

readme

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Things i keep forgetting

1 Norms

def

1. Positivity $\|x\| \geq 0$
2. Positive definiteness $\|x\| = 0 \iff x = 0$
3. Homogeneity $\|\alpha x\| = |\alpha| \|x\|$ for arbitrary scalar α
4. Triangle inequality $\|x + y\| \leq \|x\| + \|y\|$

Note: not sure if this holds for ever norm

The different matrix ones (Norms on $A \in \mathbb{R}^{m \times n}$)

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}|$$

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$$

Frobenius norm, sometimes also called the Euclidean norm

$$\|A\|_F \equiv \sqrt{\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2}$$

$$\|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$$

Vector norms (Norms on \mathbb{R}^n)

$$|\mathbf{x}|_p \equiv \left(\sum_i |x_i|^p \right)^{1/p}$$

special case:

$$|\mathbf{x}|_\infty \equiv \max_i |x_i|$$

Tips: pretty sure think $\|\cdot\|$ usually just refers to the 2-norm.

Scalar norm (Norm on $C[a, b]$)

$$\left. \begin{aligned} \|f\|_p &= \left(\int_a^b |f(\tau)|^p d\tau \right)^{\frac{1}{p}}, \quad p \in [1, \infty] \\ \|f\|_\infty &= \sup_{a \leq t \leq b} |f(t)| \end{aligned} \right\} \quad \mathcal{L}_p \text{ - norms} \quad (1)$$

$C[0, \infty), \mathcal{L}_p$ is a Banach space

$f \in \mathcal{L}_p \Leftrightarrow \|f\|_p$ is bounded, i.e. $\exists c : \|f\|_p \leq c$

2 Tensor rank

rank	object
0	scalar
1	vector
2	matrix (/Dyad)
≥ 3	tensor

Also sometimes triad, tetrad are used to refer to tensors of rank 3 and 4 respectively. Some refer to the rank of a tensor as its order or its degree.

3 SVD

Example: $A = \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

The SVD is defined as

$$A = P\Sigma Q^T$$

Method

Computing:

$$AA^T = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

$$-\lambda^3 + 10\lambda^2 - 16\lambda = -\lambda(\lambda^2 - 10\lambda + 16)$$

$$= -\lambda(\lambda - 8)(\lambda - 2) \quad (2)$$

Eigenvals of AA^T are $\lambda = 8, \lambda = 2, \lambda = 0$, thus the singular values are $\sigma_1 = 2\sqrt{2}, \sigma_2 = \sqrt{2}$ (and $\sigma_3 = 0$).

Giving out the matrix

$$\Sigma = 0_{3 \times 3} + \sigma = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Finding the eigenvectors $(A - \lambda I)\mathbf{x} = \mathbf{0}$, we get respectively to the eigenvector who is described before: $p_1 = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$, $p_2 = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ and $p_3 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$ (note: normalised vectors).

Yeilding

$$P = [p_1^T p_2^T p_3^T] = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

With the eigenvals $\lambda = 8, \lambda = 2, \lambda = 0$ with eigenvectors $q_1 = \left(\frac{1}{\sqrt{6}}, \frac{3}{\sqrt{12}}, \frac{1}{\sqrt{12}}\right)$, $q_2 = \left(\frac{1}{\sqrt{3}}, 0, -\frac{2}{\sqrt{6}}\right)$ and $q_3 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2}\right)$. (Acually can also use the the formula $p_i = \frac{1}{\sigma_i} A^T p_i$ to get the various q_i).

$$Q = [q_1^T q_2^T q_3^T] = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{12}} & 0 & -\frac{1}{2} \\ \frac{1}{\sqrt{12}} & -\frac{2}{\sqrt{6}} & \frac{1}{2} \end{bmatrix}$$

We have then the SVD defined as

$$A = P\Sigma Q^T$$

4 Covar

5 diagonalize / change of basis?

diagonalize

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$$

$$\lambda = 5 \quad \text{NUL}(A - 5I) = \text{SPAN} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$$\lambda = 3 \quad \text{NUL}(A - 3I) = \text{SPAN} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

OK, good

$$A = PDP^{-1}$$

jordan

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\lambda = 1 \quad \text{NUL}(A - 1I) = \text{SPAN} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

2 eigenvectors? :-d

$$\begin{bmatrix} \lambda & 1 & ? \\ 0 & \lambda & ? \\ 0 & 0 & ? \end{bmatrix}$$

$$AV_1 = \lambda V_1$$

$$AV_2 = V_1 + \lambda V_2$$

$$AV_2 - \lambda V_2 = V_1$$

$$(A - \lambda I)V_2 = V_1$$

V_1 is given what is V_2 ? let $V_1 = [1, 0, 0]^T$

$$(A - 1I)V_2 = V_1$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} V_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad (3)$$

Let

$$V_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (4)$$

$$V_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad (5)$$

$$A \quad \text{has the form} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (6)$$

$$P = [\quad V_1 \quad V_2 \quad V_3 \quad] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad (7)$$

then

$$A = PJP^{-1} \quad J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (8)$$

6 linear def

7 positive definite

A square matrix A is positive definite if there is a positive scalar α such that

$$x^T Ax \geq \alpha x^T x, \quad \text{for all } x \in \mathbf{R}^n \quad (9)$$

It is positive semidefinite if

$$x^T Ax \geq 0, \quad \text{for all } x \in \mathbf{R}^n \quad (10)$$

We can recognize that a symmetric matrix is positive definite by computing its eigenvalues and verifying that they are all positive, or by performing a Cholesky factorization.