A quick review of topology

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2023/06/18

1. Overview
2. Homotopy and fundamental group
3. Covering space
4. Continuous group action

In this talk we give a quick review of topology which we will use frequently, and the main topics are listed as follows:

- Homotopy and fundamental group.
- Covering spaces.
- Continuous group action.

Definition (homotopy)

Let X and Y be topological spaces and $f, g\colon X \to Y$ be continuous maps. A homotopy from f to g is a continuous map $F\colon X \times I \to Y$ such that for all $x \in X$, one has

$$F(x,0) = f(x)$$
$$F(x,1) = g(x)$$

If there exists a homotopy from f to g, then we say f and g are homotopic, and write $f \simeq g$.

Definition (stationary homotopy)

Let X and Y be topological spaces and $A \subseteq X$ an arbitrary subset. A homotopy F between continuous maps $f, g\colon X \to Y$ is said to be stationary on A if

$$F(x, t) = f(x)$$

for all $x \in A$ and $t \in I$. If there exists such a homotopy, then we say f and g are homotopic relative to A.

Remark.

If f and g are homotopic relative to A, then f must agree with g on A.

Definition (path homotopy)

Let X be a topological space and γ_1, γ_2 be two paths in X. They are said to be path homotopic if they are homotopic relative on $\{0, 1\}$, and write $\gamma_1 \simeq \gamma_2$.

Definition (loop homotopy)

Let X be a topological space and γ_1, γ_2 be two loops in X. They're called loop homotopic if they are homotopic relative on $\{0\}$, and write $\gamma_1 \simeq \gamma_2$.

Remark.

For convenience, if γ_1, γ_2 are paths (loops), then when we say γ_1 is homotopic to γ_2, we mean γ_1 is path (loop) homotopic to γ_2.

Definition (free loop homotopy)

Let X be a topological space and γ_1, γ_2 be two loops in X. They are said to be freely loop homotopic if they're homotopic through loops (but not necessarily preserving the base point), that is, there exists a homotopy $F(x,t)\colon [0,1]\times[0,1]\to X$ such that

$$F(0,t) = \gamma_1(t)$$
$$F(1,t) = \gamma_2(t)$$
$$F(s,0) = F(s,1) \text{ holds for all } s \in [0,1]$$

Lemma

Let X be a topological space. For any $p, q \in X$, path homotopy is an equivalence relation on the set of all paths in X from p to q. For any path γ in X, the path homotopy class is denoted by $[\gamma]$.

Proof.

For path $\gamma\colon I \to X$, γ is homotopic to itself by $F(s,t) = \gamma(s)$. If γ_1 is homotopic to γ_2 by F, then γ_2 is homotopic to γ_1 by $G(s,t) = F(s, 1-t)$. Finally, suppose γ_1 is homotopic to γ_2 by F, γ_2 is homotopic to γ_3 by G. Then consider

$$H = \begin{cases} F(s, 2t) & 0 \leq t \leq \frac{1}{2} \\ G(s, 2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

which is a homotopy from γ_1 to γ_3. This shows path homotopy is an equivalence relation. $\square$

Definition (reparametrization)

A reparametrization of a path $f\colon I \to X$ is of the form $f \circ \varphi$ for some continuous map $\varphi\colon I \to I$ fixing 0 and 1.

Lemma

Any reparametrization of a path f is homotopic to f.

Proof.

Suppose $f \circ \varphi$ is a reparametrization of f, and let $F\colon I \times I \to I$ denote the straight-line homotopy from the identity map to φ. Then $f \circ H$ is a path homotopy from f to $f \circ \varphi$. $\square$

Definition (product of path)

Let X be a topological space and f, g be paths. f and g are composable if $f(1) = g(0)$. If f and g are composable, their product $f \cdot g \colon I \to X$ is defined by

$$f \cdot g(s) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

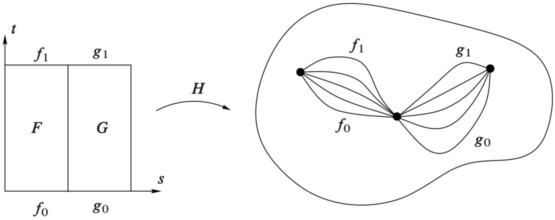
Lemma

Let X be a topological space and $p \in X$ and f_0, f_1, g_0, g_1 be loops in X based at p. If $f_0 \simeq g_0$, $f_1 \simeq g_1$, then $f_0 \cdot g_0 \simeq f_1 \cdot g_1$.

Proof.

Suppose the homotopy from f_0 to f_1 is given by F and the homotopy from g_0 to g_1 is given by G. Then the required homotopy H from $f_0 \cdot g_0$ to $f_1 \cdot g_1$ is given by

$$H(s,t) = \begin{cases} F(2s,t) & 0 \leq s \leq \frac{1}{2}, 0 \leq t \leq 1 \\ G(2s-1,t) & \frac{1}{2} \leq s \leq 1, 0 \leq t \leq 1 \end{cases}$$



Theorem

Let X be a topological space and $[f], [g], [h]$ be homotopy classes of loops based at $p \in X$.

