PRINCIPAL BUNDLE AND ITS APPLICATIONS

BOWEN LIU

Contents

1. Preface

1.1. **About this lecture.**

1.2. **Some notations.**

1.2.1. *On base manifold.*

(1) M is used to denote a smooth manifold, and $x \in M$ denotes its point.
(2) TM and Ω^k_M are used to denote tangent bundle and bundle of k-forms over M respectively.
(3) $\Omega^k_M(E)$ is used to denote bundle of k-forms over M valued E.
(4) v is used to denote vector in tangent space.
(5) X is used to denote a vector field on M, and X_x denotes the value of X at point $x \in M$.
(6) α is used to denote a k-form on M, and α_x denotes the value of α at point $x \in M$.
(7) For a vector bundle E over M, $C^\infty(E, M)$ is used to denote its sections.

1.2.2. *On principal bundle.*

(1) G is used to denote a Lie group, with Lie algebra $\mathfrak{g}$.
(2) $\pi\colon P \to M$ is used to denote a principal G-bundle over M, and $p \in P$ denotes its point.
(3) $\widetilde{X}$ is used to denote vector field on principal bundle P, so do $\widetilde{\alpha}$ and $\widetilde{v}$.
(4) ω is used to denote connection 1-form on P, with curvature 2-form Ω.

Part 1. Principal bundle and its geometry

2. Principal bundle

2.1. A glimpse of fiber bundle.

Definition 2.1.1 (fiber bundle)**.** Let F, E, B be topological spaces. A fiber bundle with fiber F over B is a surjective map $\pi\colon E \to B$ such that for any $p \in B$, there exists an open neighborhood $U \ni p$ and a homeomorphism φ such that the following diagram commutes

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_1"] \\
& U &
\end{tikzcd}$$

We always use $F \to E \xrightarrow{\pi} B$ or (E, B, π, F) to denote this fiber bundle and

(1) B is called base space.
(2) $E_x = \pi^{-1}(x)$ is called the fiber of E at x.
(3) (U, φ) is called a local trivialization at point p, and use $E|_U$ to denote $\pi^{-1}(U)$.

Example 2.1.1 (trivial bundle)**.** Consider $E = B \times F$ and $\pi\colon E \to B$ is just the projection onto the first summand.

Example 2.1.2. Consider $E = S^n$ and $B = \mathbb{RP}^n$, then natural map $\pi\colon E \to B$ is a fiber bundle with $\mathbb{Z}/2\mathbb{Z}$. It's clear that this fiber bundle is not trivial, since S^n is connected.

Example 2.1.3 (Hopf fibration)**.** Recall that

$$\mathbb{CP}^n = \{\text{the set of all complex lines through origin in } \mathbb{C}^{n+1}\}$$

Consider the canonical open covering $\{U_i\}$ of $\mathbb{CP}^n$, that is

$$U_i = \{[z_0 : \ldots : z_n] \mid z_i \neq 0\}$$

Now view $S^{2n+1} \subset \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$ as the set of all $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ with $|z_0|^2 + \cdots + |z_n|^2 = 1$. Then the projection map $\pi\colon \mathbb{C}^{n+1} - \{0\} \to \mathbb{CP}^n$ restricts to a surjective smooth map

$$\pi\colon S^{2n+1} \to \mathbb{CP}^n$$

We claim that it's a fiber bundle with fiber S^1. Indeed, by definition we have

$$\pi^{-1}(U_i) = \{(z_0, \ldots, z_n) \in S^{2n+1} \mid z_i \neq 0\}$$

and local trivialization map can be taken as

$$\begin{aligned}
\varphi_i : \pi^{-1}(U_i) &\to U_i \times S^1 \\
z &\mapsto ([z_0 : \ldots : z_n], \frac{z_i}{|z_i|})
\end{aligned}$$

It's also not trivial which can be seen by considering their fundamental groups.

Example 2.1.4. The covering space is a fiber bundle with discrete set as fiber.

2.2. Principal bundle.

2.2.1. *Definitions.* Briefly speaking, given a Lie group G and a smooth manifold M, a principal G-bundle P is a fiber bundle with fiber G equipped with a suitable smooth right G-action on it. For a smooth right G-action we mean a smooth map

$$\begin{aligned} P \times G &\to P \\ (p, g) &\mapsto pg \end{aligned}$$

Definition 2.2.1 (principal G-bundle)**.** A principal G-bundle is a surjective smooth map $\pi\colon P \to M$ between smooth manifolds such that:

(1) There is a smooth right G-action on P.
(2) For all $x \in M$, $\pi^{-1}(x)$ is a G-orbit.
(3) For all $x \in M$, there exists an open subset U_α and a G-equivariant diffeomorphism φ_α, which is called a local trivialization, such that the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times G \arrow[dl, "\pi_{U_\alpha}"] \\ & U_\alpha & \end{tikzcd}$$

Notation 2.2.1. $\mathcal{P}_G M$ is used to denoted the set of all principal G-bundles over M up to isomorphism.

Remark 2.2.1. If we write $\varphi_\alpha(p) = (\pi(p), g_\alpha(p))$, then φ_α is G-equivariant if and only if $g_\alpha(pg) = g_\alpha(p)g$ for any $g \in G$.

Proposition 2.2.1. Let P be a principal G-bundle, then G acts on P freely and transitively.

Proof. It's clear from local trivialization. $\square$

Example 2.2.1. $S^n \to \mathbb{RP}^n$ is a $\mathbb{Z}/2\mathbb{Z}$-principal bundle, where $\mathbb{Z}/2\mathbb{Z}$ acts on S^n via $x \mapsto -x$.

Example 2.2.2. $S^{2n+1} \to \mathbb{CP}^n$ is a $U(1)$-principal bundle, where $U(1)$ acts on S^{2n+1} via $(z_0, z_1, \dots, z_n) \mapsto (z_0 e^{i\theta}, z_1 e^{i\theta}, \dots, z_n e^{i\theta})$.

Definition 2.2.2 (morphism between principal G-bundle)**.** For two principal G-bundles $(P, M, \pi), (P', M, \pi')$, a morphism between them is a G-equivariant smooth map $\varphi\colon P' \to P$ making the following diagram commute

$$\begin{tikzcd} P \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

Proposition 2.2.2. A morphism $\varphi\colon P \to P'$ between principal G-bundles over M is an isomorphism.

Proof. All information are encoded in the G-equivariance of φ and properties of principal G-bundle:

(1) φ is injective: For any $p_1, p_2 \in P$, if $\varphi(p_1) = \varphi(p_2)$, then p_1, p_2 lie in same fiber, since above diagram commutes. If $p_1 = p_2 g$ for $g \in G$, then $\varphi(p_1) = \varphi(p_2)g$, which impiles $g = e$, since G acts on P' freely, that is $p_1 = p_2$.

(2) φ is surjective: For any $p' \in P'$, if $\pi'(p') = x$, then $p' \in P'_x$. So choose an arbitrary element $p \in P_x$, there must be some $g \in G$ such that $\varphi(pg) = p'$, since P'_x is a G-orbit and φ is G-equivariant.

□

Definition 2.2.3 (trivial principal bundle)**.** A principal G-bundle P is called trivial principal bundle, if there exists a principal G-bundle isomorphism $\varphi\colon P \to M \times G$.

2.2.2. *Structure group.* Let $\{U_\alpha, \varphi_\alpha\}$ be a local trivialization of principal G-bundle P. If $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \varnothing$, then transition functions $g_{\alpha\beta}\colon U_{\alpha\beta} \to \operatorname{Diff} G$ is defined by

$$\begin{aligned} \varphi_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1}\colon U_{\alpha\beta} \times G &\to U_{\alpha\beta} \times G \\ (x, h) &\mapsto (x, g_{\alpha\beta}(x)h) \end{aligned}$$

Note that

$$\begin{aligned} (\pi(p), g_\alpha(p)) &= \varphi_\alpha \circ \varphi_\beta^{-1} \circ \varphi_\beta(p) \\ &= \varphi_{\alpha\beta}(\pi(p), g_\beta(p)) \end{aligned}$$

This shows

$$g_{\alpha\beta}(x) g_\beta(p) = g_\alpha(p) \tag{2.1}$$

where $p \in \pi^{-1}(x)$. Fix $x \in U_{\alpha\beta}$, it's clear

$$g_{\alpha\beta}(x)(h_1 h_2) = g_{\alpha\beta}(h_1)h_2$$

holds for arbitrary $h_1, h_2 \in G$, then $g_{\alpha\beta}(x)$ must take the form $h \mapsto gh$ for some $g \in G$. This shows the transition functions of principal G-bundle valued in G, that is

$$g_{\alpha\beta}\colon U_{\alpha\beta} \to G$$

That is to say, the structure group of a principal G-bundle is G.

2.2.3. *Section.*

Definition 2.2.4 (global section)**.** A global section of principal G-bundle $\pi\colon P \to M$ is a smooth map $s\colon M \to P$ such that $\pi \circ s = \mathrm{id}$.

Proposition 2.2.3. A principal G-bundle P over M admits a section if and only if it is trivial[1].

[1]This is in sharp contrast with vector bundles, which always admit sections.

Proof. If $s\colon M\to P$ is a smooth section, consider

$$\begin{aligned}\varphi\colon P&\to M\times G\\ p&\mapsto(\pi(p),g(p))\end{aligned}$$

where $g(p)\in G$ such that $p=s(\pi(p))g(p)$, it always exsits since the right action of G is transitive on each fiber and it is unique since the action is free on each fiber. Clearly, it's G-equivariant, since

$$\varphi(ph)=(\pi(ph),g(ph))=(\pi(p),g(p)h)$$

and the last equality holds since

$$ph=s(\pi(ph))g(ph)=s(\pi(p))g(ph)=pg^{-1}(p)g(ph)\implies h=g^{-1}(p)g(ph)$$

Thus $\varphi\colon P\to M\times G$ is a morphism between principal G-bundles over M, so by Proposition 2.2.2, P is isomorphic to $M\times G$, that is P is trivial principal G-bundle. □

Example 2.2.3. Although P may not admit global section, it always admits local section σ_α over local trivialization $\{U_\alpha,\varphi_\alpha\}$, which is given by

$$\begin{aligned}\sigma_\alpha\colon U_\alpha&\to\pi^{-1}(U_\alpha)\\ x&\mapsto\varphi_\alpha^{-1}(x,e)\end{aligned}$$

Proposition 2.2.4.

$$\sigma_\beta(x)=\sigma_\alpha(x)g_{\alpha\beta}(x)$$

where $x\in U_{\alpha\beta}$.

Proof. Direct computation shows

$$\begin{aligned}\varphi_\beta(\sigma_\alpha(x)g_{\alpha\beta}(x))&=\varphi_\beta\circ\varphi_\alpha^{-1}(x,e)g_{\alpha\beta}(x)\\&=(x,g_{\beta\alpha}(x)g_{\alpha\beta}(x))\\&=(x,e)\end{aligned}$$

that is $\sigma_\alpha(x)g_{\alpha\beta}(x)=\varphi_\beta^{-1}(x,e)=\sigma_\beta(x)$. □

2.3. **Associated fiber bundle.** Given a principal G-bundle $\pi\colon P\to M$ and a smooth manifold F admitting a smooth left G-action on it, that is there is a group homomorphism $\rho\colon G\to\mathrm{Diff}(F)$.

Proposition 2.1. The set $P\times_\rho F:=P\times F/_\sim$, where $(p,f)\sim(p',f')$ if and only if $p'=pg,f'=g^{-1}f$, admits a fiber bundle structure over M with fiber F.

Proof. Consider the map taking an equivalence class $[p,f]$ to $\pi(p)$. To see the local structure, since we already have the local structure of principal bundle P, i.e. for any $x\in M$, there exists open $U_\alpha\ni x$ and $\varphi_\alpha\colon\pi^{-1}(U_\alpha)\to U_\alpha\times G$. Now we define the local trivialization of $P\times_G V$ as

$$\begin{aligned}\varphi_\alpha^F:(P\times_\rho F)|_{U_\alpha}&\to U_\alpha\times F\\ (p,f)&\mapsto(\pi(p),g_\alpha(p)f)\end{aligned}$$

First note that this is well-defined, since

$$(pg, g^{-1}f) \mapsto (\pi(pg), g_\alpha(pg)g^{-1}f) = (\pi(p), g_\alpha(p)gg^{-1}f) = (\pi(p), g_\alpha(p)f)$$

And this map gives a diffeomorphism, since g_α is smooth and taking inverse is a smooth operation of Lie groups. □

Remark 2.3.1 (transition function of associated bundle). Though we've found the local trivialization of $P \times_\rho F$, it's also necessary to see what does the transition functions look like. Let $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ be local trivializations, with transition functions

$$\begin{aligned} \varphi_\alpha \circ \varphi_\beta^{-1} : U_{\alpha\beta} \times G &\to U_{\alpha\beta} \times G \\ (x, g) &\mapsto (x, g_{\alpha\beta}(x)g) \end{aligned}$$

then we can compute the transition functions of associated vector bundles as follows

$$\begin{aligned} \varphi_\alpha^F \circ (\varphi_\beta^F)^{-1} \colon U_{\alpha\beta} \times F &\to U_{\alpha\beta} \times F \\ (x, f) &\mapsto (x, g_\alpha(p)(g_\beta(p))^{-1}f) \end{aligned}$$

Then by equation (2.1), it's clear to see transition functions of associated fiber bundle is exactly $\{\rho(g_{\alpha\beta})\}$.

Example 2.3.1 (associated vector bundle)**.** Now let's consider a special case, that is associated vector bundles. Given a representation of G, that is a group homomorphism $\rho \colon G \to \mathrm{GL}(V)$, thus you can construct a vector bundle $P \times_\rho V$. However, there is a more simple way to construct in transition functions viewpoint: By Remark 2.3.1, we can see the transition function of this associated vector bundle is $\{\rho(g_{\alpha\beta})\}$, where $\{g_{\alpha\beta}\}$ is transition function of P.

Remark 2.3.2 (relations between vector bundle and principal bundle). For real vector bundles endowed with Riemannian metric, consider

$$\begin{aligned} \Phi \colon \mathcal{P}_{\mathrm{O}(n)}M &\to \mathrm{Vect}_n^{\mathbb{R}} M \\ P &\mapsto P \times_\rho \mathbb{R}^n \end{aligned}$$

where $\rho \colon \mathrm{O}(n) \to \mathrm{GL}(n, \mathbb{R})$ is trivial representation, that is inclusion. Φ is bijective with inverse Ψ is given by considering frame bundle of vector bundle. thus we have the following one to one correspondence up to isomorphism

$$\mathcal{P}_{\mathrm{O}(n)}M \longleftrightarrow \mathrm{Vect}_n^{\mathbb{R}} M$$

Similarly we also have

$$\mathcal{P}_{\mathrm{U}(n)}M \longleftrightarrow \mathrm{Vect}_n^{\mathbb{C}} M$$

In this viewpoint, principal G-bundles generalize the conception of vector bundles.

Example 2.3.2. There are two important examples of associated bundles that we will use later.

(1) The associated bundle obtained from conjugate action Conj of G acting on G, denoted by $P \times_{\mathrm{Conj}} G$.
(2) The associated vector bundle obtained from adjoint action Ad of G acting on $\mathfrak{g}$, denoted by $P \times_{\mathrm{Ad}} \mathfrak{g}$.

Remark 2.3.3. For a principal G-bundle, you can obtain a vector bundle from a representation of G. However, there are too many representations of G, so special representations may correspond to special vector bundles.

Proposition 2.3.1. There is a one to one correspondence

$$C^\infty(M, P \times_\rho F) \xleftrightarrow{1-1} \{f \colon P \to F \mid f \text{ is smooth and } f(xg) = g^{-1}f(x)\}$$

Proof. For a G-equivariant smooth function $f \colon P \to F$, consider $s_f \in C^\infty(M, P \times_\rho F)$ given by

$$s_f(x) = \{(p, f(p)) \mid \pi(p) = x\}$$

where $x \in M$. It's well-defined, since if we choose pg instead of p for some $g \in G$, then $s_f(x) = (pg, f(pg)) = (pg, g^{-1}f(p)) \sim (p, f(p)) \in P \times_\rho F$. Conversely, given $s \in C^\infty(M, P \times_\rho F)$, then for any $p \in P$, we consider $\pi(p) = x \in M$ and write $s(x) = [(p, v)]$, then we define $f(p) = v$. It's clear $f(pg) = g^{-1}f(p)$, since $[(p, v)] = [(pg, g^{-1}v)]$. $\square$

Remark 2.3.4. In fact, this proposition is not a coincidence, and it's a quite important motivation which shows why we introduce principal G-bundles. If $\pi \colon P \to M$ is a principal G-bundle, and E is a vector bundle over M such that E is an associated vector bundle of P, then if we use π to pull E back to P, we claim that the vector bundle π^*E is the trivial bundle $P \times V$ over P. Indeed, we define the following bundle map

$$\begin{aligned} \psi \colon P \times V &\to P \times_G V \\ (p, v) &\mapsto [p, v] \end{aligned}$$

and consider the following diagram

$$\begin{tikzcd} P \times V \arrow[r] \arrow[d, "\psi"] & P \arrow[d, "\pi"] \\ E = P \times_G V \arrow[r] & M \end{tikzcd}$$

Clearly $P \times V$ satisfies the universal property of pullback, thus by uniqueness we obtain $\pi^*E \cong P \times V$.

In general case, we can use π to pull $(P \times_G V) \otimes E'$ back to P, and prove it's $(P \times V) \otimes \pi^*E'$ by the same method. The cases we will encounter are $E' = T^*M$ or $E' = \bigwedge^k T^*M$. We use $\Omega^k_M(P \times_G V)$ to denote $(P \times_G V) \otimes \bigwedge^k T^*M$, the generalization tells that we have the one to one correspondence between sections of $\Omega^k_M(P \times_G V)$ and sections of $(P \times V) \otimes \pi^* \bigwedge^k T^*M$ with equivariant conditions, we will call such forms basic forms, a conception we will define in section 4.2.

2.4. **Reduction of principal bundle.** Given a principal G-bundle $\pi\colon P \to M$ and a H-principal bundle $\pi' : P' \to M$. Furthermore, there is a Lie group homomorphism $\alpha\colon H \to G$.

Definition 2.4.1 (reduction)**.** If there exists a smooth map $\varphi\colon P' \to P$ such that the following diagram commutes

$$\begin{tikzcd} P' \arrow[rr, "\varphi"] \arrow[dr, "\pi_F"'] & & P \arrow[dl, "\pi_E"] \\ & M & \end{tikzcd}$$

and φ is H-equivariant, that is for any $p \in F, h \in H$

$$\varphi(ph) = \varphi(p)\alpha(h)$$

Then P is called an extension of P' from H to G and P' is called an reduction of P from G to H.

Remark 2.4.1. Here are two cases we're concern about:

(1) $H < G$ is a subgroup, α is an inclusion.
(2) $\alpha\colon H \to G$ is surjective, for example, H is universal covering of G.

Extension of principal bundle always exists, and it's unique, according to the following proposition.

