Solutions to Homework

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Contents

Chapter 1

Solutions to Homework-2

Exercise. *Show that two different reduced row echelon system of linear equations have different solutions (if the solutions exist). Derive that the reduced row echelon matrix associated to a given matrix is unique.*

Proof. For the first part, suppose reduced row echelon systems $A_1X = 0, A_2X = 0$ have the same solutions and show the row echelon system. Then the number of pivots and free unknowns are same, and the solutions of $A_1X = 0, A_2X = 0$ are given by the combinations of these unknowns and entries of A_1 and A_2 respectively. As a result $A_1 = A_2$ since $A_1X = 0$ and $A_2X = 0$ have the same solutions.

For the second part, if A is reduced to A_1 and A_2, then $A_1X = 0$ has the same solutions as $A_2X = 0$, since both of them have the same solutions as $AX = 0$, and thus $A_1 = A_2$. □

Exercise. *Find all solutions of the equation $x_1 + x_2 + 2x_3 - x_4 = 3$.*

Proof. All possible solutions in $\mathbb{R}$ are given by

$$x = \begin{pmatrix} a \\ b \\ c \\ a+b+2c-3 \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$. □

Exercise. *Find all solutions of the system of equations $AX = B$ when*

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 0 & 0 & 4 \\ 1 & -4 & -2 & 2 \end{pmatrix}$$

and

(a)

$$B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(b)

$$B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

(c)

$$B = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

Proof. For (a). By Gaussian elimination one has

$$[A \mid B] = \begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 3 & 0 & 0 & 4 & 0 \\ 1 & 4 & -2 & 2 & 0 \end{pmatrix} \implies \begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & -6 & -3 & 1 & 0 \\ 0 & 2 & -3 & 1 & 0 \end{pmatrix} \implies \begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & -6 & -3 & 1 & 0 \\ 0 & 0 & -4 & \frac{4}{3} & 0 \end{pmatrix} \implies \begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & -6 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This shows all possible solutions over $\mathbb{R}$ is given by

$$x = \begin{pmatrix} -\frac{3}{4}b \\ \frac{1}{6}(-3a+b) \\ a \\ b \end{pmatrix},$$

where $a, b \in \mathbb{R}$. Similarly one can show the solutions of (b) is empty set and (c) are given by

$$x = \begin{pmatrix} \frac{3}{2} - \frac{4}{3}b \\ -\frac{1}{3} - \frac{1}{2}a + \frac{1}{6}b \\ a \\ b \end{pmatrix},$$

where $a, b \in \mathbb{R}$. □

Exercise. *Find the inverse of the following matrix by elementary row reduction:*

$$\begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}$$

Proof. The inverse of above matrix is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}$$

□

Exercise. *Write the following permutations as products of transpositions, and determine their sign.*

(a) $1, 3, 5, 2, 4, 8, 6, 7,$

(b) $9, 5, 3, 8, 4, 6, 2, 1, 7,$

(c) $7, 1, 6, 2, 5, 3, 4.$

Proof. For (a). Firstly note that the permutation

$$\begin{pmatrix} 1,2,3,4,5,6,7,8 \\ 1,3,5,2,4,8,6,7 \end{pmatrix}$$

can be written as $(1)(2,3,5,4)(6,8,7)$, and thus it can be written as products of transpositions as follows

$$(1)(23)(25)(24)(68)(67),$$

which implies it's an odd permutation. By the same method one can write the other permutations as products of transpositions. $\square$

Exercise. *Evaluate the following determinants:*

(a) $\begin{pmatrix} 1 & i \\ 2-i & 3 \end{pmatrix}$,

(b) $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$,

(c) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$,

(d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 5 & 2 & 0 & 0 \\ 8 & 6 & 3 & 0 \\ 0 & 9 & 7 & 4 \end{pmatrix}$.

Proof. The determinants are $2-2i, -2, 3, 24$ respectively. $\square$

Exercise. *Compute the determinant of the following $n \times n$ matrix using induction on n :*

$$\begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & \cdot & & & \\ & & & & \cdot & & \\ & & & & & 2 & -1 \\ & & & & & -1 & 2 \end{pmatrix}.$$

Proof. Let T_n denote above $n \times n$ matrix. It's clear $\det T_1 = 2$ and $\det T_2 = 3$. Now let's prove $\det T_n = n+1$ by induction. Suppose it holds for $n < k$. Then for $n = k$, by expansion along the first row one has

$$\det T_k = 2\det T_{k-1} - (-1) \times (-1)\det T_{k-2} = 2k - (k-1) = k+1.$$

$\square$

Exercise. *Suppose $R = K[t]$ is the polynomial ring and $A = (a_{ij}) \in M_n(R)$. Show that*

(a)

$$\frac{\partial \det A}{\partial t} = \det A_1 + \cdots + \det A_n,$$

where A_i is obtained from A by taking the derivative of the i-th row and keep the other rows.

(b)

$$\frac{\partial \det A}{\partial t} = \sum_{i,j} (-1)^{i+j} \frac{\partial a_{ij}}{\partial t} \det a_{ij},$$

where $a_{ij} = A^c \begin{pmatrix} i \\ j \end{pmatrix}$.

Proof. For (a). Note that

$$\begin{aligned}\frac{\partial \det A}{\partial t} &= \frac{\partial}{\partial t}\left(\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \dots a_{n\sigma(n)}\right) \\ &= \sum_k \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\left(a_{1\sigma(1)} \dots \frac{\partial a_{k\sigma(k)}}{\partial t} \dots a_{n\sigma(n)}\right) \\ &= \det A_1 + \dots + \det A_n.\end{aligned}$$

For (b). It suffices to note that

$$A_k = \sum_i (-1)^{k+j} \frac{\partial a_{kj}}{\partial t} A_{k,j}.$$

Then by (a), one has

$$\det A = \det A_1 + \dots + \det A_n = \sum_{i,j} (-1)^{i+j} \frac{\partial a_{ij}}{\partial t} A_{i,j}$$

$\square$

Exercise. *Suppose* $\det \begin{pmatrix} x & y & z \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} = 1$. *Compute the following determinant.*

(1) $\begin{pmatrix} 2x & 2y & 2z \\ \frac{3}{2} & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(2) $\begin{pmatrix} x & y & z \\ 3x+3 & 3y & 3z+2 \\ x+1 & y+1 & z+1 \end{pmatrix}$.

(3) $\begin{pmatrix} x-1 & y-1 & z-1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}$.

