PRINCIPAL BUNDLE AND ITS APPLICATIONS

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Abstract.

Contents

0. Preface

0.1. **About this lecture.**

0.2. **Some notations.**

1. M is used to denote a smooth manifold, and $x \in M$ denotes its point.
2. TM and Ω_M^k denote tangent bundle and bundle of k-forms over M respectively.
3. v is used to denote vector in tangent space.
4. X is used to denote a vector field on M, then X_x denote the value of X at point $x \in M$, similarly for a k-form ω.
5. For a vector bundle E over M, $C^\infty(M, E)$ denotes its sections.
6. G is used to denote a Lie group, with Lie algebra $\mathfrak{g}$.
7. $\pi\colon P \to M$ is used to denote a principal G-bundle over M, and $p \in P$ denotes its point.
8. $\widetilde{X}$ is used to denote vector field on principal bundle P, so do $\widetilde{\omega}$ and $\widetilde{v}$.

Part 1. Principal bundle and its geometry

1. Principal bundle

1.1. A glimpse of fiber bundle.

Fix topological spaces E, B, F.

Definition 1.1.1 (fiber bundle)**.** A fiber bundle with fiber F over B is a surjective map $\pi\colon E \to B$ such that for any $p \in B$, there exists an open neighborhood $U \ni p$ and a homeomorphism φ such that the following diagram commutes

$$
\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_1"] \\
& U &
\end{tikzcd}
$$

We always use $F \to E \xrightarrow{\pi} B$ to denote this fiber bundle and

1. B is called base space;
2. $E_x = \pi^{-1}(x)$ is called the fiber of E at x;
3. (U, φ) is called a local trivialization at point p, and use $E|_U$ to denote $\pi^{-1}(U)$.

Example 1.1.1 (trivial bundle)**.** Consider $E = B \times F$ and $\pi\colon E \to B$ is just the projection onto the first summand.

Example 1.1.2. Consider $E = S^n$ and $B = \mathbb{RP}^n$, then natural map $\pi\colon E \to B$ is a fiber bundle with $\mathbb{Z}/2\mathbb{Z}$. It's clear that this fiber bundle is not trivial, since S^n is connected.

Example 1.1.3 (Hopf fibration)**.** Recall that

$$\mathbb{CP}^n = \{\text{the set of all complex lines through origin in } \mathbb{C}^{n+1}\}$$

Consider the canonical open covering $\{U_i\}$ of $\mathbb{CP}^n$, that is

$$U_i = \{[z_0 : \ldots : z_n] \mid z_i \neq 0\}$$

Now view $S^{2n+1} \subset \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$ as the set of all $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ with $|z_0|^2 + \cdots + |z_n|^2 = 1$. Then the projection map $\pi\colon \mathbb{C}^{n+1} - \{0\} \to \mathbb{CP}^n$ restricts to a surjective smooth map

$$\pi\colon S^{2n+1} \to \mathbb{CP}^n$$

We claim that it's a fiber bundle with fiber S^1. Indeed, by definition we have

$$\pi^{-1}(U_i) = \{(z_0, \ldots, z_n) \in S^{2n+1} \mid z_i \neq 0\}$$

and local trivialization map can be taken as

$$\begin{aligned} \varphi_i : \pi^{-1}(U_i) &\to U_i \times S^1 \\ z &\mapsto ([z_0 : \ldots : z_n], \frac{z_i}{|z_i|}) \end{aligned}$$

It's also not trivial which can be seen by considering their fundamental groups.

Example 1.1.4. The covering space is a fiber bundle with discrete set as fiber.

1.2. **Principal bundle.**

1.2.1. *Definitions.* Briefly speaking, given a Lie group G and a smooth manifold M, a principal G-bundle P is a fiber bundle with fiber G equipped with a suitable smooth right G-action on it. For a smooth right G-action we mean a smooth map

$$\begin{aligned} P \times G &\to P \\ (p, g) &\mapsto pg \end{aligned}$$

Definition 1.2.1 (principal G-bundle)**.** A principal G-bundle is a surjective smooth map $\pi\colon P \to M$ between smooth manifolds such that:

1. There is a smooth right G-action on P;
2. For all $x \in M$, $\pi^{-1}(x)$ is a G-orbit;
3. For all $x \in M$, there exists an open subset U_α and a diffeomorphism φ_α, which is called a local trivialization, such that the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times G \arrow[dl, "\pi_{U_\alpha}"] \\ & U_\alpha & \end{tikzcd}$$

   Furthermore, if we write $\varphi_\alpha(p) = (\pi(p), g_\alpha(p))$, then we require $g_\alpha(pg) = g_\alpha(p)g$ for any $g \in G$.

Remark 1.2.1. G acts on P freely and transitively, which can be seen from local trivialization.

Notation 1.2.1. $\mathcal{P}_G M$ is used to denoted the set of all principal G-bundles over M up to isomorphism.

Example 1.2.1. $S^n \to \mathbb{RP}^n$ is a $\mathbb{Z}/2\mathbb{Z}$-principal bundle, where $\mathbb{Z}/2\mathbb{Z}$ acts on S^n via $x \mapsto -x$.

Example 1.2.2. $S^{2n+1} \to \mathbb{CP}^n$ is a $U(1)$-principal bundle, where $U(1)$ acts on S^{2n+1} via $(z_0, z_1, \ldots, z_n) \mapsto (z_0 e^{i\theta}, z_1 e^{i\theta}, \ldots, z_n e^{i\theta})$.

Definition 1.2.2 (morphism between principal G-bundle)**.** For two principal G-bundles $(P, M, \pi), (P', M, \pi')$, a morphism between them is a G-equivariant smooth map $\varphi\colon P' \to P$ making the following diagram commute

$$\begin{tikzcd} P \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

Proposition 1.2.1. A morphism $\varphi\colon P \to P'$ between principal G-bundles over M is an isomorphism.

Proof. All information are encoded in the equivariance of φ and properties of principal G-bundle:

1. φ is injective: For any $p_1, p_2 \in P$, if $\varphi(p_1) = \varphi(p_2)$, then p_1, p_2 lie in same fiber, since above diagram commutes. If $p_1 = p_2 g$ for $g \in G$, then $\varphi(p_1) = \varphi(p_2)g$, which impiles $g = e$, since G acts on P' freely, that is $p_1 = p_2$;
2. φ is surjective: For any $p' \in P'$, if $\pi'(p') = x$, then $p' \in P'_x$. So choose an arbitrary element $p \in P_x$, there must be some $g \in G$ such that $\varphi(pg) = p'$, since P'_x is a G-orbit and φ is G-equivariant.

□

Definition 1.2.3 (trivial principal bundle)**.** A principal G-bundle P is called trivial, if there exists a principal G-bundle isomorphism $\varphi\colon P \to M \times G$.

1.2.2. *Structure group.* By (3) of Definition 1.2.1, there exists an open covering $\{U_\alpha\}$ together with G-equivariant diffeomorphism $\varphi_\alpha\colon U_\alpha \to U_\alpha \times G$. If $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \varnothing$, then

$$\begin{aligned} \varphi_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1}\colon U_{\alpha\beta} \times G &\to U_{\alpha\beta} \times G \\ (x, g) &\mapsto (x, g_\alpha(\varphi_\beta^{-1}(x, g))) \end{aligned}$$

If we denote

$$g_{\alpha\beta}(x) = g_\alpha(\varphi_\beta^{-1}(x, g))$$

then $g_{\alpha\beta}(x)$ is a G-equivariant diffeomorphism of G. But it's easy to show

$$\begin{aligned} G &\to \{f\colon G \to G \mid f \text{ is a } G\text{-equivariant diffeomorphism}\} \\ g &\mapsto (x \mapsto gx) \end{aligned}$$

is bijective, which impiles the transition functions of principal G-bundle valued in G, that is

$$g_{\alpha\beta}\colon U_{\alpha\beta} \to G$$

That is to say, the structure group of a principal G-bundle is G.

1.2.3. *Section.*

Definition 1.2.4 (global section)**.** A global section of principal G-bundle $\pi\colon P \to M$ is a smooth map $s\colon M \to P$ such that $\pi \circ s = \mathrm{id}$.

Proposition 1.2.2. A principal G-bundle P over M admits a section if and only if it is trivial[1].

Proof. If $s\colon M \to P$ is a smooth section, consider

$$\begin{aligned} \varphi\colon P &\to M \times G \\ p &\mapsto (\pi(p), g(p)) \end{aligned}$$

[1]This is in sharp contrast with vector bundles, which always admit sections.

where $g(p) \in G$ such that $p = s(\pi(p))g(p)$, it always exsits since the right action of G is transitive on each fiber and it is unique since the action is free on each fiber. Clearly, it's G-equivariant, since

$$\varphi(ph) = (\pi(ph), g(ph)) = (\pi(p), g(p)h)$$

and the last equality holds since

$$ph = s(\pi(ph))g(ph) = s(\pi(p))g(ph) = pg^{-1}(p)g(ph) \implies h = g^{-1}(p)g(ph)$$

Thus $\varphi\colon P \to M \times G$ is a morphism between principal G-bundles over M, so by Proposition 1.2.1, P is isomorphic to $M \times G$, that is P is trivial principal G-bundle. □

1.3. **Associated fiber bundle.**

Given a principal G-bundle $\pi\colon P \to M$ and a smooth manifold F admitting a smooth left G-action on it.

Proposition 1.1. The set $P \times_G F := P \times F/\sim$, where $(p, f) \sim (p', f')$ if and only if $p' = pg, f' = g^{-1}f$, admits a fiber bundle structure over M with fiber F.

Proof. Consider the map taking an equivalence class $[p, f]$ to $\pi(p)$. To see the local structure, since we already have the local structure of principal bundle P, i.e. for any $x \in M$, there exists open $U_\alpha \ni x$ and $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$. Now we define the local trivialization of $P \times_G V$ as

$$\begin{aligned} \varphi_\alpha^V : (P \times_G V)|_{U_\alpha} &\to U_\alpha \times V \\ (p, v) &\mapsto (\pi(p), g_\alpha(p)v) \end{aligned}$$

First note that this is well-defined, since

$$(pg, g^{-1}v) \mapsto (\pi(pg), g_\alpha(pg)g^{-1}v) = (\pi(p), g_\alpha(p)gg^{-1}v) = (\pi(p), g_\alpha(p)v)$$

And this map is one to one, and invertible, its inverse sends $(x, v) \in U_\alpha \times V$ to the equivalence class of $(\varphi_\alpha^{-1}(x, e), v)$. Directly check as follows

$$\begin{aligned} \varphi_\alpha^V(\varphi_\alpha^{-1}(x, e), v) &= (x, ev) \\ &= (x, v) \end{aligned}$$

since $\pi(\varphi_\alpha^{-1}(x, e)) = x$ and $g_\alpha(\varphi_\alpha^{-1}(x, e)) = e$. □

Remark 1.3.1 (transition function of associated bundle). Though we've found the local trivialization of $P \times_G V$, it's also necessary to see what does the transition functions look like.

Let $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ be local trivializations, with transition functions

$$\begin{aligned} \varphi_\alpha \circ \varphi_\beta^{-1} : U_{\alpha\beta} \times G &\to U_{\alpha\beta} \times G \\ (x, g) &\mapsto (x, g_{\alpha\beta}(x)g) \end{aligned}$$

then we can compute the transition functions of associated vector bundles as follows

$$\begin{aligned} \varphi_\alpha^V \circ (\varphi_\beta^V)^{-1} : U_{\alpha\beta} \times V &\to U_{\alpha\beta} \times V \\ (x, v) &\mapsto (\varphi_\beta^{-1}(x, e), v) \mapsto (x, g_{\alpha\beta}(x)v) \end{aligned}$$

Example 1.3.1 (associated vector bundle)**.** Now let's consider a special case, that is associated vector bundles. Given a representation of G, that is a group homomorphism $\rho : G \to \mathrm{GL}(V)$, thus you can construct a vector bundle $P \times_G V$. However, there is a more simple way to construct in transition functions viewpoint: By Remark 1.3.1, we can see the transition function of this associated vector bundle is $\{\rho \circ g_{\alpha\beta}\}$, where $\{g_{\alpha\beta}\}$ is transition function of P.

Remark 1.3.2 (relations between vector bundle and principal bundle)*.* If we consider real vector bundles, we have the following one to one correspondence

$$\begin{aligned}\Phi\colon \mathcal{P}_{\mathrm{GL}(n,\mathbb{R})}M &\to \mathrm{Vect}_n^{\mathbb{R}} M\\ P &\mapsto P \times_\rho \mathbb{R}^n\end{aligned}$$

where $\rho : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ is trivial representation, that is the identity map. The inverse ψ is given by considering frame bundle of vector bundle. Furthermore, if we endow vector bundle a Riemannian metric, then it can be regarded as a $\mathrm{O}(n)$-principal bundle, and one can show it's independent of the choice of Riemannian metric, thus in fact we have the following one to one correspondence

$$\mathcal{P}_{\mathrm{O}(n)}M \longleftrightarrow \mathrm{Vect}_n^{\mathbb{R}} M$$

Similarly we also have

$$\mathcal{P}_{\mathrm{U}(n)}M \longleftrightarrow \mathrm{Vect}_n^{\mathbb{C}} M$$

In this viewpoint, principal G-bundles generalize the ideal of vector bundles.

Example 1.3.2. There are two important examples of associated bundles that we will use later.

1. The associated bundle obtained from G acts on G by conjugation, denoted by $\mathrm{Ad}\, P$.
2. The associated vector bundle obtained from G acts on $\mathfrak{g}$ by adjoint representation, denoted by $\mathrm{ad}\, P$.

Remark 1.3.3. A philosophy of geometry is that we can study the objects lying over this geometric objects to study this geometric object itself, and that's why we study the vector bundle over a smooth manifold. Note that for a principal G-bundle, you can obtain a vector bundle from a representation of G. However, there are too many representations of G, so special representations may correspond to special vector bundles.

Proposition 1.3.1. There is a one to one correspondence

$$C^\infty(M, P \times_G F) \xleftrightarrow{1-1} \{f\colon P \to F \mid f \text{ is smooth and } f(xg) = g^{-1}f(x)\}$$

Proof. For smooth function $f\colon P \to F$ which is G-equivariant, consider $s_f \in C^\infty(M, P \times_G F)$ given by

$$s_f(x) = \{(p, f(p)) \mid \pi(p) = x\}$$

where $x \in M$. It's well-defined, since if we choose pg instead of p for some $g \in G$, then $s_f(x) = (pg, f(pg)) = (pg, g^{-1}f(p)) \sim (p, f(p)) \in P \times_G F$. Conversely, given $s \in C^\infty(M, P \times_G F)$, then for any $p \in P$, we consider $\pi(p) = x \in M$ and write $s(x) = [(p, v)]$, then we define $f(p) = v$. It's clear $f(pg) = g^{-1}f(p)$, since $[(p, v)] = [(pg, g^{-1}v)]$. $\square$

In fact, this proposition is not a coincidence, and it's a quite important motivation which shows why we introduce principal G-bundles. If $\pi\colon P \to M$ is a principal G-bundle, and E is a vector bundle over M such that E is an associated vector bundle of P, then if we use π to pull E back to P, we claim that the vector bundle π^*E is the trivial bundle $P \times V$ over P. Indeed, we define the following bundle map

$$\begin{aligned} \psi : P \times V &\to P \times_G V \\ (p, v) &\mapsto [p, v] \end{aligned}$$

and consider the following diagram

$$\begin{tikzcd} P \times V \arrow[r] \arrow[d, "\psi"] & P \arrow[d, "\pi"] \\ E = P \times_G V \arrow[r] & M \end{tikzcd}$$

Clearly $P \times V$ satisfies the universal property of pullback, thus by uniqueness we obtain $\pi^*E \cong P \times V$.

It's clear sections of trivial bundle $P \times V$ can be regard as smooth functions $f\colon P \to V$, and by relation between sections of bundle and its pullback bundle, there is no surprise you have one to one correspondence in Proposition 1.3.1.

Remark 1.3.4. More generally, we can use π to pull $(P \times_G V) \otimes E'$ back to P, and prove it's $(P \times V) \otimes \pi^*E'$ by the same method. The cases we will encounter are $E' = T^*M$ or $E' = \bigwedge^k T^*M$. We use $\Omega^k_M(P \times_G V)$ to denote $(P \times_G V) \otimes \bigwedge^k T^*M$, the generalization tells that we have the one to one correspondence between sections of $\Omega^k_M(P \times_G V)$ and sections of $(P \times V) \otimes \pi^* \bigwedge^k T^*M$ with equivariant conditions, we will call such forms basic forms, a conception we will define in section 3.3.

1.4. **Reduction of principal bundle.** Given a principal G-bundle $\pi\colon P \to M$ and a H-principal bundle $\pi' : P' \to M$. Furthermore, there is a Lie group homomorphism $\alpha : H \to G$.

Definition 1.4.1 (reduction)**.** If there exists a smooth map $\varphi\colon P' \to P$ such that the following diagram commutes

$$\begin{tikzcd} P' \arrow[rr, "\varphi"] \arrow[dr, "\pi_F"'] & & P \arrow[dl, "\pi_E"] \\ & M & \end{tikzcd}$$

and φ is H-equivariant, that is for any $p \in F, h \in H$

$$\varphi(ph) = \varphi(p)\alpha(h)$$

Then P is called an extension of P' from H to G and P' is called an reduction of P from G to H.

Remark 1.4.1. Here are two cases we're concern about:

1. $H < G$ is a subgroup, α is an inclusion.
2. $\alpha : H \to G$ is surjective, for example, H is universal covering of G.

Extension of principal bundle always exists, and it's unique, according to the following proposition.

Proposition 1.4.1. Given a Lie group homomorphism $\alpha : H \to G$ and a H-principal bundle P', there exists a unique extension of P' from H to G.

