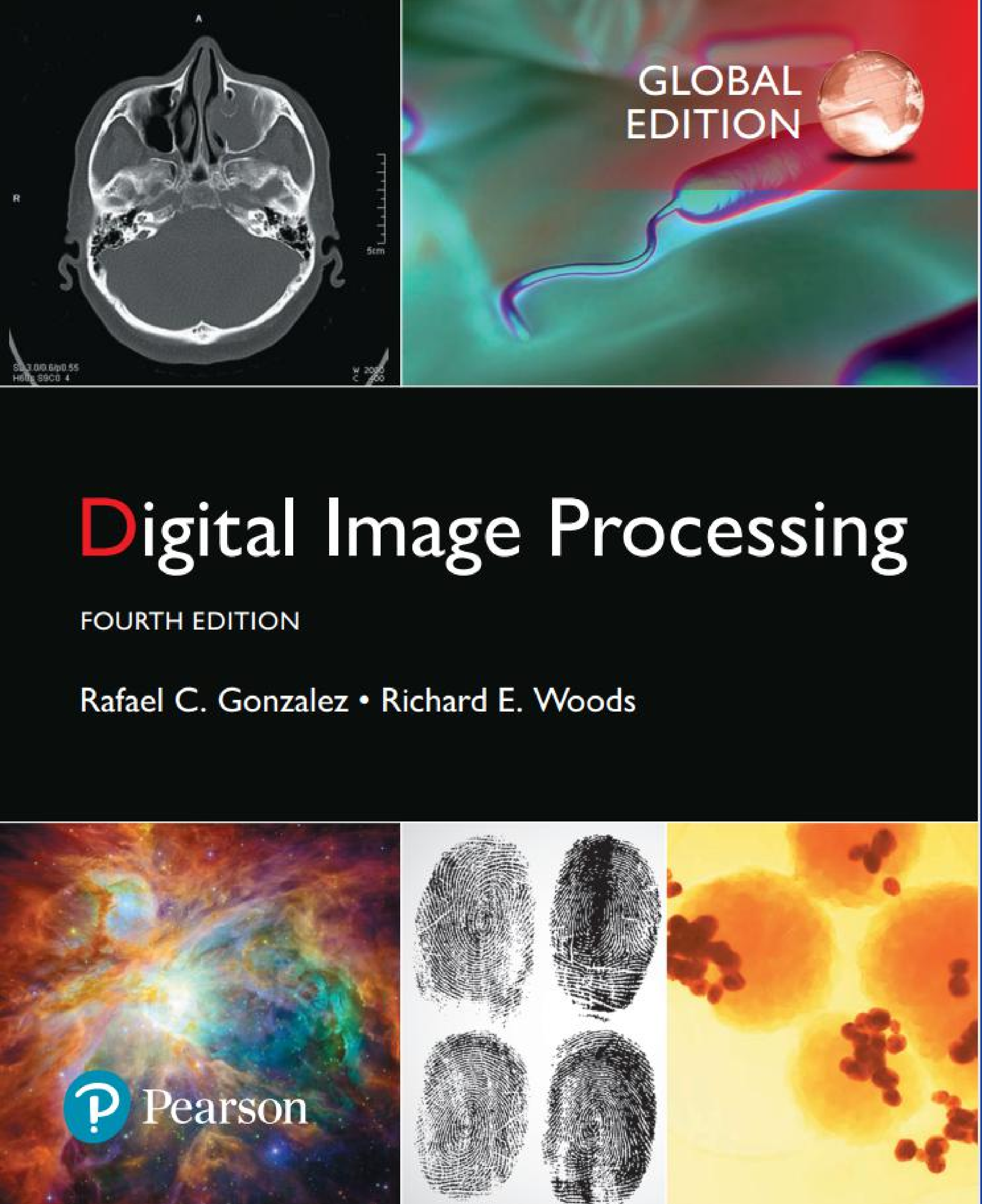

杭州电子科技大学
HANGZHOU DIANZI UNIVERSITY

频域滤波
1D傅里叶变换
Filtering in the Frequency Domain

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学习目标

Upon completion of this chapter, readers should:

- Understand the meaning of frequency domain filtering, and how it differs from filtering in the spatial domain.
- Be familiar with the concepts of sampling, function reconstruction, and aliasing.
- Understand convolution in the frequency domain, and how it is related to filtering.
- Know how to obtain frequency domain filter functions from spatial kernels, and vice versa.
- Be able to construct filter transfer functions directly in the frequency domain.
- Understand why image padding is important.
- Know the steps required to perform filtering in the frequency domain.
- Understand when frequency domain filtering is superior to filtering in the spatial domain.
- Be familiar with other filtering techniques in the frequency domain, such as unsharp masking and homomorphic filtering.
- Understand the origin and mechanics of the fast Fourier transform, and how to use it effectively in image processing.

目 录

傅里叶

- **吉恩·巴普提斯特·约瑟夫·傅里叶**
  - 1768-1830，男爵，法国数学家、物理学家
  - 主要成就：热的传导理论、傅里叶级数
  - 小行星10101号傅里叶星，名字被刻在埃菲尔铁塔

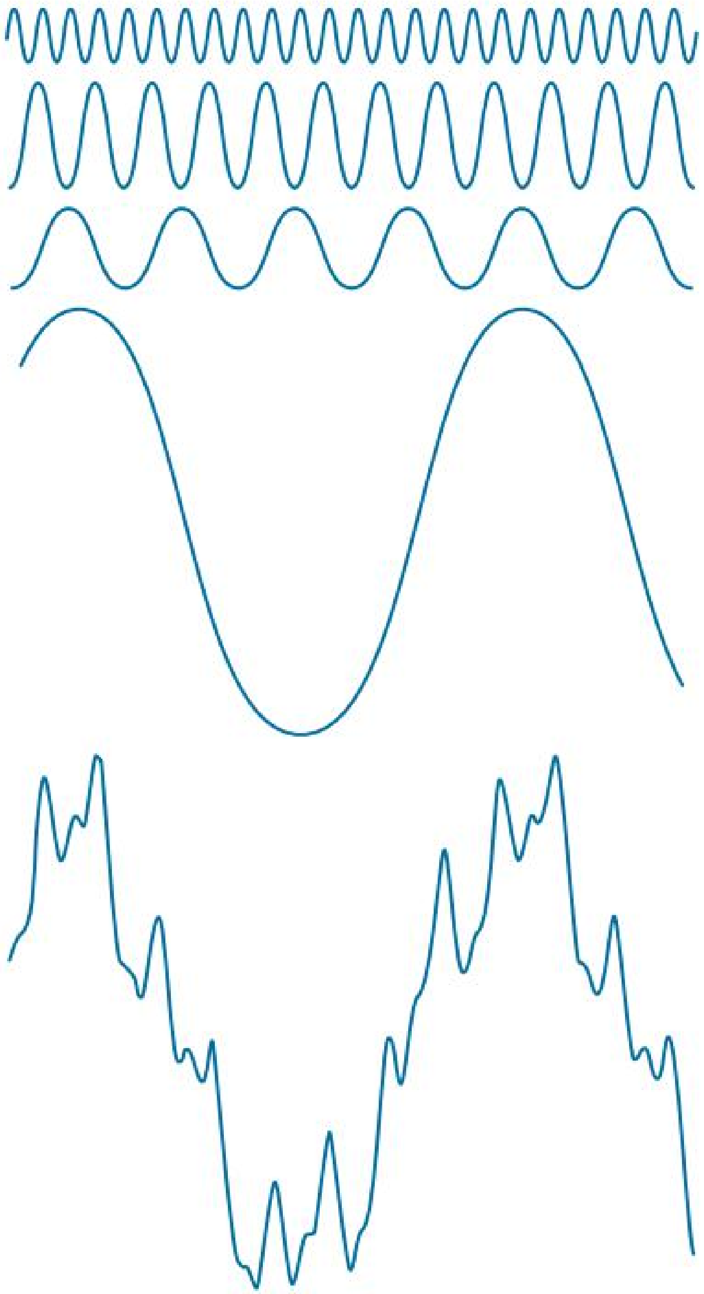
- **傅里叶级数**
  - 任何周期函数都可以表示为不同频率的正弦和/或余弦之和的形式，每一项都乘以不同的系数

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x_n e^{i2\pi k\frac{n}{N}}$$

To find the energy at a particular frequency, spin your signal around a circle at that frequency, and average a bunch of points along that path.

傅里叶变换
Fourier Transform

FIGURE 4.1
The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.



基本概念
PRELIMINARY CONCEPTS

复数 COMPLEX NUMBERS

- **复数**

$$C = R + jI \qquad j = \sqrt{-1}$$

幅度: $|C| = \sqrt{R^2 + I^2}$ $C = |C|(\cos\theta + j\sin\theta)$

相位: $\theta = \arctan\frac{I}{R}$ $C = |C|e^{j\theta}$

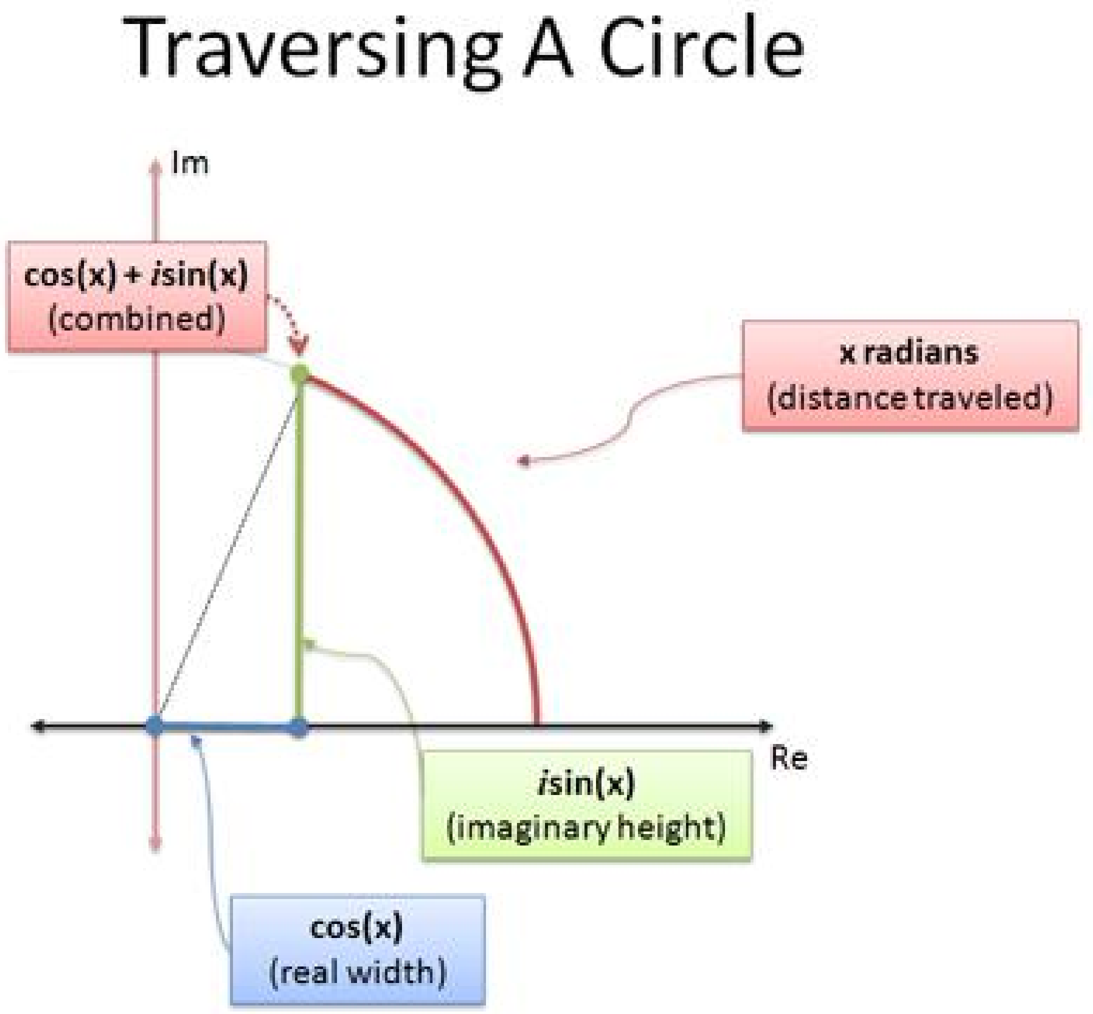
- **欧拉公式 Euler's formula**

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$C = |C|e^{j\theta}$$

  - 假设 θ 是时间 t 的函数，I 会如何?