Proof. By elementary operations, one can see all of above three determinants equal to the determinant of

$$\begin{pmatrix} x & y & z \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$\square$

Exercise. *Calculate the Vandermonde determinant*

$$\det \begin{pmatrix} 1 & 1 & \dots & 1 \\ \theta_1 & \theta_2 & \dots & \theta_n \\ \vdots & \vdots & & \vdots \\ \theta_1^{n-1} & \theta_2^{n-1} & \dots & \theta_n^{n-1} \end{pmatrix}.$$

Proof. Now let's prove the Vandermonde determinant equals $\prod_{1\le i<j\le n}(\theta_j-\theta_i)$ by induction. It holds for $n=2$, and suppose it holds for $n<k$. Let V denote the Vandermonde matrix. Then

$$\begin{aligned}
\det V &= \det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & \theta_2-\theta_1 & \cdots & \theta_k-\theta_1 \\ \vdots & \vdots & & \vdots \\ 0 & \theta_2^{k-1}-\theta_1^{k-1} & \cdots & \theta_k^{k-1}-\theta_1^{k-1} \end{pmatrix} \\
&= \det\begin{pmatrix} \theta_2-\theta_1 & \cdots & \theta_k-\theta_1 \\ \theta_2^2-\theta_1^2 & \cdots & \theta_k^2-\theta_1^2 \\ \vdots & \vdots & \vdots \\ \theta_2^{k-1}-\theta_1^{k-1} & \cdots & \theta_k^{k-1}-\theta_1^{k-1} \end{pmatrix} \\
&= \det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \theta_2+\theta_1 & \theta_3+\theta_1 & \cdots & \theta_k+\theta_1 \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{i=0}^{k-2}\theta_2^{k-2-i}\theta_1^i & \sum_{i=0}^{k-2}\theta_3^{k-2-i}\theta_1^i & \cdots & \sum_{i=0}^{k-2}\theta_k^{k-2-i}\theta_1^i \end{pmatrix}\begin{pmatrix} \theta_2-\theta_1 & 0 & \cdots & 0 \\ 0 & \theta_3-\theta_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \theta_k-\theta_1 \end{pmatrix}
\end{aligned}$$

Note that

$$\begin{aligned}
&\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \theta_2+\theta_1 & \theta_3+\theta_1 & \cdots & \theta_k+\theta_1 \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{i=0}^{k-2}\theta_2^{k-2-i}\theta_1^i & \sum_{i=0}^{k-2}\theta_3^{k-2-i}\theta_1^i & \cdots & \sum_{i=0}^{k-2}\theta_k^{k-2-i}\theta_1^i \end{pmatrix} \\
=&\begin{pmatrix} 1 & 0 & \cdots & 0 \\ \theta_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ \theta_1^{n-2} & \theta_1^{n-3} & \cdots & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \theta_2 & \theta_3 & \cdots & \theta_k \\ \vdots & \vdots & & \vdots \\ \theta_2^{k-2} & \theta_3^{k-2} & \cdots & \theta_k^{k-2} \end{pmatrix}.
\end{aligned}$$

Then by induction hypothesis one has

$$\det V = \prod_{j=2}^{k}(x_j-x_1)\prod_{2\le i<j\le k}(x_j-x_i) = \prod_{1\le i<j\le k}(x_j-x_i)$$

as desired. □

Exercise. *Let a $2n\times 2n$ matrix be given in the form $M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}$, where each block is an $n\times n$ matrix. Suppose that A is invertible and that $AC=CA$. Use block multiplication to prove that $\det M=\det(AD-CB)$. Give an example to show that this formula need not hold if $AC\ne CA$.*

Proof. Note that

$$\begin{aligned}
\det M &= \det\begin{pmatrix} A & B \\ C & D\end{pmatrix} \\
&= \begin{pmatrix} A & B \\ O & D-CA^{-1}B\end{pmatrix} \\
&= \det A\cdot\det(D-CA^{-1}B) \\
&= \det(AD-ACA^{-1}B) \\
&= \det(AD-CB).
\end{aligned}$$

□

Exercise. *Suppose* $a_{ii} > 0$ *and* $a_{ij} < 0$ *for* $i \neq j$. *Suppose in addition that* $\sum_{i=1}^n a_{ij} > 0$ *for all* j. *Show that* $\det(a_{ij}) > 0$.

Proof. Let's prove this by induction. For $n = 1$,

$$\det(a_{ij}) = a_{11} > 0.$$

Now suppose it holds for $n < k$. Then for $n = k$, note that

$$\begin{aligned}\det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix} &= \det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ 0 & a_{22} - \frac{a_{12}}{a_{11}}a_{21} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ 0 & a_{k2} - \frac{a_{12}}{a_{11}}a_{k1} & \cdots & a_{kk} \end{pmatrix} \\ &= a_{11}\det\begin{pmatrix} a_{22} - \frac{a_{12}}{a_{11}}a_{21} & \cdots & a_{2k} \\ \vdots & & \vdots \\ a_{k2} - \frac{a_{12}}{a_{11}}a_{k1} & \cdots & a_{kk} \end{pmatrix}\end{aligned}$$

Note that $a_{22} - \dfrac{a_{12}}{a_{11}}a_{21} > 0$ and $-\dfrac{a_{12}}{a_{11}}(a_{21} + \cdots + a_{k1}) > -\dfrac{a_{12}}{a_{11}}(-a_{11}) = a_{12}$. Then by induction hypothesis

$$\det\begin{pmatrix} a_{22} - \frac{a_{12}}{a_{11}}a_{21} & \cdots & a_{2k} \\ \vdots & & \vdots \\ a_{k2} - \frac{a_{12}}{a_{11}}a_{k1} & \cdots & a_{kk} \end{pmatrix} > 0.$$

This completes the proof. $\square$

Exercise. *Suppose* $A \in M_{n\times s}(R)$ *and* $B \in M_{s\times n}(R)$. *Prove*

$$\det(AB) = \begin{cases} 0, & n > s; \\ \det A \cdot \det B, & n = s; \\ \sum_{1 \leqslant k_1 < k_2 < \cdots < k_n \leqslant s} \det A\begin{pmatrix} 12\cdots n \\ k_1k_2\cdots k_n \end{pmatrix} \cdot \det B\begin{pmatrix} k_1k_2\cdots k_n \\ 12\cdots n \end{pmatrix}, & n < s. \end{cases}$$

Proof. If $n > s$, then there exists a non-zero x such that $Bx = 0$, and thus $ABx = 0$. This shows the system of linear equations $ABX = 0$ has a non-zero solution, and thus $\det AB = 0$. If $n = s$, then both A, B are square matrices, so

$$\det AB = \det A \cdot \det B$$

by properties of determinants. If $n < s$, consider the matrix

$$M = \begin{pmatrix} A & O \\ I_s & B \end{pmatrix},$$

which is a $(n+s) \times (n+s)$ matrix. On one hand, one has

$$\det M = \det\begin{pmatrix} A & O \\ I_s & B \end{pmatrix} = \det\begin{pmatrix} O & -AB \\ I_s & B \end{pmatrix} = (-1)^{ns+n}\det AB.$$

