TORIC VARIETY

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Contents

Part 1. Basic theories of toric varieties

1. Preliminaries

1.1. Torus.

Definition 1.1.1 (torus)**.** A torus T is an affine variety isomorphic to $(\mathbb{C}^*)^n$, where T inherits a group structure from the isomorphism.

Definition 1.1.2 (character)**.** A character of a torus T is a morphism $\chi\colon T \to \mathbb{C}^*$ that is a group homomorphism.

Definition 1.1.3 (one-parameter subgroup)**.** A one-parameter subgroup of a torus T is a morphism $\lambda\colon \mathbb{C}^* \to T$ that is a group homomorphism.

Example 1.1.1. All characters of $(\mathbb{C}^*)^n$ arise from

$$\chi^{(a_1,\dots,a_n)}\colon (t_1,\dots,t_n) \mapsto t_1^{a_1}\dots t_n^{a_n},$$

and all one-parameter subgroups of $(\mathbb{C}^*)^n$ arise from

$$\lambda^{(b_1,\dots,b_n)}\colon t \mapsto (t^{b_1},\dots,t^{b_n}),$$

where $(a_1,\dots,a_n),(b_1,\dots,b_n) \in \mathbb{Z}^n$.

Theorem 1.1.1. Let T_N be a n-torus with group M consisting of characters and group N consisting of one-parameter subgroups. Then

(1) M, N are lattices of rank n.
(2) M, N are dual lattices, that is $N \cong \mathrm{Hom}(M,\mathbb{Z})$ and $N \cong \mathrm{Hom}(N,\mathbb{Z})$.
(3) $T_N \cong \operatorname{Spec}\mathbb{C}[M]$ as varieties.
(4) $T_N \cong N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong \mathrm{Hom}(M,\mathbb{C}^*)$ as groups.

1.2. Affine semigroups.

Definition 1.2.1 (affine semigroup)**.** An affine semigroup S is a semigroup group such that

(1) The binary operation on S is communicative.
(2) The semigroup is finitely generated.
(3) The semigroup can be embedded in a lattice M.

Example 1.2.1. $\mathbb{N}^n \subseteq \mathbb{Z}^n$ is an affine semigroup.

Example 1.2.2. Given a finite set $\mathcal{A}$ of a lattice M, $\mathbb{N}\mathcal{A} \subseteq M$ is an affine semigroup.

Definition 1.2.2 (semigroup algebra)**.** Let $S \subseteq M$ be an affine semigroup. The semigroup algebra $\mathbb{C}[S]$ is the vector space over $\mathbb{C}$ with S as basis and multiplication is induced by the semigroup structure.

Remark 1.2.1. To make this precise, we write

$$\mathbb{C}[S] = \{\sum_{m\in S} c_m\chi^m \mid c_m \in C \text{ and } c_m = 0 \text{ for all but finitely many } m\}$$

with multiplication given by

$$\chi^m \cdot \chi^{m'} = \chi^{m+m'}.$$

If $S = \mathbb{N}\mathcal{A}$ for $\mathcal{A} = \{m_1, \ldots, m_s\}$, then $\mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, \ldots, \chi^{m_s}]$.

Example 1.2.3. The affine semigroup $\mathbb{N}^n \subseteq \mathbb{Z}^n$ gives the polynomial ring

$$\mathbb{C}[\mathbb{N}^n] = \mathbb{C}[x_1, \ldots, x_n]$$

where $x_i = \chi^{e_i}$ and $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{Z}^n$.

Example 1.2.4. If $e_1, \ldots, e_n$ is a basis of a lattice M, then M is generated by $\mathcal{A} = \{\pm e_1, \ldots, \pm e_n\}$ as an affine semigroup, and the semigroup algebra gives the Laurent polynomial ring

$$\mathbb{C}[M] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

where $x_i = \chi^{e_i}$.

For torus T_N with character group M, there is a natural action of T_N on the semigroup algebra $\mathbb{C}[M]$ as follows: For $t \in T_N$ and $\chi^m \in M$, $t \cdot \chi^m$ is defined by $p \mapsto \chi^m(t^{-1}p)$ for $p \in T_N$.

Theorem 1.2.1. Let $A \subseteq \mathbb{C}[M]$ be a subspace stable under the action of T_N. Then

$$A = \bigoplus_{\chi^m \in A} \mathbb{C} \cdot \chi^m.$$

Proof. See Lemma 1.1.16 in [CLS11]. ☐

1.3. **Strongly convex rational polyhedral cones.** From now on, unless otherwise specified, we always assume M, N are dual lattices with associated $\mathbb{R}$-vector spaces $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, and the pairing between M and N is denoted by $\langle \text{-}, \text{-} \rangle$.