$$I = |C|\sin\theta$$



复数 COMPLEX NUMBERS

- **正弦波就是一个圆周运动在一条直线上的投影**

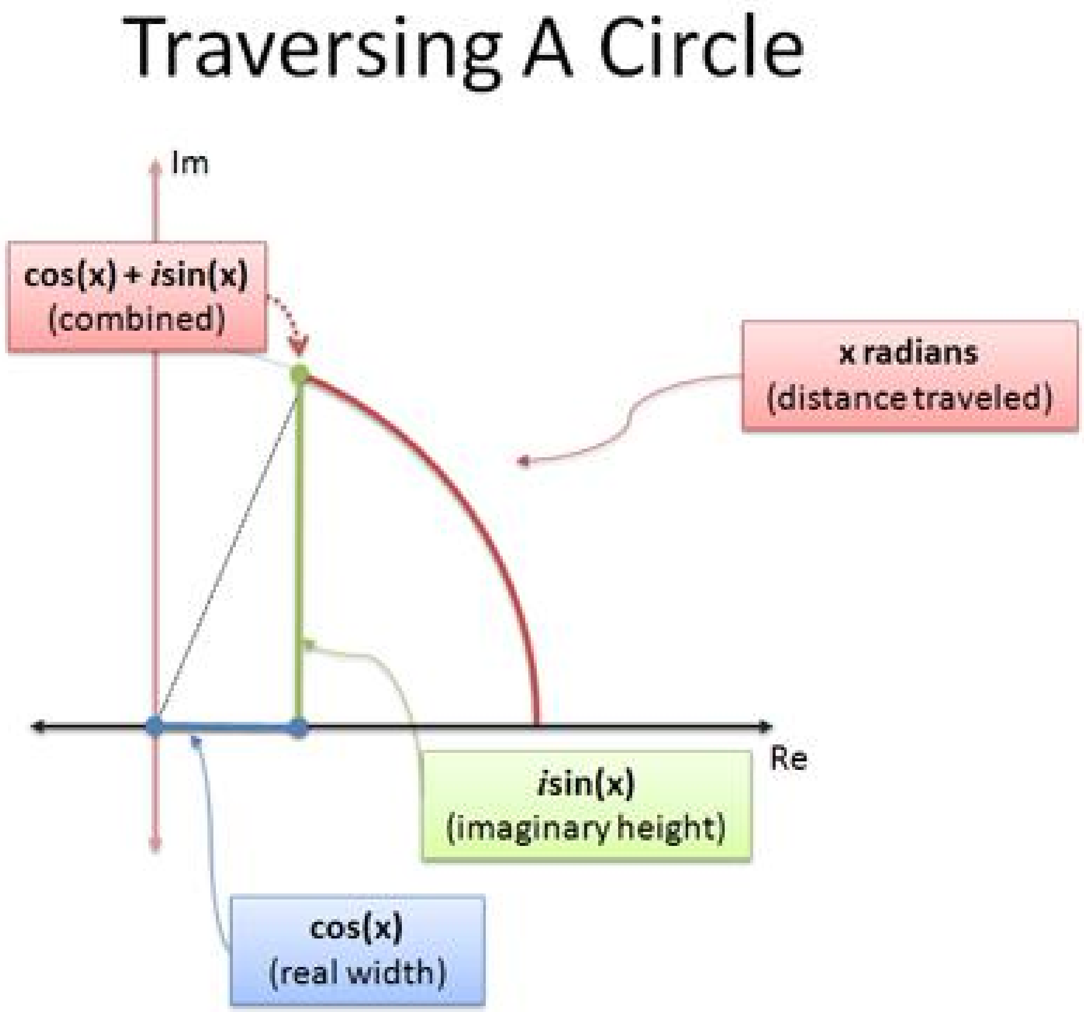
$$I = |C| \sin\theta$$

幅度：$|C| = \sqrt{R^2 + I^2}$

相位：$\theta = \arctan\frac{I}{R}$

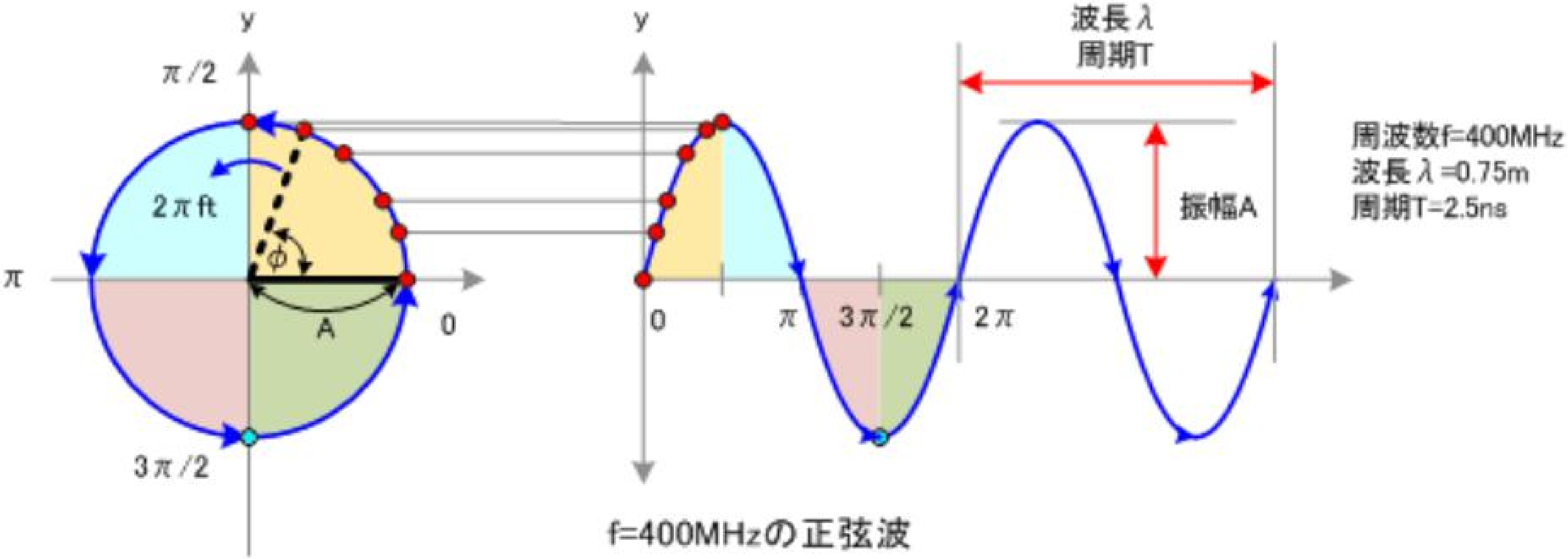
$$C = |C|(\cos\theta + j\sin\theta)$$

$$C = |C| e^{j\theta}$$



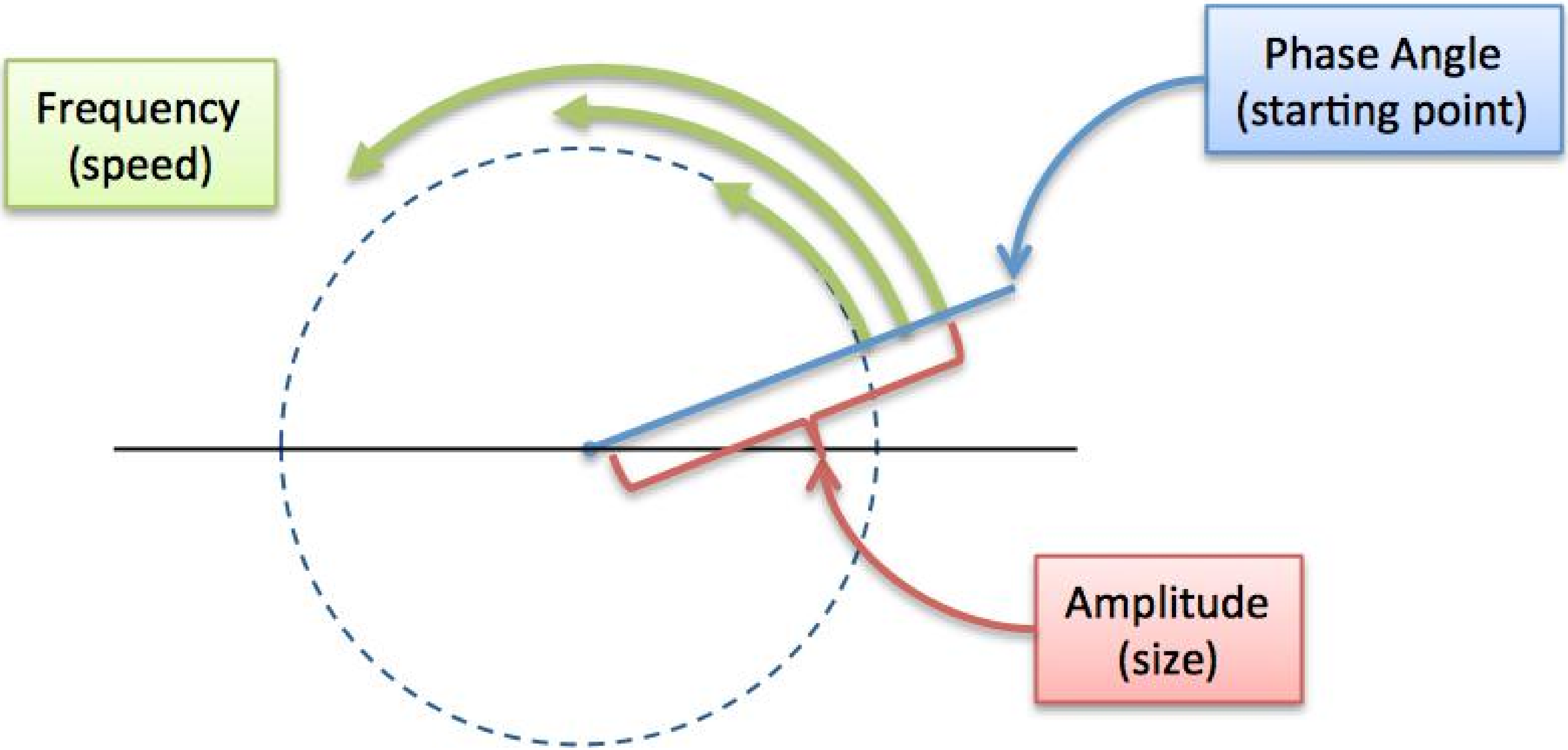
复数 COMPLEX NUMBERS

- **正弦波就是一个圆周运动在一条直线上的投影。**



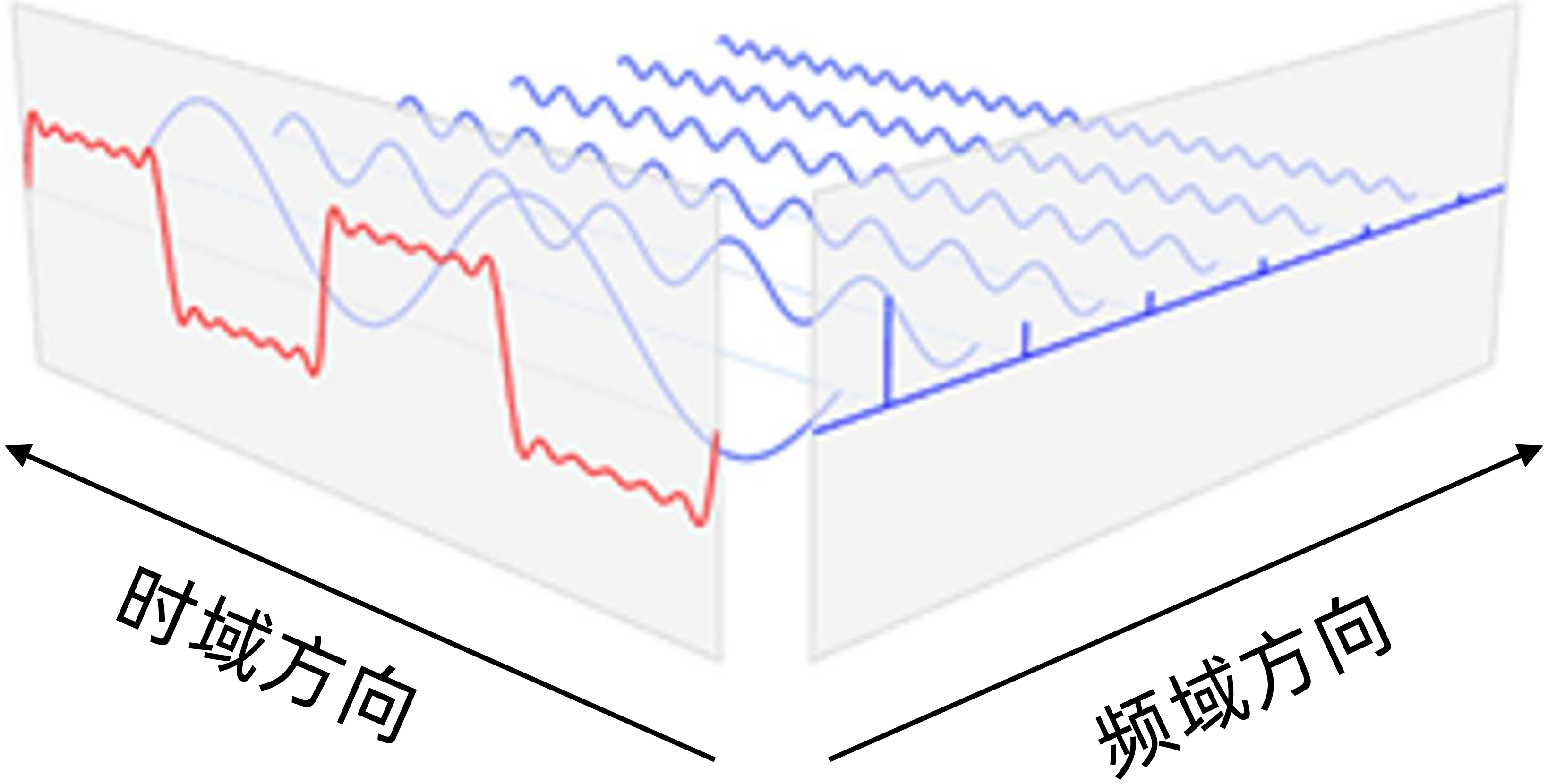
频域

- **频率F、幅度A、相位P**



矩形波的
频域表示

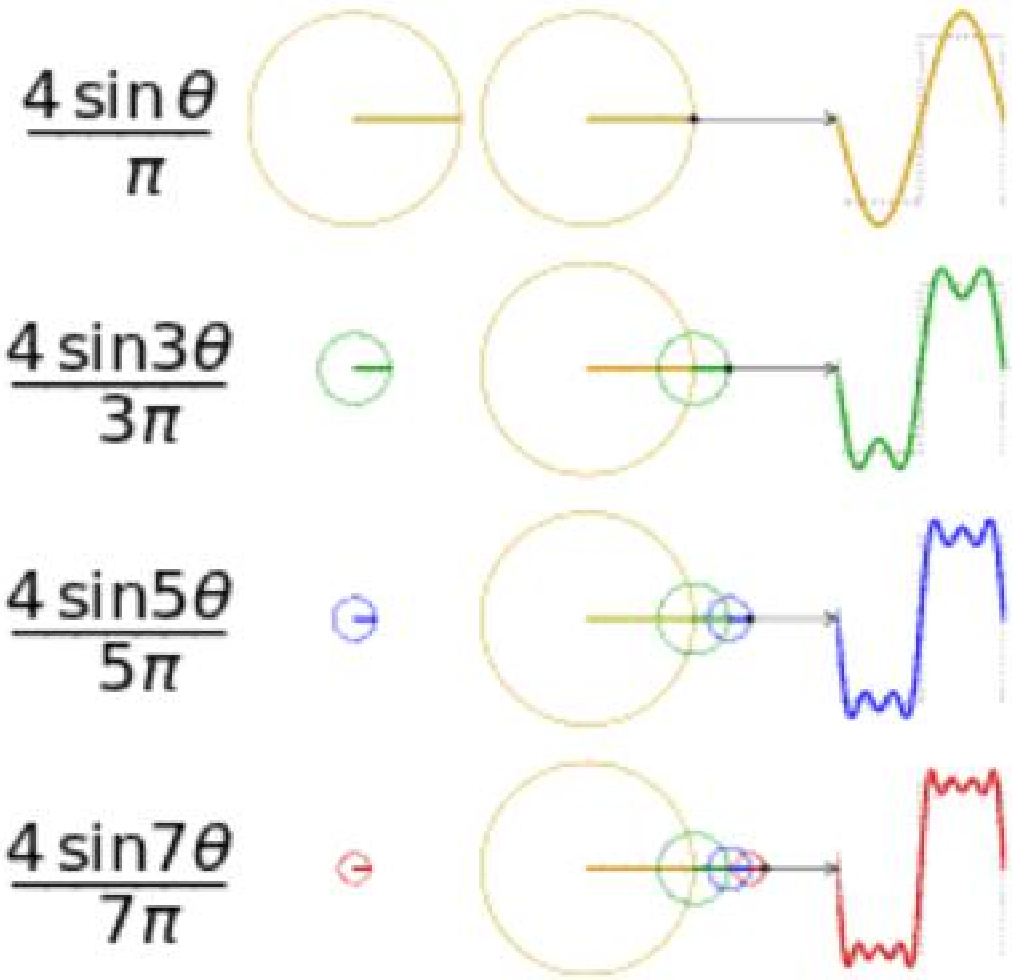
矩形波的频域表示



相位：每个波形开始的时间差

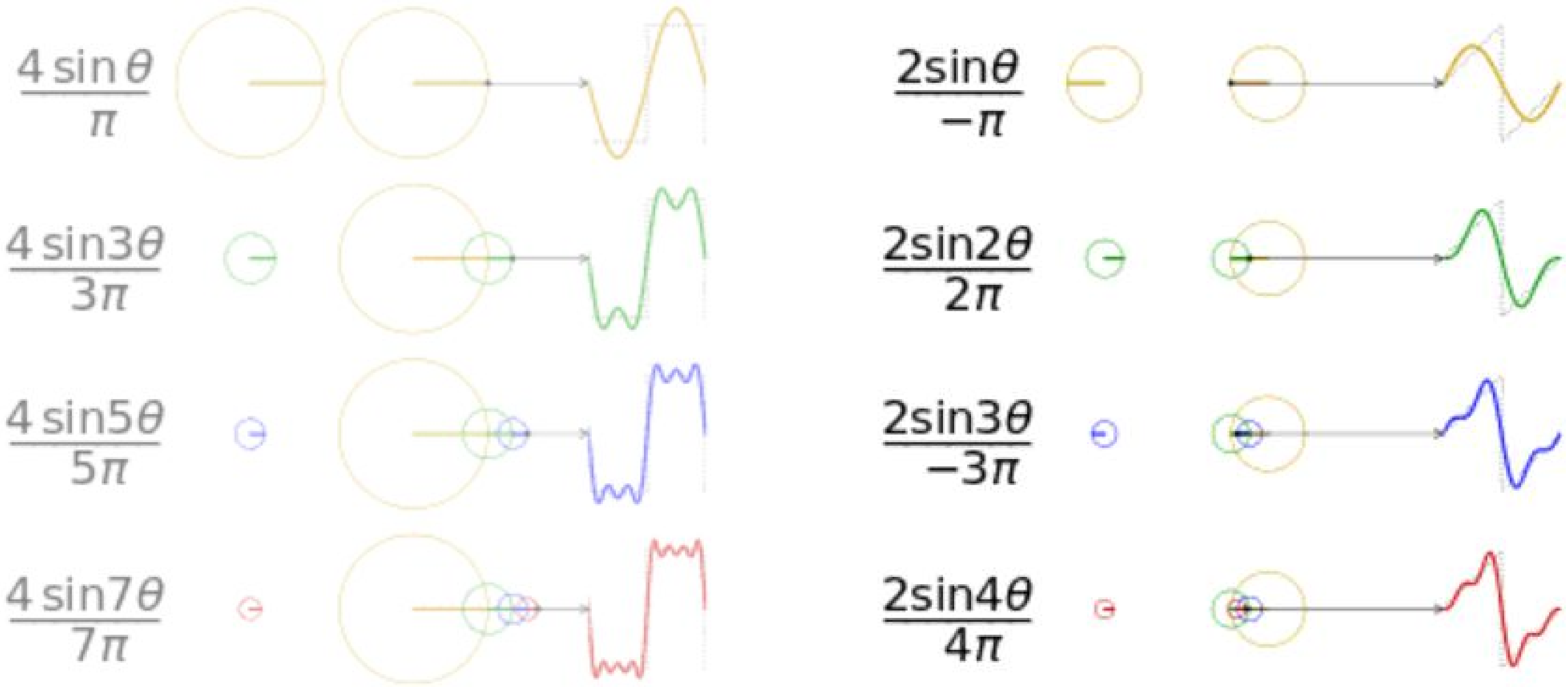
频域

- **频域的基本单元也可以理解为一个始终在旋转的圆**



频域

- 所以频域的基本单元也可以理解为一个始终在旋转的圆

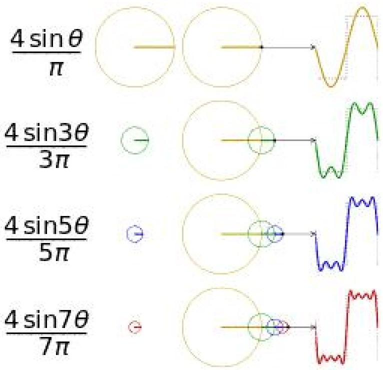


傅里叶序列 Fourier series

FOURIER SERIES

As indicated in the previous section, a function $f(t)$ of a continuous variable, t, that is periodic with a period, T, can be expressed as the sum of sines and cosines multiplied by appropriate coefficients. This sum, known as a *Fourier series*, has the form

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t} \tag{4-8}$$



where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi n}{T}t}\,dt \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots \tag{4-9}$$

are the coefficients. The fact that Eq. (4-8) is an expansion of sines and cosines follows from Euler's formula, Eq. (4-6).