Proof. Existence: Note that $\alpha : H \to G$ gives a smooth left H-action on G, then consider associated fiber bundle $P' \times_H G$, it's a principal G-bundle, and if we define

$$\begin{aligned} \varphi\colon P' &\to P' \times_H G \\ p' &\mapsto [p', 1] \end{aligned}$$

Then φ is desired equivariant map which makes diagram commutes.

Uniqueness: If there is another extension $\varphi' : P' \to P$, in order to make the following diagram commutes

$$\begin{tikzcd} & P' \times_H G \arrow[d, dashed, "\psi"] \\ P' \arrow[ur, "\varphi"] \arrow[r, "\varphi'"'] & P \end{tikzcd}$$

we define ψ by $\psi([p, 1]) = \varphi'(p)$. Thus principal G-bundles $P' \times_H G$ and P are isomorphic to each other. □

However, reduction may not exist.

Lemma 1.4.1. Let $\alpha : H \to G$ be a Lie group homomorphism, P is a principal G-bundle with transition functions $\psi_{\alpha\beta} : U_{\alpha\beta} \to G$. The following statements are equivalent:

1. There exists reduction of P from G to H;
2. There exists $\varphi_{\alpha\beta} : U_{\alpha\beta} \to H$ such that $\alpha \circ \varphi_{\alpha\beta} = \psi_{\alpha\beta}$.

Corollary 1.4.1. Let P be a principal G-bundle and H is a Lie subgroup of G, then there exists a reduction of P from G to H if and only if there exists transition functions of P valued in H.

Example 1.4.1. If $E \to M$ is a complex vector bundle with a hermitian inner product, then a local trivialization

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^n$$

gives a hermitian inner product on $\mathbb{C}^n$. Thus a transition function must preserve the inner product, thus

$$\begin{tikzcd}
U_\alpha \cap U_\beta \arrow[r] \arrow[dr, dashed] & \mathrm{GL}_n(\mathbb{C}) \\
& \mathrm{U}(n) \arrow[u, hook]
\end{tikzcd}$$

This gives a reduction of $\mathrm{GL}_n(\mathbb{C})$-principal bundle to a $\mathrm{U}(n)$-principal bundle.

Example 1.4.2. If $E \to M$ is a real vector bundle, by the same argument we can always reduce its frame bundle P, that is from a $\mathrm{GL}_n(\mathbb{R})$-principal bundle, to a $\mathrm{O}(n)$-principal bundle. Furthermore,

1. P can be reduced to a $\mathrm{SO}(n)$-principal bundle if and only if E is orientable;
2. P can be reduced to a $\{e\}$-principal bundle if and only if E is trivial.

Example 1.4.3. Let (M, g) be an oriented Riemannian manifold, then TM is a $\mathrm{SO}(n)$-principal bundle. Consider universal covering[2] $\mathrm{Spin}(n) \xrightarrow{2:1} \mathrm{SO}(n)$. If there exists a reduction from $\mathrm{SO}(n)$ to $\mathrm{Spin}(n)$, then we say M admits a spin structure.

[2]See section 8.2.2 for more details about spin groups and this universal covering.

2. Connection of principal bundle

2.1. **Forms valued in vector space.** In this section, let M be a smooth manifold, V a vector space with basis $\{e_\alpha\}$ and G a Lie group with Lie algebra $\mathfrak{g}$. A k-form valued in vector space V can be written as

$$\omega = \omega^\alpha e_\alpha$$

where ω^α are k-forms. We use $\Omega^k_M(V)$ to denote the bundle of k-forms valued in V. It's an easy generalization of differential forms, just by replacing $\mathbb{R}$ with a general vector space, and properties of k-forms also hold for k-forms value in V.

Definition 2.1.1 (exterior derivative)**.** Let $\omega = \omega^\alpha e_\alpha$ be a k-form valued in V, then its exterior derivative is defined as

$$\mathrm{d}\omega = \mathrm{d}\omega^\alpha e_\alpha$$

Proposition 2.1.1 (Cartan's formula)**.** Let $\omega = \omega^\alpha e_\alpha$ be a k-form valued in V, then

$$\begin{aligned}(\mathrm{d}\omega)(X_1,\ldots,X_{k+1}) =& \sum_{i=1}^{k+1}(-1)^{i+1}X_i\omega(X_1,\ldots,\widehat{X_i},\ldots,X_{k+1}) \\ &+\sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_{k+1})\end{aligned}$$

where X_i are vector fields.

Definition 2.1.2 (wedge product)**.** Let ω_1,ω_2 be forms valued in V with degree k and l respectively, then

$$(\omega_1\wedge\omega_2)(X_1,\ldots,X_{k+l}) := \frac{1}{k!\times l!}\sum_{\sigma\in S_{k+l}}(-1)^{|\sigma|}\omega_1(X_{\sigma(1)},\ldots,X_{\sigma(k)})\otimes\omega_2(X_{\sigma(k+1)},\ldots,X_{\sigma(k+l)})$$

where X_i are vector fields.

Proposition 2.1.2. Let ω_i, where $i = 1, 2, 3$, be forms valued in V, then
1. $(\omega_1\wedge\omega_2)\wedge\omega_3 = \omega_1\wedge(\omega_2\wedge\omega_3)$;
2. $\mathrm{d}(\omega_1\wedge\omega_2) = \mathrm{d}\omega_1\wedge\omega_2 + (-1)^{\deg\omega_1}\omega_1\wedge\mathrm{d}\omega_2$.

Definition 2.1.3. Let $T: V\to W$ be a linear map between vector spaces, and ω is a k-form valued in V, then $T\omega$ is a k-form valued in W, which is defined as

$$T\omega(X_1,\ldots,X_k) := T(\omega(X_1,\ldots,X_k))$$

where X_i are vector fields.

Example 2.1.1. Let ω_1,ω_2 be forms with degree k and l respectively, then by our definition one has $\omega_1\wedge\omega_2\in\Omega^{k+l}_M(\mathbb{R}\otimes\mathbb{R})$. It's a little bit different from standard definition of wedge product, since $\omega_1\wedge\omega_2$ should be a $(k+l)$-form. If we consider

$$\begin{aligned}T:\mathbb{R}\otimes\mathbb{R}&\to\mathbb{R}\\ a\otimes b&\mapsto ab\end{aligned}$$

Then $T(\omega_1 \wedge \omega_2)$ is a $(k+l)$-form, coincides with standard definition, so we just denote $T(\omega_1 \wedge \omega_2)$ by $\omega_1 \wedge \omega_2$ for convenience.

Example 2.1.2. Let ω_1 be a k-form valued in $\mathfrak{g}$, and ω_2 a l-form valued in V. Given a representation $\rho : G \to \mathrm{GL}(V)$, it induces a representation of Lie algebra, that is $\rho_* : \mathfrak{g} \to \mathfrak{gl}(V)$. If we consider

$$\begin{aligned} T : \mathfrak{g} \otimes V &\to V \\ \xi \otimes v &\mapsto \rho_*(\xi)v \end{aligned}$$

Then we have $T(\omega_1 \wedge \omega_2)$ is a $(k+l)$-form valued in V, we just denote it by $\omega_1 \wedge \omega_2$ for convenience.

Example 2.1.3. Let ω_1, ω_2 be forms valued in $\mathfrak{g}$ with degree k and l respectively, by our definition $\omega_1 \wedge \omega_2$ is a $(k+l)$-form valued in $\mathfrak{g}$. If we consider

$$\begin{aligned} T : \mathfrak{g} \otimes \mathfrak{g} &\to \mathfrak{g} \\ \xi \otimes \eta &\mapsto [\xi, \eta] \end{aligned}$$

Then we have $T(\omega_1 \wedge \omega_2)$ is a $(k+l)$-form valued in $\mathfrak{g}$, we just denote it by $\omega_1 \wedge \omega_2$ for convenience.

Remark 2.1.1. If Lie group $G = \mathrm{GL}(n, \mathbb{R})$, then $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R})$ consists of matrix. Thus in this case for any $\xi, \eta \in \mathfrak{g}$, we can define T as multiplying them together to obtain an element in $\mathfrak{gl}(n, \mathbb{R})$. However, these two definitions may cause some misunderstandings.

Example 2.1.4. Let ω be a 1-form valued in $\mathfrak{g}$, then for vector fields X, Y, one has

$$\begin{aligned} \omega \wedge \omega(X, Y) &= T((\omega \wedge \omega)(X, Y)) \\ &= T(\frac{1}{1! \times 1!}(\omega(X) \otimes \omega(Y) - \omega(Y) \otimes \omega(X))) \\ &= [\omega(X), \omega(Y)] - [\omega(Y), \omega(X)] \\ &= 2[\omega(X), \omega(Y)] \end{aligned}$$

Remark 2.1.2. If T is choose as in Remark 2.1.1, then in this case we have

$$\omega \wedge \omega(X, Y) = [\omega(X), \omega(Y)]$$

That's where misunderstanding lies. Different authors may use different notations, so be careful!

Proposition 2.1.3. Let ω be a 1-form valued in $\mathfrak{g}$, then

$$(\omega \wedge \omega) \wedge \omega = \omega \wedge (\omega \wedge \omega) = 0$$

Proof. For arbitrary vector fields X, Y and Z, one has

$$\begin{aligned}(\omega\wedge\omega)\wedge\omega(X,Y,Z) &= \frac{1}{2!\times 1!}\{[\omega\wedge\omega(X,Y),\omega(Z)]+[\omega\wedge\omega(Y,Z),\omega(X)]+[\omega\wedge\omega(Z,X),\omega(Y)]\\ &\quad -[\omega\wedge\omega(Y,X),\omega(Z)]+[\omega\wedge\omega(Z,Y),\omega(X)]+[\omega\wedge\omega(X,Z),\omega(Y)]\}\\ &= \frac{2}{2!\times 1!}\{[[\omega(X),\omega(Y)],\omega(Z)]+[[\omega(Y),\omega(Z)],\omega(X)]+[[\omega(Z),\omega(X)],\omega(Y)]\\ &\quad -[[\omega(Y),\omega(X)],\omega(Z)]+[[\omega(Z),\omega(Y)],\omega(X)]+[[\omega(X),\omega(Z)],\omega(Y)]\}\end{aligned}$$

This equals to zero according to Jacobi identity of Lie bracket. □

Proposition 2.1.4. Let ω_1, ω_2 be forms valued in $\mathfrak{g}$ with degree k and l respectively, then

$$\omega_1\wedge\omega_2 = (-1)^{kl+1}\omega_2\wedge\omega_1$$

Proof. Note that for a k-form ω_1 and a l-form ω_2, we have

$$\omega_1\wedge\omega_2 = (-1)^{kl}\omega_2\wedge\omega_1$$

But in this case, there is one more -1 coming from Lie bracket. □

2.2. **Maurer-Cartan form.**

Example 2.2.1 (Maurer-Cartan form)**.** The Maurer-Cartan form θ, which is defined by

$$\theta_g := (L_{g^{-1}})_*$$

is a $\mathfrak{g}$-valued 1-form on G. Indeed, since tangent bundle of Lie group is trivial, so we may assume vector field X is left-invariant, then

$$\theta_g(X_g) = (L_{g^{-1}})_*(L_g)_*X_e = X_e \in \mathfrak{g}$$

where X_g means value of X at point $g \in G$.

Remark 2.2.1. If G is a matrix group, we also use $g^{-1}\mathrm{d}g$ to denote its Maurer-Cartan form, which is easy to compute. For example, consider $G = \mathrm{SO}(2) \subset \mathrm{GL}(2,\mathbb{R})$. We may parametrize $\mathrm{SO}(2)$ by

$$g(\theta) = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

where $\theta \in \mathbb{R}$. Then directly compute we have

$$\begin{aligned}\omega &= g^{-1}\mathrm{d}g\\ &= \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}-\sin\theta\mathrm{d}\theta & -\cos\theta\mathrm{d}\theta\\ \cos\theta\mathrm{d}\theta & -\sin\theta\mathrm{d}\theta\end{pmatrix}\\ &= \begin{pmatrix}0 & -\mathrm{d}\theta\\ \mathrm{d}\theta & 0\end{pmatrix}\end{aligned}$$

2.3. **Motivation.** One motivation is that connection of principal G-bundles can be used as a tool to study connection of vector bundle E, if E is an associated vector bundle of P. Recall a connection on E is defined as the following $\mathbb{R}$-linear operator

$$\nabla : C^\infty(M,E) \to C^\infty(M,\Omega^1_M(E))$$

satisfying Leibniz rule.

Suppose vector bundle E is associated to principal G-bundle $\pi\colon P \to M$, and written as $P \times_G V$, then from Proposition 1.3.1, we have a one to one correspondence between sections of E with G-equivariant maps from P to V. Given a section s of E, if we use s^P to denote the G-equivariant map obtained from one to one correspondence, it's easy to take derivatives of s^P to obtain a 1-form on P valued in V, that is a G-equivariant fiber-wise linear map from TP to V.

However, this 1-form does not by itself define a covariant derivative of s. Since by definition of connection, $\nabla s \in C^\infty(M,\Omega^1_M(E))$, so by Remark 1.3.4, a covariant derivative appears upstairs on P is supposed to be a G-equivariant section over $(P\times V)\otimes \pi^*T^*M$, that is a G-equivariant fiber-wise linear map from π^*TM to V.

To see what is missing, it is important to keep in mind that TP has some properties that arise from the fact that P is a principal bundle over M. In fact, we have the following exact sequence

$$0 \to \ker\pi_* \to TP \to \pi^*TM \to 0 \tag{2.1}$$

This exact sequence is quite important, let's make following remarks:

Remark 2.3.1. The map from $\ker\pi_*$ is clearly an inclusion. And the surjective map from TP to π^*TM is characterized as follows

$$\begin{aligned} TP &\to \pi^*TM \subset P\times TM \\ v &\mapsto (p,\pi_* v) \end{aligned}$$

where $v \in T_pP$.

Remark 2.3.2. $\ker\pi_*$ is isomorphic to trivial bundle $P\times\mathfrak{g}$. Indeed, we have the following bundle isomorphism

$$\begin{aligned} \psi : P\times\mathfrak{g} &\to \ker\pi_* \\ (p,X) &\mapsto \sigma(X)_p := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} pe^{tX} \end{aligned}$$

where $\sigma(X)$ is also called fundamental vector field of X, which will be used frequently later. It's clear $\sigma(X)\in\ker\pi_*$, since for each $p\in P$,

$$\begin{aligned} \pi_*(\sigma(X)_p) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi(pe^{tX}) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi(p) \\ &= 0 \end{aligned}$$

Remark 2.3.3 (G-equivariance of exact sequence). The action of G on P can be lifted to the exact sequence (2.1). Let $R_g : P \to P$ denote the action of $g \in G$ on P, given by $p \mapsto pg$.

1. The G action on TP is given by $(R_g)_* : TP \to TP$, and it descends to $\ker \pi_*$ since if $v \in \ker \pi_*$, then
$$\begin{aligned}\pi_*((R_g)_* v) &= (\pi \circ R_g)_*(v)\\ &= \pi_*(v)\\ &= 0\end{aligned}$$
2. The G action on $\pi^* TM$ is given by sending defined by sending a pair $(p, v) \in P \times TM$ to the pair (pg, v). It's well-defined, that is $(pg, v) \in \pi^* TM$, since $\pi(pg) = \pi(p) = \pi(v)$.

Furthermore, we claim the exact sequence (2.1) is equivariant with respect to the lifts.

1. It automatically holds for inclusion map from $\ker \pi_*$ to TP, since G action on $\ker \pi_*$ is obtain from descending the one on TP;
2. It holds for the map from TP to $\pi^* TM$, since for $v \in TP$ we have $(R_g)_* v$ is sent to $(pg, \pi_*(R_g)_* v)$, that is exactly $(pg, \pi_* v)$, since $\pi \circ R_g = \pi$.

If we want to identify $\ker \pi_*$ as $P \times \mathfrak{g}$, we need to choose an appropriate G-action on $\mathfrak{g}$ properly such that the isomorphism ψ is G-equivariant. It turns out to be adjoint representation. Indeed, direct computation shows
$$\begin{aligned}(R_g)_* \psi(p, X) &= (R_g)_* \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} p \exp(tX) \right)\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} p \exp(tX) g\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (pg) \left(g^{-1} \exp(tX) g\right)\\ &= \psi(pg, \mathrm{Ad}(g^{-1}) X)\end{aligned}$$

2.4. **Connection on principal bundle.** So if we want to obtain a fiber-wise linear map $\pi^* TM \to V$ from a fiber-wise linear map $TP \to V$, one way is to desire exact sequence (2.1) splitting. In other words, we desire there exists a G-equivariant $\omega : TP \to P \times \mathfrak{g}$, such that $\omega|_{P \times \mathfrak{g}}$ is identity. Such ω is called a connection on principal G-bundle P.

Definition 2.4.1 (connection on principal G-bundle)**.** Let $\pi \colon P \to M$ be a principal G-bundle. If $\omega \in C^\infty(P, \Omega^1_P(\mathfrak{g}))$ satisfies

1. For any $X \in \mathfrak{g}$, $\omega(\sigma(X)) = X$;
2. For any $g \in G$, $R_g^* \omega = \mathrm{Ad}(g^{-1}) \circ \omega$.

Then ω is called a connection on P.

Notation 2.4.1. $\mathcal{A}(P)$ denotes the set of all connections on P.

Remark 2.4.1 (horizontal distribution viewpoint). If we define $H = \ker\omega$, then

$$TP = H \oplus (P \times \mathfrak{g})$$

such that $(R_g)_* H_p = H_{pg}$. H is called a horizontal distribution and $P \times \mathfrak{g}$ is called vertical distribution. Conversely, give a horizontal distribution, one can also construct a connection, they're the same things.