1. $[c_p] \cdot [f] = [f] \cdot [c_p] = [f]$, *where c_p is constant loop based at p.*
2. $[f] \cdot [\overline{f}] = [c_p]$ *and* $[\overline{f}] \cdot [f] = [c_p]$, *where $\overline{f}$ is the loop based at p obtained from reversing f.*
3. $[f] \cdot ([g] \cdot [h]) = ([f] \cdot [g]) \cdot [h]$.

Proof.

For (1). Let us show that $c_p \cdot f \simeq f$, and the other case is similar. Define $H \colon I \times I \to X$ by

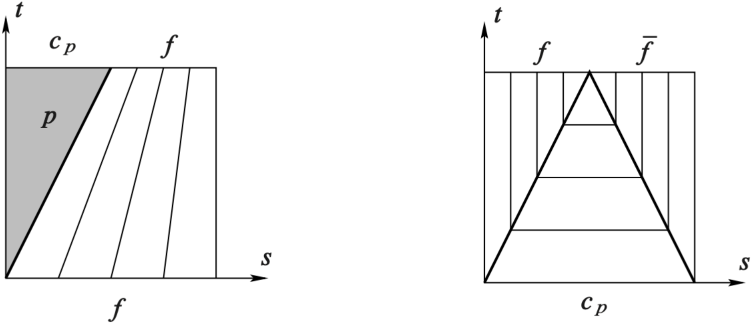
$$H(s,t) = \begin{cases} p & t \geq 2s \\ f\left(\frac{2s-t}{2-t}\right) & t \leq 2s \end{cases}$$

Continuation.

For (2). It suffices to show that $f \cdot \overline{f} \simeq c_p$, since the reverse path of $\overline{f}$ is f, the other relation follows by interchanging the roles of f and $\overline{f}$. Define

$$H(s,t) = \begin{cases} f(2s) & 0 \leq s \leq \frac{t}{2} \\ f(t) & \frac{t}{2} \leq s \leq 1 - \frac{t}{2} \\ f(2-2s) & 1 - \frac{t}{2} \leq s \leq 1 \end{cases}$$

It is easy to check that H is a homotopy from c_p to $f \cdot \overline{f}$.
For (3). It suffices to show $(f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$. The first path follows f and then g at quadruple speed for $s \in [0, \frac{1}{2}]$, and then follows h at double speed for $s \in [\frac{1}{2}, 1]$, while the second follows f at double speed and then g and h at quadruple speed. The two paths are therefore reparametrizations of each other and thus homotopic by Lemma 8. □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Definition (fundamental group)

Let X be a topological space. The fundamental group of X based at p, denoted by $\pi_1(X,p)$, is the set of path homotopy classes of loops based at p equipped with composition as its group structure.

Theorem (base point change)

Let X be a topological space, $p, q \in X$ and g is any path from p to q. The map

$$\begin{aligned}\Phi_g\colon \pi_1(X,p) &\to \pi_1(X,q)\\ [f] &\mapsto [\overline{g}]\cdot[f]\cdot[g]\end{aligned}$$

is a group isomorphism with inverse $\Phi_{\overline{g}}$.

Proof.

It suffices to show Θ_g is a group homomorphism, since it's clear $\Phi_g \circ \Phi_{\overline{g}} = \Phi_{\overline{g}} \circ \Phi_g = \mathrm{id}$. For $[\gamma_1], [\gamma_2] \in \pi_1(X, p)$, one has

$$\begin{aligned}
\Phi_g[\gamma_1] \cdot \Phi[\gamma_2] &= [\overline{g}] \cdot [\gamma_1] \cdot [g] \cdot [\overline{g}] \cdot [\gamma_2] \cdot [g] \\
&= [\overline{g}] \cdot [\gamma_1] \cdot [c_p] \cdot [\gamma_2] \cdot [g] \\
&= [\overline{g}] \cdot [\gamma_1] \cdot [\gamma_2] \cdot [g] \\
&= \Phi_g([\gamma_1] \cdot [\gamma_2])
\end{aligned}$$

□

Corollary

If X is a path-connected topological space, then its fundamental is independent of the choice of base point, and denoted by $\pi_1(X)$ for convenience.

In this section we assume all spaces are connected and locally path connected topological spaces, and all maps are continuous. We are including these hypotheses[1] since most of the interesting results (such as lifting criterion) require them, and most of the interesting topological space (such as connected topological manifold) satisfy them.

[1] In fact, it's almost the strongest connected hypotheses, since if a topological space is connected and locally path-connected, then it's also path connected.

Definition (covering space)

A covering space of X is a map $\pi\colon \widetilde{X} \to X$ such that there exists a discrete space D and for each $x \in X$ an open neighborhood $U \subseteq X$, such that $\pi^{-1}(U) = \coprod_{d\in D} V_d$ and $\pi|_{V_d}\colon V_d \to U$ is a homeomorphism for each $d \in D$.

1. Such a U is called evenly covered by $\{V_d\}$.
2. The open sets $\{V_d\}$ are called sheets.
3. For each $x \in X$, the discrete subset $\pi^{-1}(x)$ is called the fiber of x.
4. The degree of the covering is the cardinality of the space D.

Definition (isomorphism between covering spaces)

Let $\pi_1\colon \widetilde{X}_1 \to X$ and $\pi_2\colon \widetilde{X}_2 \to X$ be two covering spaces. An isomorphism between covering spaces is a homeomorphism $f\colon \widetilde{X}_1 \to \widetilde{X}_2$ such that $\pi_1 = \pi_2 \circ f$.

