ANALYTIC COMPLEX GEOMETRY

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Contents

0. Preface

0.1. **To readers.** It's a lecture note about analytic complex geometry, and the main reference is [Dem97].

0.2. **Basic notations.**

1. M denotes a smooth real manifold, with tangent bundle TM and cotangent bundle T^*M.
2. ${}^s\mathcal{E}^p(M)$ denotes the space of C^s-global sections of $\bigwedge^p T^*M$, and $\mathcal{E}^p(M)$ denotes the space of smooth global sections of $\bigwedge^p T^*M$.
3. X denotes a smooth complex manifold, with tangent bundle TX and cotangent bundle T^*X.
4. ${}^s\mathcal{E}^{p,q}(X)$ denotes the space of C^s-global sections of $\bigwedge^{p,q} T^*X$, and $\mathcal{E}^{p,q}(X)$ denotes the space of smooth global sections of $\bigwedge^{p,q} T^*X$.

1. Currents

In this section, M is assumed to be an oriented smooth real manifold with dimension n.

1.1. **Currents on smooth manifold.**

Firstly we want to give a topology on the space of ${}^s\mathcal{E}^p(M)$ to make it to be a topological vector space. For $u \in {}^s\mathcal{E}^p(M)$, on coordinate open set $\Omega \subset M$ it can be written as

$$u = \sum_{|I|=p} u_I \mathrm{d}x^I$$

To each $L \Subset \Omega$ and every integer $s \in \mathbb{N}$, we associate a seminorm

$$P_{L,\Omega} = \sup_{x \in L} \max_{|\alpha| \leq s, |I|=p} |D^\alpha u_I(x)|$$

Since our manifolds are suppose to be Hausdorff, then M can be covered by countable coordinate set, that is $M = \bigcup_{k=1}^\infty \Omega_k$, and consider exhaustion for ech k

$$L_{k_1} \Subset L_{k_2} \Subset \cdots \Subset \Omega_k$$

then seminorms $\{P_{L_{k_m}, \Omega_k}\}$ gives a topology on ${}^s\mathcal{E}^p(M)$. Then according to Remark A.2.1, this topology is given by a translation invariant metric, and in this case it's complete, which makes ${}^s\mathcal{E}^p(M)$ a Fréchet space.

Let $K \Subset M$, ${}^s\mathcal{D}^p(K)$ is the subspace of elements $u \in {}^s\mathcal{E}^p(M)$ with compact support in K, together with induced topology. The ${}^s\mathcal{D}^p(M)$ denotes the set of all elements of ${}^s\mathcal{E}^p(M)$ with compact support, that is

$${}^s\mathcal{D}^p(M) = \bigcup_{K \Subset M} {}^s\mathcal{D}^p(K)$$

A sequence $u_l \to 0$ in ${}^s\mathcal{D}^p(M)$ if there exists $K \Subset M$ such that $\operatorname{supp} u_l \subset K$ for all $l \geq 1$ and $u_l \to 0$ in ${}^s\mathcal{E}^p(M)$.

Remark 1.1.1. Similarly one can define $\mathcal{D}^p(K), \mathcal{D}^p(M)$, in particular, if $p = 0$ and $M = \mathbb{R}^n$, then $\mathcal{D}^0(\mathbb{R}^n)$ is exactly the space of test functions.

Definition 1.1.1 (current)**.** The space of current of dimension p or degree $n-p$, denoted by $\mathcal{D}'_p(M) = \mathcal{D}'^{n-p}(M)$, is the space of linear functionals on $\mathcal{D}^p(M)$ such that the restriction on any $\mathcal{D}^p(K)$ is continous, where $K \Subset M$.

Notation 1.1.1. For a current $T \in \mathcal{D}'_p(M)$, $\langle T, u \rangle$ denotes the pairing between a current T and test form $u \in \mathcal{D}^p(M)$.

Remark 1.1.2. If a current T extends continuously to ${}^s\mathcal{D}^p(M)$, then T is called of order s.

Definition 1.1.2. For a current $T \in \mathcal{D}'_p(M)$, the support of T, denoted by $\operatorname{supp}(T)$, is the smallest closed set A such that $T|_{\mathcal{D}^p(M \setminus A)} = 0$.

The following two basic examples explains the terminology used for dimension and degree.

Example 1.1.1 (current of integration)**.** Let $Z \subseteq M$ be an oriented closed submanifold with dimension p. The current of integration $[Z]$ is given by

$$\langle [Z], u\rangle := \int_Z u$$

where $u \in \mathcal{D}^p(M)$. It's clear that $[Z]$ is a current with $\operatorname{supp}[Z] = Z$, and its dimension is exactly the dimension of Z as a manifold.

Example 1.1.2 (current of form)**.** Let f be a p-form with L^1_{loc} coefficients, the T_f given by

$$\langle T_f, u\rangle = \int_M f \wedge u$$

where $u \in \mathcal{D}^{n-p}(M)$, is a current of degree p.

1.2. **Exterior derivative and wedge product on currents.**

1.2.1. *Exterior derivative.* As we have seen in Example 1.1.2, currents generalize the ideal of forms, and in this viewpoint, many of the operations for forms can also be extended to currents. Let $T \in \mathcal{D}'^p(M)$, the exterior derivative $\mathrm{d}T$ is given by

$$\langle \mathrm{d}T, u\rangle := (-1)^{p+1}\langle T, \mathrm{d}u\rangle$$

where $u \in \mathcal{D}^{n-p-1}(M)$. The continuity of the linear functional $\mathrm{d}T$ follows from the exterior derivative d is continous, thus $\mathrm{d}T$ is a current of degree $p+1$.

Remark 1.2.1. If $T \in \mathcal{E}^p(M)$,

$$\langle \mathrm{d}T, u\rangle = \int_M \mathrm{d}T \wedge u = \int_M \mathrm{d}(T\wedge u) + (-1)^{p+1} T \wedge \mathrm{d}u = (-1)^{p+1}\int_M T \wedge \mathrm{d}u$$

That's why we define exterior derivative like this.

Example 1.2.1. Consider current T_f given by p-form with L^1_{loc} coefficients, then

$$\begin{aligned}\langle T_{\mathrm{d}f}, u\rangle &= \int_M \mathrm{d}f \wedge u \\ &= \int_M \mathrm{d}(f\wedge u) + (-1)^{p+1} f \wedge \mathrm{d}u \\ &= (-1)^{p+1}\int_M f\wedge \mathrm{d}u \\ &= \langle \mathrm{d}T_f, u\rangle\end{aligned}$$

This shows $T_{\mathrm{d}f} = \mathrm{d}T_f$, and that's why exterior derivative is defined like this.

Example 1.2.2. Consider current $T=[Z]$ given by a oriented closed submanifold of M with dimension p, then

$$\begin{aligned}\langle \mathrm{d}T, u\rangle &= (-1)^{n-p+1}\langle T, \mathrm{d}u\rangle \\ &= (-1)^{n-p+1}\int_Z \mathrm{d}u \\ &= (-1)^{n-p+1}\int_{\partial Z} u\end{aligned}$$

that is $\mathrm{d}T = (-1)^{n-p+1}[\partial Z]$.

1.2.2. *Wedge product.* Let $T\in \mathcal{D}'^p(M)$, $g\in\mathcal{E}^q(M)$, the wedge product $T\wedge g$ is a current of degree $p+q$, given by

$$\langle T\wedge g, u\rangle := \langle T, g\wedge u\rangle$$

where $u\in\mathcal{D}^{n-p-q}(M)$.

Proposition 1.2.1. Let $T\in \mathcal{D}'^p(M)$, $g\in\mathcal{E}^q(M)$, then

$$\mathrm{d}(T\wedge g) = \mathrm{d}T\wedge g + (-1)^p T\wedge \mathrm{d}g$$

1.3. **Currents on complex manifold.** Let X be a complex n-manifold, in particular, it's also a real manifold, then what we have talked can be used here, but there is a more explicit structure on $\mathcal{E}^k(X)$, that is

$$\mathcal{E}^k(X) = \bigoplus_{k=p+q}\mathcal{E}^{p,q}(X)$$

So $\mathcal{E}^{p,q}(X)$ is also endowed wtih a topology which is induced by seminorms, and $\mathcal{D}^{p,q}(X)$ is the space of smooth (p,q)-forms with compact supports.

Definition 1.3.1. The space of currents of bidimension (p,q) or bidegree $(n-p,n-q)$, denoted by

$$\mathcal{D}'_{p,q}(X) = \mathcal{D}'^{(n-p,n-q)}(X)$$

is the topological dual of $\mathcal{D}^{p,q}(X)$.

Now let's try to explains a motivation for why we need currents. Let $\mathcal{E}^{p,q}$ be the locally free sheaf associated to $\bigwedge^{p,q}T^*X$, and Dolbeault-Grothendieck lemma says it gives a resolution of holomorphic p-forms, that is

$$0\to\Omega^p\to\mathcal{E}^{p,0}\xrightarrow{\overline{\partial}}\mathcal{E}^{p,1}\xrightarrow{\overline{\partial}}\dots$$

is an exact sequence of sheaves. Thus the Dolbeault cohomology in fact computes the sheaf cohomology of sheaf Ω^p.

The same story works for currents, in the setting of currents we also have $\overline{\partial}$-operator, and there is the following sequences

$$0\to\Omega^p\to\mathcal{D}'^{p,0}\xrightarrow{\overline{\partial}}\mathcal{D}'^{p,1}\xrightarrow{\overline{\partial}}\dots$$

A non-trivial fact says it's also an exact sequence, thus there is a new way to compute sheaf cohomology of sheaf of holomorphic p-forms.