On the other hand, by Laplacian expansion one has

$$\det M = \sum_{1 \leq k_1 < \cdots < k_n \leq s} (-1)^{\frac{n(n+1)}{2} + k_1 + \cdots + k_n} \det A\begin{pmatrix} 12\cdots n \\ k_1k_2\cdots k_n \end{pmatrix} \det M\begin{pmatrix} n+1n+2\cdots n+s \\ k_{n+1}k_{n+2}\cdots k_{n+s} \end{pmatrix},$$

where $\{k_1, \ldots, k_{n+s}\}$ is a permutation of $\{1, \ldots, n+s\}$. Note that among the first n rows, the last n columns are zeros, so if

$$\det A \begin{pmatrix} 12 \ldots n \\ k_1 k_2 \ldots k_n \end{pmatrix} \neq 0,$$

we must have $1 \leq k_1, \ldots, k_n \leq s$. In particular, one has

$$M \begin{pmatrix} n+1 n+2 \cdots n+s \\ k_{n+1} k_{n+2} \cdots k_{n+s} \end{pmatrix} = \begin{pmatrix} I_s \begin{pmatrix} 12 \ldots s \\ \mu_1, \ldots, \mu_{s-n} \end{pmatrix} & B \end{pmatrix},$$

where $\{k_1, \ldots, k_n\} \cup \{\mu_1, \ldots, \mu_{s-n}\} = \{1, 2, \ldots, s\}$. Again by Laplacian expansion one has

$$\det M \begin{pmatrix} n+1 n+2 \cdots n+s \\ k_{n+1} k_{n+2} \cdots k_{n+s} \end{pmatrix} = (-1)^{\frac{(s-n)(s-n+1)}{2} + \mu_1 + \cdots + \mu_{s-n}} \det B \begin{pmatrix} k_1 k_2 \ldots k_n \\ 12 \ldots n \end{pmatrix}.$$

Then

$$\det M = (-1)^{\frac{n(n+1)}{2} + \frac{(s-n)(s-n+1)}{2} + k_1 + \cdots + k_n + \mu_1 + \cdots + \mu_{s-n}} \sum_{1 \leq k_1 < \cdots < k_n \leq s} \det A \begin{pmatrix} 12 \cdots n \\ k_1 k_2 \cdots k_n \end{pmatrix} \cdot \det B \begin{pmatrix} k_1 k_2 \cdots k_n \\ 12 \cdots n \end{pmatrix}.$$

Note that

$$\begin{aligned} \frac{n(n+1)}{2} + \frac{(s-n)(s-n+1)}{2} + k_1 + \cdots + k_n + \mu_1 + \cdots + \mu_{s-n} &= n^2 + s^2 + s - ns \\ &\equiv ns + n \pmod 2. \end{aligned}$$

This completes the proof. $\square$

Chapter 2

Solutions to Homework-4

Exercise. *Show that* $\operatorname{rank}(AB) = \operatorname{rank} B$ *if and only if the solution space of* $AB\underline{x} = \underline{0}$ *is the same as the solution space of* $B\underline{x} = \underline{0}$*. Moreover, show that in this case, for any* C*, we have* $\operatorname{rank}(ABC) = \operatorname{rank}(BC)$ *whenever the product is well defined.*

Proof. Firstly it's clear the solution space of $Bx = 0$ is included in the one of $ABx = 0$. In other words, $\ker B \subseteq \ker AB$. Then $\ker A = \ker AB$ if and only if $\dim\ker B = \dim\ker AB$, which is equivalent to $\operatorname{rank} B = \operatorname{rank} AB$ since $\operatorname{rank} B = n - \dim\ker B$ and $\operatorname{rank} AB = n - \dim\ker AB$, where n is the number of columns of B.

In the case of $\ker B = \ker AB$, suppose $C = (v_1, \ldots, v_m)$ with $v_i \in \mathbb{R}^n$. Then $Bv_{i_1}, \ldots, Bv_{i_k}$ are linearly independent if and only if

$$c_1 v_{i_1} + \cdots + c_k v_{i_k} \in \ker B \implies c_1 = \cdots = c_k = 0.$$

But $\ker B = \ker AB$, this shows that $Bv_{i_1}, \ldots, Bv_{i_k}$ are linearly independent if and only if $ABv_{i_1}, \ldots, ABv_{i_k}$ are linearly independent. This shows $\operatorname{rank} ABC = \operatorname{rank} BC$. □

Exercise. *Suppose* $A \in M_n(\mathbb{R})$*. Show that* $\operatorname{rank}(A^t A) = \operatorname{rank} A$*.*

Proof. It suffices to show that the equations $Ax = 0$ and $A^t Ax = 0$ have the same solution space. It's clear that $Ax = 0$ implies $A^t Ax = 0$. On the other hand, if $A^t Ax = 0$, then

$$x^t A^t Ax = (Ax)^t Ax = 0.$$

This shows $Ax = 0$. □

Exercise. *Find a basis of the space of symmetric and skew-symmetric matrices over a field* K*, and compute their dimensions.*

Proof. It depends on the characteristic of the field K. If $\operatorname{char} K \neq 2$, then we have already shown in the Homework3 that

$$\{E_{ij} + E_{ji}\}_{i \neq j} \cup \{E_{ii}\}$$

gives a basis of the space of symmetric matrices, and thus the dimension of the space of symmetric matrices is $n(n+1)/2$. On the other hand,

$$\{E_{ij} + E_{ji}\}_{i \neq j}$$

gives a basis of the space of skew-symmetric matrices, and thus the dimension of the space of skewsymmetric matrices is $n(n-1)/2$. However, if $\operatorname{char} K = 2$, then the space of symmetric matrices and skew-symmetric matrices concide, both have dimension $n(n+1)/2$. □

Exercise. *Let* $K = \mathbb{Z}_p$ *be a finite field with* p *elements, where* p *is a prime. For positive integer* n*, compute the number of different basis of* K^n*.*

Proof. Note that it suffices to compute $\sharp\,\mathrm{GL}(n,\mathbb{Z}_p)$, since any two basis of K^n differs a unique element in $\mathrm{GL}(n,\mathbb{Z}_p)$. (In other words if you like, $\mathrm{GL}(n,\mathbb{Z}_p)$ acts on the set of basis of K^n transitively with trivial stabilizer.)

