PRINCIPAL BUNDLE AND ITS APPLICATIONS

BOWEN LIU

Abstract.

Contents

0. Preface

0.1. About this lecture.

Part 1. The Yang-Mills equations on Riemann surface

1. The Yang-Mills equations

In this section we assume G is a compact Lie group, since we desire Killing form of G is non-degenerate, and (M, g) is an oriented compact Riemannian manifold, since we need to consider integration.

1.1. The Yang-Mills functional.

Let P be a principal G-bundle, V is a vector space and $\rho : G \to \mathrm{GL}(V)$ is a representation of G. If we want to construct an inner product on $\Omega^k_M(P \times_\rho V)$, firstly on each local trivialization U_α, view such forms as forms with values in V, so all we need is an inner product on V, since we already have a Riemannian metric g on M, which induces an inner product on forms.

But if we desire such inner product $\langle \text{-}, \text{-} \rangle$ can be glued well on overlaps, we need to require that it is G-invariant, that is, for all $g \in G, v, w \in V$,

$$\langle \rho(g)w, \rho(g)w \rangle = \langle v, w \rangle$$

since if $\omega \in C^\infty(M, \Omega^k_M(P \times_\rho V))$ is represented locally by $\omega_\alpha \in C^\infty(U_\alpha, \Omega^k_{U_\alpha}(V))$, then on a non-empty overlap $U_{\alpha\beta}$, we have $\omega_\alpha = \rho(g_{\alpha\beta})\omega_\beta$.

The case we're most interested in is $V = \mathfrak{g}$, since curvature of a connection is a section of $\Omega^2_M(\operatorname{Ad} \mathfrak{g})$. So we what we need is an inner product on Lie algebra $\mathfrak{g}$ which is invariant under the adjoint action. Since G is compact, its Killing form is a non-degenerate inner product, that's what we're looking for!

Thus we have an pointwise inner product on the bundle $\Omega^k_M(\operatorname{Ad} \mathfrak{g})$, and denote it by $\langle \text{-}, \text{-} \rangle$, and define a global inner product on $\Omega^k_M(\operatorname{Ad} \mathfrak{g})$ as

$$(\alpha, \beta) := \int_M \langle \alpha, \beta \rangle \operatorname{vol}$$

where $\alpha, \beta \in C^\infty(M, \Omega^k_M(\operatorname{Ad} \mathfrak{g}))$.

Definition 1.1.1 (Hodge star operator)**.** There exists an operator

$$* : C^\infty(M, \Omega^k_M(\operatorname{Ad} \mathfrak{g})) \to C^\infty(M, \Omega^{n-k}_M(\operatorname{Ad} \mathfrak{g}))$$

For $\beta \in C^\infty(M, \Omega^k_M(\operatorname{Ad} \mathfrak{g}))$, $*\beta$ is given by

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \operatorname{vol}, \quad \forall \alpha \in C^\infty(M, \Omega^k_M(\operatorname{Ad} \mathfrak{g}))$$

With some of the preliminary results established, we arrive at the Yang-Mills functional.

Definition 1.1.2 (Yang-Mills functional)**.** The Yang-Mills functional is the map $YM : \mathscr{A}(P) \to \mathbb{R}$ given by

$$YM(\omega) := \|F_\omega\|^2 = \int_M \langle F_\omega, F_\omega \rangle \operatorname{vol}$$

where F_ω is curvature of connection ω, which is a section of $\Omega^2_M(\operatorname{Ad}\mathfrak{g})$.

Remark 1.1.1. By using Hodge star operator, we may rewrite Yang-Mills functional as follows

$$YM(\omega) = \int_M F_\omega \wedge *F_\omega$$

The advantages of writing Yang-Mills functional in this way is that we can use some properties of Hodge operator to simplify our computations

Proposition 1.1.1. Yang-Mills functional YM is gauge invariant, that is for any gauge transformation $\Phi \in \mathscr{G}(P)$, one has $YM(\Phi^*\omega) = YM(\omega)$ holds for connection ω.

Proof. On each local trivialization U_α, the curvature of $\Phi^*\omega$ is given by $\operatorname{Ad}(\phi^{-1}) \circ F_\alpha$, where ϕ is given by $\Phi|_{U_\alpha}(x,g) = (x, \phi(x)g)$, thus Yang-Mills functional is gauge invariant follows from inner product $\langle \text{-}, \text{-} \rangle$ is adjoint invariant. $\square$

Definition 1.1.3 (Yang-Mills connection)**.** A Yang-Mills connection is a connection $A \in \mathscr{A}(P)$ which is a local extremum of Yang-Mills functional.

Notation 1.1.1. $\mathscr{A}_{YM}(P)$, or briefly $\mathscr{A}_{YM}$ denotes the set of all Yang-Mills connections.

1.2. **The variational problem.** Let's see how to use a second-order partial differential equation to characterize Yang-Mills connection. Recall that $\mathscr{A}(P)$ is an affine space modelled on $\Omega^1_M(\operatorname{Ad}\mathfrak{g})$. This means the tangent space to $\mathscr{A}(P)$ at any point is isomorphic to $\Omega^1_M(\operatorname{Ad}\mathfrak{g})$.

Given $\omega \in \mathscr{A}(P)$ and $\tau \in C^\infty(M, \Omega^1_M(\operatorname{Ad}\mathfrak{g}))$. The directional derivative of Yang-Mills functional at ω in the direction τ is given by

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} YM(\omega + t\tau)$$

And Yang-Mills condition states that this vanishes for all τ. In order to see what this means, firstly we need the following lemma.

Lemma 1.2.1. Given $\omega \in \mathscr{A}(P)$ and $\tau \in C^\infty(M, \Omega^1_M(\operatorname{Ad}\mathfrak{g}))$, then

$$F_{\omega+\tau} = F_\omega + \mathrm{d}_\omega \tau + \frac{1}{2}\tau \wedge \tau$$

where d_ω is connection induced by ω on $\Omega^1_M(\operatorname{Ad}\mathfrak{g})$.

Proof. On local trivialization U_α one has

$$\begin{aligned}(F_{\omega+\tau})_\alpha &= \mathrm{d}(A_\alpha+\tau_\alpha)+\frac{1}{2}(A_\alpha+\tau_\alpha)\wedge(A_\alpha+\tau_\alpha)\\ &= (F_\omega)_\alpha+\mathrm{d}\tau_\alpha+\frac{1}{2}(A_\alpha\wedge\tau_\alpha+\tau_\alpha\wedge A_\alpha)+\frac{1}{2}\tau_\alpha\wedge\tau_\alpha\\ &\overset{(1)}{=} (F_\omega)_\alpha+\mathrm{d}\tau_\alpha+A_\alpha\wedge\tau_\alpha+\frac{1}{2}\tau_\alpha\wedge\tau_\alpha\\ &\overset{(2)}{=} (F_\omega)_\alpha+\mathrm{d}_\omega\tau_\alpha+\frac{1}{2}\tau_\alpha\wedge\tau_\alpha\end{aligned}$$

where

(1) holds from both A_α, τ_α are 1-form valued in $\mathfrak{g}$;
(2) holds from (**??**).

