

YANG-MILLS EQUATIONS ON RIEMANN SURFACES AND MODULI SPACES

BOWEN LIU

ABSTRACT. This article is mainly divided into three parts. In the first part, as a preparation for further theories, I studied the principal bundles and the geometry of vector bundle and principal bundle, such as covariant derivative or connection and curvature on them. I introduced the motivations of studying covariant derivative and what roles does connection of principal bundle play. In order to a better understanding of it, I studied three different views of connection. I mainly followed “Differential Geometry” written by Taubes [2], and refer [3], [4] for some supplements.

In the second part, I reviewed some basic definitions in geometry, such as volume form or Hodge star operator, since they’re crucial in defining Yang-Mills functional. Then I gave an introduction about Yang-Mills functional and its variations.

In the third part, I mainly followed [3], which give a neat introduction about Yang-Mills equations’ applications on the stability of holomorphic vector bundles.

Key words: principle bundle, connection, curvature, Yang-Mills equations, stability of holomorphic vector bundles.

CONTENTS

1. Geometry of Principle Bundles	3
1.1. Principle bundles	3
1.2. Associated vector bundles	5
1.3. Covariant derivatives and connections	8
1.4. Gauge transformations	18
1.5. Curvatures	21
1.6. Relations of connections and covariant derivatives	25
1.7. Flat connections and holonomy	27
2. The Yang-Mills Equations	29
2.1. Some geometry	29
2.2. The variational problem	31
3. Holomorphic Vector Bundles and Yang-Mills Connections	34
3.1. Stability of holomorphic vector bundles	34
3.2. Episode: Fubini-Study metric	37
3.3. Symplectic reduction	40
3.4. Narasimhan-Sechadri	42
4. Further research	42

5. Acknowledgement	43
References	43

1. GEOMETRY OF PRINCIPLE BUNDLES

1.1. Principle bundles.

Definition 1.1.1 (principle G -bundle). *A principle G -bundle is a smooth manifold P , with the following data:*

1. A Lie group G acting freely and transitively on P on the right:

$$\begin{aligned} R_g : P \times G &\rightarrow P \\ (p, g) &\mapsto pg \end{aligned}$$

2. A surjective map $\pi : P \rightarrow M$ that is G -invariant, that is, $\pi(pg) = \pi(p), \forall p \in P, g \in G$.
3. For every point $x \in M$, there exists a neighborhood U of x and a diffeomorphism $\varphi : P|_U := \pi^{-1}(U) \rightarrow U \times G$ that is G -equivariant, and the following diagram commutes

$$\begin{array}{ccc} P|_U & \xrightarrow{\varphi} & U \times G \\ & \searrow \pi \quad \swarrow & \\ & U & \end{array}$$

where $U \times G \rightarrow U$ is projection onto the first factor.

Remark 1.1.2. From 3, we know that M admits an open cover $\{U_\alpha\}$ and G -equivariant diffeomorphisms $\varphi_\alpha : \pi^{-1}U_\alpha \rightarrow U_\alpha \times G$. More explicitly, this means $\varphi_\alpha(p) = (\pi(p), g_\alpha(p))$ for some G -equivariant map $g_\alpha : \pi^{-1}U_\alpha \rightarrow G$ which is a fiberwise diffeomorphism. Equivariance means that

$$g_\alpha(pg) = g_\alpha(p)g, \quad \forall g \in G$$

Remark 1.1.3. If we consider two local trivialization $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ with non-empty intersection $U_{\alpha\beta}$, then¹

$$\varphi_\alpha \circ \varphi_\beta^{-1} : U_\beta \times G \rightarrow U_\alpha \times G$$

then for any $(m, h) \in U_{\alpha\beta} \times G$, we will $\varphi_\alpha \circ \varphi_\beta^{-1}((m, h)) = (m, g_{\alpha\beta}(m)h) \in U_{\alpha\beta} \times G$, where $g_{\alpha\beta} : U_{\alpha\beta} \rightarrow \text{Aut}(G)$.

Furthermore, since local trivialization is G -equivariant, then in fact

$$g_{\alpha\beta} \in \text{Aut}_G(G) := \{f : G \rightarrow G \mid f(xg) = f(x)g, \forall x, g \in G, f \text{ is a diffeomorphism.}\}$$

and we have $\text{Aut}_G(G) \cong G$. Indeed, $G \subseteq \text{Aut}_G(G)$ automatically holds, since for any $g \in G$, we can define a map $x \mapsto gx$. Conversely, note that such f is completely determined by its value at any point. If we let g to denote $f(e)$, where $e \in G$ is the identity. Then $f(x) = f(ex) = gx$ for all $x \in G$, that is, f is just $x \mapsto gx$.

So not similar to case of vector bundle with fiber V , in which case gluing data are given by

$$g_{\alpha\beta} : U_{\alpha\beta} \rightarrow \text{GL}(V)$$

¹Here may be a little misleading, $\text{Aut}(G)$ is always reserved for group automorphisms, but here f is not a group automorphism, but just a diffeomorphism.

The gluing data of principal bundles are given by G -equivariant maps

$$g_{\alpha\beta} : U_{\alpha\beta} \rightarrow G$$

satisfying cocycle conditions. Conversely, this determines the principal bundles uniquely as case of vector bundles.

Many principal bundles arise from vector bundles, such as frame bundles we will discuss immediately.

Example 1.1.4 (frame bundle). Let $E \rightarrow M$ be a real vector bundle of rank k . For $x \in M$, let \mathcal{B}_x denote the set of all bases of the fiber E_x , i.e. the set of linear isomorphisms $\mathbb{R}^k \rightarrow E_x$. This has a natural right action of $\mathrm{GL}_k(\mathbb{R})$ by precomposition. Then let

$$\mathcal{B}_{\mathrm{GL}_k(\mathbb{R})}(E) := \coprod_{x \in M} \mathcal{B}_x$$

Using local trivialization of the vector bundle E , we equip $\mathcal{B}_{\mathrm{GL}_k(\mathbb{R})}(E)$ with the structure of a smooth manifold such that $\pi : \mathcal{B}_{\mathrm{GL}_k(\mathbb{R})}(E) \rightarrow M$ taking \mathcal{B}_x to x . This gives $\mathcal{B}_{\mathrm{GL}_k(\mathbb{R})}(E)$ the structure of $\mathrm{GL}_k(\mathbb{R})$ -bundle, called the frame bundle of E .

Example 1.1.5 (orthonormal frame bundle). Let $E \rightarrow M$ be a rank k vector bundle equipped with a fiber metric, i.e. a smoothly varying inner product on the fibers E_x . Then the orthonormal frame bundle of E , denoted $\mathcal{B}_O(E)$, is the principal O_k -bundle where the fiber over $x \in M$ is the linear isometries $\mathbb{R}^k \rightarrow E_x$, where we use the standard inner product on \mathbb{R}^k and the fiber metric restricted to E_x .

Remark 1.1.6. A near identical story holds for complex vector bundles: from any complex vector bundle we a principal $\mathrm{GL}_k(\mathbb{C})$ -bundle of frames, and if we fix a Hermitian fiber metric, we a principle U_k -bundle of orthogonal frames, which is quite important in future.

Definition 1.1.7 (trivial principal bundles). *A principal bundle P is called trivial, if it admits a global trivialization. In other words, if there exists a diffeomorphism $\varphi : P \rightarrow M \times G, \varphi(p) = (\pi(p), g(p))$ such that $g(ph) = g(p)h, \forall h \in G$.*

Remark 1.1.8. In other words, a principal bundle is trivial if and only if it admits a global trivialization.

Definition 1.1.9 (section). *A section is a smooth map $s : M \rightarrow P$ such that $\pi \circ s = \mathrm{id}$.*

Remark 1.1.10. In other words, a section is a smooth assignment to each point in the base of a point in the fiber over it. However, sections are quite rare.

Proposition 1.1.11. *A principal bundle admits a section if and only if it is trivial.*²

²This is in sharp contrast with vector bundles, which always admit sections.

Proof. If $s : M \rightarrow P$ is a smooth section, we define

$$\begin{aligned}\varphi : P &\rightarrow M \times G \\ p &\mapsto (\pi(p), g(p))\end{aligned}$$

where $g(p) \in G$ is such that $p = s(\pi(p))g(p)$, it always exists since the right action of G is transitive on each fiber and it is unique since the action is free on each fiber.

Clearly, it's G -equivariant, since

$$\varphi(ph) = (\pi(ph), g(ph)) = (\pi(p), g(p)h)$$

and the last equality holds since

$$ph = s(\pi(ph))g(ph) = s(\pi(p))g(ph) = pg^{-1}(p)g(ph) \implies h = g^{-1}(p)g(ph)$$

And it's easy to see φ is a bijection, with inverse map

$$\begin{aligned}\varphi^{-1} : M \times G &\rightarrow P \\ (p, g) &\mapsto s(p)g\end{aligned}$$

The smoothness of the section and of the G -action on P imply smoothness. \square

However, since P is locally trivial, local sections do exist. In fact, there are local sections $s_\alpha : U_\alpha \rightarrow \pi^{-1}U_\alpha$ canonically associated to the trivialization, defined so that for every $m \in U_\alpha$, $\varphi_\alpha(s_\alpha(m)) = (m, e)$, where $e \in G$ is the identity element. In other words, $g_\alpha \circ s_\alpha : U_\alpha \rightarrow G$ is the constant function sending every point to the identity.

1.2. Associated vector bundles. We have seen that for a vector bundle, we can construct its frame principal bundle or orthogonal frames bundle. As explained here, this construction has an inverse of sorts.

To set the stage, suppose that G is a Lie group and $\pi : P \rightarrow M$ is a principal G bundle. The construction of a vector bundle from P requires an additional input, that is a representation ρ of the group G into $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$, and we use V to denote representation space.

Definition 1.2.1 (associated vector bundle). *Let $P \rightarrow M$ be a principal G -bundle, and $\rho : G \rightarrow \mathrm{GL}(V)$ be a representation of G . The associated vector bundle, denoted by $P \times_G V$ is the space*

$$P \times_G V := (P \times V)/G$$

where the G -action on $P \times_G V$ is the diagonal action, i.e. $(p, v) \cdot g := (pg, \rho(g^{-1})v)$.

Remark 1.2.2. More generally, V need not be a vector space, it might be any manifold admitting a G action.

The following proposition shows that why associated fiber bundle indeed gives us a fiber bundle with model fiber.

Proposition 1.2.3. *$P \times_G V$ is a vector bundle over M with model fiber V .*

Proof. Consider the map taking an equivalence class $[p, f]$ to $\pi(p)$. To see the local structure, since we already have the local structure of principal bundle P , i.e. for any $x \in M$, there exists open $U_\alpha \ni x$ and $\varphi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times G$. Now we define the local trivialization of $P \times_G V$ as

$$\begin{aligned} \varphi_\alpha^V : (P \times_G V)|_{U_\alpha} &\rightarrow U_\alpha \times V \\ (p, v) &\mapsto (\pi(p), \rho(g_\alpha(p))v) \end{aligned}$$

First note that this is well-defined, since

$$(pg, \rho(g^{-1})v) \mapsto (\pi(pg), \rho(g_\alpha(pg))\rho(g^{-1})v) = (\pi(p), \rho(g_\alpha(p)gg^{-1})v) = (\pi(p), \rho(g_\alpha(p))v)$$

And this map is one to one, and invertible, its inverse sends $(m, v) \in U_\alpha \times V$ to the equivalence class of $(\varphi_\alpha^{-1}(m, e), v)$. Directly check as follows

$$\begin{aligned} \varphi_\alpha^V(\varphi_\alpha^{-1}(m, e)) &= (m, \rho(e)v) \\ &= (m, v) \end{aligned}$$

since $\pi(\varphi_\alpha^{-1}(m, e)) = m$ and $g_\alpha(\varphi_\alpha^{-1}(m, e)) = e$. \square

Remark 1.2.4. Though we've find the local trivialization of $P \times_G V$, it's also necessary to see what does the transition functions look like.

Let U_α, U_β be open sets with non-empty intersection $U_{\alpha\beta}$, and $\varphi_\alpha, \varphi_\beta$ be local trivializations of principal bundles, with transition functions

$$\begin{aligned} \varphi_\alpha \circ \varphi_\beta^{-1} : U_{\alpha\beta} \times G &\rightarrow U_{\alpha\beta} \times G \\ (m, h) &\mapsto (m, g_{\alpha\beta}(m)h) \end{aligned}$$

then we can compute the transition functions of associated vector bundles as follows

$$\begin{aligned} \varphi_\alpha^V \circ (\varphi_\beta^V)^{-1} : U_{\alpha\beta} \times V &\rightarrow U_{\alpha\beta} \times V \\ (m, v) &\mapsto (m, \rho(g_{\alpha\beta}(m))v) \end{aligned}$$

Example 1.2.5. If $E \rightarrow M$ is a given vector bundle, with fiber $V = \mathbb{R}^n$, then we will have its frame principal G bundle P with $G = \text{GL}(n, \mathbb{R})$. Let ρ be the defining representation of G on V , i.e. the representation $\rho : G \rightarrow \text{GL}(n, \mathbb{R})$ is identity map. Then we claim that $P \times_G V$ is canonically isomorphic to E , since from Remark 1.2.3 we can directly see that they have the same transition functions.

Remark 1.2.6. The above example shows that we can recover a vector bundle from its frame bundle using associated vector bundle. But we need to use defining representation, so that what “of sorts” means, i.e. principal bundles indeed encode more information.

Example 1.2.7. There are two important examples of associated bundles that we will need to discuss the Yang-Mills equations.

1. The bundle $\text{Ad } P := P \times_G G$, where G acts on G by conjugation.
2. The bundle $\text{ad } P := P \times_G \mathfrak{g}$, where the action is the adjoint action. This bundle is sometimes denoted by \mathfrak{g}_P .

Remark 1.2.8. Let's recall that's adjoint action, and fix some notations. For any $g \in G$. consider the conjugation $\text{Ad}_g : G \rightarrow G$, defined by $h \mapsto ghg^{-1}$ and its differential. That's

$$\begin{aligned} \text{ad}_g : \mathfrak{g} &\rightarrow \mathfrak{g} \\ X &\mapsto \left. \frac{d}{dt} \right|_{t=0} ge^{tX}g^{-1} \end{aligned}$$

So we have

$$\begin{aligned} \text{ad} : G &\rightarrow \text{GL}(\mathfrak{g}) \\ g &\mapsto \text{ad}_g \end{aligned}$$

This representation is called adjoint representation.

