HODGE THOERY AND COMPLEX ALGEBRAIC GEOMETRY

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ABSTRACT. It's a lecture note I typed for "Hodge theory and complex algebraic geometry" taught by Qizheng Yin, in spring 2022. This note mainly follows the blackboard-writing of Prof. Yin. I also add some details and my understandings in it.

Attention: there may be a considerable number of mistakes in this note, and that's all my fault, since I still have too many problems to work out.

CONTENTS

0. Overview

In this course, we will introduce two parts:

I Kähler manifold and Hodge decomposition;

II Hodge theory in algebra geometry.

For the first part, if X is a compact complex manifold, we can consider the following structures:

(1) Topology: $H^*_B(X, \mathbb{Z})$, singular cohomology, where B means "Betti".

(2) C^∞-structure: $H^*_{dR}(X, \mathbb{R}) = H^*(X, \Omega^\bullet_{X,\mathbb{R}})$, de Rham cohomology. In fact, de Rham theorem implies that

$$H^*_{dR}(X, \mathbb{R}) \cong H^*_B(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$$

(3) Complex structure: For $x \in X$, we use $T_{X,x}$ to denote its tangent space at x, its real dimension is $2n$. And there is a linear map $J_x : T_{X,x} \to T_{X,x}$ for any $x \in M$, such that $J_x^2 = -\mathrm{id}$. If we complexificate $T_{X,x}$, then we can decompose it into

$$T_{X,x} \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}_{X,x} \oplus T^{0,1}_{X,x}$$

where $T^{1,0}_{X,x}$ is the eigenspace belonging to eigenvalue $\sqrt{-1}$, and $T^{0,1}_{X,x}$ is the eigenspace belonging to eigenvalue $-\sqrt{-1}$.

If we consider its dual, we will get bundle/sheaf of differential forms, and we can also decompose them as follows

$$\Omega^1_{X,\mathbb{C}} = \Omega^1_{X,\mathbb{R}} \otimes \mathbb{C} = \Omega^{1,0}_X \oplus \Omega^{0,1}_X$$

and

$$\Omega^k_{X,\mathbb{C}} = \Omega^k_{X,\mathbb{R}} \otimes \mathbb{C} = \bigoplus_{p+q=k} \Omega^{p,q}_X$$

where $\Omega^{p,q}_X = \bigwedge^p \Omega^{1,0}_X \otimes \wedge^q \Omega^{0,1}_X$

Since we have such decomposition for differential forms, it's natural to ask if there is a similar decomposition for de Rham cohomology? In other words, do we have

$$H^k_{dR}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

such that $\overline{H^{p,q}(X)} = H^{q,p}(X)$?

The Hodge decomposition says it's true for compact Kähler manifolds. It's a very beautiful result, connecting "Topology" and "Complex geometry", since de Rham cohomology reflects the topology information and

$$H^{p,q}(X) \cong H^q_{Dol}(X, \Omega^p_X)$$

where "Dol" means Dolbeault cohomology, reflects the holomorphic information of a complex manifold.

Here is some examples of Kähler manifolds. In fact, almost every interesting manifold is Kähler manifold:

Example 0.0.1. Riemann surfaces, complex torus, projective manifolds are Kähler manifolds.

We also need to know an example that is not Kähler manifold:

Example 0.0.2 (Hopf surface)**.** Consider $\mathbb{Z}$ acts on $\mathbb{C}^2\backslash\{0\}$ by $mz = \lambda^m z, m \in \mathbb{Z}$ for some $\lambda \in (0,1)$, then we define Hopf surface as follows

$$S = \mathbb{C}^2\backslash\{0\}/\mathbb{Z}$$

As we can see, S is diffeomorphic to $S^3 \times S^1$,then $\dim_{\mathbb{C}} H^1(S, \mathbb{C}) = 1$, so S can not be a Kähler manifold by Hodge's decomposition, since for a Kähler manifold, $\dim_{\mathbb{C}} H^1$ must be even.

Remark 0.0.3. By Chow's theorem/GAGA, a compact complex manifold X admitting an embedding into projective space can be defined by polynomial equations, i.e. X is a projective variety, so here comes the forth structure, and that's the second part of this course, we want to apply Hodge theory in algebraic geometry.

(4) Algebraic structure.

Part 1. Preliminaries

1. Complex manifold

1.1. Manifolds and vector bundles.

1.1.1. *Definitions and Examples.*

Definition 1.1.1 (complex manifold)**.** *A complex manifold consists of $(X, \{U_i, \phi_i\}_{i\in I})$, where X is a connected, Hausdorff topological space, $\{U_i\}_{i\in I}$ is an open cover of X such that the index set I is countable, and ϕ_i is a homeomorphism from U_i to an open subset V_i of $\mathbb{C}^n$, such that*

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is biholomorphic.

Remark 1.1.2. Such $\phi_i \circ \phi_j^{-1}$ is called transition function; n is called the dimension of X, denoted by $\dim_{\mathbb{C}} C$; $\{U_i, \phi_i\}_{i\in I}$ is called complex atlas. Two atlas are called equivalent, if the union of them is still an atlas.

Definition 1.1.3 (complex structure)**.** *A complex structure is an equivalence class of a complex atlas.*

Remark 1.1.4. Replace $\mathbb{C}^n$ by $\mathbb{R}^n$, and biholomorphism is replaced by homeomorphism or diffeomorphism, then we get topological manifold or smooth manifold. It's a philosophy, since these things are all trivial locally, so how does we glue them together really matters.

Remark 1.1.5. $V_i \subset \mathbb{C}^n$ usually can not be the whole $\mathbb{C}^n$. For example, there is no non-constant holomorphism from $\mathbb{C}$ to unit disk $\mathbb{D}$. More generally, X is called Brody hyperbodic if there is no non-constant holomorphism from $\mathbb{C}$ to X.

Example 1.1.6. Projective space $\mathbb{P}^n$ is a complex manifold. Atlas are $U_i = \{[z] \in \mathbb{P}^n \mid z_j \neq 0\}, 0 \leq i \leq n, \phi_i : U_i \to \mathbb{C}^n$ is defined as follows

$$[z] \mapsto (\frac{z_0}{z_i}, \ldots, \frac{\widehat{z_i}}{z_i}, \ldots, \frac{z_n}{z_i})$$

Transition functions are calculated as follows, for $i < j$

$$\phi_i \circ \phi_j^{-1} : (u_1, \ldots, u_n) \mapsto (\frac{u_1}{u_i}, \ldots, \frac{\widehat{u_i}}{u_i}, \ldots, \frac{u_{j-1}}{u_i}, \frac{1}{u_i}, \frac{u_{j+1}}{u_i}, \ldots, \frac{u_n}{u_i})$$

In fact, $\mathbb{P}^n$ is a compact complex manifold, since $\mathbb{P}^n$ is diffeomorphic to S^{2n+1}/S^1.

Example 1.1.7. Grassmannian manifold

$$Gr(k, n) = \{k\text{-dimensional subspace of } \mathbb{C}^n\}$$

Now we're going to show $Gr(k, n)$ is a manifold of dimension $k(n-k)$. An atlas for $Gr(k, n)$ is given as follows: The idea is to present linear space of dimension k as graphs of linear maps. To be precise, let's introduce some notations.

For any subset $I \subset \{1, \ldots, n\}$ of the set of indices, let

$$I' = \{1, 2, \ldots, n\} \backslash I$$

be its complement. And define $\mathbb{C}^I = \{x \in \mathbb{C}^n \mid x^i = 0, \forall i \in I'\}$. Clearly, if $|I| = k$, then $\mathbb{C}^I \in Gr(k, n)$. Note that $\mathbb{C}^{I'} = (\mathbb{C}^I)^\perp$. Let

$$U_I = \{E \in Gr(k, n) \mid E \cap \mathbb{C}^{I'} = \{0\}\}$$

Thus each $E \in U_I$ can be described as the graph of a unique linear map $A_I : \mathbb{C}^I \to \mathbb{C}^{I'}$, that is

$$E = \{y + A_I(y) \mid y \in \mathbb{C}^I\}$$

This gives a bijection

$$\begin{aligned} \varphi_I : U_I &\to \operatorname{Hom}(\mathbb{C}^I, \mathbb{C}^{I'}) \cong \mathbb{C}^{k(n-k)} \\ E &\mapsto \varphi_I(E) = A_I \end{aligned}$$

Now it suffices to show this really is an atlas.

Example 1.1.8. Complex torus is $\mathbb{C}^n/\Lambda$ where Λ is a free abelian subgroup of $\mathbb{C}^n$ with rank $2n$, called a lattice.

Definition 1.1.9 (holomorphic map)**.** *Let X, Y be complex manifolds of dimension n, m, with atlas $(U_i, \phi_i : U_i \to V_i)$ and $(M_j, \psi_j : M_j \to N_j)$ respectively. A continous map $f : X \to Y$ is called holomorphic, if for any two charts, we have*

$$\psi_j \circ f \circ \phi_i^{-1} : V_i \to \psi_j(f(U_i) \cap M_j)$$

is holomorphic.

Definition 1.1.10 (holomorphic function)**.** *A holomorphic function on X is a holomorphic map $f : X \to \mathbb{C}$.*

Example 1.1.11. Let $S = (\mathbb{C}^2 \backslash \{0\})/\mathbb{Z}$ be Hopf surface, then

$$f : S \to \mathbb{P}^1 = (\mathbb{C}^2 \backslash \{0\})/\mathbb{C}^*$$

is a holomorphic map. The fibers of f are biholomorphic to 1-dimensional complex torus.

Proposition 1.1. *If X is a compact complex manifold, then every holomorphic function on X is constant.*

Proof. Standard conclusion in complex analysis. □

Definition 1.1.12 (immersion/submersion)**.** *A holomorphic map $f : X \to Y$ is called an immersion(resp submersion), if for all $x \in X$, there exists $(x \in U_i, \phi_i), (f(x) \in M_j, \psi_j)$, such that*

$$J_{\psi_j \circ f \circ \phi_i^{-1}}(\phi_i(x))$$

has the max rank $\dim X$(resp $\dim Y$)

Definition 1.1.13 (embedding)**.** *$f : X \to Y$ is an embedding, if it is immersion and $f : X \to f(X) \subset Y$ is homeomorphism.*

Definition 1.1.14 (submanifold)**.** *A closed connected subset Y of X is called a submanifold, if for all $x \in Y$, there exists $x \in U \subset X$ and a holomorphic submersion $f : U \to \mathbb{D}^k$ such that*

$$U \cap Y = f^{-1}(0)$$

where k is the codimension of Y in X.

Example 1.1.15 (regular value theorem)**.** Let X, Y be complex manifold with dimension n, m, If $y \in Y$ such that $\mathrm{rank} J_{f(x)}$ reaches maximum m for all $x \in f^{-1}(y)$, then $f^{-1}(y)$ is a submanifold of codimension m.

Definition 1.1.16 (projective manifold)**.** *A projective manifold*[1] *X is a submanifold of $\mathbb{P}^N$ of the form*

$$X = \{[z] \in \mathbb{P}^N \mid f_1(z) = \cdots = f_m(z) = 0\}$$

where f_i is a homogenous polynomial in $\mathbb{C}[z_0, \ldots, z_n]$

Remark 1.1.17. Here we always assume $(f_1, \ldots, f_m) \subset \mathbb{C}[z_0, \ldots, z_n]$ is a prime ideal, so the case that X is defined by polynomials like $f^2 = 0$ is not allowed, what's more, the condition that X is a manifold implies the following cases won't happen:

[1] Chow's theorem claims that every submanifold of $\mathbb{P}^n$ must be defined by a set of homogenous polynomials, so we can use this property to define a projective manifold, in convenient.

1. $f_1 f_2 = 0$;
2. X has a singular point.

Definition 1.1.18 (complete intersection)**.** *Let $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ be the projection, then X is a submanifold of codimension k if and only if*

$$J = (\frac{\partial f_i}{\partial z_j})_{\substack{1\le i\le m\\ 0\le j\le N}}$$

has rank k, for all $x \in \pi^{-1}(X)$. Then X is called a complete intersection, if $m = k$.

Example 1.1.19. Consider $C \subset \mathbb{P}^n$ defined by

$$xw - yz = y^2 - xz = z^2 - yw = 0$$

is not a complete intersection, called twisted cubic.

Example 1.1.20. Plücker embedding

$$\Phi : G(k, V) \hookrightarrow \mathbb{P}(\wedge^k V)$$

defined by $S \subset V$ with basis $s_1, \dots, s_k$ is mapped to $[s_1 \wedge \cdots \wedge s_k]$. It's easy to check it's well-defined, this fact follows from the following lemma

Lemma 1.1.21. *Let W be a subspace of a finite dimensional vector space, and let $\mathcal{B}_1 = \{w_1, \dots, w_k\}$ and $\mathcal{B}_2 = \{v_1, \dots, v_k\}$ be two basis for W. Then $v_1 \wedge \cdots \wedge v_k = \lambda w_1 \wedge \cdots \wedge w_k$ for some λ lying in the base field.*

Proof. Write $w_j = a_{1j}v_1 + \cdots + a_{kj}v_k$. Then one can directly compute that

$$\begin{aligned} w_1 \wedge \cdots \wedge w_k &= (a_{11}v_1 + \cdots + a_{k1}v_k) \wedge \cdots \wedge (a_{1k}v_1 + \cdots + a_{kk}v_k) \\ &= \sum_{\sigma\in S_k} \epsilon(\sigma) a_{1\sigma(1)} \cdots a_{k\sigma(k)} v_1 \wedge \cdots \wedge v_k \\ &= \lambda v_1 \wedge \cdots \wedge v_k \end{aligned}$$

Note that λ we need is exactly the determinant of the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$. □

However, it's a little bit complicated to check it's injective. I will add the proof if I'm not tooo lazy in future.(smile)

Remark 1.1.22. Together above embedding with Chow's theorem, we have the fact that Grassmannian manifold is a variety, in fact.

1.1.2. *Vector bundle.*

Definition 1.1.23 (complex vector bundle)**.** *Let X be a differential manifold, E is a complex vector bundle of rank r on X*

1. (*Via total space*) *E is a differential manifold with surjective map $\pi : E \to X$, such that*
   (1) *For all $x \in X$, fiber E_x is a $\mathbb{C}$-vector space of dimension r;*
   (2) *For all $x \in X$, there exists $x \in U \subset X$ and $\pi^{-1}(U)$ is diffeomorphic to $U \times \mathbb{C}^r$ via h such that*

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\pi"] \arrow[dr, "h"'] & & U \\
& U \times \mathbb{C}^r \arrow[ur, "p_1"'] \arrow[u, phantom, "\circlearrowright"] \arrow[r, "p_2"] & \mathbb{C}^r
\end{tikzcd}$$

and for all $y \in U$, $E_y \xrightarrow{p_2 \circ h} \mathbb{C}^r$ is a $\mathbb{C}$-vector space isomorphism.

Remark 1.1.24. Consider two local trivialization $(U_\alpha, h_\alpha), (U_\beta, h_\beta)$, then $h_\alpha \circ h_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$, this induces

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{\text{diff}} \mathrm{GL}(r, \mathbb{C})$$

such $g_{\alpha\beta}$ are called transition function[2], such that

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = \mathrm{id} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma$$

$$g_{\alpha\alpha} = \mathrm{id} \quad \text{on } U_\alpha$$

2. (*Via transition function*) *E is the data of*
   (1) *open covering $\{U_\alpha\}$ of X;*
   (2) *transition functions $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{\text{diff}} \mathrm{GL}(r, \mathbb{C})\}$, satisfies*

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = \mathrm{id} \quad \textit{on } U_\alpha \cap U_\beta \cap U_\gamma$$

$$g_{\alpha\alpha} = \mathrm{id} \quad \textit{on } U_\alpha$$

Remark 1.1.25. The two definitions above are equivalent. The first definition implies the second clearly. The converse is a standard constructive method, which tell us how to glue things together using gluing data:

If we already have an open covering and a set of transition functions, the vector bundle E is defined to be the quotient of the disjoint union $\coprod_{U_\alpha}(U \times \mathbb{C}^r)$ by the equivalence relation that puts $(p', v') \in U_\beta \times \mathbb{C}^r$ equivalent to $(p, v) \in U_\alpha \times \mathbb{C}^r$ if and only if $p = p'$ and $v' = g_{\alpha\beta}(p)v$. To connect this definition with the previous one, define the map π to send the equivalence class of any given (p, v) to p.

Definition 1.1.26 (holomorphic vector bundle)**.** *X is a complex manifold, $\pi : E \to X$ is a complex vector bundle, given by $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{C})$, E is called holomorphic if $g_{\alpha\beta}(x)$ is holomorphic function on $\mathbb{C}^r$ for each $x \in U_\alpha \cap U_\beta$.*

Exercise 1.1.27. Show that the total space of a holomorphic vector bundle E is a complex manifold.

Proof. Since we already have a complex structure on X, we need to pull it back to E using π and use the holomorphic transition functions to show it indeed gives a complex structure on E. □

[2] Note that here "diff" means for each $x \in U_\alpha \cap U_\beta$, we have $g_{\alpha\beta}(x)$ is a smooth function on $\mathbb{C}^r$.

Definition 1.1.28 (morphism between vector bundles)**.** *ϕ is a diffeomorphic/holomorphic morphism of vector bundle on X of rank k, if $\phi : E \to F$ is diffeomorphic/holomorphic map and fiberwise $\mathbb{C}$-linear of rank k.*

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi_1"'] & {} \arrow[d, phantom, "\circlearrowleft"] & F \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

Example 1.1.29. X is a differential/complex manifold, then $X \times \mathbb{C}^r$ is the trivial rank r complex/holomorphic vector bundle on X.

Example 1.1.30. E, F are complex/holomorphic vector bundles on X, then $E \oplus F, E \otimes F, \mathrm{Hom}(E, F), E^* = \mathrm{Hom}(E, \mathbb{C}), \mathrm{Sym}^k E, \bigwedge^k E, \det E$ are complex/holomorphic vector bundles.

Though all of these are basic constructions of linear algebra, we need to make it clear here in order to avoid further confusion.

If E, F are given by transition functions $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$. WLOG, we may assume they use the same open covering $\{U_\alpha\}$, otherwise we can take their common refinement.

Then, for direct sum, we can define transition functions as

$$\begin{aligned} g''_{\alpha\beta} : U_\alpha \cap U_\beta &\to \mathrm{GL}(2r, \mathbb{C}) \\ x &\mapsto \mathrm{diag}(g_{\alpha\beta}(x), g'_{\alpha\beta}(x)) \end{aligned}$$

and similarly for tensor $E \otimes F$, we can define transition functions as $g_{\alpha\beta} \otimes g'_{\alpha\beta}$.

Now if we can define the dual vector bundle E^*, then in fact we can define $\mathrm{Hom}(E, F)$ as

$$\mathrm{Hom}(E, F) = E^* \otimes F$$

For dual vector bundle defined by $\{g_{\alpha\beta}\}$, in fact the transition functions are $\{(g_{\alpha\beta}^{-1})^T\}$, i.e. the transpose of the inverse. But it may be difficult to understand why? In fact, you will find it's just a fact in linear algebra.

Let's back to the definition via total space, it's natural to define the dual vector bundle of E, by defining all fibers to be the dual space of E_x. To elaborate, E^* is, first of all, the set of pairs $\{(p, l) \mid p \in X, \text{ and } l : E_x \to \mathbb{C} \text{ is a linear map.}\}$, and π maps (p, l) to $p \in X$. Furthermore, it's important to know what is the trivialization. If (U_α, h_α) is the trivialization of vector bundle E, defined by $E_p \ni (p, e) \mapsto (p, \lambda_\alpha(e)) \in U \times \mathbb{C}^r$, then we can define the trivialization of E^* as

$$\begin{aligned} h^*_\alpha : \pi^{-1}(U) &\to U \times \mathbb{C}^r \\ (p, l) &\mapsto (p, \lambda^*_\alpha(l)) \end{aligned}$$

where $\lambda^*_\alpha(l)$ can be seen as a functional on $\mathbb{C}^r$, such that $\lambda^*_\alpha(l)(\lambda_\alpha(e)) = l(e)$. It's quite natural to require that.

So in the language of linear algebra, if you have a matrix $A : \mathbb{C}^r \to \mathbb{C}^r$, then it induces a matrix of dual spaces $A' : (\mathbb{C}^r)^* \to (\mathbb{C}^r)^*$, then facts in linear algebra tells you $A' = (A^{-1})^T$, that's why here the relationship between $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ is transpose of inverse.

Remark 1.1.31. We should always hold such an ideal, all information of a vector bundle is encoded in its transition functions. So if the transition are trivial, i.e. identity matrix, then the vector bundle is just trivial one, or product bundle. So from the relationship between transition functions of vector bundle and its dual, if the vector bundle is a line bundle, i.e. $g_{\alpha\beta} \in \mathbb{C}\backslash\{0\}$, then

$$(g_{\alpha\beta}^{-1})^T g_{\alpha\beta} = g_{\alpha\beta}^{-1} g_{\alpha\beta} = \mathrm{id}$$

So the vector bundle $\mathrm{End}(L) = L^* \otimes L$ is the trivial bundle, but in general $\mathrm{End}(E)$ is not trivial. We will use this fact later.

Definition 1.1.32 (line bundle)**.** *A holomorphic line bundle L is a rank 1 vector bundle, i.e.,*

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{holo} \mathbb{C}^*$$

Exercise 1.1.33. L is a trivial line bundle $X \times \mathbb{C}$ if and only if up to refinement, there exists $s_\alpha : U_\alpha \to \mathbb{C}^*$, such that $g_{\alpha\beta} = s_\alpha / s_\beta$ on $U_\alpha \cap U_\beta$

Proof. □

Definition 1.1.34 (picard group)**.** *X is a complex manifold, then*

$$\mathrm{Pic}(X) = (\{\textit{holomorphic line bundles on } X\}/\textit{isomorphism}, \otimes)$$

called the picard group of X.

Remark 1.1.35. Clearly the identity of this group is trivial line bundle, and from Remark 1.1.31 we can see that the inverse element of line bundle L is its dual bundle L^*.

Example 1.1.36 (tautological line bundle)**.** Here is a special line bundle on $\mathbb{P}^n$

$$\begin{array}{c} L = \{([l], x) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid x \in l\} \subset \mathbb{P}^n \times \mathbb{C}^{n+1} \\ \Big\downarrow \pi \\ \mathbb{P}^n \end{array}$$

is called tautological line bundle. Consider open coverings

$$U_i = \{[l] = [l_1, \dots, l_n] \in \mathbb{P}^n \mid l_i \neq 0\}$$

there is a map $U_i \to \pi^{-1}(U_i)$, defined as

$$[l] \mapsto ([l], (\frac{l_0}{l_i}, \dots, 1, \dots, \frac{l_n}{l_i}))$$

and local trivialization $h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ defined as

$$([l], x) \mapsto ([l], \lambda)$$

where
$$x = \lambda(\frac{l_0}{l_i}, \dots, 1, \dots, \frac{l_n}{l_i})$$
so we can calculate transition function
$$h_i \circ h_j^{-1} : (U_i \cap U_j) \times \mathbb{C} \longrightarrow (U_i \cap U_j) \times \mathbb{C}$$
$$([l], \lambda_j) \mapsto ([l], \lambda_j(\frac{l_0}{l_j}, \dots, \frac{l_n}{l_j})) \mapsto ([l], \lambda_i)$$
such that
$$\lambda_j(\frac{l_0}{l_j}, \dots, \frac{l_n}{l_j}) = \lambda_i(\frac{l_0}{l_i}, \dots, \frac{l_n}{l_i})$$
which implies
$$\lambda_i = \lambda_j \frac{l_i}{l_j}$$
so we can see transition function $g_{ij} = l_i/l_j \in \mathbb{C}^*$. This line bundle L will be denoted by $\mathcal{O}_{\mathbb{P}^n}(-1)$.

Definition 1.1.37 (line bundles on $\mathbb{P}^n$)**.** *We can define*
$$\mathcal{O}_{\mathbb{P}^n}(-k) = \mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes k}, \quad k \in \mathbb{N}^+$$
$$\mathcal{O}_{\mathbb{P}^n}(k) = (\mathcal{O}_{\mathbb{P}^n}(-k))^*, \quad k \in \mathbb{N}^+$$
$$\mathcal{O}_{\mathbb{P}^n}(0) = \mathbb{P}^n \times \mathbb{C}, \quad \textit{trivial line bundle.}$$

Remark 1.1.38. In fact, line bundles listed above contain all possible line bundles over $\mathbb{P}^n$.

Example 1.1.39. More generally, consider
$$\begin{array}{c} E = \{([S], x) \in Gr(k,n) \times \mathbb{C}^n \mid x \in S\} \subset Gr(k,n) \times \mathbb{C}^n \\ \Big\downarrow \pi \\ Gr(k,n) \end{array}$$

Definition 1.1.40 (section)**.** *$\pi : E \to X$ is a complex/holomorphic vector bundle. A (global) section of E is a differential/holomorphic map $s : X \to E$, such that $\pi \circ s = \mathrm{id}_X$, denoted by $C^\infty(X, E)$ / $\Gamma(X, E)$.*

Example 1.1.41. Global holomorphic sections of trivial holomorphic vector bundle are exactly holomorphic functions $f : X \to \mathbb{C}^r$.

Remark 1.1.42. In fact, global holomorphic sections are very rare, as we can seen from the above example, if X is a compact complex manifold, then all global holomorphic functions are only constant.

Definition 1.1.43 (subbundle)**.** *$\pi : E \to X$ is a complex/holomorphic vector bundle. $F \subset E$ is called a subbundle of rank s, if*

1. *For all $x \in X$, $F \cap E_x$ is a subvector space of dimension s.*
2. *$\pi|_F : F \to X$ induces a complex/holomorphic vector bundle.*

Remark 1.1.44. If F is a subbundle of E, then given a section of F, i.e. $\sigma : X \to F$ such that $\pi|_F \circ \sigma = \mathrm{id}_X$, then clearly we can extend it to a section of E.

Example 1.1.45. $\mathcal{O}_{\mathbb{P}^n}(-1) \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$, is a subbundle.

