Derivation of Intersections from Two Parametric Equations

KrashKart

Suppose we have a parametric equation,

$$
\begin{aligned}
x &= x_0 + v_x t \\
y &= y_0 + v_y t
\end{aligned}
$$

We first convert this into the equivalent linear equation,

$$
\begin{aligned}
\frac{x - x_0}{v_x} &= t \\
\frac{y - y_0}{v_y} &= t \\
\frac{y - y_0}{v_y} &= \frac{x - x_0}{v_x} \\
y - y_0 &= \frac{v_y}{v_x}(x - x_0) \\
y &= \frac{v_y}{v_x}x + \left(y_0 - \frac{v_y}{v_x}x_0\right) \\
y &= mx + c
\end{aligned}
$$

where

$$
\begin{aligned}
m &= \frac{v_y}{v_x} \\
c &= y_0 - \frac{v_y}{v_x}x_0
\end{aligned}
$$

Now, at an intersection between two parametric equations Π_1 and Π_2,

$$\begin{aligned}
y &= y \\
\frac{v_{y,1}}{v_{x,1}}x + \left(y_{0,1} - \frac{v_{y,1}}{v_{x,1}}x_{0,1}\right) &= \frac{v_{y,2}}{v_{x,2}}x + \left(y_{0,2} - \frac{v_{y,2}}{v_{x,2}}x_{0,2}\right) \\
\frac{v_{y,1}}{v_{x,1}}x - \frac{v_{y,2}}{v_{x,2}}x &= y_{0,2} - \frac{v_{y,2}}{v_{x,2}}x_{0,2} - y_{0,1} + \frac{v_{y,1}}{v_{x,1}}x_{0,1} \\
\left(\frac{v_{y,1}}{v_{x,1}} - \frac{v_{y,2}}{v_{x,2}}\right)x &= y_{0,2} - y_{0,1} + \frac{v_{y,1}v_{x,2}x_{0,1} - v_{y,2}v_{x,1}x_{0,2}}{v_{x,1}v_{x,2}} \\
\left(v_{y,1}v_{x,2} - v_{y,2}v_{x,1}\right)x &= v_{x,1}v_{x,2}y_{0,2} - v_{x,1}v_{x,2}y_{0,1} + v_{y,1}v_{x,2}x_{0,1} - v_{y,2}v_{x,1}x_{0,2} \\
\Rightarrow x &= \frac{v_{x,1}v_{x,2}\left(y_{0,2} - y_{0,1}\right) + v_{y,1}v_{x,2}x_{0,1} - v_{y,2}v_{x,1}x_{0,2}}{v_{y,1}v_{x,2} - v_{y,2}v_{x,1}}
\end{aligned}$$

Without loss of generality, we can apply the same steps to y to arrive at:

$$\begin{aligned}
x &= \frac{v_{x,1}v_{x,2}\left(y_{0,2} - y_{0,1}\right) + v_{y,1}v_{x,2}x_{0,1} - v_{y,2}v_{x,1}x_{0,2}}{v_{y,1}v_{x,2} - v_{y,2}v_{x,1}} \\
y &= \frac{v_{y,1}v_{y,2}\left(x_{0,2} - x_{0,1}\right) + v_{x,1}v_{y,2}y_{0,1} - v_{x,2}v_{y,1}y_{0,2}}{v_{x,1}v_{y,2} - v_{x,2}v_{y,1}}
\end{aligned}$$

which are the equations used in the algorithm.