8

The Bernoulli-Euler Beam

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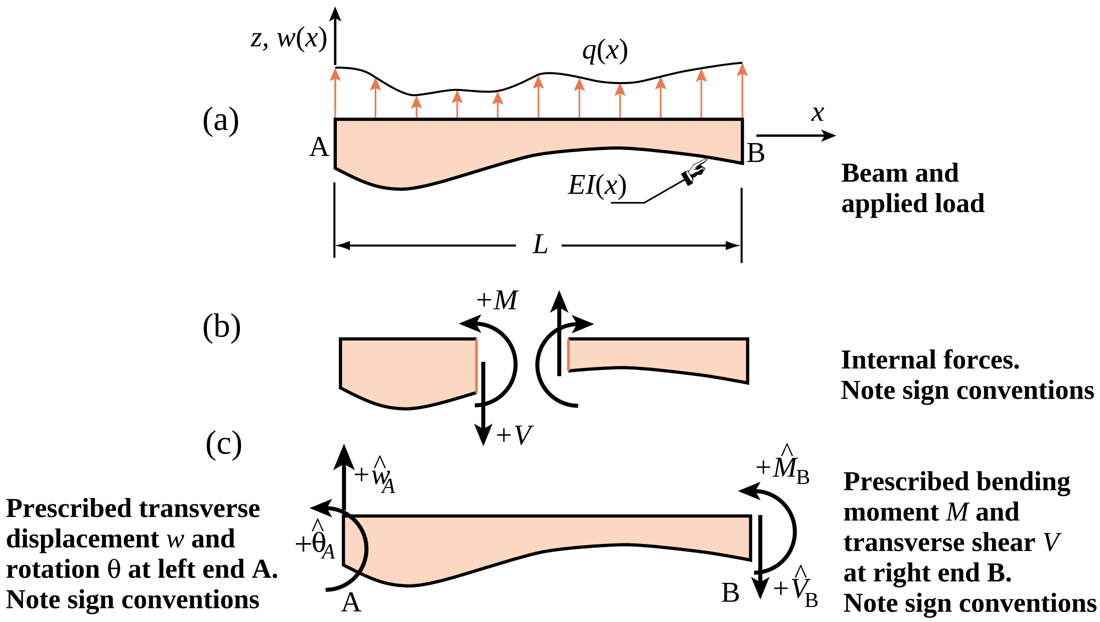

FIGURE 8.1. The Bernoulli-Euler beam model: (a) beam and transverse load; (b) positive convention for moment and shear; (c) boundary conditions.

§8.1. Introduction

From the Poisson equation we move to elasticity and structural mechanics. Rather than tackling the full 3D problem first this Chapter illustrates, in a tutorial style, the derivation of Variational Forms for a one-dimensional model: the Bernoulli-Euler beam.

Despite the restriction to 1D, the mathematics offers a new and challenging ingredient: the handling of functionals with second space derivatives of displacements. Physically these are curvatures of deflected shapes. In structural mechanics curvatures appear in problems involving beams, plates and shells. In fluid mechanics second derivatives appear in slow viscous flows.

§8.2. The Beam Model

The beam under consideration extends from $x = 0$ to $x = L$ and has a bending rigidity EI, which may be a function of x. See Figure 8.1(a). The transverse load $q(x)$ is given in units of force per length. The unknown fields are the transverse displacement $w(x)$, rotation $\theta(x)$, curvature $\kappa(x)$, bending moment $M(x)$ and transverse shear $V(x)$. Positive sign conventions for M and V are illustrated in Figure 8.1(b).

Boundary conditions are applied only at A and B. For definiteness the end conditions shown in Figure 8.1(c) will be used.

§8.2.1. Field Equations

In what follows a prime denotes derivative with respect to x. The field equations over $0 \le x \le L$ are as follows.

(KE) Kinematic equations:

$$\boxed{\theta = \frac{dw}{dx} = w', \qquad \kappa = \frac{d^2w}{dx^2} = w'' = \theta',} \tag{8.1}$$

where θ is the rotation of a cross section and κ the curvature of the deflected longitudinal axis. Relations (8.1) express the kinematics of an Bernoulli-Euler beam: plane sections remain plane and normal to the deflected neutral axis.

(CE) Constitutive equation:

$$\boxed{M = EI\kappa,} \tag{8.2}$$

where I is the second moment of inertia of the cross section with respect to y (the neutral axis). This moment-curvature relation is a consequence of assuming a linear distribution of strains and stresses across the cross section. It is derived in elementary courses of Mechanics of Materials.