Theorem (unique lifting property)

Let $\pi\colon \widetilde{X} \to X$ be a covering space and a map $f\colon Y \to X$. If two lifts $\widetilde{f}_1, \widetilde{f}_2\colon Y \to \widetilde{X}$ of f agree at one point of Y, then $\widetilde{f}_1$ and $\widetilde{f}_2$ agree on all of Y.

Proof.

Let A be the set consisting of points of Y where $\widetilde{f}_1$ and $\widetilde{f}_2$ agree. If $\widetilde{f}_1$ agrees with $\widetilde{f}_2$ at some point of Y, then A is not empty, and we may assume $A \neq Y$, otherwise there is nothing to prove. For $y \notin A$, let $\widetilde{U}_1$ and $\widetilde{U}_2$ be the sheets containing $\widetilde{f}_1(y)$ and $\widetilde{f}_2(y)$ respectively. By continuity of $\widetilde{f}_1$ and $\widetilde{f}_2$, there exists a neighborhood N of y mapped into $\widetilde{U}_1$ by $\widetilde{f}_1$ and mapped into $\widetilde{U}_2$ by $\widetilde{f}_2$. Since $\widetilde{f}_1(y) \neq \widetilde{f}_2(y)$, then $\widetilde{U}_1 \cap \widetilde{U}_2 = \varnothing$. This shows $\widetilde{f}_2 \neq \widetilde{f}_2$ throughout the neighborhood N, and thus $Y \backslash A$ is open, that is A is closed. To see A is open, for $y \in A$ one has $\widetilde{f}_1(y) = \widetilde{f}_2(y)$, and thus $\widetilde{U}_1 = \widetilde{U}_2$. Since $\pi|_{\widetilde{U}_1}$ is a diffeomorphism, one has $\widetilde{f}_1 = \pi^{-1} \circ f = \widetilde{f}_2$ on $\widetilde{U}_i$. This shows the set A is open, and thus $A = Y$ since Y is connected. □

Theorem (homotopy lifting property)

Let $\pi\colon \widetilde{X} \to X$ be a covering space and $F\colon Y \times I \to X$ be a homotopy. If there exists a map $\widetilde{F}\colon Y \times \{0\} \to \widetilde{X}$ which lifts $F|_{Y\times\{0\}}$, then there exists a unique homotopy $\widetilde{F}\colon Y \times I \to \widetilde{X}$ which lifts F and restricting to the given $\widetilde{F}$ on $Y \times \{0\}$. Furthermore, if F is stationary on A, so is $\widetilde{F}$.

Corollary (path lifting property)

Let $\pi\colon \widetilde{X} \to X$ be a covering space. Suppose $\gamma\colon I \to X$ is any path, and $\widetilde{x} \in \widetilde{X}$ is any point in the fiber of $\pi^{-1}(\gamma(0))$. Then there exists a unique lift $\widetilde{\gamma}\colon I \to \widetilde{X}$ of γ such that $\widetilde{\gamma}(0) = \widetilde{x}$.

Corollary (monodromy theorem)

Let $\pi\colon \widetilde{X} \to X$ be a covering space. Suppose γ_1 and γ_2 are paths in X with the same initial point and the same terminal point, and $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ are their lifts with the same initial point. Then $\widetilde{\gamma}_1$ is homotopic to $\widetilde{\gamma}_2$.

Proof.

Suppose $F\colon I \times I \to X$ is the homotopy from γ_1 to γ_2 which is stationary on $\{0, 1\}$ and $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ are lifts of γ_1, γ_2 with the same initial point. Then by Theorem 18 there exists a homotopy $\widetilde{F}\colon I \times I \to \widetilde{X}$ from $\widetilde{\gamma}_1$ to $\widetilde{\gamma}_2$ which is also stationary on $\{0, 1\}$, which shows $\widetilde{\gamma}_1$ is homotopic to $\widetilde{\gamma}_2$. $\square$

Corollary

Let $\pi\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering space. Then

1. *The map $\pi_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective.*
2. *$\pi_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ consists of the homotopy class of loops in X whose lifts to $\widetilde{X}$ are still loops.*
3. *The index of $\pi_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ in $\pi_1(X, x_0)$ is the degree of covering. In particular, the degree of universal covering equals $|\pi_1(X, x_0)|$.*

Proof.

For (1). An element of $\ker \pi_*$ is represented by a loop $\widetilde{\gamma}_0\colon I \to \widetilde{X}$ with a homotopy F of $\gamma_0 = \pi \circ \widetilde{\gamma}_0$ to the trivial loop γ_1. By Theorem 18 there is a lifted homotopy of loops $\widetilde{F}$ starting with $\widetilde{\gamma}_0$ and ending with a constant loop. Hence $[\widetilde{\gamma}_0] = 0$ in $\pi_1(\widetilde{X}, \widetilde{x}_0)$ and π_* is injective.

Continuation.

For (2). The loops at x_0 lifting to loops at $\widetilde{x}_0$ certainly represent elements of the image of $\pi_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$. Conversely, a loop representing an element of the image of π_* is homotopic to a loop having such a lift, so by Theorem 18, the loop itself must have such a lift.