Exercise 1.1.46.

$$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \begin{cases} 0, & k < 0 \\ \mathbb{C}, & k = 0 \\ \text{homogeneous polynomials in } n+1 \text{ variables of deg } k, & k > 0 \end{cases}$$

Proof. Let's see what happened for $k = -1$, the tautological line bundle. Since we have $\mathcal{O}_{\mathbb{P}^n}(-1)$ is a subbundle of trivial bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$. So we have a global section $\sigma \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-1))$ must be a global section of $\Gamma(\mathbb{P}^n, \mathbb{P}^n \times \mathbb{C}^{n+1})$. However, since $\mathbb{P}^n$ is a compact complex manifold, we have that global sections $\Gamma(\mathbb{P}^n, \mathbb{P}^n \times \mathbb{C}^{n+1})$ must be constant, i.e. for any $x \in \mathbb{P}^n$, $\sigma(x) = v$ is a constant. However, $v \in [l]$, for all $[l] \in \mathbb{P}^n$, which forces $v = 0$. Similarly we will get the result for case $k < 0$.

And for case $k = 0$, global sections are exactly constant so we have $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(0)) = \mathbb{C}$.

Now consider what will happen when $k > 0$. Take $k = 1$ for an example. Life is like a seesaw, so is mathmatics. If something is defined concisely, it must be quite difficult to compute. Since sections of a trivial bundle is easy to compute, so in practice, we always compute the sections of the trivialization of a vector bundle, and glue them together to get a global one, that's what we always do.

For projective space $\mathbb{P}^n$, there exists a canonical affine cover $\{U_i\}$, where $U_i = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n \mid z_i \neq 0\} \cong \mathbb{C}^n$. And the transition functions g_{ij} of $\mathcal{O}_{\mathbb{P}^n}(1)$ is given by z_j/z_i, according to Example 1.1.34. So a global section is given by $s_i \in \Gamma(U_i, \mathcal{O}_{\mathbb{P}^n}(1))$ such that $s_i = z_j s_j / z_i$ on $U_i \cap U_j$. We claim that the only possible functions are of form $s_i = L/z_i$, where $L = a_0 z_0 + \cdots + a_n z_n$, that's a homogenous polynomial in $n+1$ variables of degree 1.

We compute explictly for $n = 1$ in order to really understand what's going on: A global section s of $\mathcal{O}_{\mathbb{P}^1}(1)$ is given by a function $s_0(z) = a + bz + cz^2 + \cdots \in \Gamma(U_0, \mathcal{O}_{\mathbb{P}^1}(1)) = k[z]$, where $z = z_1/z_0$, and a function $s_1(w) = \alpha + \beta w + \gamma w^2 + \cdots \in \Gamma(U_1, \mathcal{O}_{\mathbb{P}^1}(1)) = k[w]$ such that on $U_0 \cap U_1$

$$s_0(z) = a + bz + cz^2 + \ldots = z(\alpha + \beta w + \gamma w^2 + \ldots) = \alpha z + \beta + \gamma(1/z) + \ldots$$

This implies $a = \beta, b = \alpha$ and that all other coefficients are zero. $\square$

Example 1.1.47. For a morphism between vector bundles $\phi : E \to F$, $\operatorname{Ker}\phi \subset E, \operatorname{Im}\phi \subset F$ are subbundles.

Definition 1.1.48 (exact)**.** *A sequence of vector bundles*

$$S \xrightarrow{\phi} E \xrightarrow{\psi} Q$$

is called exact at E if $\operatorname{Ker}\psi = \operatorname{Im}\phi$*;*

Definition 1.1.49 (pullback)**.** *$f : X \to Y$ is a differential/holomorphic map, $\pi : E \to Y$ is a vector bundle, define*

$$f^*E = \{(x, e) \in X \times E \mid f(x) = \pi(e)\} \subset X \times E$$

is called the pullback of π.

Remark 1.1.50. To be somewhat more explicit, suppose $U \subset Y$ is a local trivialization, i.e. $\varphi_U : E|_U \to U \times \mathbb{C}^r$ with $\varphi_U(e) = (\pi(e), \lambda_U(e))$. Then we can define a local trivialization of f^*E on $f^{-1}(U) \subset X$, by

$$\begin{aligned} \varphi_U^* : f^*E|_{f^{-1}(U)} &\to f^{-1}(U) \times \mathbb{C}^r \\ (x, e) &\mapsto (x, \lambda_U(e)) \end{aligned}$$

and transition functions on $f^{-1}(U_\alpha) \cap f^{-1}(U_\beta)$ is given by $g_{\alpha\beta} \circ f$.

1.2. **Episode: sheaves.** Why we need sheaves here? As we have seen in the last section, the global sections of holomorphic vector bundle are very rare, but there are many local sections, we need to keep these information and learn the connection between global and local systemically. Sheaf is a power language for us to manage global and local at the same time. However, sheaf gives more information. Indeed, as we can see in Exercise 1.2.5, vector bundles are exactly locally free sheaves.

Definition 1.2.1 (sheaf)**.** *X is a topological space. A sheaf*[3] *of abelian group $\mathscr{F}$ on X is the data of:*

1. *For any open subset U of X, $\mathscr{F}(U)$ is an abelian group.*
2. *If $U \subset V$ are two open subsets of X, then there is a group homomorphism $r_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$, such that*
   (1) $\mathscr{F}(\varnothing) = 0$
   (2) $r_{UU} = \mathrm{id}$
   (3) *If $W \subset U \subset V$, then $r_{UW} = r_{VW} \circ r_{UV}$*
   (4) *$\{V_i\}$ is an open covering of $U \subset X$, and $s \in \mathscr{F}(U)$. If $s|_{V_i} := r_{UV_i}(s) = 0, \forall i$, then $s = 0$.*
   (5) *$\{V_i\}$ is an open covering of $U \subset X$, and $s_i \in \mathscr{F}(V_i)$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there exists $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$.*

Example 1.2.2 (sheaf of sections of a holomorphic vector bundle)**.** If $\pi : E \to X$ is a holomorphic vector bundle, then define

$$\mathscr{F}(U) = \Gamma(U, E|_U), \quad \forall U \subset_{\text{open}} X$$

This $\mathscr{F}$ will be denoted by $\mathcal{O}_X(E)$. In particular, E is a trivial vector bundle, then $\mathcal{O}_X(E) = \mathcal{O}_X$, the sheaf of holomorphic function, also called the structure sheaf of X.

Definition 1.2.3 (morphism of sheaves on X)**.** *$\phi : \mathscr{F} \to \mathscr{G}$ is called a morphism of sheaves, if for any open subset U of X, there is a group homomorphism $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$, such that if $U \subset V$ are two open subsets of X, the the following diagram commutes*

[3] A sheaf which fails to meet $(4), (5)$ is called a presheaf.

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\phi(U)} & \mathscr{G}(U) \\ \Big\downarrow r_{UV} & & \Big\downarrow r_{UV} \\ \mathscr{F}(V) & \xrightarrow{\phi(V)} & \mathscr{G}(V) \end{array}$$

Example 1.2.4 (locally free sheaves)**.** A sheaf is called locally free, if there exists covering $\{U_\alpha\}$ such that $\mathscr{F}|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}^{\oplus r}$ of rank r. For $r = 1$, it is called invertible sheaf.

Exercise 1.2.5. There are correspondences:

$$\{\text{holomorphic vector bundles}\} \xleftrightarrow{1-1} \{\text{locally free sheaves}\}$$

$$\{\text{holomorphic line bundles}\} \xleftrightarrow{1-1} \{\text{invertible sheaves}\}$$

Proof. It suffices to prove the first correspondence. If we have a holomorphic vector bundle $\pi : E \to X$. Then consider the sheaf of sections $\mathcal{O}_X(E)$, We claim it's a locally free sheaf. Since we have local trivialization of holomorphic vector bundle $\{U_\alpha\}$. Then consider what's $\mathcal{O}_X(E)|_{U_\alpha}$. Since $E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^r$, then holomorphic sections of $U_\alpha \times \mathbb{C}^r \to U_\alpha$ are just holomorphic functions $f : U \to \mathbb{C}^r$, i.e. $\mathcal{O}_X(E|_{U_\alpha}) = \mathcal{O}_{U_\alpha}^{\oplus r}$. So sheaf $\mathcal{O}_X(E)$ is a locally free sheaf.

Conversely, if we have a locally free sheaf $\mathcal{E}$, how can we get a holomorphic vector bundle? Assume $\mathcal{E}$ is locally free over an open covering $\{U_\alpha\}$ of X, then we just need to glue $U_\alpha \times \mathbb{C}^r \to U_\alpha$ together to get a vector bundle. Therefore we need a family of gluing data $g_{\alpha\beta} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$. Since $\mathcal{E}$ is locally free, we have local isomorphism $f_\alpha : \mathcal{E}|_{U_\alpha} \to \mathcal{O}_{U_\alpha}^{\oplus r}$. Restricting to intersection $U_\alpha \cap U_\beta$, we get

$$f_{\alpha\beta} = f_\alpha|_{U_\alpha \cap U_\beta} \circ f_\beta^{-1}|_{U_\alpha \cap U_\beta} : \mathcal{O}_{U_\beta}^{\oplus r}|_{U_\alpha \cap U_\beta} \to \mathcal{O}_{U_\alpha}^{\oplus r}|_{U_\alpha \cap U_\beta}$$

Every such map is induced by a map

$$g_{\alpha\beta} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$$

that's gluing data we desire. □

1.3. **Tangent bundle.**

Definition 1.3.1 (tangent bundle)**.** *X is a differential manifold,* $\dim_{\mathbb{R}} X = n$*, and $\{U_\alpha, \phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n\}$ is a atlas of X. The* (*real*) *tangent bundle $T_{X,\mathbb{R}}$ is defined through transition functions*

$$\begin{aligned} g_{\alpha\beta} : U_\alpha \cap U_\beta &\xrightarrow{\text{diff}} \mathrm{GL}(n, \mathbb{R}) \\ x &\mapsto J_{\phi_\alpha \circ \phi_\beta^{-1}}(\phi_\beta(x)) \end{aligned}$$

Then $T_{X,\mathbb{C}} = T_{X,\mathbb{R}} \otimes \mathbb{C}$ is a complex vector bundle, called the complexified tangent vector bundle.

Remark 1.3.2. The following statement may be a little bit boring, I write it down just to make myself more clear and to get familiar with two definition of vector bundle.

The tangent bundle $T_{X,\mathbb{R}}$ can be defined as the set

$$T_{X,\mathbb{R}} = \coprod_{x \in X} T_{X,x}$$

and note that there is a natural projection $\pi : T_{X,\mathbb{R}} \to X$, sending $v \in T_{X,x}$ to $x \in X$. Now we want to give a chart on $T_{X,\mathbb{R}}$ to make it into a differential manifold. Let $\{(U_i, \phi_i = (x_i^1, \dots, x_i^n)\}$ be a chart of X, then we can define a chart on X by considering $\{(\pi^{-1}(U_i), \widetilde{\phi}_i)\}$, where $\widetilde{\phi}_i$ is defined through

$$\widetilde{\phi}_i(v) = (\phi_i(\pi(v)), (\mathrm{d}x_i^1)_{\pi(v)}(v), \dots, (\mathrm{d}x_i^n)_{\pi(v)}(v)) \subset \mathbb{R}^n \times \mathbb{R}^n$$

note that such $\widetilde{\varphi}_i$ is bijective. And it's easy to equip $T_{X,\mathbb{R}}$ with a topology such that $\widetilde{\varphi}_i$ is diffeomorphism.

Now I need to calculate transition function to confirm myself as follows: For two charts $(U, \phi = (x_1, \dots, x_n)), (V, \psi = (y_1, \dots, y_n))$, then calculate

$$\widetilde{\psi} \circ \widetilde{\phi}^{-1} : \phi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$$

Note that

$$\widetilde{\phi}^{-1}(r_1, \cdots, r_n, u_1, \cdots, u_n) = \sum_i u_i \frac{\partial}{\partial x_i}|_{\phi^{-1}(r_1, \cdots, r_n)} \in T_{\phi^{-1}(r_1, \cdots, r_n)} M$$

But

$$\mathrm{d}y_j(\sum_i u_i \frac{\partial}{\partial x_i}) = \sum_i u_i (\frac{\partial}{\partial x_i}(y_j)) = \sum_i \frac{\partial y_i}{\partial x_j} u_i$$

Thus transition functions are

$$\begin{aligned} \widetilde{\psi} \circ \widetilde{\phi}^{-1}(r_1, \cdots, r_n, u_1, \cdots, u_n) &= (\psi \circ \phi^{-1}(r), (\sum_i \frac{\partial y_1}{\partial x_i}(r) u_i, \dots, \sum_i \frac{\partial y_n}{\partial x_i}(r) u_i)) \\ &= (\psi \circ \phi^{-1}(r), (\frac{\partial y_j}{\partial x_i}(r))(\begin{matrix} u_1 \\ \vdots \\ u_n \end{matrix})) \end{aligned}$$

So transition function $g_{\alpha\beta}$ are exactly Jacobian of $\varphi_\alpha \circ \varphi_\beta^{-1}$.

Definition 1.3.3 (holomorphic tangent bundle)**.** *X is a complex manifold, $\dim_{\mathbb{C}} X = n$, and $\{U_\alpha, \phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^n\}$ is a atlas of X. The holomorphic tangent bundle T_X is defined through transition functions*

$$\begin{aligned} g_{\alpha\beta} : U_\alpha \cap U_\beta &\to \mathrm{GL}(n, \mathbb{C}) \\ z &\mapsto J^{holo}_{\phi_\alpha \circ \phi_\beta^{-1}}(\phi_\beta(x)) \end{aligned}$$

where J^{holo} is holomorphic Jacobian.

Remark 1.3.4. Since $\phi_\alpha \circ \phi_\beta^{-1} : V_\beta \to V_\alpha$ is a holomorphic functions, then holomorphic Jacobian means the matrix

$$(\frac{\partial(\phi_\alpha \circ \phi_\beta^{-1})^j}{\partial z_i})_{1\leq i,j\leq n}$$

Remark 1.3.5. Clearly, $T_X \neq T_{X,\mathbb{C}}$, even they don't have the same rank! For example, if X is a n-dimensional complex manifold, then

$$\dim T_X = n \neq 2n = \dim T_{X,\mathbb{C}}$$

Later we will see the relationship between them.

Remark 1.3.6 (sheaf viewpoint)**.** $\mathcal{O}_X$ is the sheaf of holomorphic function, then define the stalk at x is

$$\mathcal{O}_{X,x} := \varinjlim_{x\in U\subset X} \mathcal{O}_X(U)$$

The elements of $\mathcal{O}_{X,x}$ are called germs. For a tangent vector, we can take derivation in this direction, so

$$\text{tangent vector} \longrightarrow \text{derivation } D : \mathcal{O}_{X,x} \to \mathbb{C}$$

where a derivation is a map which satisfies

1. $\mathbb{C}$-linear
2. Leibniz rule $D(fg) = D(f)g + fD(g)$

In fact, the above correspondence is $1-1$, that is, every derivation arises from a tangent vector. So we have $T_{X,x} \cong$ space of derivation of $\mathcal{O}_{X,x}$, that's also a nice definition many authors prefer.

Definition 1.3.7 (cotangent bundle/(anti)canonical bundle)**.** $\Omega_X = T^*_X$ *is called holomorphic cotangent bundle;* $K_X = \det\Omega_X$ *is called canonical bundle;* $K^*_X = \det T_X$ *is called the anticanonical bundle.*

Exercise 1.3.8. We calculate tangent bundle of $\mathbb{P}^n$ through the following exact sequence called Euler sequence[4].

$$0 \to \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \xrightarrow{\psi} T_{\mathbb{P}^n} \to 0$$

Let's clearify what does the map look like in a geometry viewpoint:

Let $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ denote the canonical projection from $\mathbb{C}^n\backslash\{0\} \to \mathbb{P}^n$. We know that basis of tangent vector at $z \in \mathbb{C}^{n+1}\backslash\{0\}$ is $\{\frac{\partial}{\partial z_0}, \dots, \frac{\partial}{\partial z_n}\}$, but these are not tangent vector for $\mathbb{P}^n$. Since function $f([z])$ defined on $\mathbb{P}^n$ satisfies $f([z]) = f([\lambda z]), \forall\lambda \in \mathbb{C}\backslash\{0\}$, regard it as a function defined on $\mathbb{C}^n\backslash\{0\}$ and use tangent vector $\frac{\partial}{\partial z_i}$ to act on both sides of this equation, we have

$$\frac{\partial f}{\partial z_i} = \lambda\frac{\partial f}{\partial z_i}$$

A contradiction. However, $z_i\frac{\partial}{\partial z_i}$ will descend to tangent vector at $[z] \in \mathbb{P}^n$.

[4]Refer to pages 408-409 of Griffiths-Harris for more details.

Recall $z_0, \dots, z_n$ form basis of $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and span $(\mathcal{O}_{\mathbb{P}^n}(1))_{[z]}$, so can define

$$\psi : (\mathcal{O}_{\mathbb{P}^n}(1))^{\oplus n+1}_{[z]} \to T_{\mathbb{P}^n,[z]}$$
$$(0, \dots, \underbrace{z_i}_{j-th}, \dots, 0) \mapsto z_i \frac{\partial}{\partial z_j}$$

But $\sum_{i=0}^n z_i \frac{\partial}{\partial z_i}$ tangent to the fibers of π, so descends to zero at $[z] \in \mathbb{P}^n$, so we can define ϕ as

$$\phi : \mathcal{O}_{\mathbb{P}^n,[z]} \to (\mathcal{O}_{\mathbb{P}^n}(1))^{\oplus n+1}_{[z]}$$
$$1 \to (z_0, \dots, z_n)$$

In fact, for a homogenous polynomial f with degree d, we have the famous relation

$$\sum_{i=0}^n x_i \frac{\partial f}{\partial x_i} = df$$

which is discovered by Euler, and that's why this sequence is called Euler sequence.

Exercise 1.3.9. For Grassmannian manifold $Gr(k,n)$, we have

$$0 \to E \to Gr(k,n) \otimes \mathbb{C}^n \to Q \to 0$$

Show that

$$T_{Gr(k,n)} \cong \mathrm{Hom}(E, Q)$$

Exercise 1.3.10. Let $\pi : L \to X$ is a holomorphic line bundle, given $s \in \Gamma(X, L)$, suppose that $D = \{x \in X \mid s(x) = 0\}$ is a smooth submanifold of codimensional 1. Show that the following sequence is exact:

$$0 \to T_D \to T_X|_D \to L|_D \to 0$$

then we can get

$$K_X^*|_D \cong K_D^* \otimes L|_D$$

which is called adjunction formula.

In particular, let $X = \mathbb{P}^n$ and $L = \mathcal{O}_{\mathbb{P}^n}$, then Exercise 1.1.46 tells us that $D \subset \mathbb{P}^n$ is a smooth hypersurface defined by a homogenous polynomial with degree d. Then we have

$$K_D^* \cong \mathcal{O}_{\mathbb{P}^n}(n+1-d)$$

and we call it

$$\begin{cases} \text{Fano}, \quad d < n+1 \\ \text{Calabi-Yau}, \quad d = n+1 \\ \text{General type}, \quad d > n+1 \end{cases}$$

1.4. **Almost complex structure and integrable theorem.** Now let us talk about some linear algebra:

Consider a $2n$-dimensional real vector space V, a almost complex structure on V is a $\mathbb{R}$-linear transformation $J : V \to V$ such that $J^2 = -\operatorname{id}$. We can regard V as a complex vector space, by

$$(a + bi)v = av + bJ(v), \quad a, b \in \mathbb{R}$$

Indeed, the $\mathbb{R}$-linearity of J and the assumption $J^2 = -\operatorname{id}$ yield associative law, i.e

$$((a + ib)(c + id))v = (a + ib)((c + id)v), \quad v \in V$$

In particular, we have $i^2 v = -v$.

If we consider $V \otimes \mathbb{C}$, then J can extend to $V \otimes \mathbb{C}$ by $J(v \otimes \alpha) = J(v) \otimes \alpha$, then we can decompose $V \otimes \mathbb{C}$ into

$$\begin{aligned} V_{\mathbb{C}} = V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1} \\ &= \{v \in V_{\mathbb{C}} \mid J(v) = iv\} \oplus \{v \in V_{\mathbb{C}} \mid J(v) = -iv\} \end{aligned}$$

such that $\overline{V^{1,0}} = V^{0,1}$, where conjugate means $\overline{v \otimes \alpha} = v \otimes \overline{\alpha}$. More explictly, for any $v \in V_{\mathbb{C}}$, we can decompose it into

$$v = \frac{1}{2}(v - iJ(v)) + \frac{1}{2}(v + iJ(v))$$

since the first part do lies in $V^{1,0}$, checked as follows

$$\begin{aligned} J(\frac{1}{2}(v - iJ(v))) &= \frac{1}{2}(J(v) + iv) \\ i(\frac{1}{2}(v - iJ(v))) &= \frac{1}{2}(iv + J(v)) \end{aligned}$$

and the latter part holds similarly.

Definition 1.4.1 (almost complex structure)**.** *X is a differential manifold of $\dim_{\mathbb{R}} X = 2n$. An almost complex structure on X is a complex structure on $T_{X,\mathbb{R}}$, i.e. an isomorphism of differential vector bundles $J : T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$ such that $J^2 = -\operatorname{id}$.*

It's natural to ask, if X is a complex manifold, and forget its complex structure and just regard it as a differential manifold, can we give a natural almost complex structure on it? That's the following example:

Example 1.4.2. X is a complex manifold, and $T_{X,\mathbb{R}}$ is its (real) tangent bundle if we just see X as a differential manifold. Locally we have

$$T_{X,\mathbb{R}}|_U \cong U \times \mathbb{C}^n$$

for open subset U in X, where we regard $\mathbb{C}^n$ as a $2n$ dimension real vector space. But there is a natural almost complex structure on it arising from multiplying i. So we get $J : T_{X,\mathbb{R}}|_U \to T_{X,\mathbb{R}}|_U$. More explictly, since we can write $T_{X,\mathbb{R}} = \operatorname{span}_{\mathbb{R}}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\}$, we can write J explictly as follows

$$J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}$$
$$J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i}$$

If we want to get a global one, it suffices to glue them together. So we consider

For two charts $(U_1, \varphi_1) = (z_1 = x_1 + iy_1, , \ldots, z_n = x_n + iy_n)), (U_2, \varphi_2) = (g_1 = u_1 + iv_1, \ldots, g_n = u_n + iv_n))$ with $U_1 \cap U_2 \neq \varnothing$, there are two ways to define J on $U_1 \cap U_2$

$$\begin{tikzcd}
U_1 \cap U_2 \times \mathbb{C}^n \arrow[ddd, "\times i"] & & & U_1 \cap U_2 \times \mathbb{C}^n \arrow[ddd, leftrightarrow, "\times i"] \\
& T_X|_{U_1 \cap U_2} \arrow[ul, "\widetilde{\varphi}_1"] \arrow[d, "J"] & T_X|_{U_1 \cap U_2} \arrow[l] \arrow[ur, "\widetilde{\varphi}_2"] \arrow[d, "J"] & \\
& T_X|_{U_1 \cap U_2} \arrow[dl, "\widetilde{\varphi}_1"] & T_X|_{U_1 \cap U_2} \arrow[l] \arrow[dr, "\widetilde{\varphi}_2"] & \\
U_1 \cap U_2 \times \mathbb{C}^n & & & U_1 \cap U_2 \times \mathbb{C}^n
\end{tikzcd}$$

We need to check transition functions of tangent bundle $T_{X,\mathbb{R}}$ commute with J, calculated in a local chart as follows: For a 2×2 part, Jacobian of $\varphi_2 \circ \varphi_1^{-1}$ is

$$\begin{pmatrix} \dfrac{\partial u_j}{\partial x_k} & \dfrac{\partial u_j}{\partial y_k} \\ \dfrac{\partial v_j}{\partial x_k} & \dfrac{\partial v_j}{\partial y_k} \end{pmatrix} \overset{\text{C-R}}{=} \begin{pmatrix} \dfrac{\partial v_j}{\partial y_k} & \dfrac{\partial u_j}{\partial y_k} \\ -\dfrac{\partial u_j}{\partial y_k} & \dfrac{\partial v_j}{\partial y_k} \end{pmatrix}$$

and J is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

So they commute with each other.

So complex structure gives a almost complex structure (naturally), but the question is: Does every complex structure on a complex manifold can be induced from a almost complex structure on a even-dimensional differential manifold? Unfortunately, it's false in general, but we have the following theorem.

Theorem 1.4.3 (Newlander-Nirenberg)**.** *Let (X, J) be a complex manifold, J is induced by a almost complex structure on X is equivalent to*

$$[T_X^{1,0}, T_X^{1,0}] \subset T_X^{1,0}$$

which is called an integrable condition.

1.5. **Operator ∂ and $\overline{\partial}$.** In this section, we will discuss the relationship between $T_X, T_{X,\mathbb{R}}, T_{X,\mathbb{C}}$ and so on, for a complex manifold X.[5]

First, we have $T_X \hookrightarrow T_{X,\mathbb{C}}$ as complex vector bundle[6], with image $T^{1,0}_{X,\mathbb{C}}$. In fact, if we consider locally, take $(z_1,\dots,z_n)\in V\subset\mathbb{C}^n, z_j = x_j+iy_j$ as a coordinate, then we can do the following identification

$$T_X \ni \frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}) \in T^{1,0}_{X,\mathbb{C}}$$

moreover, we can define the conjugation as

$$\frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}) \in T^{0,1}_{X,\mathbb{C}}$$

that's non holomorphic part of $T_{X,\mathbb{C}}$.