For $A\in\mathrm{GL}(n,\mathbb{Z}_p)$, there are p^n-1 choices for the first row, and if we have fixed the first columns, there are p^n-p choices for the second row since the second row has to be linearly independent with the first column. Repeat above arguments one can see there are

$$(p^n-1)(p^n-p)\dots(p^n-p^{n-1}).$$

□

Exercise.

(*a*) *Prove that the set* $\mathbf{B}=\big((1,2,0)^{\mathrm{t}},(2,1,2)^{\mathrm{t}},(3,1,1)^{\mathrm{t}}\big)$ *is a basis of* $\mathbb{R}^3$.

(*b*) *Find the coordinate vector of the vector* $v=(1,2,3)^t$ *with respect to this basis.*

(*c*) *Let* $\mathbf{B}'=\big((0,1,0)^t,(1,0,1)^t,(2,1,0)^t\big)$. *Determine the basechange matrix* P *from* $\mathbf{B}$ *to* $\mathbf{B}'$.

Proof. For (a). It suffices to compute the determinant of the matrix given by this basis.

For (b). It suffices to solve a system of linear equations.

For (c). It suffices to solve three systems of linear equations. □

Exercise. *Let* U,V,W *be three subspaces of a vector space. Is the following formula correct? Find a proof or a counterexample.*

$$\begin{aligned}\dim(U+V+W)=&\dim(U)+\dim(V)+\dim(W)\\&-\dim(U\cap V)-\dim(U\cap W)-\dim(V\cap W)\\&+\dim(U\cap V\cap W)\end{aligned}$$

Proof. It's wrong. Just consider the

$$U=\{(x,0)\mid x\in\mathbb{R}\},\quad V=\{(x,0)\mid x\in\mathbb{R}\},\quad W=\{(x,x)\mid x\in\mathbb{R}\}.$$

Then

$$U\cap V=U\cap W=V\cap W=U\cap V\cap W=\{0\}$$

But

$$\dim(U+V+W)=2\neq 3=\dim U+\dim V+\dim W.$$

□

Exercise. *Consider the linear transform:* $T\colon\ \mathbb{R}^3\to\mathbb{R}^2$ *such that* $(x_1,x_2,x_3)^t\mapsto(x_1+2x_2,x_1-x_2)^t$. *Compute the matrix of* T *with respect to the basis* $\alpha_1,\alpha_2,\alpha_3$ *of* $\mathbb{R}^3$ *and* β_1,β_2 *of* $\mathbb{R}^2$*:*

(*a*) $\alpha_1=(1,0,0)^{\mathrm{T}},\alpha_2=(0,1,0)^{\mathrm{T}},\alpha_3=(0,0,1)^{\mathrm{T}};\beta_1=(1,0)^{\mathrm{T}},\beta_2=(0,1)^{\mathrm{T}}$;

(*b*) $\alpha_1=(1,1,1)^{\mathrm{T}},\alpha_2=(0,1,1)^{\mathrm{T}},\alpha_3=(0,0,1)^{\mathrm{T}};\beta_1=(1,1)^{\mathrm{T}},\beta_2=(1,0)^{\mathrm{T}}$;

(*c*) $\alpha_1=(1,2,3)^{\mathrm{T}},\alpha_2=(0,1,-1)^{\mathrm{T}},\alpha_3=(-1,-2,3)^{\mathrm{T}};\beta_1=(1,2)^{\mathrm{T}},\beta_2=(2,1)^{\mathrm{T}}$.

Proof. A routine computation. □

Exercise. *Let* θ *be a real number. Consider the complex matrices*

$$A=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix},\quad B=\begin{pmatrix}\mathrm{e}^{\sqrt{-1}\theta} & 0\\ 0 & \mathrm{e}^{-\sqrt{-1}\theta}\end{pmatrix}.$$

Find a complex matrix P *such that* $P^{-1}AP=B$.

Proof. Note that
$$e^{\sqrt{-1}\theta} = \cos\theta + \sqrt{-1}\sin\theta.$$
Then
$$\begin{pmatrix} \sqrt{-1} & 1 \\ 1 & \sqrt{-1} \end{pmatrix}^{-1} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sqrt{-1} & 1 \\ 1 & \sqrt{-1} \end{pmatrix} = \begin{pmatrix} e^{\sqrt{-1}\theta} & 0 \\ 0 & e^{-\sqrt{-1}\theta} \end{pmatrix}.$$
□

Exercise. *Let W be a subspace of V, let $\pi\colon V \to V/W$ be the projection map. Let $g\colon V/W \to V/W$ be a linear transformation. Is there always a linear transformation $f\colon V \to V$ such that $g \circ \pi = \pi \circ f$?*

Proof. Suppose $\dim W = k$ with basis $\{w_1, \ldots, w_k\}$. Firstly we extend the basis of W to a basis of V as $\{e_1, \ldots, e_{n-k}, w_1, \ldots, w_k\}$, and then $\{e_1 + W, \ldots, e_{n-k} + W\}$ is a basis of V/W.

Given a linear transformation $g\colon V/W \to V/W$, one has
$$\begin{aligned} g \circ \pi(e_i) &= g(e_i + W) = g(e_i) + W \\ g \circ \pi(w_j) &= g(W) = W. \end{aligned}$$
Then we define a linear transformation $f\colon V \to V$ by evaluating on basis $\{e_1, \ldots, e_{n-k}, w_1, \ldots, w_k\}$ as
$$\begin{aligned} f(e_i) &= g(e_i) \\ f(w_i) &= w_i. \end{aligned}$$
Then it's a linear transformation which extends g. □

Exercise. *Let $f(x) \neq 0 \in K[x]$, where K is a field. Let $f(x) \cdot K[x]$ be the subspace of $K[x]$ consisting of polynomials divisible by $f(x)$.*

(a) *Find a basis of $V = K[x]/(f(x) \cdot K[x])$ and compute its dimension.*

(b) *Consider the linear transformation $T\colon V \to V$ such that $\bar{g}(x) \mapsto \bar{x} \cdot \bar{g}(x)$. Find the matrix representing T with respect to your basis.*

Proof. For (a). Suppose the degree of $f(x)$ is n. Then
$$\{1, x, x^2, \ldots, x^{n-1}\}$$
is a basis of $K[x]/(f(x)K[x])$. Indeed, it's clear above elements are linearly independent over K, and for any element $g(x)$ in $K[x]$ with degree higher than n, we can use division with remainders to write
$$g(x) = q(x)f(x) + r(x)$$
where $\deg r(x) < n$. This shows in $K[x]/(f(x)K[x])$ one has $g(x)$ is the same as $r(x)$, which implies $g(x)$ is a linear combination of $\{1, x, x^2, \ldots, x^{n-1}\}$.

