RIEMANNIAN SYMMETRIC SPACE

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CONTENTS

Part 1. Riemannian symmetric space

1. Geometric viewpoints

1.A. Basic definitions and properties.

1.A.1. *Riemannian symmetric space.*

Definition 1.1 (Riemannian symmetric space)**.** *A Riemannian manifold (M, g) is called a Riemannian symmetric space if for each $p \in M$ there exists an isometry $\varphi : M \to M$, which is called a symmetry at p, such that $\varphi(p) = p$ and $(\mathrm{d}\varphi)_p = -\mathrm{id}$.*

Remark 1.2. Theorem B.8 implies if symmetry at point p exists, then it's unique.

Proposition 1.3. *The following statements are equivalent:*

(1) (M, g) is a Riemannian symmetric space.

(2) For each $p \in M$, there exists an isometry $\varphi : M \to M$ such that $\varphi^2 = \mathrm{id}$ and p is an isolated fixed point of φ.

Proof. From (1) to (2). Let φ be a symmetry at $p \in M$. Since $(\mathrm{d}\varphi^2)_p = (\mathrm{d}\varphi)_p \circ (\mathrm{d}\varphi)_p = \mathrm{id}$ and $\varphi^2(p) = p$, one has $\varphi^2 = \mathrm{id}$ by Theorem B.8. If p is not an isolated fixed point, then there exists a sequence $\{p_i\}_{i=1}^\infty$ converging to p such that $\varphi(p_i) = p_i$. For $0 < \delta < \mathrm{inj}(p)$, there exists sufficiently large k such that $p_k \in B(p, \delta)$, and we denote $v = \exp_p^{-1}(p_k)$. Since φ is an isometry, one has $\varphi(\exp_p(tv))$ and $\exp_p(tv)$ are two geodesics connecting p and p_k, and thus

$$\varphi(\exp_p(tv)) = \exp_p(tv)$$

by uniqueness. In particular, one has $v = (\mathrm{d}\varphi)_p v$, which is a contradiction.

From (2) to (1). From $\varphi^2 = \mathrm{id}$ we have $(\mathrm{d}\varphi)_p^2 = \mathrm{id}$, so only possible eigenvalues of $(\mathrm{d}\varphi)_p$ are ± 1. Now it suffices to show all eigenvalues of $(\mathrm{d}\varphi)_p$ are -1. Otherwise if it has an eigenvalue 1, there exists some non-zero $v \in T_p M$ such that $(\mathrm{d}\varphi)_p v = v$. Since φ is an isometry, one has $\varphi(\exp_p(tv))$ and $\exp_p(tv)$ are geodesics with the same direction at p. Thus

$$\varphi(\exp_p(tv)) = \exp_p(tv)$$

for $0 < t < \mathrm{inj}(p)$. In particular, p is not an isolated fixed point, which is a contradiction. $\square$

Proposition 1.4. *The fundamental group of a Riemannian symmetric space is abelian.*

Corollary 1.5. *A surface of genus $g \geq 2$ does not admit a Riemannian metric with respect to which it is a symmetric space.*

1.A.2. *Locally Riemannian symmetric space.*

Definition 1.6 (locally Riemannian symmetric space)**.** *A Riemannian manifold (M, g) is called a locally Riemannian symmetric space if each $p \in M$ has a neighborhood U such that there exists an isometry $\varphi : U \to U$ such that $\varphi(p) = p$ and $(\mathrm{d}\varphi)_p = -\mathrm{id}$.*

Theorem 1.7. *Let (M, g) be a Riemannian manifold. The following statements are equivalent:*

(1) (M, g) is a locally Riemannian symmetric space.

(2) $\nabla R = 0$.

Proof. From (1) to (2). If φ is the symmetry at point $p \in M$, then it's an isometry such that $(\mathrm{d}\varphi)_p = -\mathrm{id}$, and thus for $u, v, w, z \in T_pM$, one has

$$\begin{aligned} -\nabla_u R(v,w)z &= (\mathrm{d}\varphi)_p\left(\nabla_u R(v,w)z\right) \\ &= \nabla_{(\mathrm{d}\varphi)_p u}((\mathrm{d}\varphi)_p)v, (\mathrm{d}\varphi)_p w)(\mathrm{d}\varphi)_p z \\ &= \nabla_u R(v,w)z \end{aligned}$$

This shows $(\nabla R)_p = 0$, and thus $\nabla R = 0$ since p is arbitrary.

From (2) to (1). For arbitrary $p \in M$, it suffices to show

$$\varphi = \exp_p \circ \Phi_0 \circ \exp_p^{-1} : B(p,\delta) \to B(p,\delta)$$

is an isometry, where $0 < \delta < \mathrm{inj}(p)$ and $\Phi_0 = -\mathrm{id} : T_pM \to T_pM$. For $v \in T_pM$ with $|v| < \delta$ and $\gamma(t) = \exp_p(tv), \widetilde{\gamma}(t) = \exp_p(t\Phi_0(v))$, if we define

$$\Phi_t = P_{0,t;\widetilde{\gamma}} \circ \Phi_0 \circ P_{t,0;\gamma}$$

then direct computation shows

$$\begin{aligned} \Phi_t^* R_{\widetilde{\gamma}(t)} &= (P_{t,0;\gamma})^* \circ \Phi_0^* \circ (P_{0,t;\widetilde{\gamma}})^* R_{\widetilde{\gamma}(t)} \\ &\overset{(a)}{=} (P_{t,0;\gamma})^* \circ \Phi_0^* R_{\widetilde{\gamma}(0)} \\ &\overset{(b)}{=} (P_{t,0;\gamma})^* R_{\gamma(0)} \\ &\overset{(c)}{=} R_{\gamma(t)} \end{aligned}$$

where

(a) and (c) holds from Proposition B.12.

(b) holds from $\widetilde{\gamma}(0) = \gamma(0)$ and R is a $(0,4)$-tensor.

Then by Theorem B.9, that is Cartan-Ambrose-Hicks's theorem, φ is an isometry, which completes the proof. □

Remark 1.8. The proof for locally Riemannian symmetric space has parallel curvature tensor can be applied to other situations. For example, one can easy show if a p-form ω is invariant under isometries, that is $\varphi^*\omega = \omega$ for arbitrary isometry, then $\mathrm{d}\omega = 0$, and in Section 7 we will use this idea to show any almost Hermitian symmetric space is Kähler.

1.B. **Transvection.**

Definition 1.9 (transvection). *Let (M,g) be a Riemannian symmetric space and γ be a geodesic. The transvection along γ is defined as*

$$T_t = s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)},$$

where s_p is the symmetry at point p.

Proposition 1.10. *Let (M,g) be a Riemannian symmetric space and T_t be the transvection along geodesic γ. Then*

(1) For any $a, t \in \mathbb{R}$, $s_{\gamma(a)}(\gamma(t)) = \gamma(2a - t)$.

(2) T_t translates the geodesic γ, that is $T_t(\gamma(s)) = \gamma(t+s)$.

(3) $(\mathrm{d}T_t)_{\gamma(s)}: T_{\gamma(s)}M \to T_{\gamma(t+s)}M$ *is the parallel transport* $P_{s,t+s;\gamma}$*.*
(4) T_t *is one-parameter subgroup of* $\mathrm{Iso}(M,g)$*.*

Proof. For (1). It follows from the uniqueness of geodesics with given initial value.

For (2). By (1) one has

$$\begin{aligned} T_t(\gamma(s)) &= s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}(\gamma(s)) \\ &= s_{\gamma(\frac{t}{2})}(\gamma(-s)) \\ &= \gamma(t+s). \end{aligned}$$

For (3). Let X be a parallel vector field along γ. By uniqueness of parallel vector fields with given initial data, we have $(\mathrm{d}s_{\gamma(0)})_{\gamma(s)} X_{\gamma(s)} = -X_{\gamma(-s)}$ for all s, since $(\mathrm{d}s_{\gamma(0)})_{\gamma(0)} X_{\gamma(0)} = -X_{\gamma(0)}$. Thus

$$\begin{aligned} (\mathrm{d}T_t)_{\gamma(s)} X_{\gamma(s)} &= (\mathrm{d}s_{\gamma(\frac{t}{2})})_{\gamma(-s)}(-X_{\gamma(-s)}) \\ &= X_{\gamma(t+s)}. \end{aligned}$$

This shows $(\mathrm{d}T_t)_{\gamma(s)} = P_{s,t+s;\gamma}$.

For (4). In order to show $T_{t+s} = T_t \circ T_s$, it suffices to check they're same at some point, so do their derivatives, since isometry can be determined by these two values. Note that

$$\begin{aligned} T_{t+s}(\gamma(0)) &= \gamma(t+s) \\ &= T_t \circ T_s(\gamma(0)), \\ (\mathrm{d}T_{t+s})_{\gamma(0)} &= P_{0,t+s;\gamma} \\ &= P_{s,t+s;\gamma} \circ P_{0,s;\gamma} \\ &= (\mathrm{d}T_t)_{\gamma(s)} \circ (\mathrm{d}T_s)_{\gamma(0)} \\ &= (\mathrm{d}(T_t \circ T_s))_{\gamma(0)}. \end{aligned}$$

This completes the proof. □

1.C. **Symmetric space, locally symmetric space and homogeneous space.** In this section, we will show any complete locally Riemannian symmetric space is a quotient of Riemannian symmetric space (Corollary 1.12), and any Riemannian symmetric space is a Riemannian homogeneous space (Corollary 1.16).

1.C.1. *Riemannian symmetric space and locally Riemannian symmetric space.*

Theorem 1.11. *Let* (M,g) *be a complete, simply-connected locally Riemannian symmetric space. Then* (M,g) *is a Riemannian symmetric space.*

Proof. For $p \in M$ and $0 < \delta < \mathrm{inj}(p)$, suppose $\varphi: B(p,\delta) \to B(p,\delta)$ is an isometry such that $\varphi(p) = p$ and $(\mathrm{d}\varphi)_p = -\mathrm{id}$. For arbitrary $q \in M$, we use $\Omega_{p,q}$ to denote all curves γ with $\gamma(0) = p, \gamma(1) = q$, and for $c \in \Omega_{p,q}$ we choose[1] a covering $\{B(p_i,\delta_i)\}_{i=0}^k$ of c such that

(1) $0 < \delta_i < \mathrm{inj}(p_i)$.
(2) $B(p_0,\delta_0) = B(p,\delta)$ and $p_k = q$.
(3) $p_{i+1} \in B(p_i,\delta_i)$.

[1]Since injective radius is a continuous function, it has a positive minimum on curve c, so such covering exists.

