DIFFERENTIAL FORMS IN ALGEBRAIC TOPOLOGY

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Abstract. Dedicated to my youth.

Contents

1. Introduction

The most intuitively evident topological invariant of a space is the number of connected pieces into which it falls. But we wonder is there any high-dimensional analogues. These are the homotopy and cohomology groups of the space.

The evolution of the higher homotopy groups from the component concept is deceptively simple and essentially unique. To describe it, let $\pi_0(X)$ denote the set of path connected components of X and if p is a point of X, we let $\pi_0(X)$ denote the set $\pi_0(X)$ with the path connected component of p singled out. Also, corresponding to such a point p, let $\Omega_p X$ denote the space of maps[1] of the unit circle which send 1 to p, made into a topological space via the compact open topology[2]. The path components of this so called loop space $\Omega_p X$ are now taken to be the elements of $\pi_1(X,p)$:

$$\pi_1(X,p) = \pi_0(\Omega_p X, \bar{p})$$

The composition of loops induces a group structure on $\pi_1(X,p)$ in which the constant map $\bar{p}$ of the circle to p plays the role of identity; so endowed, $\pi_1(X,p)$ is called the fundamental group of X at p. In general, it's not abelian.

To get general case, all the higher homotopy groups $\pi_k(X,p)$ for $k \geq 2$ can now be defined through the inductive formula

$$\pi_{k+1}(X,p) = \pi_k(\Omega_p X, p)$$

For Riemannian surface, the higher $\pi_k, k \geq 2$ are all trivial, and it is in part for this reason that $\pi_1(X)$ is sufficient to classify them. The group π_k for $k \geq 2$ turn out to be abelian and therefore do not seem to have been taken seriously until the $1930's$ when W.Hurewicz defined them and showed that, far from being trivial, they constituted the basic ingredients needed to describe the homotopy-theoretic properties of a space.

Unfortunately, the great drawback of these easy defined invariants of a space is quite difficult to calculate. To this day not all the homotopy groups of 2-sphere have been computed. Nonetheless, by now much is known concerning the general properties of the homotopy groups, largely due to the formidable algebraic techniques to which the "cohomological extension" of the component concept lends itself, and the relations between homotopy and cohomology which have been discovered over the years.

This cohomological extension starts with the dual point of view in which a component is characterized by the property that on it every locally constant function is global constant. Such a component is sometimes called a connected component, to distingush it from a path component. Thus, if we

[1] In this note, map means continous function

[2] The subbasis of compact open topology is $M_{K,U} = \{f : X \to Y \mid f(K) \subset U\}$, for compact set $K \subset X$ and open set $U \subset Y$.

define $H^0(X)$ to be the vector space of real-valued locally constant functions on X, then $\dim H^0(X)$ tells us the number of connected components of X. Note that on reasonable spaces where path connected component and connected component agree, we therefore have the formula

$$|\pi_0(X)| = \dim H^0(X)$$

Still the two concepts are dual to each other, the first using map of the unit interval into X to test for connectedness and the second using map of X into $\mathbb{R}$ for the same purpose. One further difference is that the cohomology group has a natural $\mathbb{R}$-module structure.

So what is the proper high-dimensional analogues of $H^0(X)$? Unfortunately there is no decisive answer here. Many plausible definitions of $H^k(X)$ for $k > 0$ have been proposed, all with slightly different properties but all isomorphic on "reasonable spaces", Furthermore, in the realm of differential manifolds, all these theories coincide with the de Rham theory, in some sense the most perfect example of a cohomology theory. The de Rham theory is also unique in that it stands at the crossroads of topology, analysis, and physics, enriching all three disciplines.

The gist of the "de Rham extension" is comprehended most easily when M is an open set in $\mathbb{R}^n$, with coordinate $x_1, \ldots, x_n$. Then amongst the C^∞ functions on M the locally constant ones are precisely thoese whose gradient

$$\mathrm{d}f = \sum \frac{\partial f}{\partial x_i}\mathrm{d}x_i$$

vanishes identically. Thus here $H^0(M)$ appears as the space of solutions of the differential equation $\mathrm{d}f = 0$. This suggests that $H^1(M)$ should also appear as the space of solutions of some natural differential equations on the manifold M.

Consider 1-form on M:

$$\theta = \sum a_i \mathrm{d}x_i$$

where the $a_i's$ are C^∞ functions on M. Such an expression can be integrated along a smooth path γ, so that we may think of θ as a function on path γ:

$$\gamma \mapsto \int_\gamma \theta$$

It then suggests itself to seek those θ which give rise to locally constant function of γ. Stokes' theorem tells us that these line integrals are characterized by

$$\frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i} = 0$$

On the other hand, the fundamental theorem of calculus implies that $\int_\gamma \mathrm{d}f = f(Q) - f(P)$, where P and Q are the endpoints of γ, so that the gradients are trivially locally constant.

One is here irresistibly led to the definition of $H^1(M)$ as the vector space of locally constant line integrals modulo the trivially constant ones. Similarly the higher cohomology groups are defined by simply replacing line integrals with their high-dimensional analogues, the k-volume integrals.

