ALGEBRAIC CURVES

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Contents

0. Motivations

0.1. **Meromorphic functions.** Let $U \subseteq \mathbb{C}$ be an open subset with coordinate $\{z\}$. In complex analysis we learnt that a meromorphic function f is a function that is holomorphic on all of U except for a set of isolated points, which are poles of the function. In other words, a meromorphic function can be regarded as a function $f\colon U \to \mathbb{C}\cup\{\infty\}$.

Topologically speaking, $\mathbb{C}\cup\{\infty\}$ is S^2, and in fact there is a complex manifold structure on it. More precisely, we can glue two pieces of complex plane via $w = 1/z$ to obtain a complex manifold called Riemann sphere

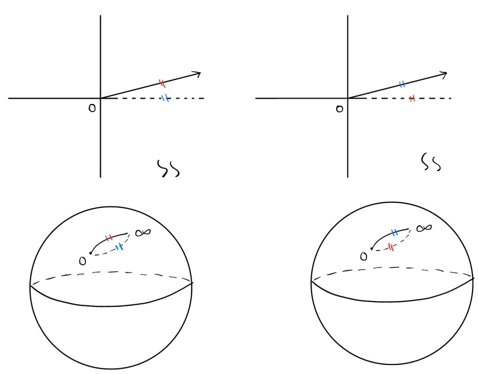
$$\mathbb{P}^1 = \mathbb{C}\cup_{\mathbb{C}^*}\mathbb{C},$$

and topologically $\mathbb{P}^1$ is exactly $\mathbb{C}\cup\{\infty\}$. By using this viewpoint, meromorphic function on U is exactly the same thing as holomorphic map from U to the Riemann sphere, and thus it gives us a lovely way to study meromorphic functions by using theories of holomorphic maps between Riemann surfaces, such as the number (counted with multiplicity) of zeros is equal to the number (counted with multiplicity) of poles.

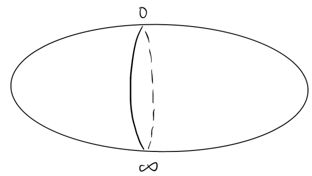
0.2. **Multivalueness of holomorphic functions.** For complex number $z = \rho e^{\sqrt{-1}\theta}$, where $\rho \in [0,\infty)$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, one has

$$(\sqrt{\rho}e^{\sqrt{-1}\theta/2})^2 = (\sqrt{\rho}e^{\sqrt{-1}\theta/2+\pi})^2 = z.$$

This shows there are two candidates for $\sqrt{z}$, and this phenomenon is called multivalueness of holomorphic function. If we define square root as $\sqrt{z} = \sqrt{\rho}e^{\sqrt{-1}\theta/2}$, then it's only well-defined on $\mathbb{C}\setminus[0,\infty)$, since it will "jump" when passing through the two sides of $[0,\infty)$, and $\mathbb{C}\setminus[0,\infty)$ is called a single value component of $\sqrt{z}$.



The ideal to solve this phenomenon is that, when passing the segment $[0,\infty)$, $\sqrt{z}$ should come into another single value component. In other words, if we want to make square root $\sqrt{z}$ defined on the whole complex plane, it should be no longer a function from $\mathbb{C}$ to $\mathbb{C}$, but a function from $\mathbb{C}$ to an object we obtained from gluing two single value components together. This construction also gives the Riemann sphere.

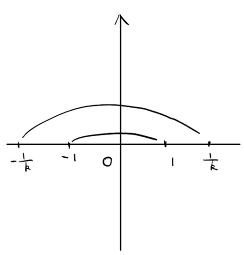


Similarly, $f(z) = \sqrt{1-z^2}$ is well-defined on $\mathbb{C} \setminus [-1,1]$, and it gives a well-defined function from $\mathbb{C}$ to something obtained by gluing two copies of $\mathbb{C} \setminus [-1,1]$, which is also the Riemann sphere.

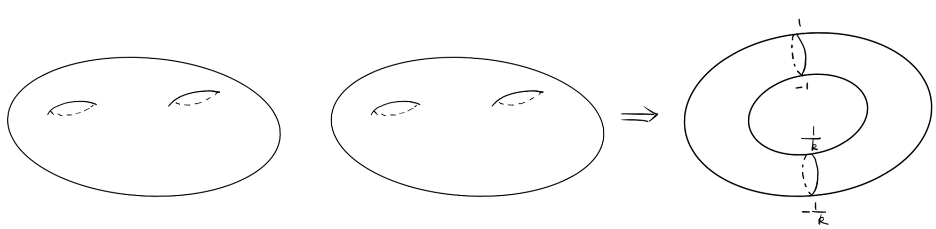
Now let's consider a more complicated example. For

$$f(z) = \sqrt{(1-z^2)(1-k^2z^2)},$$

where $k \neq \pm 1$, it gives a well-defined function on $\mathbb{C}$ minus two line segments connecting $-1, 1$ and $-1/k, 1/k$.



If we want to obtain a function defined on $\mathbb{C}$, we should glue two copies of above single value components. This gives a new Riemann surface called complex torus.



0.3. **Abelian integrals.**

0.3.1. *Arc-length of ellipse.* For ellipse given by $(x/a)^2 + (y/b)^2 = 1$, by using parameterization

$$x = a\cos\theta$$
$$y = b\sin\theta,$$

it's easy to see arc-length is given by

$$\begin{aligned}\int_{\theta_0}^{\theta_1}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\mathrm{d}\theta &= a\int_{\theta_0}^{\theta_1}\sqrt{1-k^2\sin^2\theta}\mathrm{d}\theta\\ &\overset{z=\sin\theta}{=}\int_{z_0}^{z_1}\frac{\sqrt{1-k^2z^2}}{\sqrt{1-z^2}}\mathrm{d}z\\ &=\int_{z_0}^{z_1}\frac{1-k^2z^2}{\sqrt{(1-k^2z^2)(1-z^2)}}\mathrm{d}z,\end{aligned}$$

where $k=\sqrt{1-b^2/a^2}$. For $k=0$, since $\arcsin z$ is a primitive function of $1/\sqrt{1-z^2}$, one has

$$\int_{z_0}^{z_1}\frac{1}{\sqrt{1-z^2}}\mathrm{d}z=\arcsin z_1-\arcsin z_0.$$

The classical theory of "addition formula" gives

$$\sin(\alpha+\beta)=\sin\alpha\sqrt{1-\sin^2\beta}+\sqrt{1-\sin^2\alpha}\sin\beta.$$

In terms of integration

$$\int_0^{z_1}\frac{1}{\sqrt{1-t^2}}\mathrm{d}t+\int_0^{z_2}\frac{1}{\sqrt{1-t^2}}\mathrm{d}t=\int_0^{z_1\sqrt{1-z_2^2}+z_2\sqrt{1-z_1^2}}\frac{1}{\sqrt{1-t^2}}\mathrm{d}t.$$

For analogue of above case, if we define ellipse sine sn as

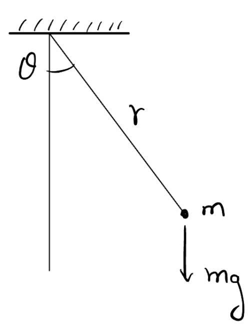
$$\int_0^{\arcsin z}\frac{1}{\sqrt{1-k^2\sin^2 t}}\mathrm{d}t=\mathrm{sn}^{-1}(z),$$

one can also show it satisfies some addition formula

$$\mathrm{sn}(\alpha+\beta)=\frac{\mathrm{sn}\,\alpha\sqrt{(1-\mathrm{sn}^2\beta)(1-k^2\,\mathrm{sn}^2\beta)}+\mathrm{sn}\,\beta\sqrt{(1-\mathrm{sn}^2\alpha)(1-k^2\,\mathrm{sn}^2\alpha)}}{1-k^2\,\mathrm{sn}^2\alpha\,\mathrm{sn}^2\beta}.$$

However, the ellipse sine cannot be expressed as an elementary function, and this is closely related to the fact that $y^2=(1-z^2)(1-k^2z^2)$ is not a Riemann sphere.