Example 2.4.1 (connection on trivial principal G-bundle)**.** Consider trivial principal G-bundle $P = M \times G$. Recall we have a Maurer-Cartan form θ, which is a 1-form valued in $\mathfrak{g}$. Then we can use $\pi_G \colon M \times G \to G$ to pull it back to P to obtain a 1-form on P valued in $\mathfrak{g}$, which is called Maurer-Cartan form on trivial principal G-bundle, and it's denoted ω_{mc}. Now let's check ω_{mc} gives a connection on trivial principal bundle.

1. For any $X \in \mathfrak{g}$, we have

$$\begin{aligned}
\omega_{mc}(\sigma(X)) &= \pi_2^*\theta(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(x,g)e^{tX}) \\
&= \theta(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} ge^{tX}) \\
&= (L_{g^{-1}})_*(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} ge^{tX}) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e^{tX} \\
&= X
\end{aligned}$$

2. It suffices to check $R_g^*\theta = \mathrm{Ad}(g^{-1}) \circ \theta$ holds for $g \in G$. For any left-invariant vector field X, recall that $\theta(X) = X_e$, thus

$$R_g^*\theta(X) = \theta((R_g)_*X) = ((R_g)_*X)_e = (L_{g^{-1}})_*(R_g)_*X_e$$

that's exactly $\mathrm{Ad}(g^{-1}) \circ \theta(X)$.

Remark 2.4.2. It's clear to see $\ker\omega_{mc} = \pi^*TM$, since ω_{mc} is pullback from a 1-form on G, thus in this case

$$TP \cong TM \oplus TG$$

that's exactly canonical splitting of TP.

2.5. **Gauge group.**

Definition 2.5.1 (gauge transformation)**.** For a principal G-bundle $\pi \colon P \to M$, the gauge transformation is a G-equivariant diffeomorphism $\Phi \colon P \to P$ such that $\pi = \pi \circ \Phi$.

Notation 2.5.1. $\mathcal{G}(P)$ denotes the set of all gauge transformation of P, which forms a group, called gauge group.

Remark 2.5.1 (terminologies). Here we make some clarifications about terminologies. A local gauge is a physicist's terminology for the choice of local trivialization, and the change of local trivialization, that is transition functions, are called gauge transformation. For physicists guage group is exactly structure group, and gauge group we defined here is sometimes called global gauge group.

Remark 2.5.2 (local expression of gauge transformation). For a gauge transformation Φ, its action on local trivialization $\varphi_\alpha\colon \pi^{-1}(U_\alpha)\to U_\alpha\times G$, given by $\varphi_\alpha(\Phi(p))=(\pi(p),g_\alpha(\Phi(p)))$, induces a map $\widetilde{\phi}_\alpha:\pi^{-1}(U_\alpha)\to G$ by

$$\widetilde{\phi}_\alpha(p)=g_\alpha(\Phi(p))g_\alpha(p)^{-1}$$

By the G-equivariance of g_α and Φ one has $\widetilde{\phi}$ is G-invariant, which impiles $\widetilde{\phi_\alpha}$ can descend to U_α, that is one can define $\phi_\alpha\colon U_\alpha\to G$ via $\widetilde{\phi}_\alpha(p)=\phi_\alpha(\pi(p))$. If we consider on the overlaps $x\in U_{\alpha\beta}$ with $p=\pi^{-1}(x)$, then

$$\begin{aligned}\phi_\alpha(x)&=g_\alpha(\Phi(p))g_\alpha(p)^{-1}\\&=g_\alpha(\Phi(p))g_\beta(\Phi(p))^{-1}g_\beta(\Phi(p))g_\beta(p)^{-1}g_\beta(p)g_\alpha(p)^{-1}\\&=g_{\alpha\beta}(x)\phi_\beta(x)g_{\alpha\beta}(x)^{-1}\end{aligned}$$

This shows $\{\phi_\alpha\}$ defines a global section of associated bundle obtained from G acts on G by conjugation, that is $\operatorname{Ad}P$ defined in Example 1.3.2. In fact, we have the following one to one correspondence.

Proposition 2.5.1. There is one to one correspondence between the group $\mathcal{G}(P)$ and $C^\infty(M,\operatorname{Ad}P)$.

Proof. We have already seen that a gauge transformation can give an element in $C^\infty(M,\operatorname{Ad}P)$. Conversely, by Proposition 1.3.1, there is a one to one correspondence between $C^\infty(M,\operatorname{Ad}P)$ and smooth functions $f\colon P\to G$ which is G-equivariant. For such f, consider $\Phi_f\colon P\to P$ given by $\Phi_f(p)=pf(p)$.

1. $\pi\circ\Phi_f=\pi$, since $\pi\circ\Phi_f(p)=\pi(pf(p))=\pi(p)$
2. It's G-equivariant since

$$\begin{aligned}\Phi_f(pg)&=pgf(pg)\\&=pgg^{-1}f(p)g\\&=pf(p)g\\&=\Phi_f(p)g\end{aligned}$$

The two maps we constructed are clearly inverse to each other, giving the desired correspondence. □

Now we're going to show $\mathcal{G}(P)$ acts on $\mathcal{A}(P)$.

Lemma 2.5.1. For any $X\in\mathfrak{g}$ and $\Phi\in\mathcal{G}(P)$, then

$$\Phi_*(\sigma(X))=\sigma(X)$$

Proof. Direct computation shows

$$\begin{aligned}\Phi_*\sigma(X) &= \Phi_*(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} pe^{tX}) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi(pe^{tX}) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi(p)e^{tX} \\ &= \sigma(X)\end{aligned}$$

□

Proposition 2.5.2. $\mathcal{G}(P)$ acts on $\mathcal{A}(P)$ via pullback.

Proof. For $\omega \in \mathcal{A}(P)$ and $\Phi \in \mathcal{G}(P)$, let's check $\Phi^*\omega \in \mathcal{A}(P)$.

1. For any $X \in \mathfrak{g}$, we have

$$\begin{aligned}\Phi^*\omega(\sigma(X)) &= \omega(\Phi_*\sigma(X)) \\ &= \omega(\sigma(X)) \\ &= X\end{aligned}$$

2. Note that $R_g^*\Phi^* = (R_g \circ \Phi)^* = (\Phi \circ R_g)^*$, thus

$$\begin{aligned}R_g^*(\Phi^*\omega) &= \Phi^*(R_g^*\omega) \\ &= \Phi^*(\mathrm{Ad}(g^{-1}) \circ \omega) \\ &= \mathrm{Ad}(g^{-1}) \circ \Phi^*\omega\end{aligned}$$

□

Remark 2.5.3. Gauge theory concerns about "space" of orbit of $\mathcal{G}(P)$, that is $\mathcal{A}(P)/\mathcal{G}(P)$.

2.6. **Local expression of connection.** Instead of considering connection 1-form living on P, we want to convert it into the one living on base manifold M, since we want to use it to study connection of vector bundle over M. To do this, we divide the process into three steps:

1. Given a connection on trivial principal G-bundle, correspond it to a 1-form on M valued $\mathfrak{g}$.
2. Figure out how does this correspondence transform under gauge transformation.
3. Since a principal G-bundle admits local trivializations, and transition functions can be regarded as gauge transformations, then we reduce the case to the first two steps.

2.6.1. *Baby case.* Fix a trivial principal G-bundle $P = M \times G$ and following notations:

1. $\pi\colon P \to M$ is natural projection, given by $p = (x, g) \mapsto x \in M$;
2. $i : M \to P$ is natural inclusion, given by $x \mapsto (x, e) \in P$.

Lemma 2.6.1. For any $A \in C^\infty(M, \Omega^1_M(\mathfrak{g}))$, there exists a unique $\widetilde{A} \in C^\infty(P, \Omega^1_P(\mathfrak{g}))$ such that

1. $i^*\widetilde{A} = A$;
2. $\widetilde{A}(\sigma(X)) = 0$, where $X \in \mathfrak{g}$;
3. $R_g^*\widetilde{A} = \mathrm{Ad}(g^{-1}) \circ \widetilde{A}$.

Proof. It suffices to construct $\widetilde{A}$ pointwise.

(a) For $p = (x, e) \in M \times G$, we have

$$T_pP = T_xM \oplus \mathfrak{g}$$

Then $\widetilde{A}$ is uniquely determined at this point according to (1) and (2).

(b) At point $p' = (x, g) \in M \times G$, it's clear $p' = pg$ and $(R_g)_* : T_pP \to T_{p'}P$ is an isomorphism, then for arbitrary $v \in T_{p'}P$, we may assume $v = (R_g)_*w$ for some $w \in T_pP$, then

$$\begin{aligned}\widetilde{A}_{p'}(v) &= \widetilde{A}_{pg}((R_g)_*w) \\ &= (R_g^*\widetilde{A})_p(w) \\ &= \mathrm{Ad}(g^{-1}) \circ \widetilde{A}(w)\end{aligned}$$

$\square$

Proposition 2.6.1. $i^* : \mathcal{A}(P) \to C^\infty(M, \Omega^1_M(\mathfrak{g}))$ is bijective, that is the following diagram commutes

$$\begin{tikzcd} C^\infty(P, \Omega^1_P(\mathfrak{g})) \arrow[r, "i^*"] & C^\infty(M, \Omega^1_M(\mathfrak{g})) \\ \mathcal{A}(P) \arrow[u, hook] \arrow[ur, "1-1"] & \end{tikzcd}$$

Proof. For any $A \in C^\infty(M, \Omega^1_M(\mathfrak{g}))$, by Lemma 2.6.1 we have $\omega_{mc} + \widetilde{A}$ is also a connection on P. Thus consider

$$\begin{aligned}\tau : \Omega^1_M(\mathfrak{g}) &\to \mathcal{A}(P) \\ A &\mapsto \omega_{mc} + \widetilde{A}\end{aligned}$$

It's clear τ is surjective, since for any $\omega \in \mathcal{A}(P)$, we have

$$\tau(i^*(\omega - \omega_{mc})) = \omega_{mc} + \omega - \omega_{mc} = \omega$$

Now it suffices to show $i^*\tau = \mathrm{id}$, which impiles τ is injective thus bijective. Indeed, for $A \in \Omega^1_M(\mathfrak{g})$,

$$i^*\tau(A) = i^*(\omega_{mc} + \widetilde{A}) = i^*\widetilde{A} = A$$

since $i^*\omega_{mc} = 0$. $\square$

2.6.2. *How to glue.* For gauge transformation Φ on trivial principal G-bundle $M \times G$, it can be written as

$$\Phi(x, g) = (x, \phi(x)g)$$

where $\phi\colon M \to G$ is smooth map. For any $\omega \in \mathcal{A}(P)$, we can write it as $\omega = \omega_{mc} + \widetilde{A}$ according to Proposition 2.6.1. Then

$$\begin{aligned} i^*\Phi^*\omega &= i^*\Phi^*(\omega_{mc} + \widetilde{A}) \\ &\overset{(1)}{=} i^*\Phi^*\pi_G^*\theta + i^*\Phi^*\widetilde{A} \\ &\overset{(2)}{=} \phi^*\theta + i^*\Phi^*\widetilde{A} \end{aligned}$$

where

(1) holds from definition of Maurer-Cartan form;
(2) holds from $\pi_G \circ \Phi \circ i(x) = \pi_G \circ \Phi(x, e) = \pi_G(x, \phi(x)) = \phi(x)$ for $x \in M$.

Now it suffices to compute $i^*\Phi^*\widetilde{A}$. For any vector field X on M, one has

$$\begin{aligned} (i^*\Phi^*\widetilde{A})(X) &= \widetilde{A}(\Phi_* i_*(X)) \\ &= \widetilde{A}(\phi_* X) \\ &= \mathrm{Ad}(\phi^{-1}) \circ \widetilde{A}(X) \end{aligned}$$

Thus we have

$$i^*(\Phi^*\omega) = \phi^*\theta + \mathrm{Ad}(\phi^{-1}) \circ \widetilde{A}$$

2.6.3. *General case.* Let $\pi\colon P \to M$ be a principal G-bundle with local trivialization $\{(U_\alpha, \varphi_\alpha)\}$, we use $i_\alpha : U_\alpha \to U_\alpha \times G$ to denote natural inclusion. For a connection $\omega \in \mathcal{A}(P)$, then we can write it locally on

$$i_\alpha^*(\varphi_\alpha^{-1})^*\omega|_{\pi^{-1}(U_\alpha)} = A_\alpha \in C^\infty(U_\alpha, \Omega^1_{U_\alpha}(\mathfrak{g}))$$

Furthermore,

$$A_\alpha = \mathrm{Ad}(g_{\alpha\beta}^{-1}) \circ A_\beta + g_{\alpha\beta}^*\theta$$

where $g_{\alpha\beta}\colon U_{\alpha\beta} \to G$ are transition functions. Thus we have the following one to one correspondence.

Proposition 2.6.2.

$$\mathcal{A}(P) \xleftrightarrow{1-1} \{(A_\alpha) \in \prod_\alpha C^\infty(U_\alpha, \Omega^1_{U_\alpha}(\mathfrak{g})) \mid A_\alpha = \mathrm{Ad}(g_{\alpha\beta}^{-1}) \circ A_\beta + g_{\alpha\beta}^*\theta\}$$

Remark 2.6.1. From this viewpoint, for two connection ω_1 and ω_2, it's clear $\omega_1 - \omega_2$ gives a global section of associated vector bundle $\operatorname{ad} P$, thus $\mathcal{A}(P)$ is an affine space modelled on $C^\infty(M, \Omega^1_M(\operatorname{ad} P))$. In particular, it's contractible.

3. Curvature of principal bundle

3.1. Definition.

Definition 3.1.1 (curvature)**.** Let P be a principal G-bundle and $\omega \in \mathcal{A}(P)$. Curvature of ω is defined as

$$\Omega := \mathrm{d}\omega + \frac{1}{2}\omega \wedge \omega \in C^{\infty}(P, \Omega^2_P(\mathfrak{g}))$$

Example 3.1.1. If $P = M \times G$, and $\omega = \omega_{mc}$, then $\Omega = 0$.

Proof. It suffices to check Maurer-Cartan form $\theta \in \Omega^1_G(\mathfrak{g})$ satisfying

$$\mathrm{d}\theta + \frac{1}{2}\theta \wedge \theta = 0$$

which is called Maurer-Cartan equation. Let X, Y are two left-invariant vector fields on G, then

$$\theta(X) = (L_{g^{-1}})_* X_g = (L_{g^{-1}})_*(L_g)_* X_e = X_e$$

is constant. Thus

$$\mathrm{d}\theta(X, Y) = -\theta([X, Y]) = -\frac{1}{2}\theta \wedge \theta(X, Y)$$

since $X(\theta(Y)) = Y(\theta(X)) = 0$. □

Theorem 3.1.1 (Bianchi identity)**.**

$$\mathrm{d}\Omega + \omega \wedge \Omega = 0$$

Proof.

$$\begin{aligned}
\mathrm{d}\Omega &= \mathrm{d}(\mathrm{d}\omega + \frac{1}{2}\omega \wedge \omega) \\
&= \frac{1}{2}\mathrm{d}\omega \wedge \omega - \frac{1}{2}\omega \wedge \mathrm{d}\omega \\
&= -\omega \wedge \mathrm{d}\omega \\
&= -\omega \wedge (\Omega - \frac{1}{2}\omega \wedge \omega) \\
&= -\omega \wedge \Omega
\end{aligned}$$

□

Definition 3.1.2 (horizontal form)**.** Let ω be a 2-form on P valued in vector space V, it's called horizontal, if $\omega(\sigma(X), \text{-}) = 0$ for arbitrary $X \in \mathfrak{g}$.

Proposition 3.1.1. Ω is a horizontal 2-form.

Proof. Divide computations into two parts:

1. If $X, Y \in \mathfrak{g}$ and write $X' = \sigma(X), Y' = \sigma(Y)$, then

$$\begin{aligned}
\mathrm{d}\omega(X', Y') &= X'(\omega(Y')) - Y'\omega(X') - \omega([X', Y']) \\
&= X'(Y) - Y'(X) - [X, Y] \\
&= -[X, Y] \\
&= -\frac{1}{2}\omega \wedge \omega(X', Y')
\end{aligned}$$

2. If $X \in \mathfrak{g}$ and Y is a horizontal vector field, note that
$$\frac{1}{2}\omega \wedge \omega(\sigma(X), Y) = 0$$
since $\omega(Y) = 0$. So it suffices to compute
$$\begin{aligned} \mathrm{d}\omega(\sigma(X), Y) &= \sigma(X)(\omega(Y)) - Y\omega(\sigma(X)) - \omega([\sigma(X), Y]) \\ &= -\omega([\sigma(X), Y]) \\ &= -\omega(\mathcal{L}_{\sigma(X)}Y) \end{aligned}$$
However, we have
$$\mathcal{L}_{\sigma(X)}Y = \lim_{t\to 0}\frac{Y \circ \phi_t - Y}{t}$$
where ϕ_t is the flow generated by $\sigma(X)$, thus it's clear $\omega(\mathcal{L}_{\sigma(X)}Y) = 0$.

□

Remark 3.1.1. Now let's give another explaination about horizontal: Given a horizontal distribution $H \subset TP$, we define the horizontal projection $h : TP \to TP$ to be the projection onto the horizontal distribution along the vertical distribution. Since both H and V are invariant under the action of G, so is h.

Then $\Omega = h^*\mathrm{d}\omega$. Indeed, it suffices to show for vector fields X, Y, one has
$$\mathrm{d}\omega(hX, hY) = \mathrm{d}\omega(X, Y) + \frac{1}{2}\omega \wedge \omega(X, Y)$$
Consider the following cases:

1. Let X, Y be horizontal. In this case there is nothing to prove, since $\omega(X) = \omega(Y) = 0$ and $hX = X, hY = Y$;
2. If one of X, Y are vertical, then it's clear both sides are zero, since both Ω and $h^*\mathrm{d}\omega$ are horizontal.

That is, $\Omega(X, Y) = 0$ if and only if $[hX, hY]$ is horizontal. In other words, the curvature of the connection measures the failure of integrability of the horizontal distribution $H \subset TP$.

