RIEMANN SURFACE

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Contents

1. Riemann surface

1.1. Definitions and Examples.

Definition 1.1.1. *If X is a surface, a (almost) complex structure is a smooth map $J : TX \to TX$, such that for any $p \in X$, $J_p : T_pX \to T_pX$ is a linear map with $J_p^2 = -\operatorname{id}$.*

Remark 1.1.2. If X admits a complex structure, then X is orientable.

Example 1.1.3. Assume X has a Riemann metric, and X is orientable. For any $v \in T_pX$, define $J(v)$ to be the tangent vector obtained by rotating v by $\pi/2$ counterclockwise.

Corollary 1.1.4. *Any orientable surface admits a complex structure.*

Example 1.1.5. If $X = \mathbb{C}$, then $T_qX \cong \mathbb{C}, \forall q \in X$, choose $v \in T_qX$, define $J(v) = iv$, then J is a complex structure on X.

Definition 1.1.6. *Assume X is a topological space. A complex chart on X is an open subset $U \subset X$ together with a homeomorphism $\varphi : U \to V \subset \mathbb{C}$, where V is an open subset. If $p \in U$, and $\varphi(p) = 0$, then (U, φ) is called a chart centered at p. For $q \in U$, $z = \varphi(q)$ is called a local coordinate of q.*

Definition 1.1.7. *If $(U_1, \varphi_1), (U_2, \varphi_2)$ are two charts on X, we say they're compatible if $U_1 \cap U_2 = \varnothing$ or*

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is holomorphic.

Definition 1.1.8. *An atlas is a collection of compatible charts $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$, such that $\bigcup_{\alpha\in I} U_\alpha = X$. Two atlas $\mathscr{A}, \mathscr{B}$ are equivalent if every chart in $\mathscr{A}$ and every chart in $\mathscr{B}$ is compatible.*

Definition 1.1.9. *A complex structure on X is an equivalent class of atlas on X.*

Remark 1.1.10. Given an atlas $\mathscr{A}$ on X, we can use charts in $\mathscr{A}$ to define $J : TX \to TX$ such that $J^2 = -\operatorname{id}$.

Definition 1.1.11. *A Riemann surface is a second countable, connected, Hausdorff topological space X together with a complex chart on X.*

Example 1.1.12. Every open subset of $\mathbb{C}$ is a Riemann surface.

Example 1.1.13. $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, consider

$$U_1 = S^2 \backslash \{(0, 0, 1)\} \xrightarrow{\varphi_1} \mathbb{C}$$

where φ_1 is defined as $\varphi_1(x, y, z) = \frac{x}{1-z} + i\frac{y}{1-z} = w$. Similarly consider

$$U_2 = S^2 \backslash \{(0, 0, -1)\} \xrightarrow{\varphi_2} \mathbb{C}$$

where φ_2 is defined as $\varphi_2(x, y, z) = \frac{x}{1+z} - i\frac{y}{1+z} = w'$. Note that $ww' = \frac{x^2+y^2}{1-z^2} = 1$. And it's easy to see the transition function is $T(w) = \frac{1}{w}$. So $\{U_1, U_2\}$ is an atlas of S^2.

Example 1.1.14. $\mathbb{CP}^1 = \{\text{complex 1-dimensional subspaces of } \mathbb{C}^2\}$, is called a 1-dimensional projective space. Given a point $(0,0) \neq (z,w) \in \mathbb{C}^2$, exists a unique point $[z,w] \in \mathbb{CP}^1$, called the homogenous coordinate of $\mathbb{CP}^1$. Consider

$$U_1 = \{[z,w] \mid z \neq 0\} \xrightarrow{\varphi_1} \mathbb{C}$$

where φ_1 is defined as $\varphi_1([z,w]) = z/w$. Similarly consider

$$U_2 = \{[z,w] \mid w \neq 0\} \xrightarrow{\varphi_2} \mathbb{C}$$

where φ_2 is defined as $\varphi_2([z,w]) = w/z$. It's easy to check $\{U_1, U_2\}$ is a atlas of $\mathbb{CP}^1$.

In fact, $\mathbb{CP}^1$ is a Riemann surface which is isomorphic to S^2.

Example 1.1.15. Given two nonzero $w_1, w_2 \in \mathbb{C}$, with $w_1 \neq aw_2$ for any $a \in \mathbb{C}$. Define lattice:

$$L = \mathbb{Z}w_1 + \mathbb{Z}w_2$$

In fact, L is a subgroup of $\mathbb{C}$ with respect to operation "+".

Then $T = \mathbb{C}/L$ is a Riemann surface called complex torus. Consider the projection $\pi : \mathbb{C} \to T$. For $p \in T$, find one of its inverse image of π, denoted by z_0. Choose $\varepsilon \in \mathbb{R}^+$ small enough such that

$$B_{2\varepsilon} \cap L = \{0\}$$

Consider

$$B_\varepsilon(z_0) \xrightarrow{\pi} \pi(B_\varepsilon(z_0)) \subset T$$

and the condition on ε implies $\pi|_{B_\varepsilon}$ is injective. So let $\{\pi(B_\varepsilon(z_0))\}$ be a open cover of T, and π^{-1} is the parametrization, this is an atlas of T.

Remark 1.1.16. The complex structure of complex torus depends on w_1, w_2. In fact, all complex structure of complex torus forms a Riemann surface in the form of $\mathbb{C}$. *

1.2. **Holomorphic function and Properties.**

Definition 1.2.1. *If X is a Riemann surface, $W \subset X$ is a open subset. The function $f : W \to \mathbb{C}$ is a complex valued function on W. f is called holomorphic at $p \in W$, if there exists a chart (U, φ) of p such that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ is holomorphic at $\varphi(p)$. f is called holomorphic on W, if it is holomorphic at any $p \in W$.*

Theorem 1.2.2 (Maximum modulus theorem)**.** *For a Riemann surface X, $W \subset X$ is an open subset, and f is a holomorphic function on W. If there exists a point $p \in W$, such that $|f(p)| \geq |f(x)|$ for all $x \in W$, then f must be a constant.*

Proof. Clear. □

*The space consists of all complex structure of a Riemann surface is called the moduli space of it.

Corollary 1.2.3. *If X is a compact Riemann surface, then any global holomorphic funtion f must be constant.*

So, it's boring to consider holomorphic funtions on a compact Riemann surface. In order to get something interesting, we need to consider meromorphic functions.