0.3.2. *Simple pendulum.* Suppose there is an object with mass m is released at $\theta=\alpha$ with zero initial velocity, and the length of pendulum is r.



The conservation of energy gives the following equation

$$\frac{1}{2}mr^2(\frac{\mathrm{d}\theta}{\mathrm{d}t})^2 = mgr\cos\theta - mgr\cos\alpha.$$

In other words,

$$(\frac{\mathrm{d}\theta}{\mathrm{d}t})^2 = 2\frac{g}{r}(\cos\theta - \cos\alpha) = 4\frac{g}{r}(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}). \tag{0.1}$$

An approximation with θ sufficiently small, one has

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \sqrt{\frac{g}{r}(\alpha^2 - \theta^2)}.$$

This shows

$$t = \int_0^\theta \sqrt{\frac{r}{g}}\frac{1}{\sqrt{\alpha^2 - s^2}}\mathrm{d}s.$$

Thus the period of the simple pendulum is given by

$$T = 4\int_0^\alpha \sqrt{\frac{r}{g}}\frac{1}{\sqrt{\alpha^2 - s^2}}\mathrm{d}s = 2\pi\sqrt{\frac{r}{g}}.$$

However, if we don't use the approximation, and use substitution

$$\sin\varphi = \frac{\sin\frac{\theta}{2}}{\sin\frac{\alpha}{2}}$$

in (0.1), one has

$$(\frac{\mathrm{d}\varphi}{\mathrm{d}t})^2 = \frac{g}{r}(1 - \sin^2\frac{\alpha}{2}\sin^2\varphi).$$

Then

$$t = \sqrt{\frac{r}{g}}\int_0^\varphi \frac{1}{\sqrt{1 - k^2\sin^2 s}}\mathrm{d}s,$$

where $k = \sin\dfrac{\alpha}{2}$, and thus explicit formula for the period of simple pendulum is

$$T = 4\sqrt{\frac{r}{g}}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2\sin^2 s}}\mathrm{d}s.$$

This is exactly ellipse integral.

0.3.3. *General.* Let P be a polynomial of two variables and $y = f(x)$ be a solution of equation $P(x, y) = 0$. Then

$$\int R(x, f(x)) = 0$$

can be expressed as elementary function if and only if $\deg P = 0, 1, 2$, and in fact $\deg P$ is closely related to the topology of algebraic curves.

1. Riemann surface and algebraic curves

1.1. Riemann Surface.

1.1.1. *Definitions.*

Definition 1.1.1 (complex atlas)**.** Let X be a topological space. A complex atlas on X consists of the following data:

(1) $\{U_i\}_{i\in I}$ is an open covering of X.
(2) For each $i \in I$, there exists a homeomorphism $\varphi_i \colon U_i \to \varphi_i(U_i) \subseteq \mathbb{C}$.
(3) For $i, j \in I$, if $U_i \cap U_j \neq \varnothing$, then the transition function

$$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} \colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

is holomorphic.

Remark 1.1.1. If $\{U_i, \varphi_i\}$ is a complex atlas on a topological space, then all transition functions φ_{ij} are not only holomorphic, but biholomorphic with inverse φ_{ji}.

Definition 1.1.2 (complex structure)**.** Two complex atlas $\mathscr{A}, \mathscr{B}$ are equivalent if $\mathscr{A} \cup \mathscr{B}$ is also a complex atlas, and a complex structure is an equivalent class of atlas on X.

Definition 1.1.3 (Riemann surface)**.** A Riemann surface is a connected, second countable, Hausdorff topological space X together with a complex structure on X.

Remark 1.1.2. A Riemann surface X is a complex manifold with $\dim_{\mathbb{C}} X = 1$, and it's called a surface since $\dim_{\mathbb{R}} X = 2$.

1.1.2. *Examples.*

Example 1.1.1 (Riemann sphere)**.** Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ be 2-sphere and $\{U_1 = S^2 \setminus (0, 0, 1), U_2 = S^2 \setminus (0, 0, -1)\}$ be an open covering of S^2. Consider

$$\begin{aligned}\varphi_1 \colon U_1 &\to \mathbb{C}\\ (x_1, x_2, x_3) &\mapsto \frac{x_1}{1 - x_3} + \sqrt{-1}\frac{x_2}{1 - x_3},\end{aligned}$$

and

$$\begin{aligned}\varphi_2 \colon U_1 &\to \mathbb{C}\\ (x_1, x_2, x_3) &\mapsto \frac{x_1}{1 + x_3} - \sqrt{-1}\frac{x_2}{1 + x_3}.\end{aligned}$$

A direct computation shows that

$$\left(\frac{x_1}{1 - x_3} + \sqrt{-1}\frac{x_2}{1 - x_3}\right)\left(\frac{x_1}{1 + x_3} - \sqrt{-1}\frac{x_2}{1 + x_3}\right) = \frac{x_1^2 + x_2^2}{1 - x_3^2} = 1,$$

and thus the transition function $\varphi_2 \circ \varphi_1^{-1}(z) = 1/z$. This shows $\{U_1, U_2\}$ is a complex atlas of S^2. It's clear as a topological space S^2 is connected, second countable and Hausdorff, and thus S^2 is a Riemann surface, called Riemann sphere.

Remark 1.1.3. There is another construction of Riemann sphere, given by gluing two complex planes together on $\mathbb{C}^*$, and the gluing data on $\mathbb{C}^*$ is given by $z \sim 1/w$. One thing to mention is that it's not clear object constructed in this way is Hausdorff. For example, if we glue two complex planes together on $\mathbb{C}^*$ by using gluing data $z \sim w$, then the object obtained is not Hausdorff.

Example 1.1.2 (complex projective line)**.** The complex projective line $\mathbb{P}^1 = \mathbb{C}^2 \setminus (0,0)/\sim$, where $(x,y) \sim (z,w)$ if and only if $(\lambda x, \lambda y) = (z,w)$ for some $\lambda \in \mathbb{C}^*$, and the equivalent class for (x,y) is denoted by $[x,y]$, called the homogenous coordinate. The quotient topology on $\mathbb{P}^1$ which makes it second countable, Hausdorff and compact. Consider

$$U_0 = \{[z,w] \mid z \neq 0\} \xrightarrow{\varphi_0} \mathbb{C}$$

where φ_0 is defined as $\varphi_1([z,w]) = z/w$. Similarly consider

$$U_1 = \{[z,w] \mid w \neq 0\} \xrightarrow{\varphi_1} \mathbb{C}$$

where φ_1 is defined as $\varphi_1([z,w]) = w/z$. For $z \in \varphi_1(U_0 \cap U_1)$, one has

$$z \xrightarrow{\varphi_1^{-1}} [z:1] = [1:\frac{1}{z}] \xrightarrow{\varphi_0} \frac{1}{z}.$$

This shows the transition function $\varphi_{01}(z) = 1/z$, which is holomorphic, and thus $\{(U_0,\varphi_0),(U_1,\varphi_1)\}$ gives a complex atlas on $\mathbb{P}^1$.