脉冲函数 IMPULSES AND THEIR SIFTING PROPERTIES

- **单位脉冲 unit impulse**

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

$$\boxed{\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\,dt = f(t_0)}$$

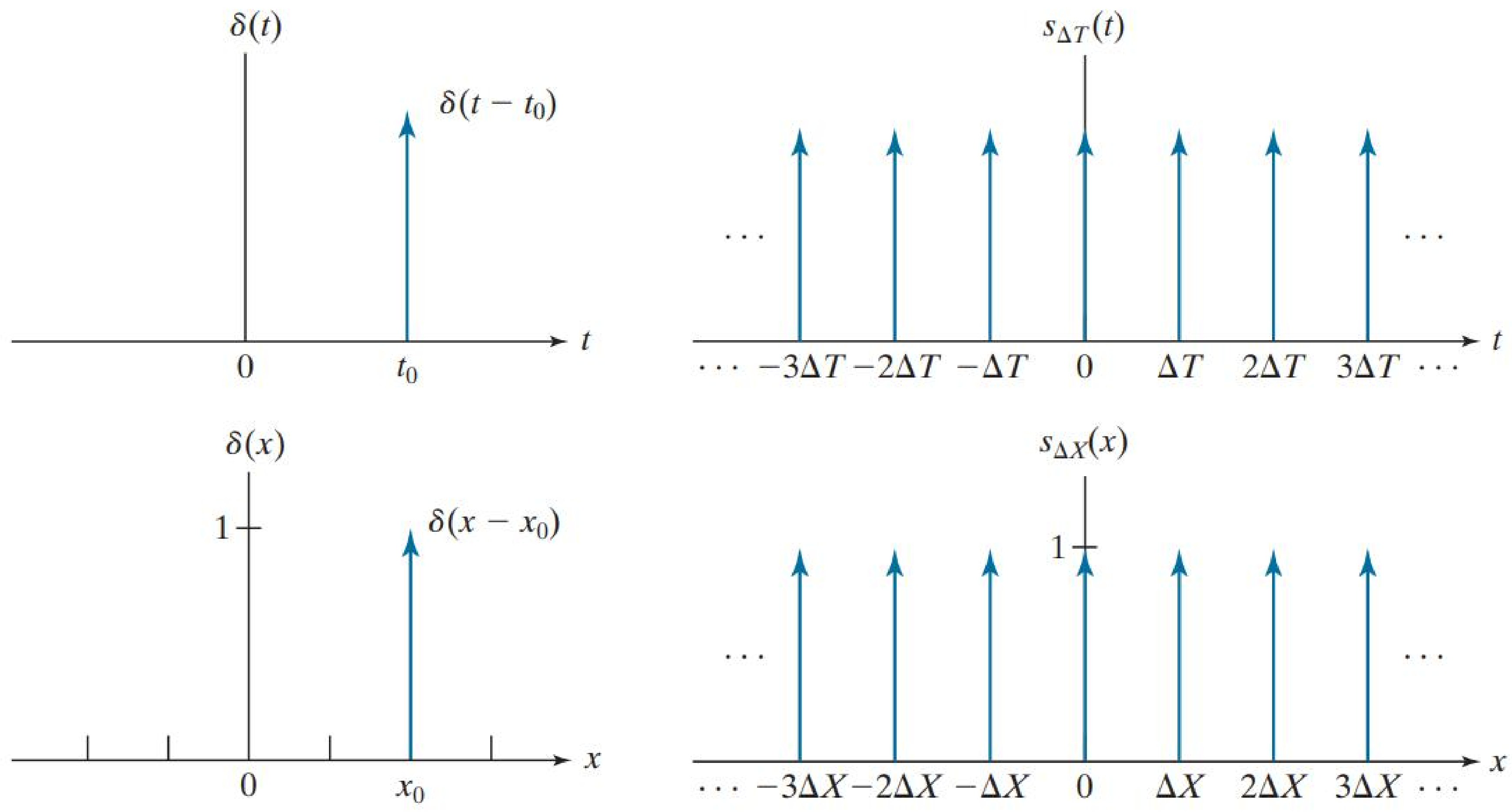


FIGURE 4.3 (a) Continuous impulse located at $t = t_0$. (b) An impulse train consisting of continuous impulses. (c) Unit discrete impulse located at $x = x_0$. (d) An impulse train consisting of discrete unit impulses.

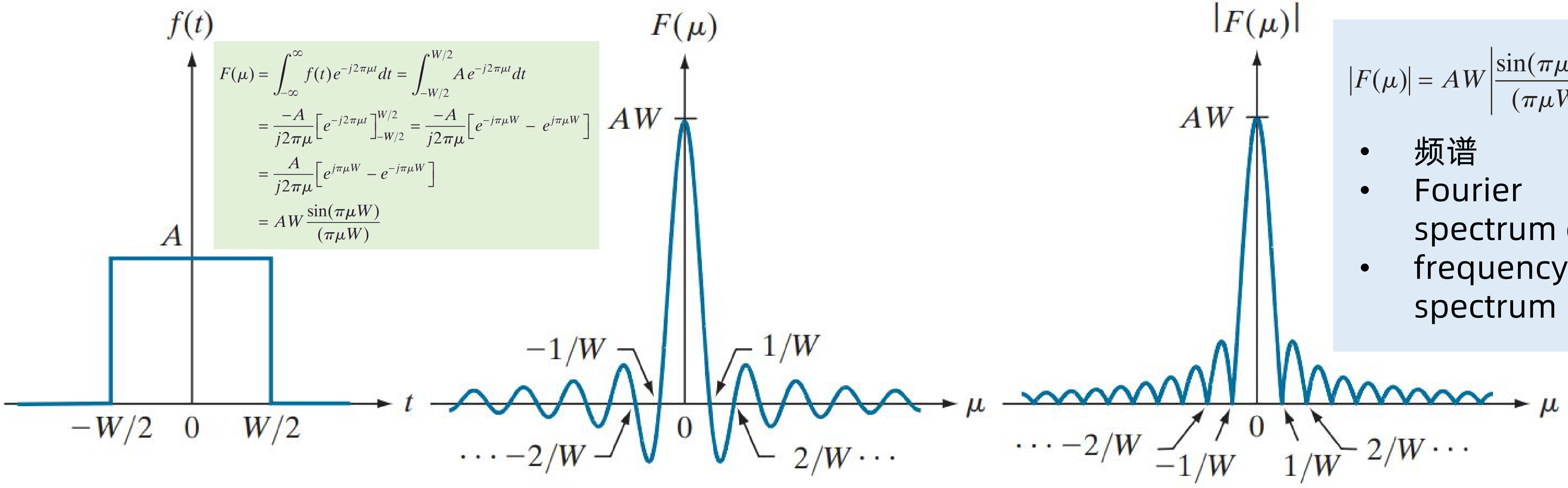
连续变量函数的傅里叶变换

- **傅里叶变换 Fourier transform**

$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t}\, dt$$

- **逆傅里叶变换 inverse Fourier transform**

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t}\, d\mu$$



$$\begin{aligned} F(\mu) &= \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t}\, dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t}\, dt \\ &= \frac{-A}{j2\pi\mu}\Big[e^{-j2\pi\mu t}\Big]_{-W/2}^{W/2} = \frac{-A}{j2\pi\mu}\Big[e^{-j\pi\mu W} - e^{j\pi\mu W}\Big] \\ &= \frac{A}{j2\pi\mu}\Big[e^{j\pi\mu W} - e^{-j\pi\mu W}\Big] \\ &= AW \frac{\sin(\pi\mu W)}{(\pi\mu W)} \end{aligned}$$

$$|F(\mu)| = AW \left| \frac{\sin(\pi\mu W)}{(\pi\mu W)} \right|$$

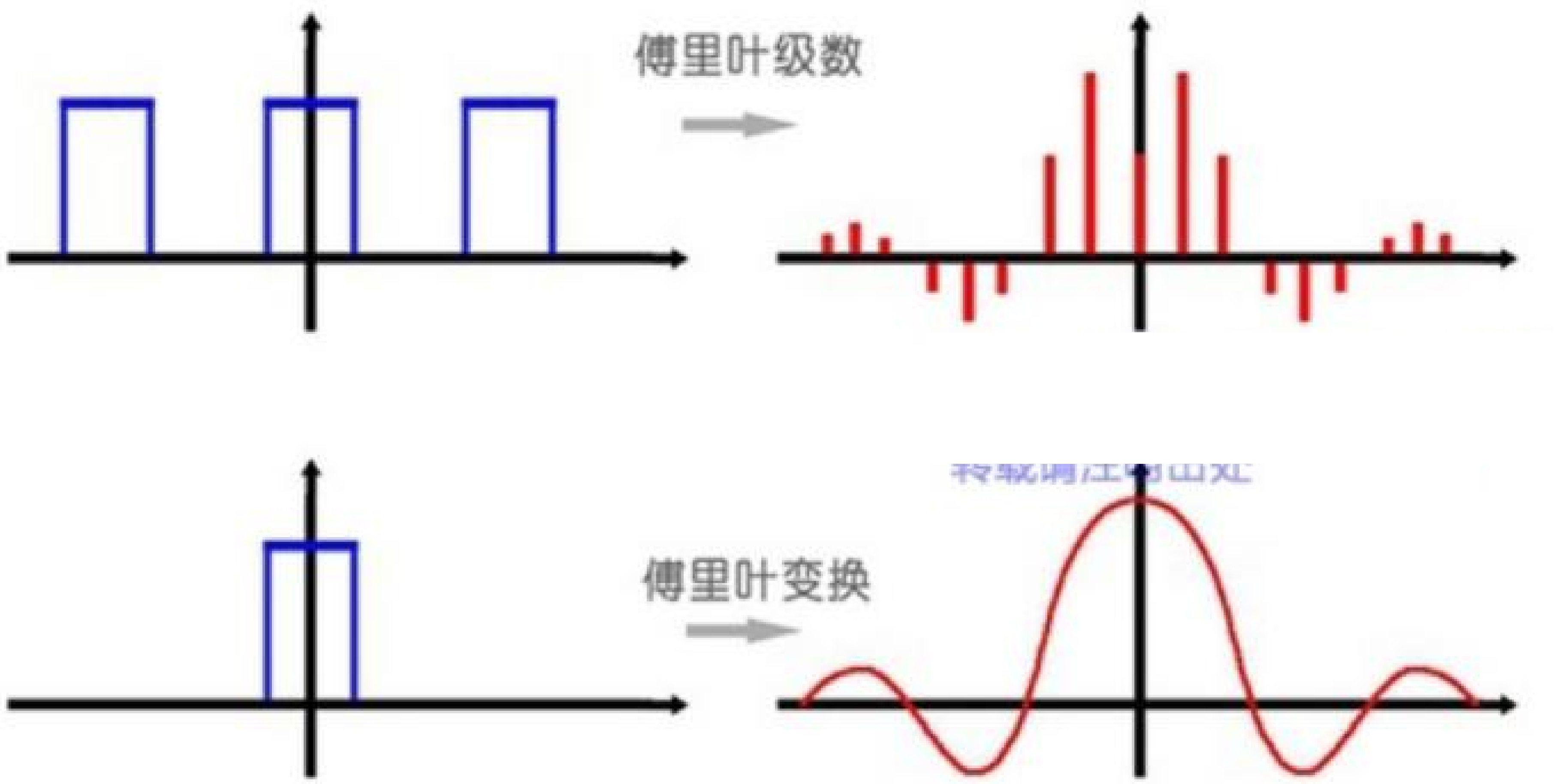
- 频谱
- Fourier spectrum or
- frequency spectrum

