

RIEMANNIAN GEOMETRY

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0. PREFACE

0.1. About this lecture.

0.2. **To readers.** This note is divided into several parts:

- (1) In the **First** part, we firstly introduce connections on a vector bundle E in different viewpoints. Holding a connection on E , one can construct connection on its dual bundle E^* , tensor product $E \otimes E^*$ and so on. When E is chosen to be tangent bundle equipped with a Riemannian metric, there is a unique connection which is compatible with metric and torsion-free, which is called Levi-Civita connection.

A section of tensor products of tangent bundle with its dual bundle is called a tensor, and tensor computation is a powerful tool of Riemannian geometry, so we collect some basic properties and operations about tensor in section 2.

However, tensor computation may be quite complicated in general. To give a neat local computation for tensor, we introduce geodesic in section 3 in order to introduce normal coordinate. By the way we also introduce Hopf-Rinow's theorem about completeness.

- (2) The **Second** part is about curvature. We introduce curvature using two different views: curvature form and curvature tensor and prove Bianchi identities in these two views. We also introduce Ricci identity for tensor, which is a crucial step in Bochner's technique. In the end we introduce some other important curvatures such as sectional curvature, Ricci curvature and scalar curvature.
- (3) The **Third** part is about Bochner's technique, which is one of the most important techniques in modern Riemannian geometry. Holding this technical, we can see how does bounded Ricci curvature appear as an obstruction to the existence of Killing fields and harmonic 1-forms. Aside these, we also introduce Hodge theory, which allows us to use harmonic 1-forms to represent elements in the first homology group. Then Bochner's technique gives a kind of vanishing theorem.
- (4) The goal of **Fourth part** is to solve the following question: "Given two points p, q , what's the length-minimizing curve connecting p, q in a Riemannian manifold?". To answer this, we consider the arc-length functional, and
 - (a) First variation formula implies geodesics are critical points of arc-length functional.
 - (b) Second variation formula implies if a geodesic contains no interior conjugate points, then it's locally minimum of arc-length functional.
 Along the way we develop the tools of index form and Jacobi fields, which are also quite important in the following parts.
- (5) The **Fifth part** generalizes geodesic and Hessian of smooth function to some extent. In this part we define what is second fundamental form,

and when a smooth map between Riemannian manifold is harmonic map. Finally, we consider its variation and Bochner's formula.

- (6) The **Sixth part** introduces how does curvature condition control the topology of the whole manifold. We mainly consider the following three cases:
- (a) A Riemannian manifold M with non-positive sectional curvature is $K(\pi_1(M), 1)$, that is M is covered by \mathbb{R}^n . A fact in topology says if a finite dimensional CW complex is a $K(G, 1)$ space for some group G , we must have G is torsion-free. Here Cartan's torsion-free theorem gives a neat proof of this fact by using deck transformations and some basic facts about Lie group action. Furthermore, $\pi_1(M)$ deserves many other interesting properties:
 - I Preissmann's theorem says if M is compact with negative sectional curvature, then any non-trivial Abelian subgroup of $\pi_1(M)$ is isomorphic to \mathbb{Z} and $\pi_1(M)$ itself is not abelian.
 - II Byers' theorem says more: if M is compact with negative sectional curvature, then any non-trivial solvable subgroup of $\pi_1(M)$ is isomorphic to \mathbb{Z} .
 - (b) A Riemannian manifold with curvature lower bounded is also quite interesting.
 - I Myers' theorem says a Riemannian manifold with positive Ricci curvature is compact, and with finite fundamental group. However, it's meaningless to consider what will happen if Ricci curvature is upper bounded, since every Riemannian n -manifold admits a complete metric with $\text{Ric} < 0$ if $n \geq 3$.
 - II Synge's theorem says a little about fundamental group of Riemannian manifold M with positive sectional curvature and even dimension: If it's orientable, then it's simply-connected, otherwise $\pi_1(M) = \mathbb{Z}_2$.
 - (c) Finally, a celebrated theorem of Hopf implies every Riemannian manifold with constant sectional curvature is covered by three basic models, which are called space forms.
- (7) The **Seventh part** is also about curvature, but it shows how to use comparison in curvatures to obtain comparison in other objects, such as length, metrics, volume and Hessian or Laplacian operators. A philosophy is that the "larger" curvature is, the "smaller" other thing is. It also gives us some rigidity theorem, an interesting result is that Cheng's theorem.
- (a) If (M, g) be a Riemannian n -manifold with $\text{Ric}(g) \geq (n-1)kg$ for some constant $k > 0$, then Myers' theorem implies $\text{diam}(M) \leq \pi/\sqrt{k}$. If $\text{diam}(M) = \pi/\sqrt{k}$, then Cheng's theorem says (M, g) is isometric to $\mathbb{S}^n(1/\sqrt{k})$ with standard metric.

0.3. Some notations and conventions.

0.3.1. *Conventions.*

1. Einstein summation is always used.
2. We always deal with the category of smooth real manifolds.

0.3.2. *Notations about smooth manifolds.*

1. M is used to denote a smooth manifold, and $x \in M$ denotes its point.
2. TM and Ω_M^k are used to denote tangent bundle and bundle of k -forms over M respectively.
3. $\Omega_M^k(E)$ is used to denote bundle of k -forms over M valued E .
4. v is used to denote vector in tangent space.
5. X is used to denote a vector field on M , and X_x denotes the value of X at point $x \in M$.
6. α is used to denote a k -form on M , and α_x denotes the value of α at point $x \in M$.
7. For a vector bundle E over M , $C^\infty(E, M)$ is used to denote its sections.
8. $\frac{\partial f}{\partial x^i}$ or $\partial_i f$ is used to denote the partial derivative of a smooth function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ with respect to x^i , where $\{x^i\}_{i=1}^n$ is a coordinate of \mathbb{R}^n .

0.3.3. *Notations about Riemannian manifolds.*

1. (M, g) is used to denote a Riemannian manifold, that is a smooth manifold M together with a Riemannian metric g .
2. $\langle -, - \rangle_g$ is used to denote a Riemannian metric g , or directly $\langle -, - \rangle$ if there is no ambiguity.

Part 1. Basic settings

1. CONNECTIONS

Connection is a very basic conception in realm of geometry of vector bundles, and there are too many ways to understand it. This part is divided into three parts:

- (1) In the first section, we will introduce one approach to connection in two different ways, the first one is often used in complex geometry, and the second is given in [Car92].
- (2) In the second section, we will put more restrictions on our connection, such as compatibility with metric and torsion-free.
- (3) In the third section, we will construct many new connections from a given connection, which play an important role in our later discuss.

1.1. Two different viewpoints to connection.

1.1.1. *First viewpoint.* When I first learn Riemannian geometry or complex geometry, I'm quite confused about why we need connection, and why we define it like this? In fact, given a vector bundle $\pi: E \rightarrow M$, connections on E are arisen to take “derivative” of a section $s: M \rightarrow E$ in a given direction. It's quite natural to ask such a question, since when we learn calculus, we already know how to take derivative of a smooth function $f: M \rightarrow \mathbb{R}^m$ to obtain a 1-form, that is a section of T^*M . In another point of view, any smooth function $f: M \rightarrow \mathbb{R}^m$ can be regarded as a section of trivial vector bundle $M \times \mathbb{R}^m$, as follows

$$x \mapsto (x, f(x))$$

we can also regard its derivative df as a section of $T^*M \otimes (M \times \mathbb{R}^m)$. So taking derivative can be seen as the following operator:

$$\nabla: C^\infty(M, M \times \mathbb{R}^m) \rightarrow C^\infty(M, T^*M \otimes (M \times \mathbb{R}^m))$$

In general, we can define a connection as follows:

Definition 1.1.1 (connection). A connection ∇ on a vector bundle E on a smooth manifold M is a linear operator

$$\nabla: C^\infty(M, E) \rightarrow C^\infty(M, T^*M \otimes E)$$

satisfying Leibniz rule $\nabla(fs) = df \otimes s + f\nabla s$, where $s \in C^\infty(M, E)$.

Remark 1.1.1 (local form). Let $\{e_\alpha\}$ be a local frame of E . For any section s of E written as $s^\alpha e_\alpha$, Leibniz rule implies

$$\nabla(s^\alpha e_\alpha) = ds^\alpha e_\alpha + s^\alpha \nabla e_\alpha$$

Thus ∇ is determined by

$$\nabla e_\alpha = \omega_\alpha^\beta e_\beta$$

where ω_α^β are 1-forms, which forms a 1-form valued matrix ω .

1.1.2. *Second viewpoint.* The following is the definition given in [Car92].

Definition 1.1.2 (connection). A connection ∇ on a vector bundle E on a smooth manifold M is a mapping

$$\begin{aligned} \nabla: C^\infty(M, TM) \times C^\infty(M, E) &\rightarrow C^\infty(M, E) \\ (X, s) &\mapsto \nabla_X s \end{aligned}$$

satisfying the following properties:

- (1) $\nabla_{fX+gY}s = f\nabla_X s + g\nabla_Y s$
- (2) $\nabla_X(s + s') = \nabla_X s + \nabla_X s'$
- (3) $\nabla_X(fs) = f\nabla_X s + X(f)s$

where $X, Y \in C^\infty(M, TM)$, $f, g \in C^\infty(M)$ and $s, s' \in C^\infty(M, E)$.

Remark 1.1.2 (local form). Let $\{\frac{\partial}{\partial x^i}\}$ and $\{e_\alpha\}$ be local frames of TX and E respectively. For vector field X and a section s of E locally written as $X = X^i \frac{\partial}{\partial x^i}$ and $e = s^\alpha e_\alpha$, one has

$$\begin{aligned} \nabla_X s &= \nabla_{X^i \frac{\partial}{\partial x^i}} s^\alpha e_\alpha \\ &= X^i \nabla_{\frac{\partial}{\partial x^i}} s^\alpha e_\alpha \\ &= X^i s^\alpha \nabla_{\frac{\partial}{\partial x^i}} e_\alpha + X^i \frac{\partial s^\alpha}{\partial x^i} e_\alpha \\ &= X^i s^\alpha \nabla_{\frac{\partial}{\partial x^i}} e_\alpha + X(s^\alpha) e_\alpha \end{aligned}$$

If we write $\nabla_{\frac{\partial}{\partial x^i}} s_\alpha = \Gamma_{i\alpha}^\beta e_\beta$, then

$$\nabla_X s = (X^i s^\alpha \Gamma_{i\alpha}^\beta + X(s^\beta)) e_\beta$$

Then $\Gamma_{i\alpha}^\beta$, which is sometimes called Christoffel symbol, completely determines connection ∇ .

Remark 1.1.3 (The equivalence between two definitions). Locally a connection in Definition 1.1.1 is a 1-form valued matrix ω , and write it as $\omega_\alpha^\beta = \Gamma_{j\alpha}^\beta dx^j$. Then

$$\begin{aligned} \nabla e_\alpha &= \omega_\alpha^\beta e_\beta \\ &= \Gamma_{i\alpha}^\beta dx^i e_\beta \end{aligned}$$

So if we want to define $\nabla_{\frac{\partial}{\partial x^i}} e_\alpha$, ∇e_α need to “eat” a vector field, and luckily dx^j can eat one, so we can define it as follows

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x^i}} e_\alpha &:= \Gamma_{j\alpha}^\beta dx^j \left(\frac{\partial}{\partial x^i} \right) e_\beta \\ &= \Gamma_{i\alpha}^\beta e_\beta \end{aligned}$$

From this we can see these two definitions are same.

Remark 1.1.4 (connection and covariant derivative). Some authors may also use terminology “covariant derivative”, here we make a clarification: Here we give two definitions of connection ∇ on a vector bundle E . Given a section s of E and a vector field X , we call $\nabla_X s$ the covariant derivative of s with respect to X . In fact, you can see connection and covariant derivative the same thing, just different terminology.

1.2. Compatibility and torsion-free.

1.2.1. *Compatibility with metric.* Now consider a vector bundle E with a metric g , which can be locally written as $g_{\alpha\beta}e^\alpha \otimes e^\beta$. So if there is a connection ∇ on E , it's natural to ask it to be compatible with our metric.

Definition 1.2.1 (compatibility). A connection ∇ on vector bundle E is compatible with metric g , if for any two section s, t of E , we have

$$dg(s, t) = g(\nabla s, t) + g(s, \nabla t)$$

Remark 1.2.1 (local form). Let $\{e_\alpha\}$ be a local frame of E . Direct computation shows

$$\begin{aligned} dg_{\alpha\beta} &= dg(e_\alpha, e_\beta) \\ &= g(\nabla e_\alpha, e_\beta) + g(e_\alpha, \nabla e_\beta) \\ &= \omega_\alpha^\gamma g_{\gamma\beta} + g_{\alpha\gamma} \omega_\beta^\gamma \end{aligned}$$

So in matrix notation we have¹

$$dg = \omega g + g \omega^t$$

In particular, we have

$$\frac{\partial}{\partial x^i} g_{\alpha\beta} = \Gamma_{i\alpha}^\gamma g_{\gamma\beta} + \Gamma_{i\beta}^\gamma g_{\alpha\gamma}$$

for all i, α, β .

1.2.2. *Torsion-free.* Now let's choose our vector bundle E to be tangent bundle of a Riemannian manifold (M, g) .

Definition 1.2.2 (torsion-free). A connection ∇ of TM is torsion-free if

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

where X, Y are vector fields.

Remark 1.2.2 (local form). If we choose $X = \frac{\partial}{\partial x^i}, Y = \frac{\partial}{\partial x^j}$, then we have

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} &= (\Gamma_{ij}^k - \Gamma_{ji}^k) \frac{\partial}{\partial x^k} \\ &= 0 \end{aligned}$$

which is equivalent to say Γ_{ij}^k is symmetric in i and j .

¹Here we need to pay more attention, although as a number $g_{\alpha\gamma}\omega_\beta^\gamma = \omega_\beta^\gamma g_{\alpha\gamma}$, we can not write this matrix notation as $dg = \omega g + g \omega^t$, since $\omega_\beta^\gamma g_{\gamma\alpha}$ is (β, α) -entry of ωg^t , but $dg_{\alpha\beta}$ and $g_{\alpha\gamma}\omega_\beta^\gamma$ are (α, β) -entries of $g \omega^t$.

1.3. Levi-Civita connection. There are infinitely many connections on tangent bundle of a Riemannian manifold, but an interesting thing is that there is only one of them which is both compatible with Riemannian metric and torsion-free.

It suffices to see a connection which is compatible with metric and torsion-free is completely determined, in other words, Γ_{ij}^k is completely determined. Note that compatibility implies

$$\begin{aligned} Xg(Y, Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Yg(Z, X) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ Zg(X, Y) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \end{aligned}$$

where X, Y and Z are vector fields. Adding first two equations, subtract the third and use torsion-free condition, we will see

$$Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) = g([X, Z], Y) + g([Y, Z], X) + g([X, Y], Z) + 2g(Z, \nabla_Y X)$$

thus

$$g(Z, \nabla_Y X) = \frac{1}{2}(Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g([X, Z], Y) - g([Y, Z], X) - g([X, Y], Z))$$

which implies $\nabla_X Y$ is uniquely determined. Above formula is also called Koszul formula.

Remark 1.3.1 (local form). Firstly, compatibility implies

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ki}^l g_{lj} + \Gamma_{kj}^l g_{il}$$

By permuting i, j, k we obtain the following two equations

$$\begin{aligned} \frac{\partial g_{jk}}{\partial x^i} &= \Gamma_{ij}^l g_{lk} + \Gamma_{ik}^l g_{jl} \\ \frac{\partial g_{ki}}{\partial x^j} &= \Gamma_{jk}^l g_{li} + \Gamma_{ji}^l g_{kl} \end{aligned}$$

By the symmetry of Γ_{ij}^l in i, j and symmetry of g_{ij} , we have

$$2\Gamma_{ij}^l g_{lk} = \frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}$$

If we use (g^{ij}) to denote the inverse matrix of (g_{ij}) , then we have

$$\Gamma_{ij}^l = \frac{1}{2}g^{kl} \left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right)$$

which implies Christoffel symbol is completely determined by Riemannian metric and its partial derivatives.

2. INDUCED CONNECTIONS

2.1. Transpose. In this section we collect some notations about transpose we will use later.

2.1.1. Transpose via duality. Let V be a vector space with dual vector space V^* . Suppose $\{e_\alpha\}$ is a basis of V with dual basis. For $B = B_\alpha^\beta e^\alpha \otimes e_\beta \in V^* \otimes V$, there is a natural way to define its transpose B^T via the following pairing

$$(Be_\alpha, e^\beta) = (e_\alpha, B^T e^\beta)$$

where $(-, -)$ is dual pairing. This shows

$$B_\alpha^\beta = (B^T)_\alpha^\beta$$

which implies

$$B^T = (B^T)_\alpha^\beta e_\beta \otimes e^\alpha = B_\alpha^\beta e_\beta \otimes e^\alpha \in V \otimes V^*$$

2.1.2. Transpose via metric. Let S, Q be two Euclidean vector spaces, that is a vector space equipped with an inner product. Suppose $\{e_\alpha\}$ is an orthonormal basis of S while $\{t_\beta\}$ is an orthonormal basis of Q . For $B = B_\alpha^\beta e^\alpha \otimes t_\beta \in S^* \otimes Q$, its transpose $B^T \in Q^* \otimes S$, given by

$$\langle Be_\alpha, t_\beta \rangle_Q = \langle e_\alpha, B^T t_\beta \rangle_S$$

where $\langle -, - \rangle$ are inner products on S, Q . This shows

$$B_\alpha^\beta = (B^T)_\beta^\alpha$$

which implies

$$B^T = (B^T)_\beta^\alpha t^\beta \otimes e_\alpha = B_\alpha^\beta t^\beta \otimes e_\alpha$$

Remark 2.1.1. The reason for Einstein summation fails is that here we choose orthonormal basis of S and Q , and there are some kronecker symbols we omit.

2.2. Induced connections on vector bundle. Given a vector bundle E over a smooth manifold M , you can construct many new vector bundles by algebraic method, such as considering its dual bundle E^* , tensor product $E \otimes E$ and so on. Now let's see if we already have a connection on E , how to construct some new connections on new vector bundles.

2.2.1. Induced connection on dual bundle. Let E be a vector bundle equipped with connection ∇^E , induced connection on dual bundle E^* is defined as follows

$$d(s, t) = (\nabla^{E^*} s, t) + (s, \nabla^E t)$$

where s, t are sections of E^* and E respectively, and $(-, -)$ denotes the dual pairing. Suppose $\{e_\alpha\}$ is a local frame of E with dual frame $\{e^\alpha\}$, then

$$0 = ((\omega^*)_\gamma^\alpha e^\gamma, e_\beta) + (e^\alpha, \omega_\beta^\gamma e_\gamma)$$

where ω and ω^* are connection 1-forms of ∇^E and ∇^{E^*} respectively. This shows

$$(\omega^*)^\alpha_\beta + \omega^\alpha_\beta = 0$$

This shows

$$\omega^* = (\omega^*)^\alpha_\beta e_\alpha \otimes e^\beta = -\omega^\alpha_\beta e_\alpha \otimes e^\beta$$

that is $\omega^* = -\omega^T$.

Remark 2.2.1 (another viewpoint of torsion-free). Let ∇ be a connection on TM , given by Christoffel symbol Γ^k_{ij} . Then induced connection on T^*M is given by

$$\nabla dx^k = -\Gamma^k_{ij} dx^i \otimes dx^j$$

Given a section s of T^*M , there is a natural 2-form obtained from taking exterior derivative ds , and note that $\bigwedge^2 T^*M$ is just the skew-symmetrization of $T^*M \otimes T^*M$, so it's natural to require the skew-symmetrization of ∇s is ds . Locally that's to say skew-symmetrization of $\nabla dx^k = 0$, that is,

$$-\Gamma^k_{ij} dx^i \wedge dx^j = 0$$

This shows torsion-free if and only if the skew-symmetrization of ∇s is ds .

2.2.2. Induced connection on tensor product. Let E, F be two vector bundles equipped with connection ∇^E, ∇^F respectively, induced connection on $E \oplus F$ is given by

$$\nabla^{E \oplus F}(s \otimes t) := \nabla^E s \otimes t + s \otimes \nabla^F t$$

where s, t are sections of E, F respectively.

2.2.3. Induced connection on wedge product. Let E be a vector bundle equipped with connection ∇ , there is an induced connection on $\bigwedge^2 E$, since it's a sub-bundle of $\bigotimes^2 E$. To be explicit

$$\begin{aligned} \nabla^{\bigwedge^2 E}(s \wedge t) &:= \nabla^{\bigotimes^2 E}(s \otimes t - t \otimes s) \\ &= \nabla s \otimes t + s \otimes \nabla t - \nabla t \otimes s - t \otimes \nabla s \\ &= \nabla s \wedge t + s \wedge \nabla t \end{aligned}$$

where s, t are sections of E .

Remark 2.2.2. In general case, there is an induced connection on $\bigotimes^k E$, given by

$$\nabla^{\bigotimes^k E}(s_1 \otimes \cdots \otimes s_k) = \sum_{i=1}^k s_1 \otimes \cdots \otimes \nabla s_i \otimes \cdots \otimes s_k$$

where s_1, \dots, s_k are sections of E . Its restriction on $\bigwedge^k E$ gives a connection on $\bigwedge^k E$, that is,

$$\nabla^{\bigwedge^k E}(s_1 \wedge \cdots \wedge s_k) = \sum_{i=1}^k s_1 \wedge \cdots \wedge \nabla s_i \wedge \cdots \wedge s_k$$

2.2.4. Induced connection on endomorphism bundle. Let E be a vector bundle equipped with connection ∇^E , there is an induced connection ∇ on $\text{End } E$, since we have $\text{End } E \cong E \otimes E^*$. Suppose $\{e_\alpha\}$ is a local frame of E with dual frame $\{e^\alpha\}$, for section s of $E \otimes E^*$ locally written as $s = s_\beta^\alpha e_\alpha \otimes e^\beta$, direct computation shows

$$\begin{aligned} \nabla^{E \otimes E^*}(s_\beta^\alpha e_\alpha \otimes e^\beta) &= ds_\beta^\alpha e_\alpha \otimes e^\beta + s_\beta^\alpha (\nabla^E e_\alpha \otimes e^\beta + e_\alpha \otimes \nabla^{E^*} e^\beta) \\ &= ds_\beta^\alpha e_\alpha \otimes e^\beta + s_\beta^\alpha \omega_\alpha^\gamma e_\gamma \otimes e^\beta - s_\beta^\alpha \omega_\gamma^\beta e_\alpha \otimes e^\gamma \\ &= (ds_\beta^\alpha + s_\beta^\alpha \omega_\alpha^\gamma - \omega_\gamma^\beta s_\beta^\alpha) e_\alpha \otimes e^\beta \end{aligned}$$

Thus in matrix notation we have

$$\nabla s = ds + s\omega - \omega s$$

Remark 2.2.3. There is another way to construct a connection on $E \otimes E^*$: For any section s of $E \otimes E^*$, we have a function $s(e^\alpha, e_\beta)$, the induced connection on $E \otimes E^*$ is defined as

$$ds(e^\alpha, e_\beta) = \nabla^{E \otimes E^*} s(e^\alpha, e_\beta) + s(\nabla^{E^*} e^\alpha, e_\beta) + s(e^\alpha, \nabla^E e_\beta)$$

Locally if we write $s = s_\beta^\alpha e_\alpha \otimes e^\beta$, then

$$\begin{aligned} d(s_\beta^\alpha) &= (\nabla s)_\beta^\alpha + s(-\omega_\gamma^\alpha e^\gamma, e_\beta) + s(e^\alpha, \omega_\beta^\gamma e_\gamma) \\ &= (\nabla s)_\beta^\alpha - s_\beta^\gamma \omega_\gamma^\alpha + \omega_\beta^\gamma s_\gamma^\alpha \end{aligned}$$

which implies connections obtained from these two ways are same. In fact, we will use this way to define induced connections of arbitrary tensor.

2.3. Induced connections on tensor. Let M be a smooth manifold.

Definition 2.3.1 (tensor). A section of $\bigotimes^s TM \otimes \bigotimes^r T^*M$ is called a (s, r) -tensor.

Example 2.3.1. A smooth function f is a $(0, 0)$ -tensor.

Example 2.3.2. A vector field X is a $(1, 0)$ -tensor.

Example 2.3.3. A 1-form ω is a $(0, 1)$ -tensor.

Example 2.3.4. The Riemannian metric g is a $(0, 2)$ -tensor.

Definition 2.3.2. For a (s, r) -tensor T , ∇T is a $(s, r+1)$ -tensor, which is defined by

$$\begin{aligned} \nabla T(dx^{j_1}, \dots, dx^{j_s}, \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_r}}) &:= \frac{\partial}{\partial x^{i_1}} T(dx^{j_1}, \dots, dx^{j_s}, \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_r}}) \\ &\quad - \sum_{l=1}^s T(dx^{j_1}, \dots, \nabla_{\frac{\partial}{\partial x^{i_l}}} dx^{j_l}, \dots, dx^{j_s}, \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_r}}) \\ &\quad - \sum_{m=1}^r T(dx^{j_1}, \dots, dx^{j_s}, \frac{\partial}{\partial x^{i_1}}, \dots, \nabla_{\frac{\partial}{\partial x^{i_m}}} \frac{\partial}{\partial x^{i_m}}, \dots, \frac{\partial}{\partial x^{i_r}}) \end{aligned}$$

Definition 2.3.3 (parallel tensor). A tensor T is called parallel if $\nabla T = 0$.

Definition 2.3.4 (covariant derivative of tensor). For a (s, r) -tensor T , the covariant derivative of T with respect to vector field X , which is a (s, r) -tensor, is defined as

$$\nabla_X T := \nabla T(\mathrm{d}x^{j_1}, \dots, \mathrm{d}x^{j_s}, X, \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_r}})$$

Remark 2.3.1 (local form). If we write a (s, r) -tensor T locally as

$$T_{i_1 \dots i_r}^{j_1 \dots j_s} \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_s}} \otimes \mathrm{d}x^{i_1} \otimes \dots \otimes \mathrm{d}x^{i_r}$$

and $(s, r+1)$ -tensor ∇T locally as

$$\nabla_i T_{i_1 \dots i_r}^{j_1 \dots j_s} \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_s}} \otimes \mathrm{d}x^i \otimes \mathrm{d}x^{i_1} \otimes \dots \otimes \mathrm{d}x^{i_r}$$

Then by definition we have

$$\nabla_i T_{i_1 \dots i_r}^{j_1 \dots j_s} = \frac{\partial T_{i_1 \dots i_r}^{j_1 \dots j_s}}{\partial x^i} + \sum_{l=1}^s \Gamma_{iq}^{jl} T_{i_1 \dots i_r}^{j_1 \dots j_{l-1} q j_{l+1} \dots j_s} - \sum_{m=1}^r \Gamma_{ii_m}^q T_{i_1 \dots i_{m-1} q i_{m+1} \dots i_r}^{j_1 \dots j_s}$$

Example 2.3.5. Consider $(0, 0)$ -tensor f , that is a smooth function. Then ∇f is a $(0, 1)$ -tensor, given by

$$\nabla f = \nabla_i f \mathrm{d}x^i$$

by definition $\nabla_i f = \frac{\partial f}{\partial x^i}$, it coincides with our usual notations.

Notation 2.3.1. For a smooth function $f: M \rightarrow \mathbb{R}$, the following notations are same in Riemannian geometry:

- (1) $\frac{\partial f}{\partial x^i}$.
- (2) $\partial_i f$.
- (3) $\nabla_i f$.

Any of them denotes the partial derivatives of f with respect to $\frac{\partial}{\partial x^i}$.

Inductively, we can define $\nabla^2 T$ to be $\nabla(\nabla T)$, which is a $(s, r+2)$ -tensor, and locally write it as

$$\nabla^2 T = \nabla_{k,i}^2 T_{i_1 \dots i_r}^{j_1 \dots j_s} \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_s}} \otimes \mathrm{d}x^k \otimes \mathrm{d}x^i \otimes \mathrm{d}x^{i_1} \otimes \dots \otimes \mathrm{d}x^{i_r}$$

Now there is a natural question: $\nabla_{k,i}^2 T$ is a (s, r) -tensor, and $\nabla_k \nabla_i T$ is also a (s, r) -tensor, do they agree? Unfortunately, it's false in general.

Example 2.3.6. For $(0, 0)$ -tensor f , by definition we have $\nabla^2 f$ is $\nabla(\nabla_i f \mathrm{d}x^i)$, which is called the Hessian of f , denoted by $\text{Hess } f$. More explicitly

$$\begin{aligned} \text{Hess } f &= \nabla(\nabla_i f \mathrm{d}x^i) \\ &= \frac{\partial \nabla_i f}{\partial x^k} \mathrm{d}x^k \otimes \mathrm{d}x^i - \nabla_i f \Gamma_{kj}^i \mathrm{d}x^k \otimes \mathrm{d}x^j \\ &= \left(\frac{\partial^2 f}{\partial x^k \partial x^i} - \Gamma_{ki}^j \frac{\partial f}{\partial x^j} \right) \mathrm{d}x^k \otimes \mathrm{d}x^i \end{aligned}$$

that is $\nabla_{k,i}^2 f = \frac{\partial^2 f}{\partial x^k \partial x^i} - \Gamma_{ki}^j \frac{\partial f}{\partial x^j}$. But $\nabla_k \nabla_i f = \frac{\partial^2 f}{\partial x^k \partial x^i}$, so $\nabla_{k,i}^2 \neq \nabla_k \nabla_i f$ in general.

Proposition 2.3.1.

$$\nabla_{k,i}^2 T_{i_1 \dots i_r}^{j_1 \dots j_s} = \nabla_k \nabla_i T_{i_1 \dots i_r}^{j_1 \dots j_s} - \Gamma_{ki}^j \nabla_j T_{i_1 \dots i_r}^{j_1 \dots j_s}$$

Proof. Direct computation shows

$$\begin{aligned} \nabla_{k,i}^2 T_{i_1 \dots i_r}^{j_1 \dots j_s} &= \nabla^2 T(dx^{j_1}, \dots, dx^{j_s}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_r}}) \\ &= \nabla_{\frac{\partial}{\partial x^k}} \nabla T(dx^{j_1}, \dots, dx^{j_s}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_r}}) \\ &= \underbrace{\frac{\partial}{\partial x^k} \nabla T(dx^{j_1}, \dots, dx^{j_s}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_r}})}_{\text{part I}} \\ &\quad - \underbrace{\nabla T(dx^{j_1}, \dots, dx^{j_s}, \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_r}})}_{\text{part II}} \\ &\quad - \underbrace{\sum_{l=1}^s \nabla T(dx^{j_1}, \dots, \nabla_{\frac{\partial}{\partial x^k}} dx^{j_l}, \dots, dx^{j_s}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_r}})}_{\text{part III}} \\ &\quad - \underbrace{\sum_{m=1}^r \nabla T(dx^{j_1}, \dots, dx^{j_s}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{i_1}}, \dots, \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^{i_m}}, \dots, \frac{\partial}{\partial x^{i_r}})}_{\text{part IV}} \end{aligned}$$

Note that

- (1) Part I+III+IV is $\nabla_k \nabla_i T_{i_1 \dots i_r}^{j_1 \dots j_s}$.
- (2) Part II is $\Gamma_{ki}^j \nabla_j T_{i_1 \dots i_r}^{j_1 \dots j_s}$.

□

Remark 2.3.2 (another viewpoint of compatibility). Note that we can regard our Riemannian metric g as a $(0, 2)$ -tensor. Recall our definition for compatibility is for any two vector fields X, Y we have

$$dg(X, Y) = g(\nabla X, Y) + g(X, \nabla Y)$$

Or more explicit for vector field Z , we have

$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

However, by definition of ∇g we have

$$\nabla_Z g(X, Y) = Zg(X, Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)$$

which shows that compatibility is equivalent to $\nabla g = 0$.

2.4. Type change of tensor. In general, for a (s, r) -tensor, we can change its type into any type of $(s - k, r + k)$ for all k such that $s - k \geq 0, r + k \geq 0$, since TM is canonically isomorphic to T^*M , which is called music isomorphism. More explicitly, for any vector field X , it gives a 1-form by

$$X^\flat: Y \mapsto g(X, Y)$$

where Y is a vector field. Locally we have

$$\begin{aligned} g\left(\frac{\partial}{\partial x^i}, Y\right) &= g\left(\frac{\partial}{\partial x^i}, dx^j(Y)\frac{\partial}{\partial x^j}\right) \\ &= dx^j(Y)g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \\ &= g_{ij}dx^j(Y) \end{aligned}$$

that is $(\frac{\partial}{\partial x^i})^\flat = g_{ij}dx^j$ of T^*M . Similarly, for any 1-form ω , it can be regarded as a section of $TM = T^{**}M$ as follows

$$\omega^\sharp: \beta \mapsto g(\omega, \beta)$$

and locally we have

$$\begin{aligned} g(dx^j, \beta) &= g(dx^j, \beta(\frac{\partial}{\partial x^i})dx^i) \\ &= \beta(\frac{\partial}{\partial x^i})g^{ij} \end{aligned}$$

that is $(dx^j)^\sharp = g^{ij}\frac{\partial}{\partial x^i}$. In a summary, we have the so-called music isomorphism locally looks like

$$\begin{aligned} \flat: TM &\rightarrow T^*M & \sharp: T^*M &\rightarrow TM \\ \frac{\partial}{\partial x^i} &\mapsto g_{ij}dx^j & dx^j &\mapsto g^{ij}\frac{\partial}{\partial x^i} \end{aligned}$$

a

Example 2.4.1. For a smooth function f , ∇f is a $(0, 1)$ -tensor, locally written as

$$\nabla f = \frac{\partial f}{\partial x^i} dx^i$$

Then we can change its type into $(1, 0)$, that is

$$\nabla f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$$

More generally, for a 1-form ω , locally looks like $\omega_i dx^i$. Then we can change it into a $(1, 0)$ -tensor, and it locally looks like

$$\omega^\sharp = g^{ij} \omega_i \frac{\partial}{\partial x^j}$$

Example 2.4.2 (Induced metric on T^*M). Recall that a Riemannian metric g is a $(0, 2)$ -tensor, locally written as

$$g = g_{ij} dx^i \otimes dx^j$$

Then we can change its type into $(2, 0)$, that is

$$g_{ij}g^{ik}g^{jl}\frac{\partial}{\partial x^k}\otimes\frac{\partial}{\partial x^l}=\delta_j^kg^{jl}\frac{\partial}{\partial x^k}\otimes\frac{\partial}{\partial x^l}=g^{kl}\frac{\partial}{\partial x^k}\otimes\frac{\partial}{\partial x^l}$$

that is a metric on T^*M .

2.5. Induced metric on tensor. If g is a Riemannian metric, then its $(2, 0)$ -type is a metric on T^*M . Now we can induce a metric on $T^*M \otimes T^*M$ as follows: Take two $(0, 2)$ -tensors T, S and write them locally as $T = T_{ij}dx^i \otimes dx^j, S = S_{kl}dx^k \otimes dx^l$. Then

$$\begin{aligned} g(T, S) &= T_{ij}S_{kl}g(dx^i \otimes dx^j, dx^k \otimes dx^l) \\ &:= T_{ij}S_{kl}g^{ik}g^{jl} \end{aligned}$$

Remark 2.5.1. In general we also have induced metric on $\bigotimes^k T^*M$, and on Ω_M^k , which will be used later in Hodge theory.

Proposition 2.5.1. If connection ∇ on vector bundle T^*M is compatible with metric g on it, then induced connection on $T^*M \otimes T^*M$ is compatible with induced metric g on it.

Proof. It suffices to check

$$\frac{\partial}{\partial x^m}g(dx^i \otimes dx^j, dx^k \otimes dx^l) = g(\nabla_{\frac{\partial}{\partial x^m}} dx^i \otimes dx^j, dx^k \otimes dx^l) + g(dx^i \otimes dx^j, \nabla_{\frac{\partial}{\partial x^m}} dx^k \otimes dx^l)$$

By compatibility of ∇ and g , we have

$$\frac{\partial g^{ij}}{\partial x^k} = -\Gamma_{kl}^i g^{lj} - \Gamma_{kl}^j g^{il}$$

Thus direct computation shows

$$\begin{aligned} \frac{\partial}{\partial x^m}g(dx^i \otimes dx^j, dx^k \otimes dx^l) &= -(\Gamma_{mn}^i g^{nk} + \Gamma_{mn}^k g^{in})g^{jl} - g^{ik}(\Gamma_{mn}^j g^{nl} + \Gamma_{mn}^l g^{jn}) \\ g(\nabla_{\frac{\partial}{\partial x^m}} dx^i \otimes dx^j, dx^k \otimes dx^l) &= -\Gamma_{mn}^i g^{nk} g^{jl} - \Gamma_{mn}^j g^{ik} g^{nl} \\ g(dx^i \otimes dx^j, \nabla_{\frac{\partial}{\partial x^m}} dx^k \otimes dx^l) &= -\Gamma_{mn}^k g^{in} g^{jl} - \Gamma_{mn}^l g^{ik} g^{jn} \end{aligned}$$

This yields the desired result. \square

2.6. Trace of tensor. Let's see a simple example: For a $(1, 1)$ -tensor T , we can define its "trace", since there is a natural isomorphism between $TM \otimes T^*M$ and $\text{End}(TM)$. Thus we can take its trace in the sense of matrix. To be explicit, if we locally write T as $T = T_j^i \frac{\partial}{\partial x^i} \otimes dx^j$, then trace of T , denoted by $\text{tr}_g T$, is defined as T_i^i .

If T is not in $(1, 1)$ -type, then we change it into $(1, 1)$ -type and then take trace:

- (1) If $T = T_{ij}dx^i \otimes dx^j$, then $T = g^{ik}T_{ij}\frac{\partial}{\partial x^k} \otimes dx^j$. Thus $\text{tr}_g T = g^{ij}T_{ij}$.
- (2) If $T = T^{ij}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$, then $T = g_{kj}T^{ij}\frac{\partial}{\partial x^i} \otimes dx^k$, that is, $\text{tr}_g T = g_{ij}T^{ij}$.

In general, if a tensor of type (r, s) with $r + s = 2n$, we can change its type into (n, n) and take trace n times to obtain a number. Later we will see we obtain Ricci curvature by taking trace of curvature, and we obtain scalar curvature by taking trace of Ricci curvature.

Remark 2.6.1 (scalar Laplacian). For a smooth function $f: M \rightarrow \mathbb{R}$, $\nabla^2 f$ is a $(0, 2)$ -form, locally looks like

$$\nabla_{i,j}^2 f dx^i \otimes dx^j$$

Then its trace looks like

$$\text{tr}_g \nabla^2 f = g^{ij} \nabla_{i,j}^2 f$$

That's called scalar Laplacian of f , denoted by Δf .

Remark 2.6.2. If g is induced metric on $(0, 2)$ -tensor, then for any $(0, 2)$ -tensor T , we have

$$\begin{aligned} g(g, T) &= g(g_{ij} dx^i \otimes dx^j, T_{kl} dx^k \otimes dx^l) \\ &= g_{ij} T_{kl} g^{ik} g^{jl} \\ &= \delta_j^k g^{jl} T_{kl} \\ &= g^{kl} T_{kl} \\ &= \text{tr}_g T \end{aligned}$$

Proposition 2.6.1 (magic formula). For a $(0, 2)$ -tensor T , we have

$$X(\text{tr}_g T) = g(g, \nabla_X T)$$

Proof. From above remark we can see $\text{tr}_g T = g(g, T)$, then ∇ is compatible with metric completes the proof. \square

Remark 2.6.3 (local form). Locally we have

$$\nabla_i (g^{jk} T_{jk}) = g^{jk} (\nabla_i T_{jk})$$

that is, g^{jk} can “pass through” taking covariant derivative, which is called “magic formula”.

3. GEODESIC AND NORMAL COORDINATE

In this section we always assume (M, g) is a Riemannian manifold equipped with Levi-Civita connection ∇ .

3.1. Geodesic.

Definition 3.1.1 (geodesic). A smooth curve $\gamma: (-\varepsilon, \varepsilon) \rightarrow M$ is called a geodesic, if for each local coordinate $\{x^i\}$, it satisfies

$$\frac{d^2\gamma^k}{dt^2} + \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} \Gamma_{ij}^k \circ \gamma = 0$$

where $\gamma^i = x^i \circ \gamma$.

Remark 3.1.1. In Section 10, we will give a definition of geodesic by using pullback connection.

Theorem 3.1.1. For any $p \in M, v \in T_p M$, there exists $\varepsilon > 0$ and a geodesic $\gamma: (-\varepsilon, \varepsilon) \rightarrow M$ such that

$$\begin{aligned}\gamma(0) &= p \\ \gamma'(0) &= v\end{aligned}$$

Moreover, any two such geodesics agree on their common domain.

Proof. Follows from standard result in ODEs' theory. \square

Remark 3.1.2. Note that standard result in ODEs' theory only guarantees the short time existence of geodesic. If we use I to denote the maximal interval such that γ can be defined on it, then in general $I \neq \mathbb{R}$.

Notation 3.1.1. For $v \in T_p M$, γ_v denotes the unique geodesic such that $\gamma(0) = p$ and $\gamma'(0) = v$.

Lemma 3.1.1. For each $p \in M, v \in T_p M$ and $c, t \in \mathbb{R}$, one has

$$\gamma_{cv}(t) = \gamma_v(ct)$$

whenever either side is defined.

Proof. It's clear by uniqueness. \square

Definition 3.1.2. For any $p \in M$, V_p is a subspace of $T_p M$ defined by

$$V_p := \{v \in T_p M \mid \gamma_v(1) \text{ is defined}\}$$

Remark 3.1.3. From Lemma 3.1.1, $v \in V_p$ if $|v| < \varepsilon$ for sufficiently small $\varepsilon > 0$.

Definition 3.1.3 (exponential map). For $p \in M$, the exponential map at point p is the map

$$\begin{aligned}\exp_p: V_p &\rightarrow M \\ v &\mapsto \gamma_v(1)\end{aligned}$$

Theorem 3.1.2. The exponential map \exp_p maps a neighborhood $0 \in T_p M$ diffeomorphically onto a neighborhood of $p \in M$.

Proof. Note that

$$(\mathrm{d} \exp_p)_0: T_0(T_p M) \rightarrow T_p M$$

and if we identify $T_0(T_p M)$ with $T_p M$ by using $0 + tv \mapsto v$, then $(\mathrm{d} \exp_p)_0$ then becomes a linear map from $T_p M$ to itself. By inverse function theorem, it suffices to check $(\mathrm{d} \exp_p)_0$ is an identity map. For all $v \in T_p M$, direct computation shows

$$\begin{aligned} (\mathrm{d} \exp_p)_0(v) &\stackrel{(1)}{=} \left. \frac{d}{dt} \right|_{t=0} \exp_p(0 + tv) \\ &= \left. \frac{d}{dt} \right|_{t=0} \gamma_{tv}(1) \\ &= \left. \frac{d}{dt} \right|_{t=0} \gamma_v(t) \\ &= \gamma'_v(0) \\ &= v \end{aligned}$$

where (1) holds from our identification $T_0(T_p M) \cong T_p M$ and definition of differential. \square

Definition 3.1.4 (geodesic ball). For $p \in M$, if $B(0, \delta)$ is a ball in $T_p M$ such that \exp_p is a diffeomorphism on $B(0, \delta)$, then $B(p, \delta) := \exp_p(B(0, \delta))$ is called a geodesic ball centered at p .

Theorem 3.1.3 (normal coordinate). For each $p \in M$, there exists a local coordinate centered at p such that $g_{ij}(0) = \delta_{ij}$, which is called normal coordinate.

Proof. Let $\{\frac{\partial}{\partial x^1}|_p, \dots, \frac{\partial}{\partial x^n}|_p\}$ be an orthonormal basis of $T_p M$ with respect to Riemannian metric g , and fix the following linear isomorphism

$$\begin{aligned} \Phi: T_p M &\rightarrow \mathbb{R}^n \\ v^i \frac{\partial}{\partial x^i} \Big|_p &\mapsto (v^1, \dots, v^n) \end{aligned}$$

By Theorem 3.1.2 there exists a neighborhood U of p (for example, the geodesic ball) which is mapped by $\Phi \circ \exp_p^{-1}$ diffeomorphically onto a neighborhood of $0 \in \mathbb{R}^n$, so $(\Phi \circ \exp_p^{-1}, U, p)$ gives a local coordinate centered at p .

Let e_i denotes $(0, \dots, \underbrace{1}_{i\text{-th}}, \dots, 0) \in \mathbb{R}^n$. Then

$$\begin{aligned}
g_{ij}(0) &= \langle d(\exp_p \circ \Phi^{-1})_0 e_i, d(\exp_p \circ \Phi^{-1})_0 e_j \rangle_p \\
&\stackrel{(1)}{=} \langle (d \exp_p)_0 \left. \frac{\partial}{\partial x^i} \right|_p, (d \exp_p)_0 \left. \frac{\partial}{\partial x^j} \right|_p \rangle_p \\
&\stackrel{(2)}{=} \langle \left. \frac{\partial}{\partial x^i} \right|_p, \left. \frac{\partial}{\partial x^j} \right|_p \rangle_p \\
&\stackrel{(3)}{=} \delta_{ij}
\end{aligned}$$

where

- (1) holds from Φ is a linear map, thus $d\Phi^{-1} = \Phi^{-1}$.
- (2) holds from Theorem 3.1.2.
- (3) holds from our choice of $\{\left. \frac{\partial}{\partial x^i} \right|_p\}$.

□

Theorem 3.1.4. In normal coordinate (x^i, U, p)

$$\Gamma_{ij}^k(0) = 0$$

Proof. For arbitrary $v = (v^1, \dots, v^n) \in \mathbb{R}^n$, $\gamma(t) = \exp_p(t\Phi^{-1}(v))$ is a geodesic with $\gamma(0) = p$ and $\gamma'(0) = \Phi^{-1}(v)$. In normal coordinate (x^i, U, p) one has $\gamma(t) = (tv^1, \dots, tv^n)$, thus geodesic equation is given by

$$\Gamma_{ij}^k(tv)v^i v^j = 0$$

By Evaluating this expression at $t = 0$ one has $\Gamma_{ij}^k(0)v^i v^j = 0$ for arbitrary index k , and take $v = \frac{1}{2}(e_i + e_j)$ to conclude $\Gamma_{ij}^k(0) = 0$ for all i, j, k . □

Corollary 3.1.1. In normal coordinate (x^i, U, p) , the Taylor expansion of g_{ij} around zero is

$$g_{ij}(x) = \delta_{ij} + O(|x|^2)$$

Proof. It follows from

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ki}^l(0)g_{lj}(0) + \Gamma_{kj}^l(0)g_{il}(0) = 0$$

□

3.2. Arts of computation. Tensor computation is one of the hallmarks of Riemannian geometry, but sometimes there is a way to avoid some complicated computations if you don't want to do it. In this section we collect some useful tools which provide a neat way to compute.

If we want to prove an identity of tensors, it suffices to check it pointwisely, since zero tensor is independent of the choice of coordinates. Thus the

normal coordinate is a good tool to use, since by Theorem 3.1.3, one has

$$\begin{aligned} x^i(p) &= 0 \\ g_{ij}(p) &= \delta_{ij} \\ \Gamma_{ij}^k(p) &= 0 \end{aligned}$$

Example 3.2.1. For a (s, r) -tensor T , from Proposition 2.3.1 one has

$$\nabla_{k,i}^2 T_{i_1 \dots i_r}^{j_1 \dots j_s} = \nabla_k \nabla_i T_{i_1 \dots i_r}^{j_1 \dots j_s}$$

under normal coordinate. In particular, Hessian of a smooth function f can be written as $\nabla_k \nabla_i f dx^k \otimes dx^i$ locally, which is relatively easier to compute.

Proposition 3.2.1. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ , and $\{\frac{\partial}{\partial x^i}\}$ is a local frame of TM with dual basis $\{dx^i\}$. Then

$$d = g^{ij} dx^i \wedge \nabla \frac{\partial}{\partial x^j}$$

Proof. Firstly note that exterior derivative is independent of the choice of coordinates, and direct computation also shows $g^{ij} dx^i \wedge \nabla \frac{\partial}{\partial x^j}$ is also independent of the choice of coordinates. Now it suffices to compute in normal coordinate, that is to show $d = dx^i \wedge \nabla \frac{\partial}{\partial x^i}$. For arbitrary k -form ω , without lose of generality we may write it as $f dx^1 \wedge \dots \wedge dx^k$. Then

$$\begin{aligned} dx^i \wedge \nabla \frac{\partial}{\partial x^i} \omega &= dx^i \wedge \nabla \frac{\partial}{\partial x^i} (f dx^1 \wedge \dots \wedge dx^k) \\ &= dx^i \wedge \frac{\partial f}{\partial x^i} dx^1 \wedge \dots \wedge dx^k \\ &= \frac{\partial f}{\partial x^i} dx^i \wedge dx^1 \wedge \dots \wedge dx^k \\ &= d\omega \end{aligned}$$

□

3.3. Hopf-Rinow's theorem. In this section we will figure out when does exponential map is defined on the whole $T_p M$.

Definition 3.3.1 (geodesically complete). A Riemannian manifold M is geodesically complete if for all $p \in M$, the exponential map \exp_p is defined on the whole $T_p M$.

At this stage it's convenient to introduce a distance function on a Riemannian manifold M which is not necessarily geodesic complete as follows: For $p, q \in M$, consider all the piecewise smooth curves joining p and q . Since M is connected, such curves always exist (cover a continuous curve joining p and q by a finite number of coordinates neighborhood and replace each piece contained in a coordinate neighborhood by a smooth one).

Definition 3.3.2 (distance). Let (M, g) be a Riemannian manifold, $p, q \in M$, the distance between p and q is defined by the infimum of the lengths of all piecewise smooth curves joining p and q , denoted by $\text{dist}(p, q)$.

Proposition 3.3.1. The topology induced by distance function on M coincides with the original topology on M .

Proof. See Proposition 2.6 in Page146 of [Car92]. \square

Theorem 3.3.1 (Hopf-Rinow). Let (M, g) be a Riemannian manifold and $p \in M$. The following statements are equivalent.

- (1) M is geodesically complete.
- (2) The closed and bounded sets of M are compact.
- (3) M is complete as a topological space.

In addition, any of statements above implies that for any $p, q \in M$, there exists a geodesic joining p and q with length $\text{dist}(p, q)$.

Proof. See Theorem 2.8 in Page146 of [Car92]. \square

Remark 3.3.1. Note that (2) is equivalent to (3) is a basic fact in general topology.

Definition 3.3.3 (complete). A Riemannian manifold is called complete, if it's geodesically complete, or it's complete as a topological space.

Corollary 3.3.1. If M is compact, then it's complete.

Part 2. Curvature

4. RIEMANNIAN CURVATURE

4.1. Curvature form. Let (M, g) be a Riemannian manifold with connection ∇ of a vector bundle E over M . Now we're going to extend connection to something called exterior derivative² defined on sections of vector bundle valued k -forms as follows

$$\begin{aligned} d^\nabla : C^\infty(M, \Omega_M^k \otimes E) &\rightarrow C^\infty(M, \Omega_M^{k+1} \otimes E) \\ \omega \otimes e &\mapsto d\omega \otimes e + (-1)^k \omega \wedge \nabla e \end{aligned}$$

Suppose $\{e_\alpha\}$ is a local frame of E , then

$$\begin{aligned} (d^\nabla)^2(s^\alpha e_\alpha) &= d^\nabla(ds^\alpha \otimes e_\alpha + s^\alpha \omega_\alpha^\beta \otimes e_\beta) \\ &= -ds^\alpha \wedge \omega_\alpha^\beta \otimes e_\beta + d(s^\alpha \omega_\alpha^\beta) \otimes e_\beta - s^\alpha \omega_\alpha^\beta \wedge \omega_\beta^\gamma \otimes e_\gamma \\ &= s^\alpha (d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta) \otimes e_\beta \\ (d^\nabla)^2(e_\alpha) &= d^\nabla(\omega_\alpha^\beta \otimes e_\beta) \\ &= d\omega_\alpha^\beta \otimes e_\beta - \omega_\alpha^\beta \wedge \nabla e_\beta \\ &= d\omega_\alpha^\beta \otimes e_\beta - \omega_\alpha^\beta \wedge \omega_\beta^\gamma \otimes e_\gamma \\ &= (d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta) \otimes e_\beta \end{aligned}$$

This shows smooth functions commutes with $(d^\nabla)^2$. In fact, it's a tensor property, which implies

- (1) $(d^\nabla)^2(e_\alpha)$ completely determines $(d^\nabla)^2$ locally, thus we can say $(d^\nabla)^2$ locally looks like $d\omega - \omega \wedge \omega$.
- (2) $(d^\nabla)^2$ is a global section of $\Omega_M^2 \otimes \text{End } E$, that is, it's compatible with change of basis. Indeed, for two local frames e, \tilde{e} such that $\tilde{e} = ge$, we will see

$$\begin{aligned} g(d^\nabla)^2 e &= (d^\nabla)^2 ge \\ &= (d^\nabla)^2 \tilde{e} \\ &= (d\tilde{\omega} - \tilde{\omega} \wedge \tilde{\omega}) \tilde{e} \\ &= (d\tilde{\omega} - \tilde{\omega} \wedge \tilde{\omega}) ge \end{aligned}$$

which implies

$$g^{-1}(d\tilde{\omega} - \tilde{\omega} \wedge \tilde{\omega})g = d\omega - \omega \wedge \omega$$

In general case, for $s \in C^\infty(M, \Omega_M^k \otimes E)$, locally written as $s = s^\alpha e_\alpha$, where s^α is a k -form, direct computation still yields

$$(d^\nabla)^2(s^\alpha e_\alpha) = s^\alpha \wedge (d^\nabla)^2(e_\alpha)$$

²Just like exterior derivative learnt in calculus.

Definition 4.1.1 (curvature form). Let E be a vector bundle over M equipped with connection ∇ , there exists a section $\Theta \in C^\infty(M, \Omega_M^2 \otimes \text{End } E)$, called curvature form, such that

$$(d^\nabla)^2 s = \Theta \wedge s$$

for all $s \in C^\infty(X, \Omega_M^k \otimes E)$.

Remark 4.1.1 (local form). Suppose $\{e_\alpha\}$ is a local frame of E and write $\Theta = \Theta_\alpha^\beta e_\beta \otimes e^\alpha$, where

$$\Theta_\alpha^\beta = \Omega_{ij\alpha}^\beta dx^i \wedge dx^j$$

Then $\Theta = d\omega - \omega \wedge \omega$ shows

$$\begin{aligned} \Theta_{ij\alpha}^\beta dx^i \wedge dx^j &= \Theta_\alpha^\beta \\ &= d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta \\ &= d(\Gamma_{i\alpha}^\beta dx^i) - (\Gamma_{i\alpha}^\gamma dx^i) \wedge (\Gamma_{j\gamma}^\beta dx^j) \\ &= (-\partial_j \Gamma_{i\alpha}^\beta - \Gamma_{i\alpha}^\gamma \Gamma_{j\gamma}^\beta) dx^i \wedge dx^j \end{aligned}$$

that is $\Theta_{ij\alpha}^\beta = -(\partial_j \Gamma_{i\alpha}^\beta + \Gamma_{i\alpha}^\gamma \Gamma_{j\gamma}^\beta)$.

Remark 4.1.2. In physicists' language, a connection is a "field", the curvature is the "strength" of the field, and choosing a local frame is called "fixing the gauge". The reason for these names comes from H. Weyl's work, rewriting Maxwell's equations.

4.2. Curvature tensor. Let M be a smooth manifold, E a vector bundle over M equipped with connection ∇ , in [Car92], the curvature of a connection ∇ is defined as follows:

$$\begin{aligned} R: C^\infty(M, TM) \times C^\infty(M, TM) \times E &\rightarrow E \\ (X, Y, s) &\mapsto R(X, Y)s \end{aligned}$$

where $R(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X, Y]} s$.

Proposition 4.2.1. The curvature R has tensorial property.

Remark 4.2.1 (local form). Suppose $\{e_\alpha\}$ is a local frame of E , and write

$$R = R_{ij\alpha}^\beta dx^i \otimes dx^j \otimes e^\alpha \otimes e_\beta$$

Note that

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} e_\alpha &= \nabla_{\frac{\partial}{\partial x^i}} (\Gamma_{j\alpha}^\beta e_\beta) \\ &= \partial_i \Gamma_{j\alpha}^\beta e_\beta + \Gamma_{j\alpha}^\beta \Gamma_{i\beta}^\gamma e_\gamma \\ &= (\partial_i \Gamma_{j\alpha}^\beta + \Gamma_{j\alpha}^\gamma \Gamma_{i\gamma}^\beta) e_\beta \end{aligned}$$

Thus

$$R_{ij\alpha}^\beta e_\beta = (\partial_i \Gamma_{j\alpha}^\beta - \partial_j \Gamma_{i\alpha}^\beta + \Gamma_{j\alpha}^\gamma \Gamma_{i\gamma}^\beta - \Gamma_{i\alpha}^\gamma \Gamma_{j\gamma}^\beta) e_\beta$$

or in other words,

$$R_\alpha^\beta = (\partial_i \Gamma_{j\alpha}^\beta - \partial_j \Gamma_{i\alpha}^\beta + \Gamma_{j\alpha}^\gamma \Gamma_{i\gamma}^\beta - \Gamma_{i\alpha}^\gamma \Gamma_{j\gamma}^\beta) dx^i \otimes dx^j$$

Recall that our curvature form Ω is a section of $\Omega_M^2 \otimes \text{End } E$, and you can regard it as a section of $T^*M \otimes T^*M \otimes \text{End } E$, that is

$$\begin{aligned}\Theta_\alpha^\beta &= (-\partial_j \Gamma_{i\alpha}^\beta - \Gamma_{i\alpha}^\gamma \Gamma_{j\gamma}^\beta) dx^i \wedge dx^j \\ &= (\partial_i \Gamma_{j\alpha}^\beta - \partial_j \Gamma_{i\alpha}^\beta + \Gamma_{j\alpha}^\gamma \Gamma_{i\gamma}^\beta - \Gamma_{i\alpha}^\gamma \Gamma_{j\gamma}^\beta) dx^i \otimes dx^j\end{aligned}$$

So if you regard curvature form as a tensor, then it's exactly curvature tensor we defined here.

If we take E to be tangent bundle, then curvature tensor R is a $(1,3)$ -tensor, locally looks like

$$R_{ijk}^r dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^r}$$

However, we always use its $(0,4)$ type, that is

$$R_{ijkl} = g_{rl} R_{ijk}^r$$

Now let's give a more explicit expression about R_{ijkl} . Direct computation shows

$$\begin{aligned}R_{ijkl} &= R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) \\ &= \left\langle \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right\rangle \\ &= \partial_i \left\langle \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right\rangle - \left\langle \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^l} \right\rangle - (\partial_j \left\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right\rangle - \left\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}, \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l} \right\rangle) \\ &= \underbrace{\partial_i \left\langle \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right\rangle - \partial_j \left\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right\rangle}_{\text{part I}} + \underbrace{\left\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}, \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l} \right\rangle - \left\langle \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^l} \right\rangle}_{\text{part II}}\end{aligned}$$

For part II, we have

$$g_{rs}(\Gamma_{ik}^r \Gamma_{jl}^s - \Gamma_{jk}^r \Gamma_{il}^s)$$

For part I, note that

$$\begin{aligned}\partial_i(\Gamma_{jk}^r g_{rl}) &= \partial_i\left(\frac{1}{2}g^{rs}(\partial_j g_{ks} + \partial_k g_{js} - \partial_s g_{jk})g_{rl}\right) \\ &= \partial_i\left(\frac{1}{2}\delta_l^s(\partial_j g_{ks} + \partial_k g_{js} - \partial_s g_{jk})\right) \\ &= \frac{1}{2}\partial_i(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk})\end{aligned}$$

Thus we have part I is

$$\partial_i(\Gamma_{jk}^r g_{rl}) - \partial_j(\Gamma_{ik}^r g_{rl}) = \frac{1}{2}(\partial_i \partial_k g_{jl} + \partial_j \partial_l g_{ik} - \partial_i \partial_l g_{jk} - \partial_j \partial_k g_{il})$$

So we have an explicit expression for R_{ijkl}

$$R_{ijkl} = \frac{1}{2}(\partial_i \partial_k g_{jl} + \partial_j \partial_l g_{ik} - \partial_i \partial_l g_{jk} - \partial_j \partial_k g_{il}) + g_{rs}(\Gamma_{ik}^r \Gamma_{jl}^s - \Gamma_{jk}^r \Gamma_{il}^s)$$

From this expression, we can see in general curvature depends on second order partial derivatives of metric. Furthermore, there are some (skew) symmetries of R_{ijkl} listed as follows:

- (1) $R_{ijkl} = -R_{jikl}$.
- (2) $R_{ijkl} = -R_{ijlk}$.
- (3) $R_{ijkl} = R_{klij}$.

4.3. Curvature of induced connections.

4.3.1. *Curvature form of induced connection on dual bundle.* Let E be a vector bundle over a smooth manifold M , equipped with connection ∇ with curvature Θ , and Θ^* is curvature form of induced connection on E^* . Suppose $\{e_\alpha\}$ is a local frame of E with dual frame $\{e^\alpha\}$, then

$$\begin{aligned} (\Theta^*)^\alpha_\beta &= d(\omega^*)^\alpha_\beta - (\omega^*)^\beta_\gamma \wedge (\omega^*)^\gamma_\alpha \\ &= -d\omega^\alpha_\beta - \omega^\beta_\gamma \wedge \omega^\gamma_\alpha \\ &= -d\omega^\alpha_\beta + \omega^\gamma_\alpha \wedge \omega^\beta_\gamma \\ &= -\Theta^\alpha_\beta \end{aligned}$$

Thus

$$\Theta^* = (\Theta^*)^\alpha_\beta e_\alpha \otimes e^\beta = -\Theta^\alpha_\beta e_\alpha \otimes e^\beta$$

that is $\Theta^* = -\Theta^T$.