If we consider its dual space, we get the differential forms

$$\Omega^1_{X,\mathbb{C}} = \Omega^1_{X,\mathbb{R}}\otimes\mathbb{C} = \Omega^{1,0}_X \oplus \Omega^{0,1}_X$$

and take wedge product k times, then we get

$$\Omega^k_{X,\mathbb{C}} = \bigwedge^k \Omega^1_{X,\mathbb{C}} = \bigoplus_{p+q=k}\Omega^{p,q}_X, \quad \text{where } \Omega^{p,q}_X = \bigwedge^p \Omega^{1,0}_X \wedge (\bigwedge^q \Omega^{0,1}_X)$$

Remark 1.5.1. We also want to decompose $\Omega^1_{X,\mathbb{C}}$ into two parts with respect to the dual map of J, so we need to elaborate how does J^* acts on $\Omega^1_{X,\mathbb{C}}$. Since $\Omega^1_{X,\mathbb{C}} = \mathrm{span}_{\mathbb{C}}\{\mathrm{d}x_1,\mathrm{d}y_1,\dots,\mathrm{d}x_n,\mathrm{d}y_n\}$, then by definition

$$\begin{aligned} J^*(\mathrm{d}x_i)(\frac{\partial}{\partial x_i}) &= \mathrm{d}x_i(J(\frac{\partial}{\partial x_i})) = \mathrm{d}x_i(\frac{\partial}{\partial y_i}) = 0 \\ J^*(\mathrm{d}x_i)(\frac{\partial}{\partial y_i}) &= \mathrm{d}x_i(J(\frac{\partial}{\partial y_i})) = \mathrm{d}x_i(-\frac{\partial}{\partial x_i}) = -1 \end{aligned}$$

so we have

$$\begin{aligned} J^*(\mathrm{d}x_i) &= -\mathrm{d}y_i \\ J^*(\mathrm{d}y_i) &= \mathrm{d}x_i \end{aligned}$$

that is,

$$\begin{aligned} \Omega^{1,0}_{X,\mathbb{C}} &= \mathrm{span}_{\mathbb{C}}\{\mathrm{d}x_1 + i\mathrm{d}y_1,\dots,\mathrm{d}x_n + i\mathrm{d}y_n\} \\ \Omega^{0,1}_{X,\mathbb{C}} &= \mathrm{span}_{\mathbb{C}}\{\mathrm{d}x_1 - i\mathrm{d}y_1,\dots,\mathrm{d}x_n - i\mathrm{d}y_n\} \end{aligned}$$

By our identification, the dual of $\dfrac{\partial}{\partial z_j}$ is

$$\mathrm{d}z_j = \mathrm{d}x_j + i\mathrm{d}y_j \in \Omega^{1,0}_X$$

[5]Keep in mind: T_X is the tangent bundle of complex manifold X, and $T_{X,\mathbb{R}}$ is the underlying real tangent bundle of T_X, and $T_{X,\mathbb{C}} = T_{X,\mathbb{R}}\otimes\mathbb{C}$.

[6]However, not as a holomorphic vector bundle, since $T_{X,\mathbb{C}}$ contains part which is not holomorphic.

and the dual of $\dfrac{\partial}{\partial \overline{z_j}}$ is

$$\mathrm{d}\overline{z_j} = \mathrm{d}x_j - i\mathrm{d}y_j \in \Omega_X^{0,1}$$

For any $\alpha \in C^\infty(X, \Omega^1_{X,\mathbb{C}})$, locally we have

$$\alpha = \sum \alpha_j \mathrm{d}x_j + \sum \beta_j \mathrm{d}y_j$$

then we can decompose it into

$$\alpha = \sum \frac{1}{2}(\alpha_j - i\beta_j)\mathrm{d}z_j + \sum \frac{1}{2}(\alpha_j + i\beta_j)\mathrm{d}\overline{z_j}$$

where the first part lies in $\Omega_X^{1,0}$ and the later part lies in $\Omega_X^{0,1}$.

Definition 1.5.2 (differential k-form)**.** *A k-form α of type (p,q) is a differential section of $\Omega_X^{p,q}$, that is*

$$\alpha \in C^\infty(X, \Omega_X^{p,q}) \subset C^\infty(X, \Omega^k_{X,\mathbb{C}})$$

Remark 1.5.3. It's quite neccessary for us to keep in mind how to distinguish a differential k-form in $C^\infty(X, \Omega^k_{X,\mathbb{C}})$ what type it is, particularly for the case $k = 2$, since later we will study the first Chern class, a special $(1,1)$ form. Let's elaborate in the case $k = 2$. For any element ω in

$$\Omega^2_{X,\mathbb{C}} = \Omega_X^{2,0} \oplus \Omega_X^{1,1} \oplus \Omega_X^{0,2}$$

it will eat two tangent vector $v, w \in T_{X,\mathbb{C}}$, and output a complex number $\omega(v,w)$.

Note that $\Omega_X^{2,0} = \Omega_X^{1,0} \wedge \Omega_X^{1,0}$ and $\Omega_X^{0,2} = \Omega_X^{0,1} \wedge \Omega_X^{0,1}$, so if ω is a $(1,1)$-form, it must eat $v, w \in T^{1,0}_{X,\mathbb{C}}$ and output zero, similar for $v, w \in T^{0,1}_{X,\mathbb{C}}$. That's a necessary sufficient condition for ω is a $(1,1)$-form.

Exercise 1.5.4. For $\mathbb{C}^n \cong \mathbb{R}^{2n}$, we have

$$w = \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}x_n = (\frac{i}{2})^n \mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\overline{z_n}$$

Proof. It suffices to show the case $n = 1$, and we can compute directly as follows

$$\begin{aligned}
(\frac{i}{2})\mathrm{d}z \wedge \mathrm{d}\overline{z} &= (\frac{i}{2})(\mathrm{d}x + i\mathrm{d}y) \wedge (\mathrm{d}x - i\mathrm{d}y) \\
&= (\frac{i}{2})(-2i\mathrm{d}x \wedge \mathrm{d}y) \\
&= \mathrm{d}x \wedge \mathrm{d}y
\end{aligned}$$

□

1.6. **Exterior differential.** Recall what we have done in the theory of de Rham cohomology: Let X be a differential manifold, with real dimension n, we have exterior differential

$$\mathrm{d}: C^\infty(X, \Omega^k_{X,\mathbb{R}}) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{R}})$$

such that

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \mathrm{d}\beta$$

Exterior differential has an important property:

$$\mathrm{d}^2 = 0$$

So we can consider such cochain complex

$$0 \to C^\infty(X, \Omega^0_{X,\mathbb{R}}) \to C^\infty(X, \Omega^1_{X,\mathbb{R}}) \to C^\infty(X, \Omega^2_{X,\mathbb{R}}) \to \cdots \to C^\infty(X, \Omega^n_{X,\mathbb{R}}) \to 0$$

with de Rham cohomology group

$$H^k(X, \mathbb{R}) := Z^k(X, \mathbb{R}) / B^k(X, \mathbb{R})$$

The following theorem implies that the de Rham cohomology is just a topological data.

Theorem 1.6.1 (comparision theorem). *$H^k(X, \mathbb{R})$ computes the singular cohomology of X with real coefficient.*

Theorem 1.6.2 (Poincaré lemma). *Let $X = B(x_0, r_0) \subset \mathbb{R}^n$ is a open ball, then $H^k(X, \mathbb{R}) = 0, \forall k > 0$.*

Remark 1.6.3. Poincaré lemma implies that for small enough open set, the cohomology groups are trivial, so only for global differential forms, de Rham cohomology tells interesting information.

So let's see what is the complex version of above theory. Now Let X be a complex manifold, with complex dimension n, then similar we also have an exterior derivative

$$\mathrm{d}: C^\infty(X, \Omega^k_{X,\mathbb{C}}) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{C}})$$

But there is also something interesting, we already know that we can decompose $\Omega^k_{X,\mathbb{C}}$, but for any $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{C}})$, we can also decompose $\mathrm{d}\alpha$.

Example 1.6.4. For $\alpha \in C^\infty(X, \Omega^0_{X,\mathbb{C}})$, then

$$\mathrm{d}\alpha \in C^\infty(X, \Omega^1_{X,\mathbb{C}}) = C^\infty(X, \Omega^{1,0}_X) \oplus C^\infty(X, \Omega^{0,1}_X)$$

Locally, we have

$$\begin{aligned}
\mathrm{d}\alpha &= \sum \frac{\partial \alpha}{\partial x_j} \mathrm{d}x_j + \sum \frac{\partial \alpha}{\partial y_j} \mathrm{d}y_j \\
&= \sum \frac{1}{2}\left(\frac{\partial \alpha}{\partial x_j} - i\frac{\partial \alpha}{\partial y_j}\right)\mathrm{d}z_j + \sum \frac{1}{2}\left(\frac{\partial \alpha}{\partial x_j} + i\frac{\partial \alpha}{\partial y_j}\right)\mathrm{d}\overline{z_j} \\
&= \sum \frac{\partial \alpha}{\partial z_j}\mathrm{d}z_j + \sum \frac{\partial \alpha}{\partial \overline{z_j}}\mathrm{d}\overline{z_j}
\end{aligned}$$

More generally, for $\alpha \in C^\infty(X, \Omega_X^{p,q})$, then locally

$$\alpha = \sum_{|J|=p,|K|=q} \alpha_{J,K} \mathrm{d}z_J \wedge \mathrm{d}\overline{z_K}$$

then

$$\mathrm{d}\alpha = \sum_{|J|=p,|K|=q} \frac{\partial \alpha_{J,K}}{\partial z_l} \mathrm{d}z_l \wedge \mathrm{d}z_J \wedge \mathrm{d}\overline{z_K} + \sum_{|J|=p,|K|=q} \frac{\partial \alpha_{J,K}}{\partial \overline{z_l}} \mathrm{d}\overline{z_l} \wedge z_J \wedge \overline{z_K}$$

that is

$$\mathrm{d}\alpha \in C^\infty(X, \Omega_X^{p+1,q}) \oplus C^\infty(X, \Omega_X^{p,q+1})$$

Definition 1.6.5 (partial operator)**.** *For $\alpha \in C^\infty(X, \Omega_X^{p,q})$, we can define partial operator and its conjugation $\partial\alpha, \overline{\partial}\alpha$ as follows*

$$\mathrm{d}\alpha = \partial\alpha + \overline{\partial}\alpha$$

where $\partial\alpha \in C^\infty(X, \Omega_X^{p+1,q}), \overline{\partial}\alpha \in C^\infty(X, \Omega_X^{p,q+1})$. More generally, if $\alpha \in C^\infty(X, \Omega_{X,\mathbb{C}}^k)$, write $\alpha = \sum \alpha^{p,q}$, then we can define

$$\partial\alpha = \sum_{p,q} \partial\alpha^{p,q}, \quad \overline{\partial}\alpha = \sum_{p,q} \overline{\partial}\alpha^{p,q}$$

Remark 1.6.6. We have the following relations

1. Leibniz rule

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \partial\beta$$

2. [7]

$$\partial^2 = \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0$$

So we can do the same thing for ∂ by consider the following cochain complex[8]

$$0 \to C^\infty(X, \Omega_X^{p,0}) \xrightarrow{\overline{\partial}} C^\infty(X, \Omega_X^{p,1}) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} C^\infty(X, \Omega_X^{p,n}) \to 0$$

Definition 1.6.7 (Dolbeault cohomology)**.**

$$H^{p,q}(X) := Z^{p,q}(X)/B^{p,q}(X) = H^q_{\overline{\partial}}(C^\infty(X, \Omega_X^{p,*}))$$

Key question: Since we have $C^\infty(X, \Omega_{X,\mathbb{C}}^k) = \bigoplus_{p+q=k} C^\infty(X, \Omega_X^{p,q})$, could we have the following decomposition?

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

[7] Hint: consider $\mathrm{d}^2 = (\partial + \overline{\partial})^2 = 0$

[8] You may wonder why don't we use ∂ to construct such cobchain complex. In fact, the two definitions are almost the same, since they conjugate to each other. However, the cohomology group of cochain complex defined by $\overline{\partial}$ is more meaningful, as we will see later.

Example 1.6.8. What is $H^{p,0}(X)$? Since $B^{p,0} = 0$, then

$$H^{p,0}(X) = Z^{p,0}(X) = \{\alpha \in C^\infty(X, \Omega_X^{p,0}) \mid \overline{\partial}\alpha = 0\}$$

Locally $\alpha = \sum_{|J|=p} \alpha_J \mathrm{d}z_J$, then

$$\overline{\partial}\alpha = \sum_{|J|=p} \frac{\partial \alpha_J}{\partial \overline{z_k}} \mathrm{d}\overline{z_k} \wedge \mathrm{d}z_J = 0 \implies \frac{\partial \alpha_J}{\partial \overline{z_k}} = 0$$

That is, α_J is holomorphic function. Since $\Omega_X^{p,0} \cong \Omega_X^p$ as complex vector bundle, we have $H^{p,0}(X) = \Gamma(X, \Omega_X^p)$. [9]

Example 1.6.9. For a holomorphic map $f : X \to Y$ between complex manifold, then

$$f^* : C^\infty(Y, \Omega_{Y,\mathbb{C}}^k) \to C^\infty(X, \Omega_{X,\mathbb{C}}^k)$$

Then[10]

$$f^* : C^\infty(Y, \Omega_{Y,\mathbb{C}}^{p,q}) \to C^\infty(X, \Omega_{X,\mathbb{C}}^{p,q})$$

and

$$f^* : H^{p,q}(Y) \to H^{p,q}(X)$$

so Dolbeault cohomology is a contravariant functor.

Example 1.6.10. Dolbeault cohomology of a holomorphic vector bundle[11] $E \to X$, we can also define

$$\overline{\partial_E} : C^\infty(X, \Omega_X^{0,q} \otimes E) \to C^\infty(X, \Omega_X^{0,q+1} \otimes E)$$

satisfies $\overline{\partial_E}^2 = 0$. Let's elaborate this construction. Since any global section is glued together by local sections, we just need to define for local sections. We can choose a local holomorphic frame $\{e_1, \dots, e_n\}$ for E on U, so any section $\sigma \in C^\infty(U, \Omega_X^{0,q} \otimes E)$ we can write $\sigma = \sum_i \varphi_i \otimes e_i$ for $\varphi_i \in \Omega_U^{0,q}$. Then we can define

$$\overline{\partial_E}(\sigma) = \sum_i \overline{\partial}\varphi_i \otimes e_i$$

We need to check it's independent of choice of local chart! And check $\overline{\partial_E}^2 = 0$.

So we can construct a cochain complex and define its cohomology, denoted by

$$H^q(X, E) = H^q_{\overline{\partial_E}}(C^\infty(X, \Omega_X^{0,*} \otimes E))$$

and we can show a result similar to Example 1.6.8.

$$H^0(X, E) = \Gamma(X, E)$$

[9] This implies that Dolbeault cohomology do computes useful information.

[10] Check this, we need back to definition, a holomorphic map induces a tangent map $T_f : T_{X,\mathbb{C}} \to f^*T_{Y,\mathbb{C}}$,and consider its dual we get cotangent map $\Omega_f : f^*\Omega_{Y,\mathbb{C}} \to \Omega_{X,\mathbb{C}}$

[11] In previous, $E = \Omega_X^p$

Theorem 1.6.11 (Dolbeault lemma)**.** *Let $X = D(z_0, r_0) \subset \mathbb{C}^n$ be a polydisk, then*

$$H^{p,q}(X) = 0, \quad \forall p \geq 0, q > 0$$

1.7. **Čech cohomology.** Let X be a topological space, and $\mathcal{U} = (U_i)_{i \in I}$ be an open covering, such that I is countable and an ordered set. For all $i_0, \dots, i_p \in I$ write

$$U_{i_0 \dots i_p} = U_{i_0} \cap \dots \cap U_{i_p}$$

Let $\mathscr{F}$ be a sheaf of abelian group, define a chain complex $C^*(\mathcal{U}, \mathscr{F})$ as

$$0 \to C^0(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathscr{F}) \xrightarrow{\delta} \dots$$

where

$$C^p = \prod_{i_0 < \dots < i_p} \mathscr{F}(U_{i_0 \dots i_p})$$

and δ is defined as

$$(\delta \alpha)_{i_0 \dots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \dots \widehat{i_k} \dots i_{p+1}}|_{U_{i_0 \dots i_{p+1}}}$$

Exercise 1.7.1 (Once and only once exercise in your whole life)**.** Check that $\delta \circ \delta = 0$

So we can define Čech cohomology as

$$\check{H}^q(\mathcal{U}, \mathscr{F}) := H^q_\delta(C^*(\mathcal{U}, \mathscr{F}))$$

Example 1.7.2. We consider

$$\check{H}^0(\mathcal{U}, \mathscr{F}) = \{\alpha \in C^0(\mathcal{U}, \mathscr{F}) \mid \delta \alpha = 0\}$$

then if $\alpha = \prod_{i_0} \alpha_{i_0}$, then $\delta \alpha = 0$ implies

$$\alpha_i|_{U_i \cap U_j} = \alpha_j|_{U_i \cap U_j}$$

then we have

$$\check{H}^0(\mathcal{U}, \mathscr{F}) = \mathscr{F}(X)$$

However, we want out definition is independent of open cover, so

Definition 1.7.3 (Čech cohomology)**.** *We define Čech cohomology as*

$$\check{H}^q(X, \mathscr{F}) = \varinjlim_{\mathcal{U}} \check{H}^q(\mathcal{U}, \mathscr{F})$$

Remark 1.7.4. In other words, $\alpha = \alpha' \in \check{H}^q(X, \mathscr{F})$ is equivalent to there exists a common refinement $\mathcal{U}''$ such that

$$\alpha = \alpha' \in \check{H}^q(\mathcal{U}'', \mathscr{F})$$

Why we want to introduce Čech cohomology here? In fact, it provides a method to compute de Rham cohomology and Dolbeault cohomology we defined before. However, we will use cohomology of sheaf to unify all de Rham cohomology and Čech cohomology together later.

Recall that if X is a complex manifold, $E \to X$ is a holomorphic vector bundle. Then we can define a sheaf of holomorphic sections, defined by

$$\mathcal{O}_X(E) : U \mapsto \Gamma(U, E|_U)$$

then we get a Čech cohomology of this sheaf

$$\check{H}^q(X, \mathcal{O}_X(E)) = \varinjlim_{\mathcal{U}} \check{H}^q(\mathcal{U}, \mathcal{O}_X(E))$$

Theorem 1.7.5 (comparision)**.** *We have the following isomorphism*

$$\check{H}^q(X, \mathcal{O}_X(E)) \cong H^q(X, E) = H^q_{\overline{\partial}_E}(C^\infty(X, \Omega^{0,*}_X \otimes E))$$

In particular, let $E = \Omega^p_X$, we have

$$\check{H}^q(X, \mathcal{O}_X(\Omega^p_X)) \cong H^{p,q}(X) = H^q_{\overline{\partial}}(C^\infty(X, \Omega^{p,*}_X))$$

Remark 1.7.6. In fact, Theorem 1.7.5 uses the sequence of sheaves

$$0 \to E \to \Omega^{0,0}_X \otimes E \xrightarrow{\overline{\partial_E}} \Omega^{0,1}_E \otimes E \xrightarrow{\overline{\partial_E}} \cdots \xrightarrow{\overline{\partial_E}} \Omega^{0,n}_X \otimes E \to 0$$

And Dolbeault lemma implies the above sequence is exact. That's what behind the comparision theorem, and tells us why Dolbeault cohomology is about holomorphic information, since here we use $\mathcal{O}_X(\Omega^p_X)$, sheaf of holomorphic sections.

Similarly, in de Rham cohomology, we have the same story. There is a sequence of sheaves

$$0 \to \underline{\mathbb{C}} \xrightarrow{i} \Omega^0_{X,\mathbb{C}} \xrightarrow{\mathrm{d}} \Omega^1_{X,\mathbb{C}} \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n_{X,\mathbb{C}} \to 0$$

where $\underline{\mathbb{C}}$ is the sheaf of locally constant functions, i.e.

$$\underline{\mathbb{C}} : U \mapsto \{\text{locally constant functions } f : U \to \mathbb{C}\}$$

Then Poincaré lemma implies the above sequence is also exact. Parallel to Theorem 1.7.5, we will get

$$\check{H}^q(X, \underline{\mathbb{C}}) \cong H^k(X, \mathbb{C}) = H^k_{dR}(C^\infty(X, \Omega^*_{X,\mathbb{C}}))$$

This also explain why de Rham cohomology is just a topological information, since here we just use $\underline{\mathbb{C}}$, a pure topological information.

Theorem 1.7.7 (Leray)**.** *Let $\mathcal{U}$ be a covering such that for all $i_0 \dots i_k \in I$, and for all $q > 0$, we have*

$$H^q(U_{i_0 \dots i_k}, E|_{U_{i_0 \dots i_k}}) = 0$$

then $\mathcal{U}$ is called acyclic for E. Then

$$\check{H}^q(\mathcal{U}, \mathcal{O}_X(E)) \cong H^q(X, E)$$

Remark 1.7.8. This provides us a practical way to compute Čech cohomology.

Example 1.7.9. Consider $\mathcal{O}_X^\times \subset \mathcal{O}_X$, the sheaf of invertible holomorphic functions. Then we have

$$\check{H}^1(X, \mathcal{O}_X^\times) \cong \operatorname{Pic}(X)$$

2. Geometry of vector bundles

2.1. Connections.

Definition 2.1.1 (connection)**.** *X is a differential manifold, and $\pi : E \to X$ is a complex vector bundle. A connection on E is a $\mathbb{C}$-linear operator*

$$D : C^\infty(X, E) \to C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes E)$$

satisfying the Leibniz rule

$$D(f\sigma) = \mathrm{d}f \otimes \sigma + fD(\sigma)$$

for $f \in C^\infty(X)$ and $\sigma \in C^\infty(X, E)$.

Remark 2.1.2. In fact, if we ask D to satisfy the Leibniz rule, it induces

$$D : C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes E) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{C}} \otimes E)$$

for any k, by setting[12]

$$D(\varphi \otimes \sigma) = \mathrm{d}\varphi \otimes \sigma + (-1)^{\deg \varphi} \varphi \wedge (D\sigma)$$

for $\varphi \in C^\infty(X, \Omega^k_{X,\mathbb{C}})$ and $\sigma \in C^\infty(X, E)$.

Remark 2.1.3. Let's see what's going on in local pointview. Locally around $x \in U \subset X$, then $\pi^{-1}(U) \cong U \times \mathbb{C}^r$, there is a basis $\{e_1, \dots, e_r\}$ for $\mathbb{C}^r$. For $\sigma \in C^\infty(U, E|_U)$, we have

$$\sigma = \sum_{j=1}^r s_j e_j, \quad s_j \in C^\infty(U)$$

By Leibniz rule, we have

$$D\sigma = \sum_{j=1}^r (\mathrm{d}s_j \otimes e_j + s_j De_j)$$

where $De_j \in C^\infty(U, \Omega^1_{U,\mathbb{C}} \otimes E)$. So we can write more explictly as

$$De_j = \sum_{i=1}^r a_{ij} \otimes e_i, \quad a_{ij} \in C^\infty(U, \Omega^1_{U,\mathbb{C}})$$

So we have

$$D\sigma = \sum_{j=1}^r (\mathrm{d}s_j \otimes e_j + \sum_{i=1}^r s_j a_{ij} \otimes e_i)$$

[12]Some authors may extend D by setting usual Leibniz rule, that is, without $(-1)^{\deg \varphi}$, it's not quit important. We will see that reason in local chart computation.

We can rewrite the above formula in frame of $\{e_1, \dots, e_r\}$ as

$$D\sigma = Ds = \mathrm{d}s + As$$

where

$$\sigma = s = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}, \quad A = (a_{ij}) \in C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes \mathrm{End}(E|_U))$$

Here we chose a local trivialization of the vector bundle E, so we may wonder what will happen if we change our choice.

If $x \in U' \subset X$ is another trivialization, so $\pi^{-1}(U') \cong U' \times \mathbb{C}^r$, and $\{e'_1, \dots, e'_r\}$ is another local frame. Then

$$D\sigma = \begin{cases} Ds = \mathrm{d}s + As \\ Ds' = \mathrm{d}s' + A's' \end{cases}$$

so we wonder the relationship between A and A'. Transition functions between U and U' are

$$g : U \cap U' \to \mathrm{GL}(r, \mathbb{C})$$

so we have $s = gs'$ and $Ds = gDs'$. We compute as follows

$$\begin{aligned} \mathrm{d}s = \mathrm{d}(gs') &= (\mathrm{d}g)s' + g(\mathrm{d}s') = g(g^{-1}(\mathrm{d}g)s' + \mathrm{d}s') \\ \mathrm{d}s + As &= g(g^{-1}(\mathrm{d}g)s' + \mathrm{d}s' + g^{-1}Ags') \\ &= g(\mathrm{d}s' + (g^{-1}\mathrm{d}g + g^{-1}Ag)s') \end{aligned}$$

Since we have

$$\mathrm{d}s + As = gDs' = g(\mathrm{d}s' + A's')$$

So we have

$$A' = g^{-1}\mathrm{d}g + g^{-1}Ag$$

You may feel quite uncomfortable since A' does not conjugate to A under the change of the frame, but if we apply D twice, something interesting may happen.

Before that, we compute what does $D : C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes E) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{C}} \otimes E)$ look like locally:

Take $\sigma \in C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes E)$, locally we can write as

$$\sigma = \sum_{j=1}^r s_j \otimes e_j, \quad s_j \in \Omega^k_{X,\mathbb{C}}$$

then

$$\begin{aligned}
D\sigma &= D(\sum_{j=1}^{r} s_j \otimes e_j) \\
&= \sum_{j=1}^{r} \mathrm{d}s_j \otimes e_j + (-1)^k s_j \wedge De_j \\
&= \sum_{j=1}^{r} (\mathrm{d}s_j \otimes e_j + (-1)^k s_j \wedge \sum_{i=1}^{r} a_{ij} \otimes e_i) \\
&= \sum_{j=1}^{r} (\mathrm{d}s_j \otimes e_j + \sum_{i=1}^{r} a_{ij} \wedge s_j \otimes e_i) \\
&= \mathrm{d}s + A \wedge s
\end{aligned}$$

So we know that $D : C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes E) \to C^\infty(X, \Omega^{k+1}_{X,\mathbb{C}} \otimes E)$ still looks like[13]

$$D\sigma = Ds = \mathrm{d}s + A \wedge s$$

Here we can see clearly what does $A \wedge s$ mean. Furthermore, we can see that $A \wedge A$ isn't trivial, unless in the case of line bundle.

