REPRESENTATION THEORY

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Abstract. In this course, we will cover the following aspects:
1. Representation of finite groups.
2. Symmetric functions.
3. Lie groups and Lie algebra.
4. Representations of complex semisimple Lie algebra.
5. Representations of compact Lie groups.

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1. Introduction and overview

Group theory is the study of symmetrics of a mathmatics object. This is the point of view of geometry: given a geometry object X, what is its group of symmetries?

But representation theory reverse this question, given a group G, what object X does it act on? Here we pay more attention on linear action, i.e., X is a vector space.

We can compare with manifolds, since every abstract manifold can be embedded into $\mathbb{R}^n$, every abstract group can be embedded into S_n, according to Cayley's theorem as follows

Theorem 1.1. *Any finite group of order n is isomorphic to a subgroup of the symmetric group S_n.*

In this course, we are interested in the following groups:

1. finite group, in particular symmetric group, Coxeters groups.
2. Lie groups over $\mathbb{R}$ and $\mathbb{C}$.

And representation theory is a very useful tool, one of the most important applications is the classification of finite simple groups, all kinds of finite simple groups are listed as follows

1. cyclic groups C_p for prime p
2. alternating groups $A_n, n \geq 5$
3. 16 simple groups of Lie type
4. 26 sporadic groups

Among those sporadic groups, the largest one is the monster M, with order $|M| \sim 8 \cdot 10^{53}$, but the number of irrducible representations is only 194. As we will see, all irreducible representations of one group will reflect all imformation about it, so it's possible for us to learn the properties of monster group, by using its irreducible representations.

It's also worth mentioning that there is a crazy conjecture about monster group, called Monstrous Monnlight conjecture, proven by Borcherds in 1992, and he got his Fields medal in 1998.

2. Basic Definitions and Irreduciblity

2.1. Basic Definitions.

Definition 2.1. *Let G be a finite group, V is a finite-dimensional vector space over k. A **representation** of G on V is a group homomorphism $\rho : G \to \mathrm{GL}(V)$.*

Notation 2.2. *We say V is a representation of G and often write gv instead of $\rho(g)v$, we also say that G acts on V.*

Remark 2.3. *We give following remarks:*

1. *ρ equips V with the G-module structure.*

2. *We will mostly work with $k = \mathbb{C}$, more generally, V can be finite-dimensional R-module for a communicative ring with 1.*
3. *Let $B = (e_1, \dots, e_n)$ be a basis of V, for $\varphi \in \operatorname{End}_k V$, write $\varphi e_i = \sum a_{ji} e_j$, and let $A = (a_{ij}) \in M_n(k)$. If ρ is a representation, the $\rho_B(g)$ is the matrix of $\rho(g)$ with respect to B. Then $g \to \rho_B(g)$ is a homomorphism from G to $\mathrm{GL}(n,k)$, called the matrix representation.*

Definition 2.4. *Let V, W be two representations of finite group G. A linear map $\varphi : V \to W$ is a* ***map of representation*** *of G if the following diagram commutes for all $g \in G$.*

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g"] & W \arrow[d, "g"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd}$$

Definition 2.5. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation. A* ***subrepresentation*** *of V is a vector subspace W of V, such that $\rho(g)W \subseteq W, \forall g \in G$. For a subrepresentation W, the map $\rho(g)(v + W) := \rho(g)v + W$ defines a representation of G on V/W, called the* ***quotient representation.***

Lemma 2.6. *For a map of representation $\varphi : V \to W$, the kernel of φ is a subrepresentation of V, image and cokernel of φ are subrepresentations of W.*

Proof. Trivial. □

By some standard linear algebra methods, we can construct new representations from old ones:

Lemma 2.7. *Let $\rho : G \to \mathrm{GL}(V), \sigma : G \to \mathrm{GL}(W)$ be a representation of G, then*

1. $\rho \oplus \sigma : G \to \mathrm{GL}(V \oplus W), g(v \oplus w) = gv \oplus gw$
2. $\rho \otimes \sigma : G \to \mathrm{GL}(V \otimes W), g(v \otimes w) = gv \otimes gw$
3. $\rho^{\otimes n} : G \to \mathrm{GL}(V^{\otimes n}), g(v^{\otimes n}) = (gv)^{\otimes n}$
4. $\wedge^n \rho : G \to \mathrm{GL}(\wedge V^n), g(v_1 \wedge \cdots \wedge v_n) = gv_1 \wedge \cdots \wedge gv_n$
5. $\operatorname{Sym}^n \rho : G \to \mathrm{GL}(\operatorname{Sym}^n V), g(v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n$
6. $\rho^\vee : G \to \mathrm{GL}(V^\vee), \rho^\vee(g) = (\rho(g)^t)^{-1}$
7. $\rho_{V,W} : G \to \operatorname{Hom}(V, W), (\rho(g)\varphi)(v) = \rho(g)\varphi(\rho(g^{-1}))$

are representations of G.

Proof. □

Lemma 2.8. *We have the following isomorphism*

$$\operatorname{Hom}_G(V, W) \cong \operatorname{Hom}(V, W)^G = G\text{-invariants of } \operatorname{Hom}(V, W)$$

Proof. □

Lemma 2.9. *The following are isomorphisms of representations U, V, W of G*

1. $\mathrm{Hom}(V, W) \cong V^\vee \otimes W$
2. $V \otimes (U \oplus W) \cong V \otimes U \oplus V \otimes W$
3. $\wedge^k(V \oplus W) \cong \bigoplus_{a+b=k} \wedge^a V \otimes \wedge^b W$
4. $\wedge^k(V^\vee) \cong (\wedge^k V)^\vee$
5. $\wedge^k(V^\vee) \cong \wedge^{n-k} V \otimes \det V^\vee$, *where* $n = \dim V, \det V = \wedge V^m$.