Proposition 2.4.1. Given a Lie group homomorphism $\alpha\colon H \to G$ and a H-principal bundle P', there exists a unique extension of P' from H to G.

Proof. Existence: Note that $\alpha\colon H \to G$ gives a smooth left H-action on G, then consider associated fiber bundle $P' \times_H G$, it's a principal G-bundle, and if we define

$$\begin{aligned} \varphi\colon P' &\to P' \times_H G \\ p' &\mapsto [p', 1] \end{aligned}$$

Then φ is desired equivariant map which makes diagram commutes.

Uniqueness: If there is another extension $\varphi' : P' \to P$, in order to make the following diagram commutes

$$\begin{tikzcd} & P' \times_H G \arrow[d, dashed, "\psi"] \\ P' \arrow[ur, "\varphi"] \arrow[r, "\varphi'"'] & P \end{tikzcd}$$

we define ψ by $\psi([p, 1]) = \varphi'(p)$. Thus principal G-bundles $P' \times_H G$ and P are isomorphic to each other. $\square$

However, reduction may not exist.

Lemma 2.4.1. Let $\alpha\colon H \to G$ be a Lie group homomorphism, P is a principal G-bundle with transition functions $\psi_{\alpha\beta} : U_{\alpha\beta} \to G$. The following statements are equivalent:

(1) There exists reduction of P from G to H.
(2) There exists $\varphi_{\alpha\beta} : U_{\alpha\beta} \to H$ such that $\alpha \circ \varphi_{\alpha\beta} = \psi_{\alpha\beta}$.

Corollary 2.4.1. Let P be a principal G-bundle and H is a Lie subgroup of G, then there exists a reduction of P from G to H if and only if there exists transition functions of P valued in H.

Example 2.4.1. If $E \to M$ is a complex vector bundle with a hermitian inner product, then a local trivialization

$$\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^n$$

gives a hermitian inner product on $\mathbb{C}^n$. Thus a transition function must preserve the inner product, thus

$$\begin{tikzcd} U_\alpha \cap U_\beta \arrow[r] \arrow[dr, dashed] & \mathrm{GL}_n(\mathbb{C}) \\ & \mathrm{U}(n) \arrow[u, hook] \end{tikzcd}$$

This gives a reduction of $\mathrm{GL}_n(\mathbb{C})$-principal bundle to a $\mathrm{U}(n)$-principal bundle.

Example 2.4.2. If $E \to M$ is a real vector bundle, by the same argument we can always reduce its frame bundle P, that is from a $\mathrm{GL}_n(\mathbb{R})$-principal bundle, to a $\mathrm{O}(n)$-principal bundle. Furthermore,

(1) P can be reduced to a $\mathrm{SO}(n)$-principal bundle if and only if E is orientable.
(2) P can be reduced to a $\{e\}$-principal bundle if and only if E is trivial.

Example 2.4.3. Let (M, g) be an oriented Riemannian manifold, then TM is a $\mathrm{SO}(n)$-principal bundle. Consider universal covering[2] $\mathrm{Spin}(n) \xrightarrow{2:1} \mathrm{SO}(n)$. If there exists a reduction from $\mathrm{SO}(n)$ to $\mathrm{Spin}(n)$, then we say M admits a spin structure.

[2]See section 10.2.2 for more details about Spin groups and this universal covering.

3. Connection of principal bundle

3.1. **Forms valued in vector space.** In this section, let M be a smooth manifold, V a vector space with basis $\{e_\alpha\}$ and G a Lie group with Lie algebra $\mathfrak{g}$. A k-form valued in vector space V can be written as

$$\omega = \omega^\alpha e_\alpha$$

where ω^α are k-forms.

Notation 3.1.1. $\Omega^k_M(V)$ denotes the bundle of k-forms valued in V.

$\Omega^k_M(V)$ is an easy generalization of differential forms, just by replacing $\mathbb{R}$ with a general vector space, and properties of k-forms also hold for k-forms value in V.

Definition 3.1.1 (exterior derivative)**.** Let $\omega = \omega^\alpha e_\alpha$ be a k-form valued in V, then its exterior derivative is defined as

$$\mathrm{d}\omega = \mathrm{d}\omega^\alpha e_\alpha$$

Proposition 3.1.1 (Cartan's formula)**.** Let $\omega = \omega^\alpha e_\alpha$ be a k-form valued in V, then

$$\begin{aligned}(\mathrm{d}\omega)(X_1,\dots,X_{k+1}) = &\sum_{i=1}^{k+1}(-1)^{i+1}X_i\omega(X_1,\dots,\widehat{X_i},\dots,X_{k+1})\\ &+\sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_1,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_{k+1})\end{aligned}$$

where X_i are vector fields.

Definition 3.1.2 (wedge product)**.** Let ω_1,ω_2 be forms valued in V with degree k and l respectively, then

$$(\omega_1\wedge\omega_2)(X_1,\dots,X_{k+l}) := \frac{1}{k!\times l!}\sum_{\sigma\in S_{k+l}}(-1)^{|\sigma|}\omega_1(X_{\sigma(1)},\dots,X_{\sigma(k)})\otimes\omega_2(X_{\sigma(k+1)},\dots,X_{\sigma(k+l)})$$

where X_i are vector fields.

Proposition 3.1.2. Let ω_i, where $i=1,2,3$, be forms valued in V, then
(1) $(\omega_1\wedge\omega_2)\wedge\omega_3 = \omega_1\wedge(\omega_2\wedge\omega_3)$.
(2) $\mathrm{d}(\omega_1\wedge\omega_2) = \mathrm{d}\omega_1\wedge\omega_2 + (-1)^{\deg\omega_1}\omega_1\wedge\mathrm{d}\omega_2$.

Definition 3.1.3. Let $T\colon V\to W$ be a linear map between vector spaces, and ω is a k-form valued in V, then $T\omega$ is a k-form valued in W, which is defined as

$$T\omega(X_1,\dots,X_k) := T(\omega(X_1,\dots,X_k))$$

where X_i are vector fields.

Example 3.1.1. Let ω_1,ω_2 be forms with degree k and l respectively, then by our definition one has $\omega_1\wedge\omega_2\in\Omega^{k+l}_M(\mathbb{R}\otimes\mathbb{R})$. It's a little bit different from standard definition of wedge product, since $\omega_1\wedge\omega_2$ should be a $(k+l)$-form. If we consider

$$\begin{aligned} T\colon\ \mathbb{R}\otimes\mathbb{R}&\to\mathbb{R}\\ a\otimes b&\mapsto ab \end{aligned}$$

Then $T(\omega_1\wedge\omega_2)$ is a $(k+l)$-form, coincides with standard definition, so we just denote $T(\omega_1\wedge\omega_2)$ by $\omega_1\wedge\omega_2$ for convenience.

Example 3.1.2. Let ω_1 be a k-form valued in $\mathfrak{g}$, and ω_2 a l-form valued in V. Given a representation $\rho\colon G\to\mathrm{GL}(V)$, it induces a representation of Lie algebra, that is $\rho_*\colon\mathfrak{g}\to\mathfrak{gl}(V)$. If we consider

$$\begin{aligned} T\colon\mathfrak{g}\otimes V&\to V\\ \xi\otimes v&\mapsto\rho_*(\xi)v \end{aligned}$$

Then we have $T(\omega_1\wedge\omega_2)$ is a $(k+l)$-form valued in V, we just denote it by $\omega_1\wedge\omega_2$ for convenience.

Example 3.1.3. Let ω_1,ω_2 be forms valued in $\mathfrak{g}$ with degree k and l respectively, by our definition $\omega_1\wedge\omega_2$ is a $(k+l)$-form valued in $\mathfrak{g}$. If we consider

$$\begin{aligned} T\colon\mathfrak{g}\otimes\mathfrak{g}&\to\mathfrak{g}\\ \xi\otimes\eta&\mapsto[\xi,\eta] \end{aligned}$$

Then we have $T(\omega_1\wedge\omega_2)$ is a $(k+l)$-form valued in $\mathfrak{g}$, we just denote it by $\omega_1\wedge\omega_2$ for convenience.

Remark 3.1.1. If Lie group $G=\mathrm{GL}(n,\mathbb{R})$, then $\mathfrak{g}=\mathfrak{gl}(n,\mathbb{R})$ consists of matrix. Thus in this case for any $\xi,\eta\in\mathfrak{g}$, we can define T as multiplying them together to obtain an element in $\mathfrak{gl}(n,\mathbb{R})$. However, these two notations may cause some misunderstandings.

Example 3.1.4. Let ω be a 1-form valued in $\mathfrak{g}$, then for vector fields X,Y, one has

$$\begin{aligned} \omega\wedge\omega(X,Y)&=T((\omega\wedge\omega)(X,Y))\\ &=T(\frac{1}{1!\times 1!}(\omega(X)\otimes\omega(Y)-\omega(Y)\otimes\omega(X)))\\ &=[\omega(X),\omega(Y)]-[\omega(Y),\omega(X)]\\ &=2[\omega(X),\omega(Y)] \end{aligned}$$

Remark 3.1.2. If T is choose as in Remark 3.1.1, then in this case we have

$$\omega\wedge\omega(X,Y)=[\omega(X),\omega(Y)]$$

So be careful about which wedge product you're using.

Proposition 3.1.3. Let ω be a 1-form valued in $\mathfrak{g}$, then

$$(\omega\wedge\omega)\wedge\omega=\omega\wedge(\omega\wedge\omega)=0$$

Proof. For arbitrary vector fields X, Y and Z, one has

$$\begin{aligned}(\omega\wedge\omega)\wedge\omega(X,Y,Z) = \frac{1}{2!\times 1!}\{&[\omega\wedge\omega(X,Y),\omega(Z)]+[\omega\wedge\omega(Y,Z),\omega(X)]+[\omega\wedge\omega(Z,X),\omega(Y)]\\ &-[\omega\wedge\omega(Y,X),\omega(Z)]+[\omega\wedge\omega(Z,Y),\omega(X)]+[\omega\wedge\omega(X,Z),\omega(Y)]\}\\ = \frac{2}{2!\times 1!}\{&[[\omega(X),\omega(Y)],\omega(Z)]+[[\omega(Y),\omega(Z)],\omega(X)]+[[\omega(Z),\omega(X)],\omega(Y)]\\ &-[[\omega(Y),\omega(X)],\omega(Z)]+[[\omega(Z),\omega(Y)],\omega(X)]+[[\omega(X),\omega(Z)],\omega(Y)]\}\end{aligned}$$

This equals to zero according to Jacobi identity of Lie bracket. □

Proposition 3.1.4. Let ω_1, ω_2 be forms valued in $\mathfrak{g}$ with degree k and l respectively, then

$$\omega_1\wedge\omega_2 = (-1)^{kl+1}\omega_2\wedge\omega_1$$

Proof. Note that for a k-form ω_1 and a l-form ω_2, we have

$$\omega_1\wedge\omega_2 = (-1)^{kl}\omega_2\wedge\omega_1$$

But in this case, there is one more -1 coming from Lie bracket. □

3.2. **Maurer-Cartan form.**

Definition 3.2.1 (Maurer-Cartan form)**.** The Maurer-Cartan form θ is a $\mathfrak{g}$-valued 1-form on G, defined by

$$\theta_g \colon = (L_{g^{-1}})_*$$

where $g\in G$.

Remark 3.2.1. For arbitrary vector $v\in T_gG$ which is given by $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} ge^{tX}$, where $X\in\mathfrak{g}$. Direct computation shows

$$\begin{aligned}\theta_g(v) &= (L_{g^{-1}})_* v\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} g^{-1}ge^{tX}\\ &= X\in\mathfrak{g}\end{aligned}$$

This shows Maurer-Cartan form is a $\mathfrak{g}$-valued 1-form.

Proposition 3.2.1. Let $G\subseteq \mathrm{GL}(n,\mathbb{R})$ be a matrix Lie group, and $g\colon M\to G$ is a smooth map, where M is a smooth manifold. Then $g^*\theta = g^{-1}\mathrm{d}g$, where θ is Maurer-Cartan form on G and $g^{-1}\mathrm{d}g$ is the multiplication of matrices.

Proof. For $v\in T_xM$, direct computation shows

$$\begin{aligned}(g^*\theta)_x v &= \theta_{g(x)}((\mathrm{d}g)_x v)\\ &= (L_{g(x)^{-1}})_*(\mathrm{d}g)_x v\end{aligned}$$

Note that

$$\begin{aligned}L_{g(x)^{-1}}\colon \mathrm{GL}(n,\mathbb{R}) &\to \mathrm{GL}(n,\mathbb{R})\\ A &\mapsto g(x)^{-1}A\end{aligned}$$

is a linear transformation, which impiles $(L_{g(x)^{-1}})_* = L_{g(x)^{-1}}$. Thus

$$(g^*\theta)_x v = g(x)^{-1}(\mathrm{d}g)_x v$$

which impiles $g^*\theta = g^{-1}\mathrm{d}g$. □

Corollary 3.2.1. Let $G \subseteq \mathrm{GL}(n, \mathbb{R})$ be a matrix Lie group. Then Maurer-Cartan form on G is given by $g^{-1}\mathrm{d}g$, where $g\colon G \to G$ is identity map and $g^{-1}\mathrm{d}g$ is the multiplication of matrices.

3.3. **Motivation for connection on principal bundle.** All in all, our motivation is that connection of principal G-bundles can be used as a tool to study connection of vector bundle E, if E is an associated vector bundle of P. Recall a connection on vector bundle E is defined as the following $\mathbb{R}$-linear operator

$$\nabla\colon C^\infty(M, E) \to C^\infty(M, \Omega^1_M(E))$$

satisfying Leibniz rule.

Suppose E is associated to principal G-bundle $\pi\colon P \to M$, written as $P \times_\rho V$, then from Proposition 2.3.1, there is an one to one correspondence between sections of E with G-equivariant maps from P to V. Given a section s of E, if we use s^P to denote the G-equivariant map obtained from one to one correspondence, it's easy to take derivatives of s^P to obtain a 1-form on P valued in V, that is a G-equivariant fiber-wise linear map from TP to V. However, this 1-form does not by itself define a covariant derivative of s. Indeed, by definition of connection, $\nabla s \in C^\infty(M, \Omega^1_M(E))$, then by Remark 2.3.4, a covariant derivative appears upstairs on P is supposed to be a G-equivariant section over $(P \times V) \otimes \pi^* T^* M$, that is a G-equivariant fiber-wise linear map from $\pi^* TM$ to V.

To see what is missing, it is important to keep in mind that TP has some properties that arise from the fact that P is a principal bundle over M. In fact, we have the following exact sequence

$$0 \to \ker \pi_* \to TP \to \pi^* TM \to 0 \tag{3.1}$$

This exact sequence is quite important, let's make following remarks:

Remark 3.3.1. The map from $\ker \pi_*$ is clearly an inclusion. And the surjective map from TP to $\pi^* TM$ is characterized as follows

$$\begin{aligned} TP &\to \pi^* TM \subset P \times TM \\ v &\mapsto (p, \pi_* v) \end{aligned}$$

where $v \in T_p P$.

Remark 3.3.2. $\ker \pi_*$ is isomorphic to trivial bundle $P \times \mathfrak{g}$. Indeed, we have the following bundle isomorphism

$$\begin{aligned} \psi\colon P \times \mathfrak{g} &\to \ker \pi_* \\ (p, X) &\mapsto \sigma(X) \end{aligned}$$

where $\sigma(X)_p := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} pe^{tX}$ is called fundamental vector field of X. It's clear $\sigma(X) \in \ker \pi_*$, since for each $p \in P$,

$$\begin{aligned}\pi_*(\sigma(X)_p) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \pi(pe^{tX}) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \pi(p) \\ &= 0\end{aligned}$$

Remark 3.3.3 (G-equivariance of exact sequence). The action of G on P can be lifted to the exact sequence (3.1). Let $R_g \colon P \to P$ denote the action of $g \in G$ on P, given by $p \mapsto pg$.

(1) The G action on TP is given by $(R_g)_* : TP \to TP$, and it descends to $\ker \pi_*$ since if $v \in \ker \pi_*$, then

$$\begin{aligned}\pi_*((R_g)_* v) &= (\pi \circ R_g)_*(v) \\ &= \pi_*(v) \\ &= 0\end{aligned}$$

(2) The G action on π^*TM is given by sending defined by sending a pair $(p, v) \in P \times TM$ to the pair (pg, v). It's well-defined, that is $(pg, v) \in \pi^*TM$, since $\pi(pg) = \pi(p) = \pi(v)$.

Furthermore, we claim the exact sequence (3.1) is equivariant with respect to the lifts.

(1) It automatically holds for inclusion map from $\ker \pi_*$ to TP, since G action on $\ker \pi_*$ is obtain from descending the one on TP.
(2) It holds for the map from TP to π^*TM, since for $v \in TP$ we have $(R_g)_* v$ is sent to $(pg, \pi_*(R_g)_* v)$, that is exactly $(pg, \pi_* v)$, since $\pi \circ R_g = \pi$.

If we want to identify $\ker \pi_*$ as $P \times \mathfrak{g}$, we need to choose an appropriate G-action on $\mathfrak{g}$ properly such that the isomorphism ψ is G-equivariant. It turns out to be adjoint representation. Indeed, direct computation shows

$$\begin{aligned}(R_g)_*\psi(p, X) &= (R_g)_* \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} p\exp(tX) \right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} p\exp(tX)g \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (pg)\left(g^{-1}\exp(tX)g\right) \\ &= \psi(pg, \mathrm{Ad}(g^{-1})X)\end{aligned}$$

3.4. **Connection on principal bundle.** So if we want to obtain a fiber-wise linear map $\pi^*TM \to V$ from a fiber-wise linear map $TP \to V$, one way is to desire exact sequence (3.1) splitting. In other words, we desire there exists a G-equivariant $\omega : TP \to P \times \mathfrak{g}$, such that $\omega|_{P\times\mathfrak{g}}$ is identity. Such ω is called a connection on principal G-bundle P.

Definition 3.4.1 (connection on principal G-bundle)**.** Let $\pi\colon P\to M$ be a principal G-bundle. $\omega\in C^\infty(P,\Omega^1_P(\mathfrak{g}))$ is called a connection on P, if it satisfies

(1) For any $X\in\mathfrak{g}$, $\omega(\sigma(X))=X$.
(2) For any $g\in G$, $(R_g)^*\omega=\mathrm{Ad}(g^{-1})\omega$, that is

$$\omega((R_g)_*X)=\mathrm{Ad}(g^{-1})\omega(X)$$

holds for all $X\in C^\infty(T,TP)$.

Notation 3.4.1. $\mathcal{A}(P)$ denotes the set of all connections on P.

Remark 3.4.1 (horizontal distribution viewpoint)*.* If we define $H=\ker\omega$, then

$$TP=H\oplus(P\times\mathfrak{g})$$

such that $(R_g)_*H_p=H_{pg}$. H is called a horizontal distribution and $P\times\mathfrak{g}$ is called vertical distribution. Conversely, give a horizontal distribution, one can also construct a connection.