So we can compute as follows.

$$\begin{aligned}
D^2\sigma = D(\mathrm{d}s + As) &= \mathrm{d}(\mathrm{d}s + As) + A \wedge (\mathrm{d}s + As) \\
&= \mathrm{d}^2 s + \mathrm{d}(As) + A \wedge \mathrm{d}s + A \wedge As \\
&= \mathrm{d}^2 s + (\mathrm{d}A)s - A \wedge \mathrm{d}s + A \wedge \mathrm{d}s + A \wedge As \\
&= (\mathrm{d}A + A \wedge A)s
\end{aligned}$$

And we check what will happen if we choose another trivialization

$$\begin{aligned}
D^2\sigma = D^2 s = (\mathrm{d}A + A \wedge A)s &= (\mathrm{d}A + A \wedge A)gs' \\
= gD^2 s' &= g(\mathrm{d}A' + A' \wedge A')s'
\end{aligned}$$

so we have

$$\mathrm{d}A' + A' \wedge A' = g^{-1}(\mathrm{d}A + A \wedge A)g$$

that is, $\mathrm{d}A + A \wedge A$ behaves "well" under the change of frame, object with such property we always call it a "tensor"[14].

From discussion above, we can give the following definition

Definition 2.1.4 (curvature)**.** *There exists a global section $H_D \in C^\infty(X, \Omega^2_{X,\mathbb{C}} \otimes \mathrm{End}(E))$ such that*

$$D^2\sigma = H_D \wedge \sigma, \quad \forall \sigma \in C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes E)$$

such H_D is called the curvature tensor of connection D.

[13] That's why we need $(-1)^{\deg \varphi}$ when we extend D.

[14] But what is a "tensor"? Here I quote a motto said by Leonard Susskind, a well-known physicist. I'm quite impressed when I first heard it in my childhood. "Tensor is something which behaves like a tensor."

Definition 2.1.5 (Hermitian metric)**.** *X is a differential manifold, and $\pi : E \to X$ is a complex vector bundle. A Hermitian metric h on E is a Hermitian inner product on each fiber E_x, such that for all open subset $U \subset X$, and $\xi, \eta \in C^\infty(U, E|_U)$, we have*

$$\begin{aligned} \langle \xi, \eta \rangle : U &\to \mathbb{C} \\ x &\mapsto \langle \xi(x), \eta(x) \rangle \end{aligned}$$

is a C^∞-function.

Example 2.1.6. Locally, for $x \in U \subset X$, we have $\pi^{-1}(U) \cong U \times \mathbb{C}^r$, and $\{e_1, \ldots, e_r\}$ is a local frame. Then our Hermitian metric is just a Hermitian matrix

$$H = (h_{\lambda\mu})$$

where $h_{\lambda\mu} \in C^\infty(U)$, defined by

$$h_{\lambda\mu}(x) = \langle e_\lambda(x), e_\mu(x) \rangle$$

Indeed, this Hermitian matrix can tell us how does the metric works. In our local frame, two sections $\xi = \sum_{i=1}^r \xi_i e_i, \eta = \sum_{i=1}^r \eta_i e_i \in C^\infty(U, E|_U)$ can be write as

$$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_r \end{pmatrix}, \eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_r \end{pmatrix}$$

Then

$$\begin{aligned} h(\xi, \eta) &= h(\sum_{i=1}^r \xi_i e_i, \sum_{i=1}^r \eta_i e_i) \\ &= \sum_{i,j=1}^r \xi_i \overline{\eta_j} h(e_i, e_j) \\ &= \sum_{i,j=1}^r \xi_i \overline{\eta_j} h_{ij} \\ &= (\xi_1, \ldots, \xi_r) H \begin{pmatrix} \overline{\eta_1} \\ \vdots \\ \overline{\eta_r} \end{pmatrix} \\ &= \xi^t H \overline{\eta} \end{aligned}$$

And take another $x \in U' \subset X$, $\pi^{-1}(U') \cong U' \times \mathbb{C}^r$, with $\{e'_1, \ldots, e'_r\}$, and g is the transition function, we have

$$H' = g^t H \overline{g}$$

Proposition 2.1.7. *Every complex vector bundle admits a Hermitian metric*

Proof. Use partition of unity. □

Now for a complex vector bundle over a differential manifold, we have two structures on it, connection and Hermitian metric, so it's natural to require them to exist in a harmony.

For a Hermitian metric h, it induces a pairing $\{\cdot,\cdot\}$

$$C^\infty(X,\Omega^p_{X,\mathbb{C}}\otimes E)\times C^\infty(X,\Omega^q_{X,\mathbb{C}}\otimes E)\to C^\infty(X,\Omega^{p+q}_{X,\mathbb{C}})$$

We describe this pairing locally, consider $x\in U\subset X$, $\pi^{-1}(U)\cong U\times\mathbb{C}^r$, and $\{e_1,\dots,e_r\}$ is a local frame. Then $\sigma\in C^\infty(X,\Omega^p_{X,\mathbb{C}}\otimes E)$ and $\eta\in C^\infty(X,\Omega^q_{X,\mathbb{C}}\otimes E)$ are in form

$$\sigma=\sum_{i=1}^r s_i\otimes e_i,\quad \tau=\sum_{j=1}^r t_j\otimes e_j$$

where s_i are p-forms and t_j are q-forms, then the pairing is locally look like

$$\begin{aligned}\{\sigma,\tau\}&=\{\sum_{i=1}^r s_i\otimes e_i,\sum_{j=1}^r t_j\otimes e_j\}\\&=\sum_{i,j=1}^r s_i\wedge t_jh(e_i,e_j)\\&=\sum_{i,j=1}^r s_i\wedge t_jh_{ij}\\&=s^tH\bar{t}\end{aligned}$$

Using this pairing, we can define when a connection is called Hermitian.

Definition 2.1.8 (Hermitian connection)**.** *(E,h) is a Hermitian vector bundle on X. A connection D on E is Hermitian if for all $\sigma\in C^\infty(X,\Omega^p_{X,\mathbb{C}}\otimes E),\eta\in C^\infty(X,\Omega^q_{X,\mathbb{C}}\otimes E)$,*

$$\mathrm{d}\{\sigma,\tau\}=\{D\sigma,\tau\}+(-1)^{\deg\sigma}\{\sigma,D\tau\}$$

Since we know that a connection locally looks like $D=\mathrm{d}+A\wedge$. Then let's compute in a local frame to show what condition A needs to satisfy for a Hermitian connection.

Fixing a local frame, then $\sigma=s=(s_1,\dots,s_r)^t,\tau=t=(t_1,\dots,t_r)^t$. WLOG, we assume $\{e_1,\dots,e_r\}$ is a orthonormal basis, i.e. H is identity matrix, then

$$\{\sigma,\tau\}=s^t\bar{t}$$

Otherwise we need to consider the derivative of H, which will make things more complicated. You can try to solve the following exercise which is also useful later.

Exercise 2.1.9. Show that

$$\mathrm{d}H=A^tH+H\overline{A}$$

If we already use orthonormal basis, we can directly compute as follows

$$\begin{aligned}
\mathrm{d}\{\sigma,\tau\} &= (\mathrm{d}s)^t \wedge \bar{t} + (-1)^{\deg\sigma} s^t \wedge \mathrm{d}\bar{t} \\
\{D\sigma,\tau\} &= (\mathrm{d}s + A\wedge s)^t \wedge \bar{t} = (\mathrm{d}s)^t \wedge \bar{t} + (-1)^{\deg\sigma} s^t \wedge A^t \wedge \bar{t} \\
\{\sigma, D\tau\} &= s^t \wedge \overline{\mathrm{d}t + A\wedge t} = s^t \wedge \mathrm{d}\bar{t} + s^t \wedge \overline{A} \wedge \bar{t}
\end{aligned}$$

then

$$\mathrm{d}\{\sigma,\tau\} - \{D\sigma,\tau\} - \{\sigma, D\tau\} = (-1)^{\deg\sigma} s^t \wedge (A^t + \overline{A}) \wedge \bar{t}$$

So D is a Hermitian connection if and only if $A^t + \overline{A} = 0$.

Let's make it more beautiful. We define D^{adj}, adjoint connection, locally given by $-\overline{A}^t$ with respect to $H = I$, then we always have

$$\mathrm{d}\{\sigma,\tau\} = \{D\sigma,\tau\} + (-1)^{\deg\sigma}\{\sigma, D^{adj}\tau\}$$

Take $\frac{1}{2}(D + D^{adj})$, which is also a connection, locally looks like

$$\frac{1}{2}(A - \overline{A}^t)$$

is a Hermitian connection. So it's easy to get a Hermitian connection, just average A with its adjoint.

Proposition 2.1.10. *Every Hermitian vector bundle admits a Hermitian connection.*

Proof. Use partition of unity to show the existence of connection, and take the average of it and its adjoint connection. ☐

2.2. **Connections and metrics on holomorphic vector bundles.** In this section, let's see when the base space is a complex manifold, and the vector bundle is holomorphic, what will happen?

Recall that for a complex manifold X, we have

$$\Omega^1_{X,\mathbb{C}} = \Omega^{1,0}_X \oplus \Omega^{0,1}_X$$

Consider $E \to X$ is a complex vector bundle, and D is a connection, then we can decompose $D = D^{1,0} + D^{0,1}$ by composing the projection as follows

$$\begin{tikzcd}
 & & C^\infty(X, \Omega^{1,0}_X \otimes E) \\
C^\infty(X,E) \arrow[r, "D"] & C^\infty(X, \Omega^1_{X,\mathbb{C}} \otimes E) \arrow[ur] \arrow[dr] & \\
 & & C^\infty(X, \Omega^{0,1}_X \otimes E)
\end{tikzcd}$$

Locally, we have $D = \mathrm{d} + A$, then

$$D^{1,0} = \partial + A^{1,0}, \quad D^{0,1} = \overline{\partial} + A^{0,1}$$

both $D^{1,0}$ and $D^{0,1}$ satisfy Leibniz rule.

Now consider X is a complex manifold, and $E \to X$ is a holomorphic vector bundle. Recall that we already have

$$\overline{\partial}_E : C^\infty(X, E) \to C^\infty(X, \Omega_X^{0,1} \otimes E)$$

We want to compare $D_E^{0,1}$ and $\overline{\partial}_E$

Theorem 2.2.1 (Chern connection)**.** *X is a complex manifold, (E, h) is a Hermitian holomorphic vector bundle, then there exists a unique Hermitian connection D_E such that $D_E^{0,1} = \overline{\partial}_E$. D_E is called the Chern connection of (E, h).*

Proof. Uniqueness: If we already have $D_E^{0,1} = \overline{\partial}_E$. Locally $x \in U \subset X$, $\{e_1, \dots, e_r\}$ is holomorphic local frame. And smooth section $\sigma = s = (s_1, \dots, s_r)^t$. Then

$$D_E^{0,1}\sigma = \overline{\partial}s + A^{0,1}s = \overline{\partial}_E\sigma$$

If s is a holomorphic section, then $\overline{\partial}_E\sigma = \overline{\partial}s = 0$, which implies $A^{0,1} = 0$.

Since we have

$$\mathrm{d}H = A^t H + H\overline{A}$$

then

$$\overline{\partial}H = H\overline{A}$$

So A is uniquely determined by

$$A = \overline{H^{-1}}\partial\overline{H}$$

Existence: It suffices to prove we can glue A together to get a global connection, i.e. compatible with holomorphic change of frames.

Consider another holomorphic local chart $x \in U' \subset X$, with frame $\{e_1', \dots, e_r'\}$. And the metric with respect to this new frame is H', we have

$$H' = g^t H\overline{g}$$

Then

$$\begin{aligned} A' = \overline{H'^{-1}}\partial\overline{H'} &= g^{-1}\overline{H^{-1}(g^t)^{-1}}\partial(\overline{g^t}\overline{H}g) \\ &= g^{-1}\overline{H^{-1}(g^t)^{-1}}((\partial\overline{g^t}) + \overline{g^t}(\partial\overline{H})g + \overline{g^t}\overline{H}\partial g) \\ &= g^{-1}\overline{H^{-1}}(\partial\overline{H}g) + g^{-1}\mathrm{d}g \\ &= g^{-1}\mathrm{d}g + g^{-1}Ag \end{aligned}$$

As we desire. □

Corollary 2.2.2. *If X is a complex manifold, (E, h) is a Hermitian holomorphic vector bundle. D_E is Chern connection on it, and H_E is Chern curvature. If A is the matrix of D_E with respect to holomorphic local frame, then*

1. *A is of type $(1, 0)$, with $\partial A = -A \wedge A$*
2. *locally we have $H_E = \overline{\partial}A$, a form of type $(1, 1)$*
3. $\overline{\partial}H_E = 0$

Proof. Locally we have $A = \overline{H}^{-1}\partial\overline{H}$, so it's of type $(1,0)$, and we compute

$$\begin{aligned}\partial A = \partial(\overline{H}^{-1}\partial\overline{H}) &= \partial\overline{H}^{-1} \wedge \partial\overline{H} \\ &= (-\overline{H}^{-1}\partial H\overline{H}^{-1}) \wedge \partial\overline{H} \\ &= -(\overline{H}^{-1}\partial\overline{H}) \wedge (\overline{H}^{-1}\partial\overline{H}) \\ &= -A \wedge A\end{aligned}$$

Chern curvature locally looks like

$$H_E = \mathrm{d}A + A \wedge A = \mathrm{d}A - \partial A = \overline{\partial}A$$

, which is of type $(1,1)$. And clearly $\overline{\partial}H_E = 0$. □

Exercise 2.2.3. (E,h) is a Hermitian holomorphic vector bundle, and $S \hookrightarrow E$ is a holomorphic subbundle. $S^\perp$ is defined by $(S^\perp)_x = (S_x)^\perp$ with respect to h. We have $E = S \oplus S^\perp$ as complex vector bundle.[15] So we can define a natural projection

$$P_s : C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes E) \to C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes S)$$

Show that $D_S = P_s \circ D_E$.

2.3. **Case of Line bundle.** In this case we will consider a special case, i.e. line bundle, to find some interesting things.

Recall that if X is a differential manifold, and $\pi : X \to L$ is a complex line bundle, D is a connection on L. Since

$$H_D \in C^\infty(X, \Omega^2_{X,\mathbb{C}} \otimes \mathrm{End}(L))$$

But for line bundle, $\mathrm{End}(L) \cong L^* \otimes L \cong X \times \mathbb{C}$ is just trivial bundle. So in fact, $H_D \in C^\infty(X, \Omega^2_{X,\mathbb{C}})$, that is, curvature of connection is exactly a 2-form, without coefficient.

Furthermore, in a local pointview, D is represented by 1-form A, then $H_D = \mathrm{d}A + A \wedge A$, and for a line bundle, we clearly have $A \wedge A = 0$, since A is just a 1×1 matrix, and forms are skew symmetric. So $H_D = \mathrm{d}A$. A immediate consequence is that $\mathrm{d}H_D = 0$, i.e. H_D is a closed form, so we get

$$[H_D] \in H^2(X, \mathbb{C})$$

an element of de Rham cohomology group $H^2(X,\mathbb{C})$.

Now it's natural to ask what's the relationship between closed form from different connections. A surprising result is that they exactly lie in the same cohomology class.

If we consider another connection $\widetilde{D}$, and $\widetilde{A}$, let's compare H_D with $H_{\widetilde{D}}$. For all $\sigma \in C^\infty(X, \Omega^k_{X,\mathbb{C}} \otimes L)$, locally $\sigma = s$ with respect to $\{e_1, \ldots, e_r\}$.

[15]But in general, $S^\perp$ may not be a holomorphic subbundle of E. That is, if we have a short exact sequence of holomorphic vector bundle

$$0 \to S \to E \to Q \to 0$$

this exact sequence generally won's split. That's why we perfer short exact sequence rather than direct sum in algebraic geometry.

Then

$$\begin{aligned} D(\sigma) - \widetilde{D}(\sigma) &= (\mathrm{d}s + A \wedge s) - (\mathrm{d}s - \widetilde{A} \wedge s) \\ &= (A - \widetilde{A}) \wedge s \\ &= B \wedge \sigma \end{aligned}$$

where $B = (A - \widetilde{A}) \in C^\infty(X, \Omega^1_{X,\mathbb{C}})$.

Then

$$H_D - H_{\widetilde{D}} = \mathrm{d}B$$

So different connections give the same cohomology class in $H^2(X, \mathbb{C})$. What a beautiful result!

Definition 2.3.1 (First Chern class)**.** *Let $\pi : E \to X$ be a complex line bundle, D is any connection. Then define*

$$c_1(L) := [\frac{i}{2\pi} H_D] \in H^2(X, \mathbb{C})$$

, which is called the first Chern class of line bundle.

Remark 2.3.2. The first Chern class is a property of line bundle itself, and independent of connections on it. In other words, it's a topological information.

Let's explain why we need coefficient $\dfrac{i}{2\pi}$ here.

Lemma 2.3.3. *(E, h) is a Hermitian line bundle, and D is a Hermitian connection, then*

$$\frac{i}{2\pi} H_D \in C^\infty(X, \Omega^2_{X,\mathbb{R}})$$

hence $c_1(L) \in H^2(X, \mathbb{R})$.

Proof. Locally we have $\overline{A} = -A$

$$\overline{\frac{i}{2\pi}} H_D = -\frac{i}{2\pi}\overline{H_D} = -\frac{i}{2\pi}\overline{\mathrm{d}A} = -\frac{i}{2\pi}\mathrm{d}\overline{A} = \frac{i}{2\pi}\mathrm{d}A = \frac{i}{2\pi}H_D$$

$\square$

Remark 2.3.4. However, why we need 2π here? Later we will see in fact $c_1(L) \in H^2(X, \mathbb{Z})$.

Exercise 2.3.5. Let $E \to X$ be a complex vector bundle of rank r, define

$$c_1(E) := c_1(\det E)$$

If $L \to X$ is a complex line bundle, Show that

$$c_1(E \otimes L) = c_1(E) + r c_1(L)$$

Now, Let's combine all we have together, to see what will happen. Let X be a complex manifold and L be a holomorphic line bundle, with a Hermitian metric. D_L is the Chern connection of L, and its Chern curvature is H_L.

By Corollary 2.2.2 and Lemma 2.3.3, we have

$$\frac{i}{2\pi}H_L \in C^\infty(X, \Omega^2_{X,\mathbb{R}}) \cap C^\infty(X, \Omega^{1,1}_X)$$

such that

$$\mathrm{d}(\frac{i}{2\pi}H_L) = \overline{\partial}(\frac{i}{2\pi}H_L) = 0$$

that is

$$[\frac{i}{2\pi}H_L] \in H^2(X, \mathbb{R}), \quad [\frac{i}{2\pi}H_L] \in H^{1,1}(X)$$

Exercise 2.3.6. Show that $[\frac{i}{2\pi}H_L] \in H^{1,1}(X)$ is independent of h.

Proof. □

Example 2.3.7. Locally we have $x \in U \subset X$, with $\pi^{-1}(U) \cong U \times \mathbb{C}$, $\{e_1\}$ is the local frame. Then Hermitian metric is

$$H(z) = \langle e_1(z), e_1(z) \rangle = \|e_1(z)\|^2_h$$

Write $\varphi(z) = -\log H(z)$, a function $U \to \mathbb{R}$. Then

$$A = \overline{H^{-1}\partial \overline{H}} = e^{\varphi(z)}\partial e^{-\varphi(z)} = -\partial\varphi(z)$$

then

$$H_L = \overline{\partial}A = -\overline{\partial}\partial\varphi(z) = \partial\overline{\partial}\varphi(z)$$

then we have

$$\begin{aligned}\frac{i}{2\pi}H_L = \frac{i}{2\pi}\partial\overline{\partial}\varphi(z) &= \frac{i}{2\pi}\partial\overline{\partial}(-\log H(z)) \\ &= \frac{1}{2\pi i}\partial\overline{\partial}\log\|e_1(z)\|^2_h\end{aligned}$$

Summarize as follows

Proposition 2.3.8. *X is a complex manifold, (L, h) is a Hermitian holomorphic line bundle. Then $c_1(L)$ is represented by a real $(1,1)$-form, given locally by*

$$\frac{i}{2\pi}H_L = \frac{1}{2\pi i}\partial\overline{\partial}\log\|e_1(z)\|^2_h$$

2.4. **Positive line bundle.** Before we go into deeper, let's discuss some facts about linear algebra we will need.

Let V be a n-dimensional complex vector space, and use $V_\mathbb{R}$ to denote the underlying real vector space, with real dimension $2n$. And J acts on $V_\mathbb{R}$ as $\times i$. Then

$$V_\mathbb{C} = V_\mathbb{R} \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$$

Consider its dual space $W_\mathbb{R} = V^*_\mathbb{R}$, and $W_\mathbb{C} = W_\mathbb{R} \otimes \mathbb{C} = W^{1,0} \oplus W^{0,1}$. Then $W^{1,1} = W^{1,0} \otimes W^{0,1} = W^{1,1} \subset \bigwedge^2 W_\mathbb{C}$.

For a Hermitian form $h : V \times V \to \mathbb{C}$, we have the following magic correspondence.

Lemma 2.4.1. *We have the following canonical correspondence*

$$\{\textit{Hermitian forms on } V\} \longleftrightarrow \{\textit{real } (1,1)\textit{-form on } V_{\mathbb{R}}\}$$

Proof. Given a Hermitian form h, then $h \mapsto \omega = -\operatorname{Im}(h)$; Conversely, given such a form ω, we define

$$h = \omega(\cdot, J\cdot) - i\omega(\cdot,\cdot)$$

Now let's check it: In one direction, write $h = \operatorname{Re} h + i \operatorname{Im} h$, then

$$\operatorname{Re} h(u,v) + i \operatorname{Im} h(u,v) = h(u,v) = \overline{h(v,u)} = \operatorname{Re} h(v,u) - i \operatorname{Im} h(v,u)$$

So $\operatorname{Im} h$ is skew symmetric, that is, $\omega = -\operatorname{Im} h$ is an alternating real 2-form.

Now we need to show ω is a $(1,1)$-form. Since $W^{1,1} = (V^{1,1})^*$, where $\bigwedge^2 V = V^{2,0} \oplus V^{1,1} \oplus V^{0,2}$. So we have $\omega \in W^{1,1}$ is equivalent to $\omega(V^{1,0}, V^{1,0}) = \omega(V^{0,1}, V^{0,1}) = 0$

Recall that $V^{1,0}$ is spanned by $u - iJ(u)$, then

$$\omega(u - iJ(u), v - iJ(v)) = \omega(u,v) - \omega(J(u),J(v)) - i(\omega(u,J(v)) + \omega(J(u),v))$$

Since

$$h(J(u),J(v)) = ih(u,J(v)) = i \times (-i) h(u,v) = h(u,v)$$

then

$$\omega(u,v) = \omega(J(u),J(v))$$

Similarly, we can check the last two terms cancel with each other.

On the other hand, if ω is a real $(1,1)$-form, then

$$\begin{aligned}
\overline{h(u,v)} := \overline{\omega(u,J(v)) - i\omega(u,v)} &= \omega(u,J(v)) + i\omega(u,v) \\
&= -\omega(J(v),u) - i\omega(v,u) \\
&= -\omega(J^2(v),J(u)) - i\omega(v,u) \\
&= \omega(v,J(u)) - i\omega(v,u) \\
&= h(v,u)
\end{aligned}$$

□

Remark 2.4.2. Though the correspondence above is canonical, we can choose a basis to see what's going on:

If we choose a basis $z_1, \ldots, z_n$ of V. Then Hermitian forms on V can be write as $h = \sum_{j,k} h_{jk} z_j^* \otimes \overline{z_k}^*$, where z_k^* is the dual basis of z_k and $h_{jk} = h(z_j, z_k)$ is a Hermitian matrix.

If we let $u = (u_1, \ldots, u_n)^T, v = (v_1, \ldots, v_n)^n$, then $h(u,v) = \sum_{jk} u_j h_{jk} \overline{v_k}$. By definition, we have

$$\begin{aligned}
\omega(u,v) &= -\operatorname{Im} h(u,v) \\
&= \frac{i}{2}\left(\sum h_{jk} u_j \overline{v_k} - \sum \overline{h_{jk}}\,\overline{u_j} v_k\right) \\
&= \frac{i}{2} \sum (h_{jk} u_j \overline{v_k} - h_{jk} v_j \overline{u_k}) \\
&= \frac{i}{2} \sum h_{jk} (u_j \overline{v_k} - v_j \overline{u_k})
\end{aligned}$$

That is, the corresponding real $(1,1)$-form is

$$\omega = \frac{i}{2}\sum_{j,k} h_{jk} z_j^* \wedge \overline{z_k}^*$$

To be more explicit, if we choose $x_1, \dots, x_n, y_1, \dots, y_n$ to be basis of $V_{\mathbb{R}}$, then $z_i^* = x_i^* + iy_i^* \in (V_{\mathbb{C}})^*$.

Definition 2.4.3 (positive form)**.** *For a real $(1,1)$-form ω, it is called positive, if the corresponding Hermitian form h is positive definite.*

Definition 2.4.4 (positive line bundle)**.** *X is a complex manifold, L is a holomorphic line bundle. L is called positive if it admits a Hermitian metric h such that*

$$\frac{i}{2\pi} H_L$$

corresponds to a positive Hermitian metric on the holomorphic tangent bundle T_X.

Remark 2.4.5. For any $x \in X$, then

$$(\frac{i}{2\pi} H_L)_x \in (\Omega^2_{X,\mathbb{R}} \cap \Omega^{1,1}_X)_x$$

is a real $(1,1)$-form on $(T_{X,\mathbb{R}})_x$. Then by Lemma 2.4.1, we know that there is a one to one correspondence with Hermitian form on $T_{X,x}$. So globally we have that $\frac{i}{2\pi} H_L$ will correspond to a Hermitian metric on T_X.

Locally, we have

$$\frac{i}{2\pi} H_L = \frac{i}{2\pi}\partial\overline{\partial}\varphi(z) = \frac{i}{2\pi}\sum_{j,k}\frac{\partial^2\varphi}{\partial z_j \partial \overline{z_k}} \mathrm{d}z_j \wedge \mathrm{d}\overline{z_k}$$

Then L is positive is equivalent to the Hermitian metric

$$(\frac{\partial \varphi^2}{\partial z_j \partial \overline{z_k}})$$

is everywhere positive definite.

