ALGEBRAIC DE RHAM COHOMOLOGY

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Abstract. In this note, we will introduce the algebraic de Rham cohomology for a smooth complex variety X. To be explicit, we mainly concern about the following aspects.

On one hand, we will introduce the comparison theorem between the algebraic de Rham cohomology of X and the singular cohomology of its underlying manifold, and we will also introduce the local system valued version of comparison theorem.

On the other hand, we also introduce the so-called Hodge to de Rham spectral sequences, some results and consequences about its E_1-degenerations in both characteristic zero and positive characteristic.

Contents

1. Introduction

1.1. **Comparison theorems.** In the intersection between topology and geometry, there are some so-called comparison theorems, which explain the cohomology groups defined in different settings reflect the same information of the manifold in fact.

Let X be a smooth manifold. In the setting of topology, we may consider its singular cohomology $H^*_{sing}(X, \mathbb{C})$, which is given by the cohomology of singular cochain complex with $\mathbb{C}$ coefficients. On the other hand, in the setting of differential geometry, we can use the de Rham theory to define its de Rham cohomology $H^*_{dR}(X)$, which is defined by the cohomology of the following complex

$$0 \to \Gamma(X, \Omega^0_X) \xrightarrow{\mathrm{d}} \Gamma(X, \Omega^1_X) \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Gamma(X, \Omega^n_X) \to 0,$$

where $\Gamma(X, \Omega^k_X)$ is the $\mathbb{C}$-vector space of differential k-forms on X, and n is the dimension of X.

The classical comparison theorem says that the singular cohomology of X is isomorphic to its de Rham cohomology.

Theorem 1.1.1. Let X be a smooth manifold. Then there is the following isomorphism

$$H^*_{sing}(X, \mathbb{C}) \cong H^*_{dR}(X).$$

The proof of above comparison theorem is based on the sheaf theory. Given any sheaf $\mathcal{F}$ on a smooth manifold X, we may choose an injective resolution of $\mathcal{F} \to \mathcal{I}^\bullet$. Then the sheaf cohomology is defined by

$$H^*(X, \mathcal{F}) := H^*(\Gamma(X, \mathcal{I}^\bullet)).$$

In this setting, the classical Poincaré lemma can be rephrased as the following sequence of sheaves

$$0 \to \underline{\mathbb{C}} \to \Omega^0_X \xrightarrow{\mathrm{d}} \Omega^1_X \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Omega^n_X \to 0$$

is exact, and thus it gives a fine resolution of the constant sheaf $\underline{\mathbb{C}}$.

In other words, in the sheaf-theoretic setting, the de Rham cohomology is exactly the sheaf cohomology of the constant sheaf $\underline{\mathbb{C}}$, and similarly the singular cochain also gives[1] a resolution which can be used to compute the cohomology of constant sheaf $\underline{\mathbb{C}}$. Thus the sheaf cohomology builds a bridge which connects singular cohomology and de Rham cohomology.

In the algebraic setting, there is also an algebraic version of de Rham complex. Let X be a smooth complex variety (with Zariski topology). The algebraic de Rham complex $\Omega^\bullet_X$ is defined to be the complex of sheaves of regular differentials as follows

$$\mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega^1_X \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Omega^n_X \to 0.$$

[1] A good reference is [Wel80].

On the other hand, X has an underlying structure of complex manifold, denoted by X^{an}. So it's natural for us to conjecture that there is a comparison theorem, but in general

$$H^*_{sing}(X^{an}, \mathbb{C}) \ncong H^*(\Gamma(X, \Omega^\bullet_X)).$$

In order to obtain the "right" comparison theorem, we need to consider the hypercohomology of the complex instead of simply taking global sections and then taking its cohomology.

The first goal of this note is to introduce the following comparison theorem and its local system valued version.

Theorem 1.1.2 ([Gro66a])**.** Let X be a smooth complex variety and X^{an} be the underlying complex manifold. Then there is the following isomorphism

$$H^*(X^{an}, \mathbb{C}) \cong \mathbb{H}^*(X, \Omega^\bullet_X).$$

Theorem 1.1.3 ([Del70])**.** Let X be a smooth complex variety and X^{an} be the underlying complex manifold. Let $\mathcal{E}$ be a vector bundle on X equipped with a regular integrable connection ∇ and $\mathcal{L} = \mathcal{E}^{\nabla=0}$ be the local system of horizontal sections on the underlying complex manifold X^{an}. Then there is the following isomorphism

$$H^*(X^{an}, \mathcal{L}) \cong \mathbb{H}^*(X, \Omega^\bullet_X(\mathcal{E})).$$

1.2. **Hodge to de Rham spectral sequences.** In the setting of differential geometry, we don't concern about the cohomology of the sheaf of differential k-forms, since they're so-called fine sheaves and thus they don't have any cohomology but zero degree. But in the setting of complex geometry, things become more rigid and we also concern about the cohomology of sheaves of holomorphic p-forms, also denoted by Ω^p_X. The holomorphic Poincaré lemma implies that the following complex

$$0 \to \Omega^p_X \to \Omega^{p,0}_X \xrightarrow{\overline{\partial}} \Omega^{p,1}_X \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} \Omega^{p,n}_X \to 0$$

is a fine resolution of Ω^p_X, and thus the Dolbeault cohomology $H^{p,q}(X) := H^q(\Gamma(X, \Omega^{p,\bullet}_X))$ computes the sheaf cohomology of Ω^p_X.

One of the most important results in complex geometry is the following Hodge decomposition theorem, which relates the topology information and holomorphic information together.

Theorem 1.2.1 (Hodge)**.** Let (X, ω) be a compact Kähler manifold. Then there is a decomposition

$$H^k_{dR}(X) \cong \bigoplus_{p+q=k} H^{p,q}(X).$$

A natural equestion is to ask is there any analogy of Hodge decomposition in the setting of algebraic geometry. The answer is affirmative, and that's the E_1-degenration of Hodge to de Rham spectral spectral we're going to introduce.

Let X be a smooth complex variety of dimension n. The Hodge filteration on the algebraic de Rham complex $\Omega_X^\bullet$ is given by

$$\Omega_X^\bullet = F^0\Omega_X^\bullet \supset F^1\Omega_X^\bullet \supset \cdots \supset F^n\Omega_X^\bullet \supset \{0\},$$

where

$$F^p\Omega_X^\bullet \colon 0 \to \cdots \to 0 \to \Omega_X^p \to \cdots \to \Omega_X^n.$$

By the standard theory of spectral sequence, this filteration gives a spectral sequence, which is called the Hodge to de Rham spectral sequence.

Theorem 1.2.2 (E_1-degeneration)**.** Let X be a smooth projective complex variety. The Hodge to de Rham spectral sequence attached to algebraic de Rham complex

$$E_1^{p,q} = H^q(X, \Omega_X^p) \Longrightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet)$$

degenerates at E_1-page.

It's clear that the Hodge decomposition theorem implies the E_1-degeneration of Hodge to de Rham spectral sequence since every projective manifold is Kähler. However, for a long time no algebraic proof of E_1-degeneration was known until P. Deligne and L. Illusie's work in positive characteristic, and the standard degeneration arguments allow to deduce the degeneration of the Hodge to de Rham spectral sequence in characteristic zero.

Theorem 1.2.3 ([DI87])**.** Let k be an algebraically closed field with characteristic $p > 0$. Let X/k be a smooth proper variety which is $W_2(k)$-liftable and of dimension $< p$. Then the Hodge to de Rham spectral sequence degenerates attached to algebraic de Rham complex at E_1-page.

Part 1. Preliminaries

In this part we select some basic definitions and techniques which may be used later, it's independent of our main topics, so feel free to skip this part and return back when you're not familiar with some certain properties.

2. Spectral sequence

In this section, we always assume $\mathcal{C}$ is an abelian category.

Definition 2.1. A **double complex** $K^{\bullet\bullet}$ consists of objects $K^{p,q}\in\mathcal{C}$ for $(p,q)\in\mathbb{Z}^2$, together with two sorts of differentials

(1) The horizontal differential δ, which maps $K^{p,q}$ to $K^{p+1,q}$.
(2) The vertical differential d, which maps $K^{p,q}$ to $K^{p,q+1}$.

such that $\delta^2=\mathrm{d}^2=0$ and $\mathrm{d}\circ\delta=\delta\circ\mathrm{d}$.

Definition 2.2. A **morphism** $\phi\colon K^{\bullet\bullet}\to L^{\bullet\bullet}$ of double complexes consists of homomorphisms $\phi^{p,q}\colon K^{p,q}\to L^{p,q}$ which are compatible with differentials.

Definition 2.3. Let $K^{\bullet\bullet}$ be a double complex. Then the **total complex** $(K^\bullet,D)$ of $K^{\bullet\bullet}$ is defined by

$$K^n=\bigoplus_{p+q=n}K^{p,q},$$

where $D=\delta+(-1)^p\mathrm{d}$.

Definition 2.4. Let $K^{\bullet\bullet}$ be a double complex and $K^\bullet$ be its total complex. The **first filteration** $F_H^\bullet$ is given by the subcomplexes $F_H^p=\bigoplus_{i\geq p}K^{i,j}$, with degree n component $\bigoplus_{i\geq p}K^{i,n-i}$.

Definition 2.5. Let $K^{\bullet\bullet}$ be a double complex and $K^\bullet$ be its total complex. The **second filteration** $F_H^\bullet$ is given by the subcomplexes $F_V^p=\bigoplus_{j\geq p}K^{i,j}$, with degree n component $\bigoplus_{j\geq p}K^{n-j,j}$.

Let $(K^{\bullet\bullet},\delta,\mathrm{d})$ be a double complex and $(K^\bullet,D)$ be its total complex equipped with a filteration F. For any integer $r\geq 1$, we define a new filteration

$$\cdots\subseteq Z_r^{p+1}\subseteq Z_r^p\subseteq\dots,$$

where $Z_r^p=\{x\in F^p\mid Dx\in F^{p+r}\}$, and the $(p+q)$-component of Z_r^p is denoted by $Z_r^{p,q}$. The E_r-page of the **spectral sequence** is defined to be

$$E_r^{p,q}=\frac{Z_r^{p,q}}{F^{p+1}K^{p+q}+D\big(F^{p-r+1}K^{p+q-1}\big)}.$$

Now let's construct the differential

$$\mathrm{d}_r\colon E_r^{p,q}\to E_r^{p+r,q-r+1}.$$

Let $a\in E_r^{p,q}$ be represented by some $x\in F^p$ such that $Dx\in F^{p+r}$. Then $Dx\in F^{p+r}$ will be a representative of $\mathrm{d}_r(a)$. To see it's independent of the choice of x, it suffices to show for $x \in F^{p+1}K^{p+q}$, the class of Dx in $E_r^{p+r,q-r+1}$ is trivial. It's clear, since $Dx \in D(F^{p+1}K^{p+q})$, which is zero in $E_r^{p+r,q-r+1}$.

Remark 2.1. The begining terms of a spectral sequence are easy to understand, that is,

$$E_1^{p,q} = H^{p+q}(F^p/F^{p+1}),$$

and the differential $\mathrm{d}_1 \colon H^{p+q}(F^p/F^{p+1}) \to H^{p+q+1}(F^{p+1}/F^{p+2})$ is the connecting homomorphism in the exact sequence of complexes

$$0 \to F^{p+1}/F^{p+2} \to F^p/F^{p+2} \to F^p/F^{p+1} \to 0.$$

Proposition 2.1. The maps $\{\mathrm{d}_r\}$ satisfy $\mathrm{d}_r^2 = 0$, and the cohomology

$$\ker(\mathrm{d}_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}) / \operatorname{im}(\mathrm{d}_r \colon E_r^{p-r,q+r-1} \to E_r^{p,q})$$

identifies canonically with $E_{r+1}^{p,q}$.

Proof. It's clear that $\mathrm{d}_r^2 = 0$ as $D^2 = 0$. Now let's describe the group

$$A = \ker(\mathrm{d}_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}).$$

It consists of classes in $E_r^{p,q}$ represented by some $x \in F^pK^{p+q}$ such that Dx is of the form $y + Dz$, where $y \in F^{p+r+1}K^{p+r+q}$ and $z \in F^{p+1}K^{p+q}$.

As the class of z in $E_r^{p,q}$ is zero, the class of x is represented also by $x - z$, which satisfies $D(x - z) = y \in F^{p+r+1}$. This gives a well-defined map $f \colon A \to E_{r+1}^{p,q}$, mapping the class of x to the class of $x - z$, which is also surjective.

The kernel of f consists of the classes of $x \in F^pK^{p+q}$ such that $x - z$ is of the form $u + Dv$, where $u \in F^{p+1}K^{p+q}$ and $v \in F^{p-r}K^{p+q-1}$. In other words, one has

$$x - z - u = Dv.$$

Since the class of x is also represented by $x - z - u$, this shows $[x] \in E_r^{p,q}$ is equal to $\mathrm{d}_r([v])$, and thus the kernel of f is contained in the image of d_r. The same argument proves the inverse inclusion, which completes the proof. □

Proposition 2.2. If the filteration F of $K^\bullet$ satisfies that for every n there exist $m_n \leq q_n$ such that $K^n \cap F^{q_n} = 0$ and $K^n \subseteq F^{m_n}$, then the spectral sequence converges to the $H_D^*(K^\bullet)$, that is,

$$E_\infty^{p,q} = F^pH^{p+q}(K^\bullet).$$

Proof. See Proposition 1.2.2 in [Bry08]. □

Definition 2.6. A spectral sequence **degenerates** at E_r if $E_r = E_{r+1} = \cdots = E_\infty$.

Definition 2.7. Let $E_r^{\bullet\bullet}$ and $(E_r')^{\bullet\bullet}$ be two spectral sequences. A **morphism of spectral sequence** means for each r, there is a homomorphism $\phi_r \colon E_r^{p,q} \to (E')_r^{p,q}$ such that $\mathrm{d}_r \circ \phi_r = \phi_r \circ \mathrm{d}_r$

Proposition 2.3. Let $E_r^{\bullet\bullet}$ and $(E_r')^{\bullet\bullet}$ be convergent spectral sequences and $\phi_r\colon E_r^{p,q} \to (E')_r^{p,q}$ be a morphism of spectral sequences. If for some s, the morphism ϕ_s is an isomorphism, then ϕ_r is an isomorphism for all $r \geq s$, including $r = \infty$.

Proof. See Proposition 1.2.4 in [Bry08]. $\square$

3. Sheaf and its cohomology

Along this section, X denotes a topological space unless otherwise specified.

3.1. **Sheaves.**

3.1.1. *Sheaves and its morphisms.*

Definition 3.1.1. A **presheaf** of abelian group $\mathcal{F}$ on X consisting of the following data:

(1) For any open subset U of X, $\mathcal{F}(U)$ is an abelian group.
(2) If $V \subseteq U$ are two open subsets of X, then there is a group homomorphism $r_{UV}\colon \mathcal{F}(U) \to \mathcal{F}(V)$. Moreover, above data satisfy
    I $\mathcal{F}(\varnothing) = 0$.
    II $r_{UU} = \mathrm{id}$.
    III If $W \subseteq V \subseteq U$ are open subsets of X, then $r_{UW} = r_{VW} \circ r_{UV}$.

Moreover, $\mathcal{F}$ is called a sheaf if it satisfies the following extra conditions

IV Let $\{V_i\}_{i\in I}$ be an open covering of open subset $U \subseteq X$ and $s \in \mathcal{F}(U)$. If $s|_{V_i} := r_{UV_i}(s) = 0$ for all $i \in I$, then $s = 0$.
V Let $\{V_i\}_{i\in I}$ be an open covering of open subset $U \subseteq X$ and $s_i \in \mathcal{F}(V_i)$. If $s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j}$ for all $i, j \in I$, then there exists $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for all $i \in I$.

Example 3.1.1. For an abelian group G, the **constant presheaf** assign each open subset U the group G itself, but in general it's not a sheaf.

Definition 3.1.2. A **morphism** $\varphi\colon \mathcal{F} \to \mathcal{G}$ between presheaves consisting of the following data:

(1) For any open subset U of X, there is a group homomorphism $\varphi(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$.
(2) If $U \subseteq V$ are two open subsets of X, then the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow{\scriptstyle r_{UV}} & & \downarrow{\scriptstyle r_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V). \end{array}$$

Notation 3.1.1. For convenience, for $s \in \mathcal{F}(U)$, we often write $\varphi(s)$ instead of $\varphi(U)(s)$.

Remark 3.1.1. The morphisms between sheaves are defined as morphisms of presheaves.

Definition 3.1.3. A morphism of presheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is called an **isomorphism** if it has two-sided inverse, that is, there exists a morphism of presheaves $\psi\colon \mathcal{G} \to \mathcal{F}$ such that $\psi \circ \varphi = \mathrm{id}_{\mathcal{F}}$ and $\varphi \circ \psi = \mathrm{id}_{\mathcal{G}}$.

Remark 3.1.2. A morphism of presheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if for every open subset $U \subseteq X$, $\varphi(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism of abelian groups.

3.1.2. *Stalks.*

Definition 3.1.4. For a presheaf $\mathcal{F}$ and $p \in X$, the **stalk** at p is defined as

$$\mathcal{F}_p = \varinjlim_{p\in U} \mathcal{F}(U)$$

Remark 3.1.3. In order to avoid language of direct limit, we give a more useful but equivalent description of stalk: For $p \in U \cap V$, $s_U \in \mathcal{F}(U)$ and $s_V \in \mathcal{F}(V)$ are equivalent if there exists $p \in W \subseteq U \cap V$ such that $s_U|_W = s_V|_W$. An element $s_p \in \mathcal{F}_p$, which is called a germ, is an equivalence class $[s_U]$.

Notation 3.1.2.
(1) For $s \in \mathcal{F}(U)$ and $p \in U$, $s|_p$ denotes the equivalent class it gives.
(2) For $s_p \in \mathcal{F}_p$, $s \in \mathcal{F}(U)$ denotes the section such that $s|_p = s_p$.

Definition 3.1.5. Given a morphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$, it induces a **morphism of stalks** $\varphi_p\colon \mathcal{F}_p \to \mathcal{G}_p$ as follows:

$$\begin{aligned}\varphi_p\colon \mathcal{F}_p &\to \mathcal{G}_p\\ s_p &\mapsto \varphi(s)|_p.\end{aligned}$$

Remark 3.1.4. It's neccessary to check the φ_p is well-defined since there are different choices s such that $s|_p = s_p$.

Proposition 3.1.1. Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism between sheaves. Then φ is an isomorphism if and only if the induced map $\varphi_p\colon \mathcal{F}_p \to \mathcal{G}_p$ is an isomorphism for every $p \in X$.

Proof. It's clear if φ is an isomorphism between sheaves, then it induces an isomorphism between stalks. Conversely, it suffices to show $\varphi(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for every open subset $U \subseteq X$.

(1) Injectivity: For $s, s' \in \mathcal{F}(U)$ such that $\varphi(s) = \varphi(s')$, by passing to stalks one has $\varphi_p(s|_p) = \varphi_p(s'|_p)$ for every $p \in U$, and thus $s|_p = s'|_p$ since φ_p is an isomorphism. By definition of stalks there exists an open subset $V_p \subseteq U$ containing p such that s agrees with s' on V_p. Then it gives an open covering $\{V_p\}$ of U, and by axiom (IV) one has $s = s'$ on U.
(2) Surjectivity: For $t \in \mathcal{G}(U)$, by passing to stalks there exists $s_p \in \mathcal{F}_p$ such that $\varphi_p(s_p) = t|_p$ for every $p \in U$ since φ_p is surjective. By definition of stalks there exists an open subset $V_p \subseteq U$ containing p and $s \in \mathcal{F}(V_p)$ such that $\varphi(s) = t$ on V_p. This gives a collection of sections defined on an open covering $\{V_p\}$ of U, and by injectivity we proved above one has these sections agree with each other on the intersections. Then by axiom (V) there exists a section $s \in \mathcal{F}(U)$ such that $\varphi(s) = t$.

$\square$

3.1.3. *Sheafification.* In Example 3.1.1, we come across a presheaf that is not a sheaf. To obtain a sheaf from a presheaf, we require a process known as sheafification. One approach to defining sheafification is through its universal property.