Example 3.4.1 (connection on trivial principal G-bundle)**.** Consider trivial principal G-bundle $P=M\times G$. Recall we have a Maurer-Cartan form θ, which is a 1-form valued in $\mathfrak{g}$. Then we can use $\pi_G\colon M\times G\to G$ to pull it back to P to obtain a 1-form on P valued in $\mathfrak{g}$, which is called Maurer-Cartan form on trivial principal G-bundle, and it's denoted ω_{mc}. Now let's check ω_{mc} gives a connection on trivial principal bundle.

(1) For any $X\in\mathfrak{g}$, we have

$$\begin{aligned}\omega_{mc}(\sigma(X))&=\pi_G^*\theta(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(x,g)e^{tX})\\&=\theta(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}ge^{tX})\\&=(L_{g^{-1}})_*(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}ge^{tX})\\&=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}e^{tX}\\&=X\end{aligned}$$

(2) It suffices to check $(R_g)^*\theta=\mathrm{Ad}(g^{-1})\theta$ holds for $g\in G$. At point $h\in G$, and $v\in T_hG$ given by $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}he^{tX}$, where $X\in\mathfrak{g}$. Direct computation

shows

$$\begin{aligned}(R_g)^*\theta_h(v) &= \theta_{hg}(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} he^{tX}g) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (hg)^{-1}he^{tX}g \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g^{-1}e^{tX}g \\ &= \mathrm{Ad}(g^{-1})\theta_h(v)\end{aligned}$$

Remark 3.4.2. It's clear to see $\ker \omega_{mc} = \pi^* TM$, since ω_{mc} is pullback from a 1-form on G, thus in this case

$$TP \cong TM \oplus TG$$

that's exactly canonical splitting of TP.

3.5. **Gauge group.**

Definition 3.5.1 (gauge transformation)**.** For a principal G-bundle $\pi\colon P \to M$, the gauge transformation is a G-equivariant diffeomorphism $\Phi\colon P \to P$ such that $\pi = \pi \circ \Phi$.

Notation 3.5.1. $\mathcal{G}(P)$ denotes the set of all gauge transformation of P, which forms a group, called gauge group.

Remark 3.5.1 (terminologies). Here we make some clarifications about terminologies. A local gauge is a physicist's terminology for the choice of local trivialization, and the change of local trivialization, that is transition functions, are called gauge transformation. For physicists gauge group is exactly structure group, and gauge group we defined here is sometimes called global gauge group.

Remark 3.5.2 (local expression of gauge transformation). For a gauge transformation Φ, its action on local trivialization $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$, given by $\varphi_\alpha(\Phi(p)) = (\pi(p), g_\alpha(\Phi(p)))$, induces a map $\widetilde{\phi}_\alpha : \pi^{-1}(U_\alpha) \to G$ by

$$\widetilde{\phi}_\alpha(p) = g_\alpha(\Phi(p))g_\alpha(p)^{-1}$$

By the G-equivariance of g_α and Φ one has $\widetilde{\phi}$ is G-invariant, which impiles $\widetilde{\phi_\alpha}$ can descend to U_α, that is one can define $\phi_\alpha\colon U_\alpha \to G$ via $\widetilde{\phi}_\alpha(p) = \phi_\alpha(\pi(p))$. If we consider on the overlaps $x \in U_{\alpha\beta}$ with $p = \pi^{-1}(x)$, then

$$\begin{aligned}\phi_\alpha(x) &= g_\alpha(\Phi(p))g_\alpha(p)^{-1} \\ &= g_\alpha(\Phi(p))g_\beta(\Phi(p))^{-1}g_\beta(\Phi(p))g_\beta(p)^{-1}g_\beta(p)g_\alpha(p)^{-1} \\ &= g_{\alpha\beta}(x)\phi_\beta(x)g_{\alpha\beta}(x)^{-1}\end{aligned}$$

This shows $\{\phi_\alpha\}$ defines a global section of associated bundle obtained from G acts on G by conjugation, that is $P \times_{\mathrm{Conj}} G$ defined in Example 2.3.2. In fact, we have the following one to one correspondence.

Proposition 3.5.1. There is one to one correspondence between the group $\mathcal{G}(P)$ and $C^\infty(M, P\times_{\mathrm{Conj}} G)$.

Proof. We have already seen that a gauge transformation can give an element in $C^\infty(M, P\times_{\mathrm{Conj}} G)$. Conversely, by Proposition 2.3.1, there is a one to one correspondence between $C^\infty(M, P\times_{\mathrm{Conj}} G)$ and smooth functions $f\colon P\to G$ which is G-equivariant. For such f, consider $\Phi_f\colon P\to P$ given by $\Phi_f(p)=pf(p)$.

(1) $\pi\circ\Phi_f=\pi$, since $\pi\circ\Phi_f(p)=\pi(pf(p))=\pi(p)$
(2) It's G-equivariant since

$$\begin{aligned}\Phi_f(pg)&=pgf(pg)\\&=pgg^{-1}f(p)g\\&=pf(p)g\\&=\Phi_f(p)g\end{aligned}$$

The two maps we constructed are clearly inverse to each other, giving the desired correspondence. □

Now we're going to show $\mathcal{G}(P)$ acts on $\mathcal{A}(P)$.

Lemma 3.5.1. For any $X\in\mathfrak{g}$ and $\Phi\in\mathcal{G}(P)$, then

$$\Phi_*(\sigma(X))=\sigma(X)$$

Proof. Direct computation shows

$$\begin{aligned}\Phi_*\sigma(X)&=\Phi_*(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}pe^{tX})\\&=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\Phi(pe^{tX})\\&=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\Phi(p)e^{tX}\\&=\sigma(X)\end{aligned}$$

□

Proposition 3.5.2. $\mathcal{G}(P)$ acts on $\mathcal{A}(P)$ via pullback.

Proof. For $\omega\in\mathcal{A}(P)$ and $\Phi\in\mathcal{G}(P)$, let's check $\Phi^*\omega\in\mathcal{A}(P)$.

(1) For any $X\in\mathfrak{g}$, we have

$$\begin{aligned}\Phi^*\omega(\sigma(X))&=\omega(\Phi_*\sigma(X))\\&=\omega(\sigma(X))\\&=X\end{aligned}$$

(2) Note that $(R_g)^*\Phi^* = (R_g \circ \Phi)^* = (\Phi \circ R_g)^*$, since Φ is G-equivariant, thus

$$\begin{aligned}(R_g)^*(\Phi^*\omega) &= \Phi^*((R_g)^*\omega)\\ &= \Phi^*(\mathrm{Ad}(g^{-1})\omega)\\ &= \mathrm{Ad}(g^{-1})\Phi^*\omega\end{aligned}$$

□

Remark 3.5.3. Gauge theory concerns about "space" of orbit of $\mathcal{G}(P)$, that is $\mathcal{A}(P)/\mathcal{G}(P)$.

3.6. **Local expression of connection.** Instead of considering connection 1-form living on P, we want to convert it into the one living on base manifold M, since we want to use it to study connection of vector bundle over M. To do this, we divide the process into three steps:

(1) Given a connection on trivial principal G-bundle, correspond it to a 1-form on M valued $\mathfrak{g}$.
(2) Figure out how does this correspondence transform under gauge transformation.
(3) Since a principal G-bundle admits local trivializations, and transition functions can be regarded as gauge transformations, then we reduce the case to the first two steps.

3.6.1. *Baby case.* Fix a trivial principal G-bundle $P = M \times G$ and following notations:

(1) $\pi\colon P \to M$ is natural projection, given by $p = (x, g) \mapsto x \in M$.
(2) $i\colon M \to P$ is natural inclusion, given by $x \mapsto (x, e) \in P$.

Lemma 3.6.1. For any $A \in C^\infty(M, \Omega^1_M(\mathfrak{g}))$, there exists a unique $\widetilde{A} \in C^\infty(P, \Omega^1_P(\mathfrak{g}))$ such that

(1) $i^*\widetilde{A} = A$.
(2) $\widetilde{A}(\sigma(X)) = 0$, where $X \in \mathfrak{g}$.
(3) $(R_g)^*\widetilde{A} = \mathrm{Ad}(g^{-1})\widetilde{A}$.

Proof. It suffices to construct $\widetilde{A}$ pointwisely.

(a) For $p = (x, e) \in M \times G$, we have

$$T_pP = T_xM \oplus \mathfrak{g}$$

Then $\widetilde{A}$ is uniquely determined at this point according to (1) and (2).

(b) At point $p' = (x, g) \in M \times G$, it's clear $p' = pg$ and $(R_g)_* : T_pP \to T_{p'}P$ is an isomorphism, then for arbitrary $v \in T_{p'}P$, we may assume $v = (R_g)_*w$ for some $w \in T_pP$, then

$$\begin{aligned}\widetilde{A}_{p'}(v) &= \widetilde{A}_{pg}((R_g)_*w)\\ &= ((R_g)^*\widetilde{A})_p(w)\\ &= \mathrm{Ad}(g^{-1})\widetilde{A}(w)\end{aligned}$$

□

Proposition 3.6.1. $i^*: \mathcal{A}(P) \to C^\infty(M, \Omega^1_M(\mathfrak{g}))$ is bijective, that is the following diagram commutes

$$\begin{tikzcd} C^\infty(P, \Omega^1_P(\mathfrak{g})) \arrow[r, "i^*"] & C^\infty(M, \Omega^1_M(\mathfrak{g})) \\ \mathcal{A}(P) \arrow[u, hook] \arrow[ur, "1-1"] & \end{tikzcd}$$

Proof. For any $A \in C^\infty(M, \Omega^1_M(\mathfrak{g}))$, by Lemma 3.6.1 we have $\omega_{mc} + \widetilde{A}$ is also a connection on P, thus we consider

$$\begin{aligned} \tau\colon C^\infty(M, \Omega^1_M(\mathfrak{g})) &\to \mathcal{A}(P) \\ A &\mapsto \omega_{mc} + \widetilde{A} \end{aligned}$$

It's clear τ is surjective, since for any $\omega \in \mathcal{A}(P)$, we have

$$\tau(i^*(\omega - \omega_{mc})) = \omega_{mc} + \omega - \omega_{mc} = \omega$$

Now it suffices to show $i^*\tau = \mathrm{id}$, which impiles τ is injective thus bijective. Indeed, for $A \in C^\infty(M, \Omega^1_M(\mathfrak{g}))$,

$$i^*\tau(A) = i^*(\omega_{mc} + \widetilde{A}) = i^*\widetilde{A} = A$$

since $i^*\omega_{mc} = 0$. □

3.6.2. *How to glue.* Any gauge transformation Φ on trivial principal G-bundle $P = M \times G$ can be written as

$$\Phi(x, g) = (x, \phi(x)g)$$

where $\phi\colon M \to G$ is smooth map.

Proposition 3.6.2. For $\omega \in \mathcal{A}(P)$

$$i^*\Phi^*\omega = \mathrm{Ad}(\phi^{-1})i^*\omega + \phi^*\theta$$

where θ is Maurer-Cartan form.

Proof. For any $\omega \in \mathcal{A}(P)$, it can be written as $\omega = \omega_{mc} + \widetilde{A}$ according to Proposition 3.6.1. Then

$$\begin{aligned} i^*\Phi^*\omega &= i^*\Phi^*(\omega_{mc} + \widetilde{A}) \\ &\overset{(1)}{=} i^*\Phi^*\pi_G^*\theta + i^*\Phi^*\widetilde{A} \\ &\overset{(2)}{=} \phi^*\theta + i^*\Phi^*\widetilde{A} \end{aligned}$$

where

(1) holds from definition of Maurer-Cartan form.
(2) holds from $\pi_G \circ \Phi \circ i(x) = \pi_G \circ \Phi(x, e) = \pi_G(x, \phi(x)) = \phi(x)$ for $x \in M$.

Now it suffices to compute $i^*\Phi^*\widetilde{A}$. For $v \in T_xM$, one has

$$
\begin{aligned}
(i^*\Phi^*\widetilde{A})_x(v) &= \Phi^*\widetilde{A}_{(x,e)}(v,0) \\
&= \widetilde{A}_{(x,\phi(x))}(v,0) \\
&= (R_{\phi(x)})^*\widetilde{A}_{(x,e)}(v,0) \\
&= \mathrm{Ad}(\phi^{-1}(x))(i^*\widetilde{A})_x(v)
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
i^*(\Phi^*\omega) &= \phi^*\theta + \mathrm{Ad}(\phi^{-1})i^*\widetilde{A} \\
&\overset{(3)}{=} \phi^*\theta + \mathrm{Ad}(\phi^{-1})i^*\omega
\end{aligned}
$$

where (3) holds from $i^*\omega_{mc} = 0$ and $\omega = \omega_{mc} + \widetilde{A}$. □

3.6.3. *General case.* Let $\pi\colon P \to M$ be a principal G-bundle with local trivializations $\{U_\alpha, \varphi_\alpha\}$, and $i_\alpha\colon U_\alpha \to U_\alpha \times G$ sends x to (x, e). For a connection $\omega \in \mathcal{A}(P)$, we define $\omega_\alpha := (\varphi_\alpha^{-1})^*\omega_{\pi^{-1}(U_\alpha)}$, which is a $\mathfrak{g}$-valued 1-form on $U_\alpha \times G$, and

$$A_\alpha := i_\alpha^*\omega_\alpha \in C^\infty(U_\alpha, \Omega^1_{U_\alpha}(\mathfrak{g}))$$

Remark 3.6.1. In Example 2.2.3 we introduce local section σ_α with respect to local trivialization $\{U_\alpha, \varphi_\alpha\}$, it's clear to see $A_\alpha = \sigma_\alpha^*(\omega|_{\pi^{-1}(U_\alpha)})$.

Proposition 3.6.3.

$$\mathcal{A}(P) \overset{1-1}{\longleftrightarrow} \{(A_\alpha) \in \prod_\alpha C^\infty(U_\alpha, \Omega^1_M(\mathfrak{g})) \mid A_\beta = \mathrm{Ad}(g_{\alpha\beta}^{-1})A_\alpha + g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta}\}$$

Proof. Note that

$$
\begin{aligned}
\Phi\colon U_{\alpha\beta} \times G &\to U_{\alpha\beta} \times G \\
(x, h) &\mapsto (x, g_{\alpha\beta}(x)h)
\end{aligned}
$$

gives a gauge transformation of trivial principal G-bundle $U_{\alpha\beta} \times G$. Then for $\omega \in \mathcal{A}(P)$, one has

$$
\begin{aligned}
i_\beta^*\Phi^*\omega_\alpha &\overset{(1)}{=} \mathrm{Ad}(g_{\alpha\beta}^{-1})(i_\alpha^*\omega_\alpha) + g_{\alpha\beta}^*\theta \\
&\overset{(2)}{=} \mathrm{Ad}(g_{\alpha\beta}^{-1})A_\alpha + g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta}
\end{aligned}
$$

where $g_{\alpha\beta}\colon U_{\alpha\beta} \to G$ are transition functions, and

(1) holds from Proposition 3.6.2.
(2) holds from Proposition 3.2.1.

Note that

$$
\begin{aligned}
\omega_\alpha &= (\varphi_\alpha^{-1})^*\omega|_{\pi^{-1}(U_\alpha)} \\
&= (\varphi_\alpha^{-1})(\varphi_\beta)^*(\varphi_\beta^{-1})^*\omega|_{\pi^{-1}(U_\alpha)} \\
&= (\varphi_\beta \circ \varphi_\alpha^{-1})^*\omega_\beta \\
&= (\Phi^{-1})^*\omega_\beta
\end{aligned}
$$

This shows

$$i_\beta^*\Phi^*\omega_\alpha = i_\beta^*\omega_\beta = A_\beta$$

Conversely, suppose $\{A_\alpha\}$ is a set of $\mathfrak{g}$-valued 1-form satisfying

$$A_\beta = \mathrm{Ad}(g_{\alpha\beta}^{-1})A_\alpha + g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta}$$

By Lemma 3.6.1 there exists a $\mathfrak{g}$-valued 1-form $\widetilde{A}_\alpha$ on $\pi^{-1}(U_\alpha)$ such that

(1) $(\sigma_\alpha)^*\widetilde{A}_\alpha = A_\alpha$.
(2) $\widetilde{A}_\alpha(\sigma(X)) = 0$ for $X \in \mathfrak{g}$.
(3) $(R_g)^*\widetilde{A}_\alpha = \mathrm{Ad}(g^{-1})\widetilde{A}_\alpha$.

Direct computation shows $\{\widetilde{A}_\alpha\}$ gives a $\mathfrak{g}$-valued 1-form $\widetilde{A}$ defined on P, and then $\widetilde{A} + \omega_{mc}$ gives a connection ω on P. Furthermore, these two constructions are inverse to each other, which completes the proof. $\square$

Corollary 3.6.1. $\mathcal{A}(P)$ is an affine space modelled on $C^\infty(M, \Omega_M^1(P \times_{\mathrm{Ad}} \mathfrak{g}))$.

4. Curvature of principal bundle

4.1. Definition.

Definition 4.1.1 (curvature)**.** Let P be a principal G-bundle and $\omega \in \mathcal{A}(P)$. Curvature of ω is defined as

$$\Omega := \mathrm{d}\omega + \frac{1}{2}\omega \wedge \omega \in C^\infty(P, \Omega^2_P(\mathfrak{g}))$$

Proposition 4.1.1.

$$(R_g)^*\Omega = \mathrm{Ad}(g^{-1})\Omega$$

where $g \in G$.

Proof. It follows from pullback commutes with exterior derivative and wedge product. □

Proposition 4.1.2. Let $P = M \times G$ be trivial principal G-bundle equipped with connection ω_{mc}, then $\Omega = 0$.

Proof. It suffices to check Maurer-Cartan form $\theta \in C^\infty(G, \Omega^1_G(\mathfrak{g}))$ satisfying

$$\mathrm{d}\theta + \frac{1}{2}\theta \wedge \theta = 0$$

which is called Maurer-Cartan equation. Firstly we suppose X, Y are left-invariant vector fields, then

$$\theta(X) = (L_{g^{-1}})_* X_g = (L_{g^{-1}})_*(L_g)_* X_e = X_e$$

is constant. Thus

$$\mathrm{d}\theta(X, Y) = -\theta([X, Y]) = -\frac{1}{2}\theta \wedge \theta(X, Y)$$

since $X(\theta(Y)) = Y(\theta(X)) = 0$. But left-invariant vector fields span the tangent space at any point, thus Maurer-Cartan equation holds for arbitrary vector fields X, Y. □

Theorem 4.1.1 (Bianchi identity)**.**

$$\mathrm{d}\Omega + \omega \wedge \Omega = 0$$

Proof.

$$\begin{aligned}
\mathrm{d}\Omega &= \mathrm{d}(\mathrm{d}\omega + \frac{1}{2}\omega \wedge \omega) \\
&= \frac{1}{2}\mathrm{d}\omega \wedge \omega - \frac{1}{2}\omega \wedge \mathrm{d}\omega \\
&= -\omega \wedge \mathrm{d}\omega \\
&= -\omega \wedge (\Omega - \frac{1}{2}\omega \wedge \omega) \\
&= -\omega \wedge \Omega
\end{aligned}$$

□

Definition 4.1.2 (horizontal form)**.** Let α be a k-form on P valued in vector space V, it's called horizontal if $\iota_{\sigma(X)}\alpha = 0$ for arbitrary $X \in \mathfrak{g}$.

