Symmetric space

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In this talk we give an introduction about Riemannian symmetric space, and it contains the following parts:

- Firstly we give a quick review of basic facts in Riemannian geometry we used.
- Basic definitons and properties of Riemannian symmetric space, and the relations between symmetric, locally symmetric and homogenous spaces.
- The Cartan decomposition of Lie algebra, and how to use Killing form to compute curvatures.
- A brief introduction to the classification of Riemannian symmetric space and some basic examples.

Theorem

Let $\varphi, \psi \colon (M, g_M) \to (N, g_N)$ be two local isometries between Riemannian manifolds, and M is connected. If there exists $p \in M$ such that

$$\varphi(p) = \psi(p)$$
$$(\mathrm{d}\varphi)_p = (\mathrm{d}\psi)_p$$

then $\varphi = \psi$.

Theorem (Myers-Steenrod)

Let (M, g) be a Riemannian manifold and $G = \mathrm{Iso}(M, g)$. Then

1. *G is a Lie group with respect to compact-open topology.*
2. *for each $p \in M$, the isotropy group G_p is compact.*
3. *G is compact if M is compact.*

Theorem (Cartan-Ambrose-Hicks)

Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be two Riemannian manifold, and $\Phi_0\colon T_pM \to T_{\widetilde{p}}\widetilde{M}$ is a linear isometry, where $p \in M, \widetilde{p} \in \widetilde{M}$. For $0 < \delta < \min\{\mathrm{inj}_p(M), \mathrm{inj}_{\widetilde{p}}(\widetilde{M})\}$, The following statements are equivalent.

1. *There exists an isometry $\varphi\colon B(p, \delta) \to B(\widetilde{p}, \delta)$ such that $\varphi(p) = \widetilde{p}$ and $(\mathrm{d}\varphi)_p = \Phi_0$.*
2. *For $v \in T_pM, |v| < \delta, \gamma(t) = \exp_p(tv), \widetilde{\gamma}(t) = \exp_{\widetilde{p}}(t\Phi_0(v))$, if we define*

$$\Phi_t = P_{0,t;\widetilde{\gamma}} \circ \Phi_0 \circ P_{t,0;\gamma}\colon T_{\gamma(t)}M \to T_{\widetilde{\gamma}(t)}\widetilde{M}$$

then Φ_t preserves curvature, that is $(\Phi_t)^*R = R$.

Lemma

Let (M, g) be a Riemannian manifold, $\gamma\colon I \to M$ a smooth curve and $P_{s,t;\gamma}\colon T_{\gamma(s)}M \to T_{\gamma(t)}M$ is the parallel transport along γ. For any $s \in I$ with $v = \gamma'(s)$, one has

$$\nabla_v R = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=s} (P_{s,t;\gamma})^* R_{\gamma(t)}$$

In particular, if $\nabla R = 0$ then

$$(P_{s,t;\gamma})^* R_{\gamma(t)} = R_{\gamma(s)}$$

holds for arbitrary $t, s \in I$.

Lemma

If $\pi\colon (\widetilde{M}, \widetilde{g}) \to (M, g)$ is a Riemannian covering, then M is complete if and only if $\widetilde{M}$ is.

Lemma

Let (M, g_M) be a complete Riemannian manifold and $f: (M, g_M) \to (N, g_N)$ be a local isometry. Then f is a Riemannian covering map.

Definition (symmetric space)

A Riemannian manifold (M, g) is called a Riemannian symmetric space if for each $p \in M$ there exists an isometry $\varphi \colon M \to M$, which is called a symmetry at p, such that $\varphi(p) = p$ and $(\mathrm{d}\varphi)_p = -\,\mathrm{id}$.

Remark.

Note that Theorem 1, that is rigidity property of isometry, implies if symmetry at point p exists, then it's unique.

Example

Let g_{can} be the Euclidean metric on $\mathbb{R}^n$. For each $p \in \mathbb{R}^n$, the reflection

$$\varphi(x) = 2p - x$$

is a symmetric at point p. Thus $(\mathbb{R}^n, g_{\text{can}})$ is a Riemannian symmetric space.

Example

Let g_{can} be the metric of S^n induced from $(\mathbb{R}^{n+1}, g_{\text{can}})$. For each $p \in S^n$, the reflection

$$\varphi(x) = 2\langle x, p\rangle p - x$$

is a symmetric at point p. Thus (S^n, g_{can}) is a Riemannian symmetric space.

Lemma

The following statements are equivalent.

1. (M, g) *is a Riemannian symmetric space.*
2. *For each* $p \in M$, *there exists an isometry* $\varphi \colon M \to M$ *such that* $\varphi^2 = \mathrm{id}$ *and* p *is an isolated fixed point of* φ.

Proof.

From (1) to (2). Let φ be a symmetry at $p \in M$. Since $(\mathrm{d}\varphi^2)_p = (\mathrm{d}\varphi)_p \circ (\mathrm{d}\varphi)_p = \mathrm{id}$ and $\varphi^2(p) = p$, one has $\varphi^2 = \mathrm{id}$ by Theorem 1. If p is not an isolated fixed point, then there exists a sequence $\{p_i\}_{i=1}^{\infty}$ converging to p such that $\varphi(p_i) = p_i$. For $0 < \delta < \mathrm{inj}(p)$, there exists sufficiently large k such that $p_k \in B(p, \delta)$, and we denote $v = \exp_p^{-1}(p_k)$. Since φ is an isometry, one has $\varphi(\exp_p(tv))$ and $\exp_p(tv)$ are two geodesics connecting p and p_k.

Continuation.

By uniqueness of geodesic, one has

$$\varphi(\exp_p(tv)) = \exp_p(tv)$$

In particular, one has $v = (\mathrm{d}\varphi)_p v$, which is a contradiction.
From (2) to (1). From $\varphi^2 = \mathrm{id}$ we have $(\mathrm{d}\varphi)_p^2 = \mathrm{id}$, so only possible eigenvalues of $(\mathrm{d}\varphi)_p$ are ± 1. Now it suffices to show all eigenvalues of $(\mathrm{d}\varphi)_p$ are -1. Otherwise if it has an eigenvalue 1, there exists some non-zero $v \in T_pM$ such that $(\mathrm{d}\varphi)_p v = v$. Since φ is an isometry, one has $\varphi(\exp_p(tv))$ and $\exp_p(tv)$ are geodesics with the same direction at p. Thus

$$\varphi(\exp_p(tv)) = \exp_p(tv)$$

for $0 < t < \mathrm{inj}(p)$. In particular, p is not an isolated fixed point, which is a contradiction. $\square$

Lemma

The fundamental group of a Riemannian symmetric space is abelian.

Corollary

A surface of genus $g \geq 2$ does not admit a Riemannian metric with respect to which it is a symmetric space.

Definition (locally Riemannian symmetric space)

A Riemannian manifold (M, g) is called a locally Riemannian symmetric space if each $p \in M$ has a neighborhood U such that there exists an isometry $\varphi\colon U \to U$ such that $\varphi(p) = p$ and $(\mathrm{d}\varphi)_p = -\operatorname{id}$.