For (b). Suppose $f(x) = a_n x^n + \cdots + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Then for x^i with $i < n-1$, one has $T(x^i) = x^{i+1}$, and
$$T(x^{n-1}) = x^n = -\frac{1}{a_n}(a_{n-1}x^{n-1} + \cdots + a_1 x + a_0).$$
This shows T has the matrix representation as
$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -\frac{a_0}{a_n} \\ 1 & 0 & 0 & \cdots & 0 & -\frac{a_1}{a_n} \\ 0 & 1 & 0 & \cdots & 0 & -\frac{a_2}{a_n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\frac{a_{n-1}}{a_n} \end{pmatrix}.$$
□

Exercise. *Let V be a vector space.*

(a) *Let V_1, V_2 and V_1', V_2' be subspaces of V such that* $\dim V_i \cap V_j = \dim V_i' \cap V_j'$ *for every possible i, j (in particular,* $\dim V_i = \dim V_i'$*). Show that there is an isomorphism $T: V \to V$ such that $T(V_i) = V_i'$.*

(b) **(open question, no need to submit) Let V_1, V_2, V_3 and V_1', V_2', V_3' be subspaces of V such that* $\dim V_i \cap V_j \cap V_k = \dim V_i' \cap V_j' \cap V_k'$ *and* $\dim V_i \cap (V_j + V_k) = \dim V_i' \cap \left(V_j' + V_k'\right)$ *for every possible i, j, k. Is there always an isomorphism $T: V \to V$ such that $T(V_i) = V_i'$?*

(c) **(open question, no need to submit) What about subspaces V_1, V_2, V_3, V_4, and more?*

Proof. For (a). Let $\{e_1, \ldots, e_k\}$ be a basis of $V_1 \cap V_2$ and $\{e_1', \ldots, e_k'\}$ be a basis of $V_1' \cap V_2'$. Then we extend $\{e_1, \ldots, e_k\}$ to a basis of V_1 by adding vectors $u_1, \ldots, u_m$ and extend $\{e_1, \ldots, e_k\}$ to a basis of V_2 by adding vectors $v_1, \ldots, v_n$. Finally we extend

$$\{e_1, \ldots, e_k, u_1, \ldots, u_m, v_1, \ldots, v_n\}$$

to a basis of V by adding vectors $\varphi_1, \ldots, \varphi_l$. Similarly, we can do the same thing to the basis $\{e_1', \ldots, e_k'\}$ and obtain a basis

$$\{e_1', \ldots, e_k', u_1', \ldots, u_m', v_1', \ldots, v_n', \varphi_1', \ldots, \varphi_l'\}$$

of V. Then we define T as follows

$$\begin{aligned} T(e_\alpha) &= e_\alpha' \\ T(u_\beta) &= u_\beta' \\ T(v_\gamma) &= v_\gamma' \\ T(\varphi_\delta) &= T(\varphi_\delta'). \end{aligned}$$

□

Chapter 3

Solutions to Homework-6

Exercise. *Suppose $A \in M_{m\times n}(K)$ and $B \in M_{n\times m}(K)$. Show that the nonzero eigenvalues of AB are the same as the nonzero eigenvalues of BA. If $m = n$, show that the eigenvalues of AB are the same as the eigenvalues of BA.*

Proof. Note that

$$\begin{pmatrix} \lambda I_m & A \\ B & I_n \end{pmatrix} \to \begin{pmatrix} \lambda I_m - AB & A \\ 0 & I_n \end{pmatrix} \to \begin{pmatrix} \lambda I_m - AB & 0 \\ 0 & I_n \end{pmatrix}$$

and

$$\begin{pmatrix} \lambda I_m & A \\ B & I_n \end{pmatrix} \to \begin{pmatrix} \lambda I_m & A \\ 0 & I_n - \frac{1}{\lambda}BA \end{pmatrix} \to \begin{pmatrix} \lambda I_m & 0 \\ 0 & I_n - \frac{1}{\lambda}BA \end{pmatrix}$$

This shows that

$$\det|\lambda I_m - AB| = \det|\lambda I_m| \det\left|I_n - \frac{1}{\lambda}BA\right| = \lambda^{m-n}\det|\lambda I_n - BA|$$

As a consequence, they have the same nonzero eigenvalues. In particular, if $m = n$, then their characteristic polynomials are the same, and thus have the same eigenvalues. □

Exercise. *Find $\lim_{n\to\infty} A^n$, where $A = \begin{pmatrix} \frac{1}{7} & \frac{3}{7} & \frac{3}{7} \\ \frac{3}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{3}{7} & \frac{3}{7} & \frac{1}{7} \end{pmatrix}$.*

Proof. Note that

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{2}{7} & & \\ & -\frac{2}{7} & \\ & & 1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Then

$$\begin{aligned} \lim_{n\to\infty} A^n &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{aligned}$$

□

Exercise. *Find the inverse matrix of the matrix $A = \begin{pmatrix} 1 & 1 & 2 \\ 9 & 2 & 0 \\ 5 & 0 & 3 \end{pmatrix}$ using the Cayley-Hamilton theorem.*

Proof. Note that the characteristic polynomials of A is $\lambda^3 - 6\lambda^2 - 8\lambda + 41$, and thus

$$A\left(A^2 - 6A - 8I_3\right) = -41I_3$$

As a consequence

$$A^{-1} = -\frac{1}{41}\left(A^2 - 6A - 8I_3\right) = \frac{1}{41}\begin{pmatrix} -6 & 3 & 4 \\ 27 & 7 & -18 \\ 10 & -5 & 7 \end{pmatrix}$$

□

Exercise. *Let $A \in M_3(\mathbb{R})$ such that* $\det A = 1$ *and* $(-1+\sqrt{-3})/2$ *is an eigenvalue of* A.

(1) *Find all eigenvalues of A.*

(2) *Suppose $A^{100} = aA^2 + bA + cI$, determine a, b, c.*

Proof. For (1). The characteristic polynomial of A is of real coefficient, and $(-1+\sqrt{-3})/2$ is a root. Then $(-1-\sqrt{-3})/2$ is also a root which is also an eigenvalue. Since the product of all eigenvalues are $\det A = 1$, one has all eigenvalues of A are

$$\frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2}, 1.$$

For (2). Note that the characteristic polynomial of A is $\lambda^3 - 1$. By Cayley-Hamilton theorem one has $A^3 = I_3$. Therefore

$$A^{100} = A = aA^2 + bA + cI_3 \Rightarrow aA^2 + (b-1)A + cI_3 = 0$$

However, the eigenvalues of A are distinct, and thus the characteristic polynomial of A is the minimal polynomial of A as well. Then

$$a = 0, \quad b = 1, \quad c = 0.$$

□

Exercise. *Let V be a vector space over K and $f, g \in V^*$ such that $f(v) = 0$ if and only if $g(v) = 0$. Show that $f = cg$ for some $0 \neq c \in K$.*

Proof. If $f = cg$ for some $0 \neq c \in K$, it's clear $f(v) = 0$ if and only if $g(v) = 0$. On the other hand, assume that $f(v) = 0$ if and only if $g(v) = 0$.