2. de Rham theory

2.1. **The Mayer-Vietoris Argument.** The Mayer-Vietoris sequences relates the cohomology of a union to those of the subsets. Together with the Five lemma, this gives a method of proof which proceeds by induction on the cardinality of an open cover, called the Mayer-Vietoris argument. As evidence of its power and versatility, we derive from it the finite dimensionality of the de Rham cohomology, Poincaré duality, the Künneth formula, the Leray-Hirsch theorem, and the Thom isomorphism, all for manifold with finite good cover.

2.1.1. *Existence of Good Cover.* Let M be a manifold of dimensional n. An open cover $\mathfrak{U} = \{U_\alpha\}$ of M is called a good cover if all nonempty finite intersections $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ are diffeomorphic to $\mathbb{R}^n$. A manifold which has a finite good cover is said to be of finite type.

Theorem 2.1.1. *Every manifold has a good cover. If the manifold is compact, then the cover may be chosen to be finite.*

Proof. Endow M with a Riemannian structure. Now we quote the theorem in differential geometry that every point in a Riemannian manifold has a geodesically convex neighborhood. The intersection of any two such neighborhood is again geodesically convex. Since a geodesically convex neighborhood in a Riemannian manifold of dimension n is diffeomorphic to $\mathbb{R}^n$, an open cover consisting of geodesically convex neighborhoods will be a good cover. □

Given two covers $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ and $\mathfrak{B} = \{V_\beta\}_{\beta \in J}$, if every V_β is contained in some U_α, we say that $\mathfrak{B}$ is a refinement of $\mathfrak{U}$ and write $\mathfrak{U} < \mathfrak{B}$. In fact, for any open cover on a manifold has a refinement which is a good cover: simply take geodesically convex neighborhoods around each point to be inside some open set of the given cover.

The set of open covers on a manifold is a directed set, since any two open cover always have a common refinement. A subset J of a directed set I is cofinal in I if for every $i \in I$, there is a $j \in J$ such that $i < j$. It is clear that J is also a directed set.

Corollary 2.1.2. *The good covers are cofinal in the set of all covers of a manifold.*

2.1.2. *Finite Dimensional of de Rham Cohomology.*

Proposition 2.1.3. *If the manifold M has a finite good cover, then its cohomology is finite dimensional*

Proof. From the Mayer-Vietoris sequence

$$\cdots \to H^{q-1}(U \cap V) \xrightarrow{d^*} H^q(U \cup V) \xrightarrow{r} H^q(U) \oplus H^q(V) \to \dots$$

we get

$$H^q(U \cup V) \cong \ker r \oplus \operatorname{im} r \cong \operatorname{im} d^* \oplus \operatorname{im} r$$

Thus, if $H^q(U), H^q(V)$ and $H^{q-1}(U \cap V)$ are finite-dimensional, then so is $H^q(U \cup V)$.

For a manifold which is diffeomorphic to $\mathbb{R}^n$, the finite dimensionality of $H^*(M)$ follows from Poincaré lemma. We now procced by induction on the cardinality of a good cover. Suppose the cohomology of any manifold having a good cover with at most p open sets is finite dimensional. Consider a manifold having a good cover $\{U_0, \dots, U_p\}$ with $p+1$ open sets. Now $(U_0 \cup \dots \cup U_{p-1}) \cap U_p$ has a good cover with p open sets. By hypothesis, the q-th cohomology of $U_0 \cup \dots \cup U_{p-1}, U_p$ and $(U_0 \cup \dots \cup U_{p-1}) \cap U_p$ are finite-dimensional, so is the q-th cohomology of $U_0 \cup \dots \cup U_p$. This completes the induction. □

2.1.3. *Poincaré Duality on an Orientable Manifold.* Consider the following pair

$$\int : H^q(M) \otimes H_c^{n-q}(M) \to \mathbb{R}$$

given by the integral of the wedge product of two forms. Our first version of Poincaré duality asserts that this pairing is nondegenerate whenever M is a orientable and has a finite good cover, or equivalently

$$H^q(M) \cong (H_c^{n-q})^*$$

Note that both of these groups are finite-dimensional.

Lemma 2.1.4. *The two Mayer-Vietoris sequences may be paired together to form a sign-commutative diagram*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^q(U \cup V) & \xrightarrow{\textit{restriction}} & H^q(U) \oplus H^q(V) & \xrightarrow{\textit{difference}} & H^q(U \cap V) & \xrightarrow{d^*} & H^{q+1}(U \cup V) \\
 & \otimes & & \otimes & & \otimes & & \otimes \\
\cdots \longleftarrow & H_c^{n-q}(U \cup V) & \xleftarrow[\textit{sum}]{} & H_c^{n-q}(U) \oplus H_c^{n-q}(V) & \longleftarrow & H_c^{n-q}(U \cap V) & \xleftarrow[d_*]{} & H_c^{n-q-1}(U \cup V) \\
 & \Big\downarrow{\scriptstyle \int_{U \cup V}} & & \Big\downarrow{\scriptstyle \int_U + \int_V} & & \Big\downarrow{\scriptstyle \int_{U \cap V}} & & \Big\downarrow{\scriptstyle \int_{U \cup V}} \\
 & \mathbb{R} & & \mathbb{R} & & \mathbb{R} & & \mathbb{R}
\end{array}$$