The advantage of a smooth manifold is everything is smooth, but that's also its disadvantages, in the setting of currents, tools of functional analysis can be used to compute, and that's where advantages of currents lie.

2. Positivity and pluri-subharmonic functions

2.1. **Positive** $(1,1)$ **current.** Let X be a complex n-manifold, a real $(1,1)$-form u on X is locally given by

$$u = \sqrt{-1}u_{ij}\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j$$

where $(u_{ij})_{n\times n}$ is a hermitian matrix. u is called positive if matrix $(u_{ij}(x))_{i\times j}$ is semi-positive. Since $(1,1)$-form and $(n-1,n-1)$-form are dual to each other, then we can also define what is positive for $(n-1,n-1)$-form. More explicitly, for a $(n-1,n-1)$-form v locally given by

$$v = v_{ij}\widehat{\mathrm{d}z^i} \wedge \widehat{\mathrm{d}\overline{z}^j}$$

where $\widehat{\mathrm{d}z^i} \wedge \widehat{\mathrm{d}\overline{z}^j}$ is a $(n-1,n-1)$-form such that

$$\mathrm{d}\widehat{z^i \wedge \mathrm{d}\overline{z}}^j \wedge \mathrm{d}z^i \wedge \mathrm{d}z^j = (\sqrt{-1})^n \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n \wedge \mathrm{d}\overline{z}^1 \wedge \cdots \wedge \mathrm{d}\overline{z}^n$$

Then v is called positive if matrix $(v_{ij})_{n\times n}$ is semi-positive.

Definition 2.1.1 (positive current)**.** Let T be a real $(1,1)$-current over X, it's called positive if

$$\langle T, v\rangle \geq 0$$

for any positive $(n-1,n-1)$-form $v \in \mathcal{D}^{n-1,n-1}(X)$.

2.2. **Pluri-subharmonic functions.**

Definition 2.2.1 (pluri-subharmonic)**.** $u\colon \Omega \to [-\infty, \infty]$ defined on an open subsect $\Omega \subset \mathbb{C}^n$ is called pluri-subharmonic function, if

1. u is upper semi-continous;
2. For any complex line $L \subseteq \mathbb{C}^n$, $u|_{\Omega\cap L}$ is subharmonic.

Notation 2.2.1. The space of pluri-subharmonic functions on Ω is denoted by $\mathrm{Psh}(\Omega)$.

Remark 2.2.1. An equivalent statement of property (2) is: For all $a \in \Omega, \xi \in \mathbb{C}^n$ with $|\xi| \ll 1$, then

$$u(a) \leq \frac{1}{2\pi}\int_0^{2\pi} u(a + e^{\sqrt{-1}\theta}\xi)\mathrm{d}\theta$$

Proposition 2.2.1.

1. The pluri-subharmonic function is subharmonic.
2. $u \in \mathrm{Psh}(\Omega)$, if Ω is connected, then $u \equiv -\infty$ or $u \in L^1_{\mathrm{loc}}(\Omega)$.
3. If $\{u_k\}$ is a sequence of pluri-subharmonic functions, and u_k descends to u, then u is pluri-subharmonic.
4. Let $u \in \mathrm{Psh}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$ and $(\rho_\varepsilon)_{\epsilon>0}$ be a family of modifiers, then $u_\varepsilon := u * \rho_\varepsilon \in C^\infty(U_\varepsilon) \cap \mathrm{Psh}(U_\varepsilon)$, and u_ε descends to u as $\varepsilon \to 0$.

Proposition 2.2.2. Let $u_1, \ldots, u_k \in \mathrm{Psh}(\Omega)$, $\chi\colon \mathbb{R}^k \to \mathbb{R}$ a convex function, which is increasing with respect to each variable, then $\chi(u_1, \ldots, u_k) \in \mathrm{Psh}(\Omega)$.

Corollary 2.2.1. Let $u_1, \dots, u_k \in \mathrm{Psh}(\Omega)$, the following functions are pluri-subharmonic:

1. $\sum_{i=1}^k u_i$;
2. $\max\{u_1, \dots, u_k\}$;
3. $\log(e^{u_1} + \cdots + e^{u_k})$.

Proposition 2.2.3. Let $f\colon \Omega_1 \to \Omega_2$ be holomorphic, for $u \in \mathrm{Psh}(\Omega_2)$, then $f^*u \in \mathrm{Psh}(\Omega_1)$.

Example 2.2.1.

1. $\log|z| \in \mathrm{Psh}(\mathbb{C})$, since $\frac{\sqrt{-1}}{\pi}\partial\overline{\partial}\log|z|$ is Dirac measure;
2. Let $g\colon \Omega \to \mathbb{C}$ be holomorphic function, then $\log|g(z)| \in \mathrm{Psh}(\Omega)$;
3. Let $f_1, \dots, f_k\colon \Omega \to \mathbb{C}$ be holomorphic functions, $\alpha_1, \dots, \alpha_k > 0$, then $\log(|f_1|^{\alpha_1} + \cdots + |f_k|^{\alpha_k}) \in \mathrm{Psh}(\Omega)$, since
$$\log(|f_1|^{\alpha_1} + \cdots + |f_k|^{\alpha_k}) = \log(e^{\alpha_1 \log|f_1|} + \cdots + e^{\alpha_k \log|f_k|})$$

Definition 2.2.2 (pluri-subharmonic function on complex manifold)**.** A function f is called a pluri-subharmonic function on a complex manifold X, if it's pluri-subharmonic on each holomorphic chart.

Remark 2.2.2. If X is compact, then there is no non-trivial pluri-subharmonic by maximal principle, that is $\mathrm{Psh}(X) = \mathbb{R}$.

Proposition 2.2.4.

1. Let $u \in \mathrm{Psh}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$, then $\sqrt{-1}\partial\overline{\partial}u$ is a positive $(1,1)$-current.
2. Given a distribution φ on Ω, then $\sqrt{-1}\partial\overline{\partial}\varphi \geq 0$ in the sense of current, then $\varphi = u$ for some $u \in \mathrm{Psh}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$.

3. Lelong numbers

3.1. **Poincaré-Lelong formula.** Currents of integration in Example 1.1.1 can be generalized to currents of subvariety as follows, since for an irreducible analytic subvariety V, its regular part V_{reg} is smooth and dense.

Definition 3.1.1 (current of subvariety)**.** Let X be a complex manifold, V an irreducible analytic subvariety of X with dimension $n-1$, then current of V is a $(1,1)$-current given by

$$\langle [V], \varphi \rangle := \int_{V_{\text{reg}}} \varphi$$

where $\varphi \in \mathcal{D}^{n-1,n-1}(X)$.

Theorem 3.1.1 (Lelong)**.** Let X be a complex manifold, V an irreducible analytic subvariety of X with dimension $n-1$, then $[V]$ is a d-closed positive $(1,1)$-current on X.

Definition 3.1.2 (current of divisor)**.** Let X be a complex manifold, $D = \sum_{a_k \in \mathbb{R}} a_k D_k \in \operatorname{Div}(X)$, where D_k is prime divisor, the current of divisor is defined as

$$[D] := \sum_{a_k \in \mathbb{R}} a_k [D_k]$$

Theorem 3.1.2 (Poincaré-Lelong formula)**.** Let X be a complex manifold, $f\colon X \to \mathbb{C}$ a holomorphic functions with zero divisor $\operatorname{div} f$, then

$$\frac{\sqrt{-1}}{\pi} \partial\overline{\partial} \log |f| = [\operatorname{div} f]$$

Proof. For arbitrary regular point $p \in Z_{\text{reg}}$, consider a local coordinate $(z_1, \ldots, z_n)$ around p such that $f(z) = z_1^{m_1}$, then

$$\frac{\sqrt{-1}}{\pi} \partial\overline{\partial} \log |z_1|^m = m[z_1 = 0] = [\operatorname{div} f]$$

Note that

$$\operatorname{supp}(\frac{\sqrt{-1}}{\pi} \partial\overline{\partial} \log |f| - [\operatorname{div} f]) \subseteq Z_{\text{sing}}$$

then support theorem yields the desired result. □

3.2. **Lelong numbers.** Let's consider the following baby case: Consider the holomorphic function $f(z) = z^m$. On one hand, we have a pluri-subharmonic function $\log |f| = m \log |z|$, while on the other hand we have a divisor $\operatorname{div} f = m[0]$. The number m has the following means:

1. m is order of f at $z = 0$;
2. m is the multiplicity of $\operatorname{div} f$ at $[0]$.

Thus we want to define something for a pluri-subharmonic function, such that it also equals to m. This motivation leads to the definition of Lelong number of pluri-subharmonic function.

3.2.1. *Lelong number of pluri-subharmonic function.*

Lemma 3.2.1. Let $S \subseteq \mathbb{R}^n$ be an open convex subset, a function u on S is convex if and only if $u(z) := u(x + \sqrt{-1}y)$ is pluri-subharmonic on $S + \sqrt{-1}\,\mathbb{R}^n$.

Lemma 3.2.2. For open subset $\Omega \subseteq \mathbb{C}^n$, $u \in \mathrm{Psh}(\Omega)$ and $a \in \Omega$, then

$$t \mapsto \sup_{B(a,e^t)} u$$

is convex with respect to t.