Definition 1.2.4. *If X is a Riemann surface, let f be a holomorphic function defined on $U\backslash\{p\}$ where $U \subset X$ is an open subset. p is called a removbale singularity/pole/essential singularity, if there exists a chart (U, φ) of p, such that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ has $\varphi(p)$ as a removbale singularity/pole/essential singularity.*

Remark 1.2.5. We have the following criterions:

1. If $|f(x)|$ is bounded in a punctured neighborhood of p, then p is a removable singularity. And we can cancel the singularity by defining $f(p) = \lim_{x\to p} f(x)$.
2. If $\lim_{x\to p} |f(x)| = \infty$, then p is a pole.
3. If $\lim_{x\to p} |f(x)|$ doesn't exist, then p is a essential singularity.

Definition 1.2.6. *f is called a meromorphic function at p if p is either a removbale singularity or a pole, or f is holomorphic at p; f is called a meromorphic function on W, if it's meromorphic at any point $p \in W$.*

Remark 1.2.7. If f, g are meromorphic on W, then $f \pm g, fg$ are also meromorphic on W. If in addition, $g \not\equiv 0$, then f/g is also meromorphic on W.

Example 1.2.8. Consider f, g are two polynomials in variable z with $g \not\equiv 0$, then f/g is a meromorphic function on $S^2 = \mathbb{C} \cup \{\infty\}$. In fact, all meromorphic functions on S^2 are in this form.

Theorem 1.2.9 (Singularities and zeros)**.** *Let X be a Riemann surface and $W \subset X$ is an open subset, f is a meromorphic function on W, then set of singularities and zeros of f is discrete, unless $f \equiv 0$.*

Corollary 1.2.10. *If X is compact, $f \not\equiv 0$, then f has finitely many poles and zeros on X. As a consequence, if f, g are two meromorphic functions on an open subset $W \subset X$, and f agrees with g on a set with limit point in W, then $f \equiv g$.*

Definition 1.2.11. *Let X, Y be two Riemann surfaces, $F : X \to Y$. For a point $p \in X$, f is called holomorphic at p, if there exists a chart (U, φ) of p, and a chart (V, ψ) of $F(p)$, such that*

$$\psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V \cap F(U))$$

is holomorphic at $\varphi(p)$; F is called holomorphic in W, if F is holomorphic at any point in W.

Remark 1.2.12. $\psi \circ F \circ \varphi^{-1}$ is called the local representation of F at p.

Example 1.2.13. Any meromorphic function on X can be seen as a holomorphic map from X to S^2; Conversely, we can construct a meromorphic function from a holomorphic map from X to S^2.

Definition 1.2.14. *Two Riemann surfaces are called biholomorphic or isomorphic to each other, if there are two holomorphic map $F : X \to Y, G : Y \to X$, such that $F \circ G = G \circ F = \mathrm{id}$.*

Example 1.2.15. S^2 is biholomorphic to $\mathbb{RP}^2$.

Theorem 1.2.16 (Open mapping theorem)**.** *$F : X \to Y$ is a non-constant holomorphic map, then F is an open map.*

Corollary 1.2.17. *If X is compact, and Y is connected, $F : X \to Y$ is a non-constant holomorphic map, then Y is compact and $F(X) = Y$.*

Proof. By open mapping theorem, $F(X)$ is an open subset of Y, and $F(X)$ is compact in Y, since continous function maps compact set to compact set. Then $F(X)$ is both open and closed in Y, then $F(X) = Y$. $\square$

1.3. **Ramification covering.**

Theorem 1.3.1. *$F : X \to Y$ is a non-constant holomorphic function on Riemann surfaces, then for any $p \in Y$, $F^{-1}(y)$ is a discrete set. Furthemore, if X is compact, then $F^{-1}(y)$ only contains finite many points.*

So we wonder what's exact number of $F^{-1}(y)$, the local normal form tells you answer.

Theorem 1.3.2 (Local normal form)**.** *$F : X \to Y$ is a non-constant holomorphic function on X, then there is a local representation of F at $p \in X$, such that*

$$\psi \circ F \circ \varphi^{-1}(z) = z^k, \quad \forall z \in \varphi(U \cap F^{-1}(V))$$

k is called the multiplicity† *of F at p, denoted by $\mathrm{mult}_p(F)$. In fact, k is independent of the choice of charts.*

Proof. Fix a chart (U_2, φ_2) of $F(p)$, choose an arbitary local chart (U, ψ) of p such that $F(U) \subset U_2$, denote $\varphi_2 \circ F \circ \psi^{-1} = T$, then $T(0) = 0$. Consider the Taylor expansion of T at $w = 0$ has

$$T(w) = \sum_{k=m}^{\infty} a_k w^k, \quad a_m \neq 0$$

So $T(w) = w^m S(w)$, where $S(w)$ is a holomorphic function with $S(0) \neq 0$, then there exists a holomorphic function $R(w)$ such that $R^m(w) = S(w)$.

Then $T(w) = (wR(w))^m = (\eta(w))^m$, so $\eta(0) = 0, \eta'(0) = R(0) \neq 0$, so η is invertible near $w = 0$ by inverse funtion theorem. So there exists another chart of $p \in U_1 \subset U$, with

$$U \supset U_1 \xrightarrow{\psi} V \xrightarrow{\eta} V_1 \subset \mathbb{C}$$

†Sometimes this number is also called ramification of F at p.

then we can define a local chart $(U_1, \varphi_1 = \eta \circ \psi)$, and check

$$\varphi_2 \circ F \circ \varphi_1^{-1}(z) = \varphi_2 \circ F \circ \psi^{-1} \circ \eta^{-1}(z) = T(w) = (\eta(w))^m = z^m$$

What's more, we can see from the local normal form that for any $q \in Y, q \neq F(p)$ and q lies in a small neighborhood of p such that $F^{-1}(q)$ lies in a small neighborhood of p, then $F^{-1}(q)$ consists of exactly k points. So the ramification index is independent of the charts we choose. $\square$

Definition 1.3.3. *p is called a ramification point of a holomorphic map $F : X \to Y$, if $\mathrm{mult}_p(F) > 1$, such $F(p)$ is called a ramification value.*

Lemma 1.3.4. *p is a ramification point of a holomorphic map $F : X \to Y$ if $T'(w) = 0$, for any local representation of F.*

Corollary 1.3.5. *The set of ramification points of a holomorphic map is a discrete set.*

Theorem 1.3.6. *Assume X, Y are complex Riemann surface, $F : X \to Y$ is non-constant holomorphic function, for $q \in Y$, let*

$$d_q(F) = \sum_{p \in F^{-1}(q)} \mathrm{mult}_p(F)$$

then $d_q(F)$ is independent of $q \in Y$, and denoted by $\deg(F)$.