4.3.2. *Curvature form of induced connection on tensor product.* Let E be a vector bundles over a smooth manifold M , equipped with connection ∇ with curvature Θ . Suppose $\{e_\alpha\}$ is a local frame of E , here we define

$$\begin{aligned} \otimes : C^\infty(M, \Omega_M^p \otimes E) \times C^\infty(M, \Omega_M^q \otimes E) &\rightarrow C^\infty(M, \Omega_M^{p+q} \otimes E \otimes E) \\ (s^\alpha e_\alpha, t^\beta e_\beta) &\mapsto s^\alpha \wedge t^\beta e_\alpha \otimes e_\beta \end{aligned}$$

Then for $s \in C^\infty(M, \Omega_M^p \otimes E), t \in C^\infty(M, \Omega_M^q \otimes E)$, direct computation shows

$$\begin{aligned} \nabla^{E \otimes E}(s \otimes t) &= \nabla^{E \otimes E}(s^\alpha \wedge t^\beta e_\alpha \otimes e_\beta) \\ &= d(s^\alpha \wedge t^\beta) e_\alpha \otimes e_\beta + (-1)^{p+q} s^\alpha \wedge t^\beta (\nabla e_\alpha \otimes e_\beta + e_\alpha \otimes \nabla e_\beta) \\ &= (ds^\alpha \wedge t^\beta + (-1)^p s^\alpha \wedge dt^\beta) e_\alpha \otimes e_\beta + (-1)^{p+q} s^\alpha \wedge t^\beta (\nabla e_\alpha \otimes e_\beta + e_\alpha \otimes \nabla e_\beta) \\ &= (ds^\alpha \wedge t^\beta e_\alpha \otimes e_\beta + (-1)^{p+q} s^\alpha \wedge t^\beta \nabla e_\alpha \otimes e_\beta) \\ &\quad + (-1)^p s^\alpha \wedge dt^\beta e_\alpha \otimes e_\beta + (-1)^{p+q} s^\alpha \wedge t^\beta e_\alpha \otimes \nabla e_\beta \\ &= (ds^\alpha \wedge t^\beta e_\alpha \otimes e_\beta + (-1)^p s^\alpha \wedge \nabla e_\alpha \otimes t) + (-1)^p \{s \otimes dt^\beta e_\beta + (-1)^q s \otimes (t^\beta \wedge \nabla e_\beta)\} \\ &= \nabla s \otimes t + (-1)^p s \otimes \nabla t \end{aligned}$$

Then curvature of $\nabla^{E \otimes E}$ can be computed as

$$\begin{aligned} (d^{\nabla^{E \otimes E}})^2(s \otimes t) &= d^{\nabla^{E \otimes E}}(\nabla s \otimes t + s \otimes \nabla t) \\ &\stackrel{(1)}{=} \Theta s \otimes t - \nabla s \otimes \nabla t + \nabla s \otimes \nabla t + s \otimes \Theta t \\ &= \Theta s \otimes t + s \otimes \Theta t \end{aligned}$$

where (1) holds from above computation. This shows

$$\Theta^{E \otimes E} = \Theta \otimes \text{id} + \text{id} \otimes \Theta$$

In general case, by induction one can show

$$\Theta^{\otimes^k E} = \sum_{i=1}^k \text{id} \otimes \cdots \otimes \underbrace{\Theta}_{i\text{-th}} \otimes \cdots \otimes \text{id}$$

4.3.3. Curvature form of induced connection on wedge product. Let E be a vector bundles over a smooth manifold M , equipped with connection ∇ with curvature Θ . Suppose $\{e_\alpha\}$ is a local frame of E , here we define

$$\begin{aligned} \wedge: C^\infty(M, \Omega_M^p \otimes E) \times C^\infty(M, \Omega_M^q \otimes E) &\rightarrow C^\infty(M, \Omega_M^{p+q} \otimes \wedge^2 E) \\ (s^\alpha e_\alpha, t^\beta e_\beta) &\mapsto s^\alpha \wedge t^\beta e_\alpha \wedge e_\beta \end{aligned}$$

By definition, for $s \in C^\infty(M, \Omega_M^p \otimes E), t \in C^\infty(M, \Omega_M^q \otimes E)$ one has

$$\begin{aligned} s \wedge t &= s^\alpha \wedge t^\beta e_\alpha \wedge e_\beta \\ &= s^\alpha \wedge t^\beta (e_\alpha \otimes e_\beta - e_\beta \otimes e_\alpha) \\ &= s \otimes t - (-1)^{pq} t \otimes s \end{aligned}$$

Thus

$$\begin{aligned} \nabla^{\wedge^2 E}(s \wedge t) &:= \nabla^{E \otimes E}(s \otimes t - (-1)^{pq} t \otimes s) \\ &= \nabla s \otimes t + (-1)^p s \otimes \nabla t - (-1)^{pq} \{\nabla t \otimes s + (-1)^q t \otimes \nabla s\} \\ &= \nabla s \otimes t - (-1)^{(p+1)q} s \otimes \nabla t + (-1)^p \{s \otimes \nabla t - (-1)^{p(q+1)} \nabla t \otimes s\} \\ &= \nabla s \wedge t + (-1)^p s \wedge \nabla t \end{aligned}$$

Remark 4.3.1. In general case, by induction one can show

$$\nabla^{\wedge^k E}(s_1 \wedge \cdots \wedge s_k) = \sum_{i=1}^k (-1)^{p_1 + \cdots + p_{i-1}} s_1 \wedge \cdots \wedge \nabla s_i \wedge \cdots \wedge s_k$$

where $s_i \in C^\infty(M, \Omega_M^{p_i} \otimes E)$. Now let's show curvature of $\nabla^{\wedge^k E}$

4.3.4. Curvature form of induced connection on determinant. Let E be a vector bundle of rank r over a smooth manifold M , equipped with connection ∇ with curvature Θ , there is an induced connection on canonical line bundle $\det E$, since $\det E = \bigwedge^r E$. In this case $\Theta^{\det E} \in C^\infty(X, \Omega_M^2)$, now we're

going to show $\Theta^{\det E} = \text{tr}_E \Theta$. By tensorial of curvature form, it suffices to check on local frame $\{e_\alpha\}$, and

$$\begin{aligned} \Theta^{\det E}(e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_r}) &= \sum_{i=1}^n e_{\alpha_1} \wedge \cdots \wedge \Theta^E e_{\alpha_i} \wedge \cdots \wedge e_{\alpha_r} \\ &= \sum_{i=1}^n e_{\alpha_1} \wedge \cdots \wedge (R_{ij\alpha_i}^\beta dx^i \otimes e_\beta) \wedge \cdots \wedge e_{\alpha_r} \\ &= R_{ij\alpha}^\alpha dx^i \otimes e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_r} \\ &= \text{tr}_E \Theta \otimes e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_r} \end{aligned}$$

This yields the desired result.

4.4. Ricci identity for tensor.

Theorem 4.4.1 (Ricci identity). Let (M, g) be a Riemannian manifold and T a (s, r) -tensor, locally written as $T_{i_1 \dots i_r}^{j_1 \dots j_s} \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_s}} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_r}$. Then

$$\nabla_{k,i}^2 T_{i_1 \dots i_r}^{j_1 \dots j_s} - \nabla_{i,k}^2 T_{i_1 \dots i_r}^{j_1 \dots j_s} = \sum_{l=1}^s R_{k i q}^{j l} T_{i_1 \dots i_r}^{j_1 \dots j_{l-1} q j_{l+1} \dots j_s} - \sum_{m=1}^r R_{k i i_m}^q T_{i_1 \dots i_{m-1} q i_{m+1} \dots i_r}^{j_1 \dots j_s}$$

Proof. Without lose of generality, we may choose normal coordinate, by Proposition 2.3.1 and Remark 2.3.1, one has

$$\begin{aligned} \nabla_{k,i}^2 T_{i_1 \dots i_r}^{j_1 \dots j_s} &= \nabla_k \nabla_i T_{i_1 \dots i_r}^{j_1 \dots j_s} \\ &= \nabla_k \left(\frac{\partial T_{i_1 \dots i_r}^{j_1 \dots j_s}}{\partial x^i} + \sum_{l=1}^s \Gamma_{i q}^{j l} T_{i_1 \dots i_r}^{j_1 \dots j_{l-1} q j_{l+1} \dots j_s} - \sum_{m=1}^r \Gamma_{i i_m}^q T_{i_1 \dots i_{m-1} q i_{m+1} \dots i_r}^{j_1 \dots j_s} \right) \\ &= \frac{\partial^2 T_{i_1 \dots i_r}^{j_1 \dots j_s}}{\partial x^k \partial x^i} + \sum_{l=1}^s \frac{\partial \Gamma_{i q}^{j l}}{\partial x^k} T_{i_1 \dots i_r}^{j_1 \dots j_{l-1} q j_{l+1} \dots j_s} - \sum_{m=1}^r \frac{\partial \Gamma_{i i_m}^q}{\partial x^k} T_{i_1 \dots i_{m-1} q i_{m+1} \dots i_r}^{j_1 \dots j_s} \end{aligned}$$

This completes the proof, since in normal coordinate one has

$$R_{ijk}^l = \frac{\partial \Gamma_{jk}^l}{\partial x^i} - \frac{\partial \Gamma_{ik}^l}{\partial x^j}$$

□

5. BIANCHI IDENTITIES

There are two famous Bianchi identities in Riemannian geometry, in [Car92] they are stated as follows

- (1) First Bianchi identity: $R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0$.
- (2) Second Bianchi identity: $\nabla_X R(Y, Z, W, R) + \nabla_Y R(Z, X, W, R) + \nabla_Z R(X, Y, W, R) = 0$.

5.1. First Bianchi. Locally we have first Bianchi identity as

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0$$

In order to compute we use (1, 3) type as follows

$$R_{ijk}^r + R_{jki}^r + R_{kij}^r = 0$$

since we have

$$R_{ijk}^r = \underbrace{\partial_i \Gamma_{jk}^r - \partial_j \Gamma_{ik}^r}_{\text{part I}} + \underbrace{\Gamma_{jk}^s \Gamma_{is}^r - \Gamma_{ik}^s \Gamma_{js}^r}_{\text{part II}}$$

- (1) For the first part, if we permute i, j, k , we have

$$\partial_i \Gamma_{jk}^r - \partial_j \Gamma_{ik}^r + \partial_j \Gamma_{ki}^r - \partial_k \Gamma_{ji}^r + \partial_k \Gamma_{ij}^r - \partial_i \Gamma_{kj}^r = 0$$

since $\Gamma_{ij}^r = \Gamma_{ji}^r$ by torsion-free.

- (2) For the second part, if we permute i, j, k , we have

$$\Gamma_{jk}^s \Gamma_{is}^r - \Gamma_{ik}^s \Gamma_{js}^r + \Gamma_{ki}^s \Gamma_{js}^r - \Gamma_{ji}^s \Gamma_{ks}^r + \Gamma_{ij}^s \Gamma_{ks}^r - \Gamma_{kj}^s \Gamma_{is}^r = 0$$

by the same reason.

Thus, we obtain first Bianchi identity, which is just a consequence of torsion-free.

Remark 5.1.1. If we consider connection on arbitrary vector bundle E , there is no first Bianchi identity, since e_α is just a section of E , not a section of TM , so $R(e_\alpha, -)$ or $R(-, e_\alpha)$ is nonsense.

5.2. Second Bianchi. In fact, we can write second Bianchi identity for arbitrary vector bundle E as follows

$$\nabla_X R(Y, Z, s, t) + \nabla_Y R(Z, X, s, t) + \nabla_Z R(X, Y, s, t) = 0$$

where $s, t \in C^\infty(M, E)$ and $X, Y, Z \in C^\infty(M, TM)$. That is to say

$$\nabla_i R_{jk\alpha\beta} + \nabla_j R_{ki\alpha\beta} + \nabla_k R_{ij\alpha\beta} = 0$$

holds for arbitrary indices.

5.2.1. *The first approach.* To prove it, for convenience we consider normal coordinate. Then

$$\nabla_{\frac{\partial}{\partial x^i}} g(\nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^m}) = g(\nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^m})$$

By permuting i, j, k we have

$$\begin{aligned} & \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l} \\ & + \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^l} \\ & + \nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^l} \\ & = R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^l} + R(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^l} + R(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^i}) \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l} \\ & = 0 \end{aligned}$$

This completes the computation of second Bianchi identity.

5.2.2. *The second approach.* From another approach, recall that our curvature form Θ is a section of $\Omega_M^2 \otimes \text{End } E$, which can be written as $\Theta_\beta^\alpha e_\alpha \otimes e^\beta$ locally. According to Section 2.2.4, $\nabla \Theta$ can be written as

$$\nabla \Theta = d\Theta + \Theta \wedge \omega - \omega \wedge \Theta$$

However, $\nabla \Theta = 0$, since

$$\begin{aligned} \nabla \Theta &= d\Theta + \Theta \wedge \omega - \omega \wedge \Theta \\ &= d(d\omega - \omega \wedge \omega) + (d\omega - \omega \wedge \omega) \wedge \omega - \omega \wedge (d\omega - \omega \wedge \omega) \\ &= d^2\omega - d\omega \wedge \omega + \omega \wedge d\omega + d\omega \wedge \omega - \omega \wedge \omega \wedge \omega - \omega \wedge d\omega + \omega \wedge \omega \wedge \omega \\ &= 0 \end{aligned}$$

If we back to local form, we have

$$d\Theta_\alpha^\beta + \Theta_\alpha^\gamma \wedge \omega_\gamma^\beta - \omega_\alpha^\gamma \wedge \Theta_\gamma^\beta = 0$$

More explicitly, if we write $\Theta_\alpha^\beta = \Omega_{ij\alpha}^\beta dx^i \wedge dx^j$, we obtain

$$(\partial_k \Theta_{ij\alpha}^\beta + \Theta_{ij\alpha}^\gamma \Gamma_{k\gamma}^\beta - \Gamma_{k\alpha}^\gamma \Theta_{ij\gamma}^\beta) dx^k \wedge dx^i \wedge dx^j = 0$$

In other words

$$\begin{aligned} & \partial_k \Theta_{ij\alpha}^\beta + \Theta_{ij\alpha}^\gamma \Gamma_{k\gamma}^\beta - \Gamma_{k\alpha}^\gamma \Theta_{ij\gamma}^\beta \\ & + \partial_i \Theta_{jk\alpha}^\beta + \Theta_{jk\alpha}^\gamma \Gamma_{i\gamma}^\beta - \Gamma_{i\alpha}^\gamma \Theta_{jk\gamma}^\beta \\ & + \partial_j \Theta_{ki\alpha}^\beta + \Theta_{ki\alpha}^\gamma \Gamma_{j\gamma}^\beta - \Gamma_{j\alpha}^\gamma \Theta_{ki\gamma}^\beta = 0 \end{aligned}$$

Note that $2\Theta_{ij\alpha}^\beta = R_{ij\alpha}^\beta$, and

$$\nabla_k R_{ij\alpha}^\beta = \partial_k R_{ij\alpha}^\beta + \Gamma_{k\gamma}^\beta R_{ij\alpha}^\gamma - \Gamma_{k\alpha}^\gamma R_{ij\gamma}^\beta$$

So $\nabla \Theta = 0$ locally looks like

$$\nabla_k R_{ij\alpha}^\beta + \nabla_i R_{jk\alpha}^\beta + \nabla_j R_{ki\alpha}^\beta = 0$$

This shows two Bianchi identities are same.

6. OTHER CURVATURES

6.1. Sectional curvature. Closely related to Riemannian curvature is sectional curvature that we're going to define, which is used to characterize a two-dimensional subspace of tangent space.

Fix $p \in M$ and let x, y are two linearly independent tangent vectors in $T_p M$. Then sectional curvature for these two vectors are defined as

$$K_p(x, y) = \frac{R(x, y, y, x)}{g(x, x)g(y, y) - g(x, y)^2}$$

In order to show it's an invariant defined for a two-dimensional subspace, we need to check if $\text{spann}_{\mathbb{R}}\{x, y\} = \text{spann}_{\mathbb{R}}\{z, w\}$, then

$$K_p(x, y) = K_p(z, w)$$

Indeed, if we write

$$\begin{cases} z = ax + by \\ w = cx + dy \end{cases}$$

Then by symmetry and skew symmetry properties of R we have

$$\begin{aligned} R(z, w, w, z) &= R(ax + by, cx + dy, cx + dy, ax + by) \\ &= R(ax, dy, dy, ax) + R(ax, dy, cx, by) + R(by, cx, dy, ax) + R(by, cx, cx, by) \\ &= a^2 d^2 R(x, y, y, x) - abcdR(x, y, y, x) - abcdR(x, y, y, x) + b^2 c^2 R(x, y, y, x) \\ &= (ad - bc)^2 R(x, y, y, x) \end{aligned}$$

And by the same computations we have

$$g(z, z)g(w, w) - g(z, w)^2 = (ad - bc)^2 \{g(x, x)g(y, y) - g(x, y)^2\}$$

Thus

$$K_p(x, y) = K_p(z, w)$$

So the following definition is well-defined:

Definition 6.1.1 (sectional curvature). The sectional curvature $K_p(\sigma)$ for two-dimensional subspace $\sigma \subseteq T_p M$ is defined as

$$K_p(\sigma) := K_p(x, y)$$

where $\{x, y\}$ is a basis of σ .

Definition 6.1.2 (isotropic). A Riemannian manifold (M, g) is called isotropic, if for each point $p \in M$, the sectional curvature $K_p(\sigma)$ is independent of σ .

Definition 6.1.3 (constant sectional curvature). A Riemannian manifold (M, g) has constant sectional curvature, if $K_p(\sigma)$ is constant for arbitrary $\sigma \subseteq T_p M, p \in M$.

Remark 6.1.1. By definition, we can see if a Riemannian manifold has constant sectional curvature. Then it must be isotropic. Conversely, if the dimension of a Riemannian manifold ≥ 3 . Then isotropic is equivalent to constant sectional curvature, see Corollary 7.1.1.

Lemma 6.1.1.

$$-6R(X, Y, Z, W) = \frac{\partial^2}{\partial s \partial t} \Big|_{s=t=0} \{R(X + sZ, Y + tW, Y + tW, X + sZ) - R(X + sW, Y + tZ, Y + tZ, X + sW)\}$$

where X, Y, Z, W are vector fields.

Proof. It suffices to compute coefficients of st of $R(X + sZ, Y + tW, Y + tW, X + sZ)$ and exchange Z with W to obtain coefficients of st of $R(X + sW, Y + tZ, Y + tZ, X + sW)$.

It's easy to see coefficients of st of $R(X + sZ, Y + tW, Y + tW, X + sZ)$ is

$$R(Z, W, Y, X) + R(Z, Y, W, X) + R(X, W, Y, Z) + R(X, Y, W, Z)$$

So coefficients of st of $R(X + sZ, Y + tW, Y + tW, X + sZ)$ is

$$R(W, Z, Y, X) + R(W, Y, Z, X) + R(X, Z, Y, W) + R(X, Y, Z, W)$$

Thus the right hand of our desired identity is

$$-4R(X, Y, Z, W) - (R(Y, Z, W, X) + R(W, Y, Z, X)) - (R(W, X, Y, Z) + R(W, Y, Z, X))$$

By first Bianchi identity we have

$$\begin{aligned} R(Y, Z, W, X) + R(W, Y, Z, X) &= R(Y, Z, W, X) + R(Z, X, W, Y) \\ &= R(X, Y, Z, W) \\ R(W, X, Y, Z) + R(W, Y, Z, X) &= R(Y, Z, W, X) + R(Z, X, W, Y) \\ &= R(X, Y, Z, W) \end{aligned}$$

This completes the proof. \square

Notation 6.1.1. For convenience, we use $R_0(X, Y, Z, W)$ to denote

$$R_0(X, Y, Z, W) = g(X, W)g(Y, Z) - g(X, Z)g(Y, W)$$

where X, Y, Z, W are vector fields. Then we can write sectional curvature as

$$K_p(\sigma) = \frac{R(x, y, y, x)}{R_0(x, y, y, x)}$$

where $\sigma \subseteq T_p M$ is spanned by x, y .

Proposition 6.1.1. A Riemannian manifold has constant sectional curvature K_p at point $p \in M$ if and only if $R = K_p R_0$, where K_p is a constant (may depend on p), R is curvature tensor.

Proof. If $R = K_p R_0$. Then for an arbitrary x, y , we have

$$K_p(x, y) = \frac{R(x, y, y, x)}{R_0(x, y, y, x)} = K_p$$

Conversely, if $K(\sigma)$ is constant at point $p \in M$, that is for arbitrary x, y we have

$$\frac{R(x, y, y, x)}{R_0(x, y, y, x)} = K_p$$

If we denote

$$\begin{aligned} F(s, t) &= R(x + sz, y + tw, y + tw, x + sz) - R(x + sw, y + tz, y + tz, x + sw) \\ F_0(s, t) &= R_0(x + sz, y + tw, y + tw, x + sz) - R_0(x + sw, y + tz, y + tz, x + sw) \end{aligned}$$

we still have $F(s, t) = K_p F_0(s, t)$. By Lemma 6.1.1, we have

$$R(x, y, z, w) = -\frac{1}{6} \frac{\partial^2}{\partial s \partial t} \Big|_{s=t=0} F(s, t)$$

and it's easy to see

$$R_0(x, y, z, w) = -\frac{1}{6} \frac{\partial^2}{\partial s \partial t} \Big|_{s=t=0} F_0(s, t)$$

This completes the proof. \square

Corollary 6.1.1. A Riemannian manifold is isotropic if and only if $R = KR_0$, where K is a smooth function.

Corollary 6.1.2. A Riemannian manifold has constant sectional curvature K if and only if $R = KR_0$, where K is a constant.

Remark 6.1.2. An important corollary is that curvature tensor of Riemannian manifold with constant sectional curvature K is quite simple, since

$$R_{ijkl} = K(g_{il}g_{jk} - g_{ik}g_{jl})$$

that is, curvature is completely determined by metric, not second order in general.

Remark 6.1.3. Suppose the dimension of Riemannian manifold (M, g) is 2, and $\{e_1, e_2\}$ is a basis of $T_p M$. Then

$$K_p = K_p(e_1, e_2) = \frac{R(e_1, e_2, e_2, e_1)}{|e_1|^2|e_2|^2 - |g(e_1, e_2)|^2}$$

is exactly Gauss curvature we learnt in theory of surface.

6.2. Ricci curvature and scalar curvature.

Definition 6.2.1 (Ricci curvature). For a Riemannian manifold (M, g) , the Ricci curvature is a $(0, 2)$ -tensor, which is defined as

$$\text{Ric}(X, Y) := \text{tr}_g(Z \mapsto R(Z, X)Y)$$

where X, Y are vector fields.

Remark 6.2.1 (local form). The trace of above endomorphism is exactly R_{ijk}^i , and it can be written as

$$g^{il} R_{ijkl}$$

In other words, Ricci curvature tensor is the contracted tensor of curvature with respect to the first and fourth index.

Definition 6.2.2 (Ricci curvature in one direction). For a point $p \in M$, and $x \in T_p M$, Ricci curvature in the direction x is defined as

$$\text{Ric}_p(x) := \text{Ric}(x, x)$$

Remark 6.2.2. For $x \in T_p M$, if we write it as $x = x^i e_i$ with respect to basis $\{e_i\}$ of $T_p M$. Then

$$\text{Ric}_p(x) = R_{jk} x^j x^k$$

Definition 6.2.3 (scalar curvature). For a Riemannian manifold (M, g) , the scalar curvature S at $p \in M$ is defined as $\text{tr}_g \text{Ric}$.

Remark 6.2.3 (local form). Locally we have

$$S = g^{jk} R_{jk}$$

Proposition 6.2.1 (contracted Bianchi identity).

$$g^{jk} \nabla_k R_{ij} = \frac{1}{2} \nabla_i S$$

where R_{ij} is Ricci curvature and S is scalar curvature.

Proof. Direct computation shows

$$\begin{aligned} g^{jk} \nabla_k R_{ij} &= g^{jk} \nabla_k g^{pq} R_{pijq} \\ &= g^{jk} g^{pq} \nabla_k R_{pijq} \\ &= g^{jk} g^{pq} (-\nabla_p R_{ikjq} - \nabla_i R_{kpjq}) \\ &= -g^{pq} \nabla_p R_{iq} + \nabla_i S \\ &= -g^{jk} \nabla_k R_{ij} + \nabla_i S \end{aligned}$$

This completes the proof. \square

Lemma 6.2.1.

$$S(p) = \sum_{i=1}^n \text{Ric}_p(e_i)$$

where $\{e_1, \dots, e_n\}$ is an orthonormal basis of $T_p M$.

Proposition 6.2.2. The scalar curvature S at $p \in M$ is given by

$$S(p) = \frac{1}{\alpha_n} \int_{\mathbb{S}^{n-1}} \text{Ric}_p(x) d\mathbb{S}^{n-1}$$

where α_n is the volume of n -dimension unit ball in \mathbb{R}^{n+1} .

Proof. Choose an orthonormal basis $\{e_1, \dots, e_n\}$ in $T_p M$ and write $x = x^i e_i$. Then

$$\begin{aligned} \text{Ric}_p(x) &= \text{Ric}_p(x^i e_i) \\ &= (x^i)^2 \text{Ric}_p(e_i) \end{aligned}$$

Since $|x| = 1$. Then the vector $\mu = (x^1, \dots, x^n)$ is a unit normal vector on \mathbb{S}^{n-1} . Denoting $V = (x^1 \text{Ric}_p(e_1), \dots, x^n \text{Ric}_p(e_n))$. Then

$$\begin{aligned}
\frac{1}{\alpha_n} \int_{\mathbb{S}^{n-1}} (x^i)^2 \text{Ric}_p(e_i) d\mathbb{S}^{n-1} &= \frac{1}{\alpha_n} \int_{\mathbb{S}^{n-1}} \langle V, \mu \rangle d\mathbb{S}^{n-1} \\
&\stackrel{(1)}{=} \frac{1}{\alpha_n} \int_{B^n} \text{div } V dB^n \\
&= \text{div } V \\
&\stackrel{(2)}{=} \sum_{i=1}^n \text{Ric}_p(e_i) \\
&= S(p)
\end{aligned}$$

where

(1) holds from Stokes theorem.

(2) holds from Lemma 6.2.1.

□

7. BASIC MODELS

7.1. Einstein manifold.

Definition 7.1.1 (Einstein manifold). A Riemannian manifold (M, g) is called Einstein manifold, if its Ricci curvature satisfies $R_{ij} = \lambda g_{ij}$ for some $\lambda \in \mathbb{R}$.

Lemma 7.1.1 (Schur's lemma). Let (M, g) be a Riemannian manifold with $\dim M \geq 3$, suppose $R_{ij} = f g_{ij}$, where f is a smooth function. Then (M, g) is an Einstein manifold.

Proof. If $R_{ij} = f g_{ij}$. Then contracted Bianchi identity shows

$$\begin{aligned} \frac{n}{2} \nabla_i f &= g^{jk} \nabla_k f g_{ij} \\ &= \nabla_i f \end{aligned}$$

for arbitrary i , which implies f is constant, since $n \geq 3$. \square

Corollary 7.1.1. For a Riemannian manifold (M, g) with $\dim M \geq 3$, it is isotropic if and only if it has constant sectional curvature.

Proof. By Remark 6.1.2, it suffices to show if M is isotropic then it has constant sectional curvature. If M is isotropic. Then there exists a smooth function K such that

$$R_{ijkl} = K(g_{il}g_{jk} - g_{ik}g_{jl})$$

Consider its Ricci curvature, that is

$$R_{jk} = (n - 1)K g_{jk}$$

Then Schur's lemma implies $(n - 1)K$ is constant, that is K is constant. \square

Proposition 7.1.1. Let (M, g) be an Einstein 3-manifold. Then (M, g) has constant sectional curvature.

Proof. For arbitrary point $p \in M$, without lose of generality we consider normal coordinate, that is $g_{ij} = \delta_{ij}$. Then

$$R_{11} = g^{ij} R_{i11j} = R_{2112} + R_{3113} = \lambda$$

Similarly, we have

$$R_{1221} + R_{3223} = \lambda$$

$$R_{1331} + R_{2332} = \lambda$$

Thus we can conclude

$$R_{1221} = R_{1331} = R_{2332} = \frac{\lambda}{2}$$

that is

$$R_{ijkl} = \frac{\lambda}{2}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})$$

This shows (M, g) has constant sectional curvature $\lambda/2$. \square

Remark 7.1.1. In fact, it's a special case of Ricci curvature controls curvature. For an n -dimensional Riemannian manifold, it's easy to see R_{jk} has $n(n+1)/2$ independent components. But for R_{ijkl} , this counting problem becomes a little complicated, it has

$$\frac{n^2(n^2-1)}{12}$$

independent components. Indeed, since R_{ijkl} is skew symmetric in ij and kl , this means that these pair of indices can take

$$m = \binom{n}{2} = \frac{n(n-1)}{2}$$

R_{ijkl} is also symmetric when you swap ij with kl , this means there would be

$$\frac{m(m+1)}{2} = \frac{n^4 - 2n^3 + 3n^2 - 2n}{8}$$

choices. However, these are not independent, since there is first Bianchi identity

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0$$

and it provides

$$\binom{n}{4} = \frac{n^4 - 6n^3 + 11n^2 - 6n}{24}$$

relations between these components, thus the number of independent components of R_{ijkl} is

$$\frac{n^4 - 2n^3 + 3n^2 - 2n}{8} - \frac{n^4 - 6n^3 + 11n^2 - 6n}{24} = \frac{n^4 - n^2}{12} = \frac{n^2(n^2-1)}{12}$$

Therefore curvature is fully determined by the Ricci curvature if and only if

$$\frac{n^2(n^2-1)}{12} \leq \frac{n(n+1)}{2}$$

or in other words, $n \leq 3$.

7.2. Sphere.

Example 7.2.1 (sphere). Let $\mathbb{S}^n(R)$ denote n -dimensional sphere with radius R . There is a natural inclusion $f: \mathbb{S}^n(R) \hookrightarrow (\mathbb{R}^{n+1}, g_{\text{can}})$, and we can use f to pullback g_{can} to obtain a metric on $\mathbb{S}^n(R)$, denoted by g . Given a local chart (x^i, U) , we can write

$$f(x^1, \dots, x^n) = (x^1, \dots, x^n, \sqrt{R^2 - \sum_{i=1}^n (x^i)^2})$$

For any $\frac{\partial}{\partial x^i}$, we have

$$\begin{aligned} df\left(\frac{\partial}{\partial x^i}\right) &= \frac{\partial f^j}{\partial x^i} \frac{\partial}{\partial x^j} \\ &= \frac{\partial}{\partial x^i} - \frac{x^i}{\sqrt{K^2 - \sum_{i=1}^n (x^i)^2}} \frac{\partial}{\partial x^{n+1}} \end{aligned}$$

Thus for any two $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}$ we have

$$\begin{aligned} g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) &= g_{\text{can}}(df\left(\frac{\partial}{\partial x^i}\right), df\left(\frac{\partial}{\partial x^j}\right)) \\ &= g_{\text{can}}\left(\frac{\partial}{\partial x^i} - \frac{x^i}{\sqrt{R^2 - \sum_{i=1}^n (x^i)^2}} \frac{\partial}{\partial x^{n+1}}, \frac{\partial}{\partial x^j} - \frac{x^j}{\sqrt{R^2 - \sum_{i=1}^n (x^i)^2}} \frac{\partial}{\partial x^{n+1}}\right) \\ &= \delta_{ij} + \frac{x^i x^j}{R^2 - \sum_{i=1}^n (x^i)^2} \end{aligned}$$

which implies

$$g_{ij} = \delta_{ij} + \frac{x^i x^j}{T^2}$$

where $T^2 = R^2 - \sum (x^i)^2$. Thus we have

$$\begin{aligned} g^{ij} &= \delta^{ij} - \frac{x^i x^j}{R^2} \\ \frac{\partial g_{ij}}{\partial x^k} &= \frac{\delta_{ki} x^j + \delta_{kj} x^i}{T^2} + \frac{2x^i x^j x^k}{T^4} \end{aligned}$$

So Christoffel symbol can be computed as

$$\begin{aligned} \Gamma_{ij}^k &= \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right) \\ &= \sum_l \frac{1}{2} \left(\delta^{kl} - \frac{x^k x^l}{R^2} \right) \left(\frac{\delta_{ij} x^l + \delta_{il} x^j}{T^2} + \frac{2x^i x^j x^l}{T^4} + \frac{\delta_{ji} x^l + \delta_{jl} x^i}{T^2} + \frac{2x^i x^j x^l}{T^4} - \frac{\delta_{li} x^j + \delta_{kj} x^i}{T^2} - \frac{2x^i x^j x^l}{T^4} \right) \\ &= \sum_l \frac{x^l}{T^2} \left(\delta_{ij} + \frac{x^i x^j}{T^2} \right) \left(\delta^{kl} - \frac{x^k x^l}{R^2} \right) \\ &= \frac{g_{ij}}{T^2} x^k \left(1 - \frac{\sum_{l=1}^n (x^l)^2}{R^2} \right) \\ &= \frac{x^k}{R^2} g_{ij} \end{aligned}$$

Direct computation shows curvature can be written as³

$$\begin{aligned} R_{ijkl} &= \frac{1}{2} (\partial_i \partial_k g_{jl} + \partial_j \partial_l g_{ik} - \partial_i \partial_l g_{jk} - \partial_j \partial_k g_{il}) + g_{rs} (\Gamma_{ik}^r \Gamma_{jl}^s - \Gamma_{jk}^r \Gamma_{il}^s) \\ &= \frac{1}{R^2} (g_{il} g_{jk} - g_{ik} g_{jl}) \end{aligned}$$

³Here I omit a huge computation, and I suggest you compute it by yourself. Maybe first it's quite tough for you to do this first time, but you should try.

So Ricci curvature and scalar curvature can be computed as follows

$$\begin{aligned}
R_{jk} &= g^{il} R_{ijkl} \\
&= \frac{1}{R^2} g^{il} (g_{il} g_{jk} - g_{ik} g_{jl}) \\
&= \frac{1}{R^2} (n g_{jk} - \delta_k^l g_{jl}) \\
&= \frac{n-1}{R^2} g_{jk} \\
S &= g^{jk} R_{jk} \\
&= \frac{n(n-1)}{R^2}
\end{aligned}$$

7.3. Hyperbolic space.

Example 7.3.1 (hyperbolic upper plane). Let $\mathbb{H}^n(R) = \{(x^1, \dots, x^{n-1}, y) \in \mathbb{R}^n \mid y > 0\}$ with metric

$$g = R^2 \frac{\delta_{ij} dx^i \otimes dx^j + dy \otimes dy}{y^2}$$

Example 7.3.2 (Poincaré disk). Let $\mathbb{D}^n(R) = \{x \in \mathbb{R}^n \mid |x| < R\}$ with metric

$$g = 4R^4 \frac{\delta_{ij} dx^i \otimes dx^j}{(R^2 - |x|^2)^2}$$

7.4. Lie group.

7.4.1. Invariant metrics.

Definition 7.4.1 (left-invariant metric). A Riemannian metric $\langle -, - \rangle$ on a Lie group G is called left-invariant if

$$\langle (L_g)_* X, (L_g)_* Y \rangle = \langle X, Y \rangle$$

holds for arbitrary $g \in G$ and vector fields X, Y .

Definition 7.4.2 (right-invariant metric). A Riemannian metric $\langle -, - \rangle$ on a Lie group G is called right-invariant if

$$\langle (R_g)_* X, (R_g)_* Y \rangle = \langle X, Y \rangle$$

holds for arbitrary $g \in G$ and vector fields X, Y .

Definition 7.4.3 (bi-invariant metric). A Riemannian metric $\langle -, - \rangle$ on a Lie group G is called bi-invariant if it's both left-invariant and right-invariant.

Proposition 7.4.1. Any compact Lie group G admits a bi-invariant metric.

Proposition 7.4.2. There is a bijective correspondence between left-invariant metrics on a Lie group G and inner products on the Lie algebra \mathfrak{g} of G .

Proof. Given an inner product $\langle -, - \rangle_e$ on Lie algebra \mathfrak{g} , we have a Riemannian metric on G defined as follows

$$\langle X_g, Y_g \rangle := \langle (L_{g^{-1}})_* X_g, (L_{g^{-1}})_* Y_g \rangle_e$$

where X, Y are two vector fields on G . It's left-invariant, since

$$\begin{aligned} \langle (L_h)_* X_g, (L_h)_* Y_g \rangle &= \langle (L_{hg^{-1}})_* (L_h)_* X_g, (L_{hg^{-1}})_* (L_h)_* Y_g \rangle_e \\ &= \langle (L_{g^{-1}})_* X_g, (L_{g^{-1}})_* Y_g \rangle_e \\ &= \langle X_g, Y_g \rangle \end{aligned}$$

Conversely, if we have a left-invariant metric $\langle -, - \rangle$ on G , then it's clear we have an inner product on \mathfrak{g} , by just considering its value at identity. These two constructions give the desired one to one correspondence. \square

Proposition 7.4.3. There is a bijective correspondence between bi-invariant metrics on a Lie group G , and Ad-invariant inner products on the Lie algebra \mathfrak{g} of G .

Proof. Given a Ad-invariant inner product $\langle -, - \rangle_e$ on the Lie algebra \mathfrak{g} , by Proposition 7.4.2, there is a left-invariant metric $\langle -, - \rangle$ on G , it suffices to check it's also right-invariant:

$$\begin{aligned} \langle (R_h)_* X_g, (R_h)_* Y_g \rangle &= \langle (L_{hg^{-1}})_* (R_h)_* X_g, (L_{hg^{-1}})_* (R_h)_* Y_g \rangle_e \\ &= \langle \text{Ad}(h^{-1})(L_{g^{-1}})_* X_g, \text{Ad}(h^{-1})(L_{g^{-1}})_* Y_g \rangle_e \\ &= \langle (L_{g^{-1}})_* X_g, (L_{g^{-1}})_* Y_g \rangle_e \\ &= \langle X_g, Y_g \rangle \end{aligned}$$

Conversely, if we start with a bi-invariant metric, then it's restriction to the Lie algebra is an Ad-invariant, since $\text{Ad}(g)$ is exactly the differential of $L_g \circ R_{g^{-1}}$. \square

Lemma 7.4.1. Let G be a Lie group equipped with left-invariant metric $\langle -, - \rangle$, and ∇ the Levi-Civita connection with respect to it. Then for all left-invariant vector fields X, Y, Z ,

$$\langle X, \nabla_Y Y \rangle = \langle Y, [X, Y] \rangle$$

Proof. By Koszul formula one has

$$\langle X, \nabla_Y Z \rangle = \frac{1}{2} (Y \langle Z, X \rangle + Z \langle X, Y \rangle - X \langle Y, Z \rangle - \langle [Y, X], Z \rangle - \langle [Z, X], Y \rangle - \langle [Y, Z], X \rangle)$$

But $Y \langle Z, X \rangle = Z \langle X, Y \rangle = X \langle Y, Z \rangle = 0$ since both metric and X, Y, Z are left-invariant. Thus

$$\langle X, \nabla_Y Z \rangle = \frac{1}{2} \{ \langle Z, [X, Y] \rangle + \langle Y, [X, Z] \rangle + \langle X, [Z, Y] \rangle \}$$

Now set $Y = Z$ to conclude. \square

Proposition 7.4.4. Let G be a Lie group equipped with bi-invariant metric $\langle -, - \rangle$, and ∇ the Levi-Civita connection with respect to it. Then for all left-invariant vector fields X, Y, Z ,

$$\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle$$

Proof. Let ϕ_t be the flow of Y . Then

$$[X, Y] = \lim_{t \rightarrow 0} \frac{(\phi_t)_* X - X}{t}$$

On the other hand, since Y is left-invariant, one has L_g commutes with ϕ_t for all $g \in G$, and thus

$$\phi_t(g) = \phi_t(L_g(e)) = L_g \phi_t(e) = g \phi_t(e) = R_{\phi_t(e)}(g)$$

As a consequence, one has $(\phi_t)_* = (R_{\phi_t(e)})_*$. Then

$$[X, Y] = \lim_{t \rightarrow 0} \frac{(R_{\phi_t(e)})_* X - X}{t}$$

Note that the metric is bi-invariant, one has

$$\begin{aligned} \langle X, Z \rangle &= \langle (R_{\phi_t(e)})_*(L_{\phi_t^{-1}(e)})_* X, (R_{\phi_t(e)})_*(L_{\phi_t^{-1}(e)})_* Z \rangle \\ &= \langle (R_{\phi_t(e)})_* X, (R_{\phi_t(e)})_* Z \rangle \end{aligned}$$

By differentiating the expression above with respect to t and setting $t = 0$ we conclude

$$0 = \langle [X, Y], Z \rangle + \langle X, [Z, Y] \rangle$$

□

7.4.2. Levi-Civita connection of bi-invariant metric.

Theorem 7.4.1. Let G be a Lie group equipped with bi-invariant metric $\langle -, - \rangle$, and ∇ the Levi-Civita connection with respect to it. Then for every left-invariant vector field X on G . Then $\nabla_X X = 0$.

Proof. From Lemma 7.4.1, we have

$$\langle Y, \nabla_X X \rangle = \langle X, [Y, X] \rangle$$

From Proposition 7.4.4, we have

$$\langle X, [Y, X] \rangle = \langle [X, Y], X \rangle = -\langle X, [Y, X] \rangle$$

that is $\langle Y, \nabla_X X \rangle = 0$ for arbitrary vector field Y , which implies $\nabla_X X = 0$. □

Corollary 7.4.1. The assumptions are as above. If X, Y are left-invariant vector fields, then $\nabla_X Y = \frac{1}{2}[X, Y]$.

Proof. Note that

$$\begin{aligned}
 0 &= \nabla_{X+Y}(X+Y) \\
 &= \nabla_X Y + \nabla_Y X + \nabla_X X + \nabla_Y Y \\
 &= \nabla_X Y + \nabla_Y X \\
 &= 2\nabla_X Y - [X, Y]
 \end{aligned}$$

Division by two finally yields

$$\nabla_X Y = \frac{1}{2}[X, Y]$$

□

Corollary 7.4.2. The assumptions are as above. If X, Y, Z are left-invariant vector fields, then $R(X, Y)Z = -\frac{1}{4}[[X, Y], Z]$.

Proof. Directly from $\nabla_X Y = \frac{1}{2}[X, Y]$ and Jacobi's identity. □

Corollary 7.4.3. The assumptions are as above. If X, Y are left-invariant vector fields which are orthogonal, and σ is the plane generated by X and Y , then

$$K(\sigma) = \frac{1}{4}\|[X, Y]\|^2$$

Proof.

$$K(\sigma) = -\frac{1}{4}\langle [[X, Y], Y], X \rangle = -\frac{1}{4}\langle [X, Y], [Y, X] \rangle = \frac{1}{4}\|[X, Y]\|^2$$

□

Remark 7.4.1. Therefore, sectional curvature of a Lie group with bi-invariant metric is non-negative. Furthermore, if the center of Lie algebra \mathfrak{g} is trivial, then the sectional curvature is positive.

Part 3. Bochner's technique

8. HODGE THEORY ON RIEMANNIAN MANIFOLD

For convenience, in this section we assume (M, g) is a compact oriented Riemannian n -manifold, since we need to consider integral.

8.1. Inner product on Ω_M^k . Before we talk about Hodge theory on (M, g) , let's recall some basic facts about differential k -forms. For a k -form φ , locally it can be written as

$$\varphi = \sum_{1 \leq i_1 < \dots < i_k \leq n} \varphi_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

where $\varphi_{i_1 \dots i_k} := \varphi(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}})$, is skew-symmetric. If we don't want indices are arranged in order, we can write

$$\varphi = \frac{1}{k!} \varphi_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

where the summation runs over arbitrary different k indices. It's clear these two expressions are same, since both $\varphi_{i_1 \dots i_k}$ and $dx^{i_1} \wedge \dots \wedge dx^{i_k}$ are skew-symmetric.

Notation 8.1.1. $\varphi_I dx^I$ denotes $\varphi_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$.

Recall that we already have an induced metric g on $\bigotimes^k T^*M$, and Ω_M^k is a subbundle of $\bigotimes^k T^*M$. Thus we can define a metric on Ω_M^k as follows

Definition 8.1.1. Let φ, ψ be two k -forms, define

$$\langle \varphi, \psi \rangle := \frac{1}{k!} g(\varphi, \psi)$$

where g is induced metric on $\bigotimes^k T^*M$.

Lemma 8.1.1. For $\varphi = \varphi_I dx^I, \psi = \psi_J dx^J$. Then

$$\langle \varphi, \psi \rangle = \varphi_I \psi_J g^{IJ}$$

where

$$g^{IJ} = \frac{1}{k!} g(dx^I, dx^J) = \det \begin{pmatrix} g^{i_1 j_1} & \dots & g^{i_1 j_k} \\ \dots & \dots & \dots \\ g^{i_k j_1} & \dots & g^{i_k j_k} \end{pmatrix}$$

Proof. It suffices to show

$$g(dx^I, dx^J) = k! \det \begin{pmatrix} g^{i_1 j_1} & \dots & g^{i_1 j_k} \\ \dots & \dots & \dots \\ g^{i_k j_1} & \dots & g^{i_k j_k} \end{pmatrix}$$

By definition one has

$$dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_k} = \sum_{\sigma \in S_k} (-1)^{|\sigma|} e_{i_{\sigma(1)}} \otimes \dots \otimes e_{i_{\sigma(k)}}$$

Then

$$\begin{aligned}
g(\mathrm{d}x^I, \mathrm{d}x^J) &= \sum_{\sigma, \tau} (-1)^{|\sigma|} (-1)^{|\tau|} g(\mathrm{d}x^{i_{\sigma(1)}} \otimes \cdots \otimes \mathrm{d}x^{i_{\sigma(k)}}, \mathrm{d}x^{j_{\tau(1)}} \otimes \cdots \otimes \mathrm{d}x^{j_{\tau(k)}}) \\
&= \sum_{\sigma, \tau} (-1)^{|\sigma|} (-1)^{|\tau|} g^{i_{\sigma(1)} j_{\tau(1)}} \cdots g^{i_{\sigma(k)} j_{\tau(k)}} \\
&= \sum_{\sigma, \tau} (-1)^{|\sigma\tau^{-1}|} g^{i_{\sigma\tau^{-1}(1)} j_1} \cdots g^{i_{\sigma\tau^{-1}(k)} j_k} \\
&= \sum_{\sigma} \sum_{\rho} (-1)^{|\rho|} g^{i_{\rho(1)} j_1} \cdots g^{i_{\rho(k)} j_k} \\
&= \sum_{\sigma} \det(g^{i_p j_q}) \\
&= k! \det(g^{i_p j_q})
\end{aligned}$$

□

Remark 8.1.1. Note that here we don't assume φ_I, ψ_I is skew-symmetric, they can be arbitrary functions.

Corollary 8.1.1. For two k -forms φ, ψ , locally written as

$$\begin{aligned}
\varphi &= \sum_{1 \leq i_1 < \cdots < i_k \leq n} \varphi_{i_1 \dots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \\
\psi &= \sum_{1 \leq j_1 < \cdots < j_k \leq n} \psi_{j_1 \dots j_k} \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_k}
\end{aligned}$$

with φ_I, ψ_J is skew-symmetric. Then

$$\langle \varphi, \psi \rangle = \sum_{\substack{1 \leq i_1 < \cdots < i_k \leq n \\ 1 \leq j_1 < \cdots < j_k \leq n}} \varphi_{i_1 \dots i_k} \psi_{j_1 \dots j_k} \det \begin{pmatrix} g^{i_1 j_1} & \cdots & g^{i_1 j_k} \\ \vdots & \ddots & \vdots \\ g^{i_k j_1} & \cdots & g^{i_k j_k} \end{pmatrix}$$

Example 8.1.1. Let φ, ψ be two 2-forms, locally written as

$$\begin{aligned}
\varphi &= \varphi_{i_1 i_2} \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \\
\psi &= \psi_{j_1 j_2} \mathrm{d}x^{j_1} \wedge \mathrm{d}x^{j_2}
\end{aligned}$$

where $i_1 < i_2, j_1 < j_2$. Then

$$\begin{aligned}
\langle \varphi, \psi \rangle &= \frac{1}{2} \varphi_{i_1 i_2} \psi_{j_1 j_2} g(\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2}, \mathrm{d}x^{j_1} \wedge \mathrm{d}x^{j_2}) \\
&= \frac{1}{2} \varphi_{i_1 i_2} \psi_{j_1 j_2} g(\mathrm{d}x^{i_1} \otimes \mathrm{d}x^{i_2} - \mathrm{d}x^{i_2} \otimes \mathrm{d}x^{i_1}, \mathrm{d}x^{j_1} \otimes \mathrm{d}x^{j_2} - \mathrm{d}x^{j_2} \otimes \mathrm{d}x^{j_1}) \\
&= \frac{1}{2} \varphi_{i_1 i_2} \psi_{j_1 j_2} (g^{i_1 j_1} g^{i_2 j_2} - g^{i_1 j_2} g^{i_2 j_1} - g^{i_2 j_1} g^{i_1 j_2} + g^{i_2 j_2} g^{i_1 j_1}) \\
&= \varphi_{i_1 i_2} \psi_{j_1 j_2} (g^{i_1 j_1} g^{i_2 j_2} - g^{i_1 j_2} g^{i_2 j_1}) \\
&= \varphi_{i_1 i_2} \psi_{j_1 j_2} \det \begin{pmatrix} g^{i_1 j_1} & g^{i_1 j_2} \\ g^{i_2 j_1} & g^{i_2 j_2} \end{pmatrix}
\end{aligned}$$

Definition 8.1.2 (volume form). The volume form vol is an n -form such that $\langle \text{vol}, \text{vol} \rangle = 1$.

Remark 8.1.2 (local form). Locally the volume form is given by $\sqrt{\det g} dx^1 \wedge \cdots \wedge dx^n$.

Definition 8.1.3 (inner product on Ω_M^k). For two k -forms φ, ψ , their inner product is defined as

$$(\varphi, \psi) := \int_M \langle \varphi, \psi \rangle \text{vol}$$

Definition 8.1.4 (formal adjoint). For a k -form φ and a $(k+1)$ -form ψ , the formal adjoint of d is an operator $d^* : C^\infty(M, \Omega_M^{k+1}) \rightarrow C^\infty(M, \Omega_M^k)$ such that

$$(d\varphi, \psi) = (\varphi, d^*\psi)$$

Remark 8.1.3. There is no guarantee for existence, but later we will see such d^* do exist, and give an explicit formula.

Definition 8.1.5 (Laplace-Beltrami operator). The Laplace-Beltrami operator $\Delta_g : C^\infty(M, \Omega_M^k) \rightarrow C^\infty(M, \Omega_M^k)$ is defined as

$$\Delta_g = dd^* + d^*d$$

Definition 8.1.6 (harmonic). A k -form α is called harmonic, if $\Delta_g \alpha = 0$.

Notation 8.1.2. The space of all harmonic forms is denoted by $\mathcal{H}^k(M)$.

Lemma 8.1.2. A k -form α is harmonic if and only if $d\alpha = 0$ and $d^*\alpha = 0$.

Proof. Note that

$$\begin{aligned} (\alpha, \Delta\alpha) &= (\alpha, dd^*\alpha) + (\alpha, d^*d\alpha) \\ &= \|d^*\alpha\|^2 + \|d\alpha\|^2 \end{aligned}$$

□

8.2. Hodge star operator. Although we have defined an inner product on Ω_M^k , it's still quite difficult to compute it. However, inner product on Ω_M^k is independent of the choice of local frame, so we can use normal coordinate to give a local frame, and define Hodge star operator on it, which will gives us an effective method to compute.

8.2.1. Baby case. Recall that for a \mathbb{F} -vector space V with inner product $\langle -, - \rangle$, and $\{e_1, \dots, e_n\}$ is a basis of V . For any $0 \leq k \leq n$, there is a natural basis of $\bigwedge^k V$, consisting of $\{e_I := e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$.

Here are two special cases:

- (1) For $k = 0$, we regard $\bigwedge^0 V^k$ as base field \mathbb{F} , and $e_I = 1$.
- (2) For $k = n$, we use vol to denote basis $e_1 \wedge \cdots \wedge e_n$.

With respect to this basis, we can write down the induced metric on $\bigwedge^k V$ as

$$\langle e_{i_1} \wedge \cdots \wedge e_{i_k}, e_{j_1} \wedge \cdots \wedge e_{j_k} \rangle = \det \begin{pmatrix} \langle e_{i_1}, e_{j_1} \rangle & \cdots & \langle e_{i_1}, e_{j_k} \rangle \\ \vdots & & \vdots \\ \langle e_{i_k}, e_{j_1} \rangle & \cdots & \langle e_{i_k}, e_{j_k} \rangle \end{pmatrix}$$

It's clear if $\{e_1, \dots, e_n\}$ is an orthonormal basis of V . Then $\{e_I\}$ is an orthonormal basis of $\bigwedge^k V$. From now on, we assume $\{e_I\}$ is an orthonormal basis of $\bigwedge^k V$.

Definition 8.2.1 (Hodge star). Hodge star operator is defined as

$$\begin{aligned} \star : \bigwedge^k V &\rightarrow \bigwedge^{n-k} V \\ e_I &\mapsto \text{sign}(I, I^c) e_{I^c} \end{aligned}$$

where I^c is $[n] - I = \{i'_1, \dots, i'_{n-k}\}$ and $\text{sign}(I, I^c)$ is the sign of the permutation $(i_1, \dots, i_k, i'_1, \dots, i'_{n-k})$.

Example 8.2.1. It's clear $\star 1 = \text{vol}$ and $\star \text{vol} = 1$.

Proposition 8.2.1.

$$\star^2 = (-1)^{k(n-k)} \text{id}$$

holds on $\bigwedge^k V$.

Proof. It suffices to check on basis e_I as follows

$$\begin{aligned} \star^2 e_I &= \star(\text{sign}(I, I^c) e_{I^c}) \\ &= \text{sign}(I, I^c) \text{sign}(I^c, I) e_I \\ &= (-1)^{k(n-k)} e_I \end{aligned}$$

□

Proposition 8.2.2. For $u \in \bigwedge^k V, v \in \bigwedge^{n-k} V$, we have

$$\star(u \wedge v) = (-1)^{k(n-k)} \langle u, \star v \rangle$$

Proof. It suffices to check on basis $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}, e_J = e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}$. Furthermore, it's clear $e_I \wedge e_J = 0$, if $J \neq I^c$, so we may assume $J = I^c$.

$$\begin{aligned} \star(e_I \wedge e_{I^c}) &= \star(\text{sign}(I, I^c) \text{vol}) \\ &= \text{sign}(I, I^c) \\ \langle e_I, \star e_{I^c} \rangle &= \langle e_I, \text{sign}(I, I^c) e_I \rangle \\ &= \text{sign}(I, I^c) \langle e_I, e_I \rangle \\ &= \text{sign}(I, I^c) \end{aligned}$$

□

Corollary 8.2.1. For $u, v \in \bigwedge^k V$, we have

$$(1) \quad u \wedge \star v = v \wedge \star u = \langle u, v \rangle \text{vol}.$$

$$(2) \quad \langle \star u, \star v \rangle = \langle u, v \rangle.$$

Proof. For (1).

$$\star(u \wedge \star v) = (-1)^{k(n-k)} \langle u, \star^2 v \rangle = \langle u, v \rangle$$

which implies $u \wedge \star v = \langle u, v \rangle \text{ vol}$. Since $\langle u, v \rangle = \langle v, u \rangle$, we obtain $u \wedge \star v = v \wedge \star u$.

For (2).

$$\begin{aligned} \langle \star u, \star v \rangle &= (-1)^{k(n-k)} \star(\star u \wedge v) \\ &= (-1)^{2k(n-k)} \star(v \wedge \star u) \\ &= (-1)^{3k(n-k)} \langle v \wedge \star^2 u \rangle \\ &= (-1)^{4k(n-k)} \langle v, u \rangle \\ &= \langle u, v \rangle \end{aligned}$$

□

Remark 8.2.1.

(1) gives us a method to compute inner product, that's why we define Hodge star, some authors also use this property to denote Hodge star operator.

(2) implies that Hodge star operator is a linear isometry between $\bigwedge^k V$ and $\bigwedge^{n-k} V$.

Corollary 8.2.2 (almost self-adjoint). For $u \in \bigwedge^k V, v \in \bigwedge^{n-k} V$, we have

$$\langle u, \star v \rangle = (-1)^{k(n-k)} \langle \star u, v \rangle$$

Proof.

$$\langle u, \star v \rangle = \langle \star u, \star^2 v \rangle = (-1)^{k(n-k)} \langle \star u, v \rangle$$

□

Remark 8.2.2. This corollary implies the adjoint operator of \star is $(-1)^{k(n-k)} \star$, so here I call it almost self-adjoint.

8.2.2. General case. Now we're going to define Hodge star operator on M , that's an operator from $C^\infty(M, \Omega_M^k)$ to $C^\infty(M, \Omega_M^{n-k})$. If we define it point-wise. Then everything reduces to the baby case. For each point $p \in M$, consider the local frame $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ of TM given by normal coordinate, with dual frame $\{dx^1, \dots, dx^n\}$. Then

$$\star(dx^I) := \text{sign}(I, I^c) dx^{I^c}$$

Theorem 8.2.1.

- (1) $\star 1 = \text{vol}, \star \text{vol} = 1$.
- (2) $\star^2 = (-1)^{k(n-k)}$ on k -forms.
- (3) If u is a k -form and v a $(n-k)$ -form. Then

$$\begin{aligned} \star(u \wedge v) &= (-1)^{k(n-k)} \langle u, \star v \rangle \\ \langle u, \star v \rangle &= (-1)^{k(n-k)} \langle \star u, v \rangle \end{aligned}$$

(4) For any two k -forms u, v . Then

$$\begin{aligned} u \wedge \star v &= v \wedge \star u = \langle u, v \rangle \text{vol} = \langle v, u \rangle \text{vol} \\ \langle \star u, \star v \rangle &= \langle u, v \rangle \end{aligned}$$

(5) $d^* = (-1)^{nk+n+1} \star d \star$ on k -forms.

Proof. It suffices to check (5), other cases we have already solved in the case of linear algebra. Take any $(k-1)$ -form α and k -form β , we need to show

$$(d\alpha, \beta) = (\alpha, d^*\beta)$$

that is to show

$$\int_M d\alpha \wedge \star \beta = \int_M \alpha \wedge \star d^*\beta$$

By Stokes theorem and Leibniz rule we have

$$0 = \int_M d(\alpha \wedge \star \beta) = \int_M d\alpha \wedge \star \beta + (-1)^{k-1} \int_M \alpha \wedge d \star \beta$$

Since $\star^2 = (-1)^{(n-k+1)(k-1)}$ on $(n-k+1)$ -forms. Then

$$(-1)^{k-1} \int_M \alpha \wedge d \star \beta = (-1)^{k-1+(n-k+1)(k-1)} \int_M \alpha \wedge \star^2 d \star \beta$$

Therefore

$$\begin{aligned} (d\alpha, \beta) &= \int_M d\alpha \wedge \star \beta \\ &= (-1)^{k+(n-k+1)(k-1)} \int_M \alpha \wedge \star \star d \star \beta \\ &= (-1)^{nk+k+1} \int_M \alpha \wedge \star (\star d \star \beta) \end{aligned}$$

which implies

$$d^*\beta = (-1)^{nk+k+1} \star d \star \beta$$

□

Remark 8.2.3. (4) allows us to give a new expression for inner product (φ, ψ) , where φ, ψ are two k -forms, that is

$$(\varphi, \psi) := \int_M \langle \varphi, \psi \rangle \text{vol} = \int_M \varphi \wedge \star \psi$$

Some authors use this expression to define Hodge star operator, and prove its properties shown in Theorem 8.2.1.

8.3. Computations of adjoint operator.

Lemma 8.3.1 (Jacobi's formula). For a function $(a_{ij}(t))$ valued in $\text{GL}(n, \mathbb{R})$, we have

$$\frac{d}{dt} \det(a_{ij}(t)) = \det(a_{ij}(t)) a^{ij}(t) \frac{da_{ij}(t)}{dt}$$

where $(a^{ij}(t))$ is the inverse matrix of $(a_{ij}(t))$.

Lemma 8.3.2. Let (M, g) be a Riemannian manifold, for any two vector fields X, Y , one has

$$\nabla_Y \circ \iota_X = \iota_X \circ \nabla_Y + \iota_{\nabla_Y X}$$

Proof. Let ω be a $(k+1)$ -form. Then for vector fields Y_1, \dots, Y_k , direct computation shows

$$\begin{aligned} \nabla_Y \circ \iota_X \omega(Y_1, \dots, Y_k) &= \nabla_Y \omega(X, Y_1, \dots, Y_k) \\ &= Y \omega(X, Y_1, \dots, Y_k) - \omega(\nabla_X Y, Y_1, \dots, Y_k) \\ &\quad - \sum_{i=1}^k \omega(X, Y_1, \dots, Y_{i-1}, \nabla_X Y_i, Y_{i+1}, \dots, Y_k) \\ &= (\iota_X \circ \nabla_Y \omega + \iota_{\nabla_X Y} \omega)(Y_1, \dots, Y_k) \end{aligned}$$

□

Lemma 8.3.3. Let (M, g) be a compact Riemannian manifold. Then

$$\langle dx^i \wedge \alpha, \beta \rangle = \langle \alpha, g^{ij} \iota_{\frac{\partial}{\partial x^j}} \beta \rangle$$

where $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ is a local frame of TM , and α, β are forms with appropriate degrees.

Proof. It suffices to check with respect to normal coordinate. If we locally write $\alpha = \alpha_I dx^I$ and $\beta = \beta_J dx^J$, it suffices to check the case there is no dx^i in dx^I and there is dx^i in dx^J , since other cases are trivial. Suppose $|I| = k$ and $|J| = k+1$, and dx^i in the m -th position of dx^J . Then

$$\langle \alpha, \iota_{\frac{\partial}{\partial x^i}} \beta \rangle = (-1)^{m+2} \alpha_I \beta_J \det G$$

where G is a $k \times k$ matrix. By definition if we write

$$\langle dx^i \wedge \alpha, \beta \rangle = \alpha_I \beta_J \det G'$$

where G' is a $(k+1) \times (k+1)$ matrix. It's clear $\det G' = (-1)^{m+2} \det G$ by expansion of $\det G'$ by the first row. □

Lemma 8.3.4. Let (M, g) be a Riemannian manifold with volume form vol . Then

$$\begin{aligned} \mathcal{L}_X \text{vol} &= \left(\frac{\partial X^i}{\partial x^i} + \frac{1}{2} X^i g^{pq} \frac{\partial g_{pq}}{\partial x^i} \right) \text{vol} \\ &= \left(\frac{\partial X^i}{\partial x^i} + \Gamma_{ij}^j X^i \right) \text{vol} \end{aligned}$$

Proof. Cartan's magic formula shows that

$$\mathcal{L}_X = \iota_X \circ d + d \circ \iota_X$$

So

$$\begin{aligned}
\mathcal{L}_X \text{vol} &= (\iota_X \circ d + d \circ \iota_X) \text{vol} \\
&= d \circ \iota_X \text{vol} \\
&= d\{(-1)^{i-1} X^i \sqrt{\det g} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n\} \\
&= \frac{1}{\sqrt{\det g}} \frac{\partial(X^i \sqrt{\det g})}{\partial x^i} \text{vol} \\
&= \frac{1}{\sqrt{\det g}} \left(\frac{\partial X^i}{\partial x^i} \sqrt{\det g} + X^i \frac{\partial \sqrt{\det g}}{\partial x^i} \right) \text{vol} \\
&= \left(\frac{\partial X^i}{\partial x^i} + X^i \frac{\partial \log \sqrt{\det g}}{\partial x^i} \right) \text{vol} \\
&= \left(\frac{\partial X^i}{\partial x^i} + \frac{1}{2} X^i \frac{\partial \log \det g}{\partial x^i} \right) \text{vol}
\end{aligned}$$

Then the following Jacobi's formula shows the first equality

$$\frac{\partial \log \det g}{\partial x^i} = \frac{1}{\det g} \frac{\partial \det g}{\partial x^i} = g^{pq} \frac{\partial g_{pq}}{\partial x^i}$$

and the second equality holds from the formula of Christoffel in terms of metric, that is $g^{jk}(\Gamma_{ij}^l g_{lk} + \Gamma_{ik}^l g_{jl}) = 2\Gamma_{ij}^j$. \square

Proposition 8.3.1. Let (M, g) be a compact Riemannian manifold equipped with Levi-Civita connection ∇ . Then

$$d^* = -g^{ij} \iota_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}}$$

where $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ is a local frame of TM .

Proof. Direct computation shows

$$\begin{aligned}
0 &= \int_M d(\alpha \wedge \star \beta) \\
&= \int_M \mathcal{L}_{\frac{\partial}{\partial x^i}} (dx^i \wedge \alpha \wedge \star \beta) \\
&= \int_M \mathcal{L}_{\frac{\partial}{\partial x^i}} (\langle dx^i \wedge \alpha, \beta \rangle \text{vol}) \\
&\stackrel{(1)}{=} \int_M \mathcal{L}_{\frac{\partial}{\partial x^i}} (\langle \alpha, g^{ij} \iota_{\frac{\partial}{\partial x^j}} \beta \rangle \text{vol}) \\
&\stackrel{(2)}{=} \int_M (\langle \nabla_i \alpha, g^{ij} \iota_{\frac{\partial}{\partial x^j}} \beta \rangle + \langle \alpha, \nabla_i (g^{ij} \iota_{\frac{\partial}{\partial x^j}} \beta) \rangle + \frac{1}{2} g^{pq} \frac{\partial g_{pq}}{\partial x^i} \langle \alpha, g^{ij} \iota_{\frac{\partial}{\partial x^j}} \beta \rangle) \text{vol} \\
&= \int_M (\langle dx^i \wedge \nabla_i \alpha, \beta \rangle + \langle \alpha, \frac{\partial g^{ij}}{\partial x^i} \iota_{\frac{\partial}{\partial x^j}} \beta \rangle + \langle \alpha, g^{ij} \nabla_i (\iota_{\frac{\partial}{\partial x^j}} \beta) \rangle + \frac{1}{2} g^{pq} \frac{\partial g_{pq}}{\partial x^i} \langle \alpha, g^{ij} \iota_{\frac{\partial}{\partial x^j}} \beta \rangle) \text{vol} \\
&\stackrel{(3)}{=} \int_M (\langle dx^i \wedge \nabla_i \alpha, \beta \rangle + \langle \alpha, \frac{\partial g^{ij}}{\partial x^i} \iota_{\frac{\partial}{\partial x^j}} \beta \rangle + \langle \alpha, g^{ij} \iota_{\frac{\partial}{\partial x^j}} (\nabla_i \beta) \rangle + \langle \alpha, g^{il} \Gamma_{il}^j \iota_{\frac{\partial}{\partial x^j}} \beta \rangle \\
&\quad + \frac{1}{2} g^{pq} \frac{\partial g_{pq}}{\partial x^i} \langle \alpha, g^{ij} \iota_{\frac{\partial}{\partial x^j}} \beta \rangle) \text{vol}
\end{aligned}$$

where

- (1) holds from Lemma 8.3.3.
- (2) holds from Lemma 8.3.4.
- (3) holds from Lemma 8.3.2.