a b c

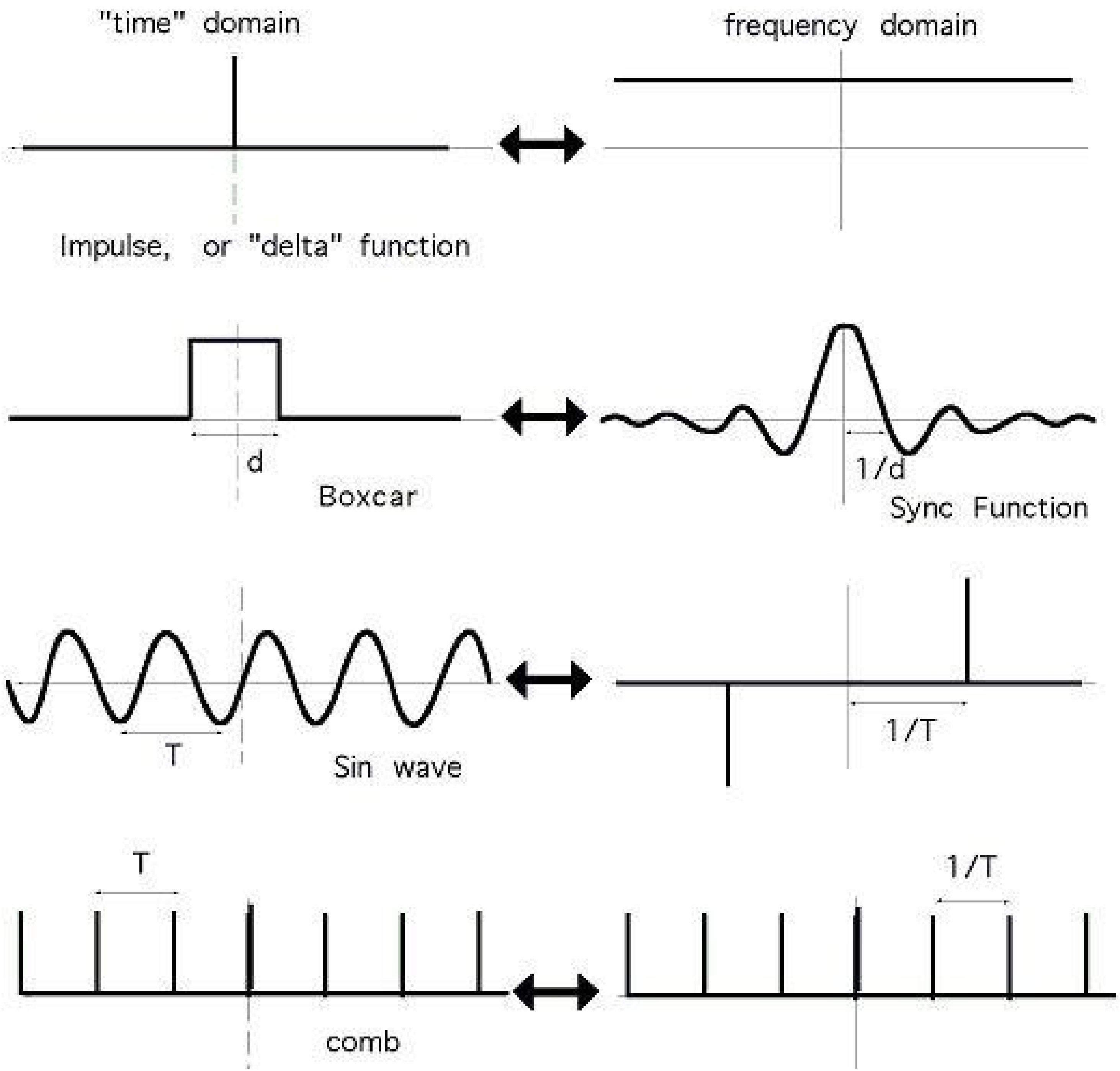
FIGURE 4.4 (a) A box function, (b) its Fourier transform, and (c) its spectrum. All functions extend to infinity in both directions. Note the inverse relationship between the width, W, of the function and the zeros of the transform.

频域

- **连续/离散、周期/非周期**



时域 vs. 频域


"time" domain
frequency domain
Impulse, or "delta" function
d
Boxcar
1/d
Sync Function
T
Sin wave
1/T
T
1/T
comb

傅里叶变换 vs. 卷积 CONVOLUTION

- **卷积**

$$(f \star h)(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau$$

- **其对应的傅里叶变换**

$$\begin{aligned}\Im\{(f \star h)(t)\} &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau \right] e^{-j2\pi\mu t} dt \\ &= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau) e^{-j2\pi\mu t} dt \right] d\tau\end{aligned}$$

傅里叶变换 vs. 卷积 CONVOLUTION

- **卷积**

$$(f \star h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

- **其对应的傅里叶变换**

$$\begin{aligned}\Im\{(f \star h)(t)\} &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau\right] e^{-j2\pi\mu t}dt \\ &= \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi\mu t}dt\right]d\tau\end{aligned}$$

$$\boxed{\Im\{h(t-\tau)\} = H(\mu)e^{-j2\pi\mu\tau}}$$

傅里叶变换 vs. 卷积 CONVOLUTION

- **卷积**

$$(f \star h)(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau$$

- **其对应的傅里叶变换**

$$\Im\{(f \star h)(t)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau \right] e^{-j2\pi\mu t} dt$$

$$= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau) e^{-j2\pi\mu t} dt \right] d\tau$$

$$\boxed{\Im\{h(t-\tau)\} = H(\mu) e^{-j2\pi\mu\tau}}$$

$$\Im\{(f \star h)(t)\} = \int_{-\infty}^{\infty} f(\tau) \left[H(\mu) e^{-j2\pi\mu\tau} \right] d\tau$$

$$= H(\mu) \int_{-\infty}^{\infty} f(\tau) e^{-j2\pi\mu\tau} d\tau$$

$$= H(\mu) F(\mu)$$

$$= (H \bullet F)(\mu)$$

傅里叶变换 vs. 卷积 CONVOLUTION

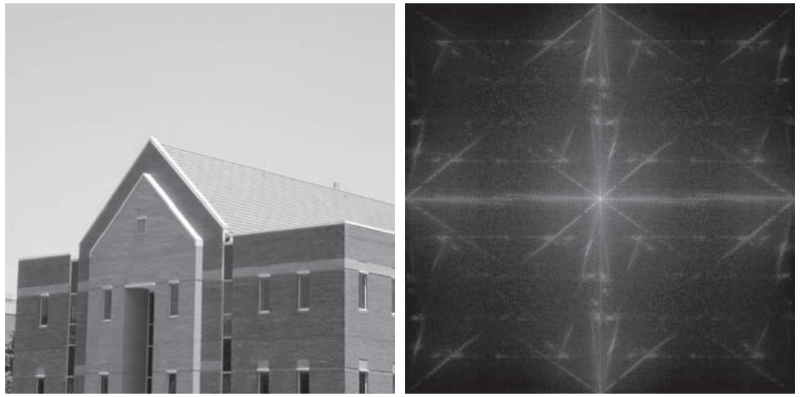
● 卷积定理 convolution theorem

$$(f \star h)(t) \Leftrightarrow (H \bullet F)(\mu)$$

$$(f \bullet h)(t) \Leftrightarrow (H \star F)(\mu)$$

a b

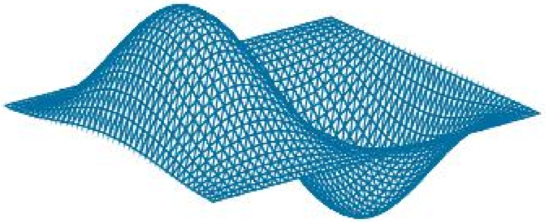
FIGURE 4.37
(a) Image of a building, and (b) its Fourier spectrum.

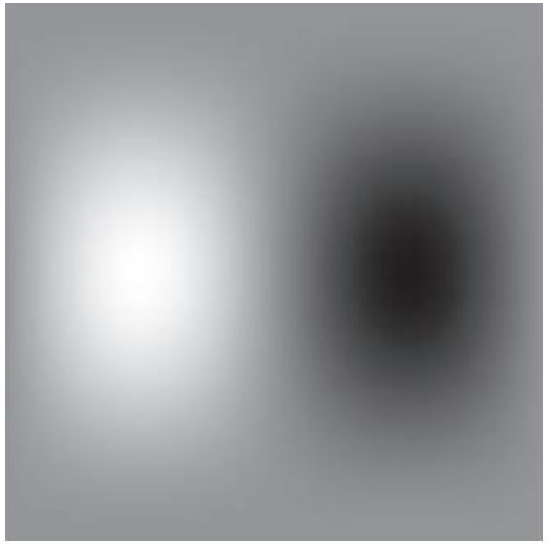


a b
c d

FIGURE 4.38
(a) A spatial kernel and perspective plot of its corresponding frequency domain filter transfer function.
(b) Transfer function shown as an image.
(c) Result of filtering Fig. 4.37(a) in the frequency domain with the transfer function in (b).
(d) Result of filtering the same image in the spatial domain with the kernel in (a). The results are identical.

−1	0	1
−2	0	2
−1	0	1





基本概念

- **尺度定理**

$$f(x,y) \Leftrightarrow F(u,v) \quad \Rightarrow \quad \begin{aligned} & af(x,y) \Leftrightarrow aF(u,v) \\ & f(ax,by) \Leftrightarrow \frac{1}{|ab|}F\left(\frac{u}{a},\frac{v}{b}\right) \end{aligned}$$

- **含义**
  - 幅度的尺度变化趋势一致；
  - 空间尺度变化趋势相反，且引起幅度变化；

$$F(\mu) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}\,dt$$

$$f(t) = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}\,d\mu$$

- **证明？**

取样及取样函数的傅里叶变换
FOURIER TRANSFORM OF SAMPLED FUNCTIONS

取样函数的傅里叶变换

- **冲激及其取样特性**
  - 单位离散冲激

$$\delta(x)=\begin{cases}1, & x=0\\ 0, & x\neq 0\end{cases}$$

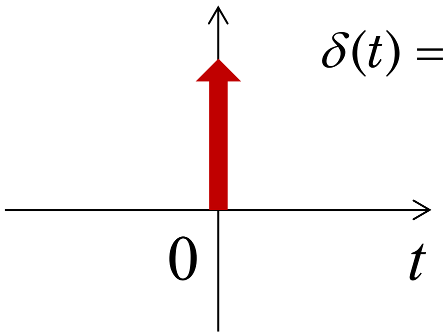
  - 满足

$$\sum_{x=-\infty}^{\infty}\delta(x)=1$$

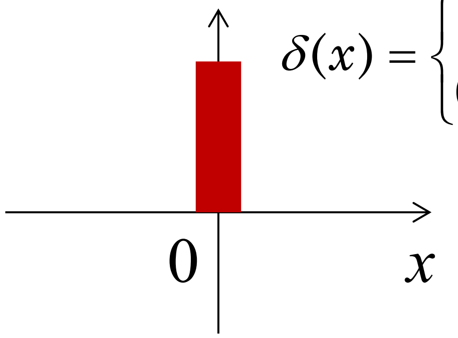
  - 取样特性

$$f(x)\delta(x)=f(0)$$

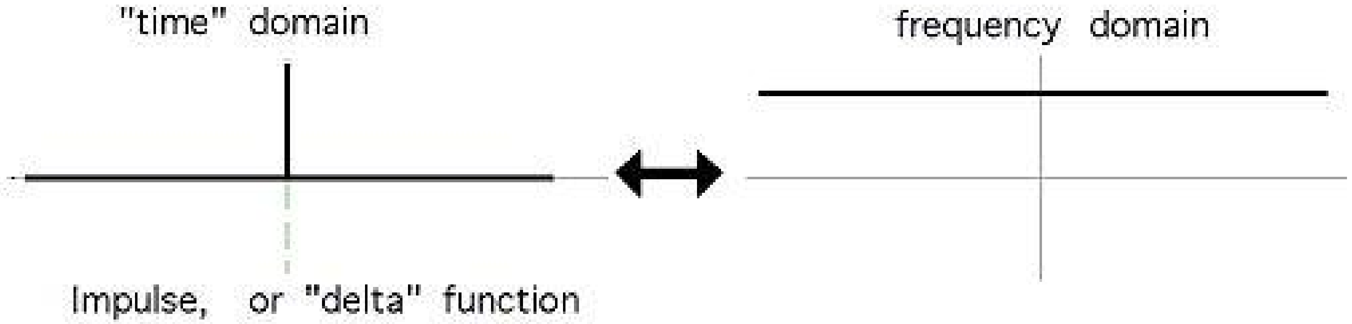
  - 如何取 x_0 处的数值?