Definition 3.1.6. Given a presheaf $\mathcal{F}$ there is a sheaf $\mathcal{F}^+$ and a morphism $\theta\colon \mathcal{F} \to \mathcal{F}^+$, called **sheafification** with the property that for any sheaf $\mathcal{G}$ and any morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ there is a unique morphism $\overline{\varphi}\colon \mathcal{F}^+ \to \mathcal{G}$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\theta"] & \mathcal{G} \\ \mathcal{F}^+ \arrow[ur, "\overline{\varphi}"] & \end{tikzcd}$$

The universal property shows that if the sheafification exists, then it's unique up to a unique isomorphism. One way to give an explicit construction of sheafification is to glue stalks together in a suitable way. Let $\mathcal{F}^+(U)$ be a set of functions

$$f\colon U \to \coprod_{p\in U} \mathcal{F}_p$$

such that $f(p) \in \mathcal{F}_p$ and for every $p \in U$ there is an open subset $V_p \subseteq U$ containing p and $t \in \mathcal{F}(V_p)$ such that $t|_q = f(q)$ for all $q \in V_p$.

Proposition 3.1.2. $\mathcal{F}^+$ is the sheafication of $\mathcal{F}$.

Proof. Firstly let's show $\mathcal{F}^+$ is a sheaf: It's clear $\mathcal{F}^+$ is a presheaf, so it suffices to check conditions (IV) and (V) in the definition. Let $U \subseteq X$ be an open subset and $\{V_i\}$ be an open covering of U.

(1) If $s \in \mathcal{F}^+(U)$ such that $s|_{V_i} = 0$ for all i, then s must be zero: It suffices to show $s(p) = 0$ for all $p \in U$. For any $p \in U$, then there exists an open subset V_i contains p, hence $s(p) = s|_{V_i}(p) = 0$.
(2) Suppose there exists a collection of sections $\{s_i \in \mathcal{F}^+(V_i)\}_{i\in I}$ such that

$$s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j}$$

holds for all $i, j \in I$. Now we construct $s \in \mathcal{F}^+(U)$ as follows: For $p \in U$ and V_i containing p, we define $s(p) = s_i(p)$. This is well-defined since s_i agree on the intersections, so it remains to show $s \in \mathcal{F}^+(U)$. It's clear $s(p) \in \mathcal{F}_p$. For $p \in U$, there exists V_i containing p, and thus there exists $W_i \subseteq V_i$ containing p and $t \in \mathcal{F}(W_i)$ such that $t|_q = s_i(q) = s(q)$ for all $q \in V_p$.

There is a canonical morphism $\theta\colon \mathcal{F} \to \mathcal{F}^+$ as follows: For open subset $U \subseteq X$, and $s \in \mathcal{F}(U)$, $\theta(s)$ is defined by

$$\begin{aligned} \theta(s)\colon U &\to \coprod_{p\in U} \mathcal{F}_p \\ p &\mapsto s|_p. \end{aligned}$$

Note that if $\mathcal{F}$ is a sheaf, the canonical morphism $\theta\colon \mathcal{F} \to \mathcal{F}^+$ is an isomorphism.

(1) Injectivity: If $s \in \mathcal{F}(U)$ such that $s|_p = 0$ for all $p \in U$, then there exists an open covering $\{V_i\}_{i\in I}$ of U such that $s|_{V_i} = 0$, by axiom (IV) of sheaf one has $s = 0$.

(2) Surjectivity: For $f \in \mathcal{F}^+(U)$ and $p \in U$, there exists $p \in V_p \subseteq U$ and $t \in \mathcal{F}(V_p)$ such that $f(p) = t|_p$ by construction of $\mathcal{F}^+$. Then glue these sections together to get our desired s such that $\theta(s) = f$.

Finally let's show $\mathcal{F}^+$ statisfies the universal property of sheafification. A morphism of presheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ induces a map on stalks

$$\varphi_p\colon \mathcal{F}_p \to \mathcal{G}_p.$$

For $f \in \mathcal{F}^+(U)$, the composite of f with the map

$$\coprod_{p\in U} \varphi_p\colon \coprod_{p\in U} \mathcal{F}_p \to \coprod_{p\in U} \mathcal{G}_p$$

gives a map $\widetilde{\varphi}(f)\colon U \to \coprod_{p\in U} \mathcal{G}_p$, and in fact $\widetilde{\varphi}(f) \in \mathcal{G}^+(U)$: For $p \in U$, $\widetilde{\varphi}(f)(p) \in \mathcal{G}_p$ since $f(p) \in \mathcal{F}_p$ and $\varphi_p\colon \mathcal{F}_p \to \mathcal{G}_p$. If for all $q \in V_p$ we have $t|_q = f(q)$, then

$$\widetilde{\varphi}(f)(q) = \varphi_q(f(q)) = \varphi_q(t|_q) = \varphi(t)|_q.$$

Since $\mathcal{G}$ is a sheaf, the canonical morphism $\theta'\colon \mathcal{G} \to \mathcal{G}^+$ is an isomorphism, so we can define $\overline{\varphi}\colon = \theta'^{-1} \circ \widetilde{\varphi}$. Now let's show $\varphi = \overline{\varphi} \circ \theta = \theta'^{-1} \circ \widetilde{\varphi} \circ \theta$. It's easy to show they coincide on each stalk since $\varphi_p = \theta_p'^{-1} \circ \widetilde{\varphi}_p \circ \theta_p$, and thus $\varphi = \overline{\varphi} \circ \theta$ by Proposition 3.1.1. Furthermore, uniqueness follows from the fact that $\overline{\varphi}_p$ is uniquely determined by φ_p. $\square$

Remark 3.1.5. From the construction, one can see the stalk of $\mathcal{F}^+$ at p is exactly $\mathcal{F}_p$.

Remark 3.1.6. The sheafification can be described in a more fancy language: Since we have sheaf of abelian groups on X as a category, denote it by $\underline{Ab}_X$, and presheaf is a full subcategory of $\underline{Ab}_X$, there is a natural inclusion functor ι from category of sheaf to category of presheaf. The sheafification is the adjoint functor of ι.

Example 3.1.2. For an abelian group G, the associated **constant sheaf** $\underline{G}$ is the sheafication of the constant presheaf. By the construction of sheafification, $\underline{G}$ can be explicitly expressed as

$$\underline{G}(U) = \{\text{locally constant function } f\colon U \to G\}$$

3.1.4. *Exact sequence of sheaf.* Given a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ between sheaves of abelian groups, there are the following presheaves

$$U \mapsto \ker \varphi(U)$$
$$U \mapsto \operatorname{im} \varphi(U)$$
$$U \mapsto \operatorname{coker} \varphi(U),$$

since $\varphi(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is a group homomorphism.

Proposition 3.1.3. The kernel of a morphism between sheaves is a sheaf.

Proof. Let $\{V_i\}_{i\in I}$ be an open covering of U.

(1) For $s \in \ker\varphi(U)$, if $s|_{V_i} = 0$, then $s = 0$ since s is also in $\mathcal{F}(U)$.
(2) If there exists $s_i \in \ker\varphi(V_i)$ such that $s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j}$, then they glue together to get $s \in \mathcal{F}(U)$. Note that

$$\varphi(U)(s)|_{V_i} = \varphi(V_i)(s|_{V_i}) = \varphi(V_i)(s_i) = 0$$

Then $s \in \ker\varphi(U)$.

□

But the image of morphism may not be a sheaf. Although we can prove the first requirement in the same way, the proof for the second requirement fails: If there exists $s_i \in \operatorname{im}\varphi(V_i)$, and we can glue them together to get a $s \in \mathcal{G}(U)$, but s may not be the image of some $t \in \mathcal{F}(U)$. The cokernel fails to be a sheaf for the same reason.

Definition 3.1.7. Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism between sheaves of abelian groups. Then the **image** and **cokernel** of φ is defined to be the sheafification of the following presheaves

$$U \mapsto \operatorname{im}\varphi(U)$$
$$U \mapsto \operatorname{coker}\varphi(U)$$

respectively.

Definition 3.1.8. For a sequence of sheaves:

$$\cdots \to \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^{i} \xrightarrow{\varphi^{i}} \mathcal{F}^{i+1} \to \dots$$

It's called **exact** at $\mathcal{F}^i$, if $\ker\varphi^i = \operatorname{im}\varphi^{i-1}$. If a sequence is exact at everywhere, then it's an exact sequence of sheaves.

Definition 3.1.9. An exact sequence of sheaves is called a **short exact sequence** if it's of the form

$$0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \to 0.$$

Proposition 3.1.4. Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism between sheaves of abelian groups. Then for any $p \in X$, one has

$$(\ker\varphi)_p = \ker\varphi_p$$
$$(\operatorname{im}\varphi)_p = \operatorname{im}\varphi_p.$$

Proof. For (1). It's clear $(\ker\varphi)_p \subseteq \ker\varphi_p$. Conversely, if $s_p \in \ker\varphi_p$, then $\varphi_p(s_p) = 0 \in \mathcal{G}_p$. In other words, there exists an open subset U containing p and $s \in \mathcal{F}(U)$ such that $s|_p = s_p$ and $\varphi(s)|_p = 0$, which implies there is another open subset V containing p such that $\varphi(s)|_V = 0$. Hence $\varphi(s|_V) = 0$, that is, $s|_V \in \ker\varphi(V)$. Thus $s_p = (s|_V)|_p \in (\ker\varphi)_p$.

For (2). It's clear $(\operatorname{im}\varphi)_p \subseteq \operatorname{im}\varphi_p$ since the sheafication doesn't change stalk. Conversely, if $s_p \in \operatorname{im}\varphi_p$, then there exists $t_p \in \mathcal{F}_p$ such that $\varphi_p(t_p) = s_p$. Suppose $t \in \mathcal{F}(U)$ is a section of some open subset U containing p such that $t|_p = t_p$. Then $\varphi(t)|_p = \varphi_p(t_p) = s_p$. In other words, s_p is in the stalk of the image presheaf at p, but the sheafication doesn't change stalk, so we have $s_p \in (\operatorname{im}\varphi)_p$. □

Corollary 3.1.1. The sequence of sheaves

$$\cdots \to \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^i \xrightarrow{\varphi^i} \mathcal{F}^{i+1} \to \cdots$$

is exact if and only if the sequence of abelian groups are exact

$$\cdots \to \mathcal{F}_p^{i-1} \xrightarrow{\varphi_p^{i-1}} \mathcal{F}_p^i \xrightarrow{\varphi_p^i} \mathcal{F}_p^{i+1} \to \cdots$$

for all $p \in X$.

Corollary 3.1.2. The the sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{G}$$

is exact if and only if for any open subset U, the following sequence of abelian groups is exact

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U).$$

Proof. For each $p \in U$, we have

$$\varphi_p \colon \mathcal{F}_p \to \mathcal{G}_p$$

is injective. That is $\ker\varphi_p = 0$. So we obtain $(\ker\varphi(U))_p = 0$ for all $p \in U$. But for a section $s \in \mathcal{F}(U)$ if we have $s|_p = 0$, then we must have $s = 0$, and thus $\ker\varphi(U) = 0$. □

Example 3.1.3. Let X be a complex manifold and $\mathcal{O}_X$ be its holomorphic function sheaf. Then

$$0 \to 2\pi\sqrt{-1}\underline{\mathbb{Z}} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

is an exact sequence of sheaves, which is called **exponential sequence**.

Proof. The difficulty is to show exponential map is surjective on stalks at $p \in X$. That is we need to construct logarithms of functions $g \in \mathcal{O}_X^*(U)$ for U, a neighborhood of p. We may choose U is simply-connected, then define

$$\log g(q) = \log g(p) + \int_{\gamma_q} \frac{\mathrm{d}g}{g}$$

for $q \in U$, where γ_q is a path from p to q in U, and the definition of $\log g(q)$ is independent of the choice of γ_q since U is simply-connected. □

Remark 3.1.7. In fact, U is simply-connected is crucial for constructing logarithm. If we consider $X = \mathbb{C}$ and $U = \mathbb{C}\setminus\{0\}$, then

$$\exp\colon \mathcal{O}_X(U) \to \mathcal{O}_X^*(U)$$

cannot be surjective.

3.1.5. *Direct image.*

Definition 3.1.10. Let $f\colon X \to Y$ be continuous map between topological spaces and $\mathcal{F}$ be a sheaf of abelian groups on X. The **direct image** of $\mathcal{F}$, denoted by $f_*\mathcal{F}$, is a sheaf on Y defined by

$$f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U)).$$

Proposition 3.1.5. $f_*\colon \underline{Ab}_X \to \underline{Ab}_Y$ is a left exact functor.

Proof. Given an exact sequence of sheaves on X

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''.$$

Then we need to show

$$0 \to f_*\mathcal{F}' \to f_*\mathcal{F} \to f_*\mathcal{F}''$$

is also an exact sequence of sheaves on Y. By Remark **??** it suffices to show that for any open subset $V \subseteq Y$, we have the following exact sequence

$$0 \to f_*\mathcal{F}'(V) \to f_*\mathcal{F}(V) \to f_*\mathcal{F}''(V),$$

and that's exactly

$$0 \to \mathcal{F}'(f^{-1}(V)) \to \mathcal{F}(f^{-1}(V)) \to \mathcal{F}''(f^{-1}(V)).$$

Since f is continuous, then $f^{-1}(V)$ is an open subset in X, and thus above sequence of abelian is exact since $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ is exact. $\square$

3.2. **Sheaf cohomology.**

3.2.1. *Derived functor formulation.* Let $\underline{Ab}_X$ denote the category of sheaves of abelian groups on X. In this section we will introduce sheaf cohomology by considering it as a derived functor.

Given an exact sequence of sheaf as follows

$$0 \to \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''.$$

By taking its section over open subset U, we obtain a sequence of abelian groups

$$0 \to \mathcal{F}'(U) \xrightarrow{\phi(U)} \mathcal{F}(U) \xrightarrow{\psi(U)} \mathcal{F}''(U).$$

Above sequence is not only exact at $\mathcal{F}'(U)$, but also is exact at $\mathcal{F}(U)$. In other words, the functor given by taking section over open subset is a left exact functor.

(1) Firstly let's show $\ker\psi(U) \supseteq \operatorname{im}\phi(U)$. For $s \in \mathcal{F}'(U)$, if we want to show $\psi \circ \phi(s) = 0$, it suffices to show $(\psi \circ \phi(s))|_p = 0$ for all $p \in U$ since $\mathcal{F}''$ is a sheaf. For any $p \in U$, by considering stalk at p we obtain an exact sequence of abelian groups

$$0 \to \mathcal{F}'_p \xrightarrow{\phi_p} \mathcal{F}_p \xrightarrow{\psi_p} \mathcal{F}''_p.$$

Then we obtain $\psi_p \circ \phi_p(s|_p) = 0$, which implies $(\psi \circ \phi(s))|_p = 0$.

(2) Conversely, Given $s \in \ker \psi(U)$, we have $s|_p \in \ker \psi_p$ for any $p \in U$. By exactness of stalks, there exists $t_p \in \mathcal{F}'_p$ such that $\phi_p(t_p) = s|_p$. Thus there exists an open subset V_i containing p and $t_i \in \mathcal{F}'(V_i)$ such that $\phi(t_i) = s|_{V_i}$. Now it suffices to show these t_i can be glued together to obtain $t \in \mathcal{F}(U)$, and since $\mathcal{F}$ is a sheaf, it suffices to check these t_i agree on intersections $V_i \cap V_j$. Note that $\phi(t_i - t_j|_{V_i \cap V_i}) = s|_{V_i \cap V_j} - s|_{V_i \cap V_j} = 0$, then these t_i agree on intersections since ϕ is injective.

In homological algebra, we always consider the derived functor of a left or right-exact functor. In particular, the functor of taking global section is a left exact functor, and its right derived functor defines the cohomology of a sheaf. Before we come into the definition of derived functor, firstly let's define the injective resolution of a sheaf.

Definition 3.2.1. A sheaf $\mathcal{I}$ is **injective** if $\mathrm{Hom}(-, \mathcal{I})$ is an exact functor.

Definition 3.2.2. Let $\mathcal{F}$ be a sheaf. An **injective resolution** of $\mathcal{F}$ is an exact sequence

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \dots$$

where $\mathcal{I}^i$ are injective for all i.

Theorem 3.2.1.

(1) Every sheaf admits an injective resolution.
(2) Let $\mathcal{F} \to \mathcal{I}^\bullet$ and $\mathcal{G} \to \mathcal{G}^\bullet$ are two resolutions and $\phi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then there exists a morphism $\widetilde{\phi}\colon \mathcal{I}^\bullet \to \mathcal{G}^\bullet$ which lifts ϕ, which is unique up to homotopy.

Proof. See Proposition 1.1.15 in [Bry08]. □

Theorem 3.2.2. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{Q} \to 0$ be an exact sequence of sheaves of abelian groups. Let $\mathcal{F} \to \mathcal{I}^\bullet$ and $\mathcal{Q} \to \mathcal{J}^\bullet$ be injective resolutions. Then there exists an injective resolution $\mathcal{G} \to \mathcal{K}^\bullet$ such that the following diagram commutes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{I}^0 \arrow[r] \arrow[d] & \mathcal{I}^1 \arrow[r] \arrow[d] & \dots \\
0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{K}^0 \arrow[r] \arrow[d] & \mathcal{K}^1 \arrow[r] \arrow[d] & \dots \\
0 \arrow[r] & \mathcal{Q} \arrow[r] \arrow[d] & \mathcal{J}^0 \arrow[r] \arrow[d] & \mathcal{J}^1 \arrow[r] \arrow[d] & \dots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with each sequence $0 \to \mathcal{I}^n \to \mathcal{K}^n \to \mathcal{J}^n \to 0$ is exact.

Proof. See Proposition 1.1.18 of [Bry08]. □

Definition 3.2.3. Let $\mathcal{F}$ be a sheaf of abelian groups. Then its **cohomology**

$$H^p(X,\mathcal{F}) := H^p(\Gamma(X,\mathcal{I}^\bullet)).$$

Remark 3.2.1. The Theorem 3.2.1 shows that the definition of sheaf cohomology is independent of the choice of injective resolution.

Example 3.2.1. The cohomology of zero degree consists of the global sections.

Example 3.2.2. If $\mathcal{F}$ is a injective sheaf, then $H^i(X,\mathcal{F}) = 0$ for all $i > 0$, since the sheaf cohomology of injective sheaf can be computed by using the following special injective resolution

$$0 \to \mathcal{F} \xrightarrow{\mathrm{id}} \mathcal{F} \to 0 \to 0 \to \dots$$

Theorem 3.2.3 (zig-zag)**.** If

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is a short sequence of sheaves, then there is an induced long exact sequence of abelian groups

$$0 \to H^0(X,\mathcal{F}) \to H^0(X,\mathcal{G}) \to H^0(X,\mathcal{H}) \to H^1(X,\mathcal{F}) \to H^1(X,\mathcal{G}) \to \dots$$

Proof. See Corollary 1.1.19 in [Bry08]. □

3.2.2. *Hypercohomology.* In homological algebra, the hypercohomology is a generalization of cohomology functor which takes as input not objects in abelian category but instead chain complexes of objects.

Definition 3.2.4. Let $\mathcal{F}^\bullet$ be a bounded below[2] complex of sheaves on X. The **hypercohomology** of the complex $\mathcal{F}^\bullet$ is defined by the total cohomology of the double complex $\Gamma(X,\mathcal{I}^{\bullet\bullet})$, where $\mathcal{I}^{p,\bullet}$ is an injective resolution of $\mathcal{F}^p$ for each p.

In fact, we can find the following injective resolution with better properties.

Proposition 3.2.1. Let $(\mathcal{F}^\bullet, \mathrm{d}_\mathcal{F})$ be a bounded below complex of sheaves on X. There exists a double complex $(\mathcal{I}^{\bullet\bullet}, \delta, \mathrm{d})$ with $\mathcal{I}^{p,q} = 0$ for $q < 0$, and a morphism of complexes $u\colon \mathcal{F}^\bullet \to (\mathcal{I}^{\bullet,0}, \mathrm{d})$ such that

(1) For each $p \in \mathbb{Z}$, the complex of sheaves $(\mathcal{I}^{p,\bullet}, \mathrm{d})$ is an injective resolution of $\mathcal{F}^p$.
(2) For each $p \in \mathbb{Z}$, the complex of sheaves $\mathrm{im}\{\delta\colon \mathcal{I}^{p-1,\bullet} \to \mathcal{I}^{p,\bullet}\} \subseteq \mathcal{I}^{p,\bullet}$ is an injective resolution of $\mathrm{im}\{\mathrm{d}_\mathcal{F}\colon \mathcal{F}^{p-1} \to \mathcal{F}^p\}$.
(3) For each $p \in \mathbb{Z}$, the complex of sheaves $\ker\{\delta\colon \mathcal{I}^{p,\bullet} \to \mathcal{I}^{p+1,\bullet}\} \subseteq \mathcal{I}^{p,\bullet}$ is an injective resolution of $\ker\{\mathrm{d}_\mathcal{F}\colon \mathcal{F}^p \to \mathcal{F}^{p+1}\}$.
(4) For each $p \in \mathbb{Z}$, the complex of sheaves $\mathcal{H}^{p,\bullet}(\mathcal{I}^{\bullet\bullet})$ (the horizontal cohomology) is an injective resolution of $\mathcal{H}^p(\mathcal{F}^\bullet)$.