Exercise 2.4.6 (positive line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$)**.** First consider function

$$\varphi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{R}$$
$$z \mapsto \log(\sum_{j=0}^n |z_j|^2)$$

then $(\frac{\partial^2\varphi}{\partial z_j \partial \overline{z_k}})$ is positive definite. Show that

1. $\frac{i}{2\pi}\partial\overline{\partial}\varphi(z)$ induces a real $(1,1)$-form on $\mathbb{P}^n$ such that $\mathrm{d}\omega = 0$.
2. φ comes from a Hermitian metric h on $\mathcal{O}_{\mathbb{P}^n}(1)$ called Fubini-Study metric.

Exercise 2.4.7. L is positive if and only if $L^{\otimes m}$ is positive for some $m \in \mathbb{N}$.

Exercise 2.4.8. L is positive, and M is any holomorphic line bundle, then there exists $N_0 \in \mathbb{N}$ such that $M \otimes L^{\otimes N}$ positive for $N \geq N_0$.

In fact, If X is a complex complex manifold, then positive is equivalent to ample.

2.5. **Lefschetz $(1,1)$-theorem.** Now we know that given a Hermitian holomorphic line bundle (L, h), then consider its Chern curvature we will get a real $(1,1)$-form. So we may wonder the converse of this statement. Is there any real $(1,1)$-form comes from such a Hermitian holomorphic line bundle? That's main theorem for this section.

Theorem 2.5.1 (Lefschetz $(1,1)$-theorem)**.** *X is a complex manifold, $\omega \in C^\infty(X, \Omega^2_{X,\mathbb{R}}) \cap C^\infty(X, \Omega^{1,1}_X)$, a real $(1,1)$-form, such that $\mathrm{d}\omega = 0$. And*

$$[\omega] \in \mathrm{Im}(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R}))$$

Then there exists a Hermitian holomorphic line bundle (L, h) such that

$$\frac{i}{2\pi} H_L = \omega$$

Before proving this theorem, let's elaborate what does the following map mean

$$H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R})$$

since in de Rham cohomology, it's meaningless to say cohomology with $\mathbb{Z}$ coefficient. Here we use comparision $H^2(X,\mathbb{R}) \cong \check{H}^2(X,\underline{\mathbb{R}})$, where $\underline{\mathbb{R}}$ is the sheaf of local constant $\mathbb{R}$-functions, and prove in an explict method, since later we will use it.

In sketch, the philosophy of this method is that we descend the degree of differential forms, but the price is we need to consider functions defined on intersections of many open subsets.

X is a differential manifold, and $Z^1 \subset \Omega^1_{X,\mathbb{R}}$, sheaf of closed 1-form. Then we have the following exact sequence of sheaves.

$$0 \to \underline{\mathbb{R}} \to C^\infty \xrightarrow{d} Z^1 \to 0$$

Locally constant functions are clearly C^∞ functions, such that d acts on them is zero, so the exactness for the first two is trivial. But for the last one, it is equivalent to that a closed form locally must be an exact form, that's Poincaré lemma.

Similarly, define $Z^2 \subset \Omega^2_{X,\mathbb{R}}$, sheaf of closed 2-forms. then

$$0 \to Z^1 \to \Omega^1_{X,\mathbb{R}} \xrightarrow{\mathrm{d}} Z^2 \to 0$$

This sequence is exact for the same reason.

By the definition of de Rham cohomology, we have

$$H^2(X,\mathbb{R}) = \frac{C^\infty(X, Z^2)}{\mathrm{d}C^\infty(X, \Omega^1_{X,\mathbb{R}})}$$

In order to avoid the limit in the definition of Čech cohomology, we take open covering $\mathcal{U}=\{U_\alpha\}$ good enough, such that

$$\mathrm{d}: C^\infty(U_\alpha,\Omega^1_{U_\alpha,\mathbb{R}})\to C^\infty(U_\alpha,Z^2)$$

is surjective for any α. And

$$\mathrm{d}: C^\infty(U_\alpha\cap U_\beta)\to C^\infty(U_\alpha\cap U_\beta,Z^1)$$

is surjective for any α,β.

If ω is a closed real 2-form, i.e. $[\omega]\in H^2(X,\mathbb{R})$. For any α, choose $A_\alpha\in C^\infty(U_\alpha,\Omega^1_{U_\alpha,\mathbb{R}})$ such that

$$\omega|_{U_\alpha}=\mathrm{d}A_\alpha$$

then

$$\prod_{\alpha,\beta}(A_\alpha-A_\beta)$$

is a Čech 1-cocchain in $C^1(\mathcal{U},Z^1)$, it's d closed since $\mathrm{d}(A_\alpha-A_\beta)|_{U_\alpha\cap U_\beta}=\omega-\omega=0$.

For any α,β, choose $f_{\alpha\beta}\in C^\infty(U_\alpha\cap U_\beta)$, such that

$$(A_\alpha-A_\beta)_{\alpha\beta}=\mathrm{d}f_{\alpha\beta}$$

then

$$f_{\beta\gamma}-f_{\alpha\gamma}+f_{\alpha\beta}|_{U_\alpha\cap U_\beta\cap U_\gamma}$$

is closed, hence locally constant,

$$\breve{\omega}=\prod_{\alpha,\beta,\gamma}(f_{\beta\gamma}-f_{\alpha\gamma}+f_{\alpha\beta})$$

is a Čech 2-cocycle, in $C^2(\mathcal{U},\underline{\mathbb{R}})$. We have $\delta\breve{\omega}=0$, and $[\omega]$ corresponds to $[\breve{\omega}]$, that's the explict construction for comparision theorem in dimension 2. In fact, the general case is proved in the same method.

And we also the some lemmas in multiply complex analysis.

Lemma 2.5.2. *Locally on a polydisk $D\subset\mathbb{C}^n$, and $\omega\in C^\infty(D,\Omega^2_{D,\mathbb{R}}\cap C^\infty(D,\Omega^{1,1}_D)$ is a* d *closed real* $(1,1)$*-form. Then there exists a smooth function $\varphi: D\to\mathbb{R}$ such that*

$$\omega=i\partial\overline{\partial}\varphi$$

Proof. Poincaré lemma implies that $\omega=\mathrm{d}A=\mathrm{d}(A^{1,0}+A^{0,1})=(\partial+\overline{\partial})(A^{1,0}+A^{0,1})$, and since A is real, then $\overline{A^{1,0}}=A^{0,1}$.

Since ω is a $(1,1)$-form, then

$$\begin{cases}\partial A^{1,0}=0\\ \overline{\partial}A^{0,1}=0\\ \omega=\overline{\partial}A^{1,0}+\partial A^{0,1}\end{cases}$$

Dolbeault lemma implies that $A^{0,1} = \overline{\partial} f$, so $A^{1,0} = \partial \overline{f}$, so we have

$$\begin{aligned} \omega &= \overline{\partial}\partial\overline{f} + \partial\overline{\partial} f \\ &= \partial\overline{\partial}(f - \overline{f}) \\ &= i\partial\overline{\partial}\varphi \end{aligned}$$

□

Lemma 2.5.3. *Locally on $U \subset \mathbb{C}^n$, a simply connected open subset, and a smooth function $\varphi : U \to \mathbb{R}$, such that $\partial\overline{\partial}\varphi = 0$[16]. Then there exists a holomorphic functions $f : U \to \mathbb{C}$, such that $\varphi = \mathrm{Re}(f)$.*

Now let's prove Lefschetz $(1,1)$-theorem

Proof. Let's first see how does the above two lemmas play a role in our proof. We will choose a good enough open cover $\mathcal{U} = \{U_\alpha\}$ of open polydisk such that for all α, β, we have $U_\alpha \cap U_\beta$ is simply connected.

Since ω is a d closed real $(1,1)$-form, Lemma 2.4.2 implies that there exists smooth function $\varphi_\alpha : U_\alpha \to \mathbb{R}$ such that

$$\omega|_{U_\alpha} = \frac{i}{2\pi}\partial\overline{\partial}\varphi_\alpha$$

On any two intersection $U_\alpha \cap U_\beta$, we have $\partial\overline{\partial}(\varphi_\alpha - \varphi_\beta) = 0$, then Lemma 2.4.3 implies that there exists a holomorphic function $f_{\alpha\beta}$, such that

$$(\varphi_\alpha - \varphi_\beta)|_{U_\alpha \cap U_\beta} = 2\,\mathrm{Re}(f_{\alpha\beta}) = f_{\alpha\beta} + \overline{f_{\alpha\beta}}$$

Consider $\prod f_{\alpha\beta} \in C^1(\mathcal{U}, \mathcal{O}_X)$, then

$$(\delta f)_{\alpha\beta\gamma} = (f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta})|_{U_\alpha \cap U_\beta \cap U_\gamma}$$

Note that $2\,\mathrm{Re}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta})_{\alpha\beta\gamma} = 0$, so it must be a locally constant imaginary number, i.e. it lies in $2\pi i\underline{\mathbb{R}}(U_\alpha \cap U_\beta \cap U_\gamma)$.

Consider real form[17]

$$A_\alpha = \frac{i}{4\pi}(\overline{\partial}\varphi_\alpha - \partial\varphi_\alpha)$$

and by directly computing, we can note that $\omega|_{U_\alpha} = \mathrm{d}A_\alpha$, and that's why we define A_α in this method.

Similar to what we have done in the proof of comparision theorem, we want to consider $A_\alpha - A_\beta$ on the intersection $U_\alpha \cap U_\beta$. So we compute the difference of each term of A_α and A_β as follows

$$\begin{aligned} \partial(\varphi_\beta - \varphi_\alpha) &= \partial(f_{\alpha\beta} + \overline{f_{\alpha\beta}}) \\ &= \partial f_{\alpha\beta} \\ &= \mathrm{d} f_{\alpha\beta} \end{aligned}$$

[16] Such φ is called pluriharmonic

[17] Here we need to consider some queer coefficients, in order to get a beautiful result. In fact, we need to use $e^{2\pi i} = 1$, a god given formula.

Similarly we have

$$\overline{\partial}(\varphi_\beta - \varphi_\alpha) = \mathrm{d}\overline{f_{\alpha\beta}}$$

then

$$(A_\beta - A_\alpha)_{\alpha\beta} = \frac{i}{4\pi}\mathrm{d}(\overline{f_{\alpha\beta}} - f_{\alpha\beta}) = \frac{1}{2\pi}\mathrm{d}(\mathrm{Im}(f_{\alpha\beta}))$$

Via $H^2(X,\mathbb{R}) \cong \check{H}^2(X,\mathbb{R})$, $[\omega]$ corresponds to $[\check{\omega}]$, above process is just what we have done in the proof of comparision theorem, so we have

$$\begin{aligned}\check{\omega} &= \prod(\frac{1}{2\pi}\mathrm{Im}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}))_{\alpha\beta\gamma} \\ &= \prod(\frac{1}{2\pi i}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}))_{\alpha\beta\gamma}\end{aligned}$$

Hypothesis tells that $[\check{\omega}]$ is an image of $[\prod n_{\alpha\beta\gamma}] \in \check{H}^2(X,\underline{\mathbb{Z}})$. However, it doesn't mean that $f_{\alpha\beta}$ are exactly integers, but not too bad, we just need some correction terms, that is

$$\prod(\frac{1}{2\pi i}(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}))_{\alpha\beta\gamma} = \prod n_{\alpha\beta\gamma} + \delta(\prod c_{\alpha\beta})$$

where $\prod(c_{\alpha\beta})$ is real 1-cochain.

So we set $f'_{\alpha\beta} = f_{\alpha\beta} - 2\pi i c_{\alpha\beta}$. Then

$$\frac{1}{2\pi i}(f'_{\beta\gamma} - f'_{\alpha\gamma} + f'_{\alpha\beta})_{\alpha\beta\gamma} = 2\pi i n_{\alpha\beta\gamma} \in 2\pi i\underline{\mathbb{Z}}(U_\alpha \cap U_\beta \cap U_\gamma)$$

Note that $e^{2\pi i} = 1$, Then consider $g_{\alpha\beta} = \exp(-f'_{\alpha\beta})$, a holomorphic from $U_\alpha \cap U_\beta$ to $\mathbb{C}^*$, it satisfies the cocycle condition

$$g_{\beta\gamma}g_{\alpha\gamma}^{-1}g_{\alpha\beta} = 1$$

so we get a holomorphic line bundle L.

Remark 2.5.4. It's important to keep in mind vector bundles are encoded in their gluing data, and you can regard it as an element in $\check{H}^1$. So if we want to get a holomorphic line bundle, we need to determine its transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ satisfying the cocycle conditions, that is, $\prod g_{\alpha\beta} \in C^1(\mathcal{U},\mathcal{O}_X^*)$ such that $\delta(\prod g_{\alpha\beta}) = 1$, i.e. $[\prod g_{\alpha\beta}] \in \check{H}^1(X,\mathcal{O}_X^*) = \mathrm{Pic}(X)$. That's what Example 1.7.9 already tells us.

Now we need to give a Hermitian metric on this holomorphic line bundle H, and calculate its curvature to complete the proof.

Note that

$$(\varphi_\alpha - \varphi_\beta)_{U_\alpha\cap U_\beta} = 2\,\mathrm{Re}(f_{\alpha\beta}) = 2\,\mathrm{Re}(f_{\alpha\beta})' \quad = -\log|g_{\alpha\beta}|^2$$

then we get a Hermitian metric

$$H_\alpha = -\exp(-\varphi_\alpha), \quad \text{on } U_\alpha$$

Indeed, since $H_\beta = |g_{\alpha\beta}|^2 H_\alpha = g_{\alpha\beta}^T H_\alpha \overline{g_{\alpha\beta}}$.

Finally,

$$\frac{i}{2\pi}H_L = \frac{i}{2\pi}\partial\overline{\partial}\varphi_\alpha = \omega$$

This completes the proof. $\square$

Remark 2.5.5. Now if we already have a holomorphic line bundle L, determined by its transition functions $g_{\alpha\beta}$. We can try to reverse what we have done above. That is, take its logarithm, consider its alternating sum and divide it by $2\pi i$, then we get an element in $\check{H}^2(X, \mathbb{R})$, and that's exact $-c_1$, where c_1 is the first Chern class.

However, we can rephrase it as a basical operation in homological algebra, consider the exponential sequence

$$0 \to \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1$$

Taking cohomology we will get a boundary map ∂

$$\mathrm{Pic}(X) \cong \check{H}^1(X, \mathcal{O}_X^*) \xrightarrow{\partial} \check{H}^2(X, \underline{\mathbb{Z}}) \to \check{H}^2(X, \underline{\mathbb{R}})$$

that's just what we have done, so this boundary map sometimes is denoted by $-c_1$.

2.6. **Hypersurface and Divisors.** This section we will briefly introduce some definitions and theorems about hypersurfaces and divisors wihout proofs.

Definition 2.6.1 (hypersurface)**.** *X is a complex manifold, a hypersurface of X is a closed subset $D \subset X$ such that for all $x \in D$, there exists an open subset $U \subset X$ containing x and a nonzero holomorphic function $f : U \to \mathbb{C}$ such that*

$$D \cap U = \{x \in U \mid f(x) = 0\}$$

Remark 2.6.2. $x \in D$ is called smooth, if we can choose $f : D \to \mathbb{C}$ is a submersion. The set of all smooth point is denoted by D_{sm}; If $D_{sm} = D$, then $D_{sm} = D$. And note that we do not assume D is connected.

Exercise 2.6.3. Let $D \subset X$ be a smooth hypersurface, then there exists an open covering $\{U_\alpha : f_\alpha \to \mathbb{C}\}$ where f_α is holomorphic submersion, such that

$$D \cap U_\alpha = \{x \in U_\alpha \mid f_\alpha(x) = 0\}$$

Then $g_{\alpha\beta} = f_\alpha / f_\beta : U_\alpha \cap U_\beta \to \mathbb{C}^*$ is a holomorphic function. Then we get a holomorphic line bundle $\mathcal{O}_X(D)$ on X. In particular, if we take $X = \mathbb{P}^n, D = \mathbb{P}^{n-1}$, then $\mathcal{O}_X(D) = \mathcal{O}_{\mathbb{P}^n}(1)$.

In fact, we can drop the assumption of "smoothness" by some technical method.

Lemma 2.6.4. *If $D \subset X$ is a hypersurface, then $D_{sm} \subset D$ is open and dense.*

Proposition 2.1. *If $D \subset X$ is a hypersurface, then there exists a holomorphic line bundle $\mathcal{O}_X(D)$ on X with global section σ, such that $D = \{x \in X \mid \sigma(x) = 0\}$*

Proof. Sketch. Set $Z = D\backslash D_{sm}$. Then $D_{sm} \subset X\backslash Z$ is a smooth hypersurface. Then we get transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \backslash Z \to \mathbb{C}^*$, so we get a holomorphic line bundle over $X\backslash Z$. Then Lemma 3.5.4 tells us Z contains no hypersurface of X, and Hartgos theorem tells us we can extend this holomorphic function $g_{\alpha\beta}$ to $U_\alpha \cap U_\beta$. So we define a holomorphic line bundle. □

Definition 2.6.5 (irreducible hypersurface)**.** *A hypersurface $D \subset X$ is called irreducible, if it can not be written as union of two hypersurface.*

Definition 2.6.6 (divisor)**.** *X is a complex manifold, a divisor on X is a finite formal sum*

$$D = \sum_i a_i D_i$$

where $a_i \in \mathbb{Z}$ and D_i is a irreducible hypersurface. And formally we can define

$$\mathcal{O}_X(D) := \bigotimes_i \mathcal{O}_X(D_i)^{\otimes a_i}$$

Remark 2.6.7. We have seen that from divisors we can get a holomorphic line bundle. So it's natural to guess all holomorphic line bundle are arised in this form. However, it's false.

Example 2.6.8. There exists a complex torus with no hypersurface, but there is non trivial holomorphic line bundle on it.

But

Theorem 2.6.9. *If X is a projective manifold, then for any holomorphic line bundle L, there exists a divisor D ($D = D_1 - D_2$, and D_1, D_2 are hypersurface in X), such that*

$$L \cong \mathcal{O}_X(D)$$

Let X be a compact complex manifold of dimension n, and $D \subset X$ is a smooth hypersurface, we can define

$$\begin{aligned} D : C^\infty(X, \Omega^{2n-2}_{X,\mathbb{R}}) &\to \mathbb{R} \\ \omega &\mapsto \int_D \omega|_D \end{aligned}$$

and if ω is a exact form, then $\int_D \omega|_D = 0$ by Stokes.

So we get a $[D] : H^{2n-2}(X, \mathbb{R}) \to \mathbb{R}$, then Poincaré duality tells us we get $[D] \in H^2(X, \mathbb{R})$.

Surprisingly,

Theorem 2.6.10 (Lelong-Poincaré)**.** *X is a compact complex manifold, $D \subset X$ is a smooth hypersurface, then*

$$[D] = c_1(\mathcal{O}_X(D)) \in H^2(X, \mathbb{R})$$

Remark 2.6.11. We can also drop smoothness condition. We need to make sense of $[D] \in H^2(X, \mathbb{R})$, i.e. we need to check the following integral make sense:

$$\int_{D_{sm}} \omega|_{D_{sm}}$$

It's not trivial, since D_{sm} is just an open subset, and integral over an open subset may be quite bad.

Part 2. Hodge theory

3. Kähler manifold

3.1. Definitions and Examples.

Definition 3.1.1 (Kähler manifold)**.** *Let X be a complex manifold, h is a positive Hermitian metric on TX, and ω is the real $(1,1)$-form corresponding to h. X is called a Kähler manifold, if $\mathrm{d}\omega = 0$.*

Remark 3.1.2. Note that the condition $\mathrm{d}\omega = 0$ is equivalent to $\partial\omega = 0$, and is also equivalent to $\overline{\partial}\omega = 0$.

Remark 3.1.3. Our definition of Kähler manifold is the complex Hermitian viewpoint. But Kähler manifold in fact is an intersection of three interesting objects: Complex manifold, Symplectic manifold and Riemannian manifold.

Let's see from the symplectic viewpoint: If (X, ω) is a symplectic manifold, where X is a differential manifold and ω is a d closed real non-degenerate symplectic form. (X, ω) is called a Kähler manifold, if there exists a integrable almost complex structure J on $T_{X,\mathbb{R}}$ such that $g(u, v) = \omega(w, Ju)$ is positive definite, that is g is a Riemannian metric.

Also, we can define Kähler manifold from the differential geometry viewpoint. Let (X, g) be a Riemannian manifold, where g is a Riemannian metric. (X, g) is called Kähler if there exists a integrable almost complex structure J on $T_{X,\mathbb{R}}$, satisfying $g(Ju, Jv) = g(u, v)$ and preserved by parallel transport with respect to Levi-Civita connection.

Anyway, the hallmarks of a Kähler manifold are "complex", "positive" and "closed".

Remark 3.1.4. For a Kähler manifold X, we have J, h, ω, g on it. Now we want to elaborate to how do these things connect with each other. One thing we need to keep in mind all the way is the identification $T_{X,x} \longleftrightarrow T_{X,\mathbb{R},x}$. We draw a diagram as follows

$$\begin{tikzcd}
& h \arrow[dl, bend right, "1"'] \arrow[dr, bend left, "3"] & \\
\omega \arrow[ur, bend right, "2"] \arrow[rr, "5"] & & g \arrow[ul, bend left, "4"'] \arrow[ll, bend left, "6"]
\end{tikzcd}$$

And the explict correspondence is listed as follows

$$\begin{array}{ll} 1 & \omega(u,v) = -\operatorname{Im} h(u,v) \\ 2 & h(u,v) = \omega(u,Jv) - i\omega(u,v) \\ 3 & g(u,v) = \operatorname{Re} h(u,v) \\ 4 & h(u,v) = g(u,v) - ig(Ju,v) \\ 5 & g(u,v) = \omega(u,Jv) \\ 6 & \omega(u,v) = g(Ju,v) \end{array}$$

Example 3.1.5. Any complex curve[18] X is Kähler. Since $\mathrm{d}\omega = 0$ automatically holds.

Example 3.1.6. If X admits a positive holomorphic line bundle, then X is Kähler, since we can take ω to be its first Chern class. In particular, $\mathbb{P}^n$ is Kähler, since $\mathcal{O}_{\mathbb{P}^n}(1)$ is a positive holomorphic line bundle of it, with respect to Fubini-Study metric.

Exercise 3.1.7. Show that a submanifold of a Kähler manifold is still Kähler. In particular, any projective manifold is Kähler.

Exercise 3.1.8. If (X,ω) is a Kähler manifold with $\dim_{\mathbb{C}} X = n$. Show that $\dfrac{\omega^n}{n!}$ is the volume form of X as a Riemannian manifold with respect to g. Furthermore, since $\mathrm{d}\omega = 0$, then $\mathrm{d}(\omega^k) = 0, 0 \leq k \leq n$, then

$$[\omega^k] \in H^{2k}(X,\mathbb{R})$$

Deduce that if X is a compact Kähler manifold, then $[\omega^k] \neq 0$ for all $0 \leq k \leq n$. So $H^{2k}(X,\mathbb{R}) \neq 0$.

Proof. In fact, $\dfrac{\omega^n}{n!} = \mathrm{vol}$ holds for any complex manifold with a positive Hermitian metric h, the Kähler condition does not really come into play, since it's just a problem of linear algebra. Let's begin by fixing a point $p \in X$ and a frame for the T_pX, orthonormal with respect to the Riemannian metric g defined by h on X, call it $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}$, and use $\mathrm{d}x_1, \mathrm{d}y_1, \dots, \mathrm{d}x_n, \mathrm{d}y_n$ to denote its dual, and $\frac{\partial}{\partial z_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}), \mathrm{d}z_i = \mathrm{d}x_i + i\mathrm{d}y_i$ By Remark 2.3.10, we have

$$\omega_p = \frac{i}{2}\sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\overline{z_j}$$

Since the volume form is just

$$\mathrm{vol}_p = \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n$$

and compare it with ω^n, directly compute

$$\omega_p^n = n!\frac{i^n}{2^n}\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\overline{z_n}$$

[18]In other words, a Riemann surface

we can compute $n = 2$ to feel what's going on:

$$\begin{aligned}\omega_p^2 &= (\frac{i}{2})^2(\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} + \mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2}) \wedge (\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} + \mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2}) \\ &= (\frac{i}{2})^2 \mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} \wedge \mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2} + \mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2} \wedge \mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} \\ &= 2(\frac{i}{2})^n \mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} \wedge \mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2}\end{aligned}$$

Use the relation

$$\mathrm{d}z_j \wedge \mathrm{d}\overline{z_j} = -2i\mathrm{d}x_j \wedge \mathrm{d}y_j$$

we get the desired result. □

As we can see from the definition of Kähler manifold, all of the requirements are local, but from the above exercise, we can see a surprising thing, that is the cohomology groups with even dimension must be non-trivial, it's a global result.

To some extent, this reflects the philosophy of Hodge theory, that is how does locally good property control global cohomology. Kähler is a locally good property, and the following theorem may cultivate you such an intuition.