Lemma 4.1.1. For $X \in \mathfrak{g}$, the flow of $\sigma(X)$ is given by

$$\phi_t(p) = pe^{tX}$$

where $p \in P$.

Proposition 4.1.3. Ω is a horizontal 2-form.

Proof. Direct computation shows

(1) For $X, Y \in \mathfrak{g}$, one has

$$\begin{aligned} d\omega(\sigma(X), \sigma(Y)) &= \sigma(X)(\omega(\sigma(Y))) - \sigma(Y)(\omega(\sigma(X))) - \omega([\sigma(X), \sigma(Y)]) \\ &\overset{(1)}{=} -[\omega(\sigma(X)), \omega(\sigma(Y))] \\ &\overset{(2)}{=} -\frac{1}{2}\omega \wedge \omega(\sigma(X), \sigma(Y)) \end{aligned}$$

where

(1) holds from $\omega(\sigma(Y))$ and $\omega(\sigma(X))$ are constant functions valued Y and X respectively.

(2) holds from Example 3.1.3.

(2) If $X \in \mathfrak{g}$ and Y is a horizontal vector field, note that

$$\frac{1}{2}\omega \wedge \omega(\sigma(X), Y) = 0$$

since $\omega(Y) = 0$, and direct computation shows

$$\begin{aligned} d\omega(\sigma(X), Y) &= \sigma(X)(\omega(Y)) - Y\omega(\sigma(X)) - \omega([\sigma(X), Y]) \\ &\overset{(3)}{=} -\omega([\sigma(X), Y]) \\ &\overset{(4)}{=} -\omega(\mathcal{L}_{\sigma(X)}Y) \end{aligned}$$

where

(3) holds from $\omega(Y) = 0$ and $\omega(\sigma(X))$ is a constant function valued X.

(4) holds from property of Lie derivative.

By definition one has

$$(\mathcal{L}_{\sigma(X)}Y)_p = \lim_{t\to 0} \frac{(\phi_{-t})_* Y_{\phi_t(p)} - Y_p}{t}$$

where ϕ_t is the flow generated by $\sigma(X)$ and $p \in P$. Thus

$$\begin{aligned} \omega_p((\mathcal{L}_{\sigma(X)}Y)_p) &\overset{(5)}{=} \omega_p(\lim_{t\to 0} \frac{(\phi_{-t})_* Y_{\phi_t(p)} - Y_p}{t}) \\ &\overset{(6)}{=} \lim_{t\to 0} \frac{1}{t}\left\{\omega_p((\phi_{-t})_* Y_{\phi_t(p)}) - \omega_p(Y_p)\right\} \\ &\overset{(7)}{=} \lim_{t\to 0} \frac{1}{t}\left\{((R_{e^{-tX}})^*\omega_p)(Y_{pe^{tX}}) - \omega_p(Y_p)\right\} \\ &= \lim_{t\to 0} \frac{1}{t}\left\{\mathrm{Ad}(e^{tX})\omega_{pe^{tX}}(Y_{pe^{tX}}) - \omega_p(Y_p)\right\} \\ &= 0 \end{aligned}$$

where

(5) holds from definition of Lie derivative.
(6) holds from ω is a smooth form.
(7) holds from Lemma 4.1.1.

□

Remark 4.1.1 (curvature vanishes and integrability). Given a horizontal distribution $H \subset TP$, we define the horizontal projection $h : TP \to TP$ to be the projection onto the horizontal distribution along the vertical distribution. Since both vertical and horizontal distribution are invariant under the action of G, so is h. Then $\Omega = h^* \mathrm{d}\omega$. Indeed, it suffices to show for vector fields X, Y, one has

$$\mathrm{d}\omega(hX, hY) = \mathrm{d}\omega(X, Y) + \frac{1}{2}\omega \wedge \omega(X, Y)$$

Consider the following cases:

(1) Let X, Y be horizontal. In this case there is nothing to prove, since $\omega(X) = \omega(Y) = 0$ and $hX = X, hY = Y$.
(2) If one of X, Y is vertical, then it's clear both sides are zero, since both Ω and $h^*\mathrm{d}\omega$ are horizontal.

As a consequence one has

$$\begin{aligned}\Omega(X, Y) &= \mathrm{d}\omega(hX, hY) \\ &= -\omega([hX, hY])\end{aligned}$$

where X, Y are two vector fields on P. This shows $\Omega(X, Y) = 0$ if and only if $[hX, hY]$ is horizontal. In other words, the curvature of the connection measures the failure of integrability of the horizontal distribution $H \subset TP$.

4.2. **Local expression of curvature and basic form.** Let $\pi \colon P \to M$ be a principal G-bundle with local trivializations $\{U_\alpha, \varphi_\alpha\}$. If we define

$$\Theta_\alpha = \sigma_\alpha^*(\Omega|_{\pi^{-1}(U_\alpha)}) \in C^\infty(U_\alpha, \Omega^2_{U_\alpha}(\mathfrak{g}))$$

By definition one has

$$\Theta_\alpha = \mathrm{d}A_\alpha + \frac{1}{2}A_\alpha \wedge A_\alpha$$

Lemma 4.2.1. For $x \in U_{\alpha\beta}$ and $v \in T_x M$

$$(\sigma_\beta)_*(v) = (R_{g_{\alpha\beta}(x)})_*((\sigma_\alpha)_* v) + (\sigma_\alpha(x))_*((g_{\alpha\beta})_* v)$$

where $(\sigma(x))_*$ is the differential of the following map

$$\begin{aligned}G &\to P \\ h &\mapsto \sigma_\alpha(x)h\end{aligned}$$

Proof. Let $\gamma(t)$ be a curve with $\gamma(0) = x$ and $\gamma'(0) = v$. Direct computation shows

$$\begin{aligned}(\sigma_\beta)_*(v) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \sigma_\beta(\gamma(t)) \\ &\overset{(1)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \sigma_\alpha(\gamma(t))g_{\alpha\beta}(\gamma(t)) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \sigma_\alpha(\gamma(t))g_{\alpha\beta}(x) + \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \sigma_\alpha(x)g_{\alpha\beta}(\gamma(t)) \\ &= (R_{g_{\alpha\beta}(x)})_*((\sigma_\alpha)_* v) + (\sigma_\alpha(x))_*((g_{\alpha\beta})_* v)\end{aligned}$$

where (1) follows from Proposition 2.2.4. □

Remark 4.2.1. From above proof it's clear to see $(\sigma_\alpha(x))_*((g_{\alpha\beta})_* v)$ is a vertical vector, which is a crucial property.

Proposition 4.2.1.

$$\Theta_\beta = \mathrm{Ad}(g_{\alpha\beta}^{-1})\Theta_\alpha$$

where $g_{\alpha\beta}\colon U_{\alpha\beta} \to G$ is transition function.

Proof. For $x \in U_{\alpha\beta}$ and $v, w \in T_x M$, direct computation shows

$$\begin{aligned}(\Theta_\beta)_x(v, w) &= \Omega_{\sigma_\beta(x)}((\sigma_\beta)_* v, (\sigma_\beta)_* w) \\ &\overset{(1)}{=} \Omega_{\sigma_\beta(x)}((R_{g_{\alpha\beta}(x)})_*(\sigma_\alpha)_* v, (R_{g_{\alpha\beta}(x)})_*(\sigma_\alpha)_* w) \\ &= ((R_{g_{\alpha\beta}(x)})^*\Omega)_{\sigma_\alpha(x)}((\sigma_\alpha)_* v, (\sigma_\alpha)_* w) \\ &\overset{(2)}{=} \mathrm{Ad}(g_{\alpha\beta}(x)^{-1})\Omega_{\sigma_\alpha(x)}((\sigma_\alpha)_* v, (\sigma_\alpha)_* w) \\ &= \mathrm{Ad}(g_{\alpha\beta}(x)^{-1})(\Theta_\alpha)_x(v, w)\end{aligned}$$

where

(1) holds from Ω is horizontal and remark of Lemma 4.2.1.
(2) holds from Ω is Proposition 4.1.1.

□

Definition 4.2.1 (basic form)**.** Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G, a k-form α on P valued in V is called a basic form, if it satisfies

(1) α is horizontal.
(2) It's ρ-equivariant, that is
$$(R_g)^*\alpha = \rho(g^{-1})\alpha$$
where $g \in G$.

Notation 4.2.1. The set of all basic k-forms on P valued V is denoted by $C^\infty(P, \Omega^k_P(V))^{\mathrm{basic}}$.

Theorem 4.2.1. Let $\rho\colon G \to \mathrm{GL}(V)$ be a linear representation, and $E = P \times_\rho V$. Then

$$C^\infty(M, \Omega^k_M(E)) \overset{1-1}{\longleftrightarrow} C^\infty(P, \Omega^k_P(V))^{\mathrm{basic}}$$

Example 4.2.1. For $k = 0$, one has

$$C^\infty(P, \Omega_P^0(V))^{\text{basic}} = \{f \colon P \to V \mid f(xg) = \rho(g^{-1})f(x)\}$$

Thus Theorem 4.2.1 recovers Proposition 2.3.1.

5. From connection on principal to connection on vector bundle

5.1. **Connection on vector bundle.** Let $\pi\colon E\to M$ be a vector bundle of rank n, and $\{U_\alpha,\varphi_\alpha\}$ is a local trivialization of E with transition functions $\{g_{\alpha\beta}\}$. If $\{e_i\}$ is the standard basis of $\mathbb{R}^n$ consisting of row vectors[3], then there is a local frame over U_α given by

$$e_i^\alpha:=\varphi_\alpha^{-1}((x,e_i))$$

Direct computation shows

$$\begin{aligned} e_i^\beta&=\varphi_\alpha^{-1}\circ\varphi_\alpha\circ\varphi_\beta^{-1}((x,e_i))\\ &=\varphi_\alpha^{-1}((x,g_{\alpha\beta}e_i))\\ &=(g_{\alpha\beta})_i^je_j^\alpha \end{aligned}$$

where j is row index and i is column index of $(g_{\alpha\beta})_i^j$. Let ∇ be a connection on E, which is locally given by $\{A_\alpha\}\in\prod C^\infty(U_\alpha,\Omega_M^1(\mathfrak{gl}(n,\mathbb{R})))$, that is

$$\nabla e_i^\alpha=(A_\alpha)_i^j\otimes e_j^\alpha$$

Direct computation shows

$$\begin{aligned} \nabla e_i^\beta&=\nabla((g_{\alpha\beta})_i^je_j^\alpha)\\ &=\mathrm{d}(g_{\alpha\beta})_i^j\otimes e_j^\alpha+(g_{\alpha\beta})_i^j(A_\alpha)_j^k\otimes e_k^\alpha\\ &=(\mathrm{d}(g_{\alpha\beta})_i^k+(g_{\alpha\beta})_i^j(A_\alpha)_j^k)\otimes e_k^\alpha \end{aligned}$$

On the other hand, one has

$$\begin{aligned} \nabla e_i^\beta&=(A_\beta)_i^j\otimes e_j^\beta\\ &=(A_\beta)_i^j(g_{\alpha\beta})_j^k\otimes e_k^\alpha \end{aligned}$$

This shows

$$(A_\beta)_i^j(g_{\alpha\beta})_j^k=\mathrm{d}(g_{\alpha\beta})_i^k+(g_{\alpha\beta})_i^j(A_\alpha)_j^k$$

and in matrix notation one has

$$A_\beta=g_{\alpha\beta}^{-1}A_\alpha g_{\alpha\beta}+g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta} \tag{5.1}$$

That is to say, if we want to give a connection on E, it suffices to gives $\{A_\alpha\}\in\prod C^\infty(U_\alpha,\Omega_M^1(\mathfrak{gl}(n,\mathbb{R})))$ satisfying relation (5.1).

5.2. **Connection on associated vector bundle.** In this section we will show if E is an associated vector bundle of principal G-bunlde P over M, then connection ω on P gives a connection on E.

[3]To be explicit, $e_i=(0,\dots,\underbrace{1}_{i\text{-th}},\dots,0)$.

5.2.1. *Baby version.* Let E be a vector bundle over M, and it's realized as an associated vector bundle of principal $\mathrm{GL}(n,\mathbb{R})$-bundle P by trivial representation. Let $\{U_\alpha\}$ be a local trivialization of P with transition functions $\{g_{\alpha\beta}\}$. For connection $\omega \in \mathcal{A}(P)$, by Proposition 3.6.3 one has a set of $\mathfrak{gl}(n,\mathbb{R})$-valued 1-forms $\{A_\alpha\}$ with

$$A_\beta = \mathrm{Ad}(g_{\alpha\beta}^{-1})A_\alpha + g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta}$$

Note that A_α is a 1-form valued $\mathfrak{gl}(n,\mathbb{R})$, and in matrix group adjoint representation can be expressed explicitly, that is

$$\mathrm{Ad}(g_{\alpha\beta}^{-1})A_\alpha = g_{\alpha\beta}^{-1}A_\alpha g_{\alpha\beta}$$

This shows $\{A_\alpha\}$ which is obtained from ω satisfies relation (5.1), and thus it gives a connection on E.

5.2.2. *General case.* Let P be a principal G-bundle with local trivializations $\{U_\alpha\}$ and transition functions $\{g_{\alpha\beta}\}$, and suppose $E = P \times_\rho \mathbb{R}^n$ is an associated vector bundle given by representation $\rho\colon G \to \mathrm{GL}(n,\mathbb{R})$. For connection $\omega \in \mathcal{A}(P)$, by Proposition 3.6.3 one has a set of $\mathfrak{g}$-valued 1-forms $\{A_\alpha\}$ with

$$A_\alpha = \mathrm{Ad}(g_{\alpha\beta}^{-1})A_\beta + g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta}$$

Let $\rho_*\colon \mathfrak{g} \to \mathfrak{gl}(n,\mathbb{R})$ be the differential of ρ, and note that the following diagram commutes

$$\begin{tikzcd}
G \arrow[r, "\mathrm{Ad}"] \arrow[d, "\rho"] & \mathrm{Aut}(\mathfrak{g}) \arrow[d, "\rho_*"] \\
\mathrm{GL}(n,\mathbb{R}) \arrow[r, "\mathrm{Ad}"] & \mathfrak{gl}(n,\mathbb{R})
\end{tikzcd}$$

Then $\{\rho_*(A_\alpha)\}$ is a set of $\mathfrak{gl}(n,\mathbb{R})$-valued 1-forms satisfying

$$\begin{aligned}
\rho_*(A_\alpha) &= \rho_*(\mathrm{Ad}(g_{\alpha\beta}^{-1})A_\beta) + \rho_*(g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta}) \\
&= \rho(g_{\alpha\beta})^{-1}\rho_*(A_\beta)\rho(g_{\alpha\beta}) + \rho_*(g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta}) \\
&= \rho(g_{\alpha\beta})^{-1}\rho_*(A_\beta)\rho(g_{\alpha\beta}) + \rho(g_{\alpha\beta})^{-1}\mathrm{d}\rho(g_{\alpha\beta})
\end{aligned}$$

This shows $\{\rho_*(A_\alpha)\}$ gives a connection on E, since the transition function[4] of E is $\{\rho(g_{\alpha\beta})\}$.

[4]See Remark 2.3.1.

6. Flat connection and holonomy

6.1. **Lifting of curves.** Let $\pi\colon P\to M$ be a principal G-bundle equipped with connection ω, consider smooth curve $\gamma\colon[0,1]\to M$ and a point $p\in\pi^{-1}(\gamma(0))$, we claim there exists a unique smooth map $\widetilde{\gamma}:[0,1]\to P$ such that

(1) The following diagram commutes:

$$\begin{tikzcd} & P \arrow[d, "\pi"] \\ {[0,1]} \arrow[ur, dashed, "\widetilde{\gamma}"] \arrow[r, "\gamma"'] & M \end{tikzcd}$$

(2) $\widetilde{\gamma}'(t)$ is horizontal.
(3) $\widetilde{\gamma}(0)=p$.

Proof. For convenience we assume G is a matrix group, and without lose of generality, we may assume P is trivial principal G-bundle $M\times G$, since it's a local problem. In this case we write $\widetilde{\gamma}=(\gamma(t),g(t))$, it's clear $\pi\circ\widetilde{\gamma}=\gamma$. For conditions (2) and (3), it's an ODE with initial value in fact: Note that we can write connection $\omega=\omega_{mc}+\widetilde{A}$, so $\widetilde{\gamma}'(t)$ is horizontal if and only if

$$\begin{aligned}(\omega_{mc}+\widetilde{A})(\widetilde{\gamma}'(t))&=(\omega_{mc}+\widetilde{A})((\gamma'(t),g'(t))\\&=g^{-1}(t)g'(t)+\widetilde{A}((\gamma'(t),g'(t)))\\&=g^{-1}(t)g'(t)+\mathrm{Ad}(g^{-1}(t))A_{\gamma(t)}(\gamma'(t))\\&=g^{-1}(t)g'(t)+g^{-1}(t)A_{\gamma(t)}(\gamma'(t))g(t)\\&=0\end{aligned}$$

This completes the proof. □

6.2. **Flat connection.**

Definition 6.2.1 (flat connection)**.** Let P be a principal G-bundle, a connection $\omega\in\mathcal{A}(P)$ is called flat, if its curvature form $\Omega=0$.

Theorem 6.2.1. The following statements are equivalent:

(1) ω is flat.
(2) There exists a local trivialization $\{U_\alpha,\varphi_\alpha\}$ such that $\omega|_{\pi^{-1}(U_\alpha)}=(\varphi_\alpha)^*\omega_{mc}$.

Hint. The curvature vanishes if and only if horizontal distribution is integrable. □

Corollary 6.2.1. The following statements are equivalent:

(1) There is a flat connection on P.
(2) There is a local trivializations $\{U_\alpha,\varphi_\alpha\}$ such that transition functions $\{g_{\alpha\beta}\colon U_\alpha\cap U_\beta\to G\}$ are locally constant functions.

Proof. From (2) to (1). By Proposition 3.6.3 a connection $\omega\in\mathcal{A}(P)$ is given by $\{A_\alpha\}$ such that

$$A_\beta=\mathrm{Ad}(g_{\alpha\beta}^{-1})A_\alpha+g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta}$$

If $g_{\alpha\beta}$ are locally constant functions, then $\mathrm{d}g_{\alpha\beta}=0$, and thus $A_\alpha=0$ gives a flat connection.

From (1) to (2). If ω is a flat connection, by Theorem 6.2.1 there exists a local trivialization $\{U_\alpha,\varphi_\alpha\}$ of P such that $\omega|_{\pi^{-1}(U_\alpha)}$ are $(\varphi_\alpha)^*\omega_{mc}$. Then with respect to this local trivialization, one has

$$A_\alpha=(\sigma_\alpha)^*(\varphi_\alpha)^*\omega_{mc}=0$$

for all α. This shows $g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta}=0$ for all α,β, that is $g_{\alpha\beta}$ are locally constant functions. □

Corollary 6.2.2. The flat connection is equivalent to $\mathbb{R}$-valued local systems.