[2] From now on, we always assume our complex of sheaves of abelian groups is bounded below for convenience.

Furthermore, the morphism between complexes of sheaves induces morphism of double complexes, which is unique up to homotopy.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & \mathcal{I}^{p-1,1} \arrow[r] \arrow[u] & \mathcal{I}^{p,1} \arrow[r] \arrow[u] & \mathcal{I}^{p+1,1} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & \mathcal{I}^{p-1,0} \arrow[r] \arrow[u] & \mathcal{I}^{p,0} \arrow[r] \arrow[u] & \mathcal{I}^{p+1,0} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & \mathcal{F}^{p-1} \arrow[r] \arrow[u] & \mathcal{F}^{p} \arrow[r] \arrow[u] & \mathcal{F}^{p+1} \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

Proof. Since $\mathcal{F}^\bullet$ is bounded from below, so we may choose k such that $\mathcal{F}^p = 0$ for $p < k$. Firstly we construct an injective resolution $\mathcal{H}^{p,\bullet}$ of $\mathcal{H}^p(\mathcal{F}^\bullet)$ for all p, and an injective resolution $\mathcal{R}^{\bullet,p}$ for $\mathrm{im}\{\mathrm{d}_{\mathcal{F}}\colon \mathcal{F}^{p-1} \to \mathcal{F}^p\}$. Then by Theorem 3.2.2 there is an injective resolution $\mathcal{S}^{p,\bullet}$ of $\ker\{\mathrm{d}_{\mathcal{F}}\colon \mathcal{F}^q \to \mathcal{F}^{q+1}\}$ such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{R}^{p,\bullet} \arrow[r] & \mathcal{S}^{p,\bullet} \arrow[r] & \mathcal{H}^{p,\bullet} \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{im}\{\mathrm{d}_{\mathcal{F}}\colon \mathcal{F}^{p-1} \to \mathcal{F}^p\} \arrow[r] \arrow[u] & \ker\{\mathrm{d}_{\mathcal{F}}\colon \mathcal{F}^q \to \mathcal{F}^{q+1}\} \arrow[r] \arrow[u] & \mathcal{H}^p(\mathcal{F}^\bullet) \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

Then by induction on p, we may use Theorem 3.2.2 again to construct an injective resolution $\mathcal{I}^{p,\bullet}$ of $\mathcal{F}^p$ such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{S}^{p,\bullet} \arrow[r] & \mathcal{I}^{p,\bullet} \arrow[r] & \mathcal{R}^{p+1,\bullet} \arrow[r] & 0 \\
0 \arrow[r] & \ker\{\mathrm{d}_{\mathcal{F}}\colon \mathcal{F}^q \to \mathcal{F}^{q+1}\} \arrow[r] \arrow[u] & \mathcal{F}^p \arrow[r] \arrow[u] & \mathrm{im}\{\mathrm{d}_{\mathcal{F}}\colon \mathcal{F}^{p} \to \mathcal{F}^{p+1}\} \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

The horizontal differential $\mathrm{d}\colon \mathcal{I}^{p,\bullet} \to \mathcal{I}^{p+1,\bullet}$ is defined to be the composition of the maps $\mathcal{I}^{p,\bullet} \to \mathcal{R}^{p+1,\bullet} \to \mathcal{S}^{p+1,\bullet} \to \mathcal{I}^{p+1,\bullet}$. This gives the desired double complex $\mathcal{I}^{\bullet\bullet}$. ☐

Theorem 3.2.4. Let $\mathcal{F}^\bullet$ be a bounded below complex of sheaves on X. Then there is a spectral sequence converging to the hypercohomology $\mathbb{H}^*(X, \mathcal{F}^\bullet)$ with $E_1^{p,q} = H^q(X, \mathcal{F}^p)$. The differential $\mathrm{d}_1\colon H^q(X, \mathcal{F}^p) \to H^q(X, \mathcal{F}^{p+1})$ is induced by the morphism of sheaves $\mathcal{F}^p \to \mathcal{F}^{p+1}$.

Proof. Let $\mathcal{I}^{\bullet\bullet}$ be an injective resolution of $\mathcal{F}^\bullet$ and consider the first filteration of the double complex $\Gamma(X, \mathcal{I}^{\bullet\bullet})$. Note that the E_1-page is the vertical cohomology, that is, $E_1^{p,q}$ is the degree q-cohomology of the complex $\Gamma(X, \mathcal{I}^{p,\bullet})$. Since $\mathcal{I}^{\bullet,p}$ is an injective resolution of the sheaf $\mathcal{F}^p$, we have $E_1^{p,q} = H^q(X, \mathcal{F}^p)$.

As the complex $\mathcal{F}^\bullet$ is bounded below, by Proposition 2.2 one has the spectral sequence converges to the total complex of $\Gamma(X, \mathcal{I}^{\bullet\bullet})$, that is, the hypercohomology $\mathbb{H}^*(X, \mathcal{F}^\bullet)$. $\square$

In fact, the hypercohomology generalize the usual sheaf cohomology, since for any sheaf $\mathcal{F}$, it gives a complex of sheaves $\mathcal{F}^\bullet[0]$, and compute its hypercohomology, which will recover the sheaf cohomology of $\mathcal{F}$.

Definition 3.2.5. For a sheaf $\mathcal{F}$, the **shifted complex** $\mathcal{F}^\bullet[n]$ is a sheaf of complex defined by

$$(\mathcal{F}^\bullet[n])^i = \begin{cases} \mathcal{F}, & i = n \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 3.2.1. Let $\mathcal{F}^\bullet$ be a complex of sheaves on X. If each sheaf in $\mathcal{F}^\bullet$ is acyclic, then there is an isomorphism between the hypercohomology $\mathbb{H}^*(X, \mathcal{F}^\bullet)$ and the cohomology groups of the complex

$$\cdots \to \Gamma(X, \mathcal{F}^{p-1}) \to \Gamma(X, \mathcal{F}^p) \to \dots$$

In particular, the hypercohomology is a generalization of the usual cohomology.

Proof. Note that the assumption means that the spectral sequence attached to the complex $\mathcal{F}^\bullet$ satisfies $E_1^{p,q} = 0$ for $q > 0$, and thus it degenerates at E_2-page. This shows $\mathbb{H}^p(X, \mathcal{F}^\bullet) = E_2^{p,0}$, which is the cohomology groups of the complex

$$\cdots \to \Gamma(X, \mathcal{F}^{p-1}) \to \Gamma(X, \mathcal{F}^p) \to \dots$$

as desired. $\square$

3.3. **Čech cohomology.**

3.3.1. *Čech cohomology of presheaf.* Let $\mathcal{F}$ be a presheaf on X and $\mathfrak{U} = \{U_\alpha\}_{\alpha \in J}$ be an open covering of X. For convenience, we use $U_{\alpha\beta}$ to denote the intersection $U_\alpha \cap U_\beta$, and similarly for $U_{\alpha\beta\gamma}$. Now consider the following complex

$$0 \to C^0(\mathfrak{U}, \mathcal{F}) \xrightarrow{\delta} C^1(\mathfrak{U}, \mathcal{F}) \xrightarrow{\delta} C^2(\mathfrak{U}, \mathcal{F}) \to \dots,$$

where $C^p(\mathfrak{U}, \mathcal{F}) = \prod \mathcal{F}(U_{\alpha_0 \dots \alpha_p})$, and the differential δ is defined by

$$\begin{aligned} \delta\colon C^p(\mathfrak{U}, \mathcal{F}) &\to C^{p+1}(\mathfrak{U}, \mathcal{F}) \\ \omega &\mapsto \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \dots \widehat{\alpha_i} \dots \alpha_{p+1}}. \end{aligned}$$

A routine computation shows that $(C^p(\mathfrak{U}, \mathcal{F}), \delta)$ forms a complex.

Definition 3.3.1. The **Čech cohomology** of presheaf $\mathcal{F}$ with repest to open covering $\mathfrak{U}$ is defined by the cohomology of the complex $(C^p(\mathfrak{U}, \mathcal{F}), \delta)$.

Note that the definition of Čech cohomology depends on the choice of the open coverings, so it's natural to ask what will happen if we consider different open coverings.

Lemma 3.3.1. Given $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ an open cover and $\mathfrak{B} = \{V_\beta\}_{\beta\in J}$ a refinement, if ϕ, ψ are two refinement maps $J \to I$, then there is a homotopy operator between $\phi^\#$ and $\psi^\#$.

Proof. Define $K\colon C^q(\mathfrak{U}, \mathcal{F}) \to C^{q-1}(\mathfrak{B}, \mathcal{F})$ by

$$(K\omega)(V_{\beta_0\dots\beta_{q-1}}) = \sum(-1)^i\omega(U_{\phi(\beta_0)\dots\phi(\beta_i)\psi(\beta_i)\dots\psi(\beta_{q-1})}).$$

Now we claim the following formula holds

$$\psi^\# - \phi^\# = \delta K + K\delta.$$

Indeed, take a cochain $\omega \in C^q(\mathfrak{U}, \mathcal{F})$, and an intersection of open covering $V_{\beta_0\dots\beta_q}$, then it's easy to see

$$\psi^\# - \phi^\#(\omega)(V_{\beta_0\dots\beta_q}) = \omega(U_{\psi(\beta_0)\dots\psi(\beta_q)}) - \omega(U_{\phi(\beta_0)\dots\phi(\beta_q)}).$$

Now let's compute $\delta K\omega$ as follows:

$$\begin{aligned}
\delta K(\omega)(V_{\beta_0\dots\beta_q}) &= \sum_i(-1)^iK\omega(V_{\beta_0\dots\widehat{\beta}_i\dots\beta_q}) \\
&= \underbrace{\sum_{i\leq j}(-1)^{i+j}\omega(U_{\phi(\beta_0)\dots\widehat{\phi(\beta_i)}\dots\phi(\beta_{j+1})\psi(\beta_{j+1})\dots\psi(\beta_q)})}_{\text{part I}} \\
&\quad + \underbrace{\sum_{i>j}(-1)^{i+j}\omega(U_{\phi(\beta_0)\dots\phi(\beta_j)\psi(\beta_j)\dots\widehat{\psi(\beta_j)}\dots\psi(\beta_q)})}_{\text{part II}}.
\end{aligned}$$

Similarly we have $K\delta\omega$ as follows

$$\begin{aligned}
K\delta\omega(V_{\beta_0\dots\beta_q}) &= \sum_j(-1)^j\delta\omega(U_{\phi(\beta_0)\dots\phi(\beta_j)\psi(\beta_j)\dots\psi(\beta_q)}) \\
&= \underbrace{\sum_{i<j}(-1)^{i+j}\omega(U_{\phi(\beta_0)\dots\widehat{\phi(\beta_i)}\dots\phi(\beta_j)\psi(\beta_j)\dots\psi(\beta_q)})}_{\text{part III}} \\
&\quad + \underbrace{\sum_{i>j}(-1)^{i+j}\omega(U_{\phi(\beta_0)\dots\phi(\beta_j)\psi(\beta_j)\dots\widehat{\psi(\beta_{i-1})}\dots\psi(\beta_q)})}_{\text{part IV}} \\
&\quad + \underbrace{\sum_j\omega(U_{\phi(\beta_0)\dots\widehat{\phi(\beta_j)}\psi(\beta_j)\dots\psi(\beta_q)})}_{\text{part V}}.
\end{aligned}$$

Note that part I cancels with part III, since if you fix i, you will find j-th terms of part I and part III are equal but differ a sign. Similarly you can find part II and part IV almost cancel each other, but

$$\text{part II} + \text{part IV} = \underbrace{\sum_j -\omega(U_{\phi(\beta_0)\dots\phi(\beta_j)\widehat{\psi(\beta_j)}\psi(\beta_{j+1})\dots\psi(\beta_q)})}_{\text{part VI}},$$

and it's clear to see that

$$\text{part V} + \text{part VI} = \omega(U_{\psi(\beta_0)\dots\psi(\beta_q)}) - \omega(U_{\phi(\beta_0)\dots\phi(\beta_q)})$$

as desired. This completes the proof. □

Thus for two different open coverings $\mathfrak{U}, \mathfrak{B}$ such that $\mathfrak{B}$ is a refinement of $\mathfrak{U}$, there is a natural homomorphism

$$f_{\mathfrak{U}\mathfrak{B}} \colon H^*(\mathfrak{U}, \mathcal{F}) \to H^*(\mathfrak{B}, \mathcal{F}).$$

Furthermore, if there are three open covering such that $\mathfrak{C}$ is a refinement of $\mathfrak{B}$, and $\mathfrak{B}$ is a refinement of $\mathfrak{U}$. then we have

$$f_{\mathfrak{U}\mathfrak{C}} = f_{\mathfrak{U}\mathfrak{B}} \circ f_{\mathfrak{B}\mathfrak{C}}.$$

If we give a partial order on set of all open coverings, that is, $\mathfrak{U} < \mathfrak{B}$ if and only if $\mathfrak{B}$ is a refinement of $\mathfrak{U}$, then $\{H^*(\mathfrak{U}, \mathcal{F}), f_{\mathfrak{U}\mathfrak{B}}\}$ is a direct system.

Definition 3.3.2. The direct limit of direct system $\{H^*(\mathfrak{U}, \mathcal{F}), f_{\mathfrak{U}\mathfrak{B}}\}$ is called **Čech cohomology** of X with values in the presheaf $\mathcal{F}$, that is,

$$\check{H}^*(X, \mathcal{F}) := \varinjlim_{\mathfrak{U}} H^*(\mathfrak{U}, \mathcal{F}).$$

3.3.2. *Comparison theorem for Čech cohomology.*

Proposition 3.3.1. Let $\mathcal{F}$ be a sheaf on X and $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open covering of X. For every intersection $U_{i_1\dots i_k}$, we always use j to denote the inclusion $U_{i_1\dots i_k} \to X$. Then

$$0 \to \mathcal{F} \to \prod_i j_*\mathcal{F}|_{U_i} \to \prod_{i,j} j_*\mathcal{F}|_{U_{i,j}} \to \dots$$

is a resolution of $\mathcal{F}$, which is called **Čech resolution**.

Proof. See Proposition 4.17 in [Voi02]. □

Definition 3.3.3. Let $\mathcal{F}$ be a sheaf on X and $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open covering of X. Then $\mathcal{F}$ is called to be **acyclic** on $\mathfrak{U}$, if $H^q(U_{i_0\dots i_p}, \mathcal{F}) = 0$ for all $q > 0$ and $i_0, \dots, i_p \in I$.

Theorem 3.3.1. Let $\mathcal{F}$ be a sheaf which is acyclic on the open covering $\mathfrak{U} = \{U_i\}_{i\in I}$ of X. Then there is an isomorphism

$$\check{H}^*(\mathfrak{U}, \mathcal{F}) \cong H^*(X, \mathcal{F}).$$

Proof. Let $\mathcal{I}^\bullet$ be a flabby resolution of $\mathcal{F}$. For each $p \geq 0$, let $0 \to \mathcal{I}^p \to \mathcal{I}^{p,\bullet}$ be the Čech resolution, which is also a flabby resolution, since $\mathcal{I}^p$ is flabby and flabby is stable under direct image. Then $H^*(X, \mathcal{F}) \cong \mathbb{H}^*(X, \mathcal{I}^\bullet)$ is given by the total cohomology of the double complex $\Gamma(X, \mathcal{I}^{\bullet\bullet})$.

By the assumption, the E_1-page of spectral sequence given by the second filteration is

$$E_1^{p,q} = \begin{cases} C^q(\mathfrak{U}, \mathcal{F}), & p = 0 \\ 0, & p > 0. \end{cases}$$

In particular, the spectral sequence degenerates at E_2-page and

$$H^*(X, \mathcal{F}) \cong \mathbb{H}^*(X, \mathcal{I}^\bullet) \cong \check{H}^*(\mathfrak{U}, \mathcal{F}).$$

□

3.3.3. *Comparison theorem for hypercohomology.*

Definition 3.3.4. Let $\mathcal{F}^\bullet$ be a complex of sheaves and $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open covering of X. The **Čech hypercohomology** $\check{\mathbb{H}}(\mathfrak{U}, \mathcal{F}^\bullet)$ is defined to be the total cohomology of the following double complex

$$\begin{array}{ccccccc}
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & C^p(\mathfrak{U}, \mathcal{F}^{q+1}) & \longrightarrow & C^{p+1}(\mathfrak{U}, \mathcal{F}^{q+1}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & C^p(\mathfrak{U}, \mathcal{F}^{q}) & \longrightarrow & C^{p+1}(\mathfrak{U}, \mathcal{F}^{q}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & &
\end{array}$$

Theorem 3.3.2. Let $\mathcal{F}^\bullet$ be a complex of sheaves such that each $\mathcal{F}^q$ is acyclic on the open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of X. Then there is an isomorphism

$$\check{\mathbb{H}}^*(\mathfrak{U}, \mathcal{F}^\bullet) \cong \mathbb{H}^*(X, \mathcal{F}^\bullet).$$

4. Algebraic and analytic geometry

4.1. Algebraic and analytic variety.

Definition 4.1.1. A **complex algebraic variety**[3] is a topological space X together with a sheaf $\mathcal{O}_X$ such that

(1) There exists a finite open covering $\{U_i\}_{i\in I}$ of X such that each U_i together with $\mathcal{O}_X|_{U_i}$ is isomorphic to some affine variety over $\mathbb{C}$ with its sheaf of regular functions.
(2) The diagonal Δ of $X \times X$ is closed in $X \times X$.

Definition 4.1.2. A **complex analytic variety**[4] is a topological space X together with a sheaf $\mathcal{O}_X$ such that

(1) There exists an open covering $\{U_i\}_{i\in I}$ of X such that each U_i together with $\mathcal{O}_X|_{U_i}$ is isomorphic to some analytic subset[5] in $\mathbb{C}^n$ equipped with sheaf of holomorphic functions.
(2) The topology of X is Hausdorff.

Definition 4.1.3. Let X be a complex algebraic variety.

(1) A point $x \in X$ is called **non-singular** if $\mathcal{O}_{X,x}$ is a regular local ring.
(2) X is called **smooth** if every point in X is non-singular.

4.2. Algebraic and analytic coherent sheaves.

4.2.1. *Definitions.* Let X be a topological space and $\mathcal{A}$ a sheaf of rings on X.

Definition 4.2.1. A sheaf of $\mathcal{A}$-modules $\mathcal{F}$ is said to be **coherent** if

(1) $\mathcal{F}$ is of finite type.
(2) If $s_1, \ldots, s_p$ are sections of $\mathcal{F}$ over an open subset $U \subseteq X$, the sheaf of relations between the s_i is of finite type.

Proposition 4.2.1. Every coherent sheaves is locally isomorphic to the cokernel of a homomorphism $\phi\colon \mathcal{A}^q \to \mathcal{A}^p$.

Proof. See Proposition 2 in [Ser55], n°12. □

Definition 4.2.2. Let X be a complex algebraic variety. The sheaf of $\mathcal{O}_X$-modules is called an **algebraic sheaf**.

Definition 4.2.3. Let X be a complex analytic variety. The sheaf of $\mathcal{O}_X$-modules is called an **analytic sheaf**.

Definition 4.2.4. Let X be a complex algebraic variety. An algebraic sheaf is called **coherent** if it's a coherent sheaf of $\mathcal{O}_X$-modules.

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[3] in the sense of [Ser55], n°34.

[4] in the sense of [Ser56], n°2.

[5] A subset U of $\mathbb{C}^n$ is analytic if for each $x \in U$, there are functions $f_1, \ldots, f_k$, holomorphic in a neighborhood of W of x, such that $U \cap W$ is given by the zero locus of $f_i(z) = 0$, where $i = 1, \ldots, k$.

Definition 4.2.5. Let X be a complex analytic variety. An analytic sheaf is called **coherent** if it's a coherent sheaf of $\mathcal{O}_X$-modules.

Theorem 4.2.1 ([Car53])**.** Let X be a Stein manifold[6] and $\mathcal{F}$ be an coherent analytic sheaf on X. Then

$$H^q(X, \mathcal{F}) = 0$$

for all $q > 0$.