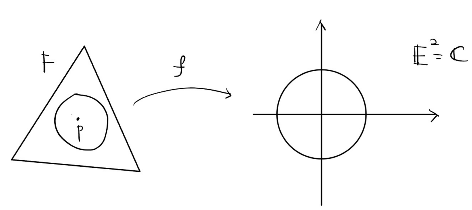
Remark 1.1.4 (complex projective space). In general, the complex projective space $\mathbb{P}^n$ is defined by $\mathbb{P}^n = \mathbb{C}^{n+1} \setminus (0,0)/\sim$, where $(x_0,x_1,\dots,x_n) \sim (y_0,y_1,\dots,y_n)$ if and only if there exists $\lambda \in \mathbb{C}^*$ such that $y_i = \lambda x_i$ holds for all $i = 0,1,\dots,n$, and the equivalent class $[x_0 : x_1 : \cdots : x_n]$ is call the homogenous coordinate of $\mathbb{P}^n$. There is a canonical affine open covering $\{(U_i,\varphi_i)\}$ of $\mathbb{P}^n$ defined by

$$U_i = \{[x_0 : x_1 : \cdots : x_n] \mid x_i \neq 0\} \xrightarrow{\varphi_i} \mathbb{C}^n,$$

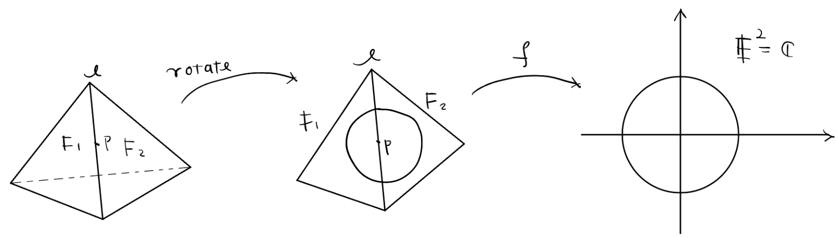
where $\varphi_i([x_0 : x_1 : \cdots : x_n]) = (x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i)$, and it makes $\mathbb{P}^n$ to be a complex n-manifold.

Example 1.1.3. Let P be a convex polyhedra in Euclidean 3-dimensional space $\mathbb{E}^3$. Topologically P is S^2, and let's construct a complex atlas on it.

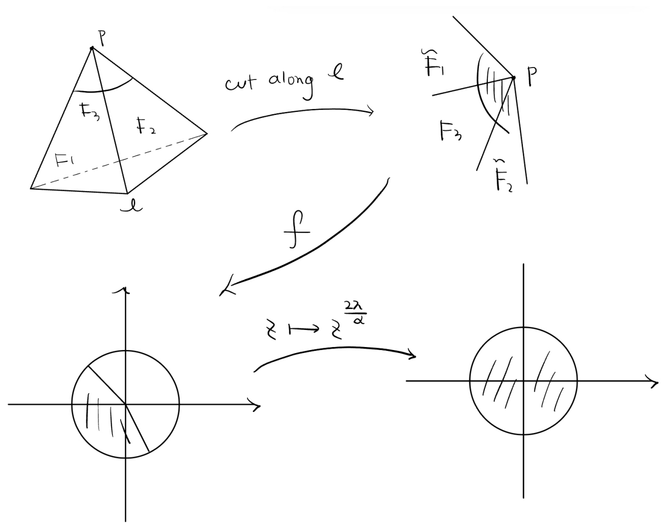
(1) Suppose p is the interior point of some face F. Since F can be isometrically embedded into $\mathbb{E}^2$, we choose an orientation-preserving, isometric embedding f which maps an open neighborhood U of p into $\mathbb{E}^2 = \mathbb{C}$.



(2) Suppose p is the interior point of some edge $l = F_1 \cap F_2$. Firstly we rotate F_2 along l to the plane of F_1, and then choose an orientation-preserving, isometric embedding f which maps an open neighborhood U of p into $\mathbb{E}^2 = \mathbb{C}$.



(3) Suppose p is an vertex which is the intersection of three faces F_1, F_2 and F_3. Firstly we cut it along some edge $l = F_1 \cap F_2$, and then rotate F_1, F_2 to the plane of F_3. Then we use some orientation-preserving, isometric embedding f to map it into $\mathbb{E}^2$, and finally composite it with $z \mapsto z^{2\pi/\alpha}$.



Exercise 1.1.1. Prove that above constructions give a complex atlas on convex polyhedra.

Remark 1.1.5. All of above three examples give complex structure on topological sphere S^2, and we will see all of them are the "same" after we define the isomorphism between Riemann surfaces. In fact, there is only one complex structure on S^2.

Example 1.1.4 (complex torus)**.** For non-zero $w_1, w_2 \in \mathbb{C}$ such that w_1, w_2 are $\mathbb{R}$-linearly independent, $L = \mathbb{Z}\, w_1 + \mathbb{Z}\, w_2$ is a discrete subgroup of $(\mathbb{C}, +)$. Then $T = \mathbb{C}/L$ equipped with quotient topology is a connected, Hausdorff and second countable topological space. Let $\pi\colon \mathbb{C} \to T$ be the natural projection. For $p \in T$, suppose z_0 is an inverse image of p. For $\varepsilon \in \mathbb{R}_{>0}$ such that

$$B_{2\varepsilon}(0) \cap L = \{0\},$$

one has $B_\varepsilon(z_0) \xrightarrow{\pi} \pi(B_\varepsilon(z_0)) \subseteq T$ is injective, and thus $\pi^{-1}\colon \pi(B_\epsilon(z_0)) \to B_\epsilon(z_0) \subseteq \mathbb{C}$ is a homeomorphism. Then $\{\pi(B_\varepsilon(\pi^{-1}(p))\}_{p \in T}$ gives an open covering of T, and together with π^{-1} it gives a complex atlas of T.

Remark 1.1.6. It's clear complex structure constructed above depends on the choice of w_1, w_2, but it's not obvious to see whether w_1, w_2 and w_1', w_2' give the same complex structure or not. Moreover, all complex structure on torus are arisen in this way.

1.1.3. *Morphisms.*

Definition 1.1.4 (holomorphic map)**.** Let X, Y be two Riemann surfaces and $F\colon X \to Y$ be a continous map. For $p \in X$, F is called holomorphic at p, if there exists a chart (U, φ) of p, and a chart (V, ψ) of $F(p)$, such that

$$\psi \circ F \circ \varphi^{-1}\colon \varphi\left(U \cap F^{-1}(V)\right) \to \psi\left(V \cap F(U)\right)$$

is holomorphic at $\varphi(p)$. Moreover, F is called holomorphic in an open subset $W \subseteq X$, if F is holomorphic at any point in W.

Exercise 1.1.2. Show that the definition of holomorphic map is independent of the choice of charts.

Definition 1.1.5 (isomorphism)**.** Let $F\colon X \to Y$ be a holomorphic map between Riemann surfaces. F is called an isomorphism if it's bijective and holomorphic.

Proposition 1.1.1. Let $F\colon X \to Y$ be a holomorphic map between Riemann surfaces. F is an isomorphism if and only if F has an two-side inverse G, and G is holomorphic.

Proposition 1.1.2. The complex projective space is isomorphic to Riemann sphere.

Theorem 1.1.1 (open mapping theorem)**.** Let $F\colon X \to Y$ be a non-constant holomorphic map between Riemann surfaces. Then F is an open map.

Corollary 1.1.1. Let $F\colon X \to Y$ be a non-constant holomorphic map between Riemann surfaces and X is compact. Then $F(X) = Y$, and thus Y is compact.

