated bundles that we will need to discuss the Yang-Mills equations.

1. The bundle $\mathrm{Ad}\, P := P \times_G G$, where G acts on G by conjugation.
2. The bundle $\mathrm{ad}\, P := P \times_G \mathfrak{g}$, where the action is the adjoint action. This bundle is sometimes denoted by $\mathfrak{g}_P$.

Associated bundles have another nice feature, their sections have a nice interpretation in terms of G-equivariant maps.

Proposition 1.2.8. *Let $E = P \times_G V$ be an associated fiber bundle. Then there is a bijective correspondence*

$$C^\infty(M, E) \longleftrightarrow \{G\text{-equivariant maps } P \to V\}$$

Proof. First, if we have a G-equivariant map $s^P : P \to V$, that is

$$s^p(pg) = \rho(g^{-1})s^p(p)$$

then the corresponding section s associates to any given point $x \in M$ the equivalence class $(p, s^P(p))$, where p is any point in the fiber P_x. Clearly the definition is independent of the choice of p, since

$$(pg, s^P(pg)) = (pg, \rho(g^{-1})s^P(p)) = (p, s^P(p))$$

Conversely, suppose s is a section of $P \times_G V$, by definition, s associates an equivalence class in $P \times_G V$ to every point of M. That equivalence class gives a G-equivariant map $P \to V$. □

Remark 1.2.9. In fact, this proposition is not a coincidence, and it's a quite important motivation which explains why we need principal bundles.

If $\pi : P \to M$ is a principal G bundle, and $\pi' : E \to M$ is a vector bundle such that E is an associated vector bundle of P, then if we use π to pull E back to P, we claim that the vector bundle π^*E is the trivial bundle $P \times V$ over P. Indeed, note that we have an isomorphism between $(P \times V)_p$ and $(P \times_G V)_{\pi(p)}$, since both of them are isomorphic to V.

That's a quite beautiful result. And there is no wonder that section s of E corresponds to a G-equivariant map $s^P : P \to V$, since that's exactly what sections of $P \times V$ look like.

Example 1.2.10. Let $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ be the canonical projection, and we can regard $\mathbb{C}^{n+1}\backslash\{0\}$ as a principal $\mathbb{C}^*$ bundle. Consider tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$ on $\mathbb{P}^n$, and consider its pullback by π. It will be a trivial line bundle over $\mathbb{C}^{n+1}\backslash\{0\}$. Indeed, by definition

$$\pi^*(\mathcal{O}(-1)) = \{(x, v) \in \mathbb{C}^n\backslash\{0\} \times \mathcal{O}(-1) \mid \pi(x) = \pi_L(v)\}$$

Note that $\pi(x) = [x] = \pi_L(v)$ implies that v is an element of $\mathbb{C}^{n+1}$ spanned by x. This means that the fiber of the pullback bundle $\pi^*(\mathcal{O}_{\mathbb{P}^n}(-1))$ over x is exactly $\{\lambda x \mid \lambda \in \mathbb{C}\}$. This happens globally so there is an obvious vector bundle isomorphism with $\mathbb{C}^{n+1}\backslash\{0\} \times \mathbb{C}$.

This pullback is trivial is not a coincidence, since we can regard tautological line bundle as an associated vector bundle, by considering $\mathbb{C}^*$ acts on $\mathbb{C}$ by multiplying.

Remark 1.2.11. This proposition shows the power of principal bundles. Instead of considering sections of $P \times_G V$, we just need to consider G-equivariant functions from P to V, that is sections of trivial bundle $P \times V$ with some equivariant conditions. And it makes non-trivial things into trivial one. What a powerful result.

This view has the following useful generalization: Suppose $\pi' : E' \to M$ is a second vector bundle, and perhaps unrelated to P. A section of $(P \times_G V) \otimes E'$ appears upstairs on P as a suitably G-equivariant section over P of $(P \times V) \otimes \pi^*E'$.

For example, if we take E' to be cotangent bundle T^*M or its wedge products $\bigwedge^k T^*M$. We sometimes use $\Omega^k_M(P \times_G V)$ to denote $(P \times_G V) \otimes \bigwedge^k T^*M$, the generalization tells that we have the one to one correspondence between sections of $\Omega^k_M(P \times_G V)$ and sections of $(P \times V) \otimes \pi^* \bigwedge^k T^*M$ with equivariant conditions, we will call such forms obtained by pullback basic, and denote it by $\Omega^k_G(P;V)$, we will explain these in more details later.

1.3. **Covariant derivatives and connections.** Let $\pi : E \to M$ be a vector bundle, and use $C^\infty(M,E)$ to denote the vector space of smooth sections of E. The questions arises as to how to take the derivative of a section $s : M \to E$ in a given direction.

It's quite natural to ask such questions. When we learn calculus, we know how to take derivative of a smooth function $f : M \to \mathbb{R}^m$, and this gives $\mathrm{d}f : TM \to \mathbb{R}^m$, or a section of T^*M. Furthermore, any smooth function $f : M \to \mathbb{R}^m$ can be regarded as a section of trivial vector bundle $M \times \mathbb{R}^m$, as follows

$$x \mapsto (x, f(x))$$

and we can also regard its derivative $\mathrm{d}f$ as a section of $T^*M \otimes M \times \mathbb{R}^m$.

Recall how we define $\mathrm{d}f$ in the model case $f : \mathbb{R}^n \to \mathbb{R}^m$. In this setting the derivative $\mathrm{d}f$ at x in the direction v is defined by the standard formula

$$\mathrm{d}f(v)(x) = \lim_{t\to 0} \frac{f(x+tv) - f(x)}{t}$$

But when passing to a section s of vector bundle E, one encounters two key issues with this definition. Firstly, since there is no linear structure on a manifold, the term $x + tv$ makes no sense on M. However, this issue is relatively easy to handle. Instead one takes a path $\gamma : (-1,1) \to M$ such that $\gamma(0) = x, \gamma'(0) = v$ and computes

$$\mathrm{d}s(v)(x) = \lim_{t\to 0} \frac{s(\gamma(t)) - s(\gamma(0))}{t}$$

and that's what we do in the case of functions on manifold. However, this still does not make sense in the case of vector bundles, since $f(\gamma(t))$ and $f(\gamma(0))$ are elements of distinct fibers $E_{\gamma(t)}$ and $E_{\gamma(0)}$. This means that substraction of these two terms is not naturally defined.

One naive ideal is that if we can "transport" vectors in $E_{\gamma(t)}$ "parallel" to those in $E_{\gamma(0)}$, then we can do substraction. That's one viewpoint to think connections. To be more explicit, a connection can be viewed as assigning to every differentiable path γ a linear isomorphisms $P^\gamma_t : E_{\gamma(t)} \to E_x$ for all t, and we can define

$$\nabla_v f = \lim_{t\to} \frac{P^\gamma_t f(\gamma(t)) - f(\gamma(0))}{t}$$

Another way to solve this issue is what we're going to use in the following. Since $\mathrm{d}f$ can be regarded as a section of Ω^1_M, we can directly define ∇s is also a section of $\Omega^1_M(E)$. Obviously, there are some other requirements.

As explained in what follows, we will define covariant derivatives on vector bundles and conventions on principal bundles in this section. Furthermore, we will see how these things blend with each other, and see the motivations of why do we need principal bundles to some extend.

1.3.1. *Covariant derivatives.* As it's known to all that derivative satisfies the Leibniz rule, a quite meaningful rule. It says that

$$\mathrm{d}(fg) = \mathrm{d}fg + f\mathrm{d}g$$

However, we can regard f multiply g as a $C^\infty(M)$-module structure of $C^\infty(M)$. Since on the space of smooth sections of E, we can also give such a module structure. The action is such that a given function f acts on $s \in C^\infty(M, E)$ to give fs. So if we want to define a derivative on it, it should be compatible with such structure, like Leibniz rule tells us.

Definition 1.3.1 (covariant derivative)**.** *A covariant derivative for $C^\infty(M, E)$ is a linear map*

$$\nabla : C^\infty(M, E) \to C^\infty(M, \Omega^1_M(E))$$

such that

$$\nabla(fs) = f\nabla s + s \otimes \mathrm{d}f$$

for all $f \in C^\infty(M)$.

Remark 1.3.2. We need to make sure that there indeed exists such a covariant derivative. We can do it locally and use partition of unity to get a global one. For a local trivialization (U, φ_U) of E, we have

$$\varphi_U : E|_U \to U \times V$$

then section v of such trivial bundle $U \times V$ is just like $x \mapsto (x, v(x))$. Now define the covariant derivative as $\mathrm{d}v : x \mapsto (x, \mathrm{d}v|_x)$.

Then define covariant derivative for any given section s of E as

$$\nabla s = \sum_U \chi_U \varphi_U^*(\mathrm{d}(\varphi_U \circ s|_U))$$

1.3.2. *The space of covariant derivatives.* In fact, there are lots of covariant derivatives. As we will explain next, the space of covariant derivatives is an affine modeled on $C^\infty(M, \mathrm{Hom}(E, \Omega^1_M(E)))$.

To see this, first note that if $\mathfrak{a} \in C^\infty(M, \mathrm{Hom}(E, \Omega^1_M(E)))$ and ∇ is any given covariant derivative, then $\nabla + \mathfrak{a}$ is also a covariant derivative. Meanwhile, if ∇ and ∇' are both covariant derivatives, then their difference $\nabla - \nabla'$, is a section of $\mathrm{Hom}(E, \Omega^1_M(E))$. Indeed, since their difference is linear over the action of $C^\infty(M)$, i.e.

$$\nabla - \nabla'(fs) = (f\nabla s + s \otimes \mathrm{d}f) - (f\nabla' s + s \otimes \mathrm{d}f) = f(\nabla - \nabla')(s)$$

and consider the following lemma

Lemma 1.3.3. *Suppose that E and E' are two vector bundles and $\mathcal{L}$ is a linear map that takes a section of E to one of E'. Suppose in addition that $\mathcal{L}(fs) = f\mathcal{L}(s)$ for all functions. Then there exists a unique section L of $\mathrm{Hom}(E, E')$ such that $\mathcal{L} = L$.*

Proof. The proof is quite easy. To find L, fix an open set $U \subset M$ where both E and E' has a basis of sections. Denote the basis for E and E' as $\{e_i\}_{1\leq i\leq d}$ and $\{e'_j\}_{1\leq j\leq d'}$, where d and d' denote here the respective fiber dimensions of E and E'. Then we have functions $\{L_{ij}\}$ such that

$$\mathcal{L}e_i = \sum_{1\leq j\leq d'} L_{ij}e'_j$$

This understood, the homomorphism L is defined over U as follows: Let s denote a section of E over U, and write $s = \sum_i s_i e_i$ in terms of the basis. Then Ls is defined to be the section of E' as

$$Ls = \sum_{i,j} L_{ij}s_i e'_j$$

The identity $\mathcal{L}s = Ls$ holds by the fact that $\mathcal{L}(fs) = f\mathcal{L}(s)$ for any functions. And the homomorphism L does not depend on the choice of the basis of sections by the same reason, since $\mathcal{L}$ can commute with transition functions, this is to say that any two choices give the same section of $\mathrm{Hom}(E, E')$. $\square$

However, it's more traditional to view $\nabla - \nabla'$ as a section of $\Omega^1_M(\mathrm{End}(E))$ rather than $\mathrm{Hom}(E, \Omega^1_M(E))$, since they're canonically isomorphic to each other.

