COMPLEX GEOMETRY

BOWEN LIU

Abstract.

Contents

Part 1. Local theory

In this part, we mainly follows [Huy05] and [Dem12].

1. Review of complex analysis

1.1. One variable case.

We first give a quick review about basic results in holomorphic functions of one variable. Fix an open subset $U \subset \mathbb{C}$. There are too many ways to define a holomorphic function, and all of them are equivalent.

Definition 1.1.1 (holomorphic)**.** A function $f : U \to \mathbb{C}$ is called holomorphic at $z_0 \in U$, if there exists an open ball $B_\varepsilon(z_0) \subset U$ with $\varepsilon > 0$ such that $f|_{B_\varepsilon(z_0)}$ can be written as convergent power series, that is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad z \in B_\varepsilon(z_0)$$

f is holomorphic om U, if f is holomorphic at any point of U.

Remark 1.1.1 (Cauchy-Riemann equation)*.* The second definition is given by Cauchy-Riemann equation. To be explicit, for a function $f : U \to \mathbb{C}$, we can regard it as a function defined on $\mathbb{R}^2$, and write it as $f(x, y) = u(x, y) + \sqrt{-1}v(x, y)$, where u, v are real-valued functions, then f is holomorphic if and only if u, v are continuously differentiable and satisfy the following C-R equations:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}$$

If we introduce the following two operators

$$\begin{aligned}\frac{\partial}{\partial z} &:= \frac{1}{2}\left(\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}\right)\\ \frac{\partial}{\partial \bar{z}} &:= \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right)\end{aligned}$$

Then C-R equation is equivalent to $\frac{\partial f}{\partial \bar{z}} = 0$. Indeed,

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + \sqrt{-1}\frac{\partial f}{\partial y}\right)\\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \sqrt{-1}\frac{\partial v}{\partial x} + \sqrt{-1}\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right)\\ &= 0\end{aligned}$$

Remark 1.1.2 (Cauchy integral formula)*.* The third definition is given by Cauchy integral formula. To be explicit, a function $f : U \to \mathbb{C}$ is holomorphic if and only if f is continuously differentiable and for any $B_\varepsilon(z_0) \subset U$, the following formula holds

$$f(z_0) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial B_\varepsilon(z_0)} \frac{f(z)}{z - z_0} \mathrm{d}z$$

Here are some standard facts in complex analysis, which can be found in any textbook.

Theorem 1.1.1 (maximum principle)**.** Let $U \subset \mathbb{C}$ be open and connected. If $f : U \to \mathbb{C}$ is holomorphic and non-constant, then $|f|$ has no local maximum in U.

Theorem 1.1.2 (identity theorem)**.** If $f, g : U \to \mathbb{C}$ are two holomorphic functions a connected open subset $U \subset \mathbb{C}$ such that $f(z) = g(z)$ for all z in a non-empty subset V of U, then $f = g$.

Theorem 1.1.3 (Riemann extension theorem)**.** Let $f : B_\varepsilon(z_0) - \{z_0\} \to \mathbb{C}$ be a bounded holomorphic function, then f can be extended to a holomorphic function $f : B_\varepsilon(z_0) \to \mathbb{C}$.

Theorem 1.1.4 (Riemann mapping theorem)**.** Let $U \subset \mathbb{C}$ be a simply connected proper open subset. Then U is biholomorphic to the unit ball.

Theorem 1.1.5 (Liouville)**.** Every bounded holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is constant.

Remark 1.1.3. Liouville theorem implies that $\mathbb{C}$ is not biholomorphic to the unit ball. It's a striking difference to the real case, since we know unit ball is homeomorphic to $\mathbb{R}$.

1.2. **Several variables case.** Now let U be an open subset of $\mathbb{C}^n$. For any $w \in U$, a polydisc $B_\varepsilon(w) = \{z \mid |z_i - w_i| < \varepsilon_i\}$, where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$. Similar to one variable case, we can define holomorphic function as follows.

Definition 1.2.1 (holomorphic)**.** A function $f : U \to \mathbb{C}$ is called holomorphic at point $w \in U$, if there exists a polydisc $B_\varepsilon(w) \subset U$ such that the restriction of $f|_{B_\varepsilon(w)}$ is given by power series

$$\sum_{i_1, \ldots, i_n = 0}^{\infty} a_{i_1 \ldots i_n} (z_1 - w_1)^{i_1} \ldots (z_n - w_n)^{i_n}$$

Remark 1.2.1. Also, we can define holomorphicity as follows:

1. A function $f : U \to \mathbb{C}$ is holomorphic, if it satisfies C-R equations for all coordinates $z_i = x_i + \sqrt{-1} y_i$, that is

$$\frac{\partial f}{\partial \overline{z}_i} = 0, \quad i = 1, 2, \ldots, n$$

where $\frac{\partial}{\partial z_i} := \frac{1}{2}(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y_i})$

2. A function $f : U \to \mathbb{C}$ is holomorphic if and only if f is continuously differentiable and for any $z_0 \in U$, the following formula holds
$$f(z) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\partial B_\varepsilon(w)} \frac{f(w)}{(z_1 - w_1)\dots(z_n - w_n)} \mathrm{d}w_1 \dots \mathrm{d}w_n$$

Remark 1.2.2. Other results such as maximum theorem, identity theorem and Liouville theorem generalize easily to the higher dimension. A version of Riemann extension still holds true. However, Riemann mapping theorem fails.

Exercise 1.2.1. Show that polydisc $B_{(1,1)}(0) \subset \mathbb{C}^2$ is not biholomorphic to the unit disk $D = \{z \in \mathbb{C}^2 \mid \|z\| < 1\}$.

Proof. □

The next result is only valid in dimension at least two.

Theorem 1.2.1 (Hartogs' theorem)**.** Suppose $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ and $\varepsilon' = (\varepsilon'_1, \dots, \varepsilon'_n)$ are given such that for all i one has $\varepsilon'_i < \varepsilon_i$. If $n > 1$, then any holomorphic map $f : B_\varepsilon(0) \backslash \overline{B_{\varepsilon'}(0)} \to \mathbb{C}$ can be uniquely extended to a holomorphic map $f : B_\varepsilon(0) \to \mathbb{C}$.

Definition 1.2.2 (holomorphic)**.** Let $U \subset \mathbb{C}^n$ be an open subset. A function $f : U \to \mathbb{C}^n$ is called holomorphic if all coordinate functions $f_1, \dots, f_n$ are holomorphic functions $U \to \mathbb{C}$.

Definition 1.2.3 (biholomorphic)**.** A holomorphic map $f : U \to V$ between two open subsets $U, V \subset \mathbb{C}^n$ is biholomorphic if f is bijective and its inverse f^{-1} is also holomorphic.

Definition 1.2.4 (complex Jacobian)**.** Let $U \subset \mathbb{C}^m$ be an open subset and let $f : U \to \mathbb{C}^n$ be a holomorphic map, the complex Jacobian of f at point $z \in U$ is the matrix
$$J(f)(z) := \left(\frac{\partial f_i}{\partial z_j}(z)\right)_{\substack{1 \le i \le n \\ 1 \le j \le m}}$$

Remark 1.2.3. The smooth map $f : U \subset \mathbb{C}^m = \mathbb{R}^{2m} \to \mathbb{C}^n = \mathbb{R}^{2n}$, then it induces for $z \in U$ a $\mathbb{R}$-linear map, which is denoted by $J_{\mathbb{R}}(f)(z) : T_z \mathbb{R}^{2m} \to T_{f(z)} R^{2n}$. With respect to basis $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_m}\}$ and $\{\frac{\partial}{\partial r_1}, \dots, \frac{\partial}{\partial r_n}, \frac{\partial}{\partial s_1}, \dots, \frac{\partial}{\partial s_n}\}$, $\mathrm{d}f(z)$ is given by
$$J_{\mathbb{R}}(f) = \begin{pmatrix} (\frac{\partial u_i}{\partial x_j}) & (\frac{\partial u_i}{\partial y_j}) \\ (\frac{\partial v_i}{\partial x_j}) & (\frac{\partial v_i}{\partial y_j}) \end{pmatrix}$$
If we consider its $\mathbb{C}$-linear extension, with respect to basis $\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_m}, \frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_m}\}$ and $\{\frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial w_n}, \frac{\partial}{\partial \bar{w}_1}, \dots, \frac{\partial}{\partial \bar{w}_n}\}$, it can be written as
$$\begin{pmatrix} (\frac{\partial f_i}{\partial z_j}) & (\frac{\partial f_i}{\partial \bar{z}_j}) \\ (\frac{\partial \bar{f}_i}{\partial x_j}) & (\frac{\partial \bar{f}_i}{\partial y_j}) \end{pmatrix}$$

In particular, if f is holomorphic, then $\det J_{\mathbb{R}}(f) = \det J(f) \det \overline{J(f)} = |\det J(f)|^2$, which is non-negative.

Definition 1.2.5 (regular value). Let $U \subset \mathbb{C}^m$ be an open subset and let $f : U \to \mathbb{C}^n$ be a holomorphic map, $z \in U$ is called regular point, if $J(f)(z)$ is surjective. If every point $z \in f^{-1}(w)$ is regular point, then w is called a regular value.

Remark 1.2.4. In particular, if $f^{-1}(w) = \varnothing$, then w is also called a regular value.

Theorem 1.2.2 (inverse function theorem). Let $f : U \to V$ be a holomorphic map between two open subsets $U, V \subset \mathbb{C}^n$. If $z \in U$ is a regular point, then there exist open subsets $z \in U' \subset U$ and $f(z) \in V' \subset V$ such that f induces a biholomorphic map $f : U' \to V'$.

Theorem 1.2.3 (implicit function theorem). Let $U \subset \mathbb{C}^m$ be an open subset and let $f : U \to \mathbb{C}^n$ be a holomorphic map, where $m \geq n$. Suppose $z_0 \in U$ is a point such that

$$\det \left(\frac{\partial f_i}{\partial z_j}(z_0) \right)_{1 \leq i,j \leq n} \neq 0$$

Then there exist open subsets $U_1 \subset \mathbb{C}^{m-n}, U_2 \subset \mathbb{C}^n$ and a holomorphic map $g : U_1 \to U_2$ such that $U_1 \times U_2 \to U$ and $f(z) = f(z_0)$ if and only if $g(z_{n+1}, \ldots, z_m) = (z_1, \ldots, z_n)$.

Corollary 1.2.1. Let $U \subset \mathbb{C}^m$ be an open subset and let $f : U \to \mathbb{C}^n$ be a holomorphic map. Assume that $z_0 \in U$ such that $J(f)(z_0)$ has maximal rank, then

1. If $m \geq n$, then there exists a biholomorphic map $h : V \to U'$, where U' is an open subset of U containing z_0, and V is an open subset of $\mathbb{C}^n$ containing $f(z_0)$, such that $f(h(z_1, \ldots, z_n)) = (z_1, \ldots, z_n)$.
2. If $m \leq n$, then there exists a biholomorphic map $g : V \to V'$, where V, V' are open subsets of $\mathbb{C}^n$ containing $f(z_0)$, such that $g(f(z)) = (z_1, \ldots, z_m, 0, \ldots, 0)$.

2. Algebraic and analytic

2.1. **Weierstrass' theorems.** Let $f : B_\varepsilon(0) \to \mathbb{C}$ be a holomorphic function defined on polydisc $B_\varepsilon(0)$. For any $w = (z_2, \ldots, z_n)$ we denote $f_w(z_1)$ the function $f(z_1, \ldots, z_n)$. Now we're going to show that all zeros of f are caused by a factor of f which has the form of a Weierstrass polynomial.

Definition 2.1.1 (Weierstrass polynomial)**.** A Weierstrass polynomial is a polynomial in z_1 of the form

$$z_1^d + \alpha_1(w) z_1^{d-1} + \cdots + \alpha_d(w)$$

where coefficients $\alpha_i(w)$ are holomorphic functions on some small disc in $\mathbb{C}^{n-1}$ vanishing at the origin.

Remark 2.1.1. Recall the one variable case, any holomorphic function $f(z)$ with a zero of order d at the origin can be written as $z^d h(z)$, where $h(0) \neq 0$. In fact, z^d is a Weierstrass polynomial, since in this case, α_i are constants which vanish at origin, that's exactly zero.

Theorem 2.1.1 (Weierstrass preparation theorem)**.** Let $f : B_\varepsilon(0) \to \mathbb{C}$ be a holomorphic function on the polydisc $B_\varepsilon(0)$. Assume $f(0) = 0$ and $f_0(z_1) \neq 0$. Then there exists a unique Weierstrass polynomial $g_w(z_1)$ and a holomorphic function h on some smaller polydisc $B_{\varepsilon'}(0) \subset B_\varepsilon(0)$ such that $f = gh$ and $h(0) \neq 0$.

Proof. See Proposition 1.1.6 in Page8 of [Huy05].

Theorem 2.1.2 (Weierstrass division theorem)**.** Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ and let $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ be a Weierstrass polynomial of degree d. Then there exist $r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ of degree $< d$ and $h \in \mathcal{O}_{\mathbb{C}^n,0}$ such that $f = gh + r$. The functions h and r are uniquely determined.

Proof. See Proposition 1.1.17 in Page15 of [Huy05]. □

□

2.2. **Stalk of sheaf of holomorphic functions.** If we use $\mathcal{O}_{\mathbb{C}^n}$ to denote the sheaf[1] of holomorphic functions on $\mathbb{C}^n$, and $\mathcal{O}_{\mathbb{C}^n,0}$ to denote its stalk at origin. It's clear $\mathcal{O}_{\mathbb{C}^n,0}$ is a local ring with maximal ideal $\mathfrak{m}$ consisting of all functions that vanish at origin, which implies units in $\mathcal{O}_{\mathbb{C}^n,0}$ are functions that don't vanish at origin.

Then Weierstrass preparation theorem can be rephrased by saying that after an appropriate coordinate choice any function $f \in \mathcal{O}_{\mathbb{C}^n,0}$ can be uniquely written as $f = gh$, where $h \in \mathcal{O}_{\mathbb{C}^n,0}$ is a unit and $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is a Weierstrass polynomial.

By using Weierstrass preparation theorem, one can say more about algebraic property of $\mathcal{O}_{\mathbb{C}^n,0}$. A algebraic lemma we need is that if a ring R is UFD, so is $R[x]$.

[1]Sheaf and its cohomology are important tools we will use once and again, if you're not familiar with it, see Appendix A.

Theorem 2.2.1. The local ring $\mathcal{O}_{\mathbb{C}^n,0}$ is a UFD.

Proof. We prove the assumption by induction on n. For $n = 0$, the ring $\mathcal{O}_{\mathbb{C}^n,0} = \mathbb{C}$ is a field, and thus a UFD. Suppose that $\mathcal{O}_{\mathbb{C}^{n-1},0}$ is a UFD, for $f \in \mathcal{O}_{\mathbb{C}^n,0}$ we choose coordinates such that Weierstrass preparation theorem is applied, that is $f = gh$, where $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is a Weierstrass polynomial and h is a unit in $\mathcal{O}_{\mathbb{C}^n,0}$. By induction we have $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is UFD, then g can be written as a product of irreducible elements of $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$. All that is left to show is that any irreducible element in $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is also irreducible as an element in $\mathcal{O}_{\mathbb{C}^n,0}$.

Assume $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is a Weierstrass polynomial which is written as the product of non-units $g_i \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$. There are two cases:

1. $g_i \in \mathcal{O}_{\mathbb{C}^{n-1},0}$. By induction hypothesis, g_i can be written as the product of irreducible elements of $\mathcal{O}_{\mathbb{C}^{n-1},0}$, which are also irreducible in $\mathcal{O}_{\mathbb{C}^n,0}$.
2. $g_i \notin \mathcal{O}_{\mathbb{C}^{n-1},0}$. In this case, g_i satisfies the hypothesis of Weierstrass preparation theorem, since g is a Weierstrass polynomial, then g_i is non-trivial on the z_1-line. So we can write $g_i = \widetilde{g}_i h_i$, where $\widetilde{g}_i$ is also Weierstrass polynomial.

Note that degree of g as a polynomial in z_1 is finite, then repeating above process leads to a decomposition, with factors are either irreducible Weierstrass polynomials or elements in $\mathcal{O}_{\mathbb{C}^{n-1},0}$.

Now it suffices to show any irreducible Weierstrass polynomial g is actually irreducible as an element of $\mathcal{O}_{\mathbb{C}^n,0}$. Suppose $g = f_1 f_2$, where $f_1, f_2 \in \mathcal{O}_{\mathbb{C}^n,0}$ are non-units. We apply Weierstrass preparation theorem to obtain $f_i = g_i h_i, i = 1, 2$, and thus $g = (g_1 g_2)(f_1 f_2)$. By uniqueness one has $g = g_1 g_2$, which contradicts to the irreducibility of g as an element of $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$. $\square$

Another important fact is that $\mathcal{O}_{\mathbb{C}^n,0}$ is noetherian, based Weierstrass division theorem.

Theorem 2.2.2. The local UFD $\mathcal{O}_{\mathbb{C}^n,0}$ is noetherian.

Proof. We prove the assumption by induction on n. For $n = 0$, it's clear since any field is noetherian. Suppose that $\mathcal{O}_{\mathbb{C}^{n-1},0}$ is noetherian, then Hilbert's basis theorem implies $\mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is also noetherian. Let $I \subset \mathcal{O}_{\mathbb{C}^n,0}$ be a non-trivial idea and choose $0 \neq f \in I$. Changing coordinates if neccessary, we may assume Weierstrass preparation theorem is applied, that is $f = gh$, where $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is a Weierstrass polynomial and h is a unit in $\mathcal{O}_{\mathbb{C}^n,0}$, hence $g \in I$. Furthermore, we assume $I \cap \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is generated by $g_1, \dots, g_k$.

For any other $\widetilde{f} \in I$, the Weierstrass division theorem implies $\widetilde{f} = g\widetilde{h} + r$ for some $r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$. Since $\widetilde{f}, g\widetilde{h} \in I$, we have $r \in I$ and therefore $r \in I \cap \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$. Thus $\widetilde{f} = g\widetilde{h} + \sum_{i=1}^{k} a_i g_i$. This shows I is finitely generated by elements $g, g_1, \dots, g_k$. $\square$

Corollary 2.2.1. Let $g \in \mathcal{O}_{\mathbb{C}^n,0}$ be an irreducible element. If $f \in \mathcal{O}_{\mathbb{C}^n,0}$ vanishes on $Z(g) = \{z \mid g(z) = 0\}$, then g divides f.

Proof. By Weierstrass preparation theorem we may assume $g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is a Weierstrass polynomial with degree d. By the Weierstrass division theorem one finds $h \in \mathcal{O}_{\mathbb{C}^n,0}$ and $r \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ of degree $< d$ such that $f = gh + r$. For $w \in \mathbb{C}^{n-1}$, by assumption r_w vanishes on the zero set g_w. If all of zeros of g_w have multiplicity one, then $r_w \equiv 0$, since r_w is of degree $< d$. Now it suffices to show the set $w \in \mathbb{C}^{n-1}$ such that g_w has zeros with multiplicity > 1 is quite "small".

Since g is irreducible and $\frac{\partial g}{\partial z_1}$ is of degree $d-1$, there exist elements $h_1, h_2 \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ and $0 \neq \gamma \in \mathcal{O}_{\mathbb{C}^{n-1},0}$ such that $h_1 g + h_2 \frac{\partial g}{\partial z_1} = \gamma$. So if g_w has a zero ξ of multiplicity > 1, then $\gamma(w) = h_1(\xi, w) g_w(\xi) + h_2(\xi, w) \frac{\partial g_w}{\partial z_1}(\xi) = 0$. This shows such w is contained in the zero set of a non-trivial holomorphic function $\gamma \in \mathcal{O}_{\mathbb{C}^{n-1},0}$. Then the following exercise completes the proof. □

Exercise 2.2.1. Let $U \subset \mathbb{C}^n$ be open and connected. Show that for any non-trivial holomorphic function $f : U \to \mathbb{C}$ the complement $U \backslash Z(f)$ of the zero set of f is connected and dense in U.

Proof. □

2.3. **Analatic germ.** For any $f \in \mathcal{O}_{\mathbb{C}^n,0}$, $Z(f)$ is not well-defined in fact, since for another $g \in \mathcal{O}_{\mathbb{C}^n,0}$, which represents the same element with f, $Z(f)$ may not equal to $Z(g)$. However, there always exists an open neighborhood $0 \in U \subset \mathbb{C}^n$ such that $Z(f) \cap U = Z(g) \cap U$.

Definition 2.3.1 (germ of a set)**.** The germ of a set in the origin $0 \in \mathbb{C}^n$ is given by a subset $X \subset \mathbb{C}^n$. Two germs of a set in the origin $X, Y \subset \mathbb{C}^n$ are same if there exists an open neighborhood $0 \in U \subset \mathbb{C}^n$ such that $X \cap U = Y \cap U$.

Remark 2.3.1. In this section we only consider germ of a set in the origin, and for convenience we just call it a germ

Example 2.3.1. For $f \in \mathcal{O}_{\mathbb{C}^n,0}$, $Z(f)$ is a well-defined germ.

Definition 2.3.2 (analytic germ)**.** A germ $X \subset \mathbb{C}^n$ is called analytic if there exist elements $f_1, \ldots, f_k \in \mathcal{O}_{\mathbb{C}^n,0}$ such that $X = Z(f_1, \ldots, f_k) := \bigcap_{i=1}^k Z(f_i)$.

Example 2.3.2. Let A be a subset of $\mathcal{O}_{\mathbb{C}^n,0}$, if we use (A) to denote the idea generated by A, then (A) is finitely generated since $\mathcal{O}_{\mathbb{C}^n,0}$ is noetherian. Thus $Z((A))$ is an analytic germ.

Definition 2.3.3. Let $X \subset \mathbb{C}^n$ be a germ. Then $I(X)$ denotes the set of all elements $f \in \mathcal{O}_{\mathbb{C}^n,0}$ with $X \subset Z(f)$.

Remark 2.3.2. It's clear $I(X)$ is an idea of $\mathcal{O}_{\mathbb{C}^n,0}$.

Lemma 2.3.1. We have the following relations:

1. If $X_1 \subset X_2$ are germs, then $I(X_2) \subset I(X_1)$;
2. If $\mathfrak{a}_1 \subset \mathfrak{a}_2$ are two ideas of $\mathcal{O}_{\mathbb{C}^n,0}$, then $Z(\mathfrak{a}_2) \subset Z(\mathfrak{a}_1)$;
3. For any analytic germ one has $Z(I(X)) = X$;
4. For any idea $\mathfrak{a}$ of $\mathcal{O}_{\mathbb{C}^n,0}$ one has $\mathfrak{a} \subset I(Z(\mathfrak{a}))$

Proof. (1),(2) and (4) are clear;

For (3). It's clear $X \subset Z(I(X))$; On the other hand, since X is analytic germ there exist elements $f_1, \dots, f_k \in \mathcal{O}_{\mathbb{C}^n,0}$ such that $X = Z(f_1, \dots, f_k)$ as germs, thus $f_1, \dots, f_k \in I(X)$, so by (2) we have $Z(I(X)) \subset X = Z(f_1, \dots, f_k)$. This completes the proof of (3). □

Definition 2.3.4 (irreducible germ)**.** An analytic germ is irreducible if the following condition is satisfied: Let $X = X_1 \cup X_2$, where X_1, X_2 are analytic germs, then $X = X_1$ or $X = X_2$.

Lemma 2.3.2. An analytic germ X is irreducible if and only if $I(X) \subset \mathcal{O}_{\mathbb{C}^n,0}$ is a prime ideal.

Proof. If X is irreducible and $f_1 f_2 \in I(X)$, then $X \subset Z(f_1 f_2) = Z(f_1) \cup Z(f_2)$, so we have $X = (X \cap Z(f_1)) \cup (X \cap Z(f_2))$ is a union of analytic germs. Then by irreducibility one has $X = X \cap Z(f_i)$ for some $i = 1$ or $i = 2$. Hence at least one of functions f_1 or f_2 vanishes on X, this shows $I(X)$ is prime.

Conversely, if $I(X)$ is a prime ideal and let $X = X_1 \cup X_2$ with X_1 and X_2 are analytic. If $f_i \in I(X_i), i = 1, 2$, then $f_1 f_2 \in I(X)$, since

$$X = X_1 \cup X_2 \subset Z(f_1) \cup Z(f_2) = Z(f_1 f_2)$$

Hence $f_1 \in I(X)$ or $f_2 \in I(X)$. Thus it suffices to shows that if $X \neq X_1$ and $X \neq X_2$, there exist elements $f_1 \in I(X_1) \backslash I(X)$ and $f_2 \in I(X_2) \backslash I(X)$. This follows immediately from (1) of Lemma 2.3.1. □

Corollary 2.3.1. For $f \in \mathcal{O}_{\mathbb{C}^n,0}$, $Z(f)$ is irreducible if and only if there exists an irreducible $g \in \mathcal{O}_{\mathbb{C}^n,0}$ such that $f = g^k$ for some $k \in \mathbb{Z}_{>0}$.

Proof. If $f = g^k$ with g irreducible, then $Z(f) = Z(g)$ and if $h \in I(Z(g))$, then g divides h by Corollary 2.2.1, this shows $I(Z(g)) = (g)$ and thus $Z(f)$ is irreducible, since $I(Z(f))$ also equals to (g), which is prime.

Conversely, if $f = \prod g_i^{n_i}$, then $Z(f) = \bigcup Z(g_i)$, which cannot be irreducible except for the case $f = g^k$ for some irreducible g. □

Lemma 2.3.3. Every decresing sequences of germs $\{X_i\}$ is stationary.

Proof. Consider its corresponding sequence $\{I(X_i)\}$, it's an increasing sequence, thus it's stationary, since $\mathcal{O}_{\mathbb{C}^n,0}$ is noetherian, this completes the proof, since for each i, $Z(I(X_i)) = X_i$. □

Theorem 2.3.1. Every germ X admits a finite decomposition $X = \bigcup_{i=1}^N X_i$, where X_i is irreducible for each i and $X_i \subsetneq X_j$ for $i \neq j$. The decomposition is unique apart from the ordering.

Proof. It suffices to show uniqueness, since existence follows from above lemma. Assume $X = \bigcup_{l=1}^{N'} X'_l$ is another decomposition, note that $X_i = \bigcup_{l=1}^{N'} X_i \cap X'_l$, we must have $X_i = X_i \cap X'_{l(i)}$, since X_i is irreducible. Likewise one has $X'_{l(i)} \cap X_j$, then we $i = j$, since $X_i \subsetneq X_j$ for $i \neq j$, and this shows $X_i = X'_{l(j)}$ □

Definition 2.3.5 (dimension)**.** Let X be an irreducible analytic germ defined by a prime ideal $\mathfrak{p} \subset \mathcal{O}_{\mathbb{C}^n,0}$, then the dimension of X is defined by $n - \mathrm{ht}\mathfrak{p}$, where $\mathrm{ht}\mathfrak{p}$ is the height of $\mathfrak{p}$.

Remark 2.3.3. For arbitrary analytic germ is of dimension d if all its irreducible components are of the same dimension d.

Remark 2.3.4. Let $X \subset \mathbb{C}^n$ be an irreducible analytic germ of codimensional 1, then the prime ideal $\mathfrak{p}$ defining X is of height 1. A basic result in communicative algebra says any prime ideal of height 1 in a UFD is principle. Therefore, $\mathfrak{p} = (f)$ for some irreducible $f \in \mathcal{O}_{\mathbb{C}^n,0}$.

2.4. **Meromorphic functions and relatively prime.**

Definition 2.4.1. Let $U \subset \mathbb{C}^n$ be an open subset. A meromorphic function f on U is a function on the complement of a nowhere dense subset $S \subset U$ with the following property: There exist an open covering $\{U_i\}$ of U and holomorphic functions $g_i, h_i : U \to \mathbb{C}$ with $h_i|_{U_i \setminus S} \cdot f|_{U_i \setminus S} = g_i|_{U_i \setminus S}$.

Remark 2.4.1. For any $z \in U$, the meromorphic function f in a neighborhood of z is given by g/h, where $g, h \in \mathcal{O}_{\mathbb{C}^n,z}$. If we assume g, h are chosen to be relatively prime, then they're unique up to units.

Proposition 2.4.1. Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$ be irreducible, then for sufficiently small ε and $z \in B_\varepsilon(0)$ the induced element $f \in \mathcal{O}_{\mathbb{C}^n,z}$ is irreducible.

Proof. Suppose $f \in \mathcal{O}_{\mathbb{C}^n,z}$ is reducible, that is $f = f_1 f_2$ where $f_i \in \mathcal{O}_{\mathbb{C}^n,z}$ non-units, i.e. $f_1(z) = f_2(z) = 0$. Thus $\frac{\partial f}{\partial z_1}(z) = \frac{\partial f_1}{\partial z_1}(z) f_2(z) + f_1(z) \frac{\partial f_2}{\partial z_1}(z) = 0$.

Thus the set of points $z \in B_\varepsilon(0)$ where f as an element of $\mathcal{O}_{\mathbb{C}^n,z}$ is reducible is contained in the analytic set $Z(f, \frac{\partial f}{\partial z_1})$. Now it suffices to show it's a proper subset of $Z(f)$, since f is irreducible, so is $Z(f)$. If not, then $\frac{\partial}{\partial z_1}$ would vanish on $Z(f)$. Since f is irreducible, we can apply Corollary 2.2.1 to obtain $\frac{\partial f}{\partial z_1}$ divides f, a contradiction. □

Proposition 2.4.2. If $f, g \in \mathcal{O}_{\mathbb{C}^n,0}$ are relatively prime, then they're relatively prime in $\mathcal{O}_{\mathbb{C}^n,z}$, for z in a sufficiently small neighborhood of 0.

Proof. Without lose of generality, we may assume $f, g \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ are Weierstrass polynomials, then f and g are relatively prime if and only if their resultant $R \in \mathcal{O}_{\mathbb{C}^{n-1}}$ has non-zero germ at 0, therefore the germ of R is also non-zero in a sufficiently small neighborhood of 0. □

Part 2. Complex manifold and vector bundle

3. Complex manifold

3.1. Definition and first properties.

Definition 3.1.1 (holomorphic atlas)**.** A holomorphic atlas on a smooth manifold is an atlas $\{(U_\alpha, \varphi_\alpha)\}$ of the form $\varphi_\alpha : U_\alpha \cong \varphi_\alpha(U_\alpha) \subset \mathbb{C}^n$ such that transition functions $\varphi_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_{\alpha\beta}) \to \varphi_\alpha(U_{\alpha\beta})$ are holomorphic functions. Furthermore,

1. The pair $(U_\alpha, \varphi_\alpha)$ is called a holomorphic chart;
2. Two holomorphic atlases are called equivalent, if the union of them is still a holomorphic atlas.

Definition 3.1.2 (complex manifold)**.** A complex n-manifold X is a smooth $2n$-manifold admitting an equivalence class of holomorphic atlases.

Remark 3.1.1. A complex manifold is called connected, compact, simply-connected and so on, if its underlying smooth manifold has this property.

Definition 3.1.3 (submanifold)**.** Let X be a complex n-manifold, let $Y \subset Y$ be a smooth manifold of (real) dimension $2k$. Then Y is a complex submanifold if there exists a holomorphic atlas $\{(U_i, \varphi_i)\}$ of X such that $\varphi_i : U_i \cap Y \cong \varphi_i(U_i) \cap \mathbb{C}^k$.

Definition 3.1.4 (holomorphic map)**.** Let X, Y be complex manifolds. A continuous map $f : X \to Y$ is a holomorphic map if for any holomorphic charts (U, φ) and (U', φ') of X and Y respectively, the map $\varphi' \circ f \circ \varphi^{-1} : \varphi(f^{-1}(U') \cap U) \to \varphi'(U')$ is holomorphic.

Definition 3.1.5 (biholomorphic)**.** Two complex manifolds X, Y are called biholomorphic, if there exists a holomorphic homeomorphism $f : X \to Y$.

Definition 3.1.6 (holomorphic function)**.** A holomorphic function on complex manifold X is a holomorphic map $f : X \to \mathbb{C}$.

Remark 3.1.2. We always use $\mathcal{O}_X$ to denote the sheaf of holomorphic functions on complex manifold X, and use $\Gamma(U, \mathcal{O}_X)$ to denote sections over open subset $U \subset X$.

Proposition 3.1. Let X be a compact connected complex manifold. Then $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$.

Proof. It's clear from maximum principle. □

Definition 3.1.7 (meromorphic function)**.** A meromorphic function on a complex manifold X is a map $f : X \to \coprod_{x \in X} Q(\mathcal{O}_{X,x})$ which associates to any $x \in X$ an element $f_x \in Q(\mathcal{O}_{X,x})$ such that for any $x_0 \in X$ there exists a neighborhood $x_0 \in U \subset X$ and two holomorphic functions $g, h : U \to \mathbb{C}$ with $f_x = g/h$ for all $x \in U$.

Remark 3.1.3. We always use $\mathcal{K}_X$ to denote the sheaf of meromorphic functions on complex manifold X, and use $K(X)$ to denote $\Gamma(X, \mathcal{K}_X)$.

Proposition 3.1.1 (Siegel)**.** Let X be a compact connected complex n-manifold. Then

$$\operatorname{trdeg}_{\mathbb{C}} K(X) \leq n$$

Proof. See Proposition 2.1.9 in Page54 of [Huy05]. □

Definition 3.1.8 (algebraic dimension)**.** The algebraic dimension of a compact connected complex manifold X is $a(X) := \operatorname{trdeg}_{\mathbb{C}} K(X)$.

