

# HODGE THEORY AND COMPLEX ALGEBRAIC GEOMETRY

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## 1. OVERVIEW

In this course, we will introduce two parts:

- I Kähler manifold and Hodge decomposition;
- II Hodge theory in algebra geometry.

For the first part, if  $X$  is a compact complex manifold, we can consider the following structures:

- (1) Topology:  $H_B^*(X, \mathbb{Z})$ , singular cohomology, where  $B$  means “Betti”.
- (2)  $C^\infty$ -structure:  $H_{dR}^*(X, \mathbb{R}) = H^*(X, \Omega_{X, \mathbb{R}})$ , de Rham cohomology. In fact, de Rham theorem implies that

$$H_{dR}^*(X, \mathbb{R}) \cong H_B^*(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$$

- (3) Complex structure: For  $x \in X$ ,  $T_{X,x}$  is tangent space at  $x$ , its real dimension is  $2n$ , that is  $\dim_{\mathbb{R}} T_{X,x} = 2n$ . And there is a linear map  $J_x : T_{X,x} \rightarrow T_{X,x}$ , such that  $J_x^2 = -\text{id}$ . If we complexificate  $T_{X,x}$ , then we can decompose it into

$$T_{X,x} \otimes \mathbb{C} = T_{X,x}^{1,0} \oplus T_{X,x}^{0,1}$$

where  $T_{X,x}^{1,0}$  is the eigenspace belonging to eigenvalue  $\sqrt{-1}$ , and  $T_{X,x}^{0,1}$  is the eigenspace belonging to eigenvalue  $-\sqrt{-1}$ .

If we consider its dual, we will get bundle/sheaf of differential forms, and we can also decompose them as follows

$$\Omega_{X, \mathbb{C}}^1 = \Omega_{X, \mathbb{R}}^1 \otimes \mathbb{C} = \Omega_X^{1,0} \oplus \Omega_X^{0,1}$$

and

$$\Omega_{X, \mathbb{C}}^k = \Omega_{X, \mathbb{R}}^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Omega_X^{p,q}$$

where  $\Omega_X^{p,q} = \bigwedge^p \Omega_X^{1,0} \otimes \bigwedge^q \Omega_X^{0,1}$

Since we have such decomposition for differential forms, it's natural to ask if there is a similar decomposition for de Rham cohomology? that is, do we have

$$H_{dR}^k(X, \mathbb{R}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

such that  $\overline{H^{p,q}(X)} = H^{q,p}(X)$ ?

The Hodge decomposition says it's true for compact Kähler manifolds. It's a very beautiful result, connecting "Topology" and "Complex geometry", since de Rham cohomology reflects the topology information and

$$H^{p,q}(X) \cong H_{Dol}^q(X, \Omega_X^p)$$

where "Dol" means Dolbeault cohomology.

Here is some examples of Kähler manifolds, in fact, almost every interesting manifold is Kähler manifold:

**Example 1.1.** Riemann surfaces, complex torus, projective manifolds are Kähler manifolds.

We also need to know an example that is not Kähler manifold:

**Example 1.2** (Hopf surface). Consider  $\mathbb{Z}$  acts on  $\mathbb{C}^2 \setminus \{0\}$  by  $mz = \lambda^m z, m \in \mathbb{Z}$  for some  $\lambda \in (0, 1)$ , then we define Hopf surface as follows

$$S = \mathbb{C}^2 \setminus \{0\} / \mathbb{Z}$$

As we can see,  $S$  is diffeomorphic to  $S^3 \times S^1$ , then  $\dim_{\mathbb{C}} H^1(S, \mathbb{C}) = 1$ , so  $S$  can not be a Kähler manifold by Hodge's decomposition, since for a Kähler manifold,  $\dim_{\mathbb{C}} H^1$  must be even.

**Remark 1.3.** By Chow's theorem/GAGA\*, a compact complex manifold  $X$  admitting an embedding into projective space can be defined by polynomial equations, i.e.,  $X$  is a projective variety, so here comes the forth structure, and that's the second part of this course, we want to apply Hodge theory in algebraic geometry.

(4) Algebraic structure.

## 2. COMPLEX MANIFOLD

**Definition 2.1** (complex manifold). A complex manifold consists of  $(X, \{U_i, \phi_i\}_{i \in I})$ , where  $X$  is a connected, Hausdorff topological space,  $\{U_i\}_{i \in I}$  is an open cover of  $X$  such that the index set  $I$  is countable, and  $\phi_i$  is a homeomorphism from  $U_i$  to an open subset  $V_i$  of  $\mathbb{C}^n$ , such that

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j)$$

is biholomorphic.

**Definition 2.2** (transition function). Such  $\phi_i \circ \phi_j^{-1}$  is called transition function;  $n$  is called the dimension of  $X$ , denoted by  $\dim_{\mathbb{C}} X$ ;  $\{U_i, \phi_i\}_{i \in I}$  is called complex atlas.