What was just said about the affine property has the following implication: Let ∇ be a covariant derivative on sections of E, and s be a section of E. Suppose U is a local trivialization of E, that is, $\varphi_U : E|_U \to U \times V$ is an bundle isomorphism. Write $\varphi_U s$ as $x \mapsto (x, s_U(x))$ with $s_U : U \to V$, then $\varphi_U(\nabla s)$ appears as the section

$$x \mapsto (x, (\nabla s)_U), \quad (\nabla s)_U = \mathrm{d}s_U + a_U s_U$$

where a_U is some s-independent section of $\Omega^1_U(\mathrm{End}(V))$.

But $U \mapsto a_U$ does not define a section over M of $\Omega^1_M(\mathrm{End}(E))$. Indeed, suppose U' is another trivialization, and $g_{U'U}$ is the transition function. Thus $s_{U'} = g_{U'U}s_U$. Meanwhile, ∇s is a bonafide section of $\Omega^1_M(E)$, so $(\nabla s)_{U'} = g_{U'U}(\nabla s)_U$. This requires that

$$(\nabla s)_{U'} = \mathrm{d}(g_{U'U}s_U) + a_{U'}g_{U'U}s_U = g_{U'U}(\mathrm{d}s_U + a_U s_U)$$

that is

$$a_{U'} = g_{U'U}a_U g^{-1}_{U'U} - (\mathrm{d}g_{U'U})g^{-1}_{U'U}$$

Conversely, we can say a covariant derivative is defined by such data, that is a collection $\{a_U\}$ satisfying above equations, where a_U is a section of $\Omega^1_U(\mathrm{End}(V))$.

1.3.3. *Connections on principal bundles.* This part we will define the notion of a connection on a principal bundle. This notion is of central importance in its own right. In any event, connections are used to give an alternate and very useful definition of the covariant derivative.

Suppose vector bundle E is associated to some principal bundle $\pi : P \to M$, and written as $P \times_G V$. Such principal bundles do exist, since we already know that E is associated to its frame principal bundle.

The reason we do this is from Proposition 1.2.8, we have a one to one correspondence between sections of E with G-equivariant maps from P to V. It's quite easy to take derivatives of s^P, it's a vector of differential 1-forms on P, and it defines a fiber-wise linear map $(s^P)_* : TP \to V$.

However, it does not by itself define a covariant derivative. As what we've defined, $\nabla s \in C^\infty(M, \Omega^1_M(E))$. so by Remark 1.2.10, a covariant derivative appears upstairs on P is supposed to be a G-equivariant section over $(P \times V) \otimes \pi^* T^* M$, that is a G-equivariant homomorphism from $\pi^* TM$ to V.

To see what $(s^P)_*$ is missing, it is important to keep in mind that TP has some properties that arise from the fact that P is a principal bundle over M. In fact, we have the following exact sequence

$$0 \to \ker(\pi_*) \to TP \to \pi^* TM \to 0 \tag{1.1}$$

This exact sequence is quite important, let's make following remarks:

Remark 1.3.4. The map from $\ker(\pi_*)$ is clearly an inclusion. And the map from TP to $\pi^* TM$ is characterized as follows

$$\begin{aligned} TP &\to \pi^* TM \subset P \times TM \\ v &\mapsto (p, \pi_* v) \end{aligned}$$

Remark 1.3.5. We can identify $\ker(\pi_*)$ with something quite simply, using the special property of principal bundle. Note that $\ker(\pi_*)$ designates the subbundle in TP that is sent by π_* to the zero sections in TM. That is to say that the vectors in the $\ker(\pi_*)$ are thoes that are tangent to the fibers of π. Thus, $\ker(\pi_*)$ over P_x is canonically isomorphic to $T(P_x)$, and that's Lie algebra $\mathfrak{g}$ of Lie group G.

In fact, we have $\ker(\pi_*)$ is isomorphic to trivial bundle $P \times \mathfrak{g}$. Indeed, we have the following bundle isomorphism

$$\begin{aligned} \psi : P \times \mathfrak{g} &\to \ker(\pi_*) \\ (p, X) &\mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} p e^{tX} \end{aligned}$$

In other words, we can define a map $\sigma : \mathfrak{g} \to TP$ assigning to each $X \in \mathfrak{g}$, the fundamental vector field $\sigma(X)$ whose value at p is given by $\psi(p, X)$.

Remark 1.3.6. Since there is a G action on P. So you may wonder if there is a G action on this exact sequence? The answer is yes. The action of G on P can be lifted to the exact sequence (1.1), as follows:

Let $R_g : P \to P$ denote the action of $g \in G$ on P, given by $p \mapsto pg$. This action lifts up to TP so as the pushout $(R_g)_* : TP \to TP$. And this action descends to $\ker(\pi_*)$. Indeed, if $v \in TP$ is in $\ker(\pi_*)$, then so is $(R_g)_* v$. We can check directly as follows, take $v \in \ker(\pi_*)$

$$\begin{aligned}\pi_*((R_g)_* v) &= (\pi \circ R_g)_*(v) \\ &= \pi_*(v) \\ &= 0\end{aligned}$$

Here we use the fact that $\pi \circ R_g = \pi$.

The action lift of R_g to $\pi^* TM$ is defined by viewing the latter in the manner described above as a subset of $P \times TM$. Viewed in this way, R_g act so as to send a pair $(p, v) \in P \times TM$ to the pair (pg, v). Clearly $(pg, v) \in \pi^* TM$, since $\pi(pg) = \pi(p)$.

Furthermore, the exact sequence (1.1) is equivariant with respect to the lifts. Indeed, this automatically holds for inclusion map from $\ker(\pi_*)$ to TP. And it holds for the map from TP to $\pi^* TM$, since for $v \in TP$ we have $(R_g)_* v$ is sent to $(p, (R_g)_* v)$, that is exactly (pg, v).

By Remark 1.3.5, we know that we can view $\ker(\pi_*)$ as the trivial bundle $P \times \mathfrak{g}$, so it's know how G acts on it if we do this identification. That is we need to choose a G action on $\mathfrak{g}$ properly such that the isomorphism ψ is G-equivariant.

This requires G acts on $\mathfrak{g}$ by adjoint representation. In other words, G acts on X by sending X to $g^{-1} X g$ for $X \in \mathfrak{g}$. Indeed, we compute as follows

$$\begin{aligned}(R_g)_* \psi(p, X) &= (R_g)_* \left(\frac{d}{dt}\bigg|_{t=0} p \exp(tX) \right) \\ &= \frac{d}{dt}\bigg|_{t=0} p \exp(tX) g \\ &= \frac{d}{dt}\bigg|_{t=0} (pg) \left(g^{-1} \exp(tX) g\right) \\ &= \psi(pg, g^{-1} X g)\end{aligned}$$

So, as we have seen, there is a canonical isomorphism of $\ker(\pi_*)$ with $P \times \mathfrak{g}$. We call $\ker(\pi_*)$ vertical subbundle of TP, and sometimes denote it by V. But in the absence of any extra structure, there is no natural complement to $\ker(\pi_*)$.

A connection on principal bundle P is neither more nor less than a G-equivariant splitting of the exact sequence (1.1). Since $(\nabla s)^P$ is supposed to be a G-equivariant fiber-wise linear map $(\nabla s)^P : \pi^* TM \to V$. So if the above sequence splits, then we have

$$TP \cong \pi^* TM \oplus \ker(\pi_*)$$

By restricting $(s^P)_*$ to $\pi^* TM$, we will define a covariant derivative of s. This explains why we need to define connections on principal bundle as follows.

Definition 1.3.7 (connection)**.** *A connection A is a linear map*

$$A : TP \to \ker(\pi_*)$$

that equals the identity on the kernel of π_**, and is* G*-equivariant with respect to the* G *action on* TP*.*

Notation 1.3.8. We use $\mathscr{A}(P)$ to denote the set of all connections on P.

Remark 1.3.9 (horizontal distribution viewpoint)**.** In other words, instead of defining a connection to be a linear projection, we can define it as a choice of horizontal distribution $H \subset TP$ such that

$$TP = H \oplus V$$

Furthermore, it satisfies some G-equivariant condition, that is $(R_g)_* H_p = H_{pg}$. In fact, it's a viewpoint the same as what we give in the definition.

Remark 1.3.10 ($\mathfrak{g}$-valued 1-form viewpoint)**.** Now, we give another description of connections. It is often more convenient for computations to rephrase a connection in terms of $\mathfrak{g}$-valued forms.

Since we have isomorphism $\psi : P \times \mathfrak{g} \to \ker(\pi_*)$, then we can define connections as

$$\omega : TP \to \mathfrak{g}$$

that is, a 1-form valued $\mathfrak{g}$. Furthermore, it satisfies

1. $\omega(\psi(p, m)) = m$.
2. If $g \in G$, then $(R_g)^* \omega = g^{-1} \omega g$.

Viewed this way, a connection is a section of $T^*P \otimes P \times \mathfrak{g}$ with certain additional properties that concern the G-action on P and the sequence (1.1). That is, we can regard connections as a $\mathfrak{g}$-valued differential 1-form, and denote it by $\Omega^1_P(\mathfrak{g})$.

Furthermore, these two different views blend with each other. Given a horizontal distribution $H \subset TP$, we can define its connection form as follows

$$\omega(v) = \begin{cases} X, & v = \sigma(X) \\ 0, & v \in H \end{cases}$$

Conversely, if we have a connection form ω, then we define $H = \ker(\omega)$.