3.2. **Local expression of curvature.** Let $\pi\colon P \to M$ be a principal G-bundle with local trivialization $\{(U_\alpha, \varphi_\alpha)\}$, we use $i_\alpha : U_\alpha \to U_\alpha \times G$ to denote natural inclusion. For connection $\omega \in \mathcal{A}(P)$, its curvature is defined as
$$\Omega = \mathrm{d}\omega + \frac{1}{2}\omega \wedge \omega$$
If we define $\Omega_\alpha = (\varphi_\alpha^{-1})^*\Omega$, which is a 2-form on $U_\alpha \times G$, and
$$F_\alpha := i_\alpha^*\Omega_\alpha \in C^\infty(U_\alpha, \Omega^1_{U_\alpha}(\mathfrak{g}))$$
By definition one has
$$F_\alpha = \mathrm{d}A_\alpha + \frac{1}{2}A_\alpha \wedge A_\alpha$$

Now we're going to show on $U_{\alpha\beta}$, one has

$$F_\beta = \mathrm{Ad}(g_{\alpha\beta}^{-1}) \circ F_\alpha$$

where $g_{\alpha\beta} : U_{\alpha\beta} \to G$ are transition function. Note that

$$\begin{aligned}F_\alpha &= \mathrm{d}A_\alpha + \frac{1}{2}A_\alpha \wedge A_\alpha \\ &= \mathrm{d}(\mathrm{Ad}(g_{\alpha\beta}^{-1}) \circ A_\beta + g_{\alpha\beta}^*\theta) + \frac{1}{2}(\mathrm{Ad}(g_{\alpha\beta}^{-1}) \circ A_\beta + g_{\alpha\beta}^*\theta) \wedge (\mathrm{Ad}(g_{\alpha\beta}^{-1}) \circ A_\beta + g_{\alpha\beta}^*\theta)\end{aligned}$$

Since θ satisfies Maurer-Cartan equation, one has

$$g_{\alpha\beta}^*(\mathrm{d}\theta + \frac{1}{2}\theta \wedge \theta) = 0$$

In order to give a neat computation of Ad, we here assume G is a matrix group[3]. Then

$$\begin{aligned}\mathrm{d}(\mathrm{Ad}(g_{\alpha\beta}^{-1}) \circ A_\beta) &= \mathrm{d}(g_{\alpha\beta}^{-1}A_\beta g_{\alpha\beta}) \\ &= \mathrm{d}g_{\alpha\beta}^{-1}A_\beta g_{\alpha\beta} + g_{\alpha\beta}^{-1}\mathrm{d}A_\beta g_{\alpha\beta} + g_{\alpha\beta}^{-1}A_\beta \mathrm{d}g_{\alpha\beta} \\ &= \mathrm{d}g_{\alpha\beta}^{-1}A_\beta g_{\alpha\beta} + g_{\alpha\beta}^{-1}A_\beta \mathrm{d}g_{\alpha\beta} + \mathrm{Ad}(g_{\alpha\beta}^{-1}) \circ \mathrm{d}A_\beta\end{aligned}$$

And

$$\begin{aligned}\mathrm{d}g_{\alpha\beta}^{-1}A_\beta g_{\alpha\beta} + g_{\alpha\beta}^{-1}A_\beta \mathrm{d}g_{\alpha\beta} &= -g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta} \wedge g_{\alpha\beta}^{-1}A_\beta g_{\alpha\beta} + g_{\alpha\beta}^{-1}A_\beta g_{\alpha\beta} g_{\alpha\beta}^{-1}\mathrm{d}g_{\alpha\beta} \\ &= -g_{\alpha\beta}^*\theta \wedge \mathrm{Ad}(g_{\alpha\beta}^{-1})A_\beta + \mathrm{Ad}(g_{\alpha\beta}^{-1})A_\beta \wedge g_{\alpha\beta}^*\theta\end{aligned}$$

Remark 3.2.1. In order words, $\{F_\alpha\}$ gives a global section of $\Omega^2_M(\mathrm{ad}\,P)$, which is denoted by F_ω.

3.3. **Basic forms.** Recall our curvature form $\Omega \in C^\infty(P, \Omega^2_P(\mathfrak{g}))$ has the following properties:

1. Ω is horizontal;
2. It's Ad-equivariant, that is

$$R_g^*\Omega = \mathrm{Ad}(g^{-1}) \circ \Omega$$

Definition 3.3.1 (basic form)**.** Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G, a k-form on P valued in V is called basic, if it satisfies

1. Ω is horizontal;
2. It's ρ-equivariant, that is

$$R_g^*\Omega = \rho(g^{-1}) \circ \Omega$$

Notation 3.3.1. The set of all basic k-forms on P valued V is denoted by $C^\infty(P, \Omega^k_P(V))^{\mathrm{basic}}$.

Example 3.3.1. The curvature form is a basic 2-form valued $\mathfrak{g}$.

[3]In fact, most interesting cases we're concern about are matrix group

Proposition 3.3.1. Let $\rho: G \to \mathrm{GL}(V)$ be a representation of G, E denotes associated vector bundle $P\times_\rho V$, then there is an one to one correspondence

$$C^\infty(P,\Omega^k_P(V))^{\text{basic}} \xleftrightarrow{1-1} C^\infty(M,\Omega^k_M(E))$$

Proof. Given $\widetilde{\omega}\in C^\infty(P,\Omega^k_P(V))^{\text{basic}}$, now we're going to construct a $\omega \in C^\infty(M,\Omega^k_M(E))$ pointwise, that is for arbitrary $x\in M$ and $v_1,\dots,v_k\in T_xM$, we give an assignment:

$$\omega_x(v_1,\dots,v_k)\in E_x$$

Recall that E_x is an equivalent class $[p,v]$, where $p\in P, v\in V$. Choose arbitrary $p\in\pi^{-1}(x)\in P$, since $\pi_*: T_pP\to T_xM$ is surjective, we can choose $\widetilde{v}_i\in T_pP$ such that $\pi_*(\widetilde{v}_i)=v_i, i=1,\dots,k$.

Now we define

$$\omega_x(v_1,\dots,v_k):=[(p,\widetilde{\omega}_x(\widetilde{v}_1,\dots,\widetilde{v}_k))]$$

It's well-defined, that is it is independent of the choice of p and $\widetilde{v}_1,\dots,\widetilde{v}_k$. Indeed, choose $p'=pg\in\pi^{-1}(x)$ and $\widetilde{v}'_1,\dots,\widetilde{v}'_k\in T_{p'}P$ with $\pi_*(\widetilde{v}'_i)=X_i, i=1,\dots,k$. Note that for each i, one has $(R_g)_*\widetilde{v}_i-\widetilde{v}'_i$ is vertical, since $\pi_*((R_g)_*\widetilde{v}_i-\widetilde{v}'_i)=0$. Thus

$$\begin{aligned}
\widetilde{\omega}_{p'}(\widetilde{v}'_1,\dots,\widetilde{v}'_k) &\overset{(1)}{=} \widetilde{\omega}_{p'}((R_g)_*\widetilde{v}_1,\dots,(R_g)_*\widetilde{v}_k)\\
&= (R_g^*\omega)_p(\widetilde{v}_1,\dots,\widetilde{v}_k)\\
&\overset{(2)}{=} \rho(g^{-1})\circ\omega_p(\widetilde{v}_1,\dots,\widetilde{v}_k)
\end{aligned}$$

where

(1) holds from $\widetilde{\omega}$ is horizontal;
(2) holds from $\widetilde{\omega}$ is G-equivariant.

This shows ω is well-defined, since $[(p,\widetilde{\omega}_x(\widetilde{v}_1,\dots,\widetilde{v}_k))]=[(p',\rho(g^{-1})\circ\omega_p(\widetilde{v}_1,\dots,\widetilde{v}_k))]$ in E. Conversely, from above construction, there is a formula

$$\omega_x(X_1,\dots,X_k)=[(p,\widetilde{\omega}_p(\widetilde{X}_1,\dots,\widetilde{X}_k))] \tag{3.1}$$

So it's clear how to construct $\widetilde{\omega}$ when you have $\omega\in C^\infty(M,\Omega^k_M(E))$. $\square$

Remark 3.3.1. Above proposition is the key tool to study the geometry of vector bundle E via principal bundle, if E can be constructed as an associated vector bundle $P\times_\rho V$. Furthermore, it gives a explicit proof of motivation we said in Remark 1.3.4. In particular, if we consider the case $k=0$, then we have

$$\{f\colon P\to V\mid f(xg)=\rho(g^{-1})f(x)\}\xleftrightarrow{1-1} C^\infty(M,E)$$

since the former is exactly $\Omega^0_P(V)^{\text{basic}}$. This shows Proposition 1.3.1 again.

3.4. **Relations between connections on principal bundle and its associated bundle.** Now we're going to define connection on vector bundle $E = P \times_\rho V$ using connection ω on principal G-bundle P. Thanks to Proposition 3.3.1, it suffices to construct

$$\mathrm{d}_\omega : C^\infty(P, \Omega^0_P(V))^{\text{basic}} \to C^\infty(P, \Omega^1_P(V))^{\text{basic}}$$

Note that there is a natural exterior derivative

$$\mathrm{d} : C^\infty(P, \Omega^0_P(V)) \to C^\infty(P, \Omega^1_P(V))$$

However, it may not descend down to basic forms. Here we define

$$\begin{aligned} \mathrm{d}_\omega : C^\infty(P, \Omega^0_P(V))^{\text{basic}} &\to C^\infty(P, \Omega^1_P(V))^{\text{basic}} \\ s &\mapsto \mathrm{d}s + \rho_*(\omega)s \end{aligned}$$

where $\rho_* : \mathfrak{g} \to \mathfrak{gl}(V)$ is representation of Lie algebra induced by ρ. Let's show d_ω is well-defined.

1. It's G-equivariant:

$$\begin{aligned} (R_g)^*(\mathrm{d}_\omega s) &= R_g^*(\mathrm{d}s + \rho_*(\omega)s) \\ &= \mathrm{d}R_g^* s + R_g^*(\rho_*(\omega)s) \\ &= \mathrm{d}R_g^* s + \rho_*(R_g^*\omega)R_g^* s \\ &= \mathrm{d}R_g^* s + \rho_*(\mathrm{Ad}(g^{-1})\omega)R_g^* s \\ &= \mathrm{d}R_g^* s + \mathrm{Ad}(\rho(g^{-1}))(\rho_*\omega)\rho_g^* s \\ &= \mathrm{d}\rho(g^{-1})s + \mathrm{Ad}(\rho(g^{-1}))(\rho_*\omega)\rho(g^{-1})s \\ &= \mathrm{d}(\rho(g^{-1})s) + \rho(g^{-1})(\rho_*(\omega)s) \\ &= \rho(g^{-1})\mathrm{d}_\omega s \end{aligned}$$

2. It's horizontal: For arbitrary vertical $\sigma(X)$

$$\begin{aligned} \mathrm{d}_\omega(s) &= \mathrm{d}s(\sigma(X)) + \rho_*(\omega(\sigma(X)))s \\ &= \mathrm{d}s(\sigma(X)) + \rho_*(X)s \end{aligned}$$

So it suffices to check

$$\sigma(X)(s) = -\rho_*(X)s$$

Indeed,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} se^{tX} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \rho(e^{-tX})s$$

More generally, we can define

$$\begin{aligned} \mathrm{d}_\omega : C^\infty(P, \Omega^k_P(V))^{\text{basic}} &\to C^\infty(P, \Omega^{k+1}_P(V))^{\text{basic}} \\ s &\mapsto \mathrm{d}s + \rho_*(\omega) \wedge s \end{aligned}$$

And one can check it's well-defined by the same method as above.

Remark 3.4.1. The case we're most interested in is $V = \mathfrak{g}$ and ρ is adjoint representation. Since in this case $\rho_*(X)$ acts on Y is exactly $[X, Y]$, where $X, Y \in \mathfrak{g}$. In particular,

$$\mathrm{d}_\omega(s) = \mathrm{d}s + \omega \wedge s$$

where $s \in C^\infty(P, \Omega^k_P(\mathfrak{g}))^{\text{basic}}$ and above wedge is wedge of forms valued in $\mathfrak{g}$.

Now we're going to back to base manifold M to give description of connection ∇ on E. Here are two methods:

1. Take $E = \Omega^1_M(P \times_{\mathrm{Ad}} \mathfrak{g})$ as an example, since we need this example later. If U_α is a local trivialization and $\pi_\alpha : U_\alpha \times G \to U_\alpha$ is the projection to the first factor, section s of E on U_α is a section of $\Omega^1_{U_\alpha}(\mathfrak{g})$, denoted by s_α, then $\widetilde{s}_\alpha := \pi^*_\alpha s_\alpha$ is a basic 1-form valued in $\mathfrak{g}$, and $\mathrm{d}_\omega \widetilde{s}_\alpha = \mathrm{d}\widetilde{s}_\alpha + \omega \wedge \widetilde{s}_\alpha$, then by using $i_\alpha : U_\alpha \to U_\alpha \times G$ to pullback, one has
$$\mathrm{d}s_\alpha + A_\alpha \wedge s_\alpha \tag{3.2}$$
where $A_\alpha \in C^\infty(U_\alpha, \Omega^1_{U_\alpha}(\mathfrak{g}))$ is given by ω.
2. Let X be a vector field on M, s a section of E, we can give an explicit formula of $\nabla_X s$ via formula (3.1). For $x \in M$, choose $p \in \pi^{-1}(x)$, $\widetilde{X}$ is horizontal such that $\pi_*(\widetilde{X}) = X$ and $\widetilde{s} : P \to V$ is G-equivariant map which corresponds to s, then
$$\begin{aligned}
(\nabla_X s)_x &= [(p, \widetilde{\nabla}\widetilde{s}(\widetilde{X}_p)) \\
&= [(p, (\mathrm{d}\widetilde{s} + \rho_*(\omega)\widetilde{s})(\widetilde{X}_p))] \\
&\overset{(1)}{=} [(p, \mathrm{d}\widetilde{s}(\widetilde{X}_p))] \\
&= [(p, \widetilde{X}_p(\widetilde{s}))]
\end{aligned}$$
where (1) holds from $\widetilde{X}$ is horizontal.

From the second method, it's easy to see write

$$([\nabla_X, \nabla_Y]s - \nabla_{[X,Y]}s)_x = [(p, -\rho_*(\omega([\widetilde{X}, \widetilde{Y}]_p))\widetilde{s})]$$

here X, Y are vector fields on M. A natural question is what's the relation of curvature of ω and ∇.

Recall curvature of ω, denoted by Ω, is basic 2-form, and it can be regarded as a section of $\Omega^2_M(\mathrm{ad}\, P)$, we use $\Omega \mapsto \Theta$ to denote this correspondence. Now let's compute Θ via formula (3.1): For $x \in M, v, w \in T_x M$, choose $\widetilde{v}, \widetilde{w}$ such that $\pi_*(\widetilde{v}) = v$ and $\pi_*(\widetilde{w}) = w$. Without lose of generality, we may assume $\widetilde{v}, \widetilde{w}$ are horizontal, then

$$\Theta_x(v, w) = [(p, \Omega_p(\widetilde{v}, \widetilde{w}))] \in (\mathrm{ad}\, P)_x$$

Note that

$$\begin{aligned}\Omega_p(\widetilde{v},\widetilde{w}) &= \mathrm{d}\omega(\widetilde{v},\widetilde{w}) + \frac{1}{2}\omega\wedge\omega(\widetilde{v},\widetilde{w})\\ &\overset{(1)}{=} \mathrm{d}\omega(\widetilde{v},\widetilde{w})\\ &\overset{(2)}{=} -\omega([\widetilde{v},\widetilde{w}])\end{aligned}$$

where

(1) holds from $\widetilde{v},\widetilde{w}$ are horizontal;
(2) holds from Cartan's formula.

Note that $\operatorname{ad} P$ can act on $P\times_\rho V$ as follows

$$[(p,X)]\times[(p,v)]\mapsto[(p,\rho_*(X)v)]$$

So $\Theta_x(v,w)$ can act on E_x, that is $\Theta\in C^\infty(M,\Omega^2_M(\operatorname{End}E))$. Thus we have the following theorem.

Theorem 3.4.1. Let $\pi\colon P\to M$ be a principal G-bundle, $E=P\times_\rho V$ an associated vector bundle of P. For vector fields X,Y over M and section s of E, then

$$[\nabla_X,\nabla_Y]s-\nabla_{[X,Y]}s=\Theta(X,Y)s$$

4. Flat connection and holonomy

4.1. **Lifting of curves.** Let $\pi\colon P \to M$ be a principal G-bundle equipped with connection ω, consider smooth curve $\gamma : [0,1] \to M$ and a point $p \in \pi^{-1}(\gamma(0))$, we claim there exists a unique smooth map $\widetilde{\gamma} : [0,1] \to P$ such that

1. The following diagram commutes:

$$\begin{tikzcd} & P \arrow[d, "\pi"] \\ {[0,1]} \arrow[ur, dashed, "\widetilde{\gamma}"] \arrow[r, "\gamma"] & M \end{tikzcd}$$

2. $\widetilde{\gamma}'(t)$ is horizontal;
3. $\widetilde{\gamma}(0) = p$.

Proof. For convenience we assume G is a matrix group, and without lose of generality, we may assume P is trivial principal G-bundle $M \times G$, since it's a local problem. In this case we write $\widetilde{\gamma} = (\gamma(t), g(t))$, it's clear $\pi \circ \widetilde{\gamma} = \gamma$.