Proof. □

Definition 2.10. *Let G be a group and X be a set. A* ***group action*** *of G on X is a map $\sigma : G \to \mathrm{Aut}(X)$, such that*

1. $\sigma(g)x \in X, \forall x \in X$
2. $\sigma(gh)x = \sigma(g)\sigma(h)x, \forall x \in X$
3. $\sigma(e)x = x, \forall x \in X$

If we have such a group action, we can construct many useful representations

Example 2.11. *Let V be a finite-dimensional over $\mathbb{C}$ with basis X, and G acts on X via σ, we define $R_X : G \to \mathrm{GL}(V)$ as follows*

$$R_X(g)(\sum_{x \in X} a_x e_x) = \sum_{x \in X} a_x e_{\sigma(g)x}$$

Here R_X is called ***permutation representation****.*

And the following examples are based on above one.

Example 2.12. *Choose X to be G considered as a set, and G acts on G by left multiply, then $R = R_G$ is called* ***regular representation****, in this case V is denoted by $k[G]$, called group algebra.*

Example 2.13. *Let V be the group algebra of G, and consider the map $\rho : G \to \mathrm{GL}(V)$ defined as follows*

$$\rho(g)(\sum_{x \in X} a_x e_x) = \sum_{x \in X} \mathrm{sgn}(\sigma(g)) a_x e_{\sigma(g)x}$$

is called the ***alternating representation****.*

Example 2.14. *Let H be subgroup of G, and $X = \{g_1, \dots, g_n\}$ be a complete set of representatives of G/H, G acts on X by $g(g_i H) = g g_i H$. In this case, R_X is called the* ***coset representation*** *of G with respect to H.*

Now we consider some concrete examples which we will use later.

Example 2.15. *Consider $G = S_n$ and $X = \{1, 2, \dots, n\}$. Let $V = \mathbb{C}X$, and $W = \mathbb{C}(e_1 + \cdots + e_n) \subset V$. Consider the permutation representation R_X, then it's easy to see that $R_X|_W$ is trivial representation.*

Example 2.16. *Regular representation for $X = \{1, 2, 3\}$, we can write down explictly as follows*

$$R(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R((12)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R((13)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$R((23)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad R((132)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad R((123)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Example 2.17. *A 2-dimension representation of S_3 : the symmetry of triangle, denoted by V*

$$V(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V((12)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad V((13)) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$V((23)) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad V((132)) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad V((123)) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

2.2. Irreduciblity.

Definition 2.18. *A representation of V is called **irreducible** if there is no proper invariant subspace W of V; A representation of V is called **indecomposable** if it can not be written as a direct sum of two nonzero subrepresentation.*

In fact, when we consider complex representation, the irreduciblility and indecomposablity coincides, stated as follows

Theorem 2.19 (Maschke's theorem)**.** *Let V be a representation of a finite group of $\mathbb{C}$, $W \subseteq V$ is a subrepresentation, then there is a complementary invariant subrepresentation W' of G, such that $V = W \oplus W'$.*

Remark 2.20. *Maschke theorem still holds when* $\operatorname{char} k \nmid |G|$

Remark 2.21. *Any continous representation of a compact group has this property, but group $(\mathbb{R}, +)$ does not, consider $a \mapsto \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$ fixes the x-axis, but there is no complementary subspace.*

Lemma 2.22 (Shur lemma)**.** *Let V, W be irreducible representations of finite group G, and $\varphi \in \operatorname{Hom}_G(V, W)$, then*

1. *either φ is isomorphism, or $\varphi = 0$*
2. *If $V = W$, then $\varphi = \lambda I, \lambda \in \mathbb{C}$*

Proposition 2.23. *Let $\rho : G \to \mathrm{GL}(V)$ be representation of finite group, then there is a unique decomposition*

$$V = \bigoplus_{i=1}^{N} V_i^{a_i}$$

where V_i is distinct irrducible representations.

2.3. Representation of abelian groups and S_3.

Proposition 2.24. *Let G be a finite abelian group, then every irrducible representation of G is 1-dimensional.*

Remark 2.25. *Let $\rho : G \to \mathrm{GL}(V)$ be any representation, then map $\rho(g) : V \to V$ is in general not a map of representations, i.e., for $h \in G$,*

$$\rho(g)(hv) \neq h(\rho(g)v)$$

In fact, we can prove $\rho(g) \in \mathrm{End}_G V$ if and only if $g \in \mathrm{Z}(G)$.

Remark 2.26. *The converse statement also holds, see corollary*3.20.

Definition 2.27. *Let G be a finite group, then $G^\vee = \mathrm{Hom}_G(G, \mathbb{C}^*)$ is called the dual group.*

Corollary 2.28. *Let G be a finite abelian group, then $\mathrm{Irr}\, G \overset{1:1}{\Longleftrightarrow} G^\vee$*

Proof. By the remark 2.25, if G is abelian, then $G = \mathrm{Z}(G)$, then $\rho(g) \in \mathrm{End}_G V = \mathbb{C}^*, \forall g \in G$ and $V \in \mathrm{Irr}(G)$. □

For S_3, we have already seen the following representations:

1. trivial representation U, with dimension 1.
2. alternating representation U', with dimension 1.
3. the regular representation R, with dimension 3.
4. the symmetric of the triangle V, with dimension 2.

And we also note that R has a 1-dimensional subrepresentation $V' = \mathbb{C}(e_1 + e_2 + e_3)$, in fact, it's a trivial representation, hence it is isomorphic to U.

Consider the complementary subspace of V' in R, denoted by $V'' = \{(v_1, v_2, v_2) \in V \mid v_1 + v_2 + v_2 = 0\}$, we can choose a basis $(\omega, 1, \omega^2), (1, \omega, \omega^2)$, where $\omega^3 = 1$.

Now, let W be an arbitrary representation of S_3, consider $\mathbb{Z}/3\mathbb{Z} = \langle \sigma \rangle \subset S_3$, and decompose W into

$$W = \bigoplus_{i=1}^{3} V_i^{\oplus a_i}, \quad V_i = \mathbb{C}v_i, \sigma v_i = \omega^i v_i$$

Let $\tau \in S_3$ be a transposition, such that

$$S_3 = \langle \sigma, \tau \rangle / (\tau\sigma\tau = \sigma^2)$$

then

$$\sigma(\tau v_i) = \tau(\sigma^2 v_i) = \tau(\omega^{2i} v_i) = \omega^{2i} \tau v_i$$

3. Character theory

In this section, G denotes a finite group.