Proof. Consider $F : \mathbb{D} \to \mathbb{D}$, defined by $z \mapsto z^m$, it's easy to check $d_q(F) = m$, for all $q \in \mathbb{D}$.

For general case, for $q \in Y$, let $F^{-1}(q) = \{p_1, \dots, p_k\} \subset X$. Fix a chart (U_2, φ_2) centered at $q \in Y$, for any $i = 1, \dots, k$, we can find local chart $(U_{1,q}, \psi_i$ centered at $p_i \in X$, such that

$$\varphi_2 \circ F \circ \psi_i^{-1}(z) = z^{m_i}, \quad z \in \psi_i(U_{1,i})$$

where $m_i = \mathrm{mult}_{p_i}(F)$. Choose $q \in W \subset U_2$ such that $F^{-1}(W) \subset \bigcup_{i=1}^k U_{1,i}$, then for any $q \in W$

$$d_q(F) = \sum_{i=1}^k m_i$$

which can be seen from trivial case we discuss firstly. Then $d_q(F)$ is a locally constant function, then $d_q(F)$ must be global constant, since Y is connected. $\square$

Corollary 1.3.7. *X is a compact Riemann surface, and f is a meromorphic function on X, then the number (with multiplicity) of zeros is equal to the number (with multiplicity) of poles.*

Proof. Note that meromorphic function on X is equivalent to the holomorphic map from X to S^2. $\square$

1.4. **Hurwitz Formula.** Now let us forget the complex structure of Riemann surface, and recall some facts about topological invariants.

Let X be a compact oriented surface, we can say the genus of X is the number of "holes" which X has, informally. We can use genus to classify all oriented compact surfaces: any two surfaces which have the same genus are diffeomorphic to each other.

We can also define Euler characterisitic of X, as

$$\chi(X) := \sum_i (-1)^i \dim H_i(X)$$

And there is a connection between genus of X and $\chi(X)$,

$$\chi(X) = 2 - 2\,\mathrm{genus}(X)$$

so we can also use $\chi(X)$ to classify oriented compact surface.

Theorem 1.4.1 (Hurwitz Formula)**.** *Let X, Y be two compact Riemann surfaces, and $F : X \to Y$ be a non-constant holomorphic map, then*

$$2\,\mathrm{genus}(Y) - 2 = \deg(F)(2\,\mathrm{genus}(X) - 2) + \sum_{p \in X} (\mathrm{mult}_p(F) - 1)$$

Note that the set of ramification points is finite, then $\sum_{p\in X}(\mathrm{mult}_p(F) - 1)$ is a finite sum, and denoted by $B(F)$.

Proof. Choose a triangulation of Y such that its vertex are exactly ramification values of F. Let v denote the number of vertices of Δ, c and t denote the number of edges and triangles of Δ, where Δ denotes a triangulation of Y. We can get a triangulation Δ' of X, by pulling back Δ through F, and use v', c' and t' to denote the same thing in Δ'.

Then we have the following obvious relations

$$t' = td, \quad e' = ed$$

where $d = \deg(F)$. The relation between v and v' is a little bit complicated, consider $q \in Y$, then

$$|F^{-1}(q)| = \sum_{p \in F^{-1}(q)} 1 = d + \sum_{p \in F^{-1}(q)} (1 - \mathrm{mult}_p(F))$$

then

$$\begin{aligned} v' = & \sum_{\text{vertex } q \text{ of } \Delta} |F^{-1}(q)| \\ = & \sum_{\text{vertex } q \text{ of } \Delta} (d + \sum_{p \in F^{-1}(q)} (1 - \mathrm{mult}_p(F))) \\ = & vd + \sum_{\text{vertex } q \text{ of } \Delta} \sum_{p \in F^{-1}(q)} (1 - \mathrm{mult}_p(F)) \\ = & vd + \sum_{p \in X} (1 - \mathrm{mult}_p(F)) \end{aligned}$$

Then by the relation between Euler characterisitic and triangulation, we get the desired conclusion. □

Definition 1.4.2. *A holomorphic map F is called ramified if $B(F) > 0$, this is equivalent to F has at least one ramification point; A holomorphic map F is called unramified if $B(F) = 0$, this is equivalent to F is a covering map.*

Corollary 1.4.3. *Let X, Y be two compact Riemann surfaces, and $F : X \to Y$ is a non-constant holomorphic map, then consider*

1. *If $Y = S^2$, and $\deg(F) > 1$, then F must be ramified.*
2. *If $\operatorname{genus}(X) = \operatorname{genus}(Y) = 1$, then F must be unramified.*
3. $\operatorname{genus}(X) \geq \operatorname{genus}(Y)$.
4. *If $\operatorname{genus}(X) = \operatorname{genus}(Y) > 1$, then F must be an isomorphism.*

Proof. All of them are simple applications of Hurwitz Formula.

1. By Hurwitz Formula we have

$$B(F) = 2(\deg(F) - 1) + 2\operatorname{genus}(X) > 0$$

2. By Hurwitz Formula we have

$$0 = 0 + B(F)$$

3. If $\operatorname{genus}(Y) = 0$, it's trivial. Otherwise, we have

$$2\operatorname{genus}(X) - 2 \geq 2\operatorname{genus}(Y) - 2 + B(F)$$

since $\deg F \geq 1$.

4. By Hurwitz Formula we have

$$(1 - \deg(F))(2\operatorname{genus}(X) - 2) = B(F)$$

Then $\deg(F) = 1$, since $\deg(F) \geq 1$, $2\operatorname{genus}(X) - 2 > 0$ and $B(F) \geq 0$. □

Remark 1.4.4. From above corollary, we can see that genus, as a topological invariants, controls geometric properties heavily.

