TOPCIS IN COMPLEX ALGEBRAIC GEOMETRY

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Contents

0. Introduction

In this lecture, the object we're most interested in is the complex variety.

Definition 0.1 (complex variety)**.** A complex algebraic variety or simply a complex variety is a quasi-projective[1] variety over $\mathbb{C}$.

Definition 0.2 (non-singular)**.** A complex variety X is non-singular if the sheaf of Kähler differentials $\Omega_{X/\mathbb{C}}$ is locally free.

Given any non-singular projective complex variety X, one can show that $X \subseteq \mathbb{CP}^n$ is a submanifold by using inverse function theorem. Conversely, Chow showed that

Theorem 0.1 (Chow)**.** Any compact complex submanifold[2] of $\mathbb{CP}^n$ must be a complex variety.

Chow's theorem implies that there is a deep connection between complex manifolds and complex varieties, and thus techniques from complex differential geometry may be used to solve some questions in algebraic geometry, such as corollaries of Calabi-Yau theorem. On the other hand, motivated by Chow's theorem, it's natural to ask whether a compact complex manifold can be (holomorphically) embedded into $\mathbb{CP}^n$ or not.

Theorem 0.2 (Riemann)**.** Any compact Riemann surface can be embedded into $\mathbb{CP}^n$.

Theorem 0.3 (Kodaira)**.** A compact complex manifold with a positive holomorphic line can be embedded into $\mathbb{CP}^n$.

Remark 0.1*.* In fact, Riemann's result can be obtained from Kodaira's embedding. Given a Hermitian holomorphic line bundle (L, h), its Chern curvature $\sqrt{-1}\Theta_h$ represents the first Chern class $c_1(L)$, and $\partial\overline{\partial}$-lemma shows that any real $(1,1)$-form which represents $c_1(L)$ can be realized as the Chern curvature of some Hermitian metric h. Thus if we want to see whether a holomorphic line bundle is positive or not, it suffice to compute its first Chern class, and there always exists holomorphic line with positive first Chern class[3].

Another important conception in complex differential geometry is Kähler. The Kähler manifold lies in the intersection of Riemannian manifold, complex manifold and symplectic manifold, and has many elegant properties.

[1] $X \subseteq \mathbb{CP}^n$ is a projective variety if it's the zero-locus of (some) finite family of homogeneous polynomials, that generate a prime ideal, and it's called quasi-projective if it's an open subset of a projective variety.

[2] In fact, "submanifold" can be replaced by analytic subvariety, that is, we allow some singularities.

[3] For holomorphic line bundle L over Riemann surface, the "positivity" of first Chern class is determined by its degree, that is, holomorphic line bundle with positive degree has positive first Chern class.

Theorem 0.4 (Hodge)**.** Let (X, ω) be a compact Kähler manifold. Then there is a decomposition

$$H^n(X) \cong \bigoplus_{p+q=n} H^{p,q}(X),$$

where $H^{p,q}(X)$ is the Dolbeault cohomology of X.

Remark 0.2. The Hodge decomposition is independent of the choice of Kähler form ω, but for the proof, we need to use theory of harmonic forms and Kähler identities.

The Hodge decomposition has lots of algebraic consequences. Let X be a non-singular projective complex variety. The algebraic de Rham complex is defined by

$$\Omega^\bullet_{X/\mathbb{C}}\colon\ \mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega_{X/\mathbb{C}} \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Omega^n_{X/\mathbb{C}},$$

where $n = \dim X$, and the algebraic de Rham cohomology is defined by the hypercohomology of above complex as follows

$$H^k_{alg}(X) = \mathbb{H}^k(\Omega^\bullet_{X/\mathbb{C}}),$$

where $k \in \mathbb{Z}_{\geq 0}$. Note that there is a natural filteration on algebraic de Rham complex

$$\Omega^\bullet_{X/\mathbb{C}} = F^0\Omega^\bullet_{X/\mathbb{C}} \supseteq F^1\Omega^\bullet_{X/\mathbb{C}} \supseteq \cdots \supseteq F^n\Omega^\bullet_{X/\mathbb{C}} = \{0\},$$

where

$$F^p\Omega^\bullet_{X/\mathbb{C}}\colon 0 \to \cdots \to 0 \to \Omega^p_{X/\mathbb{C}} \to \cdots \to \Omega^n_{X/\mathbb{C}}.$$

This filteration induces a filteration on de Rham cohomology

$$F^pH^k_{alg}(X) = \mathrm{im}\left(\mathbb{H}^k(F^p\Omega^\bullet_{X/\mathbb{C}}) \to \mathbb{H}^k(\Omega^\bullet_{X/\mathbb{C}})\right).$$