3.2. **Analatic subvariety.**

Definition 3.2.1 (analytic subvariety)**.** Let X be a complex manifold. An analytic subvariety of X is a closed subset $Y \subset X$ such that for any $x \in Y$ there exists an open neighborhood $x \in U \subset X$ such that $Y \cap U$ is a zero set of finitely many holomorphic functions $f_1, \ldots, f_k \in \mathcal{O}(U)$.

Definition 3.2.2. An analytic subvariety Y is called irreducible, if it can't be written as the union $Y = Y_1 \cup Y_2$ of two proper analytic subvarieties $Y_i \subset Y, i = 1, 2$.

Given an analytic subvariety Y of a complex manifold X.

Definition 3.2.3 (regular)**.** A point $x \in Y$ is called regular point, if the functions $f_1, \ldots, f_k$ can be chosen such that $\varphi(x) \in \varphi(U)$ is a regular point of holomorphic map $f := (f_1 \circ \varphi^{-1}, \ldots, f_k \circ \varphi^{-1}) : \varphi(U) \to \mathbb{C}^k$, where (U, φ) is a local chart of x.

Definition 3.2.4 (singular)**.** A point $x \in Y$ is singular, if it's not regular.

Proposition 3.2.1. The set of regular points $Y_{\text{reg}} = Y \backslash Y_{\text{sing}}$ is a non-empty submanifold of X. Furthermore, if Y is irreducible, then Y_{reg} is connected.

Definition 3.2.5. The dimension of an irreducible analytic subvariety Y is defined $\dim Y = \dim Y_{\text{reg}}$.

3.3. **Basic examples.**

Example 3.3.1 (affine space)**.** The n-dimensional complex plane $\mathbb{C}^n$ is a complex manifold.

Example 3.3.2 (complex tori)**.** If V is a complex vector space of dimension n and $\Gamma \subset V$ is a free abelian, discrete subgroup of order $2n$, then $X = V/\Gamma$ is a complex manifold, which is called complex tori.

Remark 3.3.1. The underlying manifolds of complex tori with different Γ are not very interesting, since they are all diffeomorphic to $(S^1)^{2n}$. However, if you pick two lattices Γ_1, Γ_2 randomly, then $\mathbb{C}^n / \Gamma_1$ and $\mathbb{C}^n / \Gamma_2$ will not be biholomorphic to each other.

Example 3.3.3 (projective space)**.** The projective space $\mathbb{CP}^n$ is a complex manifold. Indeed, atlas are given by $U_i = \{[z] \in \mathbb{CP}^n \mid z_j \neq 0\}, 0 \leq i \leq n$, and φ_i is defined as

$$\begin{aligned}\varphi_i \colon U_i &\to \mathbb{C}^n \\ [z] &\mapsto (\frac{z_0}{z_i}, \dots, \frac{\widehat{z_i}}{z_i}, \dots, \frac{z_n}{z_i})\end{aligned}$$

The transition functions are calculated as follows: For $i < j$

$$\varphi_i \circ \varphi_j^{-1} : (u_1, \dots, u_n) \mapsto (\frac{u_1}{u_i}, \dots, \frac{\widehat{u_i}}{u_i}, \dots, \frac{u_{j-1}}{u_i}, \frac{1}{u_i}, \frac{u_{j+1}}{u_i}, \dots, \frac{u_n}{u_i})$$

It's holomorphic on $U_i \cap U_j$.

Remark 3.3.2. $\mathbb{CP}^n$ is compact, since $\mathbb{CP}^n$ is diffeomorphic to S^{2n+1}/S^1, which is called Hopf fiberation.

Definition 3.3.1 (projective manifold)**.** A complex manifold X is called projective if X is biholomorphic to a closed complex submanifold of some projective space $\mathbb{CP}^N$.

Example 3.3.4 (Grassmannian manifold)**.** The Grassmannian manifold

$$Gr(k, n+1) = \{k\text{-dimensional subspace of } \mathbb{C}^{n+1}\}$$

Now we're going to show $Gr(k, n+1)$ is a manifold of dimension $k(n+1-k)$. Any $W \in Gr(k, n+1)$ is generated by the rows of a $k \times (n+1)$ matrix A of rank k. Let us denote the set of these matrices by $M_{k,n+1}$, which is an open subset of the set of all $k \times (n+1)$ matrices. The latter space is a complex manifold which is canonically isomorphic to $\mathbb{C}^{k(n+1)}$. Thus we obtain a natural surjection $\pi \colon M_{k,n+1} \to Gr(k, n+1)$, which is the quotient by the natural action of $\mathrm{GL}(k, \mathbb{C})$ on $M_{k,n+1}$.

Let's fix an ordering $\{B_1, \dots, B_m\}$ of all $k \times k$-minors of matrices $A \in M_{k,n+1}$. Define an open covering $Gr(k, n+1) = \bigcup_{i=1}^m U_i$, where U_i is the open subset $\{\pi(A) \mid \det(B_i) \neq 0\}$. Note that U_i is well-defined, since if $\pi(A) = \pi(A')$, then A and A' differs an action of $\mathrm{GL}(k, \mathbb{C})$, so $\det(B_i) \neq 0$ if and only if $\det(B_i') \neq 0$. So without lose of generality, we may assume A is of form (B_i, C_i), where C_i is a $k \times (n+1-k)$ matrix. Then the map $\varphi_i : U_i \to \mathbb{C}^{k(n+1-k)}$, given by $\pi(A) \to B_i^{-1}C_i$ is well-defined, and $\{(U_i, \varphi_i)\}$ will give atlas of $Gr(k, n+1)$, sicne all operations are matrix operation, thus they're holomorphic. This shows $Gr(k, n+1)$ is a complex manifold with dimension $k(n+1-k)$.

Remark 3.3.3. If V is a complex vector space of dimension $n+1$, then $Gr(k, V)$ is defined as the set consisting of all k-dimensional subspaces of V, which is biholomorphic to $Gr(k, n+1)$.

Example 3.3.5 (Plücker embedding)**.** Let V be a complex vector space of dimension $n+1$, then

$$\Phi : G(k, V) \hookrightarrow \mathbb{CP}(\bigwedge^k V)$$

defined by $W \subset V$ with basis $w_1, \dots, w_k$ is mapped to $[w_1 \wedge \cdots \wedge w_k]$, is called Plücker embedding. It's well-defined, thanks to the following lemma.

Lemma 3.3.1. Let W be a complex vector space of dimension k, and $\mathcal{B}_1 = \{w_1, \dots, w_k\}$ and $\mathcal{B}_2 = \{v_1, \dots, v_k\}$ are two basis for W. Then $v_1 \wedge \cdots \wedge v_k = \lambda w_1 \wedge \cdots \wedge w_k$ for some $\lambda \in \mathbb{C}^*$.

Proof. If we express $w_j = a_{1j}v_1 + \cdots + a_{kj}v_k$, then direct computation shows that

$$\begin{aligned}
w_1 \wedge \cdots \wedge w_k &= (a_{11}v_1 + \cdots + a_{k1}v_k) \wedge \cdots \wedge (a_{1k}v_1 + \cdots + a_{kk}v_k) \\
&= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{k\sigma(k)} v_1 \wedge \cdots \wedge v_k \\
&= \lambda v_1 \wedge \cdots \wedge v_k
\end{aligned}$$

Note that λ is exactly the determinant of the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$. $\square$

Remark 3.3.4. It's a little bit complicated to check it's injective. I will add the proof if I'm not tooo lazy in future(smile).

3.4. **Vector bundle.**

3.4.1. *Definitions.*

Definition 3.4.1 (complex vector bundle)**.** Let X be a smooth manifold, a complex vector bundle E of rank r on X consists of the following data

1. E is a smooth manifold with surjective map $\pi\colon E \to X$, such that
   (1) For all $x \in X$, fibre E_x is a $\mathbb{C}$-vector space of dimension r;
   (2) For all $x \in X$, there exists $x \in U \subset X$ and $\pi^{-1}(U)$ is diffeomorphic to $U \times \mathbb{C}^r$ via φ such that

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\pi"] \arrow[dr, "\varphi"'] & & U \\
& U \times \mathbb{C}^r \arrow[ur, "p_1"'] \arrow[r, "p_2"] \arrow[u, phantom, "\circlearrowright"] & \mathbb{C}^r
\end{tikzcd}$$

   and for all $y \in U$, $E_y \xrightarrow{p_2 \circ \varphi} \mathbb{C}^r$ is a $\mathbb{C}$-vector space isomorphism. (U, φ) is called a trivialization of E over U.

Remark 3.4.1 (transition functions)*.* Consider two local trivialization $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$, then $\varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$ induces

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{C})$$

where $g_{\alpha\beta}$ is called transition function. Furthermore, it satisfies

$$\begin{aligned} g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} &= \mathrm{id} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma \\ g_{\alpha\alpha} &= \mathrm{id} \quad \text{on } U_\alpha \end{aligned}$$

In fact, transition functions contain all information about this vector bundle, since a vector bundle is locally trivial, so how are these trivial pieces glued together really matters.

Definition 3.4.2 (complex vector bundle)**.** Let X be a smooth manifold, a complex vector bundle E of rank r on X consists of the following data

(1) open covering $\{U_\alpha\}$ of X;
(2) smooth functions $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{\text{diff}} \mathrm{GL}(r, \mathbb{C})\}$ satisfies

$$\begin{aligned} g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} &= \mathrm{id} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma \\ g_{\alpha\alpha} &= \mathrm{id} \quad \text{on } U_\alpha \end{aligned}$$

Remark 3.4.2*.* The two definitions above are equivalent. The first definition implies the second clearly. The converse is a standard constructive method: If we already have an open covering and a set of transition functions, the vector bundle E is defined to be the quotient of the disjoint union $\coprod_{U_\alpha}(U \times \mathbb{C}^r)$ by the equivalence relation that puts $(p', v') \in U_\beta \times \mathbb{C}^r$ equivalent to $(p, v) \in U_\alpha \times \mathbb{C}^r$ if and only if $p = p'$ and $v' = g_{\alpha\beta}(p)v$. To connect this definition with the previous one, define the map π to send the equivalence class of any given (p, v) to p.

Definition 3.4.3 (holomorphic vector bundle)**.** Let X be a complex manifold, $\pi\colon E \to X$ is a complex vector bundle given by transition functions $\{g_{\alpha\beta}\}$. E is called a holomorphic vector bundle if $\{g_{\alpha\beta}\}$ is a holomorphic map.

Exercise 3.4.1. Show that the total space of a holomorphic vector bundle E is a complex manifold.

Proof. Since we already have a complex structure on X, we need to pull it back to E using π and use the holomorphic transition functions to show it do gives a complex structure on E. □

3.4.2. *Morphism and exactness.*

Definition 3.4.4 (morphism between vector bundles)**.** ϕ is a smooth/holomorphic morphism of vector bundle on X of rank k, if $\phi : E \to F$ is smooth/holomorphic map and fibrewise $\mathbb{C}$-linear of rank k.

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi_1"'] & {} \arrow[d, phantom, "\circlearrowleft"] & F \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

Example 3.4.1. X is a smooth/complex manifold, then $X \times \mathbb{C}^r$ is the trivial rank r complex/holomorphic vector bundle on X.

Definition 3.4.5 (subbundle)**.** $\pi\colon E \to X$ is a complex/holomorphic vector bundle. $F \subset E$ is called a subbundle of rank s, if

1. For all $x \in X$, $F \cap E_x$ is a subvector space of dimension s.
2. $\pi|_F : F \to X$ induces a complex/holomorphic vector bundle.

Definition 3.4.6 (exact)**.** A sequence of vector bundles

$$S \xrightarrow{\phi} E \xrightarrow{\psi} Q$$

is called exact at E if $\ker\psi = \operatorname{im}\phi$;

3.4.3. *Algebraic construction.* E, F are complex/holomorphic vector bundles on X with transition functions $\{g_{\alpha\beta}\}, \{h_{\alpha\beta}\}$, then by algebraic construction we have

1. $E \oplus F$, given by transition functions $\{\mathrm{diag}(g_{\alpha\beta}, h_{\alpha\beta})\}$
2. $E \otimes F$, given by transition functions $\{g_{\alpha\beta} \otimes h_{\alpha\beta}\}$;
3. E^*, given by transition functions $\{(g_{\alpha\beta}^{-1})^T\}$;
4. $\mathrm{Hom}(E, F) := E^* \otimes F$;
5. $\bigwedge^k E$, given by transition functions $\{\bigwedge^r g_{\alpha\beta}\}$;
6. Let $f : X \to Y$ be a smooth/holomorphic map, $\pi\colon E \to Y$ is a vector bundle with transition functions $\{g_{\alpha\beta}\}$, then transition functions of pullback bundle f^*E is given by $\{g_{\alpha\beta} \circ f\}$.

Remark 3.4.3. Here is an explicit construction of pullback bundle defined by

$$f^*E = \{(x,e) \in X \times E \mid f(x) = \pi(e)\} \subset X \times E$$

In fact, you can regard it as a fiber product as

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow{f} & Y \end{array}$$

3.4.4. *Hermitian structure.* In this section X is a smooth manifold, and $\pi\colon E \to X$ is a complex vector bundle.

Definition 3.4.7 (hermitian metric)**.** A hermitian metric h on E is a hermitian inner product on each fibre E_x such that for all open subset $U \subset X$, and $\xi, \eta \in C^\infty(U, E|_U)$, we have

$$\begin{aligned} \langle \xi, \eta \rangle : U &\to \mathbb{C} \\ x &\mapsto \langle \xi(x), \eta(x) \rangle \end{aligned}$$

is a smooth function.

Remark 3.4.4 (local form)*.* Given a local frame $\{e_\alpha\}$ of E, hermitian metric can be written as a hermitian matrix $H = (h_{\alpha\beta})$, where $h_{\alpha\beta} \in C^\infty(U)$, defined by

$$h_{\alpha\beta}(x) = \langle e_\alpha(x), e_\beta(x) \rangle$$

For two sections s, t of E, if we write them as $s = s^\alpha e_\alpha, t = t^\beta e_\beta$ with respect to local frame, then

$$\begin{aligned} h(s,t) &= h(s^\alpha e_\alpha, t^\beta e_\beta) \\ &= s^\alpha \overline{t^\beta} h_{\alpha\beta} \end{aligned}$$

In matrix notation we have

$$h(s,t) = s^T H \bar{t}$$

Proposition 3.4.1. Every complex vector bundle admits a hermitian metric.

Proof. Use partition of unity. $\square$

3.4.5. *In viewpoint of sheaf.*

Definition 3.4.8 (section)**.** Let X be a complex manifold, $\pi\colon E \to X$ a complex/holomorphic vector bundle, and U is an open subset of X. A section of E on U is a smooth/holomorphic map $s : U \to E$, such that $\pi \circ s = \mathrm{id}_U$. The set of all smooth/holomorphic sections over U is denoted by $C^\infty(U, E)$ / $\Gamma(U, E)$.

One reason why sheaf plays an important role of study of complex geometry is that you can regard a vector bundle as a special sheaf.

Definition 3.4.9 (sheaf of sections)**.** Let X be a complex manifold and $\pi\colon E \to X$ a holomorphic vector bundle, then its sheaf of sections, denoted by $\mathcal{O}_X(E)$ is defined as

$$\mathcal{O}_X(E)(U) = \Gamma(U, E|_U)$$

Example 3.4.2. If $E \to X$ is trivial holomorphic vector bundle, then $\mathcal{O}_X(E)$ is exactly sheaf of holomorphic functions.

Example 3.4.3 (locally free sheaf)**.** A sheaf $\mathscr{F}$ is called locally free, if there exists an open covering $\{U_\alpha\}$ such that $\mathscr{F}|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}^{\oplus r}$ of rank r.

Exercise 3.4.2. There is one to one correspondence:

$$\{\text{holomorphic vector bundles}\} \xleftrightarrow{1-1} \{\text{locally free sheaves}\}$$

Proof. If $\pi\colon E \to X$ is a holomorphic vector bundle, we claim $\mathcal{O}_X(E)$ is a locally free sheaf. Since we have local trivialization of holomorphic vector bundle $\{U_\alpha\}$. Then consider what's $\mathcal{O}_X(E)|_{U_\alpha}$. Since $E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^r$, then holomorphic sections of $U_\alpha \times \mathbb{C}^r \to U_\alpha$ are just holomorphic functions $f : U \to \mathbb{C}^r$, i.e. $\mathcal{O}_X(E|_{U_\alpha}) = \mathcal{O}_{U_\alpha}^{\oplus r}$. So sheaf $\mathcal{O}_X(E)$ is a locally free sheaf.

Conversely, assume $\mathcal{E}$ is locally free over an open covering $\{U_\alpha\}$ of X, then we just need to glue $U_\alpha \times \mathbb{C}^r \to U_\alpha$ together to get a vector bundle. Therefore we need a family of gluing data $g_{\alpha\beta} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$. Since $\mathcal{E}$ is locally free, we have local isomorphism $f_\alpha : \mathcal{E}|_{U_\alpha} \to \mathcal{O}_{U_\alpha}^{\oplus r}$. Restricting to intersection $U_\alpha \cap U_\beta$, we get

$$f_{\alpha\beta} = f_\alpha|_{U_\alpha \cap U_\beta} \circ f_\beta^{-1}|_{U_\alpha \cap U_\beta} : \mathcal{O}_{U_\beta}^{\oplus r}|_{U_\alpha \cap U_\beta} \to \mathcal{O}_{U_\alpha}^{\oplus r}|_{U_\alpha \cap U_\beta}$$

Every such map is induced by a map

$$g_{\alpha\beta} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$$

that's gluing data we desire. $\square$

Definition 3.4.10. If E is a holomorphic vector bundle on a complex manifold X, then $H^q(X, E)$ denotes the q-th cohomology of its sheaf of sections.

4. Divisor and line bundle

In this section, X is a complex manifold.

4.1. Line bundle.

Definition 4.1.1 (line bundle)**.** A complex/holomorphic line bundle L is a rank 1 complex/holomorphic vector bundle.

Exercise 4.1.1. Let $E \to X$ be a line bundle (no matter complex or holomorphic), then E is a trivial line bundle if and ony if there exists a non-vanishing global section s.

Proof. It's clear there exists a non-vanishing global section if E is trivial; Conversely, if there exists a non-vanishing global section s. Define the following map

$$\begin{aligned} \varphi : X \times \mathbb{C} &\to E \\ (x, \lambda) &\mapsto \lambda s(x) \end{aligned}$$

Now it suffices to show it's an isomorphism, i.e. the map $\varphi_x \to \{x\} \times \mathbb{C} \to E_x$ is an isomorphism of vector spaces. The map φ_x is given by $\lambda s(x)$, it's injective thus an isomorphism. Indeed, if $\lambda s(x) = 0$ then we have $\lambda = 0$ since $s(x) \neq 0$. □

Remark 4.1.1. Note that for a line bundle L with transition functions $\{g_{\alpha\beta}\}$, then the transition functions of $L^* \otimes L$ is

$$(g_{\alpha\beta}^{-1})^T g_{\alpha\beta} = g_{\alpha\beta}^{-1} g_{\alpha\beta} = \mathrm{id}$$

So the vector bundle $L^* \otimes L$ is the trivial bundle.

Definition 4.1.2 (picard group)**.** The picard group $\mathrm{Pic}(X)$ is defined as the set of all holomorphic line bundles on X up to isomorphism, where group structure is given by tensor product.

Proposition 4.1.1. There is a natural isomorphism $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$.

Proof. For a line bundle L, it's completely determined by its transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}^*$, which is holomorphic functions. It gives rise to an element in $\check{H}^1(X, \mathcal{O}_X^*)$, since $g_{\alpha\beta}$ satisfies cocycle conditions. Furthermore, Čech cohomology[2] computes the sheaf cohomology for reasonable topological space, e.g. for manifolds. □

Remark 4.1.2. This proposition gives us a method to compute Picard group of a complex manifold, since there is exponential sequence as follows

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

which is a exact sequence of sheaves, then it gives a long exact sequence of cohomology groups as follows

$$\cdots \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to \dots$$

[2]For more details, see Appendix A.

Thus $\operatorname{Pic}(X)$ can in principle be computed by above exact sequence. Roughly speaking, $\operatorname{Pic}(X)$ has two parts:

1. A discrete part, measured by its image in $H^2(X, \mathbb{Z})$;
2. A continuous part coming from the $H^1(X, \mathcal{O}_X)$, which is possibly trivial.

4.2. **Divisor.**

Definition 4.2.1 (analytic hypersurface)**.** An analytic hypersurface of X is an analytic subvariety $Y \subset X$ of codimensional one.

Remark 4.2.1. A hypersurface is locally given as the zero set of a non-trivial holomorphic function.

Definition 4.2.2 (divisor)**.** A divisor D on X is a locally finite formal linear combination $D = \sum a_i[Y_i]$ with $Y_i \subset X$ are irreducible hypersurfaces and $a_i \in \mathbb{Z}$.

Remark 4.2.2. The above sum is called locally finite, if for any $x \in X$, there exists an open neighborhood $x \in U \subset X$ such that only finite many coefficients $a_i \neq 0$ with $Y_i \cap U \neq \varnothing$.

Definition 4.2.3 (divisor group)**.** The divisor group $\operatorname{Div}(X)$ is the set of all divisors endowed with the natural group structure.

Definition 4.2.4 (effective)**.** A divisor $D = \sum a_i[Y_i]$ is called effective, if $a_i \geq 0$ for all i. In this case, we write $D \geq 0$.

Example 4.2.1. Every hypersurfaces Y defines an effective divisor $\sum[Y_i] \in \operatorname{Div}(X)$, where Y_i are irreducible component of Y.

Let $Y \subset X$ be a hypersurface and let $x \in Y$. Suppose that Y defines an irreducible germ in x, that is this germ is the zero set of an irreducible $g \in \mathcal{O}_{X,x}$.

Definition 4.2.5 (order)**.** Let f be a meromorphic function in a neighborhood of $x \in Y$, then the order $\operatorname{ord}_{Y,x}(f)$ of f in x with respect to Y is given by

$$f = g^{\operatorname{ord}_{Y,x}(f)} h$$

where $h \in \mathcal{O}^*_{X,x}$.

Remark 4.2.3. For order, we have the following remarks:

1. The order of f in x with respect to Y is independent of the choice of g, since any two irreducible $g, g' \in \mathcal{O}_{X,x}$ with $Z(g) = Z(g')$ only differs by an element in $\mathcal{O}^*_{X,x}$.
2. More globally, one can define order $\operatorname{ord}_Y(f)$ as $\operatorname{ord}_Y(f) = \operatorname{ord}_{Y,x}(f)$ for $x \in Y$ such that Y defines an irreducible germ in x. Such a point $x \in Y$ always exists, for example, one can choose a regular point $x \in Y_{\mathrm{reg}}$. Moreover, it's independent of the choice of x, since

Definition 4.2.6 (zeros and poles)**.** Let f be a meromorphic function on X, then

1. f has zeros of order $d \geq 0$ along Y if $\operatorname{ord}_Y(f) = d$;
2. f has poles of order $d \geq 0$ along Y if $\operatorname{ord}_Y(f) = -d$;

Definition 4.2.7. Let $f \in K(X)$. Then the divisor associated to f is

$$(f) := \sum \operatorname{ord}_Y(f)[Y]$$

where the sum is taken over all irreducible hypersurfaces $Y \subset X$. A divisor of this form is called principal.

Remark 4.2.4. The divisor (f) can be written as the difference of two effective divisors $(f) = Z(f) - P(f)$, where

$$Z(f) = \sum_{\operatorname{ord}_Y(f)>0} \operatorname{ord}_Y(f)[Y], \quad P(f) = \sum_{\operatorname{ord}_Y(f)<0} \operatorname{ord}_Y(f)[Y]$$

Proposition 4.2.1. There exists a natural isomorphism

$$H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \cong \operatorname{Div}(X)$$

Proof. An element $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ is given by non-trivial meromorphic functions $f_i \in K_X^*(U_i)$ such that $f_i f_j^{-1}$ is a holomorphic function without zeros on $U_i \cap U_j$, where $\{U_i\}$ is an open covering of X. Thus for any irreducible hypersurface $Y \subset X$ with $Y \cap U_i \cap U_j \neq \varnothing$, one has $\operatorname{ord}_Y(f_i) = \operatorname{ord}_Y(f_j)$. Hence $\operatorname{ord}_Y(f)$ is well-defined for any irreducible hypersurface Y. Then one associates to f the divisor $(f) = \sum \operatorname{ord}_Y(f)[Y] \in \operatorname{Div}(X)$.

It's clear this map is a group homomorphism. To see it's bijective, we define the inverse as follows. If $D = \sum a_i[Y_i] \in \operatorname{Div}(X)$ is given, then there exists an open covering $\{U_i\}$ of X such that $Y_i \cap U_j$ is defined by $g_{ij} \in \mathcal{O}(U_j)$ which is unique up to elements in $\mathcal{O}^*(U_j)$. Let $f_j := \prod_i g_{ij}^{a_i} \in \mathcal{K}_X^*(U_j)$, since g_{ij} and g_{ik} defines the same irreducible hypersurface, they only differ by an element in $\mathcal{O}^*(U_j \cap U_k)$. Thus f glue to an element $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. It's clear these two maps are inverse to each other. $\square$

Remark 4.2.5. In algebraic geometry, elements in $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ are called Cartier divisors and elements in $\operatorname{Div}(X)$ are called Weil divisors. Above isomorphism still holds in the algebraic setting under a weak smoothness assumption on X.

Corollary 4.2.1. There exists a natural group homomorphism

$$\begin{aligned} \operatorname{Div}(X) &\to \operatorname{Pic}(X) \\ D &\mapsto \mathcal{O}(D) \end{aligned}$$

where $\mathcal{O}(D)$ is defined in the proof.

Proof. If $D = \sum a_i[Y_i] \in \operatorname{Div}(X)$ corresponds to $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$, which in turn is given by functions $f_i \in \mathcal{K}_X^*(U_i)$ for an open covering $\{U_i\}$. Then we define $\mathcal{O}(D) \in \operatorname{Div}(X)$ with transition functions $\psi_{ij} := f_i f_j^{-1} \in \mathcal{O}_X^*(U_{ij})$.

If D, D' are two divisors, without lose of generality we may assume they're given by $\{f_i\}$ and $\{f_i'\}$ respectively on the same open covering, then $D + D'$, then $D+D'$ corresponds to $\{f_i+f'_i\}$. By definition $\mathcal{O}(D+D')$ is described by $\{\psi_{ij}\psi'_{ij}\}$, hence $\mathcal{O}(D+D')=\mathcal{O}(D)\otimes\mathcal{O}(D')$. □

Remark 4.2.6. In fact, above corollary can be derived from the following exact sequence of sheaves

$$0\to\mathcal{O}_X^*\to\mathcal{K}_X^*\to\mathcal{K}_X^*/\mathcal{O}_X^*\to 0$$

Then above group homomorphism is exactly the boundary map, the kernel of which coincides with the image of $H^0(X,\mathcal{K}_X^*)\to H^0(X,\mathcal{K}_X^*/\mathcal{O}_X^*)$, and the latter by definition is the set of principal divisors.

Definition 4.2.8 (linearly equivalent)**.** Two divisors D,D' are called linearly equivalent, denoted by $D\sim D'$, if $D-D'$ is a principal divisor.

Corollary 4.2.2. The group homomorphism $\mathrm{Div}(X)\to\mathrm{Pic}(X)$ factorizes over an injection

$$\mathrm{Div}(X)/_\sim\hookrightarrow\mathrm{Pic}(X)$$

4.3. **Divisor and line bundle.** In general, $\mathrm{Div}(X)/_\sim\hookrightarrow\mathrm{Pic}(X)$ is a strict inclusion, but we will see if a line bundle admits a non-trivial global section, then it's contained in the image. In order to show this, we need to construct a canonical map

$$H^0(X,L)\backslash\{0\}\to\mathrm{Div}(X)$$
$$s\mapsto Z(s)$$

The map is constructed as follows: Let $L\in\mathrm{Pic}(X)$ on open covering $\{U_i\}$ be trivialized by $\psi_i: L|_{U_i}\to\mathcal{O}_{U_i}$, then divisor $Z(s)$ is given by $f:=\{f_i:=\psi_i(s|_U\}\in H^0(X,\mathcal{K}_X^*/\mathcal{O}_X^*)$.

Proposition 4.3.1. Let $0\neq s\in H^0(X,L)$, the line bundle $\mathcal{O}(Z(s))$ is isomorphic to L.

Proposition 4.3.2. For any effective divisor $D\in\mathrm{Div}(X)$, there exists a section $0\neq s\in H^0(X,\mathcal{O}(D))$ with $Z(s)=D$.

Corollary 4.3.1. Non-trivial sections $s_1\in H^0(X,L_1)$ and $s_2\in H^0(X,L_2)$ define linearly equivalent divisors $Z(s_1)\sim Z(s_2)$ if and only if $L_1\cong L_2$.

Proof. If $L_1\cong L_2$, then

If $Z(s_1)\sim Z(s_2)$, then by Corollary 4.2.2 one has $\mathcal{O}(Z(s_1))\cong\mathcal{O}(Z(s_2))$, then this shows $L_1\cong L_2$ since $\mathcal{O}(Z(s_i))=L_i, i=1,2$. □

Corollary 4.3.2. The image of the natural map $\mathrm{Div}(X)\to\mathrm{Pic}(X)$ is generated by those line bundles $L\in\mathrm{Pic}(X)$ with $H^0(X,L)\neq 0$.

Proof. We have already seen if $H^0(X,L)\neq 0$, then L is contained in the image. Conversely, any divisor $D=\sum a_i[Y_i]$ can be written as $D=\sum a_i^+[Y_i]-\sum a_j^-[Y_j]$ with $a_k^\pm\geq 0$, thus $\mathcal{O}(D)\cong\mathcal{O}(\sum a_i^+[Y_i])\otimes\mathcal{O}(\sum a_j^-[Y_j])^*$. Both $\mathcal{O}(\sum a_i^+[Y_i])$ and $\mathcal{O}(\sum a_j^-[Y_j])$ are associated to effective divisors, and therefore admit non-trivial global sections. □

Remark 4.3.1. For projective manifolds, the map $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ is surjective, but note that even for very easy manifolds, such as complex tori, this is no longer the case.

4.4. **Ample line bundle.**

Definition 4.4.1 (base point). Let L be a holomorphic line bundle on a complex manifold X. A point $x \in X$ is a base point of L if $s(x) = 0$ for all $s \in H^0(X, L)$. The base locus $\mathrm{Bs}(L)$ is the set of all base points of L.

Remark 4.4.1. If $\dim H^0(X, L) < \infty$, we can choose a basis of global sections $s_1, \dots, s_N$ of it, then $\mathrm{Bs}(L) = Z(s_1) \cap \cdots \cap Z(s_N)$ is an analytic subvariety. Later we will see if X is compact, then $\dim H^0(X, L) < \infty$.

Proposition 4.4.1. Let L be a holomorphic line bundle on a complex manifold X and suppose $s_1, \dots, s_N \in H^0(X, L)$ is a basis, then

$$\begin{aligned} \varphi_L \colon X \setminus \mathrm{Bs}(L) &\to \mathbb{CP}^N \\ x &\mapsto (s_0(x) : \cdots : s_N(x)) \end{aligned}$$

defines a holomorphic map such that $\varphi_L^* \mathcal{O}_{\mathbb{CP}^N}(-1) \cong L|_{X \setminus \mathrm{Bs}(L)}$.

Definition 4.4.2 (ample line bundle). A holomorphic line bundle L on a complex manifold X is called ample if for some $k > 0$ and some linear system in $H^0(X, L^k)$ the associated map φ is an embedding.

Remark 4.4.2. By definition, a compact complex manifold is projective if and only if it admits an ample line bundle.

5. Tangent and cotangent bundle

5.1. Complex and holomorphic tangent bundle.

Definition 5.1.1 (complex tangent bundle)**.** Let X be a smooth n-manifold, with an atlas $\{U_\alpha, \varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n\}$, then real tangent bundle $T_{X,\mathbb{R}}$ is defined through transition functions

$$\begin{aligned} g_{\alpha\beta} : U_\alpha \cap U_\beta &\stackrel{\text{diff}}{\to} \mathrm{GL}(n, \mathbb{R}) \\ x &\mapsto J(\varphi_\alpha \circ \varphi_\beta^{-1})(\varphi_\beta(x)) \end{aligned}$$

The complex tangent bundle $T_{X,\mathbb{C}}$ is defined as the complexified (real) tangent vector bundle, that is $T_{X,\mathbb{R}} \otimes \mathbb{C}$.

Definition 5.1.2 (holomorphic tangent bundle)**.** X is a complex n-manifold, with an atlas $\{U_\alpha, \varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^n\}$, then holomorphic tangent bundle T_X is defined through transition functions

$$\begin{aligned} g_{\alpha\beta} : U_\alpha \cap U_\beta &\to \mathrm{GL}(n, \mathbb{C}) \\ z &\mapsto J(\varphi_\alpha \circ \varphi_\beta^{-1})(\varphi_\beta(x)) \end{aligned}$$

Remark 5.1.1 (relations between complex tangent bundle and holomorphic tangent bundle)*.* Now here comes a natural question: For a complex manifold X, if we consider its underlying smooth manifold X, then we have a complex tangent bundle $T_{X,\mathbb{C}}$; On the other hand, we have a holomorphic tangent bundle T_X. Now we're going to show T_X is isomorphic to some component of $T_{X,\mathbb{C}}$ as complex bundle, but not as holomorphic bundle.