If we set $\varphi = \varphi_0$, then we can define isometries $\varphi_i : B(p_i, \delta_i) \to M$ such that $\varphi_i(p_i) = \varphi_{i-1}(p_i)$ and $(\mathrm{d}\varphi_i)_{p_i} = (\mathrm{d}\varphi_{i-1})_{p_i}$ by using Cartan-Ambrose-Hicks's theorem successively, and by Theorem B.8 one has φ_i and φ_{i+1} coincide on $B(p_i, \delta_i) \cap B(p_{i+1}, \delta_i)$. The covering together with isometries we construct is denoted by $\mathcal{A} = \{B(p_i, \delta_i), \varphi_i\}_{i=0}^k$. For arbitrary $x \in [0,1]$, if $c(x) \in B(p_m, \delta_m)$, we may define

$$\begin{aligned} \varphi_{\mathcal{A}}(c(x)) &:= \varphi_m(c(x)), \\ (\mathrm{d}\varphi_{\mathcal{A}})_{c(x)} &:= (\mathrm{d}\varphi_m)_{c(x)}. \end{aligned}$$

In particular, $\varphi_{\mathcal{A}}(q) := \varphi_k(q)$. If $\mathcal{B} = \{\widetilde{B}(\widetilde{p}_i, \widetilde{\delta}_i), \widetilde{\varphi}_i\}_{i=0}^l$ is another covering of c, let's show $\varphi_{\mathcal{A}}(q) = \varphi_{\mathcal{B}}(q)$. Consider

$$I = \{x \in [0,1] \mid \varphi_{\mathcal{A}}(c(x)) = \varphi_{\mathcal{B}}(c(x)), (\mathrm{d}\varphi_{\mathcal{A}})_{c(x)} = (\mathrm{d}\varphi_{\mathcal{B}})_{c(x)}\}.$$

It's clear $I \neq \varnothing$, since $0 \in I$. Now it suffices to show it's both open and closed to conclude $1 \in I$.

(a) It's open: For $x \in I$, we assume $c(x) \in B(p_m, \delta_m) \cap \widetilde{B}(\widetilde{p}_n, \widetilde{\delta}_n)$, that is

$$\begin{aligned} \varphi_m(c(x)) &= \widetilde{\varphi}_n(c(x)), \\ (\mathrm{d}\varphi_m)_{c(x)} &= (\mathrm{d}\widetilde{\varphi}_n)_{c(x)}. \end{aligned}$$

Then one has

$$\begin{aligned} \varphi_m \circ \exp_{c(x)}(v) &= \exp_{\varphi_m(c(x))} \circ (\mathrm{d}\varphi_m)_{c(x)}(v) \\ &= \exp_{\widetilde{\varphi}_n(c(x))} \circ (\mathrm{d}\widetilde{\varphi}_n)_{c(x)}(v) \\ &= \widetilde{\varphi}_n \circ \exp_{c(x)}(v). \end{aligned}$$

Since $\exp_{c(x)}$ maps onto a neighborhood of $c(x)$, it follows that some neighborhood of x also lies in I, and thus I is open.

(b) It's closed: Let $\{x_i\}_{i=1}^{\infty} \subseteq I$ be a sequence converging to x. Without lose of generality we may assume $\{x_i\}_{i=1}^{\infty} \subseteq B(p_m, \delta_m) \cap \widetilde{B}(\widetilde{p}_n, \widetilde{\delta}_n)$, then one has

$$\begin{aligned} \varphi_m(c(x_i)) &= \widetilde{\varphi}_n(c(x_i)), \\ (\mathrm{d}\varphi_m)_{c(x_i)} &= (\mathrm{d}\widetilde{\varphi}_n)_{c(x_i)}. \end{aligned}$$

By taking limit we obtain the desired results.

Since $\varphi_{\mathcal{A}}(q)$ is independent of the choice of coverings, we use $\varphi(q)$ to denote it for convenience, and as a consequence we obtain the following map

$$\begin{aligned} F : \Omega_{p,q} &\to M \\ c &\mapsto \varphi(q). \end{aligned}$$

Note that $F(c)$ is locally constant, and thus it's independent of the choice of homotopy classes of c. Since M is simply-connected, one has $F : \Omega_{p,q} \to M$ is constant, so we obtain a local isometry $\varphi : M \to M$ which extends $\varphi : B(p,\delta) \to B(p,\delta)$. By Proposition B.10 φ is a Riemannian covering map since M is complete, and thus φ is a diffeomorphism since M is simply-connected, which implies φ is an isometry. □

Corollary 1.12. *Let (M, g) be a complete locally Riemannian symmetric space. Then it's isometric to $(\widetilde{M}/\Gamma, \widetilde{g})$ where $(\widetilde{M}, \widetilde{g})$ is a Riemannian symmetric space and $\Gamma \cong \pi_1(M)$ is a discrete Lie group acting on $\widetilde{M}$ freely, properly and isometrically.*

Proof. Let $(\widetilde{M}, \widetilde{g})$ be the universal covering of (M, g) with pullback metric. Then $(\widetilde{M}, \widetilde{g})$ is a simply-connected Riemannian manifold with parallel curvature tensor. Furthermore, by Proposition B.13 it's complete, hence it is symmetric. $\square$

1.C.2. *Riemannian symmetric space and Riemannian homogeneous space.*

Definition 1.13 (Riemannian homogeneous space)**.** *A Riemannian manifold (M, g) is called a Riemannian homogeneous space, if* $\mathrm{Iso}(M, g)$ *acts on M transitively.*

Proposition 1.14. *Let (M, g) be a Riemannian homogeneous space. If there exists a symmetry at some point $p \in M$, then (M, g) is a Riemannian symmetric space.*

Proof. Let φ be a symmetry at $p \in M$. For arbitrary $q \in M$, there exists an isometry $\psi : M \to M$ such that $\psi(p) = q$ since (M, g) is a Riemannian homogeneous space. Then

$$\varphi_q := \psi \circ \varphi \circ \psi^{-1}$$

is the desired symmetry at q. $\square$

Theorem 1.15. *Let (M, g) be a Riemannian symmetric space. Then*

(1) (M, g) is complete.

(2) the identity component of isometry group acts transitively on M.

Proof. For (1). For arbitrary geodesic $\gamma : [0, 1] \to M$ with $\gamma(0) = p, \gamma'(0) = v$, the curve $\beta(t) = \varphi(\gamma(t)) : [0, 1] \to M$ is also a geodesic with $\beta(0) = p$ and $\beta'(0) = -v$. Now we obtain a smooth extension $\gamma' : [0, 2] \to M$ of γ, given by

$$\gamma'(t) = \begin{cases} \beta(1-t), & t \in [0, 1] \\ \gamma(t-1), & t \in [1, 2]. \end{cases}$$

Repeat above process to extend γ to a geodesic defined on $\mathbb{R}$, which shows completeness.

For (2). For $p, q \in M$, let γ be a geodesic connecting p, q. Then the transvection along γ gives an isometry which maps p to q. Since the transvection lies in the identity component of isometry group, one has the identity component of isometry group acts transitively on M. $\square$

Corollary 1.16. *The Riemannian symmetric space (M, g) is a Riemannian homogeneous space.*

2.A. **Riemannian symmetric space as a Lie group quotient.**

Definition 2.1 (involution)**.** *An automorphism σ of a Lie group G is called an involution if $\sigma^2 = \mathrm{id}_G$.*

Definition 2.2 (Cartan decomposition)**.** *Let G be a Lie group and σ be an involution of G. The eigen-decomposition of $\mathfrak{g}$ given by $(\mathrm{d}\sigma)_e$ is called Cartan decomposition, that is,*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m},$$

where

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid (\mathrm{d}\sigma)_e(X) = X\},$$
$$\mathfrak{m} = \{X \in \mathfrak{g} \mid (\mathrm{d}\sigma)_e(X) = -X\}.$$

Proposition 2.3. *Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the Cartan decomposition given by σ. Then*

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}.$$

Proof. Since σ is a Lie group homomorphism, $(\mathrm{d}\sigma)_e$ gives a Lie algebra homomorphism, and thus

$$(\mathrm{d}\sigma)_e([X, Y]) = [(\mathrm{d}\sigma)_e(X), (\mathrm{d}\sigma)_e(Y)],$$

where $X, Y \in \mathfrak{g}$. □

Lemma 2.4. *Let G be a Lie group and $K \subseteq G$ be a closed subgroup. A left invariant metric on G which is also right invariant under K gives a left-invariant metric on G/K.*

Theorem 2.5. *Let (M, g) be a Riemannian symmetric space and G be the identity component of $\mathrm{Iso}(M, g)$. For $p \in M$, K denotes the isotropic group of G_p.*

(1) The mapping $\sigma : G \to G$, given by $\sigma(g) = s_p g s_p$ is an involution automorphism of G.
(2) If G^σ is the set of fixed points of σ in G, and $(G^\sigma)_0$ is the identity component of G^σ, then $(G^\sigma)_0 \subseteq K \subseteq G^\sigma$.
(3) If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the Cartan decomposition given by σ, then $\mathfrak{k}$ is the Lie algebra of K, and thus $\mathfrak{m} \cong T_pM$ as vector spaces.
(4) There is a left invariant metric on G/K such that G/K with this metric is isometric to (M, g).

Proof. For (1). It's clear σ maps G to G, and it's an involution since for arbitrary $g \in G$, one has $\sigma^2(g) = \sigma(s_p g s_p) = s_p^2 g s_p^2 = g$.

For (2). It follows from the following two steps:

(a) To show $K \subseteq G^\sigma$. For any $k \in K$, in order to show $k = s_p k s_p$, it suffices to show they and their differentials agree at some point by Theorem B.8, since both of them are isometries, and p is exactly the point we desired.

(b) To see $(G^\sigma)_0 \subseteq K$. Suppose $\exp(tX) \subseteq (G^\sigma)_0$ is a one-parameter subgroup. Since $\sigma(\exp(tX)) = \exp(tX)$, one has

$$\exp(tX)(p) = s_p \exp(tX) s_p(p) = s_p \exp(tX)(p).$$

But p is an isolated fixed point of s_p, which implies $\exp(tX)(p) = p$ for all t. This shows the one-parameter subgroup lies in K. Since exponential map of Lie group is a diffeomorphism in a small neighborhood of identity element e and $(G^\sigma)_0$ can be generated by a neighborhood of e, which implies the whole $(G^\sigma)_0 \subseteq K$.

For (3). Note that $(G^\sigma)_0 \subseteq K \subseteq G^\sigma$, it suffices to show $\mathfrak{k} \cong \operatorname{Lie} G^\sigma$. For $X \in \mathfrak{k}$, we claim $\gamma_2(t) = \sigma(\exp(tX))\colon \mathbb{R} \to G$ is a one-parameter subgroup. Indeed, note that

$$\begin{aligned}\gamma_2(t)\cdot\gamma_2(s) &= s_p \exp(tX)s_p \cdot s_p \exp(sX)s_p \\ &= \sigma(\exp(tX + sX)) \\ &= \gamma_2(t+s).\end{aligned}$$

Furthermore, $\gamma_2(t) = \sigma(\exp(tX))$ and $\gamma_1(t) = \exp(tX)$ are two one-parameter subgroups of G such that $\gamma_1(0) = \gamma_2(0)$ and $\gamma_2'(0) = (\mathrm{d}\sigma)_e(X) = X = \gamma_1'(0)$. Then $\gamma_1(t) = \gamma_2(t)$, and thus $\exp(tX) \in G^\sigma$ for all $t \in \mathbb{R}$. This shows $\mathfrak{k} \subseteq \operatorname{Lie} G^\sigma$, and the converse inclusion is clear, so one has $\mathfrak{k} = \operatorname{Lie} G^\sigma$.