**Definition 2.3** (atlas). Two atlas are equivalent, if the union of them is still an atlas.

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\*Chow's theorem/GAGA: Every compact complex manifold  $X$  admitting embedding  $X \hookrightarrow \mathbb{P}^n$  is defined by homogenous polynomials.

**Definition 2.4** (complex structure). *A complex structure is an equivalence class of a complex atlas.*

**Remark 2.5.** Replace  $\mathbb{C}^n$  by  $\mathbb{R}^n$ , and biholomorphism is replaced by homeomorphism or diffeomorphism, then we get topological manifold or smooth manifold.

**Remark 2.6.**  $V_i \subset \mathbb{C}^n$  usually can not be the whole  $\mathbb{C}^n$ . For example, there is no non-constant holomorphism from  $\mathbb{C}$  to unit disk  $\mathbb{D}$ . More generally,  $X$  is called Brody hyperbolic if there is no non-constant holomorphism from  $\mathbb{C}$  to  $X$ .

**Example 2.7.** Projective space  $\mathbb{P}^n$  is a complex manifold. Atlas are  $U_i = \{[z] \in \mathbb{P}^n \mid z_j \neq 0\}, 0 \leq i \leq n, \phi_i : U_i \rightarrow \mathbb{C}^n$  is defined as follows

$$[z] \mapsto \left( \frac{z_0}{z_i}, \dots, \frac{\widehat{z_i}}{z_i}, \dots, \frac{z_n}{z_i} \right)$$

Transition functions are calculated as follows, for  $i < j$

$$\phi_i \circ \phi_j^{-1} : (u_1, \dots, u_n) \mapsto \left( \frac{u_1}{u_i}, \dots, \frac{\widehat{u_i}}{u_i}, \dots, \frac{u_{j-1}}{u_i}, \frac{1}{u_i}, \frac{u_{j+1}}{u_i}, \dots, \frac{u_n}{u_i} \right)$$

In fact,  $\mathbb{P}^n$  is a compact complex manifold, since  $\mathbb{P}^n$  is diffeomorphic to  $S^{2n+1}/S^1$ .

**Example 2.8.** Grassmannian manifold

$$G(r, n) = \{\text{r-dimensional subspace of } \mathbb{C}^n\}$$

Atlas: given  $T_i \subset \mathbb{C}^n$  of dimension  $n - r$ , set  $U_i = \{S \subset \mathbb{C}^n \text{ of dimension } r \mid S \cap T_i = 0\}$ . Choose  $S_i \in U_i$ , define

$$\phi_i : U_i \rightarrow \text{Hom}(S_i, T_i) \cong \mathbb{C}^{r(n-r)}$$

as  $S \mapsto f$ , such that  $S$  is the graph of  $f$ .

**Example 2.9.** Complex torus is  $\mathbb{C}^n/\Lambda$  where  $\Lambda$  is a free abelian subgroup of  $\mathbb{C}^n$  with rank  $2n$ , called a lattice.

**Definition 2.10** (holomorphic map). *Let  $X, Y$  be complex manifolds of dimension  $n, m$ , with atlas  $(U_i, \phi_i : U_i \rightarrow V_i)$  and  $(M_j, \psi_j : M_j \rightarrow N_j)$  respectively. A continous map  $f : X \rightarrow Y$  is called holomorphic, if for any two charts, we have*

$$\psi_j \circ f \circ \phi_i^{-1} : V_i \rightarrow \psi_j(f(U_i) \cap M_j)$$

*is holomorphic.*

**Definition 2.11** (holomorphic function). *A holomorphic function on  $X$  is a holomorphic map  $f : X \rightarrow \mathbb{C}$ .*

**Example 2.12.** Let  $S = (\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}$  be Hopf surface, then

$$f : S \rightarrow \mathbb{P}^1 = (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*$$

is a holomorphic map. The fibers of  $f$  are biholomorphic to 1-dimensional complex torus.

**Proposition 2.13.** *If  $X$  is a compact complex manifold, then every holomorphic function on  $X$  is constant.*

*Proof.* Standard conclusion in complex analysis.  $\square$

**Definition 2.14** (immersion/submersion). *A holomorphic map  $f : X \rightarrow Y$  is called an immersion (resp submersion), if for all  $x \in X$ , there exists  $(x \in U_i, \phi_i), (f(x) \in M_j, \psi_j)$ , such that*

$$J_{\psi_j \circ f \circ \phi_i^{-1}}(\phi_i(x))$$

*has the max rank  $\dim X$  (resp  $\dim Y$ )*

**Definition 2.15** (embedding).  *$f : X \rightarrow Y$  is an embedding, if it is immersion and  $f : X \rightarrow f(X) \subset Y$  is homeomorphism.*

