KEMPF-NESS THEOREM

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Abstract.

Contents

0. Preface

0.1. **About this lecture.** It's a lecture note about geometric invariant theory and its applications. The main reference is [HOS15]

0.2. **Notations and conventions.**

1. For convenience, all schemes are assumed to be finite type over algebraic closed field k, and this category is denoted by Sch.
2. A variety is a reduced, seperated scheme of finite type over k.
3. Set denotes the category of sets.
4.
5.
6.

Part 1. Preliminary

1. Moduli problem

1.1. Functors of points.

Definition 1.1.1 (functor of points)**.** The functor of points of a scheme X is a contravariant functor $h_X := \operatorname{Hom}(-, X)\colon \mathrm{Sch} \to \mathrm{Set}$.

Remark 1.1.1. A morphism of schemes $f\colon X \to Y$ induces a natural transformation of funtors $h_f\colon h_X \to h_Y$, given by

$$\begin{aligned} h_f(Z)\colon h_X(Z) &\to h_Y(Z) \\ g &\mapsto f \circ g \end{aligned}$$

where Z is a scheme.

Definition 1.1.2 (presheaf)**.** The contravariant functors from schemes to sets are called presheaves on Sch and form a category, which is denoted by $\mathrm{Psh}(\mathrm{Sch}) = \mathrm{Fun}(\mathrm{Sch}^{\mathrm{op}}, \mathrm{Set})$.

Example 1.1.1. For a scheme X, $h_X(\operatorname{Spec} k) := \operatorname{Hom}(\operatorname{Spec} k, X)$ is the set of k-points of X, maybe that's why h_X is called functor of point.

Lemma 1.1.1 (Yoneda lemma)**.** Let $\mathcal{C}$ be any category, then for any $C \in \mathcal{C}$ and any presheaf $F \in \mathrm{Psh}(\mathcal{C})$, there is a bijection

$$\{\text{natural transformsations } \eta\colon h_C \to F\} \longleftrightarrow F(C)$$

given by $\eta \mapsto \eta_C(\mathrm{id}_{\mathcal{C}})$.

Proof. To see it's surjective: For an object $s \in F(C)$, we define $\eta\colon h_C \to F$ defined as follows: For $C' \in \mathcal{C}$, consider

$$\begin{aligned} \eta_{C'}\colon h_C(C') &\to F(C') \\ f &\mapsto F(f)(s) \end{aligned}$$

It's well-defined, since F is a contravariant functor, thus for $f\colon C' \to C$, we have $F(f)\colon F(C) \to F(C')$, thus $F(f)(s) \in F(C')$. It's indeed a natural transformation, since if we take $g\colon C'' \to C'$, and consider the following diagram

$$\begin{tikzcd} h_C(C') \arrow[r, "\eta_{C'}"] \arrow[d, "h_C(g)"] & F(C') \arrow[d, "F(g)"] \\ h_C(C'') \arrow[r, "\eta_{C''}"] & F(C'') \end{tikzcd}$$

For arbitrary $f\colon C' \to C \in h_C(C')$, note that

$$\begin{aligned} \eta_{C''} \circ h_C(g) &= \eta_{C''}(f \circ g) \\ &= F(f \circ g)(s) \\ &= F(g) \circ F(f)(s) \\ &= F(g) \circ \eta_{C'}(f) \end{aligned}$$

Thus above diagram commutes, that is η is a natural transformation. By construction, we have

$$\eta_C(\mathrm{id}_C) = F(\mathrm{id}_C)(s) = s$$

To see it's injective: Suppose we have two natural transformation $\eta, \eta' \colon h_C \to F$ such that $\eta_C(\mathrm{id}_C) = \eta'_C(\mathrm{id}_C)$, in order to show $\eta = \eta'$, it suffices to show for arbitrary $C' \in \mathcal{C}$, we have $\eta_{C'} = \eta'_{C'}$. Let $g \colon C' \to C$, then we have the following commutative diagram

$$\begin{tikzcd}
h_C(C) \arrow[r, "\eta_C"] \arrow[d, "h_C(g)"] & F(C) \arrow[d, "F(g)"] \\
h_C(C') \arrow[r, "\eta_{C'}"] & F(C')
\end{tikzcd}$$

It follows that

$$F(g) \circ \eta_C(\mathrm{id}_C) = \eta_{C'} \circ h_C(g)(\mathrm{id}_C) = \eta_{C'}(g)$$

and by the same argument one has $F(g) \circ \eta'_C(\mathrm{id}_C) = \eta'_{C'}(g)$. Hence

$$\eta_{C'}(g) = F(g) \circ \eta_C(\mathrm{id}_C) = F(g) \circ \eta'_C(\mathrm{id}_C) = \eta'_{C'}(g)$$

This completes the proof. □

Corollary 1.1.1. The functor $h \colon \mathcal{C} \to \mathrm{Psh}(\mathcal{C})$ is fully faithful.

Proof. Recall that a functor is fully faithful if for every $C, C' \in \mathcal{C}$, we have the following bijection

$$\mathrm{Hom}_{\mathcal{C}}\left(C, C'\right) \leftrightarrow \mathrm{Hom}_{\mathrm{Psh}(\mathcal{C})}\left(h_C, h_{C'}\right)$$

then take $F = h_{C'}$ in Yoneda lemma to conclude. □

Definition 1.1.3 (reprensentable functor)**.** A presheaf $F \in \mathrm{Psh}(\mathcal{C})$ is called reprensentable if there exists an object $C \in \mathcal{C}$ and a natural isomorphism $F \cong h_C$.

So it's natural to ask if every presheaf F is reprensentable by a scheme X? The answer is negative, as we will see. However, we are quite interested in answering this question for special functors, known as moduli functor. Before we introduce moduli functor, we start with some naive notion of a moduli problem.