For (3). For a loop γ in X based at x_0, let $\widetilde{\gamma}$ be its lift to $\widetilde{X}$ starting at $\widetilde{x}_0$. A product $h \cdot \gamma$ with $[h] \in H = \pi_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ has the lift $\widetilde{h} \cdot \widetilde{\gamma}$ ending at the same point as $\widetilde{\gamma}$ since $\widetilde{h}$ is a loop. Thus we may define a function Φ from cosets $H[\gamma]$ to $\pi^{-1}(x_0)$ by sending $H[\gamma]$ to $\widetilde{\gamma}(1)$. The path-connectedness of $\widetilde{X}$ implies that Φ is surjective since $\widetilde{x}_0$ can be joined to any point in $\pi^{-1}(x_0)$ by a path $\widetilde{\gamma}$ projecting to a loop γ at x_0. To see that Φ is injective, observe that $\Phi(H[\gamma_1]) = \Phi(H[\gamma_2])$ implies that $\gamma_1 \cdot \overline{\gamma}_2$ lifts to a loop in $\widetilde{X}$ based at $\widetilde{x}_0$, so $[\gamma_1][\gamma_2]^{-1} \in H$ and hence $H[\gamma_1] = H[\gamma_2]$. Thus the index of H is the same as $|\pi^{-1}(x_0)|$. □

Theorem (lifting criterion)

Let $\pi\colon(\widetilde{X},\widetilde{x}_0)\to(X,x_0)$ be a covering space and $f\colon(Y,y_0)\to(X,x_0)$ be a map. A lift $\widetilde{f}\colon(Y,y_0)\to(\widetilde{X},\widetilde{x}_0)$ of f exists if and only if $f_*(\pi_1(Y,y_0))\subseteq\pi_*(\pi_1(\widetilde{X},\widetilde{x}_0))$.

Proof.

The only if" statement is obvious since $f_*=\pi_*\circ f_*$. Conversely, let $y\in Y$ and let γ be a path in Y from y_0 to y. By Corollary 19, the path $f\gamma$ in X starting at x_0 has a unique lift $\widetilde{f\gamma}$ starting at $\widetilde{x}_0$, and we define $\widetilde{f}(y)=\widetilde{f\gamma}(1)$.

To see it's well-defined, let γ' be another path from y_0 to y. Then $(f\gamma')\cdot(\overline{f\gamma})$ is a loop h_0 at x_0 with $[h_0]\in f_*(\pi_1(Y,y_0))\subseteq\pi_*(\pi_1(\widetilde{X},\widetilde{x}_0))$. This means there is a homotopy H of h_0 to a loop h_1 that lifts to a loop $\widetilde{h}_1$ in $\widetilde{X}$ based at $\widetilde{x}_0$.

Continuation.

Apply Theorem 18 to H to get a lifting $\widetilde{H}$. Since $\widetilde{h}_1$ is a loop at $\widetilde{x}_0$, so is $\widetilde{h}_0$. By Theorem 17, that is uniqueness of lifted paths, the first half of $\widetilde{h}_0$ is $\widetilde{f\gamma'}$ and the second half is $\widetilde{f\gamma}$ traversed backwards, with the common midpoint $\widetilde{f\gamma}(1) = \widetilde{f\gamma'}(1)$. This shows $\widetilde{f}$ is well-defined.

To see $\widetilde{f}$ is continuous, let $U \subseteq X$ be an open neighborhood of $f(y)$ having a lift $\widetilde{U} \subseteq \widetilde{X}$ containing $\widetilde{f}(y)$ such that $\pi \colon \widetilde{U} \to U$ is a homeomorphism. Choose a path-connected open neighborhood V of y with $f(V) \subseteq V$. For paths from y_0 to points $y' \in V$, we can take a fixed path γ from y_0 to y followed by paths η in V from y to points y'. Then the paths $(f\gamma) \cdot (f\eta)$ in X have lifts $(\widetilde{f\gamma}) \cdot (\widetilde{f\eta})$ where $\widetilde{f\eta} = \pi^{-1} f\eta$. Thus $\widetilde{f}(V) \subseteq \widetilde{U}$ and $\widetilde{f}|_V = \pi^{-1} f$, so $\widetilde{f}$ is continuous at y. $\square$

Definition (universal covering)

A simply-connected covering space of X is called universal covering.

Definition (semilocally simply-connected)

A topological space X is called semilocally simply-connected if each $x \in X$ has a neighborhood U such that the inclusion induced map $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.

Theorem

If X is a semilocally simply-connected topological space, then X has a universal covering $\widetilde{X}$.

Theorem

Suppose X is a semilocally simply-connected topological space. Then for every subgroup $H \subseteq \pi_1(X, x_0)$, there exists a covering space $\pi \colon X_H \to X$ such that $\pi_*(\pi_1(X_H, \widetilde{x}_0)) = H$ for a suitably chosen based point $\widetilde{x}_0 \in X_H$.

Lemma

Let $\pi_1\colon \widetilde{X}_1 \to X$ and $\pi_2\colon \widetilde{X}_2 \to X$ be two coverings. There exists an isomorphism $f\colon \widetilde{X}_1 \to \widetilde{X}_2$ taking a basepoint $\widetilde{x}_1 \in \pi_1^{-1}(x_0)$ to a basepoint $\widetilde{x}_2 \in \pi_2^{-1}(x_0)$ if and only if $\pi_{1*}(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = \pi_{2*}(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$.