1.5. **Automorphism groups of lower genus surface.**

1.5.1. *Automorphism group of Riemann sphere.* Firstly we determine what does the holomorphic maps $f : S^2 \to S^2$ look like

Proposition 1.5.1. *Let $f : S^2 \to S^2$ be a holomorphic map. Then f is a rational function, i.e.*

$$f(z) = \frac{p(z)}{q(z)}$$

where $p(z), q(z) \in \mathbb{C}[z]$, and $q(z) \neq 0$.

Proof. Consider f as a meromorphic from S^2 to $\mathbb{C}$. Since the Riemann sphere is compact, f can have only finitely many poles, for otherwise a sequence of poles would cluster somewhere, giving a non-isolated singularity.

Especially, f has only finitely many poles in the plane. Let the poles occur at the plane z_1 through z_n with multiplicities e_1 through e_n. Define a polynomial

$$q(z) = \prod_{i=1}^{n} (z - z_i)^{e_j}$$

Then the function

$$p(z) = f(z)q(z)$$

has removbale singularities at the poles of f in $\mathbb{C}$, i.e. it is entire. So p has a power series representation on all of $\mathbb{C}$. Also, p is meromorphic at ∞, since both f and q are. This forces p to be a polynomial. This completes the proof. $\square$

Corollary 1.1. *The biholomorphic maps on S^2 take the form*

$$f(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C}$$

Proof. If the numerator or denominator of f were to have degree greater than 1 then by the local normal form, f would not be bijective. $\square$

Furthermore, we assume that f is expressed in the lowest term, i.e. the numerator is not a scalar multiple of denominator. This discussion narrows our considerations to functions of the form

$$f(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C}, ad - bc \neq 0$$

Then there is a surjective map

$$\mathrm{GL}_2(\mathbb{C}) \longrightarrow \mathrm{Aut}(S^2), \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto f(z) = \frac{az+b}{cz+d}$$

And after an direct check we will see it's a group homomorphism. But this homomorphism is clearly not injective, since all nonzero scalar multiples of a given matrix are taken to the same automorphism. The kernel of this homomorphism is

$$\mathbb{C}^\times I = \left\{ \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} : \lambda \in \mathbb{C}, \lambda \neq 0 \right\}$$

And by the first isomorphism theorem we have

$$\mathrm{GL}_2(\mathbb{C})/\mathbb{C}^\times \,\mathrm{I}_2 \xrightarrow{\sim} \mathrm{Aut}(S^2)$$

Furthermore, we have

$$\mathrm{GL}_2(\mathbb{C})/\mathbb{C}^\times \,\mathrm{I}_2 \cong \mathrm{PSL}_2(\mathbb{C})$$

And we have its complex dimension is 3, as a complex manifold.

1.5.2. *Automorphism group of complex torus.* Consider a lattice $L = \mathbb{Z}w_1 + \mathbb{Z}w_2$, let X denote the complex torus $X = \mathbb{C}/L$, a Riemann surface with genus 1. Moreover, there is a group structure on X, induced by $(\mathbb{C}, +)$ through natural projection $\pi : \mathbb{C} \to X$, defined as follows

$$[z_1] + [z_2] := [z_1 + z_2]$$

So, inversion map

$$[z] \mapsto [-z]$$

gives an automorphism.

For $a \in \mathbb{C}$,we can define a transformation

$$T_a : X \to X, \quad [z] \mapsto [z + a]$$

which is also an automorphism.

So, as we can see, there are too many automorphism on X, let $\mathrm{Aut}(X)$ denote all automorphisms on X, which forms a group which can reflect the symmetry of X.

Obviously, we have the following inclusion

$$\mathrm{Aut}(X) \supset \{T_{[a]} \mid [a] \in X\} \cup \{\text{inversion}\}$$

In fact, we will see later that $\mathrm{Aut}(X)$ is a complex manifold with $\dim_{\mathbb{C}} \mathrm{Aut}(X) = 1$, but for now, we can only conclude that $\dim_{\mathbb{C}} \mathrm{Aut}(X) \geq 1$.

Before we come to see what is the automorphism group of X, we consider a more general case, holomorphic map between complex torus.

Assume L, M are two different lattices in $\mathbb{C}$, $X = \mathbb{C}/L, Y = \mathbb{C}/M$ are two complex torus.

Let $F : X \to Y$ be a non-constant holomorphic map, after composing some translation T_a, we can assume that $F([0]) = [0]$. Since $\mathrm{genus}(X) = \mathrm{genus}(Y)$, then by Hurwitz formula F must be a covering map.

Let $\pi_X : \mathbb{C} \to X, \pi_Y : \mathbb{C}$ are natural projection. In fact, they're universal covering map.

Consider

$$\mathbb{C} \xrightarrow{\pi_X} X \xrightarrow{F} Y$$

then $F \circ \pi_X$ is also a universal covering of Y. By uniqueness of universal covering, there exists a holomorphic map[‡] $G : \mathbb{C} \to \mathbb{C}$ such that the following diagram commutes

$$\begin{tikzcd} \mathbb{C} \arrow[r, "G"] \arrow[d, "\pi_X"] & \mathbb{C} \arrow[d, "\pi_Y"] \\ X \arrow[r, "F"] & Y \end{tikzcd}$$

Since $F([0]) = [0]$, then $G(0) \in M$. After composing with a translation in $\mathbb{C}$ with respect to $-G(0)$, we can assume $G(0) = 0$.

For any $z \in \mathbb{C}, l \in L$, consider the difference $w(z, l) := G(z + l) - G(z)$. First note that $w(z, l)$ is a holomorphic with respect to z. What's more,

[‡]Clearly, G is not unique.

$w(z,l)$ must lie in M. So $w(z,l)$ must be a constant with respect to z, since M is discrete. So

$$\frac{\partial}{\partial z}w(z,l) = G'(z+l) - G'(z) = 0$$

That is, $G'(z)$ is periodic with respect to L, so $|G'(z)|$ is bounded. By Liouville's theorem, we have $G'(z)$ is constant.

So G must have the form $G(z) = \gamma z, \gamma \in \mathbb{C}$, since we assume $G(0) = 0$. Since $G(L) \subset G(M)$, we have

$$\gamma L \subset M$$

Since $G(z) = \gamma z$ is a group homomorphism, then F is also a group homomorphism between X and Y.

Clearly, F is an isomorphism if and only if $\gamma L = M$.