For (4). Let $\pi\colon G \to M$ be the natural projection given by $\pi(g) = gp$. Then for $k \in K$ and $X \in \mathfrak{g}$ one has

$$\begin{aligned}(\mathrm{d}\pi)_e(\operatorname{Ad}(k)X) &= (\mathrm{d}\pi)_e\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} k\exp(tX)k^{-1}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi(k\exp(tX)k^{-1}) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} k\exp(tX)k^{-1}\cdot p \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} k\exp(tX)\cdot p \\ &= (\mathrm{d}L_k)_p(\mathrm{d}\pi)_e(X).\end{aligned}$$

By using the equivalent isomorphism $(\mathrm{d}\pi)_e|_{\mathfrak{m}}\colon \mathfrak{m} \to T_pM$, one has an $\operatorname{Ad}(K)$-invariant metric on $\mathfrak{m}$, and then we can extend it to an $\operatorname{Ad}(K)$-invariant metric on $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ by choosing[2] arbitrary $\operatorname{Ad}(K)$-invariant metric on $\mathfrak{k}$ such that $\mathfrak{m} \perp \mathfrak{k}$. This shows one has a left-invariant metric on G which is also right invariant with respect to K, and by Lemma 2.4 it gives a left-invariant metric on G/K. Now it suffices to show G/K with this metric is isometric to (M, g). For any $gK \in G/K$, consider the following communicative diagram

$$\begin{tikzcd} \mathfrak{m} = T_{eK}G/K \arrow[r, "(\mathrm{d}\pi)_e|_{\mathfrak{m}}"] \arrow[d, "\mathrm{d}L_g"'] & T_pM \arrow[d, "\mathrm{d}L_g"] \\ T_{gK}G/K \arrow[r] & T_{gp}M \end{tikzcd}$$

Since both $(\mathrm{d}\pi)_e|_{\mathfrak{m}}$ and $(\mathrm{d}L_g)$ are linear isometries, one has $T_{gK}G/K$ is isometric to $T_{gp}M$, and thus G/K with this metric is isometric to (M, g). □

2.B. **Riemannian symmetric pair.** In Theorem 2.5 one can see that if (M, g) is a symmetric space, then it gives a pair of Lie groups (G, K) with an involution σ on G such that

$$(G^\sigma)_0 \subseteq K \subseteq G^\sigma.$$

[2] Such metric exists since K is compact.

Furthermore, there exists a left-invariant metric on G/K such that G/K with this metric is isometric to (M, g). This motivates us a useful way to construct Riemannian symmetric spaces from a pair of Lie groups with certain properties, and such a pair is called a Riemannian symmetric pair.

Definition 2.6 (Riemannian symmetric pair)**.** *Let G be a connected Lie group and $K \subseteq G$ be a closed subgroup. The pair (G, K) is called a symmetric pair if there exists an involution $\sigma : G \to G$ with $(G^\sigma)_0 \subseteq K \subseteq G^\sigma$. If, in addition, the group $\mathrm{Ad}(K) \subseteq \mathrm{GL}(\mathfrak{g})$ is compact, then (G, K) is said to be a Riemannian symmetric pair.*

Remark 2.7. The first condition of above definition means K is compact up to the center of G since the kernel of Ad is the center of G. Thus, for a Riemannian symmetric space (M, g), if G is the identity component of the isometry group and K is the isotropy group G_p at some point $p \in M$, then (G, K) gives a Riemannian symmetric pair.

Proposition 2.8. *Let (G, K) be a symmetric pair given by σ. Then there is an isomorphism as Lie algebras*

$$\mathfrak{k} \cong \mathrm{Lie}\, K,$$

and an isomorphism as vector spaces

$$\mathfrak{m} \cong T_{eK}G/K$$

Proof. It's the same as proof of (3) in Theorem 2.5. □

Corollary 2.9. *Let $\widetilde{\sigma} : G/K \to G/K$ be the automorphism given by $\widetilde{\sigma}(gK) = \sigma(g)K$. Then $(\mathrm{d}\widetilde{\sigma})_{eK} = -\mathrm{id}_{G/K}$.*

Proof. $\widetilde{\sigma}$ is well-defined since $K \subseteq G^\sigma$, and by construction one has $(\mathrm{d}\widetilde{\sigma})_{eK} = (\mathrm{d}\sigma)_e|_{\mathfrak{m}}$. Then $(\mathrm{d}\widetilde{\sigma})_{eK} = -\mathrm{id}_{G/K}$ since $\mathfrak{m} = \{X \in \mathfrak{g} \mid (\mathrm{d}\sigma)_e X = -X\}$. □

Theorem 2.10. *Let (G, K) be a Riemannian symmetric pair given by σ. Then there exists a left-invariant metric on $M = G/K$ making it to be a Riemannian symmetric space.*

Proof. Since $\mathrm{Ad}(K) \subseteq \mathrm{GL}(\mathfrak{g})$ is a compact subgroup, by averaging trick there exists an inner product on $\mathfrak{g}$ which is also $\mathrm{Ad}(K)$-invariant, and thus it gives a left-invariant metric on M by Lemma 2.4. Furthermore, by Corollary 2.9 one has $(\mathrm{d}\widetilde{\sigma})_{eK} = -\mathrm{id}_M$.

Now it suffices to show for any $gK \in M$, $(\mathrm{d}\widetilde{\sigma})_{gK} : T_{gK}M \to T_{\sigma(g)K}M$ is an isometry. Note that $\widetilde{\sigma}(ghK) = \sigma(g)\sigma(h)K = \sigma(g)\widetilde{\sigma}(hK)$ holds for all $h \in G$. This shows $\widetilde{\sigma} \circ L_g = L_{\sigma(g)} \circ \widetilde{\sigma}$, where $L_g : M \to M$ is given by $L_g(hK) = ghK$. By taking differential one has the following communicative diagram

$$\begin{tikzcd}
T_{eK}M \arrow[r, "(\mathrm{d}\widetilde{\sigma})_{eK}"] \arrow[d, "(\mathrm{d}L_g)_{eK}"'] & T_{eK}M \arrow[d, "(\mathrm{d}L_{\sigma(g)})_{eK}"] \\
T_{gK}M \arrow[r, "(\mathrm{d}\widetilde{\sigma})_{gK}"] & T_{\sigma(g)K}M
\end{tikzcd}$$

Since $(\mathrm{d}L_g)_{eK}, (\mathrm{d}L_{\sigma(g)})_{eK}, (\mathrm{d}\widetilde{\sigma})_{eK}$ are isometries, one has $(\mathrm{d}\widetilde{\sigma})_{gK}$ is also an isometry as desired. □

Remark 2.11. In Theorem 3.4 we will see the curvature tensor of G/K is independent of the choice of the left-invariant metric on it, so here we only care about existence, which is guaranteed by $\mathrm{Ad}(K)$ is compact.

2.C. **Examples.**

Example 2.12. $G = \mathrm{SL}(n, \mathbb{R})$ together with $K = \mathrm{SO}(n)$ gives a Riemannian symmetric pair, where σ is defined by

$$\begin{aligned} \sigma : \ \mathrm{SL}(n, \mathbb{R}) &\to \mathrm{SL}(n, \mathbb{R}) \\ g &\mapsto (g^{-1})^T. \end{aligned}$$

Indeed, note that

$$(\mathrm{SL}(n, \mathbb{R}))^\sigma = \mathrm{SO}(n).$$

Thus $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ is a Riemannian symmetric space, and it can be viewed as a generalization of hyperbolic plane $\mathbb{H}^2$, since $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2) \cong \mathbb{H}^2$.

Example 2.13. $G = \mathrm{SO}(n+1)$ together with $K = \mathrm{SO}(n)$ gives a Riemannian symmetric pair, where σ is defined by

$$\begin{aligned} \sigma : \ \mathrm{SO}(n+1) &\to \mathrm{SO}(n+1) \\ a &\mapsto I_{1,n} a I_{1,n}^{-1}, \end{aligned}$$

where $I_{1,n} = diag\{-1, 1, \dots, 1\}$. Indeed, a direct computation shows

$$\mathrm{SO}(n+1)^\sigma = \{a \in \mathrm{SO}(n+1) \mid I_{1,n} a = a I_{1,n}\} = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & b \end{pmatrix} \in \mathrm{SO}(n+1) \mid b \in \mathrm{O}(n) \right\},$$

which implies $(\mathrm{SO}(n+1)^\sigma)_0 = \mathrm{SO}(n) \subseteq \mathrm{SO}(n+1)$. Thus $S^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$ is a Riemannian symmetric space.

Example 2.14 (compact Grassmannian)**.** Consider the Grassmannian of oriented k-planes in $\mathbb{R}^{k+l}$, denoted by $M = \widehat{Gr}_k(\mathbb{R}^{k+l})$. It's clear that $\mathrm{SO}(k+l)$ acts on M transitively with isotropy group $\mathrm{SO}(k) \times \mathrm{SO}(l)$, and thus $M \cong \mathrm{SO}(k+l)/\mathrm{SO}(k) \times \mathrm{SO}(l)$. Consider the involution

$$\begin{aligned} \sigma : \ \mathrm{SO}(k+l) &\to \mathrm{SO}(k+l) \\ a &\mapsto I_{k,l} a I_{k,l}^{-1}, \end{aligned}$$

where $I_{k,l} = diag\{\underbrace{-1, \dots, -1}_{k \text{ times}}, \underbrace{1, \dots, 1}_{l \text{ times}}\}$. A direct computation shows

$$\mathrm{SO}(k+l)^\sigma = S(\mathrm{O}(k) \times \mathrm{O}(l)).$$

Then $(\mathrm{SO}(k+l)^\sigma)_0 = \mathrm{SO}(k) \times \mathrm{SO}(l) \subseteq \mathrm{SO}(k+l)^\sigma$, and thus M is a Riemannian symmetric space, called compact Grassmannian. In particular, $S^n = \widehat{Gr}_1(\mathbb{R}^{n+1})$.

Example 2.15 (hyperbolic Grassmannian)**.** In $\mathbb{R}^{k,l}$ with $k \geq 2, l \geq 1$, let's consider the following quadratic form

$$v^t I_{k,l} w = v^t \begin{pmatrix} I_k & 0 \\ 0 & -I_l \end{pmatrix} w = \sum_{i=1}^{k} v_i w_i - \sum_{j=k+1}^{k+l} v_j w_j.$$

The group of linear transformation X that preserves this quadratic form is denoted by $\mathrm{O}(k, l)$, that is

$$X I_{k,l} X^t = I_{k,l},$$

and $\mathrm{SO}(k, l)$ are those with positive determinant. Now consider set consisting of those oriented k-dimensional subspaces of $\mathbb{R}^{k,l}$ on which quadratic form $I_{k,l}$ are positive definite. This space is called the hyperbolic Grassmannian $M = \widehat{Gr}_k(\mathbb{R}^{k,l})$, which is also an open subset of $\widehat{Gr}_k(\mathbb{R}^{k+l})$. It's clear $G = \mathrm{SO}(k,l)$ acting transitively on M with isotropy group $G_p = \mathrm{SO}(k)\times\mathrm{SO}(l)$. As in Example 2.14 one can also construct an involution σ to show $\widehat{Gr}_k(\mathbb{R}^{k,l})$ is a Riemannian symmetric space.