Theorem

Let (M, g) be a complete Riemannian manifold. The following statements are equivalent.

1. *(M, g) is a locally Riemannian symmetric space.*
2. $\nabla R = 0$.

Proof.

From (1) to (2). If φ is the symmetry at point $p \in M$, then it's an isometry such that $(\mathrm{d}\varphi)_p = -\operatorname{id}$, and thus for $u, v, w, z \in T_pM$, one has

$$\begin{aligned}
-\nabla_u R(v, w)z &= (\mathrm{d}\varphi)_p (\nabla_u R(v, w)z) \\
&= \nabla_{(\mathrm{d}\varphi)_p u}((\mathrm{d}\varphi)_p)v, (\mathrm{d}\varphi)_p w)(\mathrm{d}\varphi)_p z \\
&= \nabla_u R(v, w)z
\end{aligned}$$

This shows $(\nabla R)_p = 0$, and thus $\nabla R = 0$ since p is arbitrary.
From (2) to (1). For arbitrary $p \in M$, it suffices to show

$$\varphi = \exp_p \circ \Phi_0 \circ \exp_p^{-1} \colon B(p, \delta) \to B(p, \delta)$$

is an isometry, where $0 < \delta < \operatorname{inj}(p)$ and $\Phi_0 \colon T_pM \to T_pM$ is $-\operatorname{id}$.

Continuation.

For $v \in T_pM$ with $|v| < \delta$ and $\gamma(t) = \exp_p(tv), \widetilde{\gamma}(t) = \exp_p(t\Phi_0(v))$, if we define $\Phi_t = P_{0,t;\widetilde{\gamma}} \circ \Phi_0 \circ P_{t,0;\gamma}$, then direct computation shows

$$\begin{aligned}
\Phi_t^* R_{\widetilde{\gamma}(t)} &= (P_{t,0;\gamma})^* \circ \Phi_0^* \circ (P_{0,t;\widetilde{\gamma}})^* R_{\widetilde{\gamma}(t)} \\
&\overset{(a)}{=} (P_{t,0;\gamma})^* \circ \Phi_0^* R_{\widetilde{\gamma}(0)} \\
&\overset{(b)}{=} (P_{t,0;\gamma})^* R_{\gamma(0)} \\
&\overset{(c)}{=} R_{\gamma(t)}
\end{aligned}$$

where (a) and (c) holds from Lemma 4, and (b) holds from $\widetilde{\gamma}(0) = \gamma(0)$ and R is a $(0,4)$-tensor.
Then by Theorem 3, that is Cartan-Ambrose-Hicks's theorem, φ is an isometry, which completes the proof.

Theorem

Let (M,g) be a complete, simply-connected locally Riemannian symmetric space. Then (M,g) is a Riemannian symmetric space.

Proof.

For $p\in M$ and $0<\delta<\mathrm{inj}(p)$, suppose $\varphi\colon B(p,\delta)\to B(p,\delta)$ is an isometry such that $\varphi(p)=p$ and $(\mathrm{d}\varphi)_p=-\mathrm{id}$. For arbitrary $q\in M$, we use $\Omega_{p,q}$ to denote all curves γ with $\gamma(0)=p,\gamma(1)=q$, and for $c\in\Omega_{p,q}$ we choose[1] a covering $\{B(p_i,\delta_i)\}_{i=0}^k$ of c such that

1. $0<\delta_i<\mathrm{inj}(p_i)$.
2. $B(p_0,\delta_0)=B(p,\delta)$ and $p_k=q$.
3. $p_{i+1}\in B(p_i,\delta_i)$.

[1] Since injective radius is a continuous function, it has a positive minimum on curve c, so such covering exists.

Continuation.

If we set $\varphi = \varphi_0$, then we can define isometries $\varphi_i\colon B(p_i, \delta_i) \to M$ such that $\varphi_i(p_i) = \varphi_{i-1}(p_i)$ and $(\mathrm{d}\varphi_i)_{p_i} = (\mathrm{d}\varphi_{i-1})_{p_i}$ by using Cartan-Ambrose-Hicks's theorem successively, and by Theorem 1 one has φ_i and φ_{i+1} coincide on $B(p_i, \delta_i) \cap B(p_{i+1}, \delta_i)$. The covering together with isometries we construct is denoted by $\mathcal{A} = \{B(p_i, \delta_i), \varphi_i\}_{i=0}^{k}$. For arbitrary $x \in [0, 1]$, if $c(x) \in B(p_m, \delta_m)$, we may define

$$
\begin{aligned}
\varphi_{\mathcal{A}}(c(x)) &:= \varphi_m(c(x)) \\
(\mathrm{d}\varphi_{\mathcal{A}})_{c(x)} &:= (\mathrm{d}\varphi_m)_{c(x)}
\end{aligned}
$$

In particular, $\varphi_{\mathcal{A}}(q) := \varphi_k(q)$. If $\mathcal{B} = \{\widetilde{B}(\widetilde{p}_i, \widetilde{\delta}_i), \widetilde{\varphi}_i\}_{i=0}^{l}$ is another covering of c, let's show $\varphi_{\mathcal{A}}(q) = \varphi_{\mathcal{B}}(q)$. Consider

$$
I = \{x \in [0, 1] \mid \varphi_{\mathcal{A}}(c(x)) = \varphi_{\mathcal{B}}(c(x)), (\mathrm{d}\varphi_{\mathcal{A}})_{c(x)} = (\mathrm{d}\varphi_{\mathcal{B}})_{c(x)}\}
$$

Continuation.

It's clear $I \neq \varnothing$, since $0 \in I$. Now it suffices to show it's both open and closed to conclude $1 \in I$.

(a) It's open: For $x \in I$, we assume $c(x) \in B(p_m, \delta_m) \cap \widetilde{B}(\widetilde{p}_n, \widetilde{\delta}_n)$, that is

$$\varphi_m(c(x)) = \widetilde{\varphi}_n(c(x))$$
$$(\mathrm{d}\varphi_m)_{c(x)} = (\mathrm{d}\widetilde{\varphi}_n)_{c(x)}$$

Then one has

$$\begin{aligned}\varphi_m \circ \exp_{c(x)}(v) &= \exp_{\varphi_m(c(x))} \circ (\mathrm{d}\varphi_m)_{c(x)}(v) \\ &= \exp_{\widetilde{\varphi}_n(c(x))} \circ (\mathrm{d}\widetilde{\varphi}_n)_{c(x)}(v) \\ &= \widetilde{\varphi}_n \circ \exp_{c(x)}(v)\end{aligned}$$

Since $\exp_{c(x)}$ maps onto a neighborhood of $c(x)$, it follows that some neighborhood of x also lies in I, and thus I is open.

Continuation.