Here sign-commutativity means, for instance, that

$$\int_{U \cap V} \omega \wedge \mathrm{d}_* \tau = \pm \int_{U \cup V} (\mathrm{d}^*) \wedge \tau$$

for $\omega \in H^q(U\cap V), \tau \in H_c^{n-q-1}(U\cap V)$. *This lemma is equivalent to saying that the pairing induces a map from the upper exact sequence to the dual of the lower exact sequence such that the following diagram is sign-commutative:*

$$\begin{tikzcd}
{} \arrow[r] & H^q(U\cup V) \arrow[r] \arrow[d] & H^q(U)\oplus H^q(V) \arrow[r] \arrow[d] & H^q(U\cap V) \arrow[r] \arrow[d] & {} \\
{} & H_c^{n-q}(U\cup V)^* \arrow[l] & H_c^{n-q}(U)^*\oplus H_c^{n-q}(V)^* \arrow[l] & H_c^{n-q}(U\cap V)^* \arrow[l] & {} \arrow[l]
\end{tikzcd}$$

Proof. The first two squares are in fact commutative as is straightforward to check. We will show the sign-commutativity of the third square.

In fact, we need to make how d_* or d^* map clearly. First, $\mathrm{d}^*\omega$ is a form in $H^{q+1}(U\cup V)$ such that

$$\begin{aligned}\mathrm{d}^*\omega|_U &= -\mathrm{d}(\rho_V\omega)\\ \mathrm{d}^*\omega|_V &= \mathrm{d}(\rho_U\omega)\end{aligned}$$

and $\mathrm{d}_*\tau$ is a form in $H_c^{n-q}(U\cap V)$ such that

$$(-(\text{extension by 0 of } \mathrm{d}_*\tau \text{ to } U), (\text{extension by 0 of } \mathrm{d}_*\tau \text{ to } V)) = (\mathrm{d}(\rho_U\tau), \mathrm{d}(\rho_V\tau))$$

Note that $\mathrm{d}(\rho_V\tau) = (\mathrm{d}\rho_V)\tau$ because τ is closed; similarly, $\mathrm{d}(\rho_V\omega) = (\mathrm{d}\rho_V)\omega$.

$$\int_{U\cap V}\omega\wedge \mathrm{d}_*\tau = \int_{U\cap V}\omega\wedge(\mathrm{d}\rho_V)\tau = (-1)^{\deg\omega}\int_{U\cap V}(\mathrm{d}\rho_V)\omega\wedge\tau.$$

Since $\mathrm{d}^*\omega$ has support in $U\cap V$,

$$\int_{U\cup V}\mathrm{d}^*\omega\wedge\tau = -\int_{U\cap V}(\mathrm{d}\rho_V)\omega\wedge\tau$$

Therefore,

$$\int_{U\cap V}\omega\wedge \mathrm{d}_*\tau = (-1)^{\deg\omega+1}\int_{U\cup V}(\mathrm{d}^*)\wedge\tau$$

□

By the Five lemma if Poincaré duality holds for U, V and $U\cap V$, then it holds for $U\cup V$. We now proceed by induction on the cardinality of a good cover. For M is diffeomorphic to $\mathbb{R}^n$, Poincaré duality holds from Poincaré lemmas. Next suppose Poincaré duality holds for any manifold having a good cover with at most p open sets, and consider a manifold having a good cover $\{U_0,\dots,U_p\}$ with $p+1$ open sets. Now $(U_0\cup\cdots\cup U_{p-1})\cap U_p$ has a good cover with p open sets. By hypothesis, Poincaré duality holds for $U_0\cup\cdots\cup U_{p-1}, U_p, (U_0\cup\cdots\cup U_{p-1})\cap U_p$, so it holds for $U_0\cup\cdots\cup U_p$ as well. This induction argument proves Poincaré duality for any orientable manifold having a finite good cover.

Remark 2.1.5. *The finiteness assumption on the good cover is in facts not necessary. The statement is as follows: If M is a orientable manifold of dimension n, whose cohomology is not necessarily finite dimensional, then*

$$H^q(M)\cong (H_c^{n-q}(M))^*, \quad \textit{for any integer } q$$

However, the reverse implication $H_c^q(M) \cong (H^{n-q}(M))^*$ is not always true.

2.1.4. *The Künneth formula and the Leray-Hirsch theorem.* The Künneth formula states that the cohomology of the product of two manifolds M and F is the tensor product

$$H^*(M \times F) = H^*(M) \otimes H^*(F)$$

This means

$$H^n(M \times F) = \bigoplus_{p+q=n} H^p(M) \otimes H^q(F)$$

More generally we are interesting in the cohomology of a fiber bundle.

Definition 2.1.6. *Let G be a topological group which acts effectively on a space F on the left. A surjection $\pi : E \to B$ between topological space is a fiber bundle with fiber F and structure group G if B has an open cover $\{U_\alpha\}$ such that there are fiber-preseving homeomorphisms*

$$\phi_\alpha : E|_{U_\alpha} = \pi^{-1}(U_\alpha) \cong U_\alpha \times F$$

and the transition functions are continous with values in G:

$$g_{\alpha\beta}(x) = \phi_\alpha \phi_\beta^{-1}|_{\{x\}\times F} \in G$$

Sometimes the total space E is referred to as the fiber bundle. A fiber bundle with structure group G is also called a G-bundle. If $x \in B$, the set $E_x = \pi^{-1}(x)$ is called the fiber at x.