Example 2.16. Suppose K is a compact connected Lie group. Then $(K\times K, \Delta K)$ is a Riemannian symmetric pair given by σ, where $\sigma : K\times K \to K\times K$ is given by $(x,y)\mapsto (y,x)$, since

$$(K\times K)^\sigma = \{(a,a) \mid a\in K\} = \Delta K.$$

Then any compact Lie group is a Riemannian symmetric space.

3. Curvature of Riemannian symmetric space

3.A. **Formulas.** Let (M,g) be a Riemannian symmetric space with isometry group G and isotropy group G_p. On one hand, there is a Cartan decomposition of Lie algebra $\mathfrak{g}$ given by involution $\sigma: G \to G$, that is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m},$$

where $\mathfrak{m} \cong T_pM$ as vector spaces, and $\mathfrak{k}$ is the Lie algebra of isotropy group G_p. On the other hand, by Corollary B.6 there is another decomposition of $\mathfrak{g}$ given by

$$\mathfrak{g} = \mathfrak{k}' \oplus \mathfrak{m}',$$

where

$$\begin{aligned}\mathfrak{k}' &= \{X \in \mathfrak{g} \mid X_p = 0\},\\ \mathfrak{m}' &= \{X \in \mathfrak{g} \mid (\nabla X)_p = 0\}.\end{aligned}$$

In fact, for any complete Riemannian manifold, the following proposition shows $\mathfrak{k} \cong \mathfrak{k}'$, and thus above two Cartan decompositions are exactly the same.

Proposition 3.1. *Let (M,g) be a complete Riemannian manifold with isometry group G and isotropy group G_p. Then the Lie algebra of G_p is*

$$\{X \in \mathfrak{g} \mid X_p = 0\}.$$

Proof. Let $X \in \mathfrak{g}$ with $X_p = 0$ and $\varphi_t: M \to M$ be the flow of X. If we denote $\gamma_p(t) = \varphi_t(p)$, then it suffices to show $\gamma_p(t) \equiv p$. For any smooth function $f: M \to \mathbb{R}$, one has

$$\begin{aligned}\gamma_p'(s)f &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=s} f\circ\gamma_p(t)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f\circ\gamma_p(s+t)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (f\circ\varphi_s)(\gamma_p(t))\\ &= X_p(f\circ\varphi_s)\\ &= 0\end{aligned}$$

□

Proposition 3.2. *Let (M,g) be a Riemannian symmetric space and $G = \mathrm{Iso}(M,g)$ with Lie algebra $\mathfrak{g}$. For any $p \in M$, one has Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then for any $S \in \mathfrak{k}$, one has*

$$B(S,S) \leq 0,$$

where B is the Killing form of $\mathfrak{g}$. Furthermore, the identity holds if and only if $S = 0$.

Proof. Since a Killing field is determined by X_p and $(\nabla X)_p$, one has elements in $\mathfrak{k}$ are determined by $(\nabla X)_p$, and note that ∇X is a skew-symmetric matrice, so

$$\mathfrak{k} \cong \{(\nabla X) \in \mathfrak{so}(T_pM) \mid X \in \mathfrak{k}\}.$$

By using this identification, there is a natural inner product on $\mathfrak{k}$ given by

$$\langle S_1, S_2\rangle = \mathrm{tr}(S_1S_2^T) = -\mathrm{tr}(S_1S_2).$$

By adding inner product on $\mathfrak{m}$ obtained from $\mathfrak{m} \cong T_pM$ and the one on $\mathfrak{k}$ constructed as above, one can construct an inner product on $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is orthogonal. For any $S \in \mathfrak{k}$, we claim with respect to this metric, $\mathrm{ad}_S : \mathfrak{g} \to \mathfrak{g}$ is skew-symmetric. Indeed, for $X_1, X_2 \in \mathfrak{k}$, one has

$$\begin{aligned}\langle \mathrm{ad}_S X_1, X_2 \rangle &= -\mathrm{tr}(\mathrm{ad}_S X_1 X_2) \\ &= -\mathrm{tr}((SX_1 - X_1 S)X_2) \\ &= \mathrm{tr}(X_1(SX_2 - X_2 S)) \\ &= -\langle X_1, \mathrm{ad}_S X_2 \rangle.\end{aligned}$$

For $Y_1, Y_2 \in \mathfrak{m}$, since $S_p = 0$ and $(\nabla S)_p$ is skew-symmetric, one has

$$\begin{aligned}\langle \mathrm{ad}_S Y_1, Y_2 \rangle &= \langle \nabla_S Y_1 - \nabla_{Y_1} S, Y_2 \rangle \\ &= -\langle \nabla_{Y_1} S, Y_2 \rangle \\ &= \langle \nabla_{Y_2} S, Y_1 \rangle \\ &= -\langle Y_1, \nabla_S Y_2 - \nabla_{Y_2} S \rangle \\ &= -\langle Y_1, \mathrm{ad}_S Y_2 \rangle.\end{aligned}$$

If $X \in \mathfrak{k}$ and $Y \in \mathfrak{m}$, since $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$, one has

$$\begin{aligned}\langle \mathrm{ad}_S X, Y \rangle &= 0, \\ \langle X, \mathrm{ad}_S Y \rangle &= 0.\end{aligned}$$

Similarly one has

$$\begin{aligned}\langle \mathrm{ad}_S Y, X \rangle &= 0, \\ \langle Y, \mathrm{ad}_S X \rangle &= 0.\end{aligned}$$

This completes the proof of our claim. Then one has

$$B(S,S) = \mathrm{tr}(\mathrm{ad}_S \circ \mathrm{ad}_S) = \sum_i \langle \mathrm{ad}_S \circ \mathrm{ad}_S(e_i), e_i \rangle = -\sum_i \langle \mathrm{ad}_S(e_i), \mathrm{ad}_S(e_i) \rangle \leq 0.$$

Furthermore, if $B(S,S) = 0$, then $\mathrm{ad}_S = 0$ and for any $X \in \mathfrak{g}$, one has

$$0 = \mathrm{ad}_S(X) = [S, X] = \nabla_S X - \nabla_X S = -\nabla_X S,$$

since $S_p = 0$. This implies $(\nabla S)_p = 0$, and thus $S = 0$. □

Remark 3.3. For $S \in \mathfrak{k}$, the most important part of the proof of $B(S,S) = 0$ if and only if $S = 0$ is $\mathrm{ad}_S = 0$ if and only if $S = 0$. In other words, $\mathfrak{k} \cap \mathfrak{z} = \{0\}$, where $\mathfrak{z}$ is the Lie algebra of center of G.

Theorem 3.4. *Let (M, g) be a Riemannian symmetric space and $G = \mathrm{Iso}(M, g)$. For any $p \in M$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ with $\mathfrak{m} \cong T_pM$.*

(1) For any $X, Y, Z \in \mathfrak{m}$, there holds

$$\begin{aligned}R(X,Y)Z &= -[Z, [Y, X]], \\ \mathrm{Ric}(Y, Z) &= -\frac{1}{2} B(Y, Z).\end{aligned}$$

(2) If $\mathrm{Ric}(g) = \lambda g$, then for $X, Y \in \mathfrak{m}$, one has

$$2\lambda R(X, Y, Y, X) = -B([X, Y], [X, Y]).$$

Proof. For (1). For any $X, Y, Z \in \mathfrak{m}$, direct computation shows

$$\begin{aligned} R(X,Y)Z &\overset{(a)}{=} R(X,Z)Y - R(Y,Z)X \\ &\overset{(b)}{=} \nabla_Z \nabla_Y X - \nabla_{\nabla_Z Y} X - \nabla_Z \nabla_X Y + \nabla_{\nabla_Z X} Y \\ &\overset{(c)}{=} -\nabla_Z [X,Y] \\ &\overset{(d)}{=} -[Z[X,Y]], \end{aligned}$$

where

(a) holds from the first Bianchi identity.
(b) holds from (2) of Proposition B.1.
(c) holds from $X, Y \in \mathfrak{m}$, and thus $(\nabla X)_p = (\nabla Y)_p = 0$.
(d) holds from

$$\nabla_Z [X,Y] - \nabla_{[X,Y]} Z = [Z, [X,Y]],$$

and $(\nabla Z)_p = 0$.

To see Ricci curvature, note that for $Y \in \mathfrak{m}$

$$\mathrm{ad}_Y : \mathfrak{k} \to \mathfrak{m}, \quad \mathrm{ad}_Y : \mathfrak{m} \to \mathfrak{k}.$$

Thus $\mathrm{ad}_Z \circ \mathrm{ad}_Y$ preserves the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ if $Y, Z \in \mathfrak{m}$. Then

$$\begin{aligned} \mathrm{tr}(\mathrm{ad}_Z \circ \mathrm{ad}_Y |_{\mathfrak{m}}) &= \mathrm{tr}(\mathrm{ad}_Z |_{\mathfrak{k}} \circ \mathrm{ad}_Y |_{\mathfrak{m}}) \\ &= \mathrm{tr}(\mathrm{ad}_Y |_{\mathfrak{m}} \circ \mathrm{ad}_Z |_{\mathfrak{k}}) \\ &= \mathrm{tr}(\mathrm{ad}_Y \circ \mathrm{ad}_Z |_{\mathfrak{k}}). \end{aligned}$$

Hence we obtain

$$B(Y,Y) = \mathrm{tr}(\mathrm{ad}_Y \circ \mathrm{ad}\, Y|_{\mathfrak{k}}) + \mathrm{tr}(\mathrm{ad}_Y \circ \mathrm{ad}\, Y|_{\mathfrak{m}}) = 2\mathrm{tr}(\mathrm{ad}_Y \circ \mathrm{ad}_Y |_{\mathfrak{m}}).$$

Since Ricci tensor is trace of curvature tensor, and thus

$$\mathrm{Ric}(Y,Y) = -\mathrm{tr}(\mathrm{ad}_Y \circ \mathrm{ad}_Y |_{\mathfrak{m}}) = -\frac{1}{2} B(Y,Y).$$

Then by using polarization identity, one has $\mathrm{Ric}(Y,Z) = -B(Y,Z)/2$.