(b) It's closed: Let $\{x_i\}_{i=1}^\infty \subseteq I$ be a sequence converging to x. Without lose of generality we may assume $\{x_i\}_{i=1}^\infty \subseteq B(p_m,\delta_m)\cap\widetilde{B}(\widetilde{p}_n,\widetilde{\delta}_n)$, then one has

$$\begin{aligned}\varphi_m(c(x_i)) &= \widetilde{\varphi}_n(c(x_i))\\ (\mathrm{d}\varphi_m)_{c(x_i)} &= (\mathrm{d}\widetilde{\varphi}_n)_{c(x_i)}\end{aligned}$$

By taking limit we obtain the desired results.

Since $\varphi_{\mathcal{A}}(q)$ is independent of the choice of covering, we denote it as $\varphi(q)$ for convenience, and as a consequence we obtain the following map

$$\begin{aligned}F\colon \Omega_{p,q} &\to M\\ c &\mapsto \varphi(q)\end{aligned}$$

Note that $F(c)$ is locally constant, and thus it's independent of the choice of homotopy classes of c.

Continuation.

Since M is simply-connected, one has $F\colon \Omega_{p,q} \to M$ is constant, so we obtain a local isometry $\varphi\colon M \to M$ which extends $\varphi\colon B(p,\delta) \to B(p,\delta)$. By Lemma 6 φ is a Riemannian covering map since M is complete, and thus φ is a diffeomorphism since M is simply-connected, which implies φ is an isometry. $\square$

Corollary

Let (M,g) be a complete locally Riemannian symmetric space. Then it's isometric to $(\widetilde{M}/\Gamma,\widetilde{g})$ where $(\widetilde{M},\widetilde{g})$ is a Riemannian symmetric space and Γ is a discrete Lie group acting on $\widetilde{M}$ freely, properly and isometrically.

As a consequence, above argument about analytic continuation can be used to give a proof of Hopf's theorem.

Theorem (Hopf)

Let (M, g) be a complete, simply-connected Riemannian manifold with constant sectional curvature K. Then (M, g) is isometric to

$$(\widetilde{M}, g_{can}) = \begin{cases} (\mathbb{S}^n(\frac{1}{\sqrt{K}}), g_{can}) & K > 0 \\ (\mathbb{R}^n, g_{can}) & K = 0 \\ (\mathbb{H}^n(\frac{1}{\sqrt{-K}}), g_{can}) & K < 0 \end{cases}$$

Proof.

For $p \in M, \widetilde{p} \in \widetilde{M}$ and $\delta < \min\{\mathrm{inj}(p), \mathrm{inj}(\widetilde{p})\}$. By Cartan-Ambrose-Hicks's theorem, there exists an isometry $\varphi\colon B(p,\delta) \to B(\widetilde{p},\delta)$ such that $\varphi(p) = \widetilde{p}$ and $(\mathrm{d}\varphi)_p$ equals to a given linear isometry, since both (M,g) and $(\widetilde{M},\widetilde{g})$ have constant sectional curvature K. By the same argument in proof of Theorem 15, there is an isometry $\varphi\colon (M,g) \to (\widetilde{M},\widetilde{g})$ which extends $\varphi\colon B(p,\delta) \to B(\widetilde{p},\delta)$. In particular, this completes the proof. $\square$

Definition (Riemannian homogeneous space)

A Riemannian manifold (M, g) is called a Riemannian homogeneous space, if $\mathrm{Iso}(M, g)$ acts on M transitively.

Lemma

Let (M, g) be a Riemannian homogeneous space. If there exists a symmetry at some point $p \in M$, then (M, g) is a Riemannian symmetric space.

Proof.

Let φ be a symmetry at $p \in M$. For arbitrary $q \in M$, there exists an isometry $\psi\colon M \to M$ such that $\psi(p) = q$ since (M, g) is a Riemannian homogeneous space. Then

$$\varphi_q := \psi \circ \varphi \circ \psi^{-1}$$

is the desired symmetry at q.

Theorem

Let (M, g) be a Riemannian symmetric space. Then

1. (M, g) *is complete.*
2. *for any isometry* $\varphi\colon M \to M$ *with* $(\mathrm{d}\varphi)_p = -\operatorname{id}$ *and* $\varphi(p) = p$*, if* $v \in T_pM$*, then*

$$\varphi(\exp_p(v)) = \exp_p(-v)$$

3. *the isometry group* $\operatorname{Iso}(M, g)$ *acts transitively on* M*.*

Proof.

For (1). For arbitrary geodesic $\gamma\colon [0,1] \to M$ with $\gamma(0) = p, \gamma'(0) = v$, the curve $\beta(t) = \varphi(\gamma(t))\colon [0,1] \to M$ is also a geodesic with $\beta(0) = p$ and $\beta'(0) = -v$.

Continuation.

Now we obtain a smooth extension $\gamma'\colon [0,2]\to M$ of γ, given by

$$\gamma'(t)=\begin{cases}\beta(1-t), & t\in[0,1]\\ \gamma(t-1), & t\in[1,2]\end{cases}$$

Repeat above process to extend γ to a geodesic defined on $\mathbb{R}$, this shows completeness.

For (2). Note that $\varphi(\exp_p(tv)$ and $\exp_p(-tv)$ are geodesics starting at p with the same direction since φ is an isometry, and thus $\varphi(\exp_p(tv))=\exp_p(-tv)$. Furthermore, since (M,g) is complete, one has $\varphi(\exp_p(tv)$ and $\exp_p(-tv)$ are defined on $\mathbb{R}$. In particular, one has $\varphi(\exp_p(v))=\exp_p(-v)$ by considering $t=1$.

Continuation.

For (3). Let $\gamma\colon [0,1] \to M$ be a geodesic connecting $p, q \in M$, and $\varphi_m\colon M \to M$ is the symmetry at $m = \gamma(\frac{1}{2})$. If we consider $\beta(t) = \varphi_m(\gamma(\frac{1}{2} - t))$, then $\beta(0) = m, \beta'(0) = \gamma'(\frac{1}{2})$, which implies $\beta(t) = \gamma(\frac{1}{2} + t)$. Therefore
$q = \gamma(1) = \beta(\frac{1}{2}) = \varphi_m(\gamma(0)) = \varphi_m(p)$. □

Corollary

The Riemannian symmetric space (M, g) is a Riemannian homogeneous space.

Firstly let's review some basic facts about Killing fields in Riemannian geometry.

Lemma

Let (M, g) be a Riemannian manifold and X be a Killing field.

1. *If γ is a geodesic, then $J(t) = X(\gamma(t))$ is a Jacobi field.*
2. *For any two vector fields Y, Z,*

$$\nabla_Y \nabla_Z X - \nabla_{\nabla_Y Z} X + R(X, Y)Z = 0$$

Corollary

Let (M, g) be a complete Riemannian manifold and $p \in M$. Then a Killing field X is determined by the values X_p and $(\nabla X)_p$ for arbitrary $p \in M$.

Proof.