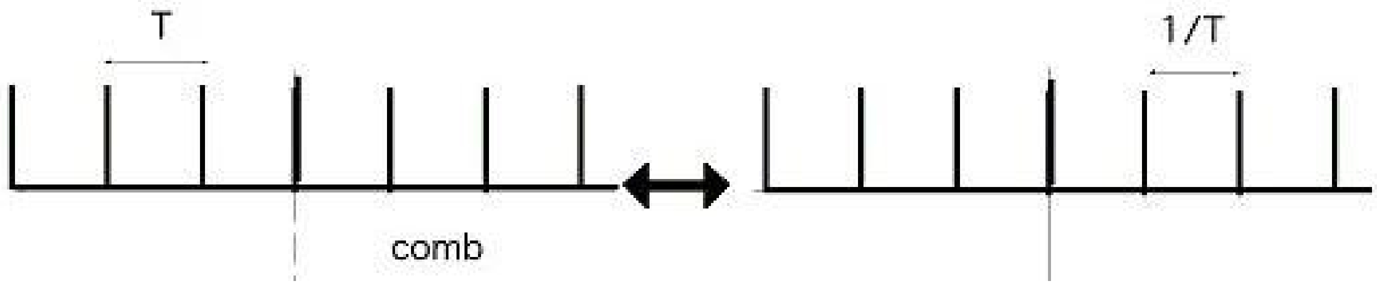


$$\delta(t)=\begin{cases}\infty, & t=0\\ 0, & t\neq 0\end{cases}$$



$$\delta(x)=\begin{cases}1, & x=0\\ 0, & x\neq 0\end{cases}$$





$$f_k = \int_{-\infty}^{\infty} f(t)\delta(t-k\Delta T)\,dt$$
$$= f(k\Delta T)$$

采样
SAMPLING

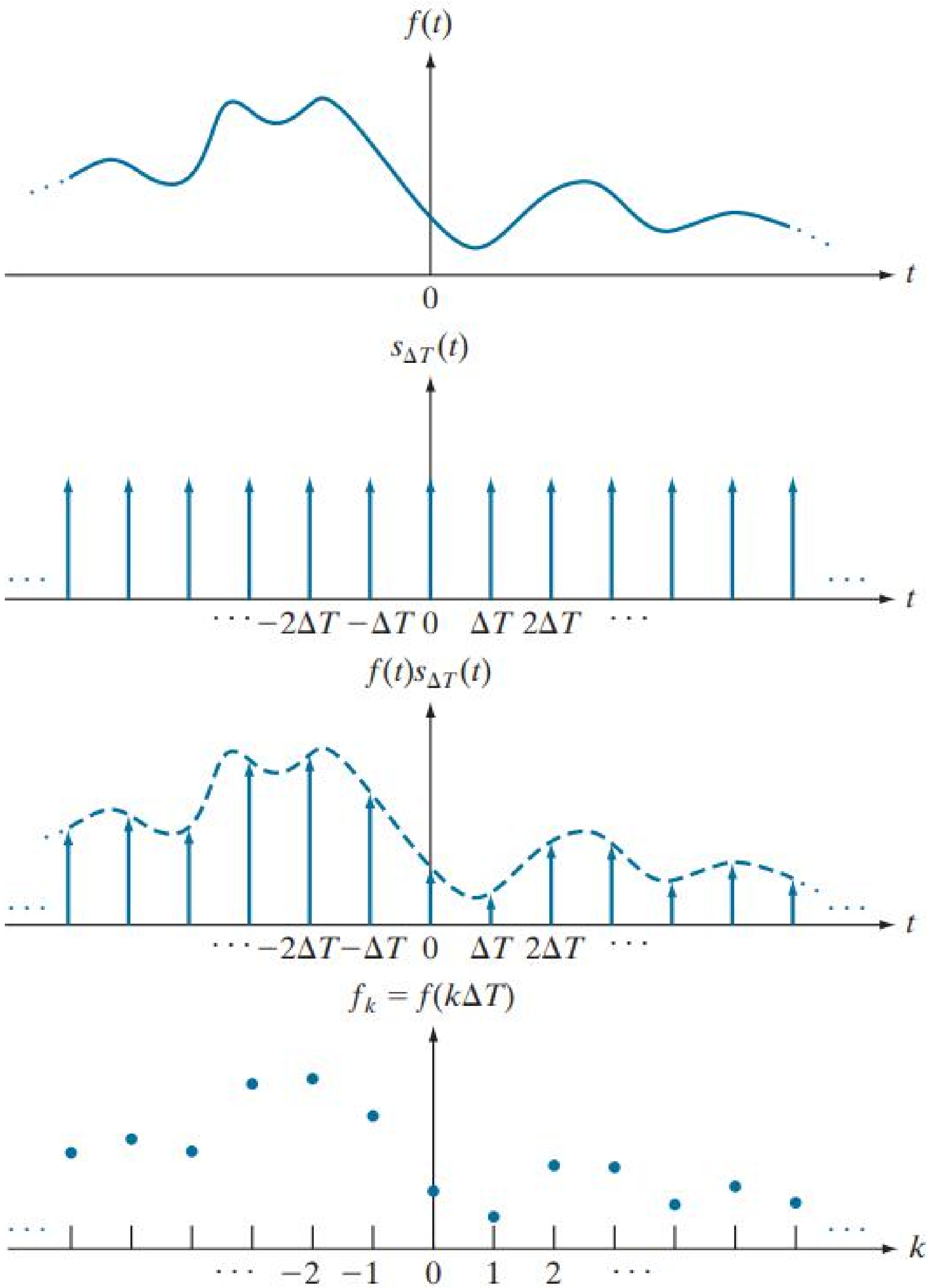
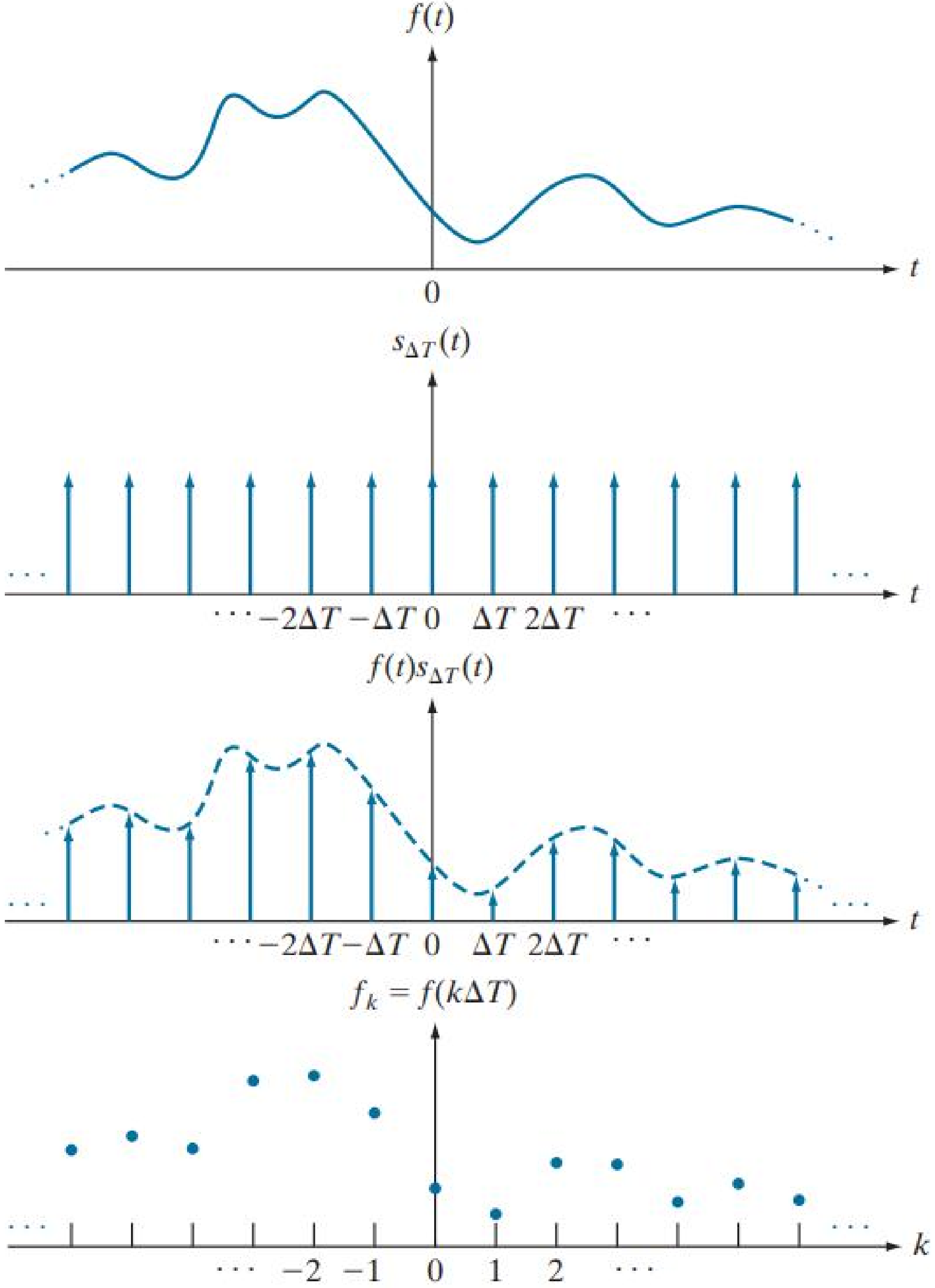
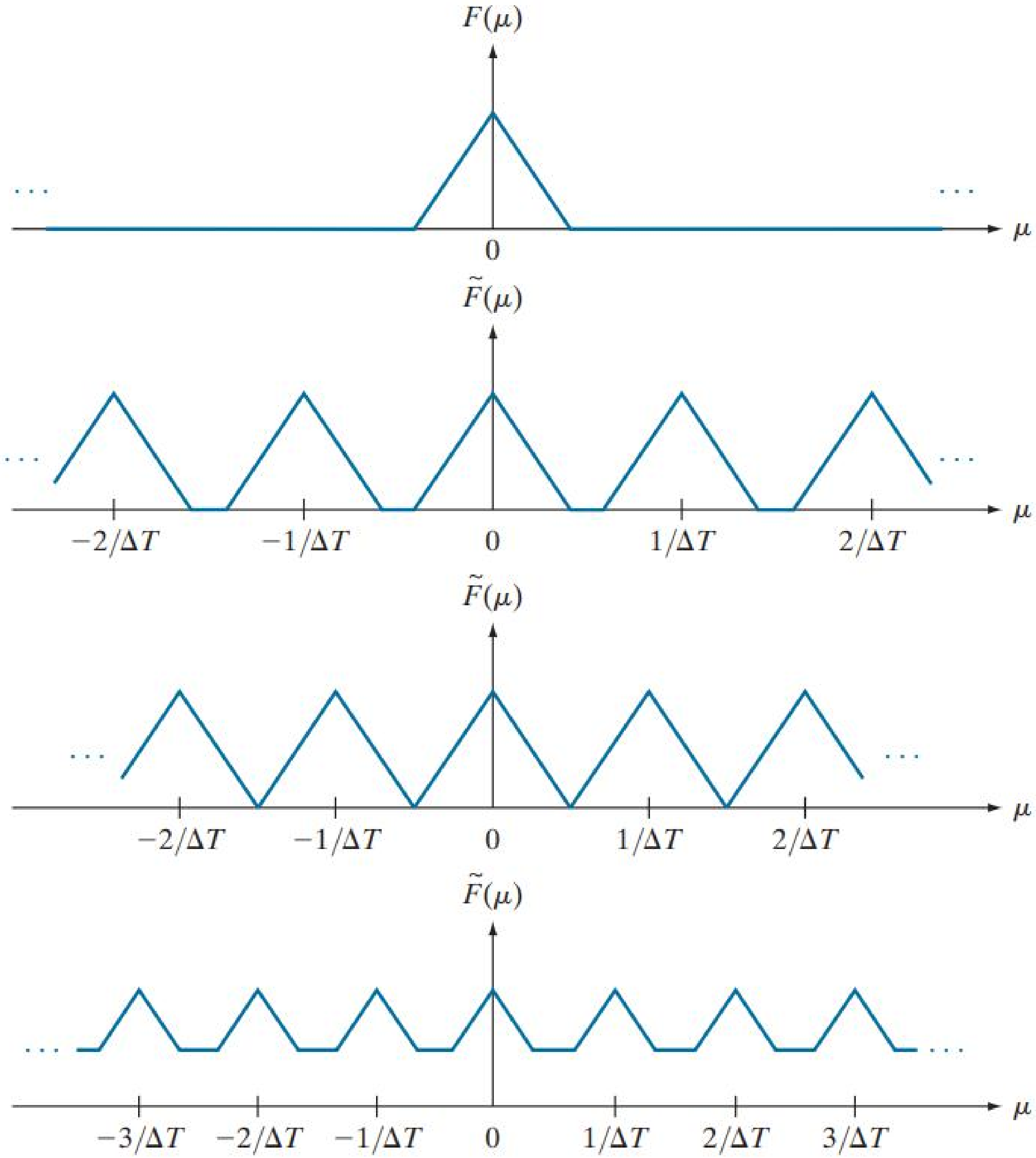


FIGURE 4.5
(a) A continuous function. (b) Train of impulses used to model sampling. (c) Sampled function formed as the product of (a) and (b). (d) Sample values obtained by integration and using the sifting property of impulses. (The dashed line in (c) is shown for reference. It is not part of the data.)



f(t)
t
0
s_ΔT(t)
t
··· −2ΔT −ΔT 0 ΔT 2ΔT ···
f(t)s_ΔT(t)
t
··· −2ΔT −ΔT 0 ΔT 2ΔT ···
f_k = f(kΔT)
k
··· −2 −1 0 1 2 ···

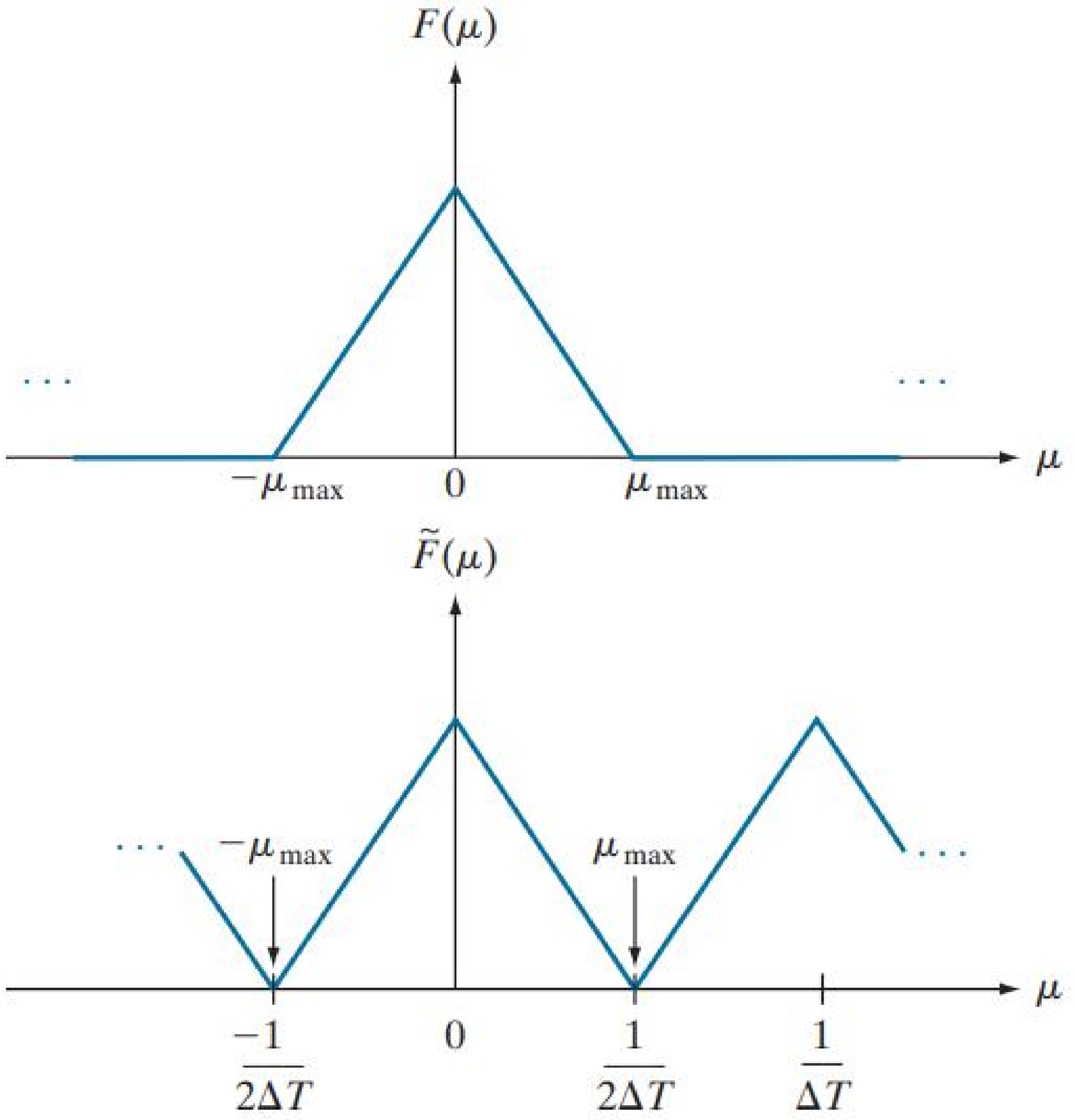


F(μ)
μ
0
F̃(μ)
μ
−2/ΔT −1/ΔT 0 1/ΔT 2/ΔT
F̃(μ)
μ
−2/ΔT −1/ΔT 0 1/ΔT 2/ΔT
F̃(μ)
μ
−3/ΔT −2/ΔT −1/ΔT 0 1/ΔT 2/ΔT 3/ΔT

取样定理 THE SAMPLING THEOREM

- **带宽**

$$[-\mu_{\max},\mu_{\max}]$$



a
b

FIGURE 4.7
(a) Illustrative sketch of the Fourier transform of a band-limited function.
(b) Transform resulting from critically sampling that band-limited function.

