Eulerian convection diffusion explicit elements

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These notes discuss the implementation of the eulerian convection diffusion explicit element.

1 Explicit time integration methods

Explicit time integration methods are faster than implicit strategies, since they do not require the solution of non-linear or linear systems. Additionally, explicit time scheme are easy to implement. The main disadvantage of explicit strategies is that they are conditionally stable. Their stability depends on the time step, which must be properly estimated.

Next, Forward Euler and Runge-Kutta methods are discussed, considering the general problem

$$\frac{\partial y}{\partial t} = f(y(t), t). \tag{1}$$

1.1 The Forward Euler method

The Forward Euler method solving equation (1) is

$$\frac{y_{n+1} - y_n}{\Delta t} = f(y_n, t_n). \tag{2}$$

This scheme is really easy to implement, but it is only one order accurate in time. For this reason, other strategies time strategies are considered.

1.2 The Runge-Kutta 4 method

Runge-Kutta is a family of explicit time integration methods, characterized by evaluating the residual at a number of intermediate points. It is known that a Runge-Kutta n method gives an accuracy of order n, up to $n = 4$. Since for $n > 4$ the order is less than n, the Runge-Kutta 4 method is a really popular method [1].

The Runge-Kutta 4 method evaluates the residual at four different intermediate points in order to consequently update the solution from step t_n to step t_{n+1}. The unknown of the problem must be updated after each Runge-Kutta step. For equation (1), the Runge-Kutta 4 method reads

$$\frac{y_{n+1} - y_n}{\Delta t} = \frac{1}{6}\left[k_1 + k_2 + k_3 + k_4\right], \tag{3}$$

where $k_i, i \in 1, 2, 3, 4$ are the equation residuals. Such residuals are computed as

$$k_1 = f(t_n, y_n) \tag{4}$$
$$k_2 = f(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2}k_1) \tag{5}$$
$$k_3 = f(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2}k_2) \tag{6}$$
$$k_4 = f(t_n + \Delta t, y_n + \Delta t k_3). \tag{7}$$

In order to appreciate the difference between the two integration schemes, we solve a simple ordinary differential equation, defined as

$$\frac{\partial y}{\partial t} = -\sin(t) \qquad \text{in } [0, T] \tag{8}$$
$$y = 1 \qquad \text{in } t = 0, \tag{9}$$

where $T = 2$ seconds. Let u_h be the numerical solution and $u = \cos(t)$ be the analytical solution. Then, the l^2 norm of the error is

$$\varepsilon := \sqrt{(u_h - u)^2}. \tag{10}$$

We solve the above system with both the Forward Euler method and the Runge-Kutta 4 method, and in figure 1.1 we can observe the different order of the two schemes.

2 Explicit unsteady convection diffusion problem

We are interested in solving the convection diffusion equation

$$\frac{\partial \phi}{\partial t} + v \cdot \nabla\phi + \phi\nabla \cdot v - \nabla \cdot k\nabla\phi = f \qquad \text{in } \Omega \times [0, T] \tag{11}$$
$$\phi = 0 \qquad \text{on } \partial\Omega \times [0, T] \tag{12}$$
$$\phi = \phi^0 \qquad \text{in } \Omega, t = 0, \tag{13}$$