(BE) Balance (equilibrium) equations:

$$\boxed{V = \frac{dM}{dx} = M', \qquad \frac{dV}{dx} - q = V' - q = M'' - q = 0.} \tag{8.3}$$

Equations (8.3) are established by elementary means in Mechanics of Materials courses.[1]

§8.2.2. Boundary Conditions

For the sake of specificity, the boundary conditions assumed for the example beam are of primary-variable (PBC) type on the left and of flux type (FBC) on the right:

$$\boxed{\begin{array}{llll} (PBC) & \text{at } A\,(x=0): & w = \hat{w}_A, & \theta = \hat{\theta}_A, \\ (FBC) & \text{at } B\,(x=L): & M = \hat{M}_B, & V = \hat{V}_B, \end{array}} \tag{8.4}$$

in which $\hat{w}_A$, $\hat{\theta}_A$, $\hat{M}_B$ and $\hat{V}_B$ are prescribed. If $\hat{w}_A = \hat{\theta}_A = 0$, these conditions physically correspond to a cantilever (fixed-free) beam. We will let $\hat{w}_A$ and $\hat{\theta}_A$ be arbitrary, however, to further illuminate their role in the functionals.

Remark 8.1. Note that a positive $\hat{V}_B$ acts downward (in the $-z$ direction) as can be seen from Figure 8.1(b), so it disagrees with the positive deflection $+w$. On the other hand a positive $\hat{M}_B$ acts counterclockwise, which agrees with the positive rotation $+\theta$.

[1] In those courses, however, $+q(x)$ is sometimes taken to act downward, leading to $V' + q = 0$ and $M'' + q = 0$.

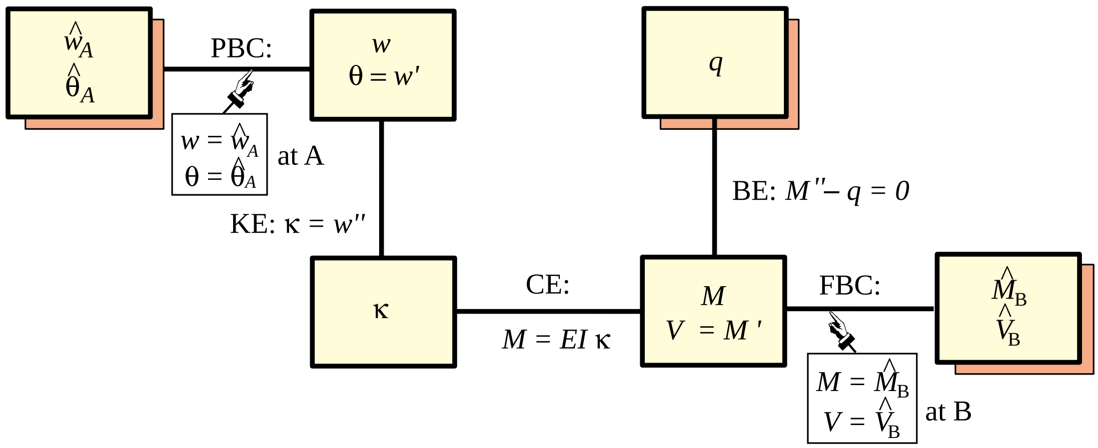

FIGURE 8.2. Tonti diagram of Strong Form of Bernoulli-Euler beam model.

§8.2.3. The SF Tonti Diagram

The Strong Form Tonti diagram for the Bernoulli-Euler model of Figure 8.1 is drawn in Figure 8.2. The diagram lists the three field equations (KE, CE, BE) and the boundary conditions (PBC, FBC). The latter are chosen in the very specific manner indicated above to simplify the boundary terms.[2]

Note than in this beam model the rotation $\theta = w'$ and the transverse shear $V = M'$ play the role of auxiliary variables that *are not constitutively related*. The only constitutive equation is the moment-curvature equation $M = EI\kappa$. The reason for the presence of such auxiliary variables is their direct appearance in boundary conditions.[3]

§8.3. The TPE (Primal) Functional

Select w as only master field. Weaken the BE and FBC connections to get the Weak Form (WF) diagram of Figure 8.4 as departure point. Choose the weighting functions on the weak links BE, FBC on M and FBC on V to be δw, $\delta\theta^w$ and δw, respectively. The weak links are combined as follows:

$$\delta\Pi = \int_0^L \Big[(M^w)'' - q\Big]\,\delta w\,dx + (M^w - \hat{M})\,\delta\theta^w\Big|_B - (V^w - \hat{V})\,\delta w\Big|_B = 0. \tag{8.5}$$

Why the different signs for the moment and shear boundary terms? If confused, read Remark 4.1. Next, integrate $\int_0^L (M^w)''\delta w\,dx$ twice by parts:

$$\begin{aligned}\int_0^L (M^w)''\,\delta w\,dx &= -\int_0^L (M^w)'\,\delta w'\,dx + \Big[(M^w)'\,\delta w\Big]_A^B \\ &= \int_0^L M^w\,\delta w''\,dx + \Big[(M^w)'\,\delta w\Big]_A^B - \Big[M^w\,\delta w'\Big]_A^B \\ &= \int_0^L M^w\,\delta\kappa^w\,dx + V^w\,\delta w\Big|_B - M^w\,\delta\theta^w\Big|_B.\end{aligned} \tag{8.6}$$

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[2] In fact $2^4 = 16$ boundary condition combinations are mathematically possible. Some of these correspond to physically realizable support conditions, for example simply supports, whereas others do not.