Theorem 4.2.2 ([Ser55])**.** Let X be an affine variety and $\mathcal{F}$ be an coherent algebraic sheaf on X. Then

$$H^q(X, \mathcal{F}) = 0$$

for all $q > 0$.

4.2.2. *The analytic variety associated to an algebraic variety.* Let $(X, \mathcal{O}_X)$ be a complex algebraic variety equipped with Zariski topology. Then there is a complex analytic variety structure on X, and X together with this complex analytic structure is denoted by $(X^{an}, \mathcal{O}_{X^{an}})$.

Example 4.2.1. If X is a smooth complex algebraic variety, then X^{an} is a complex manifold.

Example 4.2.2. If X is the affine space of dimension n, then $X^{an} = \mathbb{C}^n$.

Example 4.2.3. If X is a smooth affine variety, then X^{an} is a Stein manifold.

Definition 4.2.6. Let $\mathcal{F}$ be an algebraic sheaf on X. The **analytic sheaf associated to** $\mathcal{F}$ is defined by

$$\mathcal{F}^{an} := \epsilon^*\mathcal{F} = \epsilon^{-1}\mathcal{F} \otimes_{\epsilon^{-1}\mathcal{O}_X} \mathcal{O}_{X^{an}},$$

where $\epsilon\colon X^{an} \to X$ is the identity map, which is continuous.

Example 4.2.4. $\mathcal{O}_X^{an} = \mathcal{O}_{X^{an}}$.

Proposition 4.2.2.

(1) The functor $\mathcal{F}^{an}$ is an exact functor.
(2) If $\mathcal{F}$ is a coherent algebraic sheaf, then $\mathcal{F}^{an}$ is a coherent analytic sheaf.

Proof. See Proposition 10 in [Ser56], n°9. □

4.2.3. *Homomorphism induced on cohomology.* Let X be an complex algebraic variety and $\epsilon\colon X^{an} \to X$ be the identity map. Let $\mathcal{F}$ be an algebraic sheaf on X. If U is a Zariski open subset of X, and s is a section of $\mathcal{F}$ over U, one can consider s as a section of s' of $\epsilon^{-1}\mathcal{F}$ over the open subset U^{an} of X^{an}, and $\alpha(s') = s' \otimes 1$ is a section of $\mathcal{F}^{an} = \epsilon^{-1}\mathcal{F} \otimes \mathcal{O}_{X^{an}}$. The map $s \mapsto \alpha(s')$ is a homomorphism

$$\epsilon\colon \Gamma(U, \mathcal{F}) \to \Gamma(U^{an}, \mathcal{F}^{an}).$$

[6] A complex manifold is called Stein if it can be embedded into some $\mathbb{C}^N$.

Now let $\mathfrak{U} = \{U_i\}_{i\in I}$ be a Zariski open covering of X and thus $\{U_i^{an}\}$ forms an open covering of X^{an}, which we denote by $\mathfrak{U}^{an}$.

For all induces $i_0, \ldots, i_q \in I$, there is a canonical homomorphism

$$\epsilon\colon \Gamma(U_{i_0} \cap \ldots U_{i_q}, \mathcal{F}) \to \Gamma(U_{i_0}^{an} \cap \ldots U_{i_q}^{an}, \mathcal{F}^{an}),$$

and hence a homomorphism

$$\epsilon\colon C^*(\mathfrak{U}, \mathcal{F}) \to C^*(\mathfrak{U}^{an}, \mathcal{F}^{an}).$$

This homomorphism commutes with the coboundary δ, and so defines, by passing to cohomology, the homomorphisms

$$\epsilon\colon H^*(\mathfrak{U}, \mathcal{F}) \to H^*(\mathfrak{U}^{an}, \mathcal{F}^{an}).$$

Finally, by passing to the direct limit over $\mathfrak{U}$, one obtains the homomorphisms induced on cohomology groups

$$\epsilon\colon H^*(X, \mathcal{F}) \to H^*(X^{an}, \mathcal{F}^{an}). \tag{4.1}$$

Remark 4.2.1. These homomorphisms also induce homomorphisms between hypercohomology groups of complexes of sheaves.

4.2.4. *GAGA principle.*

Theorem 4.2.3 ([Ser56])**.** Let X be a projective variety. For any coherent algebraic sheaf on X, the homomorphism in (4.1)

$$\epsilon\colon H^*(X, \mathcal{F}) \to H^*(X^{an}, \mathcal{F}^{an})$$

is an isomorphism.

Theorem 4.2.4 ([Ser56])**.** Let X be a projective variety. If $\mathcal{F}$ and $\mathcal{G}$ are two coherent algebraic sheaves on X, every analytic homomorphism of $\mathcal{F}^{an}$ to $\mathcal{G}^{an}$ comes from a unique algebraic homomorphism of $\mathcal{F}$ to $\mathcal{G}$.

Theorem 4.2.5 ([Ser56])**.** Let X be a projective variety. For every coherent analytic sheaf $\mathcal{G}$ on X^{an}, there exists a coherent algebraic sheaf on X such that $\mathcal{F}^{an}$ is isomorphic to $\mathcal{G}$. Moreover, this property determines $\mathcal{F}$ up to isomorphisms.

4.3. **Algebraic de Rham complex.**

4.3.1. *Kähler differentials.* Let R be a $\mathbb{C}$-algebra.

Definition 4.3.1. The **module of Kähler differentials** of R over $\mathbb{C}$ is a R-module $\Omega^1_{R/\mathbb{C}}$, together with a derivation $\mathrm{d}\colon R \to \Omega^1_{R/\mathbb{C}}$ satisfies the following universal property: For any R-module M and any derivation $\mathrm{d}'\colon R \to M$, there exists a unique R-module morphism $f\colon \Omega^1_{R/\mathbb{C}} \to M$ such that $\mathrm{d}' = f \circ \mathrm{d}$.

Remark 4.3.1. The module of Kähler differentials of R over $\mathbb{C}$ can be constructed as free R-module generated by the symbol $\{\mathrm{d}f \mid f \in R\}$, and to divide out by the submodule generated by all expression of the form

(1) $\mathrm{d}c$ for all $c \in \mathbb{C}$.

(2) $\mathrm{d}(cf+g) = c\mathrm{d}f + \mathrm{d}g$ for all $c \in \mathbb{C}, f, g \in R$.
(3) $\mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f$ for all $f, g \in R$.

Example 4.3.1. If $R = \mathbb{C}[x_1, \dots, x_n]$, then

$$\Omega^1_{R/\mathbb{C}} \cong \bigoplus_{i=1}^n R\mathrm{d}x_i.$$

Proposition 4.3.1. Let $R \to S$ be a surjective morphism of $\mathbb{C}$-algebras with kernel I. Then there is an exact sequence of S-modules

$$I/I^2 \to S \otimes_R \Omega^1_{R/\mathbb{C}} \to \Omega^1_{S/\mathbb{C}} \to 0,$$

where $[f] \in I/I^2$ maps to $1 \otimes \mathrm{d}f \in S \otimes_R \Omega^1_{R/\mathbb{C}}$.

Proof. See Theorem 25.2 in [Mat86]. □

Corollary 4.3.1. Let $R = \mathbb{C}[x_1, \dots, x_n]$ and $S = R/I$, where $I = \langle f_1, \dots, f_s \rangle$. Then there is an exact sequence

$$I/I^2 \to S^n \to \Omega^1_{S/\mathbb{C}} \to 0$$

4.3.2. *Sheaf of Kähler differentials.* Let X be a complex algebraic variety of dimension n.

Definition 4.3.2. The **cotangent sheaf** Ω^1_X is the sheaf of $\mathcal{O}_X$-modules defined by

$$\Omega^1_X(U) := \Omega^1_{\mathcal{O}_X(U)/\mathbb{C}}$$

on affine open subsets U.

Definition 4.3.3. The **sheaf of k-forms** Ω^k_X is the sheaf of $\mathcal{O}_X$-modules defined by the wedge product[7] $\bigwedge^k \Omega^1_X$.

Definition 4.3.4. The derivation $\mathrm{d}\colon \mathcal{O}_X \to \Omega^1_X$ extends to a complex

$$\mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega^1_X \xrightarrow{\mathrm{d}} \Omega^2_X \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Omega^n_X,$$

which is called the **algebraic de Rham complex**.

Theorem 4.3.1 ([Har75])**.** Let X be the affine space of dimension n. Then

$$0 \to \mathbb{C} \to \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \Omega^1_{X/\mathbb{C}}) \to \cdots \to \Gamma(X, \Omega^n_{X/\mathbb{C}}) \to 0$$

is an exact sequence of $\mathbb{C}$-vector spaces.

Theorem 4.3.2. The algebraic variety X is smooth if and only if Ω^1_X is locally free of rank n.

Proof. See Theorem 8.15 in ChapterII of [Har77]. □

[7] For any sheaf $\mathcal{F}$, the wedge product $\bigwedge^k \mathcal{F}$ is the sheaf associated to the presheaf which to each open set U assigns the $\bigwedge^k \mathcal{F}(U)$.

5. Local system and integrable connection

In this section we always assume X is a complex manifold and $\mathcal{A}_X^p$ is the locally free sheaf of smooth p-forms.

Definition 5.1. A sheaf $\mathcal{V}$ on X is called a locally constant sheaf of rank r valued in $\mathbb{C}$, if for each point $x \in X$, there is an open subset U containing x such that $\mathcal{V}|_U$ is constant sheaf $\underline{\mathbb{C}^r}$.

Remark 5.1. In other words, there exists an open covering $\{U_\alpha\}$ such that $\mathcal{V}|_{U_\alpha}$ is isomorphic to constant sheaf $\underline{\mathbb{C}^r}$. Then the local system $\mathcal{V}$ is completely determined by the transition functions $g_{\alpha\beta}\colon U_{\alpha\beta} \to \mathrm{GL}_n(\mathbb{C})$, which are locally constant functions.

Definition 5.2. Let $\mathcal{E}$ be a locally free sheaf on X. A **connection** is a $\mathbb{C}$-linear map

$$\nabla\colon \mathcal{E} \to \mathcal{A}_X^1 \otimes \mathcal{E}$$

satisfying the following condition

$$\nabla(\varphi \otimes e) = \mathrm{d}\varphi \otimes e + \varphi \nabla e$$

for all sections e of $\mathcal{E}$ and φ of $\mathcal{O}_X$.

Remark 5.2. The definition of ∇ extends to $\nabla\colon \mathcal{A}_X^p \otimes \mathcal{E} \to \mathcal{A}_X^{p+1} \otimes \mathcal{E}$ by defining

$$\nabla(\omega \otimes e) = \mathrm{d}\omega \otimes e + (-1)^p \omega \wedge \nabla e$$

for all sections ω of $\mathcal{A}_X^p$ and sections e of $\mathcal{E}$.

Remark 5.3. Let $\{e_\alpha\}$ be a local frame of $\mathcal{E}$. For any section $s = s^\alpha e_\alpha$ of $\mathcal{E}$, one has

$$\nabla(s^\alpha e_\alpha) = \mathrm{d}s^\alpha e_\alpha + s^\alpha \nabla e_\alpha.$$

Thus the connection ∇ is completely determined by

$$\nabla e_\alpha = \omega_\alpha^\beta e_\beta,$$

where ω_α^β are 1-forms, which forms a (smooth) 1-form valued matrix ω.

Definition 5.3. A connection ∇ is **integrable** if its curvature $\nabla^2\colon \mathcal{E} \to \mathcal{A}_X^2 \otimes \mathcal{E}$ vanishes.

Remark 5.4. Let $\{e_\alpha\}$ be a local frame of $\mathcal{E}$. For any section $s = s^\alpha e_\alpha$ of $\mathcal{E}$, one has

$$\begin{aligned}
\nabla^2(s^\alpha e_\alpha) &= \nabla(\mathrm{d}s^\alpha \otimes e_\alpha + s^\alpha \omega_\alpha^\beta \otimes e_\beta) \\
&= -\mathrm{d}s^\alpha \wedge \omega_\alpha^\beta \otimes e_\beta + \mathrm{d}(s^\alpha \omega_\alpha^\beta) \otimes e_\beta - s^\alpha \omega_\alpha^\beta \wedge \omega_\beta^\gamma \otimes e_\gamma \\
&= s^\alpha (\mathrm{d}\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta) \otimes e_\beta, \\
\nabla^2(e_\alpha) &= \nabla(\omega_\alpha^\beta \otimes e_\beta) \\
&= \mathrm{d}\omega_\alpha^\beta \otimes e_\beta - \omega_\alpha^\beta \wedge \nabla e_\beta \\
&= \mathrm{d}\omega_\alpha^\beta \otimes e_\beta - \omega_\alpha^\beta \wedge \omega_\beta^\gamma \otimes e_\gamma \\
&= (\mathrm{d}\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta) \otimes e_\beta.
\end{aligned}$$

This shows ∇^2 is a global section of $\mathcal{A}_X^2 \otimes \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E})$, which is locally given by $\mathrm{d}\omega - \omega \wedge \omega$.

Definition 5.4. A locally free sheaf together with an integrable connection is called a **flat bundle**.

Proposition 5.1. Let ∇ be a integrable connection on locally free sheaf $\mathcal{E}$ on X. Then the horizontal section $\mathcal{E}^{\nabla=0}$ is a local system.

Proposition 5.2. Let $\mathcal{L}$ be a local system on X. Then the locally free sheaf $\mathcal{E} := \mathcal{O}_X \otimes \mathcal{L}$ together with canonical connection $\nabla_{\mathrm{can}}(f \otimes s) := \mathrm{d}f \otimes s$ is a flat bundle.

Theorem 5.1. The functor $(\mathcal{E}, \nabla) \mapsto \mathcal{E}^{\nabla=0}$ is an equivalence between category of flat bundles and the category of the complex local system with quasi-inverse $\mathcal{L} \mapsto (\mathcal{O}_X \otimes \mathcal{L}, \nabla_{\mathrm{can}})$.

Proposition 5.3. Let $\mathcal{L}$ be a local system on X. Then

$$H^*(X, \mathcal{L}) \cong \mathbb{H}^*(X, \mathcal{A}_X^\bullet \otimes \mathcal{L}).$$

Proof. Note the following complex of sheaves

$$0 \to \mathcal{L} \to \mathcal{A}_X^\bullet \otimes \mathcal{L}$$

gives a resolution of $\mathcal{L}$ by coherent sheaves. $\square$

Part 2. Comparison theorems

6. Grothendieck's version

6.1. **Introduction.** In this section, we will prove the following comparison theorem.

Theorem 6.1.1 ([Gro66a])**.** Let X be a smooth complex variety and X^{an} be the underlying complex manifold. Then there is the following isomorphism

$$H^*(X^{an}, \mathbb{C}) \cong \mathbb{H}^*(X, \Omega_X^\bullet),$$

where $\Omega_X^\bullet$ is the algebraic de Rham complex.

The following is the sketch of the proof. Firstly we use Čech cohomology arguments to reduce to the case X is a smooth affine variety, and then we embed X into some smooth projective variety Y by $j\colon X \to Y$, since for smooth projective variety, there is GAGA principle (Theorem 4.2.3) which relates algebraic geometry and analytic geometry closely.

To be precise, let X be an smooth affine variety. Then its projectivization $\overline{X}$ gives a projective variety, but it may admit some singularities. By Hironaka's resolution of singularities ([Hir64]), there is a surjective regular map $\pi\colon Y \to \overline{X}$, where Y is a smooth projective variety such that $\pi^{-1}(\overline{X} - X)$ is a simple normal crossing divisor[8] $D \subset Y$, and the restriction $\pi|_{Y-D}$ is an isomorphism.

The proof of the affine case is divided into the following three steps.

(1) Firstly, we establish the following two isomorphisms.

Theorem 6.1.2. There is an isomorphism between hypercohomology groups

$$\mathbb{H}^*(Y, j_*\Omega_X^\bullet) \cong \mathbb{H}^*(X, \Omega_X^\bullet).$$

Proof. Let $\mathfrak{V} = \{V_i\}_{i\in I}$ be an affine open covering of Y. As j is an affine morphism, the direct image $j_*\Omega_X^\bullet$ is a complex of coherent sheaves on Y, and thus by Serre's theorem (Theorem 4.2.2) and Theorem 3.3.2, one has

$$\mathbb{H}^*(Y, j_*\Omega_X^\bullet) = \check{H}^*(\mathfrak{V}, j_*\Omega_X^\bullet).$$

On the other hand, $\mathfrak{U} = \{V_i \cap X\}_{i\in I}$ is also an affine open covering of X as X is affine and the intersection of affine varieties is still affine. By the same argument one has

$$\mathbb{H}^*(X, \Omega_X^\bullet) = \check{H}(\mathfrak{U}, \Omega_X^\bullet).$$

This completes the proof, since by definition one has

$$\check{H}^*(\mathfrak{V}, j_*\Omega_X^\bullet) = \check{H}(\mathfrak{U}, \Omega_X^\bullet).$$

□

[8] A divisor D is called a **simple normal crossing divisor**, if locally there exists a coordinate $\{z_1, \dots, z_n\}$ on X such that D is defined by the equation $z_1 \dots z_r = 0$ for an integer r which naturally depends on the considered open set.

Theorem 6.1.3.

$$\mathbb{H}^*(Y^{an}, j_*^{an}\Omega^\bullet_{X^{an}}) \cong H^*(X^{an}, \mathbb{C}).$$

Proof. Since there is Cartan's theorem (4.2.1) which is the analytic analogy of Serre's theorem, so by the same argument as above we can prove

$$\mathbb{H}^*(Y^{an}, j_*^{an}\Omega^\bullet_{X^{an}}) \cong \mathbb{H}^*(X^{an}, \Omega^\bullet_{X^{an}}).$$

On the other hand, by holomorphic Poincaré lemma $\Omega^\bullet_{X^{an}}$ gives an resolution of constant sheaf $\underline{\mathbb{C}}$, and thus $\mathbb{H}^*(X^{an}, \Omega^\bullet_{X^{an}}) \cong H^*(X^{an}, \mathbb{C})$. $\square$

(2) Secondly, by using direct limit arguments and GAGA principle to prove the following isomorphism.

Theorem 6.1.4.

$$\mathbb{H}^*(Y^{an}, j_*^{m}\Omega^\bullet_{X^{an}}) \cong \mathbb{H}^*(Y, j_*\Omega^\bullet_X),$$

where $j_*^m\Omega^\bullet_{X^{an}} := (j_*\Omega^\bullet_X)^{an}$ is exactly the sheaf of sections of $\Omega^\bullet_{X^{an}}$ that are meromorphic along D.

Proof. For each p, one has $j_*\Omega^p_X = \varinjlim_n \Omega^p_Y(nD)$, where $\Omega^p_Y(nD)$ is the sheaves of p-forms on Y which are regular on X and admit poles along D with order $\leq n$. Similarly one has $j_*^m\Omega^p_{X^{an}} = \varinjlim_n \Omega^p_{Y^{an}}(nD)$.