- **Q：如何采样才不会重叠？**

取样定理 THE SAMPLING THEOREM

- **带宽**

$$[-\mu_{\max}, \mu_{\max}]$$

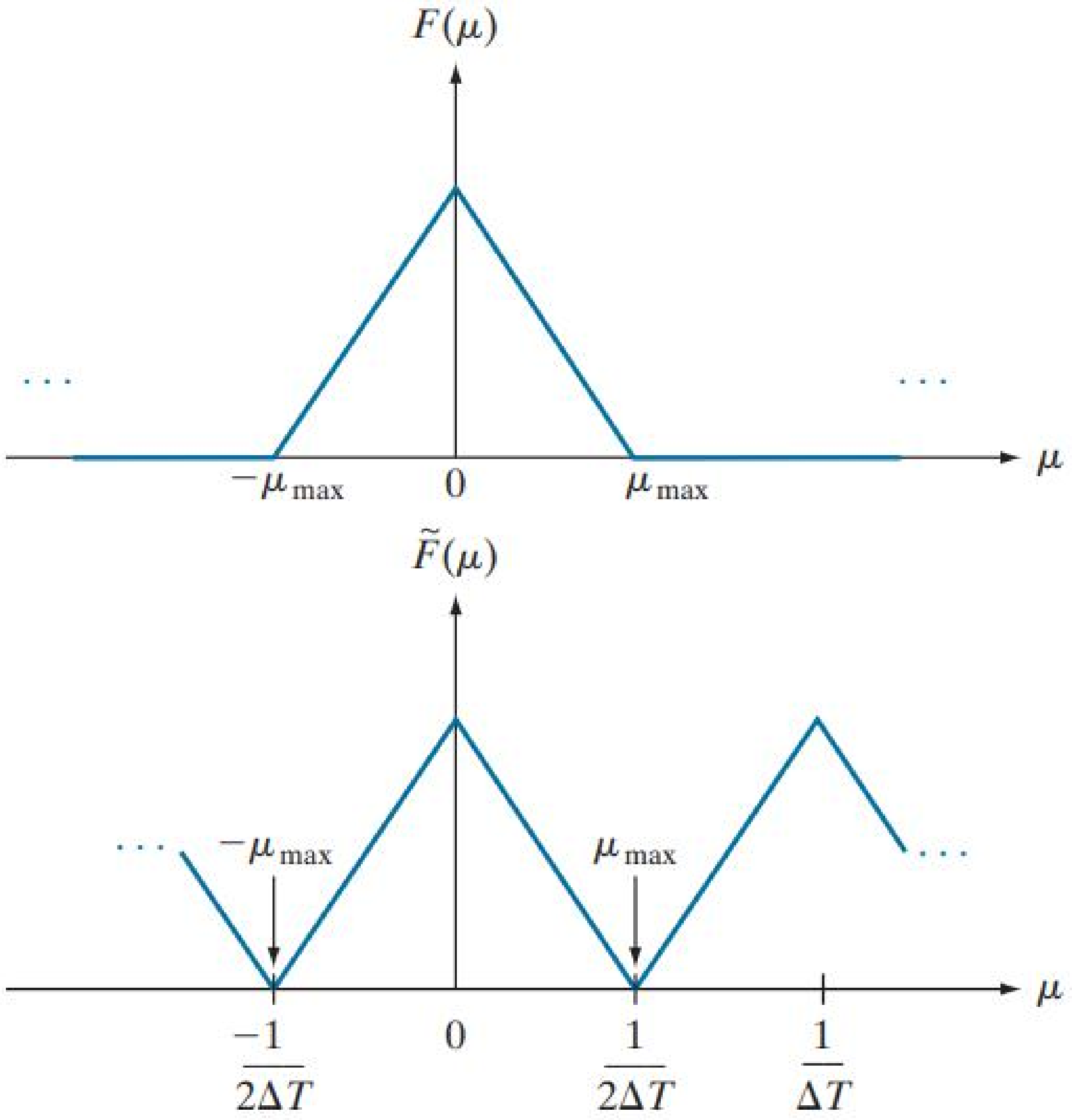
- **奈奎斯特采样率**

$$\mu_{\max} = \frac{1}{2\Delta T}$$

$$\left(\frac{1}{\Delta T} > 2\mu_{\max}\right)$$

a
b

FIGURE 4.7
(a) Illustrative sketch of the Fourier transform of a band-limited function.
(b) Transform resulting from critically sampling that band-limited function.



取样定理 THE SAMPLING THEOREM

- **带宽**

$$[-\mu_{\max}, \mu_{\max}]$$

- **奈奎斯特采样率**

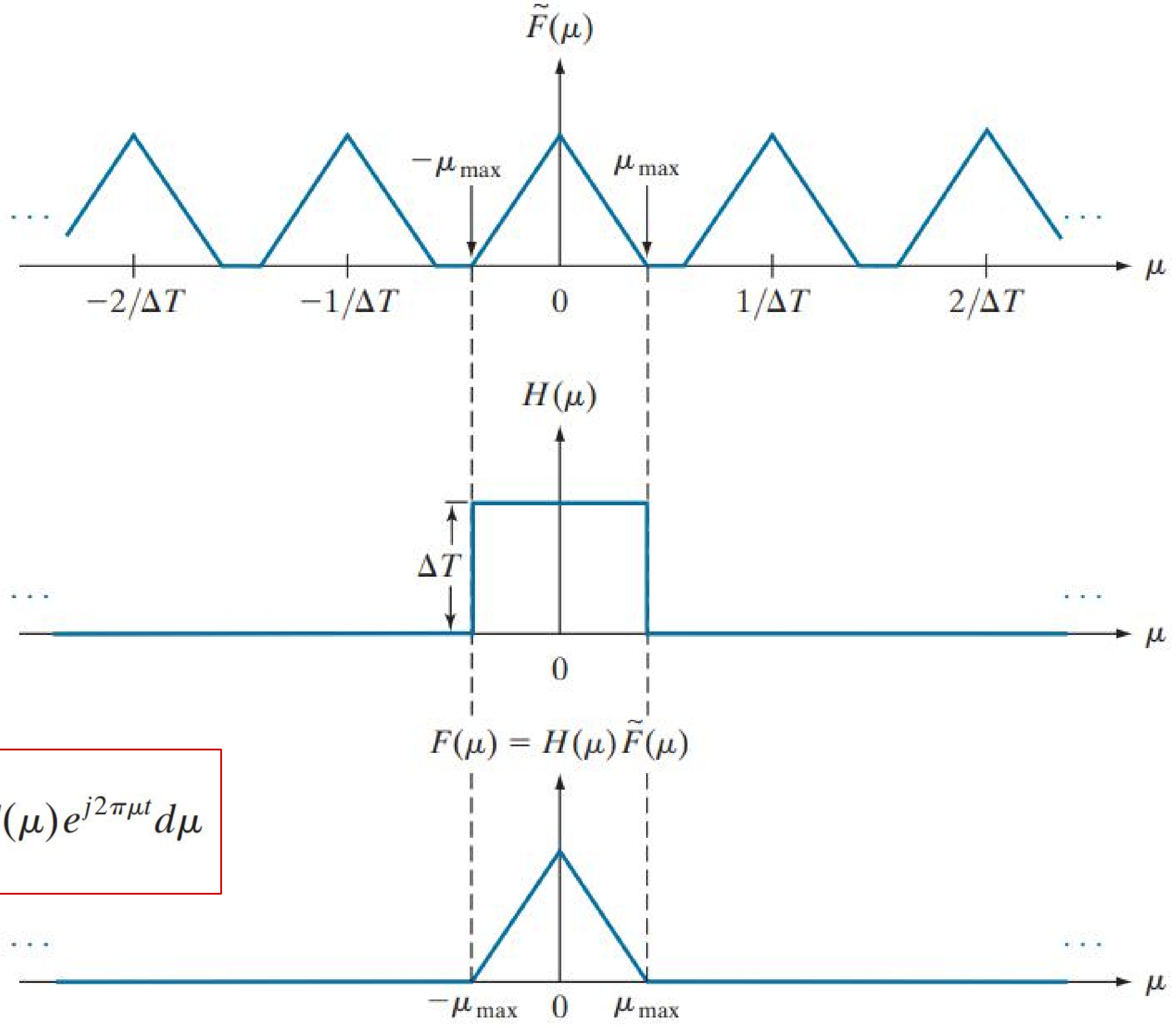
$$\mu_{\max} = \frac{1}{2\Delta T}$$

$$\left(\frac{1}{\Delta T} > 2\mu_{\max}\right)$$

- **图4.8 以略高于奈奎斯特采样率的频率采样**

a
b
c

FIGURE 4.8 (a) Fourier transform of a sampled, band-limited function. (b) Ideal lowpass filter transfer function. (c) The product of (b) and (a), used to extract one period of the infinitely periodic sequence in (a).



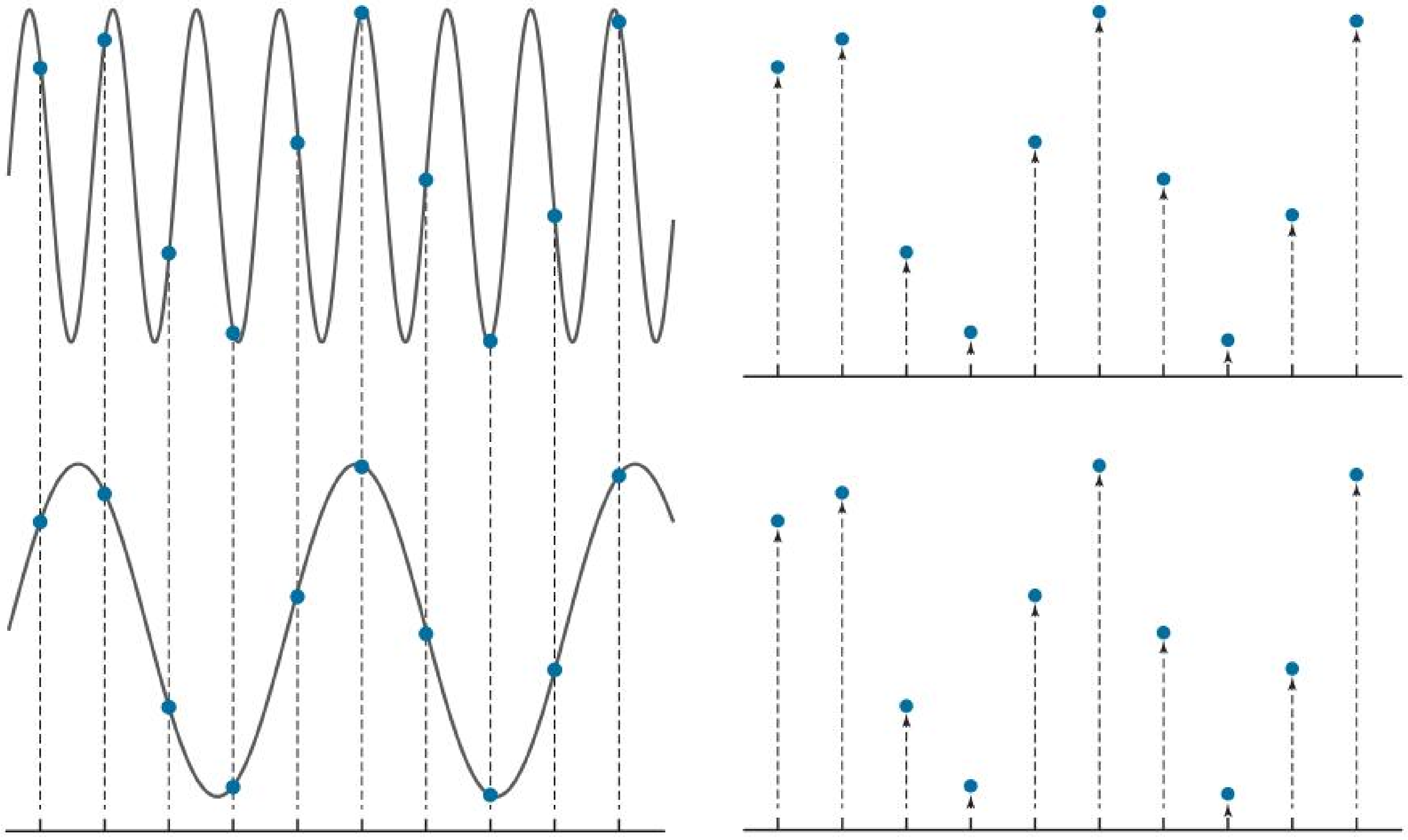
$$\boxed{f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu}$$