Definition 3.1. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation, $\chi_V : G \to \mathbb{C}, g \mapsto \chi_V(g) = \mathrm{tr}(\rho(g))$ is a character of ρ.*

Remark 3.2. *In fact, χ_V is a class function, i.e.,*

$$\chi_V \in \mathscr{C}_G = \{f : G \to \mathbb{C} \mid f|_K = \mathit{constant}, \forall K \in \mathrm{Conj}(G)\}$$

The dimension of $\mathscr{C}_G = |\mathrm{Conj}(G)|$, and we have the following isomorphism

$$\mathscr{C}_G \cong \mathrm{Z}(\mathbb{C}[G])$$

defined by

$$f \mapsto \sum_{g \in G} f(g)g$$

Proposition 3.3. *Let V, W be representations of G, then*

1. $\chi_{V \oplus W} = \chi_V + \chi_W$
2. $\chi_{V \otimes W} = \chi_V \chi_W$
3. $\chi_{V^\vee} = \overline{\chi_V}$
4. $\chi_{\mathrm{Sym}^2 V}(g) = \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2))$
5. $\chi_{\wedge^2 V}(g) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$

Proof. Note that $\{\lambda_i\lambda_j \mid i \leq j\}, \{\lambda_i\lambda_j \mid i < j\}$ are the eigenvalues of g on $\mathrm{Sym}^2 V, \wedge^2 V$ respectively, then

$$\sum_{i \leq j} \lambda_i\lambda_j = \frac{1}{2}(\sum_{i,j} \lambda_i\lambda_j + \sum_i \lambda_i^2)$$

$$\sum_{i < j} \lambda_i\lambda_j = \frac{1}{2}(\sum_{i,j} \lambda_i\lambda_j - \sum_i \lambda_i^2)$$

□

Theorem 3.4 (The fixed point formula)**.** *Let X be a finie set with an action by G. Let V be the permutation representation. Let $X^g = \{x \in X \mid gx = x\}, g \in G$. Then $\chi_V(g) = |X^g|$*

Proof. Since $\mathrm{Aut}(X) \cong S_{|X|}$, the matrix A representing $\rho(g)$ is a permutation matrix: if $ge_{x_i} = e_{x_j}$ for some $x_i, x_j \in X$, then

$$A_{ik} = \begin{cases} 1, & k = j \\ 0, & \text{otherwise} \end{cases}$$

Then, if $x_i \in X^g$, then $ge_{x_i} = e_{gx_i} = e_{x_i}$, that is $A_{ii} = 1$, so

$$\mathrm{tr}(\rho(g)) = \sum_{i: x_i \in X^g} A_{ii} = \sum_{i: x_i \in X^g} 1 = |X^g|$$

□

Definition 3.5. *The character table of G is a table with the conjugacy classes listed a cross, the irreducible representations listed on the left.*

Example 3.6. *Character table for S_3*

	1	(12)	(123)
trivial U	1	1	1
alternating U'	1	-1	1
standard V	2	0	-1
permutation P	3	1	0

Observe $W = U^{\oplus a} \oplus U'^{\oplus b} \oplus V^{\oplus c}$, then

$$\chi_W = a\chi_U + b\chi_{U'} + c\chi_V$$

Since $\chi_U, \chi_{U'}, \chi_V$ is independent, later we will see that W is determined by χ_W up to isomorphism.

We can use this fact to get some interesting results. For example, since we can decompose

$$\chi_{V\otimes V} = (4, 0, 1) = (2, 0, -1) + (1, 1, 1) + (1, -1, 1)$$

So we can decompose

$$V \otimes V = U \oplus U' \oplus V$$

Similarly, we can decompose any representation of S_3 in the above way, if we know what does its character look like.

Remark 3.7. *Note that different groups can have identical character tables, e.g., dihedral group*

$$D_{4n} = \langle a, b \mid a^2 = b^{2n} = (ab)^2 = e\rangle$$

and quaternianic group

$$Q_{4n} = \langle a, b \mid a^2 = b^{2n}, (ab)^2 = e\rangle$$

have the same character table.

Remark 3.8. *Nevertheless, characters can characterize the group G: order of G, order of all its normal subgroups, whether G is simple or not.*

Proposition 3.9. *Let V be a representation of G. The map $\varphi = \frac{1}{|G|}\sum_{g\in G} g \in \operatorname{End} V$ as a projection from V to $V^G = \{v \in V \mid gv = v, \forall g \in G\}$*

Proof. Let $w \in W$, $v = \varphi(w) = \frac{1}{|G|}\sum_{g\in G} gw$, then for any $h \in G$, we have

$$hv = \frac{1}{|G|}\sum_{g\in G} hgw = \frac{1}{|G|}\sum_{g\in G} gw = v$$

So $\operatorname{im}\varphi \subset V^G$.

Conversely, if $v \in V^G$, then $\varphi(v) = \frac{1}{|G|}\sum_{g\in G} gv = v$, this implies $V^G \subset \operatorname{im}\varphi$. Moreover, $\varphi \circ \varphi = \varphi$. □

Definition 3.10. *We let* $(\alpha, \beta) = \sum_{g \in G} \overline{\alpha(g)}\beta(g)$ *denote a Hermitian inner product on* $\mathscr{C}_G$.

Theorem 3.11 (First orthogonality relation)**.** *Let* $V, W \in \operatorname{Irr}(G)$*, then*

$$(\chi_V, \chi_W) = \begin{cases} 1, & V \cong W \\ 0, & \textit{otherwise} \end{cases}$$

Proof. If V, W are irreducible representations, then Shur's lemma implies

$$\dim \operatorname{Hom}(V, W)^G = \dim \operatorname{Hom}_G(V, W) = \begin{cases} 1, & V \cong W \\ 0, & \text{otherwise} \end{cases}$$

However, $\chi_{\operatorname{Hom}(V,W)} = \chi_{V^\vee \otimes W} = \chi_{V^\vee}\chi_W = \overline{\chi_V}\chi_W$.