**Definition 2.16** (submanifold). *A closed connected subset  $Y$  of  $X$  is called a submanifold, if for all  $x \in Y$ , there exists  $x \in U \subset X$  and a holomorphic submersion  $f : U \rightarrow \mathbb{D}^k$  such that*

$$U \cap Y = f^{-1}(0)$$

*where  $k$  is the codimension of  $Y$  in  $X$ .*

**Example 2.17** (regular value theorem). *Let  $X, Y$  be complex manifold with dimension  $n, m$ . If  $y \in Y$  such that  $\text{rank } J_{f(x)}$  reaches maximum  $m$  for all  $x \in f^{-1}(y)$ , then  $f^{-1}(y)$  is a submanifold of codimension  $m$ .*

**Definition 2.18** (projective manifold). *A projective manifold<sup>†</sup>  $X$  is a submanifold of  $\mathbb{P}^N$  of the form*

$$X = \{[z] \in \mathbb{P}^N \mid f_1(z) = \cdots = f_m(z) = 0\}$$

*where  $f_i$  is a homogenous polynomial in  $\mathbb{C}[z_0, \dots, z_n]$*

**Remark 2.19.** Here we always assume  $(f_1, \dots, f_m) \subset \mathbb{C}[z_0, \dots, z_n]$  is a prime ideal, so the case that  $X$  is defined by polynomials like  $f^2 = 0$  is not allowed, what's more, the condition that  $X$  is a manifold implies the following cases won't happen:

1.  $f_1 f_2 = 0$ ;
2.  $X$  has a singular point.

**Definition 2.20** (complete intersection). *Let  $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$  be the projection, then  $X$  is a submanifold of codimension  $k$  if and only if*

$$J = \left( \frac{\partial f_i}{\partial z_j} \right)_{\substack{1 \leq i \leq m \\ 0 \leq j \leq N}}$$

*has rank  $k$ , for all  $x \in \pi^{-1}(X)$ . Then  $X$  is called a complete intersection, if  $m = k$ .*

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<sup>†</sup>Chow's theorem claims that every submanifold of  $\mathbb{P}^n$  must be defined by a set of homogenous polynomials, so we can use this property to define a projective manifold, in convenient.

**Example 2.21.** Consider  $C \subset \mathbb{P}^3$  defined by

$$xw - yz = y^2 - xz = z^2 - yw = 0$$

is not a complete intersection, called twisted cubic.

**Example 2.22.** Plücker embedding

$$\Phi : G(r, V) \hookrightarrow \mathbb{P}(\wedge^r V)$$

defined by  $S \subset V$  with basis  $s_1, \dots, s_r$  is mapped to  $[s_1 \wedge \dots \wedge s_r]$ . Check  $\Phi$  is well-defined embedding.

### 3. VECTOR BUNDLE

**Definition 3.1** (complex vector bundle). *Let  $X$  be a differential manifold,  $E$  is a complex vector bundle of rank  $r$  on  $X$*

1. (Via total space)  $E$  is a differential manifold with surjective map  $\pi : E \rightarrow X$ , such that
  - (1) For all  $x \in X$ , fiber  $E_x$  is a  $\mathbb{C}$ -vector space of dimension  $r$ .
  - (2) For all  $x \in X$ , there exists  $U \subset X$  and  $\pi^{-1}(U)$  is diffeomorphic to  $U \times \mathbb{C}^r$  via  $h$ , such that

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\pi} & U \\ & \searrow h \quad \curvearrowright \quad p_1 & \nearrow \\ & U \times \mathbb{C}^r & \xrightarrow{p_2} \mathbb{C}^r \end{array}$$

and for all  $y \in U$ ,  $E_y \xrightarrow{p_2 \circ h} \mathbb{C}^r$  is a  $\mathbb{C}$ -vector space isomorphism.  $(U, h)$  is called a local trivialization.

**Remark 3.2.** Consider two local trivialization  $(U_\alpha, h_\alpha), (U_\beta, h_\beta)$ , then  $h_\alpha \circ h_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \rightarrow (U_\alpha \cap U_\beta) \times \mathbb{C}^r$ , this induces

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{\text{diff}} \text{GL}(r, \mathbb{C})$$

such  $g_{\alpha\beta}$  are called transition function, such that

$$\begin{aligned} g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} &= \text{id} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma \\ g_{\alpha\alpha} &= \text{id} \quad \text{on } U_\alpha \end{aligned}$$

2. (Via transition function)  $E$  is the data of

- (1) open covering  $\{U_\alpha\}$
- (2) transition functions  $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{\text{diff}} \text{GL}(r, \mathbb{C})\}$ , satisfies

$$\begin{aligned} g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} &= \text{id} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma \\ g_{\alpha\alpha} &= \text{id} \quad \text{on } U_\alpha \end{aligned}$$

**Remark 3.3.** The two definitions above are equivalent. The first definition implies the second clearly. The converse is a little bit complicated

**Definition 3.4** (holomorphic vector bundle).  $X$  is a complex manifold,  $\pi : E \rightarrow X$  is a complex vector bundle, given by  $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(r, \mathbb{C})$ ,  $E$  is called holomorphic if  $g_{\alpha\beta}$  is holomorphic.