Here we use the symbol ad to denote the adjoint representation and Ad to denote the conjugation of G , in order to in a harmony with our symbol $\text{Ad } P$ and $\text{ad } P$.

However, this may lead some confusions, since in other references authors always use Ad to denote adjoint representation, and use ad to denote its differential. Here if neccessary, we use Lie bracket $[X, Y]$ to denote $\text{ad}_X Y$ appeared in other references.

Associated bundles have another nice feature, their sections have a nice interpretation in terms of G -equivariant maps.

Proposition 1.2.9. *Let $E = P \times_G V$ be an associated fiber bundle. Then there is a bijective correspondence*

$$C^\infty(M, E) \longleftrightarrow \{G\text{-equivariant maps } P \rightarrow V\}$$

Proof. First, if we have a G -equivariant map $s^P : P \rightarrow V$, that is

$$s^P(pg) = \rho(g^{-1})s^P(p)$$

then the corresponding section s associates to any given point $x \in M$ the equivalence class $(p, s^P(p))$, where p is any point in the fiber P_x . Clearly the definition is independent of the choice of p , since

$$(pg, s^P(pg)) = (pg, \rho(g^{-1})s^P(p)) = (p, s^P(p))$$

Conversely, suppose s is a section of $P \times_G V$, by definition, s associates an equivalence class in $P \times_G V$ to every point of M . That equivalence class gives a G -equivariant map $P \rightarrow V$. \square

Remark 1.2.10. In fact, this proposition is not a coincidence, and it's a quite important motivation which explains why we need principal bundles.

If $\pi : P \rightarrow M$ is a principal G bundle, and $\pi' : E \rightarrow M$ is a vector bundle such that E is an associated vector bundle of P , then if we use π to pull E back to P , we claim that the vector bundle π^*E is the trivial bundle $P \times V$ over P .

Indeed, we define the following bundle map

$$\begin{aligned} P \times V &\rightarrow P \times_G V \\ (p, v) &\mapsto [p, v] \end{aligned}$$

Clearly, it map the fiber over p isomorphically to the fiber over $\pi(p)$, thus by the universal property of pullback, we know that $\pi^*E \cong P \times V$.

That's a quite beautiful result. And there is no wonder that section s of E corresponds to a G -equivariant map $s^P : P \rightarrow V$, since that's exactly what sections of $P \times V$ look like.

Example 1.2.11. Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$ be the canonical projection, and we can regard $\mathbb{C}^{n+1} \setminus \{0\}$ as a principal \mathbb{C}^* bundle of \mathbb{P}^n . Consider tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$ on \mathbb{P}^n , and consider its pullback by π . It will be a trivial line bundle over $\mathbb{C}^{n+1} \setminus \{0\}$. Indeed, by definition

$$\pi^*(\mathcal{O}(-1)) = \{(x, v) \in \mathbb{C}^{n+1} \setminus \{0\} \times \mathcal{O}(-1) \mid \pi(x) = \pi_L(v)\}$$

where π_L is the projection of line bundle L . Note that $\pi(x) = [x] = \pi_L(v)$ implies that v is an element of \mathbb{C}^{n+1} spanned by x . This means that the fiber of the pullback bundle $\pi^*(\mathcal{O}_{\mathbb{P}^n}(-1))$ over x is exactly $\{\lambda x \mid \lambda \in \mathbb{C}\}$. This happens globally so there is an obvious vector bundle isomorphism with $\mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}$.

This pullback is trivial is not a coincidence, since we can regard tautological line bundle as an associated vector bundle, by considering the definition representation of \mathbb{C}^* on \mathbb{C} .

Remark 1.2.12. This proposition shows the power of principal bundles. Instead of considering sections of $P \times_G V$, we just need to consider G -equivariant functions from P to V , that is sections of trivial bundle $P \times V$ with some equivariant conditions. And it makes non-trivial things into trivial one. What a powerful result.

This view has the following useful generalization: Suppose $\pi' : E' \rightarrow M$ is a second vector bundle, and perhaps unrelated to P . A section of $(P \times_G V) \otimes E'$ appears upstairs on P as a suitably G -equivariant section over P of $(P \times V) \otimes \pi^*E'$. We omit the proof here, but we will revisit this thing later in another viewpoint. What we need now is just need an intuition to see why we define connections on principal bundle in such a form.

The cases we will encounter are taking E' to be cotangent bundle T^*M or its wedge products $\bigwedge^k T^*M$. We sometimes use $\Omega_M^k(P \times_G V)$ to denote $(P \times_G V) \otimes \bigwedge^k T^*M$, the generalization tells that we have the one to one correspondence between sections of $\Omega_M^k(P \times_G V)$ and sections of $(P \times V) \otimes \pi^* \bigwedge^k T^*M$ with equivariant conditions, we will call such forms obtained by pullback basic, and denote it by $\Omega_G^k(P; V)$.

1.3. Covariant derivatives and connections. Let $\pi : E \rightarrow M$ be a vector bundle, and use $C^\infty(M, E)$ to denote the vector space of smooth sections of E . The questions arises as to how to take the derivative of a section $s : M \rightarrow E$ in a given direction.

It's quite natural to ask such questions. When we learn calculus, we know how to take derivative of a smooth function $f : M \rightarrow \mathbb{R}^m$, and this gives $df : TM \rightarrow \mathbb{R}^m$, or a section of T^*M . Furthermore, any smooth function

$f : M \rightarrow \mathbb{R}^m$ can be regarded as a section of trivial vector bundle $M \times \mathbb{R}^m$, as follows

$$x \mapsto (x, f(x))$$

and we can also regard its derivative df as a section of $T^*M \otimes M \times \mathbb{R}^m$.

Recall how we define df in the model case $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$. In this setting the derivative df at x in the direction v is defined by the standard formula

$$df(v)(x) = \lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}$$

But when passing to a section s of vector bundle E , one encounters two key issues with this definition. Firstly, since there is no linear structure on a manifold, the term $x + tv$ makes no sense on M . However, this issue is relatively easy to handle. Instead one takes a path $\gamma : (-1, 1) \rightarrow M$ such that $\gamma(0) = x, \gamma'(0) = v$ and computes

$$ds(v)(x) = \lim_{t \rightarrow 0} \frac{s(\gamma(t)) - s(\gamma(0))}{t}$$

and that's what we do in the case of functions on manifold. However, this still does not make sense in the case of vector bundles, since $f(\gamma(t))$ and $f(\gamma(0))$ are elements of distinct fibers $E_{\gamma(t)}$ and $E_{\gamma(0)}$. This means that subtraction of these two terms is not naturally defined.

One naive ideal is that if we can “transport” vectors in $E_{\gamma(t)}$ “parallel” to those in $E_{\gamma(0)}$, then we can do subtraction. That's one viewpoint to think connections. To be more explicit, a connection can be viewed as assigning to every differentiable path γ a linear isomorphisms $P_t^\gamma : E_{\gamma(t)} \rightarrow E_x$ for all t , and we can define

$$\nabla_v f = \lim_{t \rightarrow 0} \frac{P_t^\gamma f(\gamma(t)) - f(\gamma(0))}{t}$$

Another way to solve this issue is what we're going to use in the following. Since df can be regarded as a section of Ω_M^1 , we can directly define ∇s is also a section of $\Omega_M^1(E)$. Obviously, there are some other requirements.

As explained in what follows, we will define covariant derivatives on vector bundles and conventions on principal bundles in this section. Furthermore, we will see how these things blend with each other, and see the motivations of why do we need principal bundles to some extend.

1.3.1. Covariant derivatives. As it's known to all that derivative satisfies the Leibniz rule, a quite meaningful rule. It says that

$$d(fg) = df g + f dg$$

However, we can regard f multiply g as a $C^\infty(M)$ -module structure of $C^\infty(M)$. Since on the space of smooth sections of E , we can also give such a module structure. The action is such that a given function f acts on $s \in C^\infty(M, E)$ to give fs . So if we want to define a derivative on it, it should be compatible with such structure, like Leibniz rule tells us.

Definition 1.3.1 (covariant derivative). *A covariant derivative for $C^\infty(M, E)$ is a linear map*

$$\nabla : C^\infty(M, E) \rightarrow C^\infty(M, \Omega_M^1(E))$$

such that

$$\nabla(fs) = f\nabla s + s \otimes df$$

for all $f \in C^\infty(M)$.

Remark 1.3.2. We need to make sure that there do exists such a covariant derivative. We can do it locally and use partition of unity to a global one. For a local trivialization (U, φ_U) of E , we have

$$\varphi_U : E|_U \rightarrow U \times V$$

then section v of such trivial bundle $U \times V$ is just like $x \mapsto (x, v(x))$. Now define the covariant derivative as $dv : x \mapsto (x, dv|_x)$.

Then define covariant derivative for any given section s of E as

$$\nabla s = \sum_U \chi_U \varphi_U^*(d(\varphi_U \circ s|_U))$$

1.3.2. *The space of covariant derivatives.* In fact, there are lots of covariant derivatives. As we will explain next, the space of covariant derivatives is an affine modeled on $C^\infty(M, \text{Hom}(E, \Omega_M^1(E)))$.

To see this, first note that if $\alpha \in C^\infty(M, \text{Hom}(E, \Omega_M^1(E)))$ and ∇ is any given covariant derivative, then $\nabla + \alpha$ is also a covariant derivative. Meanwhile, if ∇ and ∇' are both covariant derivatives, then their difference $\nabla - \nabla'$, is a section of $\text{Hom}(E, \Omega_M^1(E))$. Indeed, since their difference is linear over the action of $C^\infty(M)$, i.e.

$$\nabla - \nabla'(fs) = (f\nabla s + s \otimes df) - (f\nabla' s + s \otimes df) = f(\nabla - \nabla')(s)$$

and consider the following lemma

Lemma 1.3.3. *Suppose that E and E' are two vector bundles and \mathcal{L} is a linear map that takes a section of E to one of E' . Suppose in addition that $\mathcal{L}(fs) = f\mathcal{L}(s)$ for all functions. Then there exists a unique section L of $\text{Hom}(E, E')$ such that $\mathcal{L} = L$.*

Proof. The proof is quite easy. To find L , fix an open set $U \subset M$ where both E and E' has a basis of sections. Denote the basis for E and E' as $\{e_i\}_{1 \leq i \leq d}$ and $\{e'_j\}_{1 \leq j \leq d'}$, where d and d' denote here the respective fiber dimensions of E and E' . Then we have functions $\{L_{ij}\}$ such that

$$\mathcal{L}e_i = \sum_{1 \leq j \leq d'} L_{ij} e'_j$$

This understood, the homomorphism L is defined over U as follows: Let s denote a section of E over U , and write $s = \sum_i s_i e_i$ in terms of the basis. Then Ls is defined to be the section of E' as

$$Ls = \sum_{i,j} L_{ij} s_i e'_j$$

The identity $\mathcal{L}s = Ls$ holds by the fact that $\mathcal{L}(fs) = f\mathcal{L}(s)$ for any functions. And the homomorphism L does not depend on the choice of the basis of sections by the same reason, since \mathcal{L} can commute with transition functions, this is to say that any two choices give the same section of $\text{Hom}(E, E')$. \square

However, it's more traditional to view $\nabla - \nabla'$ as a section of $\Omega_M^1(\text{End}(E))$ rather than $\text{Hom}(E, \Omega_M^1(E))$, since they're canonically isomorphic to each other.

What was just said about the affine property has the following implication: Let ∇ be a covariant derivative on sections of E , and s be a section of E . Suppose U is a local trivialization of E , that is, $\varphi_U : E|_U \rightarrow U \times V$ is an bundle isomorphism. Write $\varphi_U s$ as $x \mapsto (x, s_U(x))$ with $s_U : U \rightarrow V$, then $\varphi_U(\nabla s)$ appears as the section

$$x \mapsto (x, (\nabla s)_U), \quad (\nabla s)_U = ds_U + a_U s_U$$

where a_U is some s -independent section of $\Omega_U^1(\text{End}(V))$.

But $U \mapsto a_U$ does not define a section over M of $\Omega_M^1(\text{End}(E))$. Indeed, suppose U' is another trivialization, and $g_{U'U}$ is the transition function. Thus $s_{U'} = g_{U'U}s_U$. Meanwhile, ∇s is a bonafide section of $\Omega_M^1(E)$, so $(\nabla s)_{U'} = g_{U'U}(\nabla s)_U$. This requires that

$$(\nabla s)_{U'} = d(g_{U'U}s_U) + a_{U'}g_{U'U}s_U = g_{U'U}(ds_U + a_U s_U)$$

that is

$$a_{U'} = g_{U'U}a_U g_{U'U}^{-1} - (dg_{U'U})g_{U'U}^{-1}$$

Conversely, we can say a covariant derivative is defined by such data, that is a collection $\{a_U\}$ satisfying above equations, where a_U is a section of $\Omega_U^1(\text{End}(V))$.

1.3.3. Connections on principal bundles. This part we will define the notion of a connection on a principal bundle. This notion is of central importance in its own right. In any event, connections are used to give an alternate and very useful definition of the covariant derivative.

Suppose vector bundle E is associated to some principal bundle $\pi : P \rightarrow M$, and written as $P \times_G V$. Such principal bundles do exist, since we already know that E is associated to its frame principal bundle.

The reason we do this is from Proposition 1.2.9, we have a one to one correspondence between sections of E with G -equivariant maps from P to V . It's quite easy to take derivatives of s^P , it's a vector of differential 1-forms on P , and it defines a fiber-wise linear map $(s^P)_* : TP \rightarrow V$.

However, it does not by itself define a covariant derivative. As what we've defined, $\nabla s \in C^\infty(M, \Omega_M^1(E))$. so by Remark 1.2.12, a covariant derivative appears upstairs on P is supposed to be a G -equivariant section over $(P \times V) \otimes \pi^* T^*M$, that is a G -equivariant homomorphism from $\pi^* TM$ to V .

To see what $(s^P)_*$ is missing, it is important to keep in mind that TP has some properties that arise from the fact that P is a principal bundle over M . In fact, we have the following exact sequence

$$(1.1) \quad 0 \rightarrow \ker(\pi_*) \rightarrow TP \rightarrow \pi^*TM \rightarrow 0$$

This exact sequence is quite important, let's make following remarks:

Remark 1.3.4. The map from $\ker(\pi_*)$ is clearly an inclusion. And the map from TP to π^*TM is characterized as follows

$$\begin{aligned} TP &\rightarrow \pi^*TM \subset P \times TM \\ v &\mapsto (p, \pi_*v) \end{aligned}$$

where v is the tangent vector lying over p .