[3] In the Timoshenko beam model, which accounts for transverse shear energy, θ appears in the constitutive equations.

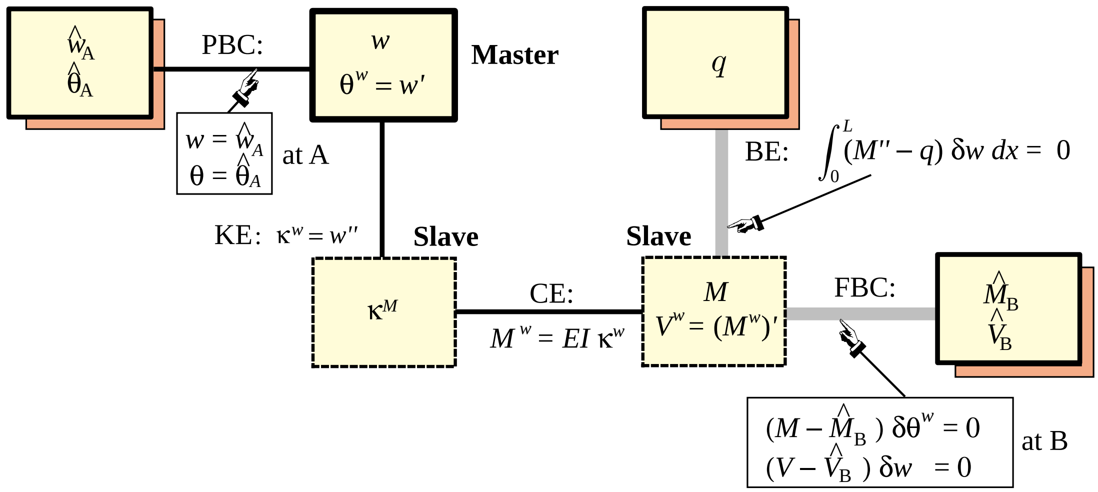

FIGURE 8.3. Weak Form used as departure point to derive the TPE functional for the Bernoulli-Euler beam

The disappearance of boundary terms at A in the last equation results from $\delta w_A = 0$, $\delta w'_A = \delta\theta^w_A = 0$ on account of the strong PBC connection at $x = 0$. Inserting (8.6) into (8.5) gives

$$\begin{aligned}\delta\Pi &= \int_0^L (M^w\,\delta\kappa^w - q\,\delta w)\,dx - \hat{M}\,\delta\theta^w\Big|_B + \hat{V}\,\delta w\Big|_B \\ &= \int_0^L (EI\,w''\,\delta w'' - q\,\delta w)\,dx - \hat{M}\,\delta w'\Big|_B + \hat{V}\,\delta w\Big|_B.\end{aligned} \tag{8.7}$$

This is the first variation of the functional

$$\boxed{\Pi_{\mathrm{TPE}}[w] = \tfrac{1}{2}\int_0^L EI\,(w'')^2\,dx - \int_0^L q\,w\,dx - \hat{M}_B w'_B + \hat{V}_B w_B.} \tag{8.8}$$

This is called the Total Potential Energy (TPE) functional of the Bernoulli-Euler beam. It was used in the Introduction to Finite Element Methods course to derive the well known Hermitian beam element in Chapter 12. For many developments it is customarily split into two terms

$$\Pi[w] = U[w] - W[w], \tag{8.9}$$

in which

$$U[w] = \tfrac{1}{2}\int_0^L EI\,(w'')^2\,dx, \qquad W[w] = \int_0^L q\,w\,dx + \hat{M}_B w'_B - \hat{V}_B w_B. \tag{8.10}$$

Here U is the internal energy (strain energy) of the beam due to bending deformations (bending moments working on curvatures), whereas W gathers the other terms that collectively represent the *external work* of the applied loads.[4]

[4] Recall that work and energy have opposite signs, since energy is the capacity to produce work. It is customary to write $\Pi = U - W$ instead of the equivalent $\Pi = U + V$, where $V = -W$ is the external work potential. This notational device also frees the symbol V to be used for transverse shear in beams and voltage in electromagnetics.