Let $\varphi = \frac{1}{|G|}\sum_{g \in G} g \in \operatorname{End}(\operatorname{Hom}(V, W))$, then we have

$$\begin{aligned} \dim \operatorname{Hom}(V, W)^G &= \operatorname{tr}_{\operatorname{Hom}(V,W)^G} \varphi = \frac{1}{|G|}\sum_{g \in G} \operatorname{tr}_{\operatorname{Hom}(V,W)}(g) \\ &= \frac{1}{|G|}\sum_{g \in G} \chi_{\operatorname{Hom}(V,W)}(g) \\ &= \frac{1}{|G|}\sum_{g \in G} \overline{\chi_V(g)}\chi_W(g) \end{aligned}$$

□

Corollary 3.12. *Any representation of a finite group* G *is determined by its character up to isomorphism, i.e.,* $V \cong W \iff \chi_V(g) = \chi_W(g), \forall g \in G$.

Corollary 3.13. *If* $V = \bigoplus_i V_i^{\oplus a_i}$*,* V_i *are irreducible, distinct representings, then*

$$a_i = (\chi_{V_i}, \chi_V)$$

In particular, V *is irreducible if and only if* $(\chi_V, \chi_V) = 1$.

Corollary 3.14. *The multiplicity of any irreducible representation* V *of* G *in the decomposition of the regular representation* $R = \mathbb{C}[G]$ *is equal to its dimension. In particular,* $|\operatorname{Irr}(G)| < \infty$.

Proof. Recall that $(e_g)_{g \in G}$ is a basis for R, and $ge_h = e_{gh}, \forall g, h \in G$. For the fixed point formula

$$\chi_R(g) = \begin{cases} 0, & g \neq e \\ |G|, & g = e \end{cases}$$

Then R is not irreducible unless G is trivial. Write $R = \bigoplus_i V_i^{\oplus a_i}$, then

$$a_i = (\chi_{V_i}, \chi_R) = \frac{1}{|G|}\chi_{V_i}(e)|G| = \dim V_i$$

□

Remark 3.15. *If* $R = \bigoplus_i V_i^{\oplus a_i}, a_i = \dim V_i$, *then*

$$|G| = \dim R = \sum_i (\dim V_i)^2$$

Remark 3.16. *If* $g \neq e$, *then* $0 = \chi_R(g) = \sum_i \dim V_i \chi_{V_i}(g)$. *If we know all but one row of character table, we can calculate the remaining one using this remark.*

Example 3.17. *Character table of* S_4

We already have trivial representation, alternating representation and standard representation. Since $24 = 1 + 1 + 9 + \sum_i (\dim V_i)^2$, *so there exist two** *other representation* $\widetilde{V}, W$, *such that* $\dim \widetilde{V} = 3, \dim W = 2$.

Consider $\widetilde{V} = U' \otimes V, \dim \widetilde{V} = 3$, *then*

$$\chi_{\widetilde{V}} = \chi_{U'}\chi_V = (3, -1, 0, 1, -1)$$

Then

$$(\chi_{\widetilde{V}}, \chi_{\widetilde{V}}) = 1$$

So it is irreducible. And the remaining one can be calculate from remark 3.16

	1	(12)	(123)	(1234)	(12)(34)
trivial U	1	1	1	1	1
alternating U'	1	−1	1	−1	1
standard V	3	1	0	−1	−1
$\widetilde{V}$	3	−1	0	1	−1
W	2	0	−1	0	2
permutation P	4	2	1	0	0

Proposition 3.18. *Let* $\alpha : G \to \mathbb{C}$ *be any function. Set* $\varphi_{\alpha,V} = \sum_{g\in G} \alpha(g)g : V \to V$ *for any representation* V. *Then* $\varphi_{\alpha,V} \in \mathrm{End}_G V$ *for all* V *if and only if* $\alpha \in \mathscr{C}_G$.

Proof. Condition for $\varphi_{\alpha,V}$ to be G-linear: For $h \in G$,

$$\begin{aligned} \varphi_{\alpha,V}(hv) &= \sum_g \alpha(g)g(hv) = \sum_g \alpha(h^{-1}gh)hgh^{-1}(hv) \\ &= h(\sum_g \alpha(hgh^{-1})gv) \\ &\overset{\alpha \text{ is class function}}{=} h(\sum_g \alpha(g)gv) = h\varphi_{\alpha,V}(v) \end{aligned}$$

*Why there is no other 1-dimensional representation? In fact, we will learn later that the number of irrducible representations is equal to the number of the conjuagate classes.

Conversely, Consider $\varphi_{\alpha,V}(hv) = h\varphi_{\alpha,V}(v)$ and take for V the regular representation R. For $x \in G$,

$$\varphi_{\alpha,R}(he_x) = \varphi_{\alpha,R}(e_{hx}) = \sum_g \alpha(g)e_{hx} = \sum_g \alpha(g)e_{ghx}$$

But we also have

$$h(\varphi_{\alpha,R}(e_x)) = h(\sum_g \alpha(g)ge_x) = \sum_g \alpha(g)hge_x = \sum_g \alpha(g)e_{hgx} = \sum_g \alpha(h^{-1}gh)e_{ghx}$$

Thus α is a class function by comparing the coefficient of two side. □

Proposition 3.19. *If $V = \bigoplus_i V_i^{\otimes a_i}$ is the isotypical decomposition, of a representation V. Then the projection $\pi_i : V \to V_i^{\otimes a_i}$ is given by*

$$\pi_i = \frac{\dim V_i}{|G|} \sum_{g\in G} \overline{\chi_{V_i}(g)}g$$

Proof. Let W be fixed irrducible representation, V be any representation. Since $\overline{\chi_W} \in \mathscr{C}_G$, then

$$\psi_{\overline{\chi_W},V} = \frac{1}{|G|} \sum_{g\in G} \overline{\chi_W(g)}g \in \mathrm{End}_G(V)$$

If V is irreducible, then Shur's lemma implies $\psi_{\overline{\chi_W},V} = \lambda\,\mathrm{id}$, where

$$\lambda = \frac{1}{\dim V}\mathrm{tr}_V\, \varphi_{\overline{\chi_W},V} = \frac{1}{\dim V\cdot|G|} \sum_{g\in G} \overline{\chi_W(g)}\chi_V(g) = \begin{cases} \frac{1}{\dim V}, & V \cong W \\ 0, & \text{otherwise} \end{cases}$$

If V is arbitrary, then $\dim W \psi_{\overline{\chi_W},V}$ is a projection onto W^a where a is the number of times W appears in V.