The equation $\mathcal{L}_X g \equiv 0$ is linear in X, so the space of Killing fields is a vector space. Therefore, it suffices to show if $X_p = 0$ and $(\nabla X)_p = 0$, then $X \equiv 0$. For arbitrary $q \in M$, let $\gamma \colon [0,1] \to M$ be a geodesic connecting p and q with $\gamma'(0) = v$. Since $J(t) = X(\gamma(t))$ is a Jacobi field, and a direct computation shows

$$(\nabla_v X)_p = J'(0)$$

Thus $J(t) \equiv 0$, since Jacobi field is determined by two initial values. In particular, $X_q = J(1) = 0$, and since q is arbitrary, one has $X \equiv 0$. □

Corollary

The dimension of vector space consisting of Killing fields $\leq n(n+1)/2$.

Lemma

Killing field on a complete Riemannian manifold (M, g) is complete.

Proof.

For a Killing field X, we need to show the flow $\varphi_t \colon M \to M$ generated by X is defined for $t \in \mathbb{R}$. Otherwise, we assume φ_t is defined on (a, b). Note that for each $p \in M$, curve $\varphi_t(p)$ is a curve defined on (a, b) having finite constant speed, since φ_t is isometry. Then we have $\varphi_t(p)$ can be extended to the one defined on $\mathbb{R}$, since M is complete. □

Theorem

Let (M, g) be a complete Riemannian manifold and $\mathfrak{g}$ the space of Killing fields. Then $\mathfrak{g}$ is isomorphic to the Lie algebra of $G = \mathrm{Iso}(M, g)$.

Proof.

It's clear $\mathfrak{g}$ is a Lie algebra since $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$. Now let's see it's isomorphic to Lie algebra consisting of Killing field as Lie algebra.

1. Given a Killing field X, by Lemma 25, one deduces that the flow $\varphi\colon \mathbb{R}\times M \to M$ generated by X is a one parameter subgroup $\gamma\colon \mathbb{R} \to G$, and $\gamma'(0) \in T_eG$.
2. Given $v \in T_eG$, consider the one-parameter subgroup $\gamma(t) = \exp(tv)\colon \mathbb{R} \to G$ which gives a flow by

$$
\begin{aligned}
\varphi\colon \mathbb{R}\times M &\to M \\
(t,p) &\mapsto \exp(tv)\cdot p
\end{aligned}
$$

Then the vector field X generated by this flow is a Killing field.

This gives a one to one correspondence between Killing fields and Lie algebra of G, and it's a Lie algebra isomorphism. □

Corollary (Cartan decomposition)

Let (M, g) be a complete Riemannian manifold and $G = \mathrm{Iso}(M, g)$ with Lie algebra $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ of G has the following decomposition as vector spaces

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

where

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid X_p = 0\}$$
$$\mathfrak{m} = \{X \in \mathfrak{g} \mid (\nabla X)_p = 0\}$$

and they satisfy

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$$

Proof.

The decomposition follows from Corollary 23 and Theorem 26, and it's easy to see

$$[\mathfrak{k},\mathfrak{k}]\subseteq\mathfrak{k},\quad [\mathfrak{m},\mathfrak{m}]\subseteq\mathfrak{k}$$

For arbitrary $X\in\mathfrak{k}$, $Y\in\mathfrak{m}$ and $v\in T_pM$, one has

$$\begin{aligned}\nabla_v[X,Y]&=\nabla_v\nabla_XY-\nabla_v\nabla_YX\\&=-R(Y,v)X+\nabla_{\nabla_vX}Y+R(X,v)Y-\nabla_{\nabla_vY}X\\&=0\end{aligned}$$

since $X_p=0$ and $(\nabla Y)_p=0$. This shows $[\mathfrak{k},\mathfrak{m}]\subseteq\mathfrak{m}$. □

Definition (involution)

An automorphism σ of G is called an involution if $\sigma^2 = \mathrm{id}_G$.

Definition (Cartan decomposition)

Let σ be an involution of G. The eigen-decomposition of $\mathfrak{g}$ given by $(\mathrm{d}\sigma)_e$ is called Cartan decomposition, that is,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

where

$$\begin{aligned} \mathfrak{k} &= \{X \in \mathfrak{g} \mid (\mathrm{d}\sigma)_e(X) = X\} \\ \mathfrak{m} &= \{X \in \mathfrak{g} \mid (\mathrm{d}\sigma)_e(X) = -X\} \end{aligned}$$

Lemma

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be a Cartan decomposition given by σ. Then

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$$

Proof.

It follows from $(\mathrm{d}\sigma)_e([X, Y]) = [(\mathrm{d}\sigma)_e(X), (\mathrm{d}\sigma)_e(Y)]$, where $X, Y \in \mathfrak{g}$. $\square$

Lemma

Suppose (M_1, g_1) and (M_2, g_2) are two Riemannian homogeneous spaces with the same isometry group G. If there exists a G-equivalent diffeomorphism φ such that $(\mathrm{d}\varphi)_p$ is an isometry for some $p \in M$, then (M_1, g_1) is isometric to (M_2, g_2).

Theorem

Let (M, g) be a Riemannian symmetric space and G be the identity component of $\mathrm{Iso}(M, g)$. For $p \in M$, K denotes the isotropic group of G_p.

1. *The mapping $\sigma\colon G \to G$, given by $\sigma(g) = s_p g s_p$ is an involution automorphism of G.*
2. *If G^σ is the set of fixed points of σ in G, and $(G^\sigma)_0$ is the identity component of G^σ, then $(G^\sigma)_0 \subseteq K \subseteq G^\sigma$.*
3. *If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the Cartan decomposition given by σ, then $\mathfrak{k}$ is the Lie algebra of K.*
4. *There is a left invariant metric on G which is also right invariant under K, such that G/K with the induced metric is isometric to (M, g).*

Proof.

(1) is clear. For (2). It follows from the following two steps:

(a) To show $K \subseteq G^\sigma$. For any $k \in K$, in order to show $k = s_p k s_p$, it suffices to show they and their differentials agree at p since both of them are isometries.

(b) To see $(G^\sigma)_0 \subseteq K$. Let $\exp(tX) \subseteq (G^\sigma)_0$ be a one-parameter subgroup. Since $\sigma(\exp(tX)) = \exp(tX)$, then

$$s_p \exp(tX) s_p(p) = s_p \exp(tX)(p) = \exp(tX)(p)$$

But p is an isolated fixed point of s_p, which implies $\exp(tX)(p) = p$ for all t. This shows the one-parameter subgroup lies in K. Since exponential map of Lie group is a diffeomorphism in a small neighborhood of identity element e and $(G^\sigma)_0$ can be generated by a neighborhood of e, which implies the whole $(G^\sigma)_0 \subseteq K$.

Continuation.