**Exercise 3.5.** Show that the total space of a holomorphic vector bundle  $E$  is a complex manifold.

**Definition 3.6** (morphism between vector bundles).  $\phi$  is a diffeomorphic/holomorphic morphism of vector bundle on  $X$ , if  $\phi : E \rightarrow F$  is diffeomorphic/holomorphic map such that

$$\begin{array}{ccc} E & \xrightarrow{\phi} & F \\ & \searrow \pi_1 \quad \curvearrowright \quad \swarrow \pi_2 & \\ & X & \end{array}$$

**Example 3.7.**  $X$  is a differential/complex manifold, then  $X \times \mathbb{C}^r$  is the trivial rank  $r$  complex/holomorphic vector bundle on  $X$ .

**Example 3.8.**  $E, F$  are complex/holomorphic vector bundles on  $X$ , then  $E \oplus F, E \otimes F, \text{Hom}(E, F), E^* = \text{Hom}(E, \mathbb{C}), \text{Sym}^k E, \bigwedge^k E, \det E$  are complex/holomorphic vector bundles.

**Example 3.9.** holomorphic line bundle  $L$  is a rank 1 vector bundle, i.e.,

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{\text{holo}} \mathbb{C}^*$$

**Exercise 3.10.**  $L$  is a trivial line bundle  $X \times \mathbb{C}$  if and only if up to refinement, there exists  $s_\alpha : U_\alpha \rightarrow \mathbb{C}^*$ , such that  $g_{\alpha\beta} = s_\alpha/s_\beta$  on  $U_\alpha \cap U_\beta$

**Definition 3.11** (picard group).  $X$  is a complex manifold, then

$$\text{Pic}(X) = (\{\text{holomorphic line bundles on } X\} / \text{isomorphism}, \otimes)$$

called the Picard group of  $X$ .

**Example 3.12.** Line bundle on  $\mathbb{P}^n$

$$\begin{array}{c} L = \{([l], x) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid x \in l\} \subset \mathbb{P}^n \times \mathbb{C}^{n+1} \\ \downarrow \pi \\ \mathbb{P}^n \end{array}$$

is called tautological line bundle.

Consider open covers

$$U_i = \{[l] = [l_1, \dots, l_n] \in \mathbb{P}^n \mid l_i \neq 0\}$$

there is a map  $U_i \rightarrow \pi^{-1}(U_i)$ , defined as

$$[l] \mapsto ([l], (\frac{l_0}{l_i}, \dots, 1, \dots, \frac{l_n}{l_i}))$$

and local trivialization  $h_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{C}$  defined as

$$([l], x) \mapsto ([l], \lambda)$$

where

$$x = \lambda(\frac{l_0}{l_i}, \dots, 1, \dots, \frac{l_n}{l_i})$$

so we can calculate transition function

$$\begin{aligned} h_i \circ h_j^{-1} : (U_i \cap U_j) \times \mathbb{C} &\longrightarrow (U_i \cap U_j) \times \mathbb{C} \\ ([l], \lambda_j) &\mapsto ([l], \lambda_j(\frac{l_0}{l_j}, \dots, \frac{l_n}{l_j})) \mapsto ([l], \lambda_i) \end{aligned}$$

such that

$$\lambda_j(\frac{l_0}{l_j}, \dots, \frac{l_n}{l_j}) = \lambda_i(\frac{l_0}{l_i}, \dots, \frac{l_n}{l_i})$$

which implies

$$\lambda_i = \lambda_j \frac{l_i}{l_j}$$

so we can see transition function  $g_{ij} = l_i/l_j \in \mathbb{C}^*$ . This line bundle  $L$  will be denoted by  $\mathcal{O}_{\mathbb{P}^n}(-1)$

**Definition 3.13** (line bundles on  $\mathbb{P}^n$ ). *We can define*

$$\begin{aligned} \mathcal{O}_{\mathbb{P}^n}(-k) &= \mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes k}, \quad k \in \mathbb{N}^+ \\ \mathcal{O}_{\mathbb{P}^n}(k) &= (\mathcal{O}_{\mathbb{P}^n}(-k))^*, \quad k \in \mathbb{N}^+ \\ \mathcal{O}_{\mathbb{P}^n}(0) &= \mathbb{P}^n \times \mathbb{C}, \quad \text{trivial line bundle.} \end{aligned}$$

*In fact, line bundle listed above contain all possible line bundles over  $\mathbb{P}^n$ .*

**Example 3.14.** More generally, consider

$$\begin{aligned} E &= \{([S], x) \in G(r, n) \times \mathbb{C}^n \mid x \in S\} \subset G(r, n) \times \mathbb{C}^n \\ &\downarrow \pi \\ &G(r, n) \end{aligned}$$

**Definition 3.15** (section).  $\pi : E \rightarrow X$  is a complex/holomorphic vector bundle. A (global) section of  $E$  is a differential/holomorphic map  $s : X \rightarrow E$ , such that  $\pi \circ s = \text{id}_X$ , denoted by  $C^\infty(X, E) / \Gamma(X, E)$ .