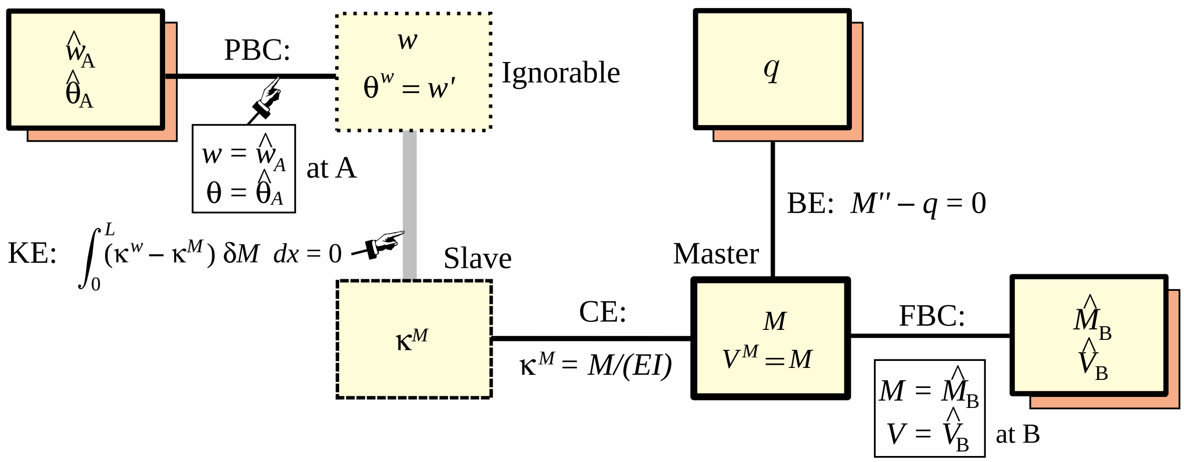

FIGURE 8.4. Weak Form used as departure point to derive the TCPE functional for the Bernoulli-Euler beam

Remark 8.2. Using integration by parts one can show that if $\delta\Pi = 0$,

$$U = \tfrac{1}{2} W. \tag{8.11}$$

In other words: at equilibrium the *internal energy is half the external work*. This property is valid for any linear elastic continuum. (It is called Clapeyron's theorem in the literature of Structural Mechanics.) It has a simple geometric interpretation for structures with finite number of degrees of freedom.

§8.4. The TCPE (Dual) Functional

The Total Complementary Potential Energy (TCPE) functional is mathematically the dual of the primal (TPE) functional.

Select the bending moment M as the only master field. Make KE weak to get the Weak Form diagram displayed in Figure 8.4. Choose the weighting function on the weakened KE to be δM. The only contribution to the variation of the functional is

$$\delta\Pi = \int_0^L (\kappa^M - \kappa^w)\, \delta M\, dx = \int_0^L (\kappa^M - w'')\, \delta M\, dx = 0. \tag{8.12}$$

Integrate $\int w''\ \delta M\, dx$ by parts twice:

$$\begin{aligned} \int_0^L w''\ \delta M\, dx &= -\int_0^L w'\, \delta M'\, dx + \Big[w'\, \delta M \Big]_A^B \\ &= \int_0^L w\, \delta M''\, dx + \Big[w'\, \delta M \Big]_A^B - \Big[w\, \delta M' \Big]_A^B \\ &= \int_0^L w\, \delta M''\, dx - \hat{\theta}\, \delta M \Big|_A + \hat{w}\, \delta V^M \Big|_A . \end{aligned} \tag{8.13}$$

The disappearance of the boundary terms at B results from enforcing strongly the free-end boundary conditions $M = \hat{M}_B$ and $V = \hat{V}_B$, whence the variations $\delta M_B = 0$, $\delta V^M = \delta M'_B = 0$. Because of

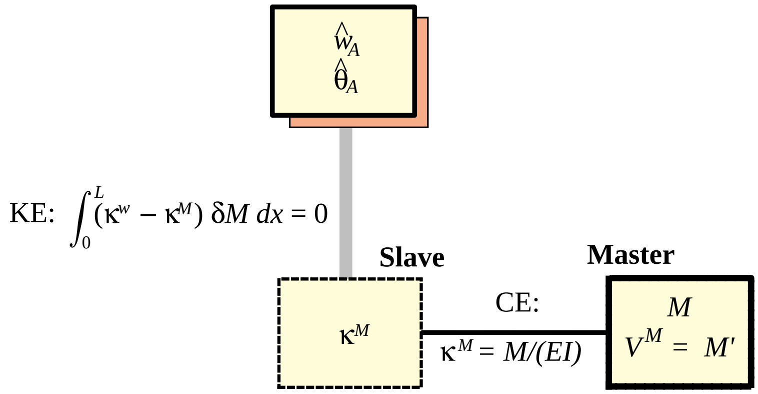

FIGURE 8.5. The collapsed WF diagram for the TCPE functional, showing only "leftovers" boxes.