For (3). Note that $(G^\sigma)_0 \subseteq K \subseteq G^\sigma$, it suffices to show $\mathfrak{k} \cong \operatorname{Lie} G^\sigma$ If $X \in \mathfrak{k}$, then $\gamma_2(t) = \sigma(\exp(tX))\colon \mathbb{R} \to G$ is a one-parameter subgroup. Indeed, note that

$$\begin{aligned} \gamma_2(t) \cdot \gamma_2(s) &= s_p \exp(tX) s_p \cdot s_p \exp(sX) s_p \\ &= \sigma(\exp(tX + sX)) \\ &= \gamma_2(t+s) \end{aligned}$$

Furthermore, $\gamma_2(t) = \sigma(\exp(tX))$ and $\gamma_1(t) = \exp(tX)$ are two one-parameter subgroups of G such that $\gamma_1(0) = \gamma_2(0)$ and $\gamma_2'(0) = (\mathrm{d}\sigma)_e(X) = X = \gamma_1'(0)$. Then $\gamma_1(t) = \gamma_2(t)$, and thus $\exp(tX) \in G^\sigma$ for all $t \in \mathbb{R}$. This shows $\mathfrak{k} \subseteq \operatorname{Lie} G^\sigma$, and the converse inclusion is clear, so one has $\mathfrak{k} = \operatorname{Lie} G^\sigma$.

Continuation.

For (4). Let $\pi\colon G\to M$ be the natural projection given by $\pi(g)=gp$. Then for $k\in K$ and $X\in\mathfrak{g}$ one has

$$\begin{aligned}(\mathrm{d}\pi)_e(\mathrm{Ad}_k X) &= (\mathrm{d}\pi)_e\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} k\exp(tX)k^{-1}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\pi(k\exp(tX)k^{-1})\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} k\exp(tX)k^{-1}\cdot p\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} k\exp(tX)\cdot p\\ &= k_*(\mathrm{d}\pi)_e(X)\end{aligned}$$

By using the equivalent isomorphism $(\mathrm{d}\pi)_e|_{\mathfrak{m}}\colon\mathfrak{m}\to T_pM$, one has an $\mathrm{Ad}(K)$-invariant metric on $\mathfrak{m}$.

Continuation.

Then we can extend it to an $\mathrm{Ad}(K)$-invariant metric on $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ by choosing arbitrary $\mathrm{Ad}(K)$-invariant metric on $\mathfrak{k}$ such that $\mathfrak{m} \perp \mathfrak{k}$. This shows one has a left-invariant metric on G which is also right invariant with respect to K. Now it suffices to show G/K with the induced metric is isometric to (M, g). For any $gK \in G/K$, consider the following communicative diagram

$$\begin{tikzcd}
\mathfrak{m} = T_{eK}G/K \arrow[r, "(\mathrm{d}\pi)_e|_{\mathfrak{m}}"] \arrow[d, "\mathrm{d}L_g"'] & T_pM \arrow[d, "\mathrm{d}L_g"] \\
T_{gK}G/K \arrow[r] & T_{gp}M
\end{tikzcd}$$

Since both $(\mathrm{d}\pi)_e|_{\mathfrak{m}}$ and $(\mathrm{d}L_g)$ are linear isometries, one has $T_{gK}G/K$ is isometric to $T_{gp}M$, and thus G/K with induced metric is isometric to (M, g). □

In Theorem 32 one can see that if (M, g) is a symmetric space, then it gives a pair of Lie groups (G, K) with an involution σ on G such that

$$(G^\sigma)_0 \subseteq K \subseteq G^\sigma$$

Then there exists a left-invariant metric on G/K such that G/K with this metric is isometric to (M, g). This motivates us an effective way to construct Riemannian symmetric spaces from a pair of Lie groups with certain properties, and such a pair is called a Riemannian symmetric pair. Unless otherwise specified, we assume G is a connected Lie group with Lie algebra $\mathfrak{g}$.

Definition (Riemannian symmetric pair)

Let K be a compact subgroup of G. The pair (G, K) is called a Riemannian symmetric pair if there exists an involution $\sigma\colon G \to G$ with $(G^\sigma)_0 \subseteq K \subseteq G^\sigma$.

Example

$G = \mathrm{SO}(n+1)$ and $K = \mathrm{SO}(n)$ is a Riemannian symmetric pair given by

$$\begin{aligned} \sigma\colon \mathrm{SO}(n+1) &\to \mathrm{SO}(n+1) \\ a &\mapsto sas^{-1} \end{aligned}$$

where $s = \mathrm{diag}\{-1, 1, \dots, 1\}$. Indeed,

$$\mathrm{SO}(n{+}1)^{\sigma} = \{a \in \mathrm{SO}(n{+}1) \mid sa = as\} = \left\{\begin{pmatrix} \pm 1 & 0 \\ 0 & b \end{pmatrix} \mid b \in \mathrm{O}(n)\right\}$$

which implies $(\mathrm{SO}(n+1)^{\sigma})_0 = \mathrm{SO}(n) \subseteq \mathrm{SO}(n+1)$.

Example

$G = \mathrm{SL}(n, \mathbb{R})$ and $K = \mathrm{SO}(n)$ is a Riemannian symmetric pair given by

$$\begin{aligned} \sigma\colon \mathrm{SL}(n, \mathbb{R}) &\to \mathrm{SL}(n, \mathbb{R}) \\ g &\mapsto (g^{-1})^T \end{aligned}$$

Indeed,

$$(\mathrm{SL}(n, \mathbb{R}))^\sigma = \mathrm{SO}(n)$$

Example

Let K be a compact Lie group and $G = K \times K$. Then (G, K) is a Riemannian symmetric pair given by σ, where $\sigma\colon G \to G$ is given by $(x, y) \mapsto (y, x)$, since

$$G^\sigma = \{(a, a) \mid a \in K\} \cong K$$

Lemma

Let (G, K) be a symmetric pair given by σ. Then there is an isomorphism as Lie algebras

$$\mathfrak{k} \cong \operatorname{Lie} K$$

and an isomorphism as vector spaces

$$\mathfrak{m} \cong T_{eK} G/K$$

Proof.

$\mathfrak{k} \cong \operatorname{Lie} K$ follows from the same as proof of (3) in Theorem 32, and $\mathfrak{m} \cong T_{eK} G/K$ is an immediate consequence. □

Corollary

Let $\widetilde{\sigma}\colon G/K \to G/K$ be the automorphism of G/K induced σ. Then $(\mathrm{d}\widetilde{\sigma})_{eK} = -\operatorname{id}_{G/K}$.

Proof.

Since $K \subseteq G^\sigma$, one has $\sigma\colon K \to K$, and thus $\widetilde{\sigma}\colon G/K \to G/K$ is well-defined. By construction one has $(\mathrm{d}\widetilde{\sigma})_{eK} = (\mathrm{d}\sigma)_e|_{\mathfrak{m}}$. Then $(\mathrm{d}\widetilde{\sigma})_{eK} = -\operatorname{id}_{G/K}$ since $\mathfrak{m} = \{X \in \mathfrak{g} \mid (\mathrm{d}\sigma)_e X = -X\}$. □

Theorem

Let (G, K) be a Riemannian symmetric pair given by σ. Then there exists a left-invariant metric on G which is also right invariant on K such that the induced metric on G/K making it to be a Riemannian symmetric space.