1.2. **Moduli Problems.** A moduli problem is a classification problem: we have a collection of objects and we want to classify them up to some equivalence. In fact, we want more than this, we want a moduli space encodes how these objects vary continously in families; this information is encoded in a moduli functor.

Definition 1.2.1 (naive moduli problem)**.** A naive moduli problem (in algebraic geometry) is a collection $\mathcal{A}$ of objects (in algebraic geometry) and an equivalence relation $\sim$ on $\mathcal{A}$.

Example 1.2.1.

1. Let $\mathcal{A}$ be the set of k-dimensional linear subspaces of an n-dimensional vector space and $\sim$ be equality.
2. Let $\mathcal{A}$ be the collection of vector bundles on a fixed scheme X and $\sim$ be the relation given by isomorphism of vector bundles.

Our aim is to find a scheme M whose k-points are in bijection with the set of equivalence classes $\mathcal{A}/_{\sim}$. Furthermore, we want M to also encode how these objects vary continously in "families".

Definition 1.2.2 (moduli problem)**.** Let $(\mathcal{A}, \sim)$ be a naive moduli problem. Then a (extended) moduli problem is given by

1. sets $\mathcal{A}_S$ of families over S and an equivalence relation $\sim_S$ on $\mathcal{A}_S$ for all schemes S.
2. pullback maps $f^*\colon \mathcal{A}_S \to \mathcal{A}_T$, for every morphism of schemes $f\colon T \to S$, such that
   (1) $(\mathcal{A}_{\operatorname{Spec} k}, \sim_{\operatorname{Spec} k}) = (\mathcal{A}, \sim)$;
   (2) For the identity $\mathrm{id}\colon S \to S$ and any family $\mathcal{F}$ over S, we have $\mathrm{id}^* \mathcal{F} = \mathcal{F}$;
   (3) For a morphism $f\colon T \to S$ and equivalent families $\mathcal{F} \sim_S \mathcal{G}$, we have $f^*\mathcal{F} \sim_T f^*\mathcal{G}$.
   (4) For morphisms $f\colon T \to S$, $g : S \to R$, and a family $\mathcal{F}$ over R, we have an equivalence
$$(g \circ f)^*\mathcal{F} \sim_T f^*g^*\mathcal{F}$$

Notation 1.2.1. For a family $\mathcal{F}$ over S and a point $s\colon \operatorname{Spec} k \to S$, $\mathcal{F}_s := s^*\mathcal{F}$ denotes the corresponding family over $\operatorname{Spec} k$.

Corollary 1.2.1. A moduli problem defines a functor $\mathcal{M} \in \mathrm{Psh}(\mathrm{Sch})$, given by

$$\begin{aligned}\mathcal{M}(S) &:= \{\text{families over } S\}/_{\sim_S} \\ \mathcal{M}(f\colon T \to S) &:= f^*\colon\ \mathcal{M}(S) \to \mathcal{M}(T)\end{aligned}$$

Proof. It's clear from the definition. □

Example 1.2.2. Consider the naive moduli problem given by vector bundles on a fixed scheme X up to isomorphism. Then this can be extended in two different ways. The natural notion for a family over S is a locally free sheaf $\mathcal{F}$ over $X \times S$ which is flat over S, but there are two possible ways to define relations:

$$\begin{aligned}\mathcal{F} \sim'_S \mathcal{G} &\iff \mathcal{F} \cong \mathcal{G} \\ \mathcal{F} \sim_S \mathcal{G} &\iff \mathcal{F} \cong \mathcal{G} \otimes \pi_S^*\mathcal{L}\end{aligned}$$

where $\mathcal{L}$ is a line bundle $\mathcal{L} \to S$ and $\pi_S\colon X \times S \to S$.

1.3. **Fine moduli spaces.** The ideal is when there is a scheme that reprensents our given moduli functor.

Definition 1.3.1 (fine moduli space)**.** Let $\mathcal{M}\colon \mathrm{Sch} \to \mathrm{Set}$ be a moduli functor, then a scheme M is a fine moduli space for $\mathcal{M}$ if it reprensents $\mathcal{M}$.

To be explicit, M is a fine moduli space for $\mathcal{M}$ if there is a natural isomorphism $\eta : \mathcal{M} \to h_M$, thus for every scheme S, we have a bijection

$$\eta_S\colon \mathcal{M}(S) = \{\text{families over } S\}/_{\sim_S} \longleftrightarrow h_M(S) = \{\text{morphisms } S \to M\}$$

In particular, let $S = \operatorname{Spec} k$, then the k-points of M are in bijection with the set $\mathcal{A}/_{\sim}$.

Definition 1.3.2 (universal family)**.** Let M be a fine moduli space for $\mathcal{M}$, then the family $\mathcal{U} \in \mathcal{M}(M)$, determined by $\mathcal{U} := \eta_M^{-1}(\mathrm{id}_M)$, is called the universal family.

Remark 1.3.1. It's called universal family, since any family $\mathcal{F}$ over a scheme S corresponds to a morphism $f\colon S \to M$ and since the families $f^*\mathcal{U}$ and $\mathcal{F}$ correspond to the same morphism $\mathrm{id}_M \circ f = f$, one has

$$f^*\mathcal{U} \sim_S \mathcal{F}$$

This shows every family is equivalent to a family obtained by pulling back the universal family.

Proposition 1.3.1. If a fine moduli space for moduli functor $\mathcal{M}$ exists, it is unique up to a unique isomorphism.

Proof. Suppose $(M, \eta), (M', \eta')$ are two fine moduli space for $\mathcal{M}$, then they are related by unique isomorphisms

$$\eta'_M((\eta_M)^{-1}(\mathrm{id}_M))\colon M \to M'$$
$$\eta_{M'}((\eta'_{M'})^{-1}(\mathrm{id}_{M'}))\colon M' \to M$$

□

Remark 1.3.2.