Remark 1.3.5. We can identify $\ker(\pi_*)$ with something quite simply, using the special property of principal bundle. Note that $\ker(\pi_*)$ designates the subbundle in TP that is sent by π_* to the zero sections in TM . That is to say that the vectors in the $\ker(\pi_*)$ are those that are tangent to the fibers of π . Thus, $\ker(\pi_*)$ over P_x is canonically isomorphic to $T(P_x)$, and that's Lie algebra \mathfrak{g} of Lie group G .

In fact, we have $\ker(\pi_*)$ is isomorphic to trivial bundle $P \times \mathfrak{g}$. Indeed, we have the following bundle isomorphism

$$\begin{aligned} \psi : P \times \mathfrak{g} &\rightarrow \ker(\pi_*) \\ (p, X) &\mapsto \left. \frac{d}{dt} \right|_{t=0} pe^{tX} \end{aligned}$$

In other words, we can define a map σ from \mathfrak{g} to $\mathfrak{X}(M)$, the set of vector fields on M , by assigning to each $X \in \mathfrak{g}$, the fundamental vector field $\sigma(X)$ whose value at p is given by $\psi(p, X)$.

Remark 1.3.6. Since there is a G action on P . So you may wonder if there is a G action on this exact sequence? The answer is yes. The action of G on P can be lifted to the exact sequence (1.1), as follows:

Let $R_g : P \rightarrow P$ denote the action of $g \in G$ on P , given by $p \mapsto pg$. This action lifts up to TP so as the pushout $(R_g)_* : TP \rightarrow TP$. And this action descends to $\ker(\pi_*)$. Indeed, if $v \in TP$ is in $\ker(\pi_*)$, then so is $(R_g)_*v$. We can check directly as follows, take $v \in \ker(\pi_*)$

$$\begin{aligned} \pi_*((R_g)_*v) &= (\pi \circ R_g)_*(v) \\ &= \pi_*(v) \\ &= 0 \end{aligned}$$

Here we use the fact that $\pi \circ R_g = \pi$.

The action lift of R_g to π^*TM is defined by viewing the latter in the manner described above as a subset of $P \times TM$. Viewed in this way, R_g act so as to send a pair $(p, v) \in P \times TM$ to the pair (pg, v) . Clearly $(pg, v) \in \pi^*TM$, since $\pi(pg) = \pi(p)$.

Furthermore, the exact sequence (1.1) is equivariant with respect to the lifts. Indeed, this automatically holds for inclusion map from $\ker(\pi_*)$ to TP . And it holds for the map from TP to π^*TM , since for $v \in TP$ we have $(R_g)_*v$ is sent to $(pg, \pi_*(R_g)_*v)$, that is exactly (pg, π_*v) , since $\pi \circ R_g = \pi$.

By Remark 1.3.6, we know that we can view $\ker(\pi_*)$ as the trivial bundle $P \times \mathfrak{g}$, so it's know how G acts on it if we do this identification. That is we need to choose a G action on \mathfrak{g} properly such that the isomorphism ψ is G -equivariant.

This requires G acts on \mathfrak{g} by adjoint representation. In other words, G acts on X by sending X to $g^{-1}Xg$ for $X \in \mathfrak{g}$. Indeed, we compute as follows

$$\begin{aligned} (R_g)_*\psi(p, X) &= (R_g)_* \left(\frac{d}{dt} \Big|_{t=0} p \exp(tX) \right) \\ &= \frac{d}{dt} \Big|_{t=0} p \exp(tX) g \\ &= \frac{d}{dt} \Big|_{t=0} (pg) (g^{-1} \exp(tX) g) \\ &= \psi(pg, g^{-1}Xg) \end{aligned}$$

So, as we have seen, there is a canonical isomorphism of $\ker(\pi_*)$ with $P \times \mathfrak{g}$. We call $\ker(\pi_*)$ vertical subbundle of TP , and sometimes denote it by V . But in the absence of any extra structure, there is no natural complement to $\ker(\pi_*)$.

A connection on principal bundle P is neither more nor less than a G -equivariant splitting of the exact sequence (1.1). Since $(\nabla s)^P$ is supposed to be a G -equivariant fiber-wise linear map $(\nabla s)^P : \pi^*TM \rightarrow V$. So if the above sequence splits, then we have

$$TP \cong \pi^*TM \oplus \ker(\pi_*)$$

By restricting $(s^P)_*$ to π^*TM , we will define a covariant derivative of s . This explains why we need to define connections on principal bundle as follows.

Definition 1.3.7 (connection). *A connection A is a linear map*

$$A : TP \rightarrow \ker(\pi_*)$$

that equals the identity on the kernel of π_ , and is G -equivariant with respect to the G -action on TP .*

Notation 1.3.8. We use $\mathcal{A}(P)$ to denote the set of all connections on P .

Remark 1.3.9 (horizontal distribution viewpoint). In other words, instead of defining a connection to be a linear projection, we can define it as a choice of horizontal distribution $H \subset TP$ such that

$$TP = H \oplus V$$

Since a projection is determined by its kernel. Furthermore, it satisfies some G -equivariant condition, that is $(R_g)_*H_p = H_{pg}$.

Remark 1.3.10 (\mathfrak{g} -valued 1-form viewpoint). Now, we give another description of connections. It is often more convenient for computations to rephrase a connection in terms of \mathfrak{g} -valued forms.

Since we have isomorphism $\psi : P \times \mathfrak{g} \rightarrow \ker(\pi_*)$, then for a connection A we defined in a horizontal distribution viewpoint, it can be identified with

$$\omega : TP \rightarrow P \times \mathfrak{g}$$

where $\omega = \psi \circ A$. However, $P \times \mathfrak{g}$ is a trivial bundle, thus we can write ω fiberwise as follows

$$\omega : TP \rightarrow \mathfrak{g}$$

that is, a 1-form valued \mathfrak{g} . Furthermore, it satisfies

1. $\omega(\psi(p, X)) = X, \forall X \in \mathfrak{g}$.
2. If $g \in G$, then $(R_g)^*\omega = g^{-1}\omega g$. In other words, $(R_g)^*\omega = \text{ad}_{g^{-1}} \circ \omega$.

Viewed this way, a connection is a section of $T^*P \otimes P \times \mathfrak{g}$ with certain additional properties that concern the G -action on P and the sequence (1.1). That is, we can regard connections as a \mathfrak{g} -valued differential 1-form, and denote it by $\Omega_P^1(\mathfrak{g})$.

Furthermore, these two different views blend with each other. Given a horizontal distribution $H \subset TP$, we can define its connection form by how does it acting on a vector field v as follows

$$\omega(v) = \begin{cases} X, & v = \sigma(X) \\ 0, & v \in H \end{cases}$$

Conversely, if we have a connection form ω , then we define $H = \ker(\omega)$.

Example 1.3.11. Now let's show a concrete example of connections on trivial principal G -bundle $M \times G$. In this case, we regard G as a Lie subgroup of $\text{GL}(n, \mathbb{R})$ ³, we use $g : G \hookrightarrow \text{GL}(n, \mathbb{R})$ to denote this inclusion. We first define a \mathfrak{g} -valued form on G as follows⁴

$$\omega = g^{-1}dg$$

Let's check it's indeed \mathfrak{g} -valued: For those vectors that are tangent to G , for $g \in G$ and $X \in T_g G$, we have

$$\begin{aligned} \omega(X) &= g^{-1}dg(X) \\ &= g^{-1} \left. \frac{d}{dt} \right|_{t=0} g e^{tX} \\ &= \left. \frac{d}{dt} \right|_{t=0} e^{tX} \\ &= X \in \mathfrak{g} \end{aligned}$$

Then we define a connection 1-form on $M \times G$ by using the projection $M \times G \rightarrow G$ to pullback ω , and still use ω to denote this form.

³In fact, every compact Lie group admits a faithful finite-dimensional representation.

⁴Immediately we will see that's exactly Maurer-Cartan form.

Clearly ω annihilates vectors that are tangent to the M factor of $M \times G$. Furthermore, it's G -equivariant, since for any $a \in G$, we have

$$\begin{aligned} (R_a)^* \omega &= (ga)^{-1} d(ga) \\ &= a^{-1} (g dg) a \\ &= a^{-1} \omega a \\ &= \text{ad}_{a^{-1}} \omega \end{aligned}$$

As desired.

This understood, the corresponding horizontal distribution H is precisely the tangents to the M factor, this factor TM in the obvious splitting of $T(M \times G) = TM \oplus TG$.

In fact, this tautological connection comes from the following form in Lie algebra.

Remark 1.3.12 (Maurer-Cartan form). The Maurer-Cartan form is a \mathfrak{g} -valued 1-form θ on G defined by

$$\theta_g = (L_{g^{-1}})_* : T_g G \rightarrow T_e G = \mathfrak{g}$$

If X is a left-invariant vector field, that is

$$X(g) = (L_g)_* X(e)$$

where $X(g)$ means the value of X at g . In this case, we have

$$\theta_g(X(g)) = (L_g)_*^{-1} X(g) = (L_g)_*^{-1} (L_g)_* X(e) = X(e)$$

which is constant. Now if X and Y are left-invariant vector fields, then it immediately that θ satisfies the structure equation

$$d\theta(X, Y) = -\theta([X, Y])$$

since $X(\theta(Y)) = Y(\theta(Y)) = 0$.

But the left-invariant fields span the tangent space at any point. (to do this we just need to pushforward a basis of $T_e G$ under diffeomorphism.) So the equation is true for any pair of vector fields X and Y . This is well known as Maurer-Cartan equation. Now we introduce the bracket of Lie algebra-valued forms, to give another form of Maurer-Cartan equation which we will often use later.

Choose a basis of TG consisting of left-invariant vector fields $\{E_i\}$, and use $\{\theta^i\}$ to denote its dual basis. Then $\{E_i(e)\}$ is a basis of Lie algebra, Then we can write $\alpha = \sum_i E_i(e) \otimes \alpha^i$ and $\beta = \sum_j E_j(e) \otimes \beta^j$. Define the bracket of Lie algebra-valued forms $[\alpha, \beta]$ as

$$[\alpha, \beta] := \sum_{i,j} [E_i(e), E_j(e)] \alpha^i \wedge \beta^j = \sum_{i,j,k} c_{ij}^k E_k(e) \alpha^i \wedge \beta^j$$

where c_{ij}^k is the structure constants of Lie algebra.

In the frame we choose, we can write $\theta = \sum_i E_i(e) \otimes \theta^i$, since by definition we have $\theta(E_i) = E_i(e)$. Note that

$$d\theta^i(E_j, E_k) = -\theta^i([E_j, E_k]) = -\sum_r c_{jk}^r \theta^i(E_r) = -c_{jk}^i = -\frac{1}{2}(c_{jk}^i - c_{kj}^i)$$

so we have

$$d\theta^i = -\frac{1}{2} \sum_{jk} c_{jk}^i \theta^j \wedge \theta^k$$

Let's take exterior derivative of Maurer-Cartan form θ

$$\begin{aligned} d\theta &= \sum_i E_i(e) \otimes d\theta^i \\ &= -\frac{1}{2} \sum_{i,j,k} c_{jk}^i E_i(e) \otimes \theta^j \wedge \theta^k \\ &= -\frac{1}{2} [\theta, \theta] \end{aligned}$$

Thus we have the structure equation, sometimes called the Maurer-Cartan equation for Maurer-Cartan forms

$$d\theta + \frac{1}{2} [\theta, \theta] = 0$$

However, review what we have done above, the most important equation about Maurer-Cartan form is that for any two vector fields X, Y we have

$$d\theta(X, Y) = -\theta([X, Y])$$

This is called the second form of Maurer-Cartan equation, since for any two vector fields X, Y , by directly computing we have:

$$[\theta, \theta](X, Y) = [\theta(X), \theta(Y)] - [\theta(Y), \theta(X)] = 2[\theta(X), \theta(Y)]$$

The above equation is an important property of the bracket of Lie algebra-valued forms. Thus the second form of Maurer-Cartan equation is equivalent to the first one. In literature, for any \mathfrak{g} -valued 1-form which satisfies Maurer-Cartan equation, we call it a Maurer-Cartan form.

Example 1.3.13. Let G be a Lie subgroup of $GL(n, \mathbb{R})$, and let $g : G \hookrightarrow GL(n, \mathbb{R})$ be the inclusion map. Then $\omega := g^{-1}dg$ is a Maurer-Cartan form. We check step by step:

We have already seen it's \mathfrak{g} -valued, it suffices to show that it do satisfies the Maurer-Cartan equation. Since $g^{-1}g = I$, then $(dg^{-1})g + g^{-1}dg = 0$. Thus we have $d(g^{-1}) = -g^{-1}(dg)g^{-1}$. Now,

$$\begin{aligned} d\omega &= d(g^{-1}) \wedge dg + g^{-1}d^2g \\ &= -g^{-1}(dg)g^{-1} \wedge dg \\ &= -g^{-1}dg \wedge g^{-1}dg \\ &= -\omega \wedge \omega \end{aligned}$$

That's it! Later we will see, that's say that the curvature of connection $g^{-1}dg$ is zero, that is, a flat connection.

Example 1.3.14. Consider $G = \mathrm{SO}(2) \subset \mathrm{GL}(2, \mathbb{R})$. We may parametrize $\mathrm{SO}(2)$ by

$$g(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad \theta \in \mathbb{R}$$

Then directly compute we have

$$\eta = g^{-1}dg = \begin{pmatrix} 0 & -d\theta \\ d\theta & 0 \end{pmatrix}$$

Remark 1.3.15 (guage field viewpoint). Now let's see connections in another pointview, which live one M instead of P . Recall that we have local sections $s_\alpha : U_\alpha \rightarrow \pi^{-1}U_\alpha$ associated canonically to the trivialization of bundle, along which we can pullback the connection 1-form ω , defining in the process the following \mathfrak{g} -valued 1-forms on U_α .

$$A_\alpha := s_\alpha^* \omega \in \Omega_{U_\alpha}^1(\mathfrak{g})$$

We claim that

Proposition 1.3.16. *The restriction of the connection 1-form ω to $\pi^{-1}U_\alpha$ agrees with*

$$\omega_\alpha = \mathrm{ad}_{g_\alpha^{-1}} \circ \pi^* A_\alpha + g_\alpha^* \theta$$

where θ is Maurer-Cartan form.