For (2). If $\mathrm{Ric}(g) = \lambda g$, then

$$\begin{aligned} 2\lambda g(R(X,Y)Y,X) &= -2\lambda g(\mathrm{ad}_Y \circ \mathrm{ad}_Y X, X) \\ &= -2\mathrm{Ric}(\mathrm{ad}_Y \circ \mathrm{ad}_Y X, X) \\ &= B(\mathrm{ad}_Y \circ \mathrm{ad}_Y X, X) \\ &= -B(\mathrm{ad}_Y X, \mathrm{ad}_Y X) \\ &= -B([X,Y],[X,Y]). \end{aligned}$$

□

Corollary 3.5. *Let (M, g) be a Riemannian symmetric space which is an Einstein manifold with Einstein constant λ. Then*

(1) If $\lambda > 0$, then (M, g) has non-negative sectional curvature.
(2) If $\lambda < 0$, then (M, g) has non-positive sectional curvature.
(3) If $\lambda = 0$, then (M, g) is flat.

Proof. By Theorem 3.4 one has

$$2\lambda R(X,Y,Y,X) = -B([X,Y],[X,Y]) \geq 0,$$

since $[X,Y] \in [\mathfrak{m},\mathfrak{m}] \subseteq \mathfrak{k}$ and B is negative definite on $\mathfrak{k}$. This shows (1) and (2). If $\lambda = 0$, one has $B([X,Y],[X,Y]) \equiv 0$ for arbitrary X, Y. Then by Proposition 3.2 one has $[X,Y] \equiv 0$ for arbitrary X, Y, and thus (M,g) is flat. □

3.B. **Computations.**

Example 3.6. In Example 2.12 we have already shown that $M = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ is a Riemannian symmetric space. Consider its Cartan decomposition

$$\mathfrak{sl}(n) = \mathfrak{so}(n) \oplus \mathfrak{m},$$

where $\mathfrak{m}$ consists of symmetric matrices and $\mathfrak{m} \cong T_pM$ for $p \in M$. On $\mathfrak{m}$ we can put the usual Euclidean metric, that is for $X, Y \in \mathfrak{m}$, we define

$$\langle X,Y\rangle = \mathrm{tr}(XY^T) = \mathrm{tr}(XY) = \frac{1}{2n}B(X,Y),$$

where B is the Killing form of $\mathfrak{sl}(n)$. By Theorem 3.4 the corresponding metric on M has the curvature formulas

$$\mathrm{Ric}(g) = -\frac{B}{2} = -ng,$$
$$R(X,Y,Y,X) = \frac{B([X,Y],[X,Y])}{2n} \leq 0.$$

Hence it has non-positive sectional curvatures. One can also show its sectional curvature is non-positive by computing curvature tensor as follows

$$\begin{aligned}
R(X,Y,Z,W) &= \mathrm{tr}([Z,[X,Y]]W)\\
&= \mathrm{tr}(Z[X,Y]W - [X,Y]ZW)\\
&= \mathrm{tr}(WZ[X,Y] - [X,Y]ZW)\\
&= \mathrm{tr}([X,Y][Z,W])\\
&= -\mathrm{tr}([X,Y][Z,W]^T)\\
&= -\langle [X,Y],[Z,W]\rangle.
\end{aligned}$$

Example 3.7 (compact Grassmannian)**.** In Example 2.14 we have already shown that $M = \widehat{Gr}_k(\mathbb{R}^{k+l})$ is a Riemannian symmetric space with Cartan decomposition

$$\mathfrak{so}(k+l) = \mathfrak{so}(k) \oplus \mathfrak{so}(l) \oplus \mathfrak{m},$$

where $\mathfrak{m} \cong T_pM$ for $p \in M$. Note that one has the block decomposition of matrices in $\mathfrak{so}(k+l)$ as follows

$$\mathfrak{so}(k+l) = \left\{ \begin{pmatrix} X_1 & B \\ -B^T & X_2 \end{pmatrix} \mid X_1 \in \mathfrak{so}(k), X_2 \in \mathfrak{so}(l), B \in M_{k\times l}(\mathbb{R}) \right\}.$$

Then one has $\mathfrak{m} \cong \left\{ \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} \mid B \in M_{k\times l}(\mathbb{R}) \right\}$. If we put the usual Euclidean metric on $\mathfrak{m}$, that is

$$\begin{aligned}
\left\langle \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} \right\rangle &= \operatorname{tr}\left(\begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}^T \right) \\
&= -\operatorname{tr}\left(\begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \right) \\
&= -\frac{1}{k+l-2} B\left(\begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} \right),
\end{aligned}$$

where B is the Killing form of $\mathfrak{so}(n)$. Then the corresponding metric on M has the curvature formulas

$$\begin{aligned}
\operatorname{Ric}(g) &= -\frac{B}{2} = \frac{k+l-2}{2} g, \\
R(X,Y,Y,X) &= -\frac{B([X,Y],[X,Y])}{k+l-2} \geq 0,
\end{aligned}$$

where $X, Y \in \mathfrak{m}$. This shows the compact Grassmannian has the non-negative sectional curvature.

Example 3.8 (hyperbolic Grassmannian)**.** In Example 2.15 we have already shown that $M = \widehat{Gr}_k(\mathbb{R}^{k,l})$ is a Riemannian symmetric space with Cartan decomposition

$$\mathfrak{so}(k,l) = \mathfrak{so}(k) \oplus \mathfrak{so}(l) \oplus \mathfrak{m},$$

where $\mathfrak{m} \cong T_pM$ for $p \in M$. Note that one has the block decomposition of matrices in $\mathfrak{so}(k,l)$ as follows

$$\mathfrak{so}(k,l) = \left\{ \begin{pmatrix} X_1 & B \\ B^T & X_2 \end{pmatrix} \mid X_1 \in \mathfrak{so}(k), X_2 \in \mathfrak{so}(l), B \in M_{k\times l}(\mathbb{R}) \right\}.$$

Then one has $\mathfrak{m} \cong \left\{ \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \mid B \in M_{k\times l}(\mathbb{R}) \right\}$. If we put the usual Euclidean metric on $\mathfrak{m}$, then

$$\left\langle \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \right\rangle = \frac{1}{k+l-2} B\left(\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \right),$$

where B is the Killing form of $\mathfrak{so}(k,l)$. Then the corresponding metric on M has the curvature formulas

$$\begin{aligned}
\operatorname{Ric}(g) &= -\frac{B}{2} = -\frac{k+l-2}{2} g, \\
R(X,Y,Y,X) &= \frac{B([X,Y],[X,Y])}{k+l-2} \leq 0,
\end{aligned}$$

where $X, Y \in \mathfrak{m}$. This shows the hyperbolic Grassmannian has non-positive sectional curvature.

Remark 3.9. Later we will see compact Grassmannian and hyperbolic Grassmannian are dual to each other in Example 4.26.

Example 3.10. In Example 2.16 one has a compact connected Lie group $G \cong G \times G/G^{\Delta}$ is a Riemannian symmetric space with Cartan decomposition $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}^{\Delta} \oplus \mathfrak{g}^{\perp}$, where

$$\mathfrak{g}^{\Delta} = \{(X, X) \mid X \in \mathfrak{g}\},$$
$$\mathfrak{g}^{\perp} = \{(X, -X) \mid X \in \mathfrak{g}\}.$$

Then one has $\mathfrak{m} \cong \mathfrak{g}^{\perp}$, and thus curvature tensor can be computed as follows

$$\begin{aligned} R(X, Y)Z &= R((X, -X), (Y, -Y))(Z, -Z) \\ &= [(Z, -Z), [(X, -X), (Y, -Y)]] \\ &= ([Z, [X, Y]], -[Z, [X, Y]]). \end{aligned}$$

Hence, we arrive at that the formula

$$R(X, Y)Z = [Z, [X, Y]].$$

Remark 3.11. If one computes the curvature tensor in the standard way using bi-invariant metric, then the formula has a factor $1/4$ on it.

Part 2. Classifications

4. Orthogonal symmetric Lie algebra

So far, we have seen that any Riemannian symmetric space (M, g) gives a Riemannian symmetric pair (G, K) with involution σ, which also gives a pair $(\mathfrak{g}, s)$ of Lie algebra $\mathfrak{g}$ and involution s of $\mathfrak{g}$ such that the eigenspace with respect to 1, denoted by $\mathfrak{k}$, is a subalgebra which is the Lie algebra of K. In this section, we will study such pairs of Lie algebras and prove decomposition theorems, which will give decomposition theorems for symmetric spaces.

4.A. Basic definitions.

Definition 4.1 (compactly embedded). *Let $\mathfrak{g}$ be a Lie algebra. A subalgebra $\mathfrak{k} \leq \mathfrak{g}$ is compactly embedded if $\mathrm{ad}(\mathfrak{k})$ is the Lie algebra of a compact subgroup of $\mathrm{GL}(\mathfrak{g})$.*

Definition 4.2 (orthogonal symmetric Lie algebra). *An orthogonal symmetric Lie algebra is a pair $(\mathfrak{g}, s)$ consisting of a real Lie algebra $\mathfrak{g}$ and an involution $s \neq \mathrm{id}$ of $\mathfrak{g}$ such that $\mathfrak{k}$ is a compactly embedded subalgebra, where $\mathfrak{k}$ is the eigenspace of eigenvalue 1.*

Remark 4.3. For an orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$, the term "orthogonal" is motivated by the fact that Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an orthogonal direct sum with respect to the Killing form of $\mathfrak{g}$.

Example 4.4. Let (G, K) be a Riemannian symmetric pair given by involution σ. Then it gives an orthogonal symmetric pair $(\mathfrak{g}, s)$ by $\mathfrak{g} = \mathrm{Lie}\, G$ and $s = (\mathrm{d}\sigma)_e$, since $\mathrm{ad}(\mathfrak{k})$ is the Lie algebra of $\mathrm{Ad}(K)$, and $\mathrm{Ad}(K)$ is compact by definition of Riemannian symmetric pair.

Definition 4.5 (effective). *An orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$ is effective if $\mathfrak{z} \cap \mathfrak{k} = 0$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$.*

Lemma 4.6. *The Riemannian symmetric pair given by Riemannian symmetric space is effective.*

Proof. It follows from Remark 3.3. □

Similar to Proposition 3.2, one also has the following proposition.

Proposition 4.7. *Let $(\mathfrak{g}, s)$ be an effective orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then the Killing form of $\mathfrak{g}$ is negative definite on $\mathfrak{k}$.*

Proof. Let B be the Killing form of $\mathfrak{g}$ and $K \subseteq \mathrm{GL}(\mathfrak{g})$ be the compact Lie group such that $\mathrm{Lie}\, K = \mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$. Without lose of generality we may assume $K \leq \mathrm{SO}(n)$, and thus $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ consisting of skew-symmetric matrices. Hence for $S \in \mathfrak{k}$,

$$B(S, S) = \mathrm{tr}(\mathrm{ad}_S \circ \mathrm{ad}_S) = \sum_i \langle \mathrm{ad}_S \circ \mathrm{ad}_S(e_i), e_i \rangle = -\sum_i \langle \mathrm{ad}_S(e_i), \mathrm{ad}_S(e_i) \rangle \leq 0,$$

and the equality holds if and only if $S \in \mathfrak{z} \cap \mathfrak{k} = 0$. □

4.B. Decomposition of Riemannian symmetric space.