So, if $V = \bigoplus_i V_i^{\otimes a_i}$ is the isotypical decomposition, then

$$\pi_i = \frac{\dim V_i}{|G|} \sum_{g\in G} \overline{\chi_{V_i}(g)}g$$

is the projection onto $V_i^{\oplus a_i}$.

□

Proposition 3.20.

$$|\,\mathrm{Irr}(G)| = |\,\mathrm{Conj}(G)|$$

In other words, $\{\chi_{V_i} \mid V_i \in \mathrm{Irr}(G)\}$ forms an orthogonal basis for $\mathscr{C}_G$.

Proof. Suppose $\alpha \in \mathscr{C}_G, (\alpha, \chi_V) = 0, \forall V \in \mathrm{Irr}(G)$, we must show $\alpha = 0$.

For any representation V, consider $\varphi_{\alpha,V}$, Shur lemma implies $\varphi_{\alpha,V} = \lambda\,\mathrm{id}_V$, let $n = \dim V$, this implies

$$\lambda = \frac{1}{n}\mathrm{tr}(\varphi_{\alpha,V}) = \frac{1}{n}\sum_g \alpha(g)\chi_V(g) = \frac{|G|}{n}\overline{(\alpha, \chi_{V^\vee})} = 0$$

Thus $\varphi_{\alpha,V} = 0$, that is,

$$\sum_g \alpha(g)g = 0, \quad \text{for any representation } V \text{ of } G.$$

In particular, for $V = R$, the set $\{\rho(g) \in \operatorname{End} R \mid g \in G\}$ consists of linearly independent elements, thus $\alpha(g) = 0, \forall g \in G$. □

Corollary 3.21. *If G is a finite group, the following are equivalent*

1. *G is abelian.*
2. *Every irrducible representation of G has dimension 1.*

Proof. (2) → (1).

$$|G| = \sum_{i=1}^{|\operatorname{Conj}(G)|} (\dim V_i)^2 = |\operatorname{Conj}(G)|$$

So $|K| = 1, \forall K \in \operatorname{Conj}(G)$, that is, G is abelian. □

Proposition 3.22 (Second orthogonality relation)**.**

$$\sum_{i:V_i \in \operatorname{Irr}(G)} \overline{\chi_{V_i}(g)}\chi_{V_i}(h) = \begin{cases} \frac{|G|}{|K_g|}, & K_g = K_h \\ 0, & \text{otherwise} \end{cases}$$

where K_g is the conjugacy class of g.

Proof. Let χ_V, χ_W be irreducible characters. First orthogonality relation implies

$$\delta_{V,W} = (\chi_V, \chi_W) = \frac{1}{|G|} = \sum_g \overline{\chi_V(g)}\chi_W(g) = \frac{1}{|G|} = \sum_{K \in \operatorname{Conj}(G)} \overline{\chi_V(K)}\chi_W(K)|K|$$

Then

$$U = (\sqrt{\frac{|K|}{|G|}}\chi_V(K))$$

is a unitary matrix. Orthogonality of the columns of U yields the claim □

Example 3.23 (Monstrous Monnlight Conjecture)**.** *Let $G = \mathbb{M}$ be the monster group, i.e., the sporadic finite simple group with $|M| \sim 8 \cdot 10^{53}$. One can show that $|\operatorname{Irr}(G)| = |\operatorname{Conj}(G)| = 194$, a relatively small number.*

To compare, $|\operatorname{Irr} S_{15}| = 176, |\operatorname{Irr} S_{16}| = 231$. Let $V_i \in \operatorname{Irr}(G)$ be ordered by their dimension.

V	V_0	V_1	V_2	V_3
$\dim V$	1	196883	21296876	842609256

Complex analysis tells Eisenstein series

$$G_k(\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau + n)^k}$$

converges for $k \geq 3$ normally and defines a holomorphic function on $\mathbb{H}$. $G_k(\tau)$ admits a Fourier expansion

$$G_k(\tau) = \sum_{n=0}^{\infty} a_k(n) q^n, \quad q = e^{2\pi i \tau}$$

Consider

$$j(\tau) = \frac{172820 G_4(\tau)^3}{20 G_4(\tau)^3 + 49 G_6(\tau)^2}$$

Then $j(\tau) - 744 = q^{-1} + 196884q + 21493690q^2 + 864299970q^3 + \dots$

Mckay 1978 *wrote a letter to Thompson*

$$196884 = 196883 + 1$$

Thompson: the next term work similarly.

Suggestion: there exists $V = \bigoplus_{i=0}^{\infty} V_i$ infinitely-dimensional graded representation of $\mathbb{M}$ such that

$$\sum_{n=0}^{\infty} \chi_{V_n} q^{n-1} = j(q) - 744$$

Moreover,

$$T_q(\tau) = \sum_{n=0}^{\infty} \chi_{V_n}(g) q^{n-1} = \textit{other well-known functions in complex analysis}$$

Corway-Norten verified this in 1979 on a computer.

Borcherds proved this conjecture in 1992 by V the structure of a module over a vertex operator algebra.

Definition 3.24. *Let G, H be finite groups, V is a representation of G, W is a representation of H, we define the external tensor product representation $V \boxtimes W$ of $G \times H$ by*

$$(g,h)(v,w) = gv \otimes hw, \quad \forall g \in G, h \in H, v \in V, w \in W.$$

and extension by linearity to $V \otimes W$.

Similarly, we define a $G \times H$ action on $\operatorname{Hom}(V,W)$ by

$$((g,h)\varphi)v = h\varphi(g^{-1}v), \quad g \in G, h \in H, v \in V, \varphi \in \operatorname{Hom}(V,W).$$

and extension by linearity.