Proof. If we can show ω_α and ω agree on the image of s_α , and they transform in the same way under the action of G , then we can our desired result. So we prove this proposition in two steps. Refer [4] for more details. \square

Now since ω is defined globally, we have that $\omega_\alpha = \omega_\beta$ on $\pi^{-1}U_{\alpha\beta}$. This allows us to relate A_α and A_β on $U_{\alpha\beta}$. Indeed, we have

$$\begin{aligned} A_\alpha &= s_\alpha^* \omega_\alpha = s_\alpha^* \omega_\beta \\ &= s_\alpha^* (\mathrm{ad}_{g_\beta^{-1}} \circ \pi^* A_\beta + g_\beta^* \theta) \end{aligned}$$

We want to compute what is exactly $g_\beta \circ s_\alpha : U_{\alpha\beta} \xrightarrow{s_\alpha} \pi^{-1}U_{\alpha\beta} \xrightarrow{g_\beta} G$. Take $m \in U_{\alpha\beta}$, then by definition we have $s_\alpha(m) \in \pi^{-1}U_{\alpha\beta}$, such that $\varphi_\alpha(s_\alpha(m)) = (m, e) \in U_{\alpha\beta} \times G$. To see $g_\beta(s_\alpha(m))$, we need to show what's $\varphi_\beta(s_\alpha(m))$. In fact, we have

$$\begin{aligned} \varphi_\beta(s_\alpha(m)) &= \varphi_\beta(\varphi_\alpha^{-1}(m, e)) \\ &= \varphi_\beta \circ \varphi_\alpha^{-1}((m, e)) \end{aligned}$$

and by the definition of $g_{\beta\alpha}$, we have $g_\beta \circ s_\alpha = g_{\beta\alpha}$. Thus

$$(1.2) \quad A_\alpha = \mathrm{ad}_{g_{\beta\alpha}} \circ A_\beta + g_{\beta\alpha}^* \theta$$

In a summary, we have already three viewpoints of connections on principal bundles, they are

1. A G -equivariant horizontal distribution $H \subset TP$;
2. A 1-form $\omega \in \Omega_P^1(\mathfrak{g})$ satisfying some requirements;
3. A family of 1-forms $A_\alpha \in \Omega_{U_\alpha}^1(\mathfrak{g})$ satisfying equation (1.2) on overlaps.

1.3.4. *The space of connections.* Since we already know what's the difference of two covariant derivatives. In this section, we explore the difference of two connections A, A' on P . Clearly $\mathfrak{a}^P := A - A'$ annihilates $\ker(\pi_*)$. As a consequence, it defines a fiber-wise linear, G -equivariant map from π^*TM to \mathfrak{g} . So it corresponds to a G -equivariant section of $\mathfrak{g}_P \otimes T^*M$, where \mathfrak{g}_P is the associated vector bundle of P defined in Example 1.2.7.

Conversely, suppose \mathfrak{a} is a section of $\mathfrak{g}_P \otimes T^*M$, then we can give a fiber-wise linear, G -equivariant map $\mathfrak{a}^P : \pi^*TM \rightarrow \mathfrak{g}$. Then if A is a connection then $A + \mathfrak{a}^P$ is another one.

So if P has one connection, then it has infinitely many, and the space of connections over P , that is $\mathcal{A}(P)$ is an affine space based on $\mathfrak{g}_P \otimes T^*M$, and this is sometimes denoted by $\Omega_M^1(\mathfrak{g}_P)$.

We can use gauge field viewpoint to see this again. Let ω, ω' be two connection 1-forms, and use τ to denote $\omega - \omega'$. Let's see what this means on M .

$$\tau_\alpha = s_\alpha^* \tau = s_\alpha^*(\omega - \omega') = A_\alpha - A'_\alpha$$

Thus we have on the overlaps $U_{\alpha\beta}$ we have

$$\tau_\alpha = \text{ad}_{g_{\alpha\beta}} \circ \tau_\beta$$

Thus $\{\tau_\alpha\}$ defines a section of $\Omega_M^1(\mathfrak{g}_P)$.

1.4. **Gauge transformations.** Now we introduce the gauge group of a principal G -bundle $P \rightarrow M$.

Definition 1.4.1 (gauge group). *The gauge group is the group of G -automorphisms of P , that is, G -equivariant diffeomorphisms $\Phi : P \rightarrow P$ such that $\pi = \pi \circ \Phi$.*

Notation 1.4.2. We use $\mathcal{G}(P)$ to denote the gauge group of principal bundle P .

Definition 1.4.3 (gauge transformation). *An element of $\mathcal{G}(P)$ is called a gauge transformation.*

Remark 1.4.4. We can describe a gauge transformation in terms of trivialization. For a gauge transformation Φ , we restrict it to a trivialization $\varphi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha$. Then consider $\varphi_\alpha(\Phi(p)) = (\pi(p), g_\alpha(\Phi(p)))$, which induces a map $\bar{\phi}_\alpha : \pi^{-1}(U_\alpha) \rightarrow G$ by

$$\bar{\phi}_\alpha(p) = g_\alpha(\Phi(p))g_\alpha(p)^{-1}$$

By the equivariance of g_α and Φ we have

$$\bar{\phi}_\alpha(pg) = \bar{\phi}_\alpha(p), \quad \forall g \in G$$

whence $\bar{\phi}_\alpha(p) = \phi_\alpha(\pi(p))$ for some $\phi_\alpha : U_\alpha \rightarrow G$. If we consider on the overlaps $m \in U_{\alpha\beta}$ with $p = \pi^{-1}(m)$. Then

$$\begin{aligned}\phi_\alpha(m) &= g_\alpha(\Phi(p))g_\alpha(p)^{-1} \\ &= g_\alpha(\Phi(p))g_\beta(\Phi(p))^{-1}g_\beta(\Phi(p))g_\beta(p)^{-1}g_\beta(p)g_\alpha(p)^{-1} \\ &= g_{\alpha\beta}(m)\phi_\beta(m)g_{\alpha\beta}(m)^{-1} \\ &= \text{Ad}_{g_{\alpha\beta}(m)} \phi_\beta(m)\end{aligned}$$

Thus $\{\phi_\alpha\}$ define a section of the associated fiber bundle $\text{Ad } P$. We give another proof as follows.

Proposition 1.4.5. *The group $\mathcal{G}(P)$ is isomorphic to the group of sections $\Gamma(M, \text{Ad } P)$, where the group operation is pointwise multiplication.*

Proof. We provide maps in both directions. Suppose we have an automorphism $\Phi : P \rightarrow P$. Since $\pi = \pi \circ \Phi$, the map Φ preserves the fibers of π . Therefore, for any $p \in P$, we have that p and $\Phi(p)$ differ by the action of some $g_p \in G$. We claim that the mapping $g_\Phi : P \rightarrow G$ taking $p \mapsto g_p$ is equivariant with respect to the conjugation action of G . Indeed, what we need to do is to compute the difference between pg and $\Phi(pg)$. Use the fact that Φ is G -equivariant and $p = \Phi(p)g_p$, we have

$$\begin{aligned}\Phi(pg) &= \Phi(p)g \\ &= pg_p^{-1}g \\ &= pgg^{-1}g_p^{-1}g\end{aligned}$$

That is

$$pg = \Phi(pg)g^{-1}g_p g$$

So by Proposition 1.2.9, it defines a section of $\text{Ad } P$.

In the other direction, given a G -equivariant map $f : P \rightarrow G$, we have a bundle automorphism $\Phi_f : P \rightarrow P$ where $\Phi_f(p) = pf(p)$. Clearly $\pi \circ \Phi_f = \pi$, since

$$\begin{aligned}\pi \circ \Phi_f(p) &= \pi(pf(p)) \\ &= \pi(p)\end{aligned}$$

And it's G -equivariant since

$$\begin{aligned}\Phi_f(pg) &= pgf(pg) \\ &= pgg^{-1}f(p)g \\ &= pf(p)g \\ &= \Phi_f(p)g\end{aligned}$$

The two maps we constructed are clearly inverse to each other, giving the desired correspondence. \square

The gauge group $\mathcal{G}(P)$ acts on the space of connections naturally. We can see this from many ways.

If we regard connections as a horizontal distribution. Let $H \subset TP$ be a connection and let $\Phi : P \rightarrow P$ be a gauge transformation. Define $H^\Phi := \Phi_* H$. We claim that H^Φ is also a connection on P . Indeed,

$$\begin{aligned} (R_g)_* H_{\Phi(p)}^\Phi &= (R_g)_* \Phi_* H_p \\ &= \Phi_* (R_g)_* H_p \\ &= \Phi_* H_{pg} \\ &= H_{\Phi(pg)}^\Phi \\ &= H_{\Phi(p)g}^\Phi \end{aligned}$$

that is, H^Φ is G -equivariant. Moreover, H^Φ is still complementary to vertical distribution V , since Φ_* is an isomorphism which preserves the vertical subspaces. Indeed,

Lemma 1.4.6. *The fundamental vector fields $\sigma(X)$ of the G -action are gauge invariant.*

Proof. We need to prove $\Phi_* \sigma(X) = \sigma(X)$ for all $\Phi \in \mathcal{G}(P)$. Compute directly by definition

$$\begin{aligned} \Phi_* \sigma(X) &= \Phi_* \left(\frac{d}{dt} \Big|_{t=0} p e^{tX} \right) \\ &= \frac{d}{dt} \Big|_{t=0} \Phi(p e^{tX}) \\ &= \frac{d}{dt} \Big|_{t=0} \Phi(p) e^{tX} \\ &= \sigma(X) \end{aligned}$$

□

If we regard connections as a \mathfrak{g} -valued 1-form ω , what we can do is to pull it back. So we define $\omega^\Phi := (\Phi^{-1})^* \omega$. You may wonder why we use Φ^{-1} to pullback instead of Φ . The reason is that we want ω^Φ is exactly the connection form of H^Φ . Indeed, note that if $v \in \ker(\omega^\Phi) \subset TP$, then

$$0 = \omega^\Phi v = (\Phi^{-1})^* \omega(v) = \omega(\Phi_*^{-1} v)$$

So we have

$$\Phi_*^{-1} v \in \ker \omega = H$$

This implies that $v \in \Phi_* H = H^\Phi$, since by Lemma 1.4.6 we have Φ_* preserves vertical distribution, so the only thing in V mapped to zero by Φ_* is zero itself.

Finally let's see the effect of gauge transformation on a gauge field. Let $m \in U_\alpha$ and $p \in \pi^{-1}(U_\alpha)$. Let A_α and A_α^Φ be the gauge fields on U_α which corresponds to ω and ω^Φ . Then they are given at p by

$$\begin{aligned} \omega_p &= \text{ad}_{g_\alpha(p)^{-1} \circ \pi^*} A_\alpha + g_\alpha^* \theta \\ \omega_p^\Phi &= \text{ad}_{g_\alpha(p)^{-1} \circ \pi^*} A_\alpha^\Phi + g_\alpha^* \theta \end{aligned}$$

Using $\omega^\Phi = (\Phi^{-1})^*\omega$ we can obtain the relation between A_α and A_α^Φ .

$$A_\alpha^\Phi = \text{ad}_{\phi_\alpha} \circ (A_\alpha - \phi_\alpha^* \theta)$$

1.5. Curvatures.

1.5.1. *Curvatures of covariant derivatives.* For any vector bundle $\pi : E \rightarrow M$, and a section s of it, we already know how to define a covariant derivative

$$\nabla : C^\infty(M, E) \rightarrow C^\infty(M, \Omega_M^1(E))$$

However, in the case of smooth function, we can extend exterior derivative to more general case

$$d : C^\infty(M, \Omega_M^k) \rightarrow C^\infty(M, \Omega_M^{k+1})$$

such that $d^2 = 0$. The reason of $d^2 = 0$ is the following hallmark of partial derivatives: if f is a smooth function on \mathbb{R}^n , then

$$\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} f = \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} f$$

With the property of d , we can construct a cochain complex, that's what de Rham cohomology concern about.

So here we wonder, can we extend a covariant derivative ∇ , to similar things? The answer is yes. Since the soul of taking derivatives is Leibniz rule, we can extend ∇ by requiring it satisfy this rule. This extension is called the exterior covariant derivative, denoted by d_∇ , and it is defined by the following rules:

1. If ω is a k -form and s is a section of E , define $d_\nabla(s\omega) = \nabla s \wedge \omega + s d\omega$.
2. If ω_1, ω_2 are sections of $\Omega_M^k(E)$, then $d_\nabla(\omega_1 + \omega_2) = d_\nabla\omega_1 + d_\nabla\omega_2$.

Although $d^2 = 0$, this is generally not the case for d_∇ . It is the case, d_∇^2 defines a section of $\text{End}(E) \otimes \bigwedge^2 T^*M$, in other words, a section of $\Omega_M^2(\text{End}(E))$. This section is sometimes denoted by F_∇ and it is characterized by the fact that

$$d_\nabla^2 m = F_\nabla \wedge m$$

Indeed, suppose ω is a k -form and s is a section of E , then

$$\begin{aligned} d_\nabla^2(s\omega) &= d_\nabla(d_\nabla s \wedge \omega) + d_\nabla(s d\omega) \\ &= d_\nabla^2 s \wedge \omega - d_\nabla s \wedge d\omega + d_\nabla s \wedge d\omega + s \wedge d^2\omega \\ &= d_\nabla^2 s \wedge \omega \end{aligned}$$

In particular, if we take $\omega = f$ and thus a function, then

$$d_\nabla(fs) = f d_\nabla s$$

so by Lemma 1.3.3, we see that d_∇^2 is given by the action of a section of $\Omega_M^2(\text{End}(E))$.

This claim can be check locally, if we choose a local trivialization U of E . And explicitly F_∇ on U can be regarded as $dA + A \wedge A$, where A is a section of $\Omega_U^1(\text{End}(E|_U))$.

The exterior covariant derivative of sections is defined as above in many textbooks. However, we will recover this from the connections on principal bundles viewpoint, and explain why $d_{\nabla} \neq 0$.

1.5.2. Curvatures of connections. In this section, we will define the curvature of a connection on a principal G -bundle and interpret it geometrically in several different ways. Along the way we will define the covariant derivative of sections of associated vector bundle, and see that how does it connect with what we have defined.