The following proof of the Künneth formula assumes that M has a finite good cover. This assumption is necessary for the induction argument.

The two natural projections

$$\begin{array}{ccc} M \times F & \xrightarrow{\rho} & F \\ \big\downarrow{\scriptstyle \pi} & & \\ M & & \end{array}$$

give rise to a map on forms

$$\omega \otimes \phi \mapsto \pi^*\omega \wedge \rho^*\phi$$

which induces a map in cohomology

$$\psi : H^*(M) \otimes H^*(F) \to H^*(M \times F)$$

If $M = \mathbb{R}^n$, then this is simply the Poincaré lemma. In the following we will regard $M \times F$ as a product bundle over M. Let U and V be open sets in M and n a fixed integer. From the Mayer-Vietoris sequence

$$\cdots \to H^p(U \cup V) \to H^p(U) \oplus H^p(V) \to H^p(U \cap V) \to \dots$$

we get an exact sequence by tensoring $H^{n-p}(F)$

$$\cdots \to H^p(U \cup V) \otimes H^{n-p}(F) \to (H^p(U) \oplus H^p(V)) \otimes H^{n-p}(F) \to H^p(U \cap V) \otimes H^{n-p}(F) \to \dots$$

since tensoring with a vector space preserves exactness. Summing over all integers p yields the exact sequence

$$\begin{aligned} \cdots \to & \bigoplus_{p=0}^{n} H^p(U\cup V)\otimes H^{n-p}(F) \\ \to & \bigoplus_{p=0}^{n} (H^q(U)\otimes H^{n-p}(F))\oplus(H^q(V)\otimes H^{n-p}(F)) \\ \to & \bigoplus_{p=0}^{n} H^p(U\cap V)\otimes H^{n-p}(F) \to \cdots \end{aligned}$$

So the following diagram is commutative[3]

$$\begin{array}{ccc} \bigoplus_{p=0}^{n} H^p(U\cup V)\otimes H^{n-p}(F) & \xrightarrow{\psi} & H^n((U\cap V)\times F) \\ \downarrow & & \downarrow \\ \bigoplus_{p=0}^{n}(H^q(U)\otimes H^{n-p}(F))\oplus(H^q(V)\otimes H^{n-p}(F)) & \xrightarrow{\psi} & H^n(U\times F)\oplus H^n(V\times F) \\ \downarrow & & \downarrow \\ \bigoplus_{p=0}^{n} H^p(U\cap V)\otimes H^{n-p}(F) & \xrightarrow{\psi} & H^n((U\cap V)\times F) \end{array}$$

The commutativity is clear except possibly for the square

$$\begin{array}{ccc} \bigoplus_{p=0}^{n} H^p(U\cap V)\otimes H^{n-p}(F) & \xrightarrow{\psi} & H^n((U\cap V)\times F) \\ \downarrow{\scriptstyle \mathrm{d}^*} & & \downarrow{\scriptstyle \mathrm{d}^*} \\ \bigoplus_{p=0}^{n} H^{p+1}(U\cup V)\otimes H^{n-p}(F) & \xrightarrow{\psi} & H^{n+1}((U\cup V)\times F) \end{array}$$

Which we need to check. Let $\omega\otimes\phi$ be in $H^q(U\cap V)\otimes H^{n-p}(F)$. Then

$$\begin{aligned} \psi\mathrm{d}^*(\omega\otimes\phi) &= \pi^*(\mathrm{d}^*\omega)\wedge\rho^*\phi \\ \mathrm{d}^*\psi(\omega\otimes\phi) &= \mathrm{d}^*(\pi^*\omega\wedge\rho^*\phi) \end{aligned}$$

Since the pull back $\{\pi^*\rho_U,\pi^*\rho_V\}$ form a partition of unity on $(U\cup V)\times F$ subordinate to the cover $\{U\times F, V\times F\}$, on $(U\cap V)\times F$

$$\begin{aligned} \mathrm{d}^*(\pi^*\omega\wedge\rho^*\phi) &= \mathrm{d}((\pi^*\rho_U)\pi^*\omega\wedge\rho^*\phi) \\ &= (\mathrm{d}\pi^*(\rho_U\omega))\wedge\rho^*\phi, \quad \text{since } \phi \text{ is closed} \\ &= \pi^*(\mathrm{d}^*\omega)\wedge\rho^*\phi \end{aligned}$$

So the diagram is commutative.

By the Five lemma if the theorem is true for U, V and $U\cap V$, then it is also true for $U\cup V$. The Künneth formula now follows by induction on the cardinality of a good cover, as in the proof before.

[3]Here I used an unusual way to draw this diagram, for there is no enough space when I using the normal way.

2.1.5. *Poincaré Dual of a Closed Oriented Submanifold.* Let M be a oriented manifold of dimension n ans S a closed oriented submanifold of dimension k; here by "closed" we mean as a subspace of M.

To every closed oriented submanifold $i : S \hookrightarrow M$ of dimension k, one can associated a unique cohomology class $[\eta_S]$ in $H^{n-k}(M)$, called its Poincaré dual, as follows.