In a summary, we have already two viewpoints of connections on principal bundles, that is

1. A G-equivariant horizontal distribution $H \subset TP$;
2. A 1-form $\omega \in \Omega^1_P(\mathfrak{g})$ satisfying some requirements;

Example 1.3.11. Now let's show a concrete example of connections on trivial principal G-bundle $M \times G$. In this case, we regard G as a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$, we use $g : G \hookrightarrow \mathrm{GL}(n, \mathbb{R})$ to denote this inclusion. We first define a $\mathfrak{g}$-valued form on G as follows

$$\omega = g^{-1} \mathrm{d} g$$

Let's check it's indeed $\mathfrak{g}$-valued: For thoes vectors that are tangent to G, for $g_0 \in G$ and $v \in T_{g_0}G$, we have

$$\begin{aligned}\omega(v) &= g_0^{-1}\mathrm{d}g(v)\\ &= g_0^{-1}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g_0e^{tv}\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{tv}\\ &= v \in \mathfrak{g}\end{aligned}$$

Then we define a connection 1-form on $M \times G$ by using the projection $M \times G \to G$ to pullback ω, and still use ω to denote this form.

Clearly ω annihilates vectors that are tangent to the M factor of $M \times G$. Furthermore, it's G-equivariant, since for any $a \in G$, we have

$$\begin{aligned}(R_a)^*\omega &= (ga)^{-1}\mathrm{d}(ga)\\ &= a^{-1}(g\mathrm{d}g)a\\ &= a^{-1}\omega a\end{aligned}$$

As desired.

This understood, the corresponding horizontal distribution H is precisely the tangents to the M factor, this factor TM in the obvious splitting of $T(M \times G) = TM \oplus TG$.

In fact, this tautological connection comes from the following form in Lie algebra.

Remark 1.3.12 (Maurer-Cartan form)**.** The Maurer-Cartan form is a $\mathfrak{g}$-valued 1-form θ on G defined by

$$\theta_g = (L_{g^{-1}})_* : T_gG \to T_eG = \mathfrak{g}$$

If X is a left-invariant vector field, that is

$$X(g) = (L_g)_*X(e)$$

where $X(g)$ means the value of X at g. In this case, we have

$$\theta_g(X(g)) = (L_g)_*^{-1}X(g) = (L_g)_*^{-1}(L_g)_*X(e) = X(e)$$

which is constant. Now if X and Y are left-invariant vector fields, then it immediately that θ satisfies the structure equation

$$\mathrm{d}\theta(X, Y) = -\theta([X, Y])$$

since $X(\theta(Y)) = Y(\theta(Y)) = 0$.

But the left-invariant fields span the tangent space at any point.(to do this we just need to pushforward a basis of T_eG under diffeomorphism.) So the equation is true for any pair of vector fields X and Y. This is well known as Maurer-Cartan equation. Now we introduce the bracket of Lie algebra-valued forms, to give another form of Maurer-Cartan equation which we will often use later.

Choose a basis of TG consisting of left-invariant vector fields $\{E_i\}$, and use $\{\theta^i\}$ to denote its dual basis. Then $\{E_i(e)\}$ is a basis of Lie algebra, Then we can write $\alpha = \sum_i E_i(e) \otimes \alpha^i$ and $\beta = \sum_j E_j(e) \otimes \beta^j$. Define the bracket of Lie algebra-valued forms $[\alpha, \beta]$ as

$$[\alpha, \beta] := \sum_{i,j} [E_i(e), E_j(e)] \alpha^i \wedge \beta^j = \sum_{i,j,k} c_{ij}^k E_k(e) \alpha^i \wedge \beta^j$$

where c_{ij}^k is the structure constants of Lie algebra.

In the frame we choose, we can write $\theta = \sum_i E_i(e) \otimes \theta^i$, since by definition we have $\theta(E_i) = E_i(e)$. Note that

$$d\theta^i (E_j, E_k) = -\theta^i ([E_j, E_k]) = -\sum_r c_{jk}^r \theta^i (E_r) = -c_{jk}^i = -\frac{1}{2} \left({c_{jk}}^i - c_{kj}^i \right)$$

so we have

$$d\theta^i = -\frac{1}{2} \sum_{jk} c_{jk}^i \theta^j \wedge \theta^k$$

Let's take exterior derivative of Maurer-Cartan form θ

$$\begin{aligned} \mathrm{d}\theta &= \sum_i E_i(e) \otimes \mathrm{d}\theta^i \\ &= -\frac{1}{2} \sum_{i,j,k} c_{jk}^i E_i(e) \otimes \theta^j \wedge \theta^k \\ &= -\frac{1}{2} [\theta, \theta] \end{aligned}$$

Thus we have the structure equation, sometimes called the Maurer-Cartan equation for Maurer-Cartan forms

$$\mathrm{d}\theta + \frac{1}{2} [\theta, \theta] = 0$$

However, there is another form, since for any two vector fields X, Y we have

$$\mathrm{d}\theta(X, Y) = -\theta([X, Y])$$

then Maurer-Cartan equation in fact is equivalent to the following one: For any two $X, Y \in TG$

$$[\theta, \theta](X, Y) = [\theta(X), \theta(Y)] - [\theta(Y), \theta(X)] = 2[\theta(X), \theta(Y)]$$

This is called the second form of Maurer-Cartan equation. In fact, for any $\mathfrak{g}$-valued 1-form which satisfies Maurer-Cartan equation, we call it a Maurer-Cartan form.

Example 1.3.13. Let G be a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$, and let $g : G \hookrightarrow \mathrm{GL}(n, \mathbb{R})$ be the inclusion map. Then $\omega := g^{-1} \mathrm{d}g$ is a Maurer-Cartan form. We check step by step:

We have already seen it's $\mathfrak{g}$-valued, it suffices to show that it do satisfies the Maurer-Cartan equation. Since $g^{-1}g = \mathrm{I}$, then $(\mathrm{d}g^{-1})g + g^{-1}\mathrm{d}g = 0$. Thus we have $\mathrm{d}(g^{-1}) = -g^{-1}(\mathrm{d}g)g^{-1}$. Now,

$$\begin{aligned}\mathrm{d}\omega &= \mathrm{d}(g^{-1}) \wedge \mathrm{d}g + g^{-1}\mathrm{d}^2 g \\ &= -g^{-1}(\mathrm{d}g)g^{-1} \wedge \mathrm{d}g \\ &= -g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \\ &= -\omega \wedge \omega\end{aligned}$$

That's it! Later we will see, that's say that the curvature of connection $g^{-1}\mathrm{d}g$ is zero, that is, a flat connection.

Example 1.3.14. Consider $G = \mathrm{SO}(2) \subset \mathrm{GL}(2, \mathbb{R})$. We may parametrize $\mathrm{SO}(2)$ by

$$g(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \theta \in \mathbb{R}$$

Then directly compute we have

$$\eta = g^{-1}\mathrm{d}g = \begin{pmatrix} 0 & -d\theta \\ d\theta & 0 \end{pmatrix}$$

1.3.4. *The space of connections.* Since we already know what's the difference of two covariant derivatives. In this section, we explore the difference of two connections A, A' on P. Clearly $\mathfrak{a}^P := A - A'$ annihilates $\ker(\pi_*)$. As a consequence, it defines a fiber-wise linear, G-equivariant map from π^*TM to $\mathfrak{g}$. So it corresponds to a G-equivariant section of $\mathfrak{g}_P \otimes T^*M$, where $\mathfrak{g}_P$ is the associated vector bundle of P defined in Example 1.2.7.

Conversely, suppose $\mathfrak{a}$ is a section of $\mathfrak{g}_P \otimes T^*M$, then we can give a fiber-wise linear, G-equivariant map $\mathfrak{a}^P : \pi^*TM \to \mathfrak{g}$, Then if A is a connection then $A + \mathfrak{a}^P$ is another one.

So if P has one connection, then it has infinitely many, and the space of connections over P, that is $\mathscr{A}(P)$ is an affine space based on $\mathfrak{g}_P \otimes T^*M$, and this is sometimes denoted by $\Omega^1_M(\mathfrak{g}_P)$.

1.4. Gauge transformations.

Now we introduce the guage group of a principal G-bundle $P \to M$.

Definition 1.4.1 (gauge group). *The gauge group is the group of G-automorphisms of P, that is, G-equivariant diffeomorphisms $\Phi : P \to P$ such that $\pi = \pi \circ \Phi$.*

Notation 1.4.2. We use $\mathscr{G}(P)$ to denote the gauge group of principal bundle P.

Definition 1.4.3 (gauge transformation). *An element of $\mathscr{G}(P)$ is called a gauge transformation.*

Proposition 1.4.4. *The group $\mathscr{G}(P)$ is isomorphic to the group of sections $\Gamma(M, \mathrm{Ad}\, P)$, where the group operation is pointwise multiplication.*

Proof. We provide maps in both directions. Suppose we have an automorphism $\Phi : P \to P$. Since $\pi = \pi \circ \Phi$, the map Φ preserves the fibers of π. Therefore, for any $p \in P$, we have that p and $\Phi(p)$ differ by the action of some $g_p \in G$. We claim that the mapping $g_\Phi : P \to G$ taking $p \mapsto g_p$ is equivariant with respect to the conjugation action of G. Indeed, what we need to do is to compute the difference between pg and $\Phi(pg)$. Use the fact that Φ is G-equivariant and $p = \Phi(p)g_p$, we have

$$\begin{aligned}\Phi(pg) &= \Phi(p)g \\ &= pg_p^{-1}g \\ &= pgg^{-1}g_p^{-1}g\end{aligned}$$

That is

$$pg = \Phi(pg)g^{-1}g_p g$$

So by Proposition 1.2.8, it do defines a section of $\operatorname{Ad} P$.

In the other direction, given a G-equivariant map $f : P \to G$, we get a bundle automorphism $\Phi_f : P \to P$ where $\Phi_f(p) = pf(p)$. Clearly $\pi \circ \Phi_f = \pi$, since

$$\begin{aligned}\pi \circ \Phi_f(p) &= \pi(pf(p)) \\ &= \pi(p)\end{aligned}$$

And it's G-equivariant since

$$\begin{aligned}\Phi_f(pg) &= pgf(pg) \\ &= pgg^{-1}f(p)g \\ &= pf(p)g \\ &= \Phi_f(p)g\end{aligned}$$

The two maps we constructed are clearly inverse to each other, giving the desired correspondence. □

The gauge group $\mathscr{G}(P)$ acts on the space of connections naturally. We can see this from many ways.