Given a connection $H \subset TP$, we can define the horizontal projection $h : TP \rightarrow TP$ to be the projection onto the horizontal distribution along the vertical distribution. In other words, $\text{im } h = H$ and $\ker h = V$. Since both H and V are invariant under the action of G , so is h .

Definition 1.5.1 (curvature 2-form). *Let $\omega \in \Omega_P^1(\mathfrak{g})$ be the connection 1-form, then we define curvature 2-form $\Omega := h^*d\omega$*

Later we will give a explicit formula for Ω , but first let us interpret the curvature geometrically. By definition

$$\begin{aligned} \Omega(u, v) &= h^*d\omega(u, v) \\ &= d\omega(hu, hv) \\ &= hu \cdot \omega(hv) - hv \cdot \omega(hu) - \omega([hu, hv]) \\ &= -\omega([hu, hv]) \end{aligned}$$

⁵That is, $\Omega(u, v) = 0$ if and only if $[hu, hv]$ is horizontal. In other words, the curvature of the connection measures the failure of integrability of the horizontal distribution $H \subset TP$.

Remark 1.5.2 (Frobenius integrability). A distribution $D \subset TP$ is said to be integrable if the Lie bracket of any two sections of D lies again in D . The theorem of Frobenius states that a distribution is integrable if every $p \in P$ lies in a unique submanifold of P whose tangent space at p agrees with the subspace $D_p \subset T_pP$. These submanifolds are said to foliate P . As we have just seen, a connection $H \subset TP$ is integrable if and only if its curvature 2-form vanishes.

In contrast, the vertical distribution $V \subset TP$ is always integrable, since the Lie bracket of two vertical vector fields is again vertical, and Frobenius's theorem guarantees that P is foliated by submanifolds whose tangent spaces are the vertical subspaces. These submanifolds are of course the fibres of $\pi : P \rightarrow M$.

For curvature Ω , it satisfies the following structure equation⁶

⁵The third equality we use the formula connecting the exterior derivative and the Lie bracket, that is $d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$. The forth equality we use the fact that $h^*\omega = 0$.

⁶That's also the definition of curvature in other references.

Proposition 1.5.3 (structure equation).

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]$$

where $[\ , \]$ is the bracket of Lie algebra-valued forms.

Proof. By definition, we need to show

$$d\omega(hu, hv) = d\omega(u, v) + [\omega(u), \omega(v)]$$

for all vector fields u, v , since we have

$$\frac{1}{2}[\omega, \omega](u, v) = [\omega(u), \omega(v)]$$

Let u, v be horizontal. In this case there is nothing to prove, since $\omega(u) = \omega(v) = 0$ and $hu = u, hv = v$.

Let u, v be vertical. So we can take $u = \sigma(X), v = \sigma(Y)$ for some $X, Y \in \mathfrak{g}$. By directly computing

$$\begin{aligned} d\omega(\sigma(X), \sigma(Y)) + [\omega(\sigma(X)), \omega(\sigma(Y))] &= \sigma(X)\omega(\sigma(Y)) - \sigma(Y)\omega(\sigma(X)) - \omega([\sigma(X), \sigma(Y)]) + [X, Y] \\ &= \sigma(X)Y - \sigma(Y)X - \omega([\sigma(X), \sigma(Y)]) + [X, Y] \\ &= -\omega([\sigma(X), \sigma(Y)]) + [X, Y] \\ &= -\omega(\sigma([X, Y])) + [X, Y] \\ &= -[X, Y] + [X, Y] \\ &= 0 \end{aligned}$$

as desired.

Finally, let u be horizontal and $v = \sigma(X)$ be vertical, then equation becomes

$$d\omega(u, \sigma(X)) = 0$$

which in turn reduces to

$$u\omega(\sigma(X)) - \omega(u)\sigma(X) + \omega([u, \sigma(X)]) = \omega([u, \sigma(X)]) = 0$$

In other words, it suffices to show that the Lie bracket of a vertical and a horizontal vector field is again horizontal. Firstly we consider a special horizontal vector field u obtained by lifting vector field u' on M . Clearly $[\sigma(X), u] = 0$, since u is G -equivariant, then it's equivariant under the action of $R_{\exp tX}$, so

$$\begin{aligned} [\sigma(X), u] &= \left. \frac{d}{dt} \right|_{t=0} (R_{\exp(tX)})_* u \\ &= 0 \end{aligned}$$

Then for any local frame u'_1, \dots, u'_n of TM , we have their liftings u_1, \dots, u_n is a frame of horizontal vector fields. Then for any horizontal vector field u ,

we have $u = \sum_{i=1}^n f_i u_i$, then

$$\begin{aligned} [\sigma(X), u] &= \left. \frac{d}{dt} \right|_{t=0} (R_{\exp(tX)})_* u \\ &= \left. \frac{d}{dt} \right|_{t=0} (R_{\exp(tX)})_* \left(\sum_{i=1}^n f_i u_i \right) \\ &= \sum_{i=1}^n (\sigma(X) f_i) u_i \end{aligned}$$

As desired. \square

Immediately we have

Proposition 1.5.4 (Bianchi identity).

$$h^* d\Omega = 0$$

Proof. Directly computes

$$\begin{aligned} h^* d\Omega &= h^* d \left(d\omega + \frac{1}{2} [\omega, \omega] \right) \\ &= h^* \left(\frac{1}{2} [d\omega, \omega] - \frac{1}{2} [\omega, d\omega] \right) \\ &= h^* [d\omega, \omega] \\ &= [h^* d\omega, h^* \omega] \\ &= 0 \end{aligned}$$

\square

Since we already see how connections transform under a gauge transformation $\Phi : P \rightarrow P$, it's natural to ask how dose curvature transform. From structure equation, it's clear that

$$\Omega \mapsto \Omega^\Phi = (\Phi^{-1})^* \Omega$$

If we pullback Ω via the canonical sections $s_\alpha : U_\alpha \rightarrow P$, then we $F_\alpha = s_\alpha^* \Omega \in \Omega_{U_\alpha}^2(\mathfrak{g})$. It follows from the structure equation that

$$F_\alpha = dA_\alpha + \frac{1}{2} [A_\alpha, A_\alpha]$$

Using the relation between A_α and A_β , we have

$$F_\alpha = \text{ad}_{g_{\alpha\beta}} \circ F_\beta$$

In other words, $\{F_\alpha\}$ define a global 2-form $F_A \in \Omega_M^2(\mathfrak{g}_P)$. And it transformed under gauge transformation as

$$F_\alpha^\Phi = \text{ad}_{\phi_\alpha} \circ F_\alpha$$

1.6. Relations of connections and covariant derivatives. Although we have seen that some relationship between connections on principal bundles and covariant derivatives on vector bundles, here we will give a more detailed explanation here, along the way we will define many definitions we will use later.

1.6.1. Basic forms. As a warm-up, we need to do is to have a better understanding of the relation between forms on P and forms on M , since later we will consider the case with coefficients.

Definition 1.6.1 (horizontal form). *A k -form $\alpha \in \Omega_P^k$ is horizontal if $h^*\alpha = \alpha$.*

Definition 1.6.2 (basic form). *A k -form $\alpha \in \Omega_P^k$ is basic, if it's horizontal and it is G -invariant.*

Remark 1.6.3. It is a fact that α is basic if and only if $\alpha = \pi^*\bar{\alpha}$ for some $\bar{\alpha} \in \Omega_M^k$. In other words, the set of basic k -forms of P are exactly $\pi^*\Omega_M^k$.

Let's check one direction for the case $k = 1$ to get a feeling, other cases are quite similar. If $\alpha = \pi^*\bar{\alpha}$ for some $\bar{\alpha} \in \Omega_M^1$, then for some $v \in TP$

$$\begin{aligned} h^*\alpha(v) &= \alpha(hv) \\ &= \pi^*\bar{\alpha}(hv) \\ &= \bar{\alpha}(\pi_*hv) \\ &= \bar{\alpha}(\pi_*v) \\ &= \pi^*\bar{\alpha}(v) \\ &= \alpha(v) \end{aligned}$$

that is, α is horizontal. Now let's check α is G -invariant, we need to show

$$R_g^*\alpha = \alpha$$

Take $v \in TP$ and check directly as follows

$$\begin{aligned} R_g^*\alpha(v) &= \alpha((R_g)_*v) \\ &= \bar{\alpha}(\pi_*(R_g)_*v) \\ &= \bar{\alpha}((\pi \circ R_g)_*v) \\ &= \bar{\alpha}(\pi_*v) \\ &= \alpha(v) \end{aligned}$$

This completes the check.

The same story happens in the case of forms on P taking value in a vector space V admitting a representation $\rho : G \rightarrow \text{GL}(V)$ of G .

Let $\alpha \in \Omega_P^k(V)$. Similarly we can define horizontal form and basic form, that is α is horizontal if $h^*\alpha = \alpha$ and α is basic if it's horizontal and G -invariant. For G -invariant it means for all $g \in G$,

$$R_g^*\alpha = \rho(g^{-1})\alpha$$

Since there is a G -action on V , and α take values in V , so $\rho(g^{-1})\alpha$ make sense.

Notation 1.6.4. We use $\Omega_G^k(P; V)$ to denote the set of basic forms on P taking values in V , that is

$$\Omega_G^k(P; V) = \left\{ \alpha \in \Omega_P^k(V) \mid h^*\alpha = \alpha \text{ and } R_g^*\alpha = \rho(g^{-1})\alpha \right\}$$

We claim that there are some things similar to Remark 1.6.3. That is, a form with coefficients in V is basic if and only if it's a form pulled back from those on M with coefficients in $P \times_G V$. Thus we have the following one to one correspondence

$$\Omega_G^k(P; V) \iff \Omega_M^k(P \times_G V)$$

With this correspondence, we can handle with V -valued forms on P with some conditions, instead of bundle-valued forms on M . This is amazing result, since forms on P valued in V is just a section of trivial bundle $P \times V \otimes T^*P$, but forms on M valued E is a section of non-trivial bundle $E \otimes T^*M$. We convert a non-trivial thing into trivial one, and the cost we pay is just some restricted conditions.

1.6.2. *The covariant derivative.* The exterior derivative $d : \Omega_P^k(V) \rightarrow \Omega_P^{k+1}(V)$ defines a V -valued de Rham complex. However, if we want to descend d to basic forms, something bad happens, since basic forms do not form a sub-complex. Indeed, since $d\alpha$ need not to be horizontal even if α is.

If we want to basic forms to form a complex, a naive way is to project $d\alpha$ onto horizontal forms, i.e.

$$\begin{aligned} d^H : \Omega_G^k(P; V) &\rightarrow \Omega_G^{k+1}(P; V) \\ \alpha &\mapsto h^*d\alpha \end{aligned}$$

but the price we pay is that $(d^H)^2 \neq 0$ in general, so we no longer have a complex. As we see soon, the failure of d^H defining a complex is measured by the curvature.

Let's give a explicit formula of d^H . Take $k = 0$ as an example. Every section $\zeta \in \Omega_M^0(P \times_G V)$ defines an equivalent function $\bar{\zeta} \in \Omega_G^0(P; V)$ obeying $R_g^*\bar{\zeta} = \rho(g^{-1})\bar{\zeta}$. By definition, the exterior covariant derivative is given by $d^H\bar{\zeta} = h^*d\bar{\zeta}$. Applying this to a vector field $u = u_V + hu$, we have

$$(d^H\bar{\zeta})(u) = d\bar{\zeta}(hu) = d\bar{\zeta}(u - u_V) = d\bar{\zeta}(u) - u_V(\bar{\zeta})$$

What we need to do is to evaluate $u_V(\bar{\zeta})$. Take $u_V = \sigma(\omega(u))$, so that

$$u_V(\bar{\zeta}) = \sigma(\omega(u))\bar{\zeta} = \left. \frac{d}{dt} \right|_{t=0} R_{g(t)}^*\bar{\zeta} \quad \text{for } g(t) = e^{t\omega(u)}$$

Since $\bar{\zeta}$ is G -equivariant, we have

$$u_V\bar{\zeta} = \left. \frac{d}{dt} \right|_{t=0} \rho(g(t)^{-1}) \circ \bar{\zeta} = -\rho(\omega(u)) \circ \bar{\zeta}$$

that is,

$$d^H \bar{\zeta} = d\zeta + \rho(\omega)\bar{\zeta}$$

If we apply twice, we have

$$\begin{aligned} (d^H)^2 \bar{\zeta} &= h^* dh^* d\bar{\zeta} \\ &= h^* d(d\bar{\zeta} + \rho(\omega) \circ \bar{\zeta}) \\ &= h^* (\rho(d\omega) \circ \bar{\zeta} - \rho(\omega) \wedge d\bar{\zeta}) \\ &= \rho(h^* d\omega) \circ \bar{\zeta} \\ &= \rho(\Omega) \circ \bar{\zeta}. \end{aligned}$$

So curvature measures the failure of d^H to be a complex.

In the following, we will change the notation and write the exterior co-variant derivative on basic forms as

$$d^\omega : \Omega_G^k(P; V) \rightarrow \Omega_G^{k+1}(P; V)$$

to show the dependence on the connection 1-form, and write the one on bundle valued forms on M by

$$d_A : \Omega_M^k(P \times_G V) \rightarrow \Omega_M^{k+1}(P \times_G V)$$

to show the dependence on the gauge field. In this notation, the Bianchi identity for the curvature can be written as $d_A F_A = 0$.

1.7. Flat connections and holonomy.

1.7.1. Flat connections.

Definition 1.7.1 (flat connection). *A connection on a principal bundle is said to be flat when its curvature 2-form is identically zero.*

Example 1.7.2. As we have seen, the simplest example is the connection on the trivial principal bundle $M \times P$ that is defined by $\omega = g^{-1}dg$.

However, there are other flat connections.

Example 1.7.3. Let $h : M \rightarrow G$ denote any given smooth map. This defines a principal bundle isomorphism $\varphi : M \times G \rightarrow M \times G$ that send (x, g) to $(x, h(x)g)$. Then consider the pullback of ω is given by

$$\varphi^* \omega = g^{-1}dg + h^{-1}dh$$

and the curvature of $\varphi^* \omega$ is zero.

So there may be quite a lot connections on $M \times G$, let's see the case $M = S^1$ for an instructive example.

1.7.2. Flat connections on bundles over the circle. In this section, $M = S^1$. Assume G is a connected and $\pi : P \rightarrow S^1$ is a principal G -bundle. Every connection on P is flat, since as all 2-forms on S^1 is zero.