On the other hand, direct limit commutes with cohomology (Proposition 2.9 in ChapterIII of [Har77]). Then for any q, one has

$$\begin{aligned} H^q(Y^{an}, j_*^m\Omega^p_{X^{an}}) &\cong H^q(Y^{an}, \varinjlim_n \Omega^p_{Y^{an}}(nD)) \\ &\cong H^q(Y, \varinjlim_n \Omega^p_Y(nD)) \\ &\cong H^q(Y, j_*\Omega^p_X). \end{aligned}$$

This gives an isomorphism between E_1-page of spectral sequences, which is compatible with the differentials in the spectral sequence, and thus it gives the isomorphism between E_∞-page (Proposition 2.3), that is,

$$\mathbb{H}^*(Y^{an}, j_*^m\Omega^\bullet_{X^{an}}) \cong \mathbb{H}^*(Y, j_*\Omega^\bullet_X).$$

$\square$

(3) The key step is to prove the following isomorphism

Theorem 6.1.5.

$$\mathbb{H}^*(Y^{an}, j_*^m\Omega^\bullet_{X^{an}}) \cong \mathbb{H}^*(Y^{an}, j_*^{an}\Omega^\bullet_{X^{an}}).$$

In a summary, we prove the following diagram.

$$\begin{tikzcd}
H^*(Y, j_*\Omega_X^\bullet) \arrow[d, "6.1.4"'] & & \mathbb{H}^*(X, \Omega_X^\bullet) \arrow[ll, "6.1.2"'] \arrow[dd, dashed, no head] \\
\mathbb{H}^*(Y^{an}, j_*^m\Omega_{X^{an}}^\bullet) \arrow[d, "6.3.2"'] & & \\
\mathbb{H}^*(Y^{an}, j_*^{an}\Omega_{X^{an}}^\bullet) \arrow[rr, "6.1.3"] & & H^*(X^{an}, \mathbb{C}).
\end{tikzcd}$$

6.2. **Reduce to affine case.** Let X be a smooth complex variety and $\mathfrak{U} = \{U_i\}_{i\in I}$ be an affine open covering of X. Since coherent sheaves are acyclic on affine pieces (Theorem 4.2.2), the algebraic de Rham complex $\Omega_X^\bullet$ is acyclic with respect to $\mathfrak{U}$, and thus by Theorem 3.3.2 one has

$$\check{\mathbb{H}}^*(\mathfrak{U}, \Omega_X^\bullet) \cong \mathbb{H}^*(X, \Omega_X^\bullet).$$

Similarly, since coherent sheaves are acyclic on Stein manifolds, there is the following isomorphism

$$\check{\mathbb{H}}^*(\mathfrak{U}^{an}, \Omega_{X^{an}}^\bullet) \cong \mathbb{H}^*(X^{an}, \Omega_{X^{an}}^\bullet).$$

Note that the E_1-page of spectral sequences converging to $\check{\mathbb{H}}^*(\mathfrak{U}, \Omega_X^\bullet)$ is

$$\begin{aligned}
E_1^{p,q} &= H_{\mathrm{d}}^q\left(C^p(\mathfrak{U}, \Omega_X^\bullet)\right) \\
&= \prod_{i_0,\dots,i_p} H_{\mathrm{d}}^q(\Gamma(U_{i_0\dots i_p}, \Omega_X^\bullet)) \\
&= \prod_{i_0,\dots,i_p} \mathbb{H}^q(\Gamma(U_{i_0\dots i_p}, \Omega_X^\bullet)),
\end{aligned}$$

where the last equality holds since $U_{i_0\dots i_p}$ is affine, and thus $\Omega_X^\bullet$ is acyclic on $U_{i_0\dots i_p}$. Similarly the E_1-page of spectral sequences converging to $\check{\mathbb{H}}^*(\mathfrak{U}^{an}, \Omega_{X^{an}}^\bullet)$ is

$$\begin{aligned}
(E_1')^{p,q} &= H_{\mathrm{d}}^q\left(C^p(\mathfrak{U}^{an}, \Omega_{X^{an}}^\bullet)\right) \\
&= \prod_{i_0,\dots,i_p} H_{\mathrm{d}}^q(\Gamma(U_{i_0\dots i_p}^{an}, \Omega_{X^{an}}^\bullet)) \\
&= \prod_{i_0,\dots,i_p} H^q(U_{i_0\dots i_p}^{an}, \mathbb{C}).
\end{aligned}$$

The proof of the affine case shows the following map

$$\epsilon\colon \mathbb{H}^*(U_{i_0\dots i_p}, \Omega_X^\bullet) \to \mathbb{H}^*(U_{i_0\dots i_p}^{an}, \Omega_{X^{an}}^\bullet) \cong H^*(U_{i_0\dots i_p}^{an}, \mathbb{C})$$

is an isomorphism, and it also commutes with differentials in spectral sequences. By Proposition 2.3, it induces an isomorphism between the E_∞-pages, that is, an isomorphism

$$\mathbb{H}^*(X, \Omega_X^\bullet) \cong H^*(X^{an}, \mathbb{C}).$$

6.3. **Logarithmic differential forms.**

6.3.1. *Logarithmic differential forms.* In order to prove the key step, firstly we introduce the differential forms with logarithmic poles along some simple normal crossing divisor in this section. Let Y be a complex manifold with a simple normal crossing divisor D and $j\colon X = Y - D \to Y$.

Definition 6.3.1. The sheaf of **differential k-forms with logarithmic poles along** D, denoted by $\Omega_Y^k(\log D)$, is the subsheaf of $j_*^m\Omega_X^k$ defined by the following condition:

- If $\alpha \in \Gamma(U, j^m\Omega_X^k)$ for some open subset $U \subseteq Y$, then $\alpha \in \Gamma(U, \Omega_Y^k(\log D))$ if and only if α admits a pole of order at most 1 along D, and the same holds for $\mathrm{d}\alpha$.

Lemma 6.3.1. Let $\{z_1, \dots, z_n\}$ be a local coordinate on an open subset $U \subseteq Y$, in which $D \cap U$ is defined by the equation $z_1 \dots z_r = 0$. For convenience we denote

$$\delta_j = \begin{cases} \mathrm{d}z_j/z_j, & j \leq r \\ \mathrm{d}z_j, & j > r, \end{cases}$$

and for $I = \{j_1, \dots, j_k\} \subseteq \{1, \dots, n\}$ with $j_1 < \dots < j_s$, we denote

$$\delta_I = \delta_{j_1} \wedge \dots \wedge \delta_{j_k}.$$

Then $\Omega_Y^k(\log D)|_U$ is a sheaf of free $\mathcal{O}_U$-modules with basis $\{\delta_I\}_{|I|=k}$

Proof. See Lemma 8.16 in [Voi02]. □

Corollary 6.3.1. The sheaves $\Omega_Y^k(\log D)$ are sheaves of locally free $\mathcal{O}_Y$-modules.

Proof. See Corollary 8.17 in [Voi02]. □

6.3.2. *The proof of the key step.* Note that there is a natural inclusion

$$\Omega_Y^\bullet(\log D) \subseteq j_*\Omega_X^\bullet, \tag{6.1}$$

which is compatible with differentials.

Theorem 6.3.1. The morphism (6.1) is a quasi-isomorphism.

Proof. In order to show $\mathcal{H}^k(\Omega_Y^\bullet(\log D)) \cong \mathcal{H}^k(j_*\Omega_X^\bullet)$ for each k, it suffices to check it stalk by stalk. If $x \in X$, it's easy to show that

$$(\mathcal{H}^k(\Omega_Y^\bullet(\log D)))_x \cong (\mathcal{H}^k(j_*\Omega_X^\bullet))_x \cong \mathbb{C}$$

unless $k = 0$ by holomorphic Poincaré lemma.

Now suppose $x \in D$, so without lose of generality we may assume $Y = D_1 \times \dots \times D_n$ is a polydisk of dimension n and $D = \{(z_1, \dots, z_n) \mid z_1 \dots z_r = 0\}$. Thus $X = Y - D = D_1^* \times \dots \times D_r^* \times D_{r+1} \times \dots \times D_n$ retracts onto the product of circles $T_r = (S^1)^r = \prod_{i \leq r} \partial D_i$ and the cohomology of such a product of circles T_r is given by

$$H^1(T_r, \mathbb{C}) \cong \mathbb{C}^r$$

$$H^k(T_r, \mathbb{C}) \cong \bigwedge^k H^1(T_r, \mathbb{C}),$$

where the second equality can be obtained by Künneth formula easily.

Firstly, let's prove the morphism (6.1) induces a surjective map in cohomology. For closed 1-forms $\omega_i = \mathrm{d}z_i/z_i \in \Gamma(Y, \Omega^1_Y(\log D))$, their integrals over the circles ∂D_i satisfy

$$\int_{\partial D_j} \omega_i = 2\pi\sqrt{-1}\delta_{ij},$$

and thus the classes of these forms generate $H^1(T_r, \mathbb{C})$. The exterior products $\omega_I = \wedge_{i\in I}\omega_i \in \Gamma(Y, \Omega^k_Y(\log D))$ also generates the cohomology of T_r in degree k since $H^k(T_r, \mathbb{C}) = \bigwedge^k H^1(T_r, \mathbb{C})$. This proves surjectivity.

For the injectivity, we need to show that if any closed $\alpha \in \Gamma(Y, \Omega^k_Y(\log D))$ is d-exact in $\Gamma(Y, j_*\Omega^\bullet_X) = \Gamma(X, \Omega^\bullet_X)$, then it's d-exact in $\Gamma(Y, \Omega^k_Y(\log D))$. The proof of injectivity is based on the induction on r.

If $r = 0$, the statement holds from the holomorphic Poincaré lemma. Now assume the statement holds for $r-1$. For any $\alpha \in \Gamma(Y, \Omega^k_Y(\log D))$, we decompose it into

$$\alpha = \frac{\mathrm{d}z_r}{z_r} \wedge \beta + \gamma,$$

where $\mathrm{d}z_r$ does not occur in β, and the coefficients of β are independent of z_r, while γ is holomorphic in z_r.

(1) If α is closed, one has

$$\mathrm{d}\alpha = \frac{\mathrm{d}z_r}{z_r} \wedge \mathrm{d}\beta + \mathrm{d}\gamma = 0.$$

Note that γ is holomorphic in z_r and the coefficients of β are independent of z_r, so it leads to $\mathrm{d}\beta = 0$ and $\mathrm{d}\gamma = 0$.

(2) If α is d-exact in $\Gamma(X, \Omega^\bullet_X)$, say $\alpha = \mathrm{d}\widetilde{\alpha}$, then we decompose as

$$\alpha = \mathrm{d}\widetilde{\alpha} = \mathrm{d}\left(\frac{\mathrm{d}z_r}{z_r} \wedge \widetilde{\beta} + \widetilde{\gamma}\right) = \frac{\mathrm{d}z_r}{z_r} \wedge \mathrm{d}\widetilde{\beta} + \mathrm{d}\widetilde{\gamma}.$$

By the same reason we have β and γ are d-exact in $\Gamma(X, \Omega^\bullet_X)$.

Now suppose $\alpha \in \Gamma(Y, \Omega^k_Y(\log D)))$ is closed and write $\alpha = \mathrm{d}z_r/z_r \wedge \beta + \gamma$. Since γ is closed and holomorphic in z_r, it may be regarded as a closed holomorphic k-form has logarithmic poles along $D' = \{z \mid z_1 \dots z_{r-1} = 0\}$, and thus by induction hypothesis it's d-exact in $\Gamma(Y, \Omega^k_Y(\log D))$. By the same argument one can show $\mathrm{d}z_r/z_r \wedge \beta$ is also also d-exact in $\Gamma(Y, \Omega^k_Y(\log D))$. This completes the proof. □

On the other hand, there is also a natural inclusion

$$\Omega^\bullet_Y(\log D) \subseteq j^m_*\Omega^\bullet_X, \tag{6.2}$$

which is compatible with differentials.

Theorem 6.3.2. The morphism (6.2) is a quasi-isomorphism.

Proof. By the same reason we may assume X and Y as before, and as seen above, the stalk of $\mathcal{H}^1(\Omega_Y^\bullet(\log D))$ is generated by $[\mathrm{d}z_1/z_1], \dots, [\mathrm{d}z_r/z_r]$. Now we prove the morphism (6.2) is a quasi-isomorphism by induction on r, and the $r=0$ case is clear.

Lemma 6.3.2. Let $\varphi \in \Gamma(Y, j_*^m \Omega_X^k)$ be a closed meromorphic k-form on Y that is holomorphic on X. Then there exists closed meromorphic forms $\varphi_0 \in \Gamma(Y, j_*^m \Omega_X^k)$ and $\alpha_1 \in \Gamma(Y, j_*^m \Omega_X^{k-1})$, which have no poles along z_r, and

$$[\varphi] = [\varphi_0] + \left[\frac{\mathrm{d}z_r}{z_r}\right] \wedge [\alpha_1].$$

Proof. Firstly we write

$$\varphi = \mathrm{d}z_r \wedge \alpha + \beta,$$

where α is meromorphic $(k-1)$-form and β do not involve $\mathrm{d}z_r$. Then let's expand α and β as Laurent series in z_r as

$$\begin{aligned} \alpha &= \alpha_0 + \alpha_1 z_r^{-1} + \cdots + \alpha_\ell z_r^{-\ell} \\ \beta &= \beta_0 + \beta_1 z_r^{-1} + \cdots + \beta_\ell z_r^{-\ell}, \end{aligned}$$

where α_i and β_i for $1 \leq i \leq \ell$ do not involve z_r or dz_r and are meromorphic in the other variables, and α_0, β_0 are holomorphic in z_r, are meromorphic in the other variables, and do not involve $\mathrm{d}z_r$. Thus

$$\varphi = \varphi_0 + \left(\mathrm{d}z_r \wedge \sum_{i=1}^r \alpha_i z_r^{-i} + \sum_{i=1}^r \beta_i z_r^{-i}\right),$$

where $\varphi_0 = \mathrm{d}z_r \wedge \alpha_0 + \beta_0$. By comparing the coefficients of $z_r^{-i}\mathrm{d}z_r$ and z_r^{-i}, one can deduce from $\mathrm{d}\varphi$,

$$\begin{aligned} &\mathrm{d}\alpha_1 = \mathrm{d}\alpha_2 + \beta_1 = \mathrm{d}\alpha_3 + 2\beta_2 = \cdots = r\beta_r = 0 \\ &\mathrm{d}\beta_1 = \mathrm{d}\beta_2 = \cdots = \mathrm{d}\beta = 0, \end{aligned}$$

and $\mathrm{d}\varphi_0 = 0$. If we write

$$\varphi = \varphi_0 + \frac{\mathrm{d}z_r}{z_r} \wedge \alpha_1 + \left(\mathrm{d}z_r \wedge \sum_{i=2}^r \alpha_i z_r^{-i} + \sum_{i=1}^r \beta_i z_r^{-i}\right)$$

It turns out

$$\theta = -\frac{\alpha_2}{z_r} - \frac{\alpha_3}{2z_r^2} - \cdots - \frac{\alpha_r}{(r-1)z_r^{r-1}}$$

satisfies

$$\varphi - \varphi_0 - \frac{\mathrm{d}z_r}{z_r} \wedge \alpha_1 = \mathrm{d}\theta.$$

□

Since φ_0 and α_1 are meromorphic forms which do not have poles along $z_r = 0$, their singularities set is contained in the simple normal crossing divisor $z_1 \dots z_{r-1} = 0$. By induction on r, the cohomology class of φ_0 and φ_1 is generated by $[\mathrm{d}z_1/z_1], \dots, [\mathrm{d}z_{r-1}/z_{r-1}]$. This completes the proof. □

Corollary 6.3.2.
$$\mathbb{H}^*(Y^{an}, j_*^m \Omega^\bullet_{X^{an}}) \cong \mathbb{H}^*(Y^{an}, j_*^{an}\Omega^\bullet_{X^{an}}).$$

Remark 6.3.1. There is also an analogy in the algebraic setting.

Theorem 6.3.3. Let X be a smooth complex algebraic variety and $j\colon X \to Y$ be an inclusion such that $D = Y - X$ is simple normal crossing. Then
$$\mathbb{H}^*(Y, \Omega^\bullet_Y(D)) \cong \mathbb{H}^*(Y, j_*\Omega^\bullet_X).$$

7. Local system valued version

7.1. **Introduction.** In this section we will prove the local system valued version of Grothendieck's comparison theorem, and the ideas are almost the same as before.

Theorem 7.1.1 ([Del70])**.** Let X be a smooth complex algebraic variety and X^{an} be the corresponding complex manifold. Let $\mathcal{E}$ be a locally free sheaf on X equipped with a regular integrable connection ∇ and $\mathcal{L} := (\mathcal{E}^{an})^{\nabla^{an}=0}$ be the local system of horizontal sections on the underlying complex manifold X^{an}. Then there is the following isomorphism

$$H^*(X^{an}, \mathcal{L}) \cong \mathbb{H}^*(X, \Omega^\bullet_X(\mathcal{E})).$$

The ideas of the proof is the same as before. Firstly by the same argument we can reduce to the case that X is smooth affine with an embedding $j\colon X \to Y$, where Y is a smooth projective variety and $D = Y - X$. Then we need to go through the following diagram

$$\begin{tikzcd}
H^*(Y, j_*\Omega^\bullet_X(\mathcal{E})) \arrow[d, "2"'] & & \mathbb{H}^*(X, \Omega^\bullet_X(\mathcal{E})) \arrow[ll, "1"'] \arrow[dd, dashed, no head] \\
\mathbb{H}^*(Y^{an}, j^m_*\Omega^\bullet_{X^{an}}(\mathcal{E}^{an})) \arrow[d, "3"'] & & \\
\mathbb{H}^*(Y^{an}, j^{an}_*\Omega^\bullet_{X^{an}}(\mathcal{E}^{an})) \arrow[rr, "4"] & & H^*(X^{an}, \mathcal{L}).
\end{tikzcd}$$

The proofs of step one, two are the same as before, and the most difficult step is to show the morphism of complexes

$$j^m_*\Omega^\bullet_{X^{an}}(\mathcal{E}) \to j_*\Omega^\bullet_{X^{an}}(\mathcal{E})$$

is a quasi-isomorphism. In this case, we need to define a canonical extension $\widetilde{\mathcal{E}}$ of $\mathcal{E}$ on Y, which is called the Deligne's extension, and prove

$$\Omega^\bullet_Y(\log D)(\widetilde{\mathcal{E}}) \to j^m_*\Omega^\bullet_{X^{an}}(\mathcal{E})$$

and

$$\Omega^\bullet_Y(\log D)(\widetilde{\mathcal{E}}) \to j_*\Omega^\bullet_{X^{an}}(\mathcal{E})$$

are quasi-isomorphisms.

7.2. **Local system valued logarithmic pole differential.** Let $j\colon X \to Y$ as before, and the simple normal crossing divisor $D = Y - X$ is written as $D_1 + \cdots + D_r$.

Definition 7.2.1. For every $k \geq 1$, the **residue map along D_i** is a map

$$\mathrm{Res}_i\colon \Omega^k_Y(\log D) \to \Omega^{k-1}_{D_i}(\log(D - D_i)|_{D_i})$$

such that if φ is a local section of $\Omega^k_Y(\log D)$, we write it as

$$\varphi = \varphi_1 + \varphi_2 \wedge \frac{\mathrm{d}z_i}{z_i},$$

where φ_1 lies in the span of the δ_I with $i \notin I$ and $\varphi_2 = \sum_{i \in I} a_I \delta_{I-\{i\}}$, then

$$\mathrm{Res}_{D_i}(\varphi) = \sum a_I \delta_{I-\{1\}}|_{D_i}.$$

Remark 7.2.1. Let $\mathcal{E}$ be a locally free sheaf on Y. Then the residue map extends to

$$\mathrm{Res}_i \colon \Omega_Y^k(\log D)(\mathcal{E}) \to \Omega_{D_i}^{k-1}(\log(D - D_i)|_{D_i})(\mathcal{E})$$

linearly.

Definition 7.2.2. Let $\mathcal{E}$ be a locally free sheaf on Y and ∇ be a integrable connection on $\mathcal{E}|_X$, and the connection form Γ of ∇ gives a section of $j_*\Omega_X^1(\underline{\mathrm{End}}(\mathcal{E}))$. The connection ∇ has **at worst logarithmic poles along** D if the connection forms present at worst logarithmic poles along D.

Theorem 7.2.1. Let $\mathcal{E}$ be a locally free sheaf on Y and ∇ be a integrable connection on $\mathcal{E}|_X$ which has at worst logarithmic poles along D. If the residues of connection form Γ along all components of D do not admit any strictly positive integer as an eigenvalue, then

$$\Omega_Y^\bullet(\log D)(\mathcal{E}) \to j_*^m \Omega_X^\bullet(\mathcal{E})$$

is a quasi-isomorphism.

Proof. See Proposition 3.13 in [Del70]. □

7.3. **Deligne's canonical extension.** Let $j \colon X \to Y$ as before, and the simple normal crossing divisor $D = Y - X$ is written as $D_1 + \cdots + D_r$.

Definition 7.3.1. A local system $\mathcal{L}$ on X is said to be **unipotent along** D if the fundamental group $\pi_1(X)$ acts on this local system by unipotent transformations.

Definition 7.3.2. A flat bundle $(\mathcal{E}, \nabla)$ on X is said to be **unipotent along** D if local system $\mathcal{E}^{\nabla=0}$ is unipotent along D.

Theorem 7.3.1. Let $(\mathcal{E}, \nabla)$ be a flat bundle on X that is unipotent along D. Then there exists a unique extension $\widetilde{\mathcal{E}}$ of $\mathcal{E}$ on Y such that the following conditions

(1) The matrix of the connection ∇, in an arbitrary local basis of $\widetilde{\mathcal{E}}$, has at worst a logarithmic pole along Y.
(2) The residue $\mathrm{Res}_i(\Gamma)$ of the connection along D_i is nilpotent.

Proof. See Proposition 5.2 in [Del70]. □

Remark 7.3.1. In general, it's still possible to make an extension of $\mathcal{E}$ on Y. See Proposition 5.4 in [Del70].