取样定理 THE SAMPLING THEOREM

- ## 混淆 ALIASING

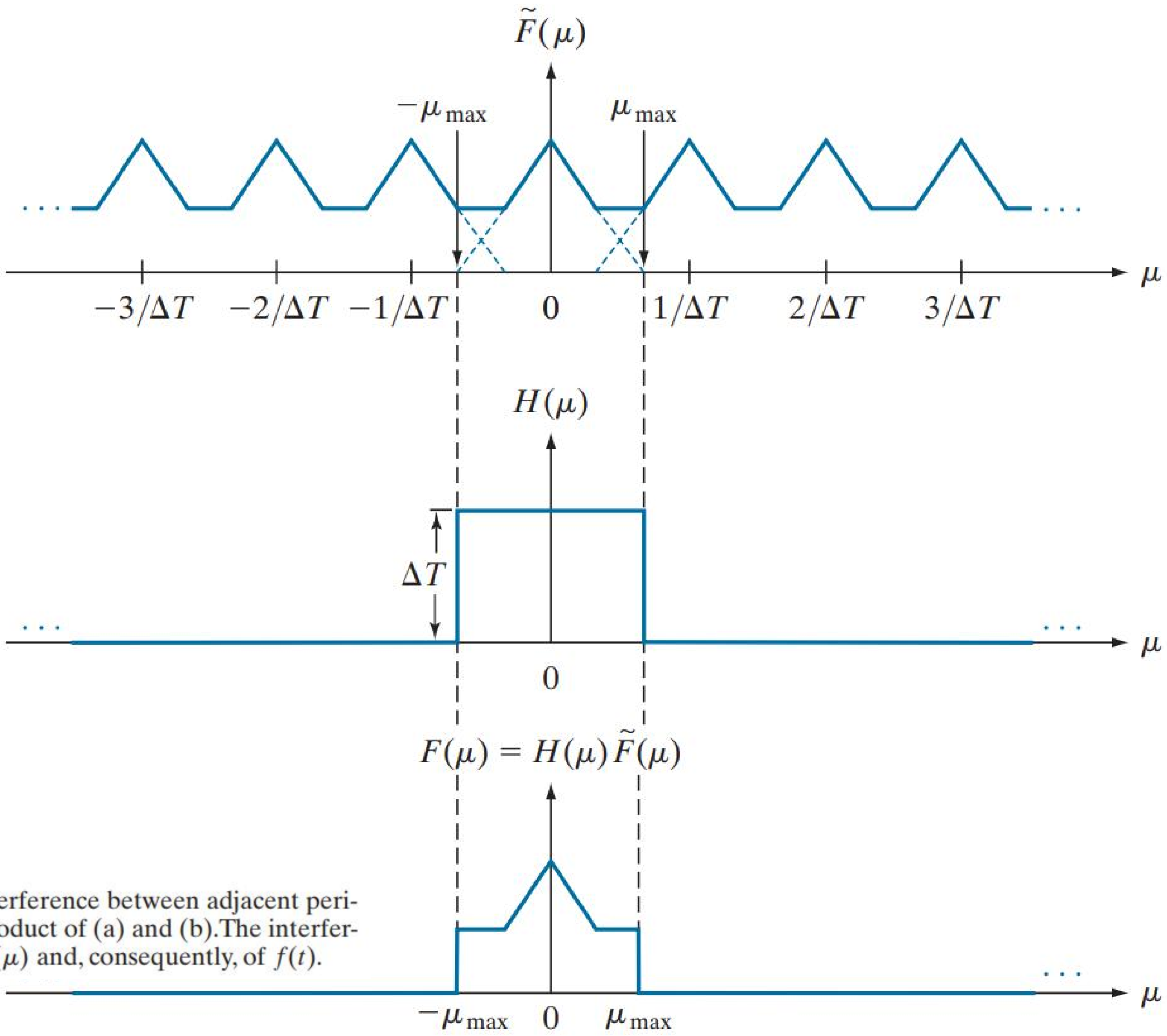
a	b
c	d

FIGURE 4.9
The functions in (a) and (c) are totally different, but their digitized versions in (b) and (d) are identical. Aliasing occurs when the samples of two or more functions coincide, but the functions are different elsewhere.



取样定理 THE SAMPLING THEOREM

- **混淆 ALIASING**
  - 带宽 $[-\mu_{\max}, \mu_{\max}]$
- **临界采样**
- **过采样**
- **欠采样**



a
b
c

FIGURE 4.10 (a) Fourier transform of an under-sampled, band-limited function. (Interference between adjacent periods is shown dashed). (b) The same ideal lowpass filter used in Fig. 4.8. (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, consequently, of $f(t)$.

取样定理 THE SAMPLING THEOREM

- ## 混淆 ALIASING

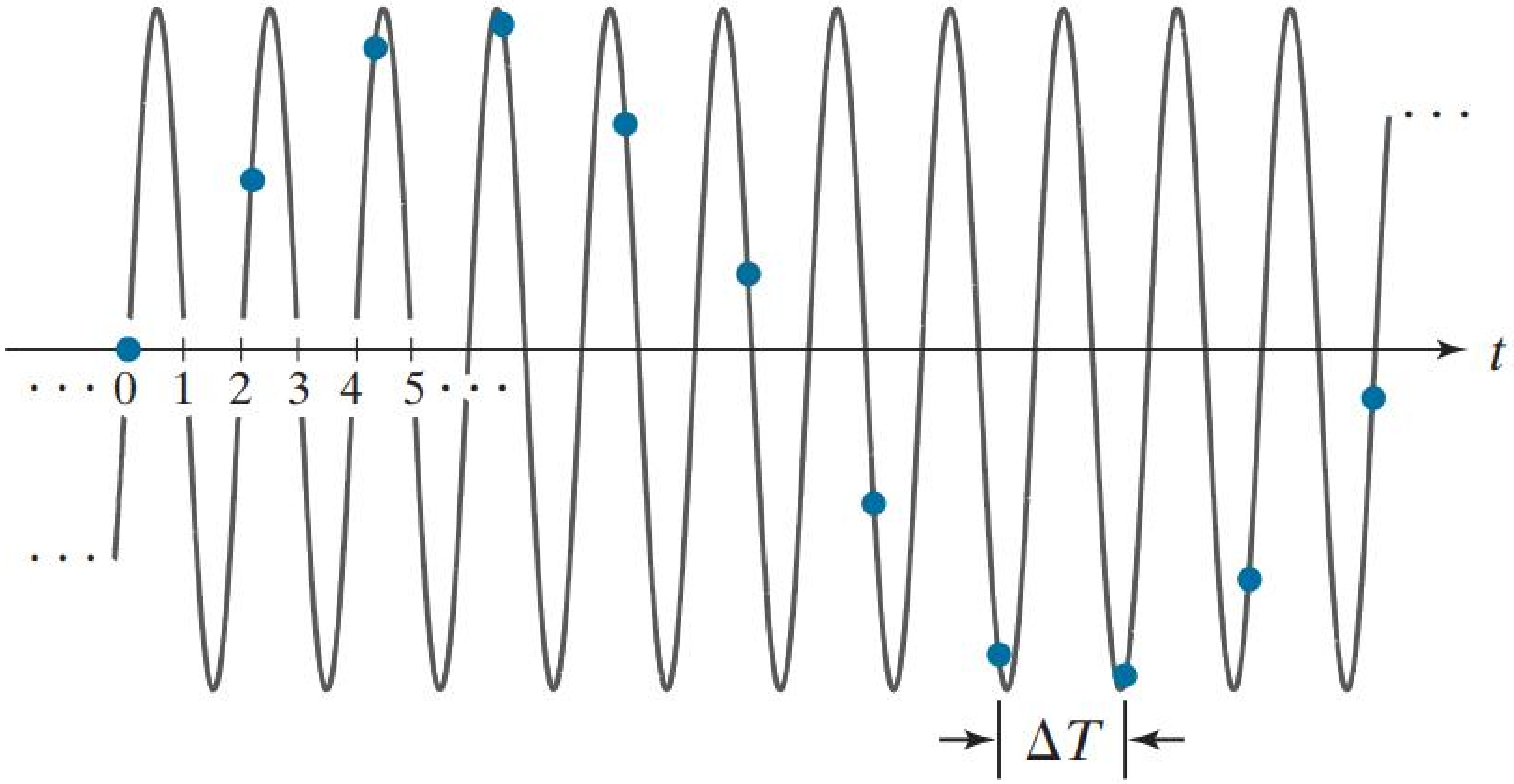


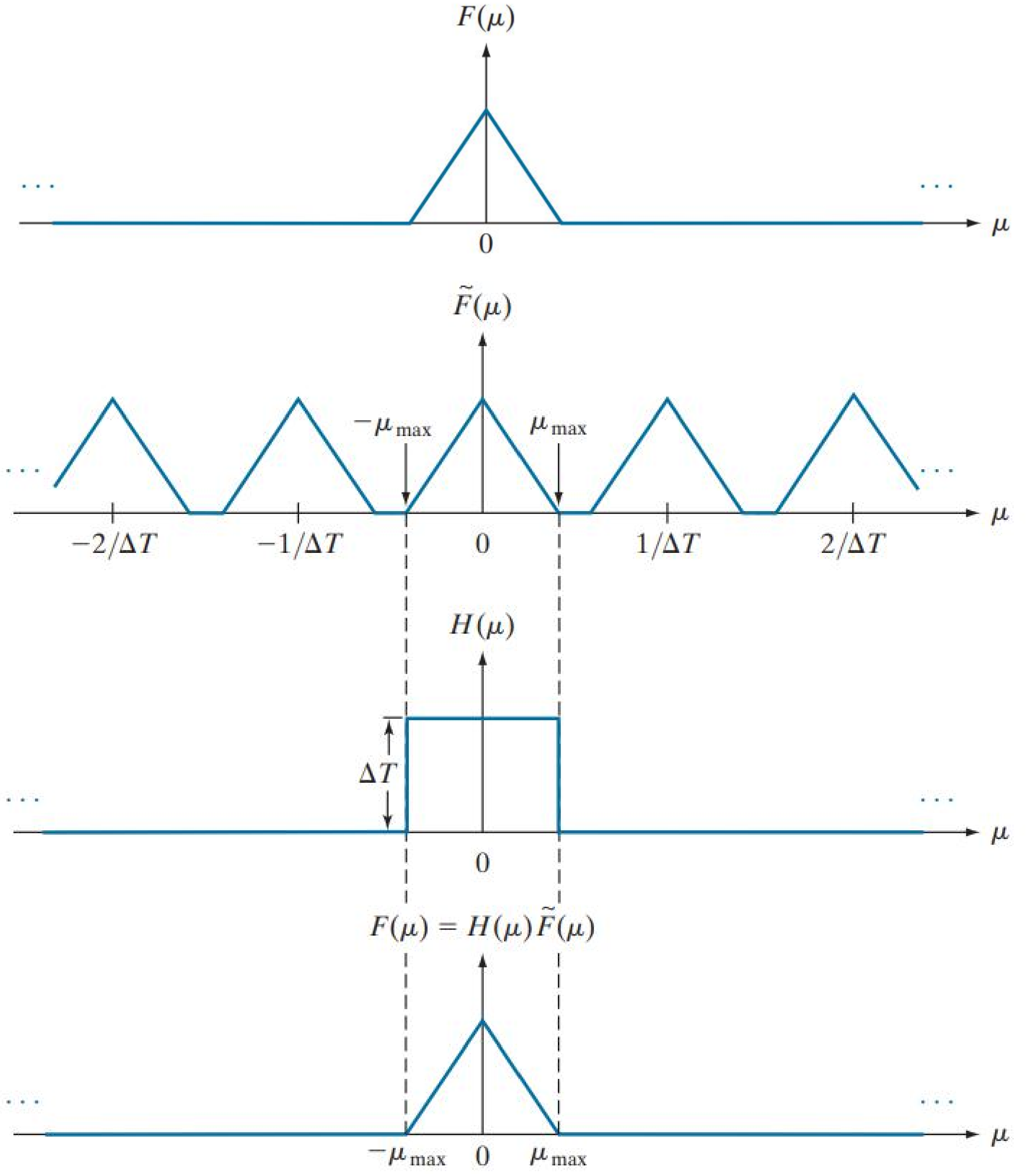
FIGURE 4.11 Illustration of aliasing. The under-sampled function (dots) looks like a sine wave having a frequency much lower than the frequency of the continuous signal. The period of the sine wave is 2 s, so the zero crossings of the horizontal axis occur every second. ΔT is the separation between samples.

采样信号重构 FUNCTION RECONSTRUCTION

- **FUNCTION RECONSTRUCTION (RECOVERY) FROM SAMPLED DATA**

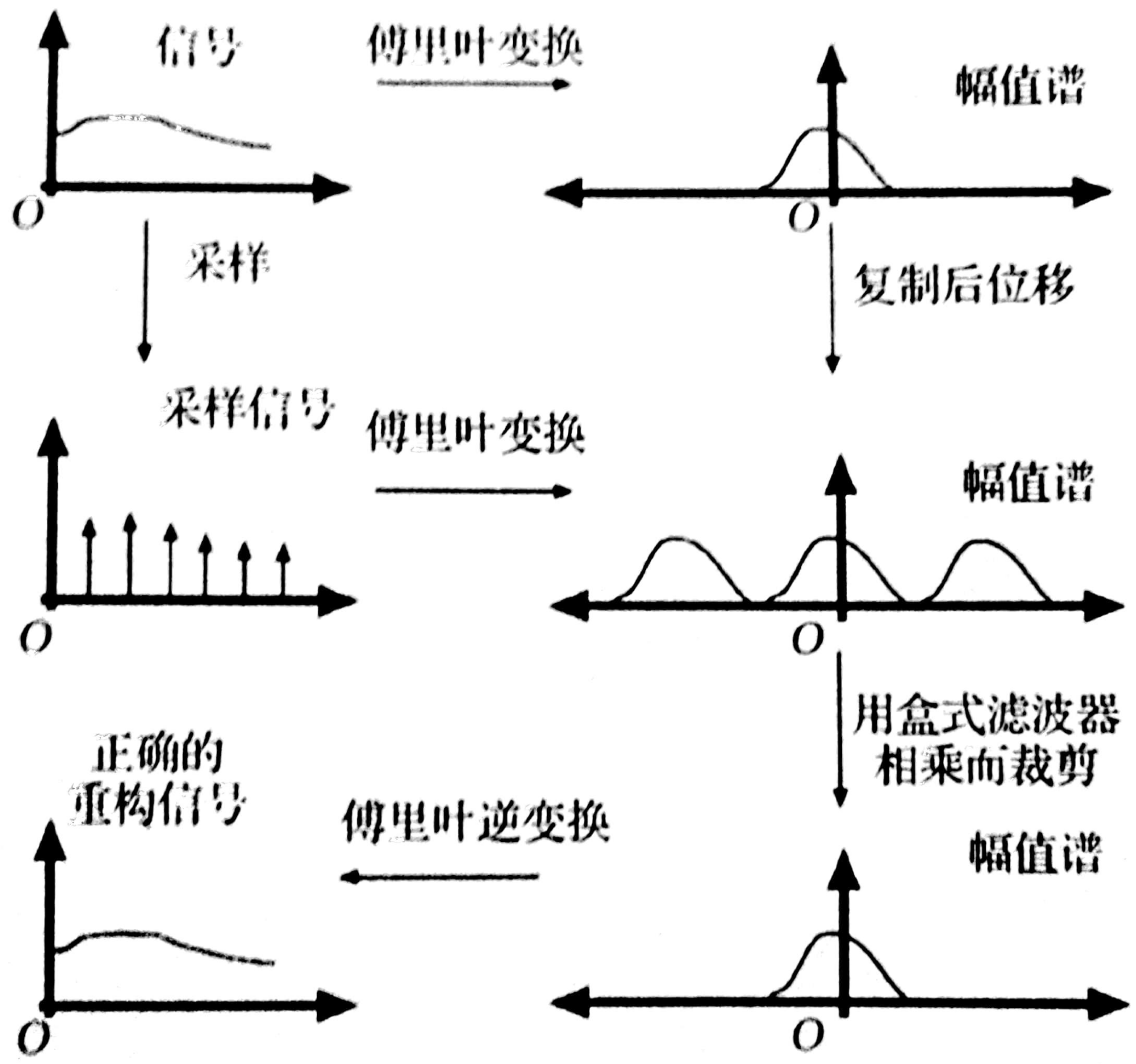
$$
\begin{aligned}
f(t) &= \Im^{-1}\left\{F(\mu)\right\} \\
&= \Im^{-1}\left\{H(\mu)\tilde{F}(\mu)\right\} \\
&= h(t) \star \tilde{f}(t)
\end{aligned}
$$

$$f(t) = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}\,d\mu$$



取样及取样函数的傅里叶变换

- **采样信号重构**



1D离散傅里叶变换
Discrete Fourier Transform (DFT)