Let ω be a closed k-form with compact support on M. Since S is closed in M, $\mathrm{Supp}(\omega|_S)$ is closed not only in S, but also in M. $i^*\omega$ also has compact support on S, so the integral $\int_S i^*\omega$ is defined. By stokes's theorem integrations over S induces a linear functional on $H_c^k(M)$, so Poincaré duality tells us that the integration over S corresponds to a unique cohomology class $[\eta_S[$ in $H^{n-k}(M)$. By definition, the Poincaré dual of S is the unique cohomology class in $H^{n-k}(M)$ satisfying

$$\int_S i^*\omega = \int_M \omega \wedge \eta_S$$

For any $\omega \in H_c^k(M)$.

Now suppose S is a compact oriented submanifold of dimension k in M. Since a compact subset of a Hausdorff space is closed, S is also a closed oriented submanifold and hence has a Poincaré dual $\eta_S \in H^{n-k}(M)$. This η_S we will call the closed Poincaré dual of S, to distinguish it from the compact Poincaré dual to be defined below.

Because S is compact, one can in fact integrate over S not only k-forms with compact support on M, but any k-form on M. In this way S defines a linear functional on $H^k(M)$ and by Poincaré duality corresponds to a unique cohomology class $[\eta_s']$ in H_c^{n-k}, the compact Poincaré dual of S. We must assume here that M has a finite good cover; otherwise, the duality $(H^k(M))^* \cong H_c^{n-k}(M)$ does not hold. The compact Poincaré dual $[\eta_S']$ is uniquely characterized by

$$\int_S i^*\omega = \int_M \omega \wedge \eta_S'$$

For any $\omega \in H^k(M)$. If the above holds for any closed k-form ω, then it certainly holds for any closed k-form with compact support. So as a form, η_S' is also the closed Poincaré dual of S, i.e., the natural map $H_c^{n-k}(M) \to H^{n-k}(M)$ sends the compact Poincaré dual to closed Poincaré dual.

However, as cohomology class, $[\eta_S] \in H^{n-k}(M)$ and $[\eta_S'] \in H_c^{n-k}(M)$ could be quite different, as the following examples demonstrate.

Example 2.1.7 (Poincaré dual of a point p on $\mathbb{R}^n$)**.** *Since $H^n(\mathbb{R}^n) = 0$, the closed Poincaré dual η_P is trivial and can be represented by any closed n-form on M, but the compact Poincaré dual is the nontrivial class in $H_c^k(\mathbb{R})$ represented by a bump form with total integral* 1*.*

2.2. **The Thom Isomorphism.** So far we have encountered two kinds of C^∞ invariants of a manifold, de Rham cohomology and compactly supported cohomology. For vector bundles there is another invariant, namely, cohomology with compact support in the vertical direction. The Thom isomorphism is a statement about this last-named cohomology.

2.2.1. *Vector Bundles and the Reduction of Structure Groups.*

Definition 2.2.1. *Let $\pi : E \to M$ be a surjective map of manifolds whose fiber $\pi^{-1}(x)$ is a vector space for every $x \in M$. The map π is a C^∞ real vector bundle of rank n if there is an open cove $\{U_\alpha\}$ of M and fiber-preserving diffeomorphisms*

$$\phi_\alpha : E|_{U_\alpha} = \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$$

which are linear isomorphisms on each other. The maps

$$\phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

are vector-space automorphisms of $\mathbb{R}^n$ in each fiber and hence give rise to maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})$$

defined as $g_{\alpha\beta}(x) = \phi_\alpha \phi_\beta^{-1}|_{\{x\}\times\mathbb{R}^n}$

Definition 2.2.2. *Let U be an open set in M. A map $s : U \to E$ is a section of the vector bundle E over M if $\pi \circ s$ is the identity on U. The space of all section over U is written $\Gamma(U, E)$.*

Definition 2.2.3. *A collection of sections $s_1, \dots, s_n$ over an open set U in M is a frame on U if for every point x in U, $s_1(x), \dots, s_n(x)$ form a basis of the vector space $\pi^{-1}(x)$.*

The transition functions $\{g_{\alpha\beta}\}$ of a vector bundle satisfy the cocycle condition. The cocycle $\{g_{\alpha\beta}\}$ depends on the choice of the trivialization.

3. Čech-de Rham Complex

3.1. **The Generalized Mayer-Vietoris Principle.** Mayer-Vietoris sequence allows us to compute the cohomology of the whole space M using an open covering $\mathfrak{U}$ consisting of two open subsets U, V. And in fact this argument can be generalized to countably many open subsets.

To make this generalization, we first reformulate the Mayer-Vietoris sequence in language of double complex $C^*(\mathfrak{U}, \Omega^*) = \bigoplus K^{p,q} = \bigoplus C^p(\mathfrak{U}, \Omega^q)$, where

$$\begin{aligned} K^{0,q} &= \Omega^q(U) \oplus \Omega^q(V) \\ K^{1,q} &= \Omega^q(U \cap V) \\ K^{2,q} &= 0, \quad p \geq 2 \end{aligned}$$

Remark 3.1.1. If you regard $\mathfrak{U}$ as a open covering consisting of countably many open subsets as the following way

$$\mathfrak{U} = \{U, V, \varnothing, \varnothing, \dots\}$$

And you can guess that we can generalize above double complex by considering the intersection of more open subsets, since here intersection of any three or more open subsets must be empty.

Note that there is two differential operators on double complex $C^*(\mathfrak{U}, \Omega^*)$, d in the vertical direction and difference operator δ in horizontal direction. Clearly d, δ commute with each other.