Example 1.3.1. Let's consider the projective space $\mathbb{P}^n = \operatorname{Proj} k[x_0, \ldots, x_n]$. This variety can be interpreted as a fine moduli space for the moduli problem of lines through the origin $V := \mathbb{A}^{n+1}$.

1.4. **Coarse moduli spaces.**

Definition 1.4.1 (coarse moduli space)**.** A coarse moduli space for a moduli functor $\mathcal{M}$ is a scheme M and a natural transformation of funtors $\eta\colon \mathcal{M} \to h_M$ such that

1. $\eta_{\operatorname{Spec} k}\colon \mathcal{M}(\operatorname{Spec} k) \to h_M(\operatorname{Spec} k)$ is bijective;
2. For any scheme N and natural transformation $\nu\colon M \to h_N$, there exists a unique morphism of schemes $f\colon M \to N$ such that $\nu = h_f \circ \eta$, where $h_f\colon h_M \to h_N$ is the corresponding natural transformation of presheaves.

Remark 1.4.1. A coarse moduli space for $\mathcal{M}$ is unique up to unique isomorphism.

Proposition 1.4.1. Let (M, η) be a corase moduli space for a moduli problem $\mathcal{M}$. Then (M, η) is a fine moduli space if and only if

1. there exists a family $\mathcal{U}$ over M such that $\eta_M(\mathcal{U}) = \mathrm{id}_M$;
2. for families $\mathcal{F}$ and $\mathcal{G}$ over a scheme S, we have $\mathcal{F} \sim_S \mathcal{G}$ if and only if $\eta_S(\mathcal{F}) = \eta_S(\mathcal{G})$.

2. Algebraic group

2.1. Definition and examples.

Definition 2.1.1 (algebraic group)**.** An algebraic group is a scheme G over k, with

1. multiplication $m\colon G \times G \to G$;
2. identity $e\colon \operatorname{Spec} k \to G$;
3. inversion $i\colon G \to G$.

such that the following diagrams commute

$$\begin{tikzcd} G \times G \times G \arrow[r] \arrow[d] & G \times G \arrow[d] \\ G \times G \arrow[r] & G \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec} k \times G \arrow[r, "e\times \mathrm{id}"] \arrow[dr] & G \times G \arrow[d, "m"] & G \times \operatorname{Spec} k \arrow[l, "\mathrm{id}\times e"'] \arrow[dl] \\ & G & \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "i\times \mathrm{id}"] \arrow[d] & G \times G \arrow[d, "m"] & G \arrow[l, "\mathrm{id}\times i"'] \arrow[d] \\ \operatorname{Spec} k \arrow[r, "e"] & G & \operatorname{Spec} k \arrow[l, "e"] \end{tikzcd}$$

Definition 2.1.2 (affine algebraic group)**.** G is an affine algebraic group, if underlying scheme is affine.

Definition 2.1.3 (group variety)**.** G is a group variety, if the underlying scheme G is a variety.

Definition 2.1.4 (homomorphism of algebraic groups)**.** A homomorphism of algebraic groups G and H is a morphism of schemes $f\colon G \to H$ such that the following diagram commute

$$\begin{tikzcd} G \times G \arrow[r, "m_G"] \arrow[d, "f\times f"'] & G \arrow[d, "f"] \\ H \times H \arrow[r, "m_H"] & H \end{tikzcd}$$

Definition 2.1.5 (algebraic subgroup)**.** An algebraic subgroup of G is a closed subscheme H such that $H \hookrightarrow G$ is a homomorphism of algebraic groups.

Theorem 2.1.1 (Hopf)**.** There exists an equivalence of categories

$$\{\text{affine algebraic groups}\} \longleftrightarrow \{\text{finitely generated Hopf algebra over } k\}$$
$$G \mapsto \mathcal{O}(G)$$

where $\mathcal{O}(G)$ is the k-algebra of regular functions on G.

Example 2.1.1 (additive group)**.** The additive group $\mathbb{G}_a = \operatorname{Spec} k[t]$ over k is an affine algebraic group, with underlying variety $\mathbb{A}^1$, and whose group structure is given by

$$\begin{aligned} m^*(t) &= t \otimes 1 + 1 \otimes t \\ e^*(t) &= 0 \\ i^*(t) &= -t \end{aligned}$$

Example 2.1.2 (multiplicative group)**.** The multiplicative group $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$ over k is the algebraic group whose underlying variety is $\mathbb{A}^1 \setminus \{0\}$, and whose group structure is given by

$$\begin{aligned} m^*(t) &= t \otimes t \\ e^*(t) &= 1 \\ i^*(t) &= t^{-1} \end{aligned}$$

Example 2.1.3 (general linear group)**.** The general linear group $\mathrm{GL}(n, k)$ is an open subvariety of $\mathbb{A}^{n^2}$ cut out by the condition that determinant is non-zero. It's an affine variety with coordinate ring $k[x_{ij} : 1 \leq i, j \leq n]_{\det(x_{ij})}$. The co-group operations are defined by

$$\begin{aligned} m^*(x_{ij}) &= \sum_{k=1}^{n} x_{ik} \otimes x_{kj} \\ e^*(x_{ij}) &= 1 \\ i^*(x_{ij}) &= (x_{ij})^{-1}_{ij} \end{aligned}$$

where $(x_{ij})^{-1}_{ij}$ is the regular function on $\mathrm{GL}(n, k)$ by taking the (i, j)-entry of the inverse of a matrix.

Definition 2.1.6 (linear algebraic group)**.** A linear algebraic group is an algebraic subgroup of $\mathrm{GL}(n, k)$.

Example 2.1.4. $\mathrm{SL}(n, k)$ and $\mathrm{O}(n)$ are linear algebraic groups.

