# RIEMANN SURFACE

BOWEN LIU

## 1. Riemann surface

### 1.1. **Definitions and Examples.**

**Definition 1.1.1.** *If $X$ is a surface, a (almost)complex structure is a smooth map $J : TX \to TX$, such that for any $p \in X$, $J_p : T_pX \to T_pX$ is a linear map with $J_p^2 = -\operatorname{id}$.*

**Remark 1.1.2.** *If $X$ admits a complex structure, then $X$ is orientable.*

**Example 1.1.3.** *Assume $X$ has a Riemann metric, and $X$ is orientable. For any $v \in T_pX$, define $J(v)$ to be the tangent vector obtained by rotating $v$ by $\pi/2$ counterclockwise.*

**Corollary 1.1.4.** *Any orientable surface admits a complex structure.*

**Example 1.1.5.** *If $X = \mathbb{C}$, then $T_qX \cong \mathbb{C}, \forall q \in X$, choose $v \in T_qX$, define $J(v) = iv$, then $J$ is a complex structure on $X$.*

**Definition 1.1.6.** *Assume $X$ is a topological space. A complex chart on $X$ is an open subset $U \subset X$ together with a homeomorphism $\varphi : U \to V \subset \mathbb{C}$, where $V$ is an open subset. If $p \in U$, and $\varphi(p) = 0$, then $(U, \varphi)$ is called a chart centered at $p$. For $q \in U$, $z = \varphi(q)$ is called a local coordinate of $q$.*

**Definition 1.1.7.** *If $(U_1, \varphi_1), (U_2, \varphi_2)$ are two charts on X, we say they're compatible if $U_1 \cap U_2 = \varnothing$ or*

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

*is holomorphic.*

**Definition 1.1.8.** *An atlas is a collection of compatible charts $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$, such that $\bigcup_{\alpha \in I} U_\alpha = X$. Two atlas $\mathscr{A}, \mathscr{B}$ are equivalent if every chart in $\mathscr{A}$ and every chart in $\mathscr{B}$ is compatible.*

**Definition 1.1.9.** *A complex structure on $X$ is an equivalent class of atlas on $X$.*

**Remark 1.1.10.** *Given an atlas $\mathscr{A}$ on $X$, we can use charts in $\mathscr{A}$ to define $J : TX \to TX$ such that $J^2 = -\operatorname{id}$.*

**Definition 1.1.11.** *A Riemann surface is a second countable, connected, Hausdorff topological space $X$ together with a complex chart on $X$.*

**Example 1.1.12.** *Every open subset of $\mathbb{C}$ is a Riemann surface.*

**Example 1.1.13.** *$S^2=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=1\}$, consider*

$$U_1=S^2\backslash\{(0,0,1)\}\xrightarrow{\varphi_1}\mathbb{C}$$

*where $\varphi_1$ is defined as $\varphi_1(x,y,z)=\frac{x}{1-z}+i\frac{y}{1-z}=w$. Similarly consider*

$$U_2=S^2\backslash\{(0,0,-1)\}\xrightarrow{\varphi_2}\mathbb{C}$$

*where $\varphi_2$ is defined as $\varphi_2(x,y,z)=\frac{x}{1+z}-i\frac{y}{1+z}=w'$. Note that $ww'=\frac{x^2+y^2}{1-z^2}=1$. And it's easy to see the transition function is $T(w)=\frac{1}{w}$. So $\{U_1,U_2\}$ is an atlas of $S^2$.*

**Example 1.1.14.** *$\mathbb{CP}^1=\{$complex 1-dimensional subspaces of $\mathbb{C}^2\}$, is called a 1-dimensional projective space. Given a point $(0,0)\neq(z,w)\in\mathbb{C}^2$, exists a unique point $[z,w]\in\mathbb{CP}^1$, called the homogenous coordinate of $\mathbb{CP}^1$. Consider*

$$U_1=\{[z,w]\mid z\neq 0\}\xrightarrow{\varphi_1}\mathbb{C}$$

*where $\varphi_1$ is defined as $\varphi_1([z,w])=z/w$. Similarly consider*

$$U_2=\{[z,w]\mid w\neq 0\}\xrightarrow{\varphi_2}\mathbb{C}$$

*where $\varphi_2$ is defined as $\varphi_2([z,w])=w/z$. It's easy to check $\{U_1,U_2\}$ is a atlas of $\mathbb{CP}^1$.*