4.B.1. *Types.*

Definition 4.8 (types)**.** *Let $(\mathfrak{g}, s)$ be an effective orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and Killing form B. Then $(\mathfrak{g}, s)$ is of*

(1) of compact type if $B|_{\mathfrak{m}} < 0$;
(2) of non-compact type if $B|_{\mathfrak{m}} > 0$;
(3) of Euclidean type if $B|_{\mathfrak{m}} = 0$;
(4) of semisimple type if $\mathfrak{g}$ is semisimple, or equivalently, B is non-degenerate.

Definition 4.9 (types)**.**

(1) A Riemannian symmetric pair is of one of above types if its corresponding orthogonal symmetric Lie algebra is.
(2) A Riemannian symmetric space is of one of above types if its corresponding Riemannian symmetric pair is.

Proposition 4.10. *Let $(\mathfrak{g}, s)$ be an effective orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. It's of Euclidean type if and only if $[\mathfrak{m}, \mathfrak{m}] = 0$.*

Proof. If $(\mathfrak{g}, s)$ is of Euclidean type, then $B(\mathfrak{k}, \mathfrak{m}) = 0$ and $B|_{\mathfrak{k}} < 0$ implies $\mathfrak{m}$ is the kernel of Killing form B, and thus $\mathfrak{m}$ is an ideal. Then

$$[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{m} \cap \mathfrak{k} = 0.$$

Conversely, if $[\mathfrak{m}, \mathfrak{m}] = 0$, then by definition of Killing form it's clear $B|_{\mathfrak{m}} = 0$. □

Proposition 4.11. *Let (G, K) be a Riemannian symmetric pair of Euclidean type. Then $M = G/K$ is flat. In particular, if M is simply-connected, then it's isometric to $\mathbb{R}^n$.*

Proof. Since $B|_{\mathfrak{m}} = 0$, by 3.4 one has M is Einstein with Einstein constant zero, and thus by 3.5 one has M is flat. □

4.B.2. *Decomposition of effective orthogonal symmetric Lie algebra.*

Theorem 4.12. *Let $(\mathfrak{g}, s)$ be an effective orthogonal symmetric Lie algebra and B be the Killing form of $\mathfrak{g}$. Then there exists ideals $\mathfrak{g}_0, \mathfrak{g}_-$ and $\mathfrak{g}_+$ with the following properties:*

(1) $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_- \oplus \mathfrak{g}_+$.
(2) $\mathfrak{g}_0, \mathfrak{g}_-$ and $\mathfrak{g}_+$ are invariant under s and orthogonal with respect to Killing form B of $\mathfrak{g}$.
(3) Let s_0, s_-, s_+ be the restrictions of s to $\mathfrak{g}_0, \mathfrak{g}_-$ and $\mathfrak{g}_+$. The pairs $(\mathfrak{g}_0, s_0), (\mathfrak{g}_-, s_-)$ and $(\mathfrak{g}_+, s_+)$ are effective orthogonal symmetric Lie algebras of the Euclidean type, compact type and non-compact type, respectively.

Proof. See Theorem 1.1 in Chapter V of [Hel78]. □

4.B.3. *Decomposition of Riemannian symmetric space.*

Theorem 4.13. *Let (M, g) be a simply-connected symmetric space. Then $M = M_0 \times M_+ \times M_-$ is the Riemannian product of symmetric space of Euclidean, non-compact and compact types respectively.*

Proof. Let (G, K) with involution σ be the effective Riemannian symmetric pair given by (M, g) and $(\mathfrak{g}, s)$ be the corresponding effective orthogonal symmetric Lie algebra. Let $\varphi: \widetilde{G} \to G$ be the universal covering and $\widetilde{K}$ be the identity component of $\varphi^{-1}(K)$. Then if ψ denotes the mapping $g\widetilde{K} \to \varphi g\widetilde{K}$ of $\widetilde{G}/\widetilde{K}$ on G/K, then it gives a covering map of $M = G/K$. Since M is simply-connected, $M = \widetilde{G}/\widetilde{K}$.

By Theorem 4.12, we obtain a decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_- \oplus \mathfrak{g}_+$. By Theorem A.3, there exists simply-connected Lie groups G_0, G_- and G_+ with Lie algebras $\mathfrak{g}_0, \mathfrak{g}_-$ and $\mathfrak{g}_+$. Then it gives a decomposition $\widetilde{G} = G_0 \times G_- \times G_+$. If $\widetilde{K} = K_0 \times K_- \times K_+$ is the corresponding decomposition, then the spaces $M_0 = G_0/K_0, M_- = G_-/K_-$ and $M_+ = G_+/K_+$ gives the desired decomposition. □

4.C. **Irreducibility.**

4.C.1. *Irreducible orthogonal symmetric Lie algebra.*

Definition 4.14 (irreducible)**.** *Suppose $(\mathfrak{g}, s)$ is an orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then $(\mathfrak{g}, s)$ is called irreducible if*

(1) $\mathfrak{g}$ is semisimple and $\mathfrak{k}$ contains no ideal of $\mathfrak{g}$;
(2) the Lie algebra $\mathrm{ad}(\mathfrak{k})$ acts irreducibly on $\mathfrak{m}$.

Remark 4.15. Any irreducible orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$ is effective, since $\mathfrak{z} \cap \mathfrak{k}$ is an ideal in $\mathfrak{k}$, and thus vanishes.

Definition 4.16 (irreducible)**.**

(1) A Riemannian symmetric pair is called irreducible if its corresponding orthogonal symmetric Lie algebra is.
(2) A Riemannian symmetric space is called irreducible if its corresponding Riemannian symmetric pair is.

Lemma 4.17 (Schur lemma)**.** *Let B_1, B_2 be two symmetric bilinear forms on a vector space V such that B_1 is positive definite. If a group K acts irreducibly on V such that B_1 and B_2 are invariant under K, then $B_2 = \lambda B_1$ for some constant λ.*

Proof. Since B_1 is positive definite, there exists an endomorphism $L : V \to V$ such that

$$B_2(u, v) = B_1(Lu, v),$$

where $u, v \in V$. Since B_1, B_2 are invariant under K, one has for any $k \in K$

$$B_1(kLu, v) = B_1(Lu, k^{-1}v) = B_2(u, k^{-1}v) = B_2(ku, v) = B_1(Lku, v),$$

holds for arbitrary $u, v \in V$, which implies $Lk = kL$ for all $k \in K$. On the other hand, the symmetry of B_1, B_2 implies

$$B_1(Lu, v) = B_2(u, v) = B_2(v, u) = B_1(Lv, u) = B_1(u, Lv).$$

Hence L is symmetric with respect to B_1, and thus the eigenvalues of L are real. If $0 \neq E \subseteq V$ is an eigenspace with eigenvalue λ, the fact $kL = Lk$ implies E is invariant under K. Since K acts irreducibly on V, one has $E = V$, that is $L = \lambda I$, which implies $B_2 = \lambda B_1$. □

Proposition 4.18. *Let (G, K) be an irreducible Riemannian symmetric pair given by σ. Then there is up to scaling a unique left-invariant metric on $M = G/K$.*

Proof. It suffices to show there is up to scaling a unique $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{m}$. Since (G, K) is an irreducible Riemannian symmetric pair, then K acts on $\mathfrak{m}$ irreducibly by adjoint representation, and thus by Lemma 4.17 any two $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{m}$ are scalar multiples of each other. In particular, $-B|_{\mathfrak{m}}$ and $B|_{\mathfrak{m}}$ give such an inner product in compact and non-compact cases respectively. □

Proposition 4.19. *Let (G,K) be a Riemannian symmetric pair and $M = G/K$.*

(1) If (G,K) is of compact type, then M has non-negative sectional curvature.
(2) If (G,K) is of non-compact type, then M has non-positive sectional curvatures.

Proof. If (G,K) is of compact type, we may assume $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{m}$ is given by $-B|_{\mathfrak{m}}$, and thus by 3.4 one has

$$\mathrm{Ric} = -\frac{1}{2}B.$$

This shows M is Einstein with Einstein constant $1/2$, and thus by Corollary 3.5 one has M has non-negative sectional curvature. Similarly one can show if (G,K) is of non-compact type, then M has non-positive sectional curvatures. □

4.C.2. *Decomposition into irreducible Riemannian symmetric spaces.*

Theorem 4.20. *Let $(\mathfrak{g}, s)$ be an effective orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ such that $\mathfrak{g}$ is semisimple and $\mathfrak{k}$ does not contain an ideal of $\mathfrak{g}$. Then there are ideals $(\mathfrak{g}_i)_{i\in I}$ of $\mathfrak{g}$ such that*

(1) $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$.
(2) The ideals $\mathfrak{g}_i$ are mutually orthogonal with respect to Killing form B of $\mathfrak{g}$, and they are invariant under s.
(3) Denoting by s_i the restriction if s to $\mathfrak{g}_i$, each $(\mathfrak{g}_i, s_i)$ is an irreducible orthogonal symmetric Lie algebra.

Proof. See Proposition 5.2 in Chapter VIII of [Hel78]. □

As Theorem 4.13, this decomposition of effective orthogonal symmetric Lie algebra gives a decomposition of Riemannian symmetric space as follows.

Theorem 4.21. *Let (M,g) be a simply-connected Riemannian symmetric space. Then M is a product*

$$(M,g) \cong (M_0, g_0) \times (M_1, g_1) \times \cdots \times (M_n, g_n),$$

where (M_0, g_0) is a Riemannian symmetric space of Euclidean type and for $i \geq 1$, the factors (M_i, g_i) are irreducible Riemannian symmetric spaces.

Proof. See Proposition 5.5 in Chapter VIII of [Hel78]. □

4.D. **Duality.** Let $\mathfrak{g}$ be a real Lie algebra. Then its complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ is a complex Lie algebra, with Lie bracket

$$[X_1 + \sqrt{-1}Y_1, X_2 + \sqrt{-1}Y_2] := [X_1, X_2] - [Y_1, Y_2] + \sqrt{-1}\left([Y_1, X_2] + [X_1, Y_2]\right)$$

Definition 4.22 (real form)**.** *Let $\mathfrak{h}$ be a complex Lie algebra. A real form of $\mathfrak{h}$ is a real Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}_{\mathbb{C}}$ is isomorphic to $\mathfrak{h}$ as complex Lie algebras.*

Remark 4.23. It's clear a real Lie algebra is a real form of its complexification but in general there are many pairwise non-isomorphic real forms.

Now suppose $(\mathfrak{g}, s)$ is an orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then there are following bracketing relations:

(1) $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$.
(2) $[\mathfrak{k}, \sqrt{-1}\mathfrak{m}] = \sqrt{-1}[\mathfrak{k}, \mathfrak{m}] \subseteq \sqrt{-1}\mathfrak{m}$.
(3) $[\sqrt{-1}\mathfrak{m}, \sqrt{-1}\mathfrak{m}] = -[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$.