To be explicit, for local coordinates $\{z^1, \ldots, z^n\}$ of X, if we write $z^i = x^i + \sqrt{-1}y^i$, then X is a smooth manifold with local coordinates $\{x^1, \ldots, x^n, y^1, \ldots, y^n\}$. Thus $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$ is a local frame of $T_{X,\mathbb{R}}$. There is a natural almost complex structure J on $T_{X,\mathbb{R}}$, locally given by

$$\begin{aligned} J(\frac{\partial}{\partial x^i}) &= \frac{\partial}{\partial y^i} \\ J(\frac{\partial}{\partial y^i}) &= -\frac{\partial}{\partial x^i} \end{aligned}$$

Thus complex tangent bundle $T_{X,\mathbb{C}}$ can be decomposed as $T_{X,\mathbb{C}} = T_X^{1,0} \oplus T_X^{0,1}$ with respect to J, with local frame as follows:

1. $\frac{\partial}{\partial z^i} := \frac{1}{2}(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i})$ is a local frame of $T_X^{1,0}$;
2. $\frac{\partial}{\partial \bar{z}^i} := \frac{1}{2}(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i})$ is a local frame of $T_X^{0,1}$.

Indeed, let's check $T_X^{1,0}$ for an example:

$$\begin{aligned}J(\frac{\partial}{\partial z^i}) &= \frac{1}{2}(\frac{\partial}{\partial y^i} + \sqrt{-1}\frac{\partial}{\partial x^i}) \\ &= \frac{\sqrt{-1}}{2}(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i}) \\ &= \sqrt{-1}\frac{\partial}{\partial z^i}\end{aligned}$$

Note that as real bundle T_X is isomorphic to $T_{X,\mathbb{R}}$, and there is a natural inclusion $T_{X,\mathbb{R}} \to T_{X,\mathbb{C}}$, if we compose it with projection $T_{X,\mathbb{C}} = T_X^{1,0} \oplus T_X^{0,1} \to T_X^{1,0}$ onto the first summand, we obtain an $\mathbb{R}$-isomorphism $T_X \to T_X^{1,0}$ with inverse map $2\operatorname{Re}(\cdot)$. Indeed, we can decompose $\frac{\partial}{\partial x^i}$ in $T_{X,\mathbb{C}}$ as

$$\frac{\partial}{\partial x^i} = \frac{1}{2}(\frac{\partial}{\partial x^i} - \sqrt{-1}J(\frac{\partial}{\partial x^i})) + \frac{1}{2}(\frac{\partial}{\partial x^i} + \sqrt{-1}J(\frac{\partial}{\partial x^i}))$$

such that the first term lies in $T_X^{1,0}$ and second term lies in $T_X^{0,1}$, so the inverse map of this composite is $2\operatorname{Re}(\cdot)$.

Furthermore, the almost complex structure J on $T_{X,\mathbb{R}}$ makes it to be a complex vector bundle, and above $\mathbb{R}$-isomorphism between $T_{X,\mathbb{R}}$ and $T_X^{1,0}$ is a $\mathbb{C}$-isomorphism in this setting. That is T_X is isomorphic to $T_X^{1,0}$ as complex bundle.

5.2. **Differential forms and operators.** Let X be a complex manifold, if we consider the complexified dual space of $T_{X,\mathbb{R}}$, it admits an analogous decomposition:

$$\Omega^1_{X,\mathbb{C}} = \Omega^1_{X,\mathbb{R}} \otimes \mathbb{C} = \Omega_X^{1,0} \oplus \Omega_X^{0,1}$$

Furthermore, there is a decomposition on its k-th wedge product:

$$\Omega^k_{X,\mathbb{C}} = \bigwedge^k \Omega^1_{X,\mathbb{C}} = \bigoplus_{p+q=k} \Omega_X^{p,q}, \quad \text{where } \Omega_X^{p,q} = \bigwedge^p \Omega_X^{1,0} \otimes (\bigwedge^q \Omega_X^{0,1})$$

Remark 5.2.1 (local form). Choose a local coordinate $(z^1, \ldots, z^n) \in U \subset \mathbb{C}^n$, $z^i = x^i + \sqrt{-1}y^i$, there is a local frame of $\Omega^1_{X,\mathbb{R}}$, consisting of $\{\mathrm{d}x^i, \mathrm{d}y^j\}$, where $\mathrm{d}x^i, \mathrm{d}y^j$ are dual basis of $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}$ respectively. By definition

$$\begin{aligned}J^*(\mathrm{d}x^i)(\frac{\partial}{\partial x^i}) &= \mathrm{d}x^i(J(\frac{\partial}{\partial x^i})) = \mathrm{d}x^i(\frac{\partial}{\partial y^i}) = 0 \\ J^*(\mathrm{d}x^i)(\frac{\partial}{\partial y^i}) &= \mathrm{d}x^i(J(\frac{\partial}{\partial y^i})) = \mathrm{d}x^i(-\frac{\partial}{\partial x^i}) = -1\end{aligned}$$

that is

$$\begin{aligned}J^*(\mathrm{d}x^i) &= -\mathrm{d}y^i \\ J^*(\mathrm{d}y^i) &= \mathrm{d}x^i\end{aligned}$$

and similarly we have

1. $\mathrm{d}z^i := \mathrm{d}x^i + \sqrt{-1}\mathrm{d}y^i$ is a local frame of $\Omega_X^{1,0}$;

2. $\mathrm{d}\overline{z}^i := \mathrm{d}x^i - \sqrt{-1}\mathrm{d}y^i$ is a local frame of $\Omega_X^{0,1}$.

Definition 5.2.1 ((p,q)-form)**.** A k-form ω of type (p,q) is a smooth section of $\Omega_X^{p,q}$, that is

$$\omega \in C^\infty(X, \Omega_X^{p,q}) \subset C^\infty(X, \Omega_{X,\mathbb{C}}^k)$$

Remark 5.2.2. It's quite neccessary for us to keep in mind how to distinguish a differential k-form what type it is, particularly for the case $k = 2$, since later we will study the first Chern class, a special $(1,1)$-form. Let's firstly see it in a local viewpoint: For a k-form omega, it locally looks like

$$\sum_{\substack{|I|=p,|J|=q \\ p+q=k}} f_{IJ}\mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J$$

So a k-form is a (p,q)-form if and only if locally it looks like

$$\sum_{|I|=p,|J|=q} f_{IJ}\mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J$$

Or we can use language of skew-symmetric bilinear form: Let's elaborate in the case $k = 2$. By the definition of wedge of cotangent bundle, any section ω of $\Omega_{X,\mathbb{C}}^2$ is a skew-symmetric bilinear form that maps $C^\infty(X, T_{X,\mathbb{C}}) \times C^\infty(X, T_{X,\mathbb{C}})$ to $\mathbb{C}$. A 2-form ω is in type $(1,1)$ if and only if

$$\omega(C^\infty(X, T_X^{1,0}), C^\infty(X, T_X^{1,0})) = \omega(C^\infty(X, T_X^{0,1}), C^\infty(X, T_X^{0,1})) = 0$$

Exercise 5.2.1. For $\mathbb{C}^n \cong \mathbb{R}^{2n}$, we have

$$\omega = \mathrm{d}x^1 \wedge \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}x^n \wedge \mathrm{d}y^n = (\frac{i}{2})^n \mathrm{d}z^1 \wedge \mathrm{d}\overline{z^1} \wedge \cdots \wedge \mathrm{d}z^n \wedge \mathrm{d}\overline{z^n}$$

Proof. It suffices to show the case $n = 1$, and we can compute directly as follows

$$\begin{aligned}(\frac{i}{2})\mathrm{d}z \wedge \mathrm{d}\overline{z} &= (\frac{i}{2})(\mathrm{d}x + i\mathrm{d}y) \wedge (\mathrm{d}x - i\mathrm{d}y) \\ &= (\frac{i}{2})(-2i\mathrm{d}x \wedge \mathrm{d}y) \\ &= \mathrm{d}x \wedge \mathrm{d}y\end{aligned}$$

□

5.3. **$\overline{\partial}$-operator.** Let X be a complex manifold, naturally there is a differential operator

$$\mathrm{d} : C^\infty(X, \Omega_{X,\mathbb{C}}^k) \to C^\infty(X, \Omega_{X,\mathbb{C}}^{k+1})$$

Since we already know that we can decompose $\Omega_{X,\mathbb{C}}^k$, so it's natural to ask how to decompose $\mathrm{d}\alpha$, for $\alpha \in C^\infty(X, \Omega_{X,\mathbb{C}}^k)$.

Example 5.3.1. For $\alpha \in C^\infty(X, \Omega_{X,\mathbb{C}}^0)$, then $\mathrm{d}\alpha$ can be written as $\partial\alpha + \overline{\partial}\alpha$, where $\partial\alpha \in C^\infty(X, \Omega_X^{1,0})$ and $\overline{\partial}\alpha \in C^\infty(X, \Omega_X^{0,1})$. It suffices to see how to decompose locally. Locally we have

$$\begin{aligned}\mathrm{d}\alpha &= \frac{\partial\alpha}{\partial x^i}\mathrm{d}x^i + \frac{\partial}{\partial y^i}\mathrm{d}y^i \\ &= \frac{1}{2}(\frac{\partial\alpha}{\partial x^i} - \sqrt{-1}\frac{\partial\alpha}{\partial y^i})\mathrm{d}z^i + \frac{1}{2}(\frac{\partial\alpha}{\partial x^i} + \sqrt{-1}\frac{\partial\alpha}{\partial y^i})\mathrm{d}\overline{z}^i \\ &= \frac{\partial\alpha}{\partial z^i}\mathrm{d}z^i + \frac{\partial\alpha}{\partial\overline{z}^i}\mathrm{d}\overline{z}^i\end{aligned}$$

that is locally $\partial\alpha$ looks like $\frac{\partial\alpha}{\partial z^i}\mathrm{d}z^i$. More generally, for $\alpha \in C^\infty(X, \Omega_X^{p,q})$, locally looks like

$$\alpha = \sum_{|I|=p,|J|=q}\alpha_{IJ}\mathrm{d}z^J \wedge \mathrm{d}\overline{z}^K$$

then

$$\mathrm{d}\alpha = \sum_{|I|=p,|J|=q}\frac{\partial\alpha_{IJ}}{\partial z^l}\mathrm{d}z^l \wedge \mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J + \sum_{|I|=p,|J|=q}\frac{\partial\alpha_{IJ}}{\partial\overline{z}^l}\mathrm{d}\overline{z}^l \wedge z^I \wedge \overline{z}^J$$

We use $\partial\alpha$ to denote the former and $\overline{\partial}\alpha$ to denote the latter, and call them partial differential operators.

Proposition 5.3.1. For ∂ and $\overline{\partial}$, we have

1. Leibniz rule

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{\deg\alpha}\alpha \wedge \partial\beta$$

2.

$$\partial^2 = \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0$$

Thus we can consider the following cochain complex[3]

$$0 \to C^\infty(X, \Omega_X^{p,0}) \xrightarrow{\overline{\partial}} C^\infty(X, \Omega_X^{p,1}) \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} C^\infty(X, \Omega_X^{p,n}) \to 0 \tag{5.1}$$

Definition 5.3.1 (Dolbeault cohomology)**.**

$$H^{p,q}(X) := Z^{p,q}(X)/B^{p,q}(X) = H^q_{\overline{\partial}}(C^\infty(X, \Omega_X^{p,*}))$$

Remark 5.3.1. Here comes a key question: Since we have $C^\infty(X, \Omega^k_{X,\mathbb{C}}) = \bigoplus_{p+q=k} C^\infty(X, \Omega_X^{p,q})$, could we have the following decomposition?

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

In fact, later we will see for compact Kähler manifold, such decomposition do holds, which is called hodge decomposition.

[3]You may wonder why don't we use ∂ to construct such cobchain complex. In fact, the two definitions are almost the same, since they conjugate to each other. However, the cohomology group of cochain complex defined by $\overline{\partial}$ is more meaningful, as we will see later.

A natural question is that what does this cohomology compute? In the setting of smooth manifold, de Rham cohomology computes the cohomology of constant sheaf, and this holds from Poincaré lemma. This complex setting similar things still hold, firstly let's see an example.

Example 5.3.2. Since $B^{p,0}=0$, then

$$H^{p,0}(X)=Z^{p,0}(X)=\{\alpha\in C^\infty(X,\Omega_X^{p,0})\mid \overline{\partial}\alpha=0\}$$

Locally we have $\alpha=\sum_{|I|=p}\alpha_I\mathrm{d}z^I$, then

$$\overline{\partial}\alpha=\sum_{|I|=p}\frac{\partial\alpha_I}{\partial\overline{z}^l}\mathrm{d}\overline{z}^l\wedge\mathrm{d}z^I=0\implies\frac{\partial\alpha_I}{\partial\overline{z}^l}=0$$

That is, α_I is holomorphic function. Since $\Omega_X^{p,0}\cong\Omega_X^p$ as complex vector bundle, we have $H^{p,0}(X)=\Gamma(X,\Omega_X^p)$.

Proposition 5.3.2 ($\overline{\partial}$-Poincaré lemma)**.** Let B be an open disc, if $\alpha\in C^\infty(B,\Omega_X^{p,q})$ is $\overline{\partial}$-closed and $q>0$, then there exists $\beta\in C^\infty(B,\Omega_X^{p,q-1})$ such that $\alpha=\overline{\partial}\beta$.

Proof. See Corollary 1.3.9 of Page47 of [Huy05]. □

Remark 5.3.2. This shows Dolbeault cohomology $H^{p,q}(X)$ computes the q-th sheaf cohomology of Ω_X^p.

Proposition 5.3.3. For a holomorphic map $f:X\to Y$ between complex manifold, then

$$f^*:C^\infty(Y,\Omega_{Y,\mathbb{C}}^k)\to C^\infty(X,\Omega_{X,\mathbb{C}}^k)$$

Then[4]

$$f^*:C^\infty(Y,\Omega_{Y,\mathbb{C}}^{p,q})\to C^\infty(X,\Omega_{X,\mathbb{C}}^{p,q})$$

and

$$f^*:H^{p,q}(Y)\to H^{p,q}(X)$$

Remark 5.3.3. This shows Dolbeault cohomology is a contravariant functor.

Example 5.3.3 (Dolbeault cohomology of a holomorphic vector bundle[5])**.** For a holomorphic vector bundle $E\to X$, we can also define

$$\overline{\partial}_E:C^\infty(X,\Omega_X^{0,q}\otimes E)\to C^\infty(X,\Omega_X^{0,q+1}\otimes E)$$

satisfies $\overline{\partial}_E^2=0$. Let's elaborate this construction: Since any global section is glued together by local sections, we just need to define $\overline{\partial}_E$ for local sections and check is well-defined under the change of local chart. We can choose a local holomorphic frame $\{e^1,\dots,e^n\}$ for E on U, so any section $\sigma\in$

[4]Check this, we need back to definition, a holomorphic map induces a tangent map $T_f:T_{X,\mathbb{C}}\to f^*T_{Y,\mathbb{C}}$,and consider its dual we get cotangent map $\Omega_f:f^*\Omega_{Y,\mathbb{C}}\to\Omega_{X,\mathbb{C}}$

[5]In previous case, $E=\Omega_X^p$

$C^\infty(U, \Omega_X^{0,q} \otimes E)$ we can write it locally as $\sigma = \varphi^i e_i$, where $\varphi^i \in C^\infty(U, \Omega_X^{0,q})$. Then we can define

$$\overline{\partial}_E(\sigma) = \overline{\partial}\varphi_i \otimes e^i$$

It's clear that this definition is independent of the choice of local chart, since the transition functions are holomorphic and $\overline{\partial}$ kills them. Furthermore, $\overline{\partial}_E^2 = 0$ holds since $\overline{\partial}^2 = 0$. So we can construct a cochain complex and define its cohomology, denoted by

$$H^q(X, E) = H^q_{\overline{\partial}_E}(C^\infty(X, \Omega_X^{0,*} \otimes E))$$

and similarly $H^q(X, E)$ computes the q-th sheaf cohomology of E.

Part 3. Geometry of vector bundle

6. Connections and its curvature

6.1. **General case.** In this section X is a complex manifold, and $\pi\colon E \to X$ is a complex vector bundle.

6.1.1. *Basic definitions.*

Definition 6.1.1 (connection)**.** A connection on E is a $\mathbb{C}$-linear operator

$$\nabla : C^\infty(X, E) \to C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes E)$$

satisfying the Leibniz rule

$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s$$

for $f \in C^\infty(X)$ and $s \in C^\infty(X, E)$.

Remark 6.1.1 (connection form)*.* If we choose a local frame $\{e_\alpha\}$ of E, then any section s of E can be written as $s = s^\alpha e_\alpha$, then

$$\nabla(s^\alpha e_\alpha) = \mathrm{d}s^\alpha e_\alpha + s^\alpha \nabla s_\alpha$$

If we write ∇e_α explicitly as follows

$$\nabla e_\alpha = \omega_\alpha^\beta e_\beta$$

where ω_α^β are 1-forms, which is called connection 1-form. Suppose there is another local frame $\widetilde{e}_\alpha$, which is related by $\widetilde{e}_\alpha = g_\alpha^\beta e_\beta$, then connection 1-form $\widetilde{\omega}$ with respect to local frame $\{\widetilde{e}_\alpha\}$ satisfies $\widetilde{\omega} = g\omega g^{-1} + \mathrm{d}gg^{-1}$. In terms of Christoffel symbol, one has

$$\omega_\alpha^\beta = \Gamma_{i\alpha}^\beta \mathrm{d}z^i + \Gamma_{\bar{i}\alpha}^\beta \mathrm{d}\bar{z}^i$$

Definition 6.1.2 (hermitian metric)**.** A hermitian metric h on E is a section of $E^* \otimes_{\mathbb{R}} \overline{E}^*$.

Remark 6.1.2 (local form)*.* Let $\{e_\alpha\}$ be a local frame of E, then a hermitian metric is determined by a positive-definite hermitian matrix $(h_{\alpha\overline{\beta}})$, where

$$h_{\alpha\overline{\beta}} = h(e_\alpha, e_\beta)$$

that is

$$h = h_{\alpha\overline{\beta}} e^\alpha \otimes \overline{e}^\beta$$

Definition 6.1.3 (hermitian vector bundle)**.** A complex vector bundle E together with a hermitian metric h is called a hermitian vector bundle (E, h).

Definition 6.1.4 (hermitian connection)**.** A connection ∇ on a hermitian vector bundle (E, h) is called a hermitian connection, if

$$\mathrm{d}h(s, t) = h(\nabla s, t) + h(s, \nabla t)$$

where s, t are sections of E.

Remark 6.1.3 (local form). If $\{e_\alpha\}$ is a local frame of E, then

$$\begin{aligned}\mathrm{d}h_{\alpha\beta} &= \mathrm{d}h(e_\alpha, e_\beta)\\ &= h(\nabla e_\alpha, e_\beta) + h(e_\alpha, \nabla e_\beta)\\ &= \omega_\alpha^\gamma h_{\gamma\overline{\beta}} + \omega_{\overline{\beta}}^{\overline{\gamma}} h_{\alpha\overline{\gamma}}\end{aligned}$$

So in matrix notation, we have

$$\mathrm{d}h = \omega h + h\overline{\omega}^T$$

In particular, if we take $\{e_\alpha\}$ to be orthogonal local frame of E with respect to h, we will find $\omega + \overline{\omega}^T = 0$, that is ω is skew-hermitian matrix.

6.1.2. *Curvature form.* Now we're going to extend connection to something called exterior derivative defined on sections of vector bundle valued k-forms as follows

$$\begin{aligned}\mathrm{d}^\nabla : C^\infty(M, \Omega^k_{X,\mathbb{C}} \otimes E) &\to C^\infty(M, \Omega^{k+1}_{X,\mathbb{C}} \otimes E)\\ \omega \otimes s &\mapsto \mathrm{d}\omega \otimes s + (-1)^k \omega \wedge \nabla s\end{aligned}$$

Note that d^∇ on $C^\infty(M, E)$ is exactly ∇. Furthermore, direct computation shows

$$\begin{aligned}(\mathrm{d}^\nabla)^2(s^\alpha e_\alpha) &= \mathrm{d}^\nabla(\mathrm{d}s^\alpha e_\alpha + s^\alpha \omega_\alpha^\beta e_\beta)\\ &= -\mathrm{d}s^\alpha \wedge \omega_\alpha^\beta e_\beta + \mathrm{d}(s^\alpha \omega_\alpha^\beta) e_\beta - s^\alpha \omega_\alpha^\beta \wedge \omega_\beta^\gamma e_\gamma\\ &= s^\alpha(\mathrm{d}\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta) e_\beta\end{aligned}$$

$$\begin{aligned}(\mathrm{d}^\nabla)^2(e_\alpha) &= \mathrm{d}^\nabla(\omega_\alpha^\beta e_\beta)\\ &= \mathrm{d}\omega_\alpha^\beta e_\beta - \omega_\alpha^\beta \wedge \nabla e_\beta\\ &= \mathrm{d}\omega_\alpha^\beta e_\beta - \omega_\alpha^\beta \wedge \omega_\beta^\gamma e_\gamma\\ &= (\mathrm{d}\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta) e_\beta\end{aligned}$$

that is smooth functions commutes with $(\mathrm{d}^\nabla)^2$. This is a quite good property, from this we can conclude:

1. $(\mathrm{d}^\nabla)^2(e_\alpha)$ completely determines $(\mathrm{d}^\nabla)^2$ locally, thus we can say $(\mathrm{d}^\nabla)^2$ locally looks like $\mathrm{d}\omega - \omega \wedge \omega$;
2. $(\mathrm{d}^\nabla)^2$ is a global section of $\Omega^2_M \otimes \operatorname{End} E$, that is it's compatible with change of basis. Indeed, for two local frame $e, \widetilde{e}$ such that $\widetilde{e} = ge$, we will see

$$\begin{aligned}g(\mathrm{d}^\nabla)^2 e &= (\mathrm{d}^\nabla)^2 ge\\ &= (\mathrm{d}^\nabla)^2 \widetilde{e}\\ &= (\mathrm{d}\widetilde{\omega} - \widetilde{\omega} \wedge \widetilde{\omega})\widetilde{e}\\ &= (\mathrm{d}\widetilde{\omega} - \widetilde{\omega} \wedge \widetilde{\omega}) ge\end{aligned}$$

   which implies

$$g^{-1}(\mathrm{d}\widetilde{\omega} - \widetilde{\omega} \wedge \widetilde{\omega}) g = \mathrm{d}\omega - \omega \wedge \omega$$

Definition 6.1.5 (curvature form)**.** Given a connection ∇ on a complex vector bundle E on a smooth manifold, there exists a global section $\Theta \in C^\infty(X, \Omega^2_{X,\mathbb{C}} \otimes \operatorname{End} E)$ such that

$$(\mathrm{d}^\nabla)^2 s = \Theta \wedge s$$

for all $s \in C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes E)$.

Remark 6.1.4 (local form)*.* In local frame the curvature form Θ can be written as

$$\Theta = R^\beta_{i\bar{j}\alpha} \mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j \otimes e^\alpha \otimes e_\beta$$

And $R^\beta_{i\bar{j}\alpha}$ can also be expressed in terms of Christoffel symbols just like what we have seen in Riemannian geometry, but there is no need to do it here, since there is nothing new. Later we will see the expression of curvature of Chern connection is quite simple.

Remark 6.1.5*.* In physicists' language, a connection is a "field", the curvature is the "strength" of the field, and choosing a local frame is called "fixing the gauge". The reason for these names comes from H. Weyl's work, rewriting Maxwell's equations.

6.1.3. *First Chern class.* In particular, if $\pi\colon X \to L$ is a complex line with connection ∇, then curvature Θ is a global section of $\Omega^2_{X,\mathbb{C}} \otimes \operatorname{End} L$ and $\operatorname{End} L$ is trivial bundle, so in this case Θ is exactly a 2-form. Furthermore, Θ locally looks like $\mathrm{d}\omega$, since for line bundle $\omega \wedge \omega = 0$. An immediate consequence is that $\mathrm{d}\Theta = 0$, that is Θ is a closed 2-form[6]. In other words,

$$[\Theta] \in H^2(X, \mathbb{C})$$

Remark 6.1.6*.* In fact, $[\Theta] \in H^2(X, \mathbb{C})$ is independent of the choice of connection. Indeed, if we consider another connection $\widetilde{\nabla}$, and $\widetilde{\omega}$, then for section s of $\Omega^k_{X,\mathbb{C}} \otimes L$, we have

$$\begin{aligned}(\nabla - \widetilde{\nabla})s &= (\mathrm{d}s + \omega \wedge s) - (\mathrm{d}s - \widetilde{\omega} \wedge s) \\ &= (\omega - \widetilde{\omega}) \wedge s\end{aligned}$$

Note that $\omega - \widetilde{\omega}$ is a global section of $\Omega^1_{X,\mathbb{C}}$, so $\Theta - \widetilde{\Theta}$ is exact.

Definition 6.1.6 (first Chern class of line bundle)**.** Let L be a complex line bundle over complex manifold X, the first Chern class of L is defined as

$$c_1(L) := [\frac{\sqrt{-1}}{2\pi}\Theta] \in H^2(X, \mathbb{C})$$

where Θ is curvature of arbitrary connection.

[6]Attention: Here Θ is just a closed form, not neccessary exact, since ω is not a globally defined object.

Remark 6.1.7. In fact, if L is a hermitian line bundle, then $c_1(L) \in H^2(X, \mathbb{R})$. Indeed, for a hermitian connection ∇, locally we have

$$\overline{\omega} = -\omega$$

Thus

$$\overline{\frac{\sqrt{-1}}{2\pi}\Theta} = -\frac{\sqrt{-1}}{2\pi}\overline{\Theta} = -\frac{\sqrt{-1}}{2\pi}\mathrm{d}\overline{\omega} = \frac{\sqrt{-1}}{2\pi}\mathrm{d}\omega = \frac{\sqrt{-1}}{2\pi}\Theta$$

Definition 6.1.7 (first Chern class of vector bundle)**.** Let E be a complex vector bundle over complex manifold X, the first Chern class of E is defined to be the first Chern class of $\det E$.

6.2. **Induced connections and its curvature.** In this section, we focus on three cases we will frequently use, that is induced connections on direct sum, tensor product and dual bundle.

6.2.1. *Induced connection on direct sum.* Let E, F be two complex vector bundles over a complex manifold X, equipped with connection ∇^E, ∇^F respectively, then there is a natural connection on bundle $E \oplus F$, given by

$$\nabla^{E\oplus F}(s \oplus t) := \nabla^E s \oplus \nabla^F t$$

where s, t are sections of E and F respectively. Furthermore, the curvature of $\nabla^{E\oplus F}$ can be expressed as

$$\Theta^{E\oplus F} = \Theta^E \oplus \Theta^F$$

6.2.2. *Induced connection on tensor product.* Let E, F be two complex vector bundles over a complex manifold X, equipped with connection ∇^E, ∇^F respectively, then there is a natural connection on bundle $E \otimes F$, given by

$$\nabla^{E\otimes F}(s \otimes t) := \nabla^E s \otimes t + s \otimes \nabla^F t$$

where s, t are sections of E and F respectively. Furthermore, the curvature of $\nabla^{E\otimes F}$ can be expressed as

$$\Theta^{E\otimes F} = \Theta^E \otimes \mathrm{id} + \mathrm{id} \otimes \Theta^F$$

6.2.3. *Induced connection on dual bundle.* Let E be a complex vector bundle over a complex manifold X, equipped with connection ∇^E, then there is a natural connection on bundle E^*, given by

$$\mathrm{d}(s, t) = (\nabla^{E^*} s, t) + (s, \nabla^E t)$$

where s, t are sections of E^* and E respectively. This is a kind of Leibniz rule. In this case, it's useful to writte down Christoffel symbols to see what's going on: If $\Gamma^\beta_{i\alpha}$ is the Christoffel symbol of ∇^E, then the Christoffel symbol of ∇^{*E} is exactly $-\Gamma^\beta_{i\alpha}$, and if curvature form of ∇^E is written as

$$\Theta^E = R^\beta_{i\bar{j}\alpha}\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j \otimes e^\alpha \otimes e_\beta$$

Then curvature form of ∇^{E^*} is given by

$$\Theta^{E^*} = -R^\beta_{i\bar{j}\alpha}\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j \otimes e^\alpha \otimes e_\beta$$

6.3. **Chern connection.** Recall that for a complex manifold X, we have

$$\Omega^1_{X,\mathbb{C}} = \Omega^{1,0}_X \oplus \Omega^{0,1}_X$$

Consider $E \to X$ is a complex vector bundle, and ∇ is a connection, then we can decompose $\nabla = \nabla^{1,0} + \nabla^{0,1}$ by composing the projection as follows

$$\begin{tikzcd} & & C^\infty(X, \Omega^{1,0}_X \otimes E) \\ C^\infty(X,E) \arrow[r, "\nabla"] & C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes E) \arrow[ur] \arrow[dr] & \\ & & C^\infty(X, \Omega^{0,1}_X \otimes E) \end{tikzcd}$$

Locally, we have $\nabla = \mathrm{d} + \omega$, then

$$\nabla^{1,0} = \partial + \omega^{1,0}, \quad \nabla^{0,1} = \overline{\partial} + \omega^{0,1}$$

both $\nabla^{1,0}$ and $\nabla^{0,1}$ satisfy Leibniz rule.

Definition 6.3.1 (complex connection)**.** A connection ∇ on a complex vector bundle E over a complex manifold X is said to be compatible with complex structure, if $\nabla^{0,1} = \overline{\partial}_E$.

Remark 6.3.1 (local form). Let $\{e_\alpha\}$ be a holomorphic local form of E, and denote

$$\nabla e_\alpha = (\Gamma^\beta_{i\alpha} \mathrm{d}z^i + \Gamma^\beta_{\bar{i}\alpha} \mathrm{d}\overline{z}^i) e_\beta$$

that is

$$\nabla^{0,1} e_\alpha = \Gamma^\beta_{\bar{i}\alpha} e_\beta \mathrm{d}\overline{z}^i$$

But since $\{e_\alpha\}$ is holomorphic, that is $\overline{\partial}_E e_\alpha = 0$, which implies ∇ is complex if and only if $\Gamma^\beta_{\bar{i}\alpha} = 0$.

Theorem 6.3.1 (Chern connection)**.** Let X be a complex manifold, (E, h) a hermitian holomorphic vector bundle, then there exists a unique hermitian connection ∇ such that $\nabla^{0,1} = \overline{\partial}_E$, which is called Chern connection.

Proof. If hermitian connection ∇ is compatible with complex structure, then the following three equations are equivalent

$$\begin{aligned} \mathrm{d}h &= \omega h + h\overline{\omega}^t \\ \partial h &= \omega h \\ \overline{\partial} h &= h\overline{\omega}^t \end{aligned}$$

since ω is a $(1,0)$-valued matrix. This shows ∇ is uniquely determined by our metric, since $\omega = (\partial h)h^{-1}$. □

Remark 6.3.2 (local form). It's neccessary to writte down local form of $\partial h = \omega h$, that is

$$\frac{\partial h_{\alpha\overline{\beta}}}{\partial z^i} = \Gamma^\gamma_{i\alpha} h_{\gamma\overline{\beta}}$$

Definition 6.3.2 (Chern curvatue)**.** Let X be a complex manifold, (E, h) a hermitian holomorphic vector bundle equipped with Chern connection ∇, the curvature of Chern connection is called Chern curvature, denoted by Θ_h.

Corollary 6.3.1. Let X be a complex manifold, (E, h) a hermitian holomorphic vector bundle equipped with Chern connection ∇, and Θ_h is its Chern curvature. Then

1. Locally $\partial\omega = \omega \wedge \omega$;
2. Locally $\Theta_h = \overline{\partial}\omega$;
3. $\overline{\partial}\Theta_h = 0$.