□

Proposition 1.2.1 (first variation formula)**.** Let ω be a Yang-Mills connection, then we have

$$\mathrm{d}_\omega^* F_\omega = 0$$

Proof. Direct computation shows

$$\begin{aligned}YM(\omega+t\tau) &= \int_M \langle F_{\omega+t\tau}, F_{\omega+t\tau}\rangle \,\mathrm{vol}\\ &= \int_M \langle F_\omega+\frac{t^2}{2}(\tau\wedge\tau)+t\mathrm{d}_\omega\tau, F_\omega+\frac{t^2}{2}(\tau\wedge\tau)+t\mathrm{d}_\omega\tau\rangle \,\mathrm{vol}\end{aligned}$$

The coefficient of linear term is

$$\int_M \langle F_\omega, \mathrm{d}_\omega\tau\rangle+\langle \mathrm{d}_\omega\tau, F_\omega\rangle \,\mathrm{vol} = 2\int_M \langle \mathrm{d}_\omega\tau, F_\omega\rangle \,\mathrm{vol}$$

Let $\mathrm{d}_\omega^* = (-1)^{2n+1} * \mathrm{d}_\omega *$ denote the formal adjoint to d_ω. Then we have

$$\int_M \langle \mathrm{d}_\omega\tau, F_\omega\rangle \,\mathrm{vol} = \int_M \langle \tau, \mathrm{d}_\omega^* F_\omega\rangle \,\mathrm{vol}$$

this shows

$$\mathrm{d}_\omega^* F_\omega = 0$$

□

Definition 1.2.1 (Yang-Mills equations)**.** A connection $\omega \in \mathscr{A}(P)$ is called satisfying Yang-Mills equations, if

$$\begin{cases}\mathrm{d}_\omega F_\omega = 0\\ \mathrm{d}_\omega^* F_\omega = 0\end{cases}$$

Remark 1.2.1. The first equation is also called Bianchi identity.

Example 1.2.1. In the case that $G = U(1)$, we have that the curvature of a connection A can be identified as a section of Ω_M^2. Indeed, the curvature form takes value in the bundle $\operatorname{Ad}\mathfrak{g}$, but here $G = U(1)$ is abelian, thus the adjoint action on $\mathfrak{u}(1)$ is trivial, so

$$\operatorname{Ad}\mathfrak{g} = M \times \mathfrak{u}(1) = M \times \mathbb{R}$$

is trivial bundle. Furthermore, ω is a Yang-Mills connection if and only if F_ω is a harmonic 2-form, that is $\Delta F_\omega = 0$, where $\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$. Indeed, thanks to $U(1)$ is abelian again, d_ω can be reduced to d, since for arbitrary form β, we have $\omega \wedge \beta = 0$. This follows from in the definition of wedge product of forms valued in Lie algebra we used Lie bracket, and abelian Lie algebra has trivial Lie bracket. Note that F_ω is harmonic if and only if

$$\begin{cases} \mathrm{d}^* F_\omega = 0 \\ \mathrm{d}F_\omega = 0 \end{cases}$$

It's a standard result in differential geometry, which can be seen from

$$\begin{aligned} 0 &= \int_M \langle \Delta F_\omega, F_\omega \rangle \operatorname{vol} \\ &= \int_M \langle \mathrm{d}\mathrm{d}^* F_\omega, F_\omega \rangle + \langle \mathrm{d}^*\mathrm{d}F_\omega, F_\omega \rangle \operatorname{vol} \\ &= \int_M \|\mathrm{d}^* F_\omega\|^2 + \|\mathrm{d}F_\omega\|^2 \operatorname{vol} \end{aligned}$$

2. GIT quotient and sympletic quotient

Note that the Yang-Mills functional is guage invariant, so if a connection ω solves the Yang-Mills equations, so does any gauge transformed $\Phi^*\omega$. In other words, the gauge group acts on $\mathscr{A}_{YM}$. The quotient $\mathscr{A}_{YM}/\mathscr{G}$ is the space of classical solutions. In general it is infinite dimensional, and the topology of this space may be quite bad. For example it may be neither Hausdorff or a smooth manifold. But adding some restrictions, we do have a good correspondence, and that's main theorem for this section.

2.1. **A Fairy Tale.**

To get a picture of the action of gauge group on $\mathscr{A}(P)$, let's study a finite-dimensional analogue: Let V be a complex vector space with a hermitian inner product $\|\cdot\|$, and S^1 acts on it by unitary matrices, that is there is a group homomorphism $S^1 \to \mathrm{U}(V)$, if we consider the complexification of this action, we obtain $\mathbb{C}^* \to \mathrm{GL}(V)$. The goal is to understand the quotient space $V/\mathbb{C}^*$, but this space can be quite unpleasant. Let's see an example:

Example 2.1.1. Let $\lambda \in \mathbb{C}^*$ acting on $\mathbb{C}^2$ by $(x, y) \mapsto (\lambda^{-1}x, \lambda y)$. The orbits are

1. the conics $xy = c, c \neq 0 \in \mathbb{C}$;
2. the axes $y = 0, x \neq 0$ or $x = 0, y \neq 0$;
3. the origin.

It's clear that the topology on the orbit space is not Hausdorff, since origin lies in the closure of axes. But note that $\mathbb{C}^2 \setminus \{\text{axes}\}/\mathbb{C}^*$ is Hausdorff. Indeed, it's homeomorphic to $\mathbb{C}$.

The problems arise from that axes are not closed orbits. More generally, if we want to form Hausdorff quotients, we just need to consider only closed orbits which are closed sets.

Definition 2.1.1 (stable)**.** A point $v \in V$ is stable if its orbit under $\mathbb{C}^*$ is closed.

Now let's see a criterion for whether an orbit is closed or not.

Theorem 2.1.1. A point $v \in V$ is stable if and only if the function $p_v : \mathbb{C}^* \to \mathbb{R}$, defined by $p_v(g) := \|g(v)\|^2, g \in \mathbb{C}^*$, attains its minimum.

Remark 2.1.1. Note that since the norm is $\mathrm{U}(V)$-invariant, then the function p_v is S^1-invariant and descend to a function on $\mathbb{C}^*/S^1$, given by

$$p_v(x) := \|e^x(v)\|^2$$

where $x \in \mathbb{C}^*/S^1$. In Example 2.1.1, we have $e^x(v_1, v_2) = (e^{-x}v_1, e^x v_2)$, so we have

$$p_v(x) = \|v_1\|^2 e^{-2x} + \|v_2\|^2 e^{2x}$$

Take its derivative and let it equal zero

$$\frac{\mathrm{d}p_v(x)}{\mathrm{d}x} = -2\|v_1\|^2 e^{-2x} + 2\|v_2\|^2 e^{2x} = 0$$

we have this function take its minimum at

$$\frac{1}{2}(\log(\|v_1\|) - \log(\|v_2\|))$$

if both v_1 and v_2 are not zero, and at 0 if $v = 0$. Furthermore, the minimum is not attained along the two punctured axes. In fact, Example 2.1.1 is quite representative.