We summarize as follows:

Theorem 1.5.2. *Any holomorphic map $F : \mathbb{C}/L \to \mathbb{C}/M$ is induced by a linear map*

$$G : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \gamma z + a, \quad \gamma, a \in \mathbb{C}$$

such that $\gamma L \subseteq M$. Moreover, F is a biholomorphic map if and only if $\gamma L = M$, for some $\gamma \in \mathbb{C}$.

Now consider non-constant biholomorphic map $F : X \to X$, where $X = \mathbb{C}/L$. After composing some translation, we may assume $F([0]) = [0]$. Then F is induced by a map $z \mapsto \gamma z$ such that $\gamma L = L$.

Note that this condition is a quite strong for γ. We list some facts as follows

1. $|\gamma| = 1$, otherwise the shortest length of non-zero element in L and γL will be different.
2. There exists integers $m \geq 1$ such that

$$\gamma^m = 1$$

otherwise L contains infty many points in a circle, a contradiction to the discreteness.

Note that $\gamma = \pm 1$ is allowed, $\gamma = 1$ is equivalent to F is identity and $\gamma = -1$ is equivalent to the inversion. Assume $\gamma \notin \mathbb{R}$, choose $w \in L\backslash\{0\}$, such that

$$|w| \leq |v|, \quad \text{for all } v \in L\backslash\{0\}$$

We claim that:

Lemma 1.5.3. *$L = \mathbb{Z}w + \mathbb{Z}\gamma w$, for w we choose above.*

Proof. Let $L' = \mathbb{Z}w + \mathbb{Z}\gamma w \subset L$. If $L' \neq L$, we will find an elment $v \in L\backslash L'$. Adding an element in L' if necessary, we may assume v lies in the parallelogram spanned by w and γw. Then

$$|v - w| + |v - \gamma w| < |w| + |\gamma w| = 2|v|$$

So either $|v - w|$ or $|v - \gamma w|$ is less than $|v|$, a contradiction. ∎

Since $\gamma L = L$, then $\gamma^2 w \in L = \mathbb{Z}w + \mathbb{Z}\gamma w$, so

$$\gamma^2 w = mw + n\gamma w, \quad m, n \in \mathbb{Z}$$

After canceling w we have the quadratic equation that γ must satisfy

$$\gamma^2 = m + n\gamma$$

so we have

$$\gamma = \frac{1}{2}(n \pm \sqrt{n^2 + 4m})$$

Since $\gamma \notin \mathbb{R}$, we have $n^2 + 4m < 0$. And

$$|\gamma|^2 = \frac{1}{4}(n^2 - (n^2 + 4m)) = -m$$

so we must have $m = -1$. So $n^2 < 4$ implies $n = \pm 1, 0$. Then all possible γ are listed as follows

$$\gamma = \begin{cases} \pm i, & n = 0 \\ \frac{1}{2}(\pm 1 \pm \sqrt{3}i), & n = \pm 1 \end{cases}$$

When $n = 0$, L is called a square lattice. When $\gamma = \pm 1$, $L = \mathbb{Z}w + \mathbb{Z}w \cdot e^{\frac{\pi}{3}i}$, is called a hexagonal lattice.

We summarize as follows

Theorem 1.5.4. *If we define* $\mathrm{Aut}_0(X) = \{\textit{automorphism } F : X \to X \mid F([0]) = [0]\}$*, then*

$$\mathrm{Aut}_0(X) = \begin{cases} \mathbb{Z}_4, & L \textit{ is a square lattice} \\ \mathbb{Z}_6, & L \textit{ is a hexagonal lattice} \\ \mathbb{Z}_2, & \textit{otherwise} \end{cases}$$

So we have

$$\mathrm{Aut}(X) = \mathrm{Aut}_0(X) \ltimes \{T_{[a]} \mid [a] \in X\}$$

In particular, we have

$$\dim_{\mathbb{C}} \mathrm{Aut}(X) = 1$$

Remark 1.5.5. As we can see, the three cases above are not isomorphic to each other, since Riemann surfaces which are isomorphic to each other have the same automorphism group. This is the first example we meet, surfaces with the same topological structure but different complex structures.

It's worth mentioning that automorphism groups of higher genus are very small.

Theorem 1.5.6. *For genus* ≥ 2*, the automorphism groups are finite.*

1.6. **Moduli space of complex torus.** Since the above results show some different complex structures on a topological torus, we want to ask: How many different complex structures are there on a topological torus? And in general, how many different complex structures are there on a given Riemann surfaces? That leads to the conception of Moduli space.

For any lattice $M = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, if we let $\gamma = \frac{1}{\omega_1}$, then

$$L = \gamma M = \mathbb{Z} + \mathbb{Z}\frac{\omega_2}{\omega_1}$$

So it suffices to consider the complex torus of form $X_\tau = \mathbb{C}/L_\tau$, where

$$L_\tau = \mathbb{Z} + \mathbb{Z}\tau$$

Since $L_{-\tau} = L_\tau$, so we can assume that $\operatorname{Im}\tau > 0$.

Let

$$\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$$

Given $\tau, \tau' \in \mathbb{H}$, we want to ask when $X_{\tau'}$ and $X_{\tau'}$ give the same complex structure on a topological torus. It is equivalent to that there exists $\gamma \in \mathbb{C}$, such that

$$\gamma L_\tau = L_{\tau'}$$

i.e.

$$\mathbb{Z}\gamma + \mathbb{Z}\gamma\tau = \mathbb{Z} + \mathbb{Z}\tau'$$

So there exists $a, b, c, d \in \mathbb{Z}$, such that

$$\begin{cases} \gamma = c + d\tau' \\ \gamma\tau = a + b\tau' \end{cases}$$

moreover,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible since it's a base change and its inverse matrix must have integral entries, so its determinant must be ± 1.

So, it's the famous Möbius transformation

$$\tau = \frac{a + b\tau'}{c + d\tau'}$$

Since we require both γ and γ have positive imaginary part, we compute as follows

$$\tau = \frac{(a\tau' + b)(c\bar{\tau}' + b)}{|c\tau' + d|^2} \implies \operatorname{Im}\tau = \frac{ad - bc}{|c\tau' + d|^2}\operatorname{Im}\tau'$$

So we need $A \in \mathrm{SL}_2(\mathbb{Z})$.