Proof. For (1). Since $\omega = (\partial h)h^{-1}$, then directly computation shows

$$\begin{aligned}\partial\omega &= -\partial h \wedge \partial(h^{-1}) \\ &= -\partial h \wedge (-h^{-1}\partial h h^{-1}) \\ &= (\partial h)h^{-1} \wedge (\partial h)h^{-1} \\ &= \omega \wedge \omega\end{aligned}$$

For (2). Θ_h locally looks like

$$\Theta_h = \mathrm{d}\omega - \omega \wedge \omega = \mathrm{d}\omega - \partial\omega = \overline{\partial}\omega$$

For (3). It's clear from (2). □

Remark 6.3.3 (local form). We can also express Chern curvature in terms of Christoffel symbol as follows

$$\Theta_h = R_{i\overline{j}\alpha}^{\gamma}\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j \otimes e^\alpha \otimes e_\gamma$$

where $R_{i\overline{j}\alpha}^{\gamma} = -\frac{\partial \Gamma_{i\alpha}^{\gamma}}{\partial \overline{z}^j}$. In other type one has

$$\begin{aligned}R_{i\overline{j}\alpha\overline{\beta}} &= h_{\gamma\overline{\beta}}R_{i\overline{j}\alpha}^{\gamma} \\ &= -h_{\gamma\overline{\beta}}\partial_{\overline{j}}(h^{\gamma\overline{\delta}}\frac{\partial h_{\alpha\overline{\delta}}}{\partial z^i}) \\ &= -\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j} + h^{\gamma\overline{\delta}}\frac{\partial h_{\alpha\overline{\delta}}}{\partial z^i}\frac{\partial h_{\gamma\overline{\beta}}}{\partial \overline{z}^j}\end{aligned}$$

6.3.1. *Chern class of line bundle.* Let X be a complex manifold, (L, h) a hermitian holomorphic line bundle, and ∇ is the Chern connection of L with Chern curvature Θ_h. Then by Remark 6.1.7 and Corollary 6.3.1, we have we have

$$\frac{\sqrt{-1}}{2\pi}\Theta_h \in C^\infty(X, \Omega^2_{X,\mathbb{R}}) \cap C^\infty(X, \Omega^{1,1}_X)$$

such that

$$\mathrm{d}(\frac{\sqrt{-1}}{2\pi}\Theta_h) = \overline{\partial}(\frac{\sqrt{-1}}{2\pi}\Theta_h) = 0$$

that is

$$[\frac{\sqrt{-1}}{2\pi}\Theta_h] \in H^2(X, \mathbb{R}) \cap H^{1,1}(X)$$

Remark 6.3.4 (local form). Let $\{e(x)\}$ be a local frame of L, then hermitian metric is

$$h(x) = \langle e(x), e(x)\rangle$$

If we denote $\varphi(x) = -\log h(x)$, then

$$\omega = (\partial h)h^{-1} = \partial e^{-\varphi(x)}e^{\varphi(x)} = -\partial\varphi(x)$$
$$\Theta_h = \overline{\partial}\omega = -\overline{\partial}\partial\varphi(x) = \partial\overline{\partial}\varphi(x)$$

Thus

$$\begin{aligned}\frac{\sqrt{-1}}{2\pi}\Theta_h &= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi(x)\\ &= -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h(x)\\ &= -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h(x)\end{aligned}$$

Remark 6.3.5. Recall that the first Chern class can be defined for arbitrary complex line bundle, but for a hermitian holomorphic line bundle (L, h), we can give an explicit formula for first Chern class in terms of h.

Exercise 6.3.1. Show that $[\frac{\sqrt{-1}}{2\pi}\Theta_h] \in H^{1,1}(X)$ is independent of h.

Proof. Note that any two metric on a line bundle differ a smooth function which is positive everywhere, so if h and h' are two different metrics, we can write $\|e(z)\|_{h'} = e^f\|e(z)\|_h$ for some globally defined smooth function f. So by Remark 6.3.4, we have the difference of first Chern classes coming from different metrics is $\frac{\sqrt{-1}}{\pi}\overline{\partial}\partial f$, and it's trivial in $H^{1,1}(X)$, since f is globally defined. □

6.4. **Useful formulas of Chern connection.** Let X be a complex manifold and (E, h) a hermitian holomorphic vector bundle on it with Chern connection ∇. Usually $\{z^i\}$ and $\{e_\alpha\}$ denote the local frame of X and E respectively, $\{e^\alpha\}$ and $\{\overline{e}_\alpha\}$ denote a local frame of E^* and $\overline{E}$ respectively. In this section we collect some useful formulas.

Proposition 6.4.1.

$$\Gamma^\beta_{i\alpha} = h^{\beta\overline{\gamma}}\frac{\partial h_{\alpha\overline{\gamma}}}{\partial z^i}$$

Proof. See Remark 6.3.2. □

Proposition 6.4.2.

$$\begin{aligned}R^\gamma_{i\overline{j}\alpha} &= -\frac{\partial\Gamma^\gamma_{i\alpha}}{\partial\overline{z}^j}\\ R_{i\overline{j}\alpha\overline{\beta}} &= -\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i\partial\overline{z}^j} + h^{\gamma\overline{\delta}}\frac{\partial h_{\alpha\overline{\delta}}}{\partial z^i}\frac{\partial h_{\gamma\overline{\beta}}}{\partial\overline{z}^j}\end{aligned}$$

Proof. See Remark 6.3.3. □

Proposition 6.4.3.

$$\nabla_{\frac{\partial}{\partial z^i}} e_\alpha = \Gamma^\beta_{i\alpha} e_\beta, \quad \nabla_{\frac{\partial}{\partial \overline{z}^i}} e_\alpha = 0, \quad \nabla_{\frac{\partial}{\partial z^i}} \overline{e}_\alpha = 0, \quad \nabla_{\frac{\partial}{\partial \overline{z}^i}} \overline{e}_\alpha = \Gamma^{\overline{\beta}}_{\overline{i}\overline{\alpha}} \overline{e}_\beta$$
$$\nabla_{\frac{\partial}{\partial z^i}} e^\alpha = -\Gamma^\alpha_{i\beta} e^\beta, \quad \nabla_{\frac{\partial}{\partial \overline{z}^i}} e^\alpha = 0, \quad \nabla_{\frac{\partial}{\partial z^i}} \overline{e}^\alpha = 0, \quad \nabla_{\frac{\partial}{\partial \overline{z}^i}} \overline{e}^\alpha = -\Gamma^{\overline{\alpha}}_{\overline{i}\overline{\beta}} \overline{e}^\beta$$

Proof. It suffices to show the first two equalities, and others can be obtained from taking conjugates and dualities. The first one holds from definition of Christoffel symbol, and $\Gamma^\beta_{\overline{i}\alpha} = 0$ holds from the Remark 6.3.1. □

Corollary 6.4.1.

1. For $s \in C^\infty(X, E)$, locally written as $s = s^\alpha e_\alpha$, then

$$\nabla_{\frac{\partial}{\partial z^i}} s = (\frac{\partial s^\beta}{\partial z^i} + s^\alpha \Gamma^\beta_{i\alpha}) e_\beta$$
$$\nabla_{\frac{\partial}{\partial \overline{z}^i}} s = \frac{\partial s^\beta}{\partial \overline{z}^i} e_\beta$$

2. For $s \in C^\infty(X, \overline{E})$, locally written as $s = s^{\overline{\alpha}} \overline{e}_\alpha$, then

$$\nabla_{\frac{\partial}{\partial z^i}} s = \frac{\partial s^{\overline{\beta}}}{\partial z^i} \overline{e}_\beta$$
$$\nabla_{\frac{\partial}{\partial \overline{z}^i}} s = (\frac{\partial s^{\overline{\beta}}}{\partial \overline{z}^i} + s^{\overline{\alpha}} \Gamma^{\overline{\beta}}_{\overline{i}\overline{\alpha}}) \overline{e}_\beta$$

3. For $s \in C^\infty(X, E^*)$, locally written as $s = s_\alpha e^\alpha$, then

$$\nabla_{\frac{\partial}{\partial z^i}} s = (\frac{\partial s_\beta}{\partial z^i} - s_\alpha \Gamma^\alpha_{i\beta}) e^\beta$$
$$\nabla_{\frac{\partial}{\partial \overline{z}^i}} s = \frac{\partial s_\beta}{\partial \overline{z}^i} e^\beta$$

4. For $s \in C^\infty(X, \overline{E}^*)$, locally written as $s = s_{\overline{\alpha}} \overline{e}^\alpha$, then

$$\nabla_{\frac{\partial}{\partial z^i}} s = \frac{\partial s_{\overline{\beta}}}{\partial z^i} \overline{e}^\beta$$
$$\nabla_{\frac{\partial}{\partial \overline{z}^i}} = (\frac{\partial s_{\overline{\beta}}}{\partial \overline{z}^i} - s_{\overline{\alpha}} \Gamma^{\overline{\gamma}}_{\overline{i}\overline{\beta}}) \overline{e}^\beta$$

Proposition 6.4.4 (Ricci identity)**.** For $s \in C^\infty(X, E)$, locally written as $s = s^\alpha e_\alpha$, then

$$\nabla_{\frac{\partial}{\partial z^i}} \nabla_{\frac{\partial}{\partial \overline{z}^j}} s^\beta - \nabla_{\frac{\partial}{\partial \overline{z}^j}} \nabla_{\frac{\partial}{\partial z^i}} s^\beta = R^\beta_{i\overline{j}\alpha} s^\alpha$$

Proof. Direct computation shows

$$\begin{aligned}\nabla_{\frac{\partial}{\partial z^i}} \nabla_{\frac{\partial}{\partial \overline{z}^j}} s &= \nabla_{\frac{\partial}{\partial z^i}} (\frac{\partial s^\alpha}{\partial \overline{z}^j} e_\alpha) \\ &= \frac{\partial^2 s^\beta}{\partial z^i \partial \overline{z}^j} e_\beta + \Gamma^\beta_{i\alpha} \frac{\partial s^\alpha}{\partial \overline{z}^j} e_\beta\end{aligned}$$

that is

$$\nabla_{\frac{\partial}{\partial z^i}} \nabla_{\frac{\partial}{\partial \overline{z}^j}} s^\beta = \frac{\partial^2 s^\beta}{\partial z^i \partial \overline{z}^j} + \Gamma^\beta_{i\alpha} \frac{\partial s^\alpha}{\partial \overline{z}^j}$$

Direct computation also shows

$$\nabla_{\frac{\partial}{\partial \overline{z}^j}} \nabla_{\frac{\partial}{\partial z^i}} s^\beta = \frac{\partial^2 s^\beta}{\partial \overline{z}^j \partial z^i} + \frac{\partial s^\alpha}{\partial \overline{z}^j} \Gamma^\beta_{i\alpha} + s^\alpha \frac{\partial \Gamma^\beta_{i\alpha}}{\partial \overline{z}^j}$$

Thus

$$\begin{aligned} \nabla_{\frac{\partial}{\partial z^i}} \nabla_{\frac{\partial}{\partial \overline{z}^j}} s^\beta - \nabla_{\frac{\partial}{\partial \overline{z}^j}} \nabla_{\frac{\partial}{\partial z^i}} s^\beta &= -s^\alpha \frac{\partial \Gamma^\beta_{i\alpha}}{\partial \overline{z}^j} \\ &= R^\beta_{i\bar{j}\alpha} s^\alpha \end{aligned}$$

□

Remark 6.4.1. There are also Ricci identities in other types, but the proof are same, so we omit here.

7. Hermitian geometry and Kähler geometry

7.1. **Baby version.** Let V be a finite dimensional $\mathbb{R}$-vector space together with a almost complex structure J, that is a $\mathbb{R}$-linear map from V to V such that $J^2 = -\operatorname{id}$. A almost complex structure J makes V into a $\mathbb{C}$-vector space, via

$$(a+b\sqrt{-1})v := av + bJ(v)$$

Thus one can regard (V,J) as a $\mathbb{C}$-vector space. The complexified $V_{\mathbb{C}} := V\otimes\mathbb{C}$ decomposes into $V^{1,0}\oplus V^{0,1}$ with respect to J, and there is a natural $\mathbb{C}$-isomorphism between (V,J) and $V^{1,0}$, given by $v\mapsto \frac{1}{2}(v-\sqrt{-1}J(v))$.

Notation 7.1.1. Always, we write $V^{1,0}$ instead of (V,J), if we want to regard (V,J) as a $\mathbb{C}$-vector space, otherwise we just regard V as a $\mathbb{R}$-vector space.

Example 7.1.1. The case we will encounter most frequently is: Let V be a $\mathbb{C}$-vector space, and consider its underlying real vector space $V_{\mathbb{R}}$, which admits a natural almost complex structure.

Definition 7.1.1 (compatible)**.** A $\mathbb{R}$-bilinear form g on V is called compatible with J, if $g(J(u),J(v)) = g(u,v)$ for all $u,v\in V$.

In the following of this section, suppose g is a $\mathbb{R}$-bilinear form on (V,J), which is compatible with complex structure J, and use (V,J,g) to denote this triple. Consider $\mathbb{C}$-extension of g to a hermitian form $g_{\mathbb{C}}$ on $V_{\mathbb{C}}$, which is defined as

$$g_{\mathbb{C}}(v\otimes\lambda, w\otimes\mu) := (\lambda\overline{\mu})g(v,w)$$

Direct computation shows $V_{\mathbb{C}} = V^{1,0}\oplus V^{0,1}$ is a orthogonal direct sum with respect to $g_{\mathbb{C}}$. If we restrict $g_{\mathbb{C}}$ to $V^{1,0}$, it gives a hermitian form h on $V^{1,0}$.

Proposition 7.1.1.

$$h(\text{-},\text{-}) = \frac{1}{2}g(\text{-},\text{-}) - \frac{\sqrt{-1}}{2}g(J(\text{-}),\text{-})$$

Remark 7.1.1. Be attention: the left hand is defined on $V^{1,0}$, while the right hand is defined on V, here we use the canonical isomorphism.

Proof. For $v,w\in V$, it corresponds to $\frac{1}{2}(v-\sqrt{-1}J(v)), \frac{1}{2}(w-\sqrt{-1}J(w))\in V^{1,0}$, then direct computation shows

$$\begin{aligned}\frac{1}{4}h(v-\sqrt{-1}J(v), w-\sqrt{-1}J(w)) &= \frac{1}{4}\{g(v,w)+\sqrt{-1}g(v,J(w))-\sqrt{-1}g(J(v),w)+g(J(v),J(w))\}\\ &= \frac{1}{2}g(v,w)-\frac{\sqrt{-1}}{2}g(J(v),w)\end{aligned}$$

□

Remark 7.1.2. Conversely, given a hermitian form on $V^{1,0}$, its real part gives a $\mathbb{R}$-bilinear form on V.

Notation 7.1.2. Let $\{x_1, \ldots, x_n\}$ be a $\mathbb{C}$-basis of (V, J), then $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, where $y_i = J(x_i)$, is a $\mathbb{R}$-basis of V. By canonical $\mathbb{C}$-isomorphism we have $\{z_i := \frac{1}{2}(x_i - \sqrt{-1}y_i)\}$ is a $\mathbb{C}$-basis of $V^{1,0}$. With respect to these basis, we may write

$$h = h_{i\bar{j}} z^i \otimes \overline{z}^j$$
$$g = g_{ij} x^i \otimes x^j + g_{iJ} x^i \otimes y^j + g_{Ij} y^i \otimes x^j + g_{IJ} y^i \otimes y^j$$

Holding these notations, one has

$$h_{i\bar{j}} = \frac{1}{2}(g_{ij} + \sqrt{-1} g_{iJ}) \tag{7.1}$$

The transposed inverse matrix of $(h_{i\bar{j}})$ is denoted by $(h^{i\bar{j}})$, that is $h^{i\bar{l}} h_{j\bar{l}} = \delta^i_j$. It's clear

$$h^{i\bar{j}} = 2(g^{ij} - \sqrt{-1} g^{iJ})$$

Definition 7.1.2 (fundamental form)**.** The fundamental form associated to (V, J, g) is the form $\omega := g(J(\text{-}), \text{-})$.

Remark 7.1.3. Holding this notation, one has $2h = g - \sqrt{-1}\omega$.

Lemma 7.1.1. The fundamental form ω of (V, J, g) satisfies
1. $\omega(J(u), J(v)) = \omega(u, v)$;
2. $\omega(u, J(v)) + \omega(v, J(u)) = 0$.
where $u, v \in V$.

Proposition 7.1.2. With respect to basis given in Notation 7.1.2, one has

$$\omega = \sqrt{-1} h_{i\bar{j}} z^i \wedge \overline{z}^j$$

Proof. Direct computation shows

$$\begin{aligned}
h + \frac{\sqrt{-1}}{2}\omega =& h_{i\bar{j}} z^i \otimes \overline{z}^j - \frac{1}{2} h_{i\bar{j}} z^i \wedge \overline{z}^j \\
=& h_{i\bar{j}} z^i \otimes \overline{z}^j - \frac{1}{2} h_{i\bar{j}} (z^i \otimes \overline{z}^j - \overline{z}^j \otimes z^i) \\
=& \frac{1}{2}(h_{i\bar{j}} z^i \otimes \overline{z}^j - h_{i\bar{j}} \overline{z}^j \otimes z^i) \\
=& \frac{1}{4}(g_{ij} + \sqrt{-1} g_{iJ})(x^i + \sqrt{-1} y^i) \otimes (x^j - \sqrt{-1} y^j) \\
& - \frac{1}{4}(g_{ij} + \sqrt{-1} g_{iJ})(x^j - \sqrt{-1} y^j) \otimes (x^i + \sqrt{-1} y^i) \\
=& \frac{1}{2}(g_{ij} x^i \otimes x^j + g_{iJ} x^i \otimes y^j + g_{Ij} y^i \otimes x^j + g_{IJ} y^i \otimes y^j) \\
=& \frac{1}{2} g
\end{aligned}$$

□

Remark 7.1.4. This is an important formula we will use frequently, and it shows the fundamental form of a hermitian form is a real (1,1)-form.

Remark 7.1.5. In a summary, the relations between g, ω and h on (V, J) are listed as follows:

$$\begin{tikzcd}
& h \arrow[dl, bend right, "1"'] \arrow[dr, bend left, "3"] & \\
\omega \arrow[ur, bend right, "2"'] \arrow[rr, "5"] & & g \arrow[ul, bend left, "4"'] \arrow[ll, bend left, "6"]
\end{tikzcd}$$

where

$$\begin{array}{cr}
1 & \omega(\text{-},\text{-}) = -\operatorname{Im} 2h(\text{-},\text{-}) \\
2 & 2h(\text{-},\text{-}) = \omega(\text{-}, J(\text{-})) - \sqrt{-1}\omega(\text{-},\text{-}) \\
3 & g(\text{-},\text{-}) = \operatorname{Re} 2h(\text{-},\text{-}) \\
4 & 2h(\text{-},\text{-}) = g(\text{-},\text{-}) - \sqrt{-1}g(J(\text{-}),\text{-}) \\
5 & g(\text{-},\text{-}) = \omega(\text{-}, J(\text{-})) \\
6 & \omega(\text{-},\text{-}) = g(J(\text{-}),\text{-})
\end{array}$$

7.2. **Hermitian manifold.**

Definition 7.2.1 (hermitian manifold)**.** A complex manifold X is called a hermitian manifold, if it's endowed with a hermitian metric h on T_X.

Definition 7.2.2 (first Chern class of hermitian manifold)**.** The first Chern class of a hermitian manifold (X, h) is the first Chern class of its holomorphic tangent bundle.

Let (X, h) be a hermitian manifold, then pointwisely the linear algebra in Section 7.1 can be applied here, that is, there exist a Riemannian metric g on the underlying real manifold of X and a real $(1,1)$-form ω, called fundamental form of h.

Remark 7.2.1. Since the fundamental form ω is the same as hermitian metric itself, so we sometimes say a hermitian metric ω.

Proposition 7.2.1. If (X, ω) is a hermitian n-manifold, then $\omega^n/n!$ is the volume form of the underlying real manifold of X with respect to g.

Proof. It suffices to check pointwise, and it reduces to a problem of linear algebra. Suppose $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^n}\}$ is orthonormal basis of tangent space at $p \in X$ with respect to Riemannian metric g, with dual basis $\{\mathrm{d}x^1, \mathrm{d}y^1, \dots, \mathrm{d}x^n, \mathrm{d}y^n\}$. Then the volume form is given by

$$\mathrm{vol}_p = \mathrm{d}x^1 \wedge \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}x^n \wedge \mathrm{d}y^n$$

By equation (7.1) and Proposition 7.1.2, one has

$$\omega_p = \frac{\sqrt{-1}}{2} \sum_{i=1}^n \mathrm{d}z^i \wedge \mathrm{d}\overline{z}^i$$

where $\mathrm{d}z^i = \mathrm{d}x^i + \sqrt{-1}\mathrm{d}y^i$. Now what we need to do is just computation. Here we compute the case $n = 2$ to feel what's going on:

$$\begin{aligned}
\omega_p^2 &= (\frac{\sqrt{-1}}{2})^2(\mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^1 + \mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^2) \wedge (\mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^1 + \mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^2) \\
&= (\frac{\sqrt{-1}}{2})^2(\mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^2 + \mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^2 \wedge \mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^1) \\
&= 2(\frac{\sqrt{-1}}{2})^2 \mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^2
\end{aligned}$$

By Exercise 5.2.1 we have

$$\mathrm{d}x^1 \wedge \mathrm{d}y^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}y^2 = (\frac{\sqrt{-1}}{2})^2 \mathrm{d}z^1 \wedge \mathrm{d}\overline{z}^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}\overline{z}^2$$

Thus we get desired result. □

Proposition 7.2.2 (Gauduchon metric)**.** Let X be a complex manifold, then there exists a hermitian metric ω such that

$$\partial\overline{\partial}\omega^{n-1} = 0$$

which is called Gauduchon metric.

Corollary 7.2.1. Let X be a compact complex n-manifold. If there exists $f \in C^\infty(X, \mathbb{R})$ such that

$$\sqrt{-1}\partial\overline{\partial} f \geq 0$$

then $f \equiv c$, where c is a constant.

Proof. Let ω be the Gauduchon metric on X, then

$$0 \leq \int_X \sqrt{-1}\partial\overline{\partial} f \wedge \omega^{n-1} = \int_X \sqrt{-1} f \wedge \partial\overline{\partial}\omega^{n-1} = 0$$

shows $\sqrt{-1}\partial\overline{\partial} f = 0$. Note that

$$\partial\overline{\partial} f^2 = \partial(2f\overline{\partial} f) = 2\partial f \wedge \overline{\partial} f + 2f\partial\overline{\partial} f = 2\partial f \wedge \overline{\partial} f = |\partial f|^2 \mathrm{d}z \wedge \mathrm{d}\overline{z}$$

Then

$$\begin{aligned}
0 &\overset{(1)}{=} \int_X \sqrt{-1}\partial\overline{\partial} f^2 \wedge \omega^{n-1} \\
&\overset{(2)}{=} 2\int_X \sqrt{-1}\partial f \wedge \overline{\partial f} \wedge \omega^{n-1} \\
&\overset{(3)}{=} C\int_X |\partial f|^2 \omega^n
\end{aligned}$$

where

(1) holds from ω is a Gauduchon metric.
(2) and (3) hold from above observation, where C is an appropriate constant.

This shows $|\partial f| = 0$, that is $\partial f = 0$, and since f is real-valued, this also shows $\overline{\partial} f = 0$, that is $\mathrm{d}f = 0$. This completes the proof. □

Theorem 7.2.1 (normal coordinate)**.** Let (X, h) be a hermitian manifold, for any $p \in X$, there exists a local holomorphic coordinate $\{z^i\}$ centered at p such that

$$h_{i\bar{j}}(p) = \delta_{i\bar{j}} \quad \text{and} \quad \frac{\partial h_{i\bar{j}}}{\partial \bar{z}^k} + \frac{\partial h_{i\bar{k}}}{\partial \bar{z}^j} = 0$$

Proof. Without lose of generality, we may assume

$$\omega = \sqrt{-1}(\delta_{i\bar{j}} + a_{i\bar{j}l}w^l + a_{i\bar{j}\bar{l}}\overline{w}^l + O(|w|^2))\mathrm{d}w^i \wedge \mathrm{d}\overline{w}^j$$

□

7.3. **Curvatures on hermitian manifold.** Sometimes we need to consider hermitian holomorphic vector bundles over a hermitian manifold, thus there are two hermitian metrics. In this case, in order to distinguish them, $h_{\alpha\overline{\beta}}$ denotes the hermitian metric on vector bundle and $h_{i\bar{j}}$ denotes the hermitian metric on tangent bundle.

Definition 7.3.1 (curvatures of vector bundle)**.** Let (E, h) be a hermitian holomorphic vector bundle on hermitian manifold (X, h), then

1. The first Chern-Ricci curvatue of (E, h) is locally given by
$$\mathrm{Ric}^{(1)}(h) = h^{\alpha\overline{\beta}}R_{i\bar{j}\alpha\overline{\beta}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j$$
which is a $(1, 1)$-form.
2. The second Chern-Ricci curvatue of (E, h) is locally given by
$$\mathrm{Ric}^{(2)}(h) = h^{i\bar{j}}R_{i\bar{j}\alpha\overline{\beta}}e^\alpha \otimes \bar{e}^\beta$$
which is a section of $\operatorname{End} E$.
3. The Chern scaler curvatue of (E, h) is locally given by
$$S = h^{i\bar{j}}h^{\alpha\overline{\beta}}R_{i\bar{j}\alpha\overline{\beta}}$$

Remark 7.3.1. The first Chern-Ricci curvatue can be defined for a hermitian vector bundle over a complex manifold, not neccessary hermitian manifold.

Definition 7.3.2 (curvatures of holomorphic tangent bundle)**.** Let (X, h) be a hermitian manifold, then

1. The first Chern-Ricci curvatue of (X, h) is locally given by
$$\mathrm{Ric}^{(1)}(X) = h^{k\bar{l}}R_{i\bar{j}k\bar{l}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j$$
which is a $(1, 1)$-form.
2. The second Chern-Ricci curvatue of (X, h) is locally given by
$$\mathrm{Ric}^{(2)}(X) = h^{i\bar{j}}R_{i\bar{j}k\bar{l}}\mathrm{d}z^k \otimes \mathrm{d}\bar{z}^l$$
which is a section of $\operatorname{End} E$.
3. The Chern scaler curvatue of (X, h) is locally given by
$$S = h^{i\bar{j}}h^{k\bar{l}}R_{i\bar{j}k\bar{l}}$$

Proposition 7.3.1. Let (E,h) be a hermitian holomorphic vector bundle on a complex manifold X. The first Chern-Ricci curvatue of (E,h) is the Chern curvature of $(\det E, \det h)$.

Proof. It suffices to shows

$$h^{\alpha\overline{\beta}}(-\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i\partial\overline{z}^j}+h^{\gamma\overline{\delta}}\frac{\partial h_{\alpha\overline{\delta}}}{\partial z^i}\frac{\partial h_{\gamma\overline{\beta}}}{\partial\overline{z}^j})=-\frac{\partial^2\log\det(h_{\alpha\overline{\beta}})}{\partial z^i\partial\overline{z}^j}$$

and the hallmark of the computation is Jacobi's formula. □

Corollary 7.3.1. The first Chern-Ricci curvatue of (E,h) gives the first Chern class of (E,h).

7.4. **Kähler manifold.**

Definition 7.4.1 (Kähler manifold)**.** A hermitian manifold (X,h) is called a Kähler manifold, if fundamental form ω is d-closed[7].

Remark 7.4.1. Note that $\mathrm{d}\omega=0$ is equivalent to $\partial\omega=0$, and is also equivalent to $\overline{\partial}\omega=0$.

Remark 7.4.2 (local form). By Proposition 7.1.2 one has

$$\omega=\sqrt{-1}h_{i\overline{j}}\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^j$$

So Kähler condition $\mathrm{d}\omega$ can be computed explicitly as follows

$$\begin{aligned}\mathrm{d}\omega&=\sqrt{-1}\mathrm{d}(h_{i\overline{j}}\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^j)\\&=\sqrt{-1}(\frac{\partial h_{i\overline{j}}}{\partial z^k}\mathrm{d}z^k\wedge\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^j-\frac{\partial h_{i\overline{j}}}{\partial\overline{z}^k}\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^k\wedge\mathrm{d}\overline{z}^j)\\&=0\end{aligned}$$

So locally Kähler condition can be written as follows

$$\begin{aligned}\partial_k h_{i\overline{j}}&=\partial_i h_{k\overline{j}}\\\partial_{\overline{k}}h_{i\overline{j}}&=\partial_{\overline{j}}h_{i\overline{k}}\end{aligned}$$

holds for all i,j,k.

Remark 7.4.3. Note that

$$R_{i\overline{j}k\overline{l}}=-\frac{\partial^2 h_{i\overline{j}}}{\partial z^k\partial\overline{z}^l}+h^{p\overline{q}}\frac{\partial h_{k\overline{q}}}{\partial z^i}\frac{\partial h_{p\overline{l}}}{\partial\overline{z}^j}$$

Thus if (X,h) is Kähler, then

$$R_{i\overline{j}k\overline{l}}=R_{k\overline{j}i\overline{l}}=R_{i\overline{l}k\overline{j}}$$

and

$$R_{ij}^{(1)}=h^{k\overline{l}}R_{i\overline{j}k\overline{l}}=h^{k\overline{l}}R_{k\overline{l}i\overline{j}}=R_{ij}^{(2)}$$

[7] ω is called Kähler form and h is called Kähler metric.

Remark 7.4.4. Definition of Kähler manifold given here is from the complex hermitian viewpoint. But Kähler manifold in fact is an intersection of three interesting objects: complex manifold, symplectic manifold and Riemannian manifold. Here are another two viewpoints:

1. If (X, ω) is a symplectic manifold, where X is a differential manifold and ω is a d-closed real non-degenerate symplectic form. (X, ω) is called a Kähler manifold, if there exists an integrable almost complex structure J on $T_{X,\mathbb{R}}$ such that $g(u, v) := \omega(u, Jv)$ is positive definite, that is g is a Riemannian metric.
2. Let (X, g) be a Riemannian manifold, where g is a Riemannian metric. (X, g) is called Kähler if there exists an integrable almost complex structure J on $T_{X,\mathbb{R}}$ satisfying $g(Ju, Jv) = g(u, v)$ and preserved by parallel transport with respect to Levi-Civita connection.

Anyway, the hallmarks of a Kähler manifold are "complex structure", "positive" and "closed".

Example 7.4.1. Any complex curve[8] X is Kähler. Since $\mathrm{d}\omega = 0$ automatically holds.

Proposition 7.4.1. A submanifold of a Kähler manifold is still Kähler.

Proof. If (X, ω) is a Kähler manifold and Y is a submanifold, the restriction of ω to Y gives Kähler form of Y. $\square$

Proposition 7.4.2. Let (X, h) be a compact Kähler n-manifold, then $H^{2k}(X, \mathbb{R}) \neq 0$ for $0 \leq k \leq n$.

Proof. Note that $\mathrm{d}(\omega^k) = 0$ holds for $0 \leq k \leq n$, since $\mathrm{d}\omega = 0$, that is $[\omega^k] \in H^{2k}(X, \mathbb{R})$. By Proposition 7.2.1, $\omega^n = n! \,\mathrm{vol}$, then consider the integral pairing

$$\int_X \omega^k \wedge \omega^{n-k} = n! \int_X \mathrm{vol} \neq 0$$

which implies $[\omega^k] \neq 0$ for $0 \leq k \leq n$. $\square$

As we can see from the definition of Kähler manifold, all of the requirements are local, but from the above proposition, we can see a surprising thing, that is the cohomology groups with even dimension must be nontrivial, it's a global result.

To some extent, this reflects the philosophy of Hodge theory, that is how does locally good property control global cohomology. Kähler is a locally good property, and the following theorem may cultivate you such an intuition.

Theorem 7.4.1. Let (X, h) be a Kähler manifold, then locally we can choose a holomorphic coordinate $(z^1, \ldots, z^n)$ such that

$$h_{i\bar{j}}(z) = \delta_{i\bar{j}} - R_{i\bar{j}k\bar{l}}(p) z^k \overline{z}^l + O(|z|^2)$$

[8]In other words, a Riemann surface

7.5. **When Chern connection encounters Levi-Civita connection.** Now let (X, g) be a hermitian manifold, then it gives a Riemannian metric on its underlying real manifold. Let ∇ be the Levi-Civita connection with respect to g, and extend it to

$$\nabla : C^\infty(X, (T_X)_\mathbb{C}) \to C^\infty(X, (T^*X)_\mathbb{C} \otimes (T_X)_\mathbb{C})$$

by $\mathbb{C}$-linearity. A natural question is that can we obtain a connection by restricting ∇ to $T_X^{1,0}$. If we can do this, do we get the Chern connection with respect to hermitian metric g?

If the $\mathbb{C}$-linear extension of Levi-Civita connection ∇ gives a connection on $T_X^{1,0}$, we must have $\Gamma_{ij}^{\overline{k}} = \Gamma_{\overline{i}j}^{\overline{k}} = 0$ for arbitrary i, j, k. Direct computation shows

$$\begin{aligned}
\Gamma_{ij}^{\overline{k}} &= \frac{1}{2} g^{\overline{\beta}\lambda}(\partial_\alpha g_{\lambda\gamma} + \partial_\gamma g_{\alpha\lambda} - \partial_\lambda g_{\alpha\gamma}) = 0 \\
\Gamma_{\overline{\alpha}\gamma}^{\overline{\beta}} &= \frac{1}{2} g^{\overline{\beta}\lambda}(\partial_{\overline{\alpha}} g_{\lambda\gamma} + \partial_\gamma g_{\overline{\alpha}\lambda} - \partial_\lambda g_{\overline{\alpha}\gamma}) = \frac{1}{2} g^{\overline{\beta}\lambda}(\partial_\gamma g_{\overline{\alpha}\lambda} - \partial_\lambda g_{\overline{\alpha}\gamma})
\end{aligned}$$

Thus ∇ gives a connection on $T_X^{1,0}$ if and only if

$$\partial_\gamma g_{\overline{\alpha}\lambda} = \partial_\lambda g_{\overline{\alpha}\gamma} \tag{7.2}$$

that is (X, g) is a Kähler manifold. Since Levi-Civita connection already preserves Riemannian metric, thus its restriction on $T_X^{1,0}$ also preserves hermitian metric. So if we want to show this restriction is Chern connection with respect to g, it remains to check

1. it's complex, according to Remark 6.3.1, that is $\Gamma_{\alpha\overline{\gamma}}^{\beta} = 0$.
2. it satisfies the equation $\partial h = \omega h$, according to Remark 6.3.2, that is $\Gamma_{\alpha\beta}^{\gamma} = g^{\gamma\overline{\lambda}} \partial_\alpha g_{\beta\overline{\lambda}}$.