**Example 3.16.** Global holomorphic sections of trivial holomorphic vector bundle are exactly holomorphic functions  $f : X \rightarrow \mathbb{C}^r$ .

**Remark 3.17.** In fact, global holomorphic sections are very rare, as we can see from the above example, if  $X$  is a compact complex manifold, then all global holomorphic functions are only constant.

**Exercise 3.18.**

$$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \begin{cases} 0, & k < 0 \\ \mathbb{C}, & k = 0 \\ \text{homogeneous polynomials in } n+1 \text{ variables of deg } k, & k > 0 \end{cases}$$

**Definition 3.19** (subbundle).  $\pi : E \rightarrow X$  is a complex/holomorphic vector bundle.  $F \subset E$  is called a subbundle of rank  $s$ , if

1. For all  $x \in X$ ,  $F \cap E_x$  is a subvector space of dimension  $s$ .
2.  $\pi|_F : F \rightarrow X$  induces a complex/holomorphic vector bundle.

**Example 3.20.**  $\mathcal{O}_{\mathbb{P}^n}(-1) \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$ , is a subbundle.

**Example 3.21.** For a morphism between vector bundles  $\phi : E \rightarrow F$ ,  $\text{Ker } \phi \subset E, \text{Im } \phi \subset F$  are subbundles.

**Definition 3.22** (exact). A sequence of vector bundles

$$S \xrightarrow{\phi} E \xrightarrow{\psi} Q$$

is called exact at  $E$  if  $\text{Ker } \psi = \text{Im } \phi$ ;

**Definition 3.23** (pullback).  $f : X \rightarrow Y$  is a differential/holomorphic map,  $\pi : E \rightarrow Y$  is a vector bundle, define

$$f^*E = \{(x, v) \in X \times E \mid f(x) = \pi(v)\} \subset X \times E$$

is called the pullback of  $\pi$ .

#### 4. SHEAVES

Why we need sheaves here? As we have seen in the last section, the global sections of holomorphic vector bundle are very rare, but there are many local sections, we need to keep these information and learn the connection between global and local systemically. Sheaf is a power language for us to manage global and local at the same time. However, there is nothing more that sheaf can give, it's just a different language, as we can see in Exercise 4.5.

**Definition 4.1** (sheaf).  $X$  is a topological space. A sheaf<sup>‡</sup> of abelian group  $\mathcal{F}$  on  $X$  is the data of:

1. For any open subset  $U$  of  $X$ ,  $\mathcal{F}(U)$  is an abelian group.
2. If  $U \subset V$  are two open subsets of  $X$ , then there is a group homomorphism  $r_{UV} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ , such that
  - (1)  $\mathcal{F}(\emptyset) = 0$
  - (2)  $r_{UU} = \text{id}$
  - (3) If  $W \subset U \subset V$ , then  $r_{UW} = r_{VW} \circ r_{UV}$
  - (4)  $\{V_i\}$  is an open covering of  $U \subset X$ , and  $s \in \mathcal{F}(U)$ . If  $s|_{V_i} := r_{UV_i}(s) = 0, \forall i$ , then  $s = 0$ .
  - (5)  $\{V_i\}$  is an open covering of  $U \subset X$ , and  $s_i \in \mathcal{F}(V_i)$  such that  $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ , then there exists  $s \in \mathcal{F}(U)$  such that  $s|_{V_i} = s_i$ .

**Example 4.2** (sheaf of sections of a holomorphic vector bundle). If  $\pi : E \rightarrow X$  is a holomorphic vector bundle, then define

$$\mathcal{F}(U) = \Gamma(U, E|_U), \quad \forall U \subset_{\text{open}} X$$

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<sup>‡</sup>A sheaf which fails to meet (4), (5) is called a presheaf.



This  $\mathcal{F}$  will be denoted by  $\mathcal{O}_X(E)$ . In particular,  $E$  is a trivial vector bundle, then  $\mathcal{O}_X(E) = \mathcal{O}_X$ , the sheaf of holomorphic function, also called the structure sheaf of  $X$ .

**Definition 4.3** (morphism of sheaves on  $X$ ).  $\phi : \mathcal{F} \rightarrow \mathcal{G}$  is called a morphism of sheaves, if for any open subset  $U$  of  $X$ , there is a group homomorphism  $\phi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ , such that if  $U \subset V$  are two open subsets of  $X$ , the the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \downarrow r_{UV} & & \downarrow r_{UV} \\ \mathcal{F}(V) & \xrightarrow{\phi(V)} & \mathcal{G}(V) \end{array}$$

**Example 4.4** (locally free sheaves). A sheaf is called locally free, if there exists covering  $\{U_\alpha\}$  such that  $\mathcal{F}|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}^{\oplus r}$  of rank  $r$ .

For  $r = 1$ , it is called invertible sheaf.