It's easy to see

$$\frac{\partial g^{ij}}{\partial x^i} + g^{il}\Gamma_{il}^j + \frac{1}{2}g^{ij}g^{pq}\frac{\partial g^{pq}}{\partial x^i} = 0$$

since

$$\Gamma_{ij}^l = \frac{1}{2}g^{kl}\left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

□

In the following examples, we always compute with respect to normal coordinate.

Example 8.3.1. For a 1-form ω locally written as $\omega_i dx^i$. Then

$$d^*\omega = -\sum_{i=1}^n \frac{\partial \omega_i}{\partial x^i}$$

Example 8.3.2. For a smooth function f . Then

$$\begin{aligned} \Delta_g f &= (dd^* + d^*d)f \\ &= d^*df \\ &= d^*\left(\frac{\partial f}{\partial x^i} dx^i\right) \\ &= -\sum_{i=1}^n \frac{\partial^2 f}{\partial x^i \partial x^i} \end{aligned}$$

So as you can see, Laplace-Beltrami operator differs a sign with scalar Laplacian.

Example 8.3.3. For an n -form ω written as $f \text{ vol}$, where f is a smooth function. Then

$$\begin{aligned} d^*\omega &= (-1)^n \star d \star (f \text{ vol}) \\ &= (-1)^n \star df \\ &= (-1)^n \star \left(\frac{\partial f}{\partial x^i} dx^i\right) \\ &= \sum_{i=1}^n (-1)^{n+i-1} \frac{\partial f}{\partial x^i} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \end{aligned}$$

8.4. Divergence.

Definition 8.4.1 (divergence). For any vector field X , its divergence $\text{div } X$ is defined as $\text{tr } \nabla X$.

Remark 8.4.1 (local form). If we locally write X as $X^i \frac{\partial}{\partial x^i}$. Then

$$\nabla X = \nabla_i X^j dx^i \otimes \frac{\partial}{\partial x^j}$$

Then

$$\operatorname{div} X = \nabla_i X^i$$

Proposition 8.4.1.

$$\operatorname{div} X \operatorname{vol} = \mathcal{L}_X \operatorname{vol}$$

Proof. Cartan's magic formula shows that

$$\mathcal{L}_X = \iota_X \circ d + d \circ \iota_X$$

So

$$\begin{aligned} \mathcal{L}_X \operatorname{vol} &= (\iota_X \circ d + d \circ \iota_X) \operatorname{vol} \\ &= d \circ \iota_X \operatorname{vol} \\ &= d((-1)^{i-1} X^i \sqrt{\det g} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n) \\ &= \frac{1}{\sqrt{\det g}} \frac{\partial(X^i \sqrt{\det g})}{\partial x^i} \operatorname{vol} \\ &= \frac{1}{\sqrt{\det g}} \left(\frac{\partial X^i}{\partial x^i} \sqrt{\det g} + X^i \frac{\partial \sqrt{\det g}}{\partial x^i} \right) \operatorname{vol} \\ &= \left(\frac{\partial X^i}{\partial x^i} + X^i \frac{\partial \log \sqrt{\det g}}{\partial x^i} \right) \operatorname{vol} \\ &= \left(\frac{\partial X^i}{\partial x^i} + \frac{1}{2} X^i \frac{\partial \log \det g}{\partial x^i} \right) \operatorname{vol} \end{aligned}$$

Note that Jacobi's formula says

$$\frac{\partial \log \det g}{\partial x^i} = \frac{1}{\det g} \frac{\partial \det g}{\partial x^i} = g^{jk} \frac{\partial g_{jk}}{\partial x^i} = g^{jk} (\Gamma_{ij}^l g_{lk} + \Gamma_{ik}^l g_{jl}) = 2\Gamma_{ij}^j$$

Thus

$$\begin{aligned} \mathcal{L}_X \operatorname{vol} &= \left(\frac{\partial X^i}{\partial x^i} + \frac{1}{2} X^i \frac{\partial \log \det g}{\partial x^i} \right) \operatorname{vol} \\ &= \left(\frac{\partial X^i}{\partial x^i} + \Gamma_{ij}^j X^i \right) \operatorname{vol} \\ &= \left(\frac{\partial X^i}{\partial x^i} + \Gamma_{ij}^i X^j \right) \operatorname{vol} \\ &= \nabla_i X^i \operatorname{vol} \end{aligned}$$

□

Remark 8.4.2. From the proof, we can say there is the following formula for divergence of a vector field X written as $X^i \frac{\partial}{\partial x^i}$, one has

$$\operatorname{div} X = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} (\sqrt{\det g} X^i)$$

Corollary 8.4.1 (divergence theorem).

$$\int_M \operatorname{div} X \operatorname{vol} = 0$$

Proposition 8.1. Let f be a smooth function on M . Then for any smooth function φ , one has

$$-\int_M \langle \nabla \varphi, \nabla f \rangle \operatorname{vol} = \int_M \Delta \varphi \cdot f \operatorname{vol}$$

Proof. Direct computation shows

$$\begin{aligned} \operatorname{div}(f \nabla \varphi) &= \nabla_k (f \nabla \varphi)^k \\ &= \frac{\partial (f \nabla \varphi)^k}{\partial x^k} + \Gamma_{ks}^k (f \nabla \varphi)^s \\ &= \frac{\partial (f g^{ik} \frac{\partial \varphi}{\partial x^i})}{\partial x^k} + \Gamma_{ks}^k f g^{is} \frac{\partial \varphi}{\partial x^i} \\ &= \underbrace{g^{ik} \frac{\partial f}{\partial x^k} \frac{\partial \varphi}{\partial x^i}}_{\text{part I}} + \underbrace{f \left(\frac{\partial g^{ik}}{\partial x^k} \frac{\partial \varphi}{\partial x^i} + g^{ik} \frac{\partial^2 \varphi}{\partial x^k \partial x^i} + g^{is} \Gamma_{ks}^k \frac{\partial \varphi}{\partial x^i} \right)}_{\text{part II}} \end{aligned}$$

We have the following observations:

(1) Part I equals

$$\begin{aligned} g^{ik} \frac{\partial f}{\partial x^k} \frac{\partial \varphi}{\partial x^i} &= g_{lj} g^{lk} \frac{\partial f}{\partial x^k} g^{ji} \frac{\partial \varphi}{\partial x^i} \\ &= \langle g^{lk} \frac{\partial f}{\partial x^k} \frac{\partial}{\partial x^l}, g^{ji} \frac{\partial \varphi}{\partial x^i} \frac{\partial}{\partial x^j} \rangle \\ &= \langle \nabla f, \nabla \varphi \rangle \end{aligned}$$

(2) Note

$$\begin{aligned} \frac{\partial g^{ik}}{\partial x^k} + g^{is} \Gamma_{ks}^k \frac{\partial \varphi}{\partial x^i} &= -g^{is} g^{kt} \frac{\partial g_{st}}{\partial x^k} + \frac{1}{2} g^{is} g^{kt} \left(\frac{\partial g_{kt}}{\partial x^s} + \frac{\partial g_{st}}{\partial x^k} - \frac{\partial g_{ks}}{\partial x^t} \right) \\ &= -\frac{1}{2} g^{is} g^{kt} \left(\frac{\partial g_{ks}}{\partial x^t} + \frac{\partial g_{st}}{\partial x^k} - \frac{\partial g_{kt}}{\partial x^s} \right) \\ &= -g^{kt} \Gamma_{kt}^i \end{aligned}$$

where $\frac{\partial g^{ik}}{\partial x^k} = -g^{is} g^{kt} \frac{\partial g_{st}}{\partial x^k}$ holds from the fact $g^{ik} g_{kt} = \delta_t^i$. Then take partial derivative with respect to x^k to conclude.

(3) From (2) and local expression of Δ , it's clear part II equals $f \Delta \varphi$.

Thus we have

$$\operatorname{div}(f \nabla \varphi) = \langle \nabla \varphi, \nabla f \rangle + f \Delta \varphi$$

Then divergence theorem completes the proof. \square

Proposition 8.4.2. Let ω be a 1-form. Then

$$d^* \omega = -\operatorname{div}(\omega^\sharp)$$

Proof. It suffices to check with respect to normal coordinate:

(1) Remark 8.4.2 or direct computation shows

$$\operatorname{div} \omega^\# = \sum_{i=1}^n \frac{\partial \omega_i}{\partial x^i}$$

(2) Example 8.3.1 implies

$$d^* \omega = - \sum_{i=1}^n \frac{\partial \omega_i}{\partial x^i}$$

This completes the proof. \square

8.5. Conformal Laplacian. For a smooth function u , according to Proposition 8.4.2 and Remark 8.4.2, we can write $\Delta_g u$ as follows

$$\begin{aligned} \Delta_g u &= d^* du \\ &= -\operatorname{div}(g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial}{\partial x^j}) \\ &= -\frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^j} (\sqrt{\det g} g^{ij} \frac{\partial u}{\partial x^i}) \end{aligned}$$

Thus Laplace-Beltrami Δ_g with respect to g is

$$\Delta_g = -\frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^j} (\sqrt{\det g} g^{ij} \frac{\partial}{\partial x^i})$$

So if we consider conformal transformation $\tilde{g} = e^{2f} g$ for some smooth function f , we have

$$\begin{aligned} \tilde{g}_{ij} &= e^{2f} g_{ij} \\ \tilde{g}^{ij} &= e^{-2f} g^{ij} \\ \det \tilde{g} &= e^{2nf} \det g \\ \sqrt{\tilde{g}} &= e^{nf} \sqrt{\det g} \end{aligned}$$

Thus

$$\begin{aligned} \Delta_{\tilde{g}} &= -\frac{1}{e^{nf} \sqrt{\det g}} \frac{\partial}{\partial x^j} (e^{nf} \sqrt{\det g} e^{-2f} g^{ij} \frac{\partial}{\partial x^i}) \\ &= -\frac{e^{-nf}}{\sqrt{\det g}} \frac{\partial}{\partial x^j} (e^{(n-2)f} \sqrt{\det g} g^{ij} \frac{\partial}{\partial x^i}) \\ &= -\frac{e^{-2f}}{\sqrt{\det g}} \frac{\partial}{\partial x^j} (\sqrt{\det g} g^{ij} \frac{\partial}{\partial x^i}) - \frac{(n-2)e^{-2f}}{\sqrt{\det g}} \frac{\partial f}{\partial x^j} \sqrt{\det g} g^{ij} \frac{\partial}{\partial x^i} \\ &= -e^{-2f} \Delta_g - (n-2)e^{-2f} g^{ij} \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^i} \end{aligned}$$

So we have

$$\Delta_{\tilde{g}} = -e^{-2f} \Delta_g$$

when $n = 2$. It's a kind of conformal invariance. However, this fails in higher dimension. Let's consider the following so-called conformal Laplacian when $n > 3$

$$L : C^\infty(M) \rightarrow C^\infty(M)$$

$$u \mapsto -\frac{4(n-1)}{n-2}\Delta_g u + Su$$

where S is scalar curvature. Let's show

$$\tilde{L}u = e^{-\frac{n+2}{2}f}L(e^{\frac{n-2}{2}f}u)$$

where \tilde{L} is the conformal Laplacian after conformal transformation. Divide computations into several parts:

(1)

$$\begin{aligned} \nabla^2(e^{\frac{n-2}{2}f}u) &= \nabla\left(\frac{n-2}{2}e^{\frac{n-2}{2}f}\frac{\partial f}{\partial x^i}u dx^i + e^{\frac{n-2}{2}f}\frac{\partial u}{\partial x^i}dx^i\right) \\ &= e^{\frac{n-2}{2}f}\nabla^2 u + \frac{n-2}{2}e^{\frac{n-2}{2}f}\frac{\partial f}{\partial x^j}\frac{\partial u}{\partial x^i}dx^i \otimes dx^j \\ &\quad + \left(\frac{(n-2)^2}{4}e^{\frac{n-2}{2}f}u\frac{\partial f}{\partial x^j}\frac{\partial f}{\partial x^i} + \frac{n-2}{2}e^{\frac{n-2}{2}f}\frac{\partial f}{\partial x^i}\frac{\partial u}{\partial x^j}\right)dx^i \otimes dx^j + \frac{n-2}{2}e^{\frac{n-2}{2}f}u\nabla^2 f \end{aligned}$$

(2)

$$\begin{aligned} \Delta_g(e^{\frac{n-2}{2}f}u) &= \text{tr}_g \nabla^2(e^{\frac{n-2}{2}f}u) \\ &= e^{\frac{n-2}{2}f}\Delta_g u + \frac{n-2}{2}e^{\frac{n-2}{2}f}g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial u}{\partial x^i} \\ &\quad + g^{ij}\left(\frac{(n-2)^2}{4}e^{\frac{n-2}{2}f}u\frac{\partial f}{\partial x^j}\frac{\partial f}{\partial x^i} + \frac{n-2}{2}e^{\frac{n-2}{2}f}\frac{\partial f}{\partial x^i}\frac{\partial u}{\partial x^j}\right) + \frac{n-2}{2}e^{\frac{n-2}{2}f}u\Delta_g f \end{aligned}$$

(3)

$$\begin{aligned} e^{-\frac{n+2}{2}f}L(e^{\frac{n-2}{2}f}u) &= -\frac{4(n-1)}{n-2}e^{-2f}\Delta_g u - 4(n-1)e^{-2f}g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial u}{\partial x^i} \\ &\quad - g^{ij}(n-2)(n-1)e^{-2f}u\frac{\partial f}{\partial x^j}\frac{\partial f}{\partial x^i} - 2(n-1)e^{-2f}u\Delta_g f + e^{-2f}Su \\ &= -\frac{4(n-1)}{n-2}e^{-2f}\Delta_g u - 4(n-1)e^{-2f}g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial u}{\partial x^i} \\ &\quad - (n-2)(n-1)e^{-2f}u|df|^2 - 2(n-1)e^{-2f}u\Delta_g f + e^{-2f}Su \end{aligned}$$

(4)

$$-\frac{4(n-1)}{n-2}\Delta_{\tilde{g}}u = -\frac{4(n-1)}{n-2}e^{-2f}\Delta_g u - 4(n-1)e^{-2f}g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial u}{\partial x^i}$$

(5) Note that

$$\tilde{S} = e^{-2f}S - 2(n-1)e^{-2f}\Delta_g f - (n-2)(n-1)e^{-2f}|df|^2$$

This completes the computation. In particular, in (2) if we take $u = 1$ we have

$$-\frac{4(n-1)}{n-2}\Delta_g(e^{\frac{n-2}{2}f}) = -(n-2)(n-1)e^{\frac{n-2}{2}f}|df|^2 - 2(n-1)e^{\frac{n-2}{2}f}\Delta_g f$$

Thus we have

$$\tilde{S} = e^{-\frac{n+2}{2}f}\left(-\frac{4(n-1)}{n-2}\Delta_g e^{\frac{n-2}{2}f} + S e^{\frac{n-2}{2}f}\right) = e^{-\frac{n+2}{2}f}L(e^{\frac{n-2}{2}f})$$

So if we put $e^{2f} = \varphi^{\frac{4}{n-2}}$, we have

$$\tilde{S} = \varphi^{-\frac{n+2}{n-2}}L\varphi$$

So it's clear g is conformal to \tilde{g} with constant scalar curvature λ if and only if φ is a smooth positive solution to the Yamabe equation

$$L\varphi = \lambda\varphi^{\frac{n+2}{n-2}}$$

8.6. Hodge theorem and corollaries.

Theorem 8.6.1 (Hodge theorem). Consider the Laplace operator $\Delta_g : C^\infty(M, \Omega_M^k) \rightarrow C^\infty(M, \Omega_M^k)$. Then

- (1) $\dim_{\mathbb{R}} \mathcal{H}^k(M) < \infty$.
- (2) There is an orthogonal decomposition

$$C^\infty(M, \Omega_M^k) = \mathcal{H}^k(M) \perp \text{im } \Delta_g$$

Proof. See Appendix B. □

Corollary 8.6.1. More explicitly, we have the following orthogonal decomposition

$$C^\infty(M, \Omega_M^k) = \mathcal{H}^k(M) \oplus d(C^\infty(M, \Omega_M^{k-1})) \oplus d^*(C^\infty(M, \Omega_M^{k+1}))$$

Proof. It suffices to check $d(C^\infty(M, \Omega_M^{k-1}))$ is orthogonal to $d^*(C^\infty(M, \Omega_M^{k+1}))$. Take $d\alpha$ and $d^*\beta$, where α is a $k-1$ -form and β is a $k+1$ -form. Then

$$(d\alpha, d^*\beta) = (d^2\alpha, \beta) = 0$$

□

Corollary 8.6.2.

$$\begin{aligned} \ker d &= \mathcal{H}^k(M) \oplus d(C^\infty(M, \Omega_M^{k-1})) \\ \ker d^* &= \mathcal{H}^k(M) \oplus d^*(C^\infty(M, \Omega_M^{k+1})) \end{aligned}$$

Proof. Clear from above corollary. □

Corollary 8.6.3. The natural map $\mathcal{H}^k(M) \rightarrow H^k(M, \mathbb{R})$ is an isomorphism. In other words, every element in $H^k(M, \mathbb{R})$ is represented by a unique harmonic form.

Proof. Clear from above corollary. □

Corollary 8.6.4. $\star : \mathcal{H}^k(M) \rightarrow \mathcal{H}^{n-k}(M)$ is an isomorphism.

Proof. It suffices to show $*$ maps harmonic forms to harmonic forms, since we already have $*$ maps k -forms to k -forms. By Lemma 8.1.2, we just need to show $d \star \alpha = d^* \star \alpha = 0$ for a harmonic form α . Directly compute as follows

$$\begin{aligned} d \star \alpha &= (-1)^{\bullet_1} \star d \star \alpha = (-1)^{\bullet_2} \star d^* \alpha = 0 \\ d^* \star \alpha &= (-1)^{\bullet_3} \star d \star \alpha = (-1)^{\bullet_4} \star d \alpha = 0 \end{aligned}$$

Here we use \bullet, \bullet' to denote the power of (-1) , since it's not necessary for us to know what exactly it is. \square

Remark 8.6.1. In fact, above corollary follows from the following identity

$$\Delta_g \star = \star \Delta_g$$

which can be directly checked. In other words, Hodge star commutes with Laplacian Δ . Here gives a method of computation: From what we have done in the proof, we will see

$$\begin{aligned} \star d^* d &= (-1)^{\bullet_2} d \star d = (-1)^{\bullet_2 + \bullet_4} d d^* \star \\ \star d d^* &= (-1)^{\bullet_4} d^* \star d^* = (-1)^{\bullet_2 + \bullet_4} d^* d \star \end{aligned}$$

So all we need to do is to figure out the precise number of \bullet_2, \bullet_4 and show that $\bullet_2 + \bullet_4$ is even.

Corollary 8.6.5 (Poincaré duality). $H^k(M, \mathbb{R}) \cong H^{n-k}(M, \mathbb{R})$.

Proof. Clear from Corollary 8.6.3 and Corollary 8.6.4. \square

9. BOCHNER'S TECHNIQUE

9.1. Bochner formula. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ . Recall in Example 2.3.6 and Remark 2.6.1 we have already seen Hessian of a smooth function f and its scalar Laplacian locally as follows:

$$\begin{aligned}\text{Hess } f &= \nabla_{i,j}^2 f dx^i \otimes dx^j \\ \Delta f &= g^{ij} \nabla_{i,j}^2 f\end{aligned}$$

where

$$\nabla_{i,j}^2 f = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k}$$

Example 8.3.2 shows that scalar Laplacian differs a sign with Laplace-Beltrami operator.

Remark 9.1.1. Unless otherwise specified, we use Δ to denote scalar Laplacian and Δ_g to denote Laplace-Beltrami operator.

Theorem 9.1.1. Let $f: (M, g) \rightarrow \mathbb{R}$ be a smooth function. Then

- (1) $p \in M$ is a local minimum or maximum. Then $\nabla f(p) = 0$.
- (2) $p \in M$ is a local minimum. Then

$$\begin{cases} \text{Hess } f(p) \geq 0 \\ \Delta f(p) \geq 0 \end{cases}$$

- (3) $p \in M$ is a local maximum. Then

$$\begin{cases} \text{Hess } f(p) \leq 0 \\ \Delta f(p) \leq 0 \end{cases}$$

Proposition 9.1.1 (Bochner formula). Let $f: (M, g) \rightarrow \mathbb{R}$ be a smooth function. Then

$$\frac{1}{2} \Delta |\nabla f|^2 = |\text{Hess } f|^2 + \text{Ric}(\nabla f, \nabla f) + g(\nabla \Delta f, \nabla f)$$

Proof. For a tensor, its norm is independent of which type it is, so we write $\nabla f = g^{ij} \nabla_i f \frac{\partial}{\partial x^j}$. Then

$$\begin{aligned} |\nabla f|^2 &= g(\nabla f, \nabla f) \\ &= g(g^{ij} \nabla_i f \frac{\partial}{\partial x^j}, g^{kl} \nabla_k f \frac{\partial}{\partial x^l}) \\ &= g^{ij} g^{kl} g_{jl} \nabla_i f \nabla_k f \\ &= g^{ij} \nabla_i f \nabla_j f \end{aligned}$$

In the following computation we may use normal coordinate. Then in this case

$$\begin{aligned}
\frac{1}{2}\Delta|\nabla f|^2 &\stackrel{(1)}{=} \frac{1}{2}g^{kl}\nabla_k\nabla_l(g^{ij}\nabla_i f\nabla_j f) \\
&\stackrel{(2)}{=} \frac{1}{2}g^{kl}g^{ij}\nabla_k\nabla_l(\nabla_i f\nabla_j f) \\
&\stackrel{(3)}{=} g^{kl}g^{ij}\nabla_l\nabla_i f \cdot \nabla_k\nabla_j f + g^{kl}g^{ij}\nabla_k\nabla_l\nabla_i f \cdot \nabla_j f \\
&= |\text{Hess } f|^2 + g^{kl}g^{ij}\nabla_k\nabla_l\nabla_i f \cdot \nabla_j f
\end{aligned}$$

where

- (1) holds from in normal coordinate $\Delta f = g^{ij}\nabla_i\nabla_j f$.
- (2) holds from Proposition 2.6.1, that is magic formula.
- (3) holds from the following direct computation

$$\begin{aligned}
\nabla_k\nabla_l(\nabla_i f\nabla_j f) &= \nabla_k(\nabla_l\nabla_i f \cdot \nabla_j f + \nabla_i f \cdot \nabla_l\nabla_j f) \\
&= \nabla_k\nabla_l\nabla_i f \cdot \nabla_j f + \nabla_l\nabla_i f \cdot \nabla_k\nabla_j f \\
&\quad + \nabla_k\nabla_i f \cdot \nabla_l\nabla_j f + \nabla_i f \cdot \nabla_k\nabla_l\nabla_j f \\
&= 2\nabla_l\nabla_i f \cdot \nabla_k\nabla_j f + 2\nabla_k\nabla_l\nabla_i f \cdot \nabla_j f
\end{aligned}$$

Then the following computation completes the proof:

$$\begin{aligned}
g^{kl}g^{ij}\nabla_k\nabla_l\nabla_i f \cdot \nabla_j f &\stackrel{(4)}{=} g^{kl}g^{ij}\nabla_k\nabla_i\nabla_l f \cdot \nabla_j f \\
&\stackrel{(5)}{=} g^{kl}g^{ij}(\nabla_i\nabla_k\nabla_l f - R_{kil}^s\nabla_s f) \cdot \nabla_j f \\
&= g^{ij}\nabla_i(g^{kl}\nabla_k\nabla_l f) \cdot \nabla_j f + g^{ij}R_i^s\nabla_s f \cdot \nabla_j f \\
&= g^{ij}\nabla_i(\Delta f) \cdot \nabla_j f + \text{Ric}(\nabla f, \nabla f) \\
&= g(\nabla\Delta f, \nabla f) + \text{Ric}(\nabla f, \nabla f)
\end{aligned}$$

where

- (4) holds from symmetry of Hessian.
- (5) holds from Theorem 4.4.1, that is Ricci identity.

□

9.2. Obstruction to the existence of Killing fields.

Definition 9.2.1 (Killing field). A vector field X on a Riemannian manifold (M, g) is called a Killing field, if $\mathcal{L}_X g = 0$.

Proposition 9.2.1. X is a Killing field if and only if flows generated by X acts as isometries.

Proof. If ϕ_t is the flow generated by X , then by definition of Lie derivative one has

$$\mathcal{L}_X g = \left. \frac{d}{dt} \right|_{t=0} (\phi_t)^* g$$

Thus X is a Killing vector field if and only if its flow acts as isometries. □

Theorem 9.2.1. The following statements are equivalent.

- (1) X is a Killing field.
- (2) For any two vector fields Y, Z , we have

$$\langle \nabla_Y X, Z \rangle + \langle Y, \nabla_Z X \rangle = 0$$

Proof. Note that

$$\begin{aligned} \mathcal{L}_X \langle Y, Z \rangle &= X \langle Y, Z \rangle - \langle \mathcal{L}_X Y, Z \rangle - \langle Y, \mathcal{L}_X Z \rangle \\ &= \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle - \langle [X, Y], Z \rangle - \langle Y, [X, Z] \rangle \\ &= \langle \nabla_Y X, Z \rangle + \langle Y, \nabla_Z X \rangle \end{aligned}$$

□

Remark 9.2.1. For (2) locally we have

$$g_{kj} \nabla_i X^j = -g_{ij} \nabla_k X^j$$

Thus X is a Killing vector if and only if ∇X is a skew-symmetric $(1, 1)$ -tensor, that is $\nabla_i X^j$ is skew-symmetric in i, j .

Corollary 9.2.1. If X is a Killing field, then for arbitrary vector field Y one has

$$\langle \nabla_Y X, Y \rangle = 0$$

Proof. Set $Y = Z$ in $\langle \nabla_Y X, Z \rangle + \langle Y, \nabla_Z X \rangle = 0$ to conclude. □

Corollary 9.2.2. If X is parallel, then X is Killing.

Proof. A zero matrix must be skew-symmetric. □

Corollary 9.2.3. If X is Killing, then $\operatorname{div} X = \nabla_i X^i = 0$.

Proof. The trace of a skew-symmetric matrix is zero. □

Lemma 9.2.1 (Bochner formula for Killing field). Let X be a Killing field, and $f = \frac{1}{2}|X|^2$. Then

- (1) $\nabla f = -\nabla_X X$.
- (2) $\operatorname{Hess} f(Y, Y) = \langle \nabla_Y X, \nabla_Y X \rangle - R(Y, X, X, Y)$ holds for any vector field Y .
- (3) $\Delta f = |\nabla X|^2 - \operatorname{Ric}(X, X)$.

Proof. For (1). By direct computation we have

$$\begin{aligned} \nabla f &= \langle \nabla X, X \rangle \\ &= \langle \nabla_k X^i dx^k \otimes \frac{\partial}{\partial x^i}, X^j \frac{\partial}{\partial x^j} \rangle \\ &= g_{ij} X^j \nabla_k X^i dx^k \\ &\stackrel{I}{=} -g_{ik} X^j \nabla_j X^i dx^k \\ \nabla_X X &= X^j \nabla_j X^i \frac{\partial}{\partial x^i} \\ &= g_{ik} X^j \nabla_j X^i dx^k \end{aligned}$$

where I holds from skew-symmetry of ∇X .

For (2). By direct computation we have

$$\begin{aligned} \text{Hess } f(Y, Y) &= \frac{1}{2} Y^i Y^j \nabla_i \nabla_j (g_{kl} X^k X^l) \\ &= Y^i Y^j g_{kl} (\nabla_i X^k \nabla_j X^l + \nabla_i \nabla_j X^k \cdot X^l) \\ &= \langle \nabla_Y X, \nabla_Y X \rangle + Y^i Y^j g_{kl} \nabla_i \nabla_j X^k \cdot X^l \end{aligned}$$

and

$$\begin{aligned} Y^i Y^j g_{kl} \nabla_i \nabla_j X^k \cdot X^l &= -Y^i Y^j g_{kj} \nabla_i \nabla_l X^k \cdot X^l \\ &\stackrel{\text{II}}{=} -Y^i Y^j g_{kj} X^l (\nabla_l \nabla_i X^k + R_{ilm}^k X^m) \\ &= -Y^i Y^j X^l X^m R_{ilmj} \\ &= -R(Y, X, X, Y) \end{aligned}$$

where (II) holds from $g_{kj} X^l \nabla_l \nabla_i X^k = 0$, since this expression is skew symmetric in i, j .

(3) holds from (2) directly. \square

Theorem 9.2.2 (Bochner). Let (M, g) be a compact, oriented Riemannian manifold,

- (1) If $\text{Ric}(g) \leq 0$. Then every Killing field is parallel.
- (2) If $\text{Ric}(g) \leq 0$ and $\text{Ric}(g) < 0$ at some point. Then there is no non-trivial Killing field.

Proof. For (1). Let X be a Killing field and set $f = \frac{1}{2}|X|^2$. Then

$$\begin{aligned} 0 &= \int_M \Delta f \text{ vol} \\ &= \int_M (|\nabla X|^2 - \text{Ric}(X, X)) \text{ vol} \\ &\geq \int_M |\nabla X|^2 \text{ vol} \\ &\geq 0 \end{aligned}$$

Thus $|\nabla X| \equiv 0$, that is X is parallel.

For (2). From proof of (1) one can see if $\text{Ric}(g) \leq 0$ and X is a Killing field. Then

$$\int_M \text{Ric}(X, X) = 0$$

which implies $\text{Ric}(X, X) \equiv 0$. So if $\text{Ric}(g) < 0$ at some point $p \in M$. Then $X_p = 0$, thus $X \equiv 0$, since it's parallel. \square

9.3. Obstruction to the existence of harmonic 1-forms. To some extent, Killing field is dual to harmonic 1-form. Let's explain this in more detail.

Lemma 9.3.1. For a harmonic 1-form α , locally written as $\alpha_i dx^i$, we have

$$\begin{aligned}\nabla_i \alpha_j &= \nabla_j \alpha_i \\ g^{ij} \nabla_j \alpha_i &= 0\end{aligned}$$

Proof. Recall α is harmonic if and only if

$$\begin{aligned}d\alpha &= 0 \\ d^* \alpha &= 0\end{aligned}$$

It's clear

$$d(\alpha_j dx^j) = \nabla_i \alpha_j dx^i \wedge dx^j = 0$$

implies $\nabla_i \alpha_j = \nabla_j \alpha_i$. Similarly, explicit expression for d^* implies the second identity. \square

Remark 9.3.1. Recall Killing field implies $g_{ij} \nabla_k X^j$ is skew-symmetric in i, k , we can see both Killing field and harmonic 1-form implies some (skew)symmetries.

Lemma 9.3.2. If α is a harmonic 1-form. Then

$$\frac{1}{2} \Delta |\alpha|^2 = |\nabla \alpha|^2 + \text{Ric}(\alpha^\sharp, \alpha^\sharp)$$

Proof. Routine computation as follows:

$$\begin{aligned}\frac{1}{2} \Delta |\alpha|_g^2 &= \frac{1}{2} g^{kl} \nabla_k \nabla_l (g^{ij} \alpha_i \alpha_j) \\ &= |\nabla \alpha|^2 + g^{kl} g^{ij} \nabla_k \nabla_l \alpha_i \cdot \alpha_j \\ &= |\nabla \alpha|^2 + g^{kl} g^{ij} \nabla_k \nabla_i \alpha_l \cdot \alpha_j \\ &= |\nabla \alpha|^2 + g^{kl} g^{ij} (\nabla_i \nabla_k \alpha_l - R_{kil}^s \alpha_s) \alpha_j \\ &= |\Delta \alpha|^2 - g^{kl} g^{ij} R_{kil}^s \alpha_s \cdot \alpha_j \\ &= |\Delta \alpha|^2 + \text{Ric}(\alpha^\sharp, \alpha^\sharp)\end{aligned}$$

\square

Theorem 9.3.1 (Bochner). Let (M, g) be a compact, oriented Riemannian manifold,

- (1) If $\text{Ric}(g) \geq 0$. Then every harmonic 1-form is parallel.
- (2) If $\text{Ric}(g) \geq 0$ and $\text{Ric}(g) > 0$ at some point. Then there is no non-trivial harmonic 1-form.

Proof. The same as before. \square

Corollary 9.3.1. Let (M, g) be a compact, oriented Riemannian manifold with $\text{Ric}(g) \geq 0$ and $\text{Ric}(g) > 0$ at some point. Then $b_1(M) = 0$.

Proof. It's clear from above theorem and Corollary 8.6.3. \square

Remark 9.3.2. It's a kind of vanishing theorem. In geometry, “positivity” may cause “vanishing”, that's a philosophy.

Corollary 9.3.2. Let (M, g) be a compact Riemannian n -manifold with $\text{Ric}(g) \geq 0$. Then $b_1(M) \leq n$. Moreover, if $b_1(M) = n$ if and only if (M, g) is isometric to a flat torus.

Proof. By Corollary 8.6.3 we have $b_1(M) = \dim \mathcal{H}^1(M)$. Now if $\text{Ric}(g) \geq 0$. Then any harmonic 1-form is parallel, thus linear map $\mathcal{H}^1(M) \rightarrow T_p M$ that evaluates ω at point p is injective. In particular, $\dim \mathcal{H}^1 \leq n$.

If the equality holds, we obviously have n linearly independent parallel fields $E_i, i = 1, \dots, n$. This shows M is flat. Thus the universal covering of (M, g) is $(\mathbb{R}^n, g_{\text{can}})$ with $\Gamma = \pi_1(M)$ acting by isometries. Now pull E_i back to \tilde{E}_i to \mathbb{R}^n , these vector fields are again parallel and are therefore constant vector field. This means we can see them as usual Cartesian coordinate vector field $\frac{\partial}{\partial x^i}$. In addition, they are invariant under the action of Γ . Thus Γ consists of translations, which implies Γ is finitely generated, abelian and torsion-free, thus $\Gamma = \mathbb{Z}^q$ for some q . We must have $q = n$, otherwise $\mathbb{R}^n / \mathbb{Z}^q$ is not compact. \square

Part 4. Minimal length curve problem

10. PULLBACK CONNECTION

10.1. Pullback and pushforward.

Definition 10.1.1 (pullback vector bundle). Let $f: M \rightarrow N$ be a smooth map between manifolds, E a vector bundle over N . The pullback vector bundle f^*E over M is defined by the set

$$\widehat{E} = f^*E := \{(p, v) \in M \times E \mid f(p) = \pi(v)\}$$

endowed with subspace topology.

Remark 10.1.1 (local form). A local frame of \widehat{E} can be written as

$$\widehat{e}_\alpha(x) := f^*e_\alpha(x) = e_\alpha \circ f(x)$$

where $x \in M$ and $\{e_\alpha\}$ is a local frame of E .

Let $f: M \rightarrow N$ be a smooth map between manifolds, and $df: TM \rightarrow TN$ its differential. There is another viewpoint to see it, consider

$$\begin{aligned} df: TM &\rightarrow f^*TN \subseteq M \times TN \\ X_p &\mapsto (p, df_p(X_p)) \end{aligned}$$

that is one can regard df as a section of $T^*M \otimes f^*TN$.

Definition 10.1.2 (pushforward). For vector field X over M , the pushforward of X is defined as $f_*X = df \circ X \in C^\infty(M, f^*TN)$.

Remark 10.1.2. In a word, pushforward of a vector field is no longer a vector field, but a section of pullback bundle.

Remark 10.1.3 (local form). Let $\{x^i\}$ and $\{y^m\}$ be local coordinates of M, N respectively, one has

$$f_*\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial f^m}{\partial x^i} f^*\left(\frac{\partial}{\partial y^m}\right)$$

10.2. Pullback connection. Let $f: M \rightarrow N$ be a smooth map between manifolds, E a vector bundle over N with connection ∇ . Now we want to give a connection $\widehat{\nabla}$ on pullback bundle \widehat{E} induced by ∇ . Let $\{e_\alpha\}$ be a local frame of E . Then $\{\widehat{e}_\alpha := f^*e_\alpha\}$ is a local frame of \widehat{E} . Now we define

$$\widehat{\nabla}(\widehat{e}_\alpha) := f^*(\nabla e_\alpha)$$

and

$$(10.1) \quad \widehat{\nabla}(f\widehat{s}) := df \otimes \widehat{s} + f\widehat{\nabla}\widehat{s}$$

where $\widehat{s} = f^*s$, and s is a section of E . Suppose ∇ is given by Christoffel symbol $\Gamma_{m\alpha}^\beta$. Then by definition

$$(10.2) \quad \widehat{\nabla}(\widehat{e}_\alpha) = f^*(\Gamma_{m\alpha}^\beta dy^m \otimes e_\beta) = \frac{\partial f^m}{\partial x^i} \Gamma_{m\alpha}^\beta(f) \cdot dx^i \otimes \widehat{e}_\beta$$

Then here comes a natural question, we need to check our definition is independent of the choice of local frame, that is to check if $\{\tilde{e}'_\beta\}$ is another local frame with $\hat{e}_\alpha = g_\alpha^\beta \tilde{e}'_\beta$. Then

$$\hat{\nabla}(\hat{e}_\alpha) = \hat{\nabla}(g_\alpha^\beta \tilde{e}'_\beta)$$

It's straightforward computation, since the left hand is computed by (10.2), and right hand can be computed by (10.1). Thus we obtain a linear operator

$$\hat{\nabla}: C^\infty(M, \hat{E}) \rightarrow C^\infty(M, T^*M \otimes \hat{E})$$

Now it remains to show it's an affine connection, that is to check for any smooth function $f \in C^\infty(M)$ and $s \in C^\infty(M, \hat{E})$, one has

$$\hat{\nabla}(fs) = df \otimes s + f\hat{\nabla}s$$

Note that it doesn't follow from (10.1), since here locally if we write $s = s^\alpha \hat{e}_\alpha$, and by definition we only obtain

$$\hat{\nabla}(fs) = \hat{\nabla}(fs^\alpha \hat{e}_\alpha) = d(fs^\alpha) \otimes \hat{e}_\alpha + fs^\alpha \hat{\nabla}\hat{e}_\alpha$$

and it's necessary to do a stepforward computation to show above equation equals to the following one

$$df \otimes s^\alpha \hat{e}_\alpha + f\hat{\nabla}(s^\alpha \hat{e}_\alpha)$$

Definition 10.2.1 (pullback metric). Let g be a metric on E , the pullback metric on f^*E is $\hat{g} = f^*g$.

Remark 10.2.1 (local form). On local frames one has

$$\begin{aligned} \hat{g}_{\alpha\beta} \hat{e}^\alpha \otimes \hat{e}^\beta &:= f^*(g_{\alpha\beta} e^\alpha \otimes e^\beta) \\ &= g_{\alpha\beta}(f) \cdot \hat{e}^\alpha \otimes \hat{e}^\beta \end{aligned}$$

that is $\hat{g}_{\alpha\beta} = g_{\alpha\beta}(f)$.

Proposition 10.2.1. If connection ∇ is compatible with metric g . Then pullback connection $\hat{\nabla}$ is compatible with \hat{g} , that is for any vector field X of M and section s, t of \hat{E} , we have

$$X\hat{g}(s, t) = \hat{g}(\hat{\nabla}_X s, t) + \hat{g}(s, \hat{\nabla}_X t)$$

Proof. It suffices to check on local frames, consider $X = \frac{\partial}{\partial x^i}$, $s = \hat{e}_\alpha$, $t = \hat{e}_\beta$. Then

$$\begin{aligned} \frac{\partial}{\partial x^i} \hat{g}_{\alpha\beta} &= \frac{\partial}{\partial x^i} g_{\alpha\beta}(f) \\ &= \frac{\partial f^m}{\partial x^i} \frac{\partial}{\partial y^m} g_{\alpha\beta}(f) \\ &\stackrel{(1)}{=} \frac{\partial f^m}{\partial x^i} (\Gamma_{m\alpha}^\gamma(f) \cdot g_{\gamma\beta}(f) + \Gamma_{m\beta}^\gamma(f) \cdot g_{\alpha\gamma}(f)) \end{aligned}$$

$$\begin{aligned}\widehat{g}(\widehat{\nabla}_{\frac{\partial}{\partial x^i}} \widehat{e}_\alpha, \widehat{e}_\beta) &= \Gamma_{m\alpha}^\gamma(f) \cdot \frac{\partial f^m}{\partial x^i} \cdot g_{\gamma\beta}(f) \\ \widehat{g}(\widehat{e}_\alpha, \widehat{\nabla}_{\frac{\partial}{\partial x^i}} \widehat{e}_\beta) &= \Gamma_{m\beta}^\gamma(f) \cdot \frac{\partial f^m}{\partial x^i} \cdot g_{\alpha\gamma}(f)\end{aligned}$$

where (1) holds from ∇ is compatible with g . \square

10.3. Pullback curvature.

Definition 10.3.1 (pullback curvature). Let $f: M \rightarrow N$ be a smooth map between manifolds, E a vector bundle over N with connection ∇ . The curvature tensor \widehat{R} of pullback connection $\widehat{\nabla}$ on vector bundle $\widehat{E} \rightarrow M$ is given by

$$\widehat{R}(X, Y, s, t) = \widehat{g}(\widehat{\nabla}_X \widehat{\nabla}_Y s - \widehat{\nabla}_Y \widehat{\nabla}_X s, t)$$

where X, Y are vector fields on M and s, t are sections of \widehat{E} .

Remark 10.3.1 (local form).

$$\widehat{R}_{ij\alpha\beta} = R_{mn\alpha\beta} \frac{\partial f^m}{\partial x^i} \frac{\partial f^n}{\partial x^j}$$

where $R_{mn\alpha\beta}$ is curvature of ∇ .

Proof.

$$\begin{aligned}\widehat{R}_{ij\alpha\beta} &= \widehat{R}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \widehat{e}_\alpha, \widehat{e}_\beta\right) \\ &= \widehat{g}\left(\widehat{R}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \widehat{e}_\alpha, \widehat{e}_\beta\right) \\ &= \widehat{g}\left(\widehat{\nabla}_{\frac{\partial}{\partial x^i}} \widehat{\nabla}_{\frac{\partial}{\partial x^j}} \widehat{e}_\alpha - \widehat{\nabla}_{\frac{\partial}{\partial x^j}} \widehat{\nabla}_{\frac{\partial}{\partial x^i}} \widehat{e}_\alpha, \widehat{e}_\beta\right)\end{aligned}$$

The first term can be computed as follows

$$\begin{aligned}\widehat{\nabla}_{\frac{\partial}{\partial x^i}} \widehat{\nabla}_{\frac{\partial}{\partial x^j}} \widehat{e}_\alpha &= \widehat{\nabla}_{\frac{\partial}{\partial x^i}} \left(\Gamma_{m\alpha}^\gamma(f) \cdot \frac{\partial f^m}{\partial x^j} \widehat{e}_\gamma \right) \\ &= \frac{\partial}{\partial x^i} \left(\Gamma_{m\alpha}^\gamma(f) \cdot \frac{\partial f^m}{\partial x^j} \right) \widehat{e}_\gamma + \Gamma_{m\alpha}^\gamma(f) \cdot \frac{\partial f^m}{\partial x^j} \widehat{\nabla}_{\frac{\partial}{\partial x^i}} \widehat{e}_\gamma \\ &= \left(\frac{\partial \Gamma_{m\alpha}^\gamma}{\partial y^n} \frac{\partial f^n}{\partial x^i} \frac{\partial f^m}{\partial x^j} + \Gamma_{m\alpha}^\gamma(f) \cdot \frac{\partial^2 f^m}{\partial x^i \partial x^j} \right) \widehat{e}_\gamma + \frac{\partial f^m}{\partial x^j} \frac{\partial f^n}{\partial x^i} \Gamma_{m\alpha}^\gamma \Gamma_{n\gamma}^\delta \widehat{e}_\delta \\ &= \frac{\partial f^m}{\partial x^j} \frac{\partial f^n}{\partial x^i} \left(\frac{\partial \Gamma_{m\alpha}^\gamma}{\partial y^n} + \Gamma_{m\alpha}^\delta \Gamma_{n\delta}^\gamma \right) \widehat{e}_\gamma + \Gamma_{m\alpha}^\gamma \frac{\partial^2 f^m}{\partial x^i \partial x^j} \widehat{e}_\gamma\end{aligned}$$

Thus

$$\widehat{\nabla}_{\frac{\partial}{\partial x^i}} \widehat{\nabla}_{\frac{\partial}{\partial x^j}} \widehat{e}_\alpha - \widehat{\nabla}_{\frac{\partial}{\partial x^j}} \widehat{\nabla}_{\frac{\partial}{\partial x^i}} \widehat{e}_\alpha = \frac{\partial f^m}{\partial x^j} \frac{\partial f^n}{\partial x^i} R_{mn\alpha}^\gamma \widehat{e}_\gamma$$

that is

$$\begin{aligned}\widehat{R}_{ij\alpha\beta} &= \widehat{g}\left(\frac{\partial f^m}{\partial x^j} \frac{\partial f^n}{\partial x^i} R_{mn\alpha}^\gamma \widehat{e}_\gamma, \widehat{e}_\beta\right) \\ &= \frac{\partial f^m}{\partial x^j} \frac{\partial f^n}{\partial x^i} R_{mn\alpha}^\gamma g_{\gamma\beta} \\ &= \frac{\partial f^m}{\partial x^j} \frac{\partial f^n}{\partial x^i} R_{mn\alpha\beta}\end{aligned}$$

□

10.4. Parallel transport. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ , $\gamma: I \rightarrow M$ a smooth curve, E a vector bundle over M and γ^*E endowed with pullback connection $\widehat{\nabla}$.

Definition 10.4.1 (parallel). Let s be a section of γ^*E , it's called parallel along γ , if $\widehat{\nabla} \frac{d}{dt} s = 0$.

From local form we can see $\widehat{\nabla} \frac{d}{dt} s = 0$ is a system of ODEs locally, which can always be solved uniquely in a sufficiently short interval if an initial value is given, that's how we define parallel transport.

Definition 10.4.2 (parallel transport). For $t_0, t \in I$, parallel transport $P_{t_0, t; \gamma}$ is an isomorphism⁴ between vector spaces defined by

$$\begin{aligned} P_{t_0, t; \gamma}: E_{\gamma(t_0)} &\rightarrow E_{\gamma(t)} \\ s_0 &\mapsto s(t) \end{aligned}$$

where s is the unique parallel section along γ satisfying $s(t_0) = s_0$.

Definition 10.4.3 (parallel orthonormal frame along curve). Suppose $\{e_\alpha\}$ is an orthonormal basis of $E_{\gamma(t_0)}$, then there is a local frame $\{e_\alpha(t)\}$ of $E_{\gamma(t)}$ along γ obtained by parallel transport, such that $e_\alpha(0) = e_\alpha$.

Proposition 10.4.1. A connection ∇ is compatible with metric if and only if for arbitrary curve $\gamma: I \rightarrow M$ and two parallel sections s_1, s_2 along γ we have $g(s_1, s_2)$ is constant.

Proof. It's clear if ∇ is compatible with metric g , then for any two sections s, t which are parallel along γ , one has

$$dg(s_1, s_2) = g(\nabla s_1, s_2) + g(s_1, \nabla s_2) = 0$$

which implies $g(s, t)$ is constant. Conversely, suppose $\{e_\alpha(t)\}$ is a parallel orthonormal frame with respect to g along γ and write

$$s_1(t) = s_1^\alpha(t) e_\alpha(t)$$

$$s_2(t) = s_2^\alpha(t) e_\alpha(t)$$

Direct computation shows

$$\begin{aligned} g(\nabla s_1, s_2) + g(s_1, \nabla s_2) &= \sum_\alpha \frac{ds_1^\alpha}{dt} s_2^\alpha + s_1^\alpha \frac{ds_2^\alpha}{dt} \\ &= \frac{d}{dt} \left(\sum_\alpha s_1^\alpha s_2^\alpha \right) \\ &= \frac{d}{dt} g(s_1, s_2) \end{aligned}$$

□

⁴Its inverse is $P_{t, t_0; \gamma}$.

Proposition 10.4.2. Let (M, g) be a Riemannian manifold, $\gamma: I \rightarrow M$ a smooth curve and $P_{s,t;\gamma}: T_{\gamma(s)}M \rightarrow T_{\gamma(t)}M$ is the parallel transport along γ . For any $s \in I$ with $v = \gamma'(s)$, one has

$$\nabla_v R = \left. \frac{d}{dt} \right|_{t=s} (P_{s,t;\gamma})^* R_{\gamma(t)}$$

In particular, if $\nabla R = 0$ then

$$(P_{s,t;\gamma})^* R_{\gamma(t)} = R_{\gamma(s)}$$

holds for arbitrary $t, s \in I$.

Proof. Let $v_1 = v$ and choose $v_2, \dots, v_5 \in T_{\gamma(s)}M$. For each $1 \leq i \leq 5$, we define vector fields along $\gamma(t)$ by $X_i(t) = P_{s,t}^{\gamma}(v_i)$. In particular, one has $X_1(t) = \gamma'(t)$. Direct computation shows

$$\begin{aligned} (\nabla R)_p(v_1, \dots, v_5) &= \lim_{t \rightarrow s} \nabla R(X_1, \dots, X_5) \\ &= \lim_{t \rightarrow s} X_1 R(X_2, \dots, X_5) - \sum_{i=2}^5 R(X_2, \dots, \widehat{\nabla}_{\frac{d}{dt}} X_i, \dots, X_5) \\ &\stackrel{(1)}{=} \lim_{t \rightarrow s} X_1 R(X_2, \dots, X_5) \\ &= \left. \frac{d}{dt} \right|_{t=s} R_{\gamma(t)}(X_2(t), \dots, X_5(t)) \\ &= \left. \frac{d}{dt} \right|_{t=s} (P_{s,t;\gamma})^* R_{\gamma(t)}(v_2, \dots, v_5) \end{aligned}$$

where (1) holds from $X_i(t)$ are parallel vector fields along $\gamma(t)$. \square

10.5. Second fundamental form. In this section, we fix the following notations:

- (1) Let $(M, g), (N, g')$ be Riemannian manifolds with Levi-Civita connection ∇ and ∇' respectively.
- (2) $f: M \rightarrow N$ is a smooth map between manifolds.
- (3) Γ_{ij}^k is used to denote Christoffel symbol of ∇ and Γ_{mn}^l is used to denote Christoffel symbol of ∇' .
- (4) $\widehat{\nabla}$ is the connection on f^*TN induced by ∇' .

Definition 10.5.1 (second fundamental form). The second fundamental form $B \in C^\infty(M, T^*M \otimes T^*M \otimes f^*TN)$ of f is defined as

$$B(X, Y) := \widehat{\nabla}_X(f_*Y) - f_*(\nabla_X Y) \in C^\infty(M, f^*TN)$$

where $X, Y \in C^\infty(M, TM)$.

Remark 10.5.1 (local form). Note that

$$\begin{aligned} f_*(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}) &= \Gamma_{ij}^k f_*(\frac{\partial}{\partial x^k}) = \Gamma_{ij}^k \frac{\partial f^m}{\partial x^k} f^*(\frac{\partial}{\partial y^m}) \\ \widehat{\nabla}_{\frac{\partial}{\partial x^i}} (\frac{\partial f^m}{\partial x^j} f^*(\frac{\partial}{\partial y^m})) &= \frac{\partial^2 f^m}{\partial x^i \partial x^j} f^*(\frac{\partial}{\partial y^m}) + \frac{\partial f^m}{\partial x^j} \widehat{\nabla}_{\frac{\partial}{\partial x^i}} f^*(\frac{\partial}{\partial y^m}) \\ &= (\frac{\partial^2 f^l}{\partial x^i \partial x^j} + \frac{\partial f^n}{\partial x^i} \frac{\partial f^m}{\partial x^j} \Gamma_{nm}^l(f)) f^*(\frac{\partial}{\partial y^l}) \end{aligned}$$

Therefore

$$\begin{aligned} B_{ij} &:= B(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \\ &= (\frac{\partial^2 f^l}{\partial x^i \partial x^j} + \frac{\partial f^n}{\partial x^i} \frac{\partial f^m}{\partial x^j} \Gamma_{mn}^l(f) - \Gamma_{ij}^k \frac{\partial f^l}{\partial x^k}) f^*(\frac{\partial}{\partial y^l}) \end{aligned}$$

that is

$$B = (\frac{\partial^2 f^l}{\partial x^i \partial x^j} + \frac{\partial f^n}{\partial x^i} \frac{\partial f^m}{\partial x^j} \Gamma_{mn}^l(f) - \Gamma_{ij}^k \frac{\partial f^l}{\partial x^k}) dx^i \otimes dx^j \otimes f^*(\frac{\partial}{\partial y^l})$$

Proposition 10.5.1. The second fundamental form B is symmetric.

Proof. It's clear from local expression. \square

Corollary 10.5.1 (torsion-free). For $X, Y \in C^\infty(M, TM)$, one has

$$\widehat{\nabla}_X(f_*Y) - \widehat{\nabla}_Y(f_*X) = f_*(\nabla_X Y - \nabla_Y X) = f_*([X, Y])$$

Proof. The first equality holds from the symmetry of second fundamental form and the second equality holds since ∇ is torsion-free. \square

Example 10.5.1 (geodesic). A smooth curve $\gamma: I \rightarrow M$ can be regarded as $\gamma: (I, g_{\text{can}}) \rightarrow (M, g)$, thus second fundamental form in this case is

$$B = (\frac{d^2 \gamma^k}{dt^2} + \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} \Gamma_{ij}^k \circ \gamma) dt \otimes dt \otimes \gamma^*(\frac{\partial}{\partial x^k})$$

since canonical metrics on I has vanishing Christoffel symbol. In this viewpoint, a smooth curve is a geodesic, if it has vanishing second fundamental form as smooth maps between Riemannian manifolds.

Example 10.5.2 (Hessian). A smooth function f can be regarded as $f: (M, g) \rightarrow (\mathbb{R}, g_{\text{can}})$, thus second fundamental form in this case is

$$B = (\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k}) dx^i \otimes dx^j \otimes f^*(\frac{\partial}{\partial y})$$

since canonical metrics on \mathbb{R} has vanishing Christoffel symbol. Recall Hessian of f is

$$\text{Hess } f = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k} dx^i \otimes dx^j$$

So second fundamental form generalizes Hessian of smooth function.

Proposition 10.5.2. Let B be fundamental form of f . Then

$$B = \tilde{\nabla} df$$

where $\tilde{\nabla}$ is the connection on $T^*M \otimes f^*TN$ induced by ∇ together with pullback connection $\hat{\nabla}$ on f^*TN .

Proof. Suppose df is locally written as

$$df = \frac{\partial f^m}{\partial x^i} dx^i \otimes f^*\left(\frac{\partial}{\partial y^m}\right)$$

Then direct computation shows

$$\begin{aligned} \tilde{\nabla} df &= \tilde{\nabla} \left(\frac{\partial f^m}{\partial x^i} dx^i \otimes f^*\left(\frac{\partial}{\partial y^m}\right) \right) \\ &= \frac{\partial^2 f^m}{\partial x^j \partial x^i} dx^j \otimes dx^i \otimes f^*\left(\frac{\partial}{\partial y^m}\right) - \frac{\partial f^m}{\partial x^i} \Gamma_{jk}^i dx^j \otimes dx^k \otimes f^*\left(\frac{\partial}{\partial y^m}\right) \\ &\quad + \frac{\partial f^m}{\partial x^i} \frac{\partial f^n}{\partial x^j} \Gamma_{mn}^l(f) \cdot dx^i \otimes dx^j \otimes f^*\left(\frac{\partial}{\partial y^l}\right) \\ &= \left(\frac{\partial^2 f^l}{\partial x^i \partial x^j} - \frac{\partial f^l}{\partial x^k} \Gamma_{ij}^k + \frac{\partial f^m}{\partial x^i} \frac{\partial f^n}{\partial x^j} \Gamma_{mn}^l(f) \right) dx^i \otimes dx^j \otimes f^*\left(\frac{\partial}{\partial y^l}\right) \\ &= B \end{aligned}$$

This completes the proof. \square

11. VARIATION FORMULAS OF CURVES

In this section and Section 12, we fix the following notations:

- (1) $I = [a, b] \subseteq \mathbb{R}$ is a closed interval.
- (2) (M, g) is a Riemannian manifold equipped with Levi-Civita connection ∇ .
- (3) For two different points $p, q \in M$, the space of piecewise smooth curves from p to q is denoted as $\mathcal{L}_{p,q}$.
- (4) For $\gamma \in \mathcal{L}_{p,q}$, $\gamma'(t)$ denotes $\gamma_*(\frac{d}{dt})$, which is a piecewise smooth vector field along γ .
- (5) The arc-length functional and energy functional on $\mathcal{L}_{p,q}$ are defined as follows

$$L(\gamma) = \int_a^b |\gamma'(t)|_{\hat{g}} dt$$

$$E(\gamma) = \frac{1}{2} \int_a^b |\gamma'(t)|_{\hat{g}}^2 dt$$

where \hat{g} is pullback metric on γ^*TM .

11.1. First variation formula.

Definition 11.1.1 (variation). Given $\gamma \in \mathcal{L}_{p,q}$, a variation (fixing endpoints) of γ is a map

$$\alpha: [a, b] \times (-\varepsilon, \varepsilon) \rightarrow M$$

such that

- (1) $\alpha(\cdot, s) \in \mathcal{L}_{p,q}$ for any $s \in (-\varepsilon, \varepsilon)$.
- (2) There is a subdivision $a = t_0 < t_1 < \dots < t_k = b$ of I such that α is smooth on each strip $(t_{i-1}, t_i] \times (-\varepsilon, \varepsilon)$ for $i = 1, \dots, k$.
- (3) $\alpha(t, 0) = \gamma(t)$ for any $t \in [a, b]$.

Remark 11.1.1. In general, we can consider variations of γ without fixing endpoints, but in this section we only consider variations fixing endpoints.

Notation 11.1.1. For pullback bundle α^*TM , $\bar{\nabla}$ and \bar{g} denote connection and metric pulled back from the ones on TM respectively. By definition the restriction of $\bar{\nabla}$ on γ^*TM is exactly $\hat{\nabla}$, and the restriction of \bar{g} on γ^*TM is \hat{g} .

Definition 11.1.2 (variation vector field). For a variation α of $\gamma \in \mathcal{L}_{p,q}$, $\alpha_*(\frac{\partial}{\partial s})|_{s=0}$ is called variation vector field of variation α , which is a piecewise smooth vector field along γ .

Remark 11.1.2. Note that for a variation

$$\begin{cases} \alpha(a, s) = p \\ \alpha(b, s) = q \end{cases}$$

holds for any $s \in (-\varepsilon, \varepsilon)$. Thus we have

$$\begin{cases} \alpha_*\left(\frac{\partial}{\partial s}\right)(a, s) = 0 \\ \alpha_*\left(\frac{\partial}{\partial s}\right)(b, s) = 0 \end{cases}$$

holds for any $s \in (-\varepsilon, \varepsilon)$. In particular, it holds for $s = 0$. In other words, variation vector field vanishes at endpoints.

Lemma 11.1.1. Let $\gamma \in \mathcal{L}_{p,q}$ and X a piecewise smooth vector field along γ which vanishes at endpoints. Then there exists a variation α of γ such that the variation vector field is exactly X , that is

$$\alpha_*\left(\frac{\partial}{\partial s}\right)\Big|_{s=0} = X$$

Proof. See Proposition 2.2 in Page193 of [Car92]. \square

Theorem 11.1.1 (first variation formula of smooth version). Let $\gamma: I \rightarrow (M, g)$ be a unit-speed smooth curve, α a variation of γ with variation vector fields V . Then

$$\begin{aligned} \frac{d}{ds}\Big|_{s=0} L(\alpha(-, s)) &\stackrel{(1)}{=} \frac{d}{ds}\Big|_{s=0} E(\alpha(-, s)) \stackrel{(2)}{=} \int_a^b \langle \widehat{\nabla}_{\frac{d}{dt}} V, \gamma'(t) \rangle dt \\ &\stackrel{(3)}{=} - \int_a^b \langle V, \widehat{\nabla}_{\frac{d}{dt}} \gamma'(t) \rangle dt \end{aligned}$$

Proof. Note that

$$\frac{d}{ds}\Big|_{s=0} L(\alpha(-, s)) = \int_a^b \frac{1}{2|\gamma'(t)|} \frac{\partial}{\partial s}\Big|_{s=0} |\alpha_*\left(\frac{\partial}{\partial t}\right)|^2 dt = \frac{1}{|\gamma'(t)|} \frac{d}{ds}\Big|_{s=0} E(\alpha(-, s))$$

since γ is unit-speed, This show equation marked by (1). Since $\gamma(t)$ is smooth, integration by parts shows

$$0 = \int_a^b \frac{d}{dt} \langle V, \gamma'(t) \rangle dt = \int_a^b \langle \widehat{\nabla}_{\frac{d}{dt}} V, \gamma'(t) \rangle dt + \int_a^b \langle V, \widehat{\nabla}_{\frac{d}{dt}} \gamma'(t) \rangle dt$$

This shows equation marked by (3). For equation marked by (2), direct computation shows

$$\begin{aligned} \frac{d}{ds} E(\alpha(-, s)) &= \frac{d}{ds} \frac{1}{2} \int_a^b |\alpha_*\left(\frac{\partial}{\partial t}\right)|^2 dt \\ &= \frac{1}{2} \int_a^b \frac{\partial}{\partial s} |\alpha_*\left(\frac{\partial}{\partial t}\right)|^2 dt \\ &= \frac{1}{2} \int_a^b 2 \langle \overline{\nabla}_{\frac{\partial}{\partial s}} \alpha_*\left(\frac{\partial}{\partial t}\right), \alpha_*\left(\frac{\partial}{\partial t}\right) \rangle_{\overline{g}} dt \\ &\stackrel{(4)}{=} \int_a^b \langle \overline{\nabla}_{\frac{\partial}{\partial t}} \alpha_*\left(\frac{\partial}{\partial s}\right), \alpha_*\left(\frac{\partial}{\partial t}\right) \rangle_{\overline{g}} dt \end{aligned}$$

The hallmark of above computation is the equality marked by (4), which can be seen from Corollary 10.5.1. Thus

$$\left. \frac{d}{ds} \right|_{s=0} E(\alpha(-, s)) = \int_a^b \langle \widehat{\nabla}_{\frac{d}{dt}} V, \gamma'(t) \rangle dt$$

since $\alpha_*(\frac{\partial}{\partial s})|_{s=0} = V$ and $\alpha_*(\frac{\partial}{\partial t})|_{s=0} = \gamma'(t)$. \square

Corollary 11.1.1 (first variation formula of piecewise smooth version). Let $\gamma: I \rightarrow (M, g)$ be a unit-speed piecewise smooth curve with breakpoints $a = t_0 < t_1 < \dots < t_k = b$, α a variation of γ with the variation vector field V . Then

$$\begin{aligned} \left. \frac{d}{ds} \right|_{s=0} L(\alpha(-, s)) &= \left. \frac{d}{ds} \right|_{s=0} E(\alpha(-, s)) = \int_a^b \langle \widehat{\nabla}_{\frac{d}{dt}} V, \gamma'(t) \rangle dt \\ &= - \int_a^b \langle V, \widehat{\nabla}_{\frac{d}{dt}} \gamma'(t) \rangle dt - \sum_{i=1}^{k-1} \langle V_{t_i}, \Delta_{t_i} \gamma' \rangle \end{aligned}$$

where $\Delta_t \gamma' = \gamma'(t_+) - \gamma'(t_-)$.