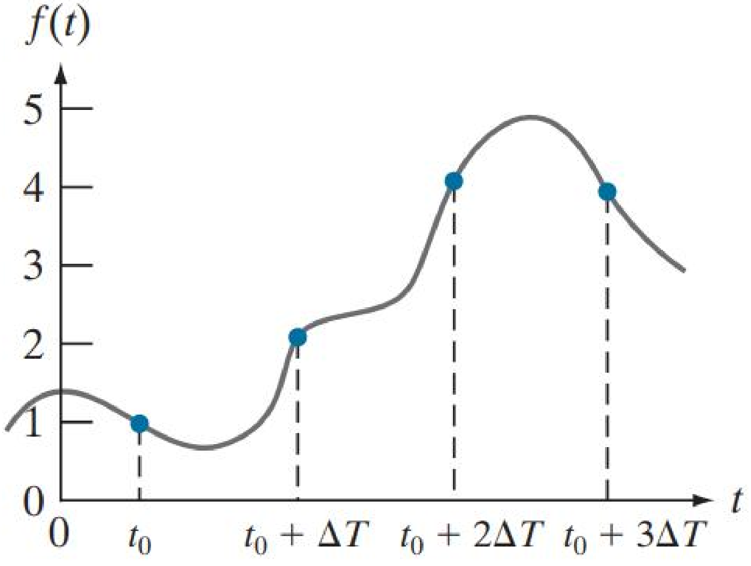
离散傅里叶变换 Discrete Fourier Transform (DFT)

- **离散信号看作连续信号的采样，其傅里叶变换连续且周期**

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t)e^{-j2\pi\mu t}dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)e^{-j2\pi\mu t}dt$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\delta(t - n\Delta T)e^{-j2\pi\mu t}dt$$

$$= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}$$



Although f_n is a discrete function, its Fourier transform, $\tilde{F}(\mu)$, is continuous and infinitely periodic with period $1/\Delta T$, as we know from Eq. (4-31). Therefore, all we need to characterize $\tilde{F}(\mu)$ is one period, and sampling one period of this function is the basis for the DFT.

Suppose that we want to obtain M equally spaced samples of $\tilde{F}(\mu)$ taken over the one period interval from $\mu = 0$ to $\mu = 1/\Delta T$ (see Fig. 4.8). This is accomplished by taking the samples at the following frequencies:

$$\mu = \frac{m}{M\Delta T} \qquad m = 0, 1, 2, \ldots, M-1 \tag{4-41}$$

Substituting this result for μ into Eq. (4-40) and letting F_m denote the result yields

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad m = 0, 1, 2, \ldots, M-1 \tag{4-42}$$

1D离散傅里叶变换

Frequency recipe = | Add up | Contributions to this frequency, from each time spike

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-i2\pi kn/N}$$

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \cdot e^{i2\pi kn/N}$$

Time point = | Add up | Contributions to this time point, from each frequency

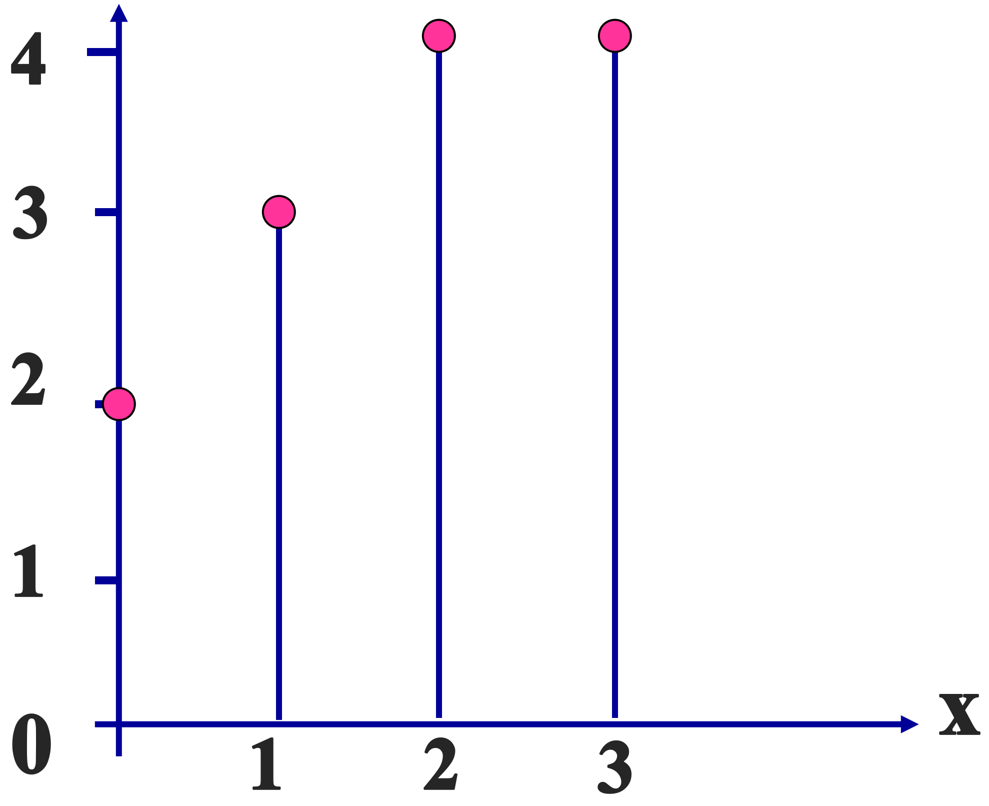
- *N = number of time samples we have*
- *n = current sample we're considering (0 .. N-1)*
- x_n *= value of the signal at time n*
- *k = current frequency we're considering (0 Hertz up to N-1 Hertz)*
- X_k *= amount of frequency k in the signal (amplitude and phase, a complex number)*

两者的差别仅在指数的符号和因子1/N.

1D离散傅里叶变换

- **离散傅里叶变换的计算**

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x_n e^{i2\pi k \frac{n}{N}}$$



$$X_0 = F(0) =$$

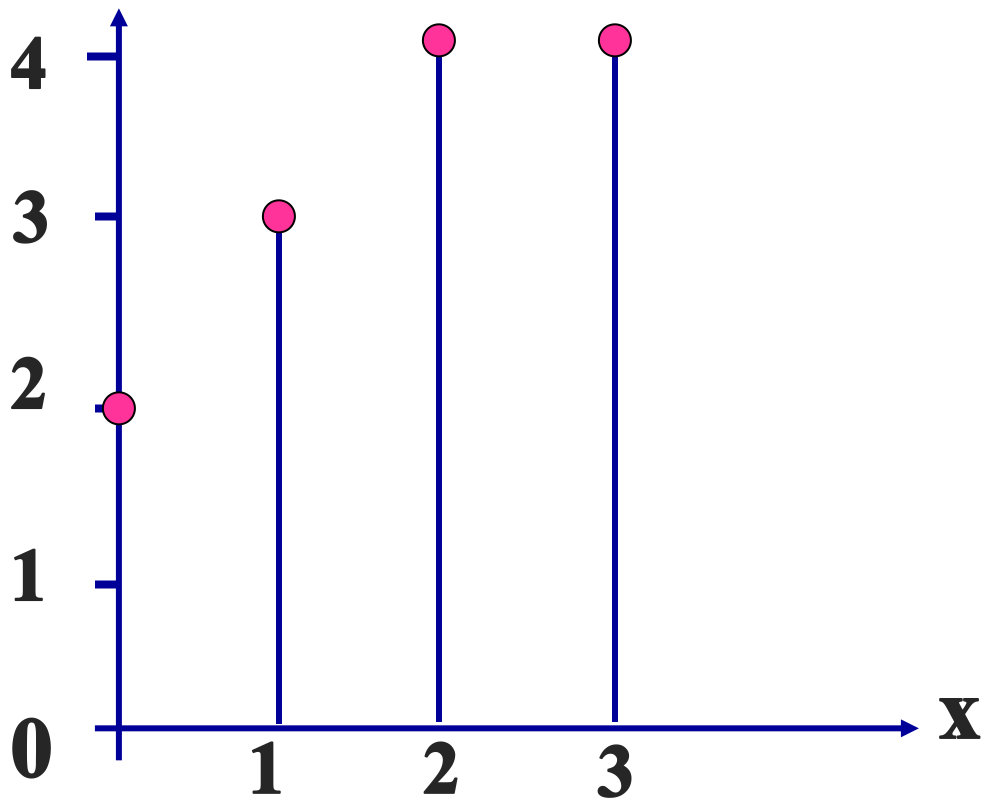
$$X_1 = \mathrm{F}(1) =$$

$$X_2 = \mathrm{F}(2) =$$

$$X_3 = \mathrm{F}(3) =$$

1D离散傅里叶变换

- **离散傅里叶变换的计算**



$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x_n e^{i2\pi k\frac{n}{N}}$$

$$X_0 = F(0) = \frac{1}{4}\sum_{n=0}^{3} x_n e^{-i2\pi 0n/4}$$

$$= \frac{1}{4}[x_0 + x_1 + x_2 + x_3] = \frac{1}{4}(2+3+4+4) = 3.25$$

$$X_1 = F(1) = \frac{1}{4}\sum_{n=0}^{3} x_n e^{-i2\pi 1n/4}$$

$$= \frac{1}{4}(2e^0 + 3e^{-i2\pi 1/4} + 4e^{-i2\pi 2/4} + 4e^{-i2\pi 3/4}) = \frac{1}{4}(-2+i)$$

$$X_2 = F(2) = \frac{1}{4}\sum_{n=0}^{3} x_n e^{-i2\pi 2n/4} = -\frac{1}{4}(1+i0)$$

$$X_3 = F(3) = \frac{1}{4}\sum_{n=0}^{3} x_n e^{-i2\pi 3n/4} = -\frac{1}{4}(2+i)$$

- **平移特性**

Demonstrate the validity of the translation (shift) properties of the following 1-D, discrete Fourier transform pairs. (*Hint:* It is easier in part (b) to work with the IDFT.)

(a)* $f(x)e^{j2\pi u_0 x/M} \Leftrightarrow F(u-u_0)$

(b) $f(x-x_0) \Leftrightarrow F(u)e^{-j2\pi u x_0/M}$

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M}$$

$$f_n = \frac{1}{M}\sum_{m=0}^{M-1} F_m e^{j2\pi mn/M}$$

小 结

小 结

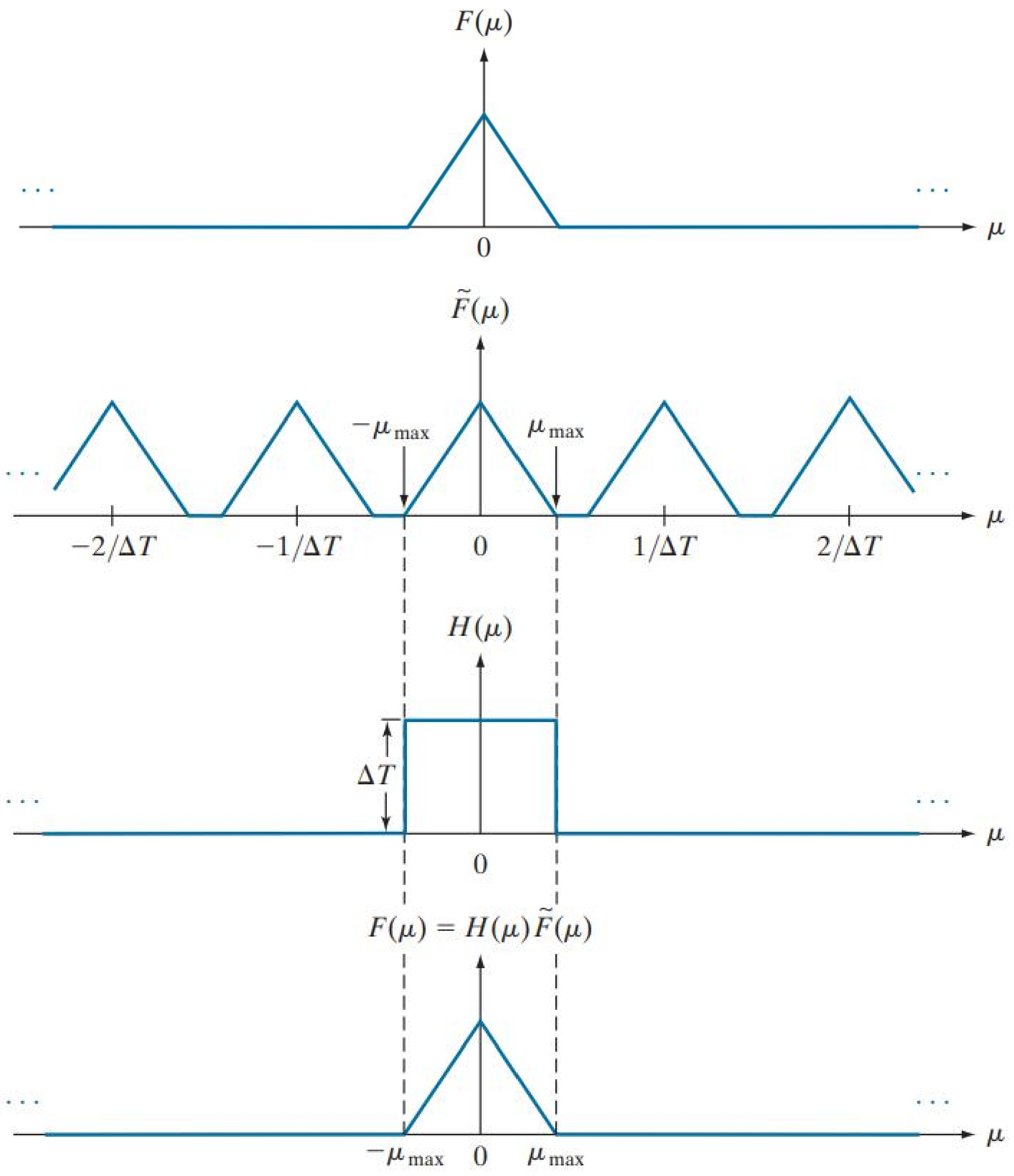
- **傅里叶变换**
  - 卷积定理

$$F(\mu)=\int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}\,dt$$

$$f(t)=\int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}\,d\mu$$

- **采样及采样定理**
  - 带宽 $[-\mu_{\max},\mu_{\max}]$
  - 奈奎斯特采样频率 $\mu_{\max}=\dfrac{1}{2\Delta T}\left(\dfrac{1}{\Delta T}>2\mu_{\max}\right)$
  - 混淆 ALIASING
  - 临界采样
  - 过采样
  - 欠采样
- **信号重构**

$$\begin{aligned} f(t) &= \Im^{-1}\{F(\mu)\} \\ &= \Im^{-1}\{H(\mu)\tilde{F}(\mu)\} \\ &= h(t)\star\tilde{f}(t) \end{aligned}$$



Q & A