Proof. By open mapping theorem, $F(X)$ is an open subset of Y, and $F(X)$ is compact in Y, since continous function maps compact set to compact set. Then $F(X)$ is both open and closed in Y, and thus $F(X) = Y$. $\square$

Theorem 1.1.2. Let $F\colon X \to Y$ be a non-constant holomorphic map between Riemann surfaces. Then for any $p \in Y$, $F^{-1}(y)$ is a discrete set. In particular, if X is compact, then $F^{-1}(y)$ is a non-empty finite set.

1.2. **Algebraic curves.**

1.2.1. *Affine plane curves.* Let $V \subseteq \mathbb{C}$ be a connected open subset and g be a holomorphic function defined on V. The graph X of g, as a subset of $\mathbb{C}^2$ is defined by

$$\{(z, g(z)) \mid z \in V\}.$$

Given X the subspace topology, and let $\pi\colon X \to V$ be the projection to the first factor. Note that π is a homeomorphism, whose inverse sends the point $z \in V$ to the ordered pair $(z, g(z))$. This makes X a Riemann surface.

A generalization of the graph of holomorphic function is that we consider "Riemann surface" which is locally a graph, but perhaps not globally. The tools we use is implicit function theorem in fact.

Theorem 1.2.1 (The implicit function theorem)**.** Let $f(z, w)\colon \mathbb{C}^2 \to \mathbb{C}$ be holomorphic function of two variables and $X = \{(z, w) \in \mathbb{C}^2 \mid f(z, w) = 0\}$ be its zero loucs. Let $p = (z_0, w_0)$ be a point of X and $\partial f/\partial z(p) \neq 0$. Then there exists a function $g(w)$ defined and holomorphic in a neighborhood of w_0 such that, near p, X is equal to the graph $z = g(w)$.

Method one. If we write $z = a + \sqrt{-1}b, w = c + \sqrt{-1}d$ and $f(z, w) = u + \sqrt{-1}v$, then u, v are smooth functions of a, b, c, d. Moreover, the Cauchy-Riemann equations give

$$\frac{\partial f}{\partial z} = \frac{\partial u}{\partial a} + \sqrt{-1}\frac{\partial v}{\partial a} = \frac{\partial v}{\partial b} - \sqrt{-1}\frac{\partial u}{\partial b} = A + \sqrt{-1}B.$$

Then

$$\frac{\partial(u, v)}{\partial(a, b)} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

and $\det \dfrac{\partial(u, v)}{\partial(a, b)} = A^2 + B^2 \neq 0$ if and only if $A + \sqrt{-1}B \neq 0$. Then the classical implicit function theorem implies the zero loucs

$$\begin{cases} u = 0 \\ v = 0 \end{cases}$$

is locally given by

$$\begin{cases} a = a(c, d) \\ b = b(c, d). \end{cases}$$

In other words, $z = g(w)$. Now it suffices to compute $\partial g/\partial \overline{w}$ to show g is holomorphic. Again by Cauchy-Riemann equations

$$\frac{\partial f}{\partial w} = \frac{\partial u}{\partial c} + \sqrt{-1}\frac{\partial v}{\partial c} = \frac{\partial v}{\partial d} - \sqrt{-1}\frac{\partial u}{\partial d} = C + \sqrt{-1}D.$$

Then by chain rule one has

$$\begin{aligned}\frac{\partial(a,b)}{\partial(c,d)} &= \left(\frac{\partial(u,v)}{\partial(a,b)}\right)^{-1}\frac{\partial(u,v)}{\partial(c,d)}\\ &= \begin{pmatrix} A & B\\ -B & A\end{pmatrix}^{-1}\begin{pmatrix} C & D\\ -D & C\end{pmatrix}\\ &= \frac{1}{A^2+B^2}\begin{pmatrix} AC+BD & AD-BC\\ BC-AD & BD+AC\end{pmatrix}.\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial g}{\partial \overline{w}} &= \frac{1}{2}\left(\frac{\partial}{\partial c}+\sqrt{-1}\frac{\partial}{\partial d}\right)(a+\sqrt{-1}b)\\ &= \frac{1}{2}\left(\frac{\partial a}{\partial c}+\sqrt{-1}\frac{\partial b}{\partial c}+\sqrt{-1}\frac{\partial a}{\partial d}-\frac{\partial b}{\partial d}\right)\\ &= 0\end{aligned}$$

□

Method two. Firstly let's recall some basic facts in complex analysis: For a holomorphic function f defined on U, the integral

$$\frac{1}{2\pi\sqrt{-1}}\oint_{\partial U}\frac{f'(z)}{f(z)}\mathrm{d}z$$

counts the number of zeros of $f(z)$ in U with multiplicity, and the integral

$$\frac{1}{2\pi\sqrt{-1}}\oint_{\partial U}z\frac{f'(z)}{f(z)}\mathrm{d}z$$

is the summation of zeros of $f(z)$ in U. Now let's prove the implicit function theorem by using above observation. Fix $w=w_0$, the holomorphic function $f(z,w_0)$ has a zero at $z=z_0$, and we may choose an open neighborhood U of z_0 such that z_0 is the only zero of $f(z,w_0)$ in U since holomorphic function has discrete zeros. Consider the integral

$$\frac{1}{2\pi\sqrt{-1}}\oint_{\partial U}\frac{f_z(z,w)}{f(z,w)}\mathrm{d}z = N(w),$$

which is well-defined on sufficiently small neighborhood D_{w_0} of w_0. It gives a continous, integer-valued function with $N(w_0)=1$. This shows $N(w)=1$ for all $w\in D_{w_0}$, and thus $f(z,w)$ has only one zero for every $w\in D_{w_0}$. Moreover, this zero point z is given by

$$\frac{1}{2\pi\sqrt{-1}}\oint_{\partial U}z\frac{f_z(z,w)}{f(z,w)}\mathrm{d}z = g(w),$$

which is holomorphic with respect to w. □

Definition 1.2.1 (affine plane curve)**.** An affine plane curve is the loucs of zeros in $\mathbb{C}^2$ of a (non-trivial) polynomial $f(z,w)$.

Definition 1.2.2 (non-singular)**.** A polynomial $f(z,w)$ is non-singular at root p if either $\partial f/\partial z$ or $\partial f/\partial w$ is not zero at p, otherwise it's called singular. The affine plane curve X defined by $f(z,w)$ is non-singular is non-singular at p if f is non-singular at p. The curve X is non-singular if it's non-singular at each of its points.

Example 1.2.1. The affine plane curve defined by z^2+w^2-1 is non-singular.

Theorem 1.2.2. A non-singular affine plane curve defined by an irreducible polynomial is a Riemann surface.

1.2.2. *Projective plane curve.*

Definition 1.2.3 (projective plane curve)**.** Let P be a homogenous polynomial in $\mathbb{C}[x,y,z]$. A projective plane curve C defined by P is the zero loucs of P, that is,
$$C=\{[x:y:z]\in\mathbb{P}^2\mid P(x,y,z)=0\}.$$

Remark 1.2.1 (Relations between affine plane curve and projective plane curve)*.* Given a projective plane curve C given by homogenous polynomial P. Consider
$$\begin{aligned}\varphi_0\colon U_0=\mathbb{C}^2&\to\mathbb{P}^2\\(y,z)&\mapsto[1:y:z]\end{aligned}$$
Then $\varphi_0^{-1}(U_0\cap C)=\{(y,z)\in\mathbb{C}^2\mid P(1,y,z)=0\}$ is an affine plane curve, and similarly there are other affine plane curves given by $\varphi_0^{-1}(U_1\cap C)$ and $\varphi_0^{-1}(U_2\cap C)$.