the strong BE connection, $M'' - q = 0$, $\delta M''$ vanishes identically in $0 \le x \le L$. Consequently

$$\int_0^L w''\,\delta M\,dx = -\hat{\theta}\,\delta M\Big|_A + \hat{w}\,\delta V^M\Big|_A. \tag{8.14}$$

Replacing into (8.12) yields

$$\delta\Pi = \int \kappa^M \delta M\,dx + \hat{\theta}\,\delta M\Big|_A - \hat{w}\,\delta V^M\Big|_A = 0. \tag{8.15}$$

This is the first variation of the functional

$$\Pi[M] = \tfrac{1}{2}\int_0^L \kappa^M M\,dx + M\hat{\theta}_A - V^M\hat{w}_A, \tag{8.16}$$

and since $\kappa^M = M/EI$, we finally get

$$\boxed{\Pi_{\mathrm{TCPE}}[M] = \tfrac{1}{2}\int_0^L \frac{M^2}{EI}\,dx + M\hat{\theta}_A - V^M\hat{w}_A.} \tag{8.17}$$

This is the TCPE functional for the Bernoulli-Euler beam model. As in the case of the TPE, it is customarily split as

$$\Pi[M] = U^*[M] - W^*[M], \tag{8.18}$$

where

$$U^* = \tfrac{1}{2}\int_0^L \frac{M^2}{EI}\,dx, \qquad W^* = -M\hat{\theta}_A + V^M\hat{w}_A. \tag{8.19}$$

Here U^* is the internal *complementary energy* stored in the beam by virtue of its deformation, and W^* is the *external complementary energy* that collects the work of the prescribed end displacements and rotations.

Note that only M (and its slaves), $\hat{\theta}_A$ and $\hat{w}_A$ remain in this functional. The transverse displacement $w(x)$ is gone and consequently is labeled as *ignorable* in Figure 8.5. Through the integration by parts process the WF diagram of Figure 8.4 collapses to the one sketched in Figure 8.5. The reduction may be obtained by invoking the following two rules:

(1) The "ignorable box" w, θ of Figure 8.4 may be replaced by the data box $\hat{w}_A$, $\hat{\theta}_A$ because only the boundary values of those quantities survive.

(2) The data boxes q and $\hat{M}$, $\hat{V}$ of Figure 8.4 may be removed because they are strongly connected to the varied field M.

The collapsed WF diagram of Figure 8.5 displays the five quantities $(M, V^M, \kappa^M, \hat{w}_A, \hat{\theta}_A)$ that survive in the TCPE functional.

Remark 8.3. One can easily show that for the actual solution of the beam problem, $U^* = U$, a property valid for any linear elastic continuum. Furthermore $U^* = \frac{1}{2} W^*$.

§8.5. The Hellinger-Reissner Functional

The TPE and TCPE functionals are *single-field*, because there is only one master field that is varied: displacements in the former and moments in the latter. We next illustrate the derivation of a two-field mixed functional, identified as the Hellinger-Reissner (HR) functional $\Pi_{\rm HR}[w, M]$, for the Bernoulli-Euler beam. Here both displacements w and moments M are picked as master fields and thus are independently varied.

The point of departure is the WF diagram of Figure 8.6. As illustrated, three links: KE, BE and FBC, have been weakened. The master (varied) fields are w and M. It is necessary to distinguish between displacement-derived curvatures $\kappa^w = w''$ and moment-derived curvatures $\kappa^M = M/EI$, as shown in the figure. The two curvature boxes are weakly connected, expressing that the equality $w'' = M/EI$ is not enforced strongly.

The mathematical expression of the WF, having chosen weights δM, δw, $\delta w' = \delta\theta^w$ and $-\delta w$ for the weak connections KE, BE, moment M in FBC and shear V in FBC, respectively, is

$$\delta\Pi[w, M] = \int_0^L (\kappa^w - \kappa^M)\,\delta M\,dx + \int_0^L (M'' - q)\,\delta w\,dx + \Big|(M - \hat{M})\,\delta\theta^w\Big|_B - \Big|(V - \hat{V})\,\delta w\Big|_B. \tag{8.20}$$

Integrating $\int_0^L M''\,\delta w$ twice by parts as in the TPE derivation, inserting in (8.20), and enforcing the strong PBCs at A, yields

$$\delta\Pi[w, M] = \int_0^L \Big[(w'' - \kappa^M)\,\delta M + M\,\delta w''\Big]\,dx - \int_0^L q\,\delta w\,dx + \hat{V}\,\delta w\Big|_B - \hat{M}\,\delta\theta^w\Big|_B, \tag{8.21}$$