Proof.

If there is an isomorphism $f\colon (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$, then from the two relations $\pi_1 = \pi_2 \circ f$ and $\pi_2 = \pi_1 \circ f^{-1}$ it follows that $\pi_{1*}(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = \pi_{2*}(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$. Conversely, suppose that $\pi_{1*}(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = \pi_{2*}(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$. By Theorem 22, that is lifting criterion, we may lift π_1 to a map $\widetilde{\pi}_1\colon (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ with $\pi_2 \circ \widetilde{\pi}_1 = \pi_1$. Similarly, one has $\widetilde{\pi}_2\colon (\widetilde{X}_2, \widetilde{x}_2) \to (\widetilde{X}_1, \widetilde{x}_1)$ with $\pi_1 \circ \widetilde{\pi}_2 = \pi_2$. Then Theorem 17, that is the unique lifting property, $\widetilde{\pi}_1 \circ \widetilde{\pi}_2 = \mathrm{id}$ and $\widetilde{\pi}_2 \circ \widetilde{\pi}_1 = \mathrm{id}$ since these composed lifts fix the basepoints. □

Lemma

For covering $\pi\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$, changing the basepoint $\widetilde{x}_0$ within $\pi^{-1}(x_0)$ corresponds exactly to changing $\pi_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ to a conjugate subgroup of $\pi_1(X, x_0)$.

Proof.

Let $\widetilde{x}_1$ be another basepoint in $\pi^{-1}(x_0)$ and $\widetilde{\gamma}$ be a path from $\widetilde{x}_0$ to $\widetilde{x}_1$. Then $\widetilde{\gamma}$ projects to a loop γ in X representing some element $g \in \pi_1(X, x_0)$. If we denote $H_i = \pi_*(\pi_1(\widetilde{X}, \widetilde{x}_i))$ for $i = 0, 1$, there is an inclusion $g^{-1}H_0 g \subseteq H_1$ since if $\widetilde{f}$ is a loop at $\widetilde{x}_0$, one has $\widetilde{\gamma}^{-1} \cdot \widetilde{f} \cdot \widetilde{\gamma}$ is a loop at $\widetilde{x}_1$. Similarly one has $gH_1 g^{-1} \subseteq H_0$. This shows changing the basepoint from $\widetilde{x}_0$ to $\widetilde{x}_1$ changes H_0 to the conjugate subgroup $H_1 = g^{-1}H_0 g$. □

Theorem

Let X be a semilocally simply-connected topological space. Then there is a bijection between the set of basepoint-preserving isomorphism classes of covering spaces $\pi : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ and the the set of subgroups of $\pi_1(X, x_0)$ obtained by associating the subgroup $\pi_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ to the covering space $(\widetilde{X}, \widetilde{x}_0)$. If basepoints are ignored, this correspondence gives a bijection between isomorphism classes of covering spaces $\pi : \widetilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$.

Proof.

Theorem 26 and Lemma 27 completes the proof of the first half, and Lemma 28 completes the proof of the last half. □

1. Overview
2. Homotopy and fundamental group
3. Covering space
   - Lifting theorems
   - The classification of the covering spaces
   - The structure of the deck transformation group
4. Continuous group action

Definition (deck transformation)

Let $\pi\colon \widetilde{X} \to X$ be a covering space. The deck transformation group is following set

$$\mathrm{Aut}_\pi(\widetilde{X}) = \{f\colon \widetilde{X} \to \widetilde{X} \text{ is homeomorphism} \mid \pi \circ f = \pi\}$$

equipped with composition as group operation.

Definition (normal)

A covering $\pi\colon \widetilde{X} \to X$ is called normal, if any deck transformation acts transitively on each fiber of $x \in X$.

Lemma

Let $\pi\colon \widetilde{X} \to X$ be a covering space. The deck transformation group $\operatorname{Aut}_\pi(\widetilde{X})$ acts on $\widetilde{X}$ freely.

Proof.

Suppose $f\colon \widetilde{X} \to \widetilde{X}$ is a deck transformation admitting a fixed point. Since $\pi \circ f = p$, we may regard f as a lift of π, and identity map of $\widetilde{X}$ is another lift of π. By Theorem 17, that is unique lifting property, one has f is exactly identity map since it agrees with identity map at fixed point. $\square$

Lemma

Let $\pi\colon \widetilde{X} \to X$ be a normal covering. Then $\widetilde{X}/\operatorname{Aut}_\pi(\widetilde{X})$ is homeomorphic to X.

Proof.

Let $\Phi\colon \widetilde{X}/\operatorname{Aut}_\pi(\widetilde{X}) \to X$ be the map sending the orbit $\mathcal{O}_{\widetilde{x}}$ to $\pi(\widetilde{x})$, where $\widetilde{x} \in \widetilde{X}$. It's clear Φ is well-defined bijection since $\operatorname{Aut}_\pi(\widetilde{X})$ acts on $\widetilde{X}$ fiberwise transitive, and the following diagram commutes

$$\begin{tikzcd} \widetilde{X} \arrow[d, "p"'] \arrow[dr, "\pi"] & \\ \widetilde{X}/\operatorname{Aut}_\pi(\widetilde{X}) \arrow[r, "\Phi"'] & X \end{tikzcd}$$

This diagram shows Φ is both continuous and open, since p is the quotient map and π is continuous and open, which shows $\widetilde{X}/\operatorname{Aut}_\pi(\widetilde{X})$ is homeomorphic to X.