Remark 3.25. *We have*

$$\operatorname{Hom}(V,W) \cong V^{\vee} \boxtimes W$$

as $G \times H$ representations.

Proposition 3.26. *We have the following well-defined bijection:*

$$\operatorname{Irr}(G) \times \operatorname{Irr}(H) \to \operatorname{Irr}(G \times H)$$
$$(V,W) \to V \boxtimes W$$

Proof. If suffices to look at characters. By property of trace we have

$$\chi_{V\boxtimes W}((g,h)) = \chi_V(g)\chi_W(h)$$

Recall that

$$\dim \mathrm{Hom}_G(V,W) = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_V(g)}\chi_W(g) = (\chi_V,\chi_W)_G$$

Then

$$\begin{aligned}(\chi_{V_1\boxtimes W_1},\chi_{V_2\boxtimes W_2}) &= \frac{1}{|G\times H|}\sum_{g,h\in G\times H}\overline{\chi_{V_1}(g)\chi_{W_1}(g)}\chi_{V_2}(g)\chi_{W_2}(g)\\ &= \frac{1}{|G|}\sum_{g}\overline{\chi_{V_1}(g)}\chi_{V_2}(g)\frac{1}{|G|}\sum_{h\in H}\overline{\chi_{W_1}(g)}\chi_{W_2}(g)\\ &= (\chi_{V_1},\chi_{V_2})_G(\chi_{W_1},\chi_{W_2})_H\end{aligned}$$

So $V\boxtimes W\in \mathrm{Irr}(G\times H)$, if $V\in\mathrm{Irr}(G), W\in\mathrm{Irr}(H)$.

By calculating the cardinality of both sides we get the desired result. □

4. Restriction and induced representation

Definition 4.1 (restriction representation)**.** *Let $H<G$ be a subgroup, V be a representation of G, we define $\mathrm{Res}\,V=\mathrm{Res}^G_H V: H\to \mathrm{GL}(V)$ to be the restriction of V onto H, $\mathrm{Res}^G_H V$ is a representation of H.*

Remark 4.2. *Restriction is transitive, i.e., for $K<H<G$, we have*

$$\mathrm{Res}^H_K\mathrm{Res}^G_H = \mathrm{Res}^G_K$$

Lemma 4.3. *Let $H<G$, $W\in\mathrm{Irr}(H)$, then there exists $V\in\mathrm{Irr}(G)$ such that*

$$(\mathrm{Res}^G_H\chi_V,\chi_W)_H\neq 0$$

Proof. Consider the regular representation R, then

$$(\mathrm{Res}^G_H\chi_R,\chi_W) = \frac{|G|}{|H|}\chi_W(e)\neq 0$$

But the left term also equals to $\sum_i \dim V_i(\mathrm{Res}^G_H\chi_{V_i},\chi_W)_H$, so there must be at least one V_i, such that

$$(\mathrm{Res}^G_H\chi_{V_i},\chi_W)\neq 0$$

□

Lemma 4.4. *Let $H<G$, $V\in\mathrm{Irr}(G)$, $\mathrm{Res}^G_H V=\bigoplus W_i^{\otimes a_i}, W_i\in\mathrm{Irr}(W)$. Then $\sum a_i^2\leq[G:H]$ with equality if and only if $\chi_V(\sigma)=0,\forall\sigma\in G/H$.*

Proof. We have

$$\frac{1}{|G|}\sum_{h\in H}|\chi_V(h)|^2 = (\mathrm{Res}^G_H V,\mathrm{Res}^G_H V) = \sum a_i^2$$

Since V is irreducible, we have

$$\begin{aligned}1 = (\chi_V, \chi_V)_G &= \frac{1}{|G|}\sum_{g\in G}|\chi_V(g)|^2 \\ &= \frac{1}{|G|}(\sum_{h\in H}|\chi_V(h)|^2 + \sum_{\sigma\in G/H}|\chi_V(\sigma)|^2) \\ &= \frac{|H|}{|G|}\sum_i a_i^2 + \frac{1}{|G|}\sum_{\sigma\in G/H}|\chi_V(\sigma)|^2 \\ &\geq \frac{|H|}{|G|}\sum_i a_i^2\end{aligned}$$

□

Proposition 4.5. *Let V, W be representation of G. Then $V \cong W \iff \operatorname{Res}_H^G V \cong \operatorname{Res}_H^G W$, for all cyclic subgroup H of G.*

Proof. One direction is obvious, consider the other: Let $g \in G, H = \langle g\rangle$, then $\chi_V(g) = \chi_{\operatorname{Res}_H^G V}(g)$, the claim follows from $V \cong W \iff \chi_V(g) = \chi_W(g), \forall g \in G$. □

Definition 4.6. *Let $H < G$ be a subgroup, $\rho : G \to \mathrm{GL}(V)$ be a representation, $W \subset V$ be a H-invariant subspace, i.e., $\psi : H \to \mathrm{GL}(W)$ is a representation. Then the subspace $gW \subset V$ depends only on gH. Therefore, for $\sigma \in G/H$, we write $\sigma W = gW, g \in \sigma$. If V has a unique decomposition $V = \bigoplus_{\sigma\in G/H}\sigma W$, we write $V = \operatorname{Ind} W = \operatorname{Ind}_H^G W$. In this case, V is called a representation induced by W.*

Remark 4.7. *Alternative formulations: for any $v \in V$, there exists a unique $v_\sigma \in \sigma W$, such that*

$$v = \sum_{\sigma\in G/H} v_\sigma$$

or if $\{g_1, \ldots, g_N\}$, $|N| = |G/H| = [G : H]$ is a complete system of representatives of G/H, then

$$V = \bigoplus_{i=1}^{N} g_i W$$

Remark 4.8.

$$\dim V = [G : H]\dim W$$

Example 4.9. *Let R be the regular representation of G, then*

$$W = \bigoplus_{h\in H}\mathbb{C}e_h$$

is H-invariant. then $\psi : H \to \mathrm{GL}(W)$ is a representation, in fact, $W \cong R_H$ and clearly $R_G = \operatorname{Ind}_H^G R_H$.