Proof. Without lose of generality, assume $B(a,1) \Subset \Omega$, $f(t) = \sup_{B(a,e^t)} u$, where $t \in (-\infty, 0]$. Note that

$$f(t) = \sup_{\xi \in B(0,1)} u(a + e^w \xi)$$

where $w = t + \sqrt{-1}s$. We have

$$w \mapsto \sup_{\xi \in B(0,1)} u(a + e^w \xi)$$

is pluri-subharmonic and independent of imaginary part of w, then by above lemma to conclude. □

Remark 3.2.1. A generalization of Hardamand's three circle theorem.

Definition 3.2.1 (Lelong number)**.** Let open subset $\Omega \subseteq \mathbb{C}^n$ and $u \in \mathrm{Psh}(\Omega)$, for $a \in \Omega$, the Lelong number of u at a, denoted by $\nu(u,a)$, is defined as

$$\lim_{t \to -\infty} \frac{\sup_{B(a,e^t)} u}{t}$$

Remark 3.2.2. The Lelong number is well-defined, since $f(t) = \sup_{B(a,e^t)} u$ is increasing and convex, thus

$$\frac{f(t) - f(0)}{t - 0} \geq 0$$

is increasing, which implies

$$\nu(u,a) = \lim_{t \to -\infty} \frac{\sup_{B(a,e^t)} u}{t} = f'(-\infty)$$

exists.

Proposition 3.2.1. Let open subset $\Omega \subseteq \mathbb{C}^n$ and $u \in \mathrm{Psh}(\Omega)$, for $a \in \Omega$,

$$u(z) \leq \sup_{B(a,r)} u \leq \nu(u,a) \log|z - a| + O(1)$$

Proof. Consider $f(t) = \sup_{B(a,e^t)} u$, then by definition one has

$$f(t) - f(0) \leq (t - 0) \lim_{t' \to -\infty} \frac{f(t') - f(0)}{t' - 0} = \nu(u,a)t$$

thus

$$f(t) \leq \nu(u,a)t + f(0)$$

which is equivalent to what we desire. □

Remark 3.2.3. In fact,

$$\nu(u,a) = \sup\{t \geq 0 \mid u(z) \leq t\log|z-a| + O(1) \text{ near } a\}$$

since

$$\begin{aligned}\nu(u,a) &= \lim_{r\to 0}\frac{\sup_{B(a,r)} u}{\log r}\\ &= \sup\{t \geq 0 \mid u(z) \leq t\log|z-a| + O(1) \text{ near } a\}\end{aligned}$$

then $\nu(u,a) > 0$ implies $u(a) = -\infty$. In general, we don't have $u(a) = -\infty$ implies $\nu(u,a) > 0$, but sometimes we can.

Example 3.2.1.

1. If $u = \log|f|$, where f is holomorphic, then $\nu(u,a) = \operatorname{ord}_a f$.
2. $\log|z| \in \mathrm{Psh}(\mathbb{D})$, let $\chi\colon \mathbb{R} \to \mathbb{R}$ be convex and increasing, then

$$u(z) = \chi(\log|z|) \in \mathrm{Psh}(\mathbb{D})$$

then

$$\begin{aligned}\nu(u,0) &= \lim_{t\to-\infty}\frac{f(t)}{t}\\ &= \lim_{t\to-\infty}\frac{\sup_{B(0,e^t)} u}{t}\\ &= \lim_{t\to-\infty}\frac{\chi(t)}{t}\\ &= \chi'(-\infty)\end{aligned}$$

If we consider the following function

$$\begin{aligned}&\chi(t) = -\log(-t)\\ &\chi(t) = -(-t)^\alpha, \quad 0 < \alpha < 1\end{aligned}$$

then $\nu(u,0)$

Remark 3.2.4. A philosophy is that one can regard $\nu(u,a)$ as the "vanishing order" of e^u at 0.

Remark 3.2.5. Lelong-Poincaré formula implies

$$\frac{\sqrt{-1}}{\pi}\partial\overline{\partial}\log|f| = [\operatorname{div} f]$$

and

$$\nu(\log|f|, a) = \operatorname{ord}_a(f) = \operatorname{mult}_a(\operatorname{div} f)$$

3.2.2. *Lelong numbers for positive currents.* For open subset $\Omega \subseteq \mathbb{C}^n$, T a d-closed positive $(1,1)$-current. $\omega = \frac{\sqrt{-1}}{2}\partial\overline{\partial}|z|^2$, then

$$\sigma_T = T \wedge \frac{\omega^{n-1}}{(n-1)!}$$

trace measure of T.

$H \subseteq \mathbb{C}^n$, linear subspace of dimension $n-1$, and

$$\sigma_{[H]} = [H] \wedge \frac{\omega^{n-1}}{(n-1)!}$$

Theorem 3.2.1 (Lelong number of positive current)**.**

$$\lim_{r\to 0} \frac{\sigma_T(B(a,r))}{\sigma_{[H]}(B(0,r))}$$

exists for any a, and called Lelong number of T at a.

Exercise 3.2.1. φ a d-closed $(1,1)$-form, $B(a,r) \Subset \Omega$, $0 < r < R$, then

$$\frac{1}{(\pi R)^{n-1}}\int_{B(a,R)} \varphi\wedge\omega^{n-1} - \frac{1}{\pi r^2}\int_{B(a,r)} \varphi\wedge\omega^{n-1} = \int_{B(a,R)\setminus B(a,r)} \varphi\wedge(\frac{\sqrt{-1}}{\pi}\partial\overline{\partial}\log|z-a|)^{n-1}$$

Remark 3.2.6. If $T = [D]$ for some effective divisor $D = \sum a_k D_k$, then

$$\nu(T,a) = \operatorname{mult}_a D$$

Remark 3.2.7. If $T = \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}u, u \in \mathrm{Psh}$, then

$$\nu(T,a) = \nu(u,a)$$

4. Multiplier ideal sheaf

4.1. Basic definition.

Definition 4.1.1 (multiplier ideal sheaf)**.** Let $\Omega \subseteq \mathbb{C}^n$, $\varphi \in \mathrm{Psh}(\Omega)$, the multiplier ideal sheaf is defined as $J(\varphi) = \bigcup_{x\in\Omega} J(\varphi)_x$, where

$$J(\varphi)_x = \{f \in \mathcal{O}_x : |f|^2 e^{-2\varphi} \in L^1_x\}$$

Notation 4.1.1. $|f|^2e^{-2\varphi} \in L^1_x$ if and only if there exists an open set $x \in V \subseteq \Omega$ such that $\int_V |f|^2 e^{2\varphi} < \infty$

Remark 4.1.1. If φ is smooth, then $J(\varphi)$ is trivial, and if $\varphi = -\infty$, then $J(\varphi)$ is also trivial.

Proposition 4.1.1. *$J(\varphi)$ is a coherent sheaf.*

Example 4.1.1. Let X be a complex manifold, $D = \sum \alpha_k D_k$ is a simple normal crossing effective $\mathbb{R}$-divisor. Locally on Ω, $D_j = \operatorname{div} g_j$, where g_j is holomorphic. consider

$$\varphi_D = \sum_j \alpha_j \log|g_j| \in \mathrm{Psh}(\Omega)$$

Then

$$J(X,D)_{\mathrm{an}} := J(\varphi_D)$$

By definition,

$$f \in J(\varphi_D)_x \longleftrightarrow \int_{V_x} |f|^2 e^{-2\varphi_D}$$

iff

$$\int_{V_x} \frac{|f|^2}{\prod |g_j|^{2\alpha_j}} < \infty$$

iff f is divisible by $\prod g_j^{m_j}$ with $m_j - \alpha_j > -1$ by SNC. In fact,

$$J(\varphi_D) = \mathcal{O}_X(-\lfloor D\rfloor) = \mathcal{O}_X(-\sum \lfloor \alpha_j \rfloor D_j)$$

Remark 4.1.2. A divisor $D = \sum_k a_k D_k$ is called simple normal crossing, if near $x \in D$, there exists coordinate systerm $(z_1, \dots, z_n)$ such that D_k is given by $\{z_k = 0\}$.

Theorem 4.1.1 (projection formula)**.** *$\mu\colon \widehat{X} \to X$ is a modification, $\varphi \in \mathrm{Psh}(X)$, then*

$$K_X \otimes J(\varphi) = \mu_*(K_{\widehat{X}} \otimes J(\varphi \circ u))$$

Proof. $\mu\colon \widehat{X}/E \cong X/S$ is an isomorphism, where $E = \mu^{-1}(S)$ and $S \subsetneq X$ a subvariety. Suppose $f \in J(\varphi)_x$, then $f\mathrm{d}z \in K_X \otimes J(\varphi)_x$, where $\mathrm{d}z = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$.

$$\int_U |f|^2 e^{-2\varphi} \mathrm{d}z \wedge \mathrm{d}\overline{z} = \int_U f \wedge \overline{f} e^{-2\varphi} < \infty$$

∎

Definition 4.1.2. The log-resolution of (X, D) is a modification $\mu\colon \widehat{X} \to X$ such that
1. $\widehat{X}$ is smooth;
2. $\mu^* D + \mathrm{Exd}(\mu)$ is simple normal crossing.