Proof. Note that $\gamma(t)$ is smooth on each $(t_{i-1}, t_i]$. Then by Theorem 11.1.1 one has

$$\left. \frac{d}{ds} \right|_{s=0} L(\alpha(-, s)) = \int_{t_{i-1}}^{t_i} \langle \widehat{\nabla}_{\frac{d}{dt}} V, \gamma'(t) \rangle dt$$

and integration by parts shows

$$\langle V, \gamma'(t) \rangle \Big|_{t_{i-1}}^{t_i} = \int_{t_{i-1}}^{t_i} \frac{d}{dt} \langle V, \gamma'(t) \rangle dt = \int_{t_{i-1}}^{t_i} \langle \widehat{\nabla}_{\frac{d}{dt}} V, \gamma'(t) \rangle dt + \int_{t_{i-1}}^{t_i} \langle V, \widehat{\nabla}_{\frac{d}{dt}} \gamma'(t) \rangle dt$$

Then we add these equations together to obtain desired equation. \square

Remark 11.1.3. Suppose α is a variation of γ without fixing endpoints with variation vector field V , from the proof of equality marked by (3), it's clear to see its first variation formula is

$$\left. \frac{d}{dt} \right|_{s=0} L(\alpha(-, s)) = \langle V(b), \gamma'(b) \rangle - \langle V(a), \gamma'(a) \rangle - \int_a^b \langle V, \widehat{\nabla}_{\frac{d}{dt}} \gamma'(t) \rangle dt - \sum_{i=1}^{k-1} \langle V_{t_i}, \Delta_{t_i} \gamma' \rangle$$

Corollary 11.1.2. Given $\gamma \in \mathcal{L}_{p,q}$. The following statements are equivalent.

- (1) γ is a critical point of energy functional $E: \mathcal{L}_{p,q} \rightarrow \mathbb{R}$.
- (2) γ has constant speed $|\gamma'(t)| = c > 0$ and γ is a critical point of arc-length functional $L: \mathcal{L}_{p,q} \rightarrow \mathbb{R}$.
- (3) γ is a geodesic. In particular, it's smooth.

Proof. From (3) to (2). Firstly a geodesic must have constant speed c , and $c > 0$ since p, q are distinct points. It's also a critical point of L since first variation formula implies

$$\left. \frac{d}{ds} \right|_{s=0} L(\alpha(-, s)) = - \int_a^b \langle V, \widehat{\nabla}_{\frac{d}{dt}} \gamma'(t) \rangle dt = 0$$

Note that there is only one term in first variation formula, since geodesic is a smooth curve.

From (2) to (1). It's clear, since from above proof we have already seen for constant speed curve, the first variation of arc-length functional and energy functional only differs a scalar.

From (1) to (3). In order to show $\widehat{\nabla}_{\frac{d}{dt}} \gamma'(t) = 0$ and it's smooth, it suffices to choose appropriate variation vector fields to conclude. \square

11.2. Second variation formula. We already know a geodesic γ is a critical point for energy functional or arc-length functional, so it suffices to compute second variation of geodesics to determine whether it's local minimum or not. To see this, we need to consider the following 2-dimensional variation

$$\alpha: [a, b] \times (-\varepsilon_1, \varepsilon_1) \times (-\varepsilon_2, \varepsilon_2)$$

such that

- (1) $\alpha(t, 0, 0) = \gamma(t)$
- (2) $\alpha(-, s_1, s_2)$ is a smooth curve connecting p and q .

11.2.1. *Second variation formula for energy.*

Theorem 11.2.1 (second variation formula for energy). Let $\gamma: [a, b] \rightarrow (M, g)$ be a smooth curve. If α is a 2-dimensional variation of γ with variation fields V, W . Then

$$\begin{aligned} \frac{\partial^2}{\partial s_1 \partial s_2} \Big|_{s_1=s_2=0} E(\alpha(-, s_1, s_2)) &= \int_a^b \langle \widehat{\nabla}_{\frac{d}{dt}} V, \widehat{\nabla}_{\frac{d}{dt}} W \rangle dt \\ &\quad - \int_a^b R(V, \gamma', \gamma', W) dt - \int_a^b \left\langle \overline{\nabla}_{\frac{\partial}{\partial s_1}} \alpha_* \left(\frac{\partial}{\partial s_2} \right) \Big|_{s_1=s_2=0}, \widehat{\nabla}_{\frac{d}{dt}} \gamma'(t) \right\rangle dt \end{aligned}$$

Proof. By first variation formula we have

$$\frac{\partial}{\partial s_2} E(\alpha(-, s_1, s_2)) = - \int_a^b \langle \alpha_* \left(\frac{\partial}{\partial s_2} \right), \overline{\nabla}_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial t} \right) \rangle_{\overline{g}} dt$$

Thus

$$\frac{\partial^2}{\partial s_1 \partial s_2} E(\alpha(-, s_1, s_2)) = - \underbrace{\int_a^b \langle \overline{\nabla}_{\frac{\partial}{\partial s_1}} \alpha_* \left(\frac{\partial}{\partial s_2} \right), \overline{\nabla}_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial t} \right) \rangle_{\overline{g}} dt}_{\text{part I}} - \underbrace{\int_a^b \langle \alpha_* \left(\frac{\partial}{\partial s_2} \right), \overline{\nabla}_{\frac{\partial}{\partial s_1}} \overline{\nabla}_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial t} \right) \rangle_{\overline{g}} dt}_{\text{part II}}$$

For part II, we have

$$\overline{\nabla}_{\frac{\partial}{\partial s_1}} \overline{\nabla}_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial t} \right) = R(\alpha_* \left(\frac{\partial}{\partial s_1} \right), \alpha_* \left(\frac{\partial}{\partial t} \right)) \alpha_* \left(\frac{\partial}{\partial t} \right) + \overline{\nabla}_{\frac{\partial}{\partial t}} \overline{\nabla}_{\frac{\partial}{\partial s_1}} \alpha_* \left(\frac{\partial}{\partial t} \right)$$

Thus we can write part II as

$$- \int_a^b \langle \alpha_* \left(\frac{\partial}{\partial s_2} \right), R \left(\frac{\partial}{\partial s_1}, \frac{\partial}{\partial t} \right) \alpha_* \left(\frac{\partial}{\partial t} \right) \rangle_{\overline{g}} dt - \underbrace{\int_a^b \langle \alpha_* \left(\frac{\partial}{\partial s_2} \right), \overline{\nabla}_{\frac{\partial}{\partial t}} \overline{\nabla}_{\frac{\partial}{\partial s_1}} \alpha_* \left(\frac{\partial}{\partial t} \right) \rangle_{\overline{g}} dt}_{\text{part III}}$$

For part III, we have

$$\begin{aligned} - \int_a^b \langle \alpha_* \left(\frac{\partial}{\partial s_2} \right), \bar{\nabla}_{\frac{\partial}{\partial t}} \bar{\nabla}_{\frac{\partial}{\partial s_1}} \alpha_* \left(\frac{\partial}{\partial t} \right) \rangle_{\bar{g}} dt &= - \int_a^b \langle \alpha_* \left(\frac{\partial}{\partial s_2} \right), \bar{\nabla}_{\frac{\partial}{\partial t}} \bar{\nabla}_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial s_1} \right) \rangle_{\bar{g}} dt \\ &= \int_a^b \langle \bar{\nabla}_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial s_2} \right), \bar{\nabla}_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial s_1} \right) \rangle_{\bar{g}} dt \end{aligned}$$

Now let's evaluate at $s_1 = s_2 = 0$. Then we have

(1) Part I

$$- \int_a^b \left\langle \bar{\nabla}_{\frac{\partial}{\partial s_1}} \alpha_* \left(\frac{\partial}{\partial s_2} \right) \right|_{s_1=s_2=0}, \hat{\nabla}_{\frac{d}{dt}} \gamma'(t) \rangle dt$$

(2) Part II

$$\int_a^b \langle \hat{\nabla}_{\frac{d}{dt}} V, \hat{\nabla}_{\frac{d}{dt}} W \rangle dt - \int_a^b R(V, \gamma', \gamma', W) dt$$

This completes the proof. \square

Corollary 11.2.1. Let $\gamma: [a, b] \rightarrow (M, g)$ be a geodesic. Then

$$\left. \frac{\partial^2}{\partial s_1 \partial s_2} \right|_{s_1=s_2=0} E(\alpha(-, s_1, s_2)) = \int_a^b \langle \hat{\nabla}_{\frac{d}{dt}} V, \hat{\nabla}_{\frac{d}{dt}} W \rangle dt - \int_a^b R(V, \gamma', \gamma', W) dt$$

11.2.2. *Second variation formula for arc-length.*

Theorem 11.2.2 (second variation formula for arc-length). Let $\gamma: [a, b] \rightarrow (M, g)$ be a unit-speed curve. If α is a 2-dimensional variation of γ with variation fields V, W . Then

$$\begin{aligned} \left. \frac{\partial^2}{\partial s_1 \partial s_2} \right|_{s_1=s_2=0} L(\alpha(-, s_1, s_2)) &= \int_a^b \langle \hat{\nabla}_{\frac{d}{dt}} V, \hat{\nabla}_{\frac{d}{dt}} W \rangle dt - \int_a^b R(V, \gamma', \gamma', W) dt \\ &\quad - \int_a^b \left\langle \bar{\nabla}_{\frac{\partial}{\partial s_1}} \alpha_* \left(\frac{\partial}{\partial s_2} \right) \right|_{s_1=s_2=0}, \hat{\nabla}_{\frac{d}{dt}} \gamma'(t) \rangle dt \\ &\quad - \int_a^b \langle \hat{\nabla}_{\frac{d}{dt}} V, \gamma' \rangle \langle \hat{\nabla}_{\frac{d}{dt}} W, \gamma' \rangle dt \end{aligned}$$

Corollary 11.2.2. Let $\gamma: [a, b] \rightarrow (M, g)$ be a unit-speed geodesic. If α is a 2-dimensional variation of γ with variation fields V, W . Then

$$\begin{aligned} \left. \frac{\partial^2}{\partial s_1 \partial s_2} \right|_{s_1=s_2=0} L(\alpha(-, s_1, s_2)) &= \int_a^b \langle \hat{\nabla}_{\frac{d}{dt}} V, \hat{\nabla}_{\frac{d}{dt}} W \rangle dt - \int_a^b R(V, \gamma', \gamma', W) dt \\ &\quad - \int_a^b \langle \hat{\nabla}_{\frac{d}{dt}} V, \gamma' \rangle \langle \hat{\nabla}_{\frac{d}{dt}} W, \gamma' \rangle dt \\ &= \int_a^b \langle \hat{\nabla}_{\frac{d}{dt}} V^\perp, \hat{\nabla}_{\frac{d}{dt}} W^\perp \rangle dt - \int_a^b R(V^\perp, \gamma', \gamma', W^\perp) dt \end{aligned}$$

where

$$V^\perp = V - \langle V, \gamma' \rangle \gamma', \quad W^\perp = W - \langle W, \gamma' \rangle \gamma'$$

Proof. It suffices to check the second equality. Direct computation shows:

$$\begin{aligned}\langle \widehat{\nabla}_{\frac{d}{dt}} V, \widehat{\nabla}_{\frac{d}{dt}} W \rangle &= \langle \widehat{\nabla}_{\frac{d}{dt}} (V^\perp + \langle V, \gamma' \rangle \gamma'), \widehat{\nabla}_{\frac{d}{dt}} (W^\perp + \langle W, \gamma' \rangle \gamma') \rangle \\ &= \langle \widehat{\nabla}_{\frac{d}{dt}} V^\perp, \widehat{\nabla}_{\frac{d}{dt}} W^\perp \rangle + \langle \widehat{\nabla}_{\frac{d}{dt}} V, \gamma' \rangle \langle \widehat{\nabla}_{\frac{d}{dt}} W, \gamma' \rangle\end{aligned}$$

Thus

$$\langle \widehat{\nabla}_{\frac{d}{dt}} V, \widehat{\nabla}_{\frac{d}{dt}} W \rangle - \langle \widehat{\nabla}_{\frac{d}{dt}} V, \gamma' \rangle \langle \widehat{\nabla}_{\frac{d}{dt}} W, \gamma' \rangle = \langle \widehat{\nabla}_{\frac{d}{dt}} V^\perp, \widehat{\nabla}_{\frac{d}{dt}} W^\perp \rangle$$

since

$$\widehat{\nabla}_{\frac{d}{dt}} (\langle V, \gamma' \rangle \gamma') = \langle \widehat{\nabla}_{\frac{d}{dt}} V, \gamma' \rangle \gamma'$$

and it's clear

$$R(V, \gamma', \gamma', W) = R(V^\perp, \gamma', \gamma', W^\perp)$$

□

So if we want to show a geodesic γ is a (locally) minimal geodesic, it suffices to show for any 2-dimensional variation α with variation vector fields, we have

$$\left. \frac{\partial^2}{\partial s_1 \partial s_2} \right|_{s_1=s_2=0} L(\alpha(-, s_1, s_2)) \geq 0$$

This motivates us to consider the following bilinear form defined on the space of variation vector fields.

Definition 11.2.1 (index form). Let $\gamma: [a, b] \rightarrow (M, g)$ be a unit-speed geodesic. The index form I_γ is defined as

$$I_\gamma(V, W) = \int_a^b \langle \widehat{\nabla}_{\frac{d}{dt}} V, \widehat{\nabla}_{\frac{d}{dt}} W \rangle dt - \int_a^b R(V, \gamma', \gamma', W) dt$$

where V, W are vector fields along γ .

By Corollary 11.2.2, a geodesic γ is locally minimal if and only if index form defined on the space of normal⁵ variation fields are semi-positive definite. In the following section, we will study when the index form defined on the normal variation vector fields along γ is positive definite, semi-positive definite or not.

⁵A vector field V along γ is called normal, if V is perpendicular to γ' .

12. JACOBI FIELD I: AS THE NULL SPACE

12.1. First properties.

Definition 12.1.1 (Jacobi field). A vector field J along geodesic γ is called a Jacobi field, if it satisfies

$$\widehat{\nabla}_{\frac{d}{dt}} \widehat{\nabla}_{\frac{d}{dt}} J + R(J, \gamma')\gamma' = 0$$

Notation 12.1.1. For convenience,

$$\begin{aligned} J' &= \widehat{\nabla}_{\frac{d}{dt}} J \\ J'' &= \widehat{\nabla}_{\frac{d}{dt}} \widehat{\nabla}_{\frac{d}{dt}} J \end{aligned}$$

Remark 12.1.1 (local form). Suppose $\{e_1, \dots, e_n\}$ is a parallel orthonormal frame along γ , and $J(t) = J^i(t)e_i(t)$, the condition for Jacobi fields becomes

$$\frac{d^2 J^k}{dt^2} + \langle J^j R(e_j, \gamma')\gamma', e_k \rangle = 0$$

Thus by standard results in ODEs, a Jacobi field J is completely determined by its initial conditions

$$J(0), J'(0) \in T_{\gamma(0)}M$$

Consequently, the set of Jacobi fields is a vector space with dimension $2n$.

Example 12.1.1. There is always a trivial Jacobi field along geodesic $\gamma: [a, b] \rightarrow M$, that is $J(t) = (at + b)\gamma'(t)$.

Proposition 12.1.1. Let $J(t)$ be a Jacobi field along geodesic γ .

- (1) If $J(t) \not\equiv 0$. Then the set consisting of zeros of $J(t)$ is discrete.
- (2) There exist constants λ, μ such that

$$J(t) = J^\perp(t) + (\lambda t + \mu)\gamma'(t)$$

where $\langle J^\perp(t), \gamma' \rangle \equiv 0$.

- (3) $J(t) \perp \gamma'$ if and only if there exist $t_1 \neq t_2$ such that

$$\langle J(t_1), \gamma'(t_1) \rangle = \langle J(t_2), \gamma'(t_2) \rangle = 0$$

- (4) If there exists t_0 such that

$$\langle J(t_0), \gamma'(t_0) \rangle = \langle J'(t_0), \gamma'(t_0) \rangle = 0$$

then $J(t) \perp \gamma'(t)$.

Proof. For (3). Note that

$$\frac{d^2}{dt^2} \langle J(t), \gamma'(t) \rangle = \langle \widehat{\nabla}_{\frac{d}{dt}} \widehat{\nabla}_{\frac{d}{dt}} J, \gamma' \rangle = \langle R(J, \gamma')\gamma', \gamma' \rangle = 0$$

Thus $\langle J(t), \gamma'(t) \rangle = \lambda t + \mu$. Note that $\langle J(t_1), \gamma'(t_1) \rangle = \langle J(t_2), \gamma'(t_2) \rangle = 0$, which implies $\langle J(t), \gamma'(t) \rangle \equiv 0$. □

Proposition 12.1.2. Let $\gamma: [a, b] \rightarrow (M, g)$ be a geodesic and $\alpha: [a, b] \times (-\varepsilon, \varepsilon) \rightarrow (M, g)$ a variation of γ consisting of geodesics. Then

$$J = \alpha_* \left(\frac{\partial}{\partial s} \right) \Big|_{s=0}$$

is a Jacobi field.

Proof. Direct computation shows

$$\begin{aligned} \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial s} \right) &\stackrel{(1)}{=} \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \alpha_* \left(\frac{\partial}{\partial t} \right) \\ &= R \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right) \alpha_* \left(\frac{\partial}{\partial t} \right) + \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \alpha_* \left(\frac{\partial}{\partial t} \right) \\ &\stackrel{(2)}{=} R \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right) \alpha_* \left(\frac{\partial}{\partial t} \right) \end{aligned}$$

where

(1) holds from Corollary 10.5.1.

(2) holds from α is a variation consisting of geodesics.

Set $s = 0$ one has

$$\widehat{\nabla}_{\frac{d}{dt}} \widehat{\nabla}_{\frac{d}{dt}} J = R(\gamma', J)\gamma' = -R(J, \gamma')\gamma'$$

which implies J is a Jacobi field. \square

Remark 12.1.2. In fact, all Jacobi fields can be obtained by above construction.

Corollary 12.1.1. Let (M, g) be a Riemannian manifold, and (x^i, U, p) is a normal coordinate centered at $p \in M$. For each $q \in U \setminus \{p\}$, and $w \in T_q M$, there exists a Jacobi field J along geodesic connecting p, q such that $J(0) = 0$, $J(0)' = w$ and $J(1) = w$.

Proof. For $q \in U \setminus \{p\}$, there exists a unique $v \in T_p M$ such that $\exp_p(v) = q$, and $\gamma(t) = \exp_p(tv)$. Consider the following variation of $\gamma(t)$ consisting of geodesics

$$\alpha(t, s) = \exp_p(t(v + sw))$$

where $w \in T_p M$. Its variation vector field $J(t) = \alpha_* \left(\frac{\partial}{\partial s} \right) \Big|_{s=0}$ is a Jacobi field along γ . In normal coordinate (x^i, U, p) , $\alpha(t, s)$ can be written explicitly as

$$\alpha(t, s) = (t(v^1 + sw^1), \dots, t(v^n + sw^n))$$

where $v = (v^1, \dots, v^n)$, $w = (w^1, \dots, w^n)$. Thus $J(t)$ is given by the formula

$$J(t) = tw^i \frac{\partial}{\partial x^i} \Big|_{\gamma(t)}$$

and it's clear $J(0) = 0$, $J(0)' = w$ and $J(1) = w$. \square

12.2. Conjugate points.

Definition 12.2.1 (conjugate point). Let $p \neq q$ be two endpoints of a geodesic γ . p and q are called conjugate along γ if there exists a non-zero Jacobi field J along γ which vanishes at endpoints.

Notation 12.2.1. The conjugate locus of p is defined as

$$\text{conj}(p) := \{q \in M \mid p \text{ and } q \text{ are conjugate along some geodesic.}\}$$

Proposition 12.2.1. There are at most $n - 1$ linearly independent Jacobi fields $J(t)$ along γ such that $J(a) = J(b) = 0$.

Proof. By Remark 12.1.1, there are at most n linearly independent Jacobi fields such that $J(a) = 0$, and the trivial Jacobi field $J(t) = (t - a)\gamma'(t)$ never vanishes at $t = b$. \square

Theorem 12.2.1. Let (M, g) be a Riemannian manifold, $p \in M$ and $v \in V_p \subseteq T_p M$. Then $(\text{d exp}_p)_v$ is not injective if and only if $q = \exp_p(v)$ is conjugate to p along $\exp_p(tv)$.

Proof. For any $0 \neq w \in T_p M$, if $(\text{d exp}_p)_v(w) = 0$, then $J(t) = (\text{d exp}_p)_{tv}(tw)$ is a Jacobi field such that $J(0) = J(1) = 0$, which implies p is conjugate to q . Conversely, if p and q are conjugate along $\exp_p(tv)$, then there exists a Jacobi field J such that $J(0) = J(1) = 0$. Note that $J(t)$ can be written as

$$J(t) = (\text{d exp}_p)_{tv}(tw)$$

where $0 \neq w = J'(0) \in T_p M$, since $J(t)$ is determined by $J(0)$ and $J'(0)$. In particular, one has

$$(\text{d exp}_p)_v(w) = J(1) = 0$$

which implies $(\text{d exp}_p)_v$ is not injective. \square

Corollary 12.2.1. Let (M, g) be a complete Riemannian manifold, $p \in M$. If the conjugate locus $\text{conj}(p) = \emptyset$, then $\exp_p: T_p M \rightarrow M$ is a local diffeomorphism.

Proof. The exponential map $\exp_p: T_p M \rightarrow M$ is surjective since M is complete. On the other hand, for arbitrary $v \in T_p M$, $(\text{d exp}_p)_v$ is injective since the conjugate locus $\text{conj}(p) = \emptyset$. Thus \exp_p is a local diffeomorphism. \square

Example 12.2.1. For $p \in \mathbb{S}^n$, $\text{conj}(p) = \{-p\}$.

Example 12.2.2. For $p \in \mathbb{S}^1 \times \mathbb{R}$, $\text{conj}(p) = \emptyset$.

12.3. Locally minimal geodesic.

Lemma 12.3.1. Let $\gamma: [a, b] \rightarrow (M, g)$ be a unit-speed geodesic with no conjugate points, there exist Jacobi fields J_2, \dots, J_n along γ such that

- (1) $J_i(a) = 0, i \geq 2$ and $\{\gamma'(b), J_2(b), \dots, J_n(b)\}$ is an orthonormal basis of $T_{\gamma(b)} M$.
- (2) $\langle J_i(t), \gamma'(t) \rangle \equiv 0$ for any $t \in [a, b]$.

(3) $\{\gamma'(t), J_2(t), \dots, J_n(t)\}$ are linearly independent for $t \in (a, b]$.

Proof. Suppose $\{\gamma'(b), e_2, \dots, e_n\}$ is an orthonormal basis of $T_{\gamma(b)}M$, since there is no conjugate points along γ , there exists a unique Jacobi field J_i such that

$$J_i(a) = 0, J_i(b) = e_i$$

for each $i = 2, \dots, n$. Now it suffices to show Jacobi fields $J_i(t)$ satisfy properties (2) and (3).

For (2). Note that $\langle J_i(a), \gamma'(a) \rangle = \langle J_i(b), \gamma'(b) \rangle = 0$. Then by (3) of Proposition 12.1.1 one $\langle J_i(t), \gamma'(t) \rangle \equiv 0$.

For (3). Suppose there exists $c \in (a, b]$ and $\lambda_i \in \mathbb{R}$ such that

$$\sum_{i=2}^n \lambda_i J_i(c) = 0$$

which implies

$$W(t) = \sum_{i=2}^n \lambda_i J_i(t) \equiv 0$$

on $(a, c]$ since there is no conjugate points. By (1) of Proposition 12.1.1 one has $W(t) \equiv 0$ on $(a, b]$, thus we have $\lambda_i = 0, i = 2, \dots, n$, since $\{\gamma'(b), J_2(b), \dots, J_n(b)\}$ is linearly independent. \square

Theorem 12.3.1. Let $\gamma: [a, b] \rightarrow (M, g)$ be a unit-speed geodesic. Then

- (1) If γ has no conjugate points. Then index form I_γ is **positive definite** on vector space consisting of normal variation fields.
- (2) If γ only has conjugate points as endpoints. Then index form is **semi-positive definite** on vector space consisting of normal variation fields. Furthermore, Jacobi field is null space.
- (3) If γ has an interior conjugate point. Then index form is **not positive definite** on vector space consisting of normal variation fields.

Proof. For (1). Suppose $\gamma(t)$ has no conjugate points, choose Jacobi fields $\{J_1(t) = \gamma'(t), J_2(t), \dots, J_n(t)\}$ in Lemma 12.3.1. Then for any normal variation vector V along γ we write it as

$$V = \sum_{i=2}^n V^i(t) J_i(t)$$

Then it's clear $V^i(b) = 0$ since $V(b) = 0$ and $\{J_2(b), \dots, J_n(b)\}$ is linearly independent. Direct computation shows

$$\begin{aligned} I_\gamma(V, V) &= \underbrace{\sum_{i,j=2}^n \int_a^b V^i V^j \langle J'_i, J'_j \rangle + \frac{dV^i}{dt} V^j \langle J_i, J'_j \rangle + V^i \frac{dV^j}{dt} \langle J'_i, J_j \rangle}_{\text{Part I}} dt \\ &\quad + \underbrace{\int_a^b \left\{ \frac{dV^i}{dt} \frac{dV^j}{dt} \langle J_i, J_j \rangle - V^i V^j R(J_i, \gamma', \gamma', J_j) \right\}}_{\text{Part II}} dt \end{aligned}$$

Note that

$$\langle J'_i, J_j \rangle = \langle J_i, J'_j \rangle$$

Then Part I is

$$\int_a^b \{ (V^i V^j \langle J'_i, J_j \rangle)' - V^i V^j \langle J''_i, J_j \rangle \} dt$$

Thus

$$\begin{aligned} I_\gamma(V, V) &= \sum_{i,j=2}^n V^i V^j \langle J'_i, J_j \rangle \Big|_a^b + \sum_{i,j=2}^n \int_a^b \frac{dV^i}{dt} \frac{dV^j}{dt} \langle J_i, J_j \rangle dt \\ &= \sum_{i,j=2}^n \int_a^b \frac{dV^i}{dt} \frac{dV^j}{dt} \langle J_i, J_j \rangle dt \\ &\geq 0 \end{aligned}$$

Furthermore, $I_\gamma(V, V) = 0$ if and only if $\sum_{i=2}^n \frac{dV^i}{dt} J_i(t) = 0$ if and only if $\frac{dV^i}{dt}(t) = 0, t \in [a, b]$, thus $V^i(t) \equiv 0$, that is $V = 0$.

For (2). For any $c \in (a, b)$, consider geodesic $\gamma_c: [a, c] \rightarrow (M, g)$. By (1) one has I_{γ_c} is positive definite on the vector space consisting of normal variation fields along γ_c . Then a standard approximation argument shows I_γ is semi-positive definite.

To see its null space: It's clear a normal variation Jacobi field V satisfies $I_\gamma(V, V) = 0$. Conversely, if a normal variation field V satisfies $I_\gamma(V, V) = 0$, then by a variation argument we have for arbitrary W we have

$$I_\gamma(V, W) = 0$$

Take appropriate W to see V satisfies the equation for Jacobi fields.

For (3). If $\gamma(a)$ is conjugate to $\gamma(c)$ for some $c \in (a, b)$, then there exists a non-zero normal Jacobi field $\tilde{J}(t)$ along γ such that $\tilde{J}(a) = \tilde{J}(c) = 0$. Consider

$$J(t) = \begin{cases} \tilde{J}(t) & t \in [a, c] \\ 0 & t \in [c, b] \end{cases}$$

Although $J(t)$ may not be a smooth vector field along γ , one still has $I_\gamma(J, J) = 0$ by integrating piecewisely. Let W be a smooth normal variation vector field along γ such that

$$W(c) = - \lim_{t \rightarrow c^-} \nabla_{\frac{d}{dt}} \tilde{J}(t)$$

It's clear $W(c) \neq 0$, otherwise $\tilde{J}(t) \equiv 0$. If we define $J_\varepsilon = J + \varepsilon W$, then one has

$$I_\gamma(J_\varepsilon, J_\varepsilon) = 2\varepsilon I_\gamma(J, W) + \varepsilon^2 I_\gamma(W, W)$$

And integration by parts implies

$$I_\gamma(J, W) = \langle \widehat{\nabla}_{\frac{d}{dt}} \tilde{J}, W \rangle \Big|_a^c = -W(c)^2 < 0$$

So for sufficiently small ε we have $I_\gamma(J_\varepsilon, J_\varepsilon) < 0$, and by approximation argument we can show there exists a smooth normal variation field such that $I_\gamma(V, V) < 0$, a contradiction. \square

Corollary 12.3.1. Let $\gamma: [a, b] \rightarrow (M, g)$ be a unit-speed geodesic with no conjugate points, and V, W are normal vector fields satisfying $V(a) = W(a), V(b) = W(b)$. If V is a Jacobi field. Then $I_\gamma(V, V) \leq I_\gamma(W, W)$, and the equality holds if and only if $V = W$.

Proof. Since V, W agree at end points, then $V - W$ is a normal variation field. Thus we have

$$0 \leq I_\gamma(V - W, V - W) = I_\gamma(V, V) + I_\gamma(W, W) - 2I_\gamma(V, W)$$

Since V is a Jacobi field, then integration by parts shows

$$I_\gamma(V, V) = \left\langle \widehat{\nabla}_{\frac{d}{dt}} V, V \right\rangle \Big|_a^b = \left\langle \widehat{\nabla}_{\frac{d}{dt}} V, W \right\rangle \Big|_a^b = I_\gamma(V, W)$$

Hence we get $I_\gamma(V, V) \leq I_\gamma(W, W)$, and the equality holds if and only if $V = W$. \square

Remark 12.3.1. From second variation formula, we can conclude that a geodesic γ is a **locally minimal geodesic** if and only if it has no interior conjugate points. However, it may not be **globally minimal geodesic**. For example, consider $M = \mathbb{S}^1 \times \mathbb{R}$. It's clear there is no conjugate points for any geodesic on M , and thus for geodesic $\gamma: [a, b] \rightarrow M$ starting at $(x, y) \in M$, it's locally minimal. But if there exists $c \in (a, b)$ such that $\gamma(c) \in \{-x\} \times \mathbb{R}$, then γ is not a minimal curve.

13. CUT LOCUS AND INJECTIVE RADIUS

13.1. Cut locus.

Definition 13.1.1 (cut time/point/locus). Let (M, g) be a complete Riemannian manifold, $p \in M$ and $v \in T_p M$.

- (1) The cut time of (p, v) is defined as

$$t_{\text{cut}}(p, v) = \sup\{c > 0 \mid \gamma_v|_{[0, c]} \text{ is a minimal geodesic}\}$$

- (2) If $t_{\text{cut}}(p, v) < \infty$, the cut point of p along γ_v is defined as $\gamma_v(t_{\text{cut}}(p, v))$.
 (3) The cut locus of p is defined as

$$\text{cut}(p) = \{q \in M \mid \exists v \in T_p M \text{ such that } q \text{ is a cut point of } p \text{ along } \gamma_v.\}$$

Remark 13.1.1.

- (1) It's possible for $t_{\text{cut}}(p, v) = +\infty$. For example, consider $(M, g) = (\mathbb{R}^n, g_{\text{can}})$.
 (2) If cut point exists, it occurs at or before the first conjugate point.
 (3) It's clear that $t_{\text{cut}}(p, v)$ depends on the $|v|$, but $\gamma_v(t_{\text{cut}}(p, v))$ is independent of $|v|$.

Theorem 13.1.1. Let (M, g) be a complete Riemannian manifold, $p \in M$, $v \in T_p M$ with $|v| = 1$. Let $c = t_{\text{cut}}(p, v) \in (0, \infty]$.

- (1) If $0 < b < c$ and b is finite, then $\gamma_v|_{[0, b]}$ has no conjugate point and it is the unique minimal unit-speed geodesic between endpoints.
 (2) If $c < \infty$, then $\gamma_v|_{[0, c]}$ is a minimal geodesic.
 (3) In the case of (2), one or both of the following holds.
 (a) $\gamma_v(c)$ is conjugate to p along γ_v .
 (b) There are two or more different unit-speed minimal geodesic connecting p and $\gamma_v(c)$.

Proof. For (1). It's clear $\gamma_v|_{[0, b]}$ has no conjugate point since cut point occurs at or before the first conjugate point, and it's minimal by definition. To see it's unique, suppose $\sigma: [0, b] \rightarrow M$ is another minimal unit-speed geodesic. Note that $\gamma'_v(b) \neq \sigma'(b)$, otherwise by uniqueness one has $\gamma_v(t) = \sigma(t)$ in $t \in [0, b]$. Now we take $b' \in (b, c)$, and construct a new unit-speed curve as follows

$$\tilde{\gamma}(t) = \begin{cases} \sigma(t), & t \in [0, b] \\ \gamma_v(t), & t \in (b, b'] \end{cases}$$

Then $\tilde{\gamma}$ has length b' , so it's also a minimal curve from p to $\gamma_v(b')$, since $\text{dist}(p, \gamma_v(b')) = b'$. However, $\tilde{\gamma}(t)$ is not smooth at $t = b$, which contradicts to the fact that minimal curve are smooth geodesics.

For (2). By definition, there exists a sequence $\{c_i\}$ increasing to c such that $\gamma_v|_{[0, c_i]}$ is minimal. By continuity of distance function, one has

$$\text{dist}(p, \gamma_v(c)) = \lim_{i \rightarrow \infty} \text{dist}(p, \gamma_v(c_i)) = \lim_{i \rightarrow \infty} c_i = c$$

which implies γ_v is minimal on $[0, c]$.

For (3). Assume $\gamma_v(c)$ is not conjugate to p along γ_v , we shall prove the existence of another unit-speed minimal geodesic from p to $\gamma_v(c)$. Let $\{c_i\}$ be a sequence descending to c . By definition $\gamma_v: [0, c_i] \rightarrow M$ is not a minimal geodesic, so there exists a unit-speed minimal geodesic $\gamma_i: [0, a_i] \rightarrow M$ connecting p and $\gamma_v(c_i)$. By construction one has $\gamma_i(a_i) = \gamma_v(c_i)$ and $a_i < c_i$. If we denote $\omega_i = \gamma'_i(0) \in T_p M$, by compactness of unit sphere in $T_p M$ and the fact $\{a_i\}$ is bounded, by taking subsequence twice we can find a subsequence of $\{\gamma_i(t)\}$, still denoted by $\{\gamma_i(t)\}$, such that ω_i converging to some $w \in T_p M$ with $|w| = 1$, and $\lim_{i \rightarrow \infty} a_i = a$. In fact $a = c$, since

$$\begin{aligned} c &= \text{dist}(p, \gamma_v(c)) \\ &= \lim_{i \rightarrow \infty} \text{dist}(p, \gamma_v(c_i)) \\ &= \lim_{i \rightarrow \infty} \text{dist}(p, \gamma_i(a_i)) \\ &= \lim_{i \rightarrow \infty} a_i = a \end{aligned}$$

Since $\gamma_v(c)$ is not conjugate to p , $(d \exp_p)_{cv}$ is non-degenerated, which implies \exp_p is injective in $B_\varepsilon(cv)$ for sufficiently small $\varepsilon > 0$. On one hand we have $a_i w_i \neq c_i v$ since $a_i < c_i$, but on the other hand we have

$$\exp_p(c_i v) = \gamma_v(c_i) = \gamma_i(a_i) = \exp_p(a_i w_i)$$

Thus injectivity implies $a_i w_i \notin B_\varepsilon(cv)$ for sufficiently large i , since $c_i v \in B_\varepsilon(cv)$. By taking limits one has

$$aw \neq cv$$

This shows $w \neq v$ since $a = c$, and thus $\exp_p(tw)$ gives another unit-speed minimal geodesic connecting p and $\gamma_v(c)$, since $\exp_p(cw) = \exp_p(aw) = \exp_p(av)$. \square

Corollary 13.1.1. If $\gamma: [0, b] \rightarrow M$ is a minimal geodesic connecting $\gamma(0)$ and $\gamma(b)$, then it's the unique minimal geodesic connecting any two points strictly between $\gamma(0)$ and $\gamma(b)$.

Proof. It follows from the proof of (1). \square

Corollary 13.1.2. Let (M, g) be a complete Riemannian manifold with $p, q \in M$.

- (1) If $q \in \text{cut}(p)$, then $p \in \text{cut}(q)$.
- (2) If $q \notin \text{cut}(p)$, then there exists a unique minimal geodesic connecting p and q .

Proof. For (1). If q is cut point of p along geodesic γ , then γ is a minimal geodesic connecting p and q , and by Theorem 13.1.1, there are the following two cases.

- (a) q is conjugate to p along γ .
- (b) There are two more different unit-speed geodesic connecting p and q .

Note that we already have γ^{-1} is a minimal geodesic connecting q and p , so if we want to show $p \in \text{cut}(q)$, it suffices to show γ^{-1} is no longer minimal after p . Let's discuss case by case.

- (a) It's clear γ^{-1} is no longer minimal after p in the first case, since if q is conjugate to p , then p is also conjugate to q .
- (b) In the second case, if γ^{-1} is still minimal after p , then by Corollary 13.1.1, one has γ^{-1} is the unique minimal geodesic connecting q and p , which contradicts to the second case.

For (2). If there exist two or more minimal geodesic connecting p and q . Then for any minimal geodesic γ connecting p and q , it's no longer minimal after q by Corollary 13.1.1, which contradicts to $q \notin \text{cut}(p)$. \square

Example 13.1.1.

- (1) For $p \in \mathbb{S}^n$, $\text{cut}(p) = \text{conj}(p) = \{-p\}$. In this case both (a), (b) hold in Theorem 13.1.1.
- (2) For $p \in \mathbb{S}^1 \times \mathbb{R}$, $\text{cut}(p) = \{-p\} \times \mathbb{R}$ and there is no conjugate point. In this case (a) fails and (b) holds in Theorem 13.1.1.

Definition 13.1.2 (tangent cut locus and injectivity domain). Let (M, g) be a complete Riemannian manifold, given $p \in M$, we define

- (1) the tangent cut locus

$$\text{TCL}(p) := \{v \in T_p M : |v| = t_{\text{cut}}(p, v/|v|)\}$$

- (2) the injectivity domain

$$\text{ID}(p) := \{v \in T_p M : |v| < t_{\text{cut}}(p, v/|v|)\}$$

It's clear that $\text{TCL}(p) = \partial \text{ID}(p)$ and $\text{cut}(p) = \exp_p(\text{TCL}(p))$. Furthermore, we have the following properties.

Proposition 13.1.1. Let (M, g) be a complete Riemannian manifold and $p \in M$. Then

- (1) The cut locus of p is a closed subset of M of measure zero.
- (2) The restriction of \exp_p to $\text{ID}(p)$ is a diffeomorphism onto $M \setminus \text{cut}(p)$.

Proof. See Theorem 10.34 of Page 311 of [Lee18]. \square

13.2. Injective radius.

Definition 13.2.1 (injective radius). Let (M, g) be a Riemannian manifold, $p \in M$. The injective radius of p is defined as

$$\text{inj}(p) := \sup\{\rho > 0 : \exp_p \text{ is defined on } B(0, \rho) \subseteq T_p M \text{ and injective}\}$$

The injectivity radius of M is

$$\text{inj}(M) := \inf_{p \in M} \text{inj}(p)$$

Theorem 13.2.1. Let (M, g) be a complete Riemannian manifold. Then

$$\text{inj}(p) = \begin{cases} \text{dist}(p, \text{cut}(p)) & \text{cut}(p) \neq \emptyset \\ \infty & \text{cut}(p) = \emptyset \end{cases}$$

Proof. See [Lee18, Proposition 10.36]. \square

Proposition 13.2.1. Let (M, g) be a complete Riemannian manifold and $p \in M$. Suppose there exists some point $q \in \text{cut}(p)$ such that $\text{dist}(p, q) = \text{dist}(p, \text{cut}(p))$. Then

- (1) Either q is a conjugate point of p along some minimizing geodesic from p to q , or there are exactly two minimizing geodesics from p to q , say $\gamma_1, \gamma_2: [0, b] \rightarrow M$, such that $\gamma'_1(b) = -\gamma'_2(b)$.
- (2) If in addition that $\text{inj}(p) = \text{inj}(M)$, and q is not conjugate to p along any minimizing geodesic. Then there is a closed unit-speed geodesic $\gamma: [0, 2b] \rightarrow M$ such that $\gamma(0) = \gamma(2b) = p$ and $\gamma(b) = q$ where $b = \text{dist}(p, q)$.

Proof. For (1). Suppose q is not conjugate to p along any minimizing geodesic. Then by Theorem 13.1.1 there are at least two unit-speed minimal geodesics $\gamma_1(t), \gamma_2(t)$ such that $\gamma_1(b) = \gamma_2(b) = q$. Suppose $\gamma'_1(b) \neq -\gamma'_2(b)$. Then there exists a unit vector $v \in T_q M$ such that

$$\langle v, \gamma'_1(b) \rangle < 0, \quad \langle v, \gamma'_2(b) \rangle < 0$$

Since q is not conjugate to p along γ_1 , there exists a neighborhood U_1 of $b\gamma'_1(0)$ in $T_p M$ such that $\exp_p|_{U_1}$ is diffeomorphism. Now choose a sufficiently small s and let

$$\xi_1(s) = (\exp_p|_{U_1})^{-1} \exp_q(sv)$$

Consider the following variation of γ_1 consisting of geodesics:

$$\alpha_1(t, s) = \exp\left(\frac{t}{b}\xi_1(s)\right)$$

It's clear $\alpha_1(t, 0) = \gamma_1(t)$, since $\xi_1(0) = (\exp_p|_{U_1})^{-1} \exp_q(0) = (\exp_p|_{U_1})^{-1}(q) = b\gamma'_1(0)$. Then by Remark 11.1.3, that is the first variation formula of general variation, one has

$$\left. \frac{dL(\gamma_s)}{ds} \right|_{s=0} = \langle v, \gamma'_1(b) \rangle < 0$$

which implies for sufficiently small s we have $L(\alpha_1(t, s)) < L(\gamma_1(t))$. For γ_2 we can do the same construction and the same argument implies for sufficiently small s we have $L(\alpha_2(t, s)) < L(\gamma_2(t))$. Thus for each sufficiently small s we have two geodesics $\alpha_1(t, s), \alpha_2(t, s)$ from p to $\exp_q(sX_q)$. However,

$$(13.1) \quad d(p, \exp_q(sv)) \leq L(\alpha_1(t, s)) < L(\gamma_1(t)) = \text{dist}(p, q) = \text{inj}(p)$$

A contradiction to the definition of injective radius. So any two different minimizing geodesics γ_1, γ_2 from p to q satisfy $\gamma'_1(b) = -\gamma'_2(b)$, which implies there are exactly two minimizing geodesics from p to q .

For (2). By (1) we know that there exists exactly two geodesics γ_1, γ_2 such that $\gamma_1(b) = \gamma_2(b) = q$ with $\gamma_1'(b) = \gamma_2'(b)$. Consider the loop $\gamma = \gamma_1 \circ \gamma_2^{-1}$. Then it's a unit-speed geodesic such that $\gamma(0) = \gamma(2b) = p, \gamma(b) = q$, where $b = \text{dist}(p, q)$, since we have already shown $\gamma_1'(b) = -\gamma_2'(b)$. To show γ is a closed geodesic, it suffices to show $\gamma'(2b) = \gamma'(0)$, that is equivalent to show $(\gamma_1^{-1})'(b) = (\gamma_2^{-1})'(b)$. Note that in the proof of (1), condition of $\text{dist}(p, q) = \text{dist}(p, \text{cut}(p)) = \text{inj}(p)$ is used in inequality (13.1), and in fact we only need $\text{dist}(p, q) \leq \text{inj}(p)$, strict equality is not necessary. So if $\text{inj}(p) = \text{inj}(M)$, thus

$$\text{dist}(q, p) = \text{dist}(p, q) = \text{inj}(p) \leq \text{inj}(q)$$

Then (1) implies $(\gamma_1^{-1})'(b) = (\gamma_2^{-1})'(b)$. □

14. JACOBI FIELD II: A USEFUL TOOL

14.1. Taylor expansion of Jacobi field.

Proposition 14.1.1. Let (M, g) be a Riemannian manifold, and $\gamma: [0, 1] \rightarrow M$ is a geodesic with $\gamma(0) = p, \gamma'(0) = v$, where $p \in M, v \in T_p M$. For any $w \in T_p M$ with $|w| = 1$, suppose $J(t)$ is the Jacobi field along γ given by

$$J(t) = (\mathrm{d} \exp_p)_{tv}(tw)$$

Then

$$\begin{aligned} |J(t)|^2 &= t^2 - \frac{1}{3}R(J', \gamma', \gamma', J')(0)t^4 + O(t^4) \\ |J(t)| &= t - \frac{1}{6}R(J', \gamma', \gamma', J')(0)t^3 + O(t^3) \end{aligned}$$

Proof. It suffices to prove the first equality, and the second follows from the first one. Note that $J(0) = 0, J'(0) = w$, direct computation shows

$$\begin{aligned} \langle J, J \rangle(0) &= 0 \\ \langle J, J \rangle'(0) &= 2\langle J, J' \rangle(0) = 0 \\ \langle J, J \rangle''(0) &= 2\langle J', J' \rangle(0) + 2\langle J'', J \rangle(0) = 2 \\ \langle J, J \rangle'''(0) &= 6\langle J', J'' \rangle(0) + 2\langle J''', J \rangle(0) \\ &= 6\langle J', R(J, \gamma')\gamma' \rangle(0) = 0 \\ \langle J, J \rangle''''(0) &= 8\langle J', J''' \rangle(0) + 6\langle J'', J'' \rangle(0) + 2\langle J'''' , J \rangle(0) \\ &= 8\langle J', J''' \rangle(0) + 6\langle R(J, \gamma')\gamma', R(J, \gamma')\gamma' \rangle(0) \\ &= 8\langle J', J''' \rangle(0) \end{aligned}$$

Now it remains to compute J''' . For arbitrary vector field W along γ , direct computation shows

$$\begin{aligned} \langle \widehat{\nabla}_{\frac{d}{dt}} R(J, \gamma')\gamma', W \rangle &= \frac{d}{dt} \langle R(J, \gamma')\gamma', W \rangle - \langle R(J, \gamma')\gamma', W' \rangle \\ &= \frac{d}{dt} \langle R(W, \gamma')\gamma', J \rangle - \langle R(J, \gamma')\gamma', W' \rangle \\ &= \langle R(W, \gamma')\gamma', J' \rangle - \langle \widehat{\nabla}_{\frac{d}{dt}} R(W, \gamma')\gamma', J \rangle - \langle R(J, \gamma')\gamma', W' \rangle \\ &= \langle R(J', \gamma')\gamma', W \rangle - \langle \widehat{\nabla}_{\frac{d}{dt}} R(W, \gamma')\gamma', J \rangle - \langle R(J, \gamma')\gamma', W' \rangle \end{aligned}$$

By setting $t = 0$ we obtain

$$\langle J', J''' \rangle(0) = -\langle J'(0), \widehat{\nabla}_{\frac{d}{dt}} R(J, \gamma')\gamma' \Big|_{t=0} \rangle = -R(J', \gamma', \gamma', J')(0)$$

As a consequence

$$|J(t)|^2 = t^2 - \frac{1}{3}R(J', \gamma', \gamma', J')(0)t^4 + O(t^4)$$

□

Corollary 14.1.1. In normal coordinate (x^i, U, p) , one has

$$g_{ij} = \delta_{ij} - \frac{1}{3}R_{iklj}(0)x^k x^l + O(|x|^3)$$

Corollary 14.1.2. In normal coordinate (x^i, U, p) , one has

- (1) $g^{ij} = \delta_{ij} + \frac{1}{3}R_{iklj}(0)x^k x^l + O(|x|^3)$
- (2) $\det(g_{ij}) = 1 - \frac{1}{3}R_{kl}(0)x^k x^l + O(|x|^3)$
- (3) $\sqrt{\det(g_{ij})} = 1 - \frac{1}{6}R_{kl}(0)x^k x^l + O(|x|^3)$

Proof. For (1). Note that g^{ij} gives a Riemannian metric on T^*M , and Levi-Civita connection ∇ on T^*M with respect to g^{ij} is exactly the induced connection from the one on TM . Note that

$$\nabla dx^k = -\Gamma_{ij}^k dx^i \otimes dx^j$$

where Γ_{ij}^k is the Christoffel symbol for Levi-Civita connection on TM , we have curvature form in this case differs a sign since

$$R_{ijk}^l(0) = \frac{\partial \Gamma_{jk}^l}{\partial x^i} - \frac{\partial \Gamma_{ik}^l}{\partial x^j}$$

Thus all computations are same as proof above, but result differs a sign in curvature.

For (2). By Jacobi's formula, we have

$$\frac{\partial \det(g_{ij})}{\partial x^k} = \det(g_{ij}) g^{ij} \frac{\partial g_{ij}}{\partial x^k}$$

Thus $\frac{\partial \det(g_{ij})}{\partial x^k}(0) = 0$, since first-order partial derivatives of g_{ij} vanishes. Furthermore, since first-order partial derivatives of g^{ij} also vanishes, we have

$$\begin{aligned} \frac{1}{2} \frac{\partial^2 \det(g_{ij})}{\partial x^l \partial x^k} &= \det(g_{ij}) g^{ij} \frac{1}{2} \frac{\partial^2 g_{ij}}{\partial x^l \partial x^k} \\ &= \det(g_{ij}) g^{ij} \left(-\frac{1}{3} R_{iklj} x^k x^l \right) \\ &= -\frac{1}{3} \det(g_{ij}) R_{kl} x^k x^l \end{aligned}$$

which implies

$$\det(g_{ij}) = 1 - \frac{1}{3}R_{kl}(0)x^k x^l + O(|x|^3)$$

For (3). It follows from (2). □

Theorem 14.1.1. Let (M, g) be a Riemannian manifold. For all $p \in M$ and r sufficiently small, the volume of $B(p, r)$ is

$$\text{Vol}(B(p, r)) = \alpha_n r^n \left(1 - \frac{S(p)}{6(n+2)} r^2 + O(r^3) \right)$$

where α_n is the volume of n -dimension unit ball in \mathbb{R}^{n+1} .

Proof. Direct computation shows

$$\begin{aligned}
\text{Vol}(B(p, r)) &= \int_0^r \int_{\mathbb{S}^{n-1}(t)} \sqrt{\det g} dS dt \\
&\stackrel{(1)}{=} \int_0^r \int_{\mathbb{S}^{n-1}(t)} \left(1 - \frac{1}{6} \text{Ric}_p(x) + O(|x|^3)\right) dS dt \\
&\stackrel{(2)}{=} \alpha_n r^n - \frac{\alpha_n}{6} \int_0^r S(p) t^{n+1} dt + O(r^{n+3}) \\
&= \alpha_n r^n - \frac{\alpha_n S(p) r^{n+2}}{6(n+2)} + O(r^{n+3}) \\
&= \alpha_n r^n \left(1 - \frac{S(p)}{6(n+2)} r^2 + O(r^3)\right)
\end{aligned}$$

where

(1) holds from Corollary 14.1.2.

(2) holds from Proposition 6.2.2. □

14.2. Gauss lemma. Let (M, g) be a Riemannian manifold, and (x^i, U, p) is the normal coordinate centered at $p \in M$.

Definition 14.2.1 (radial distance function). The radial distance function r defined on U is given by

$$r(q) := \sqrt{\sum_{i=1}^n (q^i)^2}$$

where $q = (q^1, \dots, q^n)$ in normal coordinate (x^i, U, p) .

Definition 14.2.2 (radial vector field). The radial vector field in $U \setminus \{p\}$ is defined as

$$\partial_r = \frac{x^i}{r} \frac{\partial}{\partial x^i}$$

Proposition 14.2.1. The geodesic starting at p with unit-speed is the integral curve of radial vector field ∂_r over $U \setminus \{p\}$.

Proof. Let $\gamma: I \rightarrow U$ be a geodesic with $\gamma(0) = p, \gamma'(0) = v$, where $|v| = 1$, by definition we need to show

$$\gamma'(b) = \partial_r|_{\gamma(b)}$$

where I is an open interval and $b \in I$. In normal coordinate γ looks like $\gamma(t) = (tv^1, \dots, tv^n)$. If we denote $\gamma(b) = q = (q^1, \dots, q^n)$. Then it's clear $v^i = q^i/b$, and $r(q) = b$, since $|v| = 1$. Then

$$\gamma'(b) = v^i \frac{\partial}{\partial x^i} \Big|_q = \frac{q^i}{b} \frac{\partial}{\partial x^i} \Big|_q = \frac{q^i}{r(q)} \frac{\partial}{\partial x^i} \Big|_q = \partial_r|_q$$

□

Lemma 14.2.1. Let $f: M \rightarrow \mathbb{R}$ be a smooth function and X a vector field over M , if

- (1) $Xf = |X|^2$.
- (2) X is perpendicular to the level set of f .

Then $X = \nabla f$.

Theorem 14.2.1 (Gauss lemma). The radial vector field ∂_r is perpendicular to the level set of radial distance function r on $U \setminus \{p\}$.

Proof. For any $q \in U \setminus \{p\}$ written as $q = (q^1, \dots, q^n)$ in normal coordinate with $b = r(q)$. Given $w \in T_q M$ which is tangent to the level set of r , we need to show $\langle \partial_r|_q, w \rangle = 0$. By definition there exists a curve $c(s): (-\varepsilon, \varepsilon) \rightarrow M$ such that $c(0) = q, c'(0) = w$ with $\sum_{i=1}^n (c^i(s))^2 = b$, where c^i is the coordinates of c in normal coordinate. Taking derivative with respect to s one has

$$\sum_{i=1}^n 2c^i(s)(c^i(s))' = 0$$

In particular, one has $\sum_{i=1}^n c^i(0)(c^i)'(0) = 0$. By Corollary 12.1.1 there is a Jacobi field $J(t)$ along geodesic connecting p, q such that $J(0) = 0, J'(0) = w$ and $J(1) = w$. Note that the metric at $T_p M$ is standard metric, thus $\langle J'(0), \gamma'(0) \rangle = 0$, and by construction $\langle J(0), \gamma'(0) \rangle = 0$. Then by (4) of Proposition 12.1.1 one has $\langle J(t), \gamma'(t) \rangle \equiv 0$. In particular, one has $\langle J(1), \gamma'(1) \rangle$, which complete the proof since $\gamma'(t)$ is integral curve of ∂_r . \square

Corollary 14.2.1.

- (1) $|\partial_r|^2 = 1$.
- (2) $g^{ij} \frac{\partial r}{\partial x^i} \frac{\partial}{\partial x^j} = \nabla r = \partial_r$.

Proof. For (1). It's clear, since we have already shown geodesic with unit-speed is integral curve of ∂_r .

For (2). By Lemma 14.2.1 and Theorem 14.2.1, it suffices to show $Xr = |\partial_r|^2$, which can be seen from

$$Xr = \frac{x^i}{r} \frac{\partial r}{\partial x^i} = \sum_{i=1}^n \frac{(x^i)^2}{r^2} = 1 = |\partial_r|^2$$

\square

Corollary 14.2.2. The following identities hold in (x^i, U, p) :

- (1) $g_{ij}x^j = x^i$.
- (2) $g_{im} = \delta_{im} - \frac{\partial g_{ij}}{\partial x^m} x^j$.
- (3) $\frac{\partial g_{ij}}{\partial x^m} x^j = \frac{\partial g_{mj}}{\partial x^i} x^j$.
- (4) $\frac{\partial g_{ij}}{\partial x^m} x^j x^i = \frac{\partial g_{mj}}{\partial x^i} x^j x^i = 0$.
- (5) $\Gamma_{ij}^k x^i x^j = 0$.

Proof. For (1). On one hand by Corollary 14.2.1 we have $\partial_r = \nabla r = g^{ij} \frac{x^i}{r} \frac{\partial}{\partial x^j}$. On the other hand by definition of ∂_r we have $\partial_r = \frac{x^j}{r} \frac{\partial}{\partial x^j}$, which implies

$$g^{ij} x^i = x^j$$

This shows (1).

For (2). Take partial derivatives of (1) with respect to x^m , we have

$$\frac{\partial g_{ij}}{\partial x^m} x^j + g_{ij} \delta_{jm} = \delta_{im}$$

This shows (2).

For (3). It follows from (2), since g_{im}, δ_{im} are symmetric in i, m .

For (4). It follows from (1) and (2), since

$$\begin{aligned} \frac{\partial g_{ij}}{\partial x^m} x^j x^i &\stackrel{(2)}{=} (\delta_{im} - g_{im}) x^i = x^m - g_{im} x^i \stackrel{(1)}{=} 0 \\ \frac{\partial g_{mj}}{\partial x^i} x^j x^i &\stackrel{(2)}{=} (\delta_{mi} - g_{mi}) x^i = x^m - g_{im} x^i \stackrel{(1)}{=} 0 \end{aligned}$$

For (5). It follows from (4) and

$$\Gamma_{ij}^k = \frac{1}{2} g^{mk} \left(\frac{\partial g_{mj}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m} \right)$$

□

Corollary 14.2.3.

$$\nabla_{\partial_r} \partial_r = 0$$

holds in $U \setminus \{p\}$.

Proof. Direct computation shows

$$\begin{aligned} \nabla_{\partial_r} \partial_r &= \frac{x^k}{r} \nabla_{\frac{\partial}{\partial x^k}} \left(g^{ij} \frac{x^i}{r} \frac{\partial}{\partial x^j} \right) \\ &= g^{ij} \frac{x^k}{r} \left\{ \underbrace{\left(\frac{\delta_{ki}}{r} - \frac{x^k x^i}{r^3} \right) \frac{\partial}{\partial x^j}}_{\text{part I}} + \underbrace{\frac{x^i}{r} \Gamma_{kj}^m \frac{\partial}{\partial x^m}}_{\text{part II}} \right\} \end{aligned}$$

By (1) and (5) of Corollary 14.2.2 one has

$$g^{ij} \frac{x^k x^i}{r} \Gamma_{kj}^m = \frac{1}{r} \Gamma_{kj}^m x^k x^j = 0$$

which implies part II is zero. For part I, we have

$$\frac{1}{r^2} (g^{ij} x^k \delta_{ki} - \frac{(x^k)^2}{r^2} g^{ij} x^i) = \frac{1}{r^2} (g^{ij} x^i - g^{ij} x^i) = 0$$

□

14.3. Jacobi fields on constant sectional curvature manifold.

Proposition 14.3.1. Let (M, g) be a Riemannian manifold with constant sectional curvature k and $\gamma: [0, b] \rightarrow M$ a unit-speed geodesic. Then the normal Jacobi field with $J(0) = 0$ is of the form

$$J(t) = m \operatorname{sn}_k(t) E(t)$$

where

- (1) The constant m is determined by $J'(0) = mE(0)$.
- (2)

$$\operatorname{sn}_k(t) = \begin{cases} t, & k = 0 \\ \frac{\sin(\sqrt{k}t)}{\sqrt{k}}, & k > 0 \\ \frac{\sinh(\sqrt{-k}t)}{\sqrt{-k}}, & k < 0 \end{cases}$$

- (3) $E(t)$ is a normal parallel vector field along γ with $|E(t)| = 1$

Proof. Since (M, g) has constant sectional curvature k , thus $R_{ijkl} = k(g_{il}g_{jk} - g_{ik}g_{jl})$, so for any normal vector field J along γ we have

$$\begin{aligned} R(J, \gamma', \gamma', W) &= k(\langle J, W \rangle \langle \gamma', \gamma' \rangle - \langle J, \gamma' \rangle \langle \gamma', W \rangle) \\ &= k \langle J, W \rangle \end{aligned}$$

which implies

$$R(J, \gamma')\gamma' = kJ$$

since γ is unit-speed and J is normal. Thus equation for Jacobi field can be written as

$$0 = J'' + kJ$$

Assume $J = u(t)E(t)$. Then

$$(u''(t) + ku(t))E(t) = 0$$

So if we want to find normal Jacobi fields J , it suffices to solve

$$\begin{cases} u''(t) + ku = 0 \\ u(0) = 0 \end{cases}$$

and it's clear $\operatorname{sn}_k(t)$ is the solution of this ODE. □

14.4. Polar decomposition of metric with constant sectional curvature. Let $\pi: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{S}^{n-1}$ given by $\pi(x) = x/|x|$. We can use π to pullback canonical metric on \mathbb{S}^{n-1} , and still use $g_{\mathbb{S}^{n-1}}$ to denote it.

Lemma 14.4.1. Let \bar{g} be the Euclidean metric on $\mathbb{R}^n \setminus \{0\}$. Then

$$\bar{g} = dr \otimes dr + r^2 g_{\mathbb{S}^{n-1}}$$

where $r(x) = |x|$.

Theorem 14.4.1 (polar decomposition). Let (x^i, U, p) be a normal coordinate centered at $p \in S(n, k)$. Then in U the metric g can be written as

$$g = dr \otimes dr + \text{sn}_k^2(r) g_{\mathbb{S}^{n-1}}$$

where r is radial distance function.

Proof. We use g_c to denote metric $dr \otimes dr + \text{sn}_k^2(r) g_{\mathbb{S}^{n-1}}$ and \bar{g} to denote standard metric on Euclidean space. By Corollary 14.2.1, we have

$$g(\partial_r, \partial_r) = 1 = g_c(\partial_r, \partial_r)$$

So it remains to show for each $q \in U \setminus \{p\}$ and $w_1, w_2 \in T_q M$ such that $g(w_i, \partial_r|_q) = 0, i = 1, 2$, we have

$$g(w_1, w_2) = g_c(w_1, w_2)$$

By polarization it suffices to show that $g(w, w) = g_c(w, w)$ for every such vector w .

Suppose $\text{dist}(p, q) = b$, on one hand we have

$$|w|_{g_c}^2 \stackrel{(1)}{=} \text{sn}_k^2(b) |w|_{g_{\mathbb{S}^{n-1}}}^2 \stackrel{(2)}{=} \frac{\text{sn}_k(b)}{b^2} |w|_{\bar{g}}^2$$

where

(1) holds from definition of g_c .

(2) holds from polar decomposition of standard metric of Euclidean space, that is Lemma 14.4.1.

On the other hand, let $\gamma: [0, b] \rightarrow U$ be a unit-speed geodesic connecting p, q , and we can write it with respect to normal coordinate U as

$$\gamma(t) = \left(\frac{tq^1}{b}, \dots, \frac{tq^n}{b} \right)$$

where $q = (q^1, \dots, q^n)$ in normal coordinate U . Let J be a Jacobi field such that $J(0) = 0, J(b) = w$. Then we have

$$|w|_g^2 = |J(b)|_g^2 \stackrel{(3)}{=} \text{sn}_k^2(b) |J'(0)|_g^2 \stackrel{(4)}{=} \text{sn}_k^2(b) |J'(0)|_{\bar{g}}^2$$

where

(3) holds from the fact Jacobi field on constant sectional curvature space is of form $J(t) = |J'(0)| \text{sn}_k(t) E(t)$.

(4) holds from the metric on $T_p M$ is standard metric in normal coordinate.

Furthermore, suppose $J'(0) = a$. Then we can write it as $J(t) = \alpha_* \left(\frac{\partial}{\partial s} \right) \Big|_{s=0}$, where

$$\alpha(s, t) = \exp_p(t(\gamma'(0) + sJ'(0)))$$

In normal coordinate we can write $\alpha(s, t)$ explicitly as

$$\alpha(s, t) = \left(\frac{tq^1}{b} + tsa^1, \dots, \frac{tq^n}{b} + tsa^n \right)$$

thus $J(t) = ta^i \frac{\partial}{\partial x^i} \Big|_{\gamma(t)}$. We can conclude $a^i = \frac{w^i}{b}$ by setting $t = b$, in particular, we have $J'(0) = \frac{w^i}{b} \frac{\partial}{\partial x^i} \Big|_p$. Then

$$\text{sn}_k^2(b) |J'(0)|_g^2 = \text{sn}_k^2(b) \frac{|w|_g^2}{b^2} = |w|_{g_c}^2$$

□

Remark 14.4.1. Note that there are four points we need in the proof of above theorem, and the **key point** is (3), that is Jacobi field of $S(n, k)$ has the form of

$$J(t) = m \text{sn}_k(t) E(t)$$

So this motivates us that if on a normal neighborhood of some point, the Jacobi field has the above form. Then metric g can be written as

$$g = dr \otimes dr + \text{sn}_k(r)^2 g_{\mathbb{S}^{n-1}}$$

in U . In particular, it has constant sectional curvature k .