(1) If $f = 0$, then for any $v \in V$ one has $g(v) = 0$ and thus $g = f = 0$.

(2) If $f \neq 0$, then $f(v) = 0$ if and only if $g(v) = 0$ is equivalent to say $\ker f = \ker g = W$, where W is a linear subspace with codimension one. Suppose $V = W \oplus S$ with $0 \neq v_0 \in S$, and take $c = f(v_0)/g(v_0)$. For any $x \in V$, it can be written uniquely as

$$x = tv_0 + w, \quad w \in W.$$

Thus

$$f(x) = f\left(tv_0 + w\right) = tf\left(v_0\right) = \frac{f\left(v_0\right)}{g\left(v_0\right)} tg\left(v_0\right) = cg\left(tv_0 + w\right) = cg(x).$$

Therefore $f = cg$.

□

Exercise. *Let $\alpha_1 = (1,0,-1), \alpha_2 = (1,1,1), \alpha_3 = (2,2,0)$ be a basis of $\mathbb{C}^3$. Find the coordinates of the dual basis of α_i with respect to the dual basis of the standard basis of $\mathbb{C}^3$.*

Proof. Suppose $\{f^i\}$ is a dual basis of $\{\alpha_i\}$. For convenience we write $f^i = \sum_{j=1}^3 a_{ij}\epsilon^j$, where $\epsilon^1, \epsilon^2, \epsilon^3$ are dual basis for standard basis of $\mathbb{C}^3$. Note that $f^i(\alpha_j) = \delta_{ij}$. Then it suffices to solve several systems of linear equations to find out

$$f^1 = \epsilon^1 - \epsilon^2, \quad f^2 = \epsilon^1 - \epsilon^2 + \epsilon^3, \quad f^3 = -\frac{1}{2}\epsilon^1 + \epsilon^2 - \frac{1}{2}\epsilon^3.$$

□

Exercise. *Let V be the vector space of all polynomial functions p from $\mathbb{R}$ to $\mathbb{R}$ that have degree 2 or less:*

$$p(x) = c_0 + c_1 x + c_2 x^2.$$

Define three linear functionals on V by

$$f_1(p) = \int_0^1 p(x)\mathrm{d}x, \quad f_2(p) = \int_0^2 p(x)\mathrm{d}x, \quad f_3(p) = \int_0^{-1} p(x)\mathrm{d}x$$

Show that $\{f_1, f_2, f_3\}$ is a basis for V^* by exhibiting the basis for V of which it is dual.

Proof. For $p(x) = c_0 + c_1 x + c_2 x^2$, a direct computation shows

$$f_1(p) = c_0 + \frac{c_1}{2} + \frac{c_2}{3}, \quad f_2(p) = 2c_0 + 2c_1 + \frac{8}{3}c_2, \quad f_3(p) = -c_0 + \frac{c_1}{2} - \frac{c_2}{3}.$$

Then a dual basis can be taken as

$$p_1(x) = 1 + x - \frac{3}{2}x^2, \quad p_2(x) = -\frac{1}{6} + \frac{1}{2}x^2, \quad p_3(x) = -\frac{1}{3} + 1x - \frac{1}{2}x^2.$$

This is a basis since

$$\det\begin{pmatrix} 1 & 1 & -\frac{3}{2} \\ -\frac{1}{6} & 0 & \frac{1}{2} \\ -\frac{1}{3} & 1 & -\frac{1}{2} \end{pmatrix} = -\frac{1}{2} \neq 0$$

□

Exercise. *Let W be the supspace of $\mathbb{R}^5$ spanned by the vectors $\alpha_1 = e_1 + 2e_2 + e_3, \alpha_2 = e_2 + 3e_3 + 3e_4 + e_5, \alpha_3 = e_1 + 4e_2 + 6e_3 + 4e_4 + e_5$, where e_i are the standard basis of $\mathbb{R}^5$. Find a basis for $W^\perp$ in terms of the dual basis of e_i.*

Proof. Suppose $f \in W^\perp$ is given by $\sum_{i=1}^5 \alpha_i \epsilon^i$. Then one has the following system of linear equations

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 3 \\ 0 & 1 & 0 & -3 & -2 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} = 0$$

Thus $W^\perp$ can be viewed as a solution space, which is generated by

$$\left\{-4\epsilon^1 + 3\epsilon^2 - 2\epsilon^3 + \epsilon^4, -3\epsilon^1 + 2\epsilon^2 - \epsilon^3 + \epsilon^5\right\}.$$

□

Exercise. *Let n be a positive integer and K a field. Let W be the set of vectors $(x_1, \ldots, x_n)$ in K^n such that $\sum x_i = 0$.*

(1) *Prove that* $W^{\perp}$ *consists of all linear functionals* f *of the form* $f(x_1, \ldots, x_n) = c \sum x_i$.

(2) *Suppose* $K = \mathbb{R}$. *Show that* W^* *can be naturally identified with the set of linear functionals* $f = \sum c_i x_i$ *on* K^n *such that* $\sum c_i = 0$.

Proof. For (1). Suppose $f \in W^{\perp}$ and we denote $f(e_i) = c_i$ for convenience. Since $e_i - e_j \in W$, one has

$$c_i - c_j = f(e_i) - f(e_j) = f(e_i - e_j) = 0.$$

This shows $c_i = c$ for all i. Therefore

$$f(x_1, \ldots, x_n) = f(x_1 e_1 + \cdots + x_n e_n) = x_1 f(e_1) + \cdots + x_n f(e_n) = c \sum x_i.$$

On the other hand, it's obvious that $f(x_1, \ldots x_n) = 0$ if f is of this form.