If we regard connections as a horizontal distribution. Let $H \subset TP$ be a connection and let $\Phi : P \to P$ be a gauge transformation. Define $H^\Phi := \Phi_* H$. We claim that H^Φ is also a connection on P. Indeed,

$$\begin{aligned}(R_g)_* H^\Phi_{\Phi(p)} &= (R_g)_* \Phi_* H_p \\ &= \Phi_* (R_g)_* H_p \\ &= \Phi_* H_{pg} \\ &= H^\Phi_{\Phi(pg)} \\ &= H^\Phi_{\Phi(p)g}\end{aligned}$$

that is, H^Φ is G-equivariant. Moreover, H^Φ is still complementary to vertical distribution V, since Φ_* is an isomorphism which preserves the vertical subspaces. Indeed,

Lemma 1.4.5. *The fundamental vector fields $\sigma(X)$ of the G-action are gauge invariant.*

Proof. We need to prove $\Phi_*\sigma(X) = \sigma(X)$ for all $\Phi \in \mathscr{G}(P)$. Compute directly by definition

$$\begin{aligned}\Phi_*\sigma(X) &= \Phi_*(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} pe^{tX}) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi(pe^{tX}) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi(pe^{tX}) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi(p)e^{tX} \\ &= \sigma(X)\end{aligned}$$

□

If we regard connections as a $\mathfrak{g}$-valued 1-form ω, what we can do is to pull it back. So we define $\omega^\Phi := (\Phi^{-1})^*\omega$. You may wonder why we use Φ^{-1} to pullback instead of Φ. The reason is that we want ω^Φ is exactly the connection form of H^Φ. Indeed, note that if $v \in \ker(\omega^\Phi) \subset TP$, then

$$0 = \omega^\Phi v = (\Phi^{-1})^*\omega(v) = \omega(\Phi_*^{-1} v)$$

So we have

$$\Phi_*^{-1} v \in \ker\omega = H$$

This implies that $v \in \Phi_* H = H^\Phi$, since by Lemma 1.4.5 we have Φ_* preserves vertical distribution, so the only thing in V mapped to zero by Φ_* is zero itself.

1.5. **Curvatures.**

1.5.1. *Curvatures of covariant derivatives.* For any vector bundle $\pi : E \to M$, and a section s of it, we already know how to define a covariant derivative

$$\nabla : C^\infty(M, E) \to C^\infty(M, \Omega^1_M(E))$$

However, in the case of smooth function, we can extend exterior derivative to more general case

$$\mathrm{d} : C^\infty(M, \Omega^k_M) \to C^\infty(M, \Omega^{k+1}_M)$$

such that $\mathrm{d}^2 = 0$. The reason of $\mathrm{d}^2 = 0$ is the following hallmark of partial derivatives: if f is a smooth function on $\mathbb{R}^n$, then

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} f = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} f$$

With the property of d, we can construct a cochain complex, that's what de Rham cohomology concern about.

So here we wonder, can we extend a covariant derivative ∇, to get similar things? The answer is yes. Since the soul of taking derivatives is Leibniz rule, we can extend ∇ by requiring it satisfy this rule. This extension is called the exterior covariant derivative, denoted by d_∇, and it is defined by the following rules:

1. If ω is a k-form and s is a section of E, define $\mathrm{d}_\nabla(s\omega) = \nabla s \wedge \omega + s\mathrm{d}\omega$.
2. If ω_1, ω_2 are sections of $\Omega^k_M(E)$, then $\mathrm{d}_\nabla(\omega_1 + \omega_2) = \mathrm{d}_\nabla\omega_1 + \mathrm{d}_\nabla\omega_2$.

Although $\mathrm{d}^2 = 0$, this is generally not the case for d_∇. It is th case, d^2_∇ defines a section of $\mathrm{End}(E) \otimes \bigwedge^2 T^*M$, in other words, a section of $\Omega^2_M(\mathrm{End}(E))$. This section is sometimes denoted by F_∇ and it is characterized by the fact that

$$\mathrm{d}^2_\nabla m = F_\nabla \wedge m$$

Indeed, suppose ω is a k-form and s is a section of E, then

$$\begin{aligned}
\mathrm{d}^2_\nabla(s\omega) &= \mathrm{d}_\nabla(\mathrm{d}_\nabla s \wedge \omega) + \mathrm{d}_\nabla(s\mathrm{d}\omega) \\
&= \mathrm{d}^2_\nabla s \wedge \omega - \mathrm{d}_\nabla s \wedge \mathrm{d}\omega + \mathrm{d}_\nabla s \wedge \mathrm{d}\omega + s \wedge \mathrm{d}^2\omega \\
&= \mathrm{d}^2_\nabla s \wedge \omega
\end{aligned}$$

In particular, if we take $\omega = f$ and thus a function, then

$$\mathrm{d}_\nabla(fs) = f\mathrm{d}_\nabla s$$

so by Lemma 1.3.3, we see that d^2_∇ is given by the action of a section of $\Omega^2_M(\mathrm{End}(E))$.

This claim can be check locally, if we choose a local trivialization U of E. And explicitly F_∇ on U can be regarded as $\mathrm{d}A + A \wedge A$, where A is a section of $\Omega^1_U(\mathrm{End}(E|_U))$.

The exterior covariant derivative of sections is defined as above in many textbooks. However, we will recover this from the connections on principal bundles viewpoint, and explain why $\mathrm{d}_\nabla \neq 0$.

1.5.2. *Curvatures of connections.* In this section, we will define the curvature of a connection on a principal G-bundle and interprete it geometrically in several different ways. Along the way we will define the covariant derivative of sections of associated vector bundle, and see that how does it connect with what we have defined.

Given a connection $H \subset TP$, we can define the horizontal projection $h : TP \to TP$ to be the projection onto the horizontal distribution along the vertical distribution. In other words, $\operatorname{im} h = H$ and $\ker h = V$. Since both H and V are invariant under the action of G, so is h.

Definition 1.5.1 (curvature 2-form). *Let $\omega \in \Omega^1_P(\mathfrak{g})$ be the connection 1-form, then we define curvature 2-form $\Omega := h^*\mathrm{d}\omega$*

Later we will give a explicit formula for Ω, but first let us interpret the curvature geometrically. By definition

$$\begin{aligned}\Omega(u,v)&=h^*\mathrm{d}\omega(u,v)\\&=d\omega(hu,hv)\\&=hu\cdot\omega(hv)-hv\cdot\omega(hu)-\omega([hu,hv])\\&=-\omega([hu,hv])\end{aligned}$$

[3]That is, $\Omega(u,v)=0$ if and only if $[hu,hv]$ is horizontal. In other words, the curvature of the connection measures the failure of integrability of the horizontal distribution $H\subset TP$.

Remark 1.5.2 (Frobenius integrability)**.** A distribution $D\subset TP$ is said to be integrable if the Lie bracket of any two sections of D lies again in D. The theorem of Frobenius states that a distribution is integrable if every $p\in P$ lies in a unique submanifold of P whose tangent space at p agrees with the subspace $D_p\subset T_pP$. These submanifolds are said to foliate P. As we have just seen, a connection $H\subset TP$ is integrable if and only if its curvature 2 -form vanishes.

In contrast, the vertical distribution $V\subset TP$ is always integrable, since the Lie bracket of two vertical vector fields is again vertical, and Frobenius's theorem guarantees that P is foliated by submanifolds whose tangent spaces are the vertical subspaces. These submanifolds are of course the fibres of $\pi:P\to M$.

For curvature Ω, it satisfies the following structure equation[4]

Proposition 1.5.3 (structure equation)**.**

$$\Omega=\mathrm{d}\omega+\frac{1}{2}[\omega,\omega]$$

where $[\ ,\ ]$ *is bracket of Lie algebra-valued forms in Remark* 1.3.13*.*

Proof. By definition, we need to show

$$\mathrm{d}\omega(hu,hv)=\mathrm{d}\omega(u,v)+[\omega(u),\omega(v)]$$

for all $u,v\in TP$, since we have

$$\frac{1}{2}[\omega,\omega](u,v)=[\omega(u),\omega(v)]$$

Let u,v be horizontal. In this case there is nothing to prove, since $\omega(u)=\omega(v)=0$ and $hu=u,hv=v$.

[3]The third equality we use the formula connecting the exterior derivative and the Lie bracket, that is $\mathrm{d}\omega(X,Y)=X\omega(Y)-Y\omega(X)-\omega([X,Y])$. The forth equality we use the fact that $h^*\omega=0$.

[4]That's also the definition of curvature in other references.

Let u, v be vertical. So we can take $u = \sigma(X), v = \sigma(Y)$ for some $X, Y \in \mathfrak{g}$. By directly computing

$$\begin{aligned}
0 &= d\omega(\sigma(X), \sigma(Y)) + [\omega(\sigma(X)), \omega(\sigma(Y))] \\
&= \sigma(X)Y - \sigma(Y)X - \omega([\sigma(X), \sigma(Y)]) + [X, Y] \\
&= -\omega([\sigma(X), \sigma(Y)]) + [X, Y] \\
&= -\omega(\sigma([X, Y])) + [X, Y]
\end{aligned}$$

as desired.

Finally, let u be horizontal and $v = \sigma(X)$ be vertical, then equation becomes

$$\mathrm{d}\omega(hu, X) = 0$$

which in turn reduces to

$$hu\omega(X) - \omega(hu)X + \omega([hu, X]) = \omega([hu, X]) = 0$$

In other words, we need to show that the Lie bracket of a vertical and a horizontal vector field is again horizontal. But this is simply the infinitesimal version of the G-invariance of H.

□

Immediately we have

Proposition 1.5.4 (Bianchi identity)**.**

$$h^*\mathrm{d}\Omega = 0$$

Proof. Directly computes

$$\begin{aligned}
h^* d\Omega &= h^* d\left(d\omega + \frac{1}{2}[\omega, \omega]\right) \\
&= h^*\left(\frac{1}{2}[d\omega, \omega] - \frac{1}{2}[\omega, d\omega]\right) \\
&= h^*[d\omega, \omega] \\
&= [h^* d\omega, h^*\omega] \\
&= 0
\end{aligned}$$

□

Since we already see how connections transform under a gauge transformation $\Phi : P \to P$, it's natural to ask how dose curvature transform. From structure equation, it's clear that

$$\Omega \mapsto \Omega^\Phi = (\Phi^{-1})^*\Omega$$

1.5.3. *Relations of connections and covariant derivatives.* Although we have seen that some relationship between connections on principal bundles and covariant derivatives on vector bundles, here we will give a more detailed explaination here, along the way we will define many definitions we will use later.

As a warm-up, we need to do is to have a better understanding of the relation between forms on P and forms on M.

Definition 1.5.5 (horizontal form). *A k-form $\alpha \in \Omega^k_P$ is horizontal if $h^*\alpha = \alpha$.*

Definition 1.5.6 (basic form). *A k-form $\alpha \in \Omega^k_P$ is basic, if it's horizontal and it is G-invariant.*

Remark 1.5.7. It is a fact that α is basic if and only if $\alpha = \pi^*\overline{\alpha}$ for some $\overline{\alpha} \in \Omega^k_M$. In other words, the set of basic k-forms of P are exactly $\pi^*\Omega^k_M$.