Proof. Note that the S^1-action is reducible, so V splits into an orthogonal direct sum $V_1 \oplus \cdots \oplus V_n$ of 1-dimensional representations where S^1 acts on V_m as $v_m \mapsto \lambda^{j_m} v_m$ for some weight j_m. Therefore we have

$$p_v(x) = \sum_m \|v_m\|^2 e^{2j_m x} = \sum_{k=-\infty}^{\infty} a_k e^{kx}$$

where only finitely many $a_k \neq 0$. We divide our analysis into three cases:

1. $a_k = 0$ for all $k \neq 0$. In this case the minimum is obviously attained and the orbit is obviously closed since $j_m = 0$ so the action fixes v.
2. $a_k = 0$ for all $k < 0$ (resp. $k > 0$) and $a_k \neq 0$ for some $k > 0$ (resp. $k < 0$). In this case the minimum is obviously not attained and the orbit is obviously not closed since $e^x(v)$ tends to an orbit of the first type as $x \to -\infty$ (resp. ∞).
3. There is a $k > 0$ and a $k' < 0$ such that $a_k \neq 0$ and $a_{k'} \neq 0$. In this case the minimum is obviously attained (just do the calculus). We will now show that this implies v is stable.

Conversely, if v is not stable, then p_v does not attain its minimum. Indeed, if v is not stable then its orbit is not closed so there exists $w \in V$ such that $w \in \overline{\mathbb{C}^*(v)}$ but $w \notin \mathbb{C}^*(v)$, so either $w = \lim_{x\to\pm\infty} e^x(v)$. The corresponding limit $\lim_{x\to\pm\infty} p_v(x) = p_v(w)$ is finite and hence the j_m are either all nonpositive or all nonnegative. Since $w \neq v$ there must be one j_m which is nonzero. It's now easy to see that the function $p_v(x)$ is of type II and hence does not attain its minimum. □

So the above criterion says if we want to understand the space of stable points, it's neccessary to understand the critical point of p_v. Take derivative we have

$$\frac{\mathrm{d}p_v}{\mathrm{d}x} = 2\sum_{m=1}^{n} j_m \|v_m\|^2 e^{2j_m x}$$

Suppose v is stable and without lose of generality we assume its minimum occurs at $x = 0$. Therefore the orbit of a stable vector contains a zero of the function

$$\mu = \sum_{m=1}^{n} j_m \|v_m\|^2 : V \to \mathbb{R}$$

So we can restate above criterion as follows, and it's a fairy tale version of Kempf-Ness.

Theorem 2.1.2 (Kempf-Ness)**.** Let V^s denote the space of stable vectors under the action of $\mathbb{C}^*$. Then

$$V^s/\mathbb{C} = \mu^{-1}(0)/S^1$$

Let's define more conceptions which we will see a more abstract version later. Let V be a vector space and $Q : V \otimes V \to \mathbb{R}$ a non-degenerate bilinear form. Then we can make a 1-form ω into a vector field X by defining

$$\omega(Y) = Q(X, Y), \quad \forall Y \in TV$$

In particular, if $f : V \to \mathbb{R}$ is a function and consider its derivative 1-form $\mathrm{d}f$. Then it corresponds to a vector field $\mathrm{Qgrad}(f)$ by defining

$$\mathrm{d}f(Y) = Q(\mathrm{Qgrad}(f), Y)$$

We call f the Hamiltonian generating $\mathrm{Qgrad}(f)$, and μ the moment map for the circle action.

Example 2.1.2. Take $f(x, y) = x^2 + y^2$ and $Q = \mathrm{d}x \wedge \mathrm{d}y$. Then

$$\mathrm{Qgrad}\, f = -y\partial_x + x\partial_y$$

2.2. **Kempf-Ness Theorem.** In this section, we give an abstract version of Kempf-Ness theorem. Let X be a Kähler manifold, $K \subset G$ denote the maximal compact subgroup, which has the property that its complexification is isomorphic to G.

Suppose that the action of K on X is sympletic, i.e. the action of any $k \in K$ preserves the Kähler metric on X. Let $\mathfrak{k}$ denote the Lie algebra of K. Then the infinitesimal action of K is given by the Lie algebra homomorphism $\mathfrak{k} \to \mathfrak{X}(X)$ defined by $\xi \mapsto X_\xi$, where

$$(X_\xi)_p := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} p \exp(t\xi)$$

Definition 2.2.1 (Hamiltonian)**.** A symplectic action of K on X is Hamiltonian if for each $\xi \in \mathfrak{k}$, there exists a function $H_\xi : X \to \mathbb{R}$ such that for all $p \in X$ and $v \in T_pX$ we have

$$\omega_p((X_\xi)_p, v) = (\mathrm{d}H_\xi)_p(v)$$

and the mapping $\xi \mapsto H_\xi$ is K-equivariant with respect to the right action of K on $\mathfrak{k}$ by the adjoint action and precomposition with right translation R_k on $C^\infty(X)$. The functions H_ξ are called Hamiltonian functions.

Definition 2.2.2 (moment map)**.** Suppose we have a Hamiltonian action of K on X. A moment map for the action is a K-equivariant map $\mu : X \to \mathfrak{k}^*$(where the action on $\mathfrak{k}^*$ is the coadjoint action) such that for any $p \in X, v \in T_pX$ and $\xi \in \mathfrak{k}$, we have

$$\mathrm{d}\mu_p(v)(\xi) = \omega_p((X_\xi)_p, v)$$

Remark 2.2.1. Let's make coadjoint action more clear:

Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Let $\operatorname{ad} : G \to \operatorname{Aut}(\mathfrak{g})$ denote the adjoint representation of G. Then we can define its coadjoint representation $\operatorname{ad}^* : G \to \operatorname{Aut}(\mathfrak{g}^*)$ as

$$\langle \operatorname{ad}_g^* \mu, Y \rangle = \langle \mu, \operatorname{ad}_{g^{-1}} Y \rangle$$

for $g \in G, Y \in \mathfrak{g}, \mu \in \mathfrak{g}^*$.

Remark 2.2.2. One thing to note is that the Hamiltonian functions can be recovered by the moment maps. If a Hamiltonian action admits a moment map, then

$$H_\xi(p) = \mu(p)(\xi)$$

Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{k}^*$ that is invariant under the coadjoint action, and $\| \cdot \|$ be the induced norm. Since X is compact, then the map $\|\mu\|^2 : X \to \mathbb{R}$ attains its minimum, and without lose of generality we assume that the minimum value is 0.

Definition 2.2.3 (symplectic quotient)**.** The symplectic quotient of X by K is the quotient space

$$\mu^{-1}(0)/K$$

Remark 2.2.3. The symplectic quotient can also be referred to as the symplectic reduction. It should be noted that the symplectic quotient depends on our choice of moment map.

Theorem 2.2.1. The symplectic quotient of X by K admits a unique Kähler structure such that the Kähler metric on $\mu^{-1}(0)/K$ is induced by the Kähler metric on X.

The relationship between the GIT quotient and the symplectic quotient is given by the Kempf-Ness theorem

Theorem 2.2.2 (Kempf-Ness)**.** Suppose a complex reductive group G acts on a Kähler manifold X such that the action of the maximal compact subgroup $K \subset G$ is Hamiltonian and admits a moment map $\mu : X \to \mathfrak{k}^*$. Then the G-orbit of any semistable point contains a unique K-orbit minimizing $\|\mu\|^2$. This establish a homeomorphism

$$X_{ss}/G \longleftrightarrow \mu^{-1}(0)/K$$

3. Holomorphic Vector Bundles and Hermitian Yang-Mills Connections

In the second part, we have already established the foundations of Yang-Mills equations in a general stage, or in other words, in the stage of Riemannian manifold (M, g).

As we have seen, when the dimension of underlying space is one, all curvature forms are trivial, so there is nothing interesting. Thus the first "non-trivial" theory arises when our underlying space is of dimension two.