In particular, $\mathfrak{g}^* := \mathfrak{k} \oplus \sqrt{-1}\mathfrak{m}$ is a real Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let $s_{\mathbb{C}}$ be the $\mathbb{C}$-linear extension of s to $\mathfrak{g}_{\mathbb{C}}$ and s^* be the restriction of $s_{\mathbb{C}}$ to $\mathfrak{g}^*$. Then $(\mathfrak{g}^*, s^*)$ is also an orthogonal symmetric Lie algebra, which is defined to be the dual of $(\mathfrak{g}, s)$.

Theorem 4.24. *Let $(\mathfrak{g}, s)$ be an orthogonal symmetric Lie algebra with dual $(\mathfrak{g}^*, s^*)$.*

(1) If $(\mathfrak{g}, s)$ is of compact type, then $(\mathfrak{g}^*, s^*)$ is of non-compact type, and vice versa.
(2) If $(\mathfrak{g}, s)$ is of Euclidean type, then $(\mathfrak{g}^*, s^*)$ is of Euclidean type.
(3) $(\mathfrak{g}, s)$ is irreducible if and only if $(\mathfrak{g}^*, s^*)$ is irreducible.

Proof. For (1) and (2). It suffices to establish a relation between the respective Killing forms. Note that there is an isomorphism of vector spaces $\Psi\colon \mathfrak{g} \to \mathfrak{g}^*$ given by $X + Y \mapsto X + \sqrt{-1}Y$. For $Z_1, Z_2 \in \mathfrak{m}$, a direct computation shows

$$\begin{aligned}
\mathrm{ad}_{\mathfrak{g}^*}(\sqrt{-1}Z_1)\mathrm{ad}_{\mathfrak{g}^*}(\sqrt{-1}Z_2)(X + \sqrt{-1}Y) &= \left[\sqrt{-1}Z_1, [\sqrt{-1}Z_2, X + \sqrt{-1}Y]\right] \\
&= -[Z_1, [Z_2, X]] - \sqrt{-1}\,[Z_1, [Z_2, Y]] \\
&= -\Psi([Z_1, [Z_2, X + Y]]) \\
&= -\Psi(\mathrm{ad}_{\mathfrak{g}}(Z_1)\mathrm{ad}_{\mathfrak{g}}(Z_2)(X + Y)).
\end{aligned}$$

Therefore $B_{\mathfrak{g}^*}(\sqrt{-1}Z_1, \sqrt{-1}Z_2) = -B_{\mathfrak{g}}(Z_1, Z_2)$. As a consequence, $B_{\mathfrak{g}}|_{\mathfrak{m}} > 0$ if and only if $B_{\mathfrak{g}^*}|_{\sqrt{-1}\mathfrak{m}} < 0$ and vice versa.

For (3). Note that $\mathfrak{g}$ is semisimple if and only if its Killing form is non-degenerate, so $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g}^*$ is, and thus $(\mathfrak{g}, s)$ is irreducible if and only if $(\mathfrak{g}^*, s^*)$ is irreducible. □

4.D.1. *Examples of duality.*

Example 4.25. Consider the orthogonal symmetric Lie algebra $(\mathfrak{sl}(n, \mathbb{R}), s)$, where $s\colon X \mapsto -X^T$. Its Cartan decomposition is given by

$$\begin{aligned}
\mathfrak{k} &= \{X \in \mathfrak{sl}(n, \mathbb{R}) \mid X^T + X = 0\}, \\
\mathfrak{m} &= \{X \in \mathfrak{sl}(n, \mathbb{R}) \mid X^T = X\}.
\end{aligned}$$

Then $\mathfrak{sl}(n, \mathbb{R})_{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$ and

$$\begin{aligned}
\mathfrak{k} + \sqrt{-1}\mathfrak{m} &= \{Z \in \mathfrak{sl}(n, \mathbb{C}) \mid Z = X + \sqrt{-1}Y, X^T + X = 0, Y^T = Y\} \\
&= \{Z \in \mathfrak{sl}(n, \mathbb{C}) \mid Z + \overline{Z}^T = 0\} \\
&= \mathfrak{su}(n).
\end{aligned}$$

As a consequence, the Riemannian symmetric space $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ and $\mathrm{SU}(n)/\mathrm{SO}(n)$ are dual to each other. For $n = 2$, one has $\mathbb{H}^2$ is dual to S^2, since $\mathrm{SU}(2)$ is the universal covering of $\mathrm{SO}(3)$.

Example 4.26. Consider the orthogonal symmetric Lie algebra $(\mathfrak{so}(n), s)$, where s is given by

$$\begin{aligned} s:\ \mathfrak{so}(n) &\to \mathfrak{so}(n) \\ X &\mapsto I_{k,l} X I_{k,l} \end{aligned}$$

where $k + l = n$. Its Cartan decomposition is given by

$$\mathfrak{so}(n) = \mathfrak{so}(k) \oplus \mathfrak{so}(l) \oplus \mathfrak{m},$$

where

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} \mid B \in M_{k\times l}(\mathbb{R}) \right\}.$$

Then

$$\mathfrak{g}^* = \left\{ \begin{pmatrix} X_1 & \sqrt{-1}B \\ -\sqrt{-1}B^T & X_2 \end{pmatrix} \mid X_1 \in \mathfrak{so}(k), X_2 \in \mathfrak{so}(l), B \in M_{k\times l}(\mathbb{R}) \right\}.$$

It's easy to verify the mapping

$$\begin{pmatrix} X_1 & \sqrt{-1}B \\ -\sqrt{-1}B^T & X_2 \end{pmatrix} = \begin{pmatrix} X_1 & B \\ B^T & X_2 \end{pmatrix}$$

is a Lie algebra isomorphism of $\mathfrak{g}^*$ to $\mathfrak{so}(p, q)$. This shows compact Grassmannian and hyperbolic Grassmannian are dual to each other.

5. Non-compact type symmetric space

6. Compact type symmetric space

Part 3. Hermitian symmetric space

7. Hermitian symmetric space

Definition 7.1 (Hermitian symmetric space)**.** *Let (M, g) be a Riemannian symmetric space. (M, g) is said to be a Hermitian symmetric manifold if (M, g) is a Hermitian manifold and the symmetric at each point is a holomorphic isometry.*

Lemma 7.2. *Any almost Hermitian structure on a Riemannian symmetric space (M, g) is integrable, and any Hermitian symmetric space is Kähler.*

Proof. Suppose φ is the symmetry at point $p \in M$ and J is an almost Hermitian structure of (M, g). Since φ is a holomorphic isometry one has $(\mathrm{d}\varphi)_p \circ J = J \circ (\mathrm{d}\varphi)_p$, and thus

$$\begin{aligned} -N_J(X, Y) &= (\mathrm{d}\varphi)_p N_J(X, Y) \\ &= (\mathrm{d}\varphi)_p \left([JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]\right) \\ &= [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] \\ &= N_J(X, Y). \end{aligned}$$

This shows $N_J = 0$ at point p, and since p is arbitrary one has $N_J \equiv 0$, which implies J is integrable. By the same argument one can show $\nabla J = 0$, and thus (M, g) is Kähler. □

Proposition 7.3. *Let (G, K) be a symmetric pair with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. If $J : \mathfrak{m} \to \mathfrak{m}$ satisfies*

(1) J is orthogonal and $J^2 = -\mathrm{id}$.

(2) $J \circ \mathrm{Ad}(k) = \mathrm{Ad}(k) \circ J$ for all $k \in K$.

Then $M = G/K$ is a Hermitian symmetric space, and thus Kähler.

Corollary 7.4. *Let (G, K) be a symmetric pair. Then*

(1) (G, K) is Hermitian symmetric if and only if its dual is Hermitian symmetric.

(2) If (G, K) is irreducible and Hermitian symmetric, then it's Kähler-Einstein.

Proposition 7.5. *Let (G, K) be an irreducible symmetric pair.*

(1) If (G, K) is of compact type, then it's Hermitian symmetric if and only if $H^2(M, \mathbb{R}) \neq 0$.

(2) (G, K) is Hermitian symmetric if and only if K is not semisimple.

(3) The complex structure J is unique up to a sign.

Proof. For (1). It's clear if (G, K) is Hermitian symmetric, then $H^2(M, \mathbb{R}) \neq 0$ since its Kähler form lies in it; Conversely, for $0 \neq \omega \in H^2(M, \mathbb{R})$, we may construct a new 2-form $\widetilde{\omega}$ by

$$\widetilde{\omega}_p := \int_G \omega_{gp} \mathrm{d}g.$$

It's clear $\widetilde{\omega}$ is invariant under isometries. □

8. Bounded symmetric domains

8.A. **The Bergman metrics.**

8.B. **Classical bounded symmetric domains.**

8.C. **Curvatures of classical bounded symmetric domains.**

Part 4. **Appendix**

Appendix A. Lie group and Lie algebras

A.A. **Lie theorems.**

Theorem A.1. *If* $\Phi\colon \operatorname{Lie} G \to \operatorname{Lie} H$ *is a Lie group homomorphism and* G *is simply-connected, then there exists a unique Lie group homomorphism* $\varphi\colon G \to H$ *such that* $\Phi = (\mathrm{d}\varphi)_e$.

Theorem A.2. *If* G *is a Lie group and* $\mathfrak{h} \subseteq \operatorname{Lie} G$ *is a Lie subalgebra. then there exists a unique connected Lie subgroup* $H \subseteq G$ *with* $\operatorname{Lie} H = \mathfrak{h}$.

Theorem A.3. *Every finite-dimensional real Lie algebra is the Lie algebra of some simply-connected Lie group.*

Appendix B. Basic facts in Riemannian geometry

B.A. **Killing fields.**

B.A.1. *Basic properties.*

Proposition B.1. *Let (M,g) be a Riemannian manifold and X be a Killing field.*
(1) If γ is a geodesic, then $J(t)=X(\gamma(t))$ is a Jacobi field.
(2) For any two vector fields Y,Z,

$$\nabla_Y\nabla_Z X-\nabla_{\nabla_Y Z}X+R(X,Y)Z=0$$

Proof. For (1). Suppose φ_s is the flow generated by X. Then we obtain a variation $\alpha(s,t)=\varphi_s(\gamma(t))$ consisting of geodesics, and thus

$$X(\gamma(t))=\left.\frac{\partial\varphi_s(\gamma(t))}{\partial s}\right|_{s=0}$$

is a Jacobi field.