This is the first variation of the Hellinger-Reissner (HR) functional

$$\boxed{\Pi_{\rm HR}[w, M] = \int_0^L \left(M w'' - \frac{M^2}{2EI} - q w\right) dx + \hat{V}_B w_B - \hat{M}_B \theta^w_B.} \tag{8.22}$$

Again this can be split as $\Pi_{\rm HR} = U - W$, in which

$$U[w, M] = \int_0^L \left(M w'' - \frac{M^2}{2EI}\right) dx, \quad W[w] = \int_0^L q w\,dx - \hat{V}_B w_B + \hat{M}_B \theta^w_B, \tag{8.23}$$

represent internal energy and external work, respectively.

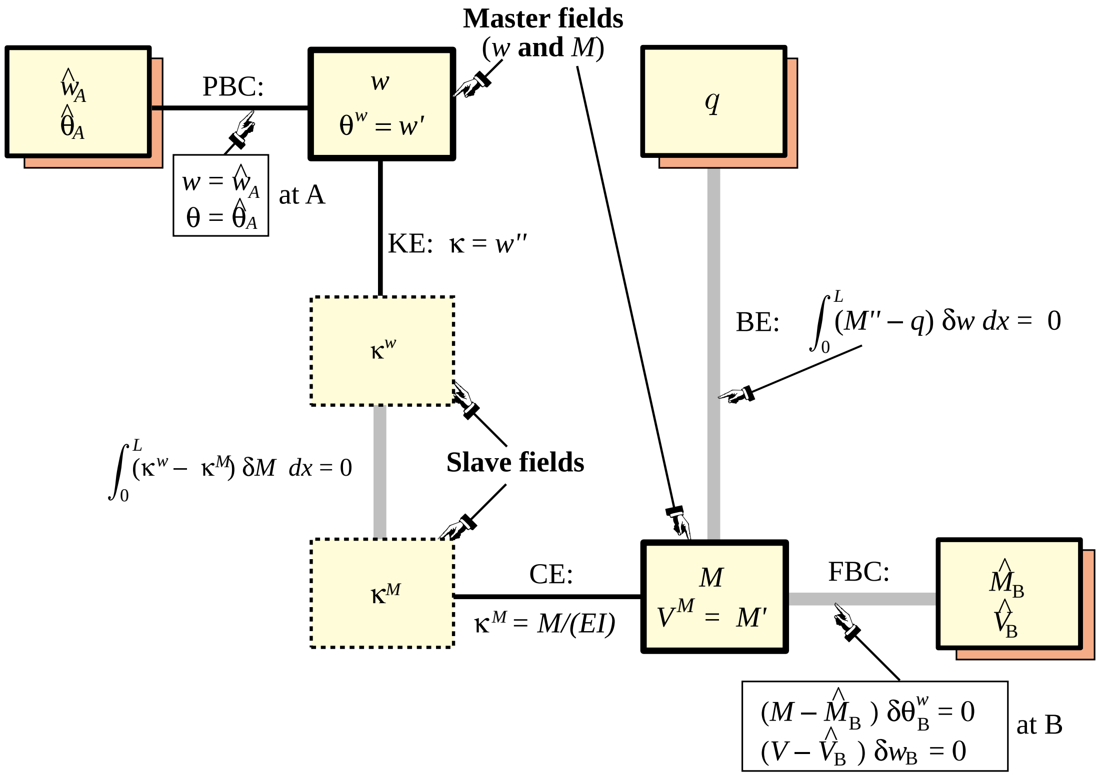

FIGURE 8.6. The WF Tonti diagram used as departure point for deriving the Hellinger-Reissner (HR) functional.

Remark 8.4. If the primal boundary conditions (PBC) at A are weakened, the functional (8.22) gains two extra boundary terms.

Remark 8.5. The $Mw'' = M\kappa^w$ term in (8.23) may be transformed by applying integration by parts once:

$$\int_0^L Mw''\,dx = Mw'\Big|_0^L - \int_0^L M'w'\,dx = \hat{M}_B\theta^w_B - M_A\hat{\theta}^w_A - \int_0^L M'w'\,dx \tag{8.24}$$

to get an alternative form of the HR equation with "balanced derivatives" in w and M. Such transformations are common in the finite element applications of mixed functionals. The objective is to exert control over interelement continuity conditions.

Homework Exercises for Chapter 8
The Bernoulli- Euler Beam

EXERCISE 8.1 [A:25] An assumed-curvature mixed functional. The WF diagram of a two-field displacement-curvature functional $\Pi(w,\kappa)$ for the Bernoulli-Euler beam is shown in Figure E8.1.