This prototype theory merits a good deal of study due to the richness of structures naturally occurring on such manifold, such as a complex structure associated to the almost complex structure determined by the Hodge star operator $* : \Omega^p_M \to \Omega^{2-p}_M$. Furthermore, smooth Hermitian vector bundle E over Riemann surface have inherent holomorphic structures due to the vacuous integrability conditions on connections on E, in other words, this gives a correspondence between unitary connections and holomorphic structure $\overline{\partial}_E$ on E. Thus the study of Yang-Mills connections on Riemann surface can be put into a complex analytic framework.

Using such ideal, we give a description of Kempf-Ness theorem which relates symplectic quotient and GIT quotient. In this section, if the underlying space is a Riemann surface, we will see there is a parallel story for the action of gauge group $\mathscr{G}$ on the space of connections $\mathscr{A}(P)$.

We will complexify the action of $\mathscr{G}$ and state a theorem analogous to Kempf-Ness theorem, which is known as Narasimhan-Seshadri theorem.

Notation 3.0.1. For complex manifold X, we use Ω^k_X to denote the space of smooth complex-valued k-forms, and use $\Omega^{p,q}_X$ to denote the space of smooth (p, q)-forms.

3.1. Moment map in Yang-Mills theory.

When X is a Riemann surface, the space of connections has a natural symplectic form. As we already knonw, $\mathscr{A}(P)$ is affine modelled on $\Omega^1_X(\mathfrak{g}_P)$, then we consider the following non-degenerate symplectic form

$$Q(\alpha, \beta) = \int_X \alpha \wedge \beta, \quad \alpha, \beta \in \Omega^1_X(\mathfrak{g}_P)$$

where this integral do make senses since the real dimension of X is two.

Take $\phi \in \Omega^0_M(\mathfrak{g}_P)$, we can get a vector field on $\mathscr{A}(P)$ by the action of ∇ on ϕ, that is $V = \nabla\phi$.

Lemma 3.1.1. The function $f : \nabla \to -\int_X F_\nabla \wedge \phi$ is a Hamiltonian functions on $\mathscr{A}(P)$ generating V.

Proof. It suffices to check

$$Q(\nabla\phi, A) = \mathrm{d}f(A), \quad \forall A \in \Omega^1_X(\mathfrak{g}_P)$$

Integration by parts we have

$$\begin{aligned} Q(\nabla\phi, A) &= \int_X \nabla\phi \wedge A \\ &= -\int_X \phi \wedge \nabla A \\ &= -\int_X \nabla A \wedge \phi \end{aligned}$$

Note that $F_{\nabla+\varepsilon A} = F_\nabla + \varepsilon\nabla A + O(\varepsilon^2)$, then

$$\begin{aligned} \mathrm{d}f(A) &= \lim_{\varepsilon\to 0} \frac{-\int_X F_{\nabla+\varepsilon A} \wedge \phi + \int_X F_\nabla \wedge \phi}{\varepsilon} \\ &= -\int_X \nabla A \wedge \phi \end{aligned}$$

As desired. $\square$

Remark 3.1.1. In our case the Lie algebra of gauge group is $\Omega^2_X(\mathfrak{g}_P)$ and the moment map is just

$$\nabla \mapsto -F_\nabla$$

The Yang-Mills functional is just the norm of the moment map.

Our ultimate goal is to relate moduli spaces of holomorphic vector bundles over X to Yang-Mills connections. Firstly, we want to consider $\mathscr{A}(P)$ as a space of holomorphic vector bundles.

4. Moduli space of semi-stable vector bundles

In thise section, the guiding problem is to classify holomorphic vector bundles on a Riemann surface with genus g, denoted by Σ_g. For the case $g = 0, 1$, there are complete classification results for holomorphic vector bundles on Σ_g, due to Grothendieck for the case of the Riemann sphere [?], and due to Atiyah for the case of elliptic curves [?]. So in the following discussion, we always assume $g \geq 2$.

4.1. Stable bundle.

Definition 4.1.1 (holomorphic vector bundle)**.** A holomorphic vector bundle is a complex vector bundle $\pi : E \to X$ such that the total space E is a complex manifold and π is holomorphic.

Definition 4.1.2 (degree)**.** Let $\pi : E \to X$ be a holomorphic vector bundle, its degree is defined as

$$\deg(E) := \int_X c_1(E)$$

where $c_1(E) \in H^2(X, \mathbb{Z})$ is the first Chern class of E.

Definition 4.1.3 (slope)**.** Let $\pi : E \to X$ be a holomorphic vector bundle, its slope is defined as

$$\mu(E) := \frac{\deg(E)}{\operatorname{rank}(E)}$$

Remark 4.1.1. One thing to note is that the slope of a holomorphic vector bundle is independent of the holomorphic structure, since both the degree and rank are topological invariants.

Definition 4.1.4 (slope stability)**.** Let $\pi : E \to X$ be a holomorphic vector bundle, it's

1. stable if for every non-trivial holomorphic subbundle F, $\mu(F) < \mu(E)$;
2. semi-stable if for every non-trivial holomorphic subbundle F, $\mu(F) \leq \mu(E)$;
3. unstable if it's not semi-stable.

Remark 4.1.2. For slope stability, we have the following remarks:

(a) It's clear that all holomorphic line bundles are stable, since they don't have non-trivial subbundles;
(b) A semi-stable vector bundle with coprime rank and degree is actually stable, since
(c) While the slope is a topological invariant, slope stability is not, since here we only consider holomorphic subbundles, which depends on the holomorphic structure.

Proposition 4.1.1. Let $E \to \Sigma_g$ be a holomorphic vector bundle, it's

1. stable if and only if for every non-trivial holomorphic subbundle F, $\mu(E/F) > \mu(E)$;

2. semi-stable if and only if for every non-trivial holomorphic subbundle F, $\mu(E/F) \geq \mu(E)$.

Proof. Denote r, r', r'' the ranks of $E, F, E/F$ respectively, and d, d', d'' their degrees respectively. From exact sequence

$$0 \to E \to E \to E/F \to 0$$

one has $r = r' + r''$ and $d = d' + d''$, thus

$$\frac{d'}{r'} < \frac{d' + d''}{r' + r''} \iff \frac{d'}{r'} < \frac{d''}{r''} \iff \frac{d' + d''}{r' + r''} < \frac{d''}{r''}$$

and likewise with the case semi-stable. □

A philosophy is that semi-stable bundles don't admit too many subbundles, since any subbundle they may have is of slope no greater than their own. This turns out to have many interesting consequences we're going to show, for example, the category of semi-stable bundles is abelian.

Lemma 4.1.1. If $\varphi : E \to E'$ is a non-zero homomorphism of holomorphic vector bundles over Σ_g, then

$$\mu(E/\ker\varphi) \leq \mu(\operatorname{im}\varphi)$$

Proposition 4.1.2. Let E, E' be two semi-stable bundles such that $\mu(E) > \mu(E')$, then any homomorphism $\varphi : E \to E'$ is zero.