Theorem

Let $\pi\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ *be a covering space and* $H = \pi_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \subseteq \pi_1(X, x_0)$. *Then*

1. π *is a normal covering if and only if* H *is a normal subgroup of* $\pi_1(X, x_0)$.
2. $\mathrm{Aut}_\pi(\widetilde{X})$ *is isomorphic to the quotient* $N(H)/H$, *where* $N(H)$ *is the normalizer of* H *in* $\pi_1(X, x_0)$.

Proof.

For (1). By proof of Lemma 28 one has changing the basepoint $\widetilde{x}_0 \in \pi^{-1}(x_0)$ to $\widetilde{x}_1 \in \pi^{-1}(x_0)$ corresponds precisely to conjugating H by an element $[\gamma] \in \pi_1(X, x_0)$ where γ lifts to a path $\widetilde{\gamma}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$. Thus $[\gamma]$ is in the normalizer $N(H)$ if and only if $\pi_*(\pi_1(\widetilde{x}, \widetilde{x}_0)) = \pi_*(\pi_1(\widetilde{x}, \widetilde{x}_1))$, which is equivalent to the existence of a deck transformation taking $\widetilde{x}_0$ to $\widetilde{x}_1$ by Lemma 27. Thus the covering space is normal if and only if $N(H) = \pi_1(X, x_0)$.

Continuation.

For (2). Define $\varphi\colon N(H) \to \mathrm{Aut}_\pi(\widetilde{X})$ by sending $[\gamma]$ to the deck transformation τ taking $\widetilde{x}_0$ to $\widetilde{x}_1$, in the notation above. Let's show φ is a homomorphism. If γ' is another loop corresponding to the deck transformation τ' taking $\widetilde{x}_0$ to $\widetilde{x}_1'$, then $\gamma \cdot \gamma'$ lifts to $\tilde{\gamma} \cdot (\tau(\tilde{\gamma}'))$, a path from $\widetilde{x}_0$ to $\tau(\widetilde{x}_1') = \tau\tau'(\widetilde{x}_0)$, so $\tau\tau'$ is the deck transformation corresponding to $[\gamma][\gamma']$. By the proof of (1) one has φ is surjective. The kernel of φ consists of classes $[\gamma]$ lifting to loops in $\widetilde{x}$, which are exactly the elements of $\pi_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = H$. □

Corollary

Let $\pi\colon \widetilde{X} \to X$ be the universal covering. Then $\mathrm{Aut}_\pi(\widetilde{X}) \cong \pi_1(X, x_0)$.

Definition (act by homeomorphisms)

Let Γ be a group and X be a topological space. The group Γ is calld acting X by homeomorphisms, if Γ acts on X, and for every $g \in \Gamma$, the map $x \mapsto gx$ is a homeomorphism.

Definition (topological group)

A group is called a topological group, if it's a topological space such that the multiplication and the inversion are continuous.

Definition (continuous action)

Let X be a topological space and G a topological group. A continuous G-action on X is given by the following data:

1. G acts on X by homeomorphisms.
2. The map $G \times X \to X$ given by $(g, x) \mapsto gx$ is continuous.

Lemma

Let X be a topological space and Γ a group acting on X by homeomorphisms. Then the quotient map $\pi\colon X \to X/\Gamma$ is an open map.

Proof.

For any $g \in \Gamma$ and any subset $U \subseteq X$, the set $gU \subseteq X$ is defined as

$$gU = \{gx \mid x \in U\}$$

If $U \subseteq X$ is open, then $\pi^{-1}(\pi(U))$ is the union of all sets of the form gU as g ranges over G. Since $p \mapsto gp$ is a homeomorphism, each set is open, and therefore $\pi^{-1}(\pi(U))$ is open in X. Since π is a quotient map, this implies $\pi(U)$ is open in X/Γ, and therefore π is an open map. $\square$

Definition (proper)

Let X be a topological space and G a topological group. A continuous G-action on X is called proper if the continuous map

$$\begin{aligned}\Theta\colon G\times X&\to X\times X\\ (g,x)&\mapsto(gx,x)\end{aligned}$$

is proper, that is, the preimage of a compact set is compact.

Lemma

Let X, Y be topological spaces and $\pi\colon X\to Y$ be an open quotient map. Then Y is Hausdorff if and only if the set $\mathcal{R}=\{(x_1,x_2)\mid\pi(x_1)=\pi(x_2)\}$ is closed in $X\times X$.

Lemma

Let X be a topological space and G a topological group acting on X continuously. If the action is also proper, then the orbit space is Hausdorff.

Proof.

Let $\Theta\colon G \times X \to X \times X$ be the proper map $\Theta(g, x) = (gx, x)$ and $\pi\colon X \to X/G$ be the quotient map. Define the orbit relation $\mathcal{O} \subseteq X \times X$ by

$$\mathcal{O} = \Theta(G \times X) = \{(gx, x) \mid x \in X, g \in G\}$$

Since proper continuous map is closed, it follows that $\mathcal{O}$ is closed in $X \times X$, and since π is open by Lemma 39, one has X/G is Hausdorff by Lemma 41. □

Theorem

Let M be a topological manifold and G a topological group acting on M continuously. The following statements are equivalent.