Let's check one direction for the case $k = 1$, other cases are quite similar. If $\alpha = \pi^*\overline{\alpha}$ for some $\overline{\alpha} \in \Omega^1_M$, then for some $v \in TP$

$$\begin{aligned} h^*\alpha(v) &= \alpha(hv) \\ &= \pi^*\overline{\alpha}(hv) \\ &= \overline{\alpha}(\pi_* hv) \\ &= \overline{\alpha}(\pi_* v) \\ &= \pi^*\overline{\alpha}(v) \\ &= \alpha(v) \end{aligned}$$

that is, α is horizontal. Now let's check α is G-invariant, we need to show

$$R_g^*\alpha = \alpha$$

Take $v \in TP$ and check directly as follows

$$\begin{aligned} R_g^*\alpha(v) &= \alpha((R_g)_* v) \\ &= \overline{\alpha}(\pi_*(R_g)_* v) \\ &= \overline{\alpha}((\pi \circ R_g)_* v) \\ &= \overline{\alpha}(\pi_* v) \\ &= \alpha(v) \end{aligned}$$

This completes the check.

The same story happens in the case of forms on P taking value in a vector space V admitting a representation $\rho : G \to \mathrm{GL}(V)$ of G.

Let $\alpha \in \Omega^k_P(V)$. Similarly we can define horizontal form and basic form, that is α is horizontal if $h^*\alpha = \alpha$ and α is basic if it's horizontal and G-invariant. For G-invariant it means for all $g \in G$,

$$R_g^*\alpha = \rho(g^{-1})\alpha$$

Since there is a G-action on V, and α take values in V, so $\rho(g^{-1})\alpha$ make sense. If we use $\Omega^k_{\mathrm{G}}(P;V)$ to denote the set of basic forms on P taking values in V, then

$$\Omega^k_{\mathrm{G}}(P;V) = \left\{\alpha \in \Omega^k_P(V) \mid h^*\alpha = \alpha \text{ and } R_g^*\alpha = \rho(g^{-1})\alpha\right\}$$

We claim that basic forms are in one to one correspondence with forms on M with values in the associated bundle $P \times_G V$. That is,

$$\Omega^k_{\mathrm{G}}(P;V) \Longleftrightarrow \Omega^k_M(P \times_G V)$$

With this isomorphism, we can handle with V-valued forms on P with some conditions, instead of bundle-valued forms on M.

This is amazing result, since forms on P valued in V is just a section of trivial bundle $P \times V \otimes T^*P$, but forms on M valued E is a section of non-trivial bundle $E \otimes T^*M$

The exterior derivative $\mathrm{d} : \Omega^k_P(V) \to \Omega^{k+1}_P(V)$ defines a V-valued de Rham complex. However, if we want to descend d to basic forms, something bad happens, since basic forms do not form a subcomplex. Indeed, since $\mathrm{d}\alpha$ need not to be horizontal even if α is. If we want to basic forms to form a complex, a naive way is to project $\mathrm{d}\alpha$ onto horizontal forms, i.e.

$$\begin{aligned} \mathrm{d}^H : \Omega^k_G(P;V) &\to \Omega^{k+1}_G(P;G) \\ \alpha &\mapsto h^*\mathrm{d}\alpha \end{aligned}$$

but the price we pay is that $(\mathrm{d}^H)^2 \neq 0$ in general, so we no longer have a complex. As we see soon, the failure of d^H defining a complex is measured by the curvature.

Let's give a explicit formula of d^H. Take $k = 0$ as an example. Every section $\zeta \in \Omega^0_M(P \times_G V)$ defines an equivalent function $\overline{\zeta} \in \Omega^0_G(P;V)$ obeying $R_g^*\overline{\zeta} = \rho(g^{-1})\bar{\zeta}$. By definition, the exterior covariant derivative is given by $\mathrm{d}^H\overline{\zeta} = h^*\mathrm{d}\overline{\zeta}$. Applying this to a vector field $u = u_V + hu$, we have

$$\left(d^H\bar{\zeta}\right)(u) = d\bar{\zeta}(hu) = d\bar{\zeta}\left(u - u_V\right) = d\bar{\zeta}(u) - u_V(\bar{\zeta})$$

What we need to do is to evaluate $u_V\bar{\zeta}$. Take $u_V = \sigma(\omega(u))$, so that

$$u_V\bar{\zeta} = \sigma(\omega(u))\bar{\zeta} = \left.\frac{d}{dt}\right|_{t=0} R^*_{g(t)}\bar{\zeta} \quad \text{for } g(t) = e^{t\omega(u)}$$

Since $\bar{\xi}$ is G-equivariant, we have

$$u_V\bar{\zeta} = \left.\frac{d}{dt}\right|_{t=0} \rho\left(g(t)^{-1}\right) \circ \bar{\zeta} = -\varrho(\omega(u)) \circ \bar{\zeta}$$

that is,

$$\mathrm{d}^H\overline{\zeta} = \mathrm{d}\zeta + \rho(\omega)\overline{\zeta}$$

If we apply twice, we have

$$\begin{aligned}\left(d^{\mathrm{H}}\right)^2 \bar{\zeta} &= h^* d h^* d\bar{\zeta} \\ &= h^* d(d\bar{\zeta} + \varrho(\omega) \circ \bar{\zeta}) \\ &= h^*(\varrho(d\omega) \circ \bar{\zeta} - \varrho(\omega) \wedge d\bar{\zeta}) \\ &= \rho\left(h^* d\omega\right) \circ \bar{\zeta} \\ &= \varrho(\Omega) \circ \bar{\zeta}.\end{aligned}$$

So curvature measures the failure of d^H to be a complex.

In the following, we will change the notation and write the exterior covariant derivative on basic forms as

$$\mathrm{d}^\omega : \Omega^k_G(P;V) \to \Omega^{k+1}_G(P;V)$$

to show the dependence on the connection 1-form, and write the one on bundle valued forms on M by

$$\mathrm{d}_A : \Omega^k_M(P \times_G V) \to \Omega^{k+1}_M(P \times_G V)$$

And we will use F_A to denote curvature of connection A. In this notation, the Bianchi identity for the curvature can be written as $\mathrm{d}_A F_A = 0$.

For the most part, we will be concerned with the case when the vector bundle is $\mathfrak{g}_P$. In which case, the formula is given by

$$\mathrm{d}_A \psi = \mathrm{d}\psi + [A \wedge \psi], \quad \psi \in \Omega^k_M(\mathfrak{g}_P)$$

Since $\mathscr{A}(P)$ is an affine space over $\Omega^1_M(\mathfrak{g}_P)$, given a connection $A \in \mathscr{A}(P)$ and a $\mathfrak{g}_p$-valued 1-form $\eta \in \Omega^1_M(g_p)$, we have that $A+\eta$ is also a connection. It can be shown that the curvature of $A+\eta$ is given by

$$F_{A+\eta} = F_A + \frac{1}{2}[\eta \wedge \eta] + d_A \eta$$

In particular, if we take a line of connections $A + t\eta$ with $t \in \mathbb{R}$, we have

$$\left.\frac{d}{dt}\right|_{t=0} F_{A+t\eta} = \left.\frac{d}{dt}\right|_{t=0} F_A + \frac{t^2}{2}[\eta \wedge \eta] + t d_A \eta = d_A \eta$$

So the exterior covariant derivative on g_p measures the infinitesimal change of the curvature of A in the direction η.

1.6. **Flat connections and holonomy.**

1.6.1. *Flat connections.*

Definition 1.6.1 (flat connection)**.** *A connection on a principal bundle is said to be flat when its curvature* 2*-form is identically zero.*

Example 1.6.2. As we have seen, the simplest example is the connection on the trivial principal bundle $M \times P$ that is defined by $\omega = g^{-1}\mathrm{d}g$.

However, there are other flat connections.

Example 1.6.3. Let $h : M \to G$ denote any given smooth map. This defines a principal bundle isomorphism $\varphi : M \times G \to M \times G$ that send (x, g) to $(x, h(x)g)$. Then consider the pullback of ω

2. The Yang-Mills equations

Untill now we have imposed no conditions on M or on G, but now things will change. To discuss the Yang-Mills equations, we will restrict to compact Lie groups G, and M will be a oriented Riemannian manifold with metric g. The orientation on M is given by a nowhere-vanishing n-form, which we will take to be the volume form of the metric.

2.1. **Some geometry.**

2.1.1. *The volume form and Hodge star operator.* Since we have an orientation and a Riemannian metric g on M. This gives us

1. A Riemannian volume form $\mathrm{vol}_g \in \Omega^n_M$.
2. A Hodge star operator $* : \Omega^k_M \to \Omega^{n-k}_M$.

How could we get a Riemannian metric and a Hodge star operator from g? If $(x^1, \ldots, x^n)$ are oriented local coordinate, then we define Riemannian volume as

$$\mathrm{vol}_n = \mathrm{d}V = \sqrt{\det(g_{ij})}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x_n$$

But for Hodge star operator, it is more complicated and interesting. In general, Let V be an n-dimensional vector space with a nondegenerate symmtric bilinear form $\langle \cdot, \cdot \rangle$. This induces an inner product on k-vectors $\alpha, \beta \in \bigwedge^k V$ for $0 \leq k \leq n$, by defining it on decomposition $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k, \beta = \beta_1 \wedge \cdots \wedge \beta_k$ to equal

$$\langle \alpha, \beta \rangle = \det(\langle \alpha_i, \beta_j \rangle_{i,j=1}^k)$$

extended to $\bigwedge^k V$ through linearity.

Define unit n-vector $\omega \in \bigwedge^n V$ in term of an oriented orthonormal basis $\{e_1, \ldots, e_n\}$ of V as

$$\omega := e_1 \wedge \cdots \wedge e_n$$

Now we can define Hodge star operator, mapping k-vector to $(n-k)$-vector, has the following property, which defines it completely:

$$\alpha \wedge (*\beta) = \langle \alpha, \beta \rangle \omega \tag{*}$$

Dually, in the space $\bigwedge^n V^*$ of n-forms, what is dual of ω? By definition,

$$\omega^*(v_1, \ldots, v_n) = \langle \omega, v_1 \wedge \cdots \wedge v_n \rangle = \det = \det((v_i^j)_{i,j=1}^n)$$

So ω^* is the volume form det, the function whose value on $v_1 \wedge \cdots \wedge v_n$ is the determinat of the $n \times n$ matrix assembled from the column vectors of v_i in e_i-coordinates.