Proof. If φ is non-zero, since E is semi-stable, then

$$\mu(\operatorname{im}\varphi) \overset{(1)}{\geq} \mu(E/\ker\varphi) \overset{(2)}{\geq} \mu(E) > \mu(E')$$

where

(1) holds from Lemma 4.1.1;

(2) holds from Proposition 4.1.1.

which contradicts to the semi-stablity of E'. □

Proposition 4.1.3. Let $\varphi : E \to E'$ be a non-zero homomorphism of semi-stable holomorphic of slope μ, then $\ker\varphi$ and $\operatorname{im}\varphi$ are semi-stable bundles of slope μ, and the natural map $E/\ker\varphi \to \operatorname{im}\varphi$ is an isomorphism.

Corollary 4.1.1. The category of semi-stable bundles of slope μ is abelian, and the simple object[1] in this category is the stable bundles of slope μ.

Proof. By Proposition 4.1.3 one has the category of semi-stable bundles of slope μ is abelian. A stable bundle E is simple in this category, since it admits no non-trivial subbundles with slope μ; Conversely, if a semi-stable bundle E is simple, then any non-trivial subbundle F satisfies $\mu(F) \leq \mu(E)$ since E is semi-stable and $\mu(F) \neq \mu(E)$ since E is simple, this shows E is stable. □

[1]Recall a simple object in an abelian category is an object with no non-trivial subobject.

Proposition 4.1.4. Let E, E' be two stable vector bundles over Σ_g with same slopes, and $\varphi : E \to E'$ be a non-zero homomorphism, then φ is an isomorphism.

Proof. Since $\varphi : E \to E'$ is a non-zero homomorphism between stable bundles with same slopes, then by Proposition 4.1.3 one has $\ker\varphi$ is either 0 or has slope $\mu(E)$, but E is actually stable, then $\ker\varphi$ must be 0, and since φ is strict, this shows φ is injective. Likewise, $\operatorname{im}\varphi \neq 0$ and has slope $\mu(E')$, then it must be E' since E' is stable. Then again by φ is strict, $\operatorname{im}\varphi = E'$ impiles φ is surjective. Therefore φ is an isomorphism. □

Proposition 4.1.5. If E is a stable bundle over Σ_g, then $\operatorname{End} E = \mathbb{C}$. In particular, $\operatorname{Aut} E = \mathbb{C}^*$.

Proof. Let φ be a non-zero endomorphism of E, by Proposition 4.1.4 one has φ is an automorphism, so $\operatorname{End} E$ is a field, which contains $\mathbb{C}$ as its subfield of scalar endomorphisms. For any $\varphi \in \operatorname{End} E$, by Cayley-Hamilton theorem one has φ is algebraic over $\mathbb{C}$, and since $\mathbb{C}$ is algebraic closed, this shows $\operatorname{End} E \cong \mathbb{C}$. □

Corollary 4.1.2. A stable bundle is indecomposable, that is it's not isomorphic to a direct sum of non-trivial subbundles.

Proof. The automorphism group of $E = E_1 \oplus E_2$ contains $\mathbb{C}^* \times \mathbb{C}^*$, so by Proposition 4.1.5 it can't be stable. □

Theorem 4.1.1 (Jordan-Hölder filteration)**.** Any semi-stable bundle of slope μ over Σ_g admits a filteration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_k = E$$

by holomorphic subbundles such that for each $1 \leq i \leq k$, one has
1. E_i/E_{i-1} is stable;
2. $\mu(E_i/E_{i-1}) = \mu(E)$.

Proposition 4.1.6 (Seshadri)**.** Any two Jordan-Hölder filterations

$$S : 0 = E_0 \subset E_1 \subset \cdots \subset E_k = E$$

and

$$S' : 0 = E'_0 \subset E'_1 \subset \cdots \subset E'_l = E$$

of a semi-stable bundle E have same length, and the associated graded objects

$$\operatorname{gr}(S) : 0 = E_1/E_0 \oplus \cdots \oplus E_k/E_{k-1}$$

and

$$\operatorname{gr}(S') : 0 = E'_1/E'_0 \oplus \cdots \oplus E'_k/E'_{k-1}$$

satisfy $E_i/E_{i-1} \cong E'_i/E'_{i-1}$ for all $1 \leq i \leq k$.

Definition 4.1.5 (poly-stable bundle)**.** A holomorphic vector bundle E over Σ_g is called poly-stable if it is isomorphic to a direct sum $E_1 \oplus \cdots \oplus E_k$ of stable bundles of the same slope.

Example 4.1.1. A stable bundle is poly-stable.

Example 4.1.2. The graded object associated to any Jordan-Hölder filteration of a semi-stable bundle E is a poly-stable, and by Proposition 4.1.6, it's unique up to isomorphism, this isomorphic class is denoted by $\mathrm{gr}(E)$.

Definition 4.1.6 (S-equivalence class)**.** The graded isomorphism class $\mathrm{gr}(E)$ associated to a semi-stable bundle E is called the S-equivalence class of E. If $\mathrm{gr}(E) \cong \mathrm{gr}(E')$, E and E' are called S-equivalent, and denoted by $E \sim_S E'$.

Definition 4.1.7. The set $\mathcal{M}_{\Sigma_g}(r, d)$ of S-equivalence classes of semi-stable bundles of rank r and degree d over Σ_g is called its moduli set, it contains the set $\mathcal{N}_{\Sigma_g}(r, d)$ of isomorphism classes of stable bundles of rank r and degree d.

Theorem 4.1.2 (Mumford-Seshadri)**.** Let $g \geq 2, r \geq 1$ and $d \in \mathbb{Z}$.

1. The set $\mathcal{N}_{\Sigma_g}(r, d)$ admits a structure of smooth, complex quasi-projective variety of dimension $r^2(g-1)+1$;
2. The set $\mathcal{M}_{\Sigma_g}(r, d)$ admits a structure of complex projective variety of dimension $r^2(g-1)+1$;
3. $\mathcal{N}_{\Sigma_g}(r, d)$ is an open dense subvariety of $\mathcal{M}_{\Sigma_g}(r, d)$.

In particular, when r and d are coprime, $\mathcal{M}_{\Sigma_g}(r, d) = \mathcal{N}_{\Sigma_g}(r, d)$ is a smooth complex projective variety.

Proof. See [**?**] and [**?**]. □

4.2. **The Harder-Narasimhan filteration.**

Theorem 4.2.1 (Harder-Narasimhan)**.** Any holomorphic vector bundle E over Σ_g has a unique filteration

$$0 = E_0 \subset E_1 \subset \dots E_k = E$$

by holomorphic subbundles such that

1. for all $1 \leq i \leq k$, E_i/E_{i-1} is semi-stable;
2. the slope $\mu_i := \mu(E_i/E_{i-1})$ of successive quotients satisfies

$$\mu_1 > \mu_2 > \cdots > \mu_k$$

This filteration is called Harder-Narasimhan filteration.