Proof.

For convenience we use M to denote G/K. Note that a left-invariant metric on G which is also right invariant on K is equivalent to a metric on $\mathfrak{g}$ which is $\mathrm{Ad}(K)$-invariant. Since K is compact, it admits a $\mathrm{Ad}(K)$-invariant metric, and it can be extended to a $\mathrm{Ad}(K)$-invariant metric on $\mathfrak{g}$ as what we have done in the proof of (4) in Theorem 32. Furthermore, by Corollary 38 one has $(\mathrm{d}\widetilde{\sigma})_{eK} = -\mathrm{id}_M$.

Continuation.

Now it suffices to show for any $gK \in M$, $(\mathrm{d}\widetilde{\sigma})_{gK}\colon T_{gK}M \to T_{\sigma(g)K}M$ is an isometry. Note that $\widetilde{\sigma}(ghK) = \sigma(g)\sigma(h)K = \sigma(g)\widetilde{\sigma}(hK)$ holds for all $h \in G$. This shows $\widetilde{\sigma} \circ L_g = L_{\sigma(g)} \circ \widetilde{\sigma}$, where $L_g\colon M \to M$ is given by $L_g(hK) = ghK$. By taking differential one has the following communicative diagram

$$\begin{tikzcd}
T_{eK}M \arrow[r, "(\mathrm{d}\widetilde{\sigma})_{eK}"] \arrow[d, "(\mathrm{d}L_g)_{eK}"'] & T_{eK}M \arrow[d, "(\mathrm{d}L_{\sigma(g)})_{eK}"] \\
T_{gK}M \arrow[r, "(\mathrm{d}\widetilde{\sigma})_{gK}"] & T_{\sigma(g)K}M
\end{tikzcd}$$

Since $(\mathrm{d}L_g)_{eK}, (\mathrm{d}L_{\sigma(g)})_{eK}, (\mathrm{d}\widetilde{\sigma})_{eK}$ are isometries, one has $(\mathrm{d}\widetilde{\sigma})_{gK}$ is also an isometry as desired. □

Let (M, g) be a Riemannian manifold and $\mathfrak{g}$ be the Lie algebra of isometry group. Recall in Corollary 27 we have the following decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

In this section we will give more explicit descriptions for this decomposition in case of Riemannian symmetric space.

Theorem

Let (M, g) be a complete Riemannian manifold with isometry group G. For any $p \in M$, the Lie algebra of the isotropy subgroup G_p is isomorphic to

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid X_p = 0\}$$

where $\mathfrak{g}$ is the Lie algebra of G.

Proof.

Let $X \in \mathfrak{g}$ with $X_p = 0$, and $\varphi_t : M \to M$ the flow of X. It suffices to show $\varphi_t(p) = p$ for all $t \in \mathbb{R}$. If we use $\gamma_p(t)$ to denote $\varphi_t(p)$, then for any smooth function $f : M \to \mathbb{R}$ and $s \in \mathbb{R}$, we have

$$\begin{aligned}
\gamma_p'(s)f &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=s} f \circ \gamma_p(t) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f \circ \gamma_p(t+s) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f \circ \varphi_s \circ \varphi_t(p) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (f \circ \varphi_s)(\gamma_p(t)) \\
&= \gamma_p'(0)(f \circ \varphi_s) \\
&= X_p(f \circ \varphi_s) = 0
\end{aligned}$$

In order to describe $\mathfrak{m}$, we need to introduce transvection.

Definition (transvection)

Let (M, g) be a Riemannian symmetric space and γ a geodesic. The transvection along γ is defined as

$$T_t = s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}$$

where s_p is the symmetry at point p.

Lemma

Let (M, g) be a Riemannian symmetric space, γ a geodesic and T_t the transvection along γ. Then

1. *For any $a, t \in \mathbb{R}$, $s_{\gamma(a)}(\gamma(t)) = \gamma(2a - t)$.*
2. *T_t translates the geodesic γ, that is $T_t(\gamma(s)) = \gamma(t + s)$.*
3. *$(\mathrm{d}T_t)_{\gamma(s)} : T_{\gamma(s)}M \to T_{\gamma(t+s)}M$ is the parallel transport $P_{s,t+s;\gamma}$.*
4. *T_t is one-parameter subgroup of $\mathrm{Iso}(M, g)$.*

Proof.

For (1). It follows from the uniqueness of geodesics with given initial value. □

Continuation.

For (2). By (1) one has

$$\begin{aligned} T_t(\gamma(s)) &= s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}(\gamma(s)) \\ &= s_{\gamma(\frac{t}{2})}(\gamma(-s)) \\ &= \gamma(t+s) \end{aligned}$$

For (3). Let X be a parallel vector field along γ. By uniqueness of parallel vector fields with given initial data, we have $(\mathrm{d}s_{\gamma(0)})_{\gamma(s)}X_{\gamma(s)} = -X_{\gamma(-s)}$ for all s, since $(\mathrm{d}s_{\gamma(0)})_{\gamma(0)}X_{\gamma(0)} = -X_{\gamma(0)}$. Thus

$$\begin{aligned} (\mathrm{d}T_t)_{\gamma(s)}X_{\gamma(s)} &= (\mathrm{d}s_{\gamma(\frac{t}{2})})_{\gamma(-s)}(-X_{\gamma(-s)}) \\ &= X_{\gamma(t+s)} \end{aligned}$$

This shows $(\mathrm{d}T_t)_{\gamma(s)} = P_{s,t+s;\gamma}$.

Continuation.

For (4). In order to show $T_{t+s} = T_t \circ T_s$, it suffices to check they're same at some point, so do their derivatives, since isometry can be determined by these two values. Note that

$$\begin{aligned} T_{t+s}(\gamma(0)) &= \gamma(t+s) \\ &= T_t \circ T_s(\gamma(0)) \\ (\mathrm{d}T_{t+s})_{\gamma(0)} &= P_{0,t+s;\gamma} \\ &= P_{s,t+s;\gamma} \circ P_{0,s;\gamma} \\ &= (\mathrm{d}T_t)_{\gamma(s)} \circ (\mathrm{d}T_s)_{\gamma(0)} \\ &= (\mathrm{d}(T_t \circ T_s))_{\gamma(0)} \end{aligned}$$

This completes the proof. $\square$

Definition (infinitesimal transvection)

Let (M, g) be a Riemannian symmetric space. For any point $p \in M$ and any $v \in T_pM$, the infinitesimal generator X of transvections T_t along γ_v is given by

$$X_p = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} T_t(p)$$

This Killing field X is called an infinitesimal transvection.

Theorem

Let (M, g) be a Riemannian symmetric space and X an infinitesimal transvection of transvection T_t along geodesic $\gamma = \exp_p(tv)$. Then

$$X_p = v, \quad (\nabla X)_p = 0$$

Proof.