Conversely, given an affine plane curve C defined by $f\in\mathbb{C}[y,z]$. Consider the homogenous polynomial $P(x,y,z)$ defined by
$$P(x,y,z)=x^d f(\frac{y}{x},\frac{z}{x})$$
where $d=\deg f$. Then P defines a projective plane curve, such that the affine plane curve it gives on affine piece U_0 is exactly C.

Definition 1.2.4 (non-singular)**.** A projective plane curve C is non-singular if the affine plane curves $\varphi_i^{-1}(U_i\cap C)$ are non-singular for $i=0,1,2$, where $\varphi_i\colon U_i\to\mathbb{P}^2$ are standard affine covering of $\mathbb{P}^2$.

Proposition 1.2.1. A projective plane curve $C=\{P(x,y,z)=0\}\subseteq\mathbb{P}^2$ is non-singular if and only if
$$\frac{\partial P}{\partial x}=\frac{\partial P}{\partial y}=\frac{\partial P}{\partial y}=0$$
has no solution in $\mathbb{P}^2$.

Proof. Since P is a homogenous polynomial, it satisfies the Euler's formula
$$dP=x\frac{\partial P}{\partial x}+y\frac{\partial P}{\partial y}+z\frac{\partial P}{\partial z},$$
where $d=\deg P$. Now let's start our proof as follows:

(1) Suppose $\partial P/\partial x = \partial P/\partial y = \partial P/\partial z = 0$ has a solution (a, b, c) with $a \neq 0$. Then

$$\frac{\partial P}{\partial y}(1, \frac{b}{a}, \frac{c}{a}) = \frac{1}{a^{d-1}} \frac{\partial P}{\partial y}(a, b, c) = 0$$
$$\frac{\partial P}{\partial z}(1, \frac{b}{a}, \frac{c}{a}) = \frac{1}{a^{d-1}} \frac{\partial P}{\partial z}(a, b, c) = 0.$$

Thus

$$P(1, \frac{b}{a}, \frac{c}{a}) = \frac{1}{a^d} P(a, b, c) = 0.$$

(2) Conversely, if the projective plane curve defined by P is singular, without lose of generality we may assume $X_0 := \varphi_i^{-1}(U_0 \cap C)$ is singular. Then there exists a solution $(b, c) \in \mathbb{C}^2$ such that

$$P(1, b, c) = \frac{\partial P}{\partial y}(1, b, c) = \frac{\partial P}{\partial z}(1, b, c) = 0.$$

By Euler's formula one has

$$\frac{\partial P}{\partial x}(1, b, c) = dP(1, b, c) - b\frac{\partial P}{\partial y} - c\frac{\partial P}{\partial z} = 0.$$

As a consequence, $(1, a, b)$ is a solution of $\partial P/\partial x = \partial P/\partial y = \partial P/\partial z = 0$.

□

Theorem 1.2.3. Any non-singular projective plane curve is a compact Riemann surface.

Remark 1.2.2. One way to understand projective plane curve is to regard it as a compactifications of affine plane curve.

Example 1.2.2 (Fermat curve)**.** $x^d + y^d = z^d$ gives a non-singular projective plane curve.

Example 1.2.3. The polynomial $f(x, y) = y^2 - (1-x^2)(1-k^2x^2), k \neq 0, \pm 1$ gives a non-singular affine plane curve C. Now we consider the compactification of C. Let $P(x, y, z)$ be the homogenous polynomial given by $f(x, y)$, that is,

$$P(x, y, z) = z^2y^2 - (z^2 - x^2)(z^2 - k^2x^2).$$

$P(x, y, z)$ gives a projective plane curve, and the affine plane curve it gives on the affine piece U_2 is exactly C, so it suffices to see the affine plane curves it gives on the other affine pieces.

(1) The affine plane curve it gives on the affine piece U_1 is defined by

$$P(x, 1, z) = z^2 - (z^2 - x^2)(z^2 - k^2x^2).$$

In this case there is a new point $[0 : 1 : 0]$, which is singular.

(2) The affine plane curve it gives on the affine piece U_0 is defined by

$$P(1,y,z) = z^2y^2 - (z^2-1)(z^2-k^2).$$

But in this case, there is no more new point since there is no solution satisfying $z=0$.

In a summary, the compactification of the affine plane curve C adds a singular point, and later we will see how to handle with singularities by resolutions.

1.2.3. *Quadratic.* A homogenous polynomial P of degree 2 can be written as

$$P = (x,y,z)A\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where $A \in M_{3\times 3}(\mathbb{C})$ is a symmetric matrix. In this section we will see the projective plane curve C defined by P is determined by the rank of A.

Proposition 1.2.2. If $\operatorname{rk} A = 3$, then P is non-singular, and C is isomorphic to $\mathbb{P}^1$.

Method one. If $\operatorname{rk} A = 3$, then there exists $P \in \mathrm{GL}(3,\mathbb{C})$ such that

$$P^TAT = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

This shows that afer a suitable change of coordinate, we may assume the projective plane curve C defined by P is $\{[x:y:z] \mid x^2+y^2-z^2=0\} \subseteq \mathbb{P}^2$. The following map gives an isomorphism between C and $\mathbb{P}^1$.

$$\begin{aligned} F\colon \mathbb{P}^1 &\to C \\ [1:t] &\mapsto [1-t^2 : 2t : 1+t^2]. \end{aligned}$$

□

Method two. Consider the following holomorphic embedding

$$\begin{aligned} F\colon \mathbb{P}^1 &\to \mathbb{P}^2 \\ [t_0:t_1] &\mapsto [t_0^2 : t_0t_1 : t_1^2]. \end{aligned}$$

Note that the image of F is a projective plane curve defined by the equation $xz=y^2$. On the other hand, after a suitable change of coordinate we may also assume C is defined by this equation since there also exists $P \in \mathrm{GL}(3,\mathbb{C})$ such that

$$P^TAT = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

□

Proposition 1.2.3. If $\operatorname{rk} A = 2$, then C is isomorphic to the union of two $\mathbb{P}^1$.

Proof. If $\operatorname{rk} A = 2$, then there exists $P \in \mathrm{GL}(3, \mathbb{C})$ such that

$$P^T A T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This shows the projective plane curve C is defined by $x^2 + y^2 = (x + \sqrt{-1}y)(x - \sqrt{-1}y)$, which is the union of two $\mathbb{P}^1$ which intersects at $[0 : 0 : 1]$. In particular, it's singular. $\square$

Proposition 1.2.4. If $\operatorname{rk} A = 1$, then C is isomorphic to a double line.

Proof. If $\operatorname{rk} A = 1$, then there exists $P \in \mathrm{GL}(3, \mathbb{C})$ such that

$$P^T A T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This shows the projective plane curve C is defined by $x^2 = 0$, which is a singular projective plane curve called double line. $\square$

2. Ramified covering

Topologically speaking a Riemann surface is an orientable 2-dimensional real manifold without boundary. In particular, the topology of a compact Riemann surface can be classified by its genus. So there is a natural question: Given a non-singular projective plane curve C defined by the homogenous polynomial $F(x,y,z)=y^2z-x(x-z)(x-\lambda z), \lambda\neq 0,1$, topologically C is a closed orientable surface, is there any way to compute its genus?