6.3. **Holonomy and Riemann-Hilbert correspondence.** Let $\gamma\colon[0,1]\to M$ be a smooth closed curve with lifting $\widetilde{\gamma}\colon[0,1]\to P$ starting at $\widetilde{\gamma}(0)\in\pi^{-1}(\gamma(0))$. Note that

$$\widetilde{\gamma}(1)\in\pi^{-1}(\gamma(1))=\pi^{-1}(\gamma(0))$$

So there exists $g\in G$ such that $\widetilde{\gamma}(1)=\widetilde{\gamma}(0)g$, since fiber is an orbit of G. The element g is called holonomy, which is denoted by $\mathrm{Hol}(\gamma,p)$, since it only depends on γ and p.

Proposition 6.3.1.

(1) For $p,pg\in P$, where $g\in G$, one has

$$\mathrm{Hol}(\gamma,pg)=g^{-1}\,\mathrm{Hol}(\gamma,p)g$$

(2) Let γ_1,γ_2 be two smooth closed curves, then

$$\mathrm{Hol}(\gamma_1\gamma_2,p)=\mathrm{Hol}(\gamma_1,p)\,\mathrm{Hol}(\gamma_2,p)$$

Proof. It's clear. □

From (2) of above proposition, Hol can be regarded as a group homomorphism to some extend, so if we want to give a homomorphism

$$\mathrm{Hol}\colon\pi_1(M)\to G$$

It suffices to check when $\mathrm{Hol}(\gamma,p)$ is independent of homotopy class. Consider the following homotopy

$$\gamma_s\colon(-\varepsilon,\varepsilon)\times[0,1]\to M$$

such that $\gamma_0=\gamma$. If we write its lifting on local trivialization as $\widetilde{\gamma}_s(t)=(\gamma_s(t),g_s(t))$, then the following equation holds

$$\frac{\partial g_s}{\partial t}(t)+A_{\gamma(t)}(\frac{\partial g_s}{\partial t}(t))g_s(t)=0$$

So if ω is a flat connection, then it reduces to for arbitrary $s\in(-\varepsilon,\varepsilon)$, one has $\frac{\partial g_s}{\partial t}(t)=0$. This shows it's independent of s.

Theorem 6.3.1 (Riemann-Hilbert correspondence)**.**

$$\{\text{flat connections on } P\}/\text{isomorphism}\xleftrightarrow{1-1}\mathrm{Hom}(\pi_1(M),G)/\text{conjugate}$$

Part 2. Chern-Weil theory

7. Chern-Weil homomorphism

7.1. Invariant polynomial.

7.1.1. *General theory.* Let V be a vector space over $\mathbb{R}$ and $\operatorname{Sym}^k V^*$ the space of symmetric k-linear mappings f from $V \times \cdots \times V$ to $\mathbb{R}$, and $\operatorname{Sym} V^* = \bigoplus_{k=0}^\infty \operatorname{Sym}^k V^*$ is a commutative algebra over $\mathbb{R}$, where the multiplication is given by

$$fg(x_1, \ldots, x_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} f(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) g(x_{\sigma(k+1)}, \ldots, x_{\sigma(k+l)})$$

where $f \in \operatorname{Sym}^k V^*, g \in \operatorname{Sym}^l V^*$ and $x_i \in V$. Let $P^k(V)$ denote the space of homogeneous polynomial functions of degree k on V. Then $P(V) = \bigoplus_{k=0}^\infty P^k(V)$ is the algebra of polynomial functions on V.

Proposition 7.1.1. The mapping $\varphi\colon \operatorname{Sym} V^* \to P(V)$ defined by

$$(\varphi f)(t) := f(t, \ldots, t)$$

for $f \in \operatorname{Sym}^k(V)$ and $t \in V$ is an isomorphism.

Proof. See Proposition 2.1 in [KN96]. $\square$

Proposition 7.1.2. Given a group of linear transformation of V, let $\operatorname{Sym}_G V^*$ and $P_G(V)$ be the subalgebra of $\operatorname{Sym} V^*$ and $P(V)$, respectively, consisting of G-invariant elements. Then isomorphism in Proposition 7.1.1 gives an isomorphism from $\operatorname{Sym}_G V^*$ to $P_G(V)$.

Proof. The proof is straightforward and is left to the reader. $\square$

7.1.2. *G-invariant polynomial.* Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\operatorname{Sym}^k \mathfrak{g}^*$ be the symmetric k-linear functionals, that is,

$$\operatorname{Sym}^k \mathfrak{g}^* = \{f\colon \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{k \text{ times}} \to \mathbb{R} \mid f \text{ is } k\text{-linear and symmetric}\}$$

Furthermore, G acts on $\operatorname{Sym}^k \mathfrak{g}^*$ as follows

$$gf(x_1, \ldots, x_k) := f(\operatorname{Ad}(g)x_1, \ldots, \operatorname{Ad}(g)x_k)$$

where $f \in \operatorname{Sym}^k g^*$ and $x_1, \ldots, x_k \in \mathfrak{g}$.

Definition 7.1.1 (G-invariant polynomial)**.** The set of G-invariant polynomials of degree k is

$$I^k(\mathfrak{g}) := \{f \in \operatorname{Sym}^k \mathfrak{g}^* \mid gf = f, \forall g \in G\}$$

and

$$I(\mathfrak{g}) := \bigoplus_{k \geq 0} I^k(\mathfrak{g})$$

which is a commutative algebra over $\mathbb{R}$.

Proposition 7.1.3. The algebra $I(\mathfrak{g})$ may be identified with the algebra of $\mathrm{Ad}(G)$-invariant polynomial functions on $\mathfrak{g}$.

7.2. **Chern-Weil homomorphism.** Let $\pi\colon P \to M$ be a principal G-bundle, and ω is a connection on P with curvature Ω.

Lemma 7.2.1. Let $\widetilde{\alpha}$ be a k-form on P such that
(1) $R_g^*(\widetilde{\alpha}) = \widetilde{\alpha}$.
(2) $\widetilde{\alpha}$ is horizontal.

Then there exists a unique k-form α on M such that $\widetilde{\alpha} = \pi^*\alpha$.

Proof. For any vector fields $X_1, \dots, X_k$ on M, there are R_g-invariant vector fields $\widetilde{X}_1, \dots, \widetilde{X}_k$ such that $\pi_*(\widetilde{X}_i = X_i$, where $i = 1, \dots, k$. Given $\widetilde{\alpha}$ as above, we define

$$\alpha(X_1, \dots, X_k) = \widetilde{\alpha}(\widetilde{X}_1, \dots, \widetilde{X}_k)$$

It suffices to check it's well-defined, that is it's independent of the choice of $\widetilde{X}_1, \dots, \widetilde{X}_k$. Indeed, suppose $\widetilde{X}'_1, \dots, \widetilde{X}'_k$ are another R_g-invariant vector fields with $\pi_*(\widetilde{X}'_i) = X_i$, where $i = 1, \dots, k$. Then

$$\begin{aligned}\widetilde{\alpha}(\widetilde{X}_1, \dots, \widetilde{X}_k) - \widetilde{\alpha}(\widetilde{X}'_1, \dots, \widetilde{X}'_k) &= \widetilde{\alpha}(\widetilde{X}_1 - \widetilde{X}'_1, \dots, \widetilde{X}_k - \widetilde{X}'_k) \\ &= 0\end{aligned}$$

since $\widetilde{X}_i - \widetilde{X}'_i$ is horizontal for $i = 1, \dots, k$. The uniqueness follows from π is a submersion. □

Proposition 7.2.1. For $f \in I^k(\mathfrak{g})$, one has

$$f(\Omega) := f(\underbrace{\Omega \wedge \cdots \wedge \Omega}_{k \text{ times}})$$

is a $2k$-form[5] on P, and
(1) $f(\Omega)$ is horizontal, G-invariant and closed.
(2) there exists a unique $2k$-form $f(\Theta)$ on M such that $\pi^*(f(\Theta)) = f(\Omega)$ and $\mathrm{d}f(\Theta) = 0$.
(3) $[f(\Theta)] \in H^{2k}(M, \mathbb{R})$ is independent of the choice of connection ω.

Proof. For (1). $f(\Omega)$ is horizontal since Ω is, and it's G-invariant since

$$\begin{aligned}(R_g)^*(f(\Omega)) &= f((R_g)^*\Omega) \\ &\overset{(a)}{=} f(\mathrm{Ad}(g^{-1})\Omega) \\ &\overset{(b)}{=} f(\Omega)\end{aligned}$$

where
(a) holds from Ω is G-equivariant.
(b) holds from f is G-invariant.

[5]Here we regard $\underbrace{\Omega \wedge \cdots \wedge \Omega}_{k \text{ times}}$ as a $\bigotimes_{i=1}^k \mathfrak{g}$ valued $2k$-form on P, so $f(\Omega)$ is well-defined.

To see it's closed, direct computation shows

$$\begin{aligned}\mathrm{d}f(\Omega) &= f(\mathrm{d}\Omega\wedge\cdots\wedge\Omega)+\cdots+f(\Omega\wedge\cdots\wedge\mathrm{d}\Omega)\\ &\overset{(c)}{=} f(-\omega\wedge\Omega\wedge\cdots\wedge\Omega)+\cdots+f(\Omega\wedge\cdots\wedge-\omega\wedge\mathrm{d}\Omega)\end{aligned}$$

where (c) holds from Bianchi identity. Since $\ker\omega$ is horizontal distribution, it suffices to show $\mathrm{d}f(\Omega)$ is horizontal to conclude $\mathrm{d}f(\Omega)=0$. Let X be a vertical vector field, by proof of Proposition 4.1.3 one has $\mathcal{L}_X f(\Omega)=0$ since $f(\Omega)$ is horizontal. Then by Cartan formula one has

$$\begin{aligned}0&=\mathcal{L}_X f(\Omega)\\ &=\mathrm{d}\circ\iota_X f(\Omega)+\iota_X\circ\mathrm{d}f(\Omega)\\ &=\iota_X\mathrm{d}f(\Omega)\end{aligned}$$

This completes the proof of (1).

For (2). The unique existence of $f(\Theta)$ follows from Lemma 7.2.1, and it's closed since

$$\pi^*(\mathrm{d}f(\Theta))=\mathrm{d}(\pi^*(f(\Theta)))=\mathrm{d}f(\Omega)=0$$

For (3). Suppose ω' is another connection on P. Let $P\times\mathbb{R}$ be a principal G-bundle over $M\times\mathbb{R}$, and $\widetilde{\omega}=(1-t)\omega+t\omega'$ is a connection on it with curvature $\widetilde{\Omega}$. Then $f(\widetilde{\Omega})$ gives a unique $2k$-form $\widetilde{\Theta}$ on $M\times\mathbb{R}$. If we use i_0,i_1 to denote maps from M to $M\times\{0\}$ and $M\times\{1\}$ respectively, then

$$\begin{aligned}f(\Theta)&=i_0^*f(\widetilde{\Theta})\\ f(\Theta')&=i_1^*f(\widetilde{\Theta})\end{aligned}$$

Since i_0 is homotopic to i_1, the homotopy invariance of de Rham cohomology impiles $i_0^*,i_1^*\colon H^{2k}(M\times\mathbb{R},\mathbb{R})\to H^{2k}(M,\mathbb{R})$ coincide, and thus $[f(\Theta)]=[f(\Theta')]$. □

Theorem 7.2.1 (Chern-Weil homomorphism)**.** There is a ring homomorphism

$$\begin{aligned}W(P,\text{-})\colon I(\mathfrak{g})&\to H^*(M,\mathbb{R})\\ f&\mapsto[f(\Theta)]\end{aligned}$$

Proof. For $f\in I^k(\mathfrak{g}),g\in I^l(\mathfrak{g})$, it suffices to show

$$fg(\Theta)=f(\Theta)\wedge g(\Theta)$$

Note that π^* is injective, so it suffices to check

$$fg(\Omega)=f(\Omega)\wedge g(\Omega)$$

which is clear. □

Remark 7.2.1.

7.3. **Transgression.** In this section we will show for a given principal G-bundle P and a connection ω on it with curvature Ω, $[f(\Omega)] = 0 \in H^{2k}(P, \mathbb{R})$, where $f \in I^k(\mathfrak{g}), k \geq 1$. To see this, let's introduce the funtorial Chern-Weil homomorphism. Given the following homomorphism between principal G-bundles

$$\begin{tikzcd} P' \arrow[r] \arrow[d, "\pi'"] & P \arrow[d, "\pi"] \\ M' \arrow[r, "\varphi"] & M \end{tikzcd}$$

where $P' = \varphi^* P$.

Proposition 7.3.1 (funtorial)**.** For all $f \in I(\mathfrak{g})$, we have

$$W(\varphi^* P, f) = \varphi^* W(P, f)$$

Proof. Given a connection $\omega \in \mathcal{A}(P)$ with curvature Ω, and use ω' to denote the pullback connection $\widetilde{\varphi}^* \omega \in \mathcal{A}(P')$ with curvature Ω'. For any $f \in I(\mathfrak{g})$, it's clear

$$f(\Omega') = \widetilde{\varphi}^* f(\Omega)$$

Then

$$(\pi')^*(f(\Theta')) = \widetilde{\varphi}^* \pi^* f(\Theta) = (\pi')^* \varphi^* f(\Theta')$$

which impiles $f(\Theta') = \varphi^* f(\Theta)$, since $(\pi')^*$ is injective. $\square$

Example 7.3.1. Let $P = M \times G$ be trivial principal G-bundle, consider

$$\begin{tikzcd} M \times G \arrow[r] \arrow[d, "\pi'"] & G \arrow[d, "\pi"] \\ M \arrow[r, "\varphi"] & \{\mathrm{pt}\} \end{tikzcd}$$

So for any $f \in I^k(\mathfrak{g}), k \geq 1$, we have

$$W(P, f) = \varphi^* W(G, f) = 0$$

since $W(G, f) \in H^{2k}(\{\mathrm{pt}\}) = 0$ if $k \geq 1$.

Remark 7.3.1. This example shows if P is a trivial principal G-bundle, then the Chern-Weil homomorphism $W(P, \text{-})$ is trivial.

Now let's consider the following case

$$\begin{tikzcd} f^* P \arrow[r] \arrow[d, "\pi'"] & P \arrow[d, "\pi"] \\ P \arrow[r, "\varphi"] & M \end{tikzcd}$$

where $\varphi = \pi$. In fact we can write $f^* P$ down as

$$\begin{aligned} \varphi^* P &= \{(x', x) \in P \times P \mid \varphi(x') = \pi(x)\} \\ &= \{(x', x) \in P \times P \mid \pi(x') = \pi(x)\} \end{aligned}$$

It's clear it has global section, given by

$$\begin{aligned} s\colon P &\to \varphi^* P \\ x &\mapsto (x, x) \end{aligned}$$

so $\varphi^* P$ is trivial principal bundle. Thus for any $f \in I^k(\mathfrak{g}), k \geq 1$, we have

$$W(\varphi^* P, f) = 0 \in H^{2k}(P)$$

However, funtorial impiles

$$\begin{aligned} W(\varphi^* P, f) &= \varphi^* W(P, f) \\ &= \varphi^* [f(\Theta)] \\ &= \pi^* [f(\Theta)] \\ &= [f(\Omega)] \end{aligned}$$

This shows $[f(\Omega)] = 0$ in $H^{2k}(P, \mathbb{R})$.

8. Characteristic class

8.1. Chern class.

Proposition 8.1.1. Let $G = \mathrm{U}(n)$ with Lie algebra $\mathfrak{g} = \mathfrak{u}(n)$. For any $X \in \mathfrak{g}$, consider

$$\det(I - \frac{t}{2\pi\sqrt{-1}}X) = \sum_{k=0}^{n} c_k(X)t^k$$

Then

(1) For each $1 \leq k \leq n$, $c_k \in I(\mathfrak{g})$.
(2) $I(\mathfrak{g})$ is generated by $c_1, \dots, c_n$.

Proof. For (1). For arbitrary $g \in G$, note that

$$\begin{aligned}\det(I - \frac{t}{2\pi\sqrt{-1}}\operatorname{Ad}(g)X) &= \det(I - \frac{t}{2\pi\sqrt{-1}}gXg^{-1}) \\ &= \det(I - \frac{t}{2\pi\sqrt{-1}}X)\end{aligned}$$

which impiles $c_k \in I(\mathfrak{g})$.

For (2). Note that any $X \in \mathfrak{g}$ is diagonalizable, so without lose of generality we may assume $X = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$. Then $I(\mathfrak{g})$ consists of symmetric polynomial of $\lambda_1, \dots, \lambda_n$. Then the proof follows since any symmetric function can be expressed in terms of elementary symmetric functions and

$$\begin{aligned}c_1 &= -\frac{1}{2\pi}\lambda_1 + \dots + \lambda_n \\ &\vdots \\ c_n &= (\frac{1}{2\pi})^n \lambda_1 \dots \lambda_n\end{aligned}$$

□

Let E be a complex vector bundle of rank n over M equipped with a hermitian metric. If we consider its frame bundle we obtain a $\mathrm{U}(n)$-principal bundle $\pi\colon P \to M$, then arbitrary connection ω on P with curvature Ω gives a unique $2k$-form $c_k(\Theta)$ on M such that $\pi^*(c_k(\Theta)) = c_k(\Omega)$ by Chern-Weil theory.

Definition 8.1.1 (Chern class)**.** The k-th Chern class of E is defined as

$$c_k := [c_k(\Theta)] \in H^{2k}(M, \mathbb{C})$$

Definition 8.1.2 (Chern polynomial)**.** The Chern polynomial is defined as

$$c(t) = \det(I - \frac{t}{2\pi\sqrt{-1}}\Theta) = \sum_{k=0}^{n} c_k t^k$$

Proposition 8.1.2.

$$c_k \in H^{2k}(M, \mathbb{R})$$

Proof. Note that $\mathfrak{u}(n)$ consists of skew-symmetric matrices, so for arbitrary $X \in \mathfrak{u}(n)$, one has

$$\begin{aligned}\det(I - \frac{t}{2\pi\sqrt{-1}}X) &= \det(I + \frac{t}{2\pi\sqrt{-1}}\overline{X}^t) \\ &= \overline{\det(I - \frac{t}{2\pi\sqrt{-1}}X)} \\ &= \sum_{k=0}^{n} \overline{c}_k t^k\end{aligned}$$

which impiles $c_k = \overline{c}_k$. □

Proposition 8.1.3 (Whitney sum formula)**.** Let E, F are two complex vector bundles. Then

$$c(E \oplus F) = c(E)c(F)$$

Proof. If ∇^E, ∇^F are connections on E, F respectively with curvature Θ_E and Θ_F, then $\nabla^E \oplus \nabla^F$ gives a connection on $E \oplus F$ with curvature $\left(\begin{smallmatrix}\Theta_E & 0 \\ 0 & \Theta_F\end{smallmatrix}\right)$. This shows

$$\begin{aligned}c(E \oplus F) &= \det\begin{pmatrix} I - \frac{1}{2\pi\sqrt{-1}}\Theta_E & 0 \\ 0 & I - \frac{1}{2\pi\sqrt{-1}}\Theta_F\end{pmatrix} \\ &= \det(I - \frac{1}{2\pi\sqrt{-1}}\Theta_E)\det(I - \frac{1}{2\pi\sqrt{-1}}\Theta_F) \\ &= c(E)c(F)\end{aligned}$$

□

8.2. **Pontrjagin class.** Now let E be a (real) vector bundle of rank n over M equipped with a Riemannian metric, then its frame bundle is a $\mathrm{O}(n)$-principal bundle P. For any $X \in \mathfrak{o}(n)$, consider

$$\det(I - \frac{t}{2\pi}X) = \sum_{k=0}^{n} q_k(X)t^k$$

By the same argument as above one can show $q_k \in I(\mathfrak{g})$, thus we pick arbitrary connection ω of P with curvature Ω, then it gives rise to a closed $2k$-form $q_k(\Theta)$ on M for each k. Note that $X + X^t = 0$, then

$$\det(I + \frac{t}{2\pi}X) = \det(I - \frac{-t}{2\pi}X)$$

which impiles

$$q_k(X) = q_k(-X) = (-1)^k q_k(X)$$

Thus we can conclude $q_k = 0$ for odd k.