It's necessary for us to consider a more general double complex, since it's crucial ingredient of spectral sequences we will discuss later. For a double complex $K^{*,*}$ we means it's a complex with two differential operators d, δ in vertical and horizontal direction, and we can make it into a singly graded complex K^*[4] in the following way: Consider $K^n = \bigoplus_{p+q=n} K^{p,q}$ and differential operator is defined by

$$D = \delta + (-1)^p \mathrm{d}, \quad \text{on } K^{p,q}$$

Let's check D is a differential operator. Indeed, since K^n is a direct sum of $K^{p,q}$, so it suffice to take $\alpha^{p,q} \in K^{p,q}$ and check $D^2(\alpha^{p,q})$:

$$\begin{aligned} D^2(\alpha^{p,q}) &= D(\delta\alpha^{p,q} + (-1)^p \mathrm{d}\alpha^{p,q}) \\ &= (-1)^{p+1} \mathrm{d}\delta\alpha^{p,q} + (-1)^p \delta\mathrm{d}\alpha^{p,q} \\ &= 0 \end{aligned}$$

Here we use the fact $\delta\alpha^{p,q} \in K^{p+1,q}$, $\mathrm{d}^2 = \delta^2 = 0$ and d commutes with δ.

Remark 3.1.2. $(-1)^p$ is crucial here, otherwise we won't get $D^2 = 0$.

Now Mayer-Vietoris sequence is reformulated as following

Theorem 3.1.3. *The double complex $C^*(\mathfrak{U}, \Omega^*)$ commutes the de Rham cohomology of M.*

[4]Sometimes call it total complex.

Proof. The hallmark of proof is we need to note the following thing: A q-cochain α in double complex may have two components

$$\alpha = \alpha_0 + \alpha_1, \quad \alpha_0 \in K^{0,q}, \alpha_1 \in K^{1,q-1}$$

And we need to show that it's D-cohomologous to a cochain with only top components. And this fact holds mainly rely on the exactness of Mayer-Vietoris sequence:

$$0 \to \Omega^*(U \cup V) \xrightarrow{r} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{\delta} \Omega^*(U \cap V) \to 0$$

By surjectivity of δ we can take $\beta \in \Omega^*(U) \oplus \Omega^*(V)$ such that $\delta\beta = \alpha$, then $\alpha - D\beta$ only contains top components. $\square$

Now we generalize our ideal as what we have mentioned in Remark 3.1.1. Fix an open covering $\mathfrak{U} = \{U_\alpha\}_{\alpha \in J}$ of M. And use $U_{\alpha\beta}$ to denote the intersection $U_\alpha \cap U_\beta$, similarly for $U_{\alpha\beta\gamma}$. Clearly there is a natural inclusion $\partial_\alpha : U_{\alpha\beta\gamma} \to U_{\beta\gamma}$. Thus we will have the following sequence of differential forms:

$$\Omega^*(M) \xrightarrow{r} \prod \Omega^*(U_{\alpha_0}) \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \prod_{\alpha_0 < \alpha_1} \Omega^*(U_{\alpha_0\alpha_1}) \overset{\delta_0}{\underset{\delta_2}{\overset{\delta_1}{\rightrightarrows}}} \prod_{\alpha_0 < \alpha_1 < \alpha_2} \Omega^*(U_{\alpha_0\alpha_1\alpha_2}) \to \dots$$

Recall difference operator δ we defined in Mayer-Vietoris is $\delta = \delta_0 - \delta_1$, here we also define generalized difference operator in a similar way: For $\omega \in \prod \Omega^q(U_{\alpha_0 \dots \alpha_p})$, we define

$$(\delta\omega)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \dots \widehat{\alpha_i} \dots \alpha_{p+1}}$$

We also need this alternating signal in order to make $\delta^2 = 0$, that is making above sequence a complex.

Remark 3.1.4. It's a little weird in our definition about difference operator δ we require indices in $\omega_{\alpha_0 \dots \alpha_p}$ are arranged in an increasing order, since by definition $U_{\alpha\beta} = U_{\beta\alpha}$. More generally we will allow indices in any order[5], by setting

$$\omega_{\dots\alpha\dots\beta\dots} = -\omega_{\dots\beta\dots\alpha\dots}$$

But we need to check it's consistent with definition of difference operator, that is Exercise 8.4 in Bott-Tu, and we check this as follows

Proof. We use the definition formula of difference operator to compute $(\delta\omega)_{\dots\beta\dots\alpha}$. For terms $\omega_{\dots\beta\dots\widehat{\alpha_i}\dots\alpha\dots}$, it's clearly that it equals to $-\omega_{\dots\alpha\dots\widehat{\alpha_i}\dots\beta\dots}$ by definition. So it suffice to show

$$(-1)^i \omega_{\dots\widehat{\beta}\dots\alpha\dots} + (-1)^j \omega_{\dots\beta\dots\widehat{\alpha}\dots} = (-1)\{(-1)^i \omega_{\dots\widehat{\alpha}\dots\beta\dots} + (-1)^j \omega_{\dots\alpha\dots\widehat{\beta}\dots}\}$$

[5]And we should do this, since sometimes it's difficult to ask these indices to be arranged in an increaing order, we will see this in later.