Definition 2.1.7 (linear representation)**.** A linear representation of algebraic group G on a finite dimensional k-vector space V is a homomorphism of algebraic groups $\rho \colon G \to \mathrm{GL}(V)$.

2.2. **Representation theory of tori.**

Definition 2.2.1 (tori)**.** Let G be an affine algebraic group over k.

1. G is a torus if $G \cong G_m^n$ for some $n > 0$.
2. A torus of G is a subgroup scheme of G which is a torus.
3. A maximal torus of G is a torus $T \subset G$ which is not contained in any other torus.

Definition 2.2.2 (character group)**.** For a torus T. $X^*(T) := \mathrm{Hom}(T, G_m)$ is called character group and $X_*(T) := \mathrm{Hom}(G_m, T)$ is called cocharacter group.

Theorem 2.2.1. The map

$$\theta\colon \mathbb{Z}^n \to X^*(G_m^n)$$
$$(a_1, \dots, a_n) \mapsto \{(t_1, \dots, t_n) \mapsto (t_1^{a_1}, \dots, t_n^{a_n})\}$$

Proof. It's clear well-defined and injective. To see surjectivity, for $\phi \in X^*(G_m^n)$, where

$$\phi^*\colon k[t, t^{-1}] \to k[t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}]$$
$$t \mapsto \sum c_a t_1^{a_1} \dots t_n^{a_n}$$

and

$$m^*\phi^*(t) = \phi^*(t) \otimes \phi^*(t)$$

where □

Proposition 2.2.1. Let $T = G_m^n$, for a finite dimensional representation $\rho\colon T \to \mathrm{GL}(V)$, there exists a weight space decomposition $V = \bigoplus_{\chi \in X^*(T)} V_\chi$, where

$$V_\chi = \{v \in V \mid tv = \chi(t)v, \forall t\}$$

Remark 2.2.1. linear representation of T is one to one correspondence to $X^*(T)$-graded V spaces.

Proof. Note that each character $\chi\colon T \to G_m \subset \mathbb{A}^1$, then $\chi \in \mathcal{O}(T) = k[t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}]$, and $\chi^*(T) = \mathbb{Z}^n$ forms a basis of $\mathcal{O}(T)$.

Given a representation $\phi\colon T \to \mathrm{GL}(V)$, choose a basis $\{e_{ij}\}$ of $\operatorname{End} V$, define $\phi_{ij}\colon T \to \mathrm{GL}(V) \subset \mathrm{End}(V) \overset{pr_{ij}}{\to} k$, that is

$$\phi = \sum \phi_{ij} e_{ij}$$

If we denote $\phi_{ij} = c_\chi^{ij} \chi$, then

$$\phi = \sum \phi_{ij} e_{ij} = \sum \sum c_\chi^{ij} \chi e_{ij} = \sum A_\chi \chi$$

where

$$A_\chi = c_\chi^{ij} e_{ij} \in \operatorname{End} V$$

Need to check

1. $\operatorname{im} A_\chi = V_\chi$.
2. $\sum A_\chi = \mathrm{id}$.
3. $\mathbb{A}_\psi A_\chi = \delta_{\chi\psi} A_\psi$.

Define $\oplus A_\chi \to V$ is an isomorphism □

2.3. **Group actions.**

Definition 2.3.1 (group action)**.** An action of an algebraic group G on a scheme X is a morphism of scheme $\sigma\colon G \times X \to X$ such that the following diagrams commute

$$
\begin{tikzcd}
\operatorname{Spec} k \times X \arrow[r, "e\times \mathrm{id}_X"] \arrow[dr] & G \times X \arrow[d, "\sigma"] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times G \times X \arrow[r, "\mathrm{id}_G \times \sigma"] \arrow[d, "m_G \times \mathrm{id}_X"'] & G \times X \arrow[d, "\sigma"] \\
G \times X \arrow[r, "\sigma"] & X
\end{tikzcd}
$$

Remark 2.3.1. If X is an affine scheme over k, and $\mathcal{O}(X)$ is its algebra of regular functions, then an action of G on X give rise to a coaction homomorphism of k-algebras, that is

$$
\begin{aligned}
\sigma^* \colon \mathcal{O}(X) &\to \mathcal{O}(G \times X) \\
f &\mapsto \sum h_i \otimes f_i
\end{aligned}
$$

This gives a homomorphism $G \to \operatorname{Aut}(\mathcal{O}(X))$ where the k-algebra automorphism of $\mathcal{O}(X)$ corresponding to $g \in G$ is given by

$$f \mapsto h_i(g) \otimes f_i$$

Definition 2.3.2 (G-equivariant morphism)**.** Let $\sigma_X \colon G \times X \to X$ and $\sigma_Y \colon G \times Y \to Y$ be group actions on schemes X and Y, a morphism $f \colon X \to Y$ is G-equivariant if the following diagram commute

$$
\begin{tikzcd}
G \times X \arrow[r, "\mathrm{id}_G \times f"] \arrow[d, "\sigma_X"'] & G \times Y \arrow[d, "\sigma_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
$$

Definition 2.3.3 (rational group action)**.** An action of G on a k-algebra A is rational if every element of A is contained in a finite dimensional G-invariant linear subspace of A.

Lemma 2.3.1. Let G be an affine algebraic group acting on an affine scheme X, then every $f \in \mathcal{O}(X)$ is contained in a finite dimensional G-invariant subspace of $\mathcal{O}(X)$. Furthermore, for any finite dimensional vector subspace W of $\mathcal{O}(G)$, there is a finite dimensional G-invariant vector subspace V of $\mathcal{O}(X)$ containing W.

Theorem 2.3.1. Any affine algebraic group over k is a linear algebraic group.

2.4. **Orbits and stabilisers.**

Definition 2.4.1. Let G be a linear algebraic group acting on a scheme X by $\sigma \colon G \times X \to X$.