14.5. A criterion for constant sectional curvature space. Recall that for a smooth function $f: M \rightarrow \mathbb{R}$, $\text{Hess } f$ is a $(0, 2)$ -tensor, we use \mathcal{H}_f to denote its $(1, 1)$ -type, that is

$$g(\mathcal{H}_f(X), Y) = \text{Hess } f(X, Y)$$

where X, Y are two vector fields.

In particular, if r is the radial distance function on a normal coordinate. Then Hessian r is a $(2, 0)$ -tensor, that is $\nabla^2 r$. Then we have

$$\mathcal{H}_r = \nabla \partial_r$$

since $(1, 0)$ -type of ∇r is ∂_r .

Proposition 14.5.1. Let (M, g) be a complete Riemannian manifold, (x^i, U, p) a normal coordinate centered at p and r the radial distance function on U . If $\gamma: [0, b] \rightarrow M$ is unit-speed geodesic with $\gamma(0) = p, \gamma'(0) = v \in T_p M$, and J is a normal Jacobi field along γ with $J(0) = 0$. Then for all $t \in (0, b]$

$$\begin{aligned} \mathcal{H}_r(J(t)) &= J'(t) \\ \mathcal{H}_r(\gamma'(t)) &= 0 \end{aligned}$$

Proof. Here we only prove the first identity, the second can be computed in the same method. Let $J'(0) = w$. Then $J(t) = tw^i \frac{\partial}{\partial x^i} \Big|_{\gamma(t)}$,

$$\begin{aligned}
J'(t) &= \widehat{\nabla}_{\frac{d}{dt}} \left(tw^i \frac{\partial}{\partial x^i} \Big|_{\gamma(t)} \right) \\
&= w^i \frac{\partial}{\partial x^i} \Big|_{\gamma(t)} + tw^i \widehat{\nabla}_{\frac{d}{dt}} \frac{\partial}{\partial x^i} \Big|_{\gamma(t)} \\
&= w^i \frac{\partial}{\partial x^i} \Big|_{\gamma(t)} + tw^i \Gamma_{ij}^k(\gamma(t)) \frac{d\gamma^j}{dt} \frac{\partial}{\partial x^k} \Big|_{\gamma(t)} \\
&= (w^k + tw^i v^j \Gamma_{ij}^k(\gamma(t))) \frac{\partial}{\partial x^k} \Big|_{\gamma(t)} \\
\mathcal{H}_r(J(t)) &= \nabla_{J(t)} \partial_r \\
&= \nabla_{tw^i \frac{\partial}{\partial x^i} \Big|_{\gamma(t)}} \left(\frac{x^j}{r} \frac{\partial}{\partial x^j} \right) \\
&= tw^i \nabla_{\frac{\partial}{\partial x^i} \Big|_{\gamma(t)}} \left(\frac{x^j}{r} \frac{\partial}{\partial x^j} \right) \\
&= tw^i \frac{x^j}{r} \Gamma_{ij}^k(\gamma(t)) \frac{\partial}{\partial x^k} \Big|_{\gamma(t)} + \sum_{i=1}^n tw^i \left(\frac{\delta_{ij}}{r} - \frac{x^i x^j}{r^3} \right) \frac{\partial}{\partial x^j} \Big|_{\gamma(t)}
\end{aligned}$$

However, we have the following observations:

- (1) $r(\gamma(t)) = t$.
- (2) $x^i = tv^i$.
- (3) $\sum_{i=1}^n a^i v^i = 0$

where the last equality holds since J is a normal vector field. Then

$$0 = \langle J(t), \gamma'(t) \rangle = \langle J(0), \gamma'(0) \rangle + \langle J'(0), \gamma'(0) \rangle t$$

implies $\langle J'(0), \gamma'(0) \rangle = \sum_{i=1}^n a^i v^i = 0$. \square

Corollary 14.5.1. With the same assumption as above proposition, for any vector field W along γ with $W(0) = 0$,

$$\begin{aligned}
\text{Hess } r(J(s), W(s)) &\stackrel{(1)}{=} g(\mathcal{H}_r(J(s), W(s))) \\
&\stackrel{(2)}{=} g(J'(t), W(s)) \\
&\stackrel{(3)}{=} \int_0^s \langle J'(t), W(t) \rangle' dt \\
&\stackrel{(4)}{=} \int_0^s \langle J'(t), W'(t) \rangle - R(J, \gamma', \gamma', W) dt
\end{aligned}$$

Proof. It's clear, since

- (1) holds from definition of \mathcal{H}_r .
- (2) holds from $\mathcal{H}_r(J(t)) = J'(t)$.

(3) holds from $W(0) = 0$.

(4) holds from J is a Jacobi field.

□

Corollary 14.5.2. Let $p \in U \subseteq S(n, k)$, where U is a normal neighborhood of p . Then the following holds in $U \setminus \{p\}$

$$\mathcal{H}_r = \frac{\text{sn}'_k(r)}{\text{sn}_k(r)} \pi_r$$

where r is the radial distance function on U , and for each $q \in U \setminus \{p\}$, $\pi_r : T_q M \rightarrow T_q M$ is the orthogonal projection onto the orthogonal complement of $\partial_r|_q$.

Proof. For $p \in U \setminus \{q\}$, it's clear

$$\mathcal{H}_r(\partial_r|_q) = 0 = \frac{\text{sn}'_k(r)}{\text{sn}_k(r)} \pi_r(\partial_r|_q)$$

For $w \in T_q M$ such that $g(w, \partial_r|_q) = 0$, choose a unit-speed geodesic $\gamma : [0, b] \rightarrow M$ connecting p and q and $J(t)$ is the Jacobi field such that $J(0) = 0, J(b) = w$. Then we must have

$$J(t) = m \text{sn}_k(t) E(t)$$

where $E(t)$ is a normal parallel vector field along γ with $|E(t)| = 1$. Then

$$\begin{aligned} m \text{sn}'_k(t) E(t) &= J'(t) \\ &= \mathcal{H}_r(J(t)) \\ &= \mathcal{H}_r(m \text{sn}_k(t) E(t)) \\ &= m \text{sn}_k(t) \mathcal{H}_r(E(t)) \end{aligned}$$

Setting $t = b$ and dividing by $\text{sn}_k(b)$ one has

$$\mathcal{H}_r(E(b)) = \frac{\text{sn}'_k(b)}{\text{sn}_k(b)} E(b)$$

Note that $w = m \text{sn}_k(b) E(b)$, this completes the proof. □

Furthermore, the converse of above corollary still holds:

Proposition 14.5.2. Let (M, g) be a Riemannian manifold and U a normal neighborhood of $p \in M$, r radial distance function. If

$$\mathcal{H}_r = \frac{\text{sn}'_k(r)}{\text{sn}_k(r)} \pi_r$$

holds in $U \setminus \{p\}$. Then (M, g) has constant sectional curvature k in U .

Proof. Let $\gamma : [0, b] \rightarrow U$ be a unit-speed geodesic $r(0) = p$, J is a normal Jacobi vector field along γ with $J(0) = 0$. Then $\mathcal{H}_r(J) = J'$ implies

$$J'(t) = \frac{\text{sn}'_k(t)}{\text{sn}_k(t)} J(t)$$

holds for $t \in (0, b]$, that is

$$\left(\frac{J(t)}{\text{sn}_k(t)}\right)' = 0$$

holds for $t \in (0, b]$. So we can write every normal Jacobi fields as $J(t) = m \text{sn}_k(t) E(t)$, where E is normal a parallel vector field with $|E| = 1$ and $t \in [0, b]$. Thus by Remark 14.4.1, g has constant sectional curvature k in U . \square

Remark 14.5.1. For convenience, we record the exact formulas for the quotient $\frac{\text{sn}'_k}{\text{sn}_k}$ as follows

$$\frac{\text{sn}'_k(t)}{\text{sn}_k(t)} = \begin{cases} \frac{1}{t}, & k = 0 \\ \frac{1}{\sqrt{k}} \cot \frac{t}{\sqrt{k}}, & k > 0 \\ \frac{1}{\sqrt{k}} \coth \frac{t}{\sqrt{k}}, & k < 0 \end{cases}$$

and we can draw the graph as follows.

Part 5. Harmonic maps

15. HARMONIC MAP

In this section, let $f: (M, g) \rightarrow (N, h)$ be a smooth map between Riemannian manifolds with second fundamental form $B \in C^\infty(M, T^*M \otimes T^*M \otimes f^*TN)$.

15.1. Harmonic map and totally geodesic.

Definition 15.1.1 (scalar Laplacian). The scalar Laplacian of f is defined as

$$\Delta f := \text{tr}_g B \in C^\infty(M, f^*TN)$$

Definition 15.1.2 (harmonic map). f is called a harmonic map if its scalar Laplacian $\Delta f = 0$.

Definition 15.1.3 (totally geodesic). f is called totally geodesic, if its second fundamental form $B = 0$.

Example 15.1.1. For a geodesic $\gamma: [a, b] \rightarrow (M, g)$, if $[a, b]$ is endowed with standard metric. Then γ is totally geodesic, thus it's harmonic.

Example 15.1.2. For a smooth function $f: (M, g) \rightarrow \mathbb{R}$, if \mathbb{R} is endowed with standard metric. Then f is a harmonic map if and only if it's a harmonic function.

Lemma 15.1.1. Let $\gamma: [a, b] \rightarrow M$ be a smooth curve and $\tilde{\gamma} = f \circ \gamma$. Then

$$\tilde{\nabla}_{\frac{d}{dt}} \tilde{\gamma}_* \left(\frac{d}{dt} \right) = f_* \left(\hat{\nabla}_{\frac{d}{dt}} \gamma_* \left(\frac{d}{dt} \right) \right) + \gamma^* B$$

where $\hat{\nabla}$ and $\tilde{\nabla}$ are the induced connection on γ^*TM and $\tilde{\gamma}^*TN$ respectively.

Proof. Direct computation shows

$$\begin{aligned} \tilde{\nabla}_{\frac{d}{dt}} \tilde{\gamma}_* \left(\frac{d}{dt} \right) &= \left(\frac{d^2 \tilde{\gamma}^l}{dt^2} + \Gamma_{mn}^l(\tilde{\gamma}) \cdot \frac{d\tilde{\gamma}^m}{dt} \frac{d\tilde{\gamma}^n}{dt} \right) \frac{\partial}{\partial y^l} \\ &= \left\{ \frac{\partial f^l}{\partial x^k} \frac{d^2 \gamma^k}{dt^2} + \left(\frac{\partial^2 f^l}{\partial x^i \partial x^j} + \Gamma_{mn}^l \frac{\partial f^m}{\partial x^i} \frac{\partial f^n}{\partial x^j} \right) \frac{\partial \gamma^i}{dt} \frac{\partial \gamma^j}{dt} \right\} \frac{\partial}{\partial y^l} \\ &= \left\{ \frac{\partial f^l}{\partial x^k} \left(\frac{d^2 \gamma^k}{dt^2} + \Gamma_{ij}^k \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} \right) + \left(\frac{\partial^2 f^l}{\partial x^i \partial x^j} + \frac{\partial f^m}{\partial x^i} \frac{\partial f^n}{\partial x^j} \Gamma_{mn}^l - \Gamma_{ij}^k \frac{\partial f^l}{\partial x^k} \right) \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} \right\} \frac{\partial}{\partial y^l} \\ &= f_* \left(\hat{\nabla}_{\frac{d}{dt}} \gamma_* \left(\frac{d}{dt} \right) \right) + \gamma^* B \end{aligned}$$

□

Theorem 15.1.1. The following statements are equivalent.

- (1) f is totally geodesic.
- (2) f maps geodesics in M to geodesics in N .

Proof. It follows from above lemma.

□

15.2. First variation of smooth map.

Definition 15.2.1 (energy functional). The energy density of f is defined as $e(f) = |df|^2$, and the energy functional of f is

$$E(f) = \frac{1}{2} \int_M e(f) \text{vol}$$

Remark 15.2.1 (local form). Suppose df is locally given by $\frac{\partial f^m}{\partial x^i} dx^i \otimes f^*(\frac{\partial}{\partial y^m})$. Then

$$\begin{aligned} e(f) &= \langle \frac{\partial f^m}{\partial x^i} dx^i \otimes f^*(\frac{\partial}{\partial y^m}), \frac{\partial f^n}{\partial x^j} dx^j \otimes f^*(\frac{\partial}{\partial y^n}) \rangle \\ &= g^{ij} \frac{\partial f^m}{\partial x^i} \frac{\partial f^n}{\partial x^j} h_{mn}(f) \end{aligned}$$

Lemma 15.2.1. $\Delta f = -\widehat{\nabla}^* df \in C^\infty(M, f^*TN)$.

Theorem 15.2.1. The critical point of energy functional is harmonic maps.

Proof. We fix the following notations in the proof:

- (1) Consider a smooth variation $f: M \times \mathbb{R} \rightarrow N$ of f , we also write $f_t(-) = F(-, t)$ for convenience.
- (2) Set $\overline{M} = M \times \mathbb{R}$ and there is a natural metric $\overline{g} = g \times g_{\mathbb{R}}$ on \overline{M} .
- (3) The pullback F^*TN bundle is denoted by W , and induced connection on W is denoted by ∇^W .
- (4) Fix $t \in \mathbb{R}$, $f_t: M \rightarrow N$. Then df_t is a section of $T^*M \otimes f_t^*TN$, and we can regard it as a section of $T^*\overline{M} \otimes W$.

Holding above notations, we have

$$\begin{aligned} \frac{d}{dt} E(f_t) &= \frac{1}{2} \frac{d}{dt} \int_M |df_t|^2 \text{vol} \\ &= \int_M \langle \nabla_{\frac{\partial}{\partial t}}^{T^*\overline{M} \otimes W} df_t, df_t \rangle \text{vol} \end{aligned}$$

Here we claim

$$\langle \nabla_{\frac{\partial}{\partial t}}^{T^*\overline{M} \otimes W} df_t, df_t \rangle \stackrel{1}{=} \langle \nabla_{\frac{\partial}{\partial t}}^{T^*\overline{M} \otimes W} dF, df_t \rangle \stackrel{2}{=} \langle \nabla^W F_*(\frac{\partial}{\partial t}), df_t \rangle$$

- (1) For equation marked 1: Note that

$$\begin{aligned} dF - df_t &= \frac{\partial F^m}{\partial x^i} dx^i \otimes \frac{\partial}{\partial y^m} + \frac{\partial F^m}{\partial t} dt \otimes \frac{\partial}{\partial y^m} - \frac{\partial f_t^m}{\partial x^i} dx^i \otimes \frac{\partial}{\partial y^m} \\ &= \frac{\partial F^m}{\partial t} dt \otimes \frac{\partial}{\partial y^m} \end{aligned}$$

since $\frac{\partial F^m}{\partial x^i} = \frac{\partial f_t^m}{\partial x^i}$. So we have

$$\begin{aligned} \nabla^{T^*\overline{M} \otimes W} (dF - df_t) &= \frac{\partial^2 F^l}{\partial t^2} dt \otimes dt \otimes \frac{\partial}{\partial y^l} + \frac{\partial F^m}{\partial t} dt \otimes (\frac{\partial F^n}{\partial t} \Gamma_{mn}^l dt \otimes \frac{\partial}{\partial y^l} + \frac{\partial F^n}{\partial x^i} \Gamma_{mn}^l dx^i \otimes \frac{\partial}{\partial y^l}) \\ &= (\frac{\partial^2 F^l}{\partial t^2} + \frac{\partial F^m}{\partial t} \frac{\partial F^n}{\partial t} \Gamma_{mn}^l) dt \otimes dt \otimes \frac{\partial}{\partial y^l} + \frac{\partial F^m}{\partial t} \frac{\partial F^n}{\partial x^i} \Gamma_{mn}^l dx^i \otimes dt \otimes \frac{\partial}{\partial y^l} \end{aligned}$$

Thus we have

$$\nabla_{\frac{\partial}{\partial t}}^{T^*\overline{M} \otimes W} (dF - df_t) = \left(\frac{\partial^2 F^l}{\partial t^2} + \frac{\partial F^m}{\partial t} \frac{\partial F^n}{\partial t} \Gamma_{mn}^l \right) dt \otimes \frac{\partial}{\partial y^l}$$

From above expression it's clear

$$\langle \nabla_{\frac{\partial}{\partial t}}^{T^*\overline{M} \otimes W} (dF - df_t), df_t \rangle = 0$$

since there is no dt in df_t , which implies equation marked 1 holds.

- (2) For equation marked 2: For arbitrary $X \in C^\infty(M, TM) \subseteq C^\infty(\overline{M}, T^*\overline{M})$, since second fundamental form is symmetric, thus

$$\begin{aligned} (\nabla_{\frac{\partial}{\partial t}}^{T^*\overline{M} \otimes W} dF)(X) &= (\nabla_X^{T^*\overline{M} \otimes W} dF)\left(\frac{\partial}{\partial t}\right) \\ &= \nabla_X^W F_*\left(\frac{\partial}{\partial t}\right) - F_*\left(\nabla_X^{\overline{M}} \frac{\partial t}{\partial t}\right) \\ &= \nabla_X^W F_*\left(\frac{\partial}{\partial t}\right) \end{aligned}$$

Now let v be an arbitrary variation vector field, that is

$$v = F_*\left(\frac{\partial}{\partial t}\right)\Big|_{t=0} \in C^\infty(M, f^*TN)$$

Hence when $t = 0$ we have

$$\left(\nabla^W F_*\left(\frac{\partial}{\partial t}\right)\right)\Big|_{t=0} = \widehat{\nabla} v$$

where $\widehat{\nabla}$ is the induced connection on f^*TN . So we have first variation formula

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} E(f_t) &= \int_M \langle \widehat{\nabla} v, df \rangle \text{vol} \\ &= \int_M \langle v, \widehat{\nabla}^* df \rangle \text{vol} = 0 \end{aligned}$$

where $\widehat{\nabla}^*$ is the formal adjoint operator of $\widehat{\nabla}$. Since v is arbitrary, we deduce $\widehat{\nabla}^* df = 0$. \square

15.3. Second variation formula of harmonic map. Consider the following variation map of f

$$f: M \times (-\varepsilon_1, \varepsilon_1) \times (-\varepsilon_2, \varepsilon_2) \rightarrow N$$

with variation fields

$$\begin{aligned} v &= F_*\left(\frac{\partial}{\partial t}\right)\Big|_{s=t=0} \in C^\infty(M, f^*TN) \\ w &= F_*\left(\frac{\partial}{\partial s}\right)\Big|_{s=t=0} \in C^\infty(M, f^*TN) \end{aligned}$$

For convenience we denote $F(-, s, t) = f_{s,t}(-)$.

Theorem 15.3.1 (second variation formula). If $f: (M, g) \rightarrow (N, h)$ is a harmonic map. Then the second variation of the harmonic map f along v, w is

$$\frac{\partial^2}{\partial s \partial t} \Big|_{s=t=0} E(f_{s,t}) = \int_M \langle \widehat{\nabla} v, \widehat{\nabla} w \rangle \text{vol} - \int_M g^{ij} R_{pmnq} v^p w^q \frac{\partial f^m}{\partial x^i} \frac{\partial f^n}{\partial x^j} \text{vol}$$

Proof. In this proof, we still use the notations in proof of first variation formula. By first variation formula, we have

$$\frac{\partial}{\partial t} E(f_{s,t}) = \int_M \langle \nabla^W F_* \left(\frac{\partial}{\partial t} \right), df_{s,t} \rangle \text{vol}$$

So

$$\begin{aligned} \frac{\partial^2}{\partial s \partial t} E(f_{s,t}) &= \underbrace{\int_M \langle \nabla_{\frac{\partial}{\partial s}}^{T^* \overline{M} \otimes W} \nabla^W F_* \left(\frac{\partial}{\partial t} \right), df_{s,t} \rangle}_{\text{part I}} \text{vol} \\ &\quad + \underbrace{\int_M \langle \nabla^W F_* \left(\frac{\partial}{\partial t} \right), \nabla_{\frac{\partial}{\partial s}}^{T^* \overline{M} \otimes W} df_{s,t} \rangle}_{\text{part II}} \text{vol} \end{aligned}$$

Note that

$$\begin{aligned} \nabla_{\frac{\partial}{\partial s}}^{T^* \overline{M} \otimes W} df_{s,t} &= \nabla_{\frac{\partial}{\partial s}}^{T^* \overline{M} \otimes W} \left(\frac{\partial F^m}{\partial x^i} dx^i \otimes \frac{\partial}{\partial y^m} \right) \\ &= \frac{\partial^2 F^m}{\partial s \partial x^i} dx^i \otimes \frac{\partial}{\partial y^m} + \frac{\partial F^m}{\partial x^i} \frac{\partial F^n}{\partial s} \Gamma_{mn}^l dx^i \otimes \frac{\partial}{\partial y^l} \\ &= \left(\frac{\partial^2 F^l}{\partial s \partial x^i} + \frac{\partial F^m}{\partial x^i} \frac{\partial F^n}{\partial s} \Gamma_{mn}^l \right) dx^i \otimes \frac{\partial}{\partial y^l} \\ \widehat{\nabla} w &= \widehat{\nabla} \frac{\partial}{\partial x^i} \left(\frac{\partial F^n}{\partial s} \Big|_{t=s=0} \right) dx^i \otimes \frac{\partial}{\partial y^n} + \frac{\partial F^m}{\partial s} \frac{\partial F^n}{\partial x^i} \Big|_{t=s=0} \Gamma_{mn}^l dx^i \otimes \frac{\partial}{\partial y^l} \\ &= \left(\frac{\partial^2 F^l}{\partial s \partial x^i} + \frac{\partial F^m}{\partial x^i} \frac{\partial F^n}{\partial s} \Big|_{t=s=0} \Gamma_{mn}^l \right) dx^i \otimes \frac{\partial}{\partial y^l} \end{aligned}$$

which implies setting $t = s = 0$ we have part II is

$$\int_M \langle \widehat{\nabla} v, \widehat{\nabla} w \rangle \text{vol}$$

For part I, take arbitrary $X \in C^\infty(M, TM) \subseteq C^\infty(\overline{M}, T^* \overline{M})$, we have
Hence we obtain

$$\nabla_{\frac{\partial}{\partial s}}^{T^* \overline{M} \otimes W} \nabla^W F_* \left(\frac{\partial}{\partial t} \right) (X) = (\nabla^{T^* \overline{M} \otimes W} \nabla^W F_* \left(\frac{\partial}{\partial t} \right) (X)) \left(\frac{\partial}{\partial s}, X \right)$$

Setting $t = s = 0$ we have

Hence

$$\begin{aligned} \frac{\partial^2}{\partial s \partial t} \Big|_{t=s=0} E(f_{s,t}) &= \int_M \langle \widehat{\nabla}(\nabla_{\frac{\partial}{\partial s}}^W F_*(\frac{\partial}{\partial t})) \Big|_{s=t=0}, df \rangle \text{vol} \\ &\quad + \int_M g^{ij} R_{pmqn} v^p w^q \frac{\partial f^m}{\partial x^i} \frac{\partial f^n}{\partial x^j} \text{vol} + \int_M \langle \widehat{\nabla} w, \widehat{\nabla} v \rangle \text{vol} \end{aligned}$$

If f is harmonic, that is $\widehat{\nabla}^* df = 0$, we obtain the desired formula. \square

15.4. Bochner formula for harmonic map. Recall that for a smooth function $f: (M, g) \rightarrow \mathbb{R}$,

$$\frac{1}{2} \Delta |df|^2 = |\text{Hess } f|^2 + \text{Ric}(\nabla f, \nabla f) + g(\nabla \Delta f, \nabla f)$$

In this section we generalize this formula to smooth map $f: (M, g) \rightarrow (N, h)$ between Riemannian manifolds, to get similar Bochner's theorem we have proven before.

Theorem 15.4.1. Let $f: (M, g) \rightarrow (N, h)$ be a smooth map between Riemannian manifolds. Then

$$\frac{1}{2} \Delta |df|^2 = |\widetilde{\nabla} df|^2 + \langle \widehat{\nabla}(df), df \rangle + g^{ik} g^{jl} R_{ij} \frac{\partial f^m}{\partial x^k} \frac{\partial f^n}{\partial x^l} h_{mn} - g^{kl} g^{ij} R_{mnpq} \frac{\partial f^m}{\partial x^i} \frac{\partial f^n}{\partial x^j} \frac{\partial f^p}{\partial x^k} \frac{\partial f^q}{\partial x^l}$$

Theorem 15.4.2. Let $f: (M, g) \rightarrow (N, h)$ be a harmonic map between Riemannian manifolds. If

- (1) M is compact with positive Ricci curvature.
- (2) N has non-positive sectional curvature.

Then f is constant.

Proof. Suppose $|df|^2$ attains its maximum at some point $p \in M$, we have

$$\Delta |df|^2(p) \leq 0$$

On the other hand,

$$\frac{1}{2} \Delta |df|^2 \geq g^{ik} g^{jl} R_{ij} \frac{\partial f^m}{\partial x^k} \frac{\partial f^n}{\partial x^l} h_{mn} - g^{kl} g^{ij} R_{mnpq} \frac{\partial f^m}{\partial x^i} \frac{\partial f^n}{\partial x^j} \frac{\partial f^p}{\partial x^k} \frac{\partial f^q}{\partial x^l}$$

since $|\widetilde{\nabla} df|^2 + \langle \widehat{\nabla}(df), df \rangle \geq 0$.

Without loss of generality, we may assume $g_{ij}(p) = \delta_{ij}$, $h_{mn}(f(p)) = \delta_{mn}$ by choosing normal coordinates. Then

$$\frac{1}{2} \Delta |df|^2 \geq \sum_{i,j,m} R_{ij} \frac{\partial f^m}{\partial x^i} \frac{\partial f^m}{\partial x^j} - \sum_{i,j} R_{mnpq} \frac{\partial f^m}{\partial x^i} \frac{\partial f^n}{\partial x^i} \frac{\partial f^p}{\partial x^j} \frac{\partial f^q}{\partial x^j} \geq 0$$

since R_{ij} is positive, which implies $df \equiv 0$, thus f is constant since we always assume M is connected. \square

Corollary 15.4.1. Let (M, g) be a compact Riemannian manifold with non-negative Ricci curvature, (N, h) a Riemannian manifold with non-positive sectional curvature, and $f: (M, g) \rightarrow (N, h)$ a harmonic map. Then

- (1) f is totally geodesic.

- (2) If $\text{Ric}(g)$ is strictly positive at some point. Then f is constant.
- (3) If sectional curvature of h is negative. Then either f is constant or its image is a closed geodesic.

Part 6. Topology of Riemannian manifold

16. ISOMETRY GROUP

16.1. Isometry and local isometry.

Definition 16.1.1 (local isometry). A smooth map $f: (M, g_M) \rightarrow (N, g_N)$ between Riemannian manifolds is called a local isometry if for each point $p \in M$ the differential $(df)_p: T_p M \rightarrow T_{f(p)} N$ is a linear isometry.

Definition 16.1.2 (isometry). A local isometry $f: (M, g_M) \rightarrow (N, g_N)$ between Riemannian manifolds is called an isometry if it's a diffeomorphism.

Proposition 16.1.1. Let $f: (M, g_M) \rightarrow (N, g_N)$ be smooth map between Riemannian manifolds. The following statements are equivalent.

- (1) f is a local isometry.
- (2) For each $p \in M$, there are open neighborhoods U of p and V of $f(p)$ such that $f|_U: U \rightarrow V$ is an isometry.

Proof. It's clear (2) implies (1), and the converse follows immediately from inverse function theorem. \square

Definition 16.1.3 (isometry group). The isometry group $\text{Iso}(M, g)$ of Riemannian manifold (M, g) is the group consisting of all isometries from (M, g) to itself.

Proposition 16.1.2. Let $f: (M, g_M) \rightarrow (N, g_N)$ be a local isometry.

- (1) f maps geodesics to geodesics.
- (2) $f \circ \exp_p = \exp_{f(p)} \circ (df)_p$ holds on V_p .
- (3) f is distance decreasing.
- (4) If f is an isometry, then f is distance preserving.

Proof. See [Pet16, Proposition 5.6.1]. \square

Remark 16.1.1. A non-trivial fact is that any bijective map which preserves distance is an isometry. See [MS39] or [Pal57].

Theorem 16.1.1. Let $\varphi, \psi: (M, g_M) \rightarrow (N, g_N)$ be two local isometries between Riemannian manifolds, and M is connected. If there exists $p \in M$ such that

$$\begin{aligned}\varphi(p) &= \psi(p) \\ (d\varphi)_p &= (d\psi)_p\end{aligned}$$

then $\varphi = \psi$.

Proof. Consider the following set

$$A = \{p \in M \mid \psi(p) = \varphi(p), (d\psi)_p = (d\varphi)_p\}$$

Since M is connected, it suffices to show it's both open and closed.

- (1) To see it's open: For $p \in A$, let $B(p, \delta)$ be a geodesic ball centered at p such that $\varphi|_{B(p, \delta)}, \psi|_{B(p, \delta)}$ are diffeomorphisms. Then

$$f := (\varphi^{-1} \circ \psi)|_{B(p, \delta)}: B(p, \delta) \rightarrow B(p, \delta)$$

satisfies $f(p) = p, (df)_p = \text{id}$. For any $q \in B(p, \delta)$, there exists a unique $v \in T_p M$ such that $\exp_p(v) = q$. Then

$$\begin{aligned} f(q) &= f \circ \exp_p(v) \\ &= \exp_{f(p)} \circ (df)_p(v) \\ &= \exp_p(v) \\ &= q \end{aligned}$$

which implies $q \in A$, and thus A is open.

- (2) To see it's closed: Suppose $\{p_i\}_{i=1}^n \subseteq A$, that is

$$\begin{aligned} \psi(p_i) &= \varphi(p_i) \\ (d\psi)_{p_i} &= (d\varphi)_{p_i} \end{aligned}$$

The desired result can be obtained from taking limits. □

Theorem 16.1.2 (Myers-Steenrod). Let (M, g) be a Riemannian manifold and $G = \text{Iso}(M, g)$. Then

- (1) G is a Lie group with respect to compact-open topology.
- (2) for each $p \in M$, the isotropy group G_p is compact.
- (3) G is compact if M is compact.

Proof. See [MS39]. □

16.2. Killing field as Lie algebra of isometry group.

Proposition 16.2.1. Let (M, g) be a Riemannian manifold and X a Killing field.

- (1) If γ is a geodesic, then $J(t) = X(\gamma(t))$ is a Jacobi field.
- (2) For any two vector fields Y, Z ,

$$\nabla_Y \nabla_Z X - \nabla_{\nabla_Y Z} X + R(X, Y)Z = 0$$

Proof. For (1). Suppose φ_s is the flow generated by X . Then we obtain a variation $\alpha(s, t) = \varphi_s(\gamma(t))$ consisting of geodesics, thus

$$X(\gamma(t)) = \left. \frac{\partial \varphi_s(\gamma(t))}{\partial s} \right|_{s=0}$$

is a Jacobi field.

For (2). It's an equation of tensors, so we check it pointwisely and use normal coordinate $\{x^i\}$ centered at p . Furthermore, we assume $X = X^i \frac{\partial}{\partial x^i}, Y =$

$\frac{\partial}{\partial x^j}, Z = \frac{\partial}{\partial x^k}$. Then

$$\begin{aligned} \nabla_Y \nabla_Z X - \nabla_{\nabla_Y Z} X + R(X, Y)Z &= \nabla_j \nabla_k X + X^i R_{ijk}^l \frac{\partial}{\partial x^l} \\ &= \left(\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma_{ki}^l}{\partial x^j} + X^i R_{ijk}^l \right) \frac{\partial}{\partial x^l} \\ &= \left(\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma_{jk}^l}{\partial x^i} \right) \frac{\partial}{\partial x^l} \end{aligned}$$

since $R_{ijk}^l = \frac{\partial \Gamma_{jk}^l}{\partial x^i} - \frac{\partial \Gamma_{ik}^l}{\partial x^j} + \Gamma_{jk}^s \Gamma_{is}^l - \Gamma_{ik}^s \Gamma_{js}^l$. Now it suffices to show $\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma_{jk}^l}{\partial x^i} \equiv 0$. In order to show this, for arbitrary $p \in M$, consider a geodesic γ starting at p and consider Jacobi field $J(t) = X(\gamma(t))$. Direct computation shows

$$\begin{aligned} J'(t) &= \left(\frac{\partial X^i}{\partial x^k} \frac{d\gamma^k}{dt} + X^i \Gamma_{ki}^l \frac{d\gamma^k}{dt} \right) \frac{\partial}{\partial x^l} \Big|_{\gamma(t)} \\ J''(0) &= \left(\frac{\partial^2 X^l}{\partial x^j \partial x^k} \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} + X^i \frac{\partial \Gamma_{ki}^l}{\partial x^j} \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} \right) \frac{\partial}{\partial x^l} \Big|_p \\ &= \left(\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma_{ki}^l}{\partial x^j} \right) \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} \frac{\partial}{\partial x^l} \Big|_p \\ &= \left(\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma_{jk}^l}{\partial x^i} + X^i \frac{\partial \Gamma_{ki}^l}{\partial x^j} - X^i \frac{\partial \Gamma_{jk}^l}{\partial x^i} \right) \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} \frac{\partial}{\partial x^l} \Big|_p \\ &= \left(\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma_{jk}^l}{\partial x^i} \right) \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} \frac{\partial}{\partial x^l} \Big|_p - R(X, \gamma')\gamma' \end{aligned}$$

which implies

$$\frac{\partial^2 X^l}{\partial x^j \partial x^k} + X^i \frac{\partial \Gamma_{jk}^l}{\partial x^i} = 0$$

holds at point p , and since p is arbitrary, this completes the proof. \square

Corollary 16.2.1. Let (M, g) be a complete Riemannian manifold and $p \in M$. Then a Killing field X is determined by the values X_p and $(\nabla X)_p$ for arbitrary $p \in M$.

Proof. The equation $\mathcal{L}_X g \equiv 0$ is linear in X , so the space of Killing vector fields is a vector space. Therefore, it suffices to show if $X_p = 0$ and $(\nabla X)_p = 0$, then $X \equiv 0$. For arbitrary $q \in M$, let $\gamma: [0, 1] \rightarrow M$ be a geodesic connecting p and q with $\gamma'(0) = v$. Since $J(t) = X(\gamma(t))$ is a Jacobi field, and a direct computation shows

$$(\nabla_v X)_p = J'(0)$$

Thus $J(t) \equiv 0$, since Jacobi field is determined by two initial values. In particular, $X_q = J(1) = 0$, and since q is arbitrary, one has $X \equiv 0$. \square

Corollary 16.2.2. The dimension of vector field consisting of Killing field $\leq \frac{n(n+1)}{2}$.

Proof. Note that ∇X is skew-symmetric, and the dimension of skew-symmetric matrices is $\frac{n(n-1)}{2}$, thus the dimension of vector field consisting of Killing field

$$\leq n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

□

Lemma 16.2.1. Killing field on a complete Riemannian manifold (M, g) is complete.

Proof. Let X be a Killing field, we need to show the flow $\varphi_t: M \rightarrow M$ generated by X is defined for $t \in \mathbb{R}$. Otherwise, we assume φ_t is defined on (a, b) . Note that for each $p \in M$, curve $\varphi_t(p)$ is a curve defined on (a, b) having finite constant speed, since φ_t is isometry. Then we have $\varphi_t(p)$ can be extended to the one defined on \mathbb{R} , since M is complete. □

Theorem 16.2.1. Let (M, g) be a complete Riemannian manifold and \mathfrak{g} the space of Killing fields. Then \mathfrak{g} is isomorphic to the Lie algebra of $G = \text{Iso}(M, g)$.

Proof. It's clear \mathfrak{g} is a Lie algebra since $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X, Y]}$. Now let's show second statement.

- (1) Given a Killing field X , by Lemma 16.2.1, one deduces that the flow $\varphi: \mathbb{R} \times M \rightarrow M$ generated by X is a one parameter subgroup $\gamma: \mathbb{R} \rightarrow G$, and $\gamma'(0) \in T_e G$.
- (2) Given $v \in T_e G$, consider the one-parameter subgroup $\gamma(t) = \exp(tv): \mathbb{R} \rightarrow G$ which gives a flow by

$$\begin{aligned} \varphi: \mathbb{R} \times M &\rightarrow M \\ (t, p) &\mapsto \exp(tv) \cdot p \end{aligned}$$

Then the vector field X generated by this flow is a Killing field.

This gives a one to one correspondence between Killing fields and Lie algebra of G , and it's a Lie algebra isomorphism. □

16.3. Cartan decomposition of Killing fields. Let (M, g) be a complete Riemannian manifold and $G = \text{Iso}(M, g)$ with Lie algebra \mathfrak{g} . Together with Corollary 16.2.1 and Theorem 16.2.1, we can decompose Lie algebra \mathfrak{g} of isometry group G as direct sum of following vector spaces

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

where

$$\begin{aligned} \mathfrak{k} &= \{X \in \mathfrak{g} \mid X_p = 0\} \\ \mathfrak{m} &= \{X \in \mathfrak{g} \mid (\nabla X)_p = 0\} \end{aligned}$$

A direct computation shows

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$$

For arbitrary $X \in \mathfrak{k}$, $Y \in \mathfrak{m}$ and $v \in T_p M$, one has

$$\begin{aligned}\nabla_v[X, Y] &= \nabla_v \nabla_X Y - \nabla_v \nabla_Y X \\ &= -R(Y, v)X + \nabla_{\nabla_v X} Y + R(X, v)Y - \nabla_{\nabla_v Y} X \\ &= 0\end{aligned}$$

since $X_p = 0$ and $(\nabla Y)_p = 0$. This shows $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$.

Theorem 16.3.1. Let (M, g) be a Riemannian symmetric space and G the isometry group. For any $p \in M$, the Lie algebra of the isotropy subgroup G_p is isomorphic to

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid X_p = 0\}$$

where \mathfrak{g} is the Lie algebra of G .

Proof. Let $X \in \mathfrak{g}$ with $X_p = 0$, and $\varphi_t : M \rightarrow M$ the flow of X . It suffices to show $\varphi_t(p) = p$ for all $t \in \mathbb{R}$. We use $\gamma_p(t)$ to denote $\varphi_t(p)$. Then for any smooth function $f : M \rightarrow \mathbb{R}$ and $s \in \mathbb{R}$, we have

$$\begin{aligned}\gamma'_p(s)f &= \left. \frac{d}{dt} \right|_{t=s} f \circ \gamma_p(t) \\ &= \left. \frac{d}{dt} \right|_{t=0} f \circ \gamma_p(t+s) \\ &= \left. \frac{d}{dt} \right|_{t=0} f \circ \varphi_s \circ \varphi_t(p) \\ &= \left. \frac{d}{dt} \right|_{t=0} (f \circ \varphi_s)(\gamma_p(t)) \\ &= \gamma'_p(0)(f \circ \varphi_s) \\ &= X_p(f \circ \varphi_s) \\ &= 0\end{aligned}$$

Hence $\gamma'_p(s) = 0$ for all $s \in \mathbb{R}$, thus $\gamma_p(s)$ is constant, which implies $\gamma_p(s) = \gamma_p(0) = p$. \square

In order to describe \mathfrak{m} , we need to introduce transvection.

Definition 16.3.1 (transvection). Let (M, g) be a Riemannian symmetric space and γ a geodesic. The transvection along γ is defined as

$$T_t = s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}$$

where s_p is the symmetry at point p .

Proposition 16.3.1. Let (M, g) be a Riemannian symmetric space, γ a geodesic and T_t the transvection along γ . Then

- (1) For any $a, t \in \mathbb{R}$, $s_{\gamma(a)}(\gamma(t)) = \gamma(2a - t)$.
- (2) T_t translates the geodesic γ , that is $T_t(\gamma(s)) = \gamma(t + s)$.
- (3) $(dT_t)_{\gamma(s)} : T_{\gamma(s)}M \rightarrow T_{\gamma(t+s)}M$ is the parallel transport $P_{s, t+s}^\gamma$.
- (4) T_t is one-parameter subgroup of $\text{Iso}(M, g)$.

Proof. For (1). It follows from the uniqueness of geodesics with given initial value.

For (2). Note that

$$\begin{aligned} T_t(\gamma(s)) &= s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}(\gamma(s)) \\ &= s_{\gamma(\frac{t}{2})}(\gamma(-s)) \\ &= \gamma(t+s) \end{aligned}$$

For (3). Let X be a parallel vector field along γ . By uniqueness of parallel vector fields with given initial data, we have $(ds_{\gamma(0)})_{\gamma(s)}X_{\gamma(s)} = -X_{\gamma(-s)}$ for all s , since $(ds_{\gamma(0)})_{\gamma(0)}X_{\gamma(0)} = -X_{\gamma(0)}$. Thus

$$\begin{aligned} (dT_t)_{\gamma(s)}X_{\gamma(s)} &= (ds_{\gamma(\frac{t}{2})})_{\gamma(-s)}(-X_{\gamma(-s)}) \\ &= X_{\gamma(t+s)} \end{aligned}$$

This shows $(dT_t)_{\gamma(s)} = P_{s,t+s}^\gamma$.

For (4). In order to show $T_{t+s} = T_t \circ T_s$, it suffices to check they're same at some point, so do their derivatives, since isometry can be determined by these two values. Note that

$$\begin{aligned} T_{t+s}(\gamma(0)) &= \gamma(t+s) \\ &= T_t \circ T_s(\gamma(0)) \\ (dT_{t+s})_{\gamma(0)} &= P_{0,t+s}^\gamma \\ &= P_{s,t+s}^\gamma \circ P_{0,s}^\gamma \\ &= (dT_t)_{\gamma(s)} \circ (dT_s)_{\gamma(0)} \\ &= (d(T_t \circ T_s))_{\gamma(0)} \end{aligned}$$

This completes the proof. \square

Definition 16.3.2 (infinitesimal transvection). Let (M, g) be a Riemannian symmetric space. For any point $p \in M$ and any $v \in T_p M$, the infinitesimal generator X of transvections T_t along γ_v is given by

$$X_q = \left. \frac{d}{dt} \right|_{t=0} T_t(q)$$

This Killing field X is called an infinitesimal transvection.

Theorem 16.3.2. Let (M, g) be a Riemannian symmetric space and X an infinitesimal transvection of transvection T_t along geodesic $\gamma = \exp_p(tv)$. Then

$$X_p = v, \quad (\nabla X)_p = 0$$

Corollary 16.3.1. The space of infinitesimal transvection is exactly \mathfrak{m} , and there is an isomorphism between $\mathfrak{m} \cong T_p M$ given by $X \mapsto X_p$.

16.4. Cartan-Ambrose-Hicks theorem.

Theorem 16.4.1 (Cartan-Ambrose-Hicks). Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be two Riemannian manifold, and $\Phi_0: T_p M \rightarrow T_{\widetilde{p}} \widetilde{M}$ is a linear isometry, where $p \in M, \widetilde{p} \in \widetilde{M}$. For $0 < \delta < \min\{\text{inj}_p(M), \text{inj}_{\widetilde{p}}(\widetilde{M})\}$, The following statements are equivalent.

- (1) There exists an isometry $\varphi: B(p, \delta) \rightarrow B(\widetilde{p}, \delta)$ such that $\varphi(p) = \widetilde{p}$ and $(d\varphi)_p = \Phi_0$.
- (2) For $v \in T_p M, |v| < \delta, \gamma(t) = \exp_p(tv), \widetilde{\gamma}(t) = \exp_{\widetilde{p}}(t\Phi_0(v))$, if we define

$$\Phi_t = P_{0,t;\widetilde{\gamma}} \circ \Phi_0 \circ P_{t,0;\gamma}: T_{\gamma(t)} M \rightarrow T_{\widetilde{\gamma}(t)} \widetilde{M}$$

then Φ_t preserves curvature, that is $(\Phi_t)^* R = R$.

Proof. From (1) to (2). If we can show $\Phi_t = (d\varphi)_{\gamma(t)}$. Then it's clear that Φ_t preserves curvature, since φ is an isometry. By definition of Φ_t , it suffices to show the following diagram commutes

$$\begin{array}{ccc} T_p M & \xrightarrow{(d\varphi)_p} & T_{\widetilde{p}} \widetilde{M} \\ \downarrow P_{0,t}^\gamma & & \downarrow P_{0,t;\widetilde{\gamma}} \\ T_{\gamma(t)} M & \xrightarrow{(d\varphi)_{\gamma(t)}} & T_{\widetilde{\gamma}(t)} \widetilde{M} \end{array}$$

since $(d\varphi)_p = \Phi_0$. Note that $\varphi(\gamma(t)) = \widetilde{\gamma}(t)$ since both of them are geodesics, and they and their derivatives agree at $t = 0$. So it's tautological that

$$P_{0,t}^{\varphi \circ \gamma} \circ (d\varphi)_p(v) = (d\varphi)_{\gamma(t)} \circ P_{0,t}^\gamma(v)$$

where $v = \gamma'(0)$, since

$$\begin{aligned} P_{0,t}^\gamma(v) &= \gamma'(t) \\ (d\varphi)_{\gamma(t)}(\gamma'(t)) &= (\varphi \circ \gamma)'(t) = P_{0,t}^{\varphi \circ \gamma} \circ (d\varphi)_p(v) \end{aligned}$$

Now consider $w \in T_p M$ which is not parallel to $v = \gamma'(0)$. Since both $(d\varphi)_{\gamma(t)}$ and parallel transport preserve angles, so $P_{0,t}^{\varphi \circ \gamma} \circ (d\varphi)_p(w)$ and $(d\varphi)_{\gamma(t)} \circ P_{0,t}^\gamma(w)$ has the same angle with $(d\varphi)_{\gamma(t)}(\gamma'(t))$, and they have the same length, so they're equal.

From (2) to (1). Define

$$\varphi = \exp_{\widetilde{p}} \circ \Phi_0 \circ \exp_p^{-1}$$

It suffices to show for any $q \in B(p, \delta)$,

$$(d\varphi)_q: T_q M \rightarrow T_{\varphi(q)} \widetilde{M}$$

is a linear isometry. For any $w \in T_q M$, by Corollary 12.1.1, there exists a geodesic $\gamma: [0, 1] \rightarrow M$ with $\gamma(0) = p, \gamma(1) = q$ and a Jacobi field J such that $J(0) = 0, J(1) = w$ along γ . Now we claim:

- (1) **Claim 1:** $\widetilde{J}(t) = \Phi_t(J(t))$ is a Jacobi field.
- (2) **Claim 2:** $\widetilde{J}(1) = (d\varphi)_q(J(1))$.

From claim 2 we have

$$|(\mathrm{d}\varphi)_q(w)| = |\tilde{J}(1)| = |J(1)| = |w|$$

since Φ_t preserves length. This completes the proof. Now let's give proofs of these two claims.

- (1) **Proof of Claim 1:** Given an orthonormal $\{e_1(0) = \frac{\gamma'(0)}{|\gamma'(0)|}, e_2(0), \dots, e_n(0)\}$ of $T_p M$ and use parallel transport to obtain a parallel frame along γ . With respect to this frame we can write $J(t) = J^i(t)e_i(t)$. Then $\tilde{J}(t) = J^i(t)\tilde{e}_i(t)$, where $\tilde{e}_i(t) = \Phi_t(e_i(t))$. Furthermore, $\tilde{e}_i(t)$ is also a parallel frame by definition of Φ_t . Then $\tilde{J}(t)$ is a Jacobi field, since

$$\begin{aligned} & \frac{\mathrm{d}^2 J^j}{\mathrm{d}t^2} + J^i(t)|\tilde{\gamma}(t)|^2 \tilde{R}(\tilde{e}_i(t), \tilde{e}_1(t), \tilde{e}_j(t), \tilde{e}_1(t)) \\ &= \frac{\mathrm{d}^2 J^j}{\mathrm{d}t^2} + J^i(t)|\gamma(t)|^2 R(e_i(t), e_1(t), e_j(t), e_1(t)) \\ &= 0 \end{aligned}$$

holds for arbitrary j , where we use the fact Φ_t preserves the length and curvature, and $J(t)$ is a Jacobi field.

- (2) **Proof of Claim 2:** Since $\tilde{J}(t) = \Phi_t(J(t))$. Then $\tilde{J}'(0) = \Phi_0 J'(0)$. On the other hand, by Corollary one has

$$\begin{aligned} J(t) &= (\mathrm{d} \exp_p)_{t\gamma'(0)}(tJ'(0)) \\ \tilde{J}(t) &= (\mathrm{d} \exp_{\tilde{p}})_{t\tilde{\gamma}'(0)}(t\tilde{J}'(0)) \end{aligned}$$

Therefore

$$\begin{aligned} \tilde{J}(1) &= (\mathrm{d} \exp_{\tilde{p}})_{\tilde{\gamma}'(0)} \circ \Phi_0(J'(0)) \\ &= (\mathrm{d} \exp_{\tilde{p}})_{\tilde{\gamma}'(0)} \circ \Phi_0 \circ (\mathrm{d} \exp_p)_{\gamma'(0)}^{-1}(J(1)) \end{aligned}$$

which completes the proof of claim 2. □

17. COVERING SPACES

17.1. Topological covering. In this section we mainly follow [Hat02], and we always assume X is a path connected topological space.

Definition 17.1.1 (covering space). A (topological) covering of X is a continuous map $\pi: \tilde{X} \rightarrow X$ such that there exists a discrete space D and for each $x \in X$ an open neighborhood $U \subseteq X$, such that $\pi^{-1}(U) = \coprod_{d \in D} V_d$ and $\pi|_{V_d}: V_d \rightarrow U$ is a homeomorphism for each $d \in D$. Furthermore,

- (1) The open sets V_d are called sheets.
- (2) For each $x \in X$, the discrete subset $x \in X$ is called the fiber of x .
- (3) The degree of the covering is the cardinality of the space D .

Proposition 17.1.1 (homotopy lifting property). Given a covering space $\pi: \tilde{X} \rightarrow X$, a homotopy $f_t: Y \rightarrow X$ and a map $\tilde{f}_0: Y \rightarrow \tilde{X}$ lifting f_0 . Then there exists a unique homotopy $\tilde{f}_t: Y \rightarrow \tilde{X}$ lifts f_t .

Remark 17.1.1. Note that above statement says if there is a lift of f_0 . Then there is a unique homotopy which lifts f_t . However, what's the existence and uniqueness of such lifts? Here are two results:

- (1) Existence: If Y is path connected and locally path connected⁶. Then a lift $\tilde{f}: Y \rightarrow \tilde{X}$ of f exists if and only if $f_*(\pi_1(Y)) \subseteq \pi_*(\pi_1(\tilde{X}))$.
- (2) Uniqueness: If two lifts $\tilde{f}, \tilde{g}: Y \rightarrow \tilde{X}$ of f agree at one point of Y and Y is connected. Then \tilde{f} and \tilde{g} agree on all Y .

Remark 17.1.2. Here are two special cases:

- (1) Taking Y to be a point gives the path lifting property.
- (2) Taking Y to be I , we see that every homotopy f_t of a path f_0 in X lifts to a homotopy \tilde{f}_t of each lift \tilde{f}_0 of f_0 .

Corollary 17.1.1. The map $\pi_*: \pi_1(\tilde{X}) \rightarrow \pi_1(X)$ induced by $\pi: \tilde{X} \rightarrow X$ is injective. Furthermore,

- (1) $\pi_*(\pi_1(\tilde{X}))$ consists of the homotopy class of loops in X whose lifts to \tilde{X} are still loops.
- (2) The index of $\pi_*(\pi_1(\tilde{X}))$ in $\pi_1(X)$ is the degree of covering.

Definition 17.1.2 (universal covering). A simply-connected covering space of X is called universal covering.

Corollary 17.1.2. The degree of universal covering equals $|\pi_1(X)|$.

Definition 17.1.3 (deck transformation). Let $\pi: \tilde{X} \rightarrow X$ be a covering, the deck transformation group of this covering is defined as

$$\text{Aut}_\pi(\tilde{X}) = \{f: \tilde{X} \rightarrow \tilde{X} \text{ is homeomorphism} \mid \pi \circ f = \pi\}$$

⁶Note that a path connected space may not be locally path connected. For example, the topologist's sine curve.

Definition 17.1.4. A covering $\pi: \tilde{X} \rightarrow X$ is called normal, if deck transformation is transitive on each fiber of $x \in X$.

Corollary 17.1.3. If $\pi: \tilde{X} \rightarrow X$ is a normal covering. Then X is homeomorphic to $\tilde{X}/\pi_1(X)$.

Proposition 17.1.2. Let $\pi: \tilde{X} \rightarrow X$ be a path-connected covering space of the path-connected, locally path-connected space X , and let H be $\pi_*(\pi_1(\tilde{X}))$. Then

- (1) This covering space is normal if and only if H is a normal subgroup of $\pi_1(X)$.
- (2) The group of deck transformation is isomorphic to the quotient $N(H)/H$, where $N(H)$ is the normalizer of H .
- (3) In particular, the group of deck transformation is isomorphic to $\pi_1(X)$, if \tilde{X} is universal covering.

Corollary 17.1.4. The universal covering is a normal covering. In particular, X is homeomorphic to $\tilde{X}/\pi_1(X)$.

The group of deck transformation is a special case of the general notation of groups acting on spaces.

Definition 17.1.5 (group acting on space). Given a group G and a topological space X . Then an action of G on X is a homomorphism ρ from G to the group $\text{Homeo}(X)$ consisting of all homeomorphisms from X to itself.

Remark 17.1.3. Thus to each $g \in G$ is associated a homeomorphism $\rho(g): X \rightarrow X$, which for notational simplicity we write simply as $g: X \rightarrow X$.

Definition 17.1.6 (properly discontinuous). An action is called properly discontinuous if each $x \in X$ has a neighborhood U such that all images $g(U)$ for varying $g \in G$ are disjoint, that is, $g_1(U) \cap g_2(U) \neq \emptyset$ implies $g_1 = g_2$.

Remark 17.1.4. If an action is properly discontinuous. Then it's free. Indeed, if G acts on X properly discontinuous and there exists $g \in G$ such that $gx = x$ for all $x \in X$. Then for arbitrary neighborhood U of x , one has $g(U) \cap U \neq \emptyset$, thus $g = e$.

Proposition 17.1.3. For a covering $\pi: \tilde{X} \rightarrow X$, the group of deck transformation $\text{Aut}_\pi(\tilde{X})$ acts on \tilde{X} properly discontinuous.

In a summary, we have the group of deck transformations acts on covering space

- (1) homeomorphically.
- (2) transitively.
- (3) properly discontinuous. In particular, freely.

17.2. Riemannian covering.

Definition 17.2.1 (smooth covering). A smooth map $\pi: \widetilde{M} \rightarrow M$ between smooth manifolds is called a smooth covering, if

- (1) π is a topological covering.
- (2) π is a local diffeomorphism.

Proposition 17.2.1. Let $f: \widetilde{M} \rightarrow M$ be a local diffeomorphism between smooth manifolds which is proper. Then f is a smooth covering.

Definition 17.2.2 (Riemannian covering). A smooth map $\pi: (\widetilde{M}, \widetilde{g}) \rightarrow (M, g)$ between Riemannian manifolds is called a Riemannian covering if

- (1) π is a smooth covering map.
- (2) π is a local isometry.

Example 17.2.1. Let (M, g) be a Riemannian manifold with smooth covering $\pi: \widetilde{M} \rightarrow M$. Then $\pi: (\widetilde{M}, \widetilde{g}) \rightarrow (M, g)$ is a Riemannian covering where $\widetilde{g} = \pi^*g$.

Proposition 17.2.2. Let $\pi: (\widetilde{M}, \widetilde{g}) \rightarrow (M, g)$ be a Riemannian universal covering with deck transformation $\Gamma \subseteq \text{Iso}(\widetilde{M}, \widetilde{g})$. Then

- (1) (M, g) is isometric to $(\widetilde{M}/\Gamma, \widetilde{g})$.
- (2) Γ acts on \widetilde{M} isometrically, transitively and properly discontinuously.

Proposition 17.2.3. If $\pi: (\widetilde{M}, \widetilde{g}) \rightarrow (M, g)$ is a Riemannian covering, then M is complete if and only if \widetilde{M} is.

Proof. □

Proposition 17.2.4. Let (M, g_M) be a complete Riemannian manifold, and $f: M \rightarrow N$ is a local diffeomorphism such that for all $p \in M$ and for all $v \in T_p M$, one has $|(df)_p v| \geq |v|$. Then f is a Riemannian covering map.

Proof. It suffices to show f has path lifting property, that is, for a smooth curve $c: [0, 1] \rightarrow N$ and $p \in M$ such that $f(p) = c(0)$, there exists a curve $\tilde{c}: [0, 1] \rightarrow M$ such that $f \circ \tilde{c} = c$ and $\tilde{c}(0) = p$. Let A be the set of values such that c can be lifted to a curve defined on A . It's clear $A \subseteq [0, 1]$ and $A \neq \emptyset$. Now it suffices to show A is both open and closed.

- (1) To see it's open: For $x \in A$, note that f is a local diffeomorphism at $c(x)$, so there exists an open interval I of x such that c can be lifted on I , this shows A is open.
- (2) To see it's closed: It suffices to show if an increasing sequence $\{x_i\}_{i=1}^\infty \subseteq A$ converging to x , then one has $x \in A$. Firstly we claim $\{\tilde{c}(x_i)\}_{i=1}^\infty$ is contained in a compact set K . If not, the distance from $\tilde{c}(x_i)$ to $\tilde{c}(0)$

can be arbitrary large since M is complete. However,

$$\begin{aligned}
 l_{0,x_i}(c) &= \int_0^{x_i} \left| \frac{dc}{dt} \right| dt \\
 &= \int_0^{x_i} |(df)_{\tilde{c}(t)}| \frac{d\tilde{c}}{dt} dt \\
 &\geq \int_0^{x_i} \left| \frac{d\tilde{c}}{dt} \right| dt \\
 &\geq \text{dist}(\tilde{c}(x_i), \tilde{c}(0))
 \end{aligned}$$

implies the length of c between $\tilde{c}(x_i)$ and $\tilde{c}(0)$ is bounded, a contradiction. Since $\{\tilde{c}(x_i)\}_{i=1}^\infty$ is contained in a compact set K , there exists a converging subsequence, still denoted by $\{\tilde{c}(x_i)\}_{i=1}^\infty$ which converges to $r \in M$, and by continuity one has $f(r) = c(x)$. Let V be a neighborhood of r such that $f|_V$ is a diffeomorphism. Then one has $c(x) \in f(V)$, and by continuity there exists an open interval I of x such that $c(I) \subseteq f(V)$. Pick an index n such that $\tilde{c}(x_n) \in V$, there is a lifting of segment $c: [x_n, x] \rightarrow M$ since $f|_V$ is a diffeomorphism, which implies c can be lifted to a curve defined on $[0, x]$, that is, $x \in A$.

□

Corollary 17.2.1. Let (M, g_M) be a complete Riemannian manifold, and $f: (M, g_M) \rightarrow (N, g_N)$ is a local isometry. Then f is a Riemannian covering map.

18. TOPOLOGY OF NON-POSITIVE SECTIONAL CURVATURE MANIFOLD

18.1. Cartan-Hadamard manifold.

Definition 18.1.1 (Cartan-Hadamard manifold). A simply-connected, complete Riemannian manifold with non-positive sectional curvature is called Cartan-Hadamard manifold.

18.1.1. Expansion property of exponential map.

Theorem 18.1.1. Let (M, g) be a simply-connected complete Riemannian manifold. The following statements are equivalent.

- (1) M is Cartan-Hadamard manifold.
- (2) For any $p \in M$ and $v, w \in T_p M$, we have

$$|(\mathrm{d} \exp_p)_v w| \geq |w|$$

- (3) For any $p \in M, T > 0$ and $v, w \in T_p M$, we have

$$|v - w| \leq \frac{\mathrm{dist}(\exp_p(tv), \exp_p(tw))}{t}$$

holds for arbitrary $t > 0$.

Proof. From (1) to (2). For all $p \in M$ and $v, w \in T_p M$, $J(t) = (\mathrm{d} \exp_p)_{tv}(tw)$ is a Jacobi field along $\exp_p(tv)$. If M has non-positive sectional curvature, direct computation shows

$$|J(t)|'' = \frac{|J|^2 |J'|^2 - \langle J, J' \rangle^2}{|J|^3} - \frac{R(J, \gamma', \gamma', J)}{|J|} \geq 0$$

for all $t > 0$. Thus consider

$$f(t) = |J(t)| - t|w|$$

It's clear $f''(t) \geq 0$ and $f'(0) = 0$, thus $f(t) \geq 0$ for all $t > 0$ since $f(0) = 0$. In particular, set $t = 1$ we have

$$|(\mathrm{d} \exp_p)_v(w)| - |w| \geq 0$$

From (2) to (1). If M has sectional curvature $K(\sigma) > 0$ at $p \in M$, where σ is the plane spanned by v, w with $|v| = |w| = 1$. Then consider geodesic $\exp_p(tv)$ and Jacobi field

$$J(t) = (\mathrm{d} \exp_p)_{tv}(tw)$$

along it. Then by Proposition 14.1.1 we have $|J(t)|'' < 0$ for sufficiently small t . If we set $f(t) = |J(t)| - t|w|$. Then we can see $f(0) = 0, f'(0) = 0$ and $f''(0) < 0$ for sufficiently small t . In particular, we have

$$|(\mathrm{d} \exp_p)_{\varepsilon v}(\varepsilon w)| - |\varepsilon w| = f(\varepsilon) < 0$$

where $\varepsilon > 0$ is sufficiently small. This leads to a contradiction.

From (2) to (3). For arbitrary $t > 0$. Let $\gamma(s) : [0, 1] \rightarrow M$ be a geodesic connecting $\exp_p(tv), \exp_p(tw)$ and choose a curve $v(s) \in T_p M$ such that

$$\exp_p(v(s)) = \gamma(s)$$

for all $s \in [0, 1]$. Hence $v(0) = tv, v(1) = tw$. Then

$$\begin{aligned} \text{dist}(\exp_p(tv), \exp_p(tw)) &= \int_0^1 |\gamma'(s)| ds \\ &= \int_0^1 |(\text{d exp}_p)_{v(s)}(v'(s))| ds \\ &\geq \left| \int_0^1 v'(s) ds \right| \\ &= t|v - w| \end{aligned}$$

This shows

$$|v - w| \leq \frac{\text{dist}(\exp_p(tv), \exp_p(tw))}{t}$$

holds for arbitrary $t > 0$.

From (3) to (2). Note that

$$\begin{aligned} |(\text{d exp}_p)_v(w)| &= \lim_{t \rightarrow 0} \frac{\text{dist}(\exp_p(v + tw), \exp_p(v))}{t} \\ &= \lim_{t \rightarrow 0} \frac{\text{dist}(\exp_p(tv' + tw), \exp_p(tv'))}{t} \\ &\geq |v' + w - v'| \\ &= |w| \end{aligned}$$

□

Remark 18.1.1. This shows the exponential map of simply-connected complete Riemannian manifold with non-positive sectional curvature has “expansion” property.

Corollary 18.1.1. Let (M, g) be a Cartan-Hadamard manifold with $a, b, c \in M$. Such points determine a unique geodesic triangle T with vertices a, b, c . Let α, β, γ be the angles of the vertices a, b, c respectively, and let A, B, C be the lengths of the side opposite the vertices a, b, c respectively. Then

- (1) $A^2 + B^2 - 2AB \cos \gamma \leq C^2 (< C^2, \text{ if } K < 0)$.
- (2) $\alpha + \beta + \gamma \leq \pi (< \pi, \text{ if } K < 0)$

Proof. See Lemma 3.1 in Page 259 of [Car92].