*In fact, $\mathbb{CP}^1$ is a Riemann surface which is isomorphic to $S^2$.*

**Example 1.1.15.** *Given two nonzero $w_1,w_2\in\mathbb{C}$, with $w_1\neq aw_2$ for any $a\in\mathbb{C}$. Define lattice:*

$$L=\mathbb{Z}w_1+\mathbb{Z}w_2$$

*In fact, $L$ is a subgroup of $\mathbb{C}$ with respect to operation "+".*

*Then $T=\mathbb{C}/L$ is a Riemann surface called complex torus. Consider the projection $\pi:\mathbb{C}\to T$. For $p\in T$, find one of its inverse image of $\pi$, denoted by $z_0$. Choose $\varepsilon\in\mathbb{R}^+$ small enough such that*

$$B_{2\varepsilon}\cap L=\{0\}$$

*Consider*

$$B_\varepsilon(z_0)\xrightarrow{\pi}\pi(B_\varepsilon(z_0))\subset T$$

*and the condition on $\varepsilon$ implies $\pi|_{B_\varepsilon}$ is injective. So let $\{\pi(B_\varepsilon(z_0))\}$ be a open cover of $T$, and $\pi^{-1}$ is the parametrization, this is an atlas of $T$.*

**Remark 1.1.16.** *The complex structure of complex torus depends on $w_1,w_2$. In fact, all complex structure of complex torus forms a Riemann surface of genus one.* *

*The space consists of all complex structure of a Riemann surface is called the moduli space of it.

## 1.2. **Holomorphic function and Properties.**

**Definition 1.2.1.** *If $X$ is a Riemann surface, $W \subset X$ is a open subset. The function $f : W \to \mathbb{C}$ is a complex valued function on $W$. $f$ is called holomorphic at $p \in W$, if there exists a chart $(U, \varphi)$ of $p$ such that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ is holomorphic at $\varphi(p)$. $f$ is called holomorphic on $W$, if it is holomorphic at any $p \in W$.*

**Theorem 1.2.2** (Maximum modulus theorem)**.** *For a Riemann surface $X$, $W \subset X$ is an open subset, and $f$ is a holomorphic function on $W$. If there exists a point $p \in W$, such that $|f(p)| \geq |f(x)|$ for all $x \in W$, then $f$ must be a constant.*

*Proof.* Clear. □

**Corollary 1.2.3.** *If $X$ is a compact Riemann surface, then any global holomorphic funtion $f$ must be constant.*

So, it's boring to consider holomorphic funtions on a compact Riemann surface. In order to get something interesting, we need to consider meromorphic functions.

**Definition 1.2.4.** *If $X$ is a Riemann surface, let $f$ be a holomorphic function defined on $U\backslash\{p\}$ where $U \subset X$ is an open subset. $p$ is called a removbale singularity/pole/essential singularity, if there exists a chart $(U, \varphi)$ of $p$, such that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ has $\varphi(p)$ as a removbale singularity/pole/essential singularity.*

**Remark 1.2.5.** *We have the following criterions:*

1. *If $|f(x)|$ is bounded in a punctured neighborhood of $p$, then $p$ is a removable singularity. And we can cancel the singularity by defining $f(p) = \lim_{x \to p} f(x)$.*
2. *If $\lim_{x \to p} |f(x)| = \infty$, then $p$ is a pole.*
3. *If $\lim_{x \to p} |f(x)|$ doesn't exist, then $p$ is a essential singularity.*

**Definition 1.2.6.** *$f$ is called a meromorphic function at $p$ if $p$ is either a removbale singularity or a pole, or $f$ is holomorphic at $p$; $f$ is called a meromorphic function on $W$, if it's meromorphic at any point $p \in W$.*

**Remark 1.2.7.** *If $f, g$ are meromorphic on $W$, then $f \pm g, fg$ are also meromorphic on W. If in addition, $g \not\equiv 0$, then $f/g$ is also meromorphic on $W$.*

**Example 1.2.8.** *Consider $f, g$ are two polynomials in variable $z$ with $g \not\equiv 0$, then $f/g$ is a meromorphic function on $S^2 = \mathbb{C} \cup \{\infty\}$. In fact, all meromorphic functions on $S^2$ are in this form.*