We summarize as follows

Theorem 1.6.1. *$X_\tau \cong X_{\tau'}$ if and only if there exists $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that*

$$\tau = \frac{a + b\tau'}{c + d\tau'}$$

For any $A \in \mathrm{SL}_2(\mathbb{Z})$, it induces a map from $\mathbb{H}$ to itself, defined by

$$\tau \mapsto \frac{a+b\tau}{c+d\tau}$$

In fact, it's an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. Furthermore, A and $-A$ gives the same action. So the above theorem can be rephrased as follows

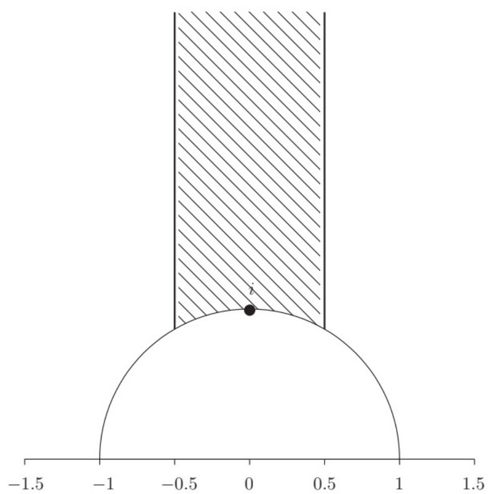
Theorem 1.6.2. *The set of isomorphism classes of complex structure on complex torus is* $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z}) = \mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$ *,where* $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm \mathrm{I}_2\}$.

Remark 1.6.3. As we have shown, all complex structures on a complex torus $\mathbb{C}/L$ are $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$. In fact, it contains all possible complex structure on surface with genus 1[§], called the moduli space of surface with genus 1, denoted by $\mathscr{M}(1)$.

So we wonder what's the fundamental domain[¶] of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{H}$. We will show that it is

$$D = \{\tau \in \mathbb{C} \mid |\tau| \geq 1, -\frac{1}{2} \leq \mathrm{Re}\,\tau \leq \frac{1}{2}\}$$

and can be drawn as follows



Theorem 1.6.4. *D is the fundamental domain of* $\mathrm{PSL}_2(\mathbb{Z})$ *action on* $\mathbb{H}$.

Definition 1.6.5. *Consider the following two matrices*

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Theorem 1.6.6. $\mathrm{SL}_2(\mathbb{Z})$ *is generated by S and T.*

§We will show this later, using Abel's theorem.

¶Fundamental domain is usually defined as a set of representatives for the orbits. However, definition we give here is sometimes called a fundamental domain with boundary.

Remark 1.6.7. Before proving the theorem, let's see what's the action of S and T on $\mathbb{H}$

$$S : \tau = re^{i\theta} \mapsto -\frac{1}{\tau} = \frac{1}{\tau} e^{i(\pi-\theta)}$$

So S preserves the upper semicircle, and $S(i) = i$.

$$T : \tau \mapsto \tau + 1$$

So T is just a translation by 1.

Proof. Proof of Theorem1.6.6 and Theorem1.6.7

Let Γ be the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by S and T. We need to show $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

Step one: For any $z \in \mathbb{H}$, there exists $A \in \Gamma$ such that $A(z) \in D$. Fix $z \in \mathbb{H}$, from the relation between $\operatorname{Im} A(z)$ and $\operatorname{Im} z$

$$\operatorname{Im}(A(z)) = \frac{1}{|cz+d|^2} \operatorname{Im}(z)$$

we have that $\{\operatorname{Im} A(z) \mid A \in \mathrm{SL}_2(\mathbb{Z})\}$ is a bounded set. Since $\mathrm{SL}_2(\mathbb{Z})$ is discrete, there exists $w \in \{A(z) \mid A \in \Gamma\}$ such that

$$\operatorname{Im} w \geq \operatorname{Im} A(z), \quad \forall A \in \Gamma$$

Since the transition by T doesn't change the imaginary part of w, so we may assume w such that

$$-\frac{1}{2} \leq \operatorname{Re} w < \frac{1}{2}$$

We claim $w \in D$ to finish step one. It suffices to show $|w| \geq 1$. If not, write $w = re^{i\theta}, r < 1, 0 < \theta < \pi$. Then $S(w) = \frac{1}{r} e^{i(\pi-\theta)}$, so we have

$$\operatorname{Im} S(w) = \frac{1}{r} \sin(\pi - \theta) > r \sin(\pi - \theta) = \operatorname{Im} w$$

a contradiction to the choice of w.

Step two: Assume $z, w \in D$, and there exists $A \in \mathrm{SL}_2(\mathbb{Z})$ such that $w = A(z)$, then

1. $A \in \Gamma$;
2. if $z \neq w$, then z and w lies in the boundary of D;
3. if $z = w \in D \backslash \partial D$, then $A = \pm \mathrm{I}_2$.

We may assume $\operatorname{Im} w \geq \operatorname{Im} z$, and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \geq 0$, then we have

$$w = \frac{az+b}{cz+d}$$

and the requirement on imaginary part implies that

$$|cz+d| \leq 1$$

Since $z \in D$, then $\operatorname{Im} z \geq \frac{\sqrt{3}}{2}$. Then

$$1 \geq |cz+d| \geq \operatorname{Im}(cz+d) = c \operatorname{Im} z \geq c\frac{\sqrt{3}}{2}$$

then c must be 0 or 1.

If $c=0$, then $\det A = ad = 1$, then $a=d=\pm 1$. Replacing A by $-A$, we may assume $A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, then $w = A(z) = z+b \in W$, then $b = 0, \pm 1$. If $b=0$, then $A = \mathrm{I}_2$. We will see later it's the only case that $z = w \in D\backslash\partial D$. If $b = \pm 1$, then $A = T$ or T^{-1}, then $A \in \Gamma$. And

$$|\operatorname{Re} z| = |\operatorname{Re} w| = \frac{1}{2}$$

implies $z = w \in \partial D$.

If $c = 1$, then

$$1 \geq |cz+d| = |z+d| = \sqrt{(\operatorname{Re} z + d)^2 + (\operatorname{Im} z)^2} \geq \sqrt{(\operatorname{Re} z + d)^2 + \frac{4}{3}}$$

Since $|\operatorname{Re} z| \leq \frac{1}{2}$, then $d = 0, \pm 1$. If $d = 0$, then $A = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} = T^a S \in \Gamma$. And

$$1 \geq |cz+d| = |z|$$

then $z \in \partial D$, since $z \in D$. Then $w = A(z) \in \partial D$. If $d = 1$, then $1 \geq |cz+d| = |z+1|$, then $z = \rho = -\frac{1}{2} + \frac{\sqrt{3}}{2} i \in \partial D$. Then $A = \begin{pmatrix} a & a-1 \\ 1 & 1 \end{pmatrix}$, and

$$A(z) = \frac{az+a-1}{z+1} = a - \frac{1}{z+1} = a - \frac{1}{2} + \frac{\sqrt{3}}{2} i \in D$$

then $a = 0, 1$, so $A(z) \in \partial D$. The case $d = -1$ is similar to $d = 1$.