1. The orbit Gx of $x \in X(k)$ is the set theoretic image of $\sigma_x \colon G \times \operatorname{Spec} k \xhookrightarrow{\mathrm{id} \times x} G \times X \to \xrightarrow{\sigma} X$
2.

Proposition 2.4.1. Let G be a linear algebraic group variety, acting on a variety X.

1. The orbits of closed points are locally closed subsets of X. Hence, it can be identified with the reduced locally closed subschemes.
2. The boundary $\overline{GX(k)}\backslash GX(k)$ is a union of orbits of strictly smaller dimension. In particular, orbits of minimal dimensional are closed and each orbit closure contains a closed orbit.

Definition 2.4.2. An action of an affine algebraic group G on a scheme X is closed if all G-orbits in X are closed.

Theorem 2.4.1. Let G be an affine algebraic group acting on a scheme X. For $x \in X(k)$, we have

$$\dim G = \dim G_x + \dim Gx$$

Proposition 2.4.2. Let G be an affine algebraic group acting on a scheme X by a morphism $\sigma \colon G \times X \to X$. Then the dimension of the stabiliser subgroup (resp. orbit) viewed as a function $X \to \mathbb{N}$ is upper semi-continous (resp. lower semi-continuous), that is for every n,

$$\{x \in X \mid \dim G_x \geq n\} \text{ and } \{x \in X \mid \dim Gx \leq n\}$$

are closed in X.

3. Quotients

3.1. Categorical quotient.

Definition 3.1.1 (invariant morphism)**.** $\varphi\colon X \to Z$ is G-invariant, if φ is G-equivariant with Z equipped with trivial G-action.

Definition 3.1.2 (categorical quotient)**.** A categorical quotient for the action of G on a scheme X is a G-invariant morphism $\varphi\colon X \to Y$ such that for any G-invariant morphism $f\colon X \to Z$, there is the following commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\varphi"'] & & Z \\ & Y \arrow[ur, dashed] & \end{tikzcd}$$

3.2. Good quotient.

Definition 3.2.1 (good quotient)**.** A morphism $\varphi\colon X \to Y$ is a good quotient for the action of G on X if

1. φ is G-invariant.
2. φ is surjective.
3. If $U \subset Y$ is an open subset, the morphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))^G \subset \mathcal{O}_X(\varphi^{-1}(U))$ is an isomorphism.
4. If W_1 and W_2 are disjoint G-invariant closed subsets, then closures of $\varphi(W_1)$ and $\varphi(W_2)$ are disjoint.
5. φ is affine.

Definition 3.2.2 (geometric quotient)**.** A good quotient is called a geometric quotient, if the preimage of each point is a single orbit.

Proposition 3.2.1. A good quotient is a categorical quotient.

Corollary 3.2.1. Let G be an affine algebraic group acting on a scheme X and $\varphi\colon X \to Y$ a good quotient, then

1. $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \varnothing$ if and only if $\varphi(x_1) = \varphi(x_2)$.
2. For each $y \in Y$, the preimage $\varphi^{-1}(y)$ contains a unique closed orbit. In particular, if the action is closed, then φ is a geometric quotient.

Proof. For (1). As φ is constant on orbit closures, then $\varphi(x_1) = \varphi(x_2)$ if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \varnothing$. Conversely, if $\varphi(x_1) = \varphi(x_2)$, $\square$

Remark 3.2.1.

(1) says that the good quotient space Y is the set of orbits modulo $\sim$, where $G \cdot x_1 \sim G \cdot x_2$ if and only if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \varnothing$.
(2) says that there exists a unique representative, that is a closed orbit in each equivalent class.

Example 3.2.1. Consider $\mathbb{G}_m$ acting on $\mathbb{A}^2$ via $t(x,y) = (tx, t^{-1}y)$, we claim

$$\begin{aligned} \varphi\colon\ \mathbb{A}^2 &\to A \\ (x,y) &\mapsto xy \end{aligned}$$

is a good quotient.

For (4). If W is G-invariant closed, then it's union of orbits:

1. finitely many orbits, then it's union of orbits closure
2. infinitely many orbits, then $I(W) \subset (xy - \alpha_i)$ for infinitely many α_i, then $b \in I(W)$ divides by infinitely many $(xy - \alpha_i)$, that is $I(W) = 0$.

   For (5).

Remark 3.2.2. Note that φ is not a geometric quotient, since $\varphi^{-1}(0)$ contains three orbits, so it gives an example for good quotient which is not geometric quotient. There is also an example for categorical quotient which is not good quotient, see [ANH99].

Proposition 3.2.2. The geometric or good quotient is local in the target.

4. Reductivity

4.1. **Reductive groups.** Let G be an algebraic group acting on a finitely generated k-algebra A, then Hilbert's 14 problem when does A^G is finitely generated. It fails for general algebraic group, and that's a motivation for reductive group.

In differential geometry, there is the following categorical equivalent

$$\{\text{real compact Lie group}\} \longleftrightarrow \{\text{complex reductive group}\}$$

From a compact Lie group K, one can construct a real algebraic group $K_{\mathbb{R}}$, then consider its $\mathbb{C}$-points to obtain a complex reductive group $K_{\mathbb{C}}$

Example 4.1.1. The complexification of $\mathrm{U}(n)$ is $\mathrm{GL}(n, \mathbb{C})$ and the complexification of $\mathrm{SU}(n)$ is $\mathrm{SL}(n, \mathbb{C})$.

Example 4.1.2. For $\mathrm{U}(1) = \{z \in \mathbb{C} \mid z\overline{z} = 1\}$, one has

$$K_{\mathbb{R}} = \operatorname{Spec} \frac{\mathbb{R}[x, y]}{(x^2 + y^2 - 1)}$$
$$K_{\mathbb{C}} = \{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 = 1\} = \mathbb{C}^*$$

Definition 4.1.1. Let G be an affine algebraic group, an element $g \in G$ is semisimple(unipotent), if there exists a faithful representation $\rho\colon G \to \mathrm{GL}(n)$ such that $\rho(g)$ is semisimple(unipotent).