Applying det to $(*)$, we have

$$\det(\alpha \wedge *\beta) = \langle \alpha, \beta \rangle$$

If we take V as tangent space T_pM with respect to a (not necessary orthonormal) basis $\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\}$, having the metric $(g_{ij})=(\langle\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\rangle)$. Then in its dual space T_p^*M we have a dual basis $\{\mathrm{d}x_1,\dots,\mathrm{d}x_n\}$ with metric matrix (g^{ij}). Then the hodge dual of a decomposable k-form is

$$*(\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k})=\frac{\sqrt{|\det(g_{ij})|}}{(n-k)!}g^{i_1j_1}\dots g^{i_kj_k}\varepsilon_{j_1\dots j_n}\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_n}$$

where $\varepsilon_{j_1\dots j_n}$ is the Levi-Civita symbol.

It's not hard to see the following relation between volume form and Hodge star operator

$$\mathrm{vol}=*(1)$$

Furthermore, on Ω^k_M, we have an identity

$$*^2=(-1)^{k(n-k)}\,\mathrm{id}$$

2.1.2. *Inner product on bundle valued forms.* Now we would also like to define a inner product on forms with values in an associated vector bundle $P\times_G V$. If we do this locally, on each U_α, view such forms as forms with values in V.

Since G is compact, its Lie algebra $\mathfrak{g}$ is semisimple, so the Killing form $\langle\cdot,\cdot\rangle$ is nondegenerate. For the rest of our discussion, Killing form can be replaced by any inner product invariant under the Adjoint action, though it does us no harm to assume that it is the Killing form.

Lemma 2.1.1. *Let $\langle\cdot,\cdot\rangle$ denote any Adjoint invariant inner product on $\mathfrak{g}$. Then for $X_1,X_2,X_3\in\mathfrak{g}$, we have*

$$\langle[X_1,X_2],X_3\rangle=\langle X_1,[X_2,X_3]\rangle$$

Proof. Just note that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathrm{Ad}_{\exp(-tX_2)}X_1&=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\exp(-tX_2)X_1\exp(tX_2)\\&=-X_2(X_1\exp(tX_2))+\exp(-tX_2)(X_1X_2)\exp(tX_2)]\Big|_{t=0}\\&=-X_2X_1+X_1X_2\\&=-[X_2,X_1]\end{aligned}$$

and use the fact that inner product is Adjoint invariant. □

The form $\langle\cdot,\cdot\rangle$ induces a fiber metric on $P\times\mathfrak{g}$, and invariance under the Adjoint action tells us that this fiber metric descends to a fiber metric on $\mathfrak{g}_p$. This gives us pairing.

$$\begin{aligned}\Omega^k_M(\mathfrak{g}_p)\otimes\Omega^\ell_M(\mathfrak{g}_p)&\to\Omega^{k+\ell}_M\\\omega\otimes\eta&\mapsto(\omega^i\wedge\eta^j)\langle\xi_i,\xi_j\rangle\end{aligned}$$

2.2. **The variational problem.** With some of the preliminary results established, we arrive at the Yang-Mills functional.

Definition 2.2.1 (Yang-Mills functional)**.** *The Yang-Mills functional is the map $L : \mathscr{A}(P) \to \mathbb{R}$ given by*

$$L(A) := \|F_A\| = \int_M \langle F_A \wedge *F_A \rangle$$

Remark 2.2.2. Note that for any gauge transformation $\varphi \in \mathscr{G}(P)$, we have $L(\varphi^* A) = L(A)$. Indeed,

$$L(\varphi^* A) = \int_X \langle \operatorname{Ad}_{g_\varphi^{-1}} F_A \wedge * \operatorname{Ad}_{g_\varphi^{-1}} F_A \rangle = \int_X \langle F_A \wedge *F_A \rangle = L(A)$$

because of this we say that L is gauge invariant.

The Yang-Mills equations are the variational equations for the Yang-Mills functional.

Proposition 2.2.3 (The first variation)**.** *Let A be a local extremum of L. Then we have*

$$\mathrm{d}_A * F_A = 0$$

Proof. Let $\eta \in \Omega^1_X(\mathfrak{g}_P)$. We then compute

$$\begin{aligned} L(A+\eta) &= \int_X \langle F_{A+t\eta} \wedge *F_{A+t\eta} \rangle \\ &= \int_X \langle F_A + \frac{t^2}{2}[\eta \wedge \eta] + t\mathrm{d}_A\eta \wedge *(F_A + \frac{t^2}{2}[\eta \wedge \eta] + t\mathrm{d}_A\eta) \end{aligned}$$

The term linear t is

$$\int_X \langle F_A \wedge *\mathrm{d}_A\eta \rangle + \langle \mathrm{d}_A\eta \wedge *F_A \rangle = 2(F_A, \mathrm{d}_A\eta)$$

Let $\mathrm{d}_A^* = (-1)^{2n+1} * \mathrm{d}_A *$ denote the formal adjoint to d_A. Since A is a local extremum, the term linear in t must vanish, so for every η we must have

$$(F_A, \mathrm{d}_A\eta) = (\mathrm{d}^* F_A, \eta) = 0$$

Then since up to sign $\mathrm{d}_A^* = *\mathrm{d}_A*$ and $*$ is an isomorphism, we have $\mathrm{d}_A * F_A = 0$. ☐

The first variation gives us what are referred to as the Yang-Mills equations

$$\begin{cases} \mathrm{d}_A F_A = 0 \\ \mathrm{d}_A^* F_A = 0 \end{cases}$$

Definition 2.2.4 (Yang-Mills connection)**.** *A Yang-Mills connection is a connection $A \in \mathscr{A}(P)$ satisfying the Yang-Mills equations, i.e. a local extremum of L.*

The first equation is simply the **Bianchi identity** and the second comes from the first variation

Proposition 2.2.5 (The second variation)**.** *Let A be a Yang-Mills connection. Then for every $\eta \in \Omega^1_M(\mathfrak{g}_P)$, we have*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{d}^*_{A+t\eta} F_{A+t\eta} = \mathrm{d}^*_A \mathrm{d}_A \eta + \star[\eta \wedge \star F_A]$$

Remark 2.2.6. If we think L as a Morse function on $\mathscr{A}(P)$, for a Yang-Mills connection A, the operator $\mathrm{d}^*_A \mathrm{d}_A \eta + *[\eta \wedge *F_A]$ can be interpreted as the Hessian of L at the critical point A.

Now we restrict ourselves to the case where M is a Riemann surface of genus $g \geq 2$. Let $\Gamma_{\mathbb{R}}$ denote the central extension of $\pi_1(X)$ by $\mathbb{R}$ where if we let J denote the element $1 \in \mathbb{R}$, we have the relation $\prod_i [a_i, b_i] = J$ where a_i and b_i are the generators for the usual presentation of a closed surface of genus g. Using this group one can prove the following theorems, though we will omit the proofs.

Theorem 2.2.7. *Every principal G-bundle $P \to X$ admits a Yang-Mills connection.*

Theorem 2.2.8. *There is a bijective correspondence*

$$\mathrm{Hom}(\Gamma_{\mathbb{R}}, G)/G \longleftrightarrow \{\textit{Principal } G\textit{-bundles } P \to X \textit{ with a Yang-Mills connection}\}/\sim$$

where the action of G is conjugation and the equivalence relation is gauge equivalence.

3. Holomorphic vector bundles and Yang-Mills Connections

3.1. **Stability of holomorphic vector bundles.**

In this section, we will restrict to case where $G = U_n$. and let X denote a complex manifold.

Definition 3.1.1 (holomorphic vector bundle)**.** *A holomorphic vector bundle is a complex bundle $\pi : E \to X$ such that the total space E is a complex manifold and π is holomorphic.*

Remark 3.1.2. Given a holomorphic vector bundle $E \to X$, we can find a trivialization of E such that the transition functions are holomorphic. In a neighborhood $U \subset X$ such that $E|_U$ is holomorphically trivial, the smooth sections can be identified with smooth functions $U \to \mathbb{C}^n$, and the holomorphic sections can be identified with holomorphic functions $U \to \mathbb{C}^n$.

We have a local operator $\overline{\partial}$, which we can apply componentwise to a local section to get an operator on smooth sections over U. Furthermore, since $\overline{\partial}$ annihilates holomorphic functions and the transition functions are holomorphic, we have that $\overline{\partial}$ glues to a well defined operator $\overline{\partial}_E : \mathcal{A}^0_X(E) \to \mathcal{A}^{0,1}_X(E)$. The holomorphic sections of E are exactly the sections annihilated by $\overline{\partial}_E$. Furthermore, the operator $\overline{\partial}_E$ extends to operator $\overline{\partial}_E : \mathcal{A}^k_X(E) \to \mathcal{A}^{k+1}_X(E)$, and satisfies the condition $\overline{\partial}^2_E = 0$, since $\overline{\partial}^2 = 0$. The punchline is that the holomorphic structure on E is entirely determined by this operator.

Theorem 3.1.3. *Let $\pi : E \to X$ be a smooth complex vector bundle, and let $D : \mathcal{A}^0_X(E) \to \mathcal{A}^{0,1}(E)$ be an operator satisfying the Leibniz rule and $D^2 = 0$. Then there exists a unique complex structure on E such that E is holomorphic and the holomorphic sections are exactly the ones annihilated by D.*

We now restrict to the case where X is a Riemann surface of genus $g \geq 2$

Definition 3.1.4 (slope)**.** *The slope of a holomorphic vector bundle $E \to X$ is*

$$\mu(E) := \frac{c_1(E)}{\operatorname{rank}(E)}$$

where we think of $c_1(E) \in H^2(X, \mathbb{Z})$ as an integer via integration over X.

Remark 3.1.5. Sometimes the integer $c_1(E)$ is also referred to as the degree of E. One thing to note is that the slope of a holomorphic vector bundle is independent of the holomorphic structure. Both of degree and rank are topological invariants, and only depend on the underlying C^∞ complex vector bundle.

Definition 3.1.6 (stablity of holomorphic bundles)**.** *A holomorphic vector bundle $E \to X$ is*

1. *Stable if for every holomorphic subbundle $F \subset E$, we have $\mu(F) < \mu(E)$.*
2. *Semistable if for every holomorphic subbundle $F \subset E$, we have $\mu(F) \leq \mu(E)$.*
3. *Unstable if E is not semistable.*

While the slope is a topological invariant, stability is not, since we only consider holomorphic subbundles which depend on the holomorphic structure. We also note that both the degree and rank are additive in exact sequences, which immediately give us

Proposition 3.1.7. *Suppose we have the short exact sequence of holomorphic bundles*

$$0 \to E \to F \to G \to 0$$

Then we have

$$\mu(F) = \frac{\deg(E) + \deg(G)}{\operatorname{rank}(E) + \operatorname{rank}(G)}$$

Corollary 3.1.8. *Given a short exact sequence of holomorphic bundles*

$$0 \to E \to F \to G \to 0$$

If $\mu(E) \geq \mu(F)$, then $\mu(F) \geq \mu(G)$. Likewise, if $\mu(E) \leq \mu(F)$, then $\mu(F) \leq \mu(G)$.