For (1)

$$\begin{aligned}
\Gamma_{\alpha\overline{\gamma}}^{\beta} &= \frac{1}{2} g^{\beta\overline{\lambda}}(\partial_\alpha g_{\overline{\lambda}\overline{\gamma}} + \partial_{\overline{\gamma}} g_{\alpha\overline{\lambda}} - \partial_{\overline{\lambda}} g_{\alpha\overline{\gamma}}) \\
&= \frac{1}{2} g^{\beta\overline{\lambda}}(\partial_{\overline{\gamma}} g_{\alpha\overline{\lambda}} - \partial_{\overline{\lambda}} g_{\alpha\overline{\gamma}}) \\
&\overset{\text{Kähler}}{=} 0
\end{aligned}$$

For (2)

$$\begin{aligned}
\Gamma_{\alpha\beta}^{\gamma} &= \frac{1}{2} g^{\gamma\overline{\lambda}}(\partial_\alpha g_{\overline{\lambda}\beta} + \partial_\beta g_{\alpha\overline{\lambda}} - \partial_{\overline{\lambda}} g_{\alpha\beta}) \\
&= \frac{1}{2} g^{\gamma\overline{\lambda}}(\partial_\alpha g_{\overline{\lambda}\beta} + \partial_\beta g_{\alpha\overline{\lambda}}) \\
&\overset{\text{Kähler}}{=} g^{\gamma\overline{\lambda}} \partial_\alpha g_{\overline{\lambda}\beta}
\end{aligned}$$

All in all, the complexified the Levi-Civita connection on T_X and restrict it to $T_X^{1,0}$ to obtain a Chern connection if and only if (X, g) is Kähler, a philosophy is that Kähler condition is a bridge to connect complex and real world.

8. Positivity

8.1. **Positive line bundle.**

Let (L, h) be a hermitian holomorphic line bundle over a complex manifold X, then $\frac{\sqrt{-1}}{2\pi}\Theta_h$ gives a real $(1,1)$-form, thus it corresponds to a hermitian form on T_X.

Definition 8.1.1 (positive line bundle)**.** Let L be a holomorphic line bundle over X. L is called positive if it admits a hermitian metric h such that the hermitian form corresponding to $\frac{\sqrt{-1}}{2\pi}\Theta_h$ gives a hermitian metric.

Remark 8.1.1 (local form)*.* Locally, one has

$$\frac{\sqrt{-1}}{2\pi}\Theta_h = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h = \frac{\sqrt{-1}}{2\pi}\frac{\partial^2\varphi}{\partial z^i\partial\overline{z}^j}\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j$$

where $\varphi = -\log h$. Thus L is positive if and only if the hermitian matrix $(\frac{\partial^2\varphi}{\partial z^i\partial\overline{z}^j})$ is positive definite everywhere.

Proposition 8.1.1. If X admits a positive holomorphic line bundle, then X is Kähler.

Proof. The first Chern class of (L, h) gives its Kähler form. □

Remark 8.1.2*.* Later we will see, by Kodaira embedding theorem, if X admits a positive holomorphic line bundle, then it's a projective manifold, thus by Corollary 8.1.2 one has X is Kähler.

Example 8.1.1 (Fubini-Study metric and positivity of line bundle $\mathcal{O}_{\mathbb{CP}^n}(1)$)**.** Let's first see a canonical metric on the projective space $\mathbb{CP}^n$. Let $\mathbb{CP}^n = \bigcup_{i=0}^n U_i$ be the standard open covering, that is $U_i = \{(z^0 : \cdots : z^n) \mid z^i \neq 0\}$. Then one defines

$$\omega_i := \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\frac{\sum_{k=0}^n |z^k|^2}{|z^i|^2})$$

If we regard U_i as $\mathbb{C}^n$ with coordinate $(w^1, \dots, w^n)$, then we can write ω_i as

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\sum_{k=1}^n |w^k|^2 + 1)$$

Now we claim that these ω_i define a global real $(1,1)$-form.

1. It's globally defined: It suffices to show $\omega_i|_{U_i\cap U_j} = \omega_j|_{U_i\cap U_j}$. Indeed,
$$\log(\frac{\sum_{j=0}^n |z^j|^2}{|z^i|^2}) = \log(\frac{|z^j|^2}{|z^i|^2}\frac{\sum_{k=0}^n |z^k|^2}{|z^j|^2})) = \log(\frac{|z^j|^2}{|z^i|^2}) + \log(\frac{\sum_{j=0}^n |z^j|^2}{|z^j|^2})$$
and note that
$$\partial\overline{\partial}\log(\frac{|z^j|^2}{|z^i|^2}) = 0$$
since $\frac{z^j}{z^i}$ is the j-th coordinate function on U_i.
2. Real: It holds from $\overline{\partial\overline{\partial}} = \overline{\partial}\partial = -\partial\overline{\partial}$.
3. ∂-closed. It's ∂-closed, since it's locally exact.

It remains to show ω is positive definite, it suffices to show on each U_i, and it's easier to compute if we regard U_i as $\mathbb{C}^n$. A straightforward computation yields

$$\partial\overline{\partial}\log(1+\sum_{i=1}^{n}|w^i|^2)=\frac{1}{(1+\sum_{i=1}^{n}|w^i|^2)^2}h_{i\overline{j}}\mathrm{d}w^i\wedge\mathrm{d}\overline{w}^j$$

where $h_{i\overline{j}}=(1+\sum_{i=1}^{n}|w^i|^2)\delta_{ij}-\overline{w}^iw^j$. To see $h_{i\overline{j}}$ is positive definite, we take a column vector $u\neq 0$ and compute as follows:

$$\begin{aligned}u^t(h_{i\overline{j}})\overline{u}&=(u,u)+(w,w)(u,u)-u^t\overline{w}w^t\overline{u}\\&=(u,u)+(w,w)(u,u)-(u,w)(w,u)\\&=(u,u)+(w,w)(u,u)-\overline{(w,u)}(w,u)\\&=(u,u)+(w,w)(u,u)-|(w,u)|^2>0\end{aligned}$$

Untill now we have shown that ω we defined above is a positive hermitian metric, but where does it come from? In fact, it comes from the line bundle $\mathcal{O}_{\mathbb{CP}^n}(1)$ on projective space. Note that line bundle $\mathcal{O}_{\mathbb{CP}^n}(1)$ can be given by data $\{U_i,g_{ij}\}$, where U_i is canonical open covering and $g_{ij}=z^j/z^i$. If we set $h_i:U_i\to\mathbb{R}_{>0}$ as

$$h_i=\frac{|z^i|^2}{\sum_{k=0}^{n}|z^k|^2}$$

Then h_i can be glued together to obtain a hermitian metric on $\mathcal{O}_{\mathbb{CP}^n}(1)$, since $h_i=h_j|g_{ij}|^2$. It's easy to see the hermitian metric corresponding to curvature of Chern connection with respect to this metric is Fubini-Study metric we defined above.

Remark 8.1.3. Here comes the last question: Why is this metric on $\mathcal{O}_{\mathbb{CP}^n}(1)$ natural? Note that if we consider the dual bundle of $\mathcal{O}_{\mathbb{CP}^n}(1)$, and that's $\mathcal{O}_{\mathbb{CP}^n}(-1)$. $\mathcal{O}_{\mathbb{CP}^n}(-1)$ is a subbundle of $\mathbb{CP}^n\times\mathbb{C}^{n+1}$, so we can obtain a natural hermitian metric of $\mathcal{O}_{\mathbb{CP}^n}(-1)$ by restricting standard hermitian metric of $\mathbb{CP}^n\times\mathbb{C}^{n+1}$. And hermitian metric on $\mathcal{O}_{\mathbb{CP}^n}(1)$ we defined before is just the dual metric of this natural metric.

Corollary 8.1.1. $\mathbb{CP}^n$ is Kähler.

Corollary 8.1.2. Any projective manifold is Kähler.

Proof. By Proposition 7.4.1, the submanifold of Kähler manifold is still Kähler. □

Exercise 8.1.1. L is positive if and only if $L^{\otimes m}$ is positive for some $m\in\mathbb{N}_{\geq 0}$.

Proof. For a line bundle L locally we have the hermitian metric corresponding to its curvature looking like

$$(\frac{\partial^2\varphi}{\partial z^i\partial\overline{z}^j})$$

and for $L^{\otimes m}, m \in \mathbb{N}_{\geq 0}$ we have

$$(m \cdot \frac{\partial^2 \varphi}{\partial z^i \partial \overline{z}^j})$$

it's clear L is positive if and only if $L^{\otimes m}$ is. □

Exercise 8.1.2. Suppose X is a compact complex manifold, L is a positive line bundle, and M is any holomorphic line bundle, then there exists $N_0 \in \mathbb{N}$ such that $M \otimes L^{\otimes N}$ positive for $N \geq N_0$.

Proof. The proof is quite similar to above exercise, we need to check locally, but compactness is neccessary here. For an open subset U_1, locally we have the hermitian metric corresponding to $M \otimes L^m$ looking like

$$(\frac{\partial^2 \varphi_M}{\partial z^i \partial \overline{z}^j} + m \cdot \frac{\partial^2 \varphi_L}{\partial z^i \partial \overline{z}^j})$$

So we can choose suffices large N_1 such that $M \otimes L^{\otimes N_1}$ is positive on U. Since X is compact, we can take a finite open covering $\{U_i\}$ of X and choose the largest N_i to be N we desired. □

Remark 8.1.4. In fact, If we use language of algebraic geometry, then a positive line bundle is equivalent to an ample divisor.

8.2. **Lefschetz $(1,1)$-theorem.** Now we know that given a hermitian holomorphic line bundle (L,h), then consider its Chern curvature we will get a real $(1,1)$-form. So we may wonder the converse of this statement. Is there any real $(1,1)$-form comes from such a hermitian holomorphic line bundle? That's main theorem for this section.

Theorem 8.2.1 (Lefschetz $(1,1)$-theorem)**.** Let X be a complex manifold, and $[\omega] \in H^2(X,\mathbb{R}) \cap H^{1,1}(X)$. If

$$[\omega] \in \operatorname{im}\{H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R})\}$$

Then there exists a hermitian holomorphic line bundle (L,h) such that

$$\frac{\sqrt{-1}}{2\pi}\Theta_h = \omega$$

Remark 8.2.1. Before proving this theorem, let's elaborate what does the following map mean

$$H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R})$$

since in de Rham cohomology, it's meaningless to say cohomology with $\mathbb{Z}$ coefficient. Here we use comparision $H^2(X,\mathbb{R}) \cong \check{H}^2(X,\underline{\mathbb{R}})$, where $\underline{\mathbb{R}}$ is constant sheaf valued $\mathbb{R}$ and consider the map in terms of Čech cohomology

$$\check{H}^2(X,\underline{\mathbb{Z}}) \to \check{H}^2(X,\underline{\mathbb{R}})$$

Generally this can be shown by using tool of spectral sequences. Here we give an explicit construction in $k=2$, since later we will use it.

In sketch, the philosophy of this construction is that we can descend the degree of differential forms, but the price we pay is to consider functions defined on intersections of many open subsets.

Let X be a smooth manifold, and $Z^1 \subset \Omega^1_{X,\mathbb{R}}$, the sheaf of closed 1-form. Then we have the following exact sequence of sheaves

$$0 \to \underline{\mathbb{R}} \to C^\infty(X) \xrightarrow{\mathrm{d}} Z^1 \to 0$$

Locally constant functions are clearly smooth functions, such that d acts on them is zero, so the exactness for the first two is trivial. But for the last one, it is equivalent to that a closed form locally must be an exact form, that's Poincaré lemma.

Similarly, define $Z^2 \subset \Omega^2_{X,\mathbb{R}}$, the sheaf of closed 2-forms. then the following sequence is also exact for the same reason

$$0 \to Z^1 \to \Omega^1_{X,\mathbb{R}} \xrightarrow{\mathrm{d}} Z^2 \to 0$$

By the definition of de Rham cohomology, we have

$$H^2(X,\mathbb{R}) = \frac{C^\infty(X,Z^2)}{\mathrm{d}C^\infty(X,\Omega^1_{X,\mathbb{R}})}$$

In order to avoid the limit in the definition of Čech cohomology, we take open covering $\mathfrak{U} = \{U_\alpha\}$ good enough, such that

$$\mathrm{d} : C^\infty(U_\alpha, \Omega^1_{U_\alpha,\mathbb{R}}) \to C^\infty(U_\alpha, Z^2)$$

is surjective for any α. And

$$\mathrm{d} : C^\infty(U_\alpha \cap U_\beta) \to C^\infty(U_\alpha \cap U_\beta, Z^1)$$

is surjective for any α, β. If ω is a closed real 2-form. For any α, choose $A_\alpha \in C^\infty(U_\alpha, \Omega^1_{U_\alpha,\mathbb{R}})$ such that

$$\omega|_{U_\alpha} = \mathrm{d}A_\alpha$$

then

$$\prod_{\alpha,\beta}(A_\alpha - A_\beta)$$

is a Čech 1-cocchain in $C^1(\mathfrak{U}, Z^1)$, it's d-closed since $\mathrm{d}(A_\alpha - A_\beta)|_{U_\alpha \cap U_\beta} = \omega - \omega = 0$. For any α, β, choose $f_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta)$, such that

$$(A_\alpha - A_\beta)_{\alpha\beta} = \mathrm{d}f_{\alpha\beta}$$

then

$$f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}|_{U_\alpha \cap U_\beta \cap U_\gamma}$$

is also d-closed by the same reason, hence locally constant. Thus

$$\check{\omega} = \prod_{\alpha,\beta,\gamma}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta})$$

is a Čech 2-cocycle in $C^2(\mathfrak{U}, \underline{\mathbb{R}})$. Thus we obtain an element in $\check{H}^2(X, \underline{\mathbb{R}})$ from an element in $H^2(X, \mathbb{R})$.

That's the explicit construction for comparision theorem in dimension 2. In fact, the general case can be proved in the same method, though it maybe quite complicated as you can imagine.

Before our proof of Lefschetz $(1,1)$-theorem, we still need two lemmas in multi-variables complex analysis.

Lemma 8.2.1 ($\partial\overline{\partial}$-lemma)**.** Locally on a polydisk $D \subset \mathbb{C}^n$, and $[\omega] \in H^2(D,\mathbb{R}) \cap H^{1,1}(D)$. Then there exists a smooth function $\varphi : D \to \mathbb{R}$ such that

$$\omega = \sqrt{-1}\partial\overline{\partial}\varphi$$

Lemma 8.2.2. Locally on $U \subset \mathbb{C}^n$, a simply connected open subset, and a smooth function[9] $\varphi : U \to \mathbb{R}$, such that $\partial\overline{\partial}\varphi = 0$. Then there exists a holomorphic functions $f : U \to \mathbb{C}$, such that $\varphi = \mathrm{Re}(f)$.

Now let's prove Lefschetz $(1,1)$-theorem

proof of theorem 8.2.1. Let's first see how does the above two lemmas play a role in our proof. Choose a good enough open cover $\mathfrak{U} = \{U_\alpha\}$ of open polydisk such that for all α, β, we have $U_\alpha \cap U_\beta$ is simply connected.

Since ω is a d-closed real $(1,1)$-form, Lemma 8.2.1 implies that there exists smooth function $\varphi_\alpha : U_\alpha \to \mathbb{R}$ such that

$$\omega|_{U_\alpha} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi_\alpha$$

On any two intersection $U_\alpha \cap U_\beta$, we have $\partial\overline{\partial}(\varphi_\alpha - \varphi_\beta) = 0$, then Lemma 8.2.2 implies that there exists a holomorphic function $f_{\alpha\beta}$, such that

$$(\varphi_\alpha - \varphi_\beta)|_{U_\alpha \cap U_\beta} = 2\,\mathrm{Re}(f_{\alpha\beta}) = f_{\alpha\beta} + \overline{f_{\alpha\beta}}$$

Consider $\prod f_{\alpha\beta} \in C^1(\mathfrak{U}, \mathcal{O}_X)$, then

$$(\delta f)_{\alpha\beta\gamma} = (f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta})|_{U_\alpha \cap U_\beta \cap U_\gamma}$$

Note that $2\,\mathrm{Re}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta})_{\alpha\beta\gamma} = 0$, so it must be a locally constant pure imaginary number, that is, it lies in $2\pi\sqrt{-1}\underline{\mathbb{R}}(U_\alpha \cap U_\beta \cap U_\gamma)$.

Consider real form[10]

$$A_\alpha = \frac{\sqrt{-1}}{4\pi}(\overline{\partial}\varphi_\alpha - \partial\varphi_\alpha)$$

and by directly computing, we can note that $\omega|_{U_\alpha} = \mathrm{d}A_\alpha$, and that's why we define A_α in this form.

Similar to what we have done in the proof of comparision theorem, if we want to find Čech cocycle which corresponding to ω, we need to consider

[9]Such φ is called pluriharmonic

[10]Here we need to consider some queer coefficients, in order to get a beautiful result. In fact, we need to use $e^{2\pi\sqrt{-1}} = 1$, a god given formula.

$A_\alpha - A_\beta$ on the intersection $U_\alpha \cap U_\beta$. Directly compute the difference of each term of A_α and A_β as follows

$$\begin{aligned}\partial(\varphi_\beta - \varphi_\alpha) &= \partial(f_{\alpha\beta} + \overline{f_{\alpha\beta}}) \\ &= \partial f_{\alpha\beta} \\ &= \mathrm{d} f_{\alpha\beta} \\ \overline{\partial}(\varphi_\beta - \varphi_\alpha) &= \mathrm{d}\overline{f_{\alpha\beta}}\end{aligned}$$

Thus

$$(A_\beta - A_\alpha)_{\alpha\beta} = \frac{\sqrt{-1}}{4\pi}\mathrm{d}(\overline{f_{\alpha\beta}} - f_{\alpha\beta}) = \frac{1}{2\pi}\mathrm{d}(\mathrm{Im}(f_{\alpha\beta}))$$

So from the explicit construction of comparision theorem, we have the Čech cocycle $\check{\omega}$ corresponding to ω is

$$\begin{aligned}\check{\omega} &= \prod\left(\frac{1}{2\pi}\,\mathrm{Im}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta})\right)_{\alpha\beta\gamma} \\ &= \prod\left(\frac{1}{2\pi\sqrt{-1}}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta})\right)_{\alpha\beta\gamma}\end{aligned}$$

Hypothesis tells that $[\check{\omega}]$ is an image of $[\prod n_{\alpha\beta\gamma}] \in \check{H}^2(X, \underline{\mathbb{Z}})$. However, it doesn't mean that $f_{\alpha\beta}$ are exactly integers, but not too bad, we just need some correction terms, that is

$$\prod\left(\frac{1}{2\pi\sqrt{-1}}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta})\right)_{\alpha\beta\gamma} = \prod n_{\alpha\beta\gamma} + \delta(\prod c_{\alpha\beta})$$

where $\prod(c_{\alpha\beta})$ is real 1-cochain. So we set $f'_{\alpha\beta} = f_{\alpha\beta} - 2\pi\sqrt{-1}c_{\alpha\beta}$. Then

$$(f'_{\beta\gamma} - f'_{\alpha\gamma} + f'_{\alpha\beta})_{\alpha\beta\gamma} = 2\pi\sqrt{-1}n_{\alpha\beta\gamma} \in 2\pi\sqrt{-1}\underline{\mathbb{Z}}(U_\alpha \cap U_\beta \cap U_\gamma)$$

Note that $e^{2\pi\sqrt{-1}} = 1$, then consider $g_{\alpha\beta} = \exp(-f'_{\alpha\beta})$, a holomorphic function from $U_\alpha \cap U_\beta$ to $\mathbb{C}^*$, it satisfies the cocycle condition

$$g_{\beta\gamma}g_{\alpha\gamma}^{-1}g_{\alpha\beta} = 1$$

so we get a holomorphic line bundle L.

Now we need to give a hermitian metric on this holomorphic line bundle H, and calculate its curvature to complete the proof.

Note that

$$(\varphi_\alpha - \varphi_\beta)_{U_\alpha \cap U_\beta} = 2\,\mathrm{Re}(f_{\alpha\beta}) = 2\,\mathrm{Re}(f_{\alpha\beta})' = -\log|g_{\alpha\beta}|^2$$

then we get a hermitian metric on U_α which is defined by

$$H_\alpha = \exp(-\varphi_\alpha)$$

Indeed, since $H_\beta = |g_{\alpha\beta}|^2 H_\alpha = g_{\alpha\beta}^T H_\alpha \overline{g_{\alpha\beta}}$. Finally,

$$\frac{\sqrt{-1}}{2\pi}\Theta_h = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi_\alpha = \omega$$

This completes the proof. $\square$

Remark 8.2.2. In fact, what we have done in the proof is just reversing the operations in constructing connecting homomorphism ∂ in

$$\cdots \to \check{H}^1(X, \mathcal{O}_X^*) \xrightarrow{\partial} \check{H}^2(X, \underline{\mathbb{Z}}) \to \check{H}^2(X, \underline{\mathbb{R}}) \to \dots$$

If we already have a holomorphic line bundle L, determined by its transition functions $g_{\alpha\beta}$. What does ∂ look like? Recall what we have learnt in homological algebra, that is, take logarithm of $g_{\alpha\beta}$, consider its alternating sum and divide it by $2\pi\sqrt{-1}$, and it turns out to be first Chern class.

So if we have some real $(1,1)$-form which may come from a holomorphic line bundle, we need to realize it as a Čech cocycle and reverse three steps above, that's what we have done in the proof.

8.3. **Positivity of vector bundles.** Let (E,h) be a hermitian holomorphic vector bundle of rank r over a complex n-manifold X with Chern connection ∇. In local frames, its Chern curvature is given by

$$\Theta_h = R_{i\bar{j}\alpha}^{\beta} \mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j \otimes e^\alpha \otimes e_\beta$$

Definition 8.3.1 (positivity)**.**

1. (E,h) is said to be Griffiths positive, if for any non-zero $(u^i) \in \mathbb{C}^n$ and $(v^\alpha) \in \mathbb{C}^r$
$$R_{i\bar{j}\alpha\overline{\beta}} u^i \overline{u}^j v^\alpha \overline{v}^\beta > 0$$
2. (E,h) is said to be Nakano positive, if for any non-zero matrix $(u^{i\alpha})$
$$R_{i\bar{j}\alpha\overline{\beta}} u^{i\alpha} \overline{u}^{j\beta} > 0$$
3. (E,h) is said to be dual Nakano positive, if for any non-zero matrix $(u^{i\alpha})$
$$R_{i\bar{j}\alpha\overline{\beta}} u^{i\beta} \overline{u}^{j\alpha} > 0$$

Remark 8.3.1. The semi-positivity and negativity can be defined in the same way.

Proposition 8.3.1.

1. If (E,h) is Nakano positive or dual Nakano positive, then (E,h) is Griffiths positive;
2. (E,h) is Nakano positive if and only if (E^*,h^*) is dual Nakano negative.

Proof. For (1). If (E,h) is Nakano positive, then for non-zero $(u^i) \in \mathbb{C}^n$ and $(v^\alpha) \in \mathbb{C}^r$, consider matrix $(u^{i\alpha})$ defined by $u^{i\alpha} := u^i v^\alpha$, then

$$R_{i\bar{j}\alpha\overline{\beta}} u^i \overline{u}^j v^\alpha \overline{v}^\beta = R_{i\bar{j}\alpha\overline{\beta}} u^{i\alpha} \overline{u}^{j\beta} > 0$$

The same argument holds for the case (E,h) is dual Nakano positive.

For (2). □

Part 4. Hodge decomposition

9. Hodge theory

9.1. Adjoint operators.

Let (X, h) be a compact hermitian manifold with fundamental form ω, with respect to local frames a (p,q)-form α can be written as

$$\alpha = \frac{1}{p! \times q!}\alpha_{i_1\dots i_p\bar{j}_1\dots\bar{j}_q}\mathrm{d}z^{i_1}\wedge\cdots\wedge\mathrm{d}z^{i_p}\wedge\mathrm{d}\bar{z}^{j_1}\wedge\cdots\wedge\mathrm{d}\bar{z}^{j_q}$$

Then for $\alpha,\beta\in C^\infty(X,\Omega_X^{p,q})$, the local inner product is defined as

$$\langle\alpha,\beta\rangle = \frac{1}{p!\times q!}h^{i_1\bar{k}_1}\dots h^{i_p\bar{k}_p}h^{l_1\bar{j}_1}\dots h^{l_q\bar{j}_q}\alpha_{i_1\dots i_p\bar{j}_1\dots\bar{j}_q}\overline{\beta_{k_1\dots k_p\bar{l}_1\dots\bar{l}_q}}$$

which is a smooth function on X.

Definition 9.1.1 (inner product on (p,q)-form)**.** An inner product on the space of (p,q)-form is defined as

$$(\alpha,\beta) := \int_X\langle\alpha,\beta\rangle\frac{\omega^n}{n!}$$

where $\alpha,\beta\in C^\infty(X,\Omega_X^{p,q})$.

Remark 9.1.1. It gives an inner product on $\Omega^k_{X,\mathbb{C}}$, since $\Omega^k_{X,\mathbb{C}} = \bigoplus_{p+q=k}\Omega_X^{p,q}$.

Holding the inner product (-,-), the adjoint operator of d is defined as an operator

$$\mathrm{d}^*\colon C^\infty(X,\Omega^{p+q}_{X,\mathbb{C}})\to C^\infty(X,\Omega^{p+q-1}_{X,\mathbb{C}})$$

satisfying $(\alpha,\mathrm{d}\beta) = (\mathrm{d}^*\alpha,\beta)$ for α,β with appropriate degrees. Note that d^* is nothing new from Riemannian geometry, but by using inner product on (p,q)-form, one can define adjoint operators of ∂ and $\bar{\partial}$.

To be explicit, the adjoint operator of ∂ is defined as an operator

$$\partial^*\colon C^\infty(X,\Omega_X^{p,q})\to C^\infty(X,\Omega_X^{p-1,q})$$

satisfying $(\alpha,\partial\beta) = (\partial^*\alpha,\beta)$ for α,β with appropriate bidegrees. In order to construct these adjoint operators, we need to introduce the well-known Hodge star operator.

Definition 9.1.2 (Hodge star operator)**.** There exists an operator

$$\star: C^\infty(X,\Omega_X^{p,q})\to C^\infty(X,\Omega_X^{n-q,n-p})$$

such that

$$(\alpha,\beta) = \int_X\alpha\wedge\star\bar{\beta}$$

Remark 9.1.2. It's well-defined, since $\bar{\beta}$ is a (q,p)-form, and thus $\star\bar{\beta}$ is a $(n-p,n-q)$-form.

Lemma 9.1.1.

1. $\star 1 = \omega^n/n!$.

2. $\star\omega = \omega^{n-1}/(n-1)!$.
3. $\overline{\star\psi} = \star\overline{\psi}$.
4. $\star\star = (-1)^{p+q}$ on $C^\infty(X, \Omega_X^{p,q})$.
5. $(\star\varphi, \star\psi) = (\varphi, \psi)$.

Proposition 9.1.1. $\mathrm{d}^* = -\star\mathrm{d}\star$.

Proof. For arbitrary $\alpha \in C^\infty(X, \Omega_{X,\mathbb{C}}^{p+q})$ and $\beta \in C^\infty(X, \Omega_{X,\mathbb{C}}^{p+q+1})$, then

$$\begin{aligned}
(\mathrm{d}\alpha, \beta) &= \int_X \mathrm{d}\alpha \wedge \star\beta \\
&= \int_X \mathrm{d}(\alpha \wedge \star\beta) - (-1)^{p+q}\alpha \wedge \mathrm{d}\star\beta \\
&= (-1)^{p+q+1}\int_X \alpha \wedge \mathrm{d}\star\beta \\
&\overset{(1)}{=} (-1)^{p+q+1}(-1)^{2n-(p+q+1)+1}\int_X \alpha \wedge \star\star\mathrm{d}\star\beta \\
&= -(\alpha, \star\mathrm{d}\star\beta)
\end{aligned}$$

where (1) holds from (4) of Lemma 9.1.1. □

Proposition 9.1.2.

$$\begin{aligned}
\partial^* &= -\star\overline{\partial}\star \\
\overline{\partial}^* &= -\star\partial\star
\end{aligned}$$

Proof. The same as above. □

Definition 9.1.3 (Lefschetz operator)**.** Let (X, ω) be a compact Kähler manifold, the Lefschetz operator is defined as

$$\begin{aligned}
L\colon C^\infty(X, \Omega_X^{p,q}) &\to C^\infty(X, \Omega_X^{p+1,q+1}) \\
\alpha &\mapsto \omega \wedge \alpha
\end{aligned}$$

where ω is fundamental form of h.

Lemma 9.1.2. $\Lambda_\omega := L^* = (-1)^{p+q}\star L\star$ on (p,q)-forms.

Proof. For $\alpha \in C^\infty(X, \Omega_{X,}^{p,q}), \beta \in C^\infty(X, \Omega_X^{p+1,q+1})$, direct computation shows

$$\begin{aligned}
(L\alpha, \beta) &= \int_X L\alpha \wedge \star\beta \\
&= \int_X \omega \wedge \alpha \wedge \star\beta \\
&\overset{(1)}{=} \int_X \alpha \wedge \omega \wedge \star\beta \\
&\overset{(2)}{=} \int_X \alpha \wedge (-1)^{p+q}\star\star\omega \wedge \star\beta \\
&= (\alpha, (-1)^{p+q}\star L\star\beta)
\end{aligned}$$

where

(1) holds from ω is a 2-form;
(2) holds from (4) of Lemma 9.1.1.

□

9.2. **Hodge theorem.**

Definition 9.2.1 (Laplacian)**.** Laplacian $\Delta_\bullet$ is an operator defined by $\Delta_\bullet := \bullet\bullet^* + \bullet^*\bullet$, where $\bullet$ can be d, ∂ and $\overline{\partial}$.

Definition 9.2.2 (harmonic)**.** A form α is called $\Delta_\bullet$-harmonic if $\Delta_\bullet\alpha = 0$, where $\bullet$ can be d, ∂ and $\overline{\partial}$.

Notation 9.2.1. $\mathcal{H}^k$ denotes the space of Δ_{d}-harmonic k-forms, and $\mathcal{H}^{p,q}$ denotes the space of $\Delta_{\overline{\partial}}$-harmonic forms of type (p,q).

Lemma 9.2.1. *α is $\Delta_\bullet$-harmonic if and only if $\bullet\alpha = 0, \bullet^*\alpha = 0$, where $\bullet$ can be d, ∂ and $\overline{\partial}$.*

Proof. Here we only check the case $\bullet = \mathrm{d}$, the other cases are same:

$$\begin{aligned}(\alpha, \Delta_{\mathrm{d}}\alpha) &= (\alpha, \mathrm{d}\mathrm{d}^*\alpha) + (\alpha, \mathrm{d}^*\mathrm{d}\alpha)\\ &= \|\mathrm{d}^*\alpha\|^2 + \|\mathrm{d}\alpha\|^2\end{aligned}$$

□

Theorem 9.2.1 (Hodge theorem)**.** *Let (X, h) be a compact hermitian n-manifold. Then*
1. *$\mathcal{H}^{p,q}$ is finite dimensional.*
2. *There is a decomposition $C^\infty(X, \Omega_X^{p,q}) = \mathcal{H}^{p,q} \oplus \Delta_{\overline{\partial}}(C^\infty(X, \Omega_X^{p,q}))$, which is orthogonal with respect to inner products in Definition 9.1.1.*

Corollary 9.2.1. *There is the following orthonormal decomposition*

$$C^\infty(X, \Omega_X^{p,q}) = \mathcal{H}^{p,q} \oplus \overline{\partial}(C^\infty(X, \Omega_X^{p,q-1}) \oplus \overline{\partial}^*(C^\infty(X, \Omega_X^{p,q+1}))$$

Corollary 9.2.2.

$$\begin{aligned}\ker\overline{\partial} &= \mathcal{H}^{p,q} \oplus \overline{\partial}^*(C^\infty(X, \Omega_X^{p,q-1}))\\ \ker\overline{\partial}^* &= \mathcal{H}^{p,q} \oplus \overline{\partial}(C^\infty(X, \Omega_X^{p,q+1}))\end{aligned}$$

Corollary 9.2.3. *The natural map $\mathcal{H}^{p,q} \to H^{p,q}(X)$ is an isomorphism. In particular, $H^{p,q}(X)$ is finite dimensional.*

In order to give the following isomorphism

$$\star\colon \mathcal{H}^{p,q} \to \mathcal{H}^{n-q,n-p}$$

Parallel to the real case[11], it suffices to have

$$\star \circ \Delta_{\overline{\partial}} = \Delta_{\overline{\partial}} \circ \star$$

But something bad happens, since we only have $\overline{\partial}^* = -\star\partial\star$, and direct computation only yields $\Delta_{\overline{\partial}} \circ \star = \star \circ \Delta_\partial$. So it fails generally since $\Delta_{\overline{\partial}} \neq \Delta_\partial$.

[11] See section Hodge theory in [Liu23]

There are two ways to deal with this gap. The first way is that we will see later if X is compact Kähler manifold, then $\Delta_\partial = \Delta_{\overline{\partial}}$, that is Theorem 9.4.1. Then

Corollary 9.2.4. If (X,ω) is a compact Kähler n-manifold, then $\star\colon \mathcal{H}^{p,q} \to \mathcal{H}^{n-q,n-p}$ is an isomorphism.