But by definition we can see

$$\omega_{\dots\widehat{\alpha}\dots\beta\dots} = (-1)^{j-i-1}\omega_{\dots\beta\dots\widehat{\alpha}\dots}$$
$$\omega_{\dots\alpha\dots\widehat{\beta}\dots} = (-1)^{j-i-1}\omega_{\dots\widehat{\beta}\dots\alpha\dots}$$

This completes the proof. □

As what we have seen in Mayer-Vietoris sequence, the exactness of Mayer-Vietoris plays an important role in proving isomorphism between cohomology of total complex and de Rham cohomology of the whole space. Here we still desire our generalized Mayer-Vietoris sequence is also exact.

Proposition 3.1.5 (The generalized Mayer-Vietoris sequence)**.** *The following sequence is exact*

$$\Omega^*(M) \xrightarrow{r} \prod \Omega^*(U_{\alpha_0}) \xrightarrow{\delta} \prod \Omega^*(U_{\alpha_0\alpha_1}) \xrightarrow{\delta} \prod \Omega^*(U_{\alpha_0\alpha_1\alpha_2}) \to \dots$$

Proof. The proof here is quite similar to what we have done in Mayer-Vietoris sequence, that is using partition of unity to construct coboundaries we desired.

But if we regard the construction process as a homotopy operator, things become interesting. We construct an operator

$$K : \prod \Omega^*(U_{\alpha_0\dots\alpha_p}) \to \prod \Omega^*(U_{\alpha_0\dots\alpha_{p-1}})$$

And we showed that

$$\delta K + K\delta = 1$$

In other words, we showed that identity map is homotopic to zero map, but homotopic chain maps induce the same map between cohomology groups. So cohomology of this complex is isomorphic to trivial group, that is it's exact. □

By the same method, we can show

Proposition 3.1.6 (Generalized Mayer-Vietoris Principle)**.** *The double complex $C^*(\mathfrak{U}, \Omega^*)$ computes the cohomology of M. Furthermore, restriction map $r : \Omega^*(M) \to C^*(\mathfrak{U}, \Omega^*)$ induces an isomorphism in cohomology.*

Remark 3.1.7. The philosophy here is that if every row of an argumented double complex is exact, then the cohomology of total complex will compute the cohomology of the first column. We will revisit this ideal in spectral sequences.

Note that the rows and columns are symmetric, so we may desire if we make this double complex into a column argumented double complex, and all columns are exact, then the cohomology of total complex will reflect the cohomology of the first row.

It's natural to argument each column by the kernel of the bottom d, it consists of the locally constant functions defined on $U_{\alpha_0\dots\alpha_p}$, and we denote it by $C^*(\mathfrak{U},\mathbb{R})$. We can write this complex explictly

$$C^0(\mathfrak{U},\mathbb{R}) \xrightarrow{\delta} C^1(\mathfrak{U},\mathbb{R}) \xrightarrow{\delta} C^2(\mathfrak{U},\mathbb{R}) \to \dots$$

We call the homology of this complex $H^*(\mathfrak{U},\mathbb{R})$, the Čech cohomology of the cover $\mathfrak{U}$. Note that it's a purely combinatorial object, since here we only care for the intersections of these open subsets, and it's computable.

But what's the condition for the columns are exact? The failure of p-th column to be exact is measured by

$$\prod H^q(U_{\alpha_0\dots\alpha_p})$$

So also by Poincaré lemma, if the open covering $\mathfrak{U}$ is a good cover, then all columns are exact, then we get

Theorem 3.1.8 (comparison theorem)**.** *If $\mathfrak{U}$ is a good cover of the manifold M, then the Čech cohomology of the cover $\mathfrak{U}$ computes the de Rham cohomology of M, that is*

$$H^*_{DR}(M) \cong H^*(\mathfrak{U},\mathbb{R})$$

3.2. **The Tic-Tac-Toe Proof of the Künneth formula.** Now let's use theory we have developed to revisit Künneth formula, but we prove it in a weaker assumption: We replace M has a finite good cover by F has a finite dimensional cohomology.

Before proving the Künneth formula, let's make some general remarks: Let $\pi : E \to M$ be a map of manifolds, and $\mathfrak{U}$ is an open covering of M, then $\pi^{-1}(\mathfrak{U})$ is clearly an open covering of E. But in general $U_\alpha \cap U_\beta = \varnothing$ is not equivalent to $\pi^{-1}U_\alpha \cap \pi^{-1}U_\beta = \varnothing$. but one direction $U_\alpha \cap U_\beta = \varnothing \implies \pi^{-1}U_\alpha \cap \pi^{-1}U_\beta = \varnothing$ always holds. Indeed, if not, take $x \in \pi^{-1}U_\alpha \cap \pi^{-1}U_\alpha$, then $\pi(x) \in U_\alpha \cap U_\beta$, a contradiction. But the other direction may fail if π is not surjective, since we can't control parts which doesn't lie in the image of π by considering its inverse.

So if π is surjective, then the combinatorial property of $\mathfrak{U}$ and $\pi^{-1}(\mathfrak{U})$ are same. The double complex of $\pi^{-1}\mathfrak{U}$ computes the cohomology of E, which can be related to the cohomology of M. This is a powerful ideal of Čech cohomology.