It's clear $X_p = v$. For any $w \in T_pM$, let c be a curve in M with $c(0) = p$ and $c'(0) = w$. Then

$$
\begin{aligned}
\nabla_w X &= \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}s}} X(c(s))\Big|_{s=0} \\
&= \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}s}} \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} T_t(c(s))\Big|_{t=s=0} \\
&= \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}s}} T_t(c(s))\Big|_{t=s=0} \\
&= \widehat{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} ((\mathrm{d}T_t)_p(w))\Big|_{t=0} \\
&= 0
\end{aligned}
$$

$\square$

Corollary

The space of infinitesimal transvection is exactly $\mathfrak{m}$*, and there is an isomorphism between* $\mathfrak{m} \cong T_pM$ *given by* $X \mapsto X_p$*.*

Lemma

Let (M, g) *be a Riemannian symmetric space and* $G = \mathrm{Iso}(M, g)$ *with Lie algebra* $\mathfrak{g}$*. For any* $p \in M$*, one has Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$*, where* $\mathfrak{k}$ *is Lie algebra of isotropy group* G_p *and* $\mathfrak{m} \cong T_pM$*. Then for any* $X \in \mathfrak{m}$*, one has*

$$B(X, X) \leq 0$$

where B *is the Killing form of* $\mathfrak{g}$*. Furthermore, the identity holds if and only if* $X = 0$*.*

Proof.

Since a Killing field is determined by X_p and $(\nabla X)_p$, one has elements in $\mathfrak{k}$ is determined by $(\nabla X)_p$, and note that ∇X is a skew-symmetric matrice, so

$$\mathfrak{k} \cong \{(\nabla X) \in \mathfrak{so}(T_pM) \mid X \in \mathfrak{k}\}$$

Continuation.

By using this identification, there is a natural metric on $\mathfrak{k}$ given by

$$\langle S_1, S_2\rangle = -\operatorname{tr}(S_1 S_2)$$

Then one has metric on $\mathfrak{g}$ since there is a metric on $\mathfrak{m}$ obtained from $\mathfrak{m} \cong T_pM$. For any $S \in \mathfrak{k}$, we claim with respect to this metric, $\operatorname{ad}_S\colon \mathfrak{g} \to \mathfrak{g}$ is skew-symmetric. Indeed, for $X_1, X_2 \in \mathfrak{k}$, one has

$$\begin{aligned}
\langle \operatorname{ad}_S X_1, X_2\rangle &= -\operatorname{tr}(\operatorname{ad}_S X_1 X_2) \\
&= -\operatorname{tr}((SX_1 - X_1 S)X_2) \\
&= \operatorname{tr}(X_1(SX_2 - X_2 S)) \\
&= -\langle X_1, \operatorname{ad}_S X_2\rangle
\end{aligned}$$

Continuation.

For $Y_1, Y_2 \in \mathfrak{m}$, since $S_p = 0$ and $(\nabla S)_p$ is skew-symmetric, one has

$$\begin{aligned}\langle \mathrm{ad}_S\, Y_1, Y_2\rangle &= \langle \nabla_S Y_1 - \nabla_{Y_1} S, Y_2\rangle \\ &= -\langle \nabla_{Y_1} S, Y_2\rangle \\ &= \langle \nabla_{Y_2} S, Y_1\rangle \\ &= -\langle Y_1, \nabla_S Y_2 - \nabla_{Y_2} S\rangle \\ &= -\langle Y_1, \mathrm{ad}_S\, Y_2\rangle\end{aligned}$$

Then one has

$$\begin{aligned}B(S,S) &= \mathrm{tr}(\mathrm{ad}_S \circ \mathrm{ad}_S) \\ &= \sum \langle \mathrm{ad}_S \circ \mathrm{ad}_S(e_i), e_i\rangle \\ &= -\sum \langle \mathrm{ad}_S(e_i), \mathrm{ad}_S(e_i)\rangle \leq 0\end{aligned}$$

Continuation.

Furthermore, if $B(S,S)=0$, then $\mathrm{ad}_S=0$ and for any $X\in\mathfrak{g}$, one has

$$0=\mathrm{ad}_S(X)=[S,X]=\nabla_S X-\nabla_X S=-\nabla_X S$$

since $S_p=0$. This implies $(\nabla S)_p=0$, and thus $S=0$. $\square$

Theorem

Let (M, g) be a Riemannian symmetric space and $G = \mathrm{Iso}(M, g)$. For any $p \in M$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ with $\mathfrak{m} \cong T_pM$.

1. *For any $X, Y, Z \in \mathfrak{m}$, there holds*

$$R(X,Y)Z = -[Z,[Y,X]]$$
$$\mathrm{Ric}(Y,Z) = -\frac{1}{2}B(Y,Z)$$

2. *If $\mathrm{Ric}(g) = \lambda g$, then for $X, Y \in \mathfrak{m}$, one has*

$$2\lambda R(X,Y,Y,X) = -B([X,Y],[X,Y])$$

Proof.

For (1). For any $X, Y, Z \in \mathfrak{m}$, direct computation shows

$$\begin{aligned} R(X,Y)Z &\overset{(a)}{=} R(X,Z)Y - R(Y,Z)X \\ &\overset{(b)}{=} \nabla_Z\nabla_Y X - \nabla_{\nabla_Z Y}X - \nabla_Z\nabla_X Y + \nabla_{\nabla_Z X}Y \\ &\overset{(c)}{=} -\nabla_Z[X,Y] \\ &\overset{(d)}{=} -[Z[X,Y]] \end{aligned}$$

where

- (a) holds from the first Bianchi identity.
- (b) holds from (2) of Lemma 22.
- (c) holds from $X, Y \in \mathfrak{m}$, and thus $(\nabla X)_p = (\nabla Y)_p = 0$.
- (d) holds from $\nabla_Z[X,Y] - \nabla_{[X,Y]}Z = [Z,[X,Y]]$, and $(\nabla Z)_p = 0$.

Continuation.

Hence we obtain

$$B(Y, Y) = \operatorname{tr}(\operatorname{ad}_Y \circ \operatorname{ad}_Y |_{\mathfrak{k}}) + \operatorname{tr}(\operatorname{ad}_Y \circ \operatorname{ad}_Y |_{\mathfrak{m}}) = 2\operatorname{tr}(\operatorname{ad}_Y \circ \operatorname{ad}_Y |_{\mathfrak{m}})$$

Since Ricci tensor is trace of curvature tensor, and thus

$$\operatorname{Ric}(Y, Y) = -\operatorname{tr}(\operatorname{ad}_Y \circ \operatorname{ad}_Y |_{\mathfrak{m}}) = -\frac{1}{2}B(Y, Y)$$

Then by using Polarization identity, one has
$\operatorname{Ric}(Y, Z) = -\frac{1}{2}B(Y, Z)$.
For (2). If $\operatorname{Ric}(g) = \lambda g$, then

$$\begin{aligned} 2\lambda g(R(X, Y)Y, X) &= -2\lambda g(\operatorname{ad}_Y \circ \operatorname{ad}_Y X, X) \\ &= -2\operatorname{Ric}(\operatorname{ad}_Y \circ \operatorname{ad}_Y X, X) = B(\operatorname{ad}_Y \circ \operatorname{ad}_Y X, X) \\ &= -B(\operatorname{ad}_Y X, \operatorname{ad}_Y X) = -B([X, Y], [X, Y]) \end{aligned}$$

Corollary

Let (M, g) be a Riemannian symmetric space which is an Einstein manifold with Einstein constant λ. Then

1. *If $\lambda > 0$, then (M, g) has non-negative sectional curvature.*
2. *If $\lambda < 0$, then (M, g) has non-positive sectional curvature.*
3. *If $\lambda = 0$, then (M, g) is flat.*

Proof.