7.4. **Regular integrable connection.**

Theorem 7.4.1. Let X be a smooth complex algebraic variety. Then the functor $\mathcal{E} \mapsto \mathcal{E}^{an}$ gives an equivalence of categories between

(1) the category of algebraic locally free sheaf on X equipped with a regular integrable connection;
(2) the category of holomorphic locally free sheaf on X^{an} endowed with an integrable connection.

Proof. See Theorem 5.9 in [Del70]. $\square$

Part 3. Hodge to de Rham spectral sequence

8. Cartier descent theorem

8.1. **Characteristic p geometry.** In this section we assume k is an algebraically closed field with positive characteristic p, $F_k \colon k \to k$ is the Frobenius map and X is a smooth algebraic variety over k.

8.1.1. *Absolute Frobenius map and relative Frobenius map.*

Definition 8.1.1. The Frobenius map F_k induces so called **absolute Frobenius map** $F_{\mathrm{abs}} \colon X \to X$, which is the identity on the underlying space of X and the p-th power on the structure sheaf.

Remark 8.1.1. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The direct image $(F_{\mathrm{abs}})_*\mathcal{F}$ equals $\mathcal{F}$ as sheaves of abelian groups, but the $\mathcal{O}_X$-module structure on $(F_{\mathrm{abs}})_*\mathcal{F}$ is given by $f \cdot s := f^p \cdot s$ for any local sections f of $\mathcal{O}_X$ and s of $\mathcal{F}$.

Definition 8.1.2. Let $X^{(p)}$ be the base change of X given by the Frobenius map F_k, that is,

$$\begin{tikzcd} X^{(p)} \arrow[r, "\pi"] \arrow[d] & X \arrow[d] \\ k \arrow[r, "F_k"] & k. \end{tikzcd}$$

By the universal property of base change there exists a morphism $F \colon X \to X^{(p)}$ such that the following diagram commutes, which is called **relative Frobenius map**.

$$\begin{tikzcd} X \arrow[dr, dotted] \arrow[drr, "F_{\mathrm{abs}}"] \arrow[ddr, "\alpha"'] & & \\ & X^{(p)} \arrow[r, "\pi"'] \arrow[d, "\alpha'"] & X \arrow[d, "\alpha"] \\ & k \arrow[r, "F_k"] & k, \end{tikzcd}$$

8.1.2. *Connections in algebraic setting.*

Definition 8.1.3. A **k-connection** on X is a pair $(\mathcal{E}, \nabla)$, which consists of the following data:

(1) $\mathcal{E}$ is a (quasi)-coherent $\mathcal{O}_X$-module.
(2) $\nabla \colon \mathcal{E} \to \Omega^1_X \otimes \mathcal{E}$ is k-linear satisfying the Leibiniz rule

$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s.$$

Definition 8.1.4. For a k-connection $(\mathcal{E}, \nabla)$ on X, the **curvature** of k-connection $(\mathcal{E}, \nabla)$ is defined by

$$\begin{aligned} \Theta_\nabla \colon \bigwedge^2 T_X &\to \mathrm{End}_k(\mathcal{E}) \\ D_1 \wedge D_2 &\mapsto [D_1, D_2] - \nabla_{[D_1, D_2]}, \end{aligned}$$

where the Lie bracket on T_X is defined by $[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$.

Remark 8.1.2. For a k-connection $(\mathcal{E}, \nabla)$ on X, it can be regarded as a "quasi-representation $\nabla\colon T_X \to \mathrm{End}_k(\mathcal{E})$, and the curvature measures the failure of ∇ to be a Lie algebra representation.

Lemma 8.1.1. The curvature Θ_∇ is $\mathcal{O}_X$-linear, that is, $\Theta_\nabla\colon \bigwedge^2 T_X \to \mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$.

8.1.3. *p-curvature.* In the case of positive characteristic, there is so called p-curvature which doesn't appear in the zero characteristic case, since the p-th power map $D \mapsto D^p := \underbrace{D \circ \cdots \circ D}_{p \text{ times}}$ gives a map between $T_X \to T_X$.

Definition 8.1.5. The ***p*-curvature** of a k-connection $(\mathcal{E}, \nabla)$ over X/k is defined by

$$\begin{aligned} \Psi_\nabla\colon T_X &\mapsto \mathrm{End}_k(\mathcal{E}) \\ D &\mapsto (\nabla_D)^p - \nabla_{D^p}. \end{aligned}$$

Proposition 8.1.1. The p-curvature Ψ_∇ is $\mathcal{O}_X$-linear, that is,

$$\Psi_\nabla\colon T_X \to \mathrm{End}_{\mathcal{O}_X}(\mathcal{E}).$$

Proof. For sections $f \in \mathcal{O}_X, s \in \mathcal{E}$ and $D \in T_X$, one has

$$(\nabla_D)^p(fs) = \sum_{i=0}^{p} \binom{p}{i} D^i(f)(\nabla_D)^{p-i}(s) = D^p(f)s + f(\nabla_D)^p(s).$$

On the other hand, it's clear $\nabla_{D^p}(fs) = D^p(f)s + f\nabla_{D^p}(s)$, and thus

$$\Psi_\nabla(D)(fs) = f\Psi_\nabla(D)(s).$$

□

Proposition 8.1.2. Let $(\mathcal{E}, \nabla)$ be a k-connection over X/k. If the curvature Θ_∇ vanishes, then

(1) $\Psi_\nabla\colon T_X \to \mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$ is additive.
(2) Ψ_∇ is F_X-linear, that is,

$$\Psi_\nabla(fD) = f^p\Psi_\nabla(D).$$

(3) Ψ_∇ is integrable, that is, $\Psi_\nabla \wedge \Psi_\nabla = 0$.

It's a highly non-trivial fact, whose proof relies on the following algebra result.

Lemma 8.1.2. Let R be a ring with characteristic $p > 0$. For $a, b \in R$,

(1) $(a+b)^p = a^p + b^p + \sum_{i=1}^{p-1} s_i(a, b)$, where

$$(\mathrm{ad}(ta+b))^p(a) = \sum_{i=0}^{p-1} is_i(a,b)t^i.$$

(2) If $\{a^{(n)}\}_{n\geq 1}$ are mutually commutative, then

$$(ab)^p = a^p b^p + a(a^{p-1})^{(p-1)}b,$$

where

$$a^{(n)} := (\operatorname{ad} b)^n(a).$$

Proof. See [Kat70]. □

Now let's begin the proof of Proposition 8.1.2.

Proof of Proposition 8.1.2. For (1). For arbitrary sections $D_1, D_2 \in T_X$, by using (1) of Lemma 8.1.2 one has

$$\begin{aligned}(\nabla_{D_1+D_2})^p &= (\nabla_{D_1} + \nabla_{D_2})^p \\ &= (\nabla_{D_1})^p + (\nabla_{D_2})^p + \sum_i s_i(D_1, D_2) \\ \nabla_{(D_1+D_2)^p} &= \nabla_{(\nabla_1^p + \nabla_2^p + \sum_i s_i(D_1,D_2))} \\ &= \nabla_{D_1^p} + \nabla_{D_2^p} + \sum_i \nabla_{s_i(D_1,D_2)}.\end{aligned}$$

Then

$$\Psi_\nabla(D_1 + D_2) = (\nabla_{D_1+D_2})^p - \nabla_{(D_1+D_2)^p} = \Psi_\nabla(D_1) + \Psi_\nabla(D_2).$$

For (2). For arbitrary $f \in \mathcal{O}_X$ and $D \in T_X$, by using (2) of Lemma 8.1.2, one has

$$\begin{aligned}(fD)^p &= f^p D^p + f(\operatorname{ad}(D))^{p-1}(f^{p-1})D \\ &= f^p D^p + f(D^{p-1}(f^{p-1}))D,\end{aligned}$$

since $\operatorname{ad}(D)(f^{p-1}) = D \circ f^{p-1} - f^{p-1}D = D(f^{p-1})$. Thus

$$\nabla_{(fD)^p} = f^p \nabla_{D^p} + f(D^{p-1}(f^{p-1}))\nabla_D.$$

Applying (2) of Lemma 8.1.2 again, one has

$$\begin{aligned}(\nabla_{fD})^p = (f\nabla_D)^p &= f^p(\nabla_D)^p + f(\operatorname{ad}(\nabla_D))^{p-1}(f^{p-1})\nabla_D \\ &= f^p(\nabla_D)^p + f(D^{p-1}(f^{p-1}))\nabla_D.\end{aligned}$$

This completes the proof of (2).

For (3). Let's check this by local computations. Suppose $\{z_1, \dots, z_n\}$ be a local coordinate and write

$$\begin{aligned}D_1 &= \sum_i a_i \partial_i \\ D_2 &= \sum_j b_j \partial_j,\end{aligned}$$

where $\partial_i = \frac{\partial}{\partial z_i}$. Note that

$$\begin{aligned}\Psi_\nabla(D_1) &= \Psi_\nabla(\sum_i a_i\partial_i) \\ &= \sum_i a_i^p \Psi_\nabla(\partial_i) \\ &= \sum_i a_i^p \left((\nabla_{\partial_i})^p - \nabla_{\partial_i^p}\right) \\ &= \sum_i a_i^p (\nabla_{\partial_i})^p.\end{aligned}$$

Then one has

$$\begin{aligned}[\Psi_\nabla(D_1), \Psi_\nabla(D_2)] &= \left[\sum_i a_i^p(\nabla_{\partial_i})^p, \sum_j b_j^p(\nabla_{\partial_j})^p\right] \\ &= \sum_{ij} a_i^p b_j^p \left[(\nabla_{\partial_i})^p, (\nabla_{\partial_j})^p\right] \\ &= 0,\end{aligned}$$

where the last equality holds since $\Theta_\nabla = 0$ implies $\nabla_{\partial_i}\nabla_{\partial_j} = \nabla_{\partial_j}\nabla_{\partial_i}$. □

8.2. **Cartier descent theorem.** In this section, we will prove the Cartier descent theorem, which is a very basic theorem in characteristic p geometry.

Theorem 8.1 (Cartier descent theorem)**.** There is a natural equivalence of categories between the category of (quasi)-coherent $\mathcal{O}_{X^{(p)}}$-module and the category of flat k-connections $(\mathcal{E}, \nabla)$ on X with vanishing p-curvatures. To be precise, the correspondence is given by

(1) For (quasi)-coherent $\mathcal{O}_{X^{(p)}}$-module $\mathcal{E}$, it corresponds to the k-connection $(F^*\mathcal{E}, \nabla_{\mathrm{can}})$, where the **canonical connection** $\nabla_{\mathrm{can}}\colon F^*\mathcal{E} \to F^*\mathcal{E} \otimes \Omega_X$ is defined by
$$\nabla_{\mathrm{can}}(e \otimes f) = e \otimes \mathrm{d}f.$$
(2) For flat k-conenction $(\mathcal{V}, \nabla)$ with vanishing p-curvature, the corresponding (quasi)-coherent $\mathcal{O}_X$-module is the $\mathcal{O}_{X^{(p)}}$-submodule[9] $\mathcal{V}^{\nabla=0} \subseteq \mathcal{V}$.

[9]For (quasi)-coherent $\mathcal{O}_X$-module $\mathcal{V}$ on X, the flat part $\mathcal{V}^{\nabla=0}$ is indeed a $\mathcal{O}_{X^{(p)}}$-module. For $f \in \mathcal{O}_{X^{(p)}}$ and $s \in \mathcal{V}^{\nabla=0}$, one has

$$\begin{aligned}\nabla(f \cdot s) &= \nabla(f^p \cdot s) \\ &= \mathrm{d}(f^p)\cdot s + f^p \cdot \nabla(s) \\ &= 0.\end{aligned}$$

9. De Rham decomposition theorem of Deligne-Illusie

In this section, we assume k is an algebraically closed field with positive characteristic p.

9.1. **Introduction.** Let X be a smooth variety over k and $F\colon X \to X^{(p)}$ denote the relative Frobenius map. Then

$$F_*\Omega_X^\bullet \colon F_*\mathcal{O}_X \to F_*\Omega_X^1 \to F_*\Omega_X^2 \to \dots$$

is a finite complex of coherent $\mathcal{O}_{X^{(p)}}$-module with $\mathcal{O}_{X^{(p)}}$-linear differential.

Theorem 9.1.1 (Deligne-Illusie)**.** Let X be a smooth variety over k. If X is $W_2(k)$-liftable and $\dim_k X = n < p$. Then there is a quasi-isomorphism

$$(F_*\Omega_X^\bullet, F_*\mathrm{d}) \cong \bigoplus_{i=0}^{n} \Omega^i_{X^{(p)}}[-i].$$

Remark 9.1.1.

(1) The condition of $W_2(k)$-liftable (Definition 9.1.2) cannot be removed, and the first counterexample is given in [Ray78] by showing Kodaira's vanishing theorem fails in positive characteristic.
(2) The statement still holds for $\dim_k X = p$, but it fails when $\dim_k X > p$, see [Pet23].

9.1.1. *Witt vectors of length two.*

Definition 9.1.1. The **Witt ring** $W_2(k)$ can be interpreted as the set $k\times k$, where the multiplication and addition for $a = (a_0, a_1)$ and $b = (b_0, b_1)$ are defined by

$$ab = (a_0a_1, b_0a_1^p + b_1a_0^p),$$

and

$$a + b = (a_0 + b_0, a_1 + b_1 - \sum_{i=1}^{p-1} p^{-1}\binom{p}{t} a_0^i b_0^{p-i}).$$

Remark 9.1.2. In fact, the operations on $W_2(k)$ makes the ghost polynomial $\Phi(a_0, a_1) = a_0^p + pa_1$ a ring homomorphism.

Proposition 9.1.1. If $k = \mathbb{Z}/p\mathbb{Z}$, then $W_2(k) = \mathbb{Z}/p^2\mathbb{Z}$.

Proof. Let $[\text{ - }]\colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z}$ be the Teichmüller lifting, that is,

$$\begin{cases} [0] = 0 \\ [i] \equiv i \pmod p \\ [i]^p \equiv [i] \pmod p \end{cases}$$

To be precise, if we write

$$[i] \equiv i + pa \pmod{p^2},$$

then

$$a \equiv \frac{1}{p}(i^p - i) \pmod p.$$

The Teichmüller lifting satisfies $[a\cdot b]=[a]\cdot[b]$, but in general $[a+b]\neq[a]+[b]$. Then the following map

$$\begin{aligned}\alpha\colon W_2(k)&\to\mathbb{Z}/p^2\mathbb{Z}\\(a_0,a_1)&\mapsto[a_0]+p[a_1].\end{aligned}$$

gives the desired isomorphism. □

Proposition 9.1.2. The set $pW_2(k)=\{(0,a)\mid a\in k\}$ is a maximal ideal of $W_2(k)$, and the following sequence is exact

$$0\to pW_2(k)\to W_2(k)\to k\to 0.$$

Proposition 9.1.3. The ring homomorphism $F_{W_2(k)}\colon W_2(k)\to W_2(k)$ given by $(a_0,a_1)\mapsto(a_0^p,a_1^p)$ reduces to the Frobenius map F_k on k modulo p.

Definition 9.1.2. Let X be a smooth variety over k. If there exists a flat morphism $\widetilde{X}\to W_2(k)$ such that the following diagram commutes

$$\begin{tikzcd}X \arrow[r, "\cong"] \arrow[dr] & \widetilde{X}\times_{W_2(k)}k \arrow[r] \arrow[d] & \widetilde{X} \arrow[d]\\ & k \arrow[r] & W_2(k),\end{tikzcd}$$

then X/k is $W_2(k)$**-liftable**.

Remark 9.1.3. Not every smooth variety X/k is $W_2(k)$-liftable. In fact, there is an obstruction in $\mathrm{ob}(\alpha)\in H^2(X,T_X)$ such that $\mathrm{ob}(\alpha)=0$ if and only if X/k is $W_2(k)$-liftable.

9.1.2. *Cartier inverse operator.*

Theorem 9.1.2. Let X be a smooth algebraic variety over k of dimension n and $F\colon X\to X^{(p)}$ be the relative Frobenius map. Then there is a unique isomorphism (called **Cartier inverse operator**) of graded $\mathcal{O}_{X^{(p)}}$-algebra

$$C^{-1}\colon\bigoplus_{i=0}^{n}\Omega^i_{X^{(p)}}\to\bigoplus_{i=0}^{n}\mathcal{H}^i(F_*\Omega^\bullet_X),$$

which is determined as follows

(1) On the 0-th degree, the operator $C^{-1}\colon\mathcal{O}_{X^{(p)}}\to\mathcal{H}^0(F_*\Omega^\bullet_X)$ is defined by the morphism $F^*\colon\mathcal{O}_{X^{(p)}}\longrightarrow F_*\mathcal{O}_X$, as $\mathcal{H}^0(F_*\Omega^\bullet_X)\subseteq F_*\mathcal{O}_X$.
(2) On the 1-st degree, there is the following commutative diagram

$$\begin{tikzcd}C^{-1}\colon\Omega^1_{X^{(p)}} \arrow[r] \arrow[dr] & \mathcal{H}^1(F_*\Omega^\bullet_X)\\ & \mathcal{Z}^1(F_*\Omega^\bullet_X) \arrow[u]\end{tikzcd}$$

such that $C^{-1}(\mathrm{d}x')=x^{p-1}\mathrm{d}x \pmod{\mathcal{B}^1}$, where $x\in\mathcal{O}_X$ and $x'=\pi^*x\in\mathcal{O}_{X^{(p)}}$.

Proof. The key observation is the global map $\mathcal{O}_{X^{(p)}} \to \mathcal{H}^1(F_*\Omega^\bullet_X)$ defined by sending x to $x^{p-1}\mathrm{d}x$ is a derivation, and thus it factors throught $\Omega^1_{X^{(p)}}$ by the universal property, that is, the following diagram commutes

$$\begin{tikzcd} \Omega^1_{X^{(p)}} \arrow[r, "C^{-1}"] & \mathcal{H}^1(F_*\Omega^\bullet_X) \\ \mathcal{O}_{X^{(p)}} \arrow[u] \arrow[ur] & \end{tikzcd}$$

Now let's prove this key observation. For arbitrary sections $x, y \in \mathcal{O}_X$, note that

$$(x+y)^p = x^p + y^p + p\sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} x^i y^{p-i}.$$

Thus

$$(x+y)^{p-1}\mathrm{d}(x+y) = x^{p-1}\mathrm{d}x + y^{p-1}\mathrm{d}y + \mathrm{d}\left(\sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} x^i y^{p-i}\right),$$

that is,

$$(x+y)^{p-1}\mathrm{d}(x+y) \equiv x^{p-1}\mathrm{d}x + y^{p-1}\mathrm{d}y \pmod{\mathcal{B}^1}.$$

On the other hand, a direct computation shows

$$(xy)^{p-1}\mathrm{d}(xy) \equiv x^p(y^{p-1}\mathrm{d}y) + y^p(x^{p-1}\mathrm{d}x) \pmod{\mathcal{B}^1}$$

This completes the proof of the observation. Now let's check C^{-1} is an isomorphism, we may assume X is affine, with local coordinate $\{z_1, \dots, z_n\}$. For simplicity we firstly assume $n = 2$, then

$$\begin{aligned} F_*\mathcal{O}_X &= \mathcal{O}_{X^{(p)}}\{z_1^{i_1} z_2^{i_2} \mid 0 \le i_1 \le p-1, 0 \le i_2 \le p-1\} \\ F_*\Omega^1_X &= \mathcal{O}_{X^{(p)}}\{z_1^{i_1} z_2^{i_2} \mid \dots\} \otimes \mathrm{d}z_1 \oplus \mathcal{O}_{X^{(p)}}\{z_1^{i_1} z_2^{i_2} \mid \dots\} \otimes \mathrm{d}z_2 \\ F_*\Omega^2_X &= \mathcal{O}_{X^{(p)}}\{z_1^{i_1} z_2^{i_2}\} \otimes \mathrm{d}z_1 \wedge \mathrm{d}z_2. \end{aligned}$$

In this case C^{-1} is given by

$$\begin{aligned} C^{-1} \colon \Omega^1_{X^{(p)}} &\to \mathcal{H}^1(F_*\Omega^\bullet_X) \\ \mathrm{d}z_i &\mapsto z_i^{p-1}\mathrm{d}z_i, \end{aligned}$$

which is clearly an isomorphism. □

9.2. **Deformation theory.**

Definition 9.2.1. Let $f \colon X \to Y$ be a morphism of schemes. Then it's called **locally finite presented** if for all $x \in X$, there exists an affine open neighborhood $x \in \operatorname{Spec} A \subseteq X$, and affine open neighborhood $f(x) \in \operatorname{Spec} B \subseteq Y$ such that $f(u) \in V$ with the following property

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
\operatorname{Spec} A \arrow[u, hook] \arrow[r] \arrow[d, hook] & \operatorname{Spec} B \arrow[u, hook] \\
\operatorname{Spec} B[t_1, \dots, t_n], \arrow[ur] &
\end{tikzcd}$$

where $I = \ker(B[t_1, \dots, t_n] \to A)$ is finitely generated.