**Theorem 1.2.9** (Singularities and zeros)**.** *Let $X$ be a Riemann surface and $W \subset X$ is an open subset, $f$ is a meromorphic function on $W$, then set of singularities and zeros of $f$ is discrete, unless $f \equiv 0$.*

**Corollary 1.2.10.** *If $X$ is compact, $f \not\equiv 0$, then $f$ has finitely many poles and zeros on $X$. As a consequence, if $f, g$ are two meromorphic functions on an open subset $W \subset X$, and $f$ agrees with $g$ on a set with limit point in $W$, then $f \equiv g$.*

**Definition 1.2.11.** *Let $X, Y$ be two Riemann surfaces, $F : X \to Y$. For a point $p \in X$, $f$ is called holomorphic at $p$, if there exists a chart $(U, \varphi)$ of $p$, and a chart $(V, \psi)$ of $F(p)$, such that*

$$\psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V \cap F(U))$$

*is holomorphic at $\varphi(p)$; $F$ is called holomorphic in $W$, if $F$ is holomorphic at any point in $W$.*

**Remark 1.2.12.** *$\psi \circ F \circ \varphi^{-1}$ is called the local representation of $F$.*

**Example 1.2.13.** *Any meromorphic function on $X$ can be seen as a holomorphic map from $X$ to $S^2$; Conversely, we can construct a meromorphic function from a holomorphic map from $X$ to $S^2$.*

**Definition 1.2.14.** *Two Riemann surfaces are called biholomorphic or isomorphic to each other, if there are two holomorphic map $F : X \to Y, G : Y \to X$, such that $F \circ G = G \circ F = \mathrm{id}$.*

**Example 1.2.15.** *$S^2$ is biholomorphic to $\mathbb{RP}^2$.*

**Theorem 1.2.16** (Open mapping theorem)**.** *$F : X \to Y$ is a non-constant holomorphic map, then $F$ is an open map.*

**Corollary 1.2.17.** *If $X$ is compact, and $Y$ is connected, $F : X \to Y$ is a non-constant holomorphic map, then $Y$ is compact and $F(X) = Y$.*

*Proof.* By open mapping theorem, $F(X)$ is an open subset of $Y$, and $F(X)$ is compact in $Y$, since continous function maps compact set to compact set. Then $F(X)$ is both open and closed in $Y$, then $F(X) = Y$. □

### 1.3. **Ramification covering.**

**Theorem 1.3.1.** *$F : X \to Y$ is a non-constant holomorphic function on Riemann surfaces, then for any $p \in Y$, $F^{-1}(y)$ is a discrete set. Furthemore, if $X$ is compact, then $F^{-1}(y)$ only contains finite many points.*

So we wonder what's exact number of $F^{-1}(y)$, and furthermore, is all these numbers are same for any $y \in Y$? In fact, it is!

**Theorem 1.3.2** (Local normal form)**.** *$F : X \to Y$ is a non-constant holomorphic function on $X$, then there is a local representation of $F$, such that*

$$\psi \circ F \circ \varphi^{-1}(z) = z^k, \quad \forall z \in \varphi(U \cap F^{-1}(V))$$

*$k$ is called the multiplicity*[†] *of $F$ at $p$, denoted by $\mathrm{mult}_p(F)$. In fact, $k$ is independent of the choice of charts.*

[†]Sometimes this number is also called ramification of $F$ at $p$.

*Proof.* Fix a chart $(U_2, \varphi_2)$ of $F(p)$, choose an arbitary local chart $(U, \psi)$ of $p$ such that $F(U) \subset U_2$, denote $\varphi_2 \circ F \circ \psi^{-1} = T$, then $T(0) = 0$. Consider the Taylor expansion of $T$ at $w = 0$ has

$$T(w) = \sum_{k=m}^{\infty} a_k w^k, \quad a_m \neq 0$$

So $T(w) = w^m S(w)$, where $S(w)$ is a holomorphic function with $S(0) \neq 0$, then there exists a holomorphic function $R(w)$ such that $R^m(w) = S(w)$.