1.3.1. *Convex polyhedral cones.*

Definition 1.3.1 (convex polyhedral cone)**.** Let $S \subseteq N_{\mathbb{R}}$ be a finite subset. A convex polyhedral cone in $N_{\mathbb{R}}$ generated by S is a set of the form

$$\sigma = \operatorname{Cone} S = \{\sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0\} \subseteq N_{\mathbb{R}}.$$

Notation 1.3.1. $\operatorname{Cone}(\varnothing) = \{0\}$.

Remark 1.3.1. A convex polyhedral cone is convex, that is $x, y \in \sigma$ implies $\lambda x + (1 - \lambda) y \in \sigma$ for all $0 \leq \sigma \leq 1$, and it's a cone, that is $x \in \sigma$ implies $\lambda x \in \sigma$ for all $\lambda \geq 0$. Since we will only consider convex cones, the cones satisfying Definition 1.3.1 will be called polyhedral cone for convenience.

Definition 1.3.2 (dimension)**.** The dimension of a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is the dimension of the smallest subspace $W \subseteq N_{\mathbb{R}}$ containing σ, and such W is called the span of σ.

Definition 1.3.3 (dual cone)**.** Let $\sigma \subseteq N_{\mathbb{R}}$ be a polyhedral. The dual cone is defined by

$$\sigma^{\vee} := \{u \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}.$$

Definition 1.3.4 (hyperplane)**.** Given $m \in M_{\mathbb{R}}$, the hyperplane given by m is defined by

$$H_m := \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle = 0\} \subseteq N_{\mathbb{R}},$$

and the closed half-space given by m is defined by

$$H_m^+ := \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle \geq 0\} \subseteq N_{\mathbb{R}}.$$

Definition 1.3.5 (supporting hyperplane)**.** The supporting hyperplane of a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is a hyperplane H_m such that $\sigma \subseteq H_m^+$, and H_m^+ is called a supporting half-space.

Remark 1.3.2. H_m is a supporting hyperplane of σ if and only if $m \in \sigma^{\vee}$, and if $m_1, \ldots, m_s$ generates $\sigma^{\vee}$, then

$$\sigma = H_{m_1}^+ \cap \cdots \cap H_{m_s}^+.$$

Thus every polyhedral cone is an intersection of finitely many closed half-spaces.

Definition 1.3.6 (face)**.** A face of a polyhedral cone σ is $\tau = H_m \cap \sigma$ for some $m \in \sigma^{\vee}$, written $\tau \preceq \sigma$. Faces $\tau \neq \sigma$ are called proper faces, written $\tau \prec \sigma$.

Definition 1.3.7 (facet and edge)**.** A facet of a polyhedral cone σ is a face of codimension one, and an edge of σ is a face of dimension one.

Theorem 1.3.1. *Suppose σ is a polyhedral cone. Then*

(1) *Every face of σ is a polyhedral cone.*
(2) *An intersection of two faces of σ is again a face of σ.*
(3) *A face of a face of σ is again a face of σ.*
(4) *If $\tau \preceq \sigma$, $v, w \in \sigma$ and $v + w \in \tau$, then $v, w \in \tau$.*
(5) *Every face of $\sigma^{\vee}$ can be uniquely written as $\sigma^{\vee} \cap \tau^{\perp}$, where $\tau \preceq \sigma$ and*

$$\tau^{\perp} = \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle = 0, \forall u \in \tau\}$$

1.3.2. *Strongly convex.*

Definition 1.3.8 (strongly convex)**.** A polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is strongly convex if $\{0\}$ is a face of σ.

Theorem 1.3.2. *Let $\sigma \subseteq N_{\mathbb{R}}$ be a polyhedral cone. Then the following statements are equivalent:*

(1) *σ is strongly convex.*
(2) *$\{0\}$ is a face of σ.*
(3) *σ contains no positive-dimensional subspace of $N_{\mathbb{R}}$.*
(4) *$\sigma \cap (-\sigma) = \{0\}$.*
(5) *$\dim \sigma^{\vee} = n$.*

1.3.3. *Rational polyhedral cones.*

Definition 1.3.9 (rational)**.** A polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is rational if $\sigma = \mathrm{Cone}(S)$ for some finite subset $S \subseteq N$.