In other words, slopes behave monitonically in short exact sequences. The terminology comes from GIT. The main result will use is

Theorem 3.1.9 (The Harder-Narasimhan Filtration). *Let $E \to X$ be a holomorphic vector bundle. Then E admits a canonical filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

by holomorphic subbundles E_i such that E_i/E_{i-1} is semistable and

$$\mu(E_1/E_0) > \mu(E_2/E_1) > \cdots > \mu(E_n/E_{n-1})$$

Proof. Sketch. The main idea is that any holomorphic vector bundle has a unique maximal semistable subbundle, which we take to be E_1. We then take E_2 to be the preimage of the maximal semistable bundle of E_1/E_0 under the quotient map, and continue inductively. $\square$

Remark 3.1.10. The slopes $\mu_i := \mu(E_i/E_{i-1})$ gives us n rational numbers. If k denotes the rank E, then we construct an element of $\mathbb{Q}^k$ by arranging the μ_i in order, and repeating the entry μ_i a total of $\operatorname{rank}(E_i/E_{i-1})$ times. We call this vector the Harder-Narasimhan type of E.

Our ultimate goal will be relate moduli spaces of holomorphic vector bundles over X to Yang-Mills connections. To see this, let $E \to X$ be a C^∞ complex vector bundle of rank n, and fix a Hermitian metric on E. Then let $P \to X$ denote the principal U_n-bundle of frames for E. We abbreviate the gauge group $\mathscr{G}(P)$ as $\mathscr{G}$.

Proposition 3.1.11. *There is a bijection $\mathscr{A}(P) \longleftrightarrow \mathscr{C}(E)$*

Proof. We provide maps in both directions. Suppose we have a connection $A \in \mathscr{A}(P)$. Then A induces a covariant derivative $\mathrm{d}_A : \mathcal{A}^0_X(E) \to \mathcal{A}^1_X(E)$. The $(0,1)$ part of d_A automatically satisfies $(\mathrm{d}^{0,1}_A)^2 = 0$, since $\mathcal{A}^2_X = 0$ by dimension reasons. Therefore, $\mathrm{d}^{0,1}_A$ defines a holomorphic structure on E.

In other direction, given a holomorphic structure $\overline{\partial_E}$, there exists a unique Hermitian connection A such that $\mathrm{d}^{0,1}_A = \overline{\partial}_E$, called the Chern connection. $\square$

Let $\mathscr{G}_{\mathbb{C}}$ denote the group of smooth bundle automorphisms of E. The space $\mathscr{C}(E)$ has a natural action by $\mathscr{G}_{\mathbb{C}}$ by conjugation. Furthermore, the orbits under this action are exactly the isomorphism classes of holomorphic structures on E. This is most easily seen by characterizing an isomorphism $\varphi : E \to F$ of holomorphic vector bundles as a smooth bundle isomorphism intertwing $\overline{\partial}_E$ and $\overline{\partial}_F$. However, the naive quotient $\mathscr{C}(E)/\mathscr{G}_{\mathbb{C}}$ is poorly behaved. To remedy this, as in GIT, we restrict our attension to semistable bundles.

The relationship between $\mathscr{G}_{\mathbb{C}}$ and $\mathscr{G}$ as well as the identification of $\mathscr{A}(P)$ and $\mathscr{C}(E)$ suggests that isomorphism classes of holomorphic bundles should have something to do with gauge equivalence classes of connections on P. This turn out to be true, and is an infinite dimensional version of the relationship between GIT quotient and a symplectic quotient.

To investigate further, we make a short digression regarding this relationship.

Let G be a reductive complex group, and X a compact Kähler manifold with Kähler metric ω, equipped with a "nice" action of G. In usually setting, X is a smooth projective variety with a fixed embedding $X \hookrightarrow \mathbb{CP}^N$, the Kähler metric ω is the restriction of the Fubini-Study form, and the G-action is induced by a homomorphism $G \to \mathrm{GL}_{N+1}(\mathbb{C})$. In general, the naíve quotient X/G is not well behaved, and one restricts the action to a subset X_{ss} consisting of semistable points to construct the GIT quotient X_{ss}/G.

3.2. **Episode: Fubini-Study metric.** This section contains some basic facts of Kähler manifold and Kähler metric on it. In particular, we will focus on $\mathbb{P}^n$ and Fubini-Study metric[5] on it.

Let $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ be the canonical coordinate cahrt. For $1 \le i, j \le n$, we define a function on $\mathbb{C}^n$ by

$$h_{ij}(z) = \frac{(1 + \sum_{k=1}^n |z^k|^2)\delta_{ij} - \overline{z}^i z^j}{(1 + \sum_{k=1}^n |z^k|^2)^2}$$

First we claim that

Proposition 3.2.1. *The matrix valued function $H(z) = (h_{ij}(z))_{i,j=1}^n$ is smooth and $H(z)$ is a positive definite Hermitian matrix for all $z \in \mathbb{C}$.*

Proof. Smooth and Hermitian is clear. We focus on its positive definiteness: Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{C}^n$ and $\|\cdot\|$ the norm induced from it. For each $z, w \in \mathbb{C}^n$, we have

$$\begin{aligned}\langle H(z)w, w\rangle &= \sum_{i=1}^n (\sum_{j=1}^n h_{ij}(z) w^j) w^i = \sum_{i=1}^n (\sum_{j=1}^n \frac{(1 + \sum_{k=1}^n |z^k|^2)\delta_{ij} - \overline{z}^i z^j}{(1 + \sum_{k=1}^n |z^k|^2)^2} w^j) w^i \\ &= \frac{(1 + \|z\|^2)\|w\|^2 - |\langle z, w\rangle|^2}{(1 + \|z\|^2)^2}\end{aligned}$$

By Cauchy-Schwarz inequality,

$$|\langle z, w\rangle|^2 \le \|z\|^2 \|w\|^2$$

Hence we find

$$\langle H(z)w, w\rangle \ge \frac{\|w\|^2}{(1 + \|z\|^2)^2} \ge 0$$

we find that $\langle H(z)w, w\rangle = 0$ implies that $\|w\|^2 = 0$. Therefore $w = 0$. This shows that $H(z)$ is positive for each $z \in \mathbb{C}^n$. ☐

Let us define a function K on $\mathbb{C}^n$ by

$$K(z) = \log(1 + \sum_{k=1}^n |z^k|^2)$$

[5]Trivia: Here the pronunciation of "Study" is "SHTOO-dee".

Let us compute $\overline{\partial} K$ as follows

$$\overline{\partial} K = \frac{\sum_{j=1}^n z^j \mathrm{d}\overline{z^j}}{1+\sum_{k=1}^n |z^k|^2}$$

Hence we have

$$\begin{aligned}
\partial\overline{\partial} K &= \sum_{i,j=1}^n \frac{\delta_{ij}(1+\|z\|^2) - \overline{z}^i z^j}{(1+\|z\|^2)^2} \mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j \\
&= \sum_{i,j=1}^n h_{ij}(z) \mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j
\end{aligned}$$

For each $0 \leq \alpha \leq n$, we define a $(1,1)$-form on U_α by

$$\omega_\alpha = \sqrt{-1} \sum_{i,j=1}^n (h_{ij} \circ \varphi_\alpha) \mathrm{d}z_\alpha^i \wedge \mathrm{d}\overline{z}_\alpha^j$$

This implies that if we denote $K_\alpha = K \circ \varphi_\alpha$ on U_α, then

$$\omega_\alpha = \sqrt{-1}\partial\overline{\partial} K_\alpha$$

Use the decomposition $\mathrm{d} = \partial + \overline{\partial}$, we can find the following result by computing directly

$$\mathrm{d}\omega_\alpha = 0$$

This shows that ω_α is a closed $(1,1)$-form on U_α for each α. In fact, we can show that the $(1,1)$-form on each U_α can be glued together to obtain a closed global $(1,1)$-form ω on $\mathbb{P}^n$.

On U_α, let us rewrite K_α as follows

$$\begin{aligned}
K_\alpha(\xi_0 : \cdots : \xi_n) &= \log(1 + \sum_{k=1}^n |z_\alpha^k(\xi_0 : \cdots : \xi_n)|^2) \\
&= \log(1 + \sum_{k=0}^n (\frac{\xi_k}{\xi_\alpha})^2) \\
&= \log(\sum_{i=0}^n |\xi_i|^2) - \log|\xi_\alpha|^2
\end{aligned}$$

Assume that $\alpha < \beta$, then on $U_\alpha \cap U_\beta$, we have

$$\begin{aligned}
K_\alpha(\xi_0 : \cdots : \xi_n) - K_\beta(\xi_0 : \cdots : \xi_n) &= \log|\xi_\beta|^2 - \log|\xi_\alpha| = \log|\frac{\xi_\beta}{\xi_\alpha}|^2 \\
&= \log|z_\alpha^\beta(\xi_0 : \cdots : \xi_n)|^2
\end{aligned}$$

In other words, on $U_\alpha \cap U_\beta$, we have

$$K_\alpha - K_\beta = \log|z_\alpha^\beta|^2$$

so we have

$$\overline{\partial}(K_\alpha - K_\beta) \implies \partial\overline{\partial}(K_\alpha - K_\beta) = 0 \quad \text{on } U_\alpha \cap U_\beta$$

Hence we can get a global $(1,1)$-form ω on $\mathbb{P}^n$. Since ω is globally defined, the 2-tensor

$$\sum_{i,j=1}^{n}(h_{ij}\circ\varphi_\alpha)\mathrm{d}z_\alpha^i\otimes\mathrm{d}\overline{z}_\alpha^j$$

is also globally defined. Since we have already verify that $H(z)$ is a positive definite matrix on $\mathbb{C}^n$, then above equation defines a Hermitian metric on $\mathbb{P}^n$ whose associated $(1,1)$-form is ω. Thus we conclude that

Theorem 3.2.2. *$\mathbb{P}^n$ is a Kähler manifold.*

The Kähler metric on $\mathbb{P}^n$ is called Fubini-Study metric.