Definition 4.1.2. An affine algebraic group is unipotent if every non-trivial reprensentation $\rho\colon G \to \mathrm{GL}(V)$ has a non-zero G-invariant vector.

Definition 4.1.3. Let $\mathcal{U}_n$ denote the upper triangular matrices with diagonamal elements 1.

Proposition 4.1.1. The following are equivalent

1. G is unipotent;
2. For all representation $\rho\colon G \to \mathrm{GL}(V)$, there exists a basis of V such that
$$\rho(G) \subset \mathcal{U}_n \subset \mathrm{GL}_n \cong \mathrm{GL}(V)$$
3. G is isomorphic to a subgroup of $\mathcal{U}_n$.

Proof. For (2) to (1): If $\{e_1, \ldots, e_n\}$ is a basis of V such that $\rho(G) \subseteq \mathcal{U}_n$, then e_1 is fixed.

For (1) to (2):

For (2) to (3): It's trivial, since we already know affine implies linear algebraic group.

For (3) to (2): □

Remark 4.1.1. If G is a unipotent affine algebraic group, then every $g \in G(k)$ is unipotent. If G is smooth, then the converse is true.

Proposition 4.1.2. An algebraic group G is seperated.

Thus group variety is a reduced algebraic group, and a reduced algebraic group is the same as smooth algebraic group.

Theorem 4.1.1 (Catier)**.** If $\operatorname{char} k = 0$, then every affine algebraic group over k is smooth.

Definition 4.1.4 (reductive)**.** An affine algebraic group G is reductive if it's smooth and every smooth connected unipotent normal subgroup of G is trivial.

Example 4.1.3. GL_n is reductive.

Example 4.1.4. $\mathbb{G}_a$ is not reductive, since $\mathbb{G}_a$ is isomorphic to a subgroup of $\mathcal{U}_2$, that is

$$c \mapsto \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

Example 4.1.5. Torus $\mathbb{G}_m^n$ is linearly reductive, since by previous result, one has

$$V = \bigoplus_{\chi \in X^*(T)} V_\chi$$

where $V_\chi = \{v \in V \mid tv = \chi(t)v\}$. So any $\langle w \rangle \subset V_\chi$ is a representation, then

$$V_\chi = \oplus_{r_i} V_\chi^{r_i}$$

This completes the proof.

Proposition 4.1.3. The following are equivalent
1. G is linearly reductive;
2. For all $\rho\colon G \to \mathrm{GL}(V)$, any G-invariant subspace $V' \subseteq V$ admits a G-invariant complement;
3. For any surjection of G-representations $\phi\colon V \to W$, the induced map $\phi^G\colon V^G \to W^G$ is surjective.
4.

Example 4.1.6. $\mathbb{G}_a$ is not linearly reductive. Use (2), consider

$$\rho\colon \mathbb{G}_a \to \mathrm{GL}_2$$
$$c \mapsto \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

Consider $V_1 = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle \subseteq k^2$, then there is no G-stable complement.

4.2. **Reynolds operator.**

Definition 4.2.1 (Reynolds operator)**.** Let G be a group acting on a k-algebra A, a G-equivariant linear map $R_A\colon A \to A^G$ is called Reynolds operator if it's projection from A to A^G.

Example 4.2.1. Let G be a finite group acting on a k-vector space V with $\operatorname{char} k \nmid |G|$, then Reynolds operator is given by

$$R_V\colon V \to V^G$$
$$v \mapsto \frac{1}{|G|} \sum_{g \in G} gv$$

Lemma 4.2.1. Let G be a linearly reductive group acting rationally on a finitely generated k-algebra, then there exists a Reynolds operator $R_A\colon A \to A^G$.

Corollary 4.2.1. Let A, B be k-algebras admitting rational action of a linearly reductive group G, with Reynolds operators R_A and R_B, then any G-equivariant homomorphism $h\colon A \to B$, the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "R_A"] & B \arrow[d, "R_B"] \\ A^G \arrow[r, "h"] & B^G \end{tikzcd}$$

Corollary 4.2.2. Let A be a k-algebras admitting a rational action of a linearly reductive group G with Reynolds operator R_A, then for $a \in A^G, b \in A$, one has $R_A(ab) = aR_A(b)$.

Proof. For $a \in A^G$, the homomorphism l_a given by

$$\begin{aligned} l_a\colon A &\to A \\ b &\mapsto ab \end{aligned}$$

is G-equivariant, then above Corollary completes the proof. $\square$

Lemma 4.2.2. Let A be a k-algebra admitting a rational action of a linearly reductive group G with Reynolds operator R_A. Then for any ideal $I \subset A^G$, we have $IA \cap A^G = I$. More generally, if $\{I_j\}_{j\in J}$ are a set of ideals in A^G, then we have

$$(\sum_{j\in J} I_j A) \cap A^G = \sum_{j\in J} I_j$$

In particular, if A is noetherian, then so is A^G.

Theorem 4.2.1 (Hilbert, Mumford)**.** Let G be a linearly reductive group acting rationally on a finitely generated k-algebra A, then A^G is finitely generated.

Theorem 4.2.2 (Popov)**.** An affine algebraic group G over k is reductive if and only if for every rational G-action on a finitely generated k-algebra A, A^G is finitely generated.

5. GIT QUOTIENT

5.1. **The affine quotient.** Let G be a reductive group acting on an affine scheme X.

Definition 5.1.1 (affine GIT quotient)**.** The affine GIT quotient is the morphism $\varphi\colon X \to X//G := \operatorname{Spec} \mathcal{O}(X)^G$ of affine schemes associated to the inclusion $\varphi^*\colon \mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$.