The E_1-degeneration problem is

$$\mathbb{H}^\bullet(\mathrm{Gr}^\bullet_F\,\Omega^\bullet_{X/\mathbb{C}}) = \mathrm{Gr}^\bullet_F\,\mathbb{H}^\bullet(\Omega^\bullet_{X/\mathbb{C}}).$$

More precisely,

$$\mathbb{H}^d(\Omega^p_{X/\mathbb{C}}[-p]) = \frac{F^pH^d_{alg}(X)}{F^{p+1}H^d_{alg}(X)}.$$

Theorem 0.5 (E_1-degeneration)**.** The Hodge to de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_{X/\mathbb{C}}) \Longrightarrow H^{p+q}_{alg}(X)$$

degenerates at E_1-page, and

$$\dim_{\mathbb{C}} H^p(X, \Omega^q_{X/\mathbb{C}}) = \dim_{\mathbb{C}} H^q(X, \Omega^p_{X/\mathbb{C}}).$$

Proof.

$$\Omega^\bullet_{X_{an}}\colon\ \mathcal{O}_{X_{an}} \xrightarrow{\partial} \Omega^1_{X_{an}} \xrightarrow{\partial} \dots \xrightarrow{\partial} \Omega^n_{X_{an}}.$$

The holomorphic de Rham complex

$$H^*_{hol}(X_{an}) = \mathbb{H}^*(\Omega^\bullet_{X_{an}}).$$

GAGA principal implies that

$$H^d_{hol}(X_{an}) \cong H^d_{alg}(X),$$

and by $\overline{\partial}$-poincaré lemma

$$H^d(X) \cong H^d_{hol}(X_{an}).$$

As a consequence

$$\dim H^d_{alg}(X) = \sum_{p+q=n} \dim H^q(X, \Omega^p_{X/\mathbb{C}})$$

□

Remark 0.3. The inequality

$$\dim H^d_{alg}(X) \leq \sum_{p+q=n} H^q(X, \Omega^p_{X/\mathbb{C}})$$

always holds, and the equality holds if and only if there is E_1-degeneration.

Remark 0.4. Although we give a proof of E_1-degeneration by using GAGA, there is also a purely algebraic proof of it.

Remark 0.5. There are lots of consequences of E_1-degeneration, such as

(1) Hodge package.
(2) Kodaira vanishing theorem[4].

There are several important developments in Kähler geometry after Hodge and Kodaira, and in this lecture we mainly talk about work on Calabi cojecture by Shing-Tung Yau, and the connection between stable vector bundles and Hermitian-Yang-Mills metrics by Uhlenbeck-Yau.

Theorem 0.6 (Calabi-Yau)**.** Let (X, ω) be a compact Kähler manifold and χ be a real $(1,1)$-form that represents the first Chern class. Then there exists a unique $\omega_h \in [\omega]$ such that $\mathrm{Ric}(\omega_h) = \chi$

Corollary 0.1. For a compact Kähler manifold with vanishing first Chern class, there exists a unique Ricci-flat Kähler metric.

Now let's introduce some algebraic consequence of Calabi-Yau theorem.

Theorem 0.7. Let X be a non-singular projective complex variety. Suppose K_X is ample. Then

$$(-1)^n \left(c_1^n(X) - \frac{2(n+1)}{n} c_1^{n-2}(X) c_2(X) \right) \leq 0.$$

Moreover, the equality holds if and only if X is a locally symmetric variety of ball type.

Corollary 0.2. If X is a locally symmetric variety of ball type, then X^σ is again a locally symmetric variety of ball type for any $\sigma \in \mathrm{Aut}(\mathbb{C})$.

[4]Let X be a non-singular projective complex variety and L be an ample line bundle. Then $H^i(X, K_X \otimes L)$ for all $i > 0$.

Theorem 0.8. Let X be a non-singular projective complex variety. Suppose $c_1(X) = 0$. Then for any ample line bundle L on X,

$$c_2(X) \cdot L^{n-2} \geq 0.$$

Moreover, the equality holds if and only if X is an abelian variety.

Theorem 0.9. Let X be a non-singular projective complex variety. Suppose $c_1(X) = 0$. Then up to a finite étale cover,

$$X \cong X_1 \times X_2 \times X_3,$$

where X_1 is the product of abelian varieties, X_2 is the product of hyper-Kähler varieties and X_3 is the product of Calabi-Yau varieties.

References

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