Remark 4.1.3. Hironaka's desingularization

$$J(\varphi_D) = \mu_*(K_{\widehat{X}/X} \otimes J(\varphi_D \circ u))$$

and $\varphi_D \circ u$ is the pluri-subharmonic function corresponding to $u^* D$ on $\widehat{X}$, and

$$J(\varphi_D \circ u) = \mathcal{O}_{\widehat{X}}(-\lfloor \mu^* D \rfloor)$$

then

$$J(X, D) = J(\varphi_D) = \mu_* \mathcal{O}_{\widehat{X}}(K_{\widehat{X}/X} - \lfloor \mu^* D \rfloor)$$

Remark 4.1.4 (Lazarfeld. Positivity in algebraic geometry II). Let X be a smooth complex variety, D an effective $\mathbb{Q}$-divisor. Fix a log-resolution of (X, D), $\mu\colon \widehat{X} \to X$, then the multiplier ideal sheaf $J(X, D)$ is defined by

$$J(X, D) = \mu_* \mathcal{O}_{\widehat{X}}(K_{\widehat{X}/X} - \lfloor \mu^* D \rfloor)$$

Example 4.1.2. $X = \mathbb{C}^2$, $D = \frac{3}{2}(A_1 + A_2 + A_3)$, where A_1, A_2, A_3 are three lines passing through 0. Take $\mu\colon \widehat{X} \to X$ be blow-up at 0, then μ is a log-resolution of (X, D)

$$\begin{aligned} \mu^* D &= \frac{3}{2}(A_1' + A_2' + A_3' + 3E) \\ &= \frac{3}{2}(A_1' + A_2' + A_3') + 2E \end{aligned}$$

and $K_{\widehat{X}/X} = E$, then

$$J(X, D) = \mu_* \mathcal{O}_{\widehat{X}}(-E)$$

Remark 4.1.5. openness of multiplier ideal sheaf, that is slightly modification won't change result. To be explicit, for $\varphi \in \mathrm{Psh}(\Omega)$, then there exists $\delta > 0$ such that

$$J((1+\varepsilon)\varphi) = J(\varphi)$$

when $0 < \varepsilon < \delta$.

Example 4.1.3. $\mathfrak{a} \subseteq \mathcal{O}_X$ is an ideal sheaf, locally given by $\mathfrak{a} = (f_1, \dots, f_N)$, consider

$$\varphi_{\mathfrak{a}} = \frac{1}{2}\log(|f_1|^2 + \cdots + |f_N|^2) \in \mathrm{Psh}$$

Then $J(\varphi_{\mathfrak{a}})$, then

$$J(\varphi_{\mathfrak{a}}) = \{g \in \mathcal{O}_x \mid \int_{V_x} |g|^2 e^{-2\varphi_{\mathfrak{a}}} < \infty\}$$

Definition 4.1.3. Let $\mu\colon \widehat{X} \to X$ be a log-resolution of $(X, \mathfrak{a})$ such that

1. $\widehat{X}$ is smooth;
2. $\mu^*\mathfrak{a} = \mathcal{O}_{\widehat{X}}(-F)$, and $F + \mathrm{exc}(\mu)$ is an effective simple normal crossing divisor, then

$$J(X, \mathfrak{a}) = \mu_* \mathcal{O}_X(K_{\widehat{X}/X} - \lfloor F \rfloor)$$

4.2. **Lelong number and multiplier ideal sheaf.** For $\varphi \in \mathrm{Psh}(\Omega), x \in \Omega$, recall

$$V(\varphi, x) = \sup\{t \geq 0 \mid \varphi(z) \leq t \log |z - x| + \text{bounded term}\}$$

one idea is that Lelong number is larger, is more singular.

Proposition 4.2.1. Assume $V(\varphi, x) \geq n + s$ for some $s \in \mathbb{Z}_{\geq 0}$, then

$$J(\varphi)_x \subseteq \mathfrak{m}_x^{s+1}$$

Theorem 4.2.1 (Skoda)**.** If $\nu(\varphi, x) < 1$, then $e^{-2\varphi} \in L^1_x$. In particular, $J(\varphi)_x = \mathcal{O}_X$.

4.3. **The singular metric.** Let L be a holomorphic line bundle, a smooth hermitian metric h is locally given by $e^{-2\varphi}$, where φ is a smooth function, which is called metric weight, then Chern curvature is given by

$$\Theta_h = 2\partial\overline{\partial}\varphi$$

and Chern class is given by

$$c_1(L, h) = \frac{\sqrt{-1}}{2\pi}\Theta_h = \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}\varphi$$

Suppose $\{g_{\alpha\beta}\}$ is transition function of L with respect to open covering $\{U_\alpha\}$, then smooth metric h is given by a collection $\{h_\alpha \in C^\infty(U_\alpha)\}$ such that $h_\alpha = |g_{\alpha\beta}|^{-2}h_\beta$. In other words, a collection of metric weights $\{\varphi_\alpha \in C^\infty(U_\alpha)\}$ such that

$$\varphi_\alpha = \varphi_\beta + \log |g_{\alpha\beta}|$$

Definition 4.3.1 (singular metric)**.** A singular metric on a holomorphic line bundle L is a collection of metric weights $\{\varphi_\alpha \in L^1_{\mathrm{loc}}(U_\alpha)\}$ such that $\varphi_\alpha = \varphi_\beta + \log |g_{\alpha\beta}|$.

Example 4.3.1. Let D be a prime divisor on a complex manifold X. Suppose $D \cap U_\alpha = \mathrm{div}(s_\alpha)$, where $s \in \mathcal{O}(U_\alpha)$ and $\{U_\alpha\}$ is a open covering of X. D gives a line bundle L_D with transition functions $g_{\alpha\beta} = s_\alpha/s_\beta$, then

$$\log |s_\alpha| = \log |s_\beta| + |g_{\alpha\beta}|$$

that is $\{\log |s_\alpha|\}$ gives a singular metric on L_D.

Example 4.3.2. Let $D = \sum_{a_k \in \mathbb{Z}} a_k D_k$ be a divisor on a complex manifold X, where D_k are prime divisors, it gives a line bundle $L_D = \bigotimes_k L_{D_k}^{\otimes a_k}$, and there is also a singular metric on L_D.

Example 4.3.3. Let L be a holomorphic line bundle over a complex manifold X with transition functions $\{g_{\alpha\beta}\}$. Consider $0 \neq s \in H^0(X, L^{\otimes m})$ locally given by $s_\alpha \in \mathcal{O}(U_\alpha)$, then $s_\alpha = g^m_{\alpha\beta} s_\beta$ implies

$$\frac{1}{m}\log|s_\alpha| = \frac{1}{m}\log|s_\beta| + \log|g_{\alpha\beta}|$$

that is $\{\frac{1}{m}\log|s_\alpha|\}$ gives a singular metric on L.

Example 4.3.4. Let L be a holomorphic line bundle over a complex manifold X with transition functions $\{g_{\alpha\beta}\}$. Consider $0 \neq s_1, \ldots, s_N \in H^0(X, L^{\otimes m})$ locally given by $s_{i,\alpha} \in \mathcal{O}(U_\alpha)$, then $s_{i,\alpha} = g^m_{\alpha\beta} s_{i,\beta}$ implies

$$\begin{aligned}\frac{1}{2}\log(\sum_{i=1}^N |s_{i,\alpha}|^{\frac{2}{m}}) &= \frac{1}{2}\log(\sum_{i=1}^N |s_{i,\beta} g^m_{\alpha\beta}|^{\frac{2}{m}}) \\ &= \frac{1}{2}\log(\sum_{i=1}^N |s_{i,\alpha}|^{\frac{2}{m}}) + \log|g_{\alpha\beta}|\end{aligned}$$

then $\{\frac{1}{2}\log(\sum_{i=1}^N |s_{i,\alpha}|^{\frac{2}{m}})\}$ gives a singular metric on L.

Example 4.3.5. Let L be a holomorphic line bundle over a complex manifold X, for each $m \geq 0$, consider $0 \neq s_1^{(m)}, \ldots, s_{N_m}^{(m)} \in H^0(X, L^{\otimes m})$, then

$$\varphi_\alpha = \frac{1}{2}\log(\sum_{m=1}^\infty \theta_m \sum_{k=1}^{N_m} |s_{k,\alpha}|^{\frac{2}{m}})$$

is a singular metric for suitable $\{\theta_m\}$, which is called Siu metric.

Definition 4.3.2 (pseudo-effective)**.** A holomorphic line bundle L equipped with singular metric h is called pseudo-effective, if every metric weight $\varphi_\alpha \in \mathrm{Psh}(U_\alpha)$.

Definition 4.3.3 (multiplier ideal sheaf)**.** Let (L, h) be a pseudo-effective line bundle, the multiplier ideal sheaf of h is defined as

$$J(h) = \bigcup_{x\in X} J(h)_x = \bigcup_{x\in X} J(\varphi_\alpha)_x$$

Proposition 4.3.1. A holomorphic line bundle L equipped with a singular metric h is pseudo-effective if and only if curvature current

$$c_1(L, h) = \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}\varphi_\alpha$$

is a positive current.

Proof. See Proposition 2.2.4. □

Definition 4.3.4 (big)**.** Let (X, ω) be a hermitian manifold. A holomorphic line bundle L is called big, if there exists a singular metric h such that $c_1(L, h) \geq \delta\omega$ in the sense of current for some $\delta > 0$.

Definition 4.3.5 (numerical effective)**.** Let (X, ω) be a hermitian manifold. A holomorphic line bundle L is called numerical effective, if for any $\varepsilon > 0$, there exists a smooth metric h_ε such that

$$c_1(L, h) + \varepsilon\omega > 0$$

Definition 4.3.6 (semi-ample)**.** Let L be a holomorphic line bundle over complex manifold X. L is called semi-ample, if there exists $m > 0$ such that $L^{\otimes m}$ is globally generated.