Lemma 5.1.1. Let G be a geometrically reductive group acting on an affine scheme X. If W_1, W_2 are disjoint G-invariant closed subsets of X, then there is an invariant function $f \in \mathcal{O}(X)^G$ which separates these sets.

Theorem 5.1.1. Let G be a reductive group acting on an affine scheme X. Then the affine GIT quotient $\varphi\colon X \to X//G$ is a good quotient, and $X//G$ is an affine scheme.

Corollary 5.1.1. Suppose a reductive group G acts on an affine scheme X and let $\varphi\colon X \to X//G$ be the affine GIT quotient. Then $\varphi(x) = \varphi(x')$ if and only if

$$\overline{G \cdot x} \cap \overline{G \cdot x'} \neq \varnothing$$

Furthermore, the preimage of each point $y \in X$ contains a unique closed orbit. In particular, if the action of G on X is closed, then φ is a geometric quotient.

Proof. See Corollary 3.2.1. □

5.2. **The projective quotient.**

6. Symplectic quotient

A good reference to this section is [Nov12].

6.1. **A quick review to symplectic geometry.** Let M be a smooth manifold admitting a Lie group G action, such manifold is often called a G-manifold. There is a one to one correspondence

$$\{\text{action of } \mathbb{R} \text{ on } M\} \longleftrightarrow \{\text{complete vector fields over } M\}$$

given by $\psi \mapsto X_p = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\psi(t,p)$. In particular, let X be an element of Lie algebra $\mathfrak{g}$, there is a complete vector field given by

$$\sigma(X) := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\exp(-tX)p$$

which is called fundamental field of X.

Definition 6.1.1 (symplectic manifold)**.** A symplectic manifold M is an even-dimensional manifold with a non-degenerate closed 2-form ω, which is called symplectic form.

Definition 6.1.2 (symplectomorphic)**.** A diffeomorphism between two symplectic manifolds $f\colon (M,\omega_M) \to (N,\omega_N)$ is called a symplectomorphic if

$$f^*\omega_N = \omega_M$$

Remark 6.1.1. The group consists of symplectomorphic of (M,ω) is denoted by $\mathrm{Sympl}(M,\omega)$, which is a subgroup of $\mathrm{Diff}(M)$.

Example 6.1.1 (standard symplectic manifold)**.** Consider $\mathbb{R}^{2n}$, there is a natural symplectic form given by

$$\omega_{\mathbb{R}^{2n}} = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}y^i$$

$(\mathbb{R}^{2n},\omega_{\mathbb{R}^{2n}})$ is called standard symplectic manifold. It's clear $(\mathbb{R}^{2n},\omega_{\mathbb{R}^{2n}})$ is symplectomorphic to $(\mathbb{C}^n,\omega_{\mathbb{C}^n})$, where $\omega_{\mathbb{C}^n} = \frac{\sqrt{-1}}{2}\sum_{i=1}^n \mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i$.

Theorem 6.1.1 (Darboux)**.** Let (M,ω) be a symplectic $2n$-manifold, around every $x \in M$, there exists a local coordinate $(x^1,\dots,x^n,y^1,\dots,y^n)$, which is sometimes caleed Darboux coordinate, such that

$$\omega = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}y^i$$

that is (M,ω) is locally symplectomorphic to the standard symplectic manifold $(\mathbb{R}^{2n},\omega_{\mathbb{R}^{2n}})$.

Proof. A good reference for the proof is □

Remark 6.1.2. In Hamiltonian mechanics, the manifold M is a cotangent bundle T^*U, the coordinates $x = (x^1,\dots,x^n)$ parameterize a point in U (the position), and the coordinates $y = (y^1,\dots,x^n)$ parameterize a point in the cotangent space T_xU (the momentum).

Definition 6.1.3. Let f be a smooth function over symplectic manifold (M, ω), then vector field X_f is defined as follows

$$\mathrm{d}f = \iota_{X_f}\omega$$

Remark 6.1.3 (local form). In Darboux coordinates, one has

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\mathrm{d}x^i + \sum_{i=1}^{n} \frac{\partial f}{\partial y^i}\mathrm{d}y^i$$
$$X_f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i} - \sum_{i=1}^{n} \frac{\partial f}{\partial y^i}\frac{\partial}{\partial y^i}$$

6.2. **Hamiltonian action.**

Definition 6.2.1 (symplectic action)**.** A symplectic action of a Lie group G over a symplectic manifold (M, ω) is a Lie group action on M which preserves ω.

Remark 6.2.1. If X is the vector field given rise from this action, then it's symplectic if and only if $\mathcal{L}_X\omega = 0$.

Let (M, ω) be a symplectic manifold, note that the non-degeneracy of ω gives an isomorphism $T_pM \to T_pM^*$ for each $p \in M$, that is we have the following one to one correspondence

$$C^\infty(M, TM) \longleftrightarrow C^\infty(M, \Omega_M)$$
$$X \mapsto \iota_X\omega$$

Cartan's formula says $\mathcal{L}_X\omega = \iota_X\mathrm{d}\omega + \mathrm{d}(\iota_X\omega)$, then by closedness of ω one has $\iota_X\omega$ is closed if and only if $\mathcal{L}_X\omega = 0$, this yields the well-defineness of following definition.

Definition 6.2.2 (symplectic vector field)**.** A vector field X on a symplectic manifold (M, ω) is symplectic if the following equivalent conditions are satisfied

1. its associated 1-form is closed;
2. its associated $\mathbb{R}$-action is symplectic;
3. $\mathcal{L}_X\omega = 0$.