Theorem 3.1.9. *X is a Kähler manifold, then locally around $x \in U \subset X$, we can choose a holomorphic coordinate $(\xi_1, \ldots, \xi_n)$ such that $h_{jk} = \delta_{jk} + O(|\xi|^2)$*

Proof. With linear change of coordinate, we can assume

$$\omega = \frac{i}{2}\sum_{jk} h_{jk}\mathrm{d}z_j \wedge \mathrm{d}\overline{z_k}$$

where $h_{jk} = \delta_{jk} + O(|\xi|)$, that is

$$h_{jk} = \delta_{jk} + \sum_l (a_{jkl}z_l + a'_{jkl}\overline{z_l}) + O(|\xi|^2)$$

What we need to do is to kill the first order term. Since h_{jk} is Hermitian, that is $h_{jk} = \overline{h_{kj}}$. So we have

$$\overline{a_{kjl}} = a'_{jkl} \tag{3.1}$$

To get above result we only use the fact that h_{jk} is Hermitian, but we also have that ω is also ∂ closed, we compute directly

$$\partial\omega = \frac{i}{2}\sum_{jkl}\frac{\partial h_{jk}}{\partial z_l}\mathrm{d}z_l \wedge \mathrm{d}z_j \wedge \mathrm{d}\overline{z_k}$$

If we want $\partial\omega = 0$, we need the coefficients of $\mathrm{d}z_l \wedge \mathrm{d}z_j$ and $\mathrm{d}z_j \wedge \mathrm{d}z_l$ are equal so that they can cancel with each other, that is

$$\frac{\partial h_{jk}}{\partial z_l} = \frac{\partial h_{lk}}{\partial z_j} \implies a_{jkl} = a_{lkj}$$

Set $\xi_k = z_k + \frac{1}{2}\sum_{jl} a_{jkl} z_j z_l$, this is a holomorphic change of coordinate, so $\xi_1, \dots, \xi_n$ is also a holomorphic coordinate. Then

$$\begin{aligned}
\mathrm{d}\xi_k &= \mathrm{d}z_k + \frac{1}{2}\sum_{jl} a_{jkl}(z_l \mathrm{d}z_j + z_j \mathrm{d}z_l) \\
&= \mathrm{d}z_k + \frac{1}{2}\sum_{jl}(a_{jkl} + a_{lkj}) z_l \mathrm{d}z_j \\
&= \mathrm{d}z_k + \sum_{jl} a_{jkl} z_l \mathrm{d}z_j
\end{aligned}$$

So we have

$$\frac{i}{2}\sum_k \mathrm{d}\xi_k \wedge \mathrm{d}\overline{\xi_k} = \frac{i}{2}\sum_k \mathrm{d}z_k \wedge \mathrm{d}\overline{z_k} + \frac{i}{2}\sum_{jkl}(\overline{a_{jkl} z_l}\mathrm{d}z_k \wedge \mathrm{d}\overline{z_j} + a_{jkl} z_l \mathrm{d}z_j \wedge \mathrm{d}\overline{z_k}) + O(|\xi|^2)$$

By (3.1), we have

$$\sum_{jkl} \overline{a_{jkl} z_l}\mathrm{d}z_k \wedge \mathrm{d}\overline{z_j} = \sum_{jkl} \overline{a_{kjl} z_l}\mathrm{d}z_j \wedge \mathrm{d}\overline{z_k} = \sum_{jkl} a'_{jkl}\overline{z_l}\mathrm{d}z_j \wedge \mathrm{d}\overline{z_k}$$

So we have

$$\begin{aligned}
\frac{i}{2}\sum_k \mathrm{d}\xi_k \wedge \mathrm{d}\overline{\xi_k} &= \frac{i}{2}\sum_{jk}(\delta_{jk} + \sum_l a_{jkl} z_l + a'_{jkl}\overline{z_l})\mathrm{d}z_j \wedge \mathrm{d}\overline{z_k} + O(|\xi|^2) \\
&= \omega + O(|\xi|^2)
\end{aligned}$$

This completes the proof. □

3.2. **Differential operators.** Now let's introduce some functionals, since we can do this when we have metrics. Indeed, we have metrics on vector bundles.

If X is an oriented differential manifold, with $\dim_{\mathbb{R}} X = n$. (E, g_E) is an Euclidean real vector bundle, that is

$$\begin{aligned}
g_E : C^\infty(X, E) \times C^\infty(X, E) &\to C^\infty(X) \\
(\alpha, \beta) &\mapsto \{\alpha, \beta\}
\end{aligned}$$

a bilinear mapping.

Suppose there exists a Riemannian metric g on X, then we have a volume form vol with respect to g. Use such volume form, we can define L^2-inner product on $C_c^\infty(X, E)$ as

$$(\alpha, \beta)_{L^2} = \int_X \{\alpha, \beta\}\,\mathrm{vol}$$

For any two Euclidean real vector bundles $(E, g_E), (F, g_F)$, and two linear operators

$$P : C_c^\infty(X, E) \to C_c^\infty(X, F), \quad P^* : C_c^\infty(X, F) \to C_c^\infty(X, E)$$

We say that P and P^* are formally adjoints[19] if

$$(P\alpha,\beta)_{L^2} = (\alpha, P^*\beta)_{L^2}, \quad \forall \alpha \in C^\infty(X,E), \beta \in C^\infty(X,F)$$

We are interested in the case $E = \Omega^k_{X,\mathbb{R}}$, and $P = \mathrm{d}$, that is

$$\mathrm{d} : C^\infty_c(X, \Omega^k_{X,\mathbb{R}}) \to C^\infty_c(X, \Omega^{k+1}_{X,\mathbb{R}})$$

Claim that in this case, its adjoint do exists

$$\mathrm{d}^* : C^\infty_c(X, \Omega^{k+1}_{X,\mathbb{R}}) \to C^\infty_c(X, \Omega^k_{X,\mathbb{R}})$$

To define this, we need the Hodge star operator. Let's first make it clear in the case of vector space.

Example 3.2.1. Let V be an oriented n-dimensional Euclidean vector space, with inner product $\langle\ ,\ \rangle$, and let $W = V^*$. There exists a canonical volume form $\mathrm{vol} \in \bigwedge^n W \cong \mathbb{R}$. More explictly, if $\{e_1,\dots,e_n\}$ is a orthonormal basis of V, and $\{e^1,\dots,e^n\}$ is the dual basis in W, then canonical volume form is $e^1 \wedge \dots \wedge e^n$.

From linear algebra we already know the wedge product

$$\bigwedge^k W \times \bigwedge^{n-k} W \xrightarrow{\wedge} \bigwedge^n W$$

is non-degenerate. So for any $\beta \in \bigwedge^k W$, we can define $*\beta \in \bigwedge^{n-k} W$ such that

$$\alpha \wedge *\beta = \langle \alpha, \beta\rangle\, \mathrm{vol}, \quad \forall \alpha \in \bigwedge^k W$$

where $\langle\ ,\ \rangle$ is the inner product induced from W, and the one on W is induced from V.

Clearly we have $*1 = \mathrm{vol}$ and $*\,\mathrm{vol} = 1$. Indeed, by definition, take $1, \alpha \in \bigwedge^0 W \cong \mathbb{R}$, then

$$\alpha * 1 = \alpha \wedge *1 = \alpha\, \mathrm{vol} \implies *1 = \mathrm{vol}$$

Similar for $*\,\mathrm{vol}$. Furthermore,

$$*e^i = (-1)^{i-1} e^1 \wedge \dots \widehat{e^i} \wedge \dots \wedge e^n$$

To show this, we also need to back to definition. For any $\alpha \in W$, write it as $\alpha = \sum_{i=1}^n a_i e^i$, then

$$\begin{aligned} \alpha \wedge *e^i &= \langle \alpha, e^i\rangle\, \mathrm{vol} \\ &= \langle \sum_{i=1}^n a_i e^i, e^i \rangle\, \mathrm{vol} \\ &= a_i\, \mathrm{vol} \\ &= a_i e^1 \wedge \dots \wedge e^n \end{aligned}$$

From this equation, it's easy to see what $*e^i$ is exactly.

[19] In order to avoid quite hard functional analysis, we use such formal definition, but we will see later, in our interested case, such adjoints really exist.

Last but not least,

$$**=*^2=(-1)^{k(n-k)}\,\mathrm{id},\quad \text{on } \bigwedge^k W$$

the proof of it is also a routine, we omit it here.

We can carry what we have done to bundles of differential forms, since differential forms are just covector spaces living on a manifold smoothly.

Definition 3.2.2 (Hodge star operator)**.** *There exists*

$$*: C^\infty(X,\Omega^k_{X,\mathbb{R}})\to C^\infty(X,\Omega^{n-k}_{X,\mathbb{R}})$$

such that

$$\alpha\wedge *\beta=\{\alpha,\beta\}\,\mathrm{vol},\quad \forall\alpha\in C^\infty(X,\Omega^k_{X,\mathbb{C}})$$

Remark 3.2.3. In particular, if $\alpha,\beta\in C^\infty_c(X,\Omega^k_{X,\mathbb{R}})$, then

$$(\alpha,\beta)_{L^2}=\int_X\alpha\wedge *\beta$$

Lemma 3.2.4. $\mathrm{d}^*=(-1)^{nk+1}*\mathrm{d}*: C^\infty(X,\Omega^{k+1}_{X,\mathbb{C}})\to C^\infty(X,\Omega^k_{X,\mathbb{C}})$

Proof. For $\alpha\in C^\infty_c(X,\Omega^k_{X,\mathbb{R}})$ and $\beta\in C^\infty_c(X,\Omega^{k+1}_{X,\mathbb{R}})$, then

$$\begin{aligned}(\mathrm{d}\alpha,\beta)_{L^2}&=\int_X \mathrm{d}\alpha\wedge *\beta\\ &=\int_X \mathrm{d}(\alpha\wedge *\beta)-(-1)^k\alpha\wedge\mathrm{d}*\beta\\ &=(-1)^{k+1}\int_X\alpha\wedge\mathrm{d}*\beta\\ &=(-1)^{k+1}(-1)^{k(n-k)}\int_X\alpha\wedge **\mathrm{d}*\beta\\ &=(-1)^{nk+1}(\alpha,*\mathrm{d}*\beta)_{L^2}\end{aligned}$$

□

Although we have a formula for d^*, we can have a more computable formula for d^* in trivial case.

Recall contraction or interior product: Given a $\theta\in C^\infty(X,T_{X,\mathbb{R}})$ and $u\in C^\infty(X,\Omega^k_{X,\mathbb{R}})$, we have

$$\iota_\theta u\in C^\infty(X,\Omega^{k-1}_{X,\mathbb{R}})$$

defined by $\iota_\theta(u)(v_1,\dots,v_{k-1})=u(\theta,v_1,\dots,v_{k-1})$. Locally, if $\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}$ is a local frame of $T_{X,\mathbb{R}}$, then

$$\iota_{\frac{\partial}{\partial x_m}}(\mathrm{d}x_{j_1}\wedge\cdots\wedge\mathrm{d}x_{j_k})=\begin{cases}0, & m\notin\{j_1,\dots,j_k\}\\ (-1)^{l-1}\mathrm{d}x_{j_1}\wedge\dots\widehat{\mathrm{d}x_{j_l}}\wedge\cdots\wedge\mathrm{d}x_{j_k}, & m=j_l\end{cases}$$

Furthermore, ι_θ satisfies the Leibniz rule.

Let $U \subset \mathbb{R}^n$ with standard Euclidean metric on $T_{U,\mathbb{R}}$, write $u = \sum_{|J|=k} u_J \mathrm{d}x_J$. Then

$$\mathrm{d}^* u = -\sum_{l=1}^{n} \sum_{|J|=k} \frac{\partial u_J}{\partial x_l} \iota_{\frac{\partial}{\partial x_l}} \mathrm{d}x_J$$

Let's check case $n = 1$ for an example. Take a 1-form $u = f\mathrm{d}x$, by definition $\mathrm{d}^* = (-1) * \mathrm{d} * = - * \mathrm{d} *$. And take any other 1-form $g\mathrm{d}x$

$$g\mathrm{d}x \wedge *(f\mathrm{d}x) = \langle g\mathrm{d}x, f\mathrm{d}x \rangle \mathrm{d}x = gf\mathrm{d}x \implies *(f\mathrm{d}x) = f$$

So we compute as follows

$$\begin{aligned} - * \mathrm{d} * u &= - * \mathrm{d} * (f\mathrm{d}x) \\ &= - * \mathrm{d}f \\ &= - * (\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x) \\ &= -\frac{\mathrm{d}f}{\mathrm{d}x} \end{aligned}$$

as desired.

Now, let's consider about n dimensional complex manifold X, endowed with a Hermitian metric h, and ω is the real $(1,1)$-form corresponds to h. As we have seen in Exercise 3.18, we have

$$\mathrm{vol} = \frac{\omega^n}{n!}$$

with respect to g.

Since we have the following decomposition

$$\Omega^k_{X,\mathbb{C}} = \bigoplus_{p+q=k} \Omega^{p,q}_X$$

You can guess that we can descend our Hodge star operator to these $\Omega^{p,q}_X$.

Definition 3.2.5 (Hodge star operator)**.** *Hodge star operator is a $\mathbb{C}$-linear operator*

$$* : C^\infty(X, \Omega^k_{X,\mathbb{C}}) \to C^\infty(X, \Omega^{2n-k}_{X,\mathbb{C}})$$

such that

$$\alpha \wedge *\overline{\beta} = \{\alpha, \beta\}\, \mathrm{vol}, \quad \forall \alpha, \beta \in C^\infty(X, \Omega^k_{X,\mathbb{C}})$$

Let's see how $*$ acts on $\Omega^{p,q}_X$. Consider $\alpha, \beta \in \Omega^{p,q}_X$, since we already know vol is a (n,n)-form. In order to get a (n,n)-form from $\alpha \wedge *\overline{\beta}$, we need $*\overline{\beta}$ is a $(n-p, n-q)$ form, that is, $*\beta$ is a $(n-q, n-p)$-form. So we will have

$$* : \Omega^{p,q}_X \cong \Omega^{n-q,n-p}_X$$

In other words, we have

$$* : C^\infty(X, \Omega^{p,q}_X) \to C^\infty(X, \Omega^{n-q,n-p}_X)$$

Use Hodge star operator, we can also define the adjoints of d, ∂ and $\overline{\partial}$.

$$\begin{aligned}
\mathrm{d}^* &= -*\mathrm{d}* \\
\partial^* &= -*\overline{\partial}* \\
\overline{\partial}^* &= -*\partial*
\end{aligned}$$

Similarly we can calculate them in a standard Hermitian metric.

Example 3.2.6. Take $U \subset \mathbb{C}^n$ with standard Hermitian metric. For any (p,q)-form u, we have

$$u = \sum_{|J|=p,|K|=q} = u_{JK}\mathrm{d}z_J \wedge \mathrm{d}\overline{z_K}$$

then we have

$$\begin{aligned}
\partial^* u &= -2\sum_{l=1}^{n}\sum_{|J|=p,|K|=q} \frac{\partial u_{JK}}{\partial \overline{z_l}} \iota_{\frac{\partial}{\partial z_l}} \mathrm{d}z_J \wedge \mathrm{d}\overline{z_K} \\
\overline{\partial}^* u &= -2\sum_{l=1}^{n}\sum_{|J|=p,|K|=q} \frac{\partial u_{JK}}{\partial z_l} \iota_{\frac{\partial}{\partial \overline{z_l}}} \mathrm{d}z_J \wedge \mathrm{d}\overline{z_K}
\end{aligned}$$

Definition 3.2.7 (Laplacians)**.**

$$\begin{aligned}
\Delta_{\mathrm{d}} &= \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d} \\
\Delta_{\partial} &= \partial\partial^* + \partial^*\partial \\
\Delta_{\overline{\partial}} &= \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}
\end{aligned}$$

Example 3.2.8. Take $U \subset \mathbb{R}^n$ with standard Euclidean metric. For $f \in C^\infty(U)$, we have

$$\begin{aligned}
\Delta_{\mathrm{d}} f &= \mathrm{d}\mathrm{d}^* f + \mathrm{d}^*\mathrm{d}f \\
&= \mathrm{d}^*\mathrm{d}f \\
&= \mathrm{d}^*(\sum_j \frac{\partial f}{\partial x_j}\mathrm{d}x_j) \\
&= -\sum_j \frac{\partial^2 f}{\partial x_j^2} \\
&= -\Delta f
\end{aligned}$$

That's why we call it Laplacian.

Definition 3.2.9 (harmonic)**.** *Let X be an oriented Riemannian manifold or a complex Hermitian manifold. A (compactly supported) form α is called $\Delta_\bullet$-harmonic if $\Delta_\bullet \alpha = 0$. Here $\bullet$ can be d, ∂ and $\overline{\partial}$.*

Lemma 3.2.10. *α is Δ_{d}-harmonic if and only if $\mathrm{d}\alpha = 0, \mathrm{d}^*\alpha = 0$. Same for ∂ and $\overline{\partial}$.*

Proof. Note that

$$\begin{aligned}(\alpha, \Delta_{\mathrm{d}}\alpha)_{L^2} &= (\alpha, \mathrm{d}\mathrm{d}^*\alpha)_{L^2} + (\alpha, \mathrm{d}^*\mathrm{d}\alpha)_{L^2} \\ &= \|\mathrm{d}^*\alpha\|^2 + \|\mathrm{d}\alpha\|^2\end{aligned}$$

□

Remark 3.2.11. This Lemma holds for ∂ and $\overline{\partial}$ using the same proof.

We're not going to study so much functionals. The reason we introduce above things is that they're neccessary for us to know what Hodge theorem is talking about.

The statement of the Hodge theorem is as follows, there are two versions, that is, real version and complex version. But for Hodge, they're the same things, just elliptic differential equations.

Firstly, let's see real version.

Theorem 3.2.12 (Hodge theorem)**.** *Let (X, g) be a compact oriented Riemannian manifold, with $\dim_{\mathbb{R}} X = n$. Let $\mathcal{H}^k$ denote the space of Δ_{d}-harmonic k-forms, a subset of $C^\infty(X, \Omega^k_{X,\mathbb{R}})$. Then*

1. *$\mathcal{H}^k$ is finite dimensional.*
2. *decomposition $C^\infty(X, \Omega^k_{X,\mathbb{R}}) = \mathcal{H}^k \oplus \Delta_{\mathrm{d}}(C^\infty(X, \Omega^k_{X,\mathbb{R}}))$. Furthermore, it is orthonormal with respect to $(\ , \ )_{L_2}$.*

Remark 3.2.13. Although we omit the proof of Hodge theorem here[20], we can see why $\mathcal{H}^k$ is orthonormal to $\Delta_d(C^\infty(X, \Omega^k_{X,\mathbb{R}}))$.

Take a harmonic k-form α and $\Delta_{\mathrm{d}}\beta$ where β is also a k-form. Then

$$\begin{aligned}(\alpha, \Delta_d\beta)_{L^2} &= (\alpha, \mathrm{d}\mathrm{d}^*\beta)_{L^2} + (\alpha, \mathrm{d}^*\mathrm{d}\beta)_{L^2} \\ &= (\mathrm{d}^*\alpha, \mathrm{d}^*\beta)_{L^2} + (\mathrm{d}\alpha, \mathrm{d}\beta)_{L^2} \\ &= (\mathrm{d}\mathrm{d}^*\alpha, \beta)_{L^2} + (\mathrm{d}^*\mathrm{d}\alpha, \beta)_{L^2} \\ &= (\Delta_{\mathrm{d}}\alpha, \beta)_{L^2} \\ &= 0\end{aligned}$$

Corollary 3.2.14. *More explictly, we have the following orthonormal decomposition*

$$C^\infty(X, \Omega^k_{X,\mathbb{R}}) = \mathcal{H}^k \oplus \mathrm{d}(C^\infty(X, \Omega^{k-1}_{X,\mathbb{R}})) \oplus \mathrm{d}^*(C^\infty(X, \Omega^{k+1}_{X,\mathbb{R}}))$$

Proof. It suffices to show $\mathrm{d}(C^\infty(X, \Omega^{k-1}_{X,\mathbb{R}}))$ is orthonormal to $\mathrm{d}^*(C^\infty(X, \Omega^{k+1}_{X,\mathbb{R}}))$, and the proof is quite easy. Take $\mathrm{d}\alpha$ and $\mathrm{d}^*\beta$, where α is a $k-1$-form and β is a $k+1$-form. Then

$$(\mathrm{d}\alpha, \mathrm{d}^*\beta)_{L_2} = (\mathrm{d}^2\alpha, \beta)_{L_2} = 0$$

□

[20] In fact, almost every textbook omits it.

Corollary 3.2.15.
$$\ker \mathrm{d} = \mathcal{H}^k \oplus \mathrm{d}^*(C^\infty(X, \Omega^{k-1}_{X,\mathbb{R}}))$$
$$\ker \mathrm{d}^* = \mathcal{H}^k \oplus \mathrm{d}(C^\infty(X, \Omega^{k-1}_{X,\mathbb{R}}))$$

Proof. Clear from Corollary 3.2.14. □

Corollary 3.2.16. *The natural map $\mathcal{H}^k \to H^k(X, \mathbb{R})$ is an isomorphism!*[21] *In other words, every element in $H^k(X, \mathbb{R})$ is represented by a unique Δ_{d}-harmonic form.*

Proof. Clear from Corollary 3.2.15. □

Corollary 3.2.17. $* : \mathcal{H}^k \to \mathcal{H}^{n-k}$ *is an isomorphism.*

Proof. It suffices to show $*$ maps harmonic forms to harmonic forms. By Lemma 3.2.10, we just need to show $\mathrm{d} * \alpha = \mathrm{d}^* * \alpha = 0$ for a harmonic form α.
$$\mathrm{d} * \alpha = (-1)^\bullet * * \mathrm{d} * \alpha = (-1)^\bullet * \mathrm{d}^*\alpha = 0$$
$$\mathrm{d}^* * \alpha = (-1)^\bullet * \mathrm{d} * * \alpha = (-1)^\bullet * \mathrm{d}\alpha = 0$$
Here we use $\bullet$ to denote the power of (-1), since it's not neccessary for us to know what exactly it is.

In fact, this corollary follows from the fact
$$\Delta_{\mathrm{d}} * = * \Delta_{\mathrm{d}}$$
which can be directly checked. □

Corollary 3.2.18 (Poincaré duality)**.** $H^k(X, \mathbb{R}) \cong H^{n-k}(X, \mathbb{R})$.

Proof. Clear from Corollary 3.2.16 and Corollary 3.2.17. □

From above corollaries of Hodge theorem, I think you've convinced yourself that Hodge theorem is really a quite meaningful theorem. Now let's elaborate complex version of it.

Theorem 3.2.19 (Hodge theorem)**.** *Let (X, h) be a compact complex Hermitian manifold with $\dim_{\mathbb{C}} X = n$. Let $\mathcal{H}^{p,q}$ be the space of $\Delta_{\overline{\partial}}$ harmonic forms of type (p, q), a subset of $C^\infty(X, \Omega^{p,q}_X)$. Then*
1. *$\mathcal{H}^{p,q}$ is finite dimensional.*
2. *decomposition $C^\infty(X, \Omega^{p,q}_X) = \mathcal{H}^{p,q} \oplus \Delta_{\overline{\partial}}(C^\infty(X, \Omega^{p,q}_X))$. Furthermore, it is orthonormal with respect to $(\ ,\ )_{L_2}$.*

The story here is the same as the real version.

Corollary 3.2.20. *More explictly, we have the following orthonormal decomposition*
$$C^\infty(X, \Omega^{p,q}_X) = \mathcal{H}^k \oplus \overline{\partial}(C^\infty(X, \Omega^{p,q-1}_X) \oplus \overline{\partial}^*(C^\infty(X, \Omega^{p,q+1}_X))$$

[21] Here we use exclamatory mark to show it's a surprising result.

Corollary 3.2.21.

$$\ker \overline{\partial} = \mathcal{H}^{p,q} \oplus \overline{\partial}^*(C^\infty(X, \Omega_X^{p,q-1}))$$
$$\ker \overline{\partial}^* = \mathcal{H}^{p,q} \oplus \overline{\partial}(C^\infty(X, \Omega_X^{p,q+1}))$$

Corollary 3.2.22. *The natural map $\mathcal{H}^{p,q} \to H^{p,q}_{Dol}(X)$ is an isomorphism! In particular, $H^{p,q}_{Dol}(X)$ is finite dimensional.*

Parallel to what have happened in the real version, we may desire that

$$* : \mathcal{H}^{p,q} \to \mathcal{H}^{n-q,n-p}$$

is an isomorphism. However, it fails generally, since we only have $\Delta_{\overline{\partial}} * = * \Delta_\partial$ and in general $\Delta_{\overline{\partial}} \neq \Delta_\partial$. There are two ways to fix it. The first way is that we will see later if X is compact Kähler manifold, then $\Delta_\partial = \Delta_{\overline{\partial}}$. Then

Corollary 3.2.23. *(X, ω) is a compact Kähler manifold with dimension n, then $* : \mathcal{H}^{p,q} \to \mathcal{H}^{n-q,n-p}$ is an isomorphism*

Another way is to fix the operator $*$, we define

$$\overline{*} : \Omega_X^{p,q} \to \Omega_X^{n-p,n-q}$$
$$\beta \mapsto * \overline{\beta}$$

then

$$\overline{*} \Delta_{\overline{\partial}} = \Delta_{\overline{\partial}} \overline{*}$$

Corollary 3.2.24. *(X, h) is a compact Hermitian manifold, then $\overline{*} : \mathcal{H}^{p,q} \to \mathcal{H}^{n-p,n-q}$ is an isomorphism.*

Corollary 3.2.25. *$H^{p,q}_{Dol}(X) \cong H^{n-p,n-q}_{Dol}(X)$.*

Remark 3.2.26. This is a special case of Serre duality.

Untill now, the same things happen simultaneously in the real world and complex world, but they do not intersect with each other, just like parallel universes. But, as we will see soon, the Kähler condition plays a role of a "wormhole", connecting these two parallel universes.[22]

3.3. **Differential operators on Kähler manifolds.** (X, ω) is a Kähler manifold,

Definition 3.3.1.

$$L : C^\infty(X, \Omega^k_{X,\mathbb{R}}) \to C^\infty(X, \Omega^{k+2}_{X,\mathbb{R}})$$
$$\alpha \mapsto \omega \wedge \alpha$$

Lemma 3.3.2. *$\Lambda := L^* = (-1)^k * L *$.*

[22]So romantic.