Example 4.10. *Let $H < G$ and V the coset representation of G, i.e., V has basis $(e_\sigma)_{\sigma \in G/H}$ and $ge_\sigma = e_{g\sigma}$. Then*

$$W = \mathbb{C}e_{eH}$$

is H-invariant, and is the trivial representation of H, then

$$V = \operatorname{Ind}_H^G W$$

In particular, if $H = \{e\}$, then V is the permutation representation P of G, and $P = \operatorname{Ind}_{\{e\}}^G \mathbb{C}$.

Example 4.11. *If $V_i = \operatorname{Ind}_H^G W_i, i = 1, 2$, then*

$$V_1 \oplus V_2 = \operatorname{Ind}_H^G(W_1 \oplus W_2)$$

Example 4.12. *If $V = \operatorname{Ind}_H^G W$, $W' \subset W$ is a H-invariant subspace, then*

$$V' = \bigoplus_{\sigma \in G/H} \sigma W' \subset V$$

is G-invariant, and $V' = \operatorname{Ind}_H^G W'$.

Proposition 4.13. *Let $H < G$ be a subgroup, $\rho : G \to \mathrm{GL}(V)$ is induced by $\psi : H \to \mathrm{GL}(W)$, let $\rho' : G \to \mathrm{GL}(V')$ be any representation, $\phi \in \operatorname{Hom}_H(W, V')$, then there exists a unique $\Phi \in \operatorname{Hom}_G(V, V')$, such that*

$$\Phi|_W = \phi$$

Proof. For uniqueness: Let $\Phi \in \operatorname{Hom}_G(V, V')$ with $\Phi|_W = \phi$, and let $w \in \rho(g)W, g \in G$, then

$$\Phi(w) = \Phi(\rho(g)\rho(g^{-1})w) = \rho'(g)\Phi(\rho(g)^{-1}w) = \rho'(g)\phi(\rho(g)^{-1}w)$$

This determines Φ on $\rho(g)W$ for all $g \in G$, hence on V.

For existence: we define

$$\Phi(w) = \rho'(g)\phi(\rho(g)^{-1}w)$$

if $w \in \rho(g)W$, this is independent of the choice of g, since

$$\begin{aligned}
\rho'(gh)\phi(\rho(gh)^{-1}w) &= \rho'(g)\rho'(h)\phi(\rho(h)^{-1}\rho(g)^{-1}w) \\
&= \rho'(g)\phi(\rho(h)\rho(h)^{-1}\rho(g)^{-1}w) \\
&= \rho'(g)\phi(\rho(g)^{-1}w)
\end{aligned}$$

□

Theorem 4.14. *Let $H < G$ be a subgroup, and $\psi : H \to \mathrm{GL}(W)$ be a representation. Then there exists a representation $\rho : G \to \mathrm{GL}(V)$ induced by W, which is unique up to isomorphism.*

Proof. For existence: By example4.11 we may assume $W \in \operatorname{Irr}(H)$, W' is isomorphic to a subrepresentation of R_H, since any $W' \in \operatorname{Irr}(H)$ appears in R_H. By example4.9 we have

$$R_G = \operatorname{Ind}_H^G R_H$$

and by example4.12 with $V = R_G, W = R_H$, we get

$$V' = \operatorname{Ind}_H^G W'$$

For uniqueness: Let $V = \operatorname{Ind}_H^G W, V' = \operatorname{Ind}_H^G W$, then proposition4.13 implies that there exists a unique $\Phi \in \operatorname{Hom}_G(V, V')$ such that $\Phi|_W = \mathrm{id}_W$, and $\Phi \circ \rho(g) = \rho'(g) \circ \Phi, \forall g \in G$. Then $\operatorname{Im} \Phi$ contains all $\rho'(g)W$, so $\operatorname{Im} \Phi = V'$.

By $\dim V = [G : H] \dim W = \dim V'$, we conclude Φ is an isomorphism.

□

Lemma 4.15. *Let V be a representation of G, and $H < G$ be a subgroup. Then*

$$V \otimes \operatorname{Ind}_H^G W = \operatorname{Ind}_H^G(\operatorname{Res}_H^G V \otimes W)$$

Proof. Note that

$$\begin{aligned} V \otimes \operatorname{Ind}_H^G W &= \bigoplus_{\sigma \in G/H} V \otimes \sigma W \\ &= \bigoplus_{\sigma \in G/H} \sigma(\operatorname{Res}_H^G V) \otimes \sigma W = \operatorname{Ind}_H^G(\operatorname{Res}_H^G V \otimes W) \end{aligned}$$

□

Corollary 4.16. *We have*

$$V \otimes P = \operatorname{Ind}_H^G(\operatorname{Res}_H^G V)$$

where P is permutation representation.

Proof. Take W as trivial representation, then this claim holds from lemma4.15.

□

Lemma 4.17. Ind *is transitive.*

Proof.

$$\begin{aligned} \operatorname{Ind}_K^H \operatorname{Ind}_H^G &= \operatorname{Ind}_K^H \bigoplus_{\tau \in G/H} \tau V \\ &= \bigoplus_{\sigma \in H/K} \bigoplus_{\tau \in G/H} \sigma\tau V \\ &= \bigoplus_{\sigma' \in G/K} \sigma' V \\ &= \operatorname{Ind}_K^G V \end{aligned}$$

□

Remark 4.18. *These results can also be obtained by looking at characters or using group algebra.*

Theorem 4.19. *Let $H < G$ be a subgroup, and $\rho : G \to \mathrm{GL}(V), \psi : H \to \mathrm{GL}(W)$ be two representations, such that $V = \mathrm{Ind}_H^G W$. Then*

$$\chi_V(g) = \sum_{\sigma \in G/H} \chi_W(g_\sigma^{-1} g g_\sigma) = \frac{1}{|H|} \sum_{\substack{x \in G, \\ x^{-1} g x \in H}} \chi_W(x^{-1} g x)$$

where g_σ is any representative of σ.