So it's natural to ask such a question: Suppose ω is a flat connection on P , is there an isomorphism $\varphi : S^1 \times G \rightarrow P$ such that

$$\varphi^*\omega = g^{-1}dg$$

To answer this question, note that in fact all principal G -bundles over S^1 are trivial one, that is we can find an isomorphism $\varphi : P \rightarrow S^1 \times G$. From Proposition 1.1.11, it suffices to show that there exists a global section on P .

To find one, it's more convenient to write S^1 as $\mathbb{Z}/2\pi\mathbb{Z}$. Fix an open interval $U \subset S^1$ containing 0 with a principal G -bundle isomorphism $\varphi : P|_U \rightarrow U \times G$. Also, fix a connection A and a point $p \in P|_0$. Let $\gamma_{A,p} : [0, 2\pi] \rightarrow P$ denote the horizontal lift of the path given by the projection map from $[0, 2\pi] \rightarrow S^1$. Here a horizontal lift means that it's a lift such that $(\gamma_{A,p})_*(dt) \in H_A$.

As $P|_{2\pi} = P|_0$, then there exists an element $h_{A,p} \in G$ such that $\gamma_{A,p}(2\pi)h_{A,p} = p$. As G is connected, we have a map $h : [0, 2\pi] \rightarrow G$ such that h joints identity $e \in G$ with $h_{A,p}$. This understood, we have a map $\sigma : [0, 2\pi] \rightarrow P$ given by $t \mapsto \gamma_{A,p}(t)h(t)$ such that $\sigma(0) = \sigma(2\pi) = p$. Thus we get a continuous section of P . Using basic tools in differential topology, we can get a smooth section of P . So as desired, we show that any principal G -bundle over S^1 is trivial.

Now let's back to our question. Choose a principal G -bundle isomorphism $P \rightarrow S^1 \times G$ and view the given connection A on P as a connection on $S^1 \times G$. If we have done this, the connection on the product can be written as the \mathfrak{g} -valued form $g^{-1}dg + g^{-1}ngdt$, where $n : S^1 \rightarrow \mathfrak{g}$. Our questions now is equivalent to the following one:

Is there a principal G -bundle isomorphism $\varphi : S^1 \times G \rightarrow S^1 \times G$ such that

$$\varphi^*(g^{-1}dg + g^{-1}ngdt) = g^{-1}dg$$

To solve this question, we use the horizontal lift of tautological path that starts at the point $(0, e) \in S^1 \times G$. Write $\gamma_{A,(0,e)}(2\pi)$ as $(0, h_{A,(0,e)})$ with $h_{A,(0,e)} \in G$.

We claim that the desired φ exists if and only if $h_{A,(0,e)}$ is the identity e . Indeed, first note that $h_{g^{-1}dg,(0,e)} = e$ and note that any given bundle automorphisms $\varphi : S^1 \times G \rightarrow S^1 \times G$ can be written as $(t, g) \mapsto (t, u(t)g)$ where $u : S^1 \rightarrow G$. This understood $h_{\varphi^*A,(0,e)} = u(0)h_{A,(0,e)}u(0)^{-1}$. Thus $\varphi^*A = g^{-1}dg$ if and only if e is conjugate to $h_{A,(0,e)}$, and e is the only element conjugate to e .

And such connection is constructable, thus our question has a positive answer.

1.7.3. Flat connections on bundles over M . The following theorem describes all of the flat connections on principal G -bundles over a given manifold M .

Let $\mathcal{F}_{M,G}$ be equivalent classes of pairs (P, A) , where $P \rightarrow M$ is a principal G -bundle and A is a flat connection with respect to gauge equivalence.

Theorem 1.7.4 (Riemann-Hilbert correspondence). *The set $\mathcal{F}_{M,G}$ is in one to one correspondence with the set $\text{Hom}(\pi_1(M), G)/G$, where G acts on $\text{Hom}(\pi_1(M), G)$ by conjugation.*

2. THE YANG-MILLS EQUATIONS

Untill now we have imposed no conditions on M or on G , but now things will change. To discuss the Yang-Mills equations, we will restrict to compact Lie groups G , and M will be a oriented Riemannian manifold with metric g . The orientation on M is given by a nowhere-vanishing n -form, which we will take to be the volume form of the metric.

2.1. Some geometry.

2.1.1. *Inner product on bundle valued forms.* Briefly speaking, the solutions of Yang-Mills equations are the critical point of Yang-Mills functional defined on the space of connections $\mathcal{A}(P)$.

However, in order to do functional analysis, we need a metric or inner product on vector bundles we are interested in, that is, differential forms with coefficients in some vector bundle.

Firstly, let's see a more general cases. We want to define an inner product on forms with values in an associated vector bundle $P \times_G V$. On each U_α , view such forms as forms with values in V , so all we need is an inner product on V , since we already have a Riemannian metric g on M , which gives us an inner product on forms.

But if we want this inner product to glue well on overlaps, we must require that it is G -invariant, that is, for all $g \in G, v, w \in V$,

$$\langle \rho(g)w, \rho(g)w \rangle = \langle v, w \rangle$$

Indeed, if $\omega \in \Omega_M^k(P \times_G V)$ is represented locally by $\omega_\alpha \in \Omega_{U_\alpha}^k(V)$. On a non-empty overlap $U_{\alpha\beta}$, we have

$$\langle \omega_\alpha, \omega_\alpha \rangle = \langle \rho(g_{\alpha\beta})\omega_\beta, \rho(g_{\alpha\beta})\omega_\beta \rangle = \langle \omega_\beta, \omega_\beta \rangle$$

whence it defines a global function $\langle \omega, \omega \rangle \in C^\infty(M)$.

The case we are interested in is $V = \mathfrak{g}$, since curvature of a connection lies in $\Omega_M^2(\mathfrak{g}_P)$. So we what we need is an inner product on Lie algebra \mathfrak{g} which is invariant under the adjoint action of G . Since G is compact, its Lie algebra \mathfrak{g} is semisimple, so the Killing form $\langle \cdot, \cdot \rangle$ is a nondegenerate inner product, we're looking for!

Thus we have an inner product on the bundle $\Omega_M^k(\mathfrak{g}_P)$, and denote it by $\langle \cdot, \cdot \rangle$.

2.1.2. *Hodge star operator.* In this section, we need the Hodge star operator defined on $\Omega_M^k(\mathfrak{g}_P)$, since we already have a metric on it.

Let's first make it clear in the case of vector space with an inner product on it.

Example 2.1.1. Let V be an oriented n -dimensional Euclidean vector space, with standard inner product $\langle \cdot, \cdot \rangle$. In other words, $\langle e_i, e_j \rangle = \delta_{ij}, \forall 1 \leq i, j \leq n$, where $\{e_1, \dots, e_n\}$ is a basis of V .

Let $W = V^*$, then we have a canonical inner product on W induced by the one on V , if we still use $\langle \cdot, \cdot \rangle$ to denote this inner product, we have

$$\langle e^i, e^j \rangle = \langle e_i, e_j \rangle, \quad \forall 1 \leq i, j \leq n$$

This induces an inner product on k -vectors $\alpha, \beta \in \bigwedge^k W$ for $0 \leq k \leq n$, by defining it on decomposition $\alpha = \alpha_1 \wedge \dots \wedge \alpha_k, \beta = \beta_1 \wedge \dots \wedge \beta_k$ to equal

$$\langle \alpha, \beta \rangle = \det(\langle \alpha_i, \beta_j \rangle_{i,j=1}^k)$$

and extend it to $\bigwedge^k V$ through linearity.

Clearly, there exists a canonical volume form $\text{vol} \in \bigwedge^n W \cong \mathbb{R}$, that is $\text{vol} = e^1 \wedge \dots \wedge e^n$, since by definition we have

$$\langle \text{vol}, \text{vol} \rangle = \det(\langle e^i, e^j \rangle_{i,j=1}^n) = \det(I_n) = 1$$

From linear algebra we already know the wedge product

$$\bigwedge^k W \times \bigwedge^{n-k} W \xrightarrow{\wedge} \bigwedge^n W$$

is non-degenerate. So for any $\beta \in \bigwedge^{n-k} W$, we can define $*\beta \in \bigwedge^k W$ such that

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \text{vol}, \quad \forall \alpha \in \bigwedge^k W$$

where $\langle \cdot, \cdot \rangle$ is the inner product induced from W , and the one on W is induced from V .

Clearly we have $*1 = \text{vol}$ and $*\text{vol} = 1$. Indeed, by definition, take $1, \alpha \in \bigwedge^0 W \cong \mathbb{R}$, then

$$\alpha * 1 = \alpha \wedge *1 = \alpha \text{vol} \implies *1 = \text{vol}$$

Similar for $*\text{vol}$. Furthermore,

$$*e^i = (-1)^{i-1} e^1 \wedge \dots \wedge \widehat{e^i} \wedge \dots \wedge e^n$$

To show this, we also need to back to definition. For any $\alpha \in W$, write it as $\alpha = \sum_{i=1}^n a_i e^i$, then

$$\begin{aligned} \alpha \wedge *e^i &= \langle \alpha, e^i \rangle \text{vol} \\ &= \left\langle \sum_{i=1}^n a_i e^i, e^i \right\rangle \text{vol} \\ &= a_i \text{vol} \\ &= a_i e^1 \wedge \cdots \wedge e^n \end{aligned}$$

From this equation, it's easy to see what $*e^i$ is exactly.

Last but not least,

$$** = *^2 = (-1)^{k(n-k)} \text{id}, \quad \text{on } \bigwedge^k W$$

the proof of it is also a routine, we omit it here.

We can carry what we have done to any vector bundles with a metric on it, since vector bundle is just vector space moving smoothly on M . In particular, we have Hodge star operator on $\Omega_M^k(\mathfrak{g}_P)$.

Definition 2.1.2 (Hodge star operator). *There exists*

$$* : C^\infty(X, \Omega_M^k(\mathfrak{g}_P)) \rightarrow C^\infty(X, \Omega_M^{n-k}(\mathfrak{g}_P))$$

such that

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \text{vol}, \quad \forall \alpha \in C^\infty(X, \Omega_M^k(\mathfrak{g}_P))$$

2.2. The variational problem. With some of the preliminary results established, we arrive at the Yang-Mills functional.

Definition 2.2.1 (Yang-Mills functional). *The Yang-Mills functional is the map $L : \mathcal{A}(P) \rightarrow \mathbb{R}$ given by*

$$L(A) := \|F_A\|^2 = \int_M \langle F_A, F_A \rangle \text{vol}$$

Remark 2.2.2. By the definition of Hodge star operator, we may rewrite Yang-Mills functional as follows

$$L(A) = \int_M F_A \wedge *F_A$$

The advantages of writing Yang-Mills functional in this way is that we can use some properties of Hodge operator to simplify our notations.

Remark 2.2.3. Note that for any gauge transformation $\Phi \in \mathcal{G}(P)$, we have $L(\Phi^*A) = L(A)$. Indeed,

$$L(\Phi^*A) = \int_M \langle \text{ad}_{\phi_\alpha} \circ F_A, \text{ad}_{\phi_\alpha} \circ F_A \rangle \text{vol} = \int_M \langle F_A, F_A \rangle \text{vol} = L(A)$$

because of this we say that L is gauge invariant.

Definition 2.2.4 (Yang-Mills connection). *A Yang-Mills connection is a connection $A \in \mathcal{A}(P)$ satisfying the Yang-Mills equations, i.e. a local extremum of L .*

Notation 2.2.5. We use $\mathcal{A}_{YM}(P)$, or briefly \mathcal{A}_{YM} to denote the set of all Yang-Mills connections.

Let's see that this Yang-Mills conditions turns into a second-order partial differential equation for A .

Recall that $\mathcal{A}(P)$ is an affine space modelled on $\Omega_M^1(\mathfrak{g}_P)$. This means the tangent space to $\mathcal{A}(P)$ at any point is isomorphic to $\Omega_M^1(\mathfrak{g}_P)$. Given any $A \in \mathcal{A}(P)$ and $\tau \in \Omega_M^1(\mathfrak{g}_P)$. The directional derivative of Yang-Mills functional at A in the direction τ is given by

$$\left. \frac{d}{dt} \right|_{t=0} L(A + t\tau)$$

And Yang-Mills condition states that this vanishes for all τ . To see what this means. From structure equation we have

$$\begin{aligned} F_{A+t\tau} &= d(A + t\tau) + \frac{1}{2}[A + t\tau, A + t\tau] \\ &= F_A + t(d\tau + \frac{1}{2}[A, \tau] + \frac{1}{2}[\tau, A]) + \frac{1}{2}t^2[\tau, \tau] \\ &= F_A + t(d\tau + [A, \tau]) + \frac{1}{2}t^2[\tau, \tau] \\ &= F_A + td_A\tau + \frac{1}{2}t^2[\tau, \tau] \end{aligned}$$

Then consider Yang-Mills functional

$$\begin{aligned} L(A + t\tau) &= \int_M \langle F_{A+t\tau}, F_{A+t\tau} \rangle \text{vol} \\ &= \int_M \langle F_A + \frac{t^2}{2}[\tau \wedge \tau] + td_A\tau, F_A + \frac{t^2}{2}[\tau \wedge \tau] + td_A\tau \rangle \text{vol} \end{aligned}$$

The term linear t is

$$\int_M \langle F_A, d_A\tau \rangle + \langle d_A\tau, F_A \rangle \text{vol} = 2 \int_M \langle d_A\tau, F_A \rangle \text{vol}$$

Let $d_A^* = (-1)^{2n+1} * d_A *$ denote the formal adjoint to d_A . Then we have

$$\int_M \langle d_A\tau, F_A \rangle \text{vol} = \int_M \langle \tau, d_A^* F_A \rangle \text{vol}$$

whence the Yang-Mills condition becomes the following differential equation

$$d_A^* F_A = 0$$

Then since up to sign $d_A^* = *d_A*$ and $*$ is an isomorphism, we have that above equation is equivalent to $d_A * F_A = 0$. Thus we have the first variation as follows

Proposition 2.2.6 (The first variation). *Let A be a local extremum of L . Then we have*

$$d_A * F_A = 0$$

Remark 2.2.7. In the case that $G = U_1$, we have that the curvature of a connection A can be identified as an element of Ω_M^2 . Indeed, the curvature form takes value in the bundle \mathfrak{g}_P , but here $G = U(1)$ is abelian, thus the adjoint action on $\mathfrak{u}(1)$ is trivial, so \mathfrak{g}_P is just the trivial bundle

$$\mathfrak{g}_P = M \times \mathfrak{u}(1) = M \times \mathbb{R}$$

Furthermore, A is a Yang-Mills connection if and only if F_A is a harmonic form, that is $\Delta F_A = 0$, where $\Delta = dd^* + d^*d$. Indeed, again thanks to $U(1)$ is abelian, d_A can be reduced to d . and Hodge theory says that if F_A is harmonic, then

$$\begin{aligned} 0 &= \int_M \langle \Delta F_A, F_A \rangle \\ &= \int_M \langle dd^* F_A, F_A \rangle + \langle d^* d F_A, F_A \rangle \\ &= \int_M |d^* F_A|^2 + |d F_A|^2 \end{aligned}$$

Thus we have

$$\begin{cases} d^* F_A = 0 \\ d F_A = 0 \end{cases}$$

That is, A satisfies the Yang-Mills equations.