Remark 4.3.1. There is a relations

$$\begin{tikzcd}
\text{ample} \arrow[r] \arrow[d] & \text{semi-ample} \arrow[r] & \text{semi-positive} \arrow[ddll] \\
\text{big} \arrow[d] & & \\
\text{pseudo-effective} & &
\end{tikzcd}$$

5. L^2-ESTIMATE

5.1. **Extension theorem.** Let (X,ω) be a complete Kähler manifold, (L,h) a holomorphic line bundle equipped with a singular metric h. Then

$$\Delta'' - \Delta' = [\sqrt{-1}\Theta_h, \Lambda] := A$$

for $u \in C^\infty(\bigwedge^{p,q} \otimes E)$ with compact support, one has

$$\|D''u\|^2 + \|D''^*\| \geq (Au, u)$$

Consider $L^2(X, \bigwedge^{p,q} \otimes E)$, then

$$\mathrm{Dom}(D') =$$
$$\mathrm{Dom}(D'') =$$

It's dense, and in the sense of Hörmander. A fact about extension is that for all $u \in \mathrm{Dom}(D'') \cap \mathrm{Dom}(D''^*)$, one has

$$\|D''u\|^2 + \|D''^*u\|^2 \geq (Au, u)$$

Let $g \in L^2(X, \bigwedge^{p,q} \otimes E)$ with $\overline{\partial} g = D''g = 0$. Assume $A \geq 0$ everywhere, and there exists $\alpha(x) \in [0, \infty)$ such that

$$|\langle g, u\rangle|^2 \leq \alpha\langle Au, u\rangle$$

Remark 5.1.1. If $A > 0$, then

$$\begin{aligned}
|\langle g, u\rangle| &= \langle A^{-\frac{1}{2}} g, A^{\frac{1}{2}} u\rangle \\
&\leq \langle A^{-\frac{1}{2}} g, A^{-\frac{1}{2}} Ag\rangle\langle A^{\frac{1}{2}} u, A^{\frac{1}{2}} u\rangle \\
&= \langle A^{-1} g, g\rangle\langle Au, u\rangle
\end{aligned}$$

that is one has $\alpha = \langle A^{-1} g, g\rangle$.

Theorem 5.1.1. Let (X,ω) be complete Kähler manifold, $g \in L^2(X, \bigwedge^{p,q} \otimes E)$ with $\overline{\partial} g = 0$ and $A \geq 0$. If $\int_X \langle A^{-1} g, g\rangle < \infty$, then there exists $f \in L^2(X, \bigwedge^{p,q} \otimes E)$ such that $g = \overline{\partial} f$ and

$$\|f\|^2 \leq \int_X \langle A^{-1} g, g\rangle$$

Proof. For arbitrary $u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*)$, then

$$\begin{aligned}
|(u, g)|^2 &= |\int \langle u, g\rangle|^2 \\
&\leq (\int |\langle u, g\rangle|)^2 \\
&\leq (\int \langle Au, u\rangle^{\frac{1}{2}} \langle A^{-1} g, g\rangle^{\frac{1}{2}})^2 \\
&\leq (A^{-1} g, g) \int \langle Au, u\rangle \\
&\leq (A^{-1} g, g)(\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2)
\end{aligned}$$

Decompose u into two parts

$$u = u_1 + u_2$$

where $u_1 \in \ker \overline{\partial}$ and $u_2 \in (\ker \overline{\partial})^{\perp} = \overline{\operatorname{im} \overline{\partial}^*}$, then $\overline{\partial} g = 0$ implies

$$(u, g) = (u_1, g)$$

then above argument shows

$$(u_1, g)^2 =$$

All in all, for all $u \in \operatorname{Dom}(\overline{\partial}) \cap \operatorname{Dom}(\overline{\partial}^*)$, one has

$$|(u, g)| \leq (A^{-1} g, g)^{\frac{1}{2}} \|\overline{\partial}^* u\|$$

This shows

$$\begin{aligned} l\colon\ \operatorname{im} \overline{\partial}^* &\to \mathbb{C} \\ \overline{\partial}^* u &\mapsto (u, g) \end{aligned}$$

is well-defined and $\|l\| \leq (A^{-1} g, g)^{\frac{1}{2}}$. By Hann-Banach theorem one has

$$\widetilde{l}\colon L^2 \to \mathbb{C}$$

then by Riesz representation theorem one has

$$\widetilde{l}(\overline{\partial}^* u) = (\overline{\partial}^* u, f)$$

for some f with $\|f\| = \|l\|$. This completes the proof. □

Example 5.1.1.

1. Any compact Kähler manifold is complete.
2. Any stein manifold is complete, eg, $\mathbb{C}^n$ and pseudo-convex domains in $\mathbb{C}^n$.

Definition 5.1.1 (weakly pseudo-convex)**.** A weakly pseudo-convex Kähler manifold, if there exists an exhaustion function ψ such that $\psi \in \operatorname{Psh} \cap C^{\infty}$.

Example 5.1.2. Let X be a compact Kähler manifold, V an analytic subvariety, then $X \backslash V$ is weakly pseudo-convex.

Proposition 5.1.1. Any weakly pseudo-convex Kähler manifold is complete.

Theorem 5.1.2. $(X, \widehat{\omega})$ a complete Kähler manifold, ω another Kähler metric. $(E, h) \to (X, \omega)$, assume

$$A = [\sqrt{-1}\Theta_h, \Lambda_{\omega}] \geq 0$$

on $\bigwedge^{n,q} \otimes E$. $g \in L^2(X, \bigwedge^{n,q} \otimes E)$ with $\overline{\partial} g = 0$ and $A \geq 0$. If $\int_X \langle A^{-1} g, g \rangle < \infty$, then there exists $f \in L^2(X, \bigwedge^{p,q} \otimes E)$ such that $g = \overline{\partial} f$ and

$$\|f\|^2 \leq \int_X \langle A^{-1} g, g \rangle$$

Theorem 5.1.3. $(L,h)\to(X,\omega)$, ω is Kähler, and (L,h) is a line bundle with a singular metric such that $c_1(L,h)\geq\delta\omega$ in the sense of current for some $\delta>0$. Assume X contains a stein Zariski open set, $g\in L^2(X,\bigwedge^{n,q}\otimes E)$ with $\overline{\partial}g=0$ and $A\geq 0$. If $\int_X\langle A^{-1}g,g\rangle_{\omega,h}<\infty$, then there exists $f\in L^2(X,\bigwedge^{p,q}\otimes E)$ such that $g=\overline{\partial}f$ and

$$\|f\|^2\leq\int_X\langle A^{-1}g,g\rangle$$

5.2. **Nadel vanishing.**

Theorem 5.2.1. Let (X,ω) be a weakly pseudo-convex Kähler manifold containing a stein Zariski open set, (L,h) a holomorphic line bundle equipped with a singular metric h such that $c_1(L,h)\geq\delta\omega$ for some $\delta>0$. Suppose $J(h)$ is the multiplier ideal sheaf of h, then

$$H^q(X,\mathcal{O}(K_X+L)\otimes J(h))=0$$

where $q\geq 1$.

Proof. For $q\geq 0$, A^q is the sheaf with germs u, where u is measurable section of $\bigwedge^{n,q}\otimes L$ such that

$$|u|^2_{\omega,h},|\overline{\partial}u|^2_{\omega,h}\in L^1_{\mathrm{loc}}$$

Consider

$$0\to\mathcal{O}(K_X+L)\otimes J(h)\to A^1\xrightarrow{\overline{\partial}}A^2\xrightarrow{\overline{\partial}}A^2\xrightarrow{\overline{\partial}}\dots$$

Note that

1. every A^q is a C^∞-module.
2. above sequence is a resolution.

where (2) holds from L^2-extension theorem. Thus

$$H^q(X,\mathcal{O}(K_X+L)\otimes J(h))=\frac{\ker\overline{\partial}\colon\Gamma(X,A^q)\to\Gamma(X,A^{q+1})}{\operatorname{im}\overline{\partial}\colon\Gamma(X,A^{q-1})\to\Gamma(X,A^q)}$$

Note that X is weakly pseudo-convex implies there exists an exhaustion function $0<\psi\in C^\infty(X)\cap\mathrm{Psh}(X)$. For $t>0$,

$$\int_{\{\psi<t\}}|g|^2_{\omega,h}<\infty$$

Take a subtle $\chi\colon\mathbb{R}_+\to\mathbb{R}_+$ convex increasing function[1] with

$$\int_X|g|^2_{\omega,h}e^{-2\chi\circ\psi}<\infty$$

Then there exists a f being a section of $\bigwedge^{n,q-1}\otimes L$ such that $\overline{\partial}f=g$, and

$$\|f\|^2_{\omega,he^{-2\chi\circ\psi}}=\int_X\|f\|^2_{\omega,h}e^{-2\chi\circ\psi}<\infty$$

this shows $f\in\Gamma(X,A^{q-1})$. □

[1]In fact, here we modify the metric, and use L^2-extension for this new metric

Corollary 5.2.1 (Kawamata-Viehweg vanishing)**.** Let X be a projective complex manifold, F a line bundle over X such that

$$mF = L + D$$

where L is nef and big, D is effective, then

$$H^q(X, \mathcal{O}(K_X + F) \otimes J(m^{-1}D)) = 0$$

where $q \geq 1$. In particular, if F is nef and big, then $H^q(X, \mathcal{O}(K_X + F)) = 0$ where $q \geq 1$.

