readme

nybo

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Contents

Things i keep forgetting 

1 Norms

def

1. Positivity $\|x\| \geq 0$
2. Positive definiteness $\|x\| = 0 \iff x = 0$
3. Homogeneity $\|\alpha x\| = |\alpha|\|x\|$ for arbitrary scalar α
4. Triangle inequality $\|x + y\| \leq \|x\| + \|y\|$

Note: not sure if this holds for ever norm

The different matrix ones (Norms on $A \in \mathbb{R}^{m\times n}$)

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}|$$

$$\|A\|_2 = \sqrt{\lambda_{\max}\left(A^T A\right)}$$

Frobenius norm, sometimes also called the Euclidean norm

$$\|\mathrm{A}\|_F \equiv \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2}$$

$$\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|$$

Vector norms (Norms on $\mathbb{R}^n$)

$$|\mathbf{x}|_p \equiv \left(\sum_i |x_i|^p\right)^{1/p}$$

special case:

$$|\mathbf{x}|_\infty \equiv \max_i |x_i|$$

Tips: pretty sure think $\|\cdot\|$ usually just refers to the 2-norm.

Scalar norm (Norm on $C[a,b]$)

$$\left.\begin{array}{ll} \|f\|_p = \left(\int_a^b |f(\tau)|^p d\tau\right)^{\frac{1}{p}}, \quad p \in [1,\infty] \\ \|f\|_\infty = \sup\limits_{a\le t\le b} |f(t)| \end{array}, \right\} \mathscr{L}_p - \text{ norms} \tag{1}$$

$$C[0,\infty), \mathscr{L}_p \text{ is a Banach space}$$

$f \in \mathscr{L}_p \Leftrightarrow \|f\|_p$ is bounded, i.e. $\exists c : \|f\|_p \le c$

2 Tensor rank

rank	object
0	scalar
1	vector
2	matrix (/Dyad)
≥ 3	tensor

Also sometimes triad, tetrad are used to refer to tensors of rank 3 and 4 respectivly. Some refer to the rank of a tensor as its order or its degree.

3 SVD

Example: $A = \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

The SVD is defined as

$$A = P\Sigma Q^T$$

Method

Computing:

$$AA^T = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\begin{aligned} -\lambda^3 + 10\lambda^2 - 16\lambda &= -\lambda\left(\lambda^2 - 10\lambda + 16\right) \\ &= -\lambda(\lambda - 8)(\lambda - 2) \end{aligned} \tag{2}$$

Eigenvals of AA^T are $\lambda = 8, \lambda = 2, \lambda = 0$, thus the singular values are $\sigma_1 = 2\sqrt{2}, \sigma_2 = \sqrt{2}$(and $\sigma_3 = 0$).

Giving out the matrix

$$\Sigma = 0_{3x3} + \sigma = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Finding the eigenvectors $(A - \lambda I)\mathbf{x} = \mathbf{0}$, we get respetivly to the egienvector who is described before: $p_1 = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$, $p_2 = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ and $p_3 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$ (note: normalised vectors).

Yeilding

$$P = \left[{p_1}^T {p_2}^T {p_3}^T\right] = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

With the eigenvals $\lambda = 8, \lambda = 2, \lambda = 0$ with eigenvectors $q_1 = \left(\frac{1}{\sqrt{6}}, \frac{3}{\sqrt{12}}, \frac{1}{\sqrt{12}}\right), q_2 = \left(\frac{1}{\sqrt{3}}, 0, -\frac{2}{\sqrt{6}}\right)$ and $q_3 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2}\right)$. (Acually can also use the the formula $p_i = \frac{1}{\sigma_i} A^T p_i$ to get the various q_i.

$$Q = \left[{q_1}^T {q_2}^T {q_3}^T\right] \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{12}} & 0 & -\frac{1}{2} \\ \frac{1}{\sqrt{12}} & -\frac{2}{\sqrt{6}} & \frac{1}{2} \end{bmatrix}$$

We have then the SVD defined as

$$A = P\Sigma Q^T$$

4 Covar

5 change of basis

eigenvalue decomposition

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$$

$$\lambda = 5 \quad \text{NUL}(A - 5I) = \text{ SPAN } \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$$\lambda = 3 \quad \text{NUL}(A - 3I) = \text{ SPAN } \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} P = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

OK, good

$$A = PDP^{-1}$$

jordan

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\lambda = 1 \quad \text{NUL}(A - 1I) = \text{ SPAN } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

2 eigenvectors? :-d (for some reason we are calling these vectors the first and third)

$$\begin{bmatrix} \lambda & 1 & ? \\ 0 & \lambda & ? \\ 0 & 0 & ? \end{bmatrix}$$

$$AV_1 = \lambda V_1$$
$$AV_2 = V_1 + \lambda V_2$$

$$AV_2 - \lambda V_2 = V_1$$
$$(A - \lambda I)V_2 = V_1$$

V_1 is given what is V_2? let $V_1 = [1, 0, 0]^T$

$$(A - 1I)V_2 = V_1$$
$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} V_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cancel{+ x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}} \tag{3}$$

Let

$$V_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4}$$

$$V_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \tag{5}$$

$$A \quad \text{has the from} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

$$P = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \tag{7}$$

then

$$A = PJP^{-1} \quad J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

6 linear def

7 positive definite

A square matrix A is positive definite if there is a positive scalar such that

$$x^T Ax \geq \alpha x^T x, \quad \text{for all } x \in \mathbf{R}^n \tag{9}$$

It is positive semidefinite if

$$x^T Ax \geq 0, \quad \text{for all } x \in \mathbb{R}^n \tag{10}$$

We can recognize that a symmetric matrix is positive definite by computing its eigenvalues and verifying that they are all positive, or by performing a Cholesky factorization.

8 rotation things

3D point as 3-vector

$$\mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

3D point using affine homogeneous

$$\begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Inverse of rotation matrix

$$\begin{aligned} \mathrm{X}' &= \mathrm{RX} + \mathrm{t} \\ \mathrm{X}' - \mathrm{t} &= \mathrm{RX} \\ \mathrm{R}^\top (\mathrm{X}' - \mathrm{t}) &= \mathrm{X} \\ \mathrm{R}^\top \mathrm{X}' - \mathrm{R}^\top \mathrm{t} &= \mathrm{X} \end{aligned} \tag{11}$$

This gives in homogenius

$$\begin{bmatrix} \mathrm{R}^\top & -\mathrm{R}^\top \mathbf{t} \\ 0^\top & 1 \end{bmatrix} \begin{bmatrix} \mathrm{X}' \\ 1 \end{bmatrix} = \begin{bmatrix} \mathrm{X} \\ 1 \end{bmatrix}$$

Rodrigues rotation

rotation matrix in vector form

$$\begin{aligned} &\theta \leftarrow norm(r) \\ &r \leftarrow r/\theta \\ &R = \cos\theta I + (1-\cos\theta) r r^T + \sin\theta \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} \end{aligned} \tag{12}$$

Inverse transformation can be also done easily, since

$$\sin\theta \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} = \frac{R - R^T}{2} \tag{13}$$

A rotation vector is a convenient and most compact representation of a rotation matrix (since any rotation matrix has just 3 degrees of freedom)