In the following, we always assume f is of locally finite presented.

Definition 9.2.2. Let $f\colon X \to Y$ be a morphism of schemes. Then f is called **smooth** if for every first order thickening[10] $i_0\colon T_0 \to T$ and $g_0\colon T_0 \to X$, there exists (Zariski locally on T) at most one $g\colon T \to X$ such that the following diagram commutes

$$\begin{tikzcd}
T_0 \arrow[r, "g_0"] \arrow[d, hook] & X \arrow[d, "f"] \\
T \arrow[r] \arrow[ur, dashed, "g"] & Y.
\end{tikzcd}$$

Remark 9.2.1. Similarly, the morphism f is called **étale** (or **unramified**), if there exists a unique (or at least one) $g\colon T \to X$ as above.

In the following, we're mainly interested in the case that $f\colon X \to Y$ is smooth.

Example 9.2.1. The projection $pr\colon \mathbb{A}^n_Y = \mathbb{A}^n \times_{\mathbb{Z}} Y \to Y$ is smooth.

Theorem 9.2.1. Let $f\colon X \to Y$ is smooth morphism. Then

(1) $\Omega_{X/Y}$ is locally free $\mathcal{O}_X$-module of finite type.

(2) Suppose $X \xrightarrow{f} Y \xrightarrow{g} S$. Then
$$0 \to g^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$
is exact, locally split

(3)

Proof. See 17.2.3 of [Gro66b]. □

Corollary 9.2.1. The morphism $f\colon X \to Y$ is smooth if and only if for all $x \in X$, there exists an open neighborhood U such that

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "f|_U"'] & \mathbb{A}^n_Y \arrow[dl, "\mathrm{pr}"] \\
Y, &
\end{tikzcd}$$

where $U \to \mathbb{A}^n_Y$ is étale.

[10]The morphism $i\colon T_0 \to T$ is a first order thickening if the ideal I of T_0 in T satisfies $I^2 = 0$.

Now we're going to state two key results in deformation theory.

Theorem 9.2.2. Let I be the ideal of T_0 in T and $f\colon X \to Y$ be smooth. Then

(1) There exists $\mathrm{ob}(g_0) \in \mathrm{Ext}^1(g_0^*\Omega^1_{X/Y}, I)$, such that $\mathrm{ob}(g_0) = 0$ if and only if there exists a Y-morphism $g\colon T \to X$ such that $g \circ i = g_0$.
(2) If $\mathrm{ob}(g_0) = 0$, then the set of all liftings g of g_0 is an affine space under $\mathrm{Hom}(g_0^*\Omega^1_{X/Y}, I)$, which is called torsor.

Theorem 9.2.3.

9.3. **Explicit quasi-isomorphism.** Let $F\colon X \to X^{(p)}$ be the relative Frobenius map for convenience.

9.3.1. *Simple case.* Firstly let's consider the case that the relative Frobenius map lifts over $W_2(k)$. In other words, there exists a morphism $\widetilde{F}\colon \widetilde{X} \to \widetilde{X}^{(p)}$ such that the following diagram commutes

$$\begin{tikzcd}
 & X^{(p)} \arrow[rr] \arrow[dd] & & \widetilde{X}^{(p)} \arrow[dd] \\
X \arrow[ur, "F"] \arrow[rr] \arrow[dr] & & \widetilde{X} \arrow[ur, dashed, "\widetilde{F}"] \arrow[dr] & \\
 & k \arrow[rr] & & W_2(k),
\end{tikzcd}$$

where $\widetilde{X}^{(p)}$ is the base change of $\widetilde{X}$ given by the Frobenius map of $W_2(k)$, that is, $(a_0, a_1) \mapsto (a_0^p, a_1^p)$.

Consider the $\mathcal{O}_{\widetilde{X}}$-linear morphism $\mathrm{d}\widetilde{F}\colon \widetilde{F}^*\Omega^1_{\widetilde{X}^{(p)}/W_2(k)} \to \Omega^1_{\widetilde{X}/W_2(k)}$. Then for any local section x of $\mathcal{O}_{\widetilde{X}^{(p)}}$, we may write $\widetilde{F}^*(x) = x^p + pa$, since $\widetilde{F}$ is a lifting of F and $F^*(x) = x^p$. Therefore,

$$\begin{aligned}
\mathrm{d}\widetilde{F}(\mathrm{d}x) &= \mathrm{d}(\widetilde{F}^*x) \\
&= \mathrm{d}(x^p + pa) \\
&= p(x^{p-1}\mathrm{d}x + \mathrm{d}a).
\end{aligned}$$

This shows $\mathrm{d}\widetilde{F}(\widetilde{F}^*\Omega^1_{\widetilde{X}^{(p)}/W_2(k)}) \subseteq p\Omega^1_{\widetilde{X}/W_2(k)}$, and thus one has the following commutative diagram

$$\begin{tikzcd}
p\widetilde{F}^*\Omega^1_{\widetilde{X}^{(p)}/W_2(k)} \arrow[d] & \\
\widetilde{F}^*\Omega^1_{\widetilde{X}^{(p)}/W_2(k)} \arrow[r, "\mathrm{d}\widetilde{F}"] \arrow[d, "(\mathrm{mod}\ p)"'] & p\Omega^1_{\widetilde{X}/W_2(k)} \arrow[d, shift right, "\frac{1}{[p]}"'] \\
F^*\Omega^1_{X^{(p)}} \arrow[r] & \Omega^1_X. \arrow[u, shift right, "\times p"']
\end{tikzcd}$$

Thus we have construct the first order derivation of $\widetilde{F}$ along p, that is,

$$\frac{\mathrm{d}\widetilde{F}}{[p]}\colon F^*\Omega^1_{X^{(p)}} \to \Omega^1_X.$$

Proposition 9.3.1. The morphism $\widetilde{F}$ induces a morphism of complexes

$$\varphi\colon \bigoplus_{i=0}^{n} \Omega^i_{X^{(p)}} \to F_*\Omega^\bullet_X$$

such that it induces the Cartier isomorphism on the cohomology sheaf. In particular, φ is a quasi-isomorphism.

Proof. Consider the following diagram

$$\begin{tikzcd}
\mathcal{O}_{X^{(p)}} \arrow[r, "0"] \arrow[d, "\varphi_0=F"'] & \Omega^1_{X^{(p)}} \arrow[r, "0"] \arrow[d, "\varphi_1=\frac{\mathrm{d}\widetilde{F}}{[p]}"'] & \dots \arrow[r, "0"] & \Omega^n_{X^{(p)}} \arrow[d, "\varphi_n"'] \\
F_*\mathcal{O}_X \arrow[r, "F_*\mathrm{d}"] & F_*\Omega^1_X \arrow[r] & \dots \arrow[r, "F_*\mathrm{d}"] & F_*\Omega^n_X.
\end{tikzcd}$$

For any $i \geq 1$, we define

$$\begin{tikzcd}
\Omega^i_{X^{(p)}} \arrow[r] \arrow[d, "\cong"'] & F_*\Omega^i_X \arrow[d, "\cong"] \\
\bigwedge^i \Omega^1_{X^{(p)}} \arrow[r, "\bigwedge^i \varphi_1"] & \bigwedge^i \Omega^1_{X'}
\end{tikzcd}$$

□

Remark 9.3.1. However, the Frobenius map seldom lifts. For example, let X/k be a smooth projective curve of genus ≥ 2, then for any $W_2(k)$-lifting $\widetilde{X}$ of X, there is no $W_2(k)$-lifting of $F\colon X \to X^{(p)}$. Now let's explain why there is no such a lifting. Note that the operator $\mathrm{d}\widetilde{F}/[p]$ induces a non-zero morphism

$$F^*\Omega^1_{X^{(p)}} \to \Omega^1_X.$$

Thus it gives a non-zero global section of $(F^*\Omega_{X^{(p)}})^\vee \otimes \Omega^1_X$. On the other hand,

$$\begin{aligned}
\deg((F^*\Omega_{X^{(p)}})^\vee \otimes \Omega^1_X) &= -p\deg\Omega^1_{X^{(p)}} + \deg\Omega^1_X \\
&= (2g-2)(1-p).
\end{aligned}$$

This shows the degree of $(F^*\Omega_{X^{(p)}})^\vee \otimes \Omega^1_X$ is non-positive when genus of X is ≥ 2, and thus it cannot admit a non-zero global section.

On the other hand, by Theorem 9.2.2, the obstruction of the lifting $\widetilde{F}$ is given by a class $\mathrm{ob}(F) \in \mathrm{Ext}^1(g_0^*\Omega^1_{\widetilde{X}^{(p)}/W_2(k)}, p\mathcal{O}_{\widetilde{X}}) \cong \mathrm{Ext}^1(F^*\Omega_{X^{(p)}}, \mathcal{O}_X) \cong H^1(X^{(p)}, F^*T_{X^{(p)}})$.

$$
\begin{tikzcd}
 & X^{(p)} \arrow[rr] \arrow[dd] & & \widetilde{X}^{(p)} \arrow[dd] \\
X \arrow[ur, "F"] \arrow[urrr, "g_0"] \arrow[rr] \arrow[dr] & & \widetilde{X} \arrow[ur, dashed, "\widetilde{F}"'] \arrow[dr] & \\
 & k \arrow[rr] & & W_2(k).
\end{tikzcd}
$$

9.3.2. *General case.* Now let's prove the Deligne-Illusie's decomposition theorem.

Theorem 9.3.1 (Deligne-Illusie)**.** *Let X/k be a smooth variety over k such that X is $W_2(k)$-liftable and $\dim_k X = n < p$. Then there exists an affine open covering $\mathfrak{U}$ of X, and an explicit a quasi-isomorphism*

$$\varphi\colon \bigoplus_{i=0}^{n} \Omega^i_{X^{(p)}}[-i] \to C^\bullet(\mathfrak{U}, F_*\Omega^\bullet_X),$$

where $C^\bullet(\mathfrak{U}, F_*\Omega^\bullet_X)$ is the total complex associated to the Čech double complex $(C^\bullet(\mathfrak{U}, F_*\Omega^\bullet_X), \delta, F_*\mathrm{d})$.

Proof. Let $\mathfrak{U} = \{U_i\}$ be an affine open covering of X. Then by Remark 9.3.1, over each U_i, we may choose a Frobenius lifting $\widetilde{F}_i\colon \widetilde{U}_i \to \widetilde{U}_i^{(p)} \hookrightarrow \widetilde{X}^{(p)}$. By the proof of Theorem 9.2.2, we get the following commutative diagram by restricting both $\widetilde{F}_i$ and $\widetilde{F}_j$ to $U_i \cap U_j$.

$$
\begin{tikzcd}
\Omega^1_{X^{(p)}} \arrow[r, "\xi_i - \xi_j"] \arrow[dr, "h_{ij}"] & F_*\Omega^1_{U_{ij}} \\
\mathcal{O}_{X^{(p)}} \arrow[u, "\mathrm{d}"] \arrow[r, "\frac{1}{[p]}(F_i^* - F_j^*)"'] & F_*\mathcal{O}_{U_{ij}}, \arrow[u, "F_*\mathrm{d}"']
\end{tikzcd}
$$

where $\xi_i = \frac{\mathrm{d}\widetilde{F}_i}{[p]}$ and $\{h_{ij}\}$ is a Čech 1-cocycle representing the class $\mathrm{ob}(F) \in H^1(X, F^*T_{X^{(p)}})$. For convenience, we collect the above diagram as follows

$$
\begin{cases}
h_{ij}\colon F^*\Omega^1_{X^{(p)}} \to \mathcal{O}_{U_{ij}} \\
\mathrm{d}h_{ij} = \xi_i - \xi_j \\
h_{ik} = h_{ij} + h_{jk}.
\end{cases} \tag{9.1}
$$

□

9.4. **Applications of de Rham decomposition.**

9.4.1. *E_1-degeneration.* Let k be an algebraically closed field with characteristic $p > 0$. In this section we will show that the decomposition of Deligne-Illusie implies the E_1-degeneration in characteristic p.

Theorem 9.4.1. Let X/k be a smooth proper variety which is $W_2(k)$-liftable and of $\dim_k X = n < p$. Then the Hodge to de Rham spectral sequence degenerates at E_1-page.

Proof. Firstly, it's clear $\dim_k H^j(X, \Omega^i_X) < \infty$ since X is proper. Note that the relative Frobenius $F\colon X \to X^{(p)}$ is an identity topologically. Then apply the Leray spectral sequence to it, one has

$$\mathbb{H}^n(X, \Omega^\bullet_X) \cong \mathbb{H}^n(X^{(p)}, \Omega^\bullet_{X^{(p)}}).$$

Let $\Omega^\bullet_X \to \mathcal{I}^\bullet$ be an injective resolution. Then $F_*\Omega^\bullet_X \to F_*\mathcal{I}^\bullet$ is still an injective resolution since F is an isomorphism as sheaves of abelian groups. Thus

$$\begin{aligned}\mathbb{H}^*(X^{(p)}, F_*\Omega^\bullet_X) &= H^*(\Gamma(X^{(p)}, F_*\mathcal{I}^\bullet)) \\ &= H^*(\Gamma(X, \mathcal{I}^\bullet)) \\ &= \mathbb{H}^*(X, \Omega^\bullet_X).\end{aligned}$$

On the other hand, by the decomposition theorem of Deligne-Illusie, one has

$$\begin{aligned}\mathbb{H}^*(X^{(p)}, F_*\Omega^\bullet_X) &= \mathbb{H}^*(X^{(p)}, \bigoplus_{i=0}^n \Omega^i_{X^{(p)}}[-i]) \\ &= \bigoplus_{i=0}^n H^{*-i}(X^{(p)}, \Omega^i_{X^{(p)}}).\end{aligned}$$

Thus

$$\begin{aligned}\dim_k \mathbb{H}^*(X, \Omega^\bullet_X) &= \sum_{i=0}^n \dim_k H^{*-i}(X^{(p)}, \Omega^i_{X^{(p)}}) \\ &= \sum_{i=0}^n \dim_k H^{*-i}(X, \Omega^i_X),\end{aligned}$$

where the last equality holds since $\pi\colon X^{(p)} \to X$ is a flat base change, and thus $\pi^*\Omega^i_X \cong \Omega^i_{X^{(p)}}$. This completes the proof. ∎

9.4.2. *Kodaira-Akizuki-Nakano theorem.*

Theorem 9.1. Let X/k be a smooth projective variety such that X/k is $W_2(k)$-liftable and $\dim_k X = n < p$. Then for any ample line bundle $\mathcal{L}$ on X, one has

$$H^j(X, \Omega^i_X \otimes \mathcal{L}) = 0$$

for all $i + j > n$.

Proof given by M.Raynaud. Note that $F^*\mathcal{L}^{-1} = (\mathcal{L}^{-1})^p$, then

$$\mathcal{L}^{-p} \xrightarrow{\nabla} \mathcal{L}^{-p} \otimes \Omega_X \xrightarrow{\nabla} \dots.$$

Then we project it

$$F_*\mathcal{L}^{-p} \xrightarrow{F_*\nabla} F_*(\mathcal{L}^{-p} \otimes \Omega^p_X) \to \dots.$$

By the projection formula one has

$$\begin{aligned} F_*\mathcal{L}^{-p} &= \mathcal{L}^{-1}\otimes F_*\mathcal{O}_X \\ F_*(\mathcal{L}^{-p}\otimes\Omega_X^p) &= \mathcal{L}^{-1}\otimes F_*\Omega_X^1. \end{aligned}$$

One can find that $F_*\nabla = \mathrm{id}\otimes F_*\mathrm{d}$, and thus above complex is $(F_*\Omega_X, F_*\mathrm{d})\otimes\mathcal{L}^{-1}$. By Deligne-Illusie's decomposition one has

$$\bigoplus_{i=0}^{n}\Omega_X^i[-i]\otimes\mathcal{L}^{-1}.$$

Then

$$\begin{aligned} \dim_k \mathbb{H}^*(\bigoplus\Omega_X^i[-i]\otimes\mathcal{L}^{-1}) &= \dim_k \mathbb{H}^*(X, F_*(\mathcal{L}^{-p}\otimes\Omega_X)) \\ &= \dim_k \mathbb{H}^*(X, \mathcal{L}^{-p}\otimes\Omega_X) \\ &\leq \sum_{i+j=*}\dim_k H^j(X,\mathcal{L}^{-p}\otimes\Omega_X^i) \\ &\leq \sum_{i+j=*}\dim_k H^j(X,\mathcal{L}^{-Np}\otimes\Omega_X^i) \end{aligned}$$

Then by Serre vanishing one has

$$H^j(X,\Omega_X^i\otimes\mathcal{L}^{-1}) = 0$$

for all $i+j<n$. □

9.5. **From characteristic p to characteristic 0.** In this section, we're going to show the following theorem.

Theorem 9.5.1. Let k be a field of characteristic zero and X is a smooth proper k-scheme. Then the Hodge to de Rham spectral sequence degenerates at E_1-page.

There're two methods to prove this, one is using the transcendental methods over $\mathbb{C}$, and the other is using the characteristic p method. But before doing anything, we must "lower" the defining field of X and even obtain an integral method of it. The following are some necessary algebraic lemmas.

Lemma 9.5.1. Let $\{A_i\}_{i\in I}$ be a direct system with direct limit A.

(1) If E is a A-module of finite presented, then there exists $i_0\in I$ and E_{i_0} is a A_{i_0}-module of finite presented such that

$$E_{i_0}\otimes_{A_{i_0}} A\cong E.$$

Moreover, for $\alpha\colon E\to F$, an A-module morphism between finite presented A-modules, there exists a finite presented A_{i_0}-module morphism $\alpha_{i_0}\colon E_{i_0}\to F_{i_0}$ such that $\alpha_{i_0}\otimes_{A_{i_0}} A\cong\alpha$.

(2) Let $f\colon X \to S = \operatorname{Spec} A$ is a finite presented morphism[11]. Then there exists $i_0 \in I$ and $f_{i_0}\colon X_{i_0} \to \operatorname{Spec} S_{i_0} = \operatorname{Spec} A_{i_0}$ such that the following diagram is Cartisian

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X_{i_0} \arrow[d, "f_{i_0}"] \\ S \arrow[r] & S_{i_0}. \end{tikzcd}$$

Moreover, if f is smooth/proper/projective, then there exists $i_0 \in I$, and $f_{i_0}\colon X_{i_0} \to S_{i_0}$ such that f_{i_0} is smooth/proper/projective.

Proof. See [Gro66b]. □

Now by above lemma, we may get a Cartisian diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[d, "f"'] & X \arrow[l] \arrow[d, "f_k"] \\ S & \operatorname{Spec} k, \arrow[l] \end{tikzcd}$$

where $\mathfrak{X}/S$ is smooth and proper, and $S = \operatorname{Spec} A$ for some finite generated $\mathbb{Z}$-subalgebra $A \subseteq k$. Therefore we can take an embedding $A \hookrightarrow \mathbb{C}$, and consider

$$\begin{tikzcd} \mathfrak{X}_\xi \arrow[r] \arrow[d, "f_{\mathbb{C}}"'] & \mathfrak{X} \arrow[d, "f"] \\ \operatorname{Spec} \mathbb{C} \arrow[r] & S. \end{tikzcd}$$

Exercise 9.5.1. Let $X/\mathbb{C}$ be a smooth and proper variety. Then use Chow's lemma and the E_1-degeneration theorem for smooth projective $\mathbb{C}$-varieties to show the Hodge to de Rham spectral sequence attached to $\Omega^\bullet_{X/\mathbb{C}}$ degenerates at E_1-page.

Now let's prove the key theorem for this section.

Proof of Theorem 9.5.1. As $\operatorname{Spec} k$ is affine, we have

$$R^j(f_k)_*\Omega^i_X \cong H^j(X, \Omega^i_X).$$

By finiteness theorem of Serre, one has $\dim_k H^i(X, \Omega^j_X) < \infty$. Then by the spectral sequence, one has

$$\dim_k \mathbb{H}^n(X, \Omega^\bullet_X) \leq \sum_{i+j=n} \dim_k H^j(X, \Omega^i_X) < \infty.$$

Likewise, by the properness of f, both $R^j f_*\Omega^i_{\mathfrak{X}/S}$ □

[11] A morphism of schemes $f\colon X \to S$ is said to be **finite presented**, if it's locally finite presented, quasi-compact and quasi-seperated.