Remark 2.2.8. Note that the Yang-Mills functional is gauge invariant, so if A solves the Yang-Mills equations, so does any gauge transformed A^Φ . In other words, the gauge group acts on \mathcal{A}_{YM} . The quotient $\mathcal{A}_{YM}/\mathcal{G}$ is the space of classical solutions. In general it is infinite dimensional, but we will see that it has interesting finite-dimensional subspaces.

Proposition 2.2.9 (The second variation). *Let A be a Yang-Mills connection. Then for every $\tau \in \Omega_M^1(\mathfrak{g}_P)$, we have*

$$\left. \frac{d}{dt} \right|_{t=0} d_{A+t\tau}^* F_{A+t\tau} = d_A^* d_A \tau + *[\tau \wedge *F_A]$$

Remark 2.2.10. If we think L as a Morse function on $\mathcal{A}(P)$, for a Yang-Mills connection A , the operator $d_A^* d_A \tau + *[\tau \wedge *F_A]$ can be interpreted as the Hessian of L at the critical point A .

Now we restrict ourselves to the case where M is a Riemann surface of genus $g \geq 2$. Let $\Gamma_{\mathbb{R}}$ denote the central extension of $\pi_1(M)$ by \mathbb{R} where if we let J denote the element $1 \in \mathbb{R}$, we have the relation $\prod_i [a_i, b_i] = J$ where a_i and b_i are the generators for the usual presentation of a closed surface of genus g . Using this group one can prove the following theorems, though we will omit the proofs.

Theorem 2.2.11. *Every principal G -bundle $P \rightarrow M$ admits a Yang-Mills connection.*

Theorem 2.2.12. *There is a bijective correspondence*

$\text{Hom}(\Gamma_{\mathbb{R}}, G)/G \longleftrightarrow \{\text{Principal } G\text{-bundles } P \rightarrow M \text{ with a Yang-Mills connection}\} / \sim$
where the action of G is conjugation and the equivalence relation is gauge equivalence.

Theorem 2.2.12 can be seen as an analogue of the classical Riemann-Hilbert correspondence we mentioned in Theorem 1.7.4.

If we take $G = U_1$, then a principal U_1 -bundle $P \rightarrow M$, then it corresponds to a Hermitian line bundle $L \rightarrow M$ by taking the associated bundle $P \times_{U_1} \mathbb{C}$ with the standard action of U_1 on \mathbb{C} .

For the rest of our discussion, we will restrict to case where $G = U_n$. and let X denote a complex manifold. We first make a remark involving the proofs of the Theorem 2.2.11 and 2.2.12. And similar to the case of U_1 , the data of a principal U_n -bundle $P \rightarrow X$ is equivalent to a rank n complex vector bundle $E \rightarrow X$ equipped with a Hermitian metric.

In the proofs, we will show that a Yang-Mills connection A is equivalent to the choice of a Lie algebra element $X \in \mathfrak{u}_n$. Write $X = -2\pi i\Lambda$ for a Hermitian metric, the Yang-Mills condition implies that the trace of Λ is equal to the first Chern class of E , thought of as an integrating over X .

So in the case of U_1 , a line bundle with trivial first Chern class always admits a Yang-Mills connection, or in other words, Yang-Mills equations are always solvable.

3. HOLOMORPHIC VECTOR BUNDLES AND YANG-MILLS CONNECTIONS

3.1. Stability of holomorphic vector bundles.

Definition 3.1.1 (holomorphic vector bundle). *A holomorphic vector bundle is a complex bundle $\pi : E \rightarrow X$ such that the total space E is a complex manifold and π is holomorphic.*

Remark 3.1.2. Given a holomorphic vector bundle $E \rightarrow X$, we can find a trivialization of E such that the transition functions are holomorphic. In a neighborhood $U \subset X$ such that $E|_U$ is holomorphically trivial, the smooth sections can be identified with smooth functions $U \rightarrow \mathbb{C}^n$, and the holomorphic sections can be identified with holomorphic functions $U \rightarrow \mathbb{C}^n$.

We have a local operator $\bar{\partial}$, which we can apply componentwise to a local section to an operator on smooth sections over U . Furthermore, since $\bar{\partial}$ annihilates holomorphic functions and the transition functions are holomorphic, we have that $\bar{\partial}$ glues to a well defined operator $\bar{\partial}_E : \mathcal{A}_X^0(E) \rightarrow \mathcal{A}_X^{0,1}(E)$. The holomorphic sections of E are exactly the sections annihilated by $\bar{\partial}_E$. Furthermore, the operator $\bar{\partial}_E$ extends to operator $\bar{\partial}_E : \mathcal{A}_X^k(E) \rightarrow \mathcal{A}_X^{k+1}(E)$, and satisfies the condition $\bar{\partial}_E^2 = 0$, since $\bar{\partial}^2 = 0$. The punchline is that the holomorphic structure on E is entirely determined by this operator.

Theorem 3.1.3. *Let $\pi : E \rightarrow X$ be a smooth complex vector bundle, and let $D : \mathcal{A}_X^0(E) \rightarrow \mathcal{A}_X^{0,1}(E)$ be an operator satisfying the Leibniz rule and $D^2 = 0$. Then there exists a unique complex structure on E such that E is holomorphic and the holomorphic sections are exactly the ones annihilated by D .*

We now restrict to the case where X is a Riemann surface of genus $g \geq 2$

Definition 3.1.4 (slope). *The slope of a holomorphic vector bundle $E \rightarrow X$ is*

$$\mu(E) := \frac{c_1(E)}{\text{rank}(E)}$$

where we think of $c_1(E) \in H^2(X, \mathbb{Z})$ as an integer via integration over X .

Remark 3.1.5. Sometimes the integer $c_1(E)$ is also referred to as the degree of E . One thing to note is that the slope of a holomorphic vector bundle is independent of the holomorphic structure. Both of degree and rank are topological invariants, and only depend on the underlying C^∞ complex vector bundle.

Definition 3.1.6 (stability of holomorphic bundles). *A holomorphic vector bundle $E \rightarrow X$ is*

1. *Stable if for every holomorphic subbundle $F \subset E$, we have $\mu(F) < \mu(E)$.*
2. *Semistable if for every holomorphic subbundle $F \subset E$, we have $\mu(F) \leq \mu(E)$.*
3. *Unstable if E is not semistable.*

While the slope is a topological invariant, stability is not, since we only consider holomorphic subbundles which depend on the holomorphic structure. We also note that both the degree and rank are additive in exact sequences, which immediately give us

Proposition 3.1.7. *Suppose we have the short exact sequence of holomorphic bundles*

$$0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$$

Then we have

$$\mu(F) = \frac{\deg(E) + \deg(G)}{\text{rank}(E) + \text{rank}(G)}$$

Corollary 3.1.8. *Given a short exact sequence of holomorphic bundles*

$$0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$$

If $\mu(E) \geq \mu(F)$, then $\mu(F) \geq \mu(G)$. Likewise, if $\mu(E) \leq \mu(F)$, then $\mu(F) \leq \mu(G)$.

In other words, slopes behave monotonically in short exact sequences. The terminology comes from GIT. The main result will use is

Theorem 3.1.9 (The Harder-Narasimhan Filtration). *Let $E \rightarrow X$ be a holomorphic vector bundle. Then E admits a canonical filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

by holomorphic subbundles E_i such that E_i/E_{i-1} is semistable and

$$\mu(E_1/E_0) > \mu(E_2/E_1) > \cdots > \mu(E_n/E_{n-1})$$

Proof. Sketch. The main idea is that any holomorphic vector bundle has a unique maximal semistable subbundle, which we take to be E_1 . We then take E_2 to be the preimage of the maximal semistable bundle of E_1/E_0 under the quotient map, and continue inductively. \square

Remark 3.1.10. The slopes $\mu_i := \mu(E_i/E_{i-1})$ gives us n rational numbers. If k denotes the rank E , then we construct an element of \mathbb{Q}^k by arranging the μ_i in order, and repeating the entry μ_i a total of $\text{rank}(E_i/E_{i-1})$ times. We call this vector the Harder-Narasimhan type of E .

Our ultimate goal will be relate moduli spaces of holomorphic vector bundles over X to Yang-Mills connections. To see this, let $E \rightarrow X$ be a C^∞ complex vector bundle of rank n , and fix a Hermitian metric on E . Then let $P \rightarrow X$ denote the principal U_n -bundle of frames for E . We abbreviate the gauge group $\mathcal{G}(P)$ as \mathcal{G} .

Proposition 3.1.11. *There is a bijection $\mathcal{A}(P) \longleftrightarrow \mathcal{C}(E)$*

Proof. We provide maps in both directions. Suppose we have a connection $A \in \mathcal{A}(P)$. Then A induces a covariant derivative $d_A : \mathcal{A}_X^0(E) \rightarrow \mathcal{A}_X^1(E)$. The $(0,1)$ part of d_A automatically satisfies $(d_A^{0,1})^2 = 0$, since $\mathcal{A}_X^2 = 0$ by dimension reasons. Therefore, $d_A^{0,1}$ defines a holomorphic structure on E .

In other direction, given a holomorphic structure $\bar{\partial}_E$, there exists a unique Hermitian connection A such that $d_A^{0,1} = \bar{\partial}_E$, called the Chern connection. \square

Let $\mathcal{G}_\mathbb{C}$ denote the group of smooth bundle automorphisms of E . The space $\mathcal{C}(E)$ has a natural action by $\mathcal{G}_\mathbb{C}$ by conjugation. Furthermore, the orbits under this action are exactly the isomorphism classes of holomorphic structures on E . This is most easily seen by characterizing an isomorphism $\varphi : E \rightarrow F$ of holomorphic vector bundles as a smooth bundle isomorphism intertwining $\bar{\partial}_E$ and $\bar{\partial}_F$. However, the naive quotient $\mathcal{C}(E)/\mathcal{G}_\mathbb{C}$ is poorly behaved. To remedy this, as in GIT, we restrict our attention to semistable bundles.

The relationship between $\mathcal{G}_\mathbb{C}$ and \mathcal{G} as well as the identification of $\mathcal{A}(P)$ and $\mathcal{C}(E)$ suggests that isomorphism classes of holomorphic bundles should have something to do with gauge equivalence classes of connections on P . This turn out to be true, and is an infinite dimensional version of the relationship between GIT quotient and a symplectic quotient.

To investigate further, we make a short digression regarding this relationship.

Let G be a reductive complex group, and X a compact Kähler manifold with Kähler metric ω , equipped with a “nice” action of G . In usually setting, X is a smooth projective variety with a fixed embedding $X \hookrightarrow \mathbb{CP}^N$, the Kähler metric ω is the restriction of the Fubini-Study form, and the G -action is induced by a homomorphism $G \rightarrow \mathrm{GL}_{N+1}(\mathbb{C})$. In general, the naïve quotient X/G is not well behaved, and one restricts the action to a subset X_{ss} consisting of semistable points to construct the GIT quotient X_{ss}/G .

3.2. Episode: Fubini-Study metric. This section contains some basic facts of Kähler manifold and Kähler metric on it. In particular, we will focus on \mathbb{P}^n and Fubini-Study metric⁷ on it.

Let $\varphi_\alpha : U_\alpha \rightarrow \mathbb{C}^n$ be the canonical coordinate chart. For $1 \leq i, j \leq n$, we define a function on \mathbb{C}^n by

$$h_{ij}(z) = \frac{(1 + \sum_{k=1}^n |z^k|^2) \delta_{ij} - \bar{z}^i z^j}{(1 + \sum_{k=1}^n |z^k|^2)^2}$$

First we claim that

Proposition 3.2.1. *The matrix valued function $H(z) = (h_{ij}(z))_{i,j=1}^n$ is smooth and $H(z)$ is a positive definite Hermitian matrix for all $z \in \mathbb{C}$.*

Proof. Smooth and Hermitian is clear. We focus on its positive definiteness: Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on \mathbb{C}^n and $\|\cdot\|$ the norm induced from it. For each $z, w \in \mathbb{C}^n$, we have

$$\begin{aligned} \langle H(z)w, w \rangle &= \sum_{i=1}^n \left(\sum_{j=1}^n h_{ij}(z) w^j \right) \bar{w}^i = \sum_{i=1}^n \left(\sum_{j=1}^n \frac{(1 + \sum_{k=1}^n |z^k|^2) \delta_{ij} - \bar{z}^i z^j}{(1 + \sum_{k=1}^n |z^k|^2)^2} w^j \right) \bar{w}^i \\ &= \frac{(1 + \|z\|^2) \|w\|^2 - |\langle z, w \rangle|^2}{(1 + \|z\|^2)^2} \end{aligned}$$

By Cauchy-Schwarz inequality,

$$|\langle z, w \rangle|^2 \leq \|z\|^2 \|w\|^2$$

Hence we find

$$\langle H(z)w, w \rangle \geq \frac{\|w\|^2}{(1 + \|z\|^2)^2} \geq 0$$

we find that $\langle H(z)w, w \rangle = 0$ implies that $\|w\|^2 = 0$. Therefore $w = 0$. This shows that $H(z)$ is positive for each $z \in \mathbb{C}^n$. \square

Let us define a function K on \mathbb{C}^n by

$$K(z) = \log(1 + \sum_{k=1}^n |z^k|^2)$$

⁷Trivia: Here the pronunciation of “Study” is “SHTOO-dee”.