Definition 1.3.10 (ray generator)**.** Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone and ρ be an edge of σ. The unique generator of semigroup $\rho \cap N$ is called ray generator of ρ, written u_{ρ}.

Remark 1.3.3. The ray generator is well-defined: Since σ is strongly convex, one has edge of σ is a ray as $\{0\}$ is its face, and since σ is rational, the semigroup $\rho \cap N$ is generated by a unique element, otherwise contradicts to the fact ρ is an edge, that is it's of dimension one.

Definition 1.3.11 (smooth and simplicial)**.** Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone.
(1) σ is smooth if its ray generators form part of a $\mathbb{Z}$-basis of N.
(2) σ is simplical if its ray generators are linearly independent over $\mathbb{R}$.

1.4. **Polytope.**

Definition 1.4.1 (polytope)**.** A polytope in $N_{\mathbb{R}}$ is a set of the form

$$P = \mathrm{Conv}(S) = \{\sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0, \sum_{u \in S} \lambda_u = 1\} \subseteq N_{\mathbb{R}},$$

where $S \subseteq N_{\mathbb{R}}$ is finite. We say P is the convex hull of S.

Remark 1.4.1. A polytope $P \subseteq N_{\mathbb{R}}$ gives a polyhedral cone $C(P) \subseteq N_{\mathbb{R}} \times \mathbb{R}$, called the cone of P and defined by

$$C(P) = \{\lambda \cdot (u, 1) \in N_{\mathbb{R}} \times \mathbb{R} \mid u \in P, \lambda \geq 0\}.$$

If $P = \mathrm{Conv}(S)$, then one can also describe this as $C(P) = \mathrm{Cone}(S \times \{1\})$.

2. Fans and toric variety

2.1. Semigroup algebras and affine toric varieties.

Definition 2.1.1 (affine toric variety)**.** An affine toric variety is an irreducible affine variety V containing a torus $T_N \cong (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T_N on itself extends to an algebraic action of T_N on V.

Proposition 2.1.1 (Gordan's lemma)**.** Let $\sigma \subseteq N_\mathbb{R}$ be a rational polyhedral cone. The semigroup $S_\sigma := \sigma^\vee \cap M$ is finitely generated.

Proof. See Proposition 1.2.17 in [CLS11]. □

Theorem 2.1.1. Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex rational polyhedral cone with semigroup $S_\sigma = \sigma^\vee \cap M$. Then

$$U_\sigma := \mathrm{Spec}(\mathbb{C}[S_\sigma])$$

is a normal affine toric variety with torus $T_N \cong \mathrm{Spec}\,\mathbb{C}[M]$. Conversely, for any normal affine toric variety X, there exists a strongly convex rational polyhedral cone $\sigma \subseteq N_\mathbb{R}$ such that $X \cong U_\sigma$.

Proof. If $\sigma \subseteq N_\mathbb{R}$ is a strongly convex rational polyhedral cone, then by Proposition 2.1.1 one has S_σ is finitely generated. Suppose $\mathcal{A} = \{m_1, \dots, m_s\}$ is a generator of S_σ. □

Proposition 2.1.2. Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex rational polyhedral cone and σ be a face of σ written as $\tau = H_m \cap \sigma$, where $m \in \sigma^\vee \cap M$. Then the semigroup algebra $\mathbb{C}[S_\tau]$ is the localization of $\mathbb{C}[S_\sigma]$ at $\chi^m \in \mathbb{C}[S_\sigma]$.

Proof. See Proposition 1.3.16 in [CLS11]. □

2.2. The toric variety of a fan.

Definition 2.2.1 (toric variety)**.** A toric variety is an irreducible variety X containing a torus $T_N \cong (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T_N on itself extends to an algebraic action of T_N on X.

Definition 2.2.2 (fan)**.** A fan Σ in $N_\mathbb{R}$ is a finite collection of cones $\sigma \subseteq N_\mathbb{R}$ such that

(1) Every $\sigma \in \Sigma$ is strongly convex rational polyhedral cone.
(2) For all $\sigma \in \Sigma$, each face of σ is also in Σ.
(3) For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each.

Notation 2.2.1. $\Sigma(r)$ is the set of r-dimensional cones of Σ.

Now let's show how the cones in any fan give the combinatorial data necessary to glue a collection of affine toric varieties to yield an abstract toric variety.