Consider the following map

$$\begin{aligned} F\colon C\setminus[0:1:0] &\to \mathbb{P}^1 \\ [x:y:z] &\mapsto [x:z]. \end{aligned}$$

It's clear that F is well-defined holomorphic map. If we desire to extend F to a holomorphic map $\widetilde{F}$ defined on C, we need to consider the behavior of C around $[0:1:0]$. On affine piece $U_1=\{[x:1:z]\mid x,z\in\mathbb{C}\}$, it gives an affine plane curve defined by

$$f(x,z)=z-x(x-z)(x-\lambda z).$$

A direct computation shows that

$$\frac{\partial f}{\partial z}\Big|_{(0,0)}=1,\quad \frac{\partial f}{\partial x}\Big|_{(0,0)}=0.$$

Then by implicit function theorem, C is given by $[x:1:z(x)]$ locally around $[0:1:0]$, and

$$z'(0)=-\frac{\partial f}{\partial x}\Big|_{(0,0)}\Big/\frac{\partial f}{\partial z}\Big|_{(0,0)}=0/1=0.$$

Thus $x=0$ is a removable singularity of $z(x)/x$, and it's reasonable to define $\widetilde{F}([0:1:0])=[1:0]$ to give an extension of F since for $x\neq 0$,

$$F([x:1:z(x)])=[x:z(x)]=[1:\frac{z(x)}{x}].$$

There are four special points for $\widetilde{F}\colon C\to\mathbb{P}^1$, listed as follows

$$\begin{aligned} [0:1:0]&\mapsto[1:0]\\ [0:0:1]&\mapsto[0:1]\\ [z:0:1]&\mapsto[z:1]\\ [\lambda z:0:1]&\mapsto[\lambda z:1]. \end{aligned}$$

Besides these points, $\widetilde{F}$ is a double covering in fact. In this section we will study such holomorphic maps, which are called ramification covering, and Hurwitz formula will gives a method to compute the genus of the ramification covering of a given space.

2.1. **Ramification covering.**

Theorem 2.1.1 (local normal form)**.** Let $F\colon X \to Y$ be a non-constant holomorphic map. Then there are local coordinates (U, φ) and (V, ψ) of p and $F(p)$ respectively, such that

$$\psi \circ F \circ \varphi^{-1}(z) = z^k$$

holds for all $z \in \varphi(U \cap F^{-1}(V))$.

Proof. Firstly we fix a local coordinate (V, ψ) of $F(p)$, and choose a local coordinate (U_1, φ_1) of p such that $F(U) \subset V$. If we denote $\psi \circ F \circ \varphi_1^{-1} = T$, then $T(0) = 0$. Suppose the Taylor expansion of T at $w = 0$ is

$$T(w) = \sum_{k=m}^{\infty} a_k w^k, \quad a_m \neq 0.$$

Then $T(w) = w^m S(w)$, where $S(w)$ is a holomorphic function with $S(0) \neq 0$, and thus there exists a holomorphic function $R(w)$ such that $R^m(w) = S(w)$.

Then $T(w) = (wR(w))^m = (\eta(w))^m$, where $\eta(0) = 0, \eta'(0) = R(0) \neq 0$. By inverse function theorem, there exists a sufficiently small neighborhood $U \subseteq U_1$ of p such that η is invertible in $\varphi_1(U)$, and thus this gives a new local coordinate of p as

$$U_1 \supseteq U \xrightarrow{\varphi_1} \varphi_1(U) \xrightarrow{\eta} \eta \circ \varphi_1(U) \subset \mathbb{C}.$$

If we define $\varphi = \eta \circ \varphi_1$, then with respect to (U, φ) and (V, ψ), the local representation of F is given by

$$\psi \circ F \circ \varphi^{-1}(z) = \psi \circ F \circ \varphi_1^{-1} \circ \eta^{-1}(z) = T(\eta^{-1}(z)) = z^m.$$

□

Definition 2.1.1 (multiplicity)**.** Let $F\colon X \to Y$ be a holomorphic map between Riemann surfaces. If its local normal form at point $p \in X$ is given by $z \mapsto z^k$, then k is called the multiplicity[1] of F at p, denoted by $\mathrm{mult}_p(F)$.

Definition 2.1.2 (ramification point and ramification value)**.** Let $F\colon X \to Y$ be a holomorphic map between Riemann surfaces. A point $p \in X$ is called a ramification point if $\mathrm{mult}_p(F) > 1$, and the image of ramification point is called a ramification value.

Lemma 2.1.1. Let $F\colon X \to Y$ be a non-constant holomorphic map between Riemann surfaces. A point $p \in X$ is a ramification point if there exists some local representation of F, denoted by T, such that $T'(0) = 0$.

Corollary 2.1.1. The set of ramification points of a holomorphic map is a discrete set.

[1] Sometimes this number is also called ramification of F at p.

Theorem 2.1.2. Let $F\colon X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces and define

$$d_q(F) = \sum_{p \in F^{-1}(q)} \operatorname{mult}_p(F).$$

Then $d_q(F)$ is independent of $q \in Y$, which is called the degree of F, and denoted by $\deg(F)$.

Proof. Suppose $X = Y = \mathbb{D}$ are unit disks and $F\colon \mathbb{D} \to \mathbb{D}$ is a holomorphic map defined by $z \mapsto z^m$. Then it's easy to show $d_q(F) = m$, for all $q \in \mathbb{D}$, since for $q = 0$, there is only one preimage of multiplicity m and for $q \neq 0$, there are m preimages of multiplicity 1.

Let's consider the general case. For $q \in Y$, since X is compact, $F^{-1}(q)$ only consists of finitely many points, denoted by $\{p_1, \dots, p_k\}$. Fix a local coordinate (V, ψ) centered at $q \in Y$, for any $i = 1, \dots, k$, there is a local coordinate (U_i, φ_i) centered at $p_i \in X$ such that

$$\psi \circ F \circ \varphi_i^{-1}(z) = z^{m_i}, \quad z \in \varphi_i(U_i),$$

where $m_i = \operatorname{mult}_{p_i}(F)$. If we choose another neighborhood $q \in W \subset V$ such that $F^{-1}(W) \subseteq \bigcup_{i=1}^k U_i$, then for any $q \in W$, from the trivial case discussed before one has

$$d_q(F) = \sum_{i=1}^k m_i.$$

This shows $d_q(F)$ is a locally constant function, and thus $d_q(F)$ is constant since Y is connected. □

Corollary 2.1.2. A holomorphic map between compact Riemann surfaces is an isomorphism if and only if it has degree one.

2.2. **Hurwitz Formula.**

In this section we talk about Hurwitz formula, which computes the genus from a given ramification covering. Before that we recall some basic facts in topology. Let X be a compact oriented surface, the Euler number of X can be defined by the triangulation of X as follows: Suppose a triangulation of X is given with v vertices, e edges and t tirangles. Then the Euler characterisitic of X is defined by $v - e + t$. On the other hand, the Euler number can also be defined as

$$\chi(X) := \sum_i (-1)^i \dim H_i(X).$$

The genus of X is defined by

$$\chi(X) = 2 - 2\operatorname{genus}(X).$$

Theorem 2.2.1 (Hurwitz Formula)**.** Let $F\colon X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces. Then

$$\chi(Y) = \deg(F)\chi(X) + \sum_{p \in X} (\operatorname{mult}_p(F) - 1)$$

Proof. Choose a triangulation Δ of Y such that its vertex are exactly ramification values of F. Let v, e, t denote the number of vertices, edges and triangles of Δ respectively. Suppose Δ' is the triangulation of X obtained by pulling back Δ through F, and use v', e' and t' to denote the number of vertices, edges and triangles of Δ' respectively.