**Exercise 4.5.** There are correspondences:

$$\begin{aligned} \{\text{holomorphic vector bundles}\} &\xleftrightarrow{1-1} \{\text{locally free sheaves}\} \\ \{\text{holomorphic line bundles}\} &\xleftrightarrow{1-1} \{\text{invertible sheaves}\} \end{aligned}$$

## 5. TANGENT BUNDLE

**Definition 5.1** (tangent bundle).  $X$  is a differential manifold,  $\dim_{\mathbb{R}} X = n$ , and  $\{U_\alpha, \phi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{R}^n\}$  is a atlas of  $X$ . The (real) tangent bundle  $T_{X,\mathbb{R}}$  is defined through transition functions

$$\begin{aligned} g_{\alpha\beta} : U_\alpha \cap U_\beta &\xrightarrow{\text{diff}} \text{GL}(n, \mathbb{R}) \\ x &\mapsto J_{\phi_\alpha \circ \phi_\beta^{-1}}(\phi_\beta(x)) \end{aligned}$$

Then  $T_{X,\mathbb{C}} = T_{X,\mathbb{R}} \otimes \mathbb{C}$  is a complex vector bundle, called the complexified tangent vector bundle.

**Remark 5.2.** The following statement may be a little bit boring, I write it down just to make myself more clear and to get familiar with two definition of vector bundle.

The tangent bundle  $T_{X,\mathbb{R}}$  can be defined as the set

$$T_{X,\mathbb{R}} = \coprod_{x \in X} T_{X,x}$$

and note that there is a natural projection  $\pi : T_{X,\mathbb{R}} \rightarrow X$ , sending  $v \in T_{X,x}$  to  $x \in X$ . Now we want to give a chart on  $T_{X,\mathbb{R}}$  to make it into a differential manifold. Let  $\{(U_i, \phi_i = (x_i^1, \dots, x_i^n))\}$  be a chart of  $X$ , then we can define a chart on  $X$  by considering  $\{(\pi^{-1}(U_i), \tilde{\phi}_i)\}$ , where  $\tilde{\phi}_i$  is defined through

$$\tilde{\phi}_i(v) = (\phi_i(\pi(v)), (dx_i^1)_{\pi(v)}(v), \dots, (dx_i^n)_{\pi(v)}(v)) \subset \mathbb{R}^n \times \mathbb{R}^n$$

note that such  $\tilde{\varphi}_i$  is bijective. And it's easy to equip  $T_{X,\mathbb{R}}$  with a topology such that  $\tilde{\varphi}_i$  is diffeomorphism.

Now I need to calculate transition function to confirm myself as follows:  
For two charts  $(U, \phi = (x_1, \dots, x_n)), (V, \psi = (y_1, \dots, y_n))$ , then calculate

$$\tilde{\psi} \circ \tilde{\phi}^{-1} : \phi(U \cap V) \times \mathbb{R}^n \rightarrow \psi(U \cap V) \times \mathbb{R}^n$$

Note that

$$\tilde{\phi}^{-1}(r_1, \dots, r_n, u_1, \dots, u_n) = \sum_i u_i \frac{\partial}{\partial x_i} |_{\phi^{-1}(r_1, \dots, r_n)} \in T_{\phi^{-1}(r_1, \dots, r_n)} M$$

But

$$dy_j(\sum_i u_i \frac{\partial}{\partial x_i}) = \sum_i u_i (\frac{\partial}{\partial x_i}(y_j)) = \sum_i \frac{\partial y_i}{\partial x_j} u_i$$

Thus transition functions are

$$\begin{aligned} \tilde{\psi} \circ \tilde{\phi}^{-1}(r_1, \dots, r_n, u_1, \dots, u_n) &= (\psi \circ \phi^{-1}(r), (\sum_i \frac{\partial y_1}{\partial x_i}(r) u_i, \dots, \sum_i \frac{\partial y_n}{\partial x_i}(r) u_i)) \\ &= (\psi \circ \phi^{-1}(r), (\frac{\partial y_j}{\partial x_i}(r)) \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}) \end{aligned}$$

So transition function  $g_{\alpha\beta}$  are exactly Jacobian of  $\varphi_\alpha \circ \varphi_\beta^{-1}$ .

**Definition 5.3** (holomorphic tangent bundle).  *$X$  is a complex manifold,  $\dim_{\mathbb{C}} X = n$ , and  $\{U_\alpha, \phi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{C}^n\}$  is a atlas of  $X$ . The holomorphic tangent bundle  $T_X$  is defined through transition functions*

$$\begin{aligned} g_{\alpha\beta} : U_\alpha \cap U_\beta &\rightarrow \text{GL}(n, \mathbb{C}) \\ z &\mapsto J_{\phi_\alpha \circ \phi_\beta^{-1}}^{\text{holo}}(\phi_\beta(x)) \end{aligned}$$

where  $J^{\text{holo}}$  is holomorphic Jacobian.