For (2). Note that there is a natural identification between W and W^* by

$$v^* \longleftrightarrow \langle v, \text{-} \rangle.$$

Under this identification, the linear functional $f = \sum_i c_i x_i$ on K^n corresponds to the vector $(c_1, \ldots, c_n)$. Thus it gives a linear functional on W if and only if $(c_1, \ldots, c_n) \in W$, that is, $\sum_i c_i = 0$. $\square$

Exercise. *Suppose* $f \in M_n(\mathbb{R})^*$ *such that* $f(AB) = f(BA)$ *for all* $A, B \in M_n(K)$ *and* $f(I) = n$. *Show that* f *is the trace function.*

Proof. Let $E_{ij} \in M_n(\mathbb{R})$ denote the matrix with (i,j) entry 1 and the others. Then

$$f(E_{ij}) = f(E_{ik} E_{kj}) = f(E_{kj} E_{ik}) = \delta_{ij} f(E_{kk}).$$

On the other hand, note that

$$f(E_{ii}) = f(E_{ij} E_{ji}) = f(E_{ji} E_{ij}) = f(E_{jj}).$$

for any i, j. Thus

$$f(I) = n f(E_{ii}) = n \Rightarrow f(E_{ii}) = 1.$$

Therefore

$$f(E_{ij}) = \delta_{ij}.$$

Then for any $A \in M_n(\mathbb{R})$, one has

$$f(A) = f\left(\sum_{i,j} a_{ij} E_{ij}\right) = \sum_{i,j} a_{ij} f(E_{ij}) = \sum_{i,j} a_{ij} \delta_{ij} = \sum_{i=1}^{n} a_{ii} = \operatorname{tr}(A).$$

Therefore f is the trace function. $\square$

Chapter 4

Solutions to Homework-8

Exercise. *Let $A = (a_{ij})$ be a symmetric real matrix. Suppose $a_{ii} > \sum_{j\neq i} |a_{ij}|$. Show that A is positive definite.*

Proof. As shown in Exercise 12 of Homework-2, one has all principle minors of A is positive, and thus A is positive definite. □

Exercise. *A real symmetric matrix $A = (a_{ij})$ of order n is semi-positive definite if $\sum_{i,j} a_{ij}x_ix_j \geq 0$ for all $(x_1, \cdots, x_n)$ in $\mathbb{R}^n$. Prove that the following are equivalent:*

(*a*) *A is semi-positive definite*

(*b*) *$A = P^t \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P$ for some real invertible matrix P*

(*c*) *$A = Q^tQ$ for some real matrix Q*

(*d*) *all principal minors of A are non-negative.*

State the corresponding conclusion for semi-positive definite Hermitian matrices. Is it equivalent to that all leading principal minors of A are non-negative?

Proof. From (a) to (b). Since A is a real symmetric matrix, there exists some invertible matrix P such that

$$A = P^t \begin{pmatrix} I_r & & \\ & -I_s & \\ & & O \end{pmatrix} P$$

for some $r, s \in \mathbb{Z}_{\geq 0}$. On the other hand, since A is semi-positive definite, s must be zero as desired.

From (b) to (c). It suffices to set $Q = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P$.

From (c) to (a). Note that

$$x^tAx = x^tQ^tQx = (Qx)^tQx \geq 0.$$

From (a) to (d). For the principal minors $A\begin{pmatrix} k_1 & k_2 & \dots & k_s \\ k_1 & k_2 & \dots & k_s \end{pmatrix}$, now we're going to show that it's semi-definite, and thus $\det A\begin{pmatrix} k_1 & k_2 & \dots & k_s \\ k_1 & k_2 & \dots & k_s \end{pmatrix} \geq 0$. Suppose $x = (x_{k_1}, \dots, x_{k_s})^t$ such that

$$x^tA\begin{pmatrix} k_1 & k_2 & \dots & k_s \\ k_1 & k_2 & \dots & k_s \end{pmatrix} x < 0.$$

Then consider $\widetilde{x} = (0, \dots, x_{k_1}, \dots, x_{k_s}, \dots, 0)$, one has $\widetilde{x}A\widetilde{x} < 0$, a contradiction.

From (d) to (a). If all principal minors of A are non-negative, then the characteristic polynomials $f(\lambda) \geq \lambda^n$ for all $\lambda > 0$ since the coefficients of $f(\lambda)$ are positive combinations of principal minors. This shows all eigenvalues of A are non-negative, and thus A is semi-positive definite.

However, the corresponding conclusion for semi-positive definite Hermitian matrices fails. □

Exercise. *Use Gram-Schimidt procedure to construct an orthonormal basis of $\mathbb{R}^4$ from the following:*

(a) $(0,0,2,1)^t, (0,3,7,2)^t, (1,1,6,2)^t, (-1,4,-1,-1)^t$;

(b) $(1,1,1,1)^t, (1,0,1,1)^t, (1,1,0,1)^t, (1,1,1,0)^t$.

Proof. It's a routine computation, and here we only show the results.

For (a).

$$\frac{1}{\sqrt{5}}(0,0,2,1)^t, \frac{1}{\sqrt{30}}(0,5,1,-2)^t, \frac{1}{\sqrt{42}}(6,-1,1,-2)^t, \frac{1}{\sqrt{7}}(1,1,-1,2)^t.$$

For (b)

$$\frac{1}{2}(1,1,1,1)^t, \frac{1}{\sqrt{12}}(1,-3,1,1)^t, \frac{1}{\sqrt{6}}(1,0,-2,1)^t, \frac{1}{\sqrt{2}}(1,0,0,-1)^t.$$

□

Exercise. *Prove that the maximal entries of a positive definite, symmetric, real matrix A are on the diagonal.*

Proof. Suppose $a_{i_0j_0} = \max_{i,j} a_{ij}$. If $i_0 \neq j_0$, then $a_{i_0i_0}a_{j_0j_0} - a_{i_0j_0}^2 > 0$, since the determinant of principal minors $A(i_0, j_0)$ is > 0. Thus $a_{i_0j_0} < \max\{a_{i_0i_0}, a_{j_0j_0}\}$ since both $a_{i_0i_0}$ and $a_{j_0j_0}$ are positive, a contradiction. □

Exercise. *Let $\langle \text{-}, \text{-} \rangle$ be a positive definite Hermitian form on a complex vector space V, and let $\{\text{-}, \text{-}\}$, and $[\text{-}, \text{-}]$ be its real and imaginary parts, the real-valued forms defined by*

$$\langle v, w \rangle = \{v, w\} + [v, w]i.$$

Prove that when V is made into a real vector space by restricting scalars to $\mathbb{R}$,$\{\text{-}, \text{-}\}$ is a positive definite symmetric form, and $[\text{-}, \text{-}]$ is a skew-symmetric form.