Let's compute a concrete case for $\mathbb{P}^1$. Let (ξ_0,ξ_1) be the standard coordinate on $\mathbb{C}^2$ and (ξ_0,ξ_1) be the corresponding coordinate on $\mathbb{P}^1$. Let U,V be the canonical covering of $\mathbb{P}^1$. and z,w be the coordinate function on U and V. In other words,

$$z(\xi_0:\xi_1)=\frac{\xi_1}{\xi_0},\quad w(\xi_0:\xi_1)=\frac{\xi_0}{\xi_1}$$

Let $K_U=\log(1+|z|^2)$ on U and $K_V=\log(1+|w|^2)$ on V. Observe that

$$\begin{aligned}K_U(\xi_0:\xi_1)&=\log(|\xi_0|^2+|\xi_1|^2)-\log|\xi_0|^2\\K_V(\xi_0:\xi_1)&=\log(|\xi_0|^2+|\xi_1|^2)-\log|\xi_1|^2\end{aligned}$$

Hence on $U\cap V$ we have

$$K_U-K_V=\log|z|^2$$

Since $|z|^2=z\overline{z}$, we have

$$\overline{\partial}(K_U-K_V)=\frac{1}{\overline{z}}\mathrm{d}\overline{z}$$

Then we have

$$\partial\overline{\partial}(K_U-K_V)=\partial(\frac{1}{\overline{z}})\wedge\mathrm{d}\overline{z}=0$$

This shows that on $U\cap V$, we indeed have

$$\partial\overline{\partial}K_U=\partial\overline{\partial}K_V$$

then global defined $(1,1)$-form ω is

$$\omega=\begin{cases}\sqrt{-1}\partial\overline{\partial}K_U & \text{on } U\\\sqrt{-1}\partial\overline{\partial}K_V & \text{on } V\end{cases}$$

Let's compute what does ω exactly look like. We know that $\overline{\partial} K_U = \frac{z}{1+|z|^2}\mathrm{d}\overline{z}$. And hence

$$\begin{aligned}\partial\overline{\partial}K_U &= \partial(\frac{z}{1+|z|^2}) \wedge \mathrm{d}\overline{z} \\ &= \frac{(1+|z|^2)\partial z - z\partial(1+|z|^2)}{(1+|z|^2)^2} \wedge \mathrm{d}\overline{z} \\ &= \frac{(1+|z|^2)\mathrm{d}z - |z|^2\mathrm{d}z}{(1+|z|^2)^2} \wedge \mathrm{d}\overline{z} \\ &= \frac{1}{(1+|z|^2)^2}\mathrm{d}z \wedge \mathrm{d}\overline{z}\end{aligned}$$

Similarly we can compute

$$\partial\overline{\partial}K_V = \frac{1}{(1+|w|^2)^2}\mathrm{d}\omega \wedge \mathrm{d}\overline{\omega}$$

So we have the Kähler metric $\mathrm{d}s^2$ on $\mathbb{P}^n$ is given by

$$ds^2 = \begin{cases} \frac{1}{(1+|z|^2)^2} dz \otimes d\overline{z} & \text{on } U \\ \frac{1}{(1+|w|^2)^2} dw \otimes d\overline{w} & \text{on } V \end{cases}$$

And check this metric is globally defined. On $U \cap V$, we have $\omega = \frac{1}{z}$ and $\overline{w} = \frac{1}{\overline{z}}$. Hence we have

$$\mathrm{d}\omega \otimes \mathrm{d}\overline{\omega} = \frac{1}{z^2}\mathrm{d}z \otimes \frac{1}{\overline{z}^2}\mathrm{d}\overline{z} = \frac{1}{|z|^4}\mathrm{d}z \otimes \mathrm{d}\overline{z}$$

then

$$\frac{1}{(1+|z|^2)^2}\mathrm{d}z \otimes \mathrm{d}\overline{z} = \frac{|z|^4}{(1+|z|^2)^2}\frac{\mathrm{d}z \otimes \mathrm{d}\overline{z}}{|z|^4} = \frac{1}{(|w|^2+1)^2}\mathrm{d}w \otimes \mathrm{d}\overline{w}$$

We prove that $\mathrm{d}s^2$ is a globally defined Hermitian metric.

3.3. **Symplectic reduction.** Now back to topical subject. Let $K \subset G$ denote the maximal compact subgroup, which has the property that its complexification is isomorphic to G. Suppose that the action of K on X is sympletic, i.e. the action of any $k \in K$ preserves the Kähler metric on X. Let $\mathfrak{k}$ denote the Lie algebra of K. Then the infinitesimal action of K is given by the Lie algebra homomorphism $\mathfrak{k} \to \mathfrak{X}(X)$ defined by $\xi \mapsto X_\xi$, where

$$(X_\xi)_p := \frac{d}{dt}\Big|_{t=0} p \cdot \exp(t\xi)$$

Definition 3.3.1 (Hamiltonian)**.** *A symplectic action of K on X is Hamiltonian if for each $\xi \in \mathfrak{k}$, there exists a function $H_\xi : X \to \mathbb{R}$ such that for all $p \in X$ and $v \in T_pX$ we have*

$$\omega_p((X_\xi)_p, v) = (\mathrm{d}H_\xi)_p(v)$$

and the mapping $\xi \mapsto H_\xi$ *is* K*-equivariant with respect to the right action of* K *on* $\mathfrak{l}$ *by the Adjoint action and precomposition with right translation* R_k *on* $C^\infty(X)$*. The functions* H_ξ *are called Hamiltonian functions.*

Definition 3.3.2 (moment map)**.** *Suppose we have a Hamiltonian action of* K *on* X*. A moment map for the action is a* K*-equivariant map* $\mu : X \to \mathfrak{l}^*$*(where the action on* $\mathfrak{l}^*$ *is the coadjoint action) such that for any* $p \in X, v \in T_pX$ *and* $\xi \in \mathfrak{l}$*, we have*

$$\mathrm{d}\mu_p(v)(\xi) = \omega_p((X_\xi)_p, v)$$

Remark 3.3.3. Let's make coadjoint action more clear:

Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Let $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ denote the adjoint representation of G. Then we can define its coadjoint representation $\mathrm{Ad}^* : G \to \mathrm{Aut}(\mathfrak{g}^*)$ as

$$\langle \mathrm{Ad}_g^* \mu, Y\rangle = \langle \mu, \mathrm{Ad}_{g^{-1}} Y\rangle$$

for $g \in G, Y \in \mathfrak{g}, \mu \in \mathfrak{g}^*$.

Remark 3.3.4. One thing to note is that the Hamiltonian functions can be recovered by the moment maps. If a Hamiltonian action admits a moment map, then

$$H_\xi(p) = \mu(p)(\xi)$$

Let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{l}^*$ that is invariant under the coadjoint action, and $\|\cdot\|$ be the induced norm. Since X is compact, then the map $\|\mu\|^2 : X \to \mathbb{R}$ attains its minimum, and WLOG we assume that the minimum value is 0.

Definition 3.3.5 (symplectic quotient)**.** *The symplectic quotient of* X *by* K *is the quotient space*

$$\mu^{-1}(0)/K$$

Remark 3.3.6. The symplectic quotient can also be referred to as the symplectic reduction. It should be noted that the symplectic quotient depends on our choice of moment map.

Theorem 3.3.7. *The symplectic quotient of* X *by* K *admits a unique Kähler structure such that the Kähler metric on* $\mu^{-1}(0)/K$ *is induced by the Kähler metric on* X*.*

The relationship between the GIT quotient and the symplectic quotient is given by the Kempf-Ness theorem

Theorem 3.3.8 (Kempf-Ness)**.** *Suppose a complex reductive group* G *acts on a Kähler manifold* X *such that the action of the maximal compact subgroup* $K \subset G$ *is Hamiltonian and admits a moment map* $\mu : X \to \mathfrak{l}^*$*. Then the* G*-orbit of any semistable point contains a unique* K*-orbit minimizing* $\|\mu\|^2$*. This establish a homeomorphism*

$$X_{ss}/G \longleftrightarrow \mu^{-1}(0)/K$$

3.4. **Narasimhan-Sechadri.** We now want to relate the previous discussion to our situation. Using the identification of $\mathscr{A}(P)$ and $\mathscr{C}(E)$, we want the action of $\mathscr{G}_{\mathbb{C}}$ to play the role of the complex reductive group G and the gauge group $\mathscr{G}$ to play the role of the maximal compact subgroup. Since the space $\mathscr{A}(P)$ is infinite dimensional, along with the group $\mathscr{G}_{\mathbb{C}}$ and $\mathscr{G}$, we are working in an infinite dimensional setting, but we will gloss over the analytic details and work with them formally.

Our first task is to realize $\mathscr{A}(P)$ as a "Kähler manifold".

Our next task is to show the action of on $\mathscr{A}(P)$ is "Hamiltonian" with respect to this Kähler structure.

To summarize, we have the following analogies:

$$\begin{aligned} \text{Kähler manifold} &\longleftrightarrow \mathscr{A}(P) \\ \text{Complex reductive group} &\longleftrightarrow \mathscr{G}_{\mathbb{C}} \\ \text{Maximal compact subgroup } K \subset G &\longleftrightarrow \mathscr{G} \\ \text{Moment map } \mu &\longleftrightarrow A \mapsto F_A \\ \text{Norm square of the moment map } \|\mu\|^2 &\longleftrightarrow L \end{aligned}$$

The last piece is something analogous to the Kempf-Ness theorem

Theorem 3.4.1 (Narasimhan-Seshadri)**.** *Let $\mathscr{A}_s(P) \subset \mathscr{A}(P)$ denote the subspace of connections that are absolute minima for the Yang-Mills functional, and correspond to irreducible representations $\Gamma_{\mathbb{R}} \to U_n$. Let $\mathscr{C}_s(E)$ denote the subspace of stable holomorphic structures on E. The isomorphism classes of holomorphic bundles in $\mathscr{C}_s(E)$ admit unique Yang-Mills connections up to gauge equivalence. In other words, there is a homeomorphism*

$$\mathscr{A}_s(P)/\mathscr{G} \longleftrightarrow \mathscr{C}_s(E)/\mathscr{G}$$

Remark 3.4.2. The original proof is more algebraic in flavor. A proof more in the spirit of the Atiyah-Bott paper was given by Donaldson in [5]. The spirit of this proof is carried on by the proof of Hermitian-Yang-Mills and the nonabelian Hodge theorem, which were both grew out of the developments from the Atiyah-Bott paper.

4. Further research

Too ideas of Atiyah and Bott's paper were extended to many directions. For example, Donaldson [6] used the space of solutions of the Yang-Mills equations to prove his celebrated theorem.

In another direction, work of Uhlenbeck and Yau [7] extended the study of slope stability to holomorphic vector bundles over higher dimensional Kähler manifolds, relating slop stability to the existence of Hermitian-Yang-Mills connections. Motivated by these ideas, Yau conjectured that the existence of Hermitian-Einstein connections on Fano manifolds would be related to another algebro-geometric notion of stability called K-stability. Recent work of Chen, Donaldson, and Sun [8] has mostly resolved this conjecture.

Another circle of ideas that grew out of the original paper have been the ideas around Higgs bundles. These were originally introduced by Hitchin [9], and have been used heavily in work by Simpson [10].

5. Acknowledgement

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