Proof. Here we need to construct a singular metric $h = e^{-2\varphi_F}$ on F such that

1. $c_1(F, h) \geq \delta\omega$
2. $J(h) = J(m^{-1}D)$.

Fact 5.2.1. If L is big, Kodaria lemma implies

$$L = A + E$$

where A is ample and E is effective, then $\varphi_L = \varphi_A + \varphi_E$, thus $c_1(\varphi_L) \geq \delta\omega$.

Define

$$\varphi_F = \frac{1}{m}(t\varphi_L + (1-t)\varphi_{L,\varepsilon} + \varphi_D)$$

Then

$$J(\varphi_F) = J(\frac{1}{m}(t\varphi_E + \varphi_D))$$

By projection formula for sufficiently small t, one has

$$J(\frac{1}{m}(t\varphi_E + \varphi_D)) = J(\frac{1}{m}\varphi_D) = J(m^{-1}D)$$

Fix $t = t_0$, one has

$$\begin{aligned} c_1(F, \varphi) &= \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\frac{1}{m}(t_0\varphi_L + (1-t_0)\varphi_{L,\varepsilon} + \varphi_D) \\ &\geq \frac{1}{m}(t_0\delta\omega - (1-t_0)\varepsilon\omega) \end{aligned}$$

Then take sufficiently small ε to conclude. □

5.3. **Analytic and algebraic positivity.**

5.3.1. *Statements.* Let X be a projective n-manifold, L a line bundle.

Definition 5.3.1 (pseudo-effective)**.** There exists a singular metric h on L such that $c_1(L, h) \geq 0$ in the sense of current.

Definition 5.3.2 (pseudo-effective)**.** $L = L_D$ is called pseudo-effective, if there exist effective divisors D_k such that $\{D_k\} \to D$ in $H^2(X, \mathbb{R})$.

Definition 5.3.3 (big)**.** There exists a singular metric h on L such that $c_1(L, h) \geq \delta\omega$ for some $\delta > 0$.

Definition 5.3.4 (big)**.** L is called big, if

$$\limsup \frac{H^0(X, L^{\otimes m})}{m^n/n!} > 0$$

Definition 5.3.5 (numerical effective)**.** Let (X, ω) be a hermitian manifold. A holomorphic line bundle L is called numerical effective, if for any $\varepsilon > 0$, there exists a smooth metric h_ε such that

$$c_1(L, h) + \varepsilon\omega > 0$$

Definition 5.3.6 (numerical effective)**.** L is numerical effective, if $c_1(L)[C] \geq 0$ for any irreducible curve $C \subset X$.

5.3.2. *Proofs.*

Lemma 5.3.1. Assume L has a singular metric $h = e^{-2\varphi}$ such that $c_1(L, h) \geq \delta\omega$, and $\nu(\varphi, x) \geq n + s$, and x is isolated in $E_1(\varphi) = \{z \in X \mid \nu(\varphi, x) \geq 1\}$, then $H^0(X, K_X + L)$ generates s-jets at x.

Proof. Take a small ball B centered at x, and a polynomial P of degree $\leq s$. Take a cut-off function χ such that $\operatorname{supp}\chi \Subset B$, $0 \leq \chi \leq 1$ and $\chi \equiv 1$ near x, then $\chi P \otimes e$ is a smooth section of $K_X + L$. We need to find a f such that
1. $\overline{\partial}(\chi P \otimes e - f) = 0$
2. $\operatorname{ord}_x(f) > s$.

□

proof from analytic psef to algebraic psef. Suppose L has a singular metric $h_L = e^{-2\varphi_L}$ such that $c_1(L, h) \geq 0$. Take $x_0 \in X$ such that $\nu(\varphi_L, x_0) = 0$, and let

$$\psi = \chi n \log |z - x_0|$$

where χ has compact support in B_x, and $\chi \equiv 1$ near x. Let A be an ample line bundle with smooth metric $h_A = e^{-2\varphi_A}$ such that $c_1(A, h_A) \geq \delta\omega$. Fix $m_0 \gg 1$ such that

$$m_0 c_1(A, h_A) + \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}\psi \geq \omega$$

Consider $kL + m_0A$ with the metric $h_L^k h_A^{m_0} e^{-2\psi}$ or metric weight $\varphi_k = k\varphi_L + m_0\varphi_A + \psi$. Claim $(kL + m_0A, \varphi_k)$ satisfies above lemma, then $H^0(X, K_X + kL + m_0A)$ generates 0-jets at x. In particular, there exists $0 \neq s_k \in H^0(X, K_X + kL + m_0A)$.

Let $D_k = \operatorname{div}(s_k)$, then Poincaré-Lelong formula one has

$$\{D_k\} = c_1(K_X) + kc_1(L) + m_0c_1(A)$$

that is

$$c_1(L) = \frac{1}{k}\{D_k\} - \frac{1}{k}(c_1(K_X) + m_0c_1(A))$$

In other words, one has $c_1(L) = \lim_{k\to\infty}\{\frac{1}{k}D_k\}$.

Now let's see claims

1.
2.
3.

□

proof from algebraic psef to analytic psef. □

5.4. Oshawa-Takegoshi extension.

Theorem 5.4.1 (Oshawa-Takegoshi extension)**.** Let $\Omega \subseteq \mathbb{C}^n$ be a bounded, pseudo-convex domain. Assume $\sup_\Omega |z_n|^2 < e^{-1}$, then there exists a constant $C_n > 0$ such that for all $\varphi \in \mathrm{Psh}(\Omega)$ and f holomorphic on $\Omega \cap \{z_n = 0\}$ with $\int_{\Omega\cap\{z_n=0\}} |f|^2 e^{-2\varphi}$, there exists holomorphic extension F of f on Ω with

$$\int_\Omega \frac{|F|^2}{|z_n|^2(\log|z_n|)^2} e^{-2\varphi} \leq C_n \int_{\Omega\cap\{z_n=0\}} |f|^2 e^{-2\varphi}$$

Proof. See [Che11]. □

6. Demailly's regularization theorem

6.1. **Local version.** Let $\Omega\subseteq\mathbb{C}^n$ be a bounded, pseudo-convex domain. Roughly speaking, Demailly's regularization theorem says

$$\{\log\sum_{k=1}^{N}|f_k|^2\mid f_k\text{ is holomorphic in }\Omega\}$$

is dense in $\mathrm{Psh}(\Omega)$ with respect to $L^1_{\mathrm{loc}}(\Omega)$.

Definition 6.1.1 (analytic singularities)**.** A function g has analytic singularities if locally

$$g=c\log\sum_{k=1}^{N}|f_k|^2+\text{bounded term}$$

where $c>0$ is a constant, and f_k is holomorphic.

Theorem 6.1.1 (Demailly's regularization)**.** Let $\Omega\subseteq\mathbb{C}^n$ be a bounded, pseudo-convex domain. A non-trivial $\varphi\in\mathrm{Psh}(\Omega)\cap L^1_{\mathrm{loc}}(\Omega)$, for $m\in\mathbb{N}$,

$$H_\Omega(m\varphi)=\{f\text{ is holomorphic in }\Omega\mid\int_\Omega|f|^2e^{-2m\varphi}<\infty\}$$

and let $\{g_{ml}\}_{l=1}^{\infty}$ be an othonormal basis of $H_\Omega(m\varphi)$. Define

$$\varphi_m=\frac{1}{2m}\log\sum_{l=1}^{\infty}|g_{ml}|^2$$

Then $\varphi_m\in\mathrm{Psh}(\Omega)$ and there exists $c_1>0,c_2>0$ independent of m, such that

1. $$\varphi(z)-\frac{c_1}{m}\leq\varphi_m(z)\leq\sup_{|\xi-z|<r}\varphi(z)+\frac{1}{m}\log\frac{c_2}{r^n}$$
   for any $z\in\Omega$ and $r>0$ with $r<\mathrm{dist}(z,\partial\Omega)$. In particular, $\varphi_m(z)\to\varphi(z)$ pointwisely, thus in $L^1_{loc}(\Omega)$.
2. $$\nu(\varphi,z)-\frac{n}{m}\leq\nu(\varphi_m,z)\leq\nu(\varphi,z)$$

Remark 6.1.1. For all $K\Subset\Omega$, there exists constant C,N depending on K such that

$$\sum_{l=1}^{\infty}|g_{ml}|^2\leq C\sum_{l=1}^{N}|g_{ml}|^2$$

on K.