(1) Firstly, by Theorem 2.1.1 one has each cone $\sigma \in \Sigma$ gives the affine toric variety
$$U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma]) = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M]).$$
If τ is a face of σ, then there exists some $m \in \sigma^\vee$ such that $\tau = \sigma \cap H_m$, and by Proposition 2.1.2 one has $\mathbb{C}[S_\tau] = (\mathbb{C}[S_\sigma])_{\chi^m}$, which implies
$$U_\tau = (U_\sigma)_{\chi^m}.$$
(2) Secondly, if $\tau = \sigma_1 \cap \sigma_2$, then there exists some $m \in \sigma_1^\vee \cap (-\sigma_2)^\vee \cap M$ such that
$$\sigma_1 \cap H_m = \tau = \sigma_2 \cap H_m.$$
This shows
$$U_{\sigma_1} \supseteq (U_{\sigma_1})_{\chi^m} = U_\tau = (U_{\sigma_2})_{\chi^{-m}} \subseteq U_{\sigma_2}.$$
Thus we have an isomorphism
$$g_{\sigma_2\sigma_1} \colon (U_{\sigma_1})_{\chi^m} \cong (U_{\sigma_2})_{\chi^{-m}}.$$
(3) Finally, we use isomorphisms in (2) to glue the collection of affine toric varieties obtained from a fan to construct the toric variety X_Σ associated to the fan Σ.

Theorem 2.2.1. Let Σ be a fan in $N_\mathbb{R}$. The variety X_Σ is normal separated toric variety.

Conversely, any normal separated toric variety comes from a fan, but it's a highly non-trivial fact.

Theorem 2.2.2. Let X be a normal separated toric variety with torus T_N. Then there exists a fan Σ in $N_\mathbb{R}$ such that $X \cong X_\Sigma$.

Proof. See [Sum74] and [Sum75]. □

2.3. **Examples.**

Example 2.3.1. Consider the fan Σ in $N_\mathbb{R} = \mathbb{R}^2$ in Figure 1, where $N = \mathbb{Z}^2$ has standard basis e_1, e_2. The fan has three 2-dimensional cones $\sigma_0 =$

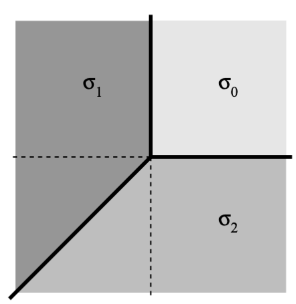

FIGURE 1. The fan Σ for $\mathbb{P}^2$

$\mathrm{Cone}(e_1, e_2), \sigma_1 = \mathrm{Cone}(-e_1 - e_2, e_2)$ and $\sigma_2 = \mathrm{Cone}(e_1, -e_1 - e_2)$, together with the three rays $\tau_{ij} = \sigma_i \cap \sigma_j$ for $i \neq j$. The toric variety X_Σ is covered by the affine opens

$$\begin{aligned} U_{\sigma_0} &= \operatorname{Spec}(\mathbb{C}[S_{\sigma_0}]) \cong \operatorname{Spec}(\mathbb{C}[x, y]) \\ U_{\sigma_1} &= \operatorname{Spec}(\mathbb{C}[S_{\sigma_1}]) \cong \operatorname{Spec}(\mathbb{C}[x^{-1}, x^{-1}y]) \\ U_{\sigma_2} &= \operatorname{Spec}(\mathbb{C}[S_{\sigma_2}]) \cong \operatorname{Spec}(\mathbb{C}[xy^{-1}, y^{-1}]). \end{aligned}$$

Moreover, the gluing data on the coordinate rings is given by

$$\begin{aligned} g_{10}^*\colon\ \mathbb{C}[x, y]_x &\cong \mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}} \\ g_{20}^*\colon\ \mathbb{C}[x, y]_y &\cong \mathbb{C}[xy^{-1}, y^{-1}]_{y^{-1}} \\ g_{21}^*\colon\ \mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}y} &\cong \mathbb{C}[xy^{-1}, y^{-1}]_{xy^{-1}} \end{aligned}$$

It's easy to see if we use usual homogenous coordinates (x_0, x_1, x_2) on $\mathbb{P}^2$, then $x \mapsto x_1/x_0$ and $y \mapsto x_2/x_0$ identify the standard affine open $U_i \subseteq \mathbb{P}^2$ with $U_{\sigma_i} \subseteq X_\Sigma$. Hence we have recovered $\mathbb{P}^2$ as a toric variety.

Example 2.3.2. Let $r \in \mathbb{N}$ and consider the fan Σ_r in $N_\mathbb{R} = \mathbb{R}^2$ consisting of the four cones σ_i shown in Figure 2, together with all of their faces.