Another way is to consider

$$\begin{aligned}\overline{\star}\colon C^\infty(X,\Omega_X^{p,q}) &\to C^\infty(X,\Omega_X^{n-p,n-q})\\ \beta &\mapsto \star\overline{\beta}\end{aligned}$$

then

$$\overline{\star}\circ\Delta_{\overline{\partial}} = \Delta_{\overline{\partial}}\circ\overline{\star}$$

Corollary 9.2.5. If (X,h) is a compact hermitian manifold, then $\overline{\star}\colon \mathcal{H}^{p,q} \to \mathcal{H}^{n-p,n-q}$ is an isomorphism.

Corollary 9.2.6. $H^{p,q}(X) \cong H^{n-p,n-q}(X)$.

Remark 9.2.1. This is a special case of Serre duality.

9.3. **Useful formulas of adjoint operators.**

Proposition 9.3.1. Let (X,h) be a hermitian manifold, then

$$\langle \mathrm{d}z^i\wedge\alpha,\beta\rangle = \langle\alpha, h^{p\bar{i}}\iota_p\beta\rangle$$

holds for forms α,β with appropriate bidegrees.

Proof. It suffices to check pointwise, and at each point we may chosoe normal coordinate in Theorem 7.2.1. For (p,q)-form α and $(p+1,q)$-form β, locally we write them as

$$\begin{aligned}\alpha &= \alpha_{I\overline{J}}\mathrm{d}z^{i_1}\wedge\dots\mathrm{d}z^{i_p}\wedge\mathrm{d}\overline{z}^{j_1}\wedge\dots\mathrm{d}\overline{z}^{j_q}\\ \beta &= \beta_{I'\overline{J'}}\mathrm{d}z^{i'_1}\wedge\dots\mathrm{d}z^{i'_{p+1}}\wedge\mathrm{d}\overline{z}^{j'_1}\wedge\dots\mathrm{d}\overline{z}^{j'_q}\end{aligned}$$

Then

$$\mathrm{d}z^i\wedge\alpha = \alpha_{I\overline{J}}\mathrm{d}z^i\wedge\mathrm{d}z^{i_1}\wedge\dots\mathrm{d}z^{i_p}\wedge\mathrm{d}\overline{z}^{j_1}\wedge\dots\mathrm{d}\overline{z}^{j_q}$$

which implies

$$\langle\mathrm{d}z^i\wedge\alpha,\beta\rangle = \frac{1}{(p+1)!\times q!}h^{i\overline{i'_1}}\dots h^{i_p\overline{i'_{p+1}}}h^{j'_1\overline{j_1}}\dots h^{j'_q\overline{j_q}}\alpha_{I\overline{J}}\overline{\beta_{I'\overline{J'}}}$$

On the other hand,

$$\langle\alpha,\iota_i\beta\rangle =$$

□

Proposition 9.3.2. Let (X,h) be a compact Kähler manifold, then locally

$$\begin{cases}\partial = \mathrm{d}z^i\wedge\nabla_i\\ \partial^* = -h^{i\bar{j}}\iota_i\circ\nabla_j = -h^{i\bar{j}}\nabla_j\circ\iota_i\end{cases}\qquad\begin{cases}\overline{\partial} = \mathrm{d}\overline{z}^i\wedge\nabla_{\bar{i}}\\ \overline{\partial}^* = -h^{i\bar{j}}\iota_{\bar{j}}\circ\nabla_i = -h^{i\bar{j}}\nabla_i\circ\iota_{\bar{j}}\end{cases}$$

Proof. Here we only give the proof of the case ∂ and ∂^*, the proof for the other two cases are same. It suffices to check pointwise, and at each point we may also choose normal coordinate in Theorem 7.4.1. For (p,q)-form α, locally written as $\alpha = \alpha_{J\overline{K}} \mathrm{d}z^J \wedge \mathrm{d}\overline{z}^K$. Then

$$\partial\alpha = \frac{\partial \alpha_{J\overline{K}}}{\partial z^i} \mathrm{d}z^i \wedge \mathrm{d}z^J \wedge \mathrm{d}\overline{z}^K$$

and

$$\begin{aligned} \mathrm{d}z^i \wedge \nabla_i \alpha &= \mathrm{d}z^i \wedge \nabla_i(\alpha_{J\overline{K}} \mathrm{d}z^J \wedge \mathrm{d}\overline{z}^K) \\ &= \mathrm{d}z^i \wedge \frac{\partial \alpha_{J\overline{K}}}{\partial z^i} \mathrm{d}z^J \wedge \mathrm{d}\overline{z}^K + \alpha_{J\overline{K}} \nabla_i (\mathrm{d}z^J \wedge \mathrm{d}\overline{z}^K) \\ &\overset{(1)}{=} \frac{\partial \alpha_{J\overline{K}}}{\partial z^i} \mathrm{d}z^i \wedge \mathrm{d}z^J \wedge \mathrm{d}\overline{z}^K \end{aligned}$$

where (1) holds from our choice of normal coordinate. To see formula of ∂^*, take arbitrary forms α, β with appropriate bidegrees, then

$$\begin{aligned} (\partial\alpha, \beta) &= (\mathrm{d}z^i \wedge \nabla_i \alpha, \beta) \\ &\overset{(2)}{=} (\nabla_i \alpha, h^{p\bar{i}} \iota_p \beta) \\ &\overset{(3)}{=} -(\alpha, h^{p\bar{i}} \nabla_i \circ \iota_p \beta) \end{aligned}$$

where

(2) holds from Proposition 9.3.1;
(3) holds from Stokes' theorem and the fact Chern connection is compatible with metric.

This shows

$$\partial^* = -h^{i\bar{j}} \nabla_j \circ \iota_i \overset{(4)}{=} -h^{i\bar{j}} \iota_i \circ \nabla_j$$

where (4) holds from $\iota_i \circ \nabla_j = \nabla_j \circ \iota_i$. □

Proposition 9.3.3. Let (X, ω) be a compact Kähler manifold, then locally

$$\Lambda_\omega = \sqrt{-1} h^{i\bar{j}} \iota_i \circ \iota_{\bar{j}} = -\sqrt{-1} h^{i\bar{j}} \iota_{\bar{j}} \circ \iota_i$$

Proof. For arbitrary forms α, β with appropriate bidegrees, direct computation shows

$$\begin{aligned} (\omega \wedge \alpha, \beta) &= (\sqrt{-1} h_{i\bar{j}} \mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j \wedge \alpha, \beta) \\ &\overset{(1)}{=} (\sqrt{-1} h_{i\bar{j}} \mathrm{d}\overline{z}^j \wedge \alpha, h^{p\bar{i}} \iota_p \beta) \\ &\overset{(2)}{=} (\sqrt{-1} h_{i\bar{j}} \alpha, h^{p\bar{i}} h^{j\bar{q}} \iota_{\bar{q}} \circ \iota_p \beta) \\ &\overset{(3)}{=} (\alpha, -\sqrt{-1} h_{j\bar{i}} h^{p\bar{i}} h^{j\bar{q}} \iota_{\bar{q}} \circ \iota_p \beta) \\ &= (\alpha, -\sqrt{-1} h^{p\bar{i}} \iota_{\bar{i}} \circ \iota_p \beta) \end{aligned}$$

where

(1) and (2) hold from Proposition 9.3.1;
(3) holds from $h_{i\bar{j}}$ is hermitian, that is $\overline{h_{i\bar{j}}} = h_{j\bar{i}}$.

This shows

$$\Lambda_\omega = -\sqrt{-1}h^{i\bar{j}}\iota_{\bar{j}} \circ \iota_i \overset{(4)}{=} \sqrt{-1}h^{i\bar{j}}\iota_i \circ \iota_{\bar{j}}$$

where (4) holds from $\iota_i \circ \iota_{\bar{j}} = -\iota_{\bar{j}} \circ \iota_i$. □

9.4. **Kähler identities.**

Definition 9.4.1 (commutor)**.** Let A, B be two differential operators, the commutor of A, B are defined as

$$[A, B] := AB - (-1)^{\deg A \deg B} BA$$

Lemma 9.4.1 (Jacobi identity)**.** Let A, B, C be differential operators, then

$$(-1)^{\deg A \deg C}[A, [B, C]] + (-1)^{\deg B \deg A}[B, [C, A]] + (-1)^{\deg C \deg B}[C, [A, B]] = 0$$

Remark 9.4.1. In our case, the degree of $\mathrm{d}, \mathrm{d}^*, \partial, \partial^*, \overline{\partial}, \overline{\partial}^*$ is one, and the degree of L and Λ is zero[12].

Proposition 9.4.1 (Kähler identities)**.** If (X, ω) is a compact Kähler manifold, then

$$\begin{aligned} [\overline{\partial}^*, L] &= \sqrt{-1}\partial \\ [\partial^*, L] &= -\sqrt{-1} \cdot \overline{\partial} \\ [\Lambda_\omega, \overline{\partial}] &= -\sqrt{-1}\partial^* \\ [\Lambda_\omega, \partial] &= \sqrt{-1} \cdot \overline{\partial}^* \end{aligned}$$

Proof. By taking conjugates and adjoints, it suffices to prove the first identity, which is a first order identity of differential equation. But by Theorem 7.4.1, locally we have $h_{i\bar{j}} = \delta_{i\bar{j}} + O(|\xi^2|)$. Thus it suffices to check Kähler identity for the case $U \subseteq \mathbb{C}^n$ equipped with standard hermitian metric.

Suppose (p, q)-form α is locally given by $\alpha = \alpha_{JK}\mathrm{d}z^J \wedge \mathrm{d}\overline{z}^K$, then by Proposition 9.3.2 one has

$$\overline{\partial}^*\alpha = -\sum_l \iota_{\frac{\partial}{\partial \overline{z}^l}} \frac{\partial \alpha}{\partial z^l}$$

---

[12] You can try to understand this thing in a following way: operators $\mathrm{d}, \mathrm{d}^*, \partial, \partial^*, \overline{\partial}, \overline{\partial}^*$ do take differentials, but L and Λ not.

Thus

$$\begin{aligned}
[\overline{\partial}^*, L]\alpha &= \overline{\partial}^*(\omega\wedge\alpha) - \omega\wedge\overline{\partial}^*\alpha \\
&= -\sum_l \iota_{\frac{\partial}{\partial\bar{z}^l}}\frac{\partial}{\partial z^l}(\omega\wedge\alpha) + \omega\wedge\sum_l \iota_{\frac{\partial}{\partial\bar{z}^l}}\frac{\partial\alpha}{\partial z^l} \\
&\overset{(1)}{=} -\sum_l \iota_{\frac{\partial}{\partial\bar{z}^l}}(\omega\wedge\frac{\partial\alpha}{\partial z^l}) + \omega\wedge\sum_l \iota_{\frac{\partial}{\partial\bar{z}^l}}\frac{\partial\alpha}{\partial z^l} \\
&= -\{\sum_l (\iota_{\frac{\partial}{\partial\bar{z}^l}}\omega)\wedge\frac{\partial\alpha}{\partial z^l} + \omega\wedge\iota_{\frac{\partial}{\partial\bar{z}^l}}\frac{\partial\alpha}{\partial z^l}\} + \sum_l \omega\wedge\iota_{\frac{\partial}{\partial\bar{z}^l}}\frac{\partial\alpha}{\partial z^l} \\
&= -\sum_l (\iota_{\frac{\partial}{\partial\bar{z}^l}}\omega)\wedge\frac{\partial\alpha}{\partial z^l} \\
&\overset{(2)}{=} \sqrt{-1}\sum_l \mathrm{d}z^l\wedge\frac{\partial\alpha}{\partial z^l} \\
&= \sqrt{-1}\partial\alpha
\end{aligned}$$

where

(1) holds from ω is a closed $(1,1)$-form;
(2) holds from Proposition 7.1.2, that is $\omega = \sqrt{-1}\sum_{i=1}^n \mathrm{d}z^i\wedge\mathrm{d}\bar{z}^i$.

□

Theorem 9.4.1. Let (X,ω) be a compact Kähler manifold, then

$$\Delta_{\mathrm{d}} = 2\Delta_{\partial} = 2\Delta_{\overline{\partial}}$$

Proof. Since

$$\Delta_{\mathrm{d}} = (\partial+\overline{\partial})(\partial^*+\overline{\partial}^*) + (\partial^*+\overline{\partial}^*)(\partial+\overline{\partial})$$

By the fourth Kähler identity, one has

1. The first term can be computed as

$$\begin{aligned}
(\partial+\overline{\partial})(\partial^*+\overline{\partial}^*) &= (\partial+\overline{\partial})(\partial^* - \sqrt{-1}\Lambda\partial + \sqrt{-1}\partial\Lambda) \\
&= \partial\partial^* - \sqrt{-1}\partial\Lambda\partial + \overline{\partial}\partial^* - \sqrt{-1}\cdot\overline{\partial}\Lambda\partial + \sqrt{-1}\cdot\overline{\partial}\partial\Lambda
\end{aligned}$$

2. The second term can be computed as

$$\begin{aligned}
(\partial^*+\overline{\partial}^*)(\partial+\overline{\partial}) &= (\partial^* - \sqrt{-1}\Lambda\partial + \sqrt{-1}\partial\Lambda)(\partial+\overline{\partial}) \\
&= \partial^*\partial + \sqrt{-1}\partial\Lambda\partial + \partial^*\overline{\partial} - \sqrt{-1}\Lambda\partial\overline{\partial} + \sqrt{-1}\partial\Lambda\overline{\partial}
\end{aligned}$$

By the third Kähler identity, one has

$$\partial^* = \sqrt{-1}[\Lambda,\overline{\partial}] = \sqrt{-1}\Lambda\overline{\partial} - \sqrt{-1}\overline{\partial}\Lambda$$

then

$$\begin{aligned}
\overline{\partial}\partial^* &= \overline{\partial}(\sqrt{-1}\Lambda\overline{\partial} - i\overline{\partial}\Lambda) = \sqrt{-1}\cdot\overline{\partial}\Lambda\overline{\partial} \\
\partial^*\overline{\partial} &= (\sqrt{-1}\Lambda\overline{\partial} - \sqrt{-1}\cdot\overline{\partial}\Lambda)\overline{\partial} = -\sqrt{-1}\cdot\overline{\partial}\Lambda\overline{\partial} = -\overline{\partial}\partial^*
\end{aligned}$$

Now we have

$$\begin{aligned}
\Delta_{\mathrm{d}} &= \Delta_\partial - \sqrt{-1}\cdot\overline{\partial}\Lambda\partial - \sqrt{-1}\Lambda\partial\overline{\partial} + i\overline{\partial}\partial\Lambda + i\partial\Lambda\overline{\partial} \\
&= \Delta_\partial + \sqrt{-1}(\Lambda\overline{\partial}\partial - \overline{\partial}\Lambda\partial) + i(\partial\Lambda\overline{\partial} - \partial\overline{\partial}\Lambda) \\
&= \Delta_\partial + \sqrt{-1}[\Lambda,\overline{\partial}]\partial + \sqrt{-1}\partial[\Lambda,\overline{\partial}] \\
&= \Delta_\partial + \partial^*\partial + \partial\partial^* \\
&= 2\Delta_\partial
\end{aligned}$$

□

Corollary 9.4.1. For a compact Kähler manifold, Δ_{d}-harmonic is equivalent to Δ_∂-harmonic, and is equivalent to $\Delta_{\overline{\partial}}$-harmonic.

Exercise 9.4.1. Show that for compact Kähler manifold we have

$$\begin{aligned}
[\Delta_{\mathrm{d}}, L] &= 0 \\
[L,\Lambda] &= (k-n)\,\mathrm{id} \quad \text{on } C^\infty(X,\Omega^k_{X,\mathbb{C}})
\end{aligned}$$

Proof. For the first one, we have $\Delta_{\mathrm{d}} = 2\Delta_\partial = 2(\partial\partial^* + \partial^*\partial)$. Thus

$$[\Delta_{\mathrm{d}}, L] = 2([\partial\partial^*, L] + [\partial^*\partial, L]) = 2(\partial[\partial^*, L] + [\partial^*, L]\partial)$$

The last equality holds by the fact that L commutes with ∂, since ω is ∂-closed. Now we use the identity $[\partial^*, L] = -\sqrt{-1}\cdot\overline{\partial}$, which anticommutes with ∂ to conclude.

For the second one, without lose of generality it suffices to check on $U\subseteq\mathbb{C}^n$ equipped with standard hermitian metric, since we are considering operators of order zero. Recall that L is the exterior product with $\omega = \frac{\sqrt{-1}}{2}\sum_i \mathrm{d}z^i\wedge\mathrm{d}\overline{z}^i$. Let A_i be the operator given by the exterior product with $\frac{\sqrt{-1}}{2}\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^i$, then $[L,\Lambda] = \sum_{i=1}^n [A_i,\Lambda]$, which motivates us to compute term by term. □

Corollary 9.4.2. Let (X,ω) be a Kähler manifold, and α a (p,q)-form, then $\Delta_{\mathrm{d}}\alpha$ is still a (p,q)-form.

Proof. $\Delta_\partial\alpha$ is still a (p,q)-form is a clear fact. □

9.5. **Hodge decomposition.**

Theorem 9.5.1. Let (X,h) be a compact Kähler manifold, $\alpha = \sum_{p+q=k}\alpha^{p,q}$. Then α is harmonic if and only if $\alpha^{p,q}$ is harmonic, that is

$$\mathcal{H}^k\otimes_{\mathbb{R}}\mathbb{C} = \bigoplus_{p+q=k}\mathcal{H}^{p,q}$$

with $\overline{\mathcal{H}^{p,q}} = \mathcal{H}^{q,p}$.

Proof. It follows from Δ_{d} preserves bidegree. □

Theorem 9.5.2 (Hodge decomposition)**.** Let (X, h) be a compact Kähler manifold. Then

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X)$$

with $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

Proof. It follows from there are natural isomorphisms $H^k(X, \mathbb{C}) \cong \mathcal{H}^k \otimes \mathbb{C}$ and $H^{p,q}(X) \cong \mathcal{H}^{p,q}$. $\square$

Corollary 9.5.1. Let (X, h) be a compact Kähler manifold, then

$$b_k = \sum_{p+q=k} h^{p,q}$$

with $h^{p,q} = h^{q,p}$, where $b_k = \dim H^k(X, \mathbb{C})$ and $h^{p,q} = \dim H^{p,q}(X)$.

Corollary 9.5.2. b_k is even when k is odd.

Corollary 9.5.3. $b_k \neq 0$ when k is even.

Proof. $h^{k,k} \neq 0$, since $0 \neq \omega^k \in H^{k,k}(X)$. $\square$

There are many relations between $h^{p,q}$, and we can draw a picture as follows, called Hodge diamond, since it has the same symmetry as a diamond.

$$\begin{array}{cccccccccc}
 & & & & h^{0,0} & & & & & b_0 \\
 & & & h^{1,0} & & h^{0,1} & & & & b_1 \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & & b_2 \\
 & \iddots & & & \vdots & & & \ddots & & \vdots \\
\text{Hodge} \updownarrow & h^{n,0} & & \cdots & \underset{\text{Serre}}{\curvearrowleft} & \cdots & & & h^{0,n} & b_n \\
 & \ddots & & & \vdots & & & \iddots & & \vdots \\
 & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & & b_{2n-2} \\
 & & & h^{n,n-1} & & h^{n-1,n} & & & & b_{2n-1} \\
 & & & & h^{n,n} & & & & & b_{2n} \\
 & & & & \underset{\text{conjugation}}{\longleftrightarrow} & & & & &
\end{array}$$

Example 9.5.1.

$$H^{p,q}(\mathbb{CP}^n) = \begin{cases} \mathbb{C} & 0 \leq p = q \leq n \\ 0 & \text{otherwise} \end{cases}$$

Proof. It's known to all that the singular cohomology of $\mathbb{CP}^n$ with complex coefficient is

$$H^k(\mathbb{CP}^n, \mathbb{C}) = \begin{cases} \mathbb{C} & k \text{ is even} \\ 0 & k \text{ is odd} \end{cases}$$

Thus it's clear to compute Dolbeault cohomology of $\mathbb{CP}^n$ using the symmetry of Hodge diamond. $\square$

9.6. **Bott-Chern cohomology.** Review what we have done: We have already proven one of the main theorems in this course, that is, Hodge decomposition. But along the way we used the Kähler metric on a Kähler manifold, a question is that: (In)denpendence of the Kähler metric? The anwser is that our decomposition is independent of the choice of Kähler metric, shown by Bott-Chern cohomology.

Definition 9.6.1 (Bott-Chern cohomology)**.** Let X be a complex manifold, the Bott-Chern cohomology is defined as

$$H^{p,q}_{\mathrm{BC}}(X) := \frac{Z^{p,q}_{\mathrm{BC}} := \{\alpha \in C^\infty(X, \Omega^{p,q}_X) \mid \mathrm{d}\alpha = 0\}}{\partial\overline{\partial} C^\infty(X, \Omega^{p-1,q-1}_X)}$$

Remark 9.6.1. There is a natural map

$$Z^{p,q}_{\mathrm{BC}}(X) \to H^{p+q}(X, \mathbb{C})$$

which descends to

$$H^{p,q}_{\mathrm{BC}}(X) \to H^{p+q}(X, \mathbb{C})$$

since $\partial\overline{\partial}\beta = \mathrm{d}\overline{\partial}\beta$. On the other hand, there is also a natural map

$$Z^{p,q}_{\mathrm{BC}}(X) \to H^{p,q}(X)$$

which descends to

$$H^{p,q}_{\mathrm{BC}}(X) \to H^{p,q}(X)$$

since $\partial\overline{\partial}\beta = -\overline{\partial}\partial\beta$. So if we can prove there are isomorphisms between

$$H^{p,q}_{\mathrm{BC}}(X) \cong H^{p,q}(X)$$

$$\bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X) \cong H^k(X, \mathbb{C})$$

then Hodge decomposition is canonical, that is independent of choice of Kähler metric, since Bott-Chern cohomology is independent of the choice of Kähler metric.

Lemma 9.6.1 ($\partial\overline{\partial}$-lemma)**.** Let (X, ω) be a compact Kähler manifold, α a d-closed (p, q)-form. If α is $\overline{\partial}$-exact or ∂-exact, then there exists a $(p-1, q-1)$-form such that

$$\alpha = \partial\overline{\partial}\beta$$

Proof. Suppose α is $\overline{\partial}$-exact, then $\alpha = \overline{\partial}\gamma$ for some $(p, q-1)$-form γ, and Hodge's theorem implies γ has decomposition

$$\gamma = a + \partial b + \partial^* c$$

where a is Δ_∂-harmonic, and b, c are forms with appropriate degrees. Direct computation shows

$$\begin{aligned}\alpha = \overline{\partial}\gamma &= \overline{\partial}a + \overline{\partial}\partial b + \overline{\partial}\partial^* c \\ &= -\partial\overline{\partial}b + \overline{\partial}\partial^* c \\ &= -\partial\overline{\partial}b - \partial^*\overline{\partial}c\end{aligned}$$

Now it suffices to show $-\partial^*\overline{\partial}c = 0$. A trick here is to note that

$$0 = \partial\alpha = -\partial\partial^*\overline{\partial}c \implies \partial^*\overline{\partial}c \in \ker\partial \cap \operatorname{im}\partial^* = 0 \implies \partial^*\overline{\partial}c = 0$$

So we have

$$\alpha = \partial\overline{\partial}(-b)$$

as desired. □

Corollary 9.6.1. Let (X, ω) be a compact Kähler manifold, then
1. $H^{p,q}_{\mathrm{BC}}(X) \to H^{p,q}(X)$ is an isomorphism;
2. $\bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X) \to H^k(X, \mathbb{C})$ is an isomorphism.

Proof. Here we only prove the first isomorphism. From Remark 9.6.1, there is a canonical map $H^{p,q}_{\mathrm{BC}}(X) \to H^{p,q}(X)$, and if we choose a Kähler metric, we have $H^{p,q}(X) \cong \mathcal{H}^{p,q}$, we will show our canonical map is both surjective and injective via this chosen metric.
1. To see surjectivity: For element in $H^{p,q}(X)$ we choose a $\Delta_{\overline{\partial}}$-harmonic representative. Since Δ_{∂}-harmonic is equivalent to Δ_{d}-harmonic, so this representative is also d-closed.
2. To see injectivity: Suppose we have $[\alpha] \in H^{p,q}_{\mathrm{BC}}(X)$ such that α is trivial in $H^{p,q}(X)$, that is $\overline{\partial}$-exact, then $\partial\overline{\partial}$-lemma implies it's trivial in Bott-Chern cohomology.

□

Corollary 9.6.2. A hermitian metric ω is Kähler if and only if it can be written locally as

$$\omega = \sqrt{-1}\partial\overline{\partial}f$$

where f is a real-valued smooth function.

Proof. It's clear if ω is locally written as $\sqrt{-1}\partial\overline{\partial}f$, then it gives a Kähler metric. Conversely, a Kähler metric ω is an element in $H^{1,1}(X)$, and we have already shown that $H^{1,1}(X) = H^{1,1}_{\mathrm{BC}}(X)$, and Dolbeault lemma implies Dolbeault cohomology vanishes on open subset which is sufficiently small, this completes the proof. □

10. Applications of Hodge theory

10.1. **Adjoint operators on bundle valued forms.** Let(E,h) be a hermitian holomorphic vector bundle on hermitian n-manifold X, we can generalize what have done before. More explicitly, for $\varphi,\psi\in C^\infty(X,\Omega_X^{p,q}\otimes E)$, locally written as $\varphi=\varphi^\alpha e_\alpha,\psi=\psi^\beta e_\beta$, then local inner product is given by

$$\langle\varphi,\psi\rangle:=h_{\alpha\overline{\beta}}\langle\varphi^\alpha,\psi^\beta\rangle$$

and $\varphi\wedge\psi$ is defined as

$$\varphi\wedge\overline{\psi}:=\varphi^\alpha\wedge\overline{\psi^\beta}h_{\alpha\overline{\beta}}$$

The inner product on $C^\infty(X,\Omega_X^{p,q}\otimes E)$ is given by

$$(\varphi,\psi):=\int_X\langle\varphi,\psi\rangle\frac{\omega^n}{n!}$$

where $\varphi,\psi\in C^\infty(X,\Omega_X^{p,q}\otimes E)$. The hodge star operator is defined as an operator

$$\star\colon C^\infty(X,\Omega_X^{p,q}\otimes E)\to C^\infty(X,\Omega_X^{n-q,n-p}\otimes E)$$

such that

$$(\varphi,\psi)=\int_X\varphi\wedge\star\overline{\psi}$$

Let ∇ be the Chern connection of (E,h), then $\nabla^{0,1}=\overline{\partial}_E$, and if we set $\nabla^{1,0}=\partial_E$, then

$$\begin{aligned}\Theta_h&=\nabla^2\\&=\partial_E^2+\partial_E\overline{\partial}_E+\overline{\partial}_E\partial_E+\overline{\partial}_E^2\\&=[\partial_E,\overline{\partial}_E]\end{aligned}$$

Exercise 10.1.1. Give formulas of ∂_E^* and $\overline{\partial}_E^*$ in terms of $\overline{\star}_E$.

Laplacians $\Delta_{\partial_E},\Delta_{\overline{\partial}_E}$ can be defined in a same way, and there is also a Hodge theorem, which gives the following decomposition

$$C^\infty(X,\Omega_X^{p,q}\otimes E)=\mathcal{H}^{p,q}(X,E)\oplus\operatorname{im}\overline{\partial}_E\oplus\operatorname{im}\overline{\partial}_E^*$$

and by the same argument we have

$$H^{p,q}(X,E)\cong\mathcal{H}^{p,q}$$

$$\begin{aligned}L\colon C^\infty(X,\Omega^{p,q}\otimes E)&\to C^\infty(X,\Omega^{p+1,q+1}\otimes E)\\\alpha&\mapsto\omega\wedge\alpha\end{aligned}$$

and Λ_ω is the formal adjoint of L. If (X,ω) is also a Kähler n-manifold, then there are also Kähler identities

$$[\overline{\partial}_E^*, L] = \sqrt{-1}\partial$$
$$[\partial_E^*, L] = -\sqrt{-1}\cdot\overline{\partial}$$
$$[\Lambda_\omega, \partial] = -\sqrt{-1}\partial^*$$
$$[\Lambda_\omega, \partial] = \sqrt{-1}\cdot\overline{\partial}^*$$

and

$$[L, \Lambda_\omega] = (p+q-n)\,\mathrm{id}$$

holds on E-valued (p,q)-forms.

10.2. **Serre duality.**

Theorem 10.2.1 (Serre duality)**.** Let X be a compact complex n-manifold, E a holomorphic vector bundle, then there exists a non-degenerate $\mathbb{C}$-linear pairing

$$H^{p,q}(X,E)\times H^{n-p,n-q}(X,E^*)\to\mathbb{C}$$
$$([\alpha],[\beta])\mapsto\int_X\alpha\wedge\beta$$

In particular, we have

$$H^{p,q}(X,E) = H^{n-p,n-q}(X,E^*)^*$$

Sketch. firstly endow E with a hermitian metric h, and show

$$\Delta_{\overline{\partial}_E^*}\circ\overline{\star}_E = \overline{\star}_E\circ\Delta_{\overline{\partial}_E}$$

then

$$\overline{\star}_E\colon\mathcal{H}^{p,q}(X,E)\xrightarrow{\cong}\mathcal{H}^{n-p,n-q}(X,E^*)$$

and Hodge theorem implies that

$$\mathcal{H}^{p,q}(X,E)\cong H^{p,q}(X,E)$$

For all $\alpha\in\mathcal{H}^{p,q}(X,E),\beta\in\mathcal{H}^{n-p,n-q}(X,E^*)$, we have $\beta=\overline{\star}_E\gamma$ for some $\gamma\in\mathcal{H}^{p,q}(X,E)$, then

$$\int_X\alpha\wedge\beta = \int_X\alpha\wedge\overline{\star}_E\gamma = \langle\alpha,\gamma\rangle$$

is non-degenerate. □

Corollary 10.2.1. Let X be a compact complex n-manifold, E a holomorphic vector bundle, then

$$H^{p,q}(X) = H^{n-p,n-q}(X)^*$$

Proof. Consider $E=\mathcal{O}_X$ in Serre duality, then desired result holds from the fact $\mathcal{O}_X^*=\mathcal{O}_X$. □

Remark 10.2.1. This recovers Corollary 9.2.6.

Corollary 10.2.2. Let X be a compact complex n-manifold, E a holomorphic vector bundle, then

$$H^q(X,E)=H^{n-q}(X,K_X\otimes E^*)^*$$

Proof. Set $p=0$ in Serre duality one has

$$H^{0,q}(X,E)\cong H^{n,n-q}(X,E^*)^*$$

which gives desired result. □

10.3. **Vanishing theorems.**

10.3.1. *Kodaira vanishing theorem.*

Theorem 10.3.1 (Bochner-Kodaira-Nakano identity)**.**

$$\Delta_{\overline{\partial}_E}=[\sqrt{-1}\Theta_h,\Lambda]+\Delta_{\partial_E}$$

Proof. Direct computation shows

$$\begin{aligned}\Delta_{\overline{\partial}_E}&=[\overline{\partial}_E,\overline{\partial}_E^*]\\&=-\sqrt{-1}[\overline{\partial}_E,[\Lambda,\partial_E]]\\&=-\sqrt{-1}[\Lambda,[\partial_E,\overline{\partial}_E]]-\sqrt{-1}[\partial_E,[\overline{\partial}_E,\Lambda]]\\&=-\sqrt{-1}[\Lambda,\Theta_h]-\sqrt{-1}[\partial_E,\sqrt{-1}\partial_E^*]\\&=[\sqrt{-1}\Theta_h,\Lambda]+\Delta_{\partial_E}\end{aligned}$$

□

Corollary 10.3.1 (Bochner-Kodaira-Nakano inequality)**.** For $\alpha\in C^\infty(X,\Omega_X^{p,q}\otimes E)$, one has

$$([\sqrt{-1}\Theta_h,\Lambda]\alpha,\alpha)\leq(\Delta_{\overline{\partial}_E}\alpha,\alpha)$$

In particular, if α is $\Delta_{\overline{\partial}_E}$-harmonic, then $([\sqrt{-1}\Theta_h,\Lambda]\alpha,\alpha)\leq 0$.

Proof. Direct computation shows

$$\begin{aligned}(\Delta_{\overline{\partial}_E}\alpha,\alpha)-([\sqrt{-1}\Theta_h,\Lambda]\alpha,\alpha)&=(\Delta_{\partial_E}\alpha,\alpha)\\&=\|\partial_E\alpha\|^2+\|\partial_E^*\alpha\|^2\geq 0\end{aligned}$$

□

Theorem 10.3.2 (Kodaira-Akizuki-Nakano vanishing)**.** Let X be a compact n-manifold, (L,h) a positive hermitian holomorphic line bundle. Then

$$H^{p,q}(X,L)=0$$

for $p+q>n$.

Proof. Let X be endowed with Kähler metric ω given by Chern curvature of L, then there is an isomorphism $H^{p,q}(X,L)\cong\mathcal{H}^{p,q}(X,L)$. For $\alpha\in\mathcal{H}^{p,q}(X,L)$, Corollary 10.3.1 implies

$$[\sqrt{-1}\Theta_h,\Lambda_\omega]\alpha\leq 0$$

On the other hand,

$$([\sqrt{-1}\Theta_h, \Lambda_\omega]\alpha, \alpha) = 2\pi(p+q-n)\|\alpha\|^2 \geq 0$$

Thus if $p+q>n$, one has $\alpha = 0$, this completes the proof. □

Corollary 10.3.2 (Kodaira vanishing)**.** Let X be a compact n-manifold, (L,h) a positive holomorphic line bundle over X. Then

$$H^q(X, K_X \otimes L) = 0$$

for $q>0$.