Starting from this form, derive the functional

$$\Pi[w,\kappa] = \int_0^L (M^w - \tfrac{1}{2}M^\kappa)\,\kappa\,dx - \int_0^L q\,w\,dx - \hat{M}\,\theta^w\Big|_B - \hat{V}\,w\Big|_B. \tag{E8.1}$$

Which extra term appears if link PBC is weakened?

Hints: The integral $\int_0^L (M^w\delta\kappa + M^\kappa\delta w'')\,dx$ is the variation of $\int_0^L M^w\,\kappa\,dx$ (work it out, it is a bit tricky) whereas the integral $\int_0^L -M^\kappa\,\delta\kappa\,dx$ is the variation of $-\frac{1}{2}\int_0^L M^\kappa\,\kappa\,dx$.

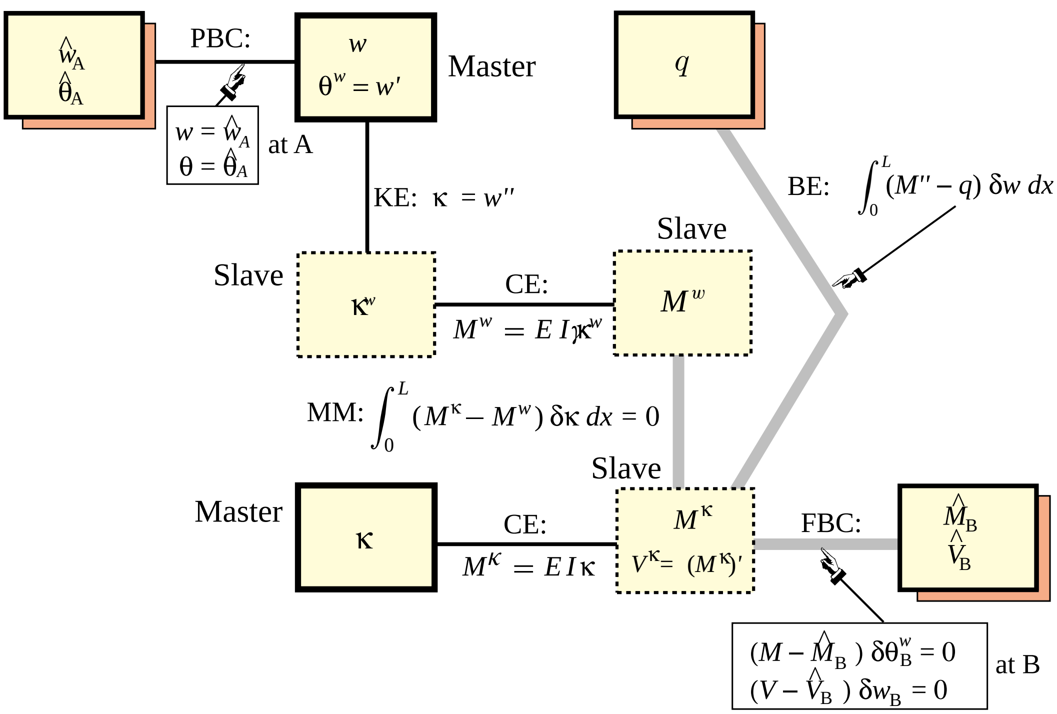

FIGURE E8.1. The WF Tonti diagram used as departure point for deriving the curvature-displacement mixed functional.

EXERCISE 8.2 [A:15] Prove the property stated in Remark 8.2.