Proof. See [**?**]. □

Remark 4.2.1. If we denote $r = \operatorname{rank} E, d = \deg E, r_i = \operatorname{rank}(E_i/E_{i-1})$ and $d_i = \deg(E_i/E_{i-1})$, one has

$$r_1 + \cdots + r_k = r, \quad d_1 + \cdots + d_k = d$$

The k-tuple

$$\vec{\mu} := (\underbrace{\mu_1, \dots, \mu_1}_{r_1 \text{ times}}, \dots, \underbrace{\mu_k, \dots, \mu_k}_{r_k \text{ times}})$$

is called the Harder-Narasimhan type of E. It's equivalent to the data of the k-tuple $(r_i, d_i)_{1\leq i\leq k}$. In the plane of coordinates (r, d), the polygonal line

$$P_{\vec{\mu}} := \{(0,0), (r_1, d_1), (r_1 + r_2, d_1 + d_2), \ldots, (r_1 + \cdots + r_k, d_1 + \cdots + d_k)\}$$

defines a convex polygon called the Harder-Narasimhan polygon of E. The slope of the line from $(0,0)$ to (r_1, d_1) is μ_1, that is the slope of E_1/E_0, and perhaps that's why it's called slope. It's indeed convex, since $\mu_1 > \cdots > \mu_k$. A vector bundle is semi-stable if and only if it is it own Harder-Narasimhan filteration, and if and only if its Harder-Narasimhan filteration is a single line from $(0,0)$ to (r,d).

Part 2. GIT quotient and sympletic quotient: the Kempf-Ness theorem

In this section, we mainly follows [?] and [?].

5. Geometric invariant theory

5.1. **Introduction.** Many objects we want to take a quotient always have some sort of geometric structures, and we desire the quotients we obtain preserve geometric structure, for example:

Example 5.1.1. Suppose G is a Lie group and X is a smooth manifold, the quotient X/G will not always have the structure of a smooth manifold (For example, the presence of non-closed orbits, usually gives a non-Hausdorff quotient). However, if G acts properly and freely, then X/G has a smooth manifold structure, such that natural projection $\pi : X \to X/G$ is a smooth submanifold.

Geometric invariant theory (GIT) is the study of such question in the context of algebraic geometry, for example:

Example 5.1.2. Let $M_n(\mathbb{C})$ be the group of all $n \times n$ matrices over $\mathbb{C}$, then it can be given a geometric structure by regarding it as an affine variety. Consider the conjugate action of $\mathrm{GL}_n(\mathbb{C})$ on $M_n(\mathbb{C})$. Can we regard $M_n(\mathbb{C})/\operatorname{GL}_n(\mathbb{C})$ as a variety?

The answer of above question is yes, but good thing does not happen always, consider

Example 5.1.3. Let $\mathbb{C}^\times$ acts on $\mathbb{C}^2$ by $\lambda(x, y) := (\lambda x, \lambda y)$. The $\mathbb{C}^\times$-orbits are $\{(\lambda x, \lambda y) : \lambda \in \mathbb{C}^\times, (x, y) \neq (0, 0)\}$ as well as the origin $\{(0, 0)\}$. Now suppose that the set of orbits is a variety, then every point must be closed

So we need to be more careful when we constructing quotients in the category of varieties. As we have seen in smooth manifold, we can guess

1. only certain types of group (compared with Lie group) are allowed;
2. only certain types of group actions (compared with properly and freely) are allowed.

5.2. **Good categorical quotient.**

Definition 5.2.1 (G-invariant morphism)**.** A morphism $f : X \to Y$ is called G-invariant morphism, if it is constant on orbits.

Definition 5.2.2 (categorical quotient)**.** In any category, we call a G-invariant morphism $\pi : X \to Y$ is categorical quotient of X by G, when for any G-invariant morphism $f : X \to Z$, we have that f factors uniquely through π, that is

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Z \\ & Y \arrow[ur, dotted, "\overline{f}"'] & \end{tikzcd}$$

Remark 5.2.1. Since categorical quotient is defined by its universal property, so it is unique when it exists.

However, for a quotient in the category of varieties, simple being a categorical quotient may not have a good geometric properties, so we need to define good categorical quotient. If G acts on a variety X, then we can get an action on the regular functions on X as follows. For $f \in \mathcal{O}(U), U \subset X$, we define

$$gf(x) = f(g^{-1}.x)$$

For the types of group action we are interested in, we require

$$gf \in \mathcal{O}(U), \quad \forall f \in \mathcal{O}(U)$$

Definition 5.2.3. A surjective G-invariant map of varieties $p : X \to Y$ is called a good categorical quotient of X by G, if the following three properties holds

1. For all open $U \subset Y$, $p^* : \mathcal{O}(U) \to \mathcal{O}(p^{-1}(U))^G$ is an isomorphism.
2. If $W \subseteq X$ is closed and G-invariant, then $p(W) \subset Y$ is closed.
3. If $V_1, V_2 \subseteq X$ are closed, G-invariants, and $V_1 \cap V_2 = \varnothing$, then $p(V_1) \cap p(V_2) = \varnothing$.

Remark 5.2.2. Note that the first requirement implies a good categorical quotient must be a categorical one: If $f : X \to Z$ is a G-invariant morphism, then $f^* : \mathcal{O}(Z) \to \mathcal{O}(X)$ must embed in $\mathcal{O}(X)^G$. If p is a good categorical quotient, then p^* is an isomorphism to $\mathcal{O}(X)^G$, so

$$\begin{tikzcd} \mathcal{O}(Z) \arrow[rr, "f^*"] \arrow[dr, dotted, "\overline{f^*}"'] & \circlearrowleft & \mathcal{O}(X)^G \arrow[r, hook] & \mathcal{O}(X) \\ & \mathcal{O}(Y) \arrow[ur, hook, two heads, "p^*"'] & & \end{tikzcd}$$

So f^* can factor through $\mathcal{O}(Y)$, and this factoring is unique since p^* is an isomorphism. By the anti-equivalence of category, the dual $f = \overline{f} \circ p$ is a unique factoring of f through p.

Remark 5.2.3. As we can see in the above Remark 5.2.2, the first requirement already implies categorical quotient, the more restrictions intend to avoid bad situation in geometry, such as Example 5.1.3

Notation 5.2.1. We denote by $X//G$ the good categorical quotient, or GIT quotient, of a variety X by a group G.

In the following, we will first construct GIT quotient in affine case, and this serves as a guide for projective case: we want to glue affine quotients to get projective one, since every projective variety admits an affine covering.

Unfortunately, we can not cover the whole of a projective variety, which leads to the concept of semistability.

It's natural to define $X//G = \operatorname{Spec}\mathcal{O}(X)^G$ in affine cases, since $X = \operatorname{Spec}\mathcal{O}(X)$, so G-invariant regular functions may representate the quotient we desire, but for this we require that $\mathcal{O}(X)^G$ is finitely generated.

Historically, whether the ring of invariants is finitely generated or not is knowns as Hilbert's 14-th problem. For general linear group over $\mathbb{C}$, Hilbert showed that the invariant rings are always finitely generated. However, Nagata gave an counterexample that $\mathcal{O}(X)^G$ is not finitely generated, and proved that for any reductive group, $\mathcal{O}^G$ is finitely generated, see Lemma 5.3.1.

5.3. **Reductive groups.** Now we focus on the reductive group which we can use to construct GIT quotient. We will define when a linear algebraic group is reductive and give some properties of it.

Definition 5.3.1 (algebraic group)**.** A (linear) algebraic group is a subgroup of $\mathrm{GL}_n(k)$ which is an affine variety, that is an irreducible algebraic set.

Example 5.3.1. The set of unitary matrices with determinant 1

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc - 1 = 0 \right\}$$

is an algebraic group[2].

Example 5.3.2. $k^\times$ is also an algebraic group, by the embedding $\lambda \to \lambda I$.

Example 5.3.3. $\mathrm{GL}_n(k)$ is an algebraic group[3].