Proof. Given $z\in\Omega$, consider

$$\begin{aligned}\mathrm{ev}_z\colon H_\Omega(m\varphi)&\to\mathbb{C}\\ f&\mapsto f(z)\end{aligned}$$

It's a bounded linear functional. Indeed, one has

$$
\begin{aligned}
|\operatorname{ev}_z(f)| &= |f(z)| \\
&\leq \frac{1}{|B(z,r)|}\int_{B(z,r)}|f| \\
&\leq \frac{1}{|B(z,r)|^{\frac{1}{2}}}(\int_{B(z,r)}|f|^2)^{\frac{1}{2}} \\
&\leq \frac{1}{|B(z,r)|^{\frac{1}{2}}}(\int_{B(z,r)}|f|^2 e^{2m(\sup_{B(z,r)}\varphi-\varphi)})^{\frac{1}{2}} \\
&\leq \frac{e^{m\sup_{B(z,r)}\varphi}}{|B(z,r)|^{\frac{1}{2}}}(\int_\Omega |f|^2 e^{-2m\varphi})^{\frac{1}{2}}
\end{aligned}
$$

By Riesz representation theorem, one has

$$\operatorname{ev}_z(f) = (f,u)$$

where $u \in H_\Omega(m\varphi)$. If we write $u = \sum_{l=1}^\infty c_l g_{ml}$, one has

$$\operatorname{ev}_z(g_{ml}) = g_{ml}(z) = (g_{ml}, \sum_{l=1}^\infty c_{l'} g_{ml'}) = \overline{c}_l$$

that is $u = \sum_{l=1}^\infty \overline{g_{ml}}(z) g_{ml}$, thus

$$\|\operatorname{ev}_z\|^2 = \sum_{l=1}^\infty |g_{ml}(z)|^2$$

One has

$$\varphi_m(z) = \frac{1}{2m}\log\|\operatorname{ev}_z\|^2 = \frac{1}{2m}\log\sup_{\|f\|_{H_\Omega(m\varphi)}\leq 1}|f(z)|^2 = \frac{1}{2m}\sup_{\|f\|_{H_\Omega(m\varphi)}\leq 1}\log|f(z)|^2$$

By mean values inequality one has

$$
\begin{aligned}
|f(z)|^2 &\leq \frac{1}{\pi^n r^{2n} n!}\int_{B(z,r)}|f(w)|^2 \\
&\leq \frac{e^{2m\sup_{B(z,r)}\varphi}}{\pi^n r^{2n} n!}\int_\Omega |f|^2 e^{-2\varphi}
\end{aligned}
$$

Thus we obtain the upper bound

$$\varphi_m(z) \leq \frac{1}{2m}\log\frac{e^{2m\sup_{B(z,r)}\varphi}}{\pi^n r^{2n} n!} = \sup_{B(z,r)}\varphi + \frac{1}{m}\log\frac{c_2}{r^n}$$

Apply Theorem 5.4.1 to the setting (Ω, z). For $a \in \mathbb{C}$, there exists a holomorphic function f in Ω such that $f(z) = a$ and

$$\int_\Omega |f|^2 e^{-2\varphi} \leq C|a|^2 e^{-2m\varphi(z)}$$

where C is a constant depending on $\operatorname{diam}\Omega$. Take a such that $C|a|^2e^{-2m\varphi(z)} = 1$, one has

$$\varphi_m(z) \geq \frac{1}{2m}\log|a|^2 = \frac{1}{2m}\log\frac{e^{2m\varphi(z)}}{C} = \varphi(z) - \frac{c_1}{m}$$

□

Corollary 6.1.1 (Siu)**.** Let φ be a pluri-subharmonic function on complex manifold X, then for $c > 0$, $E_c(\varphi) = \{z \in X \mid \nu(\varphi, z) \geq c\}$ is an analytic subvariety of X.

Proof. It's a local problem, thus without lose of generality we may assume $X = \Omega$, is a bounded, pseudo-convex domain. Note that

$$E_c(\varphi) = \bigcup_{m\geq 1} E_{c-\frac{n}{m}}(\varphi_m)$$

It suffices to show for any $c > 0$, $E_c(\varphi_m)$ is an analytic subvariety □

6.2. **Global approximation of closed positive** $(1,1)$**-current.**

Theorem 6.2.1. Let X be a compact complex manifold.

$$T = \theta + \sqrt{-1}\partial\overline{\partial}\varphi$$

is a $(1,1)$-current, where θ is a smooth $(1,1)$-form with $\mathrm{d}\theta = 0$. Assume $T \geq \gamma$, where γ is a continous $(1,1)$-form. Then there exists a sequence of currents $T_m = \theta + \sqrt{-1}\partial\overline{\partial}\varphi_m$, where φ_m is locally given by

$$\varphi_m = \frac{1}{2m}\log\sum_{l=1}^{\infty}|g_{ml}|^2 + \text{bounded term}$$

such that

1. $T_m \to T$ in the sense of current.
2. $\nu(T,x) - \frac{n}{m} \leq \nu(T_m, x) \leq \nu(T, x)$.
3. $T_m \geq \gamma - \varepsilon_m\omega$, where ε_m descends to 0.

Notation 6.2.1. T_m is calle the approximation of T with log poles.

Definition 6.2.1 (Kähler current)**.** T is called a Kähler current if $\mathrm{d}T = 0$ and $T \geq \delta\omega$, where $\delta > 0$ in the sense of current.

Corollary 6.2.1. Any Kähler current can be approximated by Kähler currents with log poles.

Proof. Pick $\gamma > 0$ in above theorem □

6.3. **Siu decompostion.**

Theorem 6.3.1. Let T be a d-closed positive current of bidegree (p,p), then for $c>0$, $E_c(T)=\{x\in X \mid \nu(T,x)\geq c\}$ is analytic, and $\dim E_c(T)\leq p$.

Theorem 6.3.2. Let X be a compact complex manifold, and $\{A_k\}$ the component of dimension p in $\bigcup_{c\in\mathbb{Q}_{>0}} E_c(T)$. Consider $\nu(T,A)=\inf_{x\in A}\nu(T,x)$, then

$$T=\sum_{k=1}^{\infty}\nu(T,A_k)[A_k]+R$$

with $R\geq 0$ and for $c>0$ one has $\dim E_c(R)<p$.

Appendix A. Topological vector spaces

In this appendix we mainly follows [Rud74].

A.1. Basic definitions and first properties.

All vector spaces are assumed to be over $\mathbb{R}$ or $\mathbb{C}$.

Definition A.1.1 (balance)**.** Let X be a vector space, a set $B \subset X$ is said to be balanced if $\alpha B \subset B$ for all scalars α with $|\alpha| < 1$.

Definition A.1.2 (invariant metric)**.** A metric d on a vector space X is called invariant, if

$$d(x+z, y+z) = d(x, y)$$

for all $x, y, z \in X$.

Definition A.1.3. A topological vector space is a vector space X with topology τ such that

1. every point of X is closed set;
2. the vector space operations are continuous with respect to τ.

Remark A.1.1. In the vector space context, the term local base always means a local base at 0, that is a collection $\mathscr{B}$ of neighborhoods of 0 such that every neighborhoods of 0 contains a member of $\mathscr{B}$.

Definition A.1.4 (types of topological vector space)**.** Let X be a topological vector space with topology τ.

1. X is locally convex if there is a local base $\mathscr{B}$ whose members are convex.
2. X is locally bounded if 0 has a bounded neighborhood.
3. X is locally compact if 0 has a neighborhood whose closure is compact.
4. X is metrizable if τ is compatible with some metric d.
5. X is a F-space if its topology is induced by a complete invariant metric d.
6. X is a Fréchet space if X is a locally convex F-space.
7. X is normable if there is a norm on X such that the metric induced by the norm is compatible with τ.
8. X has Heine-Borel property if every closed and bounded subset of X is compact.

Remark A.1.2. Here is a list of some relations between these properties of a topological vector space X.

1. If X is locally bounded, then X has a countable local base.
2. X is metrizable if and only if X has a countable local base.
3. X is normable if and only if X is locally convex and locally bounded.
4. X has finite dimension if and only if X is locally compact.
5. If a locally bounded space X has the Heine-Borel property, then X has finite dimension.

A.2. **Seminorms and local convexity.**

Definition A.2.1 (seminorm)**.** A seminorm on a vector space X is a real-valued function p on X such that

1. $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$;
2. $p(\alpha x) = |\alpha| p(x)$ for all $x \in X$ and scalars α;
3. $p(x) \neq 0$ if $x \neq 0$.

Definition A.2.2 (separating)**.** A family $\mathscr{P}$ of seminorms on X is said to be separating if to each $x \neq 0$ corresponds at least one $p \in \mathscr{P}$ with $p(x) \neq 0$.

Seminorms are closely to local convexity in two ways: In every locally convex space there exists a separating family of continous seminorms. Conversely, if $\mathscr{P}$ is a separating family of seminorms on a vector space X, then $\mathscr{P}$ can be used to define a locally convex topology on X with the property that every $p \in \mathscr{P}$ is continous.

Theorem A.2.1. Suppose $\mathscr{P}$ is a separating family of seminorms on a vector space X, associate to each $p \in \mathscr{P}$ and to each positive integer n the set

$$V(p, n) = \{x \colon p(x) < \frac{1}{n}\}$$

Let $\mathscr{B}$ be the collection of all finite intersections of the sets $V(p, n)$, then $\mathscr{B}$ is a convex balanced local base for a topology τ on X, which turns X into a locally convex space such that

1. every $p \in \mathscr{P}$ is continous;
2. a set $E \subset X$ is bounded if and only if every $p \in \mathscr{P}$ is bounded on E.

Remark A.2.1*.* If $\mathscr{P} = \{p_i \mid i = 1, 2, 3 \ldots\}$ is a countable separating family of seminorms on X, then $\mathscr{P}$ induces a topology τ with a countable local base, thus it's metrizable. However, in this case, a compatible translation invariant metric can be defined directly in terms of $\{p_i\}$, that is

$$d(x, y) = \sum_{i=1}^{\infty} \frac{2^{-i} p_i(x-y)}{1 + p_i(x-y)}$$

A.3. **Examples of Fréchet space.** In this section we introduce some function spaces which will be used in later work with distributions. The notation $C^\infty(\Omega)$ denotes the vector space consisting of smooth functions over Ω. If $K \Subset \Omega$, then $\mathcal{D}(K)$ denotes subspace of $C^\infty(\Omega)$ consisting of those smooth functions with support in K. Now we're going to define topology on it to make it into a Fréchet space such that $\mathcal{D}(K)$ is a closed subspace of $C^\infty(\Omega)$ whenever $K \Subset \Omega$, and thus $\mathcal{D}(K)$ is also a Fréchet space.