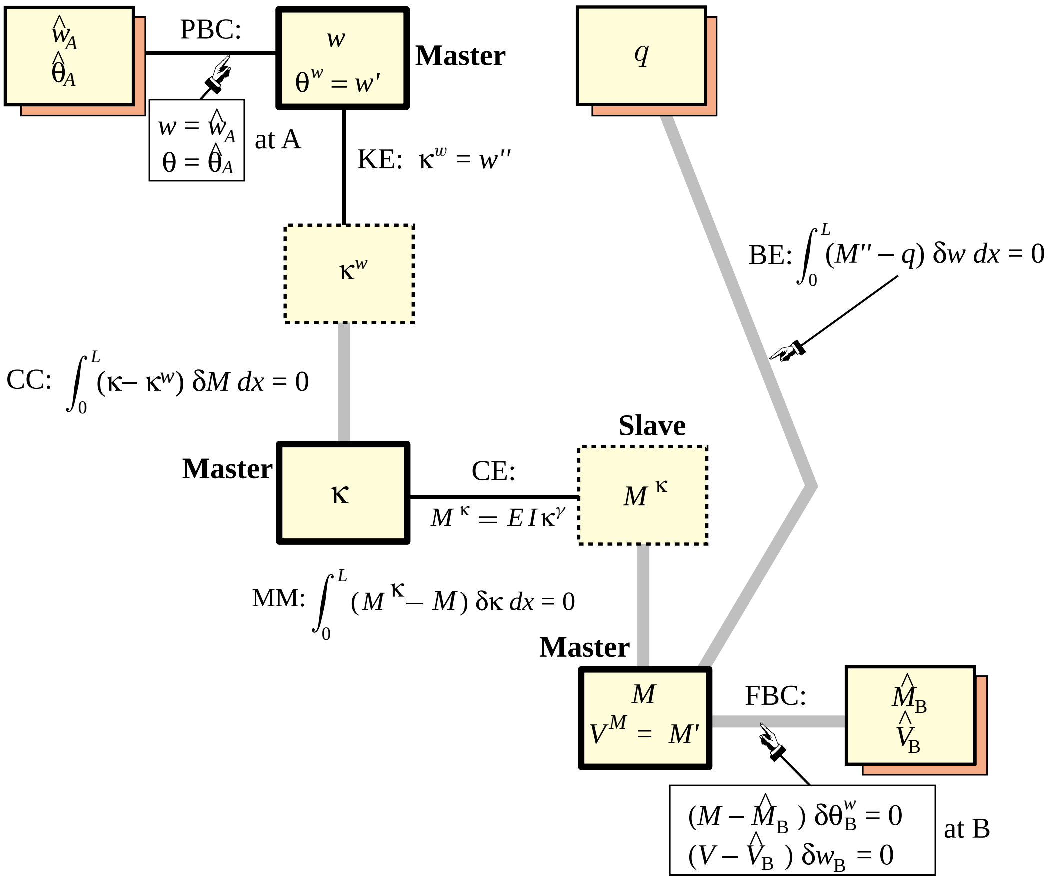

FIGURE E8.2. Starting WF diagram to derive the three-master-field Veubeke-Hu-Washizu mixed functional, which is the topic of Exercise 8.3.

EXERCISE 8.3 [A:30] The most general mixed functional in elasticity is called the Veubeke-Hu-Washizu or VHW functional. The three internal fields: displacements w, curvatures κ and moments M, are selected as masters and independently varied to get

$$\Pi[w,\kappa,M] = \int_0^L \left[\tfrac{1}{2} M^\kappa \kappa + M(\kappa^w - \kappa)\right] dx - \int_0^L q\,w\,dx - \hat{M}\theta^w\Big|_B - \hat{V}\,w\Big|_B. \tag{E8.2}$$

Derive this functional starting from the WF diagram shown in Figure E8.2.

EXERCISE 8.4 [A:35] (Advanced, research paper level) Suppose that on the beam of Figure 8.1, loaded by $q(x)$, one applies an *additional* concentrated load P at an arbitrary cross section $x = x_P$. The additional transverse displacement under that load is w_P. The additional deflection elsewhere is $w_P\,\phi(x)$, where $\phi(x)$ is called an *influence function*, whose value at $x = x_P$ is 1. For simplicity assume that the end forces at B vanish: $\hat{M}_B = \hat{V}_B = 0$. The TPE functional can be viewed as function of two arguments:

$$\Pi[w, w_P] = \tfrac{1}{2}\int_0^L EI(w'' + w_P\phi'')^2\,dx - \int_0^L \big[q + P\delta(x_P)\big](w + w_P\phi)\,dx, \tag{E8.3}$$

where $w = w(x)$ denotes here the deflection for $P = 0$ and $\Delta(x_P)$ is Dirac's delta function.[5] Show that if the

[5] This is denoted by $\Delta(.)$ instead of the usual $\delta(.)$ to avoid confusion with the variation symbol.

beam is in equilibrium (that is, $\delta\Pi = 0$):

$$P = \frac{\partial U[w, w_P]}{\partial w_P}. \tag{E8.4}$$

This is called *Castigliano's theorem on forces* (also Castigliano's first theorem).[6] In words: the partial derivative of the internal (strain) energy expressed in terms of the beam deflections with respect to the displacement under a concentrated force gives the value of that force.

Note: This energy theorem can be generalized to arbitrary elastic bodies (not just beams) but requires fancy mathematics. It also applies to concentrated couples by replacing "displacement of the load" by "rotation of the couple." This result is often used in Structural Mechanics to calculate reaction forces at supports. Castigliano's energy theorem on deflections (also called Castigliano's second theorem), which is $w_Q = \partial U^*/\partial Q$ in which U^* is the internal complementary energy, is the one normally taught in undergraduate courses for Structures. Textbooks normally prove these theorems only for systems with finite number of degrees of freedom. "Proofs" for arbitrary continua are usually faulty because singular integrals are not properly handled.

[6] Some mathematical facility with integration by parts and delta functions is needed to prove this, but it is an excellent exercise for advanced math exams.