Definition 8.2.1 (Pontrjagin class)**.** $[p_k(\Theta)] := [q_{2k}(\Theta)] \in H^{4k}(M, \mathbb{R})$ is called k-th Pontrjagin class of E.

Proposition 8.2.1. Let E be a vector bundle with its complexification $E^c = E \otimes \mathbb{C}$, which is a complex vector bundle, then

$$p_k(E) = (-1)^k c_{2k}(E^c)$$

Proof. □

If we consider oriented vector bundle E, then its frame bundle is a $\mathrm{SO}(n)$-principal bundle. Then

Lemma 8.2.1. Let E be a oriented vector bundle of rank n, then
(1) If $n = 2m+1$, then $I(\mathfrak{so}(n))$ is generated by $q_2, \dots, q_{2m}$.
(2) If $n = 2m$, then $I(\mathfrak{so}(n))$ is generated by $q_2, \dots, q_{2m}, e$, where

$$e(\mathrm{diag}\{\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \lambda_m \\ -\lambda_m & 0 \end{pmatrix}\}) = \lambda_1 \dots \lambda_m$$

Definition 8.2.2 (Euler class)**.** Let E be an oriented vector bundle of rank $2m$, then $[\frac{1}{(2\pi)^m} e(\Theta)] \in H^{2m}(M, \mathbb{R})$ is called the Euler class of E, denoted by $e(E)$.

Remark 8.2.1. For an oriented $2m$-dimensional manifold M, $e(TM)$ is the Euler number of M. See [JM74].

9. The classifying space

In last section, we have defined characteristic classes in a geometrical viewpoint. However, they're topological invariants. In this section, we work on category of topological spaces (In particular, CW-complexes) instead of smooth manifolds, and give another explaination about characteristic class.

9.1. **The universal G-bundle.**

Definition 9.1.1 (contractible)**.** A topological space is called contractible, if it's homotopic to a point.

Definition 9.1.2 (weakly homotopy)**.** Let X, Y be topological spaces. X is called weakly homotopy to Y, if there exists a continuous map $f\colon X \to Y$ such that f induces isomorphisms between homotopy groups of X and Y.

Definition 9.1.3 (weakly contractible)**.** A topological space X is called weakly contractible, if it's weakly homotopy to a point.

Example 9.1.1. A contractible space is weakly contractible.

Proposition 9.1.1. For any topological space, there exists a CW-complex which is weakly homotopic to it.

Proposition 9.1.2. A CW-complex is weakly contractible if and only if it's contractible.

Proof. It follows from Whitehead's theorem. □

Definition 9.1.4 (classifying space)**.** Let $EG \to BG$ be a principal G-bundle, where EG, BG are topological spaces. If EG is weakly contractible, then

(1) BG is called a classifying space for G.
(2) EG is called a universal G-bundle.

Proposition 9.1.3. If the classifying space for G exists, then there exists a classifying space $EG \to BG$ for G such that BG is CW-complex.

Proof. Suppose $P \to B$ is a classifying space for G, where P, B are topological spaces. By Proposition 9.1.1, there exists a CW-complex BG and a weakly homotopy $f\colon BG \to B$. By exact homotopy sequence of fiberation, one has

$$\begin{tikzcd}
\dots \to \pi_{n+1}(B) \arrow[r] \arrow[d, "\cong"] & \pi_n(G) \arrow[r] \arrow[d, "\cong"] & \pi_n(P) \arrow[r] \arrow[d] & \pi_n(B) \arrow[r] \arrow[d, "\cong"] & \pi_{n-1}(G) \to \dots \arrow[d, "\cong"] \\
\dots \to \pi_{n+1}(EG) \arrow[r] & \pi_n(G) \arrow[r] & \pi_n(f^*P) \arrow[r] & \pi_n(EB) \arrow[r] & \pi_{n-1}(G) \to \dots
\end{tikzcd}$$

Then $f^*P \to BG$ is a classifying space for G by five lemma. □

Theorem 9.1.1. Let $EG \to BG$ be a universal G-bundle. For all CW-complexes X, the following map is bijective:

$$\begin{aligned} \Phi\colon [X, BG] &\to \mathcal{P}_G X \\ f &\mapsto f^* P \end{aligned}$$

where $[X, BG]$ denotes the set of all continuous maps up to homotopy.

Proof. See [Mit01]. □

Remark 9.1.1. This theorem impiles why BG is called classifying space, since it can be used to classify principal G-bundles over a given CW-complex.

Theorem 9.1.2. For any topological group G, classifying space for G exists, and it's unique up to G-homotopy equivalence.

Proof. See [Mil56]. □

Proposition 9.1.4. For a discrete group G, $PK(G,1) \to K(G,1)$ is the universal G-bundle, and hence $K(G,1)$ is a classifying space for G.

Proof. It's clear path space $PK(G,1)$ is contractible. □

Remark 9.1.2. In [Liu22] we have already computed $K(G,1)$ for groups, for example, $K(\mathbb{Z},1) = \mathbb{S}^1$, $K(\mathbb{Z}_2,1) = \mathbb{RP}^\infty$ and so on.

Proposition 9.1.5. $V_n(\mathbb{R}^\infty) \to Gr_n(\mathbb{R}^\infty)$ is a universal $\mathrm{GL}(n,\mathbb{R})$-bundle, and hence $Gr_n(\mathbb{R}^\infty)$ is a classifying space for $\mathrm{GL}(n,\mathbb{R})$.

Proof. It suffices to show $V_n(\mathbb{R}^\infty)$ is contractible. Since we have already computed low dimensional homotopy groups of $V_n(\mathbb{R}^N)$ in [Liu22], and then telescope construction completes the proof. □

Corollary 9.1.1. For all CW-complexes X, $[X, Gr_n(\mathbb{R}^\infty)] \to \mathrm{Vect}_n^{\mathbb{R}} X$.

Proof. See Remark 2.3.2. □

Remark 9.1.3. The analogous result also holds with $\mathbb{R}$ replaced by $\mathbb{C}$.

9.2. **Homotopical properties of classifying spaces.** In this section we collect some Homotopical properties of classifying spaces.

Theorem 9.2.1. For any topological group G, G is weakly equivalent to the loop space ΩBG.

Corollary 9.2.1. For $n \geq 1$, $\pi_n(BG) = \pi_{n-1}(G)$.

Theorem 9.2.2. Let G be a topological space and H a subgroup. Then the homotopy fiber of $BH \to BG$ is G/H, up to weakly equivalent.

Theorem 9.2.3. Let G be a topological space and H a subgroup. Then there is a fiberation $BH \to BG \to B(G/H)$.

Example 9.2.1. The exact sequences $1 \to \mathrm{SO}(n) \to \mathrm{O}(n) \to \mathbb{Z}_2 \to 1$ and $1 \to \mathrm{SU}(n) \to \mathrm{U}(n) \to S^1 \to 1$ give rise to fiberation

$$B\,\mathrm{SO}(n) \to B\mathrm{O}(n) \to \mathbb{RP}^\infty$$

and

$$B\,\mathrm{SU}(n) \to B\,\mathrm{U}(n) \to \mathbb{CP}^\infty$$

9.3. **Another viewpoint to characteristice class.**

Proposition 9.1. The cohomology ring of $B\,\mathrm{U}(n)$ with integer coefficients is $\mathbb{Z}[c_1, \dots, c_n]$.

Proof. If we consider $\mathrm{U}(n-1)$ as a subgroup of $\mathrm{U}(n-1)$, then we have the following filteration

$$\begin{tikzcd} S^{2n-1} \cong \mathrm{U}(n)/\,\mathrm{U}(n-1) \arrow[r] & B\,\mathrm{U}(n) \arrow[d] \\ & B\,\mathrm{U}(n-1) \end{tikzcd}$$

Apply Leray spectral sequence this this fiberation and use the fact that the cohomology ring of $\mathbb{CP}^\infty$ is $\mathbb{Z}[c_1]$ to conclude. $\square$

Definition 9.1 (universal Chern class)**.** The generators $c_1, \cdots, c_n$ of $H^*(B\,\mathrm{U}(n), \mathbb{Z})$ are called the universal Chern classes of $\mathrm{U}(n)$-bundles.

Definition 9.2 (Chern class)**.** The k-th Chern class of the $\mathrm{U}(n)$-bundle $\pi\colon E \to M$ with classifying map $f_\pi\colon M \to B\,\mathrm{U}(n)$ is defined as

$$c_k(E) := f_\pi^*(c_k) \in H^{2k}(M, \mathbb{Z})$$

Proposition 9.2. The cohomology ring of $B\mathrm{O}(n)$ with $\mathbb{Z}_2$ coefficients is $\mathbb{Z}_2[w_1, \dots, w_n]$.

Proof. The same as above, just note that cohomology ring of $\mathbb{RP}^\infty$ with $\mathbb{Z}_2$ coefficient is $\mathbb{Z}_2[w_1]$. $\square$

Definition 9.3 (universal Steifel-Whitney class)**.** The generators $w_1, \cdots, w_n$ of $H^*(B\mathrm{O}(n), \mathbb{Z}_2)$ are called the universal Steifel-Whitney classes of $\mathrm{O}(n)$-bundles.

Definition 9.4 (Steifel-Whitney class)**.** The k-th Steifel-Whitney class of the $\mathrm{O}(n)$-bundle $\pi\colon E \to M$ with classifying map $f_\pi\colon M \to B\mathrm{O}(n)$ is defined as

$$w_k(E) := f_\pi^*(w_k) \in H^{2k}(M, \mathbb{Z}_2)$$

Part 3. Spin geometry

10. Clifford algebra, Spin group and its representations

10.1. Clifford algebra.

10.1.1. *First properties.*

Definition 10.1.1 (quadratic space)**.** Let V be a k-vector space[6] and g is a symmetric bilinear form on V, the pair (V, g) is called a quadratic space.

Definition 10.1.2 (Clifford algebra)**.** Let (V, g) be a quadratic space, then Clifford algebra $\mathrm{Cl}(V, g)$ is the quotient

$$\mathrm{Cl}(V, g) := T(V)/I_g$$

where $T(V)$ is tensor algebra of V and I_g is the ideal in $T(V)$ generated by $\{v \otimes v + g(v, v)1 \mid v \in V\}$.

Notation 10.1.1. For convenience, we sometimes omit symbol $\otimes$, that is, simply use v^2 to denote $v \otimes v$.

Notation 10.1.2. For quadratic space (V, g), we always use π to denote natural projection $\pi\colon T(V) \to \mathrm{Cl}(V, g)$.

Remark 10.1.1. There is an injection $\iota\colon V \to \mathrm{Cl}(V, g)$, and we always identify $V \cong \iota(V) \subset \mathrm{Cl}(V, g)$.

Example 10.1.1. Let (V, g) be a quadratic space with $g = 0$, then Clifford algebra satisfies $v^2 = 0$, that is $\mathrm{Cl}(V, 0) = \bigwedge V$.

Example 10.1.2. Let $\mathbb{R}^{p,q}$ denote quadratic space $(\mathbb{R}^n, g_{p,q})$ with $g_{p,q}$ is symmetric bilinear form with signature (p, q), where $p+q = n$. The Clifford algebra $\mathrm{Cl}(\mathbb{R}^{p,q})$ is denoted by $\mathrm{Cl}_{p,q}$. Furthermore, $\mathrm{Cl}_n := \mathrm{Cl}_{n,0}$.

Example 10.1.3. Let $(\mathbb{C}^n, g)$ be complex vector space with standard symmetric bilinear form, its Clifford algebra is denoted by Cl^c_n.

Example 10.1.4. By definition $\mathrm{Cl}_1 = T(\mathbb{R})/I_{g_1}$, by fixing an orthonomal basis, one has $\mathrm{Cl}_1 \cong \mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$.

Proposition 10.1.1 (universal property)**.** Given any k-algebra A and a linear map $f\colon V \to A$ such that $f(v)f(v) = -2g(v, v)$, there exists a unique algebra map $\widetilde{f}\colon \mathrm{Cl}(V, g) \to A$ such that the following diagram commutes

$$\begin{tikzcd} \mathrm{Cl}(V,g) \arrow[dr, dashed, "\widetilde{f}"] & \\ V \arrow[u, hook, "\iota"] \arrow[r, "f"] & A \end{tikzcd}$$

Corollary 10.1.1. For linear map $f\colon (V, g) \to (V, g')$ such that $f^*g' = g$, there is a unique map $\widetilde{f}\colon \mathrm{Cl}(V, g) \to \mathrm{Cl}(V', g')$.

Corollary 10.1.2. If $(V, g) \cong (V', g')$, then $\mathrm{Cl}(V, g) \cong \mathrm{Cl}(V', g')$.

---

[6] Unless otherwise specified, the base field of vector space is denoted by k.

10.1.2. *Grading of Clifford algebra.* Let (V, g) be a quadratic space, there is a natural $\mathbb{Z}$-grading on tensor algebra $T(V)$, and every $\mathbb{Z}$-grading algebra can be turned into a $\mathbb{Z}_2$-grading algebra by taking the direct sum of even and odd components. A crucial fact is that Clifford algebra does not inherit the $\mathbb{Z}$-grading, which can be seen by considering $v^{2m} \in T(V)_{2m}$, then

$$\pi(v^{2m}) = (-1)^m g(v,v)^m \in \pi(T(V)_0)$$

However, the Clifford algebra inherits the $\mathbb{Z}_2$-grading of the tensor algebra.

Proposition 10.1.2. Let (V, g) be a quadratic space, then

$$\mathrm{Cl}(V,g) = \mathrm{Cl}^0(V,g) \oplus \mathrm{Cl}^1(V,g) := \pi(T(V)_0) \oplus \pi(T(V)_1)$$

is a $\mathbb{Z}_2$-grading.

Definition 10.1.3. The map $V \to V$, given by $v \mapsto -v$, induces an involution $\alpha\colon \mathrm{Cl}(V,g) \to \mathrm{Cl}(V,g)$.

Remark 10.1.2. $\mathbb{Z}_2$-grading of $\mathrm{Cl}(V,g)$ can be also viewed as eigen-decomposition with respect to involution α, and that's why some authors call α grading operator.

10.1.3. *Transpose and norm.*

Definition 10.1.4 (transpose)**.** The map $(v_1 \dots v_m)^T\colon = v_m \dots v_1$ on $T(V)$ induces a map on $\mathrm{Cl}(V,q)$, called transpose.

Definition 10.1.5 (norm)**.** The norm is the map $N\colon \mathrm{Cl}(V,g) \to \mathrm{Cl}(V,g)$ defined by

$$N(\varphi) = \varphi\alpha(\varphi^T)$$

Remark 10.1.3. In particular, for $v \in V$, $N(v) = -v^2 = g(v,v)$. That's why it's called norm.

10.2. **Pin and Spin groups.**

10.2.1. *Twisted adjoint representation.* Let (V, g) be a quadratic space over field k and $\mathrm{Cl}(V,g)$ is its Clifford algebra. The $\mathrm{Cl}(V,g)^\times$ denotes the multiplicative group of invertible elements in $\mathrm{Cl}(V,g)$, that is

$$\mathrm{Cl}(V,g)^\times := \{\varphi \in \mathrm{Cl}(V,g) \mid \text{there exists } \varphi^{-1} \in \mathrm{Cl}(V,g) \text{ such that } \varphi^{-1}\varphi = \varphi\varphi^{-1} = 1\}$$

Note that $\mathrm{Cl}(V,g)^\times$ is an open submanifold of $\mathrm{Cl}(V,g)$, and therefore a Lie group with Lie algebra $\mathrm{Cl}(V,g)$.

Definition 10.2.1 (twisted adjoint representation)**.** The twisted adjoint representation is

$$\begin{aligned} \rho\colon \mathrm{Cl}(V,g)^\times &\to \mathrm{GL}(\mathrm{Cl}(V,g)) \\ \varphi &\mapsto (\tau \mapsto \alpha(\varphi)\tau\varphi^{-1}) \end{aligned}$$

Proposition 10.2.1. Let $v \in V$ with $g(v,v) \neq 0$. Then

(1) $v \in \mathrm{Cl}(V,g)^\times$.

(2) For any $w \in V$, one has

$$\rho(v)w = w - 2\frac{g(v,w)}{g(v,v)}v$$

that is $\rho(v)$ acts as a reflection by the hyperplane $v^{\perp}$.

(3) $\rho(v)$ stabilizes V.

Proof. For (1). It's clear, since $v^2 + g(v,v) = 0$ impiles $v^{-1} = -v/g(v,v)$ if $g(v,v) \neq 0$.

For (2). Direct computation shows

$$\begin{aligned}\rho(v)w &= \alpha(v)wv^{-1} \\ &= \frac{vwv}{g(v,v)} \\ &\overset{(a)}{=} \frac{v(-vw - 2g(v,w))}{g(v,v)} \\ &\overset{(b)}{=} w - \frac{2g(v,w)}{g(v,v)}v\end{aligned}$$

where

(a) holds from the Clifford relation $vw + wv + g(v,w) = 0$.
(b) holds from the Clifford relation $-v^2 = g(v,v)$.

For (3). It follows from (2). □

Definition 10.2.2 (Clifford group)**.** The Clifford group $\Gamma(V,g)$ is the subgroup of $\mathrm{Cl}(V,g)^{\times}$ stabilizing V in the twisted adjoint representation.

Proposition 10.2.2. Suppose g is non-degenerate, then $\ker \rho|_{\Gamma(V,g)} = k^{\times}$.