For conditions (2) and (3), it's an ODE with initial value in fact: Note that we can write connection $\omega = \omega_{mc} + \widetilde{A}$, so $\widetilde{\gamma}'(t)$ is horizontal if and only if

$$\begin{aligned}(\omega_{mc} + \widetilde{A})(\widetilde{\gamma}'(t)) &= (\omega_{mc} + \widetilde{A})((\gamma'(t), g'(t)) \\ &= g^{-1}(t)g'(t) + \widetilde{A}((\gamma'(t), g'(t))) \\ &= g^{-1}(t)g'(t) + \mathrm{Ad}(g^{-1}(t)) \circ A_{\gamma(t)}(\gamma'(t)) \\ &= g^{-1}(t)g'(t) + g^{-1}(t)A_{\gamma(t)}(\gamma'(t))g(t) \\ &= 0\end{aligned}$$

This completes the proof. □

4.2. **Flat connection.**

Definition 4.2.1 (flat connection)**.** Let P be a principal G-bundle, a connection $\omega \in \mathcal{A}(P)$ is called flat, if its curvature form $\Omega = 0$.

Theorem 4.2.1. The following are equivalent:

1. ω is flat;
2. For any $p \in M$, there exists $p \in U \subset M$ and local trivialization $\varphi\colon \pi^{-1}(U) \to U \times G$ such that $\omega|_U = \varphi^*\omega_{mc}$.

Proof. Hallmark of proof is to see curvature vanishes if and only if horizontal distribution is integrable. □

Remark 4.2.1. From this theorem, we can see a flat connection is just a topology information.

Corollary 4.2.1. The following are equivalent:

1. There is a flat connection on P;

2. There is a local trivialization $\varphi_\alpha : P|_{U_\alpha} \to U_\alpha \times G$ such that transition functions $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}$ are locally constant.

Proof. From (2) to (1): Note that a connection on P is given by the following data:

$$A_\beta = \mathrm{Ad}(g_{\alpha\beta}^{-1})A_\alpha + g_{\alpha\beta}^*\theta$$

If $g_{\alpha\beta}$ are locally constant, then $g_{\alpha\beta}^*\omega = 0$, so just take all $A_\alpha = 0$ to obtain a flat connection.

From (1) to (2): If ω is a flat connection, then we can choose a local trivialization $\{U_\alpha, \varphi_\alpha\}$ such that $\omega|_{\pi^{-1}(U_\alpha)}$ are $\varphi_\alpha^*\omega_{mc}$. In this local trivialization, we have all $A_\alpha = 0$, thus $g_{\alpha\beta}^*\theta = 0$, which impiles $g_{\alpha\beta}$ is locally constant. □

4.3. **Holonomy.** Now we give a smooth closed curve $\gamma : [0,1] \to M$ and $p \in \pi^{-1}(\gamma(0))$. Consider its lifting $\widetilde{\gamma} : [0,1] \to P$, note that

$$\widetilde{\gamma}(1) \in \pi^{-1}(\gamma(1)) = \pi^{-1}(\gamma(0))$$

So there exists $g \in G$ such that $\widetilde{\gamma}(1) = \widetilde{\gamma}(0)g$, since P_p is an orbit of G. Such element g is called holonomy, and it's denoted by $\mathrm{Hol}(\gamma, p)$, since it only depends on γ and p.

Proposition 4.3.1. For holonomy, the following properties hold.

1. If we change base point p to pg, then

$$\mathrm{Hol}(\gamma, pg) = g^{-1}\,\mathrm{Hol}(\gamma, p)g$$

2. For two smooth closed curves γ_1, γ_2, we have

$$\mathrm{Hol}(\gamma_1\gamma_2, p) = \mathrm{Hol}(\gamma_1, p)\,\mathrm{Hol}(\gamma_2, p)$$

Proof. Clear. □

From (2) of above proposition, Hol can be regarded as a group homomorphism to some extend, so if we want to give a homomorphism

$$\mathrm{Hol} : \pi_1(M) \to G$$

It suffices to check when $\mathrm{Hol}(\gamma, p)$ is independent of homotopy class. Consider the following homotopy

$$\gamma_s \colon (-\varepsilon, \varepsilon) \times [0,1] \to M$$

such that $\gamma_0 = \gamma$. If we write its lifting on local trivialization as $\widetilde{\gamma}_s(t) = (\gamma_s(t), g_s(t))$, then the following equation holds

$$\frac{\partial g_s}{\partial t}(t) + A_{\gamma(t)}(\frac{\partial g_s}{\partial t}(t))g_s(t) = 0$$

So if ω is a flat connection, then it reduces to for arbitrary $s \in (-\varepsilon, \varepsilon)$, one has $\frac{\partial g_s}{\partial t}(t) = 0$. This shows it's independent of s.

Theorem 4.3.1 (Riemann-Hilbert correspondence)**.** There is a one to one correspondence

$$\{\text{flat connections on } P\}/\text{isomorphism} \xleftrightarrow{1-1} \text{Hom}(\pi_1(M), G)/\text{conjugate}$$

Part 2. Chern-Weil theory

5. Chern-Weil theory

5.1. **G-invariant polynomial.** Let G be a Lie group with Lie algebra $\mathfrak{g}$, note that G can act on $\mathfrak{g}$ via adjoint representation, then G can also act on dual space of $\mathfrak{g}$, that is $\mathfrak{g}^*$, and thus on $\mathrm{Sym}^k \mathfrak{g}^*$. To be explicit, for $p \in \mathrm{Sym}^k \mathfrak{g}^*$ and $g \in G$, one has

$$gp(x_1, \ldots, x_k) := p(\mathrm{Ad}(g)x_1, \ldots, \mathrm{Ad}(g)x_k)$$

Definition 5.1.1 (G-invariant polynomial)**.** The set of G-invariant polynomials of degree k is

$$I^k(\mathfrak{g}) := \{p \in \mathrm{Sym}^k \mathfrak{g}^* \mid gp = p, \forall g \in G\}$$

and

$$I(\mathfrak{g}) := \bigoplus_{k \geq 0} I^k(\mathfrak{g})$$

5.2. **Chern-Weil homomorphism.** Let $\pi \colon P \to M$ be a principal G-bundle, and ω is a connection on P with curvature Ω.

Proposition 5.2.1. For $p \in I^k(\mathfrak{g})$,

$$p(\Omega) := p \circ (\underbrace{\Omega \wedge \cdots \wedge \Omega}_{k \text{ times}})$$

is a $2k$-form defined on P. Then

1. $p(\Omega)$ is horizontal, G-invariant, closed;
2. There exists a unique $2k$-form $p(\Theta)$ on M such that $\pi^*(p(\Theta)) = p(\Omega)$ and $\mathrm{d}p(\Theta) = 0$;
3. $[p(\Theta)] \in H^{2k}(M)$ is independent of the choice of connection ω.

Proof. For (1). It's clear $p(\Omega)$ is horizontal, since Ω is horizontal; To see it's G-invariant, note that

$$\begin{aligned} R_g^*(p(\Omega)) &= p(R_g^*\Omega) \\ &\overset{(a)}{=} p(\mathrm{Ad}(g^{-1}) \circ \Omega) \\ &\overset{(b)}{=} p(\Omega) \end{aligned}$$

where

(a) holds from Ω is G-equivariant;
(b) holds from p is G-invariant.

To see it's closed,

$$\mathrm{d}p(\Omega) = p(\mathrm{d}\Omega \wedge \Omega \wedge \cdots \wedge \Omega + \Omega \wedge \mathrm{d}\Omega \wedge \cdots \wedge \Omega + \ldots)$$

Bianchi identity impiles

$$\mathrm{d}\Omega + \omega \wedge \Omega = 0$$

If we substitute $\mathrm{d}\Omega$ by $-\omega\wedge\Omega$ in above, then it suffices to show $\mathrm{d}p(\Omega)$ is horizontal. To see this, given a vertical vector field X, then $\mathcal{L}_X p(\Omega)=0$, since $p(\Omega)$ is horizontal, then by Cartan formula

$$\begin{aligned} 0&=\mathcal{L}_X p(\Omega)\\ &=\mathrm{d}\iota_X p(\Omega)+\iota_X\mathrm{d}p(\Omega)\\ &=\iota_X\mathrm{d}p(\Omega)\end{aligned}$$

For (2). Note that $\operatorname{im}\pi^*=\{\tau\in C^\infty(P,\Omega_P^{2k})\mid \tau \text{ is horizontal and } G\text{-invariant}\}$ and π^* is injective impiles uniqueness. It's closed, since

$$\pi^*(\mathrm{d}p(\Theta))=\mathrm{d}\pi^*(p(\Theta))=\mathrm{d}p(\Omega)=0$$

For (3). Let ω' be another connection on P, consider principal G-bundle $P\times\mathbb{R}$ over $M\times\mathbb{R}$, and connection $\widetilde{\omega}=(1-t)\omega+t\omega'$ on it, with curvature $\widetilde{\Omega}$. If we use i_0,i_1 to denote inclusion from M to $M\times\{0\}$ and $M\times\{1\}$ respectively, then it's clear

$$\begin{aligned} p(\Theta)&=i_0^*p(\widetilde{\Omega})\\ p(\Theta')&=i_1^*p(\widetilde{\Omega})\end{aligned}$$

Furthermore, the homotopy invariance of de Rham cohomology impiles $i_0^*,i_1^*:H^{2k}(M\times\mathbb{R})\to H^{2k}(M)$ are the same map. □

Theorem 5.2.1 (Chern-Weil homomorphism)**.** There is a ring homomorphism

$$\begin{aligned} W(P,\text{-}):I(\mathfrak{g})&\to H^*(M)\\ p&\mapsto[p(\Theta)]\end{aligned}$$

Proof. It suffices to show

$$p\odot q(\Theta)=p(\Theta)\wedge q(\Theta)$$

Note that π^* is injective, thus it suffices to check

$$p\odot q(\Omega)=p(\Omega)\wedge q(\Omega)$$

and that's clear. □

5.3. **Transgression.** In this section we will show for a given principal G-bundle P and a connection ω on it with curvature Ω, $[p(\Omega)]=0\in H^{2k}(P)$, where $p\in I^k(\mathfrak{g}),k\geq 1$.

To see this, let's introduce the funtorial Chern-Weil homomorphism. Given the following homomorphism between principal G-bundles

$$\begin{tikzcd} P' \arrow[r, "\widetilde{f}"] \arrow[d, "\pi'"] & P \arrow[d, "\pi"] \\ M' \arrow[r, "f"] & M \end{tikzcd}$$

where $P'=f^*P$.

Proposition 5.3.1 (funtorial)**.** For all $p \in I(\mathfrak{g})$, we have

$$W(f^*P, p) = f^*W(P, p)$$

Proof. Given a connection $\omega \in \mathcal{A}(P)$ with curvature Ω, and use ω' to denote the pullback connection $\widetilde{f}^*\omega \in \mathcal{A}(P')$ with curvature Ω'. For any $p \in I(\mathfrak{g})$, it's clear

$$p(\Omega') = \widetilde{f}^* p(\Omega)$$

Then

$$\begin{aligned}(\pi')^*(p(\Theta')) &= \widetilde{f}^*\pi^* p(\Theta) \\ &= (\pi')^* f^* p(\Theta')\end{aligned}$$

which impiles $p(\Theta') = f^*p(\Theta)$, since $(\pi')^*$ is injective. $\square$

Example 5.3.1. Let $P = M \times G$ be trivial bundle, then we can regard it as

$$\begin{tikzcd} M \times G \arrow[r, "\widetilde{f}"] \arrow[d, "\pi'"] & G \arrow[d, "\pi"] \\ M \arrow[r, "f"] & \{\mathrm{pt}\} \end{tikzcd}$$

So for any $p \in I^k(\mathfrak{g}), k \geq 1$, we have

$$W(P, p) = f^*W(G, p) = 0$$

since $W(G, p) \in H^{2k}(\{\mathrm{pt}\}) = 0$ if $k \geq 1$.

Remark 5.3.1. This example shows if P is a trivial principal G-bundle, then the Chern-Weil homomorphism $W(P, \text{-})$ is trivial on $I(\mathfrak{g})$.

Now let's consider the following case

$$\begin{tikzcd} f^*P \arrow[r, "\widetilde{f}"] \arrow[d, "\pi'"] & P \arrow[d, "\pi"] \\ P \arrow[r, "f"] & M \end{tikzcd}$$

where $f = \pi$. In fact we can write f^*P down as

$$\begin{aligned} f^*P &= \{(x', x) \in P \times P \mid f(x') = \pi(x)\} \\ &= \{(x', x) \in P \times P \mid \pi(x') = \pi(x)\}\end{aligned}$$

It's clear it has global section, given by

$$\begin{aligned} s \colon P &\to f^*P \\ x &\mapsto (x, x)\end{aligned}$$

so f^*P is trivial principal bundle. Thus for any $p \in I^k(\mathfrak{g}), k \geq 1$, we have

$$W(f^*P, p) = 0 \in H^{2k}(P)$$

However, funtorial impiles

$$\begin{aligned} W(f^*P, p) &= f^*W(P, p) \\ &= f^*[p(\Theta)] \\ &= \pi^*[p(\Theta)] \\ &= p(\Omega) \end{aligned}$$

This shows $[p(\Omega)] = 0$ in $H^{2k}(P)$.

6. Characteristic class

6.1. Chern class.

Proposition 6.1.1. Let $G = \mathrm{U}(n)$ with Lie algebra $\mathfrak{g} = \mathfrak{u}(n)$. For any $X \in \mathfrak{g}$, consider

$$\det(I - \frac{t}{2\pi i}X) = \sum_{k=0}^{n} c_k(X)t^k$$

Then

1. For each $1 \leq k \leq n$, $c_k \in I(\mathfrak{g})$;
2. $I(\mathfrak{g})$ is generated by $c_1, \dots, c_n$

Proof. For (1). For arbitrary $g \in G$, note that

$$\begin{aligned}\det(I - \frac{t}{2\pi i}\operatorname{Ad}(g)X) &= \det(I - \frac{t}{2\pi i}gXg^{-1}) \\ &= \det(g^{-1}g - \frac{t}{2\pi i}gXg^{-1}) \\ &= \det(I - \frac{t}{2\pi i}X)\end{aligned}$$

For (2). Note that any $X \in \mathfrak{g}$ is diagonalizable, so without lose of generality we may assume $X = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$. Then $I(\mathfrak{g})$ consists of symmetric polynomial of $\lambda_1, \dots, \lambda_n$. Then the proof follows since any symmetric function can be expressed in terms of elementary symmetric functions and

$$\begin{aligned}c_1 &= -\frac{1}{2\pi}\lambda_1 + \dots + \lambda_n \\ &\vdots \\ c_n &= (\frac{1}{2\pi})^n\lambda_1 \dots \lambda_n\end{aligned}$$

□

Let E be a complex vector bundle of rank n over M equipped with a hermitian metric, then consider its frame bundle we obtain a $\mathrm{U}(n)$-principal bundle $\pi\colon P \to M$, then choose an arbitrary connection ω on it with curvature Ω, then by Chern-Weil theory there exists a unique $2k$-form $c_k(\Theta)$ on M such that $\pi^*(c_k(\Theta)) = c_k(\Omega)$.

Definition 6.1.1 (chern class)**.** The k-th Chern class of E is defined as

$$c_k := [c_k(\Theta)] \in H^{2k}(M, \mathbb{C})$$

Definition 6.1.2 (chern polynomial)**.** The Chern polynomial is defined as

$$c(t) = \det(I - \frac{t}{2\pi i}\Theta) = \sum_{k=0}^{n} c_k t^k$$

Proposition 6.1.2.

$$c_k \in H^{2k}(M, \mathbb{R})$$

Proof. Note that $\mathfrak{u}(n)$ consists of skew-symmetric matrices, then for arbitrary $X \in \mathfrak{u}(n)$, one has

$$\begin{aligned}\det(I - \frac{t}{2\pi i}X) &= \det(I + \frac{t}{2\pi i}\overline{X}^t) \\ &= \overline{\det(I - \frac{t}{2\pi i}X)} \\ &= \sum_{k=0}^{n} \overline{c}_k t^k\end{aligned}$$

which impiles $c_k = \overline{c}_k$. □

Proposition 6.1.3. Let E, F are two complex vector bundles, then

$$c(E \oplus F) = c(E)c(F)$$

Proof. If ∇_E, ∇_F are connections on E, F respectively, then $\nabla_E \oplus \nabla_F$ gives a connection on $E \oplus F$, with curvature $\begin{pmatrix} \Theta_E & 0 \\ 0 & \Theta_F \end{pmatrix}$, and thus

$$c(E \oplus F) = \det \begin{pmatrix} I - \frac{1}{2\pi i}\Theta_E & 0 \\ 0 & I - \frac{1}{2\pi i}\Theta_F \end{pmatrix} = c(E)c(F)$$

□

6.2. **Pontrjagin class.** Now let E be a (real) vector bundle of rank n over M equipped with a Riemannian metric, then its frame bundle is a $\mathrm{O}(n)$-principal bundle P. For any $X \in \mathfrak{o}(n)$, consider

$$\det(I - \frac{t}{2\pi}X) = \sum_{k=0}^{n} q_k(X)t^k$$

By the same argument as above one can show $q_k \in I(\mathfrak{g})$, thus we pick arbitrary connection ω of P with curvature Ω, then it gives rise to a closed $2k$-form $q_k(\Theta)$ on M for each k. Note that $X + X^t = 0$, then

$$\det(I + \frac{t}{2\pi}X) = \det(I - \frac{-t}{2\pi}X)$$

which impiles

$$q_k(X) = q_k(-X) = (-1)^k q_k(X)$$

Thus we can conclude $q_k = 0$ for odd k.

Definition 6.2.1 (Pontrjagin class)**.** $[p_k(\Theta)] := [q_{2k}(\Theta)] \in H^{4k}(M, \mathbb{R})$ is called k-th Pontrjagin class of E.