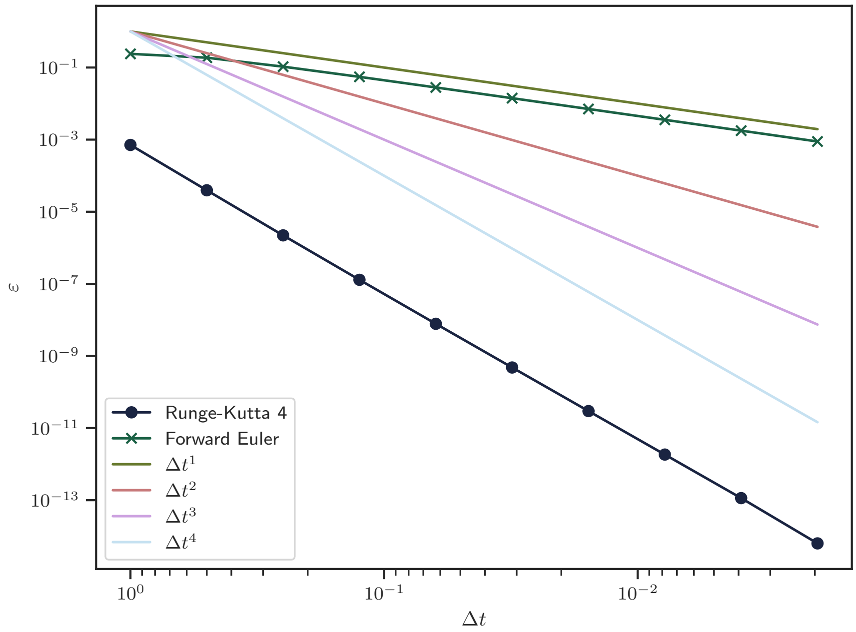


Figure 1.1: Order comparison between the Forward Euler and the Runge-Kutta 4 integration schemes.

where ϕ is the scalar solution, v the convective velocity, $k > 0$ the diffusivity coefficient, f the forcing term and $\Omega \times [0, T]$ the spatial and temporal domains. For simplicity, homogeneous Dirichlet boundary conditions are considered. The variable dependencies on time t and space x are omitted for the sake of simplicity. For example, we denote $\phi(t, x)$ as ϕ when there is no risk of misunderstanding.

Let $\mathcal{Q} = H_0^1(\Omega)$ be the space of functions whose components are square integrable, have square integrable first derivatives and vanish on $\partial\Omega$. Let $L^2(0, T; \mathcal{Q})$ be the space of functions which, for each fixed t, belong to $\mathcal{Q}$ and the $\mathcal{Q}$ norm of which is square integrable in time.

The weak form of the problem consists in finding $\phi \in L^2(0, T; \mathcal{Q})$ such that

$$\int_\Omega q\frac{\partial \phi}{\partial t}\,\mathrm{d}\Omega + \int_\Omega qv\cdot\nabla\phi\,\mathrm{d}\Omega + \int_\Omega q\phi\nabla\cdot v\,\mathrm{d}\Omega - \int_\Omega q\nabla\cdot k\nabla\phi\,\mathrm{d}\Omega = \int_\Omega qf\,\mathrm{d}\Omega \quad \forall q \in \mathcal{Q}, \tag{14}$$

where q are test functions.

We recall the two following mathematical properties

$$\nabla\cdot(u\mathbf{v}) = u\nabla\cdot\mathbf{v} + \mathbf{v}\cdot\nabla u \tag{15}$$

$$\int_\Omega \mathbf{v}\cdot\nabla u\,\mathrm{d}\phi = \int_{\partial\Omega} u\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}\partial\Omega - \int_\Omega u\nabla\cdot\mathbf{v}\,\mathrm{d}\Omega, \tag{16}$$

where u is a generic scalar, $\mathbf{v}$ a generic vector and $\mathbf{n}$ the outer normal of the boundary.

Then, integrating by part and omitting the null boundary terms (from equation (12)), the standard continuous weak form of the problem is

$$\int_\Omega q\frac{\partial \phi}{\partial t}\,\mathrm{d}\Omega+\int_\Omega qv\cdot\nabla\phi\,\mathrm{d}\Omega+\int_\Omega q\phi\nabla\cdot v\,\mathrm{d}\Omega+\int_\Omega k\nabla q\cdot\nabla\phi\,\mathrm{d}\Omega=\int_\Omega qf\,\mathrm{d}\Omega\quad\forall q\in\mathcal{Q}. \tag{17}$$

Let $\{\Omega^e\}_{e=1}^{n_{el}}$ be a finite element partition of Ω and $\mathcal{Q}_h\subset\mathcal{Q}$ the finite element space approximating $\mathcal{Q}$. The Galerkin method consists in finding $\phi_h\in L^2(0,T;\mathcal{Q}_h)$ such that

$$\begin{aligned}\int_\Omega q_h\frac{\partial \phi_h}{\partial t}\,\mathrm{d}\Omega+\int_\Omega q_hv\cdot\nabla\phi_h\,\mathrm{d}\Omega+\int_\Omega q_h\phi_h\nabla\cdot v\,\mathrm{d}\Omega+\int_\Omega k\nabla q_h\cdot\nabla\phi_h\,\mathrm{d}\Omega\\ =\int_\Omega q_hf\,\mathrm{d}\Omega\quad\forall q_h\in\mathcal{Q}_h.\end{aligned} \tag{18}$$

It is known that such a weak formulation lacks of stability when the convective part dominates the diffusive one. For this reason, stabilization finite element