Proof. For $\alpha \in C_c^\infty(X, \Omega^k_{X,\mathbb{R}}), \beta \in C_c^\infty(X, \Omega^{k+2}_{X,\mathbb{R}})$. Compute

$$\begin{aligned}
\{L\alpha, \beta\} \operatorname{vol} &= L\alpha \wedge *\beta \\
&= \omega \wedge \alpha \wedge *\beta \\
&= \alpha \wedge \omega \wedge *\beta \\
&= \alpha \wedge (-1)^{k(2n-k)} * *\omega \wedge *\beta \\
&= \alpha \wedge *((-1)^{k(2n-k)} * L * \beta) \\
&= \{\alpha, (-1)^k * L * \beta\} \operatorname{vol}
\end{aligned}$$

□

Remark 3.3.3. If A, B are two differential operators, we define the commutor of A, B as

$$[A, B] := AB - (-1)^{\deg A \deg B} BA$$

As what we have learnt in Lie algebra, commutor should satisfy Jacobi identity. Here is a similar one:

$$(-1)^{\deg A \deg C}[A, [B, C]] + (-1)^{\deg B \deg A}[B, [C, A]] + (-1)^{\deg C \deg B}[C, [A, B]] = 0$$

In our case, the degree of $\mathrm{d}, \mathrm{d}^*, \partial, \partial^*, \overline{\partial}, \overline{\partial}^*$ is one, and the degree of L and Λ is zero.

Now we have eight differential operators, and Kähler condition implies that there are some relations between them.

Proposition 3.3.4 (Kähler identities)**.** *If (X, ω) is a Kähler manifold, then we have*

$$\begin{aligned}
\left[\overline{\partial^*}, L\right] &= i\partial \\
[\partial^*, L] &= -i\overline{\partial} \\
[\Lambda, \partial] &= -i\partial^* \\
[\Lambda, \partial] &= i\bar{\partial}^*
\end{aligned}$$

Example 3.3.5. Let $U \subset \mathbb{C}^n$ be an open subset with standard Hermitian metric, then $\omega = \frac{i}{2} \sum \mathrm{d}z_j \wedge \mathrm{d}\overline{z_j}$. For any compactly supported (p, q)-form $u = \sum_{J,K} u_{JK} \mathrm{d}z_J \wedge \mathrm{d}\overline{z_K}$.

By Example 3.2.6, we have

$$\begin{aligned}
\overline{\partial}^* u &= -2 \sum_l \sum_{J,K} \frac{\partial u_{JK}}{\partial z_l} \iota_{\frac{\partial}{\partial \overline{z_l}}} \mathrm{d}z_J \wedge \mathrm{d}\overline{z_K} \\
&= -2 \sum_l \iota_{\frac{\partial}{\partial \overline{z_l}}} \frac{\partial u}{\partial z_l}
\end{aligned}$$

So we have

$$\begin{aligned}
\left[\overline{\partial^*}, L\right] u &= \overline{\partial}^*(\omega \wedge u) - \omega \wedge \overline{\partial}^* u \\
&= -2 \sum_l \iota_{\frac{\partial}{\partial \overline{z_l}}} \frac{\partial}{\partial z_l}(\omega \wedge u) + \omega \wedge 2 \sum_l \iota_{\frac{\partial}{\partial \overline{z_l}}} \frac{\partial u}{\partial z_l}
\end{aligned}$$

Since ω is a closed $(1,1)$-form, then

$$\frac{\partial}{\partial z_l}(\omega \wedge u) = \omega \wedge \frac{\partial u}{\partial z_l}$$

So we have

$$\begin{aligned}\iota_{\frac{\partial}{\partial \overline{z_l}}} \frac{\partial}{\partial z_l}(\omega \wedge u) &= \iota_{\frac{\partial}{\partial \overline{z_l}}}(\omega \wedge \frac{\partial u}{\partial z_l}) \\ &= (\iota_{\frac{\partial}{\partial \overline{z_l}}}\omega) \wedge \frac{\partial u}{\partial z_l} + \omega \wedge \iota_{\frac{\partial}{\partial \overline{z_l}}} \frac{\partial u}{\partial z_l}\end{aligned}$$

Then

$$\begin{aligned}\left[\overline{\partial^*}, L\right] u &= -2 \sum_l (\iota_{\frac{\partial}{\partial \overline{z_l}}}\omega) \wedge \frac{\partial u}{\partial z_l} \\ &= i \sum_l \mathrm{d}z_l \wedge \frac{\partial u}{\partial z_l} \\ &= i \sum_l \sum_{J,K} \frac{\partial u_{JK}}{\partial z_l} \mathrm{d}z_l \wedge \mathrm{d}z_J \wedge \mathrm{d}\overline{z_K} \\ &= i\partial u\end{aligned}$$

Proof. By conjugating and taking adjoints, it suffices to prove the first identity, that is a first order identity of differential equation

$$\left[\overline{\partial^*}, L\right] = i\partial$$

But by Theorem 3.1.9, locally we have $h_{jk} = \delta_{jk} + O(|\xi^2|)$. Thus Kähler identity holds from the $U \subset \mathbb{C}^n$ case. □

Theorem 3.3.6. (X, ω) *is a Kähler manifold. Then*

$$\Delta_{\mathrm{d}} = 2\Delta_{\partial} = 2\Delta_{\overline{\partial}}$$

In particular, Δ_{d}*-harmonic is equivalent to* Δ_{∂}*-harmonic and is equivalent to* $\Delta_{\overline{\partial}}$*-harmonic.*

Proof. Directly compute

$$\Delta_{\mathrm{d}} = (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial})$$

Use the forth Kähler identity, we firt compute the first term

$$\begin{aligned}(\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) &= (\partial + \overline{\partial})(\partial^* - i\Lambda\partial + i\partial\Lambda) \\ &= \partial\partial^* - i\partial\Lambda\partial + \overline{\partial}\partial^* - i\overline{\partial}\Lambda\partial + i\overline{\partial}\partial\Lambda\end{aligned}$$

And the second term

$$\begin{aligned}(\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) &= (\partial^* - i\Lambda\partial + i\partial\Lambda)(\partial + \overline{\partial}) \\ &= \partial^*\partial + i\partial\Lambda\partial + \partial^*\overline{\partial} - i\Lambda\partial\overline{\partial} + i\partial\Lambda\overline{\partial}\end{aligned}$$

Use the third Kähler identity, we have

$$\partial^* = i[\Lambda, \overline{\partial}] = i\Lambda\overline{\partial} - i\overline{\partial}\Lambda$$

then

$$\begin{aligned}\overline{\partial}\partial^* &= \overline{\partial}(i\Lambda\overline{\partial} - i\overline{\partial}\Lambda) = i\overline{\partial}\Lambda\overline{\partial}\\ \partial^*\overline{\partial} &= (i\Lambda\overline{\partial} - i\overline{\partial}\Lambda)\overline{\partial} = -i\overline{\partial}\Lambda\overline{\partial}\\ &= -\overline{\partial}\partial^*\end{aligned}$$

Now we have

$$\begin{aligned}\Delta_{\mathrm{d}} &= \Delta_\partial - i\overline{\partial}\Lambda\partial - i\Lambda\partial\overline{\partial} + i\overline{\partial}\partial\Lambda + i\partial\Lambda\overline{\partial}\\ &= \Delta_\partial + i(\Lambda\overline{\partial}\partial - \overline{\partial}\Lambda\partial) + i(\partial\Lambda\overline{\partial} - \partial\overline{\partial}\Lambda)\\ &= \Delta_\partial + i[\Lambda, \overline{\partial}]\partial + i\partial[\Lambda, \overline{\partial}]\\ &= \Delta_\partial + \partial^*\partial + \partial\partial^*\\ &= 2\Delta_\partial\end{aligned}$$

□

Exercise 3.3.7. Show that for Kähler manifold we have

$$\begin{aligned}&[\Delta_{\mathrm{d}}, L] = 0\\ &[L, \Lambda] = (k-n)\,\mathrm{id} \quad \text{on } C_c^\infty(X, \Omega^k_{X,\mathbb{C}})\end{aligned}$$

Proof. For the first one, we have $\Delta_{\mathrm{d}} = 2\Delta_\partial = 2(\partial\partial^* + \partial^*\partial)$. Thus

$$[\Delta_{\mathrm{d}}, L] = 2([\partial\partial^*, L] + [\partial^*\partial, L]) = 2(\partial[\partial^*, L] + [\partial^*, L]\partial)$$

The last equality holds by the fact that L commutes with ∂, since ω is ∂ closed. Now we use the identity $[\partial^*, L] = -i\overline{\partial}$, which anticommutes with ∂. Thus we have the desired result.

For the second one, since we are considering operators of order zero, thus WLOG we will assume that the metric is the standard flat metric. Recall that L is the exterior product with $\omega = \frac{i}{2}\sum_i \mathrm{d}z_i \wedge \mathrm{d}\overline{z_i}$. Let A_j be the operator given by the exterior product with $\frac{i}{2}\mathrm{d}z_j \wedge \mathrm{d}z_j$

$$[L, \Lambda]\alpha =$$

□

Corollary 3.3.8. *(X, ω) is a Kähler manifold, and α is a (p, q)-form, then $\Delta_{\mathrm{d}}\alpha$ is still a (p, q)-form.*

Proof. $\Delta_\partial\alpha$ is still a (p, q)-form is a clear fact. □

Theorem 3.3.9. *(X, ω) is a Kähler manifold, $\alpha = \sum_{p+q=k} \alpha^{p,q}$. Then α is harmonic if and only if $\alpha^{p,q}$ is harmonic. That is*

$$\mathcal{H}^k \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}$$

with $\overline{\mathcal{H}^{p,q}} = \mathcal{H}^{q,p}$.

Theorem 3.3.10 (Hodge decomposition)**.** *(X,ω) is a compact Kähler manifold. Then*

$$H^k(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X)$$

with $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

Corollary 3.3.11. *(X,ω) is a compact Kähler manifold. Set $b_k = \dim H^k(X,\mathbb{C})$ and $h^{p,q} = \dim H^{p,q}(X)$. Then*

$$b_k = \sum_{p+q=k} h^{p,q}$$

with $h^{p,q} = h^{q,p}$.

Corollary 3.3.12. *b_k is even when k is odd.*

Corollary 3.3.13. *$b_k \neq 0$ when k is even.*

Proof. $h^{k,k} \neq 0$, since $0 \neq \omega^k \in H^{k,k}(X)$. □

Example 3.3.14.

$$H^{p,q}(\mathbb{P}^n) = \begin{cases} \mathbb{C}, & 0 \leq p = q \leq n \\ 0, & \text{otherwise} \end{cases}$$

There are many relations between $h^{p,q}$, and we can draw a picture[23] as follows, called Hodge diamond, since it has the same symmetry as a diamond.

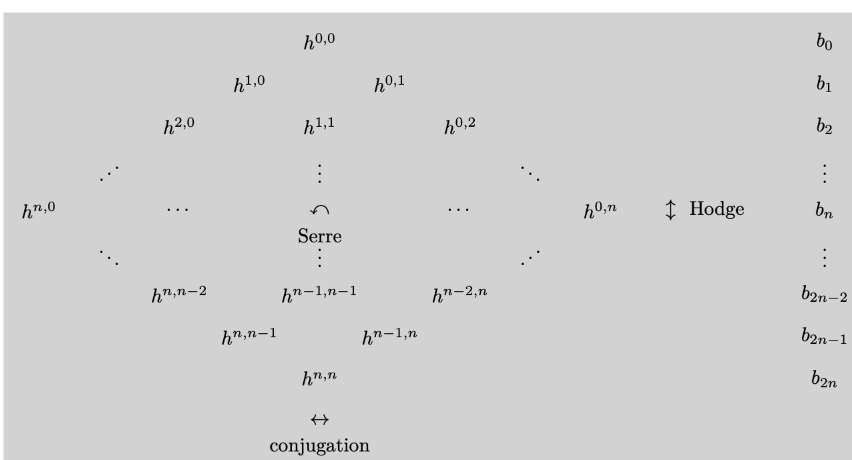

FIGURE 1. Hodge diamond

[23]Here I intended to use tex to draw a picture, but there is still something wrong with my codes. So I put a real "picture" here, I will fix it later.

3.4. **Bott-Chern cohomology.** Review what we have done: We have already proven one of the main theorems in this course, that is, Hodge decomposition. But along the way we used the Kähler metric on a Kähler manifold, so our decomposition may depend on it. A question is that: (In)denpendence of the Kähler metric? The anwser is yes, shown by Bott-Chern cohomology.

Definition 3.4.1 (Bott-Chern cohomology)**.** *X is a complex manifold, we define*

$$H^{p,q}_{BC}(X) := \frac{Z^{p,q}_{BC} := \{\alpha \in C^\infty(X, \Omega^{p,q}) \mid \mathrm{d}\alpha = 0\}}{\partial\overline{\partial}C^\infty(X, \Omega^{p-1,q-1}_X)}$$

Remark 3.4.2. There is a natural map

$$Z^{p,q}_{BC}(X) \to H^{p+q}(X, \mathbb{C})$$

and since $\partial\overline{\partial}\beta = \mathrm{d}\overline{\partial}\beta$, then it descends to

$$H^{p,q}_{BC}(X) \to H^{p+q}(X, \mathbb{C})$$

On the other hand, we also have a natural map

$$Z^{p,q}_{BC}(X) \to H^{p,q}(X)$$

and since $\partial\overline{\partial}\beta = -\overline{\partial}\partial\beta$, then it also descends to

$$H^{p,q}_{BC}(X) \to H^{p,q}(X)$$

In our definition about Bott-Chern cohomology, there's nothing relative to the choice of metrics. So if we can prove there is an isomorphism between

$$H^{p,q}_{BC}(X) \cong H^{p,q}(X), \quad \bigoplus_{p+q=k} H^{p,q}_{BC}(X) \cong H^k(X, \mathbb{C})$$

we can say our Hodge decomposition is canonical.

Lemma 3.4.3 ($\partial\overline{\partial}$-lemma)**.** *$(X, \omega)$ is a compact Kähler manifold, α is a* d*-closed (p, q)-form, i.e. $\alpha \in Z^{p,q}_{BC}(X)$. If α is $\overline{\partial}$-exact or ∂-exact, then there exists a $(p-1, q-1)$-form such that*

$$\alpha = \partial\overline{\partial}\beta$$

Proof. We have $\mathrm{d}\alpha = \partial\alpha = \overline{\partial}\alpha = 0$, and $\alpha = \overline{\partial}\gamma$. Hodge's theorem implies that we can write γ as

$$\gamma = a + \partial b + \partial^* c$$

where a is Δ_∂-harmonic. Directly compute

$$\begin{aligned} \alpha = \overline{\partial}\gamma &= \overline{\partial}a + \overline{\partial}\partial b + \overline{\partial}\partial^* c \\ &= -\partial\overline{\partial}b + \overline{\partial}\partial^* c \\ &= -\partial\overline{\partial}b - \partial^*\overline{\partial}c \end{aligned}$$

What we need to do is to show $-\partial^*\overline{\partial}c = 0$. A trick here is to note that

$$0 = \partial\alpha = -\partial\partial^*\overline{\partial}c \implies \partial^*\overline{\partial}c \in \ker(\partial) \cap \operatorname{im}(\partial^*) \implies \partial^*\overline{\partial}c = 0$$

So we have

$$\alpha = \partial\overline{\partial}(-b)$$

as desired. $\square$

Corollary 3.4.4. *$H^{p,q}_{BC}(X) \to H^{p,q}(X)$ is an isomorphism, and $\bigoplus_{p+q=k} H^{p,q}_{BC}(X) \to H^k(X,\mathbb{C})$ is an isomorphism.*

Proof. Here we only prove the first isomorphism. From Remark 3.4.2 we have a canonical map $H^{p,q}_{BC}(X) \to H^{p,q}(X)$, and if we choose a metric, we have $H^{p,q}(X) \cong \mathcal{H}^{p,q}$, we will show our canonical map is both surjective and injective using this chosen metric.

Surjectivity. For $H^{p,q}(X)$ we choose a $\Delta_{\overline{\partial}}$-harmonic representative. Since Δ_{∂}-harmonic is equivalent to Δ_{d}-harmonic, so this representative is also d-closed.

Injectivity. Suppose we have $\gamma \in H^{p,q}_{BC}(X)$, represented by $[\alpha]$, where $\alpha \in Z^{p,q}_{BC}(X)$ such that $\overline{\partial}\alpha = 0$, that is $\overline{\partial}$-exact. By $\partial\overline{\partial}$-lemma we have that $\alpha \in \partial\overline{\partial}C^\infty(X, \Omega_X^{p-1,q-1})$, that is $0 = \gamma \in H^{p,q}_{BC}(X)$. $\square$

3.5. **Lefschetz decomposition.** Let (X,ω) be a n dimensional Kähler manifold, then

$$\begin{aligned} L : \Omega^k_{X,\mathbb{R}} &\to \Omega^{k+2}_{X,\mathbb{R}} \\ \alpha &\mapsto \omega \wedge \alpha \end{aligned}$$

Proposition 3.5.1. *$L^{n-k} : \Omega^k_{X,\mathbb{R}} \to \Omega^{2n-k}_{X,\mathbb{R}}$ is an isomorphism for $k \leq n$.*

Proof. Since $\Omega^k_{X,\mathbb{R}}$ has the same rank with $\Omega^{2n-k}_{X,\mathbb{R}}$, it suffices to show L^{n-k} is injective, and it suffices to show that it's injective in each fiber. Recall that $L^* = \Lambda = (-1)^k * L* : \Omega^k_{X,\mathbb{R}} \to \Omega^{k-2}_{X,\mathbb{R}}$, and

$$[L,\Lambda] = (n-k)\alpha, \quad \forall \alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}})$$

then

$$\begin{aligned} [L^r,\Lambda] &= L^r\Lambda - \Lambda L^r \\ &= L(L^{r-1}\Lambda - \Lambda L^{r-1}) + L\Lambda L^{r-1} - \Lambda L L^{r-1} \\ &= L[L^{r-1},\Lambda] + [L,\Lambda]L^{r-1} \end{aligned}$$

So do induction on r, we have

$$[L^r,\Lambda]\alpha = (r(k-n) + r(r-1))L^{r-1}\alpha, \quad \forall \alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}})$$

Suppose $L^r\alpha = 0, r \leq n-k$, then

$$\begin{aligned} L^r\Lambda\alpha &= [L^r,\Lambda]\alpha \\ &= (r(k-n) + r(r-1))L^{r-1}\alpha \end{aligned}$$

So we have

$$L^{r-1}(L\Lambda\alpha - (r(k-n) + r(r-1))\alpha) = 0$$

that is, from a $\alpha \in \ker L^r$, we get something in $\ker L^{r-1}$. Do induction on the dimension of $\ker L^r, r \leq n-k$, we have

$$L\Lambda\alpha = (r(k-n) + r(r-1))\alpha$$

Let $\beta = \Lambda\alpha$, then

$$L^{r+1}\beta = (r(k-n) + r(r-1))L^r\alpha = 0$$

where $\beta \in C^\infty(X, \Omega^{k-2}_{X,\mathbb{R}})$. By induction on k, we have $\beta = 0$, so we have $\alpha = 0$. $\square$

Remark 3.5.2. From above proof, we can see all $L^r, r \leq n-k$ are injective.

Definition 3.5.3 (primitive form)**.** *(X, ω) is a Kähler manifold, with dimension $n \geq k$. $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}})$ is called primitive if $L^{n-k+1}\alpha = 0$.*

Exercise 3.5.4. $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}}), k \leq n$ is primitive if and only if $\Lambda\alpha = 0$.

Proof. Let's first see what will happen for a primitive n-form α. α is primitive if and only if $L\alpha = 0$. Recall that Exercise 3.3.7 implies that

$$[L, \Lambda] = (k-n)\,\mathrm{id}, \quad \text{on } C^\infty(X, \Omega^k_{X,\mathbb{C}})$$

So if $k = n$, then L and Λ commutes, so we have α is primitive if and only if $\Lambda\alpha = 0$, since

$$\Lambda\alpha = 0 \iff L\Lambda\alpha = 0 \iff \Lambda L\alpha = 0 \iff L\alpha = 0$$

and the first and last equality we use the fact that L is injective on $\Omega^k_{X,\mathbb{R}}, k \leq n$ and Λ is injective on $\Omega^{n+2}_{X,\mathbb{R}}$.

More generally, we have

$$[L^r, \Lambda]\alpha = (r(k-n) + r(r-1))L^{r-1}\alpha$$

and in particular for $r = n-k+1$ where k is the degree of α, we have

$$[L^r, \Lambda](\alpha) = 0$$

The argument can be then concluded as above. $\square$

Proposition 3.5.5. *For all $\alpha \in C^\infty(X, \Omega^k_{X,\mathbb{R}}), k \leq n$. Then there exists a unique decomposition*

$$\alpha = \sum_r L^r\alpha_r$$

with $\alpha_r \in C^\infty(X, \Omega^{k-2r}_{X,\mathbb{R}})$ is primitive.

Proof. Uniqueness. Suppose $\sum_r L^r\alpha_r = 0$ with primitive α_r. We want to show $\alpha_r = 0$. If not, then take the largest r_m such that $\alpha_{r_m} \neq 0$. By our choice, L^{n-k+r_m} kills everything in $\sum_r L^r\alpha_r$ but $L^{r_m}a_{r_m}$, so

$$0 = L^{n-k+r_m}(\sum_r L^r\alpha_r) = L^{n-k+r_m}(L^{r_m}\alpha_{r_m}) \neq 0$$

A contradiction.

Existence. For $L^{n-k+1}\alpha \in C^\infty(X, \Omega^{2n-k+2}_{X,\mathbb{R}})$, then there exists $\beta \in C^\infty(X, \Omega^{k-2}_{X,\mathbb{R}})$ such that

$$L^{n-k+1}\alpha = L^{n-k+2}\beta \implies L^{n-k+1}(\alpha - L\beta) = 0$$

So $\alpha - L\beta$ is primitive, that is

$$\alpha = (\alpha - L\beta) + L\beta$$

By induction on k, we have primitive decomposition for $\beta \in C^\infty(X, \Omega^{k-2}_{X,\mathbb{R}})$. This completes the proof. $\square$

Remark 3.5.6. Set $H = [L, \Lambda]$, we have proven that (L, H, Λ) generates an $\mathfrak{sl}_2$-action on $\bigoplus_k \Omega^k_{X,\mathbb{R}}$, hence on $\bigoplus_k C^\infty(X, \Omega^k_{X,\mathbb{R}})$.

In cohomology, we can define the following map

$$\begin{aligned} L : H^k(X, \mathbb{R}) &\to H^{k+2}(X, \mathbb{R}) \\ [\alpha] &\mapsto [\omega \wedge \alpha] \end{aligned}$$

Clearly, it's well-defined, it suffices to check that L maps closed forms to closed forms and exact forms to exact forms. If α is closed, then

$$\mathrm{d}(\omega \wedge \alpha) = \mathrm{d}\omega \wedge \alpha + \omega \wedge \mathrm{d}\alpha = 0$$

since ω is closed. And if $\alpha = \mathrm{d}\beta$, then

$$\omega \wedge \mathrm{d}\beta = \mathrm{d}\omega \wedge \beta + \omega \wedge \mathrm{d}\beta = \mathrm{d}(\omega \wedge \beta)$$

So, as you can see, L is well-defined mainly relys on the fact that ω is closed.

Theorem 3.5.7 (Hard Lefschetz theorem[24])**.** *(X, ω) is a compact Kähler manifold with dimension n, then*

$$L^{n-k} : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R})$$

is an isomorphism for $k \leq n$.

Proof. Use $[\Delta_{\mathrm{d}}, L] = 0$, we have

$$L^{n-k} : \mathcal{H}^k \to \mathcal{H}^{2n-k}$$

By the fact that L^{n-k} is injective and $\mathcal{H}^k, \mathcal{H}^{2n-k}$ have the same dimension, we have the desired result. $\square$

Definition 3.5.8 (primitive form)**.** *(X, ω) is a compact Kähler manifold, with dimension n. For $[\alpha] \in H^k(X, \mathbb{R})$, we call it primitive, if $L^{n-k+1}[\alpha] = 0$. We use $H^k(X, \mathbb{R})_{\mathrm{prim}}$ to denote the set of all primitive forms.*

---

[24]Though proof of this theorem is quite easy using tools we have, but it's quite hard for Lefschetz, since during his time, there is no Hodge! And we use L to denote this operator, in order to honor Lefschetz.

Corollary 3.5.9 (Lefschetz decomposition)**.** *For all* $[\alpha] \in H^k(X, \mathbb{R}), k \leq n$, *we have following unique decomposition*

$$[\alpha] = \sum_r L^r[\alpha_r]$$

where $[\alpha_r] \in H^{k-2r}(X, \mathbb{R})_{\text{prim}}$. *In other words*

$$H^k(X, \mathbb{R}) = \bigoplus_r L^r H^{k-2r}(X, \mathbb{R})_{\text{prim}}$$

Remark 3.5.10. If $[\omega] \in H^2(X, \mathbb{Z})$, such as ω comes from a positive holomorphic line bundle. Then we can state theorem and corollary for $H^k(X, \mathbb{Q})$.

Moreover, we have the following isomorphism

$$L^{n-k} : H^{p,q}(X) \to H^{n-q,n-p}(X)$$

for $k = p + q \leq n$.

Corollary 3.5.11. *For a compact Kähler manifold* (X, ω) *with dimension* n *Then* $b_{k-2} \leq b_k$ *and* $h^{p-1,q-1} \leq h^{p,q}$ *for* $k = p + q \leq n$.