1. *The action is proper.*
2. *If $\{p_i\}$ is a sequence in M and $\{g_i\}$ is a sequence in G such that both $\{p_i\}$ and $\{g_i p_i\}$ converge, then a subsequence of $\{g_i\}$ converges.*
3. *For every compact subset $K \subseteq M$, the set $G_K = \{g \in G \mid gK \cap K \neq \varnothing\}$ is compact.*

Proof.

Along the proof, let $\Theta\colon G \times M \to M \times M$ denote the map $(g, p) \mapsto (gp, p)$. For (1) to (2). Suppose Θ is proper, and $\{p_i\}$, $\{g_i\}$ are sequences satisfying the hypotheses of (2). Let U and V be precompact[2] neighborhoods of $p = \lim_i p_i$ and $q = \lim_i g_i p_i$. The assumption implies $\Theta(g_i, p_i)$ all lie in compact set $\overline{U} \times \overline{V}$ when i is sufficiently large, so there exists a subsequence of $\{(g_i, p_i)\}$ converges in $G \times M$ since Θ is proper. In particular, this means that a subsequence of $\{g_i\}$ converges in G.

For (2) to (3). Let K be a compact subset of M, and suppose $\{g_i\}$ is any sequence in G_K. This means for each i, there exists $p_i \in g_i K \cap K$, which is to say that $p_i \in K$ and $g_i^{-1} p_i \in K$. By passing to a subsequence twice, we may assume both $\{p_i\}$ and $\{g_i^{-1} p_i\}$ converge, and the assumption implies there exists a convergent subsequence of $\{g_i\}$. Since each sequence of G_K has a convergent subsequence, G_K is compact.

Continuation.

For (3) to (1). Suppose $L \subseteq M \times M$ is compact, and let $K = \pi_1(L) \cup \pi_2(L)$, where $\pi_1, \pi_2 \colon M \times M \to M$ are the projections onto the first and second factors, respectively. Then

$$\Theta^{-1}(L) \subseteq \Theta^{-1}(K \times K) = \{(g, p) \mid gp \in K, p \in K\} \subseteq G_K \times K$$

By assumption $G_K \times K$ is compact, and thus $\Theta^{-1}(L)$ is compact since it's a closed subset of a compact subset, which implies the action is proper. □

Corollary

Let M be a topological manifold and G a compact topological group. Then every continuous G-action on M is proper.

Definition (properly discontinuous)

Let Γ be a group acting on a topological space X by homeomorphisms. The action is called properly discontinuous, if every point $x \in X$ has a neighborhood U such that for each $g \in G$, $gU \cap U = \varnothing$ unless $g = e$.

Lemma

Suppose Γ be a group acting properly discontinuous on a topological space X. Then every subgroup of Γ still acts properly discontinuous on X.

Lemma

Let $\pi\colon \widetilde{X} \to X$ be a covering space. Then $\mathrm{Aut}_\pi(\widetilde{X})$ acts on $\widetilde{X}$ properly discontinuous.

Proof.

Let $\widetilde{U} \subseteq \widetilde{X}$ project homeomorphically to $U \subseteq X$. For $g \in \mathrm{Aut}_\pi(\widetilde{X})$, if $g(\widetilde{U}) \cap \widetilde{U} \neq \varnothing$, then $g\widetilde{x}_1 = \widetilde{x}_2$ for some $\widetilde{x}_1, \widetilde{x}_2 \in U$. Since $\widetilde{x}_1$ and $\widetilde{x}_2$ lie in the same set $\pi^{-1}(x)$, which intersects $\widetilde{U}$ in only one point, we must have $\widetilde{x}_1 = \widetilde{x}_2 = \widetilde{x}$. Then $\widetilde{x}$ is a fixed point of g, which implies $g = e$ by Lemma 32. $\square$

Theorem (covering space quotient theorem)

Let E be a topological space and Γ be a group acting on E by homeomorphisms effectively . Then the quotient map $\pi: E \to E/\Gamma$ is a covering map if and only if Γ acts on E properly discontinuous. In this case, π is a normal covering and $\mathrm{Aut}_\pi(E) = \Gamma$.

Proof.

Firstly, assume π is a covering map. Then the action of each $g \in \Gamma$ is an automorphism of the covering since it's a homeomorphism satisfying $\pi(ge) = \pi(e)$ for all $g \in \Gamma, e \in E$, so we can identify Γ with a subgroup of $\mathrm{Aut}_\pi(E)$. Then Γ acts on E properly discontinuous by Lemma 46 and Lemma 47.

Continuation.