However, things are not quite easy, since although $\pi^{-1}\mathfrak{U}$ is good cover, $\mathfrak{U}$ may not be a good cover. But for the case of vector bundle $\pi : E \to M$, the "goodness" of the cover is preserved. So we have

$$H^*_{DR}(E) \cong H^*_{DR}(M)$$

where $E \to M$ is a vector bundle.

Proposition 3.2.1 (Künneth formula)**.** *If M and F are two manifolds and F has finite dimensional cohomology, then the de Rham cohomology of the product $M \times F$ is*

$$H^*(M \times F) = H^*(M) \otimes H^*(F)$$

Proof. It suffices to show

$$\begin{aligned}\pi_{\mathfrak{U}}^* : H^*(F) \otimes C^*(\mathfrak{U}, \Omega^*) &\to C^*(\pi^{-1}\mathfrak{U}, \Omega^*) \\ [\omega_\alpha] \otimes \phi &\mapsto \rho^*\omega_\alpha \wedge \pi^*\phi\end{aligned}$$

induces an isomorphism in D-cohomology. It's clear $\pi_{\mathfrak{U}}^*$ induces an isomorphism of d-cohomology of these complexes. And the claim holds from following lemma:

Lemma 3.2.2 (Acyclic Assembly Lemma)**.** *Whenever a homomorphism $f : K \to K'$ of double complex induces H_{d}-isomorphism, it also induces H_D-isomorphism.*

Proof. It's an important lemma in homological algebra, but here we can assume only finitely many columns of K and K' are nonzero, since double complexes we concern possess this property. We prove this by induction on the number n of nonzero columns of double complex: If $n = 1$, it's trivial since H_{d} is the same as H_D. If we have proven for $n < k$, then for $n = k$, consider the following exact sequence

$$\begin{tikzcd} 0 \arrow[r] & K_1 \arrow[r] \arrow[d, "f"] & K \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & K_1' \arrow[r] & K' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

where C, C' are first columns of K and K', and K_1, K_1' are subcomplexes of K, K' obtained by cutting out the first column. Then we get a long exact sequence of cohomology groups:

$$\begin{tikzcd} \cdots \to H^{i-1}(C') \arrow[r] & H^i(K_1') \arrow[r] & H^i(K') \arrow[r] & H^i(C') \arrow[r] & H^i(K_1') \to \ldots \\ \cdots \to H^{i-1}(C) \arrow[r] \arrow[u, "f^*"] & H^i(K_1) \arrow[r] \arrow[u, "f^*"] & H^i(K) \arrow[r] \arrow[u, "f^*"] & H^i(C) \arrow[r] \arrow[u, "f^*"] & H^i(K_1) \to \ldots \arrow[u, "f^*"] \end{tikzcd}$$

And Five lemma will tell the answer. □

□

3.3. **Čech cohomology of presheaf.**

Now let's talk about the philosophy behind what we have done. When we define $C^*(\mathfrak{U}, \Omega^*)$ and difference operator δ, the only two things we used are:

1. For any open subset U, we have a group of differential forms on U;
2. For two open subsets $V \subset V$, there is a natural restriction $\Omega^*(U) \to \Omega^*(V)$.

So in fact, we don't need the property of $\Omega^*(U)$ as a differential forms, it can be any group, which implies we can consider same thing for presheaf of abelian groups.

So instead of using Ω^*, we can defined Čech complex for any contravariant presheaf $\mathscr{F}$ as follows

$$0 \to C^0(\mathfrak{U}, \mathscr{F}) \xrightarrow{\delta} C^1(\mathfrak{U}, \mathscr{F}) \xrightarrow{\delta} C^2(\mathfrak{U}, \mathscr{F}) \to \dots$$

and the cohomology of this complex is denoted by $H^*(\mathfrak{U}, \mathscr{F})$. Similarly for a covariant presheaf $\mathscr{F}$, we can define

$$0 \leftarrow C_0(\mathfrak{U}, \mathscr{F}) \xleftarrow{\delta} C_1(\mathfrak{U}, \mathscr{F}) \xleftarrow{\delta} C_2(\mathfrak{U}, \mathscr{F}) \leftarrow \dots$$

and consider its homology $H_*(\mathfrak{U}, \mathscr{F})$.

However, it's not a beautiful definition, since our definition may depend on the choice of open covering $\mathfrak{U}$. So it's necessary for us to ask what will happen when we change the choice of open covering.

Lemma 3.1. *Given $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ an open cover and $\mathfrak{V} = \{V_\beta\}_{\beta \in J}$ a refinement, if ϕ, ψ are two refinement maps $J :\to I$, then there is a homotopy operator between $\phi^\#$ and $\psi^\#$.*

Proof. Define $K : C^q(\mathfrak{U}, \mathscr{F}) \to C^{q-1}(\mathfrak{V}, \mathscr{F})$ by

$$(K\omega)(V_{\beta_0 \dots \beta_{q-1}}) = \sum (-1)^i \omega(U_{\phi(\beta_0) \dots \phi(\beta_i) \psi(\beta_i) \dots \psi(\beta_{q-1})})$$

And we can check[6]

$$\psi^\# - \phi^\# = \delta K + K\delta$$

as follows: Take a cochain $\omega \in C^q(\mathfrak{U}, \mathscr{F})$, then

□

[6]An exercise you only check once in your whole life.

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