Let us compute $\bar{\partial}K$ as follows

$$\bar{\partial}K = \frac{\sum_{j=1}^n z^j d\bar{z}^j}{1 + \sum_{k=1}^n |z^k|^2}$$

Hence we have

$$\begin{aligned} \partial\bar{\partial}K &= \sum_{i,j=1}^n \frac{\delta_{ij}(1 + \|z\|^2) - \bar{z}^i z^j}{(1 + \|z\|^2)^2} dz^i \wedge d\bar{z}^j \\ &= \sum_{i,j=1}^n h_{ij}(z) dz^i \wedge d\bar{z}^j \end{aligned}$$

For each $0 \leq \alpha \leq n$, we define a $(1,1)$ -form on U_α by

$$\omega_\alpha = \sqrt{-1} \sum_{i,j=1}^n (h_{ij} \circ \varphi_\alpha) dz_\alpha^i \wedge d\bar{z}_\alpha^j$$

This implies that if we denote $K_\alpha = K \circ \varphi_\alpha$ on U_α , then

$$\omega_\alpha = \sqrt{-1} \partial\bar{\partial}K_\alpha$$

Use the decomposition $d = \partial + \bar{\partial}$, we can find the following result by computing directly

$$d\omega_\alpha = 0$$

This shows that ω_α is a closed $(1,1)$ -form on U_α for each α . In fact, we can show that the $(1,1)$ -form on each U_α can be glued together to obtain a closed global $(1,1)$ -form ω on \mathbb{P}^n .

On U_α , let us rewrite K_α as follows

$$\begin{aligned} K_\alpha(\xi_0 : \cdots : \xi_n) &= \log(1 + \sum_{k=1}^n |z_\alpha^k(\xi_0 : \cdots : \xi_n)|^2) \\ &= \log(1 + \sum_{k=0}^n (\frac{\xi_k}{\xi_\alpha})^2) \\ &= \log(\sum_{i=0}^n |\xi_i|^2) - \log |\xi_\alpha|^2 \end{aligned}$$

Assume that $\alpha < \beta$, then on $U_\alpha \cap U_\beta$, we have

$$\begin{aligned} K_\alpha(\xi_0 : \cdots : \xi_n) - K_\beta(\xi_0 : \cdots : \xi_n) &= \log |\xi_\beta|^2 - \log |\xi_\alpha|^2 = \log \left| \frac{\xi_\beta}{\xi_\alpha} \right|^2 \\ &= \log |z_\alpha^\beta(\xi_0 : \cdots : \xi_n)|^2 \end{aligned}$$

In other words, on $U_\alpha \cap U_\beta$, we have

$$K_\alpha - K_\beta = \log |z_\alpha^\beta|^2$$

so we have

$$\bar{\partial}(K_\alpha - K_\beta) \implies \partial\bar{\partial}(K_\alpha - K_\beta) = 0 \quad \text{on } U_\alpha \cap U_\beta$$

Hence we can a global $(1, 1)$ -form ω on \mathbb{P}^n . Since ω is globally defined, the 2-tensor

$$\sum_{i,j=1}^n (h_{ij} \circ \varphi_\alpha) dz_\alpha^i \otimes d\bar{z}_\alpha^j$$

is also globally defined. Since we have already verify that $H(z)$ is a positive definite matrix on \mathbb{C}^n , then above equation defines a Hermitian metric on \mathbb{P}^n whose associated $(1, 1)$ -form is ω . Thus we conclude that

Theorem 3.2.2. \mathbb{P}^n is a Kähler manifold.

The Kähler metric on \mathbb{P}^n is called Fubini-Study metric.

Let's compute a concrete case for \mathbb{P}^1 . Let (ξ_0, ξ_1) be the standard coordinate on \mathbb{C}^2 and (ξ_0, ξ_1) be the corresponding coordinate on \mathbb{P}^1 . Let U, V be the canonical covering of \mathbb{P}^1 . and z, w be the coordinate function on U and V . In other words,

$$z(\xi_0 : \xi_1) = \frac{\xi_1}{\xi_0}, \quad w(\xi_0 : \xi_1) = \frac{\xi_0}{\xi_1}$$

Let $K_U = \log(1 + |z|^2)$ on U and $K_V = \log(1 + |w|^2)$ on V . Observe that

$$\begin{aligned} K_U(\xi_0 : \xi_1) &= \log(|\xi_0|^2 + |\xi_1|^2) - \log |\xi_0|^2 \\ K_V(\xi_0 : \xi_1) &= \log(|\xi_0|^2 + |\xi_1|^2) - \log |\xi_1|^2 \end{aligned}$$

Hence on $U \cap V$ we have

$$K_U - K_V = \log |z|^2$$

Since $|z|^2 = z\bar{z}$, we have

$$\bar{\partial}(K_U - K_V) = \frac{1}{\bar{z}} d\bar{z}$$

Then we have

$$\partial\bar{\partial}(K_U - K_V) = \partial\left(\frac{1}{\bar{z}}\right) \wedge d\bar{z} = 0$$

This shows that on $U \cap V$, we indeed have

$$\partial\bar{\partial}K_U = \partial\bar{\partial}K_V$$

then global defined $(1, 1)$ -form ω is

$$\omega = \begin{cases} \sqrt{-1}\partial\bar{\partial}K_U & \text{on } U \\ \sqrt{-1}\partial\bar{\partial}K_V & \text{on } V \end{cases}$$

Let's compute what does ω exactly look like. We know that $\bar{\partial}K_U = \frac{z}{1+|z|^2}d\bar{z}$. And hence

$$\begin{aligned}\partial\bar{\partial}K_U &= \partial\left(\frac{z}{1+|z|^2}\right) \wedge d\bar{z} \\ &= \frac{(1+|z|^2)\partial z - z\partial(1+|z|^2)}{(1+|z|^2)^2} \wedge d\bar{z} \\ &= \frac{(1+|z|^2)dz - |z|^2dz}{(1+|z|^2)^2} \wedge d\bar{z} \\ &= \frac{1}{(1+|z|^2)^2}dz \wedge d\bar{z}\end{aligned}$$

Similarly we can compute

$$\partial\bar{\partial}K_V = \frac{1}{(1+|w|^2)^2}dw \wedge d\bar{w}$$

So we have the Kähler metric ds^2 on \mathbb{P}^n is given by

$$ds^2 = \begin{cases} \frac{1}{(1+|z|^2)^2}dz \otimes d\bar{z} & \text{on } U \\ \frac{1}{(1+|w|^2)^2}dw \otimes d\bar{w} & \text{on } V \end{cases}$$

And check this metric is globally defined. On $U \cap V$, we have $\omega = \frac{1}{z}$ and $\bar{w} = \frac{1}{\bar{z}}$. Hence we have

$$dw \otimes d\bar{w} = \frac{1}{z^2}dz \otimes \frac{1}{\bar{z}^2}d\bar{z} = \frac{1}{|z|^4}dz \otimes d\bar{z}$$

then

$$\frac{1}{(1+|z|^2)^2}dz \otimes d\bar{z} = \frac{|z|^4}{(1+|z|^2)^2} \frac{dz \otimes d\bar{z}}{|z|^4} = \frac{1}{(|w|^2+1)^2}dw \otimes d\bar{w}$$

We prove that ds^2 is a globally defined Hermitian metric.

3.3. Symplectic reduction. Now back to topological subject. Let $K \subset G$ denote the maximal compact subgroup, which has the property that its complexification is isomorphic to G . Suppose that the action of K on X is symplectic, i.e. the action of any $k \in K$ preserves the Kähler metric on X . Let \mathfrak{l} denote the Lie algebra of K . Then the infinitesimal action of K is given by the Lie algebra homomorphism $\mathfrak{l} \rightarrow \mathfrak{X}(X)$ defined by $\xi \mapsto X_\xi$, where

$$(X_\xi)_p := \left. \frac{d}{dt} \right|_{t=0} p \cdot \exp(t\xi)$$

Definition 3.3.1 (Hamiltonian). *A symplectic action of K on X is Hamiltonian if for each $\xi \in \mathfrak{l}$, there exists a function $H_\xi : X \rightarrow \mathbb{R}$ such that for all $p \in X$ and $v \in T_pX$ we have*

$$\omega_p((X_\xi)_p, v) = (dH_\xi)_p(v)$$

and the mapping $\xi \mapsto H_\xi$ is K -equivariant with respect to the right action of K on \mathfrak{l} by the Adjoint action and precomposition with right translation R_k on $C^\infty(X)$. The functions H_ξ are called Hamiltonian functions.

Definition 3.3.2 (moment map). *Suppose we have a Hamiltonian action of K on X . A moment map for the action is a K -equivariant map $\mu : X \rightarrow \mathfrak{l}^*$ (where the action on \mathfrak{l}^* is the coadjoint action) such that for any $p \in X, v \in T_p X$ and $\xi \in \mathfrak{l}$, we have*

$$d\mu_p(v)(\xi) = \omega_p((X_\xi)_p, v)$$

Remark 3.3.3. Let's make coadjoint action more clear:

Let G be a Lie group and \mathfrak{g} be its Lie algebra. Let $\text{ad} : G \rightarrow \text{Aut}(\mathfrak{g})$ denote the adjoint representation of G . Then we can define its coadjoint representation $\text{ad}^* : G \rightarrow \text{Aut}(\mathfrak{g}^*)$ as

$$\langle \text{ad}_g^* \mu, Y \rangle = \langle \mu, \text{ad}_{g^{-1}} Y \rangle$$

for $g \in G, Y \in \mathfrak{g}, \mu \in \mathfrak{g}^*$.

Remark 3.3.4. One thing to note is that the Hamiltonian functions can be recovered by the moment maps. If a Hamiltonian action admits a moment map, then

$$H_\xi(p) = \mu(p)(\xi)$$

Let $\langle \cdot, \cdot \rangle$ be an inner product on \mathfrak{l}^* that is invariant under the coadjoint action, and $\| \cdot \|$ be the induced norm. Since X is compact, then the map $\| \mu \|^2 : X \rightarrow \mathbb{R}$ attains its minimum, and WLOG we assume that the minimum value is 0.

Definition 3.3.5 (symplectic quotient). *The symplectic quotient of X by K is the quotient space*

$$\mu^{-1}(0)/K$$

Remark 3.3.6. The symplectic quotient can also be referred to as the symplectic reduction. It should be noted that the symplectic quotient depends on our choice of moment map.

Theorem 3.3.7. *The symplectic quotient of X by K admits a unique Kähler structure such that the Kähler metric on $\mu^{-1}(0)/K$ is induced by the Kähler metric on X .*

The relationship between the GIT quotient and the symplectic quotient is given by the Kempf-Ness theorem

Theorem 3.3.8 (Kempf-Ness). *Suppose a complex reductive group G acts on a Kähler manifold X such that the action of the maximal compact subgroup $K \subset G$ is Hamiltonian and admits a moment map $\mu : X \rightarrow \mathfrak{l}^*$. Then the G -orbit of any semistable point contains a unique K -orbit minimizing $\| \mu \|^2$. This establish a homeomorphism*

$$X_{ss}/G \longleftrightarrow \mu^{-1}(0)/K$$

3.4. Narasimhan-Sechadri. We now want to relate the previous discussion to our situation. Using the identification of $\mathcal{A}(P)$ and $\mathcal{C}(E)$, we want the action of $\mathcal{G}_{\mathbb{C}}$ to play the role of the complex reductive group G and the gauge group \mathcal{G} to play the role of the maximal compact subgroup. Since the space $\mathcal{A}(P)$ is infinite dimensional, along with the group $\mathcal{G}_{\mathbb{C}}$ and \mathcal{G} , we are working in an infinite dimensional setting, but we will gloss over the analytic details and work with them formally.

Our first task is to realize $\mathcal{A}(P)$ as a “Kähler manifold”.

Our next task is to show the action of $\mathcal{G}_{\mathbb{C}}$ on $\mathcal{A}(P)$ is “Hamiltonian” with respect to this Kähler structure.

To summarize, we have the following analogies:

$$\begin{aligned} \text{Kähler manifold} &\longleftrightarrow \mathcal{A}(P) \\ \text{Complex reductive group} &\longleftrightarrow \mathcal{G}_{\mathbb{C}} \\ \text{Maximal compact subgroup } K \subset G &\longleftrightarrow \mathcal{G} \\ \text{Moment map } \mu &\longleftrightarrow A \mapsto F_A \\ \text{Norm square of the moment map } \|\mu\|^2 &\longleftrightarrow L \end{aligned}$$

The last piece is something analogous to the Kempf-Ness theorem

Theorem 3.4.1 (Narasimhan-Seshadri). *Let $\mathcal{A}_s(P) \subset \mathcal{A}(P)$ denote the subspace of connections that are absolute minima for the Yang-Mills functional, and correspond to irreducible representations $\Gamma_{\mathbb{R}} \rightarrow U_n$. Let $\mathcal{C}_s(E)$ denote the subspace of stable holomorphic structures on E . The isomorphism classes of holomorphic bundles in $\mathcal{C}_s(E)$ admit unique Yang-Mills connections up to gauge equivalence. In other words, there is a homeomorphism*

$$\mathcal{A}_s(P)/\mathcal{G} \longleftrightarrow \mathcal{C}_s(E)/\mathcal{G}$$

Remark 3.4.2. The original proof is more algebraic in flavor. A proof more in the spirit of the Atiyah-Bott paper was given by Donaldson in [5]. The spirit of this proof is carried on by the proof of Hermitian-Yang-Mills and the nonabelian Hodge theorem, which were both grew out of the developments from the Atiyah-Bott paper.

4. FURTHER RESEARCH

Too ideas of Atiyah and Bott’s paper were extended to many directions. For example, Donaldson [6] used the space of solutions of the Yang-Mills equations to prove his celebrated theorem.

In another direction, work of Uhlenbeck and Yau [7] extended the study of slope stability to holomorphic vector bundles over higher dimensional Kähler manifolds, relating slope stability to the existence of Hermitian-Yang-Mills connections. Motivated by these ideas, Yau conjectured that the existence of Hermitian-Einstein connections on Fano manifolds would be related to another algebro-geometric notion of stability called K -stability. Recent work of Chen, Donaldson, and Sun [8] has mostly resolved this conjecture.

Another circle of ideas that grew out of the original paper have been the ideas around Higgs bundles. These were originally introduced by Hitchin [9], and have been used heavily in work by Simpson [10].

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二零二二年五月三日
于北京大学 家园

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SCHOOL OF MATHEMATICS, SHANDONG UNIVERSITY, JINAN, 250100, P.R. CHINA,
Email address: `bowenl@mail.sdu.edu.cn`