By Theorem 47 one has

$$2\lambda R(X, Y, Y, X) = -B([X, Y], [X, Y]) \geq 0$$

since $[X, Y] \in [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{m}$ and B is negative definite on $\mathfrak{m}$. This shows (1) and (2). If $\lambda = 0$, one has $B([X, Y], [X, Y]) \equiv 0$ for arbitrary X, Y. Then by Lemma 46 one has $[X, Y] \equiv 0$ for arbitrary X, Y, and thus (M, g) is flat.

Definition (isotropy irreducible)

Let (M, g) be a Riemannian symmetric space with $G = \mathrm{Iso}(M, g)$ and $K = G_p$ for some $p \in M$. If the identity component K_0 acts irreducibly on T_pM, then M is called irreducible. Otherwise M is called reducible.

Lemma

Let B_1, B_2 be two symmetric bilinear forms on a vector space V such that B_1 is positive definite. If a group K acts irreducibly on V such that B_1 and B_2 are invariant under K, then $B_2 = \lambda B_1$ for some constant λ.

Theorem

The irreducible Riemannian symmetric space is Einstein, and the metric is unique determined up to a scalar.

Proof.

Since isometries preserves the metric and curvature, and Ricci tensor is also a symmetric bilinear form, by Lemma 50 there exists smooth function λ such that

$$\mathrm{Ric}(g) = \lambda g$$

Note that Riemannian curvature of Riemannian symmetric space is parallel, so is Ricci curvature. Thus we have λ is a constant. □

1. Overview
2. A quick review of basic facts we need
3. Geometric viewpoints of symmetric space
4. Algebraic viewpoints of symmetric space
5. Curvature of Riemannian symmetric space
6. Classifications and examples
   - Irreducible symmetric space
   - The classification of Riemannian symmetric space
   - Examples of Riemannian symmetric space

Theorem

Let (M, g) be a simply-connected Riemannian symmetric space. Then (M, g) is isometric to

$$(M_1, g_1) \times \cdots \times (M_k, g_k)$$

where (M_i, g_i) are irreducible Riemannian symmetric space for $i = 1, \ldots, k$.

Lemma

1. *For* $X, Y \in \mathfrak{gl}(n, \mathbb{R})$, *one has*
$$B(X, Y) = 2n\operatorname{tr}(XY) - 2\operatorname{tr} X \cdot \operatorname{tr} Y$$
2. *For* $X, Y \in \mathfrak{so}(n)$, *one has*
$$B(X, Y) = (n-2)\operatorname{tr}(X, Y)$$
3. *For* $X, Y \in \mathfrak{sl}(n, \mathbb{R})$, *one has*
$$B(X, Y) = 2n\operatorname{tr}(XY)$$
4. *For* $X, Y \in \mathfrak{so}(k, l)$, *one has*
$$B(X, Y) = (k+l-2)\operatorname{tr}(X, Y)$$

Example (hyperbolic Grassmannian)

In $\mathbb{R}^{k,l}$ with $k \geq 2, l \geq 1$, consider the following quadratic form

$$v^t I_{k,l} w = v^t \begin{pmatrix} I_k & 0 \\ 0 & -I_l \end{pmatrix} w = \sum_{i=1}^{k} v_i w_i - \sum_{j=k+1}^{k+l} v_j w_j$$

The group of linear transformation X that preserves this quadratic form is denoted by $\mathrm{O}(k,l)$, that is $X I_{k,l} X^t = I_{k,l}$, and $\mathrm{SO}(k,l)$ are those with positive determinant. The Lie algebra $\mathfrak{so}(k,l)$ of $\mathrm{SO}(k,l)$ is

$$\begin{aligned}&\mathfrak{so}(k,l)\\ &= \left\{X = \begin{pmatrix} X_1 & B \\ B^t & X_2 \end{pmatrix} \in \mathfrak{gl}(k+l,\mathbb{R}) \mid X_1 \in \mathfrak{so}(k), X_2 \in \mathfrak{so}(l), B \in M_{k\times l}\right\}\end{aligned}$$

Example (Continuation)

Now consider set consisting of those oriented k-dimensional subspaces of $\mathbb{R}^{k,l}$ on which quadratic form $I_{k,l}$ are positive definite. This gives a manifold which is called the hyperbolic Grassmannian $M = \widehat{Gr}(k, \mathbb{R}^{k,l})$. It's clear $G = \mathrm{O}(k, l)$ acting transitively on M with isotropy group $G_p = \mathrm{SO}(k) \times \mathrm{O}(l)$. Then we have the decomposition of Lie algebra $\mathfrak{g}$ of G as follows

$$\mathfrak{so}(k, l) \cong \mathfrak{so}(k) \oplus \mathfrak{so}(l) \oplus \mathfrak{m}$$

If we give the following metric on $\mathfrak{m} \cong T_pM$

$$\langle X, Y \rangle = \mathrm{tr}(XY) = \frac{1}{k + l - 2} B(X, Y)$$

where B is the Killing form of $\mathfrak{so}(k, l)$.

Example (Continuation)

Then the corresponding metric on M has the curvature formulas

$$\mathrm{Ric}(g) = -\frac{B}{2} = -\frac{k+l-2}{2} g$$

$$R(X, Y, Y, X) = \frac{B([X, Y], [X, Y])}{k+l-2} \leq 0$$

Hence the hyperbolic Grassmannian has non-positive curvatures.

Example

Let $G = \mathrm{SL}(n, \mathbb{R}), K = \mathrm{SO}(n)$ with Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$. Consider $M = \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$, one has

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

If we give the following metric on $\mathfrak{m} \cong T_p M$

$$\langle X, Y \rangle = \mathrm{tr}(XY) = \frac{1}{2n} B(X, Y)$$

where B is the Killing form of $\mathfrak{so}(k, l)$. Then the corresponding metric on M has the curvature formulas

$$\mathrm{Ric}(g) = -\frac{B}{2} = -ng$$

$$R(X, Y, Y, X) = \frac{B([X, Y], [X, Y])}{2n} \leq 0$$

Thanks!