**Remark 5.4.** Clearly,  $T_X \neq T_{X,\mathbb{C}}$ , even they don't have the same rank! For example, if  $X$  is a  $n$ -dimensional complex manifold, then

$$\dim T_X = n \neq 2n = \dim T_{X,\mathbb{C}}$$

Later we will see the relationship between them.

**Remark 5.5** (sheaf viewpoint).  $\mathcal{O}_X$  is the sheaf of holomorphic function, then define the stalk at  $x$  is

$$\mathcal{O}_{X,x} := \varinjlim_{x \in U \subset X} \mathcal{O}_X(U)$$

The elements of  $\mathcal{O}_{X,x}$  are called germs.

For a tangent vector, we can take derivation in this direction, so

$$\text{tangent vector} \longrightarrow \text{derivation } D : \mathcal{O}_{X,x} \rightarrow \mathbb{C}$$

where a derivation satisfies

1.  $\mathbb{C}$ -linear
2. Leibniz rule  $D(fg) = D(f)g + fD(g)$

In fact, the above correspondence is 1-1, i.e.,  $T_{X,x} \cong$  space of derivations of  $\mathcal{O}_{X,x}$ .

**Definition 5.6** (cotangent bundle/(anti)canonical bundle).  $\Omega_X = T_X^*$  is called holomorphic cotangent bundle;  $K_X = \det \Omega_X$  is called canonical bundle;  $K_X^* = \det T_X$  is called the anticanonical bundle.

**Example 5.7.** Tangent bundle of  $\mathbb{P}^n$ : we can calculate it through Euler sequence<sup>§</sup>

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \xrightarrow{\psi} T_{\mathbb{P}^n} \rightarrow 0$$

**Example 5.8.** For Grassmannian manifold  $G(r, n)$ , we have

$$0 \rightarrow E \rightarrow G(r, n) \otimes \mathbb{C}^n \rightarrow Q \rightarrow 0$$

Show that

$$T_{G(r,n)} \cong \text{Hom}(E, Q)$$

**Example 5.9.** Let  $\pi : L \rightarrow X$  is a holomorphic line bundle, given  $s \in \Gamma(X, L)$ , suppose that  $D = \{x \in X \mid s(x) = 0\}$  is a smooth submanifold of codimensional 1.

$$0 \rightarrow T_D \rightarrow T_X|_D \rightarrow L|_D \rightarrow 0$$

Then we can get

$$K_X^*|_D \cong K_D^* \otimes L|_D$$

which is called adjunct formula

Now let us talk about some linear algebra:

Consider a  $2n$ -dimensional real vector space  $V$ , a complex structure on  $V$  is a  $\mathbb{R}$ -linear transformation  $J : V \rightarrow V$  such that  $J^2 = -\text{id}$ . We can regard  $V$  as a complex vector space, by

$$(a + bi)v = av + bJ(v), \quad a, b \in \mathbb{R}$$

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<sup>§</sup>Hint: consider  $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$ . Basis of tangent vector at  $z \in \mathbb{C}^{n+1} \setminus \{0\}$  is  $\{\frac{\partial}{\partial z_0}, \dots, \frac{\partial}{\partial z_n}\}$ , but these are not tangent vector for  $\mathbb{P}^n$ . However,  $z_i \frac{\partial}{\partial z_i}$  will descend to tangent vector at  $[z] \in \mathbb{P}^n$ . Recall  $z_0, \dots, z_n$  form basis of  $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$  and span  $(\mathcal{O}_{\mathbb{P}^n}(1))_{[z]}$ , so can define

$$\begin{aligned} \psi : (\mathcal{O}_{\mathbb{P}^n}(1))_{[z]}^{\oplus n+1} &\rightarrow T_{\mathbb{P}^n, [z]} \\ (0, \dots, \underbrace{z_i}_{j\text{-th}}, \dots, 0) &\rightarrow z_i \frac{\partial}{\partial z_j} \end{aligned}$$

But  $\sum_{i=0}^n z_i \frac{\partial}{\partial z_i}$  tangent to the fibers of  $\pi$ , so descends to zero at  $[z] \in \mathbb{P}^n$ , so we can define  $\phi$  as

$$\begin{aligned} \phi : \mathcal{O}_{\mathbb{P}^n, [z]} &\rightarrow (\mathcal{O}_{\mathbb{P}^n}(1))_{[z]}^{\oplus n+1} \\ 1 &\rightarrow (z_0, \dots, z_n) \end{aligned}$$

In fact,  $\sum_{i=0}^n z_i \frac{\partial}{\partial z_i}$  is Euler vector field, that's why this sequence is called Euler sequence.

If we consider  $V \otimes \mathbb{C}$ , then  $J$  can extend to  $V \otimes \mathbb{C}$ , by  $J(v \otimes \alpha) = J(v) \otimes \alpha$ , then we can decompose  $V \otimes \mathbb{C}$  into

$$V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$$

such that  $\overline{V^{1,0}} = V^{0,1}$ , where conjugate means  $\overline{v \otimes \alpha} = v \otimes \bar{\alpha}$ .