It's clear we have the following relations

$$t' = td, \quad e' = ed$$

where $d = \deg(F)$. For $q \in Y$, note that

$$|F^{-1}(q)| = \sum_{p\in F^{-1}(q)} 1 = d + \sum_{p\in F^{-1}(q)} (1 - \mathrm{mult}_p(F)).$$

Then the relation between v and v' is given by

$$\begin{aligned} v' = & \sum_{\text{vertex } q \text{ of } \Delta} |F^{-1}(q)| \\ = & \sum_{\text{vertex } q \text{ of } \Delta} \left(d + \sum_{p\in F^{-1}(q)} (1 - \mathrm{mult}_p(F)) \right) \\ = & vd + \sum_{\text{vertex } q \text{ of } \Delta} \left(\sum_{p\in F^{-1}(q)} (1 - \mathrm{mult}_p(F)) \right) \\ = & vd + \sum_{p\in X} (1 - \mathrm{mult}_p(F)). \end{aligned}$$

Thus by the relation between Euler number and triangulation, we obtain the desired conclusion. □

Remark 2.2.1. Since the set of ramification points is finite, then $\sum_{p\in X}(\mathrm{mult}_p(F) - 1)$ is a finite number, and for convenience we denote it by $B(F)$. It describes how many ramification points of F are there on X.

Definition 2.2.1 (ramified holomorphic map)**.** A holomorphic map F is called ramified if $B(F) > 0$.

Definition 2.2.2 (unramified holomorphic map)**.** A holomorphic map F is called unramified if $B(F) = 0$.

Remark 2.2.2. A unramified holomorphic map is a covering map, and thus ramified holomorphic map is sometimes called ramified covering map.

Corollary 2.2.1. Let $F\colon X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces. Then

(1) If Y is Riemann sphere and $\deg(F) > 1$, then F must be ramified.
(2) If $\mathrm{genus}(X) = \mathrm{genus}(Y) = 1$, then F must be unramified.
(3) $\mathrm{genus}(X) \geq \mathrm{genus}(Y)$.
(4) If $\mathrm{genus}(X) = \mathrm{genus}(Y) > 1$, then F must be an isomorphism.

Proof.

(1) Since Riemann sphere has genus zero, one has
$$B(F) = 2(\deg(F) - 1) + 2\,\mathrm{genus}(X) > 0.$$

(2) By Hurwitz Formula we have
$$0 = 0 + B(F).$$

(3) If $\mathrm{genus}(Y) = 0$, it's trivial. Otherwise, we have
$$\begin{aligned} 2\,\mathrm{genus}(X) - 2 &= \deg(F)(2\,\mathrm{genus}(Y) - 2) + B(F) \\ &\geq 2\,\mathrm{genus}(Y) - 2 \end{aligned}$$
since $\deg(F) \geq 1$ and $B(F) \geq 0$.

(4) By Hurwitz Formula we have
$$(1 - \deg(F))(2\,\mathrm{genus}(X) - 2) = B(F).$$
Then $\deg(F) = 1$, since $\deg(F) \geq 1$, $2\,\mathrm{genus}(X) - 2 > 0$ and $B(F) \geq 0$.

$\square$

3. Homework

3.1. Week 1.

Exercise 3.1.1. Prove that when $\omega_1, \omega_2 \in \mathbb{C}$ are $\mathbb{R}$-linearly independent, then

(1) $\mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2$ is discrete.
(2) $\mathbb{C}\,/\,\mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2$ is Hausdorff.
(3) $\mathbb{C} \to \mathbb{C}\,/\,\mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2$ is a covering map.

Proof. For (1). Choose $0 < \epsilon < \min\{|w_1|/2, |w_2|/2, |w_1 - w_2|/2\}$. Then for any two elements u, v in $\mathbb{Z}\,w_1 + \mathbb{Z}\,w_2$, one has $B_\epsilon(u) \cap B_\epsilon(v) = \varnothing$, and thus $\mathbb{Z}\,w_1 + \mathbb{Z}\,w_2$ is discrete.

For (2). Let L denote the lattice $\mathbb{Z}\,w_1 + \mathbb{Z}\,w_2$ and $\pi\colon \mathbb{C} \to \mathbb{C}\,/L$ be the canonical projection. Suppose $\mathbb{C}\,/L$ is equipped with the quotient topology, that is, $U \subseteq \mathbb{C}\,/L$ is an open subset if and only if $\pi^{-1}(U)$ is open in $\mathbb{C}$. It's easy to show $\pi\colon \mathbb{C} \to \mathbb{C}\,/L$ is an open map, since for any open subset $U \subseteq \mathbb{C}$, one has

$$\pi^{-1}(\pi(U)) = \bigcup_{w \in L} w + U.$$

For $u, v \in \mathbb{C}\,/L$, we choose $\widetilde{u}, \widetilde{v} \in \mathbb{C}$ such that $\pi(\widetilde{u}) = u$ and $\pi(\widetilde{v}) = v$. Since $\mathbb{C}$ is Hausdorff, there exists open neighborhoods $\widetilde{U}, \widetilde{V}$ of $\widetilde{u}, \widetilde{v}$ such that $\widetilde{U} \cap \widetilde{V} = \varnothing$. Moreover, we may assume $\pi|_{\widetilde{U}}$ and $\pi|_{\widetilde{V}}$ are injective by shrinking $\widetilde{U}, \widetilde{V}$ when necessary. Then $\pi(\widetilde{U})$ and $\pi(\widetilde{V})$ are open neighborhoods of u, v respectively such that $\pi(\widetilde{U}) \cap \pi(\widetilde{V}) = \varnothing$. This shows $\mathbb{C}\,/L$ with quotient topology is Hausdorff.

For (3). For $u \in \mathbb{C}\,/L$, the preimages of u is discrete since L is discrete. For each preimage $\widetilde{u}_i$, we choose $\epsilon > 0$ small sufficiently such that $B_\epsilon(\widetilde{u}_i) \cap B_\epsilon(u_j) = \varnothing$ for $i \neq j$ and $\pi|_{B_\epsilon(\widetilde{u}_i)}$ is injective for all i. If we denote $U = \pi(B_\epsilon(\widetilde{u}_i))$, then $\pi\colon B_\epsilon(\widetilde{u}_i) \to U$ is a homeomorphism for each i and by construction $B_\epsilon(\widetilde{u}_i) \cap B_\epsilon(u_j) = \varnothing$ for $i \neq j$. This shows $\pi\colon \mathbb{C} \to \mathbb{C}\,/L$ is a covering map. $\square$

Exercise 3.1.2. Let V be a complex vector space of dimension n, with $\mathbb{C}$-basis $e_1, \ldots, e_n$, and $T\colon V \to V$ is a $\mathbb{C}$-linear transformation. Suppose T has matrix representation $X = A + \sqrt{-1}B$ where $A, B \in M_n(\mathbb{R})$ under (complex) basis $e_1, \ldots, e_n$. Prove

(1) $e_1, \ldots, e_n, \sqrt{-1}e_1, \ldots, \sqrt{-1}e_n$ is an $\mathbb{R}$-basis of V.
(2) T has matrix

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

under the $\mathbb{R}$-basis above when T is viewed as an $\mathbb{R}$-linear transformation.
(3)

$$\det \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = |\det X|^2.$$

Proof. For (1). Since $e_1, \ldots, e_n$ are $\mathbb{C}$-linearly independent and $1, \sqrt{-1}$ are $\mathbb{R}$-linearly independent, one has $e_1, \ldots, e_n, \sqrt{-1}e_1, \ldots, \sqrt{-1}e_n$ are $\mathbb{R}$-linearly independent. On the other hand, since $e_1, \ldots, e_n$ is a $\mathbb{C}$-basis, then any element $v \in V$ can be expressed as $v = v_1e_1 + \cdots + v_ne_n$, where $v_i \in \mathbb{C}$. If we write $v_i = a_i + \sqrt{-1}b_i$ with $a_i, b_i \in \mathbb{R}$, then

$$v = a_1e_1 + \cdots + a_ne_n + \sqrt{-1}b_1e_1 + \cdots + \sqrt{-1}b_ne_n.$$

This shows V as a $\mathbb{R}$-vector space is spanned by $e_1, \ldots, e_n, \sqrt{-1}e_1, \ldots, \sqrt{-1}e_n$.