Proof. For $v, w \in V$, one has

$$\langle v, w \rangle + \langle w, v \rangle = \langle v, w \rangle + \overline{\langle v, w \rangle} = 2\{v, w\}.$$

This shows $[v, w] + [w, v] = 0$, and thus $[\text{-}, \text{-}]$ is a skew-symmetric form. On the other hand, one has

$$\{v, w\} = \frac{1}{2}(\langle x + y, x + y \rangle - \langle x, x \rangle - \langle y, y \rangle) = \{y, x\}.$$

This shows $\{\text{-}, \text{-}\}$ is a symmetric form, and it's positive definite since $\{x, x\} = \langle x, x \rangle$. □

Exercise. *Let $V = \mathbb{R}^{2\times 2}$ be the vector space of real 2×2 matrices.*

(a) Determine the matrix of the bilinear form $\langle A, B \rangle = \operatorname{tr}(AB)$ on V with respect to the standard basis $\{e_{ij}\}$.

(b) Determine the signature of this form.

(c) Find an orthogonal basis for this form.

(*d*) *Determine the signature of the form trace AB on the space $\mathbb{R}^{n\times n}$ of real $n\times n$ matrices.*

Proof. For (a). Note that

$$\operatorname{tr}(AB) = a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22}.$$

A direct computation shows that the matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For (b) and (c). The orthogonal basis is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and thus the signature is $3 - 1 = 2$.

For (d). One can construct an orthogonal basis of $\mathbb{R}^{n\times n}$ by define

$$\alpha_{ij} = \begin{cases} E_{ij} & i = j \\ E_{ij} + E_{ji} & i < j \\ E_{ij} - E_{ji} & i > j. \end{cases}$$

A direct computation shows

$$\langle \alpha_{ij}, \alpha_{kl} \rangle = \begin{cases} 0, & (i,j) \neq (k,l) \\ 1, & i = j = k = l \\ 2, & i = k, j = l, i < j \\ -2, & i = k, j = l, i > j. \end{cases}$$

This shows the signature is

$$\frac{n^2+n}{2} - \frac{n^2-n}{2} = n.$$

□

Exercise. *Let W be a subspace of a Euclidean space/Hermitian space V. Show that $W = W^{\perp\perp}$*

Proof. On one hand, it's clear $W \subseteq W^{\perp\perp}$. On the other hand, one has

$$\dim W^{\perp\perp} + \dim W^{\perp} = \dim V = \dim W + \dim W^{\perp}.$$

This shows $W = W^{\perp\perp}$. □

Exercise. *Show that the Gram determinant $\det((\alpha_i, \alpha_j))$ of n real vectors $\alpha_1, \ldots, \alpha_n$ in $\mathbb{R}^n$ is non-zero if and only if the vectors are linearly independent.*

Proof. Note that $\det((\alpha_i, \alpha_j)) = \det\left(A^t A\right) = \det^2 A \neq 0$ if and only if $\det A \neq 0$, which is equivalent to say the α_i 's are linearly independent. □

Exercise. *Let V be a Euclidean space.*

(*a*) *Prove the parallelogram law $|v+w|^2 + |v-w|^2 = 2|v|^2 + 2|w|^2$.*

(*b*) *Prove that if $|v| = |w|$, then $(v+w) \perp (v-w)$.*

Proof. For (a).

$$|u+v|^2+|u-v|^2=|u|^2+2(u,v)+|v|^2+|u|^2-2|u||v|+|v|^2=2|u|^2+2|v|^2.$$

For (b).

$$(v+w,v-w)=|v|^2+(w,v)-(v,w)-|u|^2=0.$$

□

Exercise. *Let T be a linear operator on $V=\mathbb{R}^n$ whose matrix A is a real symmetric matrix.*

(a) *Prove that V is the orthogonal sum $V=(\ker T)\oplus(\operatorname{im} T)$.*

(b) *Prove that T is an orthogonal projection onto $\operatorname{im} T$ if and only if, in addition to being symmetric, $A^2=A$.*

Proof. For (a), For $v\in\ker T$ and $u=T(w)\in\operatorname{im} T$, one has

$$(v,u)=(v,T(w))=(T(v),w)=(0,w)=0$$

Therefore $\ker T\perp\operatorname{im} T$. On the other hand, one has

$$\dim\ker T+\dim\operatorname{im} T=\dim V.$$

Thus V is their orthogonal sum.

For (b). Suppose T is an orthogonal projection onto $\operatorname{im} T$. Then for every $v\in V$ with $v=v_1+v_2$, where $v_1\in\ker T, v_2\in\operatorname{im} T$, one has $Tv=v_2$ and $Tv_2=v_2$. This shows $T^2v=Tv$ for every $v\in V$, and thus $A^2=A$. Conversely, if $A^2=A$, then for every $v\in V$, one has $A^2v=Av$, and thus $Av-v\in\ker T$. This shows T is an orthogonal projection onto $\operatorname{im} T$ since $v=v-Av+Av$. □

Exercise. *Let W be the subspace of $\mathbb{R}^3$ spanned by the vectors $(1,1,0)^t$ and $(0,1,1)^t$. Determine the orthogonal projection of the vector $(1,0,0)^t$ to W.*

Proof. Note that $W^\perp$ can be spanned by $(1,-1,1)^t$. Suppose

$$(1,0,0)^t=a(1,1,0)^t+b(0,1,1)^t+c(1,-1,1)^t$$

Then $c=1/3$, and thus the orthogonal projection is

$$(1,0,0)^t-\frac{1}{3}(1,-1,1)^t=\frac{1}{3}(2,1,-1)^t.$$

□

Exercise. *Let V be the real vector space of 3×3 matrices with the bilinear form $\langle A,B\rangle=\operatorname{tr}(A^tB)$, and let W be the subspace of skew-symmetric matrices. Compute the orthogonal projection to W with respect to this form, of the matrix*

$$\begin{pmatrix}1&2&0\\0&0&1\\1&3&0\end{pmatrix}.$$

Proof. If $A\in W^\perp$, then $\operatorname{tr}(A^tB)=0$ for all $B\in W$. Since W is spanned by an orthonormal basis can be taken as

$$E_{12}-E_{21},E_{13}-E_{31},E_{23}-E_{32},$$

it reduces to

$$\operatorname{tr}(A^t(E_{12} - E_{21})) = 0$$
$$\operatorname{tr}(A^t(E_{13} - E_{31})) = 0$$
$$\operatorname{tr}(A^t(E_{23} - E_{32})) = 0.$$

This is equivalent to A is a symmetric matrix. Thus $W^\perp$ consists of the symmetric matrices. Note that

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 0 & 2 \\ \frac{1}{2} & 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -\frac{1}{2} \\ -1 & 0 & -1 \\ \frac{1}{2} & 1 & 0 \end{pmatrix}.$$

Thus the orthogonal projection is exactly

$$\begin{pmatrix} 0 & 1 & -\frac{1}{2} \\ -1 & 0 & -1 \\ \frac{1}{2} & 1 & 0 \end{pmatrix}.$$

□