Such definition may be a little abstract for someone, let's calculate in a concrete example to show how it works.

**Example 5.10.** Let  $V = \mathbb{C}^n \cong \mathbb{R}^{2n}$ , stand coordinate in  $\mathbb{C}^n$  is  $(z_1, \dots, z_n)$ , and in  $\mathbb{R}^{2n}$  we always write  $(x_1, y_1, \dots, x_n, y_n)$ . Note that complex structure on  $\mathbb{C}^n$  maps  $z_i = x_i + \sqrt{-1}y_i$  to  $\sqrt{-1}(x_i + \sqrt{-1}y_i) = -y_i + \sqrt{-1}x_i$ .

So we can define  $J : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  by  $(x_1, y_1, \dots, x_n, y_n) \mapsto (-y_1, x_1, \dots, -y_n, x_n)$ . If we complexify  $\mathbb{R}^{2n}$  into a complex vector space  $\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$ , with a basis  $\{e_1, f_1, \dots, e_n, f_n\}$ ,  $J$  is defined by  $e_j \mapsto -f_j, f_j \mapsto e_j$ , then eigenspace decomposition is as follows

$$V^{1,0} = \left\{ \frac{1}{2}(e_j + if_j) \right\}, \quad V^{0,1} = \left\{ \frac{1}{2}(e_j - if_j) \right\}$$

such that conjugation  $\overline{e_j - if_j} = e_j + if_j$

**Definition 5.11** (almost complex structure).  *$X$  is a differential manifold of  $\dim_{\mathbb{R}} X = 2n$ . An almost complex structure on  $X$  is a complex structure on  $T_{X,\mathbb{R}}$  i.e., an isomorphism of differential vector bundles  $J : T_{X,\mathbb{R}} \rightarrow T_{X,\mathbb{R}}$  such that  $J^2 = -\text{id}$*

**Example 5.12.**  $X$  is a complex manifold, and  $T_{X,\mathbb{R}}$  is its (real) tangent bundle if we just see  $X$  as a differential manifold. Locally we have

$$T_{X,\mathbb{R}}|_U \cong U \times \mathbb{C}^n$$

for open subset  $U$  in  $X$ . So we get  $J : T_{X,\mathbb{R}}|_U \rightarrow T_{X,\mathbb{R}}|_U$ , defined by multiplying  $i$ , and this is well-defined, explained as following:

For two charts  $(U_1, \varphi_1 = (z_1 = x_1 + iy_1, \dots, z_n = x_n + iy_n))$ ,  $(U_2, \varphi_2 = (g_1 = u_1 + iv_1, \dots, g_n = u_n + iv_n))$  with  $U_1 \cap U_2 \neq \emptyset$ , there are two ways to define  $J$  on  $U_1 \cap U_2$

$$\begin{array}{ccccc}
 U_1 \cap U_2 \times \mathbb{C}^n & & & & U_1 \cap U_2 \times \mathbb{C}^n \\
 \downarrow \times i & \nwarrow \tilde{\varphi}_1 & T_X|_{U_1 \cap U_2} \longleftarrow T_X|_{U_1 \cap U_2} & \nearrow \tilde{\varphi}_2 & \downarrow \times i \\
 & & \downarrow J & & \\
 & & T_X|_{U_1 \cap U_2} \longleftarrow T_X|_{U_1 \cap U_2} & & \\
 & \nwarrow \tilde{\varphi}_1 & & \searrow \tilde{\varphi}_2 & \\
 U_1 \cap U_2 \times \mathbb{C}^n & & & & U_1 \cap U_2 \times \mathbb{C}^n
 \end{array}$$

We need to check transition function commutes with  $J$ , calculated in a local chart as follows: For a  $2 \times 2$  part, Jacobian of  $\varphi_2 \circ \varphi_1^{-1}$  is

$$\begin{pmatrix} \frac{\partial u_j}{\partial x_k} & \frac{\partial u_j}{\partial y_k} \\ \frac{\partial v_j}{\partial x_k} & \frac{\partial v_j}{\partial y_k} \end{pmatrix} \stackrel{\text{C-R}}{=} \begin{pmatrix} \frac{\partial v_j}{\partial y_k} & \frac{\partial u_j}{\partial y_k} \\ -\frac{\partial u_j}{\partial y_k} & \frac{\partial v_j}{\partial y_k} \end{pmatrix}$$

and  $J$  is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

So they commute with each other.

So complex structure gives a almost complex structure(naturally), But the question is: Does every complex structure on a complex manifold can be induced from a almost complex structure on a even-dimensional differential manifold? Unfortunately, it's false in general, but we have the following theorem.

**Theorem 5.13** (Newlander-Nirenberg). *Let  $(X, J)$  be a complex manifold,  $J$  is induced by a almost complex structure on  $X$  is equivalent to*

$$[T_X^{1,0}, T_X^{1,0}] \subset T_X^{1,0}$$

*which is called an integrable condition.*

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