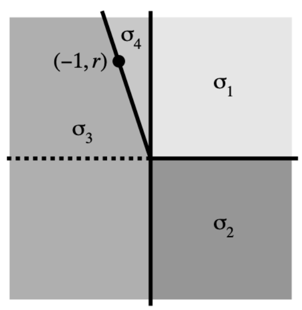

FIGURE 2. The fan Σ for Hirzebruch surface

2.4. Terminologies.

Definition 2.4.1. Let $\Sigma \subseteq N_\mathbb{R}$ be a fan.

(1) Σ is smooth if every cone in Σ is smooth.
(2) Σ is simplical if every cone in Σ is simplical.
(3) Σ is complete if its support $|\Sigma| := \bigcup_{\sigma\in\Sigma} \sigma$ is all of $N_\mathbb{R}$.

Theorem 2.4.1. Let X_Σ be the toric variety defined by a fan $\Sigma \subseteq N_\mathbb{R}$. Then

(1) X_Σ is smooth if and only if Σ is smooth.
(2) X_Σ is an orbifold[1] if and only if Σ is simplical.
(3) X_Σ is a complete variety if and only if Σ is complete.

[1] X_Σ is an orbifold if X_Σ has only finite quotient singularities.

3. The orbit-cone correspondence

3.1. Baby example.

3.2. The orbit-cone correspondence.

Theorem 3.2.1 (orbit-cone correspondence)**.** Let X_Σ be the toric variety of the fan Σ in $N_\mathbb{R}$. Then

(1) There is a bijective correspondence

$$\{\text{cones } \sigma \text{ in } \Sigma\} \longleftrightarrow \{T_N\text{-orbits in } X_\Sigma\}$$
$$\sigma \longleftrightarrow O(\sigma) \cong \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*).$$

(2) Let $n = \dim N_\mathbb{R}$. For each cone $\sigma \in \Sigma$, $\dim O(\sigma) = n - \dim \sigma$.

(3) The affine open subsets U_σ is the union of orbits

$$U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau).$$

(4) $\tau \preceq \sigma$ if and only if $O(\sigma) \subseteq \overline{O(\tau)}$, and

$$\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma),$$

where $\overline{O(\tau)}$ denotes the closure in both classical and Zariski topologies.

3.3. Orbit closure as toric varieties.

Proposition 3.3.1. Let Σ be a fan in $N_\mathbb{R}$ and $\tau \in \Sigma$. Then the orbit closure $\overline{O(\tau)}$ has a toric variety structure.

4. Divisors on toric variety

4.1. **Weil divisors on toric varieties.**

Let X_Σ be the toric variety of fan in $N_\mathbb{R}$ with $\dim N_\mathbb{R} = n$. In this section we will use torus-invariant prime divisors and characters to give a lovely description of class group of X_Σ.

4.1.1. *The divisor of a character.* By the orbit-cone correspondence, $\rho \in \Sigma(1)$ gives the codimension one orbit $O(\rho)$ whose closure $\overline{O(\rho)}$ admits a codimension one toric subvariety structure by Proposition 3.3.1. Thus $\overline{O(\rho)}$ gives a T_N-invariant prime divisor on X_Σ. To emphasize that $\overline{O(\rho)}$ is a divisor we will denote it by D_ρ for convenience. Then D_ρ gives the DVR $\mathcal{O}_{X_\Sigma, D_\rho}$ with valuation

$$\nu_\rho\colon \mathbb{C}(X_\Sigma)^* \to \mathbb{Z}.$$

Recall that any ray $\rho \in \Sigma(1)$ has a minimal generator $u_\rho \in \rho \cap N$, and also note that when $m \in M$, the character $\chi^m\colon T_N \to \mathbb{C}^*$ is a rational function in $\mathbb{C}(X_\Sigma)^*$ since T_N is Zariski open in X_Σ.

Proposition 4.1.1. Let u_ρ be the minimal generator of ray $\rho \in \Sigma(1)$ and χ^m be a character corresponding to $m \in M$. Then

$$\nu_\rho(\chi^m) = \langle m, u_\rho\rangle.$$

Proposition 4.1.2. For $m \in M$, the divisor of character χ^m is given by

$$\operatorname{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho\rangle D_\rho.$$

4.1.2. *Computing the class group.*

Theorem 4.1.1. There is the following exact sequence

$$M \to \operatorname{Div}_{T_N}(X_\Sigma) \to \operatorname{Cl}(X_\Sigma) \to 0,$$

where the first map is $m \mapsto \operatorname{div}(\chi^m)$ and the second sends a T_N-invariant divisor to its divisor class in $\operatorname{Cl}(X_\Sigma)$. Furthermore, one has the following exact sequence

$$0 \to M \to \operatorname{Div}_{T_N}(X_\Sigma) \to \operatorname{Cl}(X_\Sigma) \to 0$$

if and only if $\{u_\rho \mid \rho \in \Sigma(1)\}$ spans $N_\mathbb{R}$.