Part 4. Appendix

Appendix A. The ayclic resolution

A.1. **The ayclic sheaf.** In practice it may be difficult for us to choose an injective resolution, so we usual other resolutions to compute sheaf cohomology.

Definition A.1.1. A sheaf $\mathcal{F}$ is **acyclic** if $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Example A.1.1. Every injective sheaf is acyclic.

Definition A.1.2. Let $\mathcal{F}$ be a sheaf. An **acyclic resolution** of $\mathcal{F}$ is an exact sequence

$$0 \to \mathcal{F} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \mathcal{A}^2 \to \dots,$$

where $\mathcal{A}^i$ is acyclic for all i.

Proposition A.1.1. The cohomology of sheaf can be computed by using acyclic resolution.

In fact, it's a quite homological techniques, called dimension shifting, so we will state this technique in language of homological algebra. Let's see a baby version of it.

Example A.1.2. Let $\mathcal{F}$ be a left exact functor and $0 \to A \to M_1 \to B \to 0$ be an exact sequence with M_1 is $\mathcal{F}$-acyclic. Then $R^{i+1}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$ for $i > 0$, and $R^1\mathcal{F}(A)$ is the cokernel of $\mathcal{F}(M_1) \to \mathcal{F}(B)$.

Proof. By considering the long exact sequence induced by $0 \to A \to M^1 \to B \to 0$, one has

$$R^i\mathcal{F}(M^1) \to R^i\mathcal{F}(B) \to R^{i+1}\mathcal{F}(A) \to R^{i+1}\mathcal{F}(M^1).$$

(1) If $i > 0$, then $R^i\mathcal{F}(M^1) = R^{i+1}\mathcal{F}(M^1) = 0$ since M^1 is $\mathcal{F}$-acyclic, and thus $R^{i+1}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$ for $i > 0$.
(2) If $i = 0$, then

$$0 \to \mathcal{F}(M^1) \to \mathcal{F}(B) \to R^1\mathcal{F}(A) \to 0$$

implies $R^1\mathcal{F}(A) = \operatorname{coker}\{\mathcal{F}(M^1) \to \mathcal{F}(B)\}$.

□

Now let's prove dimension shifting in a general setting.

Lemma A.1.1 (dimension shifting)**.** If

$$0 \to A \to M^1 \to M^2 \to \cdots \to M^m \to B \to 0$$

is exact with M^i is $\mathcal{F}$-acyclic, then $R^{i+m}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$ for $i > 0$, and $R^m\mathcal{F}(A)$ is the cokernel of $\mathcal{F}(M^m) \to \mathcal{F}(B)$.

Proof. Prove it by induction on m. For $m = 1$, we already see it in Example A.1.2. Assume it holds for $m < k$, then for $m = k$, let's split $0 \to A \to M^1 \to M^2 \to \cdots \to M^k \xrightarrow{\mathrm{d}_k} B \to 0$ into two exact sequences

$$0 \to A \to M^1 \to M^2 \to \cdots \to M^{k-1} \to \ker \mathrm{d}_k \to 0$$

$$0 \to \ker \mathrm{d}_k \to M^k \xrightarrow{\mathrm{d}_k} B \to 0.$$

Then by induction hypothesis, for $i > 0$ we have

$$R^{i+k-1}\mathcal{F}(A) \cong R^i\mathcal{F}(\ker \mathrm{d}_k)$$
$$R^{i+1}\mathcal{F}(\ker \mathrm{d}_k) \cong R^i\mathcal{F}(B).$$

Combine these two isomorphisms together we obtain $R^{i+k}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$ for $i > 0$, as desired. For $i = 0$, it suffices to let $i = 1$ in $R^{i+k-1}\mathcal{F}(A) \cong R^i\mathcal{F}(\ker \mathrm{d}_k)$, then we obtain

$$R^k\mathcal{F}(A) = R^1\mathcal{F}(\ker \mathrm{d}_k) = \operatorname{coker}\{\mathcal{F}(M^k) \to \mathcal{F}(B)\}.$$

This completes the proof. □

Corollary A.1.1. If $0 \to A \to M^\bullet$ is a $\mathcal{F}$-acyclic resolution, then $R^i\mathcal{F}(A) = H^i(\mathcal{F}(M^\bullet))$.

Proof. Truncate the resolution as

$$0 \to A \to M^0 \to M^1 \to \dots M^{i-1} \to B \to 0$$
$$0 \to B \to M^i \to M^{i+1} \to \dots$$

Since we already have $R^i\mathcal{F}(A) = \operatorname{coker}\{\mathcal{F}(M^{i-1}) \to \mathcal{F}(B)\}$, and $\mathcal{F}$ is left exact, one has

$$\mathcal{F}(B) = \ker\{\mathcal{F}(M^i) \to \mathcal{F}(M^{i+1})\}.$$

Thus we obtain

$$R^i\mathcal{F}(A) = \operatorname{coker}\{\mathcal{F}(M^{i-1}) \to \ker\{\mathcal{F}(M^i) \to \mathcal{F}(M^{i+1})\}\} = H^i(\mathcal{F}(M^\bullet)).$$

□

A.2. **The flabby sheaf.** The first kind of acyclic sheaf we're going to introduce is flabby sheaf[12].

Definition A.2.1. A sheaf $\mathcal{F}$ is **flabby** if for all open $U \subseteq V$, the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective.

Example A.2.1. A constant sheaf on an irreducible topological space is flabby.

Proof. Note that the constant presheaf on a irreducible topological space is a sheaf in fact, and it's easy to see constant presheaf is flabby. □

In particular, we have

[12]Some authors also call this flasque sheaf.

Example A.2.2. Let X be an algebraic variety. Then constant sheaf $\mathbb{Z}_X$ is flabby.

Example A.2.3. If $\mathcal{F}$ is a flabby sheaf on X, and $f\colon X \to Y$ is a continuous map, then $f_*\mathcal{F}$ is a flabby sheaf on Y.

Proof. For $V \subseteq W$ in Y, it suffices to show $f_*\mathcal{F}(W) \to f_*\mathcal{F}(V)$ is surjective, and that's exactly

$$\mathcal{F}(f^{-1}W) \to \mathcal{F}(f^{-1}V).$$

Then it's surjective since $\mathcal{F}$ is flabby. $\square$

Example A.2.4. An injective sheaf is flabby.

Proof. Let $\mathcal{I}$ be an injective sheaf and $V \subseteq U$ be open subsets. Now we define sheaf $\underline{\mathbb{Z}}_U$ on X by

$$\underline{\mathbb{Z}}_U := \begin{cases} \underline{\mathbb{Z}}(W), & W \subseteq U \\ 0, & \text{otherwise} \end{cases}$$

where $\underline{\mathbb{Z}}$ is constant sheaf valued in $\mathbb{Z}$, and similarly we define sheaf $\underline{\mathbb{Z}}_V$. By construction one has $\underline{\mathbb{Z}}_U(W) = \underline{\mathbb{Z}}_V(W)$ unless $W \subseteq U$ and $W \not\subseteq V$. Thus we obtain an exact sequence

$$0 \to \underline{\mathbb{Z}}_V \to \underline{\mathbb{Z}}_U.$$

Applying the functor $\operatorname{Hom}(-, \mathcal{I})$, which is exact, we obtain an exact sequence

$$\operatorname{Hom}(\underline{\mathbb{Z}}_U, \mathcal{I}) \to \operatorname{Hom}(\underline{\mathbb{Z}}_V, \mathcal{I}) \to 0.$$

Now let's explain why we need such a weird sheaf $\underline{\mathbb{Z}}_U$. In fact, we will prove $\operatorname{Hom}(\underline{\mathbb{Z}}_U, \mathcal{I}) = \mathcal{I}(U)$. Indeed since $\varphi\colon \underline{\mathbb{Z}}_U \to \mathcal{I}$ is a sheaf morphism. Then if $W \not\subseteq U$, then $\varphi(U)$ must be zero. If $W = U$, then the group of sections of $\underline{\mathbb{Z}}_U(U)$ over any connected component is simply $\mathbb{Z}$ and hence $\varphi(U)$ on this connected component is determined by the image of $1 \in \mathbb{Z}$. Thus $\varphi(U)$ can be thought of an element of $\mathcal{I}(U)$. Now on any proper open subset of U, φ is determined by restriction maps. Hence $\operatorname{Hom}(\underline{\mathbb{Z}}_U, \mathcal{I}) = \mathcal{I}(U)$, as desired. The same argument shows $\operatorname{Hom}(\underline{\mathbb{Z}}_U, \mathcal{I}) = \mathcal{I}(V)$, and thus we obtain an exact sequence

$$\mathcal{I}(U) \to \mathcal{I}(V) \to 0,$$

which shows $\mathcal{I}$ is flabby. $\square$

Our goal is to prove a flabby sheaf is acyclic, but we still need some property of flabby sheaves.

Proposition A.2.1. If $0 \to \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \to 0$ is an exact sequence of sheaves, and $\mathcal{F}'$ is flabby, then for any open subset U, the sequence

$$0 \to \mathcal{F}'(U) \xrightarrow{\phi(U)} \mathcal{F}(U) \xrightarrow{\psi(U)} \mathcal{F}''(U) \to 0$$

is exact.

Proof. It suffices to show $\mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ is exact. Here we only gives a sketch of the proof. Since we have exact sequence on stalks for each $p \in U$ as follows

$$0 \to \mathcal{F}'_p \xrightarrow{\phi_p} \mathcal{F}_p \xrightarrow{\psi_p} \mathcal{F}''_p \to 0$$

Then for each $s \in \mathcal{F}''(U)$, there exists $t_p \in \mathcal{F}_p$ such that $\psi_p(t_p) = s|_p$, so there exists open subset $V_p \subseteq U$ containing p and $t \in \mathcal{F}(V_p)$ such that $\psi(t) = s|_{V_p}$. If we can glue these t together then we get a section in $\mathcal{F}(U)$ and is mapped to s, which completes the proof. However, they may not equal on the intersection. But things are not too bad, consider another point q and $t' \in \mathcal{F}(V_q)$ such that $\psi(t') = s|_{V_q}$, $(t' - t)|_{V_p \cap V_q} \in \ker \psi(V_p \cap V_q) = \operatorname{im} \phi(V_p \cap V_q)$. So there exists $t'' \in \mathcal{F}'(V_p \cap V_q)$ such that

$$\phi(t'') = (t' - t)|_{V_p \cap V_q}$$

Now since $\mathcal{F}'$ is flabby, then there exists $t''' \in \mathcal{F}(V_p)$ such that $t'''|_{V_p \cap V_q} = t''$. And consider $t + \phi(t''') \in \mathcal{F}(V_p)$, which will coincide with t' on $V_p \cap V_q$. After above corrections, we can glue t after correction together. $\square$

Proposition A.2.2. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathcal{F}'$ and $\mathcal{F}$ are flabby, then $\mathcal{F}''$ is flabby.

Proof. Take $V \subseteq U$ and consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathcal{F}'(U) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}''(U) \arrow[r] \arrow[d, dashed] & 0 \arrow[d] \\
0 \arrow[r] & \mathcal{F}'(V) \arrow[r] & \mathcal{F}(V) \arrow[r] & \mathcal{F}''(V) \arrow[r] & 0
\end{tikzcd}$$

Then the desired result follows from five lemma. $\square$

Proposition A.2.3. A flabby sheaf is acyclic.

Proof. Let $\mathcal{F}$ be a flabby sheaf. Since there are enough injective objects in the category of sheaf of abelian groups, there is an exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0$$

with $\mathcal{I}$ is injective. By Example A.2.4 we have $\mathcal{I}$ is flabby, and thus by Proposition A.2.2 we have $\mathcal{Q}$ is flabby. Consider the long exact sequence induced from above short exact sequence

$$\mathcal{F}(X) \to \mathcal{I}(X) \to \mathcal{Q}(X) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{I}) \to \dots$$

Note that $H^1(X, \mathcal{I}) = 0$ since $\mathcal{I}$ is injective, and thus acyclic. Then $H^1(X, \mathcal{F}) = \operatorname{coker}\{\mathcal{I}(X) \to \mathcal{Q}(X)\}$. But Proposition A.2.1 shows that $\mathcal{I}(X) \to \mathcal{Q}(X)$ is surjective since $\mathcal{F}$ is flabby, so $H^1(X, \mathcal{F}) = 0$.

Now let's prove $H^k(X, \mathcal{F}) = 0$ for $k > 0$ by induction on k, and above argument shows it's true for $k = 1$. Assume this holds for $k < n$, and consider

$$\dots \to H^{n-1}(X, \mathcal{Q}) \to H^n(X, \mathcal{F}) \to H^n(X, \mathcal{I}) \to H^n(X, \mathcal{Q}) \to \dots$$

By induction hypothesis, we can reduce above sequence to

$$\cdots \to 0 \to H^n(X, \mathcal{F}) \to 0 \to H^n(X, \mathcal{Q}) \to \dots$$

which implies $H^n(X, \mathcal{F}) = 0$. This completes the proof. $\square$

A.3. **The soft sheaf.** The second kind of acyclic sheaves is called soft sheaves, which is quit similar to flabby.

Definition A.3.1. A sheaf $\mathcal{F}$ over X is **soft** if for any closed subset $S \subseteq X$ the restriction map $\mathcal{F}(X) \to \mathcal{F}(S)$ is surjective.

Remark A.3.1. For closed subset S, the section over it is defined by

$$\mathcal{F}(S) := \varinjlim_{S \subseteq U} \mathcal{F}(U)$$

Parallel to Proposition *A.2.1* and Proposition *A.2.2*, soft sheaf has the following properties:

Proposition A.3.1. If $0 \to \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \to 0$ is an exact sequence of sheaves, and $\mathcal{F}'$ is soft, then the following sequence

$$0 \to \mathcal{F}'(X) \xrightarrow{\phi(X)} \mathcal{F}(X) \xrightarrow{\psi(X)} \mathcal{F}''(X) \to 0$$

is exact.

Proposition A.3.2. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathcal{F}'$ and $\mathcal{F}$ are soft, then $\mathcal{F}''$ is soft.

Proposition A.3.3. A soft sheaf is acyclic.

So you may wonder, what's the difference between flabby and soft since the definitions are quite similar, and both of them are acyclic. Clearly by definition of sections over a closed subset, we know that every flabby sheaf is soft, but converse fails

Example A.3.1. The sheaf of smooth functions on a smooth manifold is soft but not flabby.

Lemma A.3.1. If $\mathcal{M}$ is a sheaf of modules over a soft sheaf of rings $\mathcal{R}$, then $\mathcal{M}$ is a soft sheaf.

Proof. Let $s \in \mathcal{M}(K)$ for some closed subset $K \subseteq X$. Then s extends to some open neighborhood U of K. Let $\rho \in \mathcal{R}(K \cup (X \setminus U))$ be defined by

$$\rho = \begin{cases} 1, & \text{on } K \\ 0, & \text{on } X \setminus U \end{cases}$$

Since $\mathcal{R}$ is soft, then ρ extends to a section over X, then $\rho \circ s$ is the desired extension of s. $\square$

A.4. **The fine sheaf.** Another important kind of acyclic sheaves, which behaves like sheaf of differential forms Ω_X^k is called fine sheaf. Recall what is a partition of unity: Let $U = \{U_i\}_{i\in I}$ be a locally finite open covering of topological space X. A partition of unity subordinate to U is a collection of continuous functions $f_i\colon U_i \to [0,1]$ for each $i \in I$ such that its support lies in U_i, and for any $x \in X$

$$\sum_{i\in I} f_i(x) = 1.$$

Definition A.4.1. A **fine sheaf** $\mathcal{F}$ on X is a sheaf of $\mathcal{A}$-modules, where $\mathcal{A}$ is a sheaf of rings such that for every locally finite open covering $\{U_i\}_{i\in I}$ of X, there is a partition of unity

$$\sum_{i\in I} \rho_i = 1$$

where $\rho_i \in \mathcal{A}(X)$ and $\operatorname{supp}(\rho_i) \subseteq U_i$.

Remark A.4.1. For a sheaf $\mathcal{F}$ on X and a section $s \in \mathcal{F}(X)$, its support is defined as

$$\operatorname{supp}(s) := \overline{\{x \in X : s|_x \neq 0\}}.$$

Proposition A.4.1. A fine sheaf is acyclic.

Proof. Let $\mathcal{F}$ be a sheaf of $\mathcal{A}$-modules and a fine sheaf. And choose a injective resolution

$$0 \to \mathcal{F} \xrightarrow{\mathrm{d}} \mathcal{I}^0 \xrightarrow{\mathrm{d}} \mathcal{I}^0 \xrightarrow{\mathrm{d}} \mathcal{I}^1 \xrightarrow{\mathrm{d}} \dots$$

such that $\mathcal{I}^i$ are injective sheaves of $\mathcal{A}$-modules. Let $s \in \mathcal{I}^p(X)$ such that $\mathrm{d}s = 0$. Then by exactness of injective resolution we have X is covered by open subsets U_i such that for each i there is an element $t_i \in \mathcal{I}^{p-1}(U_i)$ such that $\mathrm{d}t_i = s|_{U_i}$. By passing to a refinement we may assume that the cover $\{U_i\}$ is locally finite. Let $\{\rho_i\}$ be a partition of unity subordinate to $\{U_i\}$. Then we have $t = \sum \rho_i t_i \in \mathcal{I}^{p-1}(X)$ such that $\mathrm{d}t = s$. This completes the proof. □

Example A.4.1. Let M be a smooth manifold and $\pi\colon E \to M$ be a vector bundle. Then the sheaf of smooth sections of E is a $C^\infty(M)$-module sheaf, which is a fine sheaf. In particular, the sheaf of tangent bundle, sheaf of differential forms Ω_M and k-forms Ω_M^k are fine sheaves.

Remark A.4.2. As a consequence, it's meaningless to compute cohomology of sheaf of differential k-forms, or any other vector bundle over a smooth manifold. But in complex version, something interesting happens. Let $(X, \mathcal{O}_X)$ be a complex manifold and $\pi\colon E \to X$ be a holomorphic vector bundle. Then the sheaf of holomorphic sections of E is not a fine sheaf since there is no partition of unity may not be holomorphic, so the cohomology of holomorphic vector bundle is not trivial, and that's what Dolbeault cohomology computes.

Lemma A.4.1. Any fine sheaf is soft.

Proof. Let $\mathcal{F}$ be a fine sheaf, $S \subseteq X$ closed and $s \in \mathcal{F}(S)$. Let $\{U_i\}$ be an open covering of S and $s_i \in \mathcal{F}(U_i)$ such that

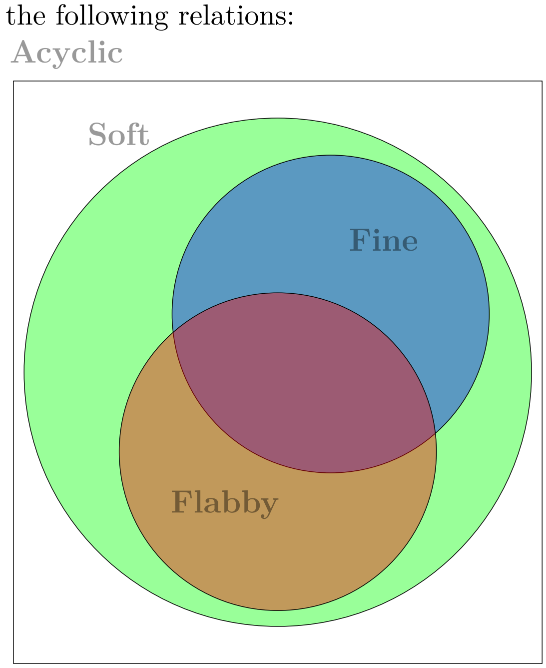
$$s_i|_{S\cap U_i} = s|_{S\cap U_i}.$$

Let $U_0 = X - S$, and $s_0 = 0$. Then $\{U_i\} \coprod \{U_0\}$ is an open covering of X. Without lose of generality, we assume this open covering is locally finite and choose a partition of unity $\{\rho_i\}$ subordinate to it. Then

$$\overline{s} := \sum_i \rho_i(s_i)$$

is a section in $\mathcal{F}(X)$ which extends s. $\square$

Remark A.4.3. Until now, we have shown that soft, fine and flabby sheaves are acyclic. Lemma A.4.1 shows fine sheaf is soft, and by definition a flabby sheaf is soft. The Example A.3.1 shows that soft sheaf may not be flabby, and constant sheaf on an irreducible space is flabby but not fine. In a summary, we have the following relations:



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