Then $T(w) = (wR(w))^m = (\eta(w))^m$, so $\eta(0) = 0, \eta'(0) = R(0) \neq 0$, so $\eta$ is invertible near $w = 0$ by inverse funtion theorem. So there exists another chart of $p \in U_1 \subset U$, with

$$U \supset U_1 \xrightarrow{\psi} V \xrightarrow{\eta} V_1 \subset \mathbb{C}$$

then we can define a local chart $(U_1, \varphi_1 = \eta \circ \psi)$, and check

$$\varphi_2 \circ F \circ \varphi_1^{-1}(z) = \varphi_2 \circ F \circ \psi^{-1} \circ \eta^{-1}(z) = T(w) = (\eta(w))^m = z^m$$

□

**Remark 1.1.** *We can see from the local normal form that for any $q \in Y, q \neq F(p)$ and $q$ lies in a small neighborhood of $p$, then $F^{-1}(q)$ consists of exactly $k$ points*

**Definition 1.3.3.** *$p$ is called a ramification point of a holomorphic map $F : X \to Y$, if $\operatorname{mult}_p(F) > 1$, such $F(p)$ is called a ramification value.*

**Lemma 1.3.4.** *$p$ is a ramification point of a holomorphic map $F : X \to Y$ if $T'(w) = 0$, for any local representation of $F$.*

**Corollary 1.3.5.** *The set of ramification points of a holomorphic map is a discrete set.*

**Theorem 1.3.6.** *Assume $X, Y$ are complex Riemann surface, $F : X \to Y$ is non-constant holomorphic function, for $q \in Y$, let*

$$d_q(F) = \sum_{p \in F^{-1}(q)} \operatorname{mult}_p(F)$$

*then $d_q(F)$ is independent of $q \in Y$, and denoted by $\deg(F)$.*

*Proof.* Consider $F : \mathbb{D} \to \mathbb{D}$, defined by $z \mapsto z^m$, it's easy to check $d_q(F) = m$, for all $q \in \mathbb{D}$.

For general case, for $q \in Y$, let $F^{-1}(q) = \{p_1, \dots, p_k\} \subset X$. Fix a chart $(U_2, \varphi_2)$ centered at $q \in Y$, for any $i = 1, \dots, k$, we can find local chart $(U_{1,q}, \psi_i$ centered at $p_i \in X$, such that

$$\varphi_2 \circ F \circ \psi_i^{-1}(z) = z^{m_i}, \quad z \in \psi_i(U_{1,i})$$

where $m_i = \mathrm{mult}_{p_i}(F)$. Choose $q \in W \subset U_2$ such that $F^{-1}(W) \subset \bigcup_{i=1}^{k} U_{1,i}$, then for any $q \in W$

$$d_q(F) = \sum_{i=1}^{k} m_i$$

which can be seen from trivial case we discuss firstly. Then $d_q(F)$ is a locally constant function, then $d_q(F)$ must be global constant, since $Y$ is connected. □

**Corollary 1.3.7.** *$X$ is a compact Riemann surface, and $f$ is a meromorphic function on $X$, then the number (with multiplicity) of zeros is equal to the number (with multiplicity) of poles.*

*Proof.* Note that meromorphic function on $X$ is equivalent to the holomorphic map from $X$ to $S^2$. □

1.4. **Hurwitz Formula.** Now let us forget the complex structure of Riemann surface, and recall some facts about topological invariants.

Let $X$ be a compact oriented surface, we can say the genus of $X$ is the number of "holes" which $X$ has, informally. We can use genus to classify all oriented compact surfaces: any two surfaces which have the same genus are diffeomorphic to each other.

We can also define Euler characterisitic of $X$, as

$$\chi(X) := \sum_i (-1)^i \dim H_i(X)$$

And there is a connection between genus of $X$ and $\chi(X)$,

$$\chi(X) = 2 - 2\,\mathrm{genus}(X)$$

so we can also use $\chi(X)$ to classify oriented compact surface.

**Theorem 1.4.1** (Hurwitz Formula)**.** *Let $X, Y$ be two compact Riemann surfaces, and $F : X \to Y$ be a non-constant holomorphic map, then*

$$2\,\mathrm{genus}(Y) - 2 = \deg(F)(2\,\mathrm{genus}(X) - 2) + \sum_{p\in X}(\mathrm{mult}_p(F) - 1)$$

*Note that the set of ramification points is finite, then $\sum_{p\in X}(\mathrm{mult}_p(F) - 1)$ is a finite sum, and denoted by $B(F)$.*

*Proof.* Choose a triangulation of $Y$ such that its vertex are exactly ramification values of $F$. Let $v$ denote the number of vertices of $\Delta$, $c$ and $t$ denote the number of edges and triangles of $\Delta$, where $\Delta$ denotes a triangulation of $Y$. We can get a triangulation $\Delta'$ of $X$, by pulling back $\Delta$ through $F$, and use $v', c'$ and $t'$ to denote the same thing in $\Delta'$.