Definition 5.3.2. A linear algebraic group G over k is reductive if every representation $\rho : G \to \mathrm{GL}_n(k)$ has a decomposition as a direct sum of irreducible representations.

Proposition 5.3.1 (Maschke)**.** Let G be a finite group, then G is reductive.

Proposition 5.3.2. The multiplicative group $\mathbb{C}^\times$ is reductive.

Proof. Let $\rho : \mathbb{C}^\times \to \mathrm{GL}_n(\mathbb{C})$ be a representation of $\mathbb{C}^\times$, we will show ρ has a decomposition as a direct sum of irreducible representations. Assume ρ is not irreducible. Let $\langle \text{-}, \text{-} \rangle$ denote the standard inner product on $V = \mathbb{C}^n$, then define

$$\langle x, y \rangle := \int_0^{2\pi} \langle \rho(e^{i\theta})x, \rho(e^{i\theta})y \rangle \mathrm{d}\theta$$

[2]In general, special linear group $\mathrm{SL}(n)$ is always an algebraic group by considering the irreducible polynomial $\det -1$.

[3]We can check this by introducing a new variable T and consider irreducible polynomial $T \cdot \det -1$ with $n^2 + 1$ variables.

This form has the following property: $\langle \rho(g)x, \rho(g)y\rangle = \langle x; y\rangle$, where $x, y \in V, g = e^{i\psi} \in S^1 = \{z \in \mathbb{C}^\times : |z| = 1\}$. Indeed,

$$\begin{aligned}
\langle \rho(e^{i\psi})x, \rho(e^{i\psi})y\rangle &= \int_0^{2\pi} \langle \rho(e^{i\theta}\rho(e^{i\psi}))x, \rho(e^{i\theta})\rho(e^{i\psi})y\rangle \mathrm{d}\theta \\
&= \int_0^{2\pi} \langle \rho(e^{i(\theta+\psi)})x, \rho(e^{i(\theta+\psi)})y\rangle \mathrm{d}\theta \\
&\overset{\phi=\theta+\psi}{=} \int_0^{2\pi} \langle \rho(e^{i\phi})x, \rho(e^{i\phi})y\rangle \mathrm{d}\phi \\
&= \langle x, y\rangle
\end{aligned}$$

And also note that $\langle \text{-}, \text{-}\rangle$ is an inner product. If ρ is not irreducible, then there exists some $\mathbb{C}^\times$-invariant subspace U of V, let $W = U^\perp$ be the orthogonal complement of U with respect to $\langle \text{-}, \text{-}\rangle$. Then we can see W is S^1-invariant as follows

$$\begin{aligned}
\langle u, \rho(g)w\rangle &= \langle \rho(g^{-1})u, \rho(g^{-1})\rho(g)w\rangle \\
&= \langle \rho(g^{-1})u, w\rangle \\
&= 0
\end{aligned}$$

where $w \in W, u \in U, g \in S$. The last equality holds since U is S^1-invariant. What we need to do is to show W is $\mathbb{C}^\times$-invaraint.

Let N be the subset of $\mathbb{C}^\times$ which leaves W invariant, it contains S obviously. We will show that this set is closed in the Zariski topology. If we can do this, since all Zariski closed subset in $\mathbb{C}^\times$ are finite sets and whole space, so we can conclude $N = \mathbb{C}^\times$, as desired.

Let $W = \operatorname{span}\{e_1, \ldots, e_r\}$, and extends this basis to a basis $\{e_1, \ldots, e_n\}$ of V. Then we can regard W as solutions of equations

$$\langle v, e_i\rangle = 0, \quad i = r+1, \ldots, n$$

these define polynomials which take the coordinate of v as variables, which we call it f_i, so we can see W as a zero set of $\{f_{r+1}, \ldots, f_n\}$.

For each $i \in \{1, \ldots, r\}, j \in \{r+1, \ldots, n\}$, consider the set $\{T \in \mathrm{GL}(V) \mid f_j(Te_i) = 0\}$. If we fix i, j, this set is the zero set of a polynomial in the coordinates of T. So it's a closed set in $\mathrm{GL}(V)$, with respect to Zariski topology. Then we have $\{T \in \mathrm{GL}(V) \mid Te_i \in W\} = \bigcap_{j=r+1}^n \{T \in \mathrm{GL}(V) \mid f_j(Te_i) = 0\}$ is closed, so

$$\{T \in \mathrm{GL}(V) \mid Te_i \in W, \forall i \in \{1, \ldots, r\}\} = \bigcap_{i=1}^r \{T \in \mathrm{GL}(V) \mid Te_i \in W\}$$

is closed, thus we have

$$\begin{aligned}
\{T \in \mathrm{GL}(V) \mid Tw \in W, \forall w \in W\} &= \{T \in \mathrm{GL}(V) : T(\lambda_1 e_1 + \ldots + \lambda_r e_r) \in U \text{ for all } \lambda_i \in \mathbb{C}\} \\
&= \{T \in \mathrm{GL}(V) : \lambda_1 (Te_1) + \ldots + \lambda_r (Te_r) \in U \text{ for all } \lambda_i \in \mathbb{C}\} \\
&= \{T \in \mathrm{GL}(V) : Te_i \in W \text{ for each } i \in \{1, 2, \ldots, r\}\}
\end{aligned}$$

is closed with respect to Zariski topology, so $N = \rho^{-1}(\{T \in \mathrm{GL}(V) \mid Tw \in W, \forall w \in W\})$ is closed, as we desired. $\square$

Remark 5.3.1. In fact, many classical groups such as $\mathrm{GL}_n(\mathbb{C}), \mathrm{SL}_n(\mathbb{C})$ are reductive, now we give a proof of $\mathbb{C}^\times$ is a reductive group.

Definition 5.3.3 (rationally)**.** For a reductive alegbraic group, we say that G acts rationally on a variety X if it acts by a morphism of varieties $G \times X \to X$.

But why we need reductive groups? and why this action? There are two key properties which might answer these questions.

Lemma 5.3.1. Let G be a reductive group acting rationally on an affine variety X, then $\mathcal{O}(X)^G$ is finitely generated.

Proof. See [**?**]. $\square$

The following lemma is used in the construction of GIT quotient. It allows us to find a G-invariant function which separates disjoint G-invariant sets.

Lemma 5.3.2. Let G be a reductive group acting rationally on an affine variety $X \subset \mathbb{A}^n$. Let Z_1, Z_2 be two closed G-invariant subsets of X with $Z_1 \cap Z_2 = \varnothing$. Then there exists a G-invariant function $F \in \mathcal{O}(X)^G$ such that $F(Z_1) = 1, F(Z_2) = 0$.

Proof. See [**?**]. $\square$

5.4. **The affine quotient.** We now have enough tools to construct the quotient of an affine variety by a reductive group. For an affine variety X, the quotient of X by a reductive group G is just $\operatorname{Spec} \mathcal{O}(X)^G$. We will prove that this construction satisfies the required conditions being a good categorical quotient.

Theorem 5.4.1. Let X be an affine variety and G be a reductive group acting rationally on X. Let $p^* : \mathcal{O}(X)^G \to \mathcal{O}(X)$ be defined by the inclusion $\mathcal{O}(X)^G \subseteq \mathcal{O}(X)$. Then the dual of this map, $p : X \to Y := \operatorname{Spec} \mathcal{O}(X)^G$ is a good categorical quotient.