Proposition 6.2.1. Let E be a vector bundle with its complexification $E^c = E \otimes \mathbb{C}$, which is a complex vector bundle, then

$$p_k(E) = (-1)^k c_{2k}(E^c)$$

Proof. □

If we consider oriented vector bundle E, then its frame bundle is a $\mathrm{SO}(n)$-principal bundle. Then

Lemma 6.2.1. Let E be a oriented vector bundle of rank n, then
1. If $n = 2m + 1$, then $I(\mathfrak{so}(n))$ is generated by $q_2, \dots, q_{2m}$;
2. If $n = 2m$, then $I(\mathfrak{so}(n))$ is generated by $q_2, \dots, q_{2m}, e$, where

$$e(\mathrm{diag}\{\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \lambda_m \\ -\lambda_m & 0 \end{pmatrix}\}) = \lambda_1 \dots \lambda_m$$

Definition 6.2.2 (Euler class)**.** Let E be an oriented vector bundle of rank $2m$, then $[\frac{1}{(2\pi)^m} e(\Theta)] \in H^{2m}(M, \mathbb{R})$ is called the Euler class of E, denoted by $e(E)$.

Remark 6.2.1. For an oriented $2m$-dimensional manifold M, $e(TM)$ is the Euler number of M. See [JM74].

7. The classifying space

In last section, we have defined characteristic classes via a geometrical method, that is we use connections. However, they're topological invariants. In this section, we will give another explaination about characteristic class, and explain why it computes the right thing.

7.1. **The universal G-bundle.**

In this section, we work on category of topological spaces (In particular, CW-complexes) instead of smooth manifolds.

Definition 7.1.1 (weakly homotopy)**.** Let X, Y be topological spaces, X is weakly homotopy to Y, if there exists a continuous map $f\colon X\to Y$ such that f induces isomorphisms between homotopy groups of X and Y.

Definition 7.1.2 (weakly contractible)**.** A topological space X is called weakly contractible, if it's weakly homotopy to a point.

Remark 7.1.1. A contractible space is weakly contractible, and by Whitehead's theorem, a CW-complex is weakly contractible if and only if it's contractible.

Definition 7.1.3 (classifying space)**.** For a principal G-bundle $EG\to BG$, where EG, BG are topological spaces. If EG is weakly contractible, then

1. BG is called a classifying space for G;
2. EG is called a universal G-bundle.

Remark 7.1.2. Note that in definition the classifying space for G is just a topological space, in fact, we can choose it as a CW-complex. Indeed, since for any topological space, there exists a CW-complex which is weakly homotopic to it. Then for a classifying space BG, there exists a CW-complex BG' and a weakly homotopy $g: BG'\to BG$, then $g^*EG\to BG'$ is also a universal G-bundle.

Theorem 7.1.1. Let $EG\to BG$ be a universal G-bundle, then for all CW-complexes X, then the following map is bijective.

$$\begin{aligned}\Phi\colon [X, BG] &\to \mathcal{P}_G X\\ f &\mapsto f^*P\end{aligned}$$

where $[X, BG]$ denotes the set of all continuous maps up to homotopy.

Proof. See [Mit01]. □

Remark 7.1.3. This theorem impiles why BG is called classifying space, since it can be used to classify principal G-bundles over a given CW-complex.

However, untill now we still don't know whether classifying space exists or not. The following theorem is due to [Mil56].

Theorem 7.1.2. Let G be any topological group, then there exists a classifying space for G.

Now let's see some examples of classifying space for special Lie group G.

Proposition 7.1.1. Let G be a discrete group, then $PK(G,1) \to K(G,1)$ is a universal G-bundle, and hence $K(G,1)$ is a classifying space for G.

Proof. It's clear path space $PK(G,1)$ is contractible. □

Remark 7.1.4. In [Liu22] we have already computed $K(G,1)$ for groups, for example, $K(\mathbb{Z},1) = \mathbb{S}^1$, $K(\mathbb{Z}_2,1) = \mathbb{RP}^\infty$ and so on.

Proposition 7.1.2. $V_n(\mathbb{R}^\infty) \to Gr_n(\mathbb{R}^\infty)$ is a universal $\mathrm{GL}(n,\mathbb{R})$-bundle, and hence $Gr_n(\mathbb{R}^\infty)$ is a classifying space for $\mathrm{GL}(n,\mathbb{R})$.

Proof. It suffices to show $V_n(\mathbb{R}^\infty)$ is contractible. Since we have already computed low dimensional homotopy groups of $V_n(\mathbb{R}^N)$ in [Liu22], and then telescope construction completes the proof. □

Corollary 7.1.1. For all CW-complexes X, $[X, Gr_n(\mathbb{R}^\infty)] \to \mathrm{Vect}_n^{\mathbb{R}} X$.

Proof. See Remark 1.3.2. □

Remark 7.1.5. The analogous result with $\mathbb{R}$ replaced by $\mathbb{C}$ also holds.

7.2. **Homotopical properties of classifying spaces.** In this section we collect some Homotopical properties of classifying spaces.

Theorem 7.2.1. Let G be any topological group, then G is weakly equivalent to the loop space ΩBG.

Corollary 7.2.1. For $n \geq 1$, $\pi_n(BG) = \pi_{n-1}(G)$.

Theorem 7.2.2. Let G be a topological space and H a subgroup, then the homotopy fiber of $BH \to BG$ is G/H, up to weakly equivalent.

Theorem 7.2.3. Let G be a topological space and H a subgroup, then there is a fiberation $BH \to BG \to B(G/H)$.

Example 7.2.1. The exact sequences $1 \to \mathrm{SO}(n) \to \mathrm{O}(n) \to \mathbb{Z}_2 \to 1$ and $1 \to \mathrm{SU}(n) \to \mathrm{U}(n) \to S^1 \to 1$ give rise to fiberation

$$B\,\mathrm{SO}(n) \to B\,\mathrm{O}(n) \to \mathbb{RP}^\infty$$

and

$$B\,\mathrm{SU}(n) \to B\,\mathrm{U}(n) \to \mathbb{CP}^\infty$$

7.3. **Another viewpoint to characteristice class.**

Proposition 7.1. The cohomology ring of $B\,\mathrm{U}(n)$ with integer coefficients is $\mathbb{Z}[c_1,\dots,c_n]$.

Proof. If we consider $\mathrm{U}(n-1)$ as a subgroup of $\mathrm{U}(n-1)$, then we have the following filteration

$$\begin{tikzcd} S^{2n-1} \cong \mathrm{U}(n)/\mathrm{U}(n-1) \arrow[r] & B\,\mathrm{U}(n) \arrow[d] \\ & B\,\mathrm{U}(n-1) \end{tikzcd}$$

Apply Leray spectral sequence this this fiberation and use the fact that the cohomology ring of $\mathbb{CP}^\infty$ is $\mathbb{Z}[c_1]$ to conclude. $\square$

Definition 7.1 (universal Chern class)**.** The generators $c_1, \cdots, c_n$ of $H^*(B\,\mathrm{U}(n), \mathbb{Z})$ are called the universal Chern classes of $\mathrm{U}(n)$-bundles.

Definition 7.2 (Chern class)**.** The k-th Chern class of the $\mathrm{U}(n)$-bundle $\pi\colon E \to M$ with classifying map $f_\pi\colon M \to B\,\mathrm{U}(n)$ is defined as

$$c_k(E) := f_\pi^*(c_k) \in H^{2k}(M, \mathbb{Z})$$

Proposition 7.2. The cohomology ring of $B\,\mathrm{O}(n)$ with $\mathbb{Z}_2$ coefficients is $\mathbb{Z}_2[w_1, \ldots, w_n]$.

Proof. The same as above, just note that cohomology ring of $\mathbb{RP}^\infty$ with $\mathbb{Z}_2$ coefficient is $\mathbb{Z}_2[w_1]$. $\square$

Definition 7.3 (universal Steifel-Whitney class)**.** The generators $w_1, \cdots, w_n$ of $H^*(B\,\mathrm{O}(n), \mathbb{Z}_2)$ are called the universal Steifel-Whitney classes of $\mathrm{O}(n)$-bundles.

Definition 7.4 (Steifel-Whitney class)**.** The k-th Steifel-Whitney class of the $\mathrm{O}(n)$-bundle $\pi\colon E \to M$ with classifying map $f_\pi\colon M \to B\,\mathrm{O}(n)$ is defined as

$$w_k(E) := f_\pi^*(w_k) \in H^{2k}(M, \mathbb{Z}_2)$$

Part 3. Spin geometry

8. Clifford algebra, spin group and its representations

8.1. Clifford algebra.

8.1.1. *First properties.*

Definition 8.1.1 (quadratic space)**.** Let V be a k-vector space[4] and g is a symmetric bilinear form on V, the pair (V, g) is called a quadratic space.

Definition 8.1.2 (Clifford algebra)**.** Let (V, g) be a quadratic space, then Clifford algebra $\mathrm{Cl}(V, g)$ is the quotient

$$\mathrm{Cl}(V, g) := T(V)/I_g$$

where $T(V)$ is tensor algebra of V and I_g is the ideal in $T(V)$ generated by $\{v \otimes v + g(v, v)1 \mid v \in V\}$.

Notation 8.1.1. For convenience, we sometimes omit symbol $\otimes$, that is, simply use v^2 to denote $v \otimes v$.

Notation 8.1.2. For quadratic space (V, g), we always use π to denote natural projection $\pi\colon T(V) \to \mathrm{Cl}(V, g)$.

Remark 8.1.1. There is an injection $\iota\colon V \to \mathrm{Cl}(V, g)$, and we always identify $V \cong \iota(V) \subset \mathrm{Cl}(V, g)$.

Example 8.1.1. Let (V, g) be a quadratic space with $g = 0$, then Clifford algebra satisfies $v^2 = 0$, that is $\mathrm{Cl}(V, 0) = \bigwedge V$.

Example 8.1.2. Let $\mathbb{R}^{p,q}$ denote quadratic space $(\mathbb{R}^n, g_{p,q})$ with $g_{p,q}$ is symmetric bilinear form with signature (p, q), where $p + q = n$. The Clifford algebra $\mathrm{Cl}(\mathbb{R}^{p,q})$ is denoted by $\mathrm{Cl}_{p,q}$. Furthermore, $\mathrm{Cl}_n := \mathrm{Cl}_{n,0}$.

Example 8.1.3. Let $(\mathbb{C}^n, g)$ be complex vector space with standard symmetric bilinear form, its Clifford algebra is denoted by Cl^c_n.

Example 8.1.4. By definition $\mathrm{Cl}_1 = T(\mathbb{R})/I_{g_1}$, by fixing an orthonomal basis, one has $\mathrm{Cl}_1 \cong \mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$.

Proposition 8.1.1 (universal property)**.** Given any k-algebra A and a linear map $f\colon V \to A$ such that $f(v)f(v) = -2g(v, v)$, there exists a unique algebra map $\widetilde{f}\colon \mathrm{Cl}(V, g) \to A$ such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Cl}(V, g) \arrow[dr, dashed, "\widetilde{f}"] & \\
V \arrow[u, hook, "\iota"] \arrow[r, "f"'] & A
\end{tikzcd}$$

Corollary 8.1.1. For linear map $f\colon (V, g) \to (V, g')$ such that $f^*g' = g$, there is a unique map $\widetilde{f}\colon \mathrm{Cl}(V, g) \to \mathrm{Cl}(V', g')$.

Corollary 8.1.2. If $(V, g) \cong (V', g')$, then $\mathrm{Cl}(V, g) \cong \mathrm{Cl}(V', g')$.

[4]Unless otherwise specified, the base field of vector space is denoted by k.

8.1.2. *Grading of Clifford algebra.* Let (V,g) be a quadratic space, there is a natural $\mathbb{Z}$-grading on tensor algebra $T(V)$, and every $\mathbb{Z}$-grading algebra can be turned into a $\mathbb{Z}_2$-grading algebra by taking the direct sum of even and odd components. A crucial fact is that Clifford algebra does not inherit the $\mathbb{Z}$-grading, which can be seen by considering $v^{2m}\in T(V)_{2m}$, then

$$\pi(v^{2m})=(-1)^m g(v,v)^m\in\pi(T(V)_0)$$

However, the Clifford algebra inherits the $\mathbb{Z}_2$-grading of the tensor algebra.

Proposition 8.1.2. Let (V,g) be a quadratic space, then

$$\mathrm{Cl}(V,g)=\mathrm{Cl}^0(V,g)\oplus\mathrm{Cl}^1(V,g):=\pi(T(V)_0)\oplus\pi(T(V)_1)$$

is a $\mathbb{Z}_2$-grading.

Definition 8.1.3. The map $V\to V$, given by $v\mapsto -v$, induces an involution $\alpha\colon\ \mathrm{Cl}(V,g)\to\mathrm{Cl}(V,g)$.

Remark 8.1.2. $\mathbb{Z}_2$-grading of $\mathrm{Cl}(V,g)$ can be also viewed as eigen-decomposition with respect to involution α, and that's why some authors call α grading operator.

8.1.3. *Transpose and norm.*

Definition 8.1.4 (transpose)**.** The map $(v_1\dots v_m)^T:=v_m\dots v_1$ on $T(V)$ induces a map on $\mathrm{Cl}(V,q)$, called transpose.

Definition 8.1.5 (norm)**.** The norm is the map $N\colon\ \mathrm{Cl}(V,g)\to\mathrm{Cl}(V,g)$ defined by

$$N(\varphi)=\varphi\alpha(\varphi^T)$$

Remark 8.1.3. In particular, for $v\in V$, $N(v)=-v^2=g(v,v)$. That's why it's called norm.

8.2. **Pin and spin groups.**

8.2.1. *Twisted adjoint representation.* Let (V,g) be a quadratic space over field k and $\mathrm{Cl}(V,g)$ is its Clifford algebra. The $\mathrm{Cl}(V,g)^\times$ denotes the multiplicative group of invertible elements in $\mathrm{Cl}(V,g)$, that is

$$\mathrm{Cl}(V,g)^\times:=\{\varphi\in\mathrm{Cl}(V,g)\mid\text{there exists }\varphi^{-1}\in\mathrm{Cl}(V,g)\text{ such that }\varphi^{-1}\varphi=\varphi\varphi^{-1}=1\}$$

Note that $\mathrm{Cl}(V,g)^\times$ is an open submanifold of $\mathrm{Cl}(V,g)$, and therefore a Lie group with Lie algebra $\mathrm{Cl}(V,g)$.

Definition 8.2.1 (twisted adjoint representation)**.** The twisted adjoint representation is

$$\begin{aligned}\rho\colon\ \mathrm{Cl}(V,g)^\times&\to\mathrm{GL}(\mathrm{Cl}(V,g))\\ \varphi&\mapsto(\tau\mapsto\alpha(\varphi)\tau\varphi^{-1})\end{aligned}$$

Proposition 8.2.1. Let $v\in V$ with $g(v,v)\neq 0$. Then
1. $v\in\mathrm{Cl}(V,g)^\times$;

2. For any $w \in V$, one has
$$\rho(v)w = w - 2\frac{g(v,w)}{g(v,v)}v$$
that is $\rho(v)$ acts as a reflection by the hyperplane $v^\perp$;
3. $\rho(v)$ stabilizes V.

Proof. For (1). It's clear, since $v^2 + g(v,v) = 0$ impiles $v^{-1} = -v/g(v,v)$ if $g(v,v) \neq 0$.

For (2). Direct computation shows
$$\begin{aligned}\rho(v)w &= \alpha(v)wv^{-1} \\ &= \frac{vwv}{g(v,v)} \\ &\overset{(a)}{=} \frac{v(-vw - 2g(v,w))}{g(v,v)} \\ &\overset{(b)}{=} w - \frac{2g(v,w)}{g(v,v)}v\end{aligned}$$
where

(a) holds from the Clifford relation $vw + wv + g(v,w) = 0$;
(b) holds from the Clifford relation $-v^2 = g(v,v)$.

For (3). It follows from (2). □

Definition 8.2.2 (Clifford group)**.** The Clifford group $\Gamma(V,g)$ is the subgroup of $\mathrm{Cl}(V,g)^\times$ stabilizing V in the twisted adjoint representation.

Proposition 8.2.2. Suppose g is non-degenerate, then $\ker \rho|_{\Gamma(V,g)} = k^\times$.

Proof. Let $\{v_1,\dots,v_n\}$ be a basis of V such that $g(v_i,v_i) \neq 0$ and $g(v_i,v_j) = 0$ for $i \neq j$. Let $0 \neq \varphi \in \ker \rho|_{\Gamma(V,g)}$, then $\alpha(\varphi)v = v\varphi$ for all $v \in V$. Decompose $\varphi = \varphi_0 + \varphi_1$ via $\mathbb{Z}_2$-grading, then
$$\begin{aligned}\varphi_0 v &= v\varphi_0 \\ \varphi_1 v &= -v\varphi_1\end{aligned}$$
holds for all $v \in V$. Suppose $\varphi_0 = a_0 + v_1 a_1$, where a_0, a_1 do not involve v_1, then
$$(a_0 + v_1 a_1)v_1 = v_1(a_0 + v_1 a_1)$$
Note that by Clifford relation a_0 commutes with v_1, while a_1 anti-commutes with v_1, that is
$$a_0 v_1 - v_1^2 = a_0 v_1 + v_1^2 a_1$$
Together with $v_1^2 = -g(v_1,v_1) \neq 0$, this shows $a_1 = 0$, that is φ_0 does not contain v_1. Proceeding with a_0 we can show in the same way that it does not contain v_2 and so on. This shows φ_0 does not contain any elements in V. The same argument shows φ_1 also does not contain any elements in V, that is $\varphi \in k^\times$. □

Proposition 8.2.3. Suppose g is non-degenerate, then $N\colon \Gamma(V,g)\to k^\times$ is a group homomorphism.