□

18.1.2. *Complete Riemannian manifold with non-positive sectional curvature is $K(G, 1)$.*

Lemma 18.1.1. If (M, g) is a complete Riemannian manifold with sectional curvature $K \leq 0$, then for any $p \in M$, $\text{conj}(p) = \emptyset$. In particular, $\exp_p: T_p M \rightarrow M$ is a local diffeomorphism.

Proof. Suppose q is conjugate to p along $\gamma: [0, 1] \rightarrow M$, and without loss of generality we may assume there is no conjugate point for $t \in (0, 1)$. Let J

be a Jacobi field along γ with $J(0) = J(1) = 0$. Then

$$\begin{aligned} \left(\frac{1}{2}|J|^2\right)' &= (g(J', J))' \\ &= g(J'', J) + g(J', J') \\ &= -R(J, \gamma', \gamma', J) + |J'|^2 \\ &\geq |J'|^2 \end{aligned}$$

Since $J'(0) \neq 0$, we have

$$\begin{aligned} g(J', J)(t) &\geq \int_0^t |J'|^2 + g(J'(0), J(0)) \\ &= \int_0^t |J'|^2 \\ &> 0 \end{aligned}$$

which implies $(\frac{1}{2}|J|^2)' = g(J', J) > 0$, a contradiction to $J(1) = 0$. \square

Theorem 18.1.2 (Cartan-Hadamard). If (M, g) is a complete Riemannian manifold with sectional curvature $K \leq 0$, then $\exp_p: T_p M \rightarrow M$ is a covering map.

Proof. Lemma 18.1.1, together with Proposition 17.2.4 and Theorem 18.1.1 completes the proof. \square

Corollary 18.1.2. Cartan-Hadamard manifold is diffeomorphic to \mathbb{R}^n .

Corollary 18.1.3. If (M, g) is a complete Riemannian manifold with $K \leq 0$, then $\pi_k(M) = 0, k \geq 2$, that is M is $K(\pi_1(M), 1)$.

Remark 18.1.2. Theory in topology says if a finite dimensional CW-complex is a $K(G, 1)$ space, then its fundamental group is torsion-free. In particular, if M is a complete Riemannian manifold with $K \leq 0$, then $\pi(M)$ is torsion-free. This fact can be proved later by tools of Riemannian manifold, called Cartan's torsion-free theorem.

Corollary 18.1.4. If M and N are two compact Riemannian manifold and one of them is simply-connected, then $M \times N$ has no metric with non-positive sectional curvature.

Proof. If both of M and N are simply-connected, and $M \times N$ admits a metric with non-positive sectional curvature, then it's diffeomorphic to \mathbb{R}^n for some positive integer n , a contradiction to compactness.

Without lose of generality, we assume M is simply-connected and N is not simply-connected with universal covering \tilde{N} . Then there is a universal covering

$$\pi: M \times \tilde{N} \rightarrow M \times N$$

If $M \times N$ admits a Riemannian metric g with non-positive sectional curvature, then π^*g is a complete metric of non-positive sectional curvature

on $M \times \tilde{N}$, so we have $M \times \tilde{N}$ is diffeomorphic to \mathbb{R}^n for some n . M is orientable since it's simply-connected, which implies $H^m(M) = \mathbb{Z}$, where $m = \dim M$. By Künneth formula $H^m(M \times \tilde{N}) \neq 0$, a contradiction to Poincaré lemma. \square

Remark 18.1.3. The condition simply-connected is crucial, for example $S^1 \times S^1$ admits a metric with flat curvature.

18.2. Cartan's torsion-free theorem.

Lemma 18.2.1. Let (M, g) be a Cartan-Hadamard manifold and $p \in M$, $v \in T_p M$. For all $q \in M$, one has

$$2 \operatorname{dist}(p, q)^2 + \operatorname{dist}(p_0, p)^2 + \operatorname{dist}(p_1, p)^2 \leq \operatorname{dist}(p_0, q)^2 + \operatorname{dist}(p_1, q)^2$$

where $p_0 = \exp_p(-v)$ and $p_1 = \exp_p(v)$.

Proof. Since $\exp_p: T_p M \rightarrow M$ is a diffeomorphism, there exists $w \in T_p M$ such that $q = \exp_p(w)$ with $\operatorname{dist}(p, q) = |w|$. Direct computation shows

$$\begin{aligned} \operatorname{dist}(p_0, q) &= \operatorname{dist}(\exp_p(-v), \exp_p(w)) \geq |w + v| \\ \operatorname{dist}(p_1, q) &= |w - v| \\ \operatorname{dist}(p, q)^2 &= |w|^2 \\ &= \frac{|w + v|^2 + |w - v|^2}{2} - |v|^2 \\ &\leq \frac{\operatorname{dist}(p_0, q)^2 + \operatorname{dist}(p_1, q)^2}{2} - \frac{\operatorname{dist}(p_0, p)^2 + \operatorname{dist}(p_1, p)^2}{2} \end{aligned}$$

\square

Lemma 18.2.2 (Serre). Let (M, g) be a Cartan-Hadamard manifold, $p \in M$ and $B(p, r)$ is the closed ball of radius r . If $\emptyset \neq \Omega \subseteq M$ is a bounded set with

$$r_\Omega = \inf\{r > 0 \mid \Omega \subseteq B(p, r), p \in M\}$$

then there exists a unique $p_\Omega \in M$ such that $\Omega \subseteq B(p_\Omega, r_\Omega)$.

Proof. Choose a bounded sequence $\{r_i > r_\Omega\}$ converging to r_Ω and $p_i \in M$ such that

$$\Omega \subseteq B(p_i, r_i)$$

For fixed $q \in \Omega$, one has $\operatorname{dist}(q, p_i) \leq r_i$ for each i , and thus $\{p_i\}$ is bounded since $\{r_i\}$ is bounded. Since M is complete, then $\{p_i\}$ has a convergent subsequence, and the limit of this convergent subsequence is p_Ω , which gives the existence of p_Ω . Suppose $p_1, p_2 \in M$ are two points such that

$$\Omega \subseteq B(p_0, r_\Omega) \cap B(p_1, r_\Omega)$$

Since \exp_{p_0} is a diffeomorphism, there exists a unique v_0 such that $p_1 = \exp_{p_0}(v_0)$. By Lemma 18.2.1, if we denote $p = \exp_{p_0}(v_0/2)$, then for all

$q \in \Omega$ one has

$$\begin{aligned} \text{dist}(p, q)^2 &\leq \frac{\text{dist}(p_0, q)^2 + \text{dist}(p_1, q)^2}{2} - \frac{\text{dist}(p_0, p)^2 + \text{dist}(p_1, p)^2}{2} \\ &\leq r_\Omega^2 - \frac{\text{dist}(p_0, p_1)^2}{4} \end{aligned}$$

since $\text{dist}(p_0, p) = \text{dist}(p_1, p) = \text{dist}(p_0, p_1)/2$. By the definition of r_Ω , one has $\text{dist}(p_0, p_1) = 0$, and thus $p_0 = p_1$. \square

Theorem 18.2.1 (Cartan's fixed point theorem). Let (M, g) be a Cartan-Hadamard manifold and G is a compact Lie group acting on M isometrically. Then G has a fixed point.

Proof. For $p \in M$, suppose Ω is the orbit of p , that is

$$\Omega = \{gp \mid g \in G\}$$

It's a bounded since M is compact, and

$$\Omega = g\Omega \subseteq B(gp_\Omega, r_\Omega)$$

since Ω is the orbit. Then by uniqueness of p_Ω , one has p_Ω is a fixed point of G . \square

Corollary 18.2.1. If (M, g) is a complete Riemannian manifold with $K \leq 0$, then $\pi_1(M)$ is torsion-free.

Proof. Let $(\widetilde{M}, \widetilde{g})$ be the universal covering of M with pullback metric. Then $(\widetilde{M}, \widetilde{g})$ is a Cartan-Hadamard manifold, and (M, g) is isometric to $(\widetilde{M}/\Gamma, \widetilde{g})$, where $\Gamma \subseteq \text{Iso}(\widetilde{M}, \widetilde{g})$ is isomorphic to $\pi_1(M)$ which acts on \widetilde{M} freely.

Now it suffices to show Γ has no torsion element. If not, suppose φ is a torsion element, then consider the finite group G generated by φ , which is a 0-dimension Lie group if it's equipped with discrete topology. By Cartan's fixed point theorem there exists a fixed point of G , which implies φ is identity, since Γ acts on \widetilde{M} freely. \square

18.3. Preissmann's Theorem.

Definition 18.3.1 (axis). Let (M, g) be a complete Riemannian manifold, and $\varphi: M \rightarrow M$ is an isometry. A non-trivial geodesic $\gamma: \mathbb{R} \rightarrow M$ is called an axis of φ if there exists $0 \neq c \in \mathbb{R}$ such that

$$\varphi(\gamma(t)) = \gamma(t + c)$$

Definition 18.3.2 (axial). An isometry without fixed points which has an axis is said to be axial.

Lemma 18.3.1. Let (M, g) be a complete Riemannian manifold, and $\varphi: M \rightarrow M$ is an isometry. If $\delta_\varphi(p) = \text{dist}(p, \varphi(p))$ has a positive minimum, then φ has an axis.

Proof. Suppose δ_φ attains its minimum at $p \in M$ and $\gamma(t) : [0, 1] \rightarrow M$ is a minimum geodesic connecting p and $\varphi(p)$. Then $\varphi \circ \gamma : [0, 1] \rightarrow M$ is also a geodesic connecting $\varphi(p)$ and $\varphi^2(p)$, since φ is an isometry. We claim these two geodesics form an angle π at point $\varphi(p)$ and thus fit together an extension of γ to $[0, 2]$. Indeed, for any $t \in [0, 1]$,

$$\begin{aligned} \delta_\varphi(p) &= \text{dist}(p, \varphi(p)) \\ &\leq \delta_\varphi(\gamma(t)) \\ &= \text{dist}(\gamma(t), \varphi \circ \gamma(t)) \\ &\leq \text{dist}(\gamma(t), \gamma(1)) + \text{dist}(\gamma(1), \varphi \circ \gamma(t)) \\ &= \text{dist}(\gamma(t), \gamma(1)) + \text{dist}(\varphi \circ \gamma(0), \varphi \circ \gamma(t)) \\ &= \text{dist}(\gamma(t), \gamma(1)) + \text{dist}(\gamma(0), \gamma(t)) \\ &= \delta_\varphi(p) \end{aligned}$$

Thus we have $\varphi(\gamma(t)) = \gamma(1+t)$ for $0 \leq t \leq 1$. Repeating this argument to obtain a geodesic $\gamma : \mathbb{R} \rightarrow M$ with period 1, and it's an axis for φ . \square

Lemma 18.3.2. Let (M, g) be a compact Riemannian manifold with universal covering \widetilde{M} . If $\varphi : \widetilde{M} \rightarrow \widetilde{M}$ is a non-trivial deck transformation, then

- (1) δ_φ has a positive minimum and $\delta_\varphi \geq 2 \text{inj}(M)$. In particular, φ has an axis $\gamma : \mathbb{R} \rightarrow \widetilde{M}$.
- (2) $\pi \circ \gamma$ is a closed geodesic in M whose length is minimal in the homotopy class $[\pi \circ \gamma]$.

Lemma 18.3.3. Let (M, g) be a Cartan-Hadamard manifold with $K < 0$. If isometry $\varphi : M \rightarrow M$ has an axis, then it's unique up to reparametrization.

Proof. Suppose $\gamma_1, \gamma_2 : \mathbb{R} \rightarrow M$ are two axes of φ , without lose of generality we may assume

$$\begin{aligned} \varphi(\gamma_1(t)) &= \gamma_1(t+1) \\ \varphi(\gamma_2(t)) &= \gamma_2(t+1) \end{aligned}$$

Suppose γ_1, γ_2 do not intersect. Then points $A = \gamma_1(0), B = \gamma_1(1) = \varphi(A), C = \gamma_2(0)$ and $D = \gamma_2(1) = \varphi(C)$ are all distinct. Let γ be a geodesic from A to C . Then $\varphi \circ \gamma$ is the geodesic from B to D . Furthermore, the geodesic quadrilateral $ABCD$ has angle sum 2π , since φ preserves angles. However, according to Lemma 18.1.1, triangle $\triangle ABC$ and $\triangle BCD$ have angle sum strictly less than π , and

$$\begin{aligned} \angle ACD &\leq \angle ACB + \angle BCD \\ \angle ABD &\leq \angle ABC + \angle CBD \end{aligned}$$

thus the angle sum of $ABCD$ is strictly less than 2π , a contradiction. Hence γ_1 and γ_2 must intersect at some point $p = \gamma_1(t_1) = \gamma_2(t_2)$. Then

$$\begin{aligned} \varphi(p) &= \varphi(\gamma_1(t_1)) = \gamma_1(t_1+1) \\ &= \varphi(\gamma_2(t_2)) = \gamma_2(t_2+1) \end{aligned}$$

is another intersection point. Since (M, g) is a Cartan-Hadamard manifold, any two points are joined by a unique geodesic, thus γ_1 is a reparametrization of γ_2 . \square

Lemma 18.3.4. If H is an additive subgroup of \mathbb{R} , then either H is dense in \mathbb{R} or $H \cong \mathbb{Z}$.

Proof. Let H be an additive subgroup of \mathbb{R} , it's clear $H \cap \mathbb{R}_{>0} \neq \emptyset$, consider

$$b := \inf\{h \in H \cap \mathbb{R}_{>0}\}$$

(1) If $b > 0$: Let $h \in H$ and $k \in \mathbb{Z}$ such that

$$kb \leq |h| < (k+1)b$$

then we have $|h| - kb \in H$, and $0 \leq |h| - kb < (k+1)b - kb = b$. By the choice of b , we have $|h| - kb = 0$, which implies $h = \pm kb$. In this case $H = b\mathbb{Z}$.

(2) If $b = 0$: For arbitrary $r \in \mathbb{R}_{\geq 0}$ and $\varepsilon > 0$, there exists $h \in H \cap (0, \varepsilon]$ since $b = 0$ and $k \in \mathbb{N}$ such that

$$kh \leq r \leq (k+1)h$$

Thus

$$0 \leq r - kh \leq (k+1)h - kh = h \leq \varepsilon$$

which implies $|r - kh| \leq \varepsilon$, that is H is dense in $\mathbb{R}_{\geq 0}$. For the same argument you can show H is also dense in $\mathbb{R}_{\leq 0}$. \square

Theorem 18.3.1 (Preissmann). If (M, g) is a compact Riemannian manifold with negative sectional curvature, then any non-trivial abelian subgroup of $\pi_1(M)$ is isomorphic to \mathbb{Z} .

Proof. Let $(\widetilde{M}, \widetilde{g})$ be the universal covering of M equipped with pullback metric. Then it's a Cartan-Hadamard manifold with negative sectional curvature. Now it suffices to show every non-trivial abelian subgroup H of group consisting of deck transformations is isomorphic to \mathbb{Z} . Let φ be a non-trivial deck transformation in H . Then Lemma 18.3.2, φ has an axis $\gamma: \mathbb{R} \rightarrow \widetilde{M}$, that is there exists $c \neq 0$ such that

$$\varphi \circ \gamma(t) = \gamma(t + c)$$

for all $t \in \mathbb{R}$. If ψ is another non-trivial element of H , then for any $t \in \mathbb{R}$ we have

$$\varphi \circ \psi(\gamma(t)) = \psi \circ \varphi(\gamma(t)) = \psi \circ \gamma(t + c)$$

which implies $\psi \circ \gamma$ is also an axis of φ . So by Lemma 18.3.3 we have $\psi \circ \gamma$ is a reparametrization of γ . Furthermore, $\psi \circ \gamma$ and γ have the same speed since ψ is an isometry, thus there are two cases:

- (1) $\psi \circ \gamma(t) = \gamma(t + a)$.
- (2) $\psi \circ \gamma(t) = \gamma(-t + a)$

(2) can't happen, otherwise $\psi \circ \gamma(\frac{a}{2}) = \gamma(\frac{a}{2})$, contradicts to deck transformation acts on \widetilde{M} freely. Consider

$$\begin{aligned} f: H &\rightarrow \mathbb{R} \\ \psi &\mapsto a \end{aligned}$$

where a is determined $\psi \circ \gamma(t) = \gamma(t + a)$. It's easy to see F is a group homomorphism with trivial kernel thus $F(H)$ is an additive subgroup of \mathbb{R} . Consider

$$b := \inf\{h \in F(H) \cap \mathbb{R}_{>0}\}$$

By Lemma 18.3.4, it suffices to show $b > 0$. If $b = 0$, then there exist $a \in (0, \text{inj}(M))$ and $\psi \in H$ such that $a = F(\psi)$, that is

$$\psi \circ \gamma(t) = \gamma(t + a)$$

Since $\pi \circ \psi = \pi$, we have $\pi \circ \gamma(t) = \pi \circ \gamma(t + a)$. Set $t = 0$ one has

$$\pi \circ \gamma(a) = \pi \circ \gamma(0)$$

A contradiction to $0 < a < \text{inj}(M)$ since $\pi \circ \gamma$ is a geodesic. \square

Corollary 18.3.1. Suppose M and N are compact smooth manifolds. Then $M \times N$ doesn't admit a Riemannian metric with negative sectional curvature.

Proof. If $M \times N$ admits a Riemannian metric with negative sectional curvature, Cartan's torsion-free theorem implies $\pi_1(M \times N)$ is torsion-free, thus for arbitrary $\alpha \in \pi_1(M), \beta \in \pi_1(N)$, unless either M or N is simply-connected, $\pi_1(M \times N)$ will contain an abelian subgroup $\mathbb{Z} \times \mathbb{Z}$ generated by α, β , which contradicts to Preissmann's theorem.

Without loss of generality, we assume M is simply-connected. Then consider the universal covering $M \times \widetilde{N}$ of $M \times N$, Cartan-Hadamard's theorem implies it's diffeomorphic to \mathbb{R}^n for $n \in \mathbb{Z}_{>0}$, but M is orientable since it's simply-connected, so $H^m(M) = \mathbb{Z}$ where $m = \dim M$. So by Künneth formula $H^n(M \times \widetilde{N}) \neq 0$, a contradiction to Poincaré lemma. \square

Lemma 18.3.5. Let (M, g) be a complete Riemannian manifold with non-positive sectional curvature and \widetilde{M} is its universal covering. If $\gamma: \mathbb{R} \rightarrow \widetilde{M}$ is a common axis for all deck transformations, then M is not compact.

Theorem 18.3.2 (Preissmann). If (M, g) is a compact Riemannian manifold with negative sectional curvature, then $\pi_1(M)$ is not abelian.

Proof. Suppose $\pi_1(M)$ is abelian. Then let γ be the axis of some deck transformation. Then it's the axis of all deck transformations since $\pi_1(M)$ is abelian, which implies M is non-compact, a contradiction. \square

18.4. Other facts.

Theorem 18.4.1 (Byers). If (M, g) is a compact Riemannian manifold with negative sectional curvature, then any non-trivial solvable subgroup of $\pi_1(M)$ is isomorphic to \mathbb{Z} .

Theorem 18.4.2 (Yau). Let (M, g) be a compact Riemannian manifold with non-positive sectional curvature. If $\pi_1(M)$ is solvable, then M is flat.

Theorem 18.4.3 (Farrell-Jones). Let $(M_i, g_i), i = 1, 2$ be two compact Riemannian manifolds with non-positive sectional curvature. If $\pi_1(M_1) = \pi_1(M_2)$, then M_1 and M_2 are homeomorphic.

19. TOPOLOGY OF POSITIVE CURVATURE MANIFOLD

19.1. Myers' theorem.

Theorem 19.1.1 (Myers). Let (M, g) be a complete Riemannian n -manifold with $\text{Ric}(g) \geq \frac{n-1}{R^2}g$. Then $\text{diam}(M) \leq \pi R$, and thus M is compact.

Proof. If $\text{diam}(M) > \pi R$, then there exists $b > \pi R$ and a minimal geodesic $\gamma: [0, b] \rightarrow M$ of unit-speed, since M is complete. Let $\{e_1(t), \dots, e_n(t)\}$ be a parallel orthonormal frame along γ with $e_1(t) = \gamma'(t)$, and for each $i = 2, \dots, n$, we define

$$V_i(t) = \sin\left(\frac{\pi t}{b}\right)e_i(t)$$

It's clear $V_i(0) = V_i(b) = 0$ for $2 \leq i \leq n$. Note that

$$\begin{aligned} I_\gamma(V_i, V_i) &= \int_0^b \langle \widehat{\nabla}_{\frac{d}{dt}} V_i, \widehat{\nabla}_{\frac{d}{dt}} V_i \rangle dt - \int_0^b R(V_i, \gamma', \gamma', V_i) dt \\ &= \int_0^b \sin^2\left(\frac{\pi t}{b}\right) \left\{ \left(\frac{\pi}{b}\right)^2 - R(e_i, e_1, e_1, e_i) \right\} dt \end{aligned}$$

Thus

$$\begin{aligned} I_\gamma\left(\sum_{i=2}^n V_i, \sum_{i=2}^n V_i\right) &= \int_0^b \sin^2\left(\frac{\pi t}{b}\right) \left\{ (n-1)\left(\frac{\pi}{b}\right)^2 - \sum_{i=2}^n R(e_i, e_1, e_1, e_i) \right\} dt \\ &\leq \int_0^b \sin^2\left(\frac{\pi t}{b}\right) \left\{ (n-1)\left(\frac{\pi}{b}\right)^2 - \frac{(n-1)}{R^2} \right\} dt \end{aligned}$$

Since γ is a minimal geodesic, one has I_γ is semi-positive definite, which implies $b \leq \pi R$. \square

Remark 19.1.1. The estimate for the diameter given by Myers' theorem can't be improved. Indeed, the unit sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ has constant sectional curvature $K = 1$ and $\text{diam}(\mathbb{S}^n) = \pi$. Furthermore, there is a rigidity theorem: Let (M, g) be a complete Riemannian n -manifold, $\text{Ric}(g) \geq \frac{n-1}{R^2}g$ and $\text{diam}(M) = \pi R$. Then (M, g) is isometric to sphere $\mathbb{S}^n(R)$ with standard metric, that's Cheng's theorem, we will see it in Theorem 22.4.2.

Corollary 19.1.1. Let M be a complete Riemannian manifold with positive Ricci curvature. Then the fundamental group $\pi_1(M)$ is finite.

Proof. Let \widetilde{M} be the universal covering of M equipped with pullback metric \widetilde{g} . It's clear $(\widetilde{M}, \widetilde{g})$ is a complete Riemannian manifold with positive Ricci curvature. By Myers' theorem \widetilde{M} is compact, which implies $\pi: \widetilde{M} \rightarrow M$ is a finite covering, that is $\pi_1(M)$ is finite, since $|\pi_1(M)|$ equals the number of sheets of covering. \square

19.2. Synge's theorem.

Lemma 19.2.1. Let A be an orthogonal linear transformation of \mathbb{R}^{n-1} and suppose $\det A = (-1)^n$. Then 1 is an eigenvalue of A .

Proof. We consider the following cases:

- (1) If n is even, then $\det(\lambda I - A)$ is a polynomial of odd degree, therefore A has at least a real eigenvalue, and it must be ± 1 since A is orthogonal. Furthermore, since $\det A = 1$ and the product of complex eigenvalue is positive, there is at least a real eigenvalue which equals 1.
- (2) If n is odd, then $\det A = -1$. Because the product of complex eigenvalue is positive, there are at least two real eigenvalues, and one of them is 1.

□

Theorem 19.2.1 (Synge). Let (M, g) be a compact Riemannian manifold with positive sectional curvature. Then

- (1) If $\dim M$ is even and orientable. Then M is simply-connected.
- (2) If $\dim M$ is odd. Then M is orientable.

Proof. Let $(\widetilde{M}, \widetilde{g})$ be the universal covering of M equipped with pullback metric.

- (1) If $\dim M$ is even, equip \widetilde{M} the pullback orientation.
- (2) If $\dim M$ is odd, equip \widetilde{M} arbitrary orientation.

Suppose the conclusions are not correct, thus $\pi_1(M)$ is non-trivial. Choose a non-trivial deck transformation $f: \widetilde{M} \rightarrow \widetilde{M}$ such that

- (1) If $\dim M$ is even, F is orientation preserving.
- (2) If $\dim M$ is odd, F is orientation reserving.

By Lemma, there exists an axis $\widetilde{\gamma}: \mathbb{R} \rightarrow \widetilde{M}$ for F and $\gamma = \pi \circ \widetilde{\gamma}$ is a closed geodesic in M that minimizes the length in $[\gamma]$,

$$F(\widetilde{\gamma}(t)) = \widetilde{\gamma}(t + 1)$$

□

Corollary 19.2.1. Let (M, g) be a compact Riemannian manifold with even dimension, if M is non-orientable. Then $\pi_1(M) = \mathbb{Z}_2$.

Proof. If M is non-orientable, it has a double-sheeted orientable covering manifold \widetilde{M} , Synge's theorem implies \widetilde{M} is simply-connected, thus it's the universal covering of M , which implies $\pi_1(M) = \mathbb{Z}_2$. □

Example 19.1. $\mathbb{RP}^n \times \mathbb{RP}^n$ admits no Riemannian metric with positive sectional curvature, since its fundamental group is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Conjecture 19.2.1 (Hopf conjecture). Does $S^2 \times S^2$ admit a Riemannian metric with positive sectional curvature?

19.3. Other facts.

Theorem 19.3.1. Let (M, g) be a compact, simply-connected Riemannian n -manifold.

- (1) (Hamilton) If $n = 3$, $\text{Ric}(g) > 0$, then M is diffeomorphism to S^3 .
- (2) (Hamilton) If $n = 4$ with curvature operator > 0 , then M is diffeomorphism to S^4 .

- (3) (Böhm-Wilking) If curvature operator > 0 , then M is diffeomorphism to S^n .

Theorem 19.3.2 (soul theorem). Let (M, g) be a complete, non-compact Riemannian n -manifold,

- (1) If M has non-negative sectional curvature, then there exists a compact totally geodesic submanifold $S \subseteq M$ (called a soul of M) such that M is diffeomorphic to the normal bundle of S in M .
- (2) If M has positive sectional curvature, then its soul is a point and M is diffeomorphic to \mathbb{R}^n .

Theorem 19.3.3 (differentiable sphere theorem). Let (M, g) be a compact, simply-connected Riemannian n -manifold with $n \geq 4$. If sectional curvature satisfies $\frac{1}{4} < K \leq 1$, then M is diffeomorphism to S^n .

20. TOPOLOGY OF CONSTANT SECTIONAL CURVATURE MANIFOLD

20.1. Hopf's theorem.

Theorem 20.1.1 (Hopf). Let (M, g) be a simply-connected complete Riemannian manifold with constant sectional curvature K . Then (M, g) is isometric to $(\widetilde{M}, g_{\text{can}})$, where

$$(\widetilde{M}, g_{\text{can}}) = \begin{cases} (\mathbb{S}^n(\frac{1}{\sqrt{K}}), g_{\text{can}}) & K > 0 \\ (\mathbb{R}^n, g_{\text{can}}) & K = 0 \\ (\mathbb{H}^n(\frac{1}{\sqrt{-K}}), g_{\text{can}}) & K < 0 \end{cases}$$

Proof. Let M be a simply-connected complete Riemannian manifold with constant sectional curvature K .

- (1) If $K \leq 0$, let $\widetilde{M} = \mathbb{R}^n$ or $\mathbb{H}^n(\frac{1}{\sqrt{-K}})$. Fix $p \in M, \tilde{p} \in \widetilde{M}$ and a linear isometry $\Phi_0: T_{\tilde{p}}\widetilde{M} \rightarrow T_pM$, Cartan-Ambrose-Hicks's theorem implies

$$\varphi = \exp_p \circ \Phi_0 \circ \exp_{\tilde{p}}^{-1}: \widetilde{M} \rightarrow M$$

is a local isometry. Furthermore, Cartan-Hadamard's theorem implies φ is a diffeomorphism, since M, \widetilde{M} are simply-connected with non-positive sectional curvature. This completes the proof of this part.

- (2) If $K > 0$, let $\widetilde{M} = \mathbb{S}^n(\frac{1}{\sqrt{K}})$. Fix $p \in M, \tilde{p} \in \widetilde{M}$ and a linear isometry $\Phi_0: T_{\tilde{p}}\widetilde{M} \rightarrow T_pM$. Consider the following smooth map

$$\varphi_1 = \exp_p \circ \Phi_0 \circ \exp_{\tilde{p}}^{-1}: \widetilde{M} \setminus \{-\tilde{p}\} \rightarrow M$$

it's well-defined since the only cut point of \tilde{p} is its antipodal point $-\tilde{p}$. Then Cartan-Ambrose-Hicks's theorem implies φ_1 is a local isometry. Choose $\tilde{q} \in \widetilde{M} \setminus \{\tilde{p}, -\tilde{p}\}$, $q = \varphi_1(\tilde{q})$ and $\Psi_0 = (d\varphi_1)_{\tilde{q}}: T_{\tilde{q}}\widetilde{M} \rightarrow T_qM$. Then the same argument shows

$$\varphi_2 = \exp_q \circ \Phi_0 \circ \exp_{\tilde{q}}^{-1}: \widetilde{M} \setminus \{-q\} \rightarrow M$$

is a well-defined local isometry defined on $\widetilde{M} \setminus \{-\tilde{q}\}$. Note that

$$\varphi_2(\tilde{q}) = q = \varphi_1(\tilde{q})$$

$$(d\varphi_2)_{\tilde{q}} = \Psi_0 = (d\varphi_1)_{\tilde{q}}$$

So by Theorem 16.1.1, we have the φ_1 agrees with φ_2 on $\widetilde{M} \setminus \{-\tilde{p}, -\tilde{q}\}$. Thus

$$\varphi(x) = \begin{cases} \varphi_1(x), & x \in \widetilde{M} \setminus \{-\tilde{p}\} \\ \varphi_2(x), & x \in \widetilde{M} \setminus \{-\tilde{q}\} \end{cases}$$

is a well-defined local isometry from $\widetilde{M} \rightarrow M$. In particular, φ is a local diffeomorphism. Then by Proposition 17.2.1 we have φ is a diffeomorphism, since \mathbb{S}^n is compact and simply-connected, thus φ is an isometry. □

Corollary 20.1.1. Let (M, g) be a Riemannian manifold with constant sectional curvature K . Then (M, g) is isometric to \widetilde{M}/Γ , where $\Gamma \subseteq \text{Iso}(\widetilde{M}, \widetilde{g})$ is isomorphic to $\pi_1(M)$ and

$$(\widetilde{M}, \widetilde{g}) = \begin{cases} (\mathbb{S}^n(\frac{1}{\sqrt{K}}), g_{\text{can}}) & K > 0 \\ (\mathbb{R}^n, g_{\text{can}}) & K = 0 \\ (\mathbb{H}^n(\frac{1}{\sqrt{-K}}), g_{\text{can}}) & K < 0 \end{cases}$$

Proof. Let $(\widetilde{M}, \widetilde{g})$ be the universal covering of M with pullback metric. Then M is isometric to \widetilde{M}/Γ , where $\Gamma \subseteq \text{Iso}(\widetilde{M}, \widetilde{g})$ is isomorphic to $\pi_1(M)$. Since $(\widetilde{M}, \widetilde{g})$ is a simply-connected Riemannian manifold with constant sectional curvature k , the Hopf's theorem completes the proof. \square

Definition 20.1.1 (space form). A complete, simply-connected Riemannian n -manifold with constant sectional curvature k is called space form, and is denoted by $S(n, k)$.

Example 20.1.1. Let (M, g) be a complete Riemannian manifold with constant sectional curvature $K = 1$. If $\dim M = 2n$. Then (M, g) is isometric to the sphere $(\mathbb{S}^{2n}, g_{\text{can}})$ or the real projective space $(\mathbb{RP}^{2n}, g_{\text{can}})$.

Proof. Note that Hopf's theorem implies (M, g) is isometric to $(\mathbb{S}^{2n}/\Gamma, g_{\text{can}})$, where Γ is isomorphic to $\pi_1(M)$, and Synge's theorem implies if $\dim M$ is even and $K > 0$. Then $\pi_1(M) = \{e\}$ or $\pi_1(M) = \mathbb{Z}_2$.

- (1) If $\pi_1(M) = \{e\}$. Then (M, g) is isometric to $(\mathbb{S}^{2n}, g_{\text{can}})$.
- (2) If $\pi_1(M) = \{e, \varphi\}$, to show (M, g) is isometric to $(\mathbb{RP}^{2n}, g_{\text{can}})$, it suffices to show φ is antipodal map. Note that only possible eigenvalues of φ is ± 1 , and if 1 is an eigenvalue of φ . Then it exists a fixed point, which implies $\varphi = e$, since $\pi_1(M)$ acts on \mathbb{S}^{2n} freely.

\square

Remark 20.1.1. In general, we have no ideal about what does $\pi_1(M)$ look like.

Part 7. Comparison theorems

21. COMPARISON THEOREMS BASED ON SECTIONAL CURVATURE

In this section, we will see the following philosophy: “The larger curvature is, the smaller the distance is.”

21.1. Rauch comparison. Rauch comparison theorem is one of the most important comparison theorems, which gives bounds on the sizes of Jacobi fields based on sectional curvature bounds. Recall that Jacobi field is a quite useful tool, based on the following observations:

- (1) Corollary 12.1.1 implies that in a normal neighborhood of p , every vector field can be represented as the value of Jacobi field that vanishes at p .
- (2) The zeros of Jacobi fields corresponds to conjugate points, beyond which geodesics can't be minimal.

Theorem 21.1.1 (Rauch comparison). Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be two Riemannian manifold with $\dim M \leq \dim \widetilde{M}$. Suppose $\gamma: [0, b] \rightarrow M$ and $\widetilde{\gamma}: [0, b] \rightarrow \widetilde{M}$ are two unit-speed geodesics such that

- (1) For all $t \in [0, b]$, and any planes $\Sigma \subseteq T_{\gamma(t)}M, \gamma'(t) \in \Sigma, \widetilde{\Sigma} \subseteq T_{\widetilde{\gamma}(t)}\widetilde{M}, \widetilde{\gamma}'(t) \in \widetilde{\Sigma}$, we have $K_{\gamma(t)}(\Sigma) \leq K_{\widetilde{\gamma}(t)}(\widetilde{\Sigma})$.
- (2) $\widetilde{\gamma}(0)$ has no conjugate points along $\widetilde{\gamma}|_{[0, b]}$.

Then for any Jacobi fields $J(t)$ and $\widetilde{J}(t)$ with

$$(1) \quad \begin{cases} J(0) = c\gamma'(0) \\ \widetilde{J}(0) = c\widetilde{\gamma}'(0) \end{cases}$$

- (2) $|J'(0)| = |\widetilde{J}'(0)|$.
- (3) $\langle J'(0), \gamma'(0) \rangle = \langle \widetilde{J}'(0), \widetilde{\gamma}'(0) \rangle$.

One has $|J(t)| \geq |\widetilde{J}(t)|$ for all $t \in [0, b]$.

Proof. Firstly we consider the following simple case:

- (1) $J(0) = \widetilde{J}(0) = 0$.
- (2) $|J'(0)| = |\widetilde{J}'(0)|$.
- (3) $\langle J'(0), \gamma'(0) \rangle = \langle \widetilde{J}'(0), \widetilde{\gamma}'(0) \rangle = 0$.

Since $\widetilde{\gamma}(0)$ has no conjugate points along $\widetilde{\gamma}|_{[0, b]}$. Then $\frac{|J(t)|^2}{|\widetilde{J}(t)|^2}$ is well-defined for all $t \in (0, b]$, and standard calculus implies

$$\lim_{t \rightarrow 0} \frac{|J|^2}{|\widetilde{J}|^2} = \lim_{t \rightarrow 0} \frac{\langle J'(t), J(t) \rangle}{\langle \widetilde{J}'(t), \widetilde{J}(t) \rangle} = \lim_{t \rightarrow 0} \frac{|J'|^2}{|\widetilde{J}'|^2} = 1$$

So it suffices to show in $(0, b]$ we have

$$\frac{d}{dt} \left(\frac{|J|^2}{|\widetilde{J}|^2} \right) \geq 0$$

Direct computation shows above inequality is equivalent to:

$$\frac{\langle J'(t), J(t) \rangle}{|J(t)|^2} \geq \frac{\langle \tilde{J}'(t), \tilde{J}(t) \rangle}{|\tilde{J}(t)|^2}$$

holds for arbitrary $t \in (0, b]$. For arbitrary $s \in (0, b]$, we can define the following Jacobi fields by scaling $J(t)$:

$$W_s(t) = \frac{J(t)}{|J(s)|}, \quad \widetilde{W}_s(t) = \frac{\tilde{J}(t)}{|\tilde{J}(s)|}$$

Then

$$\frac{\langle J'(s), J(s) \rangle}{|J(s)|^2} = \langle W'_s(s), W_s(s) \rangle$$

So it suffices to show

$$\langle W'_s(s), W_s(s) \rangle \geq \langle \widetilde{W}'_s(s), \widetilde{W}_s(s) \rangle$$

holds for arbitrary $s \in (0, b]$. Direct computation shows:

$$\begin{aligned} \langle W'_s(s), W_s(s) \rangle &= \int_0^s (\langle W_s(t), W_s(t) \rangle)' dt \\ &= \int_0^s \langle W'_s(t), W'_s(t) \rangle dt + \int_0^s \langle W''_s(t), W_s(t) \rangle dt \\ &= \int_0^s \langle W'_s(t), W'_s(t) \rangle dt - \int_0^s R(W_s(t), \gamma'(t), \gamma'(t), W_s(t)) dt \end{aligned}$$

Choose a parallel orthonormal frame $\{e_1(t), \dots, e_n(t)\}$ with $e_1(t) = \gamma'(t)$, $e_2(t) = W_s(t)$. With respect to this frame we write

$$W_s(t) = \lambda^i(t) e_i(t)$$

Similarly, we choose a parallel orthogonal frame $\{\tilde{e}_1(t), \dots, \tilde{e}_n(t)\}$ and construct the following vector field

$$\tilde{V}(t) = \lambda^i(t) \tilde{e}_i(t)$$

Then it's clear we have

$$\int_0^s \langle W'_s(t), W'_s(t) \rangle dt = \int_0^s \langle \tilde{V}'(t), \tilde{V}'(t) \rangle dt$$

and our curvature condition implies

$$\int_0^s R(W_s(t), \gamma'(t), \gamma'(t), W_s(t)) dt \leq \int_0^s \tilde{R}(\tilde{V}(t), \gamma'(t), \gamma'(t), \tilde{V}(t)) dt$$

Thus we have

$$\begin{aligned} \langle W'_s(s), W_s(s) \rangle &\leq \int_0^s \langle \tilde{V}'(t), \tilde{V}'(t) \rangle dt - \int_0^s R(\tilde{V}(t), \gamma'(t), \gamma'(t), \tilde{V}(t)) dt \\ &= \tilde{I}(\tilde{V}, \tilde{V}) \end{aligned}$$

where \tilde{I} is index form on \tilde{M} . According to Corollary 12.3.1, we have

$$\tilde{I}(\tilde{V}, \tilde{V}) \geq \tilde{I}(\tilde{W}_s, \tilde{W}_s)$$

since \widetilde{W}_s is a Jacobi field. This shows the desired result.

For general case, we consider the following decomposition

$$\begin{aligned} J(t) &= J_1(t) + \langle J(t), \gamma'(t) \rangle \gamma'(t) \\ \widetilde{J}(t) &= \widetilde{J}_1(t) + \langle \widetilde{J}(t), \widetilde{\gamma}'(t) \rangle \widetilde{\gamma}'(t) \end{aligned}$$

Then it's clear $J_1(t)$ and $\widetilde{J}_1(t)$ satisfy requirement of our simple case, that is for $t \in [0, 1]$ we have

$$|J_1(t)| \geq |\widetilde{J}_1(t)|$$

Furthermore,

$$\langle J(t), \gamma'(t) \rangle = \langle \widetilde{J}(t), \widetilde{\gamma}'(t) \rangle$$

always holds, since

$$\begin{aligned} \langle J(t), \gamma'(t) \rangle &= \langle J(0), \gamma'(0) \rangle + \langle J'(0), \gamma'(0) \rangle t \\ &\stackrel{(1)}{=} \langle \widetilde{J}(0), \widetilde{\gamma}'(0) \rangle + \langle \widetilde{J}'(0), \widetilde{\gamma}'(0) \rangle t \\ &= \langle \widetilde{J}(t), \widetilde{\gamma}'(t) \rangle \end{aligned}$$

where (1) holds from our assumption. \square

Corollary 21.1.1. Let (M, g) be a Riemannian manifold, U a normal neighborhood of $p \in M$, $\gamma: [0, b] \rightarrow U$ a unit-speed geodesic with $\gamma(0) = p$ and J a Jacobi field along γ with $J(0) = 0$.

- (1) If the sectional curvature $K \leq k$ in U . Then $|J(t)| \geq \text{sn}_k(t)|J'(0)|$, for all $t \in [0, b_0]$, where

$$b_0 = \begin{cases} b, & k \leq 0 \\ \min\{b, \pi R\}, & k = \frac{1}{R^2} > 0 \end{cases}$$

- (2) If the sectional curvature $K \geq k$ in U . Then

$$|J(t)| \leq \text{sn}_k(t)|J'(0)|$$

for all $t \in [0, b]$.

Proof. Apply Rauch comparison between M and space form $\widetilde{M} = S(n, k)$ to conclude. However, in order to avoid geodesic $\widetilde{\gamma}$ of \widetilde{M} from having conjugate points, we need to let $b_0 < \min\{b, \pi R\}$, when $k = \frac{1}{R^2} > 0$. \square

Remark 21.1.1. In particular, from above corollary, we immediately have the following corollary when $K \leq k$:

- (1) If $k \leq 0$, we have already known M has no conjugate point along any geodesic.
(2) If $k = \frac{1}{R^2} > 0$. Then there is no conjugate point along any geodesic with length $< \pi R$. Or in other words, the distance between two consecutive conjugate points is $\geq \pi R$.

Corollary 21.1.2 (metric comparison). Let (M, g) be a Riemannian n -manifold, U a normal neighborhood of $p \in M$. For all $k \in \mathbb{R}$, we use g_k to denote the metric $dr \otimes dr + \text{sn}_k(r)g_{\mathbb{S}^{n-1}}$ in $U \setminus \{p\}$.

(1) If $K \leq k$ holds for all $q \in U \setminus \{p\}$. Then for $w \in T_q M$ we have

$$g(w, w) \geq g_k(w, w)$$

holds in $U_0 \setminus \{p\}$, where

$$U_0 = \begin{cases} U, & k \leq 0 \\ U \cap B(p, \pi R), & k = \frac{1}{R^2} > 0 \end{cases}$$

(2) If $K \geq k$ holds for all $q \in U \setminus \{p\}$. Then for $w \in T_q M$ we have

$$g(w, w) \leq g_k(w, w)$$

holds in $U \setminus \{p\}$.

Proof. If $w = \partial_r|_q$, it's clear

$$g(\partial_r|_q, \partial_r|_q) = 1 = g_k(\partial_r|_q, \partial_r|_q)$$

by Gauss lemma. Then it suffices to check for $w \in T_q M$ such that $g(w, \partial_r|_q) = 0$, we have

$$g(w, w) \geq g_k(w, w)$$

Let $\gamma: [0, b] \rightarrow M$ be a unit-speed geodesic connecting p and q , and J a Jacobi field such that $J(0) = 0, J(b) = w$. In normal coordinate $J(t)$ can be written as $ta^i \frac{\partial}{\partial x^i}|_{\gamma(t)}$ for some a^i .

Since (x^i, U, p) is both normal coordinate for metric g and g_k , thus γ is also a radial geodesic for g_c , and $J(t)$ is also a Jacobi field with respect to g_c along γ . Thus we have

$$\begin{aligned} g(w, w) &= |J(b)|_g^2 \\ g_k(w, w) &= |J(b)|_{g_k}^2 \end{aligned}$$

Then by Corollary 21.1.1, this completes the proof. \square

Remark 21.1.2. The **ideal** of this proof and the proof of Theorem 14.4.1 is almost the same, that is by using Corollary 12.1.1 to construct a Jacobi field valued a given vector, and then one can use Rauch comparison to compare length of given vectors.

Corollary 21.1.3. Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be two Riemannian manifolds with $K \leq \widetilde{K}$. Fix $p \in M, \widetilde{p} \in \widetilde{M}$, linear isometry $\Phi_0: T_p M \rightarrow T_{\widetilde{p}} \widetilde{M}$ and $0 \leq \delta < \min(\text{inj}(p), \text{inj}(\widetilde{p}))$. Then for any smooth curve $\gamma: [0, 1] \rightarrow \exp_p(B(0, \delta))$ and $\widetilde{\gamma}(t) = \exp_{\widetilde{p}} \circ \Phi_0 \circ \exp_p^{-1}(\gamma(t))$, we have

$$L(\gamma) \geq L(\widetilde{\gamma})$$

Proof. Let $c(s) = \exp_p^{-1} \circ \gamma(s)$ and $\widetilde{c}(s) = \exp_{\widetilde{p}}^{-1} \circ \widetilde{\gamma}(s)$. Then $\widetilde{c}(s) = \Phi_0(c(s))$. Consider the following variations

$$\begin{aligned} \alpha(t, s) &= \exp_p(tc(s)) \\ \widetilde{\alpha}(t, s) &= \exp_{\widetilde{p}}(t\widetilde{c}(s)) \end{aligned}$$

and Jacobi fields

$$J_s(t) = \alpha_*\left(\frac{\partial}{\partial s}\right)(t, s)$$

$$\tilde{J}_s(t) = \tilde{\alpha}_*\left(\frac{\partial}{\partial s}\right)(t, s)$$

A crucial observation is for arbitrary $s_0 \in [0, 1]$, we have

$$J_{s_0}(1) = \gamma'(s_0)$$

$$\tilde{J}_{s_0}(1) = \tilde{\gamma}'(s_0)$$

So it suffices to prove $|J_{s_0}(1)| \geq |\tilde{J}_{s_0}(1)|$ holds for arbitrary $s_0 \in [0, 1]$, that is we need to use Rauch comparison to Jacobi fields $J_{s_0}(t), \tilde{J}_{s_0}(t)$ along γ_{s_0} and $\tilde{\gamma}_{s_0}$, where $\gamma_{s_0}(t) = \alpha(t, s_0)$ and $\tilde{\gamma}_{s_0}(t) = \tilde{\alpha}(t, s_0)$. Check requirements as follows:

- (1) $J_{s_0}(0) = \tilde{J}_{s_0}(0) = 0$.
- (2) $J'_{s_0}(0) = c'(s_0), \tilde{J}'_{s_0}(0) = \tilde{c}'(s_0)$, and $\tilde{c}(s_0) = \Phi_0(c(s_0))$ implies $|J'_{s_0}(0)| = |\tilde{J}'_{s_0}(0)|$, since Φ_0 is linear isometry.
- (3) $\langle \tilde{J}'_{s_0}(0), \tilde{\gamma}'_{s_0}(0) \rangle = \langle \Phi_0(c'(s_0)), \Phi_0(c(s_0)) \rangle = \langle c'(s_0), c(s_0) \rangle = \langle J'_{s_0}(0), \gamma'_{s_0}(0) \rangle$.

□

Corollary 21.1.4. Let (M, g) be a Riemannian n -manifold, $0 < k_1 \leq K \leq k_2$. Let γ be any geodesic in M and b the distance along γ between two consecutive conjugate points. Then

$$\frac{\pi}{\sqrt{k_2}} \leq b \leq \frac{\pi}{\sqrt{k_1}}$$

Proof. Without lose of generality, we assume $\gamma: [0, b] \rightarrow M$ is a unit-speed geodesic with $\gamma(0) = p, \gamma(b) = q$ and p, q are two consecutive conjugate points along γ .

- (1) By Remark 21.1.1, we have already seen $b \geq \frac{\pi}{\sqrt{k_2}}$.
- (2) Apply Rauch comparison to (M, g) and $(\mathbb{S}^n(\frac{\pi}{\sqrt{k_1}}), g_{\text{can}})$, we have

$$|J(t)| \leq |\tilde{J}(t)|$$

for $t \in [0, b]$, where $J(t), \tilde{J}(t)$ are defined the same as before. Suppose $b > \frac{\pi}{\sqrt{k_1}}$. Then take $t = \frac{\pi}{\sqrt{k_1}}$, we have

$$0 < |J(t)| \leq |\tilde{J}(t)| = 0$$

A contradiction.

□

Theorem 21.1.2. Let (M, g) be a compact Riemannian manifold with sectional curvature $K \leq k, k > 0$. If we define

$$l(M) := \inf\{L(\gamma) \mid \gamma \text{ is a closed geodesic in } M\}$$

Then either $\text{inj}(M) \geq \frac{\pi}{\sqrt{k}}$ or $\text{inj}(M) = \frac{l(M)}{2}$.

Proof. By compactness of M , there exists $p, q \in M, q \in \text{cut}(p)$ such that $\text{dist}(p, q) = \text{inj}(M) = \text{inj}(p)$. Let $\gamma: [0, b] \rightarrow M$ be a minimal geodesic connecting p and q , that is $b = \text{dist}(p, q) = \text{inj}(M)$. Then

- (1) If p and q are conjugate along γ . Then by Corollary 21.1.4 we have $\text{inj}(M) = b \geq \frac{\pi}{\sqrt{k}}$.
- (2) If p and q are not conjugate along γ . Then by Proposition 13.2.1 there exists a unit-speed closed geodesic $\gamma: [0, 2b] \rightarrow M$ with $\gamma(0) = p, \gamma(b) = q$, where $b = \text{dist}(p, q) = \text{inj}(M)$. On one hand by definition of $l(M, g)$ one has $2b \geq l(M)$. On the other hand, $l(M) \geq 2b$, since $\text{dist}(p, q) = q$. Thus in this case $\text{inj}(M) = \frac{l(M)}{2}$.

□

21.2. Hessian comparison.

Theorem 21.2.1 (Hessian comparison). Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be two Riemannian manifolds with the same dimension, $U \subseteq M, \widetilde{U} \subseteq \widetilde{M}$ normal neighborhoods around $p \in M$ and $\widetilde{p} \in \widetilde{M}$ respectively. Suppose

$$\begin{aligned} \gamma: [0, b] &\rightarrow U, \gamma(0) = p, \gamma(b) = q \\ \widetilde{\gamma}: [0, b] &\rightarrow \widetilde{U}, \widetilde{\gamma}(0) = \widetilde{p}, \widetilde{\gamma}(b) = \widetilde{q} \end{aligned}$$

are two unit-speed geodesics such that

For all $t \in [0, b]$, and any planes $\Sigma \subseteq T_{\gamma(t)}M, \gamma'(t) \in \Sigma, \widetilde{\Sigma} \subseteq T_{\widetilde{\gamma}(t)}\widetilde{M}, \widetilde{\gamma}'(t) \in \widetilde{\Sigma}$, we have $K_{\gamma(t)}(\Sigma) \leq K_{\widetilde{\gamma}(t)}(\widetilde{\Sigma})$.

Then for any $v \in T_qM, \widetilde{v} \in T_{\widetilde{q}}\widetilde{M}$ with unit length and $v \perp \gamma'(b), \widetilde{v} \perp \widetilde{\gamma}'(b)$, we have

- (1) $\text{Hess } r(v, v) \geq \text{Hess } \widetilde{r}(\widetilde{v}, \widetilde{v})$.
- (2) $\Delta r(\gamma(t)) \geq \widetilde{\Delta} \widetilde{r}(\widetilde{\gamma}(t))$ for all $t \in (0, b]$.
- (3) Moreover, the equality holds if and only if $K_{\Sigma}(\gamma(t)) = \widetilde{K}_{\widetilde{\Sigma}}(\widetilde{\gamma}(t))$.

Proof. For (1). Let $\{e_1(t), \dots, e_n(t)\}$ be a parallel orthonormal basis along γ such that $e_n(t) = \gamma'(t)$ and $\{\widetilde{e}_1(t), \dots, \widetilde{e}_n(t)\}$ a parallel orthonormal basis along $\widetilde{\gamma}$ such that $\widetilde{e}_n(t) = \widetilde{\gamma}'(t)$. Without lose of generality we may assume $\langle v, e_i(b) \rangle_g = \langle \widetilde{v}, \widetilde{e}_i(b) \rangle_{\widetilde{g}}$ for $i = 1, \dots, n-1$, it's just a trick of linear algebra.

Use Corollary 12.1.1 to construct Jacobi fields

$$\begin{cases} J(0) = 0, J(b) = v \\ \widetilde{J}(0) = 0, \widetilde{J}(b) = \widetilde{v} \end{cases}$$

With respect to $\{\tilde{e}_i(t)\}$ we can write $\tilde{J}(t)$ as $\tilde{J}(t) = \lambda^i(t)\tilde{e}_i(t)$, and construct $V(t) = \lambda^i(t)e_i(t)$. Then

$$\begin{aligned}
\text{Hess } r(v, v) &= \text{Hess } r(J(b), J(b)) \\
&\stackrel{\text{I}}{=} \int_0^b \langle J'(t), J'(t) \rangle - R(J, \gamma', \gamma', J) dt \\
&\stackrel{\text{II}}{\geq} \int_0^b \langle V'(t), V'(t) \rangle - R(V, \gamma', \gamma', V) dt \\
&\stackrel{\text{III}}{\geq} \int_0^b \langle \tilde{J}'(t), \tilde{J}'(t) \rangle - \tilde{R}(\tilde{J}, \tilde{\gamma}, \tilde{\gamma}, \tilde{J}) dt \\
&= \text{Hess } \tilde{r}(\tilde{J}(b), \tilde{J}(b)) \\
&= \text{Hess } \tilde{r}(\tilde{v}, \tilde{v})
\end{aligned}$$

where

I holds from Corollary 14.5.1.

II holds from Corollary 12.3.1.

III holds from our assumption on curvature and the choice of V .

For (2) and (3). They directly follow from (1) and proof of (1). \square

Corollary 21.2.1 (Hessian and Laplacian comparison). Let (M, g) be a Riemannian n -manifold and U a normal neighborhood of $p \in M$.

(1) If sectional curvature $K \leq k$ in $U \setminus \{p\}$. Then

$$\mathcal{H}_r \geq \frac{\text{sn}'_k(r)}{\text{sn}_k(r)} \pi_r, \quad \Delta r \geq (n-1) \frac{\text{sn}'_k(r)}{\text{sn}_k(r)}$$

holds in $U_0 \setminus \{p\}$, where

$$U_0 = \begin{cases} U, & k \leq 0 \\ U \cap B(p, \pi R), & k = \frac{1}{R^2} > 0 \end{cases}$$

(2) If sectional curvature $K \geq k$ in $U \setminus \{p\}$. Then

$$\mathcal{H}_r \leq \frac{\text{sn}'_k(r)}{\text{sn}_k(r)} \pi_r, \quad \Delta r \leq (n-1) \frac{\text{sn}'_k(r)}{\text{sn}_k(r)}$$

holds in $U \setminus \{p\}$.

(3) Moreover, if equality holds, g has constant sectional curvature k in U_0 or U .

Proof. For (1). Apply Hessian comparison to (M, g) and space form $S(n, k)$. Then we directly have

$$\text{Hess } r(v, v) \geq \text{Hess } \tilde{r}(\tilde{v}, \tilde{v})$$

for any $v \in T_q M, \tilde{v} \in T_q S(n, k)$ with unit length and $v \perp \gamma'(b), \tilde{v} \perp \tilde{\gamma}'(b)$, where

$$\gamma: [0, b] \rightarrow U, \gamma(0) = p, \gamma(b) = q$$

$$\tilde{\gamma}: [0, b] \rightarrow \tilde{U}, \tilde{\gamma}(0) = \tilde{p}, \tilde{\gamma}(b) = \tilde{q}$$

are two unit-speed geodesics, and U, \tilde{U} are normal neighborhoods of p, \tilde{p} respectively. However, we must be careful here, since if sectional curvature of M is ≤ 0 . Then b can be infinite, and in this case if $k > 0$, the diameter of \tilde{U} is $< \frac{\pi}{\sqrt{k}}$. Thus we only have

$$\text{Hess } r(v, v) \geq \text{Hess } \tilde{r}(\tilde{v}, \tilde{v})$$

for $0 < b < \frac{\pi}{\sqrt{k}}$ if $k > 0$, and there is no restriction for b if $k \leq 0$. Thus by taking different geodesics and different Jacobi fields, we can show this holds for arbitrary $v \in T_q M, \tilde{v} \in T_q S(n, k)$, where $q \in U_0 \setminus \{p\}$, that is we have

$$\mathcal{H}_r \geq \frac{\text{sn}'_k(r)}{\text{sn}_k(r)} \pi_r$$

holds in $U_0 \setminus \{p\}$. By taking trace we obtain $\Delta r \geq (n-1) \frac{\text{sn}'_k(r)}{\text{sn}_k(r)}$ holds in $U_0 \setminus \{p\}$, since π_r is a projection onto a subspace with codimension 1.

For (2), the same as (1).

For (3), if

$$\mathcal{H}_r = \frac{\text{sn}'_k(r)}{\text{sn}_k(r)} \pi_r$$

holds in $U \setminus \{p\}$. Then it's directly from Proposition 14.5.2. If

$$\Delta r \geq (n-1) \frac{\text{sn}'_k(r)}{\text{sn}_k(r)}$$

holds in $U \setminus \{p\}$, that is the trace of $\mathcal{H}_r - \frac{\text{sn}'_k(r)}{\text{sn}_k(r)} \pi_r$ vanishes identically in $U \setminus \{p\}$. Then $\mathcal{H}_r - \frac{\text{sn}'_k(r)}{\text{sn}_k(r)} \pi_r$ vanishes identically, since it's semi-positive definite. \square

22. COMPARISON THEOREMS BASED ON RICCI CURVATURE

22.1. Local Laplacian comparison.

Theorem 22.1.1 (local Laplacian comparison). Let (M, g) be a Riemannian n -manifold and U a normal coordinate of $p \in M$. If there exists $k \in \mathbb{R}$ such that $\text{Ric}(g) \geq (n-1)kg$. Then

$$\Delta r \leq (n-1) \frac{\text{sn}'_k(r)}{\text{sn}_k(r)}$$

holds in $U_0 \setminus \{p\}$, where

$$U_0 = \begin{cases} U, & k \leq 0 \\ U \cap B(p, \pi R), & k = \frac{1}{R^2} > 0 \end{cases}$$

Moreover, if equality holds. Then g has constant sectional curvature in U_0 .

22.1.1. Proof by using Jacobi fields.

Proof of Theorem 22.1.1 via Jacobi fields. For arbitrary $q \in U_0 \setminus \{p\}$, choose a unit-speed geodesic $\gamma: [0, b] \rightarrow M$ with $\gamma(0) = p, \gamma(b) = q$, and $\{e_1(t), \dots, e_n(t)\}$ is a parallel orthonormal frame along γ with $e_n(t) = \gamma'(t)$. Then by definition $\Delta r = \sum_{i=1}^n \text{Hess } r(e_i, e_i)$.

By Corollary 12.1.1 one can construct Jacobi fields $J_i(t), i = 1, \dots, n$ such that $J_i(0) = 0, J_i(b) = e_i(b)$. Then we have

$$\Delta r = \sum_{i=1}^{n-1} \text{Hess } r(J_i(b), J_i(b)) \stackrel{(1)}{=} \sum_{i=1}^{n-1} I(J_i, J_i)$$

where (1) holds from Corollary 14.5.1. Now let \widetilde{M} be the space form $S(n, k)$ and \widetilde{U} a normal coordinate of $\widetilde{p} \in \widetilde{M}$. Repeat the same process as above we have

$$(n-1) \frac{\text{sn}'_k(r)}{\text{sn}_k(r)} = \widetilde{\Delta} \widetilde{r} = \sum_{i=1}^{n-1} \widetilde{I}(\widetilde{J}_i, \widetilde{J}_i)$$

If we denote $V_i(t) = f(t)e_i(t)$, routine computation shows

$$\begin{aligned} \Delta r &= \sum_{i=1}^{n-1} I(J_i, J_i) \\ &\leq \sum_{i=1}^{n-1} I(V_i, V_i) \\ &= \sum_{i=1}^{n-1} \int_0^b \langle V'_i(t), V'_i(t) \rangle - R(V_i, \gamma', \gamma', V_i) dt \end{aligned}$$

Until now, all computations are the same as what we have done in Hessian comparison based on sectional curvature. A crucial observation is that

$\tilde{J}_i(t) = f(t)\tilde{e}_i(t)$, and the **key point** is that $f(t)$ is independent of i . Then

$$\begin{aligned}
\Delta r &\stackrel{(2)}{=} \sum_{i=1}^{n-1} \int_0^b \langle V'_i(t), V'_i(t) \rangle - f^2(t) R(e_i, e_n, e_n, e_i) dt \\
&= \sum_{i=1}^{n-1} \int_0^b \langle V'_i(t), V'_i(t) \rangle - \int_0^b f^2(t) \operatorname{Ric}(e_n, e_n) dt \\
&\leq \sum_{i=1}^{n-1} \int_0^b \langle \tilde{J}_i(t), \tilde{J}_i(t) \rangle - \int_0^b (n-1)k f^2(t) dt \\
&= \sum_{i=1}^{n-1} \tilde{I}(\tilde{J}_i, \tilde{J}_i) \\
&= (n-1) \frac{\operatorname{sn}'_k(r)}{\operatorname{sn}_k(r)}
\end{aligned}$$

the key point is used in equality marked by (2), and others are routines. \square

22.1.2. *Proof by using Bochner's technique.*

Lemma 22.1.1. Let (M, g) be a Riemannian manifold, (x^i, U, p) a normal coordinate centered at p . Then

$$\Delta r = \partial_r \log(r^{n-1} \sqrt{\det g})$$

in $U \setminus \{p\}$. Moreover, along any unit-speed geodesic $\gamma: [0, b] \rightarrow U$ with $\gamma(0) = p$, if we define $f(t) := \Delta r(\gamma(t))$. Then

$$f(t) = \frac{n-1}{t} + O(1)$$

Proof. Direct computation shows

$$\begin{aligned}
\Delta r &= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} (g^{ij} \sqrt{\det g} \frac{\partial r}{\partial x^j}) \\
&= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} (g^{ij} \sqrt{\det g} \frac{x^j}{r}) \\
&= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} (\frac{x^i}{r} \sqrt{\det g}) \\
&= \frac{\partial}{\partial x^i} (\frac{x^i}{r}) + \frac{1}{\sqrt{\det g}} \frac{x^i}{r} \frac{\partial}{\partial x^i} (\sqrt{\det g}) \\
&= \frac{n-1}{r} + \frac{1}{\sqrt{\det g}} \partial_r (\sqrt{\det g}) \\
&= \partial_r \log(r^{n-1} \sqrt{\det g})
\end{aligned}$$

Moreover, for unit-speed geodesic $\gamma: [0, b] \rightarrow U$, we have

$$f(t) = \frac{n-1}{r(\gamma(t))} + \partial_r (\log \sqrt{\det g}) \Big|_{\gamma(t)}$$

Then note that

- (1) $r(\gamma(t)) = t$, since γ is unit-speed geodesic.
- (2) Jacobi's formula implies

$$\partial_r(\log \sqrt{\det g}) \Big|_{\gamma(t)} = \frac{1}{2} g^{ij} \frac{\partial g_{ij}}{\partial x^k} \frac{d\gamma^k}{dt} = O(1)$$

we obtain the desired results. \square

Lemma 22.1.2 (Riccati comparison theorem). If $f: (0, b) \rightarrow \mathbb{R}$ is a smooth function satisfying

- (1) $f(t) = \frac{1}{t} + O(1)$.
- (2) $f' + f^2 + k \leq 0$.

Then

$$f(t) \leq \frac{\text{sn}'_k(t)}{\text{sn}_k(t)}$$

for all $t \in (0, b)$, where $k > 0, b \leq \frac{\pi}{\sqrt{k}}$.