3.6. **Hodge index.** Firstly we show the baby case: surfaces

Example 3.6.1. For open subset $U \subset \mathbb{C}^2$, we have

$$\omega = \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} + \mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2})$$

Then we have volume form

$$\frac{\omega^2}{2!} = -\frac{1}{4}(\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} \wedge \mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2})$$

Take a 2-form α, say of type $(1, 1)$, that is

$$\alpha = a_{11}\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} + a_{22}\mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2} + a_{12}\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_2} + a_{21}\mathrm{d}z_2 \wedge \mathrm{d}\overline{z_1}$$

Since $\bullet \wedge *\overline{\alpha} = \{\bullet, \alpha\}$ vol, we have

$$*\alpha = a_{22}\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} + a_{11}\mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2} - a_{12}\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_2} - a_{21}\mathrm{d}z_2 \wedge \mathrm{d}\overline{z_1}$$

Directly compute we have

$$L\alpha = \omega \wedge \alpha = \frac{i}{2}(a_{11} + a_{22})\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} \wedge \mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2}$$

So

$$L\alpha = 0 \iff a_{11} + a_{22} = 0 \iff *\alpha = -\alpha$$

Lemma 3.6.2. (X, ω) *is a Kähler surface,* $\alpha \in C^\infty(X, \Omega_X^{p,q})$ *is primitive* 2*-form, then*

$$*\alpha = (-1)^p\alpha$$

Proof. It suffices to do $U \subset \mathbb{C}^2$. □

Now let's see X is a compact Kähler surface. Poincaré duality implies that we have the following non-degenerate pairing

$$\begin{aligned} Q : H^2(X,\mathbb{R}) \times H^2(X,\mathbb{R}) &\to \mathbb{R} \\ ([\alpha],[\beta]) &\mapsto \int_X \alpha\wedge\beta \end{aligned}$$

Then we get a Hermitian form by defining

$$H([\alpha],[\beta]) = Q([\alpha],\overline{[\beta]})$$

Lemma 3.6.3. *Lefschetz decomposition for* $H^2(X,\mathbb{R}) = H^2(X,\mathbb{R})_{\mathrm{prim}} \oplus \mathbb{R}\cdot[\omega]$ *is orthonormal with respect to* Q. *If we use complex coefficient, then with respect to* H.

Proof.

$$Q([\omega],[\alpha]) = \int_X \omega\wedge\alpha = \int_X L\alpha$$

for α is primitive and harmonic. □

Theorem 3.6.4. $H^2(X,\mathbb{C})_{\mathrm{prim}} = \bigoplus_{p+q=2} H^{p,q}(X)_{\mathrm{prim}}$ *is orthonormal with respect to* H. *Furthermore,* $(-1)^p H$ *is positive definite on* $H^{p,q}(X)_{\mathrm{prim}}$.

Proof. Orthonormality is almost trivial, since only integrate $(2,2)$-form we can get a non-zero result. Take a harmonic form α such that $[\alpha]\in H^{p,q}(X)_{\mathrm{prim}}(X)\backslash\{0\}, p+q=2$.

$$\begin{aligned} (-1)^p H([\alpha],[\beta]) &= (-1)^p \int_X \alpha\wedge\overline{\alpha} \\ &= (-1)^{p+q}\int_X \alpha\wedge *\overline{\alpha} \\ &= \|\alpha\|^2 > 0 \end{aligned}$$

□

Corollary 3.6.5 (Hodge index theorem: surface case)**.** Q *on* $H^2(X,\mathbb{R})\cap H^{1,1}(X)$ *is of index* $(1, h^{1,1}-1)$.

Now let's see a more general case: (X,ω) is a compact Kähler manifold with dimension n. Then by Lefschetz decomposition we have

$$H^k(X,\mathbb{R}) = \bigoplus_r L^r H^{k-2r}(X,\mathbb{R})_{\mathrm{prim}}, \quad k\le n$$

And by Hodge decomposition we have each piece can be decomposed into smaller one

$$H^k(X,\mathbb{C})_{\mathrm{prim}} = \bigoplus_{p+q=k} H^{p,q}(X)_{\mathrm{prim}}$$

As we have seen in the case of surface, H will be positive definite or negative definite in such small piece. The same thing happens in higher dimension. Now we introduce some symbols, in order to get a neater result.

Let $\varepsilon(k) = (-1)^{\frac{k(k-1)}{2}}$, and we define

$$\begin{aligned} Q : H^k(X,\mathbb{R}) \times H^k(X,\mathbb{R}) &\to \mathbb{R} \\ ([\alpha],[\beta]) &\mapsto \varepsilon(k)\int_X \omega^{n-k}\wedge\alpha\wedge\beta \end{aligned}$$

This Q is just a bilinear form, and it is symmetric when k is even and anti-symmetric when k is odd.

Definition 3.6.6 (weil operator)**.** *We define weil operator as follows*

$$\begin{aligned} \mathsf{C} : H^k(X,\mathbb{R}) &\to H^k(X,\mathbb{R}) \\ \mathsf{C}|_{H^{p,q}(X)} &\mapsto i^{p-q}\,\mathrm{id} \end{aligned}$$

Remark 3.6.7. To see that C is a real operator, check directly as follows

$$\mathsf{C}|_{\overline{H^{p,q}(X)}} = \mathsf{C}_{H^{q,p}(X)} = i^{q-p}\,\mathrm{id} = \overline{i^{p-q}}\,\mathrm{id} = \overline{\mathsf{C}|_{H^{p,q}(X)}}$$

Now we define

$$\begin{aligned} H : H^k(X,\mathbb{C}) \times H^k(X,\mathbb{C}) &\to \mathbb{C} \\ ([\alpha],[\beta]) &\mapsto Q(\mathsf{C}[\alpha],\overline{[\beta]}) \end{aligned}$$

In other words, we have

$$H([\alpha],[\beta]) = (-1)^{\frac{k(k-1)}{2}} i^{p-q}\int_X \omega^{n-k}\wedge\alpha\wedge\overline{\beta}$$

Exercise 3.6.8. H is a Hermitian form on $H^{p,q}(X)$.

Theorem 3.6.9 (Hodge-Riemann bilinear relations)**.** *We have following results:*

1. $H^k(X,\mathbb{R}) = \bigoplus_r L^r H^{k-2r}(X,\mathbb{R})_{\mathrm{prim}}$ *is orthonormal with respect to* Q*.*
2. $H^k(X,\mathbb{C})_{\mathrm{prim}} = \bigoplus_{p+q=k} H^{p,q}(X)_{\mathrm{prim}}$ *is orthonormal with respect to* H*.*
3. H *is positive definite on* $H^{p,q}(X)_{\mathrm{prim}}$.

Proof. 1. Take $r < s$, note that

$$\omega^{n-k}\wedge L^r\gamma\wedge L^s\delta = (L^{n-k+r+s}\gamma)\wedge\delta = 0$$

since $L^{n-k+2r+1}\gamma = 0$ and $r < s$.

2. If α is a (p,q)-form, and β is (p',q')-form, and $(p,q)\neq(p',q')$, then $\omega^{n-k}\wedge\alpha\wedge\overline{\beta}$ is not a (n,n)-form.

3. We need to establish the following lemma

Lemma 3.6.10. $\alpha\in C^\infty(X,\Omega_X^{p,q})$ *is a primitive* k*-form, then*

$$*\alpha = (-1)^{\frac{k(k+1)}{2}} i^{p-q}\frac{L^{n-k}\alpha}{(n-k)!}$$

Take α which is harmonic such that it represents $[\alpha] \in H^{p,q}(X)_{\text{prim}}\backslash\{0\}$. Then

$$H([\alpha],[\alpha]) = (-1)^{\frac{k(k-1)}{2}} i^{p-q} \int_X \omega^{n-k} \wedge \alpha \wedge \overline{\alpha}$$

Note that

$$\begin{aligned} *\overline{\alpha} &= (-1)^{\frac{k(k+1)}{2}} i^{q-p} \frac{L^{n-k}\overline{\alpha}}{(n-k)!} \\ &= (-1)^{\frac{k(k-1)}{2}} i^{p-q} \frac{L^{n-k}\overline{\alpha}}{(n-k)!} \end{aligned}$$

so we have

$$H([\alpha],[\alpha]) = (n-k)! \int_X \alpha \wedge *\overline{\alpha} = (n-k)! \|\alpha\| > 0$$

□

Corollary 3.6.11 (Hodge index theorem)**.** *Let $k = n = \dim_{\mathbb{C}} X$ and k is even*[25]*. Then $\int_X \alpha \wedge \beta$ on $H^n(X,\mathbb{R})$ is of signature*

$$\sum_{p,q} (-1)^p h^{p,q}$$

where summation runs over all p, q.

Proof. Note that the signature of $\int_X \alpha \wedge \beta$ on $H^n(X,\mathbb{R})$ is the same as the signature of $int_X \alpha \wedge \overline{\beta}$ on $H^n(X,\mathbb{C})$.

We write

$$H^n(X,\mathbb{C}) = \bigoplus_{r,p,q} L^r H^{p,q}(X)_{\text{prim}}, \quad p+q+2r = n$$

Then Hodge-Riemann bilinear theorem implies that $\int_X \alpha \wedge \overline{\beta}$ is $(-1)^p$-definite on $L^r H^{p,q}(X)_{\text{prim}}$, here we need the requirement n is even.

Then we have the signature is

$$\sum_{p+q+2r=n} (-1)^p h^{p,q}_{\text{prim}}$$

We also have $h^{p,q}_{\text{prim}} = h^{p,q} - h^{p-1,q-1}$. Then signature is

$$\sum_{p+q+2r=n} (-1)^p (h^{p,q} - h^{p-1,q-1})$$

Note that $p + q = n$ counted once and $p + q < n$ counted twice, so rewrite it as

$$\sum_{p+q \text{ even}} (-1)^p h^{p,q}$$

since $h^{p,q} = h^{n-p,n-q}$. And this is also equivalent to sum all p, q, since

$$\sum_{p+q \text{ odd}} (-1)^p h^{p,q} = 0$$

[25] So that $\int_X \alpha \wedge \beta$ is symmetric on $H^n(X,\mathbb{R})$.

This completes the proof. $\square$

Example 3.6.12. For surface, we have

$$H^2(X,\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X)_{\text{prim}} \oplus \mathbb{C}[\omega] \oplus H^{0,2}(X)$$

Then this corollary implies

$$h^{0,0} + h^{2,0} - h^{1,1} + h^{0,2} + h^{2,2} = h^{2,0} + h^{0,2} + (1 - (h^{1,1} - 1))$$

recover what we have done in the case of surface.

4. Applications of Hodge theory

4.1. **Serre duality.** Let X be a compact complex manifold, and $E \to X$ be a holomorphic vector bundle. Recall

$$H^q(X,E) = H^q_{\overline{\partial}_E}(C^\infty(X, \Omega_X^{0,\bullet} \otimes E))$$

One can also define

$$H^{p,q}(X,E) = H^q_{\overline{\partial}_E}(C^\infty(X, \Omega_X^{p,\bullet} \otimes E))$$

But it does not give anything very interesting, since

$$H^{p,q}(X,E) = H^q(X, \Omega_X^p \otimes E)$$

We want to do for the forms with coefficients in E, that is, if (E,h) is a Hermitian holomorphic vector bundle, we want to define

$$\overline{*}_E : C^\infty(X, \Omega_X^{p,q} \otimes E) \to C^\infty(X, \Omega_X^{n-p,n-q} \otimes E^*)$$

We define it locally. Take $x \in U \subset X$, then we define a $\mathbb{C}$-antilinear map on each fiber

$$\begin{aligned} \tau_x : E_x &\to E_x^* \\ e_x &\mapsto \langle \bullet, e_x \rangle \end{aligned}$$

then we get a $\mathbb{C}$-antilinear map

$$\tau : E \to E^*$$

For $\varphi \otimes e \in C^\infty(U, \Omega_U^{p,q} \otimes E)$, we define

$$\overline{*}_E(\varphi \otimes e) := *\overline{\varphi} \otimes \tau(e)$$

In other words, we want to have

$$\alpha \wedge \overline{*}_E \beta = \{\alpha, \beta\} \operatorname{vol}$$

If D_E is the Chern connection, then $D_E^{0,1} = \overline{\partial}_E$, we set $D_E^{1,0} = \partial_E$, then its Chern curvature is that

$$H_E = D_E^2 = \partial_E^2 + \partial_E \overline{\partial}_E + \overline{\partial}_E \partial_E + \overline{\partial}_E^2$$

Recall that H_E is a $(1,1)$-form, with coefficients in $\operatorname{End}(X)$, then $\partial_E^2 = 0$

So,

$$H_E = \partial_E \overline{\partial}_E + \overline{\partial}_E \partial_E = [\partial_E, \overline{\partial}_E]$$

We want to define formal adjoints $\partial_E^*, \overline{\partial}_E^*$. In fact, we can write them down directly.

Exercise 4.1.1. Give formulas in terms of $\overline{*}_E$

So we can define Laplacians $\Delta_{\partial_E}, \Delta_{\overline{\partial}_E}$, and Hodge appears again! We will have

$$C^\infty(X, \Omega_X^{p,q} \otimes E) = \mathcal{H}^{p,q}(X, E) \oplus \operatorname{im} \overline{\partial}_E \oplus \operatorname{im} \overline{\partial}_E^*$$

and by the same argument we have

$$H^{p,q}(X, E) \cong \mathcal{H}^{p,q}$$

Theorem 4.1.2 (Serre duality)**.** *Let X be a compact complex manifold, E is a holomorphic vector bundle, then there exists a non-degenerate $\mathbb{C}$-linear pairing*

$$H^{p,q}(X, E) \times H^{n-p,n-q}(X, E^*) \to \mathbb{C}$$
$$([\alpha], [\beta]) \mapsto \int_X \alpha \wedge \beta$$

In particular, we have

$$H^{p,q}(X, E) = H^{n-p,n-q}(X, E^*)^*$$

Before the proof of Serre duality, let's recall how do we prove Poincaré duality using Hodge theory. Poincaré duality states that if X is a compact oriented Riemannian manifold with $\dim_{\mathbb{R}} X = n$, then the pairing

$$H^{p,q}(X, \mathbb{R}) \times H^{n-p,n-q}(X, \mathbb{R}) \to \mathbb{R}$$
$$([\alpha], [\beta]) \mapsto \int_X \alpha \wedge \beta$$

is non-degenerate. So we have $H^k(X, \mathbb{R}) = H^{n-k}(X, \mathbb{R})^*$. Our proof via Hodge theory is that

$$* : \mathcal{H}^k \xrightarrow{\cong} \mathcal{H}^{n-k}$$

and Hodge theorem imply that

$$\mathcal{H}^k \cong H^k(X, \mathbb{R})$$

For $\alpha \in \mathcal{H}^k, \beta \in \mathcal{H}^{n-k}$, we have $\beta = *\gamma$ for some $\gamma \in \mathcal{H}^k$, so

$$\int_X \alpha \wedge \beta = \int_X \alpha \wedge *\gamma = \langle \alpha, \gamma \rangle_{L^2}$$

is non-degenerate.

Our proof of Serre duality is quite similar to the one of Poincaré duality.

Proof. Sketch. Endow E with a Hermitian metric h. Firstly show

$$\Delta_{\overline{\partial}_E^*} \overline{*}_E = \overline{*}_E \Delta_{\overline{\partial}_E}$$

then

$$\overline{*}_E : \mathcal{H}^{p,q}(X, E) \xrightarrow{\cong} \mathcal{H}^{n-p,n-q}(X, E^*)$$

and Hodge theorem implies that

$$\mathcal{H}^{p,q}(X, E) \cong H^{p,q}(X, E)$$

For all $\alpha \in \mathcal{H}^{p,q}(X,E), \beta \in \mathcal{H}^{n-p,n-q}(X,E^*)$, we have $\beta = \overline{*}_E\gamma$ for some $\gamma \in \mathcal{H}^{p,q}(X,E)$, then

$$\int_X \alpha \wedge \beta = \int_X \alpha \wedge \overline{*}_E\gamma = \langle \alpha, \gamma \rangle_{L^2}$$

is non-degenerate. $\square$

Example 4.1.3. Let $E = \mathcal{O}_X$, then $H^{p,q}(X) = H^{n-p,n-q}(X)^*$. This recovers Corollary 3.2.25.

Example 4.1.4. Let $p = 0$, we have

$$H^q(X,E) = H^{n-q}(X, K_X \otimes E^*)^*$$

where n is the dimension of X. This form of Serre duality may be the one you learnt in a more serious algebraic geometry course.

4.2. **Kodaira vanishing.** Now if we let X be a compact Kähler manifold with dimension n. (E,h) as above. Similarly we can define

$$\begin{aligned} L : C^\infty(X, \Omega^{p,q} \otimes E) &\to C^\infty(X, \Omega^{p+1,q+1} \otimes E) \\ \alpha &\mapsto \omega \wedge \alpha \end{aligned}$$

and $\Lambda = L^*$, formal adjoint. We also have Kähler identities

$$\begin{aligned} \left[\overline{\partial}_E^*, L\right] &= i\partial \\ [\partial_E^*, L] &= -i\overline{\partial} \\ [\Lambda, \partial] &= -i\partial^* \\ [\Lambda, \partial] &= i\bar{\partial}^* \end{aligned}$$

and

$$[L, \Lambda] = (p+q-n)\,\mathrm{id}, \quad \text{on } C^\infty(X, \Omega_X^{p,q} \otimes E)$$

Untill now, all things are same as the things without coefficient E. But

Theorem 4.2.1 (Bochner-Kodaira-Nakano identity)**.**

$$\Delta_{\overline{\partial}_E} = [iH_E, \Lambda] + \Delta_{\partial_E}$$

Proof. By definition, we have

$$\begin{aligned} \Delta_{\overline{\partial}_E} &= [\overline{\partial}_E, \overline{\partial}_E^*] \\ &= -i[\overline{\partial}_E, [\Lambda, \partial_E]] \\ &= -i[\Lambda, [\partial_E, \overline{\partial}_E]] - i[\partial_E, [\overline{\partial}_E, \Lambda]] \\ &= -i[\Lambda, H_E] - i[\partial_E, i\partial_E^*] \\ &= [iH_E, \Lambda] + \Delta_{\partial_E} \end{aligned}$$

$\square$

Corollary 4.2.2 (Bochner-Kodaira-Nakano inequality)**.** *For* $\alpha \in C^\infty(X, \Omega_X^{p,q} \otimes E)$*. Then*

$$([iH_E, \Lambda]\alpha, \alpha)_{L^2} \leq (\Delta_{\overline{\partial}_E}\alpha, \alpha)_{L^2}$$

In particular, if α *is* $\Delta_{\overline{\partial}_E}$*-harmonic, then* $([iH_E, \Lambda]\alpha, \alpha)_{L^2} \leq 0$.

Proof.

$$\begin{aligned}(\Delta_{\overline{\partial}_E}\alpha, \alpha)_{L^2} - ([iH_E, \Lambda]\alpha, \alpha)_{L^2} &= (\Delta_{\partial_E}\alpha, \alpha) \\ &= \|\partial_E\alpha\|^2 + \|\partial_E^*\alpha\|^2 \geq 0\end{aligned}$$

□

Theorem 4.2.3 (Kodaira-Akizuki-Nakano vanishing)**.** (X, ω) *is a compact Kähler manifold with dimension* n*.* (L, h) *is a holomorphic line bundle. Then*

$$H^{p,q}(X, L) = 0, \quad p + q > n$$

In particular, if we let $p = n$*, then*

$$H^q(X, K_X \otimes L) = 0, \quad q > 0$$

Proof. Use $\Delta_{\overline{\partial}_L}$-harmonic forms, that is $H^{p,q}(X, L) \cong \mathcal{H}^{p,q}(X, L)$. For $\alpha \in \mathcal{H}^{p,q}(X, L)$, then BKN inequality says

$$([iH_E, \Lambda]\alpha, \alpha)_{L^2} \leq 0$$

On the other hand, we can choose $\omega = \dfrac{i}{2\pi} H_L$, since L is positive. So we have

$$[iH_L, \Lambda]\alpha = 2\pi(p + q - n)\alpha$$

Thus

$$([iH_E, \Lambda]\alpha, \alpha)_{L^2} = 2\pi(p + q - n)\|\alpha\|^2 \geq 0$$

Thus

$$([iH_E, \Lambda]\alpha, \alpha)_{L^2} = 0$$

That is $\alpha = 0$. □

Corollary 4.2.4 (Kodaira vanishing)**.** (X, ω) *is a compact Kähler manifold with dimension* n*.* (L, h) *is a holomorphic line bundle. Then*

$$H^q(X, K_X \otimes L) = 0, \quad q > 0$$

Proof. Let $p = n$. □

Exercise 4.2.5. Compute all $H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$ for all k, q.

Definition 4.2.6 (Fano)**.** *Fano manifold is a compact Kähler manifold with* $K_X^* = \det T_X$ *is positive.*

Exercise 4.2.7. Let X be a Fano manifold. Show that

$$H^q(X, \mathcal{O}_X) = 0, \quad \forall q > 0$$

Proof. Note that $\mathcal{O}_X = K_X \otimes K_X^*$. □

Theorem 4.2.8 (Serre vanishing)**.** *(X,ω) is a compact Kähler manifold with dimension n. (L,h) is a holomorphic line bundle. For any holomorphic vector bundle E on X. Then there exists a constant m_0 such that for all $m\geq m_0$*

$$H^q(X,E\otimes L^{\otimes m})=0,\quad q>0$$

Proof. Endow E with a Hermitian metric and consider $H^{p,q}(X,E\otimes L^{\otimes m})\cong \mathcal{H}^{p,q}(X,E\otimes L^{\otimes m})$. For $\alpha\in\mathcal{H}^{p,q}(X,E\otimes L^{\otimes m})$. BKN inequality implies that

$$([iH_{E\otimes L^{\otimes m}},\Lambda]\alpha,\alpha)_{L^2}\leq 0$$

Recall that $D_{E\otimes L^{\otimes m}}=D_E\otimes 1+1\otimes D_{L^{\otimes m}}$. So

$$H_{E\otimes L^{\otimes m}}=H_E\otimes 1+m(1\otimes H_L)\in C^\infty(X,\Omega^{1,1}_X\otimes \mathrm{End}(E\otimes L^{\otimes m}))$$

Choose $\omega=\dfrac{i}{2\pi}H_L$, since L is positive. Then

$$([iH_{E\otimes L^{\otimes m}},\Lambda]\alpha,\alpha)_{L^2}=([iH_E,\Lambda]\alpha,\alpha)_{L^2}+2\pi m(p+q-n)\|\alpha\|^2\leq 0$$

Cauchy inequality implies that

$$([iH_E,\Lambda]\alpha,\alpha)_{L^2}\geq -C\|\alpha\|^2$$

where C is the norm of $[iH_E,\Lambda]$.

So if we have $2\pi m(p+q-n)-c>0$, then

$$([iH_{E\otimes L^{\otimes m}},\Lambda]\alpha,\alpha)_{L^2}\geq 0$$

Thus $\alpha=0$ as desired.

So we take $p=n,q>0,m_0\geq\dfrac{c}{2\pi}$, we have

$$H^{n,q}(X,E\otimes L^{\otimes m})=H^q(X,K_X\otimes E\otimes L^{\otimes m})=0,\quad \forall m\geq m_0,q>0$$

Since E is arbitrary, use $K_X^*\otimes E$ instead[26]. □

4.3. **Kodaira embedding.**

Theorem 4.3.1 (Kodaira embedding)**.** *X is a compact complex manifold, the following statements are equivalent*

1. *There exists a holomorphic embedding $\varphi: X\hookrightarrow\mathbb{P}^N$.*
2. *There exists a integral Kähler form ω in X, that is, $[\omega]\in\mathrm{Im}(H^2(X,\mathbb{Z})\to H^2(X,\mathbb{R}))$*
3. *There exists a positive holomorphic line bundle on X.*

Remark 4.3.2. 1 clealy implies 2, and 2 implies 3 is Lefschetz $(1,1)$-theorem. So the heart of the proof is 3 to 1.

Remark 4.3.3. We will shall see 1 together with Chow's theorem, $X\subset\mathbb{P}^N$ can be written as a zero set of homogenous polynomials. Thus X is a projective manifold in our definition.

Before the proof of Kodaira embedding, let's see some corollaries.

[26]Note that m_0 might change in the process.

Corollary 4.3.4. *X is a compact Kähler manifold such that $H^2(X,\mathcal{O}_X) = H^{0,2}(X) = 0$. Then X is a projective manifold.*

Proof. Hodge decomposition implies that $H^{2,0}(X) = H^{0,2}(X) = 0$, so $H^2(X,\mathbb{Q})\otimes_{\mathbb{Q}} \mathbb{C} = H^2(X,\mathbb{C}) = H^{1,1}(X)$. Choose basis $[\alpha_1],\dots,[\alpha_n] \in H^2(X,\mathbb{Q})$ such that α_i is harmonic and of type $(1,1)$. Since the Kähler form ω is real, harmonic and of type $(1,1)$. It's harmonic since $[\Delta_{\mathrm{d}}, L] = 0$. Then

$$\omega = \sum_i \lambda_i\alpha_i, \quad \lambda_i \in \mathbb{R}$$

For $\mu_i \in \mathbb{Q}$ sufficiently close to λ_i, then $\sum_i \mu_i\alpha_i$ is still positive. Thus $\sum_i \mu_i\alpha_i$ gives a Kähler form. Take N sufficiently large such that $[N\sum_i \mu_i\alpha_i] \in \operatorname{Im}(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R}))$. Applying Kodaira embedding together with Chow's theorem to complete the proof. □

Corollary 4.3.5. *Fano manifold is projective.*

Proof. Since for Fano manifold, all $H^{0,p}(X) = 0, q > 0$. □

Now we give the proof of Kodaira embedding.

Proof. (Sketch.) Use holomorphic global sections $\Gamma(X, L^{\otimes m})$ for sufficiently large m to construct $\varphi : X \hookrightarrow \mathbb{P}^N$.

We need to show the following three things:

1. For sufficiently large m, $L^{\otimes m}$ is globally generated, which means for all $x \in X$, there exists a global section $s \in \Gamma(X, L^{\otimes m})$ such that $s(x) \neq 0$. Then for all $x \in X$, $H_x = \{s \in \Gamma(X, L^{\otimes m}) \mid s(x) = 0\}$ is a hypersurface. Thus we get a holomorphic map $\varphi : X \to \mathbb{P}(\Gamma(X, L^{\otimes m})^*)$, defined by $x \mapsto H_x$.
2. For more sufficiently large[27] m, $L^{\otimes m}$ separetes points, that is for all $x, y \in X$, there exists $s \in \Gamma(X, L^{\otimes m})$ such that $s(x) \neq 0, s(y) = 0$. Thus in this case our φ is injective.
3. For more more sufficiently large m, $L^{\otimes m}$ separetes tangent vectors, that is, for all $x \in X, u \in T_{X,x}$ there exists $s \in \Gamma(X, L^{\otimes m})$ such that $s(x) = 0$ and $\mathrm{d}s(u) \neq 0$. Thus in this case our φ is an immersion, together with X is compact we have φ is an embedding.

□

[27] Larger than m is step one.

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