Step three: $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. For $z \in D\backslash\partial D$. For any $B \in \mathrm{SL}_2(\mathbb{Z})$. By step one, there exists $A \in \Gamma$ such that $AB(z_0) = A(B(z_0)) \in D$, then by step two, we have $AB(z_0) = z_0$, and $AB = \pm \mathrm{I}_2$, i.e. $B = \pm A^{-1} \in \Gamma$. So we have $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. □

Remark 1.6.8. Topologically we have

$$\mathbb{H}/\mathrm{SL}_2(\mathbb{Z}) \cong S^2\backslash\{\mathrm{pt}\}$$

and we have

$$\mathbb{H}/\mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{C}$$

as Riemann surface.

2. Differential forms

Recall what we've learnt in complex analysis. Consider $\{z, \overline{z}\}$ as a coordinate on $\mathbb{C}$, smooth 1-forms on $\mathbb{C}$ have the form

$$f(z, \overline{z})\mathrm{d}z + g(z, \overline{z})\mathrm{d}z$$

Where f, g are smooth functions.

Let $z = T(w)$ be a holomorphic change of coordinate, then

$$\frac{\partial z}{\partial \overline{w}} = \frac{\partial \overline{z}}{\partial w} = 0, \quad \frac{\partial \overline{z}}{\partial \overline{w}} = \overline{\frac{\partial z}{\partial w}} = \overline{T'(w)}$$

then we have

$$\begin{aligned} \mathrm{d}z &= \frac{\partial z}{\partial w}\mathrm{d}w + \frac{\partial z}{\partial \overline{w}}\mathrm{d}\overline{w} = T'(w)\mathrm{d}w \\ \mathrm{d}\overline{z} &= \overline{T'(w)}\mathrm{d}\overline{w} \end{aligned}$$

A form $f\mathrm{d}z$ is called a $(1,0)$-form, and a form $g\mathrm{d}\overline{z}$ is called a $(0,1)$-form, and these concepts are invariant under the change of holomorphic change of coordinate, so we define them on Riemann surfaces.

Let's see deeper why it is independent of the choice of the charts. Since we have $T_p\mathbb{C} \cong \mathbb{C}$, and we identify

$$\frac{\partial}{\partial x} = 1, \quad \frac{\partial}{\partial y} = i$$

then we have J as

$$\begin{aligned} J(\frac{\partial}{\partial x}) &= \frac{\partial}{\partial y} \\ J(\frac{\partial}{\partial y}) &= -\frac{\partial}{\partial x} \end{aligned}$$

this induces linear map

$$J : T_p^*\mathbb{C} \to T_p^*\mathbb{C}$$

given by

$$\langle J(\theta), v\rangle = \langle \theta, J(v)\rangle$$

where $\theta \in T_p^*\mathbb{C}, v \in T_p\mathbb{C}$.

If we want to see what is $J(\mathrm{d}x)$, then

$$\begin{aligned} \langle J(\mathrm{d}x), \frac{\partial}{\partial x}\rangle &= \langle \mathrm{d}x, J(\frac{\partial}{\partial x})\rangle = 0 \\ \langle J(\mathrm{d}x), \frac{\partial}{\partial y}\rangle &= \langle \mathrm{d}x, J(\frac{\partial}{\partial y})\rangle = -1 \end{aligned}$$

so we have $J(\mathrm{d}x) = -\mathrm{d}y$, similarly we have $J(\mathrm{d}y) = \mathrm{d}x$. So as we can see, there is no eigenvector of J in $T_p^*\mathbb{C}$, but if we consider $T_p^*\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, and

$$\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$$

then we have

$$J(\mathrm{d}z) = J(\mathrm{d}x) + iJ(\mathrm{d}y) = -\mathrm{d}y + i\mathrm{d}x = iJ(\mathrm{d}z)$$

So, our $(1,0)$-form defined above just the eigenvectors of J with respect to the eigenvalue i, and $(0,1)$-form defined above just the eigenvectors of J with respect to the eigenvalue $-i$.

So $(1,0)$-form and $(0,1)$-form are independent of the choice of charts, since J is independent.

And what's more, we have the dual of $\mathrm{d}z$ and $\mathrm{d}\overline{z}$.

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) \in T_p\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \\ \frac{\partial}{\partial \overline{z}} &= \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}) \in T_p\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \end{aligned}$$

and J acts on them as follows

$$J(\frac{\partial}{\partial z}) = i\frac{\partial}{\partial z}$$
$$J(\frac{\partial}{\partial \overline{z}}) = -i\frac{\partial}{\partial \overline{z}}$$

For a complex function f, we have $f = u+iv$, where u and v are real-valued function, then

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}) + \frac{1}{2}i(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$$

Then we have $\dfrac{\partial f}{\partial \overline{z}} = 0$ is equivalent to the Cauthy-Riemann equations.

Now let's consider what will happen on a Riemann surface X.

Definition 2.1 (differential form)**.** *A differential* 1*-form θ on X assigns to any local chart $U \xrightarrow{\varphi} V$ a form $f\mathrm{d}z + g\mathrm{d}\overline{z}$, and compatible with the charts.*

Remark 2.2. Compatiblity means if $U' \xrightarrow{\varphi'} V'$ is another local chart, and θ is represented in this chart by

$$s\mathrm{d}w + t\mathrm{d}\overline{w}$$

and let $w = T(z) = \varphi' \circ \varphi^{-1}(z)$, then we have

$$s(T(z), \overline{T(z)})T'(z)\mathrm{d}z + t(T(z), \overline{T(z)})\overline{T'(z)}\mathrm{d}\overline{z} = f\mathrm{d}z + g\mathrm{d}\overline{z}$$

Remark 2.3. Similarly, we can define what is a 2-form on X. That is, a 2-form η on X assigns each local chart a form

$$f\mathrm{d}z \wedge \mathrm{d}\overline{z}$$

and compatible with the charts, i.e. If there is another local chart, and η is represented by

$$g\mathrm{d}w \wedge \mathrm{d}\overline{w}$$

and T is the transition function between two charts, then

$$f\mathrm{d}z\wedge\mathrm{d}\overline{z} = g(T(w), \overline{T(w)})T'(w)\overline{T'(w)}\mathrm{d}z\wedge\mathrm{d}\overline{z} = g(T(w), \overline{T(w)})|T'(z)|^2\mathrm{d}z\wedge\mathrm{d}\overline{z}$$

Since we have the differential form on a Riemann surface, then we define what is a $(1,0)$-form or a $(0,1)$-form, as what we have done.