Conversely, suppose the action is properly discontinuous. To show π is a covering map, suppose $x \in E/\Gamma$ is arbitrary. Choose $e \in \pi^{-1}(x)$, and let U be a neighborhood of e such that for each $g \in \Gamma$, $gU \cap U = \varnothing$ unless $g = 1$. Since E is locally path-connected, by passing to the component of U containing e, we may assume U is path-connected. Let $V = \pi(U)$, which is a path-connected neighborhood of x. Now $\pi^{-1}(V)$ is equal to the union of the disjoint connected open subsets gU for $g \in \Gamma$, so to show π is a covering space it remains to show π is a homeomorphism from each such set onto V. For each $g \in \Gamma$, the restriction map $g\colon U \to gU$ is a homeomorphism, and the diagram

$$\begin{tikzcd} U \arrow[rr, "g"] \arrow[dr, "\pi"'] & & gU \arrow[dl, "\pi"] \\ & V & \end{tikzcd}$$

Continuation.

Thus it suffices to show $\pi|_U : U \to V$ is a homeomorphism. It's surjective, continuous and open, and it's injective since $\pi(e) = \pi(e')$ for $e, e' \in U$ implies $e' = ge$ for some $g \in \Gamma$, so $e = e'$ by the choice of U. This shows π is a covering map.
To prove the final statement of the theorem, suppose the action is a covering space action. As noted above, each map $e \mapsto ge$ is a covering automorphism, so $\Gamma \subseteq \mathrm{Aut}_\pi(E)$. By construction, Γ acts transitively on each fiber, so $\mathrm{Aut}_\pi(E)$ does too, and thus π is a normal covering. If φ is any covering automorphism, choose $e \in E$ and let $e' = \varphi(e)$. Then there is some $g \in \Gamma$ such that $ge = e'$. Since φ and $x \mapsto gx$ are deck transformation that agree at a point, so they are equal. Thus $\Gamma = \mathrm{Aut}_\pi(E)$. $\square$

Lemma

Suppose G is a discrete topological group acting continuously and freely on a topological manifold M. The action is proper if and only if the following conditions both hold.

1. *Every point $p \in M$ has a neighborhood U such that for each $g \in G$, $gU \cap U = \varnothing$ unless $g = e$.*
2. *If $p, p' \in M$ are not in the same orbit, there exist a neighborhood V of p and V' of p' such that $gV \cap V' = \varnothing$ for all $g \in G$.*

Proof.

Firstly, suppose that the action is free and proper, and let $\pi \colon M \to M/G$ be the quotient map. By Lemma 42, the orbit space M/G is Hausdorff. If $p, p' \in M$ are not in the same orbit, we can choose disjoint neighborhoods W of $\pi(p)$ and W' of $\pi(p')$.

Continuation.

Then $V=\pi^{-1}(W)$ and $V'=\pi^{-1}(W')$ satisfy the conclusion of condition (2).

Now let's prove (1). Let $p\in M$ and V be a precompact neighborhood of p. By Theorem 43, the set $G_{\overline{V}}$ is a compact subset of G, and hence finite because G is discrete. Write $G_{\overline{V}}=\{e,g_1,\dots,g_m\}$. Shrinking V if necessary, we may assume that $g_i^{-1}p\notin\overline{V}$ for $i=1,\dots,m$. Then the open subset

$$U=V\backslash(g_1\overline{V}\cup\cdots\cup g_m\overline{V})$$

satisfies the conclusion of (1).

Conversely, assume that (1) and (2) hold. Suppose $\{g_i\}$ is a sequence in G and (p_i) is a sequence in E such that $p_i\to p$ and $g_ip_i\to p'$. If p and p' are in different orbits, there exist neighborhoods V of p and V' of p' as in (2).

Continuation.

But for large enough i, we have $p_i \in V$ and $g_i p_i \in V'$, which contradicts the fact that $(g_i V) \cap V' = \varnothing$. Thus, p and p' are in the same orbit, so there exists $g \in G$ such that $gp = p'$. This implies $g^{-1} g_i p_i \to p$. Choose a neighborhood U of p as in (i), and let i be large enough that p_i and $g^{-1} g_i p_i$ are both in U. Because $(g^{-1} g_i U) \cap U \neq \varnothing$, it follows that $g^{-1} g_i = e$. So $g_i = g$ when i is large enough, which certainly converges. By (2) of Theorem 43, the action is proper. $\square$

Theorem

Let M be a topological manifold and $\pi\colon \widetilde{M} \to M$ be a covering space. If $\mathrm{Aut}_\pi(\widetilde{M})$ is equipped with the discrete topology, then it acts on $\widetilde{M}$ continuously, freely and properly.

Proof.

By Lemma 32 one has $\mathrm{Aut}_\pi(\widetilde{M})$ acts on $\widetilde{M}$ freely and the action is also continuously since $\mathrm{Aut}_\pi(\widetilde{M})$ is equipped with discrete topology. To see the action is properly, we show the action satisfies the two conditions in Lemma 49.

Continuation.

(a) Firstly, if $\widetilde{x} \in \widetilde{M}$ is arbitrary, choose $W \subseteq M$ to be an evenly covered neighborhood of $\pi(\widetilde{x})$. If U is the component of $\pi^{-1}(W)$ containing $\widetilde{x}$, then it is easy to check that U satisfies (1).

(b) Secondly, if $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{M}$ are in different orbits, then just as in the proof Lemma 32, we can choose disjoint neighborhoods W of $\pi(\widetilde{x}_1)$ and W' of $\pi(\widetilde{x}_2)$, and it follows that $V = \pi^{-1}(W)$ and $V' = \pi^{-1}(W')$ satisfy (2).

□

Thanks!