Proof. Just note that

$$H^q(X, K_X \otimes L) = H^{n,q}(X, L)$$

□

Corollary 10.3.3. Let (X,ω) be a compact Kähler n-manifold. If (L,h) is a semi-positive line bundle and $\operatorname{rk}\Theta_h \geq k$, then

$$H^{p,q}(X, L) = 0$$

for $p+q \geq 2n-k+1$.

Exercise 10.3.1. Compute all $H^q(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(k))$ for all k, q.

Definition 10.3.1 (Fano)**.** A Fano manifold is a compact Kähler manifold with positive anti-canonical bundle $K_X^* = \det T_X$.

Proposition 10.3.1. Let X be a Fano manifold, then

$$H^q(X, \mathcal{O}_X) = 0$$

for all $q>0$.

Proof. Note that $\mathcal{O}_X = K_X \otimes K_X^*$. □

Theorem 10.3.3 (Serre vanishing)**.** Let X be a compact complex n-manifold, (L,h) a positive holomorphic line bundle over X. For any holomorphic vector bundle E on X, there exists a constant m_0 such that for all $m \geq m_0$

$$H^q(X, E\otimes L^{\otimes m}) = 0$$

for $q>0$.

Proof. X is endowed with Kähler metric ω given by Chern curvature of L and E is endowed with a hermitian metric h, then $H^{p,q}(X, E\otimes L^{\otimes m}) \cong \mathcal{H}^{p,q}(X, E\otimes L^{\otimes m})$. For $\alpha \in \mathcal{H}^{p,q}(X, E\otimes L^{\otimes m})$, one has

$$\begin{aligned}([\sqrt{-1}\Theta_h, \Lambda_\omega]\alpha, \alpha) + 2\pi m(p+q-n)\|\alpha\|^2 &\overset{(1)}{=} ([\sqrt{-1}\Theta_{E\otimes L^{\otimes m}}, \Lambda_\omega]\alpha, \alpha) \\ &\overset{(2)}{\leq} 0\end{aligned}$$

where

(1) holds from $\Theta_{E\otimes L^{\otimes m}} = \Theta^E \otimes \mathrm{id} + m(\mathrm{id}\otimes \Theta^L)$;

(2) holds from Corollary 10.3.1, that is Bochner-Kodaira-Nakano inequality.

On the other hand, Cauchy inequality implies that

$$([\sqrt{-1}\Theta_h, \Lambda_\omega]\alpha, \alpha) \geq -C\|\alpha\|^2$$

where constant C is the norm of $[\sqrt{-1}\Theta_h, \Lambda_\omega]$. So if we have $2\pi m(p+q-n) - C > 0$, the argument in proof of Kodaira vanishing theorem implies $\alpha = 0$. Consider $p = n, q > 0, m_0 \geq \frac{C}{2\pi}$, then for all $m \geq m_0$ and $q > 0$, one has

$$H^{n,q}(X, E \otimes L^{\otimes m}) = 0$$

that is to say $H^q(X, K_X \otimes E \otimes L^{\otimes m}) = 0$. So in order to show $H^q(X, E \otimes L^{\otimes m}) = 0$, it suffices to consider $K_X^* \otimes E$ at beginning, then we will obtain

$$H^q(X, K_X \otimes K_X^* \otimes E \otimes L^{\otimes m}) = H^q(X, E \otimes L^{\otimes m}) = 0$$

This completes the proof. □

10.3.2. *Nakano vanishing theorem.*

Theorem 10.3.4 (Nakano vanishing)**.** Let X be a compact complex manifold

1. If (E, h) is Nakano positive, then
$$H^{n,q}(X, E) = 0$$
for $q \geq 1$.
2. If (E, h) is dual Nakano positive, then
$$H^{p,n}(X, E) = 0$$
for $p \geq 1$.

Corollary 10.3.4. Let (X, h) be a Kähler n-manifold with $n \geq 2$, then (T_X, h) is not Nakano positive.

Proof. Suppose (T_X, h) is Nakano positive, then

$$\begin{aligned} H^{1,1}(X) &= H^{0,1}(X, T_X^*) \\ &\overset{(1)}{=} H^{n,n-1}(X, T_X) \\ &\overset{(2)}{=} 0 \end{aligned}$$

where

(1) holds from Serre duality;
(2) holds from Nakano vanishing theorem.

This leads to a contradiction, since $H^{1,1}(X) \neq 0$ for a Kähler manifold. □

Corollary 10.3.5.

Example 10.3.1. If $n \geq 2$, $(\mathbb{CP}^n, \omega_{FS})$

(1) is dual Nakano positive;
(2) is semi Nakano positive, but not Nakano positive.

Proof. Note that
$$R_{i\bar{j}k\bar{l}} = g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}$$
For (1). Direct computation shows
$$R_{i\bar{j}k\bar{l}}u^{il}\overline{u}^{jk}$$
For (2). Direct computation shows
$$\begin{aligned} R_{i\bar{j}k\bar{l}}u^{ik}\overline{u}^{jl} &= g_{i\bar{j}}g_{k\bar{l}}u^{ik}\overline{u}^{jl} + g_{i\bar{l}}g_{k\bar{j}}u^{ik}\overline{u}^{jl} \\ &= \sum |u^{ik}|^2 + \sum u^{ij}\overline{u}^{ji} \\ &= \frac{1}{2}\sum |u^{ij} + u^{ji}|^2 \end{aligned}$$
$\square$

11. Lefschetz

11.1. Lefschetz decomposition.

Proposition 11.1.1. Let (X,ω) be a Kähler n-manifold, then $L^{n-k}\colon C^\infty(X,\Omega^k_{X,\mathbb{R}})\to C^\infty(X,\Omega^{2n-k}_{X,\mathbb{R}})$ is an isomorphism for $k\leq n$.

Proof. Since $\Omega^k_{X,\mathbb{R}}$ has the same rank with $\Omega^{2n-k}_{X,\mathbb{R}}$, it suffices to show L^{n-k} is injective, and it suffices to show that it's injective in each fibre. Recall that $L^*=\Lambda=(-1)^k * L* : \Omega^k_{X,\mathbb{R}}\to\Omega^{k-2}_{X,\mathbb{R}}$, and

$$[L,\Lambda]\alpha=(n-k)\alpha,\quad \forall\alpha\in C^\infty(X,\Omega^k_{X,\mathbb{R}})$$

then

$$\begin{aligned}[L^r,\Lambda]&=L^r\Lambda-\Lambda L^r\\&=L(L^{r-1}\Lambda-\Lambda L^{r-1})+L\Lambda L^{r-1}-\Lambda LL^{r-1}\\&=L[L^{r-1},\Lambda]+[L,\Lambda]L^{r-1}\end{aligned}$$

So we can prove the following identity by induction on r:

$$[L^r,\Lambda]\alpha=(r(k-n)+r(r-1))L^{r-1}\alpha,\quad\forall\alpha\in C^\infty(X,\Omega^k_{X,\mathbb{R}})$$

Suppose $L^r\alpha=0, r\leq n-k$, then

$$\begin{aligned}L^r\Lambda\alpha&=[L^r,\Lambda]\alpha\\&=(r(k-n)+r(r-1))L^{r-1}\alpha\end{aligned}$$

So we have

$$L^{r-1}(L\Lambda\alpha-(r(k-n)+r(r-1))\alpha)=0$$

In other words, from $\alpha\in\ker L^r$, we get something in $\ker L^{r-1}$. Clearly L is injective, so we have the following identity by induction on r:

$$L\Lambda\alpha=(r(k-n)+r(r-1))\alpha$$

Let $\beta=\Lambda\alpha$, and apply L^r to both side we have

$$L^{r+1}\beta=(r(k-n)+r(r-1))L^r\alpha=0$$

where $\beta\in C^\infty(X,\Omega^{k-2}_{X,\mathbb{R}})$. By induction on k, we have $\beta=0$, so we have $\alpha=0$. □

Remark 11.1.1. From above proof, we can see all $L^r, r\leq n-k$ are injective.

Definition 11.1.1 (primitive form)**.** Let (X,ω) be a Kähler n-manifold. For $k\leq n$, $\alpha\in C^\infty(X,\Omega^k_{X,\mathbb{R}})$ is called primitive if $L^{n-k+1}\alpha=0$.

Exercise 11.1.1. $\alpha\in C^\infty(X,\Omega^k_{X,\mathbb{R}}), k\leq n$ is primitive if and only if $\Lambda\alpha=0$.

Proof. Let's first see what will happen for a primitive n-form α. α is primitive if and only if $L\alpha=0$. Recall that Exercise 9.4.1 implies that

$$[L,\Lambda]=(k-n)\,\mathrm{id},\quad\text{on } C^\infty(X,\Omega^k_{X,\mathbb{C}})$$

So if $k = n$, then L and Λ commutes, so we have α is primitive if and only if $\Lambda\alpha = 0$, since

$$\Lambda\alpha = 0 \Longleftrightarrow L\Lambda\alpha = 0 \Longleftrightarrow \Lambda L\alpha = 0 \Longleftrightarrow L\alpha = 0$$

and the first and last equality we use the fact that L is injective on $\Omega^k_{X,\mathbb{R}}, k \leq n$ and Λ is injective on $\Omega^{n+2}_{X,\mathbb{R}}$. In general case, we have

$$[L^r, \Lambda]\alpha = (r(k-n) + r(r-1))L^{r-1}\alpha$$

and in particular for $r = n - k + 1$ where k is the degree of α, we have

$$[L^r, \Lambda]\alpha = 0$$

The argument can be repeated to conclude. □

Proposition 11.1.2. For all $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}}), k \leq n$. Then there exists a unique decomposition

$$\alpha = \sum_r L^r \alpha_r$$

with $\alpha_r \in C^\infty(X, \Omega^{k-2r}_{X,\mathbb{R}})$ is primitive.

Proof. Uniqueness. Suppose $\sum_r L^r\alpha_r = 0$ with primitive α_r. We want to show $\alpha_r = 0$. If not, then take the largest r_m such that $\alpha_{r_m} \neq 0$. By our choice, L^{n-k+r_m} kills everything in $\sum_r L^r\alpha_r$ but $L^{r_m}a_{r_m}$, so

$$0 = L^{n-k+r_m}(\sum_r L^r\alpha_r) = L^{n-k+r_m}(L^{r_m}\alpha_{r_m}) \neq 0$$

A contradiction.

Existence. For $L^{n-k+1}\alpha \in C^\infty(X, \Omega^{2n-k+2}_{X,\mathbb{R}})$, then there exists $\beta \in C^\infty(X, \Omega^{k-2}_{X,\mathbb{R}})$ such that

$$L^{n-k+1}\alpha = L^{n-k+2}\beta \implies L^{n-k+1}(\alpha - L\beta) = 0$$

So $\alpha - L\beta$ is primitive, that is

$$\alpha = (\alpha - L\beta) + L\beta$$

By induction on k, we have primitive decomposition for $\beta \in C^\infty(X, \Omega^{k-2}_{X,\mathbb{R}})$. This completes the proof. □

Remark 11.1.2. Set $H = [L, \Lambda]$, we have proven that (L, H, Λ) generates an $\mathfrak{sl}_2$-action on $\bigoplus_k \Omega^k_{X,\mathbb{R}}$, hence on $\bigoplus_k C^\infty(X, \Omega^k_{X,\mathbb{R}})$.

In cohomology, we can define the following map

$$\begin{aligned} L\colon H^k(X, \mathbb{R}) &\to H^{k+2}(X, \mathbb{R}) \\ [\alpha] &\mapsto [\omega \wedge \alpha] \end{aligned}$$

Clearly, it's well-defined, it suffices to check that L maps closed forms to closed forms and exact forms to exact forms. If α is closed, then

$$\mathrm{d}(\omega \wedge \alpha) = \mathrm{d}\omega \wedge \alpha + \omega \wedge \mathrm{d}\alpha = 0$$

since ω is closed. And if $\alpha = \mathrm{d}\beta$, then

$$\omega \wedge \mathrm{d}\beta = \mathrm{d}\omega \wedge \beta + \omega \wedge \mathrm{d}\beta = \mathrm{d}(\omega \wedge \beta)$$

So, as you can see, L is well-defined mainly relys on the fact that ω is closed.

Theorem 11.1.1 (Hard Lefschetz theorem[13])**.** Let (X, ω) be a compact Kähler n-manifold, then

$$L^{n-k} : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R})$$

is an isomorphism for $k \leq n$.

Proof. In exercise we have shown $[\Delta_{\mathrm{d}}, L] = 0$, then

$$L^{n-k} \colon \mathcal{H}^k \to \mathcal{H}^{2n-k}$$

By the fact that L^{n-k} is injective and $\mathcal{H}^k, \mathcal{H}^{2n-k}$ have the same dimension, we obtain the desired result. □

Definition 11.1.2 (primitive form)**.** Let (X, ω) be a compact Kähler n-manifold. For $[\alpha] \in H^k(X, \mathbb{R})$, it's called primitive, if $L^{n-k+1}[\alpha] = 0$.

Notation 11.1.1. $H^k(X, \mathbb{R})_{\mathrm{prim}}$ denotes the set of all primitive forms.

Corollary 11.1.1 (Lefschetz decomposition)**.** There is the following decomposition

$$H^k(X, \mathbb{R}) = \bigoplus_r L^r H^{k-2r}(X, \mathbb{R})_{\mathrm{prim}}$$

Remark 11.1.3. If $[\omega] \in H^2(X, \mathbb{Z})$, such as ω comes from a positive holomorphic line bundle. Then we can state theorem and corollary for $H^k(X, \mathbb{Q})$.

Moreover, we have the following isomorphism

$$L^{n-k} : H^{p,q}(X) \to H^{n-q,n-p}(X)$$

for $k = p + q \leq n$.

Corollary 11.1.2. Let (X, ω) be a compact Kähler n-manifold, then $b_{k-2} \leq b_k$ and $h^{p-1,q-1} \leq h^{p,q}$ for $k = p + q \leq n$.

[13]Though proof of this theorem is quite easy using tools we have, but it's quite hard for Lefschetz, since during his time, there is no Hodge! And we use L to denote this operator, in order to honor Lefschetz.

Part 5. Appendix

Appendix A. Sheaf and its Cohomology

A.1. Definition and first properties.

Definition A.1.1 (sheaf)**.** Let X be a topological space. A sheaf of abelian group $\mathscr{F}$ on X is the data of:

1. For any open subset U of X, $\mathscr{F}(U)$ is an abelian group.
2. If $U \subset V$ are two open subsets of X, then there is a group homomorphism $r_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$, such that
   (*a*) $\mathscr{F}(\varnothing) = 0$;
   (*b*) $r_{UU} = \text{id}$;
   (*c*) If $W \subset U \subset V$, then $r_{UW} = r_{VW} \circ r_{UV}$;
   (*d*) $\{V_i\}$ is an open covering of $U \subset X$, and $s \in \mathscr{F}(U)$. If $s|_{V_i} := r_{UV_i}(s) = 0, \forall i$, then $s = 0$;
   (*e*) $\{V_i\}$ is an open covering of $U \subset X$, and $s_i \in \mathscr{F}(V_i)$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there exists $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$.

Definition A.1.2 (presheaf)**.** A sheaf which fails to meet $(d), (e)$ is called a presheaf.

Example A.1.1 (constant presheaf)**.** Let G be abelian group, the constant presheaf assign each open set U the group G itself. However, it's not a sheaf. Indeed, consider $U = U_1 \cup U_2$ with $U_1 \cap U_2 = \varnothing$. Consider $g_1 \in \mathscr{F}(U_1) = G, g_2 \in \mathrm{F}(U_2), g_1 \neq g_2$, then one can't find $g \in \mathscr{F}(U)$ such that $g|_{U_1} = g_1, g|_{U_2} = g_2$, since $g_1 \neq g_2$.

Definition A.1.3 (morphism of sheaves)**.** $\phi : \mathscr{F} \to \mathscr{G}$ is called a morphism of sheaves, if for any open subset U of X, there is a group homomorphism $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$, such that if $U \subset V$ are two open subsets of X, the the following diagram commutes

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\phi(U)} & \mathscr{G}(U) \\ \downarrow{\scriptstyle r_{UV}} & & \downarrow{\scriptstyle r_{UV}} \\ \mathscr{F}(V) & \xrightarrow{\phi(V)} & \mathscr{G}(V) \end{array}$$

Remark A.1.1. For convenience, for $s \in \mathscr{F}(U)$, we often write $\varphi(s)$ instead of $\varphi(U)(s)$.

A.2. **Sheafification.** In this section we will consider sheafification. Recall Example A.1.1, we encounter a presheaf which is not a sheaf. So we may wonder how can we get a sheaf from this presheaf? And that's sheafification.

There are too many ways to define sheafification. One way is to define by its universal property:

Definition A.2.1 (sheafification)**.** Given a presheaf $\mathscr{F}$ there is a sheaf $\mathscr{F}^+$ and a morphism $\theta : \mathscr{F} \to \mathscr{F}^+$ with the property that for any sheaf $\mathscr{G}$ and any morphism $\varphi : \mathscr{F} \to \mathscr{G}$ there is a unique morphism $\overline{\varphi} : \mathscr{F}^+ \to \mathscr{G}$ such that $\varphi = \overline{\varphi} \circ \theta$. In other words, the following diagram commutes:

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\varphi"] \arrow[d, "\theta"] & \mathscr{G} \\ \mathscr{F}^+ \arrow[ur, "\overline{\varphi}"] & \end{tikzcd}$$

Although the universal property shows that if the sheafification exists, it's determined uniquely up to unique isomorphism, how can we show that there do exists a sheafification?

To give an explicit construction, we need to consider stalks of a presheaf.

Definition A.2.2 (stalks)**.** For a presheaf $\mathscr{F}$ and $x \in X$, stalk at x is defined as

$$\mathscr{F}_x = \varinjlim_{x \in U} \mathscr{F}(U)$$

Remark A.2.1 (alternative definition)*.* In order to avoid language of direct limit, we give a more useful and equivalent description of stalk: an element $s_x \in \mathscr{F}_x$, which is called a germ, is an equivalence class $[s_U]$, where $s_U \in \mathscr{F}(U)$ and $x \in U$. Two such sections s_U and s_V are considered equivalent if the restrictions of the two sections coincide on some neighborhood of x. For $s \in \mathscr{F}(U), x \in U$, we use $s|_x$ to denote its equivalence class.

Remark A.2.2 (morphisms on stalks)*.* Given a morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$, it induces a morphism of abelian groups $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ as follows:

$$\begin{aligned} \varphi_p : \mathscr{F}_p &\to \mathscr{G}_p \\ s_p &\mapsto \varphi(s)|_p \end{aligned}$$

and it's easy to check φ_p is well-defined.

As you can imagine, stalks are quite local information, and the difference between sheaf and presheaf is that whether a local information can glue together uniquely or not. So stalks of presheaf and its sheafification should be the same. And one way to construct sheafification is to glue stalks together in a suitable way.

Construct $\mathscr{F}^+(U)$ as a set of functions

$$f : U \to \coprod_{p \in U} \mathscr{F}_p$$

such that $f(p) \in \mathscr{F}_p$ and for every $p \in U$ there is an open set $p \in V_p \subseteq U$ and $t \in \mathscr{F}(V_p)$ such that for all $q \in V_p$ we have the germ $t|_q = f(q)$.

$\mathscr{F}^+$ is a sheaf. Indeed:

1. Let U be an open set, $\{V_i\}$ an open covering of U, and $s \in \mathscr{F}^+(U)$ such that $s|_{V_i} = 0$ for all i, then s must be zero: It suffices to show $s(p) = 0$ for all $p \in U$. Take any $p \in U$, then there exists an open set V_i contains p, hence $s(p) = s|_{V_i}(p) = 0$;

2. Suppose for each i, we have $s_i \in \mathscr{F}^+(V_i)$ such that
$$s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$$
We can construct $s \in \mathscr{F}^+(U)$ such that $s|_{V_i} = s_i$ directly: take any $p \in U$ and V_i containing p, define $s(p) = s_i(p)$. This is well-defined since s_i agree on the intersections. All is left to check is that s satisfies the requirements of the sheafification. The first condition is trivial. For the second one, just consider that you can apply the condition to s_i, and this will give you an open neighborhood W_i contained in V_i and containing p, with $t_i \in \mathscr{F}(W_i)$ as above. Since W_i is open in V_i, which is open in U, so W_i is suitable also for the function s we have just defined.

Remark A.2.3. From this construction, you can see the stalk of $\mathscr{F}^+$ at p is exactly $\mathscr{F}_p$. Check it by definition.

Now let's define the canonical morphism $\theta : \mathscr{F} \to \mathscr{F}^+$ as follows: For open $U \subseteq X$, and $s \in \mathscr{F}(U)$, define
$$\begin{aligned} \theta(s) : U &\to \coprod_{p \in U} \mathscr{F}_p \\ p &\mapsto s|_p \end{aligned}$$
Note that if $\mathscr{F}$ is already a sheaf, we desire canonical morphism θ is an isomorphism. Indeed, if $s_p = 0$ for all $p \in U$, so there exists an open covering $\{V_i\}$ of U such that $s|_{V_i} = 0$, by axioms of sheaf we obtain $s = 0$, this is injectivity; For surjectivity: take $f \in \mathscr{F}^+(U)$. Since for each $p \in U$ there exists $p \in V_p \subseteq U$ and $t \in \mathscr{F}(V_p)$ such that $f(p) = t|_p$, then glue these t together to get our desired s such that $\theta(s) = f$.

Finally let's construct $\overline{\varphi}$: A map of presheaves $\varphi : \mathscr{F} \to \mathscr{G}$ induces a map on stalks
$$\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$$
Thus for $f \in \mathscr{F}^+(U)$, we can compose f with the map
$$\coprod_{p \in U} \varphi_p : \coprod_{p \in U} \mathscr{F}_p \to \coprod_{p \in U} \mathscr{G}_p$$
to get a map $U \to \coprod_{p \in U} \mathscr{G}_p$. Thus we get a morphism $\widetilde{\varphi} : \mathscr{F}^+ \to \mathscr{G}^+$. Indeed, $\widetilde{\varphi}(f)(p) \in \mathscr{G}_p$, since $f(p) \in \mathscr{F}_p$ and $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$; If for all $q \in V_p$ we have $t|_q = f(q)$, then
$$\widetilde{\varphi}(f)(q) = \varphi_q(f(q)) = \varphi_q(t|_q) = \varphi(t)|_q$$
So $\widetilde{\varphi}(f) \in \mathscr{G}^+$. Since $\mathscr{G}$ is assumed to be sheaf, then canonical morphism $\theta' : \mathscr{G} \to \mathscr{G}^+$ is isomorphic, so we obtain $\overline{\varphi} := \theta'^{-1} \circ \widetilde{\varphi}$. Now let's show $\varphi = \overline{\varphi} \circ \theta = \theta'^{-1} \circ \widetilde{\varphi} \circ \theta$. It's suffice to show they coincide on each stalk since both $\mathscr{F}^+$ and $\mathscr{G}$ are sheaves, and it's quite easy to see this, since $\varphi_p = \theta_p'^{-1} \circ \widetilde{\varphi}_p \circ \theta_p$. Furthermore, uniqueness follows from the fact that $\overline{\varphi}_p$ is uniquely determined by φ_p.

Remark A.2.4. We can describe sheafification in a more fancy language: Since we have sheaf of abelian groups on X as a category, denote it by $\underline{Ab}_X$, and presheaf is a full subcategory of $\underline{Ab}_X$, there is a natural inclusion functor ι from category of sheaf to category of presheaf. Then sheafification is the adjoint functor of ι.

A.3. **More examples on sheaves.**

Example A.3.1 (constant sheaf)**.** Let G be abelian group, the associated constant sheaf $\underline{G}$ is the sheafication of the presheaf

$$U \mapsto G$$

Use the construction of sheafification, we can write $\underline{G}$ more explicitly as

$$\underline{G}(U) = \{\text{locally constant function } f : U \to G\}$$

Example A.3.2 (ringed space)**.** A ringed space is the data of space + functions. For different spaces, we can define different functions:

1. Let X be a topological space, then $\mathscr{C}_X$ is defined by: For any open subset U, we define
$$\mathscr{C}_X(U) := \{\text{continuous functions } f : U \to \mathbb{R}\}$$
2. Let M be a smooth manifold, then C^∞_M is defined by: For any open subset U, we define
$$C^\infty_M(U) := \{\text{smooth functions } f : U \to \mathbb{R}\}$$
3. Let X be a complex manifold, then $\mathcal{O}_X$ is defined by: For any open subset U, we define
$$\mathcal{O}_X(U) := \{\text{holomorphic functions} f : U \to \mathbb{C}\}$$
4. Let X be an algebraic variety, then $\mathcal{O}_X$ is defined by: For any open subset U, we define
$$\mathcal{O}_X(U) := \{\text{regular functions on } U\}$$

Example A.3.3 (Sheaf of modules on a ringed space)**.** Let $(X, \mathcal{O}_X)$ be a a ringed space. A sheaf of $\mathcal{O}_X$-module is a sheaf $\mathscr{M}$ such that for any open $U \subseteq X$, $\mathscr{M}(U)$ is an $\mathcal{O}_X(U)$-module and the module structure is compatible with the restriction.

Example A.3.4. Let $(X, \mathcal{O}_X)$ be an algebraic variety, then we have (quasi) coherent sheaves of $\mathcal{O}_X$-modules.

Example A.3.5. Let $(X, \mathcal{O}_X)$ be a complex manifold and let $\pi\colon E \to M$ be a holomorphic vector bundle. Then by Example **??** we know that E defines a sheaf. In fact E is a sheaf of $\mathcal{O}_X$-modules.

A.4. **Exact sequence of sheaf.** We can consider the kernel and cokernel if we already have a morphism of objects. So it's natural to define similar conceptions for morphisms of sheaves.

Given a morphism $\varphi : \mathscr{F} \to \mathscr{G}$ between sheaves of abelian groups, we define its kernel $\ker \varphi$ as a presheaf by assigning each open subset U the abelian group $\ker \varphi(U)$, since $\varphi(U)$ is a morphism of abelian groups.

Similarly, we can define its image or cokernel as a presheaf by assigning each open subset U the abelian group $\operatorname{im} \varphi(U)$ or $\operatorname{coker} \varphi(U)$.

It's natural to ask whether the kernel, image or cokernel of morphism φ between sheaves are still sheaves or not? Unfortunately, only kernel of φ is still a sheaf, its image or cokernel may fail to be sheaf in our definition.

Why kernel of a morphism $\varphi : \mathscr{F} \to \mathscr{G}$ is still a sheaf? Let's check by definition: Take $s \in \ker \varphi(U)$, and take an open covering $\{V_i\}_{i\in I}$ of U. Then $s|_{V_i} = 0$ must imply $s = 0$ since s is also in $\mathscr{F}(U)$; If there exists $s_i \in \ker \varphi(V_i)$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, so they glue together to get $s \in \mathscr{F}(U)$, and we need to check $\varphi(U)(s) = 0$. But we have

$$\varphi(U)(s)|_{V_i} = \varphi(V_i)(s|_{V_i}) = \varphi(V_i)(s_i) = 0$$

Then we obtain $\varphi(U)(s) = 0$.

But image of morphism may not be a sheaf: Although we can prove the first requirement in a same way, for the second something bad happens. If there exists $s_i \in \operatorname{im} \varphi(V_i)$, and we can glue them together to get a $s \in \mathscr{G}(U)$, but s may not be the image of some $t \in \mathscr{F}(U)$. The cokernel fails to be a sheaf for the same reason.

So we may change our definition about image and cokernel: To define the image and cokernel of a morphism to be the sheafification of our previous definition.

For a sequence of sheaves:

$$\cdots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \dots$$

It's called exact at $\mathscr{F}^i$, if $\ker \varphi^i = \operatorname{im} \varphi^{i-1}$. If a sequence is exact at everywhere, then it's an exact sequence of sheaves.

However, there is a better description of exactness of sequence of sheaves, that is looking its stalks:

Proposition A.4.1. The sequence of sheaves

$$\cdots \to \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \dots$$

is exact if and only if the sequence of abelian groups are exact

$$\cdots \to \mathscr{F}_x^{i-1} \xrightarrow{\varphi_x^{i-1}} \mathscr{F}_x^i \xrightarrow{\varphi_x^i} \mathscr{F}_x^{i+1} \to \dots$$

for all $x \in X$.

Proof. It suffices to show for any morphism $\varphi : \mathscr{F} \to \mathscr{G}$, we have $(\ker \varphi)_p = \ker \varphi_p$, $(\operatorname{im} \varphi)_p = \operatorname{im} \varphi_p$. Let's fix $p \in X$ and check by definition.

For (1). It's clear $(\ker\varphi)_p \subseteq \ker\varphi_p$; Conversely, take $s_p \in \ker\varphi_p$, then $\varphi_p(s_p) = 0 \in \mathscr{G}_p$. In other words, there exists an open set U containing p and $s \in \mathscr{F}(U)$ such that $s|_p = s_p$ and $\varphi(s)|_p = 0$, which implies there is an open set V containing p such that $\varphi(s)|_V = 0$. Hence $\varphi(s|_V) = 0$, that is $s|_V \in \ker\varphi(V)$. Thus $s_p = (s|_V)|_p \in (\ker\varphi)_p$.

For (2). It's clear $(\operatorname{im}\varphi)_p \subseteq \operatorname{im}\varphi_p$, since $(\operatorname{im}\varphi)_p$ is the same stalk of the presheaf of image before sheafification; Conversely, if $s_p \in \operatorname{im}\varphi_p$, we have some $t_p \in \mathscr{F}_p$ such that $\varphi_p(t_p) = s_p$. Suppose $t \in \mathscr{F}(U)$ is a section of some open set U containing p such that $t|_p = t_p$. Then $\varphi(t)|_p = \varphi_p(t_p) = s_p$, so s_p is in the stalk of the image presheaf at p. But the stalk at a point remains the same after sheafification, we have $s_p \in (\operatorname{im}\varphi)_p$. $\square$

Remark A.4.1. The proof for the first part is a routine, but the proof for the half part shows the hallmark of sheafification: Stalks are not changed!.

Now let's consider a special exact sequence: short exact sequence:

$$0 \to \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{H} \to 0$$

In this case, φ is called injective and ψ is called surjective. Attention: For any open subset $U \subseteq X$, we will have

$$\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$$

is injective. Indeed, by definition we have for any open subset $U \subseteq X$, $\ker\varphi(U) = 0$, that is injectivity. Or from another point of view, for each $p \in U$, we have

$$\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$$

is injective. That is $\ker\varphi_p = 0$. So we obtain $(\ker\varphi(U))_p = 0$ for all $p \in U$. But for a section $s \in \mathscr{F}(U)$ if we have $s|_p = 0$, then we must have $s = 0$. So we obtain $\ker\varphi(U) = 0$.

The second pointview may be a little talk nonsense, but we will see it can explain why $\psi(U) : \mathscr{G}(U) \to \mathscr{H}(U)$ may not be surjective in general. Since from

$$\psi_p : \mathscr{G}_p \to \mathscr{H}_p$$

is surjective we can only get "locally surjective", and from locally surjectivity you may not get a global one. The reason for why does image fail to be a sheaf appears again.

Example A.4.1 (exponential sequence)**.** Let X be a complex manifold and $\mathcal{O}_X$ is its holomorphic function sheaf. Then

$$0 \to 2\pi\sqrt{-1}\underline{\mathbb{Z}} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

is an exact sequence of sheaves, called exponential sequence.

Proof. The difficulty is to show exp is surjective on stalks at $p \in X$. That is we need to construct logarithms of functions $g \in \mathcal{O}_X^*(U)$ for U, a neighborhood of p. We may choose U is simply-connected, then define

$$\log g(q) = \log g(p) + \int_{\gamma_q} \frac{\mathrm{d}g}{g}$$

for $q \in U$, where γ_q is a path from p to q in U, and our definition is independent of the choice of γ_q since U is simply-connected.

In fact, U is simply-connected is crucial for constructing logarithm. If we consider $X = \mathbb{C}$ and $U = \mathbb{C} \setminus \{0\}$, then

$$\exp : \mathcal{O}_X(U) \to \mathcal{O}_X^*(U)$$

won't be surjective. $\square$

A.5. **Derived functor formulation of sheaf cohomology.** The category $\underline{Ab}_X$: sheaves of abelian groups on X. In this section we will introduce sheaf cohomology by considering it as a derived functor.

For an exact sequence of sheaf:

$$0 \to \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}''$$

If we take its section on U, we get a sequence of abelian groups

$$0 \to \mathscr{F}'(U) \xrightarrow{\phi(U)} \mathscr{F}(U) \xrightarrow{\psi(U)} \mathscr{F}''(U)$$

We already know this sequence is still exact at $\mathscr{F}'(U)$, now let's it's still exact at $\mathscr{F}(U)$, that is

$$\ker \psi(U) = \operatorname{im} \phi(U)$$

Let's first show $\ker \psi(U) \supseteq \operatorname{im} \phi(U)$. Take $s \in \mathscr{F}'(U)$, and we want to show $\psi\phi(s) = 0$. It suffices to show $\psi\phi(s)|_p = 0$ for all $p \in U$, since $\mathscr{F}''$ is a sheaf. For any $p \in U$, consider its stalk we obtain an exact sequence of abelian groups

$$0 \to \mathscr{F}'_p \xrightarrow{\phi_p} \mathscr{F}_p \xrightarrow{\psi_p} \mathscr{F}''_p$$

then we obtain $\psi_p\phi_p(s|_p) = 0$, that is $\psi\phi(s)|_p$.