Then we have the following obvious relations

$$t' = td, \quad e' = ed$$

where $d = \deg(F)$. The relation between $v$ and $v'$ is a little bit complicated, consider $q \in Y$, then

$$|F^{-1}(q)| = \sum_{p \in F^{-1}(q)} 1 = d + \sum_{p \in F^{-1}(q)} (1 - \operatorname{mult}_p(F))$$

then

$$\begin{aligned}
v' &= \sum_{\text{vertex } q \text{ of } \Delta} |F^{-1}(q)| \\
&= \sum_{\text{vertex } q \text{ of } \Delta} (d + \sum_{p \in F^{-1}(q)} (1 - \operatorname{mult}_p(F))) \\
&= vd + \sum_{\text{vertex } q \text{ of } \Delta} \sum_{p \in F^{-1}(q)} (1 - \operatorname{mult}_p(F)) \\
&= vd + \sum_{p \in X} (1 - \operatorname{mult}_p(F))
\end{aligned}$$

Then by the relation between Euler characterisitic and triangulation, we get the desired conclusion. □

**Definition 1.4.2.** *A holomorphic map $F$ is called ramified if $B(F) > 0$, this is equivalent to $F$ has at least one ramification point; A holomorphic map $F$ is called unramified if $B(F) = 0$, this is equivalent to $F$ is a covering map.*

**Corollary 1.4.3.** *Let $X, Y$ be two compact Riemann surfaces, and $F : X \to Y$ is a non-constant holomorphic map, then consider*

1. *If $Y = S^2$, and $\deg(F) > 1$, then $F$ must be ramified.*
2. *If $\operatorname{genus}(X) = \operatorname{genus}(Y) = 1$, then $F$ must be unramified.*
3. *$\operatorname{genus}(X) \geq \operatorname{genus}(Y)$.*
4. *If $\operatorname{genus}(X) = \operatorname{genus}(Y) > 1$, then $F$ must be an isomorphism.*

*Proof.* All of them are simple applications of Hurwitz Formula.

1. By Hurwitz Formula we have

$$B(F) = 2(\deg(F) - 1) + 2\operatorname{genus}(X) > 0$$

2. By Hurwitz Formula we have

$$0 = 0 + B(F)$$

3. If $\operatorname{genus}(Y) = 0$, it's trivial. Otherwise, we have

$$2\operatorname{genus}(X) - 2 \geq 2\operatorname{genus}(Y) - 2 + B(F)$$

since $\deg F \geq 1$.

4. By Hurwitz Formula we have

$$(1 - \deg(F))(2\operatorname{genus}(X) - 2) = B(F)$$

Then $\deg(F) = 1$, since $\deg(F) \geq 1$, $2\operatorname{genus}(X) - 2 > 0$ and $B(F) \geq 0$. □

**Remark 1.4.4.** *From above corollary, we can see that genus, as a topological invariants, controls geometric properties heavily.*

Consider a lattice $L = \mathbb{Z}w_1 + \mathbb{Z}w_2$, let $X$ denote the complex torus $X = \mathbb{C}/L$, a Riemann surface with genus 1. Moreover, there is a group structure on $X$, induced by $(\mathbb{C}, +)$ through natural projection $\pi : \mathbb{C} \to X$, defined as follows

$$[z_1] + [z_2] := [z_1 + z_2]$$

So, inversions

$$[z] \mapsto [-z]$$

are automorphisms.

For $a \in \mathbb{C}$,we can define a transformation

$$T_a : X \to X, \quad [z] \mapsto [z + a]$$

which is also an automorphism.

So, as we can see, there are too many automorphism on $X$, let $\mathrm{Aut}(X)$ denote all automorphisms on $X$, which forms a group which can reflect the symmetry of $X$.

Obviously,

$$\mathrm{Aut}(X) \supset \{T_{[a]} \mid [a] \in X\} \cup \{\text{inversions}\}$$

In fact, $\mathrm{Aut}(X)$ is a complex manifold with $\dim_{\mathbb{C}} \mathrm{Aut}(X) \geq 1$.

School of Mathematics, Shandong University, Jinan, 250100, P.R. China,
*Email address*: `bowenl@mail.sdu.edu.cn`