Proof. Firstly let's check for $\varphi\in\Gamma(V,g)$, one has $N(\varphi)\in k^\times$. Indeeed, by definition one has $\alpha(\varphi)v\varphi^{-1}\in V$ for all $v\in V$, and note that transpose acts trivially on V, thus

$$\alpha(\varphi)v\varphi^{-1}=(\alpha(\varphi)v\varphi^{-1})^T=(\varphi^{-1})^Tv(\alpha(v))^T=(\varphi^T)^{-1}v\alpha(\varphi^T)$$

where the last equality holds, since transpose commutes with taking inverse and ε, then

$$v=\varphi^T\alpha(\varphi)v\varphi^{-1}\alpha(\varphi^T)^{-1}=\rho(\alpha(\varphi^T)\varphi)v$$

It's clear both φ^T and $\alpha(\varphi)$ lie in $\Gamma(V,g)$, since $\varphi\in\Gamma(V,g)$. In particular, $\alpha(\varphi^T)\varphi\in\Gamma(V,g)$, since it's a group. According to Proposition 8.2.2, one has

$$\alpha(\varphi^T)\varphi\in\ker\rho|_{\Gamma(V,g)}=k^\times$$

Applying α you obtain $N(\varphi^T)=\varphi^T\alpha(\varphi)\in k^\times$, which completes the proof of first part. Now let's show N is a group homomorphism. Direct computation shows

$$N(\varphi\tau)=\varphi\tau\alpha(\tau^T)\alpha(\varphi^T)=\varphi N(\tau)\alpha(\varphi^T)=N(\varphi)N(\tau)$$

□

8.2.2. *Pin and spin groups.*

Definition 8.2.3 (pin group)**.** The pin group $\mathrm{Pin}(V,g)$ is the subgroup of $\mathrm{Cl}(V,g)$ generated by elements $v\in V$ with $g(v,v)=1$.

Definition 8.2.4 (spin group)**.** The spin group $\mathrm{Spin}(V,g)$ is given by $\mathrm{Spin}(V,g):=\mathrm{Pin}(V,g)\cap\mathrm{Cl}^0(V,g)$.

Example 8.2.1. Again, standard pin and spin groups are

$$\mathrm{Pin}(p,q):=\mathrm{Pin}(\mathbb{R}^{p,q})$$
$$\mathrm{Spin}(p,q):=\mathrm{Spin}(\mathbb{R}^{p,q})$$

and

$$\mathrm{Pin}(n):=\mathrm{Pin}(n,0)$$
$$\mathrm{Spin}(n):=\mathrm{Spin}(n,0)$$

Theorem 8.2.1. There are exact sequences

$$1\to\mathbb{Z}_2\to\mathrm{Pin}(p,q)\xrightarrow{\rho}\mathrm{O}(p,q)\to 1$$
$$1\to\mathbb{Z}_2\to\mathrm{Spin}(p,q)\xrightarrow{\rho}\mathrm{SO}(p,q)\to 1$$

Proof. Here we only prove the first exact sequence, the proof for the other one is the same as this one. Note that $\mathrm{Pin}(V,g)$ is a subgroup of Clifford group according to Proposition 8.2.1, and $\rho(\varphi)$ acts on $\mathbb{R}^{p,q}$ as reflections, where $\varphi\in\mathrm{Pin}(V,g)$. By the theorem of Cartan-Dieudommé, every element of $g\in\mathrm{O}(p,q)$ is a product of reflections, hence $\mathrm{Pin}(V,g)$ surjects on $\mathrm{O}(p,q)$.

Let $\varphi \in \ker \rho \cap \mathrm{Pin}(V, g)$, then by Proposition 8.2.2 one has $\varphi \in k^\times$, thus $N(\varphi) = \varphi^2$. On the other hand, suppose $\varphi = v_1 \dots v_m$, then $N(\varphi) = N(v_1) \dots N(v_m) = 1$. This shows $\varphi \in \ker \rho \cap \mathrm{Pin}(V, g)$ if and only if $\varphi = \pm 1$. $\square$

Remark 8.2.1. Furthermore, $\rho\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is its universal covering if $n \geq 3$. Indeed, by homotopy exact sequence one has $\pi_1(\mathrm{SO}(n)) = \pi_1(\mathrm{SO}(3))$ for all $n > 3$ and $\mathrm{SO}(3)$ is exactly $\mathbb{RP}^2$, that is $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ for all $n \geq 3$. Now it suffices to show $\pm 1 \in \mathrm{Spin}(n)$ are connected by a continuous path. Such a path is given by

$$\gamma(t) = (e_1 \cos\frac{t}{2} + e_2 \sin\frac{t}{2})(-e_1 \cos\frac{t}{2} + e_2 \sin\frac{t}{2}) \in \mathrm{Spin}(n)$$

where $0 \leq t \leq \frac{\pi}{2}$ and $\{e_1, e_2, \dots, e_n\}$ is a orthonomal basis of $\mathbb{R}^{n,0}$.

Example 8.2.2. Note that $\mathrm{SO}(2) \cong S^1$ and the double covering of S^1 is exactly the map $S^1 \to S^1$, defined by $z \mapsto z^2$. This shows $\mathrm{Spin}(2) \cong S^1$.

Example 8.2.3. Note that $\mathrm{SO}(3) \cong \mathbb{RP}^3$, and $S^3 \to \mathbb{RP}^3$ is a double covering, this shows $\mathrm{Spin}(3) \cong S^3$.

Example 8.2.4. $\mathrm{Spin}(4) \cong S^3 \times S^3$.

8.2.3. *Lie algebra of spin group.*

Proposition 8.2.4. $\mathfrak{spin}(n) = \mathrm{span}\{e_ie_j \mid 1 \leq i < j \leq n\}$.

Proof. For $1 \leq i < j \leq n$, consider

$$\begin{aligned}\gamma(t) &= \cos t + e_ie_j \sin t \\ &= -(e_i \cos\frac{t}{2} + e_j \sin\frac{t}{2})(e_i \cos\frac{t}{2} - e_j \sin\frac{t}{2}) \in \mathrm{Spin}(n)\end{aligned}$$

and note that $\gamma'(0) = e_ie_j$, this shows

$$\mathrm{span}\{e_ie_j \mid 1 \leq i < j \leq n\} \subseteq \mathfrak{spin}(n)$$

Then counting dimension to conclude. $\square$

Proposition 8.2.5. The isomorphism $\rho_*\colon \mathfrak{spin}(n) \to \mathfrak{so}(n)$ is given by

$$\rho_*(e_ie_j) = 2E_{ij}$$

where E_{ij} is matrix with -1 in (i, j)-entry and 1 in (j, i)-entry.

Proof. For $1 \leq i < j \leq n$, consider $\gamma(t)$ $\square$

8.3. **Classification of real and complex Clifford algebras.**

8.3.1. *Classification of real Clifford algebras.*

Theorem 8.3.1. There are isomorphisms

$$\begin{aligned}\mathrm{Cl}_{n,0}\otimes\mathrm{Cl}_{0,2}&\cong\mathrm{Cl}_{0,n+2}\\ \mathrm{Cl}_{0,n}\otimes\mathrm{Cl}_{2,0}&\cong\mathrm{Cl}_{n+2,0}\\ \mathrm{Cl}_{p,q}\otimes\mathrm{Cl}_{1,1}&\cong\mathrm{Cl}_{p+1,q+1}\end{aligned}$$

Proposition 8.3.1.

$$\begin{aligned}\mathrm{Cl}_{1,0}&\cong\mathbb{C}\\ \mathrm{Cl}_{2,0}&\cong\mathbb{H}\\ \mathrm{Cl}_{0,1}&\cong\mathbb{R}\oplus\mathbb{R}\\ \mathrm{Cl}_{0,2}&\cong M_2(\mathbb{R})\\ \mathrm{Cl}_{1,1}&\cong M_2(\mathbb{R})\end{aligned}$$

Corollary 8.3.1. The following is table of Clifford algebras $\mathrm{Cl}_{n,0}$ and $\mathrm{Cl}_{0,n}$ for $n\leq 8$.

n	$\mathrm{Cl}_{n,0}$	$\mathrm{Cl}_{0,n}$
1	$\mathbb{C}$	$\mathbb{R}\oplus\mathbb{R}$
2	$\mathbb{H}$	$M_2(\mathbb{R})$
3	$\mathbb{H}\oplus\mathbb{H}$	$M_2(\mathbb{C})$
4	$M_2(\mathbb{H})$	$M_2(\mathbb{H})$
5	$M_4(\mathbb{C})$	$M_2(\mathbb{H})\oplus M_2(\mathbb{H})$
6	$M_8(\mathbb{R})$	$M_4(\mathbb{H})$
7	$M_8(\mathbb{R})\oplus M_8(\mathbb{R})$	$M_8(\mathbb{C})$
8	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{R})$

8.3.2. *Classification of complex Clifford algebras.*

Theorem 8.3.2. There is an isomorphism

$$\mathrm{Cl}^c_{n+2}\cong\mathrm{Cl}^c_n\otimes\mathrm{Cl}^c_2$$

Corollary 8.3.2. Let $n\in\mathbb{N}$, then

$$\mathrm{Cl}^c_n=\begin{cases}M_{2^k}(\mathbb{C})\oplus M_{2^k}(\mathbb{C}) & n=2k+1\\ M_{2^k}(\mathbb{C}) & n=2k\end{cases}$$

8.4. **Spin representation.** In this section we study some representations of Clifford algebras and spin groups, which will play an important role in associated vector bundles of principal $\mathrm{Spin}(n)$-bundle.

Definition 8.4.1 (complex spinors)**.** The vector space of complex n-spinors is defined to be

$$\Delta_n=\mathbb{C}^{2^{\lfloor n/2\rfloor}}$$

Elements of Δ_n are called complex spinors.

According to Corollary 8.3.2, one has

$$\mathrm{Cl}_n^c = \begin{cases} \mathrm{End}(\Delta_n) \oplus \mathrm{End}(\Delta_n) & n = 2k+1 \\ \mathrm{End}(\Delta_n) & n = 2k \end{cases}$$

So $\mathrm{Cl}_n^c \to \mathrm{End}(\Delta_n)$ is identity when n is even, and projection when n is odd. In this way any element Cl_n^c can act on complex spinors, this is called Clifford multiplication.

Definition 8.4.2 (Clifford multiplication)**.** The multiplication by $v \in \mathbb{R}^n$, denoted by is endomorphism $c(v) \in \mathrm{End}(\Delta_n)$ given by $\mathbb{R}^n \subset \mathrm{Cl}_n \subset \mathrm{Cl}_n^c \to \mathrm{End}(\Delta_n)$.

Definition 8.4.3 (spin representation)**.** The composition $\Delta_n \colon \mathrm{Spin}(n) \hookrightarrow \mathrm{Cl}_n \hookrightarrow \mathrm{Cl}_n^c \to \mathrm{End}(\Delta_n)$ is called the spin representation of $\mathrm{Spin}(n)$.

Definition 8.4.4 (complex volumn element)**.** The complex volumn element $\omega_{\mathbb{C}} \in \mathrm{Cl}_n^c$ is

$$\omega_{\mathbb{C}} = \sqrt{-1}^{\lfloor \frac{n+1}{2} \rfloor} e_1 \dots e_n$$

where $\{e_i\}$ is an orthonomal basis.

Remark 8.4.1*.* The volumn element is independent of the choice of orthonomal basis if we fix the orientation, and $\omega_{\mathbb{C}}^2 = 1$.

Lemma 8.4.1. If n is odd, $\omega_{\mathbb{C}}$ commutes with every element of the Clifford algebra. If n is even, $\omega_{\mathbb{C}}$ commutes with elements of Cl_n^0 and anti-commutes with Cl_n^1.

Proof. It suffices to look at the commutativity of $\omega_{\mathbb{C}}$ with a unit vector e. We extend e into a positively oriented orthonomal basis $e_1 = 1, e_2, \dots, e_n$ of Cl_n^c. In terms of this basis, $\omega_{\mathbb{C}}$ clearly commutes with e when n is odd and anti-commutes with e when n is even. $\square$

Definition 8.4.5 (Weyl spinors)**.** Elements of $\Delta_n^{\pm}$ are called Weyl spinors of $\pm$ chirality.

Theorem 8.4.1. If n is odd, Δ_n is an irreducible representation of $\mathrm{Spin}(n)$. If n is even, Δ_n decomposes into $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$ two irreducible representations of $\mathrm{Spin}(n)$. Furthermore, the Clifford multiplication interchanges $\Delta_n^{\pm}$.

9. Spin structure

9.1. **The first Steifel-Whitney class and orientablity.** Let (M,g) be a Riemannian n-manifold, $\mathfrak{U}=\{U_\alpha\}$ a good cover of M, and transition functions of TM with respect to $\mathfrak{U}$ is denoted by $\{g_{\alpha\beta}\colon U_{\alpha\beta}\to \mathrm{O}(n)\}$. Consider

$$c_{\alpha\beta}:=\det g_{\alpha\beta}=\pm 1\in\mathbb{Z}_2$$

which is continuous, and since $U_{\alpha\beta}$ is contractible, then $c_{\alpha\beta}$ is constant, and hence gives rise to a Čech 1-cochain $c\in C^1(\mathfrak{U},\mathbb{Z}_2)$. Furthermore, it defines a 1-cocycle. Indeed, direct computation shows

$$\begin{aligned}(\mathrm{d}c)_{\alpha\beta\gamma}&=c_{\beta\gamma}c_{\alpha\gamma}^{-1}c_{\alpha\beta}\\&=\det g_{\beta\gamma}\det g_{\gamma\alpha}\det g_{\alpha\beta}\\&=1\end{aligned}$$

Definition 9.1.1 (Steifel-Whitney class)**.** The cohomology class $[c]:=w_1(M)\in\check{H}^1(M,\mathbb{Z}_2)$ defined above is called the first Steifel-Whitney class of M.

Theorem 9.1.1. The first Steifel-Whitney class vanishes if and only if M is orientable.

Proof. Suppose M is orientable, for each good cover, it admits a refinement such that transition functions $g_{\alpha\beta}\colon U_{\alpha\beta}\to\mathrm{SO}(n)$, this shows first Steifel-Whitney class with respect to this cover vanishes, that is $w_1(M)=0\in\check{H}^1(M,\mathbb{Z}_2)$, since good cover is cofinal. Conversely, if first Steifel-Whitney class vanishes, then for local coordinates $(U_\alpha,\varphi_\alpha)$, without lose of generality we may assume $c_{\alpha\beta}=(\mathrm{d}s)_{\alpha\beta}=s_\beta s_\alpha^{-1}$, otherwise we can consider its refinement. Then consider coordinates $(U_\alpha,\varphi'_\alpha)$ given by $\varphi'_\alpha=s_\alpha\circ\varphi_\alpha$. With respect to this coordinates the transition functions $g'_{\alpha\beta}$ satisfy

$$\det g'_{\alpha\beta}=\det s_\beta\det g_{\alpha\beta}\det s_\alpha^{-1}=\det g_{\alpha\beta}^2=1$$

This shows M is orientable. □

9.2. **The second Steifel-Whitney class and spin structure.** Recall Example 1.4.3, we only talk about spin structure on orientable Riemannian manifold (M,g). So from now on we assume (M,g) is an orientable Riemannian n-manifold, and $\{g_{\alpha\beta}\colon U_{\alpha\beta}\to\mathrm{SO}(n)\}$ is transition functions of TM with respect to good cover $\mathfrak{U}$. Choose a lift $\widetilde{g}_{\alpha\beta}\colon U_{\alpha\beta}\to\mathrm{Spin}(n)$ such that

$$\widetilde{g}_{\alpha\beta}=\widetilde{g}_{\beta\alpha}$$

and define

$$\varepsilon_{\alpha\beta\gamma}=\widetilde{g}_{\alpha\gamma}\widetilde{g}_{\beta\alpha}\widetilde{g}_{\gamma\beta}$$

Now we're going to show this assignment gives rise to a Čech 2-cocycle valued in $\mathbb{Z}_2$ which is independent of the lift, which is divided into following lemmas.

Lemma 9.2.1. For arbitrary α, β, γ, one has

$$\varepsilon_{\alpha\beta\gamma} \in \ker \pi \cong \mathbb{Z}_2$$

where $\pi\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is double covering.

Proof. Direct computation shows

$$\begin{aligned}\pi(\varepsilon_{\alpha\beta\gamma}) &= \rho(\widetilde{g}_{\alpha\gamma}\widetilde{g}_{\beta\alpha}\widetilde{g}_{\gamma\beta})\\ &= g_{\alpha\gamma}g_{\beta\alpha}g_{\gamma\beta}\\ &= 1\end{aligned}$$

□

Corollary 9.2.1. For arbitrary α, β, γ, one has

$$\varepsilon_{\alpha\beta\gamma} = \varepsilon_{\gamma\beta\alpha}$$

Proof. Direct computation shows

$$\varepsilon_{\alpha\beta\gamma} = \varepsilon^{-1}_{\alpha\beta\gamma} = \widetilde{g}_{\gamma\alpha}\widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma} = \varepsilon_{\gamma\beta\alpha}$$

□

Lemma 9.2.2. $\varepsilon \in C^2(\mathfrak{U}, \mathbb{Z}_2)$ defines a Čech 2-cocycle.

Proof. Direct computation shows

$$\begin{aligned}(\mathrm{d}\varepsilon)_{\alpha\beta\gamma\delta} &= \varepsilon_{\beta\gamma\delta}\varepsilon^{-1}_{\alpha\gamma\delta}\varepsilon_{\alpha\beta\delta}\varepsilon^{-1}_{\alpha\beta\gamma}\\ &= \widetilde{g}_{\beta\delta}\widetilde{g}_{\gamma\beta}\widetilde{g}_{\delta\gamma}(\widetilde{g}_{\alpha\delta}\widetilde{g}_{\gamma\alpha}\widetilde{g}_{\delta\gamma})^{-1}\widetilde{g}_{\alpha\delta}\widetilde{g}_{\beta\alpha}\widetilde{g}_{\delta\beta}(\widetilde{g}_{\alpha\gamma}\widetilde{g}_{\beta\alpha}\widetilde{g}_{\gamma\beta})^{-1}\\ &= 1\end{aligned}$$

□

Lemma 9.2.3. The Čech cohomology class of ε is independent of the lift $\widetilde{g}_{\alpha\beta}$.

