

# readme

nybo

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Things i keep forgetting

## 1 Norms

def

1. Positivity  $\|x\| \geq 0$
2. Positive definiteness  $\|x\| = 0 \iff x = 0$
3. Homogeneity  $\|\alpha x\| = |\alpha| \|x\|$  for arbitrary scalar  $\alpha$
4. Triangle inequality  $\|x + y\| \leq \|x\| + \|y\|$

Note: not sure if this holds for ever norm

**The different matrix ones (Norms on  $A \in \mathbb{R}^{m \times n}$ )**

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}|$$

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$$

Frobenius norm, sometimes also called the Euclidean norm

$$\|A\|_F \equiv \sqrt{\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2}$$

$$\|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$$

**Vector norms (Norms on  $\mathbb{R}^n$ )**

$$|\mathbf{x}|_p \equiv \left( \sum_i |x_i|^p \right)^{1/p}$$

special case:

$$|\mathbf{x}|_\infty \equiv \max_i |x_i|$$

Tips: pretty sure think  $\|\cdot\|$  usually just refers to the 2-norm.

**Scalar norm (Norm on  $C[a, b]$ )**

$$\left. \begin{aligned} \|f\|_p &= \left( \int_a^b |f(\tau)|^p d\tau \right)^{\frac{1}{p}}, \quad p \in [1, \infty] \\ \|f\|_\infty &= \sup_{a \leq t \leq b} |f(t)| \end{aligned} \right\} \quad \mathcal{L}_p - \text{ norms} \quad (1)$$

$C[0, \infty), \mathcal{L}_p$  is a Banach space

$f \in \mathcal{L}_p \Leftrightarrow \|f\|_p$  is bounded, i.e.  $\exists c : \|f\|_p \leq c$

## 2 Tensor rank

rank	object
0	scalar
1	vector
2	matrix (/Dyad)
$\geq 3$	tensor

Also sometimes triad, tetrad are used to refer to tensors of rank 3 and 4 respectively. Some refer to the rank of a tensor as its order or its degree.

## 3 SVD

Example:  $A = \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

The SVD is defined as

$$A = P\Sigma Q^T$$

### Method

Computing:

$$\begin{aligned} AA^T &= \begin{bmatrix} 2 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 2 \end{bmatrix} \\ -\lambda^3 + 10\lambda^2 - 16\lambda &= -\lambda(\lambda^2 - 10\lambda + 16) \\ &= -\lambda(\lambda - 8)(\lambda - 2) \end{aligned} \quad (2)$$

Eigenvals of  $AA^T$  are  $\lambda = 8, \lambda = 2, \lambda = 0$ , thus the singular values are  $\sigma_1 = 2\sqrt{2}, \sigma_2 = \sqrt{2}$  (and  $\sigma_3 = 0$ ).

Giving out the matrix

$$\Sigma = 0_{3 \times 3} + \sigma = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Finding the eigenvectors  $(A - \lambda I)\mathbf{x} = \mathbf{0}$ , we get respectively to the eigenvector who is described before:  $p_1 = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ ,  $p_2 = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$  and  $p_3 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$  (note: normalised vectors).

Yeilding

$$P = [p_1^T p_2^T p_3^T] = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

With the eigenvals  $\lambda = 8, \lambda = 2, \lambda = 0$  with eigenvectors  $q_1 = \left(\frac{1}{\sqrt{6}}, \frac{3}{\sqrt{12}}, \frac{1}{\sqrt{12}}\right)$ ,  $q_2 = \left(\frac{1}{\sqrt{3}}, 0, -\frac{2}{\sqrt{6}}\right)$  and  $q_3 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2}\right)$ . (Acually can also use the the formula  $p_i = \frac{1}{\sigma_i} A^T p_i$  to get the various  $q_i$ ).

$$Q = [q_1^T q_2^T q_3^T] = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{12}} & 0 & -\frac{1}{2} \\ \frac{1}{\sqrt{12}} & -\frac{2}{\sqrt{6}} & \frac{1}{2} \end{bmatrix}$$

We have then the SVD defined as

$$A = P \Sigma Q^T$$

## 4 Covar

## 5 diagonalize / change of basis?

diagonalize

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$$

$$\lambda = 5 \quad \text{NUL}(A - 5I) = \text{SPAN} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$$\lambda = 3 \quad \text{NUL}(A - 3I) = \text{SPAN} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

OK, good

$$A = PDP^{-1}$$

jordan

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\lambda = 1 \quad \text{NUL}(A - 1I) = \text{SPAN} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

2 eigenvectors? :-d

$$\begin{bmatrix} \lambda & 1 & ? \\ 0 & \lambda & ? \\ 0 & 0 & ? \end{bmatrix}$$

$$AV_1 = \lambda V_1$$

$$AV_2 = V_1 + \lambda V_2$$

$$AV_2 - \lambda V_2 = V_1$$

$$(A - \lambda I)V_2 = V_1$$

$V_1$  is given what is  $V_2$ ? let  $V_1 = [1, 0, 0]^T$

$$(A - 1I)V_2 = V_1$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} V_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad (3)$$

Let

$$V_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (4)$$

$$V_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad (5)$$

$$A \quad \text{has the form} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (6)$$

$$P = [V_1 \quad V_2 \quad V_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad (7)$$

$$A = PJP^{-1} \quad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (8)$$

**linear def**

## 6 positive definite

A square matrix  $A$  is positive definite if there is a positive scalar  $\alpha$  such that

$$x^T Ax \geq \alpha x^T x, \quad \text{for all } x \in \mathbf{R}^n \quad (9)$$

It is positive semidefinite if

$$x^T Ax \geq 0, \quad \text{for all } x \in \mathbf{R}^n \quad (10)$$

We can recognize that a symmetric matrix is positive definite by computing its eigenvalues and verifying that they are all positive, or by performing a Cholesky factorization.