Definition 2.4. *A differential form θ on a Riemann surface is called a $(1,0)$-form, if it can be represented as $f\mathrm{d}z$ locally. Similarly we can define what is a $(0,1)$-form.*

Definition 2.5. *A holomorphic* 1*-form θ is a differential* 1*-form which can be locally represented as $f(z)\mathrm{d}z$, with f is holomorphic; A meromorphic* 1*-form θ is a differential* 1*-form which can be locally represented as $f(z)\mathrm{d}z$, with f is meromorphic.*

If f is a function, we can define

$$\mathrm{d}f = \frac{\partial f}{\partial z}\mathrm{d}z + \frac{\partial f}{\partial \overline{z}}\mathrm{d}\overline{z}$$

so we define

$$\partial f := \frac{\partial f}{\partial z}\mathrm{d}z$$
$$\overline{\partial} f := \frac{\partial f}{\partial \overline{z}}\mathrm{d}\overline{z}$$

For a 1-form θ, locally given by

$$\theta = f\mathrm{d}z + g\mathrm{d}\overline{z}$$

we have

$$\mathrm{d}\theta = \mathrm{d}f \wedge \mathrm{d}z + \mathrm{d}g \wedge \mathrm{d}\overline{z} = \frac{\partial f}{\partial \overline{z}}\mathrm{d}\overline{z} \wedge \mathrm{d}z + \frac{\partial g}{\partial z}\mathrm{d}z \wedge \mathrm{d}\overline{z} = (\frac{\partial g}{\partial z} - \frac{\partial f}{\partial \overline{z}})\mathrm{d}z \wedge \mathrm{d}\overline{z}$$

so we define

$$\partial\theta := \partial g \wedge \mathrm{d}\overline{z}$$
$$\overline{\partial}\theta := \overline{\partial} f \wedge \mathrm{d}z$$

Theorem 2.6. *For the exterior differential defined above, we have*

1. $\mathrm{d}^2 = \partial^2 = \overline{\partial}^2 = 0$.
2. $\partial\overline{\partial} = -\overline{\partial}\partial$.
3. *A* $(1,0)$*-form* θ *is holomorphic is equivalent to* $\overline{\partial}\theta = 0$*, and is also equivalent to* $\mathrm{d}\theta = 0$.
4. $\mathrm{d}, \partial, \overline{\partial}$ *satisfy the Leibniz rule.*

Remark 2.7. The third property implies that a $(1,0)$-form is a holomorphic form is equivalent to it's a closed form.

If X and Y are two Riemann surface, and $F : X \to Y$ is a holomorphic map, then we can pullback differential forms on Y to those on X, defined as follows.

Let (U_1, φ_1) be a local chart of X and (U_2, φ_2) be a local chart of Y, such that $F(U_1) \subseteq U_2$, and let $w = T(z) = \varphi_2 \circ F \circ \varphi_1^{-1}(z)$.

Then we define pullback F^*

$$F^*(f\mathrm{d}w + g\mathrm{d}\overline{w}) = f(T(z), \overline{T(z)})T'(z)\mathrm{d}z + g(T(z), \overline{T(z)})\overline{T'(z)}\mathrm{d}\overline{z}$$
$$F^*(f\mathrm{d}w \wedge \mathrm{d}\overline{w}) = f(T(z), \overline{T(z)})|T'(z)|^2\mathrm{d}z \wedge \mathrm{d}\overline{z}$$

Furthermore, it's easily to check F^* commutes with $\mathrm{d}, \partial, \overline{\partial}$.

If we have a differential form, then we can integral it. Let θ be a 1-form on X, and γ be a piecewise smooth curve on X, write $\gamma = \gamma_1 \cup \cdots \cup \gamma_n$, each γ_i lies in a local chart (U_i, φ_i).

Then we can define

$$\int_\gamma \theta = \sum_{i=1}^n \int_{\gamma_i} \theta = \sum_{i=1}^n \int_{a_i}^{b_i} \{f(z_i, \overline{z_i})z_i'(t) + g(z_i, \overline{z_i})\overline{z_i'(t)}\}\mathrm{d}t$$

if θ is locally given by

$$f(z_i, \overline{z_i})\mathrm{d}z_i + g(z_i, \overline{z_i})\mathrm{d}\overline{z_i}$$

and z_i is $\varphi_i \circ \gamma_i : [a_i, b_i] \to \varphi(U_i)$.

Similarly we can integral an 2-form on a reigon D on X. If η is a 2-form and D is a region on X. Write $D = D_1 \cup \cdots \cup D_n$ such that each D_i lies in a local chart (U_i, φ_i).

Note that

$$\mathrm{d}z_i \wedge \mathrm{d}\overline{z_i} = (\mathrm{d}x_i + i\mathrm{d}y_i) \wedge (\mathrm{d}x_i - i\mathrm{d}y_i) = -2i\mathrm{d}x_i \wedge \mathrm{d}y_i$$

If η is given locally by

$$f(z_i, \overline{z_i})\mathrm{d}z_i \wedge \mathrm{d}\overline{z_i}$$

then we can define

$$\int_D \eta = \sum_{j=1}^n \int_{D_j} \eta = \sum_{j=1}^n \int_{\varphi_j(D_j)} (-2i) f(x_j + iy_j, x_j - iy_j)\mathrm{d}x_j \wedge \mathrm{d}y_j$$

And we have a famous theorem

Theorem 2.8 (Stokes)**.** *If D is a compact reigon and ∂D is piecewise smooth, then*

$$\int_D \mathrm{d}\theta = \int_{\partial D} \theta$$

where θ is a smooth 1-form.

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