Proof. Suppose $\widetilde{g}_{\alpha\beta}$ and $\widetilde{g}'_{\alpha\beta}$ are lifts of $g_{\alpha\beta}$, then $\kappa_{\alpha\beta} = \widetilde{g}_{\alpha\beta}\widetilde{g}'_{\beta\alpha}$ satisfies $\rho(\kappa_{\alpha\beta} = 1)$, hence κ is a Čech 1-cochain. Direct computation shows

$$\begin{aligned}\varepsilon_{\alpha\beta\gamma}(\mathrm{d}\kappa)_{\alpha\beta\gamma} &= \widetilde{g}_{\alpha\gamma}\widetilde{g}_{\beta\alpha}\widetilde{g}_{\gamma\beta}\kappa_{\beta\gamma}\kappa^{-1}_{\alpha\gamma}\kappa_{\alpha\beta}\\ &= \widetilde{g}_{\alpha\gamma}\widetilde{g}_{\gamma\alpha}\widetilde{g}'_{\gamma\alpha}\cdot\widetilde{g}_{\beta\alpha}\widetilde{g}_{\alpha\beta}\widetilde{g}'_{\alpha\beta}\cdot\widetilde{g}_{\gamma\beta}\widetilde{g}_{\beta\gamma}\widetilde{g}'_{\beta\gamma}\\ &= \widetilde{g}'_{\gamma\alpha}\widetilde{g}'_{\alpha\beta}\widetilde{g}'_{\beta\gamma}\\ &= \varepsilon'_{\gamma\beta\alpha}\\ &= \varepsilon'_{\alpha\beta\gamma}\end{aligned}$$

This shows $\varepsilon'\varepsilon^{-1} = \mathrm{d}\kappa$, which completes the proof. □

Definition 9.2.1. The cohomology class $w_2(M) := [\varepsilon] \in \check{H}^2(M, \mathbb{Z}_2)$ is called the second Steifel-Whitney class of M.

Theorem 9.2.1. (M, g) admits a spin structure if and only if the second Steifel-Whitney class vanishes. Furthermore, if (M, g) admits spin structure, then there is an one to one correspondence

$$H^1(M, \mathbb{Z}_2) \longleftrightarrow \{\text{isomorphism classes of spin structures}\}$$

Proof. See [LM16]. $\square$

10. Spinor bundle, spin connection and Dirac operator

10.1. **Spinor bundle.** Let (M, g) be a Riemannian n-manifold admitting a spin structure P.

Definition 10.1.1 (spinor bundle)**.** The spinor bundle S_n associated to P is the associated vector bundle given by spin representation, that is

$$S_n = P \times_{\Delta_n} \Delta_n$$

Remark 10.1.1. Recall if n is even, then $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$ splits as a direct sum of irreducible representations, this impiles a splitting of the spinor bundle as $S_n = S_n^+ \oplus S_n^-$.

Definition 10.1.2 (Clifford bundle)**.** The Clifford bundle is the vector bundle over M with typical fiber the Clifford algebra $\mathrm{Cl}(M)_x := \mathrm{Cl}(T_x^*M, g_x)$.

Remark 10.1.2.

Proposition 10.1.1. The Clifford multiplication $\mathbb{R}^n \times \Delta_n \to \Delta_n$ extends to a map of sections

$$\begin{aligned} c\colon C^\infty(M, T^*M) \times S_n &\to S_n \\ (\theta, \psi) &\mapsto c(\theta)\psi \end{aligned}$$

Proof. Let $\{U_\alpha\}$ be a local trivialization for both T^*M and S_n. On U_α a section θ of T^*M is given by $\theta_\alpha\colon U_\alpha \to \mathbb{R}^n$ and a section ψ of S_n is given by $\psi_\alpha\colon U_\alpha \to \Delta_n$. Then $c(\theta)\psi$ on U_α is defined as

$$(c(\theta)\psi)_\alpha := c(\theta_\alpha)\psi_\alpha$$

Now it suffices to check it's well-defined, that is

$$(c(\theta)\psi)_\alpha = \Delta_n(\widetilde{g}_{\alpha\beta})(c(\theta)\psi)_\beta$$

since $\{\Delta_n(\widetilde{g}_{\alpha\beta})\}$ are transition functions of S_n. Direct computation shows

$$\begin{aligned} (c(\theta)\psi)_\alpha &= c(\theta_\alpha)\psi_\alpha \\ &= c(g_{\alpha\beta}\theta_\beta)\Delta_n(\widetilde{g}_{\alpha\beta})\psi_\beta \\ &= c(\rho(\widetilde{g}_{\alpha\beta})\theta_\beta)\Delta_n(\widetilde{g}_{\alpha\beta})\psi_\beta \\ &= \Delta_n(\widetilde{g}_{\alpha\beta})c(\theta_\beta)\Delta_n(\widetilde{g}_{\alpha\beta}^{-1})\Delta_n(\widetilde{g}_{\alpha\beta})\psi_\beta \\ &= \Delta_n(\widetilde{g}_{\alpha\beta})c(\theta_\beta)\psi_\beta \\ &= \Delta_n(\widetilde{g}_{\alpha\beta})(c(\theta)\psi)_\beta \end{aligned}$$

□

Remark 10.1.3. In the proof, the key point is $g_{\alpha\beta} = \rho(\widetilde{g}_{\alpha\beta})$, that is, without the spin structure, we can not define the Clifford multiplication.

10.2. **Spin connection.**

Proposition 10.2.1. Let (M, g) be a Riemannian manifold admitting a spin structure P, then any connection ∇ on principal $\mathrm{SO}(n)$-bundle TM naturally induces a connection on P, which in turns gives a connection on the spinor bundle

$$\nabla^S\colon C^\infty(M, S) \to C^\infty(M, T^*M \otimes S)$$

Furthermore, it's compatible with Clifford multiplication, that is

$$\nabla^S_X(c(v)\psi) = c(\nabla_X v)\psi + c(v)\nabla^S_X\psi$$

Proof. □

Lemma 10.2.1.

$$\nabla^S(c(\omega_{\mathbb{C}})\psi) = c(\omega_{\mathbb{C}})\nabla^S\psi$$

Corollary 10.2.1. If n is even, ∇^S is compatible with the splitting $S = S^+ \oplus S^-$. In other words, ∇^S is diagonal in this decomposition.

10.3. **Dirac operators.**

Let (M, g) be a Riemannian manifold with spin structure, and S is a spinor bundle over M.

Definition 10.3.1 (Dirac operator)**.** The Dirac operator D

$$D\colon C^\infty(M, S) \to C^\infty(M, S)$$

is the composition

$$C^\infty(M, S) \xrightarrow{\nabla^S} C^\infty(M, T^*M \otimes S) \xrightarrow{c} C^\infty(M, S)$$

Remark 10.3.1 (local form)*.* In local orthonomal basis $\{e_i\}$, one has

$$D = c(e_i)\nabla^S_{e_i}$$

where we identify e_i^* with e_i using Riemannian metric.

10.4. **Clifford module.**

Part 4. The Yang-Mills equations on Riemannian manifold

11. The Yang-Mills equations

In this section we assume G is a compact Lie group, since we desire Killing form of G is non-degenerate, and (M, g) is an oriented compact Riemannian manifold, since we need to consider integration.

11.1. **The Yang-Mills functional.** Let P be a principal G-bundle, V is a vector space and $\rho : G \to \mathrm{GL}(V)$ is a representation of G. If we want to construct an inner product on $\Omega^k_M(P \times_\rho V)$, firstly on each local trivialization U_α, view such forms as forms with values in V, so all we need is an inner product on V, since we already have a Riemannian metric g on M, which induces an inner product on forms.

But if we desire such inner product $\langle$-,-$\rangle$ can be glued well on overlaps, we need to require that it is G-invariant, that is, for all $g \in G, v, w \in V$,

$$\langle \rho(g)w, \rho(g)w\rangle = \langle v, w\rangle$$

since if $\omega \in C^\infty(M, \Omega^k_M(P \times_\rho V))$ is represented locally by $\omega_\alpha \in C^\infty(U_\alpha, \Omega^k_{U_\alpha}(V))$, then on a non-empty overlap $U_{\alpha\beta}$, we have $\omega_\alpha = \rho(g_{\alpha\beta})\omega_\beta$.

The case we're most interested in is $V = \mathfrak{g}$, since curvature of a connection is a section of $\Omega^2_M(\mathrm{ad}\,\mathfrak{g})$. So we what we need is an inner product on Lie algebra $\mathfrak{g}$ which is invariant under the adjoint action. Since G is compact, its Killing form is a non-degenerate inner product, that's what we're looking for!

Thus we have an pointwise inner product on the bundle $\Omega^k_M(\mathrm{ad}\,\mathfrak{g})$, and denote it by $\langle$-,-$\rangle$, and define a global inner product on $\Omega^k_M(\mathrm{ad}\,\mathfrak{g})$ as

$$(\alpha, \beta) := \int_M \langle \alpha, \beta\rangle \,\mathrm{vol}$$

where $\alpha, \beta \in C^\infty(M, \Omega^k_M(\mathrm{ad}\,\mathfrak{g}))$.

Definition 11.1.1 (Hodge star operator)**.** There exists an operator

$$* : C^\infty(M, \Omega^k_M(\mathrm{ad}\,\mathfrak{g})) \to C^\infty(M, \Omega^{n-k}_M(\mathrm{ad}\,\mathfrak{g}))$$

For $\beta \in C^\infty(M, \Omega^k_M(\mathrm{ad}\,\mathfrak{g}))$, $*\beta$ is given by

$$\alpha \wedge *\beta = \langle \alpha, \beta\rangle \,\mathrm{vol}, \quad \forall \alpha \in C^\infty(M, \Omega^k_M(\mathrm{ad}\,\mathfrak{g}))$$

With these preliminary results established, we arrive at the Yang-Mills functional.

Definition 11.1.2 (Yang-Mills functional)**.** The Yang-Mills functional is the map $YM : \mathcal{A}(P) \to \mathbb{R}$ given by

$$YM(\omega) := \|F_\omega\|^2 = \int_M \langle F_\omega, F_\omega\rangle \,\mathrm{vol}$$

where F_ω is curvature of connection ω, which is a section of $\Omega^2_M(\mathrm{ad}\,\mathfrak{g})$.

Remark 11.1.1. By using Hodge star operator, we may rewrite Yang-Mills functional as follows

$$YM(\omega) = \int_M F_\omega \wedge *F_\omega$$

The advantages of writing Yang-Mills functional in this way is that we can use some properties of Hodge operator to simplify our computations.

Proposition 11.1.1. Yang-Mills functional YM is gauge invariant, that is for any gauge transformation $\Phi \in \mathcal{G}(P)$, one has $YM(\Phi^*\omega) = YM(\omega)$ holds for connection ω.

Proof. On each local trivialization U_α, the curvature of $\Phi^*\omega$ is given by $\mathrm{ad}(\phi^{-1}) \circ F_\alpha$, where ϕ is given by $\Phi|_{U_\alpha}(x, g) = (x, \phi(x)g)$, thus Yang-Mills functional is gauge invariant follows from inner product $\langle -, - \rangle$ is adjoint invariant. $\square$

Definition 11.1.3 (Yang-Mills connection)**.** A Yang-Mills connection is a connection $A \in \mathcal{A}(P)$ which is a local extremum of Yang-Mills functional.

Notation 11.1.1. $\mathcal{A}_{YM}(P)$, or briefly $\mathcal{A}_{YM}$ denotes the set of all Yang-Mills connections.

11.2. **The variational problem.** Let's see how to use a second-order partial differential equation to characterize Yang-Mills connection. Recall that $\mathcal{A}(P)$ is an affine space modelled on $\Omega^1_M(\mathrm{ad}\,\mathfrak{g})$. This means the tangent space to $\mathcal{A}(P)$ at any point is isomorphic to $\Omega^1_M(\mathrm{ad}\,\mathfrak{g})$.

Given $\omega \in \mathcal{A}(P)$ and $\tau \in C^\infty(M, \Omega^1_M(\mathrm{ad}\,\mathfrak{g}))$. The directional derivative of Yang-Mills functional at ω in the direction τ is given by

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} YM(\omega + t\tau)$$

And Yang-Mills condition states that this vanishes for all τ. In order to see what this means, firstly we need the following lemma.

Lemma 11.2.1. Given $\omega \in \mathcal{A}(P)$ and $\tau \in C^\infty(M, \Omega^1_M(\mathrm{ad}\,\mathfrak{g}))$, then

$$F_{\omega+\tau} = F_\omega + \mathrm{d}_\omega \tau + \frac{1}{2}\tau \wedge \tau$$

where d_ω is connection induced by ω on $\Omega^1_M(\mathrm{ad}\,\mathfrak{g})$.

Proof. On local trivialization U_α one has

$$\begin{aligned}
(F_{\omega+\tau})_\alpha &= \mathrm{d}(A_\alpha + \tau_\alpha) + \frac{1}{2}(A_\alpha + \tau_\alpha) \wedge (A_\alpha + \tau_\alpha) \\
&= (F_\omega)_\alpha + \mathrm{d}\tau_\alpha + \frac{1}{2}(A_\alpha \wedge \tau_\alpha + \tau_\alpha \wedge A_\alpha) + \frac{1}{2}\tau_\alpha \wedge \tau_\alpha \\
&\overset{(1)}{=} (F_\omega)_\alpha + \mathrm{d}\tau_\alpha + A_\alpha \wedge \tau_\alpha + \frac{1}{2}\tau_\alpha \wedge \tau_\alpha \\
&\overset{(2)}{=} (F_\omega)_\alpha + \mathrm{d}_\omega \tau_\alpha + \frac{1}{2}\tau_\alpha \wedge \tau_\alpha
\end{aligned}$$

where

(1) holds from both A_α, τ_α are 1-form valued in $\mathfrak{g}$;
(2) holds from (3.2).

□

Proposition 11.2.1 (first variation formula)**.** Let ω be a Yang-Mills connection, then we have

$$\mathrm{d}_\omega^* F_\omega = 0$$

Proof. Direct computation shows

$$\begin{aligned} YM(\omega + t\tau) &= \int_M \langle F_{\omega+t\tau}, F_{\omega+t\tau}\rangle \,\mathrm{vol} \\ &= \int_M \langle F_\omega + \frac{t^2}{2}(\tau\wedge\tau) + t\mathrm{d}_\omega\tau, F_\omega + \frac{t^2}{2}(\tau\wedge\tau) + t\mathrm{d}_\omega\tau\rangle \,\mathrm{vol} \end{aligned}$$

The coefficient of linear term is

$$\int_M \langle F_\omega, \mathrm{d}_\omega\tau\rangle + \langle \mathrm{d}_\omega\tau, F_\omega\rangle \,\mathrm{vol} = 2\int_M \langle \mathrm{d}_\omega\tau, F_\omega\rangle \,\mathrm{vol}$$

Let $\mathrm{d}_\omega^* = (-1)^{2n+1} * \mathrm{d}_\omega *$ denote the formal adjoint to d_ω. Then we have

$$\int_M \langle \mathrm{d}_\omega\tau, F_\omega\rangle \,\mathrm{vol} = \int_M \langle \tau, \mathrm{d}_\omega^* F_\omega\rangle \,\mathrm{vol}$$

this shows

$$\mathrm{d}_\omega^* F_\omega = 0$$

□

Definition 11.2.1 (Yang-Mills equations)**.** A connection $\omega \in \mathcal{A}(P)$ is called satisfying Yang-Mills equations, if

$$\begin{cases} \mathrm{d}_\omega F_\omega = 0 \\ \mathrm{d}_\omega^* F_\omega = 0 \end{cases}$$

Remark 11.2.1*.* The first equation is also called Bianchi identity.

Example 11.2.1. In the case that $G = U(1)$, we have that the curvature of a connection A can be identified as a section of Ω_M^2. Indeed, the curvature form takes value in the bundle $\operatorname{ad}\mathfrak{g}$, but here $G = U(1)$ is abelian, thus the adjoint action on $\mathfrak{u}(1)$ is trivial, so

$$\operatorname{ad}\mathfrak{g} = M \times \mathfrak{u}(1) = M \times \mathbb{R}$$

is trivial bundle. Furthermore, ω is a Yang-Mills connection if and only if F_ω is a harmonic 2-form, that is $\Delta F_\omega = 0$, where $\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$. Indeed, thanks to $U(1)$ is abelian again, d_ω can be reduced to d, since for arbitrary form β, we have $\omega \wedge \beta = 0$. This follows from in the definition of wedge product of forms valued in Lie algebra we used Lie bracket, and abelian Lie algebra has trivial Lie bracket. Note that F_ω is harmonic if and only if

$$\begin{cases} \mathrm{d}^* F_\omega = 0 \\ \mathrm{d}F_\omega = 0 \end{cases}$$

It's a standard result in differential geometry, which can be seen from

$$\begin{aligned}0 &= \int_M \langle \Delta F_\omega, F_\omega \rangle \operatorname{vol} \\ &= \int_M \langle \mathrm{d}\mathrm{d}^* F_\omega, F_\omega \rangle + \langle \mathrm{d}^*\mathrm{d} F_\omega, F_\omega \rangle \operatorname{vol} \\ &= \int_M \|\mathrm{d}^* F_\omega\|^2 + \|\mathrm{d} F_\omega\|^2 \operatorname{vol}\end{aligned}$$

Note that the Yang-Mills functional is guage invariant, so if a connection ω solves the Yang-Mills equations, so does any gauge transformed $\Phi^*\omega$. In other words, the gauge group acts on $\mathcal{A}_{YM}$. The quotient $\mathcal{A}_{YM}/\mathcal{G}$ is the space of classical solutions. In general it is infinite dimensional, and the topology of this space may be quite bad. For example it may be neither Hausdorff or a smooth manifold. But adding some restrictions, we do have a good correspondence, and that's main theorem for next lecture.

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