For (2). It's an equation of tensors, so we check it pointwisely and use normal coordinate $\{x^i\}$ centered at p. Furthermore, we assume $X=X^i\frac{\partial}{\partial x^i},Y=\frac{\partial}{\partial x^j},Z=\frac{\partial}{\partial x^k}$. Then

$$\begin{aligned}\nabla_Y\nabla_Z X-\nabla_{\nabla_Y Z}X+R(X,Y)Z&=\nabla_j\nabla_k X+X^iR_{ijk}^l\frac{\partial}{\partial x^l}\\&=(\frac{\partial^2X^l}{\partial x^j\partial x^k}+X^i\frac{\partial\Gamma_{ki}^l}{\partial x^j}+X^iR_{ijk}^l)\frac{\partial}{\partial x^l}\\&=(\frac{\partial^2X^l}{\partial x^j\partial x^k}+X^i\frac{\partial\Gamma_{jk}^l}{\partial x^i})\frac{\partial}{\partial x^l}\end{aligned}$$

since $R_{ijk}^l=\frac{\partial\Gamma_{jk}^l}{\partial x^i}-\frac{\partial\Gamma_{ik}^l}{\partial x^j}+\Gamma_{jk}^s\Gamma_{is}^l-\Gamma_{ik}^s\Gamma_{js}^l$. Now it suffices to show $\frac{\partial^2X^l}{\partial x^j\partial x^k}+X^i\frac{\partial\Gamma_{jk}^l}{\partial x^i}\equiv 0$. In order to show this, for arbitrary $p\in M$, consider a geodesic γ starting at p and consider Jacobi field $J(t)=X(\gamma(t))$. Direct computation shows

$$\begin{aligned}J'(t)&=\left.(\frac{\partial X^i}{\partial x^k}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}+X^i\Gamma_{ki}^l\frac{\mathrm{d}\gamma^k}{\mathrm{d}t})\frac{\partial}{\partial x^l}\right|_{\gamma(t)}\\J''(0)&=\left.(\frac{\partial^2X^l}{\partial x^j\partial x^k}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}+X^i\frac{\partial\Gamma_{ki}^l}{\partial x^j}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t})\frac{\partial}{\partial x^l}\right|_p\\&=\left.(\frac{\partial^2X^l}{\partial x^j\partial x^k}+X^i\frac{\partial\Gamma_{ki}^l}{\partial x^j})\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}\frac{\partial}{\partial x^l}\right|_p\\&=\left.(\frac{\partial^2X^l}{\partial x^j\partial x^k}+X^i\frac{\partial\Gamma_{jk}^l}{\partial x^i}+X^i\frac{\partial\Gamma_{ki}^l}{\partial x^j}-X^i\frac{\partial\Gamma_{jk}^l}{\partial x^i})\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}\frac{\partial}{\partial x^l}\right|_p\\&=\left.(\frac{\partial^2X^l}{\partial x^j\partial x^k}+X^i\frac{\partial\Gamma_{jk}^l}{\partial x^i})\frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\frac{\mathrm{d}\gamma^k}{\mathrm{d}t}\frac{\partial}{\partial x^l}\right|_p-R(X,\gamma')\gamma'\end{aligned}$$

which implies

$$\frac{\partial^2X^l}{\partial x^j\partial x^k}+X^i\frac{\partial\Gamma_{jk}^l}{\partial x^i}=0$$

holds at point p, and since p is arbitrary, this completes the proof. $\square$

Corollary B.2. *Let (M, g) be a complete Riemannian manifold and $p \in M$. Then a Killing field X is determined by the values X_p and $(\nabla X)_p$ for arbitrary $p \in M$.*

Proof. The equation $\mathcal{L}_X g \equiv 0$ is linear in X, so the space of Killing fields is a vector space. Therefore, it suffices to show if $X_p = 0$ and $(\nabla X)_p = 0$, then $X \equiv 0$. For arbitrary $q \in M$, let $\gamma : [0, 1] \to M$ be a geodesic connecting p and q with $\gamma'(0) = v$. Since $J(t) = X(\gamma(t))$ is a Jacobi field, and a direct computation shows

$$(\nabla_v X)_p = J'(0)$$

Thus $J(t) \equiv 0$, since Jacobi field is determined by two initial values. In particular, $X_q = J(1) = 0$, and since q is arbitrary, one has $X \equiv 0$. $\square$

Corollary B.3. *The dimension of vector space consisting of Killing fields $\leq n(n+1)/2$.*

Proof. Note that ∇X is skew-symmetric and the dimension of skew-symmetric matrices is $n(n-1)/2$. Thus the dimension of vector space consisting of Killing fields $\leq n + n(n-1)/2 = n(n+1)/2$. $\square$

B.A.2. *Killing field as the Lie algebra of isometry group.*

Lemma B.4. *Killing field on a complete Riemannian manifold (M, g) is complete.*

Proof. For a Killing field X, we need to show the flow $\varphi_t : M \to M$ generated by X is defined for $t \in \mathbb{R}$. Otherwise, we assume φ_t is defined on (a, b). Note that for each $p \in M$, curve $\varphi_t(p)$ is a curve defined on (a, b) having finite constant speed, since φ_t is isometry. Then we have $\varphi_t(p)$ can be extended to the one defined on $\mathbb{R}$, since M is complete. $\square$

Theorem B.5. *Let (M, g) be a complete Riemannian manifold and $\mathfrak{g}$ the space of Killing fields. Then $\mathfrak{g}$ is isomorphic to the Lie algebra of $G = \mathrm{Iso}(M, g)$.*

Proof. It's clear $\mathfrak{g}$ is a Lie algebra since $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$. Now let's see it's isomorphic to Lie algebra consisting of Killing field as Lie algebra.

(1) Given a Killing field X, by Lemma B.4, one deduces that the flow $\varphi : \mathbb{R} \times M \to M$ generated by X is a one parameter subgroup $\gamma : \mathbb{R} \to G$, and $\gamma'(0) \in T_e G$.
(2) Given $v \in T_e G$, consider the one-parameter subgroup $\gamma(t) = \exp(tv) : \mathbb{R} \to G$ which gives a flow by

$$\begin{aligned} \varphi : \mathbb{R} \times M &\to M \\ (t, p) &\mapsto \exp(tv) \cdot p \end{aligned}$$

Then the vector field X generated by this flow is a Killing field.

This gives a one to one correspondence between Killing fields and Lie algebra of G, and it's a Lie algebra isomorphism. $\square$

Corollary B.6 (Cartan decomposition). *Let (M, g) be a complete Riemannian manifold and $G = \mathrm{Iso}(M, g)$ with Lie algebra $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ of G has a decomposition as vector spaces*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

where
$$\mathfrak{k} = \{X \in \mathfrak{g} \mid X_p = 0\}$$
$$\mathfrak{m} = \{X \in \mathfrak{g} \mid (\nabla X)_p = 0\}$$
and they satisfy
$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$$

Proof. The decomposition follows from Corollary B.2 and Theorem B.5, and it's easy to see
$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$$
For arbitrary $X \in \mathfrak{k}, Y \in \mathfrak{m}$ and $v \in T_pM$, one has
$$\begin{aligned}\nabla_v[X,Y] &= \nabla_v\nabla_X Y - \nabla_v\nabla_Y X \\ &= -R(Y,v)X + \nabla_{\nabla_v X}Y + R(X,v)Y - \nabla_{\nabla_v Y}X \\ &= 0\end{aligned}$$
since $X_p = 0$ and $(\nabla Y)_p = 0$. This shows $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$. □

B.B. **Hopf theorem.** The argument about analytic continuation in Theorem 1.11 can be used to give a proof of Hopf's theorem.

Theorem B.7 (Hopf)**.** *Let (M, g) be a complete, simply-connected Riemannian manifold with constant sectional curvature K. Then (M, g) is isometric to*
$$(\widetilde{M}, g_{can}) = \begin{cases}(\mathbb{S}^n(\frac{1}{\sqrt{K}}), g_{can}) & K > 0\\ (\mathbb{R}^n, g_{can}) & K = 0\\ (\mathbb{H}^n(\frac{1}{\sqrt{-K}}), g_{can}) & K < 0\end{cases}$$

Proof. For $p \in M, \widetilde{p} \in \widetilde{M}$ and $\delta < \min\{\mathrm{inj}(p), \mathrm{inj}(\widetilde{p})\}$. By Cartan-Ambrose-Hicks's theorem, there exists an isometry $\varphi : B(p, \delta) \to B(\widetilde{p}, \delta)$ such that $\varphi(p) = \widetilde{p}$ and $(\mathrm{d}\varphi)_p$ equals to a given linear isometry, since both (M, g) and $(\widetilde{M}, \widetilde{g})$ have constant sectional curvature K. By the same argument in proof of Theorem 1.11, there is an isometry $\varphi : (M, g) \to (\widetilde{M}, \widetilde{g})$ which extends $\varphi : B(p, \delta) \to B(\widetilde{p}, \delta)$. In particular, this completes the proof. □

B.C. **Other basic facts.**

Theorem B.8. *Let $\varphi, \psi : (M, g_M) \to (N, g_N)$ be two local isometries between Riemannian manifolds, and M is connected. If there exists $p \in M$ such that*
$$\varphi(p) = \psi(p)$$
$$(\mathrm{d}\varphi)_p = (\mathrm{d}\psi)_p$$
then $\varphi = \psi$.

Theorem B.9 (Cartan-Ambrose-Hicks)**.** *Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be two Riemannian manifolds and $\Phi_0 : T_pM \to T_{\widetilde{p}}\widetilde{M}$ be a linear isometry, where $p \in M, \widetilde{p} \in \widetilde{M}$. For $0 < \delta < \min\{\mathrm{inj}_p(M), \mathrm{inj}_{\widetilde{p}}(\widetilde{M})\}$, The following statements are equivalent.*

(1) There exists an isometry $\varphi : B(p, \delta) \to B(\widetilde{p}, \delta)$ such that $\varphi(p) = \widetilde{p}$ and $(\mathrm{d}\varphi)_p = \Phi_0$.

(2) For $v \in T_pM, |v| < \delta, \gamma(t) = \exp_p(tv), \widetilde{\gamma}(t) = \exp_{\widetilde{p}}(t\Phi_0(v))$, if we define

$$\Phi_t = P_{0,t;\widetilde{\gamma}} \circ \Phi_0 \circ P_{t,0;\gamma} : T_{\gamma(t)}M \to T_{\widetilde{\gamma}(t)}\widetilde{M}$$

then Φ_t preserves curvature, that is $(\Phi_t)^*R = R$.

Proposition B.10. *Let $(M, g_M), (N, g_N)$ be complete Riemannian manifolds and $f : M \to N$ be a local diffeomorphism such that for all $p \in M$ and for all $v \in T_pM$, one has $|(\mathrm{d}f)_p v| \geq |v|$. Then f is a Riemannian covering map.*

Theorem B.11 (Myers-Steenrod)**.** *Let (M, g) be a Riemannian manifold and $G = \mathrm{Iso}(M, g)$. Then*

(1) G is a Lie group with respect to compact-open topology.
(2) for each $p \in M$, the isotropy group G_p is compact.
(3) G is compact if M is compact.

Proposition B.12. *Let (M, g) be a Riemannian manifold, $\gamma : I \to M$ a smooth curve and $P_{s,t;\gamma} : T_{\gamma(s)}M \to T_{\gamma(t)}M$ is the parallel transport along γ. For any $s \in I$ with $v = \gamma'(s)$, one has*

$$\nabla_v R = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} (P_{s,t;\gamma})^* R_{\gamma(t)}$$

In particular, if $\nabla R = 0$ then

$$(P_{s,t;\gamma})^* R_{\gamma(t)} = R_{\gamma(s)}$$

holds for arbitrary $t, s \in I$.

Proposition B.13. *If $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ is a Riemannian covering, then M is complete if and only if $\widetilde{M}$ is.*

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