On the other hand. Take $s \in \ker \psi(U)$, then for any $p \in U$ we have $s|_p \in \ker \psi_p$. By exactness of stalks, there exists $t_p \in \mathscr{F}'_p$ such that $\phi_p(t_p) = s|_p$. So there exists an open subset V_i containing p and $t_i \in \mathscr{F}'(V_i)$ such that $\phi(t_i) = s|_{V_i}$. We claim that these t_i can be glued together to obtain $t \in \mathscr{F}(U)$. Since $\mathscr{F}$ is a sheaf, it suffices to check these t_i agree on intersections $V_i \cap V_j$. This follows from the injectivity of ϕ, since $\phi(t_i - t_j|_{V_i \cap V_i}) = s|_{V_i \cap V_j} - s|_{V_i \cap V_j} = 0$.

Remark A.5.1. From above argument, we can see that

$$0 \to \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}''$$

is exact if and only if for any open subset $U \subseteq X$

$$0 \to \mathscr{F}'(U) \xrightarrow{\phi(U)} \mathscr{F}(U) \xrightarrow{\psi(U)} \mathscr{F}''(U)$$

is exact.

In homological algebra, we can consider the derived functor of a left or right-exact functor. Here as we can see, take sections (in particular, global sections) of a sheaf is a left exact functor.

So, as what we did in homological, we need choose a injective resolution and consider the cohomology of the sequence of its global sections to define the sheaf cohomology.

Definition A.5.1 (injective)**.** A sheaf $\mathcal{I}$ is injective if $\mathrm{Hom}(-, \mathcal{I})$ is an exact functor.

Fact A.5.1. $\underline{Ab}_X$ is an abelian category with enough injectives. Namely, every sheaf $\mathscr{F}$ can be realized as a subsheaf of some injective sheaf.

Definition A.5.2 (injective resolution)**.** Let $\mathscr{F}$ be a sheaf, an injective resolution of $\mathscr{F}$ is an exact sequence

$$0 \to \mathscr{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \dots$$

where $\mathcal{I}^i, i = 0, 1, 2, \dots$ are injective.

Fact A.5.2. Every sheaf admits an injective resolution.

Fact A.5.3. Let $\mathscr{F} \to \mathcal{I}^\bullet$ and $\mathscr{G} \to \mathcal{G}^\bullet$ are two resolutions, and $\phi : \mathscr{F} \to \mathscr{G}$ is a homomorphism of sheaves. Then there exists $\widetilde{\phi} : \mathcal{I}^\bullet \to \mathcal{G}^\bullet$. Although the lifting of ϕ may not be unique, but they are homotopic.

Definition A.5.3 (sheaf cohomology)**.** Let $\mathscr{F}$ be a sheaf of abelian groups, then

$$H^p(X, \mathscr{F}) := H^p(\mathcal{I}^\bullet(X))$$

Remark A.5.2. This definition is independent of the choice of injective resolution thanks to $A.5.3$.

Example A.5.1. By definition,

$$H^0(X, \mathscr{F}) := \ker\{\mathcal{I}^0(X) \to \mathcal{I}^1(X)\}$$

Thus $H^0(X, \mathscr{F}) = \mathscr{F}(X)$, the global sections of sheaf.

Example A.5.2. If $\mathscr{F}$ is a injective sheaf, then $H^i(X, \mathscr{F}) = 0$ for all $i > 0$. It's clear if we choose the following injective resolution

$$0 \to \mathscr{F} \xrightarrow{\mathrm{id}} \mathscr{F} \to 0 \to 0 \to \dots$$

Proposition A.5.1 (zig-zag)**.** If

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$

is a short sequence of sheaves, then there is an induced long exact sequence of abelian groups

$$0 \to H^0(X, \mathscr{F}) \to H^0(X, \mathscr{G}) \to H^0(X, \mathscr{H}) \to H^1(X, \mathscr{F}) \to H^1(X, \mathscr{G}) \to \dots$$

Definition A.5.4 (direct image)**.** Let $f : X \to Y$ be continuous map between topological spaces. Let $\mathscr{F}$ be a sheaf of abelian groups on X. A sheaf $f_*\mathscr{F}$ on Y is defined by

$$f_*\mathscr{F}(U) := \mathscr{F}(f^{-1}(U))$$

$f_*\mathscr{F}$ is called direct image sheaf of $\mathscr{F}$.

Proposition A.5.2. $f_* : \underline{Ab}_X \to \underline{Ab}_Y$ is a left exact functor.

Proof. Given an exact sequence of sheaves on X

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$$

Then we need to show

$$0 \to f_*\mathscr{F}' \to f_*\mathscr{F} \to f_*\mathscr{F}''$$

is also an exact sequence on Y. But by Remark $A.5.1$ it suffices to check for any $V \in Y$, we have the following exact sequence

$$0 \to f_*\mathscr{F}'(V) \to f_*\mathscr{F}(V) \to f_*\mathscr{F}''(V)$$

and that's

$$0 \to \mathscr{F}'(f^{-1}(V)) \to \mathscr{F}(f^{-1}(V)) \to \mathscr{F}''(f^{-1}(V))$$

and since f is continuous, then $f^{-1}(V)$ is an open subset in X. This completes the proof. $\square$

Since we obtain another left exact functor f_*, we can consider its derived functor.

Definition A.5.5 (higher direct image sheaves)**.** Let $0 \to \mathscr{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathscr{F}$. The higher direct image sheaf is defined by: For open U,

$$R^i f_*\mathscr{F}(U) := H^i(f_*\mathcal{I}^\bullet(U))$$

For higher direct image sheaves, it has similar properties parallel to $A.5.1$, $A.5.2$ and $A.5.1$, since these are properties shared by derived functors.

A.6. **Computation for cohomology.** Since it may be difficult for us to choose an injective resolution, we usual other resolutions to compute sheaf cohomology.

Definition A.6.1 (acyclic sheaf)**.** A sheaf $\mathscr{F}$ is acyclic if $H^i(X, \mathscr{F}) = 0, \forall i > 0$.

Example A.6.1. Injective sheaf is acyclic.

Definition A.6.2 (acyclic resolution)**.** Let $\mathscr{F}$ be a sheaf, an acyclic resolution of $\mathscr{F}$ is an exact sequence

$$0 \to \mathscr{F} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \mathcal{A}^2 \to \dots$$

where $\mathcal{A}^i, i = 0, 1, 2, \dots$ are acyclic.

Proposition A.6.1. The cohomology of sheaf $\mathscr{F}$ can be computed using acyclic resolution.

In fact, it's a quite homological techniques, called dimension shifting. So we will state this technique in language of homological algebra. Let's see a baby version of it.

Example A.6.2. Let $\mathcal{F}$ be a left exact functor, and $0 \to A \to M_1 \to B \to 0$ is exact with M_1 is $\mathcal{F}$-acyclic. Then $R^{i+1}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$ for $i > 0$, and $R^1\mathcal{F}(A)$ is the cokernel of $\mathcal{F}(M_1) \to \mathcal{F}(B)$.

Proof. Consider the long exact sequence induced by $0 \to A \to M_1 \to B \to 0$, then we obtain

$$R^i\mathcal{F}(M_1) \to R^i\mathcal{F}(B) \to R^{i+1}\mathcal{F}(A) \to R^{i+1}\mathcal{F}(M_1)$$

If $i > 0$, then $R^i\mathcal{F}(M_1) = R^{i+1}\mathcal{F}(M_1) = 0$ since M_1 is $\mathcal{F}$-acyclic. So we obtain $R^{i+1}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$. If $i = 0$, then

$$0 \to \mathcal{F}(M_1) \to \mathcal{F}(B) \to R^1\mathcal{F}(A) \to 0$$

implies $R^1\mathcal{F}(A) = \operatorname{coker}\{\mathcal{F}(M_1) \to \mathcal{F}(B)\}$ □

Now let's prove dimension shifting in a general setting.

Lemma A.6.1 (dimension shifting)**.** If

$$0 \to A \to M_1 \to M_2 \to \cdots \to M_m \to B \to 0$$

is exact with M_i is $\mathcal{F}$-acyclic. Then $R^{i+m}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$ for $i > 0$, and $R^m\mathcal{F}(A)$ is the cokernel of $\mathcal{F}(M_m) \to \mathcal{F}(B)$.

Proof. Prove it by induction on m. For $m = 1$, we already see it in Example A.6.2. Assume this is holds for $m < k$, then for $m = k$, let's split $0 \to A \to M_1 \to M_2 \to \cdots \to M_k \xrightarrow{\mathrm{d}_k} B \to 0$ into two exact sequences

$$0 \to A \to M_1 \to M_2 \to \cdots \to M_{k-1} \to \ker \mathrm{d}_k \to 0$$

$$0 \to \ker \mathrm{d}_k \to M_k \xrightarrow{\mathrm{d}_k} B \to 0$$

Then by induction, for $i > 0$ we have

$$R^{i+k-1}\mathcal{F}(A) \cong R^i\mathcal{F}(\ker \mathrm{d}_k)$$
$$R^{i+1}\mathcal{F}(\ker \mathrm{d}_k) \cong R^i\mathcal{F}(B)$$

Combine these two isomorphisms together we obtain $R^{i+k}\mathcal{F}(A) \cong R^i\mathcal{F}(B)$ for $i > 0$, as desired. For $i = 0$, it suffices to let $i = 1$ in $R^{i+k-1}\mathcal{F}(A) \cong R^i\mathcal{F}(\ker \mathrm{d}_k)$, then we obtain

$$R^k\mathcal{F}(A) = R^1\mathcal{F}(\ker \mathrm{d}_k) = \operatorname{coker}\{\mathcal{F}(M_k) \to \mathcal{F}(B)\}$$

This completes the proof. □

Corollary A.6.1. $0 \to A \to M_\bullet$ is a $\mathcal{F}$-acyclic resolution, then $R^i\mathcal{F}(A) = H^i(\mathcal{F}(M))$.

Proof. Truncate the resolution as

$$0 \to A \to M_0 \to M_1 \to \dots M_{i-1} \to B \to 0$$
$$0 \to B \to M_i \to M_{i+1} \to \dots$$

Since we already have $R^i\mathcal{F}(A) = \operatorname{coker}\{\mathcal{F}(M_{i-1}) \to \mathcal{F}(B)\}$. Note $\mathcal{F}$ is left exact, then

$$\mathcal{F}(B) = \ker\{\mathcal{F}(M_i) \to \mathcal{F}(M_{i+1})\}$$

Thus we obtain

$$R^i\mathcal{F}(A) = \operatorname{coker}\{\mathcal{F}(M_{i-1}) \to \ker\{\mathcal{F}(M_i) \to \mathcal{F}(M_{i+1})\}\} = H^i(\mathcal{F}(M))$$

□

A.7. **Examples about acyclic sheaf.**

A.7.1. *Flabby sheaf.* First kind of acyclic sheaf is flabby[14] sheaf.

Definition A.7.1 (flabby)**.** A sheaf $\mathscr{F}$ is flabby if all open $U \subseteq V$, the restriction map $\mathscr{F}(V) \to \mathscr{F}(U)$ is surjective.

Now let's see some examples about flabby sheaves.

Example A.7.1. A constant sheaf on an irreducible topological space is flabby.

Proof. Note that the constant presheaf on a irreducible topological space is a sheaf in fact. And it's easy to see this constant presheaf is flabby. □

In particular, we have

Example A.7.2. Let X be an algebraic variety, so any two non-empty open sets intersect non-trivially. Then constant sheaf $\mathbb{Z}_X$ is flabby.

Example A.7.3. If $\mathscr{F}$ is a flabby sheaf on X, and $f : X \to Y$ is a continuous map, then $f_*\mathscr{F}$ is a flabby sheaf on Y.

Proof. For $V \subset W$ in Y, it suffices to show $f_*\mathscr{F}(W) \to f_*\mathscr{F}(V)$ is surjective, and that's

$$\mathscr{F}(f^{-1}W) \to \mathscr{F}(f^{-1}V)$$

it's surjective since $\mathscr{F}$ is flabby. □

Example A.7.4. An injective sheaf is flabby.

Proof. Let $\mathcal{I}$ be an injective sheaf and let $V \subseteq U$ be an inclusion of open sets. We define a sheaf $\underline{\mathbb{Z}}_U$ on X by

$$\underline{\mathbb{Z}}_U := \begin{cases} \underline{\mathbb{Z}}(W), & W \subseteq U \\ 0, & \text{otherwise} \end{cases}$$

[14]Some authors also call this flasque.

where $\underline{\mathbb{Z}}$ is constant sheaf valued in $\mathbb{Z}$. Similarly we can define $\underline{\mathbb{Z}}_V$, and we have $\underline{\mathbb{Z}}_U(W) = \underline{\mathbb{Z}}_V(W)$ unless $W \subseteq U$ and $W \not\subseteq V$. Thus we obtain an exact sequence

$$0 \to \underline{\mathbb{Z}}_V \to \underline{\mathbb{Z}}_U$$

Applying the functor $\mathrm{Hom}(-, \mathcal{I})$, which is exact, we obtain an exact sequence

$$\mathrm{Hom}(\underline{\mathbb{Z}}_U, \mathcal{I}) \to \mathrm{Hom}(\underline{\mathbb{Z}}_V, \mathcal{I}) \to 0$$

is exact. Now let's see why we need such a weird sheaf $\underline{\mathbb{Z}}_U$. In fact, we will prove $\mathrm{Hom}(\underline{\mathbb{Z}}_U, \mathcal{I}) = \mathcal{I}(U)$. Indeed, since $\varphi : \underline{\mathbb{Z}}_U \to \mathcal{I}$ is a sheaf morphism. Then if $W \not\subseteq U$, then $\varphi(U)$ must be zero. If $W = U$, then the group of sections of $\underline{\mathbb{Z}}_U(U)$ over any connected component is simply $\mathbb{Z}$ and hence $\varphi(U)$ on this connected component is determined by the image of $1 \in \mathbb{Z}$. Thus $\varphi(U)$ can be thought of an element of $\mathcal{I}(U)$. Now on any proper open subset of U, φ is determined by restriction maps. Hence $\mathrm{Hom}(\underline{\mathbb{Z}}_U, \mathcal{I}) = \mathcal{I}(U)$, as desired. We can do similar things for V, and we obtain an exact sequence

$$\mathcal{I}(U) \to \mathcal{I}(V) \to 0$$

Thus $\mathcal{I}$ is flabby. □

Our goal is to prove a flabby sheaf is acyclic, but we still need some property of flabby sheaves.

Proposition A.7.1. If $0 \to \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and $\mathscr{F}'$ is flabby, then for any open set U, the sequence

$$0 \to \mathscr{F}'(U) \xrightarrow{\phi(U)} \mathscr{F}(U) \xrightarrow{\psi(U)} \mathscr{F}''(U) \to 0$$

is exact.

Proof. It suffices to show $\mathscr{F}(U) \to \mathscr{F}''(U) \to 0$ is exact. And it may be quit hard than it looks and here we give a sketch of proof.

Since we have exact sequence on stalks for each $p \in U$ as follows

$$0 \to \mathscr{F}'_p \xrightarrow{\phi_p} \mathscr{F}_p \xrightarrow{\psi_p} \mathscr{F}''_p \to 0$$

Then for each $s \in \mathscr{F}''(U)$, then there exists $t_p \in \mathscr{F}_p$ such that $\psi_p(t_p) = s|_p$. So there exists $V_p \subseteq U$ containing p and $t \in \mathscr{F}(V_p)$ such that $\psi(t) = s|_{V_p}$. If we can glue these t together then we get a section in $\mathcal{F}(U)$ and is mapped to s, which completes the proof. However, they may not equal on the intersection. But things are not too bad, consider another point q and $t' \in \mathscr{F}(V_q)$ such that $\psi(t') = s|_{V_q}$, $(t' - t)|_{V_p \cap V_q} \in \ker \psi(V_p \cap V_q) = \operatorname{im} \phi(V_p \cap V_q)$. So there exists $t'' \in \mathscr{F}'(V_p \cap V_q)$ such that

$$\phi(t'') = (t' - t)|_{V_p \cap V_q}$$

Now since $\mathscr{F}'$ is flabby, then there exists $t''' \in \mathscr{F}(V_p)$ such that $t'''|_{V_p \cap V_q} = t''$. And consider $t + \phi(t''') \in \mathscr{F}(V_p)$, which will coincide with t' on $V_p \cap V_q$. After above corrections, we can glue t after correction together. □

Proposition A.7.2. If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ and $\mathscr{F}$ are flabby, then $\mathscr{F}''$ is flabby.

Proof. Take $V \subseteq U$ and consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathscr{F}'(U) \arrow[r] \arrow[d] & \mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{F}''(U) \arrow[r] \arrow[d, dashed] & 0 \arrow[d] \\
0 \arrow[r] & \mathscr{F}'(V) \arrow[r] & \mathscr{F}(V) \arrow[r] & \mathscr{F}''(V) \arrow[r] & 0
\end{tikzcd}$$

Then five lemma completes the proof. $\square$

Proposition A.7.3. A flabby sheaf is acyclic.

Proof. Let $\mathscr{F}$ be a flabby sheaf. Since there are enough injectives, there is an exact sequence

$$0 \to \mathscr{F} \to \mathcal{I} \to \mathscr{Q} \to 0$$

with $\mathcal{I}$ is injective. By Example $A.7.4$ we have $\mathcal{I}$ is flabby. And by Proposition $A.7.2$ we have $\mathscr{Q}$ is flabby. Consider the long exact sequence induced from above short exact sequence

$$\mathscr{F}(X) \to \mathcal{I}(X) \to \mathscr{Q}(X) \to H^1(X, \mathscr{F}) \to H^1(X, \mathcal{I}) \to \dots$$

Since injective sheaf is acyclic, then $H^1(X, \mathcal{I}) = 0$. Then $H^1(X, \mathscr{F}) = \operatorname{coker}\{\mathcal{I}(X) \to \mathscr{Q}(X)\}$. But from Proposition $A.7.1$ we know that it's surjective. So $H^1(X, \mathscr{F}) = 0$.

Now we prove $H^k(X, \mathscr{F}) = 0$ for $k > 0$ by induction on k. We already show $k = 1$. Assume this holds for $k < n$. Then consider

$$\dots \to H^{n-1}(X, \mathscr{F}) \to H^{n-1}(X, \mathcal{I}) \to H^{n-1}(X, \mathscr{Q}) \to H^n(X, \mathscr{F}) \to H^n(X, \mathcal{I}) \to H^n(X, \mathscr{Q}) \to \dots$$

From we already know, we can reduce above sequence to

$$\dots \to 0 \to 0 \to 0 \to H^n(X, \mathscr{F}) \to 0 \to H^n(X, \mathscr{Q}) \to \dots$$

which implies $H^n(X, \mathscr{F}) = 0$. This completes the proof. $\square$

A.7.2. *Soft sheaf.* The second kind of acyclic sheaves is called soft sheaves, which is quit similar to flabby.

Definition A.7.2 (soft)**.** A sheaf $\mathscr{F}$ over X is soft if for any closed subset $S \subseteq X$ the restriction map $\mathscr{F}(X) \to \mathscr{F}(S)$ is surjective.

Remark A.7.1. Here we need to know how to define sections over a closed subset: For closed subset S,

$$\mathscr{F}(S) := \varinjlim_{S \subset U} \mathscr{F}(U)$$

Quite similar to Proposition $A.7.1$ and $A.7.2$, soft sheaf has the following properties:

Proposition A.7.4. If $0 \to \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and $\mathscr{F}'$ is soft, then the following sequence

$$0 \to \mathscr{F}'(X) \xrightarrow{\phi(X)} \mathscr{F}(X) \xrightarrow{\psi(X)} \mathscr{F}''(X) \to 0$$

is exact.

Proposition A.7.5. If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ and $\mathscr{F}$ are soft, then $\mathscr{F}''$ is soft.

Proposition A.7.6. A soft sheaf is acyclic.

So you may wonder, what's the difference between flabby and soft, since the definitions are quite similar, and both of them are acyclic. Clearly by definition of sections over a closed subset, we know that every flabby sheaf is soft, but converse fails

Example A.7.5. The sheaf of smooth functions on a smooth manifold is soft but not flabby.

Lemma A.7.1. If $\mathscr{M}$ is a sheaf of modules over a soft sheaf of rings $\mathscr{R}$, then $\mathscr{M}$ is a soft sheaf.

Proof. Let $s \in \mathscr{M}(K)$ for K a closed subset of X. Then s extends to some open neighborhood U of K. Let $\rho \in \mathscr{R}(K \cup (X - U))$ be defined by

$$\rho = \begin{cases} 1, & \text{on } K \\ 0, & \text{on } X - U \end{cases}$$

Since $\mathscr{R}$ is soft, then ρ extends to a section over X, then ρs is the desired extension of s. □

A.7.3. *Fine sheaf.* Another important kind of acyclic sheaves, which behaves like sheaf of differential forms Ω^k_X is called fine sheaf. Recall what is a partition of unity: Let $U = \{U_i\}_{i \in I}$ be a locally finite open cover of X. A partition of unity subordinate to U is a collection of continuous or smooth functions $f_i : U_i \to [0, 1]$ for each $i \in I$ such that its support lies in U_i, and for any $x \in X$

$$\sum_{i \in I} f_i(x) = 1$$

Note that for a smooth manifold M, then sheaf of differential k-forms is a C^∞_M-module sheaf.

Definition A.7.3 (fine sheaf)**.** A fine sheaf $\mathscr{F}$ on X is a sheaf of $\mathscr{A}$-modules, where $\mathscr{A}$ is a sheaf of rings such that for every locally finite open cover $\{U_i\}_{i \in I}$ of X, there is a partition of unity

$$\sum_{i \in I} \rho_i = 1$$

where $\rho_i \in \mathscr{A}(X)$ and $\text{supp}(\rho_i) \subseteq U_i$.

Remark A.7.2. It' neccessary to give an explicit definition of support of a section: For a sheaf $\mathscr{F}$ on X and a section $s \in \mathscr{F}(X)$, its support is defined as

$$\operatorname{supp}(s) := \overline{\{x \in X : s|_x \neq 0\}}$$

Proposition A.7.7. A fine sheaf is acyclic.

Proof. Let $\mathscr{F}$ be a sheaf of $\mathscr{A}$-modules and a fine sheaf. And choose a injective resolution

$$0 \to \mathscr{F} \xrightarrow{\mathrm{d}} \mathcal{I}^0 \xrightarrow{\mathrm{d}} \mathcal{I}^0 \xrightarrow{\mathrm{d}} \mathcal{I}^1 \xrightarrow{\mathrm{d}} \dots$$

such that $\mathcal{I}^i$ are injective sheaves of $\mathscr{A}$-modules. Let $s \in \mathcal{I}^p(X)$ such that $\mathrm{d}s = 0$. Then by exactness of injective resolution we have X is covered by open sets U_i such that for each i there is an element $t_i \in \mathcal{I}^{p-1}(U_i)$ such that $\mathrm{d}t_i = s|_{U_i}$. By passing to a refinement we may assume that the cover $\{U_i\}$ is locally finite. Let $\{\rho_i\}$ be a partition of unity subordinate to $\{U_i\}$. Then we have $t = \sum \rho_i t_i \in \mathscr{I}^{p-1}(X)$ such that $\mathrm{d}t = s$. This completes the proof. $\square$

Example A.7.6. Let M be a manifold and let $\pi \colon E \to M$ be a vector bundle, then sheaf of smooth sections of E is also a fine sheaf.

Example A.7.7. Tangent sheaf T_M, and its tensor $T_M^{\otimes k}$, sheaf of differential forms Ω_M and k-forms Ω_M^k are fine sheaves.

Remark A.7.3. So from Example $A.7.6$, we know that it's meaningless to compute cohomology of sheaf of differential k-forms, or any other vector bundle over a smooth manifold. But in complex version, something interesting happens:

Let $(X, \mathcal{O}_X)$ be a complex manifold, and let $\pi \colon E \to X$ a holomorphic vector bundle. Then the sheaf of holomorphic sections of E is not a fine sheaf, since there is no partition of unity we use in the sense of smooth. So we may compute cohomology of some holomorphic vector bundle, and that's what does Dolbeault cohomology compute.

For fine sheaf and soft sheaf, we have

Lemma A.7.2. Fine sheaf is soft

Proof. Let $\mathscr{F}$ be a fine sheaf, $S \subseteq X$ closed and $s \in \mathscr{F}(S)$. Let $\{U_i\}$ be an open covering of S and $s_i \in \mathscr{F}(U_i)$ such that

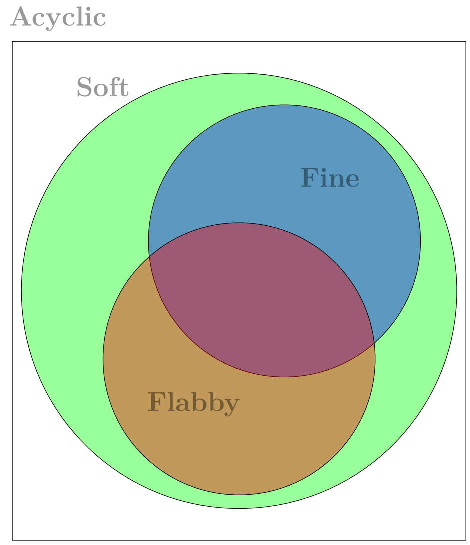
$$s_i|_{S \cap U_i} = s|_{S \cap U_i}$$

Let $U_0 = X - S$, and $s_0 = 0$. Then $\{U_i\} \coprod \{U_0\}$ is an open covering of X, WLOG we assume this open covering is locally finite and choose a partition of unity $\{\rho_i\}$ subordinate to it. Set

$$\overline{s} := \sum_i \rho_i(s_i) \in \mathscr{F}(X)$$

which extends s. $\square$

Remark A.7.4. For flabby, soft, fine and acyclic sheaves. In fact we have the following relations:



Indeed, constant sheaf on an irreducible space is flabby but not fine, sheaf of smooth functions on a smooth manifold is fine but not flabby.

A.8. **Proof of de Rham theorem using sheaf cohomology.** As we already know, for constant sheaf $\underline{\mathbb{R}}$ over a smooth manifold M, we have the following fine resolution

$$0 \to \underline{\mathbb{R}} \xrightarrow{i} \Omega^0_M \xrightarrow{\mathrm{d}} \Omega^1_M \xrightarrow{\mathrm{d}} \Omega^2_M \xrightarrow{\mathrm{d}} \dots$$

And de Rham cohomology computes the sheaf cohomology of $\underline{\mathbb{R}}$. de Rham theorem implies that de Rham cohomology equals to the singular cohomology with real coefficient. So if we can give constant sheaf another resolution using singular cochains, we may derive the de Rham cohomology.

We state this in a general setting: Let X be a topological manifold, and a constant sheaf $\underline{G}$ over X, where G is an abelian group. Let $S^p(U, G)$ be the group of singular cochains in U with coefficients in G, and let δ denote the coboundary operator.

Let $\mathscr{S}^p(G)$ be the sheaf over X generated by the presheaf $U \mapsto S^p(U, G)$, with induced differential mapping $\mathscr{S}^p(G) \xrightarrow{\delta} \mathscr{S}^{p+1}(G)$.

Similar to Poincaré lemma, we have for a unit ball U in Euclidean space, we have the following sequence

$$\dots \to S^{p-1}(U, G) \xrightarrow{\delta} S^p(U, G) \xrightarrow{\delta} S^{p+1}(U, G) \to \dots$$

is exact. So we have the following resolution of the constant sheaf $\underline{G}$

$$0 \to \underline{G} \to \mathscr{S}^0(G) \xrightarrow{\delta} \mathscr{S}^1(G) \xrightarrow{\delta} \mathscr{S}^2(G) \to \dots$$

Remark A.8.1. If M is a smooth manifold, then we can consider smooth chains, that is $f : \Delta^p \to U$, where f is a smooth function. The corresponding results above still hold, and we have a resolution by smooth cochains with coefficients in G:

$$0 \to \underline{G} \to \mathscr{S}^{\bullet}_{\infty}(G)$$

So if we choose $G = \mathbb{R}$, then it suffices to show $0 \to \underline{\mathbb{R}} \to \mathscr{S}^{\bullet}_{\infty}(\mathbb{R})$ is an acyclic resolution, then we obtain de Rham theorem.

First, note that $\mathscr{S}^{p}_{\infty}$ is a $\mathscr{S}^{0}_{\infty}$-module, given by cup product on open sets. Then by Lemma $A.7.1$ and the fact $\mathscr{S}^{0}_{\infty}$ is soft we know that it's a soft resolution. This completes the proof.

A.9. **Hypercohomology.** In homological algebra, the hypercohomology is a generalization of cohomology functor which takes as input not objects in abelian category but instead chain complexes of objects.

One of the motivations for hypercohomology comes from the fact that there isn't an obvious generalization of cohomological long exact sequences associated to short exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

It turns out hypercohomology gives techniques for constructing a similar cohomological associated long exact sequence from an arbitrary long exact sequence

$$0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \cdots \to \mathscr{F}_k \to 0$$

Now let's clearify the definition of hypercohomology: Let $\mathscr{F}^{\bullet} : \cdots \to \mathscr{F}^{i-1} \to \mathscr{F}^{i} \to \mathscr{F}^{i+1} \to \dots$ be a complex of sheaves of abelian groups. Assume $\mathscr{F}^{\bullet}$ is bounded below, i.e. $\mathscr{F}^{n} = 0$ for $n \ll 0$. Then $\mathscr{F}^{\bullet}$ admits an injective resolution $\mathscr{F}^{\bullet} \to \mathcal{I}^{\bullet}$. In other words

$$\begin{tikzcd}
\dots \arrow[r] & \mathscr{F}^{i-1} \arrow[r] \arrow[d] & \mathscr{F}^{i} \arrow[r] \arrow[d] & \mathscr{F}^{i+1} \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \mathcal{I}^{i-1} \arrow[r] & \mathcal{I}^{i} \arrow[r] & \mathcal{I}^{i+1} \arrow[r] & \dots
\end{tikzcd}$$

such that

1. All $\mathcal{I}^{i}$ are injective sheaves;
2. The induced homomorphism $H^{i}(\mathscr{F}^{\bullet}) \to H^{i}(\mathcal{I}^{\bullet})$ is an isomorphism.

The hypercohomology of $\mathscr{F}^{\bullet}$ is defined by

$$H^{i}(X, \mathscr{F}^{\bullet}) := H^{i}(\Gamma(X, \mathcal{I}^{\bullet}))$$

Notation A.9.1. Let $\mathscr{F}^{\bullet}[n]$ be the complex obtained from $\mathscr{F}$ by

$$(\mathscr{F}^{\bullet}[n])^{i} = \begin{cases} \mathscr{F}, & i = n \\ 0, & \text{otherwise} \end{cases}$$

Example A.9.1. Let $\mathscr{F}$ be a sheaf and consider $\mathscr{F}^{\bullet}[0]$, that is,

$$0 \to \underbrace{\mathscr{F}}_{\text{deg zero}} \to 0 \to 0 \to \dots$$

If $0 \to \mathscr{F} \to \mathcal{I}^{0} \to \mathcal{I}^{1} \to \dots$ is an injective resolution of $\mathscr{F}$, then

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\
0 \arrow[r] & \mathcal{I}^0 \arrow[r] & \mathcal{I}^1 \arrow[r] & \mathcal{I}^2 \arrow[r] & \dots
\end{tikzcd}$$

is an injective resolution of $\mathscr{F}^\bullet$. Indeed, $\mathcal{I}^i$ are injective for all $i \geq 0$. Furthermore,

$$H^i(\mathcal{I}^\bullet) = \begin{cases} \mathscr{F}, & n = 0 \\ 0, & \text{otherwise} \end{cases} = H^i(\mathscr{F}^\bullet[0])$$

So by definition of hypercohomology, we have $H^i(X, \mathscr{F}^\bullet[0]) = H^i(\Gamma(X, \mathcal{I}^\bullet)) = H^i(X, \mathscr{F}^\bullet)$. For general case, we have

$$H^i(X, \mathscr{F}^\bullet[n]) \cong H^{i+n}(X, \mathscr{F})$$

Proposition A.9.1 (zig-zag in hypercohomology)**.** Let $0 \to \mathscr{F}^\bullet \to \mathscr{G}^\bullet \to \mathscr{H}^\bullet \to 0$ be a short exact sequence of bounded below complex. Then there is an induced long exact sequence

$$\dots \to H^{i-1}(X, \mathscr{H}^\bullet) \to H^i(X, \mathscr{F}^\bullet) \to H^i(X, \mathscr{G}^\bullet) \to H^i(X, \mathscr{H}^\bullet) \to H^{i+1}(X, \mathscr{F}^\bullet) \to \dots$$

References

[Dem12] Jean-Pierre Demailly. *Complex Analytic and Differential Geometry*. Université de Grenoble I, 2012.

[Huy05] Daniel Huybrechts. *Complex Geometry An Introduction*. Springer Berlin, Heidelberg, 2005.

[Liu23] Bowen Liu. Riemannian geometry. `https://bowenl-math.github.io/notes/2022Fall/Riemannian_geo.pdf`, 2023.

Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, P.R. China,

Email address: `liubw22@mails.tsinghua.edu.cn`