Proof. Let $\{v_1, \ldots, v_n\}$ be a basis of V such that $g(v_i, v_i) \neq 0$ and $g(v_i, v_j) = 0$ for $i \neq j$. Let $0 \neq \varphi \in \ker \rho|_{\Gamma(V,g)}$, then $\alpha(\varphi)v = v\varphi$ for all $v \in V$. Decompose $\varphi = \varphi_0 + \varphi_1$ via $\mathbb{Z}_2$-grading, then

$$\begin{aligned}\varphi_0 v &= v\varphi_0 \\ \varphi_1 v &= -v\varphi_1\end{aligned}$$

holds for all $v \in V$. Suppose $\varphi_0 = a_0 + v_1 a_1$, where a_0, a_1 do not involve v_1, then

$$(a_0 + v_1 a_1)v_1 = v_1(a_0 + v_1 a_1)$$

Note that by Clifford relation a_0 commutes with v_1, while a_1 anti-commutes with v_1, that is

$$a_0 v_1 - v_1^2 = a_0 v_1 + v_1^2 a_1$$

Together with $v_1^2 = -g(v_1, v_1) \neq 0$, this shows $a_1 = 0$, that is φ_0 does not contain v_1. Proceeding with a_0 we can show in the same way that it does not contain v_2 and so on. This shows φ_0 does not contain any elements in V. The same argument shows φ_1 also does not contain any elements in V, that is $\varphi \in k^{\times}$. □

Proposition 10.2.3. Suppose g is non-degenerate, then $N\colon \Gamma(V,g)\to k^\times$ is a group homomorphism.

Proof. Firstly let's check for $\varphi\in\Gamma(V,g)$, one has $N(\varphi)\in k^\times$. Indeeed, by definition one has $\alpha(\varphi)v\varphi^{-1}\in V$ for all $v\in V$, and note that transpose acts trivially on V, thus

$$\alpha(\varphi)v\varphi^{-1}=(\alpha(\varphi)v\varphi^{-1})^T=(\varphi^{-1})^Tv(\alpha(v))^T=(\varphi^T)^{-1}v\alpha(\varphi^T)$$

where the last equality holds, since transpose commutes with taking inverse and ε, then

$$v=\varphi^T\alpha(\varphi)v\varphi^{-1}\alpha(\varphi^T)^{-1}=\rho(\alpha(\varphi^T)\varphi)v$$

It's clear both φ^T and $\alpha(\varphi)$ lie in $\Gamma(V,g)$, since $\varphi\in\Gamma(V,g)$. In particular, $\alpha(\varphi^T)\varphi\in\Gamma(V,g)$, since it's a group. According to Proposition 10.2.2, one has

$$\alpha(\varphi^T)\varphi\in\ker\rho|_{\Gamma(V,g)}=k^\times$$

Applying α you obtain $N(\varphi^T)=\varphi^T\alpha(\varphi)\in k^\times$, which completes the proof of first part. Now let's show N is a group homomorphism. Direct computation shows

$$N(\varphi\tau)=\varphi\tau\alpha(\tau^T)\alpha(\varphi^T)=\varphi N(\tau)\alpha(\varphi^T)=N(\varphi)N(\tau)$$

□

10.2.2. *Pin and Spin groups.*

Definition 10.2.3 (Pin group)**.** The Pin group $\mathrm{Pin}(V,g)$ is the subgroup of $\mathrm{Cl}(V,g)$ generated by elements $v\in V$ with $g(v,v)=1$.

Definition 10.2.4 (Spin group)**.** The Spin group $\mathrm{Spin}(V,g)$ is given by $\mathrm{Spin}(V,g):=\mathrm{Pin}(V,g)\cap\mathrm{Cl}^0(V,g)$.

Example 10.2.1. Again, standard pin and Spin groups are

$$\mathrm{Pin}(p,q):=\mathrm{Pin}(\mathbb{R}^{p,q})$$
$$\mathrm{Spin}(p,q):=\mathrm{Spin}(\mathbb{R}^{p,q})$$

and

$$\mathrm{Pin}(n):=\mathrm{Pin}(n,0)$$
$$\mathrm{Spin}(n):=\mathrm{Spin}(n,0)$$

Theorem 10.2.1. There are exact sequences

$$1\to\mathbb{Z}_2\to\mathrm{Pin}(p,q)\xrightarrow{\rho}\mathrm{O}(p,q)\to 1$$
$$1\to\mathbb{Z}_2\to\mathrm{Spin}(p,q)\xrightarrow{\rho}\mathrm{SO}(p,q)\to 1$$

Proof. Here we only prove the first exact sequence, the proof for the other one is the same as this one. Note that $\mathrm{Pin}(V,g)$ is a subgroup of Clifford group according to Proposition 10.2.1, and $\rho(\varphi)$ acts on $\mathbb{R}^{p,q}$ as reflections, where $\varphi\in\mathrm{Pin}(V,g)$. By the theorem of Cartan-Dieudommé, every element of $g\in\mathrm{O}(p,q)$ is a product of reflections, hence $\mathrm{Pin}(V,g)$ surjects on $\mathrm{O}(p,q)$.

Let $\varphi \in \ker\rho \cap \mathrm{Pin}(V,g)$, then by Proposition 10.2.2 one has $\varphi \in k^\times$, thus $N(\varphi) = \varphi^2$. On the other hand, suppose $\varphi = v_1 \dots v_m$, then $N(\varphi) = N(v_1) \dots N(v_m) = 1$. This shows $\varphi \in \ker\rho \cap \mathrm{Pin}(V,g)$ if and only if $\varphi = \pm 1$. □

Remark 10.2.1. Furthermore, $\rho\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is its universal covering if $n \geq 3$. Indeed, by homotopy exact sequence one has $\pi_1(\mathrm{SO}(n)) = \pi_1(\mathrm{SO}(3))$ for all $n > 3$ and $\mathrm{SO}(3)$ is exactly $\mathbb{RP}^2$, that is $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ for all $n \geq 3$. Now it suffices to show $\pm 1 \in \mathrm{Spin}(n)$ are connected by a continuous path. Such a path is given by

$$\gamma(t) = (e_1 \cos\frac{t}{2} + e_2 \sin\frac{t}{2})(-e_1 \cos\frac{t}{2} + e_2 \sin\frac{t}{2}) \in \mathrm{Spin}(n)$$

where $0 \leq t \leq \frac{\pi}{2}$ and $\{e_1, e_2, \dots, e_n\}$ is a orthonomal basis of $\mathbb{R}^{n,0}$.

Example 10.2.2. Note that $\mathrm{SO}(2) \cong S^1$ and the double covering of S^1 is exactly the map $S^1 \to S^1$, defined by $z \mapsto z^2$. This shows $\mathrm{Spin}(2) \cong S^1$.

Example 10.2.3. Note that $\mathrm{SO}(3) \cong \mathbb{RP}^3$, and $S^3 \to \mathbb{RP}^3$ is a double covering, this shows $\mathrm{Spin}(3) \cong S^3$.

Example 10.2.4. $\mathrm{Spin}(4) \cong S^3 \times S^3$.

10.2.3. *Lie algebra of Spin group.*

Proposition 10.2.4. $\mathfrak{spin}(n) = \mathrm{span}\{e_i e_j \mid 1 \leq i < j \leq n\}$.

Proof. For $1 \leq i < j \leq n$, consider

$$\begin{aligned} \gamma(t) &= \cos t + e_i e_j \sin t \\ &= -(e_i \cos\frac{t}{2} + e_j \sin\frac{t}{2})(e_i \cos\frac{t}{2} - e_j \sin\frac{t}{2}) \in \mathrm{Spin}(n) \end{aligned}$$

and note that $\gamma'(0) = e_i e_j$, this shows

$$\mathrm{span}\{e_i e_j \mid 1 \leq i < j \leq n\} \subseteq \mathfrak{spin}(n)$$

Then counting dimension to conclude. □

Proposition 10.2.5. The isomorphism $\rho_*\colon \mathfrak{spin}(n) \to \mathfrak{so}(n)$ is given by

$$\rho_*(e_i e_j) = 2E_{ij}$$

where E_{ij} is matrix with -1 in (i,j)-entry and 1 in (j,i)-entry.

Proof. For $1 \leq i < j \leq n$, consider $\gamma(t)$ □

10.3. **Classification of real and complex Clifford algebras.**

10.3.1. *Classification of real Clifford algebras.*

Theorem 10.3.1. There are isomorphisms

$$\begin{aligned}\mathrm{Cl}_{n,0}\otimes\mathrm{Cl}_{0,2}&\cong\mathrm{Cl}_{0,n+2}\\ \mathrm{Cl}_{0,n}\otimes\mathrm{Cl}_{2,0}&\cong\mathrm{Cl}_{n+2,0}\\ \mathrm{Cl}_{p,q}\otimes\mathrm{Cl}_{1,1}&\cong\mathrm{Cl}_{p+1,q+1}\end{aligned}$$

Proposition 10.3.1.

$$\begin{aligned}\mathrm{Cl}_{1,0}&\cong\mathbb{C}\\ \mathrm{Cl}_{2,0}&\cong\mathbb{H}\\ \mathrm{Cl}_{0,1}&\cong\mathbb{R}\oplus\mathbb{R}\\ \mathrm{Cl}_{0,2}&\cong M_2(\mathbb{R})\\ \mathrm{Cl}_{1,1}&\cong M_2(\mathbb{R})\end{aligned}$$

Corollary 10.3.1. The following is table of Clifford algebras $\mathrm{Cl}_{n,0}$ and $\mathrm{Cl}_{0,n}$ for $n\leq 8$.

n	$\mathrm{Cl}_{n,0}$	$\mathrm{Cl}_{0,n}$
1	$\mathbb{C}$	$\mathbb{R}\oplus\mathbb{R}$
2	$\mathbb{H}$	$M_2(\mathbb{R})$
3	$\mathbb{H}\oplus\mathbb{H}$	$M_2(\mathbb{C})$
4	$M_2(\mathbb{H})$	$M_2(\mathbb{H})$
5	$M_4(\mathbb{C})$	$M_2(\mathbb{H})\oplus M_2(\mathbb{H})$
6	$M_8(\mathbb{R})$	$M_4(\mathbb{H})$
7	$M_8(\mathbb{R})\oplus M_8(\mathbb{R})$	$M_8(\mathbb{C})$
8	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{R})$

10.3.2. *Classification of complex Clifford algebras.*

Theorem 10.3.2. There is an isomorphism

$$\mathrm{Cl}^c_{n+2}\cong\mathrm{Cl}^c_n\otimes\mathrm{Cl}^c_2$$

Corollary 10.3.2. Let $n\in\mathbb{N}$, then

$$\mathrm{Cl}^c_n=\begin{cases}M_{2^k}(\mathbb{C})\oplus M_{2^k}(\mathbb{C}) & n=2k+1\\ M_{2^k}(\mathbb{C}) & n=2k\end{cases}$$

10.4. **spin representation.** In this section we study some representations of Clifford algebras and Spin groups, which will play an important role in associated vector bundles of principal $\mathrm{Spin}(n)$-bundle.

Definition 10.4.1 (complex spinors)**.** The vector space of complex n-spinors is defined to be

$$\Delta_n=\mathbb{C}^{2^{\lfloor n/2\rfloor}}$$

Elements of Δ_n are called complex spinors.

According to Corollary 10.3.2, one has

$$\mathrm{Cl}_n^c = \begin{cases} \mathrm{End}(\Delta_n) \oplus \mathrm{End}(\Delta_n) & n = 2k+1 \\ \mathrm{End}(\Delta_n) & n = 2k \end{cases}$$

So $\mathrm{Cl}_n^c \to \mathrm{End}(\Delta_n)$ is identity when n is even, and projection when n is odd. In this way any element Cl_n^c can act on complex spinors, this is called Clifford multiplication.

Definition 10.4.2 (Clifford multiplication)**.** The multiplication by $v \in \mathbb{R}^n$, denoted by is endomorphism $c(v) \in \mathrm{End}(\Delta_n)$ given by $\mathbb{R}^n \subset \mathrm{Cl}_n \subset \mathrm{Cl}_n^c \to \mathrm{End}(\Delta_n)$.

Definition 10.4.3 (spin representation)**.** The composition $\Delta_n \colon \mathrm{Spin}(n) \hookrightarrow \mathrm{Cl}_n \hookrightarrow \mathrm{Cl}_n^c \to \mathrm{End}(\Delta_n)$ is called the spin representation of $\mathrm{Spin}(n)$.

Definition 10.4.4 (complex volumn element)**.** The complex volumn element $\omega_{\mathbb{C}} \in \mathrm{Cl}_n^c$ is

$$\omega_{\mathbb{C}} = \sqrt{-1}^{\lfloor \frac{n+1}{2} \rfloor} e_1 \dots e_n$$

where $\{e_i\}$ is an orthonomal basis.

Remark 10.4.1*.* The volumn element is independent of the choice of orthonomal basis if we fix the orientation, and $\omega_{\mathbb{C}}^2 = 1$.

Lemma 10.4.1. If n is odd, $\omega_{\mathbb{C}}$ commutes with every element of the Clifford algebra. If n is even, $\omega_{\mathbb{C}}$ commutes with elements of Cl_n^0 and anti-commutes with Cl_n^1.

Proof. It suffices to look at the commutativity of $\omega_{\mathbb{C}}$ with a unit vector e. We extend e into a positively oriented orthonomal basis $e_1 = 1, e_2, \dots, e_n$ of Cl_n^c. In terms of this basis, $\omega_{\mathbb{C}}$ clearly commutes with e when n is odd and anti-commutes with e when n is even. □

Definition 10.4.5 (Weyl spinors)**.** Elements of $\Delta_n^{\pm}$ are called Weyl spinors of $\pm$ chirality.

Theorem 10.4.1. If n is odd, Δ_n is an irreducible representation of $\mathrm{Spin}(n)$. If n is even, Δ_n decomposes into $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$ two irreducible representations of $\mathrm{Spin}(n)$. Furthermore, the Clifford multiplication interchanges $\Delta_n^{\pm}$.

11. SPIN STRUCTURE

11.1. **The first Steifel-Whitney class and orientablity.** Let (M,g) be a Riemannian n-manifold, $\mathfrak{U}=\{U_\alpha\}$ a good cover of M, and transition functions of TM with respect to $\mathfrak{U}$ is denoted by $\{g_{\alpha\beta}\colon U_{\alpha\beta}\to \mathrm{O}(n)\}$. Consider

$$c_{\alpha\beta}:=\det g_{\alpha\beta}=\pm 1\in\mathbb{Z}_2$$

which is continuous, and since $U_{\alpha\beta}$ is contractible, then $c_{\alpha\beta}$ is constant, and hence gives rise to a Čech 1-cochain $c\in C^1(\mathfrak{U},\mathbb{Z}_2)$. Furthermore, it defines a 1-cocycle. Indeed, direct computation shows

$$\begin{aligned}(\mathrm{d}c)_{\alpha\beta\gamma}&=c_{\beta\gamma}c_{\alpha\gamma}^{-1}c_{\alpha\beta}\\&=\det g_{\beta\gamma}\det g_{\gamma\alpha}\det g_{\alpha\beta}\\&=1\end{aligned}$$

Definition 11.1.1 (Steifel-Whitney class)**.** The cohomology class $[c]:=w_1(M)\in\check{H}^1(M,\mathbb{Z}_2)$ defined above is called the first Steifel-Whitney class of M.

Theorem 11.1.1. The first Steifel-Whitney class vanishes if and only if M is orientable.

Proof. Suppose M is orientable, for each good cover, it admits a refinement such that transition functions $g_{\alpha\beta}\colon U_{\alpha\beta}\to\mathrm{SO}(n)$, this shows first Steifel-Whitney class with respect to this cover vanishes, that is $w_1(M)=0\in\check{H}^1(M,\mathbb{Z}_2)$, since good cover is cofinal. Conversely, if first Steifel-Whitney class vanishes, then for local coordinates $(U_\alpha,\varphi_\alpha)$, without lose of generality we may assume $c_{\alpha\beta}=(\mathrm{d}s)_{\alpha\beta}=s_\beta s_\alpha^{-1}$, otherwise we can consider its refinement. Then consider coordinates $(U_\alpha,\varphi'_\alpha)$ given by $\varphi'_\alpha=s_\alpha\circ\varphi_\alpha$. With respect to this coordinates the transition functions $g'_{\alpha\beta}$ satisfy

$$\det g'_{\alpha\beta}=\det s_\beta\det g_{\alpha\beta}\det s_\alpha^{-1}=\det g_{\alpha\beta}^2=1$$

This shows M is orientable. □

11.2. **The second Steifel-Whitney class and spin structure.** Recall Example 2.4.3, we only talk about spin structure on orientable Riemannian manifold (M,g). So from now on we assume (M,g) is an orientable Riemannian n-manifold, and $\{g_{\alpha\beta}\colon U_{\alpha\beta}\to\mathrm{SO}(n)\}$ is transition functions of TM with respect to good cover $\mathfrak{U}$. Choose a lift $\widetilde{g}_{\alpha\beta}\colon U_{\alpha\beta}\to\mathrm{Spin}(n)$ such that

$$\widetilde{g}_{\alpha\beta}=\widetilde{g}_{\beta\alpha}$$

and define

$$\varepsilon_{\alpha\beta\gamma}=\widetilde{g}_{\alpha\gamma}\widetilde{g}_{\beta\alpha}\widetilde{g}_{\gamma\beta}$$

Now we're going to show this assignment gives rise to a Čech 2-cocycle valued in $\mathbb{Z}_2$ which is independent of the lift, which is divided into following lemmas.

Lemma 11.2.1. For arbitrary α, β, γ, one has

$$\varepsilon_{\alpha\beta\gamma} \in \ker \pi \cong \mathbb{Z}_2$$

where $\pi\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is double covering.

Proof. Direct computation shows

$$\begin{aligned}\pi(\varepsilon_{\alpha\beta\gamma}) &= \rho(\widetilde{g}_{\alpha\gamma}\widetilde{g}_{\beta\alpha}\widetilde{g}_{\gamma\beta}) \\ &= g_{\alpha\gamma}g_{\beta\alpha}g_{\gamma\beta} \\ &= 1\end{aligned}$$

$\square$

Corollary 11.2.1. For arbitrary α, β, γ, one has

$$\varepsilon_{\alpha\beta\gamma} = \varepsilon_{\gamma\beta\alpha}$$

Proof. Direct computation shows

$$\varepsilon_{\alpha\beta\gamma} = \varepsilon^{-1}_{\alpha\beta\gamma} = \widetilde{g}_{\gamma\alpha}\widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma} = \varepsilon_{\gamma\beta\alpha}$$

$\square$

Lemma 11.2.2. $\varepsilon \in C^2(\mathfrak{U}, \mathbb{Z}_2)$ defines a Čech 2-cocycle.

Proof. Direct computation shows

$$\begin{aligned}(\mathrm{d}\varepsilon)_{\alpha\beta\gamma\delta} &= \varepsilon_{\beta\gamma\delta}\varepsilon^{-1}_{\alpha\gamma\delta}\varepsilon_{\alpha\beta\delta}\varepsilon^{-1}_{\alpha\beta\gamma} \\ &= \widetilde{g}_{\beta\delta}\widetilde{g}_{\gamma\beta}\widetilde{g}_{\delta\gamma}(\widetilde{g}_{\alpha\delta}\widetilde{g}_{\gamma\alpha}\widetilde{g}_{\delta\gamma})^{-1}\widetilde{g}_{\alpha\delta}\widetilde{g}_{\beta\alpha}\widetilde{g}_{\delta\beta}(\widetilde{g}_{\alpha\gamma}\widetilde{g}_{\beta\alpha}\widetilde{g}_{\gamma\beta})^{-1} \\ &= 1\end{aligned}$$

$\square$

Lemma 11.2.3. The Čech cohomology class of ε is independent of the lift $\widetilde{g}_{\alpha\beta}$.

Proof. Suppose $\widetilde{g}_{\alpha\beta}$ and $\widetilde{g}'_{\alpha\beta}$ are lifts of $g_{\alpha\beta}$, then $\kappa_{\alpha\beta} = \widetilde{g}_{\alpha\beta}\widetilde{g}'_{\beta\alpha}$ satisfies $\rho(\kappa_{\alpha\beta} = 1)$, hence κ is a Čech 1-cochain. Direct computation shows

$$\begin{aligned}\varepsilon_{\alpha\beta\gamma}(\mathrm{d}\kappa)_{\alpha\beta\gamma} &= \widetilde{g}_{\alpha\gamma}\widetilde{g}_{\beta\alpha}\widetilde{g}_{\gamma\beta}\kappa_{\beta\gamma}\kappa^{-1}_{\alpha\gamma}\kappa_{\alpha\beta} \\ &= \widetilde{g}_{\alpha\gamma}\widetilde{g}_{\gamma\alpha}\widetilde{g}'_{\gamma\alpha} \cdot \widetilde{g}_{\beta\alpha}\widetilde{g}_{\alpha\beta}\widetilde{g}'_{\alpha\beta} \cdot \widetilde{g}_{\gamma\beta}\widetilde{g}_{\beta\gamma}\widetilde{g}'_{\beta\gamma} \\ &= \widetilde{g}'_{\gamma\alpha}\widetilde{g}'_{\alpha\beta}\widetilde{g}'_{\beta\gamma} \\ &= \varepsilon'_{\gamma\beta\alpha} \\ &= \varepsilon'_{\alpha\beta\gamma}\end{aligned}$$

This shows $\varepsilon'\varepsilon^{-1} = \mathrm{d}\kappa$, which completes the proof. $\square$

Definition 11.2.1. The cohomology class $w_2(M) := [\varepsilon] \in \check{H}^2(M, \mathbb{Z}_2)$ is called the second Steifel-Whitney class of M.

Theorem 11.2.1. (M, g) admits a spin structure if and only if the second Steifel-Whitney class vanishes. Furthermore, if (M, g) admits spin structure, then there is an one to one correspondence

$$H^1(M, \mathbb{Z}_2) \longleftrightarrow \{\text{isomorphism classes of spin structures}\}$$

Proof. See [LM16]. $\square$

12. Spinor bundle, Spin connection and Dirac operator

12.1. **Spinor bundle.** Let (M, g) be a Riemannian n-manifold admitting a spin structure P.

Definition 12.1.1 (Spinor bundle)**.** The Spinor bundle S_n associated to P is the associated vector bundle given by spin representation, that is

$$S_n = P \times_{\Delta_n} \Delta_n$$

Remark 12.1.1. Recall if n is even, then $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$ splits as a direct sum of irreducible representations, this impiles a splitting of the Spinor bundle as $S_n = S_n^+ \oplus S_n^-$.

Definition 12.1.2 (Clifford bundle)**.** The Clifford bundle is the vector bundle over M with typical fiber the Clifford algebra $\mathrm{Cl}(M)_x := \mathrm{Cl}(T_x^*M, g_x)$.

Remark 12.1.2.

Proposition 12.1.1. The Clifford multiplication $\mathbb{R}^n \times \Delta_n \to \Delta_n$ extends to a map of sections

$$\begin{aligned} c\colon C^\infty(M, T^*M) \times S_n &\to S_n \\ (\theta, \psi) &\mapsto c(\theta)\psi \end{aligned}$$

Proof. Let $\{U_\alpha\}$ be a local trivialization for both T^*M and S_n. On U_α a section θ of T^*M is given by $\theta_\alpha\colon U_\alpha \to \mathbb{R}^n$ and a section ψ of S_n is given by $\psi_\alpha\colon U_\alpha \to \Delta_n$. Then $c(\theta)\psi$ on U_α is defined as

$$(c(\theta)\psi)_\alpha := c(\theta_\alpha)\psi_\alpha$$

Now it suffices to check it's well-defined, that is

$$(c(\theta)\psi)_\alpha = \Delta_n(\widetilde{g}_{\alpha\beta})(c(\theta)\psi)_\beta$$

since $\{\Delta_n(\widetilde{g}_{\alpha\beta})\}$ are transition functions of S_n. Direct computation shows

$$\begin{aligned} (c(\theta)\psi)_\alpha &= c(\theta_\alpha)\psi_\alpha \\ &= c(g_{\alpha\beta}\theta_\beta)\Delta_n(\widetilde{g}_{\alpha\beta})\psi_\beta \\ &= c(\rho(\widetilde{g}_{\alpha\beta})\theta_\beta)\Delta_n(\widetilde{g}_{\alpha\beta})\psi_\beta \\ &= \Delta_n(\widetilde{g}_{\alpha\beta})c(\theta_\beta)\Delta_n(\widetilde{g}_{\alpha\beta}^{-1})\Delta_n(\widetilde{g}_{\alpha\beta})\psi_\beta \\ &= \Delta_n(\widetilde{g}_{\alpha\beta})c(\theta_\beta)\psi_\beta \\ &= \Delta_n(\widetilde{g}_{\alpha\beta})(c(\theta)\psi)_\beta \end{aligned}$$

□

Remark 12.1.3. In the proof, the key point is $g_{\alpha\beta} = \rho(\widetilde{g}_{\alpha\beta})$, that is, without the spin structure, we can not define the Clifford multiplication.

12.2. **Spin connection.**

Proposition 12.2.1. Let (M, g) be a Riemannian manifold admitting a spin structure P, then any connection ∇ on principal $\mathrm{SO}(n)$-bundle TM naturally induces a connection on P, which in turns gives a connection on the Spinor bundle

$$\nabla^S\colon C^\infty(M, S) \to C^\infty(M, T^*M \otimes S)$$

Furthermore, it's compatible with Clifford multiplication, that is

$$\nabla^S_X(c(v)\psi) = c(\nabla_X v)\psi + c(v)\nabla^S_X\psi$$

Proof. □

Lemma 12.2.1.

$$\nabla^S(c(\omega_{\mathbb{C}})\psi) = c(\omega_{\mathbb{C}})\nabla^S\psi$$

Corollary 12.2.1. If n is even, ∇^S is compatible with the splitting $S = S^+ \oplus S^-$. In other words, ∇^S is diagonal in this decomposition.

12.3. **Dirac operators.**

Let (M, g) be a Riemannian manifold with spin structure, and S is a Spinor bundle over M.

Definition 12.3.1 (Dirac operator)**.** The Dirac operator D

$$D\colon C^\infty(M, S) \to C^\infty(M, S)$$

is the composition

$$C^\infty(M, S) \xrightarrow{\nabla^S} C^\infty(M, T^*M \otimes S) \xrightarrow{c} C^\infty(M, S)$$

Remark 12.3.1 (local form)*.* In local orthonomal basis $\{e_i\}$, one has

$$D = c(e_i)\nabla^S_{e_i}$$

where we identify e_i^* with e_i using Riemannian metric.

12.4. **Clifford module.**

Part 4. The Yang-Mills equations on Riemannian manifold

13. The Yang-Mills equations

In this section we assume G is a compact Lie group, and (M, g) is an oriented compact Riemannian manifold.

13.1. The Yang-Mills functional.

Let P be a principal G-bundle with local trivializations $\{U_\alpha\}$, and $\rho\colon G \to \mathrm{GL}(V)$ is a linear representation of G. Firstly, on each local trivialization U_α, sections of $\Omega^k_M(P\times_\rho V)$ can be viewed forms with valued in V. Since we have a Riemannian metric g on M, if we want to construct an inner product on $\Omega^k_M(P \times_\rho V)$, it suffices to construct a G-invariant inner product $\langle\text{-},\text{-}\rangle$ on V, that is for all $g \in G, v, w \in V$, one has

$$\langle \rho(g)w, \rho(g)w\rangle = \langle v, w\rangle$$

The case we're most interested in is $V = \mathfrak{g}$, since curvature of a connection is a section of $\Omega^2_M(\mathrm{Ad}\,\mathfrak{g})$. In order to construct an inner product on $\Omega^k_M(\mathrm{Ad}\,\mathfrak{g})$, we need an inner product on $\mathfrak{g}$ which is invariant under the adjoint action. Since G is compact, its Killing form is a non-degenerate inner product, that's what we're looking for! Thus we have a pointwise inner product on the bundle $\Omega^k_M(\mathrm{Ad}\,\mathfrak{g})$, and denote it by $\langle\text{-},\text{-}\rangle$, and define a global inner product on $\Omega^k_M(\mathrm{Ad}\,\mathfrak{g})$ as

$$(\alpha, \beta) := \int_M \langle \alpha, \beta\rangle \,\mathrm{vol}$$

where $\alpha, \beta \in C^\infty(M, \Omega^k_M(\mathrm{Ad}\,\mathfrak{g}))$.

Definition 13.1.1 (Hodge star operator)**.** There exists an operator

$$* : C^\infty(M, \Omega^k_M(\mathrm{Ad}\,\mathfrak{g})) \to C^\infty(M, \Omega^{n-k}_M(\mathrm{Ad}\,\mathfrak{g}))$$

For $\beta \in C^\infty(M, \Omega^k_M(\mathrm{Ad}\,\mathfrak{g}))$, $*\beta$ is given by

$$\alpha \wedge *\beta = \langle\alpha, \beta\rangle\,\mathrm{vol}, \quad \forall \alpha \in C^\infty(M, \Omega^k_M(\mathrm{Ad}\,\mathfrak{g}))$$

With these preliminary results established, we arrive at the Yang-Mills functional.

Definition 13.1.2 (Yang-Mills functional)**.** The Yang-Mills functional is the map $YM\colon \mathcal{A}(P) \to \mathbb{R}$ given by

$$YM(\omega) := \|F_\omega\|^2 = \int_M \langle F_\omega, F_\omega\rangle\,\mathrm{vol}$$

where F_ω is curvature of connection ω, which is a section of $\Omega^2_M(\mathrm{Ad}\,\mathfrak{g})$.

Remark 13.1.1. By using Hodge star operator, we may rewrite Yang-Mills functional as follows

$$YM(\omega) = \int_M F_\omega \wedge *F_\omega$$

The advantages of writing Yang-Mills functional in this way is that we can use some properties of Hodge operator to simplify our computations.

Proposition 13.1.1. Yang-Mills functional YM is gauge invariant, that is for any gauge transformation $\Phi \in \mathcal{G}(P)$, one has $YM(\Phi^*\omega) = YM(\omega)$ holds for connection ω.

Proof. On each local trivialization U_α, the curvature of $\Phi^*\omega$ is given by $\mathrm{Ad}(\phi^{-1}) \circ F_\alpha$, where ϕ is given by $\Phi|_{U_\alpha}(x, g) = (x, \phi(x)g)$, thus Yang-Mills functional is gauge invariant follows from inner product $\langle \text{-}, \text{-} \rangle$ is adjoint invariant. $\square$

Definition 13.1.3 (Yang-Mills connection)**.** A Yang-Mills connection is a connection $A \in \mathcal{A}(P)$ which is a local extremum of Yang-Mills functional.

Notation 13.1.1. $\mathcal{A}_{YM}(P)$, or briefly $\mathcal{A}_{YM}$ denotes the set of all Yang-Mills connections.

13.2. **The variational problem.** Let's see how to use a second-order partial differential equation to characterize Yang-Mills connection. Recall that $\mathcal{A}(P)$ is an affine space modelled on $\Omega^1_M(\mathrm{Ad}\,\mathfrak{g})$. This means the tangent space to $\mathcal{A}(P)$ at any point is isomorphic to $\Omega^1_M(\mathrm{Ad}\,\mathfrak{g})$.

Given $\omega \in \mathcal{A}(P)$ and $\tau \in C^\infty(M, \Omega^1_M(\mathrm{Ad}\,\mathfrak{g}))$. The directional derivative of Yang-Mills functional at ω in the direction τ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} YM(\omega + t\tau)$$

And Yang-Mills condition states that this vanishes for all τ. In order to see what this means, firstly we need the following lemma.

Lemma 13.2.1. Given $\omega \in \mathcal{A}(P)$ and $\tau \in C^\infty(M, \Omega^1_M(\mathrm{Ad}\,\mathfrak{g}))$, then

$$F_{\omega+\tau} = F_\omega + \mathrm{d}_\omega \tau + \frac{1}{2}\tau \wedge \tau$$

where d_ω is connection induced by ω on $\Omega^1_M(\mathrm{Ad}\,\mathfrak{g})$.

Proof. On local trivialization U_α one has

$$\begin{aligned}
(F_{\omega+\tau})_\alpha &= \mathrm{d}(A_\alpha + \tau_\alpha) + \frac{1}{2}(A_\alpha + \tau_\alpha) \wedge (A_\alpha + \tau_\alpha) \\
&= (F_\omega)_\alpha + \mathrm{d}\tau_\alpha + \frac{1}{2}(A_\alpha \wedge \tau_\alpha + \tau_\alpha \wedge A_\alpha) + \frac{1}{2}\tau_\alpha \wedge \tau_\alpha \\
&\overset{(1)}{=} (F_\omega)_\alpha + \mathrm{d}\tau_\alpha + A_\alpha \wedge \tau_\alpha + \frac{1}{2}\tau_\alpha \wedge \tau_\alpha \\
&\overset{(2)}{=} (F_\omega)_\alpha + \mathrm{d}_\omega\tau_\alpha + \frac{1}{2}\tau_\alpha \wedge \tau_\alpha
\end{aligned}$$

where

(1) holds from both A_α, τ_α are 1-form valued in $\mathfrak{g}$.
(2) holds from (**??**).

$\square$

Proposition 13.2.1 (first variation formula)**.** Let ω be a Yang-Mills connection, then we have

$$\mathrm{d}_\omega^* F_\omega = 0$$

Proof. Direct computation shows

$$\begin{aligned} YM(\omega + t\tau) &= \int_M \langle F_{\omega+t\tau}, F_{\omega+t\tau} \rangle \,\mathrm{vol} \\ &= \int_M \langle F_\omega + \frac{t^2}{2}(\tau \wedge \tau) + t\mathrm{d}_\omega \tau, F_\omega + \frac{t^2}{2}(\tau \wedge \tau) + t\mathrm{d}_\omega \tau \rangle \,\mathrm{vol} \end{aligned}$$

The coefficient of linear term is

$$\int_M \langle F_\omega, \mathrm{d}_\omega \tau \rangle + \langle \mathrm{d}_\omega \tau, F_\omega \rangle \,\mathrm{vol} = 2\int_M \langle \mathrm{d}_\omega \tau, F_\omega \rangle \,\mathrm{vol}$$

Let $\mathrm{d}_\omega^* = (-1)^{2n+1} * \mathrm{d}_\omega *$ denote the formal adjoint to d_ω. Then we have

$$\int_M \langle \mathrm{d}_\omega \tau, F_\omega \rangle \,\mathrm{vol} = \int_M \langle \tau, \mathrm{d}_\omega^* F_\omega \rangle \,\mathrm{vol}$$

this shows

$$\mathrm{d}_\omega^* F_\omega = 0$$

□

Definition 13.2.1 (Yang-Mills equations)**.** A connection $\omega \in \mathcal{A}(P)$ is called satisfying Yang-Mills equations, if

$$\begin{cases} \mathrm{d}_\omega F_\omega = 0 \\ \mathrm{d}_\omega^* F_\omega = 0 \end{cases}$$

Remark 13.2.1*.* The first equation is also called Bianchi identity.

Example 13.2.1. In the case that $G = U(1)$, we have that the curvature of a connection A can be identified as a section of Ω_M^2. Indeed, the curvature form takes value in the bundle $\mathrm{Ad}\,\mathfrak{g}$, but here $G = U(1)$ is abelian, thus the adjoint action on $\mathfrak{u}(1)$ is trivial, so

$$\mathrm{Ad}\,\mathfrak{g} = M \times \mathfrak{u}(1) = M \times \mathbb{R}$$

is trivial bundle. Furthermore, ω is a Yang-Mills connection if and only if F_ω is a harmonic 2-form, that is $\Delta F_\omega = 0$, where $\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$. Indeed, thanks to $U(1)$ is abelian again, d_ω can be reduced to d, since for arbitrary form β, we have $\omega \wedge \beta = 0$. This follows from in the definition of wedge product of forms valued in Lie algebra we used Lie bracket, and abelian Lie algebra has trivial Lie bracket. Note that F_ω is harmonic if and only if

$$\begin{cases} \mathrm{d}^* F_\omega = 0 \\ \mathrm{d} F_\omega = 0 \end{cases}$$

It's a standard result in differential geometry, which can be seen from

$$\begin{aligned}
0 &= \int_M \langle \Delta F_\omega, F_\omega \rangle \operatorname{vol} \\
&= \int_M \langle \mathrm{d}\mathrm{d}^* F_\omega, F_\omega \rangle + \langle \mathrm{d}^*\mathrm{d} F_\omega, F_\omega \rangle \operatorname{vol} \\
&= \int_M \|\mathrm{d}^* F_\omega\|^2 + \|\mathrm{d} F_\omega\|^2 \operatorname{vol}
\end{aligned}$$

Note that the Yang-Mills functional is gauge invariant, so if a connection ω solves the Yang-Mills equations, so does any gauge transformed $\Phi^*\omega$. In other words, the gauge group acts on $\mathcal{A}_{YM}$. The quotient $\mathcal{A}_{YM}/\mathcal{G}$ is the space of classical solutions. In general it is infinite dimensional, and the topology of this space may be quite bad. For example it may be neither Hausdorff or a smooth manifold. But adding some restrictions, we do have a good correspondence, and that's main theorem for next lecture.

References

[JM74] James D. Stasheff John Milnor. *Characteristic Classes*. Princeton University Press, 1974.

[KN96] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of Differential Geometry, Volume 2*, volume 2. University of Texas Press, 1996.

[Liu22] Bowen Liu. Spectral sequences and its application. `https://bowenl-math.github.io/notes/2022Fall/Spectral_sequence.pdf`, 2022.

[LM16] H Blaine Lawson and Marie-Louise Michelsohn. *Spin Geometry (PMS-38), Volume 38*, volume 20. Princeton university press, 2016.

[Mil56] John W. Milnor. Construction of universal bundles, ii. *Annals of Mathematics*, 63:272, 1956.

[Mit01] Stephen A. Mitchell. Notes on principal bundles and classifying spaces. 2001.

Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, P.R. China,

Email address: `liubw22@mails.tsinghua.edu.cn`