Proof. Consider $f_k(t) = \frac{\text{sn}'_k(t)}{\text{sn}_k(t)}$, it's a smooth function defined on $(0, b)$ satisfying

- (1) $f_k(t) = \frac{1}{t} + O(1)$
- (2) $f'_k + f_k^2 + k = 0$

Choose a smooth function $f: (0, b) \rightarrow \mathbb{R}$ satisfying

- (1) $F(t) = 2 \log t + O(1)$.
- (2) $F'(t) = f + f_k$

Then

$$\begin{aligned} \frac{d}{dt}(e^F(f - f_k)) &= e^F(f^2 - f_k^2 + f' - f'_k) \leq 0 \\ \lim_{t \rightarrow 0} e^F(f - f_k) &= 0 \end{aligned}$$

Then we have $f(t) \leq f_k(t)$ holds for all $t \in (0, b)$. \square

Lemma 22.1.3.

$$|\text{Hess } r|^2 \geq \frac{(\Delta r)^2}{n-1}$$

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal frame with $e_1 = \partial_r$. Then

$$\begin{aligned} |\text{Hess } r|^2 &= \sum_{i,j=1}^n (\langle \nabla_{e_i} \partial_r, e_j \rangle)^2 \\ &= \sum_{i,j=2}^n (\langle \nabla_{e_i} \partial_r, e_j \rangle)^2 \\ &\geq \frac{1}{n-1} \sum_{i=2}^n (\langle \nabla_{e_i} \partial_r, e_i \rangle)^2 \\ &= \frac{1}{n-1} (\Delta r)^2 \end{aligned}$$

The inequality

$$|A|^2 \geq \frac{1}{k} |\operatorname{tr}(A)|^2$$

for a $k \times k$ matrix A is a direct consequence of the Cauchy-Schwarz inequality. \square

Proof of Theorem 22.1.1 by using Bochner's technique. Recall Bochner's technique says

$$\frac{1}{2} \Delta |\nabla f|^2 = |\operatorname{Hess} f|^2 + \operatorname{Ric}(\nabla f, \nabla f) + g(\nabla \Delta f, \nabla f)$$

Set $f = r$ we have

$$\begin{aligned} 0 &= |\operatorname{Hess} r|^2 + \operatorname{Ric}(\nabla r, \nabla r) + g(\nabla \Delta r, \nabla r) \\ &\stackrel{(1)}{\geq} |\operatorname{Hess} r|^2 + \partial_r(\Delta r) + (n-1)k \\ &\stackrel{(2)}{\geq} \left(\frac{\Delta r}{n-1}\right)^2 + \partial_r\left(\frac{\Delta r}{n-1}\right) + k \end{aligned}$$

where

(1) holds from $\partial_r = \nabla_r$ and lower bounded of Ricci.

(2) holds from Lemma 22.1.3 and divided by $n-1$.

Thanks to Lemma 22.1.1, we can apply Riccati comparison to $f(r) = \frac{\Delta r}{n-1}$. Then we have

$$\frac{\Delta r}{n-1} \leq \frac{\operatorname{sn}'_k(r)}{\operatorname{sn}_k(r)}$$

This shows desired comparison.

Furthermore, if equality holds

$$\frac{\Delta r}{n-1} = \frac{\operatorname{sn}'_k(r)}{\operatorname{sn}_k(r)}$$

then direct computation shows

$$\left(\frac{\Delta r}{n-1}\right)^2 + \partial_r\left(\frac{\Delta r}{n-1}\right) + k = 0$$

which implies inequalities in (1) and (2) are in fact equalities. In particular, one has

$$|\operatorname{Hess} r|^2 = \frac{(\Delta r)^2}{n-1}$$

that is inequality in Cauchy-Schwarz inequality holds, which implies

$$\mathcal{H}_r = \frac{\operatorname{sn}'_k(r)}{\operatorname{sn}_k(r)} \pi_r$$

Then g has constant sectional curvature k in U_0 by Proposition 14.5.2. \square

22.2. Maximal principle.

Proposition 22.2.1. Let (M, g) be a Riemannian manifold and f, h be two smooth functions on M . If there is a point p such that $f(p) = h(p)$ and $f(x) \geq h(x)$ for all x near p . Then

$$\nabla f(p) = \nabla h(p), \quad \text{Hess } f|_p \geq \text{Hess } h|_p, \quad \Delta f(p) \geq \Delta h(p)$$

Proof. Firstly let's consider the case $(M, g) \subseteq (\mathbb{R}^n, g_{\text{can}})$, it's a simple calculus since we can use Taylor expansion. To be explicit, for all x near p , we have

$$f(x) = f(p) + \nabla f(p)^T(x-p) + \frac{1}{2}(x-p)^T \text{Hess } f|_p(x-p) + O(|x|^3)$$

where ∇f is an n column vector and $\text{Hess } f$ is an $n \times n$ matrix in this case. Similarly, we have

$$h(x) = h(p) + \nabla h(p)^T(x-p) + \frac{1}{2}(x-p)^T \text{Hess } h|_p(x-p) + O(|x|^3)$$

Then consider

$$f(x) - h(x) = (\nabla f - \nabla h)(p)^T(x-p) + \frac{1}{2}(x-p)^T \text{Hess}(f-h)|_p(x-p) + O(|x|^3)$$

Since $f(x) - h(x) \geq 0$ for all x near p . Then we must have

$$\begin{aligned} \nabla f(p) &= \nabla h(p) \\ \text{Hess } f|_p &\geq \text{Hess } h|_p \end{aligned}$$

By taking trace we have

$$\Delta f(p) \geq \Delta h(p)$$

For general case, take $\gamma: (-\varepsilon, \varepsilon) \rightarrow M$ to be a geodesic with $\gamma(0) = p$. Then use previous case on $f \circ \gamma, h \circ \gamma$ to obtain

$$\begin{aligned} \nabla_{\gamma'(0)} f(p) &= \nabla_{\gamma'(0)} h(p) \\ \text{Hess } f_p(\gamma'(0), \gamma'(0)) &\geq \text{Hess } h_p(\gamma'(0), \gamma'(0)) \end{aligned}$$

Then it's clear this proposition holds if we let $v = \gamma'(0)$ run over all $v \in T_p M$. \square

Definition 22.2.1 (barrier sense). Let (M, g) be a Riemannian manifold and $f \in C(M)$. Suppose f_q is a C^2 function defined in a neighborhood of U of $q \in M$.

(1) f_q is called a lower barrier function of f at q if

$$f_q(q) = f(q), \quad f_q(x) \leq f(x), \quad x \in U$$

(2)

$$\Delta f(q) \geq c$$

in the barrier sense if for all $\varepsilon > 0$, there exists a lower barrier function $f_{q,\varepsilon}$ of f at q such that

$$\Delta f_{q,\varepsilon}(q) \geq c - \varepsilon$$

(3)

$$\Delta f(q) \leq c$$

in the barrier sense if for all $\varepsilon > 0$, there exists an upper barrier function $f_{q,\varepsilon}$ of f at q such that

$$\Delta f_{q,\varepsilon}(q) \leq c + \varepsilon$$

Definition 22.2.2 (distribution sense). Let (M, g) be an orientable Riemannian manifold and $f \in C(M)$.

$$\Delta f \leq h$$

in distribution sense, if

$$\int_M f \Delta \varphi \leq \int_M h \varphi$$

holds for all $\varphi \geq 0 \in C_c^\infty(M)$

Theorem 22.2.1 (maximal principle). Let (M, g) be a Riemannian manifold and $f \in C(M)$.

- (1) If $\Delta f \geq 0$ in the barrier sense or distribution sense. Then if f has a local(global) maximum. Then it's local(global) constant.
- (2) If $\Delta f \leq 0$ in the barrier sense or distribution sense. Then if f has a local(global) minimal. Then it's local(global) constant.
- (3) $\Delta f = 0$ implies $f \in C^\infty(M)$.

Proof. See Theorem 66 in Page 280 of [Pet16]. □

22.3. Global Laplacian comparison.

22.3.1. In the barrier sense.

Proposition 22.3.1. Let (M, g) be a complete Riemannian manifold and $p, q \in M$. Let $\gamma : [0, b] \rightarrow M$ be a unit-speed minimal geodesic with $\gamma(0) = p$ and $\gamma(b) = q$. For any small $\varepsilon > 0$,

$$r_\varepsilon(x) = \varepsilon + \text{dist}(\gamma(\varepsilon), x)$$

where $x \in M$. Then

- (1) $q \notin \text{cut}(\gamma(\varepsilon))$ and in particular, r_ε is smooth at q .
- (2) r_ε is an upper barrier function of $r(x) = \text{dist}(p, x)$ at point q .

Proof. For (1). If $q \in \text{cut}(\gamma(\varepsilon))$, by Corollary 13.1.2, one has $\gamma(\varepsilon) \in \text{cut}(q)$, a contradiction to γ is a minimal geodesic connecting p and q .

For (2). Firstly note that $\gamma(b) = q$. Then

$$r(q) = \text{dist}(p, q) = \text{dist}(\gamma(0), \gamma(b)) \stackrel{\text{I}}{=} \text{dist}(\gamma(0), \gamma(\varepsilon)) + \text{dist}(\gamma(\varepsilon), \gamma(b)) \stackrel{\text{II}}{=} r_\varepsilon(q)$$

where

I holds since γ is a minimal geodesic.

II holds since γ is unit-speed minimal geodesic. Then $\text{dist}(\gamma(0), \gamma(\varepsilon)) = \varepsilon$.

By triangle inequality, one has

$$r(q') = \text{dist}(p, q') \leq \varepsilon + \text{dist}(\gamma(\varepsilon), q) = r_\varepsilon(q')$$

for all q' near q . Combining these two facts together we have r_ε is an upper barrier function of r . \square

Theorem 22.3.1 (global Laplacian comparison). Let (M, g) be a complete Riemannian manifold with

$$\text{Ric}(g) \geq (n-1)kg$$

Then for $q \in M$

$$\Delta r(q) \leq (n-1) \frac{\text{sn}'_k(r(q))}{\text{sn}_k(r(q))}$$

in the barrier sense.

Proof. We consider the following three cases:

- (1) If $q \in M \setminus \{p\} \cup \text{cut}(p)$, it's exactly smooth case we have proven.
- (2) If $q = p$, it's clear, since the right hand is infinite.
- (3) For arbitrary $q \in \text{cut}(p)$, there exists a unit-speed $\gamma: [0, b] \rightarrow M$ with $\gamma(0) = p, \gamma(b) = q$. Then for each $\gamma > 0$, define

$$\gamma_\varepsilon(x) = \varepsilon + \text{dist}(\gamma(\varepsilon), x)$$

Then by Proposition 22.3.1 we have $\gamma_\varepsilon(x)$ is an upper barrier of $r(x)$ and γ_ε is smooth at q . Thus we have

$$\begin{aligned} \Delta \gamma_\varepsilon(q) &= \Delta \text{dist}(\gamma(\varepsilon), q) \\ &\leq (n-1) \frac{\text{sn}'_k(\gamma_\varepsilon(q) - \varepsilon)}{\text{sn}_k(\gamma_\varepsilon(q) - \varepsilon)} \\ &= (n-1) \frac{\text{sn}'_k(\gamma(q) - \varepsilon)}{\text{sn}_k(\gamma(q) - \varepsilon)} \end{aligned}$$

which descends to $(n-1) \frac{\text{sn}'_k(\gamma(q))}{\text{sn}_k(\gamma(q))}$ as $\varepsilon \rightarrow 0$ by monotonicity. This completes the proof. \square

22.3.2. In the distribution sense.

Proposition 22.1. Let (M, g) be an orientable Riemannian manifold and $f: M \rightarrow \mathbb{R}$ a Lipschitz function. Then for any $\varphi \in C_0^\infty(M, \mathbb{R})$, one has

$$-\int_M \langle \nabla \varphi, \nabla f \rangle d\text{vol}_g = \int_M \Delta \varphi \cdot f d\text{vol}_g$$

Theorem 22.3.2 (global Laplacian comparison II). Let (M, g) be a complete Riemannian manifold with

$$\text{Ric}(g) \geq (n-1)kg$$

Then for $x \in M$

$$\Delta r(x) \leq (n-1) \frac{\text{sn}'_k(r(x))}{\text{sn}_k(r(x))}$$

in the distribution sense.

Proof. For fixed $p \in M$, the domain $\Sigma(p)$ of injective radius $\text{inj}(p)$ is a star-shaped open subset of $T_p M$ and $M = \exp_p(\Sigma(p)) \cup \text{cut}(p)$. The boundary of $\Sigma(p)$ is locally a graph of continuous function and so there exists a family of star-shaped domains $\{U_j\}$ with smooth boundaries such that

$$U_j \subseteq U_{j+1} \subseteq \cdots \subseteq \Sigma(p), \quad \Sigma(p) = \bigcup U_j$$

If we set $\Omega = \exp_p(\Sigma(p))$. Then $\Omega = \bigcup \Omega_j$, where $\Omega_j = \exp_p(U_j)$. Since each U_j is star-shaped, by Gauss lemma, on each boundary $\partial\Omega_j$, one has $\frac{\partial r}{\partial v} = g(\nabla r, v) \geq 0$ where v is the outer normal vector on $\partial\Omega_j$.

Therefore for each $\varphi \in C_c^\infty(M)$ with $\varphi \geq 0$, one has

$$\begin{aligned} \int_M r \Delta \varphi \text{ vol} &\stackrel{(1)}{=} - \int_M \langle \nabla r, \nabla \varphi \rangle \text{ vol} \\ &\stackrel{(2)}{=} - \lim_j \int_{\Omega_j \setminus \{p\}} \langle \nabla r, \nabla \varphi \rangle \\ &\stackrel{(3)}{=} \lim_j \left(\int_{\Omega_j \setminus \{p\}} \Delta r \varphi \text{ vol} - \int_{\partial\Omega_j} \varphi \frac{\partial r}{\partial v} \right) \\ &\stackrel{(4)}{\leq} \lim_j \int_{\Omega_j \setminus \{p\}} \Delta r \varphi \text{ vol} \\ &\stackrel{(5)}{\leq} \lim_j \int_{\Omega_j \setminus \{p\}} (n-1) \frac{\text{sn}'_k(r)}{\text{sn}_k(r)} \varphi \text{ vol} \\ &\stackrel{(6)}{=} \int_{\Omega \setminus \{p\}} (n-1) \frac{\text{sn}'_k(r)}{\text{sn}_k(r)} \varphi \text{ vol} \\ &\stackrel{(7)}{=} \int_M (n-1) \frac{\text{sn}'_k(r)}{\text{sn}_k(r)} \varphi \text{ vol} \end{aligned}$$

where

- (1) holds from the fact r is Lipschitz and Proposition 22.1.
- (2) and (6) holds from dominated convergence theorem.
- (3) holds from Stokes theorem.
- (4) holds from $\varphi \geq 0$ and $\frac{\partial r}{\partial v} \geq 0$.
- (5) holds from Local Laplacian comparison theorem, that is Theorem 22.1.1.
- (7) holds from the fact $\text{cut}(p)$ is zero-measure.

□

22.4. Volume comparison.

Lemma 22.4.1. Let (M, g) be a complete, connected Riemannian manifold and $p \in M$. For any $\delta \in \mathbb{R}^+$

$$\exp_p(B(0, \delta) \cap \Sigma(p)) \subseteq B(p, \delta) \subseteq \exp_p(B(0, \delta) \cap \Sigma(p)) \cup \text{cut}(p)$$

In particular, under the map $\Phi: \mathbb{R}^+ \times \mathbb{S}^{n-1} \rightarrow T_p M \setminus \{0\}$ given by $\Phi(\rho, \omega) = \rho\omega$

$$\begin{aligned} \text{Vol}(B(p, \delta)) &= \text{Vol}(\exp_p(B(0, \delta)) \cap \Sigma(p)) \\ &= \int_{B(0, \delta) \cap \Sigma(p)} \exp_p^* \text{vol} \\ &= \int_{B(0, \delta)} \chi_{\Sigma(p)} \exp_p^* \text{vol} \\ &= \int_{\mathbb{S}^{n-1}} \int_0^\delta \chi_{\Sigma(p)} \sqrt{\det g} \circ \Phi(\rho, \omega) \rho^{n-1} d\rho \text{vol}_{\mathbb{S}^{n-1}} \end{aligned}$$

Corollary 22.4.1. Let $p \in S(n, k)$

(1) If $k \leq 0$. Then for any $\delta \in \mathbb{R}^+$

$$\text{Vol}(B(p, \delta)) = \int_{\mathbb{S}^{n-1}} \int_0^\delta \text{sn}_k^{n-1}(\rho) d\rho \text{vol}_{\mathbb{S}^{n-1}}$$

(2) If $k = \frac{1}{R^2} \geq 0$. Then for any $\delta \in \mathbb{R}^+$

$$\text{Vol}(B(p, \delta)) = \int_{\mathbb{S}^{n-1}} \int_0^\delta \chi_{B(0, \pi R)} \text{sn}_k^{n-1}(\rho) d\rho \text{vol}_{\mathbb{S}^{n-1}}$$

Lemma 22.4.2. Let (M, g) be a Riemannian manifold, and (x^i, U, p) be a geodesic ball chart of radius b around $p \in M$.

(1) If $K \leq k$. Then for each fixed $\omega \in \mathbb{S}^{n-1}$ the volume density ratio

$$\lambda(\rho, \omega) = \frac{\rho^{n-1} \sqrt{\det g} \circ \Phi(\rho, \omega)}{\text{sn}_k^{n-1}(\rho)}$$

is non-decreasing in $\rho \in (0, b_0)$ where

$$b_0 = \begin{cases} b, & k \leq 0 \\ \min\{b, \pi R\}, & k = \frac{1}{R^2} \end{cases}$$

Moreover, $\lim_{\rho \rightarrow 0} \lambda(\rho, \omega) = 1$.

(2) If $K \geq k$ or $\text{Ric}(g) \geq (n-1)kg$. Then for each fixed $\omega \in \mathbb{S}^{n-1}$ the volume density ratio $\lambda(\rho, \omega)$ is non-increasing in $\rho \in (0, b)$ and $\lim_{\rho \rightarrow 0} \lambda(\rho, \omega) = 1$.

Proof. By Corollary 21.2.1 and Lemma 22.1.1

$$\partial_r \log(r^{n-1} \sqrt{\det g}) = \Delta r \geq (n-1) \frac{\text{sn}'_k(r)}{\text{sn}_k(r)} = \partial_r \log(\text{sn}_k^{n-1}(r))$$

Hence $\log \left(\frac{r^{n-1} \sqrt{\det g}}{\operatorname{sn}_k^{n-1}(r)} \right)$ is a non-decreasing function of r along each radial geodesic γ , that is

$$\frac{d}{dt} \left(\log \left(\frac{r^{n-1} \sqrt{\det g}}{\operatorname{sn}_k^{n-1}(r)} \right) \circ \gamma(t) \right) \geq 0$$

Hence, $f(r) = \frac{r^{n-1} \sqrt{\det g}}{\operatorname{sn}_k^{n-1}(r)}$ is a non-decreasing function of r along each radial geodesic γ . It is easy to see that $r \circ \Phi = \rho$ (the exponential map is used in normal coordinate). Hence,

$$\lambda(\rho, \omega) = f \circ \Phi(\rho, \omega)$$

is nondecreasing in ρ for any fixed $\omega \in \mathbb{S}^{n-1}$. It is obvious that

$$\lim_{\rho \rightarrow 0} \sqrt{\det g} = \lim_{\rho \rightarrow 0} \frac{\rho^{n-1}}{\operatorname{sn}_k^{n-1}(\rho)} = 1$$

The proof of (2) is similar. □

Lemma 22.4.3. Let $f: [0, +\infty) \rightarrow [0, +\infty)$, $g: [0, +\infty) \rightarrow (0, +\infty)$ be two integrable functions. If

$$\lambda(t) = \frac{f(t)}{g(t)}: [0, +\infty) \rightarrow [0, +\infty)$$

is non-increasing. Then

$$F(t) = \frac{\int_0^t f(\tau) d\tau}{\int_0^t g(\tau) d\tau}: [0, +\infty) \rightarrow [0, +\infty)$$

is non-increasing. Moreover, if there exists $0 < t_1 < t_2$ such that

$$F(t_1) = F(t_2),$$

then $\lambda(t) \equiv \lambda(t_1)$ for almost all $t \in [0, t_2]$.

Proof. We can assume $f(t) > 0$ for all $t \in [0, +\infty)$, otherwise we replace it by $f(t) + \varepsilon g(t)$ for some $\varepsilon > 0$. Given $0 < t_1 < t_2$, we need to show

$$\int_0^{t_1} f(\tau) d\tau \int_0^{t_2} g(\tau) d\tau - \int_0^{t_2} f(\tau) d\tau \int_0^{t_1} g(\tau) d\tau \geq 0$$

Indeed,

$$\begin{aligned}
& \int_0^{t_1} f(\tau) d\tau \int_0^{t_2} g(\tau) d\tau - \int_0^{t_2} f(\tau) d\tau \int_0^{t_1} g(\tau) d\tau \\
&= \int_0^{t_1} f(\tau) d\tau \int_0^{t_2} g(\tau) d\tau - \int_0^{t_1} f(\tau) d\tau \int_0^{t_1} g(\tau) d\tau - \int_{t_1}^{t_2} f(\tau) d\tau \int_0^{t_1} g(\tau) d\tau \\
&= \int_0^{t_1} f(\tau) d\tau \int_{t_1}^{t_2} g(\tau) d\tau - \int_{t_1}^{t_2} f(\tau) d\tau \int_0^{t_1} g(\tau) d\tau \\
&\stackrel{(1)}{\geq} \int_0^{t_1} \frac{f(t_1)}{g(t_1)} g(\tau) d\tau \int_{t_1}^{t_2} \frac{g(t_1)}{f(t_1)} f(\tau) d\tau - \int_{t_1}^{t_2} f(\tau) d\tau \int_0^{t_1} g(\tau) d\tau \\
&= 0
\end{aligned}$$

where (1) holds from $\lambda(t)$ is non-increasing. It is clear that if $F(t_1) = F(t_2)$. Then the inequality marked by (1) is an equality, which implies for almost all $t \in [0, t_2]$, $\lambda(t) \equiv \lambda(t_1)$. \square

Remark 22.4.1. For any $0 \leq \delta_1 < \delta_2 \leq \delta_3 < \delta_4$, we can slightly adapt above proof to show

$$\frac{\int_{\delta_3}^{\delta_4} f(\tau) d\tau}{\int_{\delta_3}^{\delta_4} g(\tau) d\tau} \leq \frac{\int_{\delta_1}^{\delta_2} f(\tau) d\tau}{\int_{\delta_1}^{\delta_2} g(\tau) d\tau}$$

Indeed, just note that

$$\begin{aligned}
& \int_{\delta_3}^{\delta_4} f(\tau) d\tau \int_{\delta_2}^{\delta_1} g(\tau) d\tau - \int_{\delta_1}^{\delta_2} f(\tau) d\tau \int_{\delta_3}^{\delta_4} g(\tau) d\tau \\
&\leq \int_{\delta_3}^{\delta_4} \frac{f(\delta_3)}{g(\delta_3)} g(\tau) d\tau \int_{\delta_2}^{\delta_1} g(\tau) d\tau - \int_{\delta_1}^{\delta_2} \frac{f(\delta_2)}{g(\delta_2)} g(\tau) d\tau \int_{\delta_3}^{\delta_4} g(\tau) d\tau \\
&= \left(\frac{f(\delta_3)}{g(\delta_3)} - \frac{f(\delta_2)}{g(\delta_2)} \right) \int_{\delta_2}^{\delta_1} g(\tau) d\tau \int_{\delta_3}^{\delta_4} g(\tau) d\tau \\
&\leq 0
\end{aligned}$$

Theorem 22.4.1 (Bishop-Gromov). Let (M, g) be a complete Riemannian manifold and $p \in M$. Let $B(p, \delta)$ be the metric ball centered at p with radius δ and g_k be the metric with constant sectional curvature k on $B(p, \delta) \setminus \{p\}$.

- (1) Suppose $K \leq k$. Then the volume ratio $\frac{\text{Vol}_g(B(p, \delta))}{\text{Vol}_{g_k}(B(p, \delta))}$ is non-decreasing for any $0 < \delta \leq \delta_0$ where $\delta_0 = \text{inj}(p)$ if $k \leq 0$, and $\delta_0 = \min\{\text{inj}(p), \pi/\sqrt{k}\}$ if $k > 0$. Moreover,

$$\lim_{\delta \rightarrow 0} \frac{\text{Vol}_g(B(p, \delta))}{\text{Vol}_{g_k}(B(p, \delta))} = 1$$

In particular,

$$\text{Vol}_g(B(p, \delta)) \geq \text{Vol}_{g_k}(B(p, \delta)),$$

- (2) If $K \geq k$ or $\text{Ric}(g) \geq (n-1)kg$. Then the volume ratio $\frac{\text{Vol}_g(B(p, \delta))}{\text{Vol}_{g_k}(B(p, \delta))}$ is non-increasing for $\delta \in \mathbb{R}^+$. Moreover,

$$\lim_{\delta \rightarrow 0} \frac{\text{Vol}_g(B(p, \delta))}{\text{Vol}_{g_k}(B(p, \delta))} = 1$$

In particular,

$$\text{Vol}_g(B(p, \delta)) \leq \text{Vol}_{g_k}(B(p, \delta)),$$

- (3) Furthermore, if there exists $\delta_1 < \delta_2$ such that

$$\frac{\text{Vol}_g(B(p, \delta_1))}{\text{Vol}_{g_k}(B(p, \delta_1))} = \frac{\text{Vol}_g(B(p, \delta_2))}{\text{Vol}_{g_k}(B(p, \delta_2))}$$

then $\text{Vol}_g(B(p, \delta)) = \text{Vol}_{g_k}(B(p, \delta))$ for any $\delta \in [0, \delta_2]$ and g has constant sectional curvature k on $B(p, \delta_2)$.

Proof. For (1). By the assumption, we know the metric ball $B(p, \delta)$ is actually a geodesic ball. We have the expression

$$\begin{aligned} \frac{\text{Vol}_g(B(p, \delta))}{\text{Vol}_{g_k}(B(p, \delta))} &\stackrel{\text{I}}{=} \frac{\int_{\mathbb{S}^{n-1}} \int_0^\delta \rho^{n-1} \sqrt{\det g} \circ \Phi(\rho, \omega) d\rho d\text{Vol}_{\mathbb{S}^{n-1}}}{\int_{\mathbb{S}^{n-1}} \int_0^\delta \text{sn}_k^{n-1}(\rho) d\rho d\text{Vol}_{\mathbb{S}^{n-1}}} \\ &\stackrel{\text{II}}{=} \frac{1}{\text{Vol}(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} \left(\frac{\int_0^\delta \rho^{n-1} \sqrt{\det g} \circ \Phi(\rho, \omega) d\rho}{\int_0^\delta \text{sn}_k^{n-1}(\rho) d\rho} \right) d\text{Vol}_{\mathbb{S}^{n-1}} \end{aligned}$$

where

I holds from Lemma 22.4.1.

II holds from Fubini's theorem.

By Lemma 22.4.2, one has $\lambda(\rho, \omega) = \frac{\rho^{n-1} \sqrt{\det g} \circ \Phi(\rho, \omega)}{\text{sn}_k^{n-1}(\rho)}$ is non-decreasing in ρ .

Then by Lemma 22.4.3 we have $\frac{\text{Vol}_g(B(p, \delta))}{\text{Vol}_{g_k}(B(p, \delta))}$ is non-decreasing in ρ . On the other hand,

$$\lim_{\delta \rightarrow 0} \frac{\text{Vol}_g(B(p, \delta))}{\text{Vol}_{g_k}(B(p, \delta))} = 1$$

Hence, for any $0 < \delta \leq \delta_0$, $\text{Vol}_g(B(p, \delta)) \geq \text{Vol}_{g_k}(B(p, \delta))$

For (2). Let's divide into the following two cases:

- (a) If $k \leq 0$, for any $\delta \in \mathbb{R}^+$, we get

$$\begin{aligned} \frac{\text{Vol}_g(B(p, \delta))}{\text{Vol}_{g_k}(B(p, \delta))} &= \frac{\int_{\mathbb{S}^{n-1}} \int_0^\delta \chi_{\Sigma(p)} \rho^{n-1} \sqrt{\det g} \circ \Phi(\rho, \omega) d\rho d\text{Vol}_{\mathbb{S}^{n-1}}}{\int_{\mathbb{S}^{n-1}} \int_0^\delta \text{sn}_k^{n-1}(\rho) d\rho d\text{Vol}_{\mathbb{S}^{n-1}}} \\ &= \frac{1}{\text{Vol}(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} \left(\frac{\int_0^\delta \chi_{\Sigma(p)} \rho^{n-1} \sqrt{\det g} \circ \Phi(\rho, \omega) d\rho}{\int_0^\delta \text{sn}_k^{n-1}(\rho) d\rho} \right) d\text{Vol}_{\mathbb{S}^{n-1}} \end{aligned}$$

where these two equalities hold from the same reasons. So in this case we consider

$$\tilde{\lambda}(\rho, \omega) := \chi_{\Sigma(p)} \lambda(\rho, \omega)$$

It's clear $\tilde{\lambda}$ is also non-increasing in ρ , since $\chi_{\Sigma(p)}$ is just a cut-off function. Then the same argument implies for arbitrary $\delta \in \mathbb{R}^+$, one has $\text{Vol}_g(B(p, \delta)) \leq \text{Vol}_{g_k}(B(p, \delta))$.

(b) If $k = \frac{1}{R^2} > 0$, for any $\delta \in \mathbb{R}^+$, we get

$$\begin{aligned} \frac{\text{Vol}_g(B(p, \delta))}{\text{Vol}_{g_k}(B(p, \delta))} &= \frac{\int_{\mathbb{S}^{n-1}} \int_0^\delta \chi_{\Sigma(p)} \rho^{n-1} \sqrt{\det g} \circ \Phi(\rho, \omega) d\rho d\text{Vol}_{\mathbb{S}^{n-1}}}{\int_{\mathbb{S}^{n-1}} \int_0^\delta \chi_{B(0, \pi R)} \text{sn}_k^{n-1}(\rho) d\rho d\text{Vol}_{\mathbb{S}^{n-1}}} \\ &= \frac{1}{\text{Vol}(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} \left(\frac{\int_0^\delta \chi_{\Sigma(p)} \rho^{n-1} \sqrt{\det g} \circ \Phi(\rho, \omega) d\rho}{\int_0^\delta \chi_{B(0, \pi R)} \text{sn}_k^{n-1}(\rho) d\rho} \right) d\text{Vol}_{\mathbb{S}^{n-1}} \end{aligned}$$

So in this case we consider⁷

$$\tilde{\lambda}(\rho, \omega) := \frac{\chi_{\Sigma(p)}}{\chi_{B(0, \pi R)}} \lambda(\rho, \omega)$$

Then the same argument shows the result.

For (3).

□

Corollary 22.4.2. Let (M, g) be a complete Riemannian n -manifold with $\text{Ric}(g) \geq 0$. Then the volume growth of (M, g) satisfies

$$\text{Vol}_g(B(p, r)) \leq c_n r^n$$

where c_n is a constant > 0 depending only on n .

Proof. Consider $k = 0$ and use Theorem 22.4.1, one has

$$\text{Vol}_g(B(p, r)) \leq \text{Vol}_{g_0}(B(p, r)) = \frac{\text{Vol}_{g_1}(\mathbb{S}^{n-1}) r^n}{n}$$

where \mathbb{S}^{n-1} is the unit sphere. Thus we just set $c_n = \text{Vol}_{g_1}(\mathbb{S}^{n-1})/n$ to conclude. □

Corollary 22.4.3. Let (M, g) be a complete Riemannian n -manifold with $\text{Ric}(g) \geq 0$. If

$$\lim_{r \rightarrow \infty} \frac{\text{Vol}_g(B(p, r))}{r^n} \geq \frac{\text{Vol}_{g_1}(\mathbb{S}^{n-1})}{n}$$

where \mathbb{S}^{n-1} is unit sphere. Then (M, g) is isometric to $(\mathbb{R}^n, g_{\text{can}})$.

⁷Be careful, our notation here is a little ambiguous, since it's nonsense if $\chi_{B(0, \pi R)} = 0$. However, Myers' theorem implies $\text{diam}(M, g) \leq \pi R$, hence $\Sigma(p) \subseteq B(0, \pi R)$, so here the explicit means of $\frac{\chi_{\Sigma(p)}}{\chi_{B(0, \pi R)}}$ is as follows

$$\frac{\chi_{\Sigma(p)}}{\chi_{B(0, \pi R)}} = \begin{cases} 1, & \delta \in \Sigma(p) \\ 0, & \text{otherwise} \end{cases}$$

Proof. Note that $\text{Vol}_{g_0}(B(p, r)) = \frac{\text{Vol}_{g_1}(\mathbb{S}^{n-1})r^n}{n}$. Then our assumption is equivalent to say

$$\lim_{r \rightarrow \infty} \frac{\text{Vol}_g(B(p, r))}{\text{Vol}_{g_0}(B(p, r))} = 1$$

However, by Theorem 22.4.1 we know volume ratio $\frac{\text{Vol}_g(B(p, r))}{\text{Vol}_{g_0}(B(p, r))}$ is non-increasing, with

$$\lim_{r \rightarrow 0} \frac{\text{Vol}_g(B(p, r))}{\text{Vol}_{g_0}(B(p, r))} = 1$$

which implies $\frac{\text{Vol}_g(B(p, r))}{\text{Vol}_{g_0}(B(p, r))} = 1$ holds for arbitrary $r > 0$. By rigidity of volume comparison, we conclude g has constant sectional curvature 0 on $B(p, r)$ for arbitrary $r > 0$. Since $\overline{B(p, \infty)} = M$, we deduce (M, g) has constant sectional curvature 0.

Thanks to Hopf's theorem, now it suffices to show M is simply-connected, suppose $\pi: \mathbb{R}^n \rightarrow M$ is the universal covering, one deduces that

$$|\pi_1(M)| = \frac{\text{Vol}_{g_0}(\mathbb{R}^n)}{\text{Vol}_g(M)} = 1$$

which implies M is simply-connected. \square

Corollary 22.4.4. Let (M, g) be a complete Riemannian n -manifold with $\text{Ric}(g) \geq (n-1)kg$ for some constant $k > 0$. Then

$$\text{Vol}_g(M) \leq \text{Vol}_{g_k}(\mathbb{S}^n(\frac{1}{\sqrt{k}}))$$

If the equality holds. Then (M, g) is isometric to $\mathbb{S}^n(1/\sqrt{k})$ with standard metric.

Proof. Let $k = 1/R^2$. Then Myers' theorem implies $\text{diam}(M, g) \leq \pi R$, thus compact. Hence, for any $p \in M$ one has $\Sigma(p) \subseteq B(0, \pi R)$. Therefore

$$\text{Vol}_g(B(p, \pi R)) = \text{Vol}_g(M)$$

where $B(p, \pi R)$ is a metric ball in M . On the other hand, it is obvious that

$$\text{Vol}_{g_k}(B(p, \pi R)) = \text{Vol}_{g_k}(\mathbb{S}^n(R))$$

Hence by Theorem 22.4.1, one has

$$\text{Vol}_g(M) \leq \text{Vol}_{g_k}(\mathbb{S}^n(R))$$

Furthermore, if the equality holds, g has constant sectional curvature on $B(p, \pi R)$. Then use the argument in Corollary 22.4.3 completes the proof. \square

Corollary 22.4.5. Let (M, g) be a complete Riemannian manifold and $p \in M$. Let $B(p, \delta)$ be the metric ball centered at p with radius δ and

g_k be the metric with constant sectional curvature k on $B(p, \delta) \setminus \{p\}$. If $\text{Ric}(g) \geq (n-1)kg$. Then for any $0 \leq \delta_1 < \delta_2 \leq \delta_3 < \delta_4$

$$\frac{\text{Vol}_g(B(p, \delta_4)) - \text{Vol}_g(B(p, \delta_3))}{\text{Vol}_g(B(p, \delta_2)) - \text{Vol}_g(B(p, \delta_1))} \leq \frac{\text{Vol}_{g_k}(B(p, \delta_4)) - \text{Vol}_{g_k}(B(p, \delta_3))}{\text{Vol}_{g_k}(B(p, \delta_2)) - \text{Vol}_{g_k}(B(p, \delta_1))}$$

Proof. Just note that volume density ratio is non-decreasing. Then by Remark 22.4.1, one has

$$\frac{\text{Vol}_g(B(p, \delta_4)) - \text{Vol}_g(B(p, \delta_3))}{\text{Vol}_{g_k}(B(p, \delta_4)) - \text{Vol}_{g_k}(B(p, \delta_3))} \leq \frac{\text{Vol}_g(B(p, \delta_2)) - \text{Vol}_g(B(p, \delta_1))}{\text{Vol}_{g_k}(B(p, \delta_2)) - \text{Vol}_{g_k}(B(p, \delta_1))}$$

This gives desired result. \square

Theorem 22.4.2 (Cheng). Let (M, g) be a complete Riemannian n -manifold with $\text{Ric}(g) \geq (n-1)kg$ for some constant $k > 0$. If $\text{diam}(M) = \pi/\sqrt{k}$. Then (M, g) is isometric to $\mathbb{S}^n(1/\sqrt{k})$ with standard metric.

Proof. Let $k = 1/R^2$. Since M is complete, there exist points $p, q \in M$ and $\text{dist}(p, q) = \pi R$, thus for any $\delta \in (0, \pi R)$

$$B(p, \delta) \cap B(q, \pi R - \delta) = \emptyset$$

Then

$$\begin{aligned} \text{Vol}_g(M) &\stackrel{(1)}{\geq} \text{Vol}_g(B(p, \delta)) + \text{Vol}_g(B(q, \pi R - \delta)) \\ &\stackrel{(2)}{\geq} \text{Vol}_{g_k}(B(p, \delta)) \frac{\text{Vol}_g(B(p, \pi R))}{\text{Vol}_{g_k}(B(p, \pi R))} + \text{Vol}_{g_k}(B(q, \pi R - \delta)) \frac{\text{Vol}_g(B(q, \pi R))}{\text{Vol}_{g_k}(B(q, \pi R))} \\ &\stackrel{(3)}{=} \text{Vol}_g(M) \end{aligned}$$

where

(1) holds from $B(p, \delta) \cap B(q, \pi R - \delta) = \emptyset$.

(2) holds from Theorem 22.4.1.

(3) holds since for any $x, y \in M$, $\text{Vol}_g(B(x, \pi R)) = \text{Vol}_g(M)$ and

$$\text{Vol}_{g_k}(B(x, \pi R)) = \text{Vol}_{g_k}(\mathbb{S}^n(R))$$

$$\text{Vol}_{g_k}(B(x, \delta)) + \text{Vol}_{g_k}(B(y, \pi R - \delta)) = \text{Vol}_{g_k}(\mathbb{S}^n(R))$$

Hence, for any $0 < \delta < \pi R$.

$$\frac{\text{Vol}_g(B(p, \delta))}{\text{Vol}_{g_k}(B(p, \delta))} = \frac{\text{Vol}_g(B(p, \pi R))}{\text{Vol}_{g_k}(B(p, \pi R))} = \frac{\text{Vol}_g(M)}{\text{Vol}_{g_k}(\mathbb{S}^n(R))}$$

Let $\delta \rightarrow 0$, and we deduce $\text{Vol}_g(M) = \text{Vol}_{g_k}(\mathbb{S}^n(R))$. By Proposition 22.4.4, (M, g) is isometric to $\mathbb{S}^n(R)$ with standard metric. \square

Theorem 22.4.3 (Bishop-Yau). Let (M, g) be a complete non-compact Riemannian n -manifold with $\text{Ric}(g) \geq 0$. Then the volume growth of (M, g) satisfies

$$c_n \text{Vol}_g(B(p, 1)) \cdot r \leq \text{Vol}_g(B(p, r))$$

for $r \geq 1$, where c_n is a positive constant depending only on n .

Proof. Let $x \in \partial B(p, 1 + r)$. Then

$$B(p, 1) \subseteq B(x, 2 + r) \setminus B(x, r), \quad B(x, r) \subseteq B(p, 1 + 2r)$$

By Corollary 22.4.5, one has

$$\begin{aligned} \text{Vol}_g(B(p, 1)) &\leq \text{Vol}_g(B(x, 2 + r)) - \text{Vol}_g(B(x, r)) \\ &\leq \text{Vol}_g(B(x, r)) \cdot \frac{\text{Vol}(B(x, 2 + r)) - \text{Vol}(B(x, r))}{\text{Vol}(B(x, r))} \\ &\leq \text{Vol}_g(B(p, 1 + 2r)) \cdot \frac{(2 + r)^n - r^n}{r^n} \\ &\leq \text{Vol}_g(B(p, 1 + 2r)) \cdot \frac{1}{r} c_n \end{aligned}$$

where $r \geq 1$. By changing variable, we obtain the lower bound. \square

Proposition 22.4.1. Let (M, g) be a Cartan-Hadamard manifold with $\text{Ric}(g) \leq -kg$ for some $k > 0$. Then for any $p \in M$

$$\text{Vol}_g(B(p, r)) \geq c_n e^{\sqrt{kr}}$$

where c_n is a positive constant depending only on n .

Proposition 22.4.2 (Cheeger-Colding). For each integer $n \geq 2$, there exists a real number $\delta(n) \in (0, 1)$ with the following property: if (M, g) is a compact Riemannian manifold of dimension n with $\text{Ric}(g) \geq (n - 1)g$ and

$$\text{Vol}(M, g) \geq (1 - \delta(n)) \text{Vol}(\mathbb{S}^n)$$

then M is diffeomorphic to \mathbb{S}^n .

23. SPLITTING THEOREM

23.1. Geodesic rays.

Definition 23.1.1 (geodesic ray). A geodesic ray is a unit-speed geodesic $\gamma: [0, \infty) \rightarrow M$ such that for any $s, t \geq 0$,

$$\text{dist}(\gamma(s), \gamma(t)) = |s - t|$$

Lemma 23.1.1. Let (M, g) be a complete Riemannian manifold. Then the following statements are equivalent.

- (1) M is non-compact.
- (2) For any $p \in M$, there exists a geodesic ray $\gamma: [0, \infty) \rightarrow M$ starting from p .

Proof. From (1) to (2). If M is non-compact, for any $p \in M$, there is a sequence of points $\{p_i\}$ such that $\text{dist}(p, p_i) = i$. Let $\gamma_i(t) = \exp_p(tv_i)$ be a unit-speed minimal geodesic connecting p and p_i , that is $\gamma_i(0) = p$ and $\gamma_i(i) = p_i$. By possibly passing to a subsequence, we may assume $v_i \rightarrow v \in T_p M$. Then

$$\gamma(t) = \exp_p(tv), \quad t \in [0, +\infty)$$

is a unit-speed geodesic ray. Indeed, for any $s, t \geq 0$, and for any $k > \max\{s, t\}$, one has

$$\text{dist}(\gamma_k(s), \gamma_k(t)) = |s - t|$$

By continuity of exponential map \exp_p , one obtains

$$\text{dist}(\gamma(s), \gamma(t)) = \lim_{k \rightarrow +\infty} \text{dist}(\gamma_k(s), \gamma_k(t)) = |s - t|$$

Hence γ is a geodesic ray.

From (2) to (1). It's trivial. □

23.2. Buseman function.

Definition 23.2.1. Let (M, g) be a complete Riemannian manifold, $p \in M$ and $\gamma: [0, \infty) \rightarrow M$ be a geodesic ray starting from p . For any $t \geq 0$, $b_\gamma^t: M \rightarrow \mathbb{R}$ as

$$b_\gamma^t(x) := \text{dist}(x, \gamma(t)) - t$$

Proposition 23.2.1. Let (M, g) be a complete non-compact Riemannian manifold, $p \in M$ and γ be a geodesic ray starting from p . The function $b_\gamma^t(x): M \rightarrow \mathbb{R}$ has the following properties:

- (1) For any fixed $x \in M$, $b_\gamma^t(x)$ is non-increasing in t .
- (2) For any $x \in M$ and $t \geq 0$, $|b_\gamma^t(x)| \leq \text{dist}(x, \gamma(0))$.
- (3) For any $x, y \in M$ and $t \geq 0$, $|b_\gamma^t(x) - b_\gamma^t(y)| \leq \text{dist}(x, y)$.

Proof. For (1). Note that for $t > s > 0$, one has

$$\begin{aligned} b_\gamma^t(x) - b_\gamma^s(t) &= \text{dist}(x, \gamma(t)) - \text{dist}(x, \gamma(s)) + s - t \\ &\leq \text{dist}(\gamma(t), \gamma(s)) + s - t \\ &= |t - s| + s - t \\ &= 0 \end{aligned}$$

For (2), (3). Directly from triangle inequality. \square

Definition 23.2.2 (Buseman function). The Buseman function with respect to the geodesic ray is defined as

$$b_\gamma := \lim_{t \rightarrow \infty} b_\gamma^t(x)$$

Example 23.2.1 (Buseman function on hyperbolic plane). Note that geodesics on $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ are

- (1) Semicircles centered on \mathbb{R} .
- (2) Straight lines perpendicular to \mathbb{R} .

Given $x \in \mathbb{H}$, in order to compute Buseman function

$$b_\gamma(x) = \lim_{t \rightarrow \infty} \text{dist}(x, \gamma(t)) - \text{dist}(\gamma(0), \gamma(t))$$

It suffices to solve the following calculus: Fix $z_1, z_2 \in \mathbb{H}$ and $\alpha \in \partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$, solve

$$(23.1) \quad \lim_{q \rightarrow \alpha} \text{dist}(q, z_1) - \text{dist}(q, z_2) = ?$$

then we can set $q = \gamma(t), \alpha = \gamma(\infty), z_1 = x, z_2 = \gamma(0)$ to conclude. Let's divide into several steps:

Step one: For arbitrary $r > s > 0$, the distance between ri, si in \mathbb{H} is $\ln \frac{r}{s}$, where i is imaginary number. Indeed, since metric on this line is exactly $\frac{dy \otimes dy}{y^2}$.

Step two: In hyperbolic planes, it's possible to use isometry to translate any two points to the positive imaginary axis. To be explicit, consider the Möbius transformation V mapping Poincaré disk \mathbb{D} to \mathbb{H} with inverse V^{-1} , given by

$$\begin{aligned} z &= V(w) = \frac{-iw + 1}{w - i} \\ w &= V^{-1}(z) = \frac{iz + 1}{z + i} \end{aligned}$$

Now for arbitrary $z_1, z_2 \in \mathbb{H}$, firstly use V^{-1} to send z_1, z_2 to $w_1, w_2 \in \mathbb{D}$ respectively. Then let $S(w) = e^{i\theta} \frac{w - w_1}{1 - \overline{w_1}w}$ be transformation in \mathbb{D} that send w_1 to 0, with θ chosen carefully so that w_2 get sent to the positive imaginary axis, that is, w_2 get sent to the point ki , where $k = |S(w_2)|$. Finally apply V to this situation, 0 gets sent to i and ki get sent to $\frac{1+k}{1-k}i$.

Step three: Combine step one and two, one can conclude that for arbitrary $z_1, z_2 \in \mathbb{H}$, the distance between them are

$$\text{dist}(z_1, z_2) = \ln \frac{1+k}{1-k}$$

If we express k in terms of z_1, z_2 , one has

$$\text{dist}(z_1, z_2) = \ln \frac{|z_1 + z_2| + |z_1 - z_2|}{|z_1 + z_2| - |z_1 - z_2|}$$

Step four: Consider a special case of (23.1), that is we assume $z_1 = ri, z_2 = i$, where $\ln r = \text{dist}(z_1, z_2)$. Now we choose a sequence $q_n = u_n + iv_n$ such that $u_n \rightarrow \alpha$ and $v_n \rightarrow v$, where $v = 0$ as $n \rightarrow \infty$. Then

$$\begin{aligned} \lim_{q \rightarrow \alpha} (\text{dist}(q, ri) - \text{dist}(q, i)) &= \lim_{n \rightarrow \infty} (\ln \left(\frac{|q_n + ri| + |q_n - ri|}{|q_n + ri| - |q_n - ri|} \right) - \ln \left(\frac{|q_n + i| + |q_n - i|}{|q_n + i| - |q_n - i|} \right)) \\ &= \lim_{n \rightarrow \infty} (\ln \left(\frac{|q_n + ri| + |q_n - ri|}{|q_n + i| + |q_n - i|} \right) + \ln \left(\frac{|q_n + i| - |q_n - i|}{|q_n + ri| - |q_n - ri|} \right)) \\ &= \lim_{n \rightarrow \infty} \ln \frac{\sqrt{u_n^2 + (v_n + r)^2} + \sqrt{u_n^2 + (v_n - r)^2}}{\sqrt{u_n^2 + (v_n + 1)^2} + \sqrt{u_n^2 + (v_n - 1)^2}} \\ &\quad + \lim_{n \rightarrow \infty} \ln \frac{\sqrt{u_n^2 + (v_n + 1)^2} - \sqrt{u_n^2 + (v_n - 1)^2}}{\sqrt{u_n^2 + (v_n + r)^2} - \sqrt{u_n^2 + (v_n - r)^2}} \\ &= \lim_{v_n \rightarrow 0} \ln \underbrace{\frac{\sqrt{\alpha^2 + (v_n + r)^2} + \sqrt{\alpha^2 + (v_n - r)^2}}{\sqrt{\alpha^2 + (v_n + 1)^2} + \sqrt{\alpha^2 + (v_n - 1)^2}}}_{\text{part I}} \\ &\quad + \lim_{v_n \rightarrow 0} \ln \underbrace{\frac{\sqrt{\alpha^2 + (v_n + 1)^2} - \sqrt{\alpha^2 + (v_n - 1)^2}}{\sqrt{\alpha^2 + (v_n + r)^2} - \sqrt{\alpha^2 + (v_n - r)^2}}}_{\text{part II}} \end{aligned}$$

It's clear Part I is $\frac{\sqrt{\alpha^2 + r^2}}{\sqrt{\alpha^2 + 1}}$, and apply L'Hospital's rule to Part II one has

$$\lim_{v_n \rightarrow 0} \frac{\sqrt{\alpha^2 + (v_n + 1)^2} - \sqrt{\alpha^2 + (v_n - 1)^2}}{\sqrt{\alpha^2 + (v_n + r)^2} - \sqrt{\alpha^2 + (v_n - r)^2}} = \lim_{v_n \rightarrow 0} \frac{\frac{v_n + 1}{\sqrt{\alpha^2 + (v_n + 1)^2}} - \frac{v_n - 1}{\sqrt{\alpha^2 + (v_n - 1)^2}}}{\frac{v_n + r}{\sqrt{\alpha^2 + (v_n + r)^2}} - \frac{v_n - r}{\sqrt{\alpha^2 + (v_n - r)^2}}} = \frac{\sqrt{\alpha^2 + r^2}}{r\sqrt{\alpha^2 + 1}}$$

which implies

$$\lim_{q \rightarrow \alpha} \text{dist}(q, ri) - \text{dist}(q, i) = \ln \frac{\alpha^2 + r^2}{\alpha^2 + 1} - \ln r$$

Step five: In order to solve general case of (23.1), we can use processes in step two to translate z_1, z_2 to the positive imaginary axis. However, α is also translated into a new point α' , that is

$$\alpha' = V \circ S \circ V^{-1}(\alpha)$$

where V, V^{-1} and S are defined in step two. Thus from step four one has

$$\lim_{q \rightarrow \alpha} \text{dist}(q, z_1) - \text{dist}(q, z_2) = \ln \frac{(\alpha')^2 + r^2}{(\alpha')^2 + 1} - \ln r$$

where $\ln r = \text{dist}(z_1, z_2)$.

Proposition 23.2.2. Let (M, g) be a complete non-compact Riemannian manifold, $p \in M$ and γ be a geodesic ray starting from p . The Busemann function $b_\gamma: M \rightarrow \mathbb{R}$ is Lipschitz continuous with $\text{Lip}(b_\gamma) \leq 1$

Proof. It follows from Arzela-Ascoli lemma. \square

Proposition 23.2.3. Let (M, g) be a complete non-compact Riemannian manifold, and γ be a geodesic ray starting from $p \in M$. If $\text{Ric}(g) \geq 0$. Then

$$\Delta b_\gamma \leq 0$$

in the sense of distribution.

Proof. For any non-negative smooth function $\varphi \in C_0^\infty(M)$, one has

$$\begin{aligned} \int_M \Delta \varphi b_\gamma^t \text{vol} &= \int_M \Delta \varphi (\text{dist}(x, \gamma(t)) - t) \text{vol} \\ &\stackrel{(1)}{=} \int_M \Delta \varphi \text{dist}(x, \gamma(t)) \text{vol} \\ &\stackrel{(2)}{\leq} \int_M \frac{(n-1)\varphi}{\text{dist}(x, \gamma(t))} \text{vol} \end{aligned}$$

where

(1) holds from Stokes' theorem.

(2) holds from Theorem 22.3.2.

Then Lebesgue's dominated convergence implies

$$\int_M \Delta \varphi b_\gamma \text{vol} \leq 0$$

\square

Definition 23.2.3 (geodesic line). A geodesic line is a unit-speed geodesic $\gamma: (-\infty, \infty) \rightarrow M$ such that for any $s, t \in \mathbb{R}$,

$$\text{dist}(\gamma(s), \gamma(t)) = |s - t|$$

Lemma 23.2.1. Let (M, g) be a connected, non-compact Riemannian manifold. If M contains a compact subset K such that $M \setminus K$ has at least two unbounded components⁸. Then there is a geodesic line passing through K .

Proof. Since $M \setminus K$ has at least two unbounded components, there are two unbounded sequences of points $\{p_i\}$ and $\{q_i\}$ such that any curve from p_i to q_i passes through K . Let $\gamma_i: [-a_i, b_i] \rightarrow M$ be minimal geodesics connecting p_i and q_i with $\gamma_i(-a_i) = p_i$, $\gamma_i(b_i) = q_i$ and $\gamma_i(0) \in K$. Hence, $a_i \rightarrow +\infty$

⁸Some authors use “ends” to call such unbounded components.

and $b_i \rightarrow +\infty$. By possibly passing to subsequences, $\{\gamma_i\}$ converges to a geodesic line $\gamma_\infty : (-\infty, +\infty) \rightarrow M$. \square

Proposition 23.2.4. Let (M, g) be a complete non-compact Riemannian manifold with $\text{Ric}(g) \geq 0$. If (M, g) contains a geodesic line γ . Then $b_{\gamma_\pm} : M \rightarrow \mathbb{R}$ are smooth harmonic functions with

$$|\nabla b_{\gamma_\pm}| = 1, \quad \text{Hess } b_{\gamma_\pm} = 0$$

where $\gamma_\pm(t) = \gamma(\pm t) : [0, +\infty) \rightarrow M$.

Proof. Let $b(x) = b_{\gamma_+}(x) + b_{\gamma_-}(x)$. By the triangle inequality

$$\begin{aligned} b(x) &= \lim_{s \rightarrow +\infty} \text{dist}(x, \gamma_+(s)) + \text{dist}(x, \gamma_-(s)) - 2s \\ &= \lim_{s \rightarrow +\infty} \text{dist}(x, \gamma(s)) + \text{dist}(x, \gamma(-s)) - 2s \\ &\geq 0 \end{aligned}$$

By Proposition 23.2.3, $\Delta b \leq 0$ in the sense of distributions. On the other hand,

$$b(\gamma(t)) = \lim_{s \rightarrow +\infty} \text{dist}(\gamma(t), \gamma(s)) + \text{dist}(\gamma(t), \gamma(-s)) - 2s = 0$$

Hence the subharmonic function b attains its absolute minimum, by Theorem 22.2.1, $b \equiv 0$, that is $b_{\gamma_+} = -b_{\gamma_-}$. Hence $\Delta b_{\gamma_+} = \Delta b_{\gamma_-} = 0$, and by Wely's lemma one has b_{γ_\pm} are smooth.

Bochner's formula says

$$\frac{1}{2} \Delta |\nabla f|^2 = |\text{Hess } f|^2 + \text{Ric}(\nabla f, \nabla f) + g(\nabla \Delta f, \nabla f)$$

Let $f = b_{\gamma_+}$. Then

$$\frac{1}{2} \Delta |\nabla b_{\gamma_+}|^2 \geq |\text{Hess } b_{\gamma_+}|^2 \geq 0$$

since b_{γ_+} is harmonic and $\text{Ric}(g) \geq 0$, thus $|\nabla b_{\gamma_+}|^2$ is superharmonic. On the other hand, by Proposition 23.2.2, $\text{Lip}(b_{\gamma_+}) \leq 1$, and so $|\nabla b_{\gamma_+}| \leq 1$. Note that

$$b_{\gamma_+}(\gamma_+(t)) = \lim_{s \rightarrow +\infty} \text{dist}(\gamma_+(t), \gamma_+(s)) - s = \lim_{s \rightarrow +\infty} |t - s| - s = -t$$

For any $x = \gamma_+(t_0)$

$$|\nabla b_{\gamma_+}|(x) \stackrel{(1)}{=} |\nabla b_{\gamma_+}|(\gamma_+(t_0)) \stackrel{(2)}{\geq} |\langle \nabla b_{\gamma_+}(x), \gamma'_+(t_0) \rangle| = 1$$

where

(1) holds from the trivial fact γ_+ is unit-speed.

(2) holds from Cauchy-Schwarz inequality.

Hence, the superharmonic function $|\nabla b_{\gamma_+}|^2$ attains its absolute maximum in M , hence $|\nabla b_{\gamma_+}|^2 \equiv 1$ on M . Again by the Bochner formula, one has $\text{Hess } b_{\gamma_+} = 0$. The same argument holds for b_{γ_-} , this completes the proof. \square

Lemma 23.2.2. Let (M, g) be a complete Riemannian manifold, and V a smooth vector field with $|V|_g \leq C$ for some constant C . Then V is a complete vector field.

Proof. We need to show the integral curve of V is globally defined, that is defined on \mathbb{R} . Suppose $\gamma: (a, b) \rightarrow M$ is an integral curve of M and $b < \infty$. For arbitrary $t, s \in (a, b)$, we have

$$\gamma(t) = \gamma(s) + \int_s^t V(\gamma(\tau)) d\tau$$

By using the boundedness of V , we can conclude that

$$|\gamma(t) - \gamma(s)| \leq C|t - s|$$

which implies $\gamma(t)$ is uniformly continuous on (a, b) , thus it's possible to extend γ to $(a, b]$ since $b < \infty$, a contradiction. \square

Proposition 23.2.5. Let (M, g) be a complete Riemannian manifold. Suppose $f \in C^\infty(M, \mathbb{R})$ satisfies

$$|\nabla f| = 1 \quad \text{and} \quad \text{Hess } f = 0$$

Let Σ denote $f^{-1}(0)$, with induced metric $h := g|_\Sigma$.

- (1) (Σ, h) is a totally geodesic submanifold of (M, g) .
- (2) The map

$$f: (\mathbb{R} \times \Sigma, g_{\mathbb{R}} \oplus h) \rightarrow (M, g), \quad F(t, p) = \exp_p(t \nabla_p f)$$

is an isometry.

Proof. For (1). Recall that (Σ, h) is a totally geodesic submanifold of (M, g) if the second fundamental form of Σ vanishes, and facts in basic differential geometry says the second fundamental form of a hyperplane Σ with induced metric is given by

$$\mathbf{II}(v, w) := \langle \nabla_v n, w \rangle$$

where n is the normal vector of Σ . In this case, if we consider $\Sigma = f^{-1}(0)$. Then the normal vector of Σ is exactly ∇f , and thus

$$\mathbf{II}(v, w) := \langle \nabla_v \nabla f, w \rangle$$

Then $\text{Hess } f = \nabla^2 f = 0$ implies the second fundamental form of Σ vanishes, that is Σ is a totally geodesic submanifold of (M, g) .

For (2). For a fixed p , let $X = \nabla f$, and consider $\gamma(t) = \exp_p(tX_p)$. Since $\nabla X = 0$, we have $E(t) = X(\gamma(t))$ and $\gamma'(t)$ are two parallel vector fields along γ with the same initial value. Hence

$$\gamma'(t) = X(\gamma(t))$$

that is γ is exactly the integral curve of X . Furthermore, since $|X| = 1$, by Lemma 23.2.2 one has γ is globally defined, and one can deduce F is a global flow of X , thus it's a diffeomorphism.

Now it remains to prove that F is an isometry. For $v \in T_p \Sigma$, let J be the Jacobi field along γ with $J(0) = 0$ and $J'(0) = v$. By the radial curvature equation

$$R(-, \nabla f, \nabla f, -) = \text{Hess}\left(\frac{1}{2}|\nabla f|^2\right)(-, -) - (\nabla_{\nabla f} \text{Hess } f)(-, -) - \text{Hess } f(\nabla - \nabla f, -)$$

one has $R(-, \nabla f, \nabla f, -) = 0$, thus Jacobi equation

$$J''(t) + R(J, \gamma')\gamma' = 0$$

reduces to $J''(t) = 0$. It implies that $J'(t)$ is a parallel vector field and in particular, $|J'(t)| \equiv |J'(0)| = |v|$. By uniqueness of Jacobi fields, we deduce

$$J(t) = tJ'(t)$$

Then F is an isometry holds as follows:

- (a) It is easy to see that $(dF)_{(1,p)}v = J(1)$, thus $|(dF)_{(1,p)}v| = |J(1)| = |J'(1)| = |v|$.
- (b) $|(dF)_{(0,p)}\partial_t| = |\nabla f| = 1 = |\partial_t|$.

□

23.3. Splitting theorem and its corollaries.

Theorem 23.3.1 (splitting theorem). Let (M, g) be a complete Riemannian n -manifold with $\text{Ric}(g) \geq 0$. If there is a geodesic line in M . Then (M, g) is isometric to $(\mathbb{R} \times N, g_{\mathbb{R}} \oplus g_N)$, where $\text{Ric}(g_N) \geq 0$.

Proof. Directly from Proposition 23.2.4 and Proposition 23.2.5. □

Corollary 23.3.1. Let (M, g) be a complete Riemannian n -manifold with $\text{Ric}(g) \geq 0$

- (1) (M, g) is isometric to $(\mathbb{R}^k \times N, g_{\mathbb{R}^k} \oplus g_N)$, where N does not contain a geodesic line and $\text{Ric}(g_N) \geq 0$.
- (2) The isometry group splits

$$\text{Iso}(M, g) \cong \text{Iso}(\mathbb{R}^k) \times \text{Iso}(N, g_N)$$

Theorem 23.3.2 (structure theorem for manifold with $\text{Ric} \geq 0$). Let (M, g) be a compact Riemannian manifold with $\text{Ric}(g) \geq 0$, and $\pi: (\widetilde{M}, \widetilde{g}) \rightarrow (M, g)$ is its universal covering with the pullback metric.

- (1) There exists some integer $k \geq 0$ and a compact Riemannian manifold (N, g_N) with $\text{Ric}(g_N) \geq 0$ such that $(\widetilde{M}, \widetilde{g})$ is isometric to $(\mathbb{R}^k \times N, g_{\text{can}} \oplus g_N)$.
- (2) The isometry group splits

$$\text{Iso}(\widetilde{M}, \widetilde{g}) \cong \text{Iso}(\mathbb{R}^k) \times \text{Iso}(N, g_N)$$

Proof. For (1). Suppose to the contrary that N is non-compact. Then fix a point $x_0 \in N$, there exists a geodesic ray $\gamma: [0, \infty) \rightarrow N$ starting from x_0 . Since M is compact, there exists a compact subset $\widetilde{K} \subseteq \widetilde{M}$ such that

$$\text{Aut}_{\pi}(\widetilde{M})\widetilde{K} = \widetilde{M}$$

□

Corollary 23.3.2. $\mathbb{S}^n \times \mathbb{S}^1$ doesn't admit any Ricci flat metrics when $n = 2, 3$.

Proof. If $\mathbb{S}^n \times \mathbb{S}^1$ admits a Ricci flat metric, after splitting its universal covering we obtain a Ricci flat metric on \mathbb{S}^n . However, \mathbb{S}^n doesn't admit such a metric when $n = 2, 3$. Indeed, since any Einstein manifold with dimension 2 or 3 has constant sectional curvature, thus if $\mathbb{S}^n, n = 2, 3$ admit a Ricci flat metric. Then it has constant sectional curvature 0, and it's also simply-connected, so Hopf's theorem implies it's diffeomorphic to \mathbb{R}^n , a contradiction. □

Remark 23.3.1. It's clear $\mathbb{S}^1 \times \mathbb{S}^1$ admits a Ricci flat metric, and when $n \geq 4$, we don't know whether \mathbb{S}^n admit a Ricci flat metric or not.