Now we give a concrete example to show how powerful the GIT construction is, and gives the answer to the Example 5.1.1 we mentioned at first.

Example 5.4.1. Consider the set X of 2×2 matrices over $\mathbb{C}$, embedded in $\mathbb{C}^4$ by

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} \mapsto (w, x, y, z)$$

It is an affine variety obviously, and consider the general linear group acts on it by conjugate action, then as the theorem above implies

$$X//G = \operatorname{Spec} k[w, z, y, z]^G$$

We know that there are two important invariants under conjugate action, that is, determinant and trace. In this case they are $\det = wz - xy$ and $\mathrm{tr} = w + z$, so we have an obvious inclusion

$$k[wz - xy, w + z] \subset k[w, x, y, z]^G$$

We will show that we in fact have equality.

Let $\lambda \in \mathbb{C}^\times$ be arbitrary and consider the matrix $A = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$. For all matrices $M = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$, we can calculate as follows

$$\begin{aligned} A^{-1}MA &= \begin{pmatrix} 0 & -\frac{1}{\lambda} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \\ &= \begin{pmatrix} z & \frac{y}{\lambda} \\ \lambda x & w \end{pmatrix} \end{aligned}$$

Let $f \in k[w, x, y, z]^G$, i.e. we require f satisfy that $f(w, x, y, z) = A.f(M) = f(A.M) = f(A^{-1}MA) = f(z, \frac{y}{\lambda}, \lambda x, w)$. That is

$$f(w, x, y, z) = f\left(z, \frac{y}{\lambda}, \lambda x, w\right)$$

From this equality, we can make the following observations

1. x must appear in the form xy to cancel λ in $A.f$.
2. z and w must appear in an symmetric way, i.e. must in the forms of $z+w$ or zw.

So we conclude $f \in k[xy, wz, z+w]$. Similarly consider matrix $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. And after the same calculation we can get

$$f(w, x, y, z) = f(w - x, w + x - y - z, y, y + z)$$

As we already have $f \in k[xy, wz, w+z]$, we can reformulate this requirement into

$$f(xy, wz, w + z) = f\left(wy + xy - y^2 - z, wy + wz - y^2 - yz, w + z\right)$$

We can see that this formular holds only when extra terms in $B.f$ must cancel with each other, which implies $f \in k[wz - xy, w + z]$, as desired. So we have the construction

$$\begin{aligned} X//G &= \operatorname{Spec} k[w, x, y, z]^G \\ &= \operatorname{Spec} k[wz - xy, w + z] \\ &= \operatorname{Spec} k[u, v] \\ &= \mathbb{C}^2 \end{aligned}$$

Remark 5.4.1. There is a high-dimensional analogous: if $\mathrm{GL}_n(\mathbb{C})$ acts on $M_n(\mathbb{C})$ by conjugate action, then

$$M_n(\mathbb{C}) // \mathrm{GL}_n(\mathbb{C}) = \mathbb{C}^n$$

See [?] for more details.

5.5. **The projective quotient.** Now we construct projective quotient by gluing together affine quotients.

Let X be a projective variety, then X can be covered by some affine varieties X_{f_i}. In order to construct GIT quotient of X by G, it's natural for us to take quotient for every affine variety of G of the form $X_{f_i}//G = \mathrm{Spec}(\mathcal{O}(X_{f_i})^G)$, and cover the projective quotient by them. To do this, we need an action of G on the coordinates of X.

Our approach is to embed X in $\mathbb{P}^m$ for some m such that the action of G can be extended to a linear action on $\mathbb{A}^{m+1}$. This is called a linearisation of the action of G.

Definition 5.5.1. Let the group G act rationally on a projective variety X. Let $\varphi : X \hookrightarrow \mathbb{P}^m$ be an embedding of X that extends the group action, i.e. we have a rationally group action on $\mathbb{P}^m$ such that $\varphi(g.x) = g.\varphi(x)$. Let $\pi : \mathbb{A}^{m+1} \to \mathbb{P}^m$ be the natural projection. A linearisation of the action of G with respect to φ is a linear action of G on $\mathbb{A}^{m+1}$ that is compatible with the action of G on X in the following sense

1. For any $y \in \mathbb{A}^{m+1}, g \in G$
$$\pi(g.y) = g.(\pi(y))$$
2. For all $g \in G$, the map
$$\mathbb{A}^{m+1} \to \mathbb{A}^{m+1}, \quad y \mapsto g.y$$
is linear.

We write φ_G for a linearisation of the action of G with respect to φ.

Remark 5.5.1. Note that such action induces an action of G on $\mathcal{O}(X)$. we have $\mathcal{O}(X) \cong k[x_0, \ldots, x_m]/I$ for some homogeneous ideal I, since X is isomorphic to the image $\varphi(X) \subseteq \mathbb{P}^m$. Using the fact that G acts on $k[x_0, \ldots, k_m]$ by $g.f(x_0, \ldots, x_m) := f(g^{-1}.(x_0, \ldots, x_m))$, we can know that G also acts on $\mathcal{O}(X)$, and it's well-defined, since $g.f' \in I$ for $f' \in I$.

Example 5.5.1. Let $\mathbb{C}^\times$ act on $\mathbb{P}^1$ by $\lambda.(x_0, x_1) = (x_0 : \lambda x_1)$. A linearisation can be given by the obvious action on $\mathbb{A}^2$ with $\lambda.(x_0, x_1) = (x_0, \lambda x_1)$.

The above example illustrates a quite important issue when we are constructing projective quotient: good categorical quotient may not exist. The only possible G-invariant morphism sends all orbits to a point, since $(1, 0), (0, 1)$ are both in the closure of $(1, t)$. But this fails to seperate closed orbits, so is not a good categorical quotient.

The solution to such problem is to take an open G-invariant subset which has a good categorical quotient. We desire this subset to be covered by G-invariant open affine subsets so that we can cover the quotient by gluing together affine quotients. This leads us to the notion of semistability,

Definition 5.5.2. Let G be a reductive group acting on a projective variety X which has an embedding $\varphi : X \to \mathbb{P}^m$. A point $x \in X$ is called semistable (with respect to the linearisation φ_G) if there exists some G-invariant homogeneous polynomial f of degree greater than 0 in $\mathcal{O}(X)$, such that $f(x) \neq 0$ and X_f is affine.

Remark 5.5.2. Write $X^{\mathrm{as}}(\varphi_G)$ for the set of semistable points of X with respect to φ_G, or just X^{as} when it's not ambiguous.

For Example7.4.3, the set of semistable points of X with respect to φ_G is $X^{\mathrm{as}} = X_{x_0} = \mathbb{P}^1 \backslash \{(0:1)\}$. On this subset, the map to a point $p : X^{\mathrm{as}} \to \mathbb{P}^0$ is indeed a good categorical quotient.

Theorem 5.5.1. Let G be a reductive group acting rationally on a projective variety X embedded in $\mathbb{P}^m$ with a linearisation φ_G. Let R be the coordinate ring of X, then there is a good categorical quotient

$$p : X^{\mathrm{as}}(\varphi_G) \to X^{\mathrm{as}(\varphi_G)}//G \cong \operatorname{Proj} R^G$$

6. Symplectic quotient

7. The Kempf-Ness theorem

References

Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, P.R. China,
Email address: `liubw22@mails.tsinghua.edu.cn`