Example 4.1.1. The fan of $\mathbb{P}^n$ has ray generators given by $u_0 = -e_1 - \cdots - e_n$ and $u_1 = e_1, \ldots, u_n = e_n$. Thus the map $M \to \operatorname{Div}_{T_N}(\mathbb{P}^n)$ can be written as

$$\begin{aligned} \mathbb{Z}^n &\to \mathbb{Z}^{n+1} \\ (a_1, \ldots, a_n) &\mapsto (-a_1 - \cdots - a_n, a_1, \ldots, a_n). \end{aligned}$$

Using the map

$$\begin{aligned} \mathbb{Z}^{n+1} &\to \mathbb{Z} \\ (b_0, \ldots, b_n) &\mapsto b_0 + \cdots + b_n, \end{aligned}$$

one gets the exact sequence

$$0 \to \mathbb{Z}^n \to \mathbb{Z}^{n+1} \to \mathbb{Z} \to 0$$

which proves $\mathrm{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$.

Example 4.1.2.

4.2. **Cartier divisors on toric varieties.**

4.3. **The sheaf of a torus-invariant divisor.** Let D be a T_N-invariant divisor on a toric variety X_Σ. In this section we will give descriptions of the global sections $H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$.

Proposition 4.3.1. If D is a T_N-invariant Weil divisor on X_Σ, then

$$H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{\mathrm{div}(\chi^m)+D\geq 0} \mathbb{C}\cdot\chi^m$$

Remark 4.3.1. For $D = \sum_\rho a_\rho D_\rho$ and $m \in M$, $\mathrm{div}(\chi^m)+D \geq 0$ is equivalent to

$$\langle m, u_\rho\rangle + a_\rho \geq 0$$

for all $\rho \in \Sigma(1)$. If we define

$$P_D = \{m \in M_\mathbb{R} \mid \langle m, u_\rho\rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\},$$

then above proposition can be written as

$$H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{m\in P_D\cap M} \mathbb{C}\cdot\chi^m$$

Example 4.3.1. The fan Σ_2 for the Hirzebruch surface $\mathcal{H}_2$ has ray generators $u_1 = -e_1 + 2e_2, u_2 = e_2, u_3 = e_1$ and $u_4 = -e_2$. The corresponding divisors are D_1, D_2, D_3, D_4, and Example 4.1.2 implies that the classes of D_1 and D_2 are a basis of $\mathrm{Cl}(\mathcal{H}_2) \cong \mathbb{Z}^2$.

Example 4.3.2.

5. Line bundles on toric variety

6. Canonical divisors of toric variety

6.1. One-forms on toric varieties.

6.2. Differential forms on toric varieties.

6.3. The canonical sheaf of toric varieties.

Theorem 6.3.1. For a toric variety X_Σ, the canonical sheaf ω_{X_Σ} is given by

$$\omega_{X_\Sigma} \cong \mathcal{O}_{X_\Sigma}(-\sum_\rho D_\rho).$$

Thus $K_{X_\Sigma} = -\sum_\rho D_\rho$ is a torus-invariant canonical divisor on X_Σ.

Example 6.3.1. The canonical bundle of $\mathbb{P}^n$ is

$$\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$$

for all $n \geq 1$ since $\mathrm{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$ and thus $D_0 \sim D_1 \sim \cdots \sim D_n$.

Example 6.3.2. When we computed the class group of the Hirzebruch surface $\mathcal{H}_r$ in Example 4.1.2, we wrote divisors D_ρ as $D_1, \ldots, D_4$ and showed that

$$\begin{aligned} D_3 &\sim D_1 \\ D_4 &\sim rD_1 + D_2. \end{aligned}$$

Thus the canonical bundle can be written as

$$\omega_{\mathcal{H}_r} = \mathcal{O}_{\mathcal{H}_r}(-D_1 - D_2 - D_3 - D_4) = \mathcal{O}_{\mathcal{H}_r}(-(r+2)D_1 - 2D_2).$$

7. Sheaf cohomology of toric varieties

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