Corollary 23.3.3. Let (M, g) be a compact Riemannian manifold with $\text{Ric}(g) \geq 0$, and $(\widetilde{M}, \widetilde{g})$ is its universal covering equipped with pullback metric.

- (1) If \widetilde{M} is contractible, then $(\widetilde{M}, \widetilde{g})$ is isometric to $(\mathbb{R}^n, g_{\text{can}})$ and (M, g) is flat.
- (2) If $(\widetilde{M}, \widetilde{g})$ doesn't contain a geodesic line, then $\pi_1(M)$ is finite and $b_1(M) = 0$.

Proof. For (1). If $\widetilde{M} \cong N \times \mathbb{R}^k$ is contractible, we must have N is just a point, since it's compact,

For (2). If \widetilde{M} doesn't contain a geodesic line. Then \widetilde{M} is compact, which implies $|\pi_1(M)|$ is finite. Furthermore, since there is a natural Hurwicz surjective

$$h : \pi_1(M) \rightarrow H_1(M, \mathbb{Z})$$

thus $H_1(M, \mathbb{Z})$ can't have free part, otherwise h can't be surjective, since there is no surjective map from a finite group to an infinite one. So we have $b_1(M) = 0$. □

Corollary 23.3.4. Let (M, g) be a compact Riemannian manifold with $\text{Ric}(g) \geq 0$. If there exists a point $p \in M$ such that $\text{Ric}(g) > 0$ on $T_p M$. Then $\pi_1(M)$ is finite and $b_1(M) = 0$.

Proof. Since $\text{Ric}(g) > 0$ on the whole tangent space $T_p M$, the universal covering $(\widetilde{M}, \widetilde{g})$ can't split into a product $(\mathbb{R}^k \times N, g_{\text{can}} \oplus g_N)$, since metric on \widetilde{M} is pullback metric, and g_{can} on \mathbb{R}^k has vanishing Ricci curvature. Thus \widetilde{M} is compact, consequently, we have $|\pi_1(M)|$ is finite and $b_1(M) = 0$. □

Remark 23.3.2. We have already seen this in Bochner's technique.

Part 8. Symmetric space

24. A GEOMETRIC VIEWPOINT

24.1. Riemannian symmetric space.

Definition 24.1.1 (Riemannian symmetric space). A Riemannian manifold (M, g) is called a Riemannian symmetric space if for each $p \in M$ there exists an isometry $\varphi: M \rightarrow M$, which is called a symmetry at p , such that $\varphi(p) = p$ and $(d\varphi)_p = -\text{id}$.

Remark 24.1.1. Note that Theorem 16.1.1, that is rigidity property of isometry, implies if symmetry at point p exists, it's unique.

Definition 24.1.2 (locally Riemannian symmetric space). A Riemannian manifold (M, g) is called a locally Riemannian symmetric space if each $p \in M$ has a neighborhood U such that there exists an isometry $\varphi: U \rightarrow U$ such that $\varphi(p) = p$ and $(d\varphi)_p = -\text{id}$.

Proposition 24.1.1. The following statements are equivalent.

- (1) (M, g) is a Riemannian symmetric space.
- (2) For each $p \in M$, there exists an isometry $\varphi: M \rightarrow M$ such that $\varphi^2 = \text{id}$ and p is an isolated fixed point of φ .

Proof. From (1) to (2). Let φ be a symmetry at $p \in M$. Since $(d\varphi^2)_p = (d\varphi)_p \circ (d\varphi)_p = \text{id}$ and $\varphi^2(p) = p$, one has $\varphi^2 = \text{id}$ by Theorem 16.1.1. If p is not an isolated fixed point, then there exists a sequence $\{p_i\}_{i=1}^\infty$ converging to p such that $\varphi(p_i) = p_i$. For $0 < \delta < \text{inj}(p)$ we choose a fixed point $q \in B(0, \delta)$ of φ with $v = \exp_p^{-1}(q)$. Since φ is an isometry, one has $\varphi(\exp_p(tv))$ and $\exp_p(tv)$ are two geodesics connecting p and q , and thus

$$\varphi(\exp_p(tv)) = \exp_p(tv)$$

by uniqueness. In particular, one has $v = (d\varphi)_p v$, which is a contradiction.

From (2) to (1). From $\varphi^2 = \text{id}$ we have $(d\varphi)_p^2 = \text{id}$, so only possible eigenvalues of $(d\varphi)_p$ are ± 1 . Now it suffices to show all eigenvalues of $(d\varphi)_p$ are -1 . Otherwise if it has an eigenvalue 1 , there exists some non-zero $v \in T_p M$ such that $(d\varphi)_p v = v$. Since φ is an isometry, one has $\varphi(\exp_p(tv))$ and $\exp_p(tv)$ are geodesics with the same direction at p . Thus

$$\varphi(\exp_p(tv)) = \exp_p(tv)$$

for $0 < t < \text{inj}(p)$. In particular, p is not an isolated fixed point, which is a contradiction. \square

Lemma 24.1.1. Let (M, g) be a Riemannian symmetric space and φ is an isometry at point p , then $\varphi^*(\nabla R) = \nabla R$.

Proof. Let X_1, \dots, X_5 be vector fields. Direct computation shows

$$\begin{aligned}
 (\varphi^*(\nabla R))(X_1, \dots, X_5) &= \nabla R(\varphi_* X_1, \dots, \varphi_* X_5) \\
 &= \nabla_{\varphi_* X_1} R(\varphi_* X_2, \dots, \varphi_* X_5) \\
 &= (\varphi^*(\nabla_{\varphi_* X_1} R))(X_2, \dots, X_5) \\
 &= (\nabla_{(\varphi_*)^2 X_1} \varphi^* R)(X_2, \dots, X_5) \\
 &\stackrel{(1)}{=} \nabla_{X_1} R(X_2, \dots, X_5) \\
 &= \nabla R(X_1, \dots, X_5)
 \end{aligned}$$

where (1) holds from $\varphi^2 = \text{id}$ and $\varphi^* R = R$. \square

Theorem 24.1.1. Let (M, g) be a complete Riemannian manifold, The following statements are equivalent.

- (1) (M, g) is a locally Riemannian symmetric space.
- (2) $\nabla R = 0$.

Proof. From (1) to (2). For arbitrary $p \in M$, there exists a neighborhood U of p and an isometry $\varphi: U \rightarrow U$ such that $\varphi(p) = p$ and $(d\varphi)_p = -\text{id}$. Since φ is an isometry, it preserves curvature, and thus for any $v_i \in T_p M, i = 1, 2, \dots, 5$, one has

$$\begin{aligned}
 (\nabla R)_p(v_1, \dots, v_5) &= (\varphi^*(\nabla R))_p(v_1, \dots, v_5) \\
 &= (\nabla R)_p((d\varphi)_p v_1, \dots, (d\varphi)_p v_5) \\
 &= -(\nabla R)_p(v_1, \dots, v_5)
 \end{aligned}$$

This shows $(\nabla R)_p = 0$, and since p is arbitrary, one has $\nabla R = 0$.

From (2) to (1). For arbitrary $p \in M$, it suffices to show

$$\varphi = \exp_p \circ \Phi_0 \circ \exp_p^{-1}: B(p, \delta) \rightarrow B(p, \delta)$$

is an isometry, where $0 < \delta < \text{inj}(p)$ and $\Phi_0 = -\text{id}: T_p M \rightarrow T_p M$. For $v \in T_p M$ with $|v| < \delta$ and $\gamma(t) = \exp_p(tv)$, $\tilde{\gamma}(t) = \exp_p(t\Phi_0(v))$, if we define

$$\Phi_t = P_{0,t;\tilde{\gamma}} \circ \Phi_0 \circ P_{t,0;\gamma}$$

then direct computation shows

$$\begin{aligned}
 \Phi_t^* R_{\tilde{\gamma}(t)} &= (P_{t,0;\gamma})^* \circ \Phi_0^* \circ (P_{0,t;\tilde{\gamma}})^* R_{\tilde{\gamma}(t)} \\
 &\stackrel{(a)}{=} (P_{t,0;\gamma})^* \circ \Phi_0^* R_{\tilde{\gamma}(0)} \\
 &\stackrel{(b)}{=} (P_{t,0;\gamma})^* R_{\gamma(0)} \\
 &\stackrel{(c)}{=} R_{\gamma(t)}
 \end{aligned}$$

where

(a) and (c) holds from Proposition 10.4.2.

(b) holds from $\tilde{\gamma}(0) = \gamma(0)$ and R is a $(0, 4)$ -tensor.

Then by Theorem 16.4.1, that is Cartan-Ambrose-Hicks's theorem, φ is an isometry, which completes the proof. \square

24.2. The relations between symmetric, locally symmetric and homogeneous space. In this section, we will show any complete locally Riemannian symmetric space is a quotient of Riemannian symmetric space (in Corollary 24.2.1), and any Riemannian symmetric space is a Riemannian homogeneous space (in Corollary 24.2.2).

24.2.1. Riemannian symmetric space and locally Riemannian symmetric space.

Theorem 24.2.1. Let (M, g) be a complete, simply-connected locally Riemannian symmetric space. Then (M, g) is a Riemannian symmetric space.

Proof. For $p \in M$ and $0 < \delta < \text{inj}(p)$, suppose $\varphi: B(p, \delta) \rightarrow B(p, \delta)$ is an isometry such that $\varphi(p) = p$ and $(d\varphi)_p = -\text{id}$. For arbitrary $q \in M$, we use $\Omega_{p,q}$ to denote all curves γ with $\gamma(0) = p, \gamma(1) = q$, and for $c \in \Omega_{p,q}$ we choose⁹ a covering $\{B(p_i, \delta_i)\}_{i=0}^k$ of c such that

- (1) $0 < \delta_i < \text{inj}(p_i)$.
- (2) $B(p_0, \delta_0) = B(p, \delta)$ and $p_k = q$.
- (3) $p_{i+1} \in B(p_i, \delta_i)$.

If we set $\varphi = \varphi_0$, then we can define isometries $\varphi_i: B(p_i, \delta_i) \rightarrow M$ such that $\varphi_i(p_i) = \varphi_{i-1}(p_i)$ and $(d\varphi_i)_{p_i} = (d\varphi_{i-1})_{p_i}$ by using Cartan-Ambrose-Hicks's theorem successively, and by Theorem 16.1.1 one has φ_i and φ_{i+1} coincide on $B(p_i, \delta_i) \cap B(p_{i+1}, \delta_{i+1})$. The covering together with isometries we construct is denoted by $\mathcal{A} = \{B(p_i, \delta_i), \varphi_i\}_{i=0}^k$. For arbitrary $x \in [0, 1]$, if $c(x) \in B(p_m, \delta_m)$, we may define

$$\begin{aligned}\varphi_{\mathcal{A}}(c(x)) &:= \varphi_m(c(x)) \\ (d\varphi_{\mathcal{A}})_{c(x)} &:= (d\varphi_m)_{c(x)}\end{aligned}$$

In particular, $\varphi_{\mathcal{A}}(q) := \varphi_k(q)$.

If $\mathcal{B} = \{\tilde{B}(\tilde{p}_i, \tilde{\delta}_i), \tilde{\varphi}_i\}_{i=0}^l$ is another covering of c , let's show $\varphi_{\mathcal{A}}(q) = \varphi_{\mathcal{B}}(q)$. Consider

$$I = \{x \in [0, 1] \mid \varphi_{\mathcal{A}}(c(x)) = \varphi_{\mathcal{B}}(c(x)), (d\varphi_{\mathcal{A}})_{c(x)} = (d\varphi_{\mathcal{B}})_{c(x)}\}$$

It's clear $I \neq \emptyset$, since $0 \in I$. Now it suffices to show it's both open and closed to conclude $1 \in I$.

(a) It's open: For $x \in I$, we assume $c(x) \in B(p_m, \delta_m) \cap \tilde{B}(\tilde{p}_n, \tilde{\delta}_n)$, that is

$$\begin{aligned}\varphi_m(c(x)) &= \tilde{\varphi}_n(c(x)) \\ (d\varphi_m)_{c(x)} &= (d\tilde{\varphi}_n)_{c(x)}\end{aligned}$$

Then one has

$$\begin{aligned}\varphi_m \circ \exp_{c(x)}(v) &= \exp_{\varphi_m(c(x))} \circ (d\varphi_m)_{c(x)}(v) \\ &= \exp_{\tilde{\varphi}_n(c(x))} \circ (d\tilde{\varphi}_n)_{c(x)}(v) \\ &= \tilde{\varphi}_n \circ \exp_{c(x)}(v)\end{aligned}$$

⁹Since injective radius is a continuous function, it has a positive minimum on curve c , so such covering exists.

Since $\exp_{c(x)}$ maps onto a neighborhood of $c(x)$, it follows that some neighborhood of x also lies in I , and thus I is open.

- (b) It's closed: Let $\{x_i\}_{i=1}^\infty \subseteq I$ be a sequence converging to x . Without lose of generality we may assume $\{x_i\}_{i=1}^\infty \subseteq B(p_m, \delta_m) \cap \widetilde{B}(\widetilde{p}_n, \widetilde{\delta}_n)$, then one has

$$\varphi_m(c(x_i)) = \widetilde{\varphi}_n(c(x_i))$$

$$(d\varphi_m)_{c(x_i)} = (d\widetilde{\varphi}_n)_{c(x_i)}$$

By taking limit we obtain the desired results.

Since $\varphi_{\mathcal{A}}(q)$ is independent of the choice of covering, we denote it as $\varphi(q)$ for convenience, and as a consequence we obtain the following map

$$\begin{aligned} F: \Omega_{p,q} &\rightarrow M \\ c &\mapsto \varphi(q) \end{aligned}$$

Note that $F(c)$ is locally constant, and thus it's independent of the choice of homotopy classes of c . Since M is simply-connected, one has $F: \Omega_{p,q} \rightarrow M$ is constant, so we obtain a local isometry $\varphi: M \rightarrow M$ which extends $\varphi: B(p, \delta) \rightarrow B(p, \delta)$. By Proposition 17.2.4 φ is a Riemannian covering map since M is complete, and thus φ is a diffeomorphism since M is simply-connected, which implies φ is an isometry. \square

Corollary 24.2.1. Let (M, g) be a complete locally Riemannian symmetric space. Then it's isometric to $(\widetilde{M}/\Gamma, \widetilde{g})$, where $(\widetilde{M}, \widetilde{g})$ is a Riemannian symmetric space and $\Gamma \subseteq \text{Iso}(\widetilde{M}, \widetilde{g})$.

Proof. Let $(\widetilde{M}, \widetilde{g})$ be the universal covering of (M, g) with pullback metric. Then $(\widetilde{M}, \widetilde{g})$ is a simply-connected Riemannian manifold with parallel curvature tensor. Furthermore, by Proposition 17.2.3 it's complete, hence it is symmetric. \square

As a consequence, above argument about analytic continuation can be used to give a proof of Hopf's theorem.

Theorem 24.1 (Hopf). Let (M, g) be a complete, simply-connected Riemannian manifold with constant sectional curvature K . Then (M, g) is isometric to

$$(\widetilde{M}, g_{\text{can}}) = \begin{cases} (\mathbb{S}^n(\frac{1}{\sqrt{K}}), g_{\text{can}}) & K > 0 \\ (\mathbb{R}^n, g_{\text{can}}) & K = 0 \\ (\mathbb{H}^n(\frac{1}{\sqrt{-K}}), g_{\text{can}}) & K < 0 \end{cases}$$

Proof. For $p \in M, \widetilde{p} \in \widetilde{M}$ and $\delta < \min\{\text{inj}(p), \text{inj}(\widetilde{p})\}$. By Cartan-Ambrose-Hicks's theorem, there exists an isometry $\varphi: B(p, \delta) \rightarrow B(\widetilde{p}, \delta)$ such that $\varphi(p) = \widetilde{p}$ and $(d\varphi)_p$ equals to a given linear isometry, since both (M, g) and $(\widetilde{M}, \widetilde{g})$ have constant sectional curvature K . By the same argument in proof of Theorem 24.2.1, there is an isometry $\varphi: (M, g) \rightarrow (\widetilde{M}, \widetilde{g})$ which extends $\varphi: B(p, \delta) \rightarrow B(\widetilde{p}, \delta)$. In particular, this completes the proof. \square

24.2.2. Riemannian symmetric space and Riemannian homogeneous space. In appendix A.4, we review what is homogeneous space. Now let's consider its analogy in Riemannian geometry.

Definition 24.2.1 (Riemannian homogeneous space). A Riemannian manifold (M, g) is called a Riemannian homogeneous space, if $\text{Iso}(M, g)$ acts on M transitively.

Proposition 24.2.1. Let (M, g) be a Riemannian homogeneous space. If there exists a symmetry at some point $p \in M$, then (M, g) is a Riemannian symmetric space.

Proof. Let φ be a symmetry at $p \in M$. Since (M, g) is a Riemannian homogeneous space, then for any $q \in M$ there exists an isometry $\psi: M \rightarrow M$ such that $\psi(p) = q$. Then

$$\varphi_q := \psi \circ \varphi \circ \psi^{-1}$$

is a symmetry at q . □

Theorem 24.2.2. Let (M, g) be a Riemannian symmetric space. Then

- (1) (M, g) is complete.
- (2) for any isometry $\varphi: M \rightarrow M$ with $(d\varphi)_p = -\text{id}$ and $\varphi(p) = p$, if $v \in T_p M$, then

$$\varphi(\exp_p(v)) = \exp_p(-v)$$

- (3) the isometry group $\text{Iso}(M, g)$ acts transitively on M .

Proof. For (1). For arbitrary geodesic $\gamma: [0, 1] \rightarrow M$ with $\gamma(0) = p, \gamma'(0) = v$, the curve $\beta(t) = \varphi(\gamma(t)): [0, 1] \rightarrow M$ is also a geodesic with $\beta(0) = p$ and $\beta'(0) = -v$. Now we obtain a smooth extension $\gamma': [0, 2] \rightarrow M$ of γ , given by

$$\gamma'(t) = \begin{cases} \beta(1-t), & t \in [0, 1] \\ \gamma(t-1), & t \in [1, 2] \end{cases}$$

Repeat above process to extend γ to a geodesic defined on \mathbb{R} , this shows completeness.

For (2). Note that $\varphi(\exp_p(tv))$ and $\exp_p(-tv)$ are geodesics starting at p with the same direction since φ is an isometry, and thus $\varphi(\exp_p(tv)) = \exp_p(-tv)$. Furthermore, since (M, g) is complete, one has $\varphi(\exp_p(tv))$ and $\exp_p(-tv)$ are defined on \mathbb{R} . In particular, one has $\varphi(\exp_p(v)) = \exp_p(-v)$ by considering $t = 1$.

For (3). Let $\gamma: [0, 1] \rightarrow M$ be a geodesic connecting $p, q \in M$, and $\varphi_m: M \rightarrow M$ is the symmetry at $m = \gamma(\frac{1}{2})$. If we consider $\beta(t) = \varphi_m(\gamma(\frac{1}{2} - t))$, then $\beta(0) = m, \beta'(0) = \gamma'(\frac{1}{2})$, which implies $\beta(t) = \gamma(\frac{1}{2} + t)$. Therefore $q = \gamma(1) = \beta(\frac{1}{2}) = \varphi_m(\gamma(0)) = \varphi_m(p)$. □

Corollary 24.2.2. The Riemannian symmetric space (M, g) is a Riemannian homogeneous space.

25. AN ALGEBRAIC VIEWPOINT

25.1. Riemannian symmetric pair. In this section, G is a connected Lie group with Lie algebra \mathfrak{g} .

Definition 25.1.1 (involution). An automorphism σ of G is called an involution, if $\sigma^2 = \text{id}_G$.

Definition 25.1.2 (Cartan decomposition). Let σ be an involution of G . The eigen-decomposition of \mathfrak{g} given by $(d\sigma)_e$ is called Cartan decomposition, that is,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

where

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid (d\sigma)_e(X) = X\}$$

$$\mathfrak{m} = \{X \in \mathfrak{g} \mid (d\sigma)_e(X) = -X\}$$

Proposition 25.1.1. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be a Cartan decomposition given by σ . Then

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$$

Proof. It follows from

$$(d\sigma)_e([X, Y]) = [(d\sigma)_e(X), (d\sigma)_e(Y)]$$

where $X, Y \in \mathfrak{g}$. □

Definition 25.1.3 (Riemannian symmetric pair). Let K be a closed subgroup of G . The pair (G, K) is called a Riemannian symmetric pair if

- (1) $\text{Ad}(K) < \text{GL}(\mathfrak{g})$ is compact.
- (2) there exists an involution $\sigma: G \rightarrow G$ with $(G^\sigma)_0 \subseteq K \subseteq G^\sigma$, where G^σ is the set of fixed points of σ in G , and $(G^\sigma)_0$ is the identity component of G^σ .

Remark 25.1.1. The first condition means K is compact up to a center, since $\ker \text{Ad}$ is the center of G_0 .

Example 25.1.1. $G = \text{SO}(n+1)$ and $K = \text{SO}(n)$ is a Riemannian symmetric pair given by

$$\begin{aligned} \sigma: \text{SO}(n+1) &\rightarrow \text{SO}(n+1) \\ a &\mapsto sas^{-1} \end{aligned}$$

where $s = \text{diag}\{-1, 1, \dots, 1\}$. Indeed,

$$\text{SO}(n+1)^\sigma = \{a \in \text{SO}(n+1) \mid sa = as\} = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & b \end{pmatrix} \mid b \in \text{O}(n) \right\}$$

which implies $(\text{SO}(n+1)^\sigma)_0 = \text{SO}(n) \subseteq \text{SO}(n+1)$.

Example 25.1.2. $G = \text{SL}(n, \mathbb{R})$ and $K = \text{SO}(n)$ is a Riemannian symmetric pair given by

$$\begin{aligned} \sigma: \text{SL}(n, \mathbb{R}) &\rightarrow \text{SL}(n, \mathbb{R}) \\ g &\mapsto (g^{-1})^T \end{aligned}$$

Indeed,

$$(\mathrm{SL}(n, \mathbb{R}))^\sigma = \mathrm{SO}(n)$$

Example 25.1.3. Let K be a compact Lie group, and set $G = K \times K$. Then (G, K) is a Riemannian symmetric pair given by σ , where $\sigma: G \rightarrow G$ given by $(x, y) \mapsto (y, x)$, since

$$G^\sigma = \{(a, a) \mid a \in K\} \cong K$$

Proposition 25.1.2. Let (G, K) be a symmetric pair given by σ . Then there is an isomorphism as Lie algebras

$$\mathfrak{k} \cong \mathrm{Lie} K$$

and an isomorphism as vector spaces

$$\mathfrak{m} \cong T_{eK}G/K$$

Proof. Note that $\mathrm{Lie} G^\sigma = \mathrm{Lie}(G^\sigma)_0$, and $(G^\sigma)_0 \subseteq K \subseteq G^\sigma$ since (G, K) is a symmetric pair. Then

$$\begin{aligned} \mathrm{Lie} K &= \mathrm{Lie}(G^\sigma)_0 = \{X \in \mathfrak{g} \mid \exp(tX) \in G^\sigma, \forall t \in \mathbb{R}\} \\ &= \{X \in \mathfrak{g} \mid \sigma(\exp(tX)) = \exp(tX), \forall t \in \mathbb{R}\} \\ &= \{X \in \mathfrak{g} \mid (\mathrm{d}\sigma)_e(X) = X\} \\ &= \mathfrak{k} \end{aligned}$$

where the last equality follows from the fact σ is a Lie group homomorphism and therefore $\sigma(\exp(tX)) = \exp(t(\mathrm{d}\sigma)_e(X))$ for all $t \in \mathbb{R}$. The second isomorphism follows directly from the first one. \square

Corollary 25.1.1. Let $\tilde{\sigma}: G/K \rightarrow G/K$ be the automorphism of G/K induced σ . Then $(\mathrm{d}\tilde{\sigma})_{eK} = -\mathrm{id}_{G/K}$.

Proof. $\tilde{\sigma}: G/K \rightarrow G/K$ is well-defined, since $K \subseteq G^\sigma$, and thus $\sigma: G \rightarrow G$ descends to $G/K \rightarrow G/K$. On the other hand, $(\mathrm{d}\tilde{\sigma})_{eK} = -\mathrm{id}_{G/K}$ since $T_{eK}G/K \cong \mathfrak{m}$ as vector spaces. \square

Theorem 25.1.1. Let (G, K) be a Riemannian symmetric pair given by σ . Then there exists a left-invariant metric on G/K making G/K to be a Riemannian symmetric space.

Proof. Note that bi-invariant metric on G/K is the same as the Ad-invariant inner product on $T_{eK}G/K \cong \mathfrak{m}$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the Cartan decomposition given by σ . Since (G, K) is a Riemannian symmetric pair, then $\mathrm{Ad}(K) < \mathrm{GL}(\mathfrak{g})$ is compact, and thus $\mathrm{Ad}(K)|_{\mathfrak{m}}$ is still compact. Let $\langle -, - \rangle$ be an arbitrary inner product on \mathfrak{m} and μ be the Haar measure on $\mathrm{Ad}(K)$. Then

$$(u, v) := \int_{\mathrm{Ad}(K)|_{\mathfrak{m}}} \langle Au, Av \rangle \mu(A)$$

is a Ad-invariant inner product on \mathfrak{m} . By Corollary 25.1.1 it suffices to show $\tilde{\sigma}: G/K \rightarrow G/K$ is an isometry with respect to this bi-invariant metric.

For $gK \in G/K$ and $v, w \in T_{gK}G/K$, suppose $v = (dL_g)_{eK}v_0$ and $w = (dL_g)_{eK}w_0$. Then

$$\begin{aligned} ((d\tilde{\sigma})_{eK}v, (d\tilde{\sigma})_{eK}w)_{\tilde{\sigma}(eK)} &= ((d\tilde{\sigma})_{eK}(dL_g)_{eK}v_0, (d\tilde{\sigma})_{eK}(dL_g)_{eK}w_0)_{\tilde{\sigma}(eK)} \\ &= ((d(\tilde{\sigma} \circ L_g))_{eK}v_0, (d(\tilde{\sigma} \circ L_g))_{eK}w_0)_{\tilde{\sigma}(eK)} \\ &= ((d(L_g \circ \tilde{\sigma}))_{eK}v_0, (d(L_g \circ \tilde{\sigma}))_{eK}w_0)_{\sigma(g)K} \\ &= ((dL_g)_{eK}(-v_0), (dL_g)_{eK}(-w_0))_{\tilde{\sigma}(eK)} \\ &= (v, w)_{\tilde{\sigma}(eK)} \end{aligned}$$

This completes the proof. \square

Example 25.1.4. S^n is a Riemannian symmetric space, since $S^n \cong \text{SO}(n+1)/\text{SO}(n)$ and $(\text{SO}(n+1), \text{SO}(n))$ is a Riemannian symmetric pair.

Example 25.1.5. $\text{SL}(n, \mathbb{R})/\text{SO}(n)$ is a Riemannian symmetric pair, and it's a generalization of hyperbolic plane \mathbb{H}^2 , since $\text{SL}(2, \mathbb{R})/\text{SO}(2) \cong \mathbb{H}^2$.

Example 25.1.6. Any compact Lie group K is a Riemannian symmetric space, since $(K \times K, K)$ is a Riemannian symmetric pair.

Theorem 25.1.1 gives a method to construct lots of Riemannian symmetric spaces as examples shown above, so there is a natural question: Does every Riemannian symmetric space comes from a symmetric pair?

Proposition 25.1.3. Let (M, g) be a Riemannian symmetric space and G be the identity component of $\text{Iso}(M, g)$. For $p \in M$, let K denote the isotropic group of G at p . Then (G, K) is a symmetric pair.

Proof. Let s_p be the symmetry at p , and consider

$$\begin{aligned} \sigma: G &\rightarrow G \\ g &\mapsto s_p g s_p \end{aligned}$$

It's clear σ is an involution of G , since $s_p^2 = \text{id}$. To show (G, K) is a symmetric pair, it suffices to show

$$(G^\sigma)_0 \subseteq K \subseteq G^\sigma$$

since by Theorem 16.1.2 one has G is a Lie group and K is compact.

- (1) To show $K \subseteq G^\sigma$. For any $k \in K$, in order to show $k = s_p k s_p$, it suffices to show they and their differentials agree at some point by Theorem 16.1.1, since both of them are isometries, and p is exactly the point we desired.
- (2) To see $(G^\sigma)_0 \subseteq K$. Let $\exp(tX) \subseteq (G^\sigma)_0$ be a one-parameter subgroup. Since $\sigma(\exp(tX)) = \exp(tX)$, then

$$s_p \exp(tX) s_p(p) = s_p \exp(tX)(p) = \exp(tX)(p)$$

But p is an isolated fixed point of s_p , which implies $\exp(tX)(p) = p$ for all t . This shows the one-parameter subgroup lies in K . Since exponential map of Lie group is a diffeomorphism in a small neighborhood of identity

element e and $(G^\sigma)_0$ can be generated by a neighborhood of e , which implies the whole $(G^\sigma)_0 \subseteq K$.

□

26. CURVATURE OF RIEMANNIAN SYMMETRIC SPACE

26.1. Curvature of Riemannian symmetric space.

Proposition 26.1.1. Let (M, g) be a Riemannian symmetric space and $G = \text{Iso}(M, g)$. For any $p \in M$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where \mathfrak{k} is Lie algebra of isotropy group G_p and $\mathfrak{m} \cong T_p M$. For any $X \in \mathfrak{m}$, one has

$$B_g(X, X) \leq 0$$

and the identity holds if and only if $X = 0$.

Theorem 26.1.1. Let (M, g) be a Riemannian symmetric space and $G = \text{Iso}(M, g)$. For any $p \in M$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ with $\mathfrak{m} \cong T_p M$.

(1) For any $X, Y, Z \in \mathfrak{m}$, there holds

$$R(X, Y)Z = -[Z, [Y, X]]$$

$$\text{Ric}(Y, Z) = -\frac{1}{2}B(Y, Z)$$

(2) If $\text{Ric}(g) = \lambda g$. Then for $X, Y \in \mathfrak{m}$, one has

$$2\lambda R(X, Y, Y, X) = -B([X, Y], [X, Y])$$

Proof. For (1). For any $X, Y, Z \in \mathfrak{m}$, direct computation shows

$$\begin{aligned} R(X, Y)Z &\stackrel{\text{I}}{=} R(X, Z)Y - R(Y, Z)X \\ &\stackrel{\text{II}}{=} \nabla_Z \nabla_Y X - \nabla_{\nabla_Z Y} X - \nabla_Z \nabla_X Y + \nabla_{\nabla_Z X} Y \\ &\stackrel{\text{III}}{=} -\nabla_Z [X, Y] \\ &\stackrel{\text{IV}}{=} -[Z, [X, Y]] \end{aligned}$$

where

I holds from the first Bianchi identity.

II holds from (2) of Proposition 16.2.1.

III holds from $X, Y \in \mathfrak{m}$, thus $(\nabla X)_p = (\nabla Y)_p = 0$.

IV holds from

$$\nabla_Z [X, Y] - \nabla_{[X, Y]} Z = [Z, [X, Y]]$$

and $(\nabla Z)_p = 0$.

To see Ricci curvature, note that for $Y \in \mathfrak{m}$

$$\text{ad}_Y : \mathfrak{k} \rightarrow \mathfrak{m}, \quad \text{ad}_Y : \mathfrak{m} \rightarrow \mathfrak{k}$$

Thus $\text{ad}_Z \circ \text{ad}_Y$ preserves the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ if $Y, Z \in \mathfrak{m}$. Then

$$\begin{aligned} \text{tr}(\text{ad}_Z \circ \text{ad}_Y|_{\mathfrak{m}}) &= \text{tr}(\text{ad}_Z|_{\mathfrak{k}} \circ \text{ad}_Y|_{\mathfrak{m}}) \\ &= \text{tr}(\text{ad}_Y|_{\mathfrak{m}} \circ \text{ad}_Z|_{\mathfrak{k}}) \\ &= \text{tr}(\text{ad}_Y \circ \text{ad}_Z|_{\mathfrak{k}}) \end{aligned}$$

Hence we obtain

$$B(Y, Y) = \text{tr}(\text{ad}_Y \circ \text{ad}_Y|_{\mathfrak{k}}) + \text{tr}(\text{ad}_Y \circ \text{ad}_Y|_{\mathfrak{l}}) = 2 \text{tr}(\text{ad}_Y \circ \text{ad}_Y|_{\mathfrak{m}})$$

Since Ricci tensor is trace of curvature tensor, thus

$$\text{Ric}(Y, Y) = -\text{tr}(\text{ad}_Y \circ \text{ad}_Y |_{\mathfrak{m}}) = -\frac{1}{2}B(Y, Y)$$

By using symmetry for $Y + Z$, one has $\text{Ric}(Y, Z) = -\frac{1}{2}B(Y, Z)$.

For (2). If $\text{Ric}(g) = \lambda g$. Then

$$\begin{aligned} 2\lambda g(R(X, Y)Y, X) &= -2\lambda g(\text{ad}_Y \circ \text{ad}_Y X, X) \\ &= -2\text{Ric}(\text{ad}_Y \circ \text{ad}_Y X, X) \\ &= B(\text{ad}_Y \circ \text{ad}_Y X, X) \\ &= -B(\text{ad}_Y X, \text{ad}_Y X) \\ &= -B([X, Y], [X, Y]) \end{aligned}$$

□

Corollary 26.1.1. Let (M, g) be a Riemannian symmetric space which is an Einstein manifold with Einstein constant λ . Then

- (1) If $\lambda > 0$. Then (M, g) has non-negative sectional curvature.
- (2) If $\lambda < 0$. Then (M, g) has non-positive sectional curvature.
- (3) If $\lambda = 0$. Then (M, g) is flat.

Proof. By above theorem one has

$$2\lambda R(X, Y, Y, X) = -B([X, Y], [X, Y]) \geq 0$$

since $[X, Y] \in [\mathfrak{m}, \mathfrak{m}] \in \mathfrak{m}$ and B is negative-definite on \mathfrak{m} . □

26.2. Irreducible space.

Definition 26.2.1 (isotropy irreducible). Let (M, g) be a Riemannian symmetric space with $G = \text{Iso}(M, g)$ and $K = G_p$ for some $p \in M$. If the identity component K_0 acts irreducibly on $T_p M$. Then M is called irreducible. Otherwise M is called reducible.

Lemma 26.2.1. Let B_1, B_2 be two symmetric bilinear forms on a vector space V such that B_1 is positive definite. If a group K acts irreducibly on V such that B_1 and B_2 are invariant under K . Then $B_2 = \lambda B_1$ for some constant λ .

Proof. Since B_1 is positive definite. Then there exists an endomorphism $L : V \rightarrow V$ such that

$$B_2(u, v) = B_1(Lu, v)$$

where $u, v \in V$. Since B_1, B_2 are invariant under K . Then for any $k \in K$

$$B_1(kLu, v) = B_1(Lu, k^{-1}v) = B_2(u, k^{-1}v) = B_2(ku, v) = B_1(Lku, v)$$

holds for arbitrary $u, v \in V$, which implies $Lk = kL$ for all $k \in K$. Moreover, the symmetry of B_1, B_2 implies

$$B_1(Lu, v) = B_2(u, v) = B_2(v, u) = B_1(Lv, u) = B_1(u, Lv)$$

Hence L is symmetric with respect to B_1 , thus the eigenvalues of L are real. If $E \subseteq V$ is an eigenspace with eigenvalue λ , the fact $kL = Lk$ implies E is invariant under K . Since K acts irreducibly on V , thus $E = V$, that is $L = \lambda I$, which implies $B_2 = \lambda B_1$. \square

Theorem 26.2.1. An irreducible Riemannian symmetric space is Einstein. Moreover, the metric is unique determined up to a multiple.

Proof. Since isometries preserves the metric and curvature, and Ricci tensor is also a symmetric bilinear form, thus there exists smooth λ such that

$$\text{Ric}(g) = \lambda g$$

Since Ric is parallel, which relies on $\nabla R = 0$. Then we have λ is a constant. \square

Definition 26.2.2. Let (M, g) be an irreducible Riemannian symmetric manifold.

- (1) If the Ricci curvature is positive. Then M is called of compact type.
- (2) If the Ricci curvature is negative. Then M is called of non-compact type.
- (3) If the Ricci curvature is zero. Then M is called of Euclidean type.

Remark 26.2.1. We have the following observations:

- (1) By Myers' theorem, if M is of compact type. Then it's compact.
- (2) If M is of non-compact type. Then it's non-compact, otherwise by Bochner's technique there is no non-trivial Killing vector field.
- (3) If M is of Euclidean type. Then it is flat and so it is covered by \mathbb{R}^n .

27. EXAMPLES OF RIEMANNIAN SYMMETRIC SPACE

27.1. Compact Lie group as Riemannian symmetric space.

Theorem 27.1.1. Let G be a compact Lie group and \mathfrak{g} be its Lie algebra.

- (1) G equipped with bi-invariant g is a Riemannian symmetric space.
- (2) Every left-invariant vector field is a Killing field.
- (3) For any $X \in \mathfrak{g}$, $\text{ad}_X: \mathfrak{g} \rightarrow \mathfrak{g}$ is a skew-symmetric with respect to g .
- (4) For any $X \in \mathfrak{g}$, $B_{\mathfrak{g}}(X, X) \leq 0$ and the identity holds if and only if X lies in center of \mathfrak{g} .
- (5) (G, g) has non-negative sectional curvature.
- (6) If \mathfrak{g} has no center. Then B induces a bi-invariant metric g_B with $\text{Ric}(g_B) = \frac{1}{2}g_B$.

Proof. For (1). Consider Lie group $G \times G$ and its compact subgroup G , we claim the involution σ on $G \times G$, given by $(g, h) \mapsto (h, g)$ makes pair $(G \times G, G)$ a Riemannian symmetric pair. Indeed,

$$G \cong G^\sigma = G^\Delta := \{(g, g) \in G \times G \mid g \in G\}$$

By Theorem ??, one has $G \times G/G^\Delta$ with induced metric is a Riemannian symmetric space. Note that the following diffeomorphism

$$\begin{aligned} G \times G/G^\Delta &\rightarrow G \\ (a, b)G^\Delta &\mapsto ab^{-1} \end{aligned}$$

is an isometry. This shows G is a Riemannian symmetric space.

For (2). Since the flows of a left-invariant vector fields are left translations which are isometries, thus every left-invariant vector field is Killing.

For (3).

For (4). $B_g(X, X) = \text{tr}(\text{ad}_X \circ \text{ad}_X) \leq 0$, since

For (5) and (6). They follow from Theorem 26.1.1. □

Remark 27.1.1.

27.2. Examples.

Example 27.2.1 (hyperbolic Grassmannian). In $\mathbb{R}^{k,l}$ with $k \geq 2, l \geq 1$, consider the following quadratic form

$$v^t I_{k,l} w = v^t \begin{pmatrix} I_k & 0 \\ 0 & -I_l \end{pmatrix} w = \sum_{i=1}^k v_i w_i - \sum_{j=k+1}^{k+l} v_j w_j$$

The group of linear transformation X that preserves this quadratic form is denoted by $O(k, l)$, that is

$$X I_{k,l} X^t = I_{k,l}$$

If $k, l > 0$, $O(k, l)$ is not compact, but it contains a compact subgroup $O(k) \times O(l)$

Part 9. Appendix

APPENDIX A. REVIEW OF SMOOTH MANIFOLDS

In this section we give a quick review of facts in differential geometry we may use.

A.1. Lie group.

Definition A.1.1 (Lie group). A Lie group G is a smooth manifold which is also endowed with a group structure such that the multiplication map and the inverse map are smooth.

Proposition A.1.1. For any $g \in G$, L_g, R_g defined as follows

$$\begin{aligned} L_g(h) &= gh \\ R_g(h) &= hg \end{aligned}$$

are diffeomorphisms.

Proof. It's clear they're smooth, since multiplication map is smooth. The inverse maps are given by $L_{g^{-1}}$ and $R_{g^{-1}}$, since inverse maps are also smooth. \square

Definition A.1.2 (Lie subgroup). A Lie subgroup of a Lie group G is a subgroup $H \subseteq G$ endowed with a smooth structure such that H is a Lie group and an immersed submanifold.

Theorem A.1.1 (Cartan's closed subgroup theorem). Any closed subgroup H of a Lie group is a Lie subgroup.

Definition A.1.3 (invariant vector field). A vector field X on a Lie group G is called left-invariant, if

$$(L_g)_*X = X$$

for arbitrary $g \in G$.

Remark A.1.1. It's clear there is the following isomorphism

$$\begin{aligned} \{\text{left-invariant vector fields}\} &\rightarrow T_e G \\ X &\mapsto X_e \end{aligned}$$

where X_e is its value in $T_e G$. Furthermore, since Lie bracket of two left-invariant vector fields is still left-invariant, we can equip $T_e G$ a Lie bracket.

Definition A.1.4 (Lie algebra). The tangent space $T_e G$ of a Lie group G equipped with Lie bracket is called Lie algebra of G , denoted by \mathfrak{g} .

Definition A.1.5 (adjoint representation). The adjoint representation is defined as follows

$$\begin{aligned} \text{Ad}: G &\rightarrow \text{GL}(\mathfrak{g}) \\ g &\mapsto (L_g)_*(R_{g^{-1}})_* \end{aligned}$$

Definition A.1.6 (integral curve). Let X be a vector field of G and $g \in G$. Then an integral curve of X through the point p is a smooth curve $\gamma: I \subseteq \mathbb{R} \rightarrow G$ such that

$$\begin{aligned}\gamma(0) &= g \\ \gamma'(t) &= X(\gamma(t))\end{aligned}$$

Definition A.1.7 (complete vector field). A vector field X is called complete, if its integral curve is defined for all $t \in \mathbb{R}$.

Proposition A.1.2. Every left-invariant vector field on a Lie group G is complete.

Proof. Let X be a left-invariant vector field and γ the unique integral curve for X defined on $(-\varepsilon, \varepsilon)$ such that $\gamma(0) = e$. Then $\gamma_g(t) := L_g\gamma(t)$ is an integral curve for X such that $\gamma_g(0) = g$. Indeed,

$$\begin{aligned}\gamma'_g(t) &= (dL_g)_{\gamma(t)}(\gamma'(t)) \\ &= (dL_g)_{\gamma(t)}(X(\gamma(t))) \\ &= X(L_g\gamma(t)) \\ &= X(\gamma_g(t))\end{aligned}$$

In particular, for $t_0 \in (-\varepsilon, \varepsilon)$, the curve $t \mapsto \gamma(t_0)\gamma(t)$ is an integral curve for X starting at $\gamma(t_0)$. By uniqueness, this curve coincides with $\gamma(t_0 + t)$ for all $t \in (-\varepsilon, \varepsilon) \cap (-\varepsilon - t_0, \varepsilon - t_0)$. Define

$$\tilde{\gamma}(t) = \begin{cases} \gamma(t) & t \in (-\varepsilon, \varepsilon) \\ \gamma(t_0)\gamma(t) & t \in (-\varepsilon - t_0, \varepsilon - t_0) \end{cases}$$

Repeat above operations to get our desired extension. \square

Remark A.1.2. From this proof we can see integral curve of left-invariant vector fields through identity e is just a Lie group homomorphism $\gamma: \mathbb{R} \rightarrow G$, such homomorphism is called a one parameter subgroup.

A.2. Killing form.

Definition A.2.1 (Killing form). Let \mathfrak{g} be a Lie algebra.

- (1) For any $X \in \mathfrak{g}$, the adjoint linear map $\text{ad}_X: \mathfrak{g} \rightarrow \mathfrak{g}$ is defined as $\text{ad}_X Y = [X, Y]$.
- (2) The Killing form $B: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ is a bilinear symmetric form defined as

$$B(X, Y) = \text{tr}(\text{ad}_X \circ \text{ad}_Y)$$

Lemma A.2.1. Let B be the Killing form on Lie algebra \mathfrak{g} of Lie group G . Then for any $g \in G$ and $X, Y, Z \in \mathfrak{g}$. Then

- (1) $B(\text{Ad}_g X, \text{Ad}_g Y) = B(X, Y)$.
- (2) $B(\text{ad}_Z X, Y) = -B(X, \text{ad}_Z Y)$.

Remark A.2.1. Recall the following facts about Lie algebra:

- (1) For $g \in G$, Ad_g is the differential at identity element of inner automorphism $x \mapsto gxg^{-1}$ of G , and it's a Lie algebra homomorphism.
 (2) For $X, Y \in \mathfrak{g}$, $\left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{\exp(tX)} Y = \text{ad}_X Y$

Proof. For (1). For any $X, Y \in \mathfrak{g}$, one has

$$\begin{aligned} [\text{Ad}_g X, Y] &= [\text{Ad}_g X, \text{Ad}_g \circ \text{Ad}_{g^{-1}}(Y)] \\ &= \text{Ad}_g([X, \text{Ad}_{g^{-1}} Y]) \\ &= \text{Ad}_g \circ \text{ad}_X \circ (\text{Ad}_g)^{-1}(Y) \end{aligned}$$

If we use σ to denote Ad_g . Then $\text{ad}_{\sigma(X)} = \sigma \circ \text{ad}_X \circ \sigma^{-1}$. Hence,

$$B(\sigma(X), \sigma(Y)) = \text{tr}(\text{ad}_{\sigma(X)} \circ \text{ad}_{\sigma(Y)}) = \text{tr}(\sigma \circ \text{ad}_X \circ \text{ad}_Y \circ \sigma^{-1}) = B(X, Y)$$

For (2). For $Z \in \mathfrak{g}$, from (1) one has

$$B(\text{Ad}_{\exp(tZ)} X, \text{Ad}_{\exp(tZ)} Y) = B(X, Y)$$

By taking derivative at point $t = 0$, one has

$$B(\text{ad}_Z X, Y) + B(X, \text{ad}_Z Y) = 0$$

□

Proposition A.2.1. Let \mathfrak{g} be a Lie algebra with Killing form B , \mathfrak{h} is a Lie subalgebra of \mathfrak{g} . Then Killing form on \mathfrak{h} is exactly the restriction of B on it.

In the following computations we always use ϕ to denote $\text{ad}_X \circ \text{ad}_Y$.

Example A.2.1. Killing form $B(X, Y)$ on $\mathfrak{gl}(n)$ is $2n \text{tr}(XY) - 2 \text{tr}(X) \text{tr}(Y)$.

Proof. There is a canonical basis of $\mathfrak{gl}(n)$, that is $\{E_{ij}\}$, where E_{ij} is the matrix such that

$$(E_{ij})_{kl} = \begin{cases} 1, & (k, l) = (i, j) \\ 0, & \text{otherwise} \end{cases}$$

We compute the trace of ϕ in terms of this basis. A direct computation shows

$$\phi(E_{ij}) = \sum_{k=1}^n (XY)_{jk} E_{ik} + (XY)_{ki} E_{kj} - \sum_{k,l=1}^n (X_{ki} Y_{jl} + Y_{ki} X_{jl}) E_{kl}$$

which implies the trace of ϕ is

$$\sum_{i,j=1}^n (XY)_{jj} + (XY)_{ii} - X_{ii} Y_{jj} - Y_{ii} X_{jj} = 2n \text{tr}(XY) - 2 \text{tr}(X) \text{tr}(Y)$$

□

Example A.2.2. Killing form $B(X, Y)$ on $\mathfrak{sl}(n)$ is $2n \text{tr}(XY)$.

Proof. Note that $\mathfrak{sl}(n)$ is an ideal of $\mathfrak{gl}(n)$, which implies¹⁰ the restriction of Killing form on $\mathfrak{gl}(n)$ to $\mathfrak{sl}(n)$ is exactly the one on $\mathfrak{sl}(n)$. Thus by Example A.2.1 one has Killing form on $\mathfrak{sl}(n)$ is $2n \operatorname{tr}(XY)$, since $\mathfrak{sl}(n)$ consisting of matrices with vanishing trace. \square

Example A.2.3. Killing form $B(X, Y)$ on $\mathfrak{so}(n)$ is $(n - 2) \operatorname{tr}(XY)$.

Proof. There is a natural basis of $\mathfrak{so}(n)$, that is $\{E_{ij} - E_{ji}\}_{i < j}$. If we denote

$$\phi(E_{ij}) = a_{ij,ij}E_{ij} + a_{ij,ji}E_{ji} + \dots$$

The computation in Example A.2.1 shows

$$\begin{aligned} a_{ij,ij} &= (XY)_{jj} + (XY)_{ii} - X_{ii}Y_{jj} - Y_{ii}X_{jj} \\ a_{ij,ji} &= \delta_{ij}((XY)_{jj} + (XY)_{ii}) - X_{ji}Y_{ji} - Y_{ji}X_{ji} \end{aligned}$$

Note that

$$\phi(E_{ij} - E_{ji}) = (a_{ij,ij} - a_{ij,ji})(E_{ij} - E_{ji}) + \dots$$

Thus the Killing form on $\mathfrak{so}(n)$ is

$$\begin{aligned} B(X, Y) &= \sum_{i < j} ((XY)_{jj} + (XY)_{ii} - X_{ii}Y_{jj} - X_{jj}Y_{ii} + X_{ji}Y_{ji} + Y_{ji}X_{ji}) \\ &= \frac{1}{2} \sum_{i \neq j} ((XY)_{jj} + (XY)_{ii} - X_{ii}Y_{jj} - X_{jj}Y_{ii} + X_{ji}Y_{ji} + Y_{ji}X_{ji}) \\ &= (n - 1) \operatorname{tr}(XY) + \frac{1}{2} \sum_{i \neq j} (-X_{ii}Y_{jj} - X_{jj}Y_{ii} + X_{ji}Y_{ji} + Y_{ji}X_{ji}) \\ &\stackrel{(1)}{=} (n - 1) \operatorname{tr}(XY) - \operatorname{tr}(X) \operatorname{tr}(Y) - \frac{1}{2} \sum_{i \neq j} (X_{ji}Y_{ij} - Y_{ji}X_{ij}) \\ &\stackrel{(2)}{=} (n - 1) \operatorname{tr}(XY) - \frac{1}{2} (\operatorname{tr}(XY) + \operatorname{tr}(YX)) \\ &= (n - 2) \operatorname{tr}(XY) \end{aligned}$$

where

- (1) holds from X, Y are skew-symmetric.
- (2) holds from skew-symmetry matrix has vanishing trace.

\square

A.3. Lie group action. In this section we assume M is a smooth manifold and G is a Lie group.

Definition A.3.1 (smooth Lie group action). A smooth Lie group action on M is given by the following data:

- (1) For every $g \in G$, it gives a diffeomorphism of M , denoted by θ_g , where $p \in M$.
- (2) The map $G \times M \rightarrow M$ given by $(g, p) \mapsto gp$ is smooth.
- (3) For $g_1, g_2 \in G$ and $x \in M$, $(g_1 g_2)p = g_1(g_2 p)$.

¹⁰See Section 5.1 of [Hum12] for a proof.

Definition A.3.2. Suppose G acts on M smoothly.

- (1) For $g \in G$, if $gp = p$ for some $p \in M$ implies $g = e$, then the group action is called free.
- (2) For $g \in G$, if $gp = p$ for all $p \in M$ implies $g = e$, then the group action is called effective.
- (3) If for arbitrary $p, q \in M$, there exists $g \in G$ such that $gp = q$, then the group action is called transitive.
- (4) If the map $G \times M \rightarrow M \times M$ given by $(g, p) \rightarrow (gp, p)$ is a proper map¹¹, then the action is called proper.

Remark A.3.1. If a group action is free, then it's effective, but converse statement may not hold.

Proposition A.3.1. Suppose G acts on M smoothly. The following statements are equivalent.

- (1) The action is proper.
- (2) If $\{p_i\}$ is a sequence in M and $\{g_i\}$ is a sequence in G such that both $\{p_i\}$ and $\{g_i p_i\}$ converge, then a subsequence of $\{g_i\}$ converges.
- (3) For every compact subset $K \subseteq M$, the set $G_K = \{g \in G \mid gK \cap K \neq \emptyset\}$ is compact.

Proof.

□

Corollary A.3.1. If G is a compact Lie group, then every smooth G -action on M is proper.

Proof. Since G is compact, then every sequence in G admits a convergent subsequence, and thus the action is proper by (2) of Proposition A.3.1. □

Definition A.3.3 (G -homogeneous space). A smooth manifold M equipped with a transitive smooth G -action is called a homogeneous G -space.

Definition A.3.4 (isotropy group). Suppose G acts on M smoothly. The isotropy group at $p \in M$ is defined as

$$G_p = \{g \in G \mid gp = p\}$$

Theorem A.3.1. Let M and N be smooth manifolds and G is a Lie group. Suppose $F: M \rightarrow N$ is a smooth map which is equivalent with transitive smooth G -action on M and any smooth G -action on N . Then F has constant rank.

Theorem A.3.2 (quotient manifold theorem). Let G be a Lie group acting on M smoothly, freely and properly. Then the quotient space M/G is a topological manifold with dimension $\dim M - \dim G$, and it admits a unique smooth structure such that $\pi: M \rightarrow M/G$ is a submersion.

Proof.

□

¹¹A map is called proper if the preimage of every compact set is compact.

A.4. Homogeneous space. In this section we assume G is a Lie group.

Theorem A.4.1. Let H be a closed subgroup of G . Then

- (1) The left coset space G/H is a topological manifold of dimension $\dim G - \dim H$.
- (2) G/H admits a unique smooth structure such that the quotient map $\pi: G \rightarrow G/H$ is a smooth submersion.
- (3) The left action

$$\begin{aligned} G \times G/H &\rightarrow G/H \\ (g_1, g_2H) &\mapsto (g_1g_2)H \end{aligned}$$

turns G/H into a G -homogeneous space.

Proof. By Theorem A.3.2, it suffices to show H acting on G by left multiplication is smoothly, freely and properly, since it's clear G acting on G/H by left multiplication is transitive.

- (1) It's smooth: By Theorem A.1.1 one has H is a Lie subgroup, and thus the action is smoothly, since multiplication of Lie group is smooth.
- (2) It's free: If $hg = g$ for some $g \in G$, it's clear $h = e$.
- (3) It's proper: Note that the inclusion $H \times G \hookrightarrow G \times G$ is proper, since H is closed and the intersection of a compact subset with a closed subset is compact. On the other hand,

$$\begin{aligned} G \times G &\rightarrow G \times G \\ (g_1, g_2) &\mapsto (g_1g_2, g_2) \end{aligned}$$

is proper, since it's a diffeomorphism. Thus

$$\begin{aligned} K \times G &\rightarrow G \times G \\ (h, g) &\mapsto (hg, g) \end{aligned}$$

is proper, since it's the composition of proper maps.

□

Theorem A.4.2. Let M be a G -homogeneous space and $p \in M$. Then the isotropy group G_p is a closed subgroup of G and the map

$$\begin{aligned} F: G/G_p &\rightarrow M \\ gG_p &\mapsto gp \end{aligned}$$

is a G -equivariant diffeomorphism.

Proof. Let's prove this by the following steps:

- (1) For $p \in M$, consider the following map

$$\begin{aligned} \theta^{(g)}: G &\rightarrow M \\ g &\mapsto gp \end{aligned}$$

It's smooth since G acting on M smoothly, and $G_p = (\theta^{(g)})^{-1}(p)$ implies G_p is a closed subgroup of G .

- (2) For $g_1, g_2 \in G$ with $g_1 g_2^{-1} = g \in G_p$, it's clear $g_1 p = g_2 g p = g_2 p$, which implies F is well-defined.
- (3) If $\pi: G \rightarrow G/G_p$ is the projection map, then by Theorem A.4.1 there exists a unique smooth structure on G/G_p such that π is a surjective smooth submersion. Then F is smooth since $F \circ \pi = \theta^{(p)}$.
- (4) It's clear F is G -equivariant, then Theorem A.3.1 shows F has constant rank. So it suffices to check F is bijective to conclude F is a diffeomorphism. Given any $q \in M$, there exists $g \in G$ such that $g p = q$ since M is a G -homogeneous space, and thus $F(g G_p) = q$. On the other hand, if $F(g_1 G_p) = F(g_2 G_p)$ for $g_1, g_2 \in G$, then $g_1 p = g_2 p$ implies $g_1 g_2^{-1} \in G_p$, and thus $g_1 G_p = g_2 G_p$.

□

APPENDIX B. HODGE THEOREM

In this section, we mainly follow the Chapter 6 of [War10].

B.1. Introduction and proof of Hodge theorem. We shall use Δ^* to denote the adjoint of Laplace-Beltrami operator on Ω_M^k . This operator is precisely Δ itself, since Laplace-Beltrami operator is self-adjoint, and we usually make no distinction between Δ and Δ^* . However, this distinction will be important for the form of the following definition.

An important question is to find a necessary and sufficient condition for there to exist a solution ω of equation $\Delta\omega = \alpha$, where α is a given k -form. Suppose ω is a solution. Then

$$(\Delta\omega, \varphi) = (\alpha, \varphi)$$

holds for all k -forms φ . Equivalently we have

$$(\omega, \Delta^*\varphi) = (\alpha, \varphi)$$

holds for all k -forms φ . In this viewpoint, we can regard a solution of $\Delta\omega = \alpha$ as a certain type of linear functional on $C^\infty(M, \Omega_M^k)$, namely solution ω determines a bounded linear functional l on $C^\infty(M, \Omega_M^k)$ by

$$l(\varphi) = (\omega, \varphi), \quad \varphi \in C^\infty(M, \Omega_M^k)$$

such that

$$l(\Delta^*\varphi) = (\alpha, \varphi)$$

holds for all k -forms φ .

Definition B.1.1 (weak solution). A linear functional l on $C^\infty(M, \Omega_M^k)$ is called a weak solution of $\Delta\omega = \alpha$, if

$$l(\Delta^*\varphi) = (\alpha, \varphi)$$

holds for all k -forms φ .

We have seen that each ordinary solution of $\Delta\omega = \alpha$ determines a weak solution of it, it turns out that the major effort of this section will be to prove a regularity theorem which says that the converse of this is true, that is each weak solution determines an ordinary solution. The main step is to show if l is a weak solution of $\Delta\omega = \alpha$. Then there exists a smooth form ω such that

$$l(\varphi) = (\alpha, \varphi), \quad \varphi \in C^\infty(M, \Omega_M^k)$$

Then ω is an ordinary solution follows from

$$(\Delta, \varphi) = (\omega, \Delta^*\varphi) = l(\Delta^*\varphi) = (\alpha, \varphi)$$

holds for all k -forms φ , which implies $\Delta\omega = \alpha$.

The key theorems we will prove are listed as follows:

Theorem B.1.1 (regularity theorem). Let $\alpha \in C^\infty(M, \Omega_M^k)$, and l be a weak solution of $\Delta\omega = \alpha$. Then there exists $\omega \in C^\infty(M, \Omega_M^k)$ such that

$$l(\varphi) = (\omega, \varphi)$$

holds for every k -forms φ . In particular, $\Delta\omega = \alpha$.

Theorem B.1.2. Let $\{\alpha_n\}$ be a sequence of smooth k -forms on M such that $\|\alpha_n\| \leq c$ and $\|\Delta\alpha_n\| \leq c$ for all n and for some constant $c > 0$. Then a subsequence of $\{\alpha_n\}$ is a Cauchy sequence in $C^\infty(M, \Omega_M^k)$.

Corollary B.1.1. There exists a constant $c > 0$ such that

$$\|\psi\| \leq c\|\Delta\psi\|$$

holds for all $\psi \in (\mathcal{H}^k)^\perp$

Proof. Suppose the contrary. Then there exists a sequence $\psi_j \in (\mathcal{H}^k)^\perp$ with $\|\psi_j\| = 1$ and $\|\Delta\psi_j\| \rightarrow 0$. By Theorem B.1.2, there exists a subsequence of $\{\psi_j\}$ which for convenience we can assume to be $\{\psi_j\}$ itself, is Cauchy. Thus for each $\varphi \in C^\infty(M, \Omega_M^k)$, $\lim_{j \rightarrow \infty} (\psi_j, \varphi)$ exists. Consider the linear functional l on $C^\infty(M, \Omega_M^k)$ defined by

$$l(\varphi) := \lim_{j \rightarrow \infty} (\psi_j, \varphi), \quad \varphi \in C^\infty(M, \Omega_M^k)$$

It's clear l is bounded, and

$$l(\Delta\varphi) = \lim_{j \rightarrow \infty} (\psi_j, \Delta\varphi) = \lim_{j \rightarrow \infty} (\Delta\psi_j, \varphi) = 0$$

holds for all $\varphi \in C^\infty(M, \Omega_M^k)$, which implies l is a weak solution of $\Delta\psi = 0$. By Theorem B.1.1, there exists a k -form ψ such that $l(\varphi) = (\psi, \varphi)$, where $\psi \in C^\infty(M, \Omega_M^k)$. Consequently, $\psi_j \rightarrow \psi$, and $\psi \in (\mathcal{H}^k)^\perp$ with $\|\psi\| = 1$. However, Theorem B.1.1 implies $\psi \in \mathcal{H}^k$, a contradiction. \square

Holding above results, we can prove Hodge theorem.

Theorem B.1.3 (Hodge theorem). Consider the Laplace operator $\Delta : C^\infty(M, \Omega_M^k) \rightarrow C^\infty(M, \Omega_M^k)$. Then

- (1) $\dim_{\mathbb{R}} \mathcal{H}^k < \infty$.
- (2) There is an orthogonal direct sum decomposition

$$C^\infty(M, \Omega_M^k) = \mathcal{H}^k \oplus \text{im } \Delta$$

Proof. For (1). If \mathcal{H}^k is not finite dimensional. Then there exists an infinite orthonormal sequence. By Theorem B.1.2, this orthonormal sequence contains a Cauchy sequence, which is impossible. Thus \mathcal{H}^k is finite dimensional.

For (2). Note that we naturally have the following orthogonal decomposition

$$C^\infty(M, \Omega_M^k) = (\mathcal{H}^k)^\perp \oplus \mathcal{H}^k$$

The theorem will be proved by showing that $(\mathcal{H}^k)^\perp = \text{im } \Delta$. We use \mathcal{H} to denote the projection from $C^\infty(M, \Omega_M^k)$ to \mathcal{H}^k , that is $\mathcal{H}(\alpha)$ is the harmonic part of α .

It's easy to see $\text{im } \Delta \subseteq (\mathcal{H}^k)^\perp$, since for all $\omega \in C^\infty(M, \Omega_M^k)$ and $\alpha \in \mathcal{H}^k$, we have

$$(\Delta\omega, \alpha) = (\omega, \Delta\alpha) = 0$$

To see converse, for $\alpha \in (\mathcal{H}^k)^\perp$, we define a linear functional l on $\text{im } \Delta$ by setting

$$l(\Delta\varphi) := (\alpha, \varphi)$$

for all $\varphi \in C^\infty(M, \Omega_M^k)$.

- (1) l is well-defined, since if $\Delta\varphi_1 = \Delta\varphi_2$. Then $\varphi_1 - \varphi_2 \in \mathcal{H}^k$. Then $(\alpha, \varphi_1 - \varphi_2) = 0$.
- (2) l is bounded. Indeed, for $\varphi \in C^\infty(M, \Omega_M^k)$, let $\psi = \varphi - \mathcal{H}(\varphi)$. Then

$$\begin{aligned} |l(\Delta\varphi)| &= |l(\Delta\psi)| \\ &= |(\alpha, \psi)| \\ &\leq \|\alpha\| \|\psi\| \\ &\stackrel{*}{\leq} c \|\alpha\| \|\Delta\psi\| \\ &= c \|\alpha\| \|\Delta\varphi\| \end{aligned}$$

where $*$ holds from Corollary B.1.1.

By Hahn-Banach theorem, l extends to a bounded linear functional on $C^\infty(M, \Omega_M^k)$, thus l is a weak solution of $\Delta\omega = \alpha$. By Theorem B.1.1, there exists a k -form ω such that $\Delta\omega = \alpha$. Hence

$$(\mathcal{H}^k)^\perp = \text{im } \Delta$$

This completes the proof of Hodge theorem. □

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