TORIC VARIETY

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Contents

1. Cones and affine toric variety

1.1. Affine semigroups.

Definition 1.1.1 (affine semigroup)**.** An affine semigroup S is a semigroup group such that

(1) The binary operation on S is communicative.
(2) The semigroup is finitely generated.
(3) The semigroup can be embedded in a lattice M.

Example 1.1.1. $\mathbb{N}^n \subseteq \mathbb{Z}^n$ is an affine semigroup.

Example 1.1.2. Given a finite set $\mathcal{A}$ of a lattice M, $\mathbb{N}\mathcal{A} \subseteq M$ is an affine semigroup.

Definition 1.1.2 (semigroup algebra)**.** Let $S \subseteq M$ be an affine semigroup. The semigroup algebra $\mathbb{C}[S]$ is the vector space over $\mathbb{C}$ with S as basis and multiplication is induced by the semigroup structure.

Example 1.1.3. The affine semigroup $\mathbb{N}^n \subseteq \mathbb{Z}^n$ gives the polynomial ring

$$\mathbb{C}[\mathbb{N}^n] = \mathbb{C}[x_1, \dots, x_n]$$

Example 1.1.4. If $e_1, \dots, e_n$ is a basis of a lattice M, then M is generated by $\mathcal{A} = \{\pm e_1, \dots, \pm e_n\}$ as an affine semigroup, and the semigroup algebra gives the Laurent polynomial ring

$$\mathbb{C}[M] = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

1.2. Cones.

In this section we assume M, N are dual lattices with associated $\mathbb{R}$-vector spaces $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, and the pairing between M and N is denoted by $\langle \text{-}, \text{-} \rangle$.

1.2.1. *Convex polyhedral cones.*

Definition 1.2.1 (convex polyhedral cone)**.** Let $S \subseteq N_{\mathbb{R}}$ be a finite subset. A convex polyhedral cone in $N_{\mathbb{R}}$ generated by S is a set of the form

$$\sigma = \operatorname{Cone} S = \{\sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0\} \subseteq N_{\mathbb{R}}.$$

Notation 1.2.1. $\operatorname{Cone}(\varnothing) = \{0\}$.

Remark 1.2.1. A convex polyhedral cone is convex, that is $x, y \in \sigma$ implies $\lambda x + (1-\lambda) y \in \sigma$ for all $0 \leq \sigma \leq 1$, and it's a cone, that is $x \in \sigma$ implies $\lambda x \in \sigma$ for all $\lambda \geq 0$. Since we will only consider convex cones, the cones satisfying Definition 1.2.1 will be called polyhedral cone for convenience.

Definition 1.2.2 (dimension)**.** The dimension of a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is the dimension of the smallest subspace $W \subseteq N_{\mathbb{R}}$ containing σ.

Definition 1.2.3 (dual cone)**.** Let $\sigma \subseteq N_{\mathbb{R}}$ be a polyhedral. The dual cone is defined by

$$\sigma^{\vee} := \{u \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}.$$

Definition 1.2.4 (hyperplane)**.** Given $m \in M_{\mathbb{R}}$, the hyperplane given by m is defined by

$$H_m := \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle = 0\} \subseteq N_{\mathbb{R}},$$

and the closed half-space given by m is defined by

$$H_m^+ := \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle \geq 0\} \subseteq N_{\mathbb{R}}.$$

Definition 1.2.5 (supporting hyperplane)**.** The supporting hyperplane of a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is a hyperplane H_m such that $\sigma \subseteq H_m^+$, and H_m^+ is called a supporting half-space.

Remark 1.2.2. H_m is a supporting hyperplane of σ if and only if $m \in \sigma^\vee$, and if $m_1, \ldots, m_s$ generates $\sigma^\vee$, then

$$\sigma = H_{m_1}^+ \cap \cdots \cap H_{m_s}^+.$$

Thus every polyhedral cone is an intersection of finitely many closed half-spaces.

Definition 1.2.6 (face)**.** A face of a polyhedral cone σ is $\tau = H_m \cap \sigma$ for some $m \in \sigma^\vee$, written $\tau \preceq \sigma$. Faces $\tau \neq \sigma$ are called proper faces, written $\tau \prec \sigma$.

Definition 1.2.7 (facet and edge)**.** A facet of a polyhedral cone σ is a face of codimension one, and an edge of σ is a face of dimension one.

1.2.2. *Strongly convex.*

Definition 1.2.8 (strongly convex)**.** A polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is strongly convex if $\{0\}$ is a face of σ.

1.2.3. *Rational polyhedral cones.*

Definition 1.2.9 (rational)**.** A polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is rational if $\sigma = \mathrm{Cone}(S)$ for some finite subset $S \subseteq N$.

Definition 1.2.10 (ray generator)**.** Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone and ρ be an edge of σ. The unique generator of semigroup $\rho \cap N$ is called ray generator of ρ, written u_ρ.

Definition 1.2.11. Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone.

2. Fans and toric variety

3. Divisors on toric variety

4. Line bundles on toric variety

5. Canonical divisors of toric variety

References

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