To do this, choose a exhaustion K_i of Ω, and define seminorms $p_N, N = 1, 2, \ldots$ by setting

$$p_N(f) = \max\{|D^\alpha f(x)| : x \in K_N, |\alpha| \leq N\}$$

They define a metrizable locally convex topology on $C^\infty(\Omega)$ according Remark A.2.1. For each $x \in \Omega$, the functional $f \to f(x)$ is continous with respect to this topology, and note that $\mathcal{D}(K)$ is the intersection of the null spaces of these functionals, as x ranges over the complement of K, thus $\mathcal{D}(K)$ is closed in $C^\infty(M)$.

To show $C^\infty(\Omega)$ is a Fréchet space, now it suffices to show it's complete. Note that a local base is given by the sets

$$V_N = \{f \in C^\infty(\Omega) : p_N(f) < \frac{1}{N}\}$$

If $\{f_i\}$ is a Cauchy sequence in $C^\infty(\Omega)$ and if N is fixed, then $f_i - f_j \in V_N$ if i and j are sufficiently large. Thus $|D^\alpha f_i - D^\alpha f_j| < \frac{1}{N}$ on K_N if $|\alpha| \leq N$. It follows that each $D^\alpha f_i$ converges uniformly on compact subsets of Ω to a function g_α. In particular, f_i converges to g_0. It's clear $g_0 \in C^\infty(\Omega)$ with $g_\alpha = D^\alpha g_0$, and $f_i \to g$ with respect to the topology of $C^\infty(\Omega)$.

Appendix B. Distribution theory

B.1. **Introduction.** The theory of distributions frees differential calculus from certain problems arisen from non-smooth functions exist. This is done by extending it to a class of objects called distribution which is much larger than the class of smooth functions.

To motivate the definitions to come, let's firstly consider $n = 1$ and the integrals that follow are taken with respect to Lebesgue measure. Let $\mathcal{D} = \mathcal{D}(\mathbb{R})$ denotes the vector space consisting of all $\phi \in C^\infty(\mathbb{R})$ with compact support. Then $\int f\phi$ exists for every locally integerable f and every $\phi \in \mathcal{D}$. If f happens to be continously differentiable, then

$$\int f'\phi = -\int f\phi'$$

and if f is smooth, then

$$\int f^{(k)}\phi = (-1)^k \int f\phi^{(k)}$$

for arbitrary $k \in \mathbb{Z}_{\geq 1}$. However, the right hands of above equalities make sense whether f is smooth or not, then we can therefore assign a "k-th derivative" to every f that is locally integerable as a functional on $\mathcal{D}$ that sends ϕ to $(-1)^k \int f\phi^{(k)}$, and distributions will be defined to be those linear functionals on $\mathcal{D}$ that are continuous with respect to a certain topology we will define later.

B.2. **Test function spaces.** Let $\Omega \subset \mathbb{R}^n$ be a non-empty open set, for each compact $K \subset \Omega$, we have already seen $\mathcal{D}(K)$ is a Fréchet space in section A.3. The union of the spaces $\mathcal{D}(K)$ as K ranges over all compact subsets of Ω, is called test function space $\mathcal{D}(\Omega)$. It's clear $\mathcal{D}(\Omega)$ is a vector space, now let's introduce the norms

$$\|\phi\|_N = \max\{|D^\alpha\phi(x)| : x \in \Omega, |\alpha| \leq N\}$$

where $\phi \in \mathcal{D}(\Omega)$ and $N \geq 0$. According to Theorem A.2.1, it gives a locally convex topology on $\mathcal{D}(\Omega)$.

The restriction of these norms to any fix $\mathcal{D}(K) \subset \mathcal{D}(\Omega)$ induce the same topology on $\mathcal{D}(K)$ as topology we have given before, but this topology has the disadvantages of not being complete. For example, take $n = 1, \Omega = \mathbb{R}$ and pick $\phi \in \mathcal{D}(\Omega)$ with support in $[0, 1]$ and $\phi > 0$ in $(0, 1)$. Consider

$$\psi_m(x) = \phi(x-1) + \frac{1}{2}\phi(x-2) + \cdots + \frac{1}{m}\phi(x-m)$$

Then $\{\psi_m\}$ is a Cauchy sequence in the suggested topology of $\mathcal{D}(\Omega)$, but the limit of $\{\psi_m\}$ does not have compact support, hence not in $\mathcal{D}(\Omega)$. In particular, $\mathcal{D}(\Omega)$ is not a Fréchet space with respect to above topology.

Now we define another locally convex topology τ on $\mathcal{D}(\Omega)$ which is complete, the fact that this τ is not metrizable is only minor inconvenience, as we will see.

Definition B.2.1. Let Ω be a non-empty open set in $\mathbb{R}^n$.

1. For every compact $K \subset \Omega$, τ_K denotes the Fréchet space topology of $\mathcal{D}(K)$;
2. β is the collection of all convex balanced sets $W \subset \mathcal{D}(\Omega)$ such that $\mathcal{D}(K) \cap W \in \tau_K$ for every compact $K \subset \Omega$;
3. τ is the collection of all unions of sets of the form $\phi + W$, with $\phi \in \mathcal{D}(\Omega)$ and $W \in \beta$.

Definition B.2.2 (distribution)**.** A linear functional on $\mathcal{D}(\Omega)$ which is continous with respect to the topology τ described in Definition B.2.1 is called a distribution in Ω.

Notation B.2.1. The space of all distributions in Ω is denoted by $\mathcal{D}'(\Omega)$.

Theorem B.2.1. If Λ is a linear functional on $\mathcal{D}(\Omega)$, the following two conditions are equivalent:

1. $\Lambda \in \mathcal{D}'(\Omega)$;
2. To every compact $K \subset \Omega$ corresponds a non-negative integer N and a constant $C < \infty$ such that the inequality
$$|\Lambda\phi| \leq C\|\phi\|_N$$
holds for every $\phi \in \mathcal{D}(K)$.

Example B.2.1. Suppose f is a locally integerable function in Ω, consider
$$\Lambda_f(\phi) = \int_\Omega f\phi \mathrm{d}x$$
where $\phi \in \mathcal{D}(\Omega)$. Note that $|\Lambda_f(\phi)| \leq (\int_K |f|)\|\phi\|_0$, then according to Theorem B.2.1 one has $\Lambda_f \in \mathcal{D}'(\Omega)$. It's customary to identify the distribution Λ_f with the function f and to say that such distributions are functions.

Example B.2.2. If μ is a positive measure on Ω with $\mu(K) < \infty$ for every compact $K \subset \Omega$, then
$$\Lambda_\mu(\phi) = \int_\Omega \phi \mathrm{d}\mu$$
defines a distribution Λ_μ in Ω, which is usually identified with μ.

Appendix C. Subharmonic functions

C.1. **Definition.** Let u is a Borel function on $\overline{B}(a,r)$ which is bounded above or below, consider the mean values of u over the ball or sphere

$$\mu_B(u;a,r) = \frac{1}{\alpha_m r^m}\int_{B(a,r)} u(x)\mathrm{d}\lambda(x)$$
$$\mu_S(u;a,r) = \frac{1}{\sigma_{m-1} r^{m-1}}\int_{S(a,r)} u(x)\mathrm{d}\sigma(x)$$

As $\mathrm{d}\lambda = \mathrm{d}r\mathrm{d}\sigma$, these mean values are related by

$$\begin{aligned}\mu_B(u;a,r) &= \frac{1}{\alpha_m r^m}\int_0^r \sigma_{m-1}t^{m-1}\mu_S(u;a,t)\mathrm{d}t \\ &= m\int_0^1 t^{m-1}\mu_S(u;a,rt)\mathrm{d}t\end{aligned}$$

Theorem C.1.1 (subharmonic)**.** Let $u\colon \Omega \to [-\infty,\infty]$ be an upper semi-continous function, the following various forms of mean value inequalities are equivalent:

1. $u(a) \le \mu_S(u;a,r)$ holds for all $\overline{B}(a,r) \subset \Omega$;
2. $u(a) \le \mu_B(u;a,r)$ holds for all $\overline{B}(a,r) \subset \Omega$;
3. For $a \in \Omega$, there exists a sequence (r_n) descends to 0 such that
$$u(a) \le \mu_B(u;a,r_n)$$
holds for arbitrary n.
4. For $a \in \Omega$, there exists a sequence (r_n) descends to 0 such that
$$u(a) \le \mu_S(u;a,r_n)$$
holds for arbitrary n.

A function u satisfying one of the above properties is said to be a subharmonic on Ω.

Notation C.1.1. The set of subharmonic functions over Ω is denoted by $\mathrm{Sh}(\Omega)$

C.2. **First properties.**

Theorem C.2.1 (maximum principle)**.** If u is subharmonic in Ω, then

$$\sup_\Omega u = \limsup_{\Omega\ni z\to\partial\Omega\cup\{\infty\}} u(z)$$

and $\sup_K u = \sup_{\partial K} u(z)$ for every compact subset $K \subset \Omega$.

Theorem C.2.2. For any decreasing sequence (u_k) of subharmonic functions, the limit $u = \lim u_k$ is subharmonic.

Theorem C.2.3. If Ω is connected and $u \in \mathrm{Sh}(\Omega)$, then either $u \equiv -\infty$ or $u \in L^1_{\mathrm{loc}}(\Omega)$.

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