For (2). Since T has matrix representation $X = A + \sqrt{-1}B$ under $\mathbb{C}$-basis $e_1, \ldots, e_n$, one has

$$\begin{aligned} T(e_i) &= \sum_{j=1}^n X_{ij}e_j = \sum_{j=1}^n \left(A_{ij}e_j + B_{ij}\sqrt{-1}e_j\right) \\ T(\sqrt{-1}e_i) &= \sum_{j=1}^n X_{ij}\sqrt{-1}e_j = \sum_{j=1}^n \left(-B_{ij}e_j + A_{ij}\sqrt{-1}e_j\right). \end{aligned}$$

This shows T has matrix

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

under the $\mathbb{R}$-basis $e_1, \ldots, e_n, \sqrt{-1}e_1, \ldots, \sqrt{-1}e_n$.

For (3). By elementary operations, one has

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \longrightarrow \begin{pmatrix} A + \sqrt{-1}B & B \\ -B + \sqrt{-1}A & A \end{pmatrix} \longrightarrow \begin{pmatrix} A + \sqrt{-1}B & B \\ 0 & A + \sqrt{-1}B \end{pmatrix}$$

Since the elementary operations don't change the determinant, this shows the desired result. □

Exercise 3.1.3 (implicit function theorem)**.** Let $f(z, w)\colon \mathbb{C}^2 \to \mathbb{C}$ be holomorphic function of two variables and $X = \{(z, w) \in \mathbb{C}^2 \mid f(z, w) = 0\}$ be its zero loucs. Let $p = (z_0, w_0)$ be a point of X and $\partial f/\partial z(p) \neq 0$. Then there exists a function $g(w)$ defined and holomorphic in a neighborhood of w_0 such that, near p, X is equal to the graph $z = g(w)$.

Proof. If we write $z = a + \sqrt{-1}b, w = c + \sqrt{-1}d$ and $f(z, w) = u + \sqrt{-1}v$, then u, v are smooth functions of a, b, c, d. Moreover, the Cauchy-Riemann equations give

$$\frac{\partial f}{\partial z} = \frac{\partial u}{\partial a} + \sqrt{-1}\frac{\partial v}{\partial a} = \frac{\partial v}{\partial b} - \sqrt{-1}\frac{\partial u}{\partial b} = A + \sqrt{-1}B.$$

Then

$$\frac{\partial(u, v)}{\partial(a, b)} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

and $\det \dfrac{\partial(u,v)}{\partial(a,b)} = A^2 + B^2 \neq 0$ if and only if $A + \sqrt{-1}B \neq 0$. Then the classical implicit function theorem implies the zero loucs

$$\begin{cases} u = 0 \\ v = 0 \end{cases}$$

is locally given by

$$\begin{cases} a = a(c,d) \\ b = b(c,d). \end{cases}$$

In other words, $z = g(w)$. Now it suffices to compute $\partial g/\partial\overline{w}$ to show g is holomorphic. Again by Cauchy-Riemann equations

$$\frac{\partial f}{\partial w} = \frac{\partial u}{\partial c} + \sqrt{-1}\frac{\partial v}{\partial c} = \frac{\partial v}{\partial d} - \sqrt{-1}\frac{\partial u}{\partial d} = C + \sqrt{-1}D.$$

Then by chain rule one has

$$\begin{aligned} \frac{\partial(a,b)}{\partial(c,d)} &= \left(\frac{\partial(u,v)}{\partial(a,b)}\right)^{-1} \frac{\partial(u,v)}{\partial(c,d)} \\ &= \begin{pmatrix} A & B \\ -B & A \end{pmatrix}^{-1} \begin{pmatrix} C & D \\ -D & C \end{pmatrix} \\ &= \frac{1}{A^2+B^2} \begin{pmatrix} AC+BD & AD-BC \\ BC-AD & BD+AC \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial g}{\partial\overline{w}} &= \frac{1}{2}\left(\frac{\partial}{\partial c} + \sqrt{-1}\frac{\partial}{\partial d}\right)(a + \sqrt{-1}b) \\ &= \frac{1}{2}\left(\frac{\partial a}{\partial c} + \sqrt{-1}\frac{\partial b}{\partial c} + \sqrt{-1}\frac{\partial a}{\partial d} - \frac{\partial b}{\partial d}\right) \\ &= 0 \end{aligned}$$

□

Exercise 3.1.4. Let $x_1, \dots, x_n$ be distinct points on $\mathbb{C}$ and

$$f(x,y) = y^d - (x - x_1)\cdots(x - x_n).$$

Prove that $C = \{f(x,y) = 0\}$ defines a Riemann surface in $\mathbb{C}^2$.(Question to think about: what is the topological shape of C?)

Proof. Note that there is no common zero of $f(x,y)$ and $\partial f/\partial x$ since $x_1, \dots, x_n$ are distinct points, so the affine plane curve defined by $f(x,y)$ is non-singular, and thus it's a Riemann surface. □

Remark 3.1.1. Now let's consider its compactification. Suppose $n \geq d$, and consider the homogenous polynomial defined by $f(x,y)$ as follows

$$F(x,y,z) = z^{n-d}y^d - (x - x_1 z)\dots(x - x_n z).$$

By setting $z = 0$ we found a new point $[0:1:0]$. It suffices to see it's singular or not. A direct computation shows

$$\frac{\partial F}{\partial x} = -(x - x_2 z)\dots(x - x_n z) - \dots - (x - x_1 z)\dots(x - x_{n-1} z)$$
$$\frac{\partial F}{\partial y} = d z^{n-d} y^{d-1}$$
$$\frac{\partial F}{\partial z} = (n-d) z^{n-d-1} y^d + x_1(x - x_2 z)\dots(x - x_n z) + \dots + x_n(x - x_1 z)\dots(x - x_{n-1} z).$$

Then

(1) If $n > d+1$, then it's singular.
(2) If $n = d+1$ or $n = d$, it's non-singular.

Now we suppose $n < d$, and then the homogenous polynomial defined $f(x, y)$ is given by

$$F(x, y, z) = y^d - z^{d-n}(x - x_1 z)\dots(x - x_n z).$$

By setting $z = 0$ we find a new point $[1:0:0]$. It suffices to see it's singular or not. A direct computation shows

$$\begin{aligned}
\frac{\partial F}{\partial x} &= -z^{d-n}\left((x - x_2 z)\dots(x - x_n z) + \dots + (x - x_1 z)\dots(x - x_{n-1} z)\right)\\
\frac{\partial F}{\partial y} &= d y^{d-1}\\
\frac{\partial F}{\partial z} &= (n-d) z^{d-n-1}(x - x_1 z)\dots(x - x_n z)\\
&\quad + x_1 z^{d-n}(x - x_2 z)\dots(x - x_n z) + \dots + x_n z^{d-n}(x - x_1 z)\dots(x - x_{n-1} z).
\end{aligned}$$

Then

(1) If $n < d-1$, then it's singular.
(2) If $n = d-1$, then it's non-singular.

In a summary, only when $n = d-1, d, d+1$, the compactification is non-singular, otherwise it's singular.

References

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