Remark 6.2.2. The symplectic vector field is just like Killing field in Riemannian geometry, and by the same reason one has symplectic vector fields are closed under Lie bracket, since

$$\mathcal{L}_{[X,Y]}\omega = \mathcal{L}_X\mathcal{L}_Y\omega - \mathcal{L}_Y\mathcal{L}_X\omega$$

Example 6.2.1. Let V be a complex vector space equipped with a hermitian product $\langle \text{-}, \text{-} \rangle$, there is a natural symplectic form given by its fundamental form, that is

$$\omega = -\operatorname{Im}\langle \text{-}, \text{-} \rangle$$

Indeed, (V,ω) is symplectomorphic to $(\mathbb{C}^n, \omega_{\mathbb{C}^n})$. Suppose furthermore there is a complex linear action of a Lie group G on V, and suppose $\langle -,- \rangle$ is G-invariant, then action of G is symplectic.

Definition 6.2.3 (Hamiltonian action)**.** Let G be a Lie group and (M,ω) a symplectic G-manifold, the action of G is Hamiltonian if there exists a map $\mu\colon M \to \mathfrak{g}^*$ such that

1. For every $X \in \mathfrak{g}$, if $\mu^X : M \to \mathbb{R}$ is given by $\mu^X(p) := \langle \mu(p), X\rangle$, then
$$\iota_{\sigma(X)}\omega = \mathrm{d}\mu^X$$
2. μ is equivariant with respect to the action of G on M and co-adjoint action of G on $\mathfrak{g}^*$.

Remark 6.2.3. The function μ above is called moment map and functions μ^X are called Hamiltonian functions.

Example 6.2.2. Let K be a compact Lie group acting on a vector space V, and $\langle -,- \rangle$ a K-invariant hermitian product[1]. In Example 6.2.1 we have seen there is a symplectic structure on V and action of K is symplectic with respect to it. Now we're going to show such an action is actually Hamiltonian.

Firstly, suppose K acts through a homomorphism $\rho : K \to \mathrm{GL}(V)$, then there is an induced representation of $\mathfrak{k}$, given by differential of ρ. To be explicit, for $\xi \in \mathfrak{k}$
$$\xi v := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \rho(\exp(t\xi))v = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp(t\mathrm{d}\rho(e)(\xi))v = \mathrm{d}\rho(e)(\xi)v$$
Now we're going to show the moment map is given by
$$\langle \mu(v), \xi\rangle := \frac{1}{2}\operatorname{Im}\langle v, \xi v\rangle$$
where $v \in V$ and $\xi \in \mathfrak{k}$ as follows:

1. Direct computation shows
$$\begin{aligned}\mathrm{d}\mu^\xi(v)(w) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \operatorname{Im}\langle v+tw, \xi v + t\xi w\rangle \\ &= \frac{1}{2}\operatorname{Im}\langle w, \xi v\rangle + \frac{1}{2}\operatorname{Im}\langle v, \xi w\rangle\end{aligned}$$
Note that $\mathfrak{k}$ acts on V as skew-hermitian matrices, so we have
$$\langle v, \xi w\rangle = -\overline{\langle w, \xi v\rangle}$$
This shows
$$\begin{aligned}\mathrm{d}\mu^\xi(v)(w) &= \frac{1}{2}\operatorname{Im}(\langle w, \xi v\rangle - \overline{\langle w, \xi v\rangle}) \\ &= -\operatorname{Im}\langle \xi v, w\rangle \\ &= \omega_v(\sigma(\xi), w)\end{aligned}$$

[1]Such hermitian product can be obtained from Haar's integral.

2. To see μ is K-equivariant:

$$\begin{aligned}\langle \mu(gv), \xi \rangle &= \frac{1}{2} \operatorname{Im}\langle \rho(g)v, \mathrm{d}\rho(e)(\xi)\rho(g)v \rangle \\ &= \frac{1}{2} \operatorname{Im}\langle v, \rho(g)^* \mathrm{d}\rho(e)(\xi)\rho(g)v \rangle\end{aligned}$$

Since $\rho(g)$ is unitary, then $\rho(g)^* = \rho(g)^{-1}$, then $\rho(g)^* \mathrm{d}\rho(e)(\xi)\rho(g)v = \mathrm{ad}_g(\xi)v$, which implies

$$\langle \mu(gv), \xi \rangle = \langle \mu(v), \mathrm{ad}_g(\xi)\rho \rangle$$

This completes the proof.

6.3. **Symplectic reduction.**

Theorem 6.3.1 (Meyer, Marsden-Weinstein)**.** Let (M, ω, G) be a Hamiltonian G-manifold with moment map μ. Suppose that the action of G is free and proper on $\mu^{-1}(0)$. Then

1. $M_{\mathrm{red}} := \mu^{-1}(0)/G$ is a manifold;
2. The projection $\pi \colon \mu^{-1}(0) \to \mu^{-1}(0)/G$ is a principal G-bundle;
3. There is a symplectic form ω_{red} on M_{red} such that $i^*\omega = \pi^*\omega_{\mathrm{red}}$, where $i \colon \mu^{-1}(0) \to M$ is natural inclusion.

7. The Kempf-Ness theorem

7.1. Baby version.

7.2. Statement and proof of the Kempf-Ness theorem.

Theorem 7.2.1 (Kempf-Ness)**.** Let G be the complexification of a compact real Lie group K acting on a finite dimensional complex vector space V through a representation $\rho\colon G \to \mathrm{GL}(V)$. Suppose that the action of K is Hamiltonian with respect to the symplectic form on V induced by a K-invariant hermitian product. Let $X \subseteq V$ be a smooth G-invariant affine variety, then

1. $\mu^{-1}(0) \subseteq X^{ps}$;
2. $X^{ps} \subseteq G\mu^{-1}(0)$;
3. Every G-orbit in X^{ps} contains only one K-oribit of $\mu^{-1}(0)$;
4. There is a bijection

$$X//G \cong X^{ps}/G \to \mu^{-1}(0)/K$$

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