Proof. Let $V = \bigoplus_{\sigma \in G/H} \sigma W$, $\rho(g)$ permutes the σW among themselves, i.e., if $g_\sigma \in \sigma$ is a representative, we write $g g_\sigma = g_\tau h$ for some $\tau \in G/H, h \in H$.

$$g(g_\sigma W) = (g_\tau h) W = g_\tau(hW) = g_\tau W$$

Then we can calculate

$$\begin{aligned}
\chi_V(g) = \mathrm{tr}_V(\rho(g)) &= \sum_{\sigma \in G/H} \mathrm{tr}_{\sigma W}(\rho(g)) \\
&= \sum_{\sigma \in G/H} \chi_W(g_\sigma^{-1} g g_\sigma) = \sum_{\tau \in G/H} \chi_W(h^{-1} g_\tau^{-1} g g_\tau h) \\
&= \frac{1}{|H|} \sum_{\tau \in G/H} \sum_{h \in H} \chi_W(h^{-1} g_\tau^{-1} g g_\tau h) = \frac{1}{|H|} \sum_{\substack{x \in G, \\ x^{-1} g x \in H}} \chi_W(x^{-1} g x)
\end{aligned}$$

□

Theorem 4.20 (Frobenius reciprocity)**.** *Let $H < G$ be a subgroup, W be a representation of H, U be a representation of G. Assume that $V = \mathrm{Ind}_H^G W$, then*

$$\mathrm{Hom}_H(W, \mathrm{Res}_H^G U) \cong \mathrm{Hom}_G(V, U)$$

i.e., for $\varphi \in \mathrm{Hom}_H(W, \mathrm{Res}_H^G U)$ extends uniquely to $\tilde{\varphi} \in \mathrm{Hom}_G(V, U)$

Proof. We write $V = \bigoplus_{\sigma \in G/H} \sigma W$, define $\tilde{\phi}$ on σW by the compostion

$$\sigma W \xrightarrow{g_\sigma^{-1}} W \xrightarrow{\varphi} U \xrightarrow{g_\sigma} U$$

This is independent of the choice of g_σ since

$$g_\sigma h(\varphi(h^{-1} g_\sigma^{-1}(w))) = g_\sigma \varphi(h h^{-1} g_\sigma(w))$$

by $\varphi \in \mathrm{Hom}_H(W, \mathrm{Res}_H^G U)$ □

Corollary 4.21. *Let $H < G$ be a subgroup, W be a representation of H, U be a representation of G. Then*

$$(\chi_W, \mathrm{Res}_H^G \chi_U)_H = (\mathrm{Ind}_H^G \chi_W, \chi_U)_G$$

Proof. By linearity, we can assume W, U are irreducible representations. This claim follows from the Frobenius reciprocity and shur's lemma

$$(\chi_V, \chi_U)_G = \dim \mathrm{Hom}_G(V, U)$$

□

Example 4.22. *Let $G = S_3, H = S_2$. In S_2, the standard representation V_2 is isomorphic to the alternating representation U_2'. We have seen that U_3, U_3', V_3 are all irreducible representations of S_3.*

And we can write down their character tables as follows

	1	(12)
trivial U_2	1	1
alternating U_2'	1	-1

,

	1	(12)	(123)
trivial U_3	1	1	1
alternating U_3'	1	-1	1
standard V_3	2	0	-1

Note that

$$\operatorname{Res} U_3 = U_2, \quad \operatorname{Res} U_3' = U_2', \quad \operatorname{Res} V_3 = U_2 \oplus U_2'$$

If we want to calculate Ind*, firstly note that we have seen*

$$P \otimes U = \operatorname{Ind}(\operatorname{Res} U), \quad U \text{ is any representation of } G$$

For $U = U_3$, we have $P = U_3 \oplus V_3 = \operatorname{Ind} U_2$.

If we want to calculate $\operatorname{Ind} V_2$*, it's a little bit complicated.*

By Frobenius reciprocity

$$\operatorname{Hom}_{S_3}(\operatorname{Ind} V_2, U_3) = \operatorname{Hom}_{S_2}(V_2, \operatorname{Res} U_3 = U_2) \overset{shur}{=} 0$$

$$\operatorname{Hom}_{S_3}(\operatorname{Ind} V_2, U_3') = \operatorname{Hom}_{S_2}(V_2, \operatorname{Res} U_3' = U_2') \overset{shur}{=} \mathbb{C}$$

$$\operatorname{Hom}_{S_3}(\operatorname{Ind} V_2, V_3) = \operatorname{Hom}_{S_2}(V_2, \operatorname{Res} V_3 = U_2 \oplus U_2') \overset{shur}{=} \mathbb{C}$$

So

$$\operatorname{Ind} V_2 = U_3' \oplus V_3$$

Definition 4.23. *Let G be a finite group, and $R_k(G)$ be the free abelian group generated by all isomorphism classes of representations of G over a field k, modulo the subsgroup generated by elements of the form $V + W - (V \oplus W)$. $R(G)$ is called the representation ring of G, or the Grothendieck group of G, denoted by $K_0(G)$, elements of $R(G)$ are called virtual representations.*

The ring structure on $R(G)$ is the tensor product, defined on the generators of $R(G)$, and extended by linearity.

Remark 4.24. *We have the following remarks:*

1. *A character defines a ring homomorphism from $R(G)$ to $\mathscr{C}_G$*
2. *χ is injective is equivalent to a representation is determined by its character, the image of χ are called virtual characters.*
3. *$\chi_{\mathbb{C}} : R(G) \otimes_{\mathbb{Z}} \mathbb{C} \to \mathscr{C}_G$ is an isomorphism.*
4. *The virtual characters form a lattice $\Lambda \cong \mathbb{Z}^c \subset \mathscr{C}_G$. The actual characters form a cone $\Lambda_0 \cong \mathbb{N}^0 \subset \Lambda$.*
5. *By* 3. *we can define an inner product on $R(G)$ by*

$$(V, W) = \dim \operatorname{Hom}_G(V, W)$$

Example 4.25. *Let $G = C_n$, then $R(C_n) = \mathbb{Z}[x]/(x^n - 1)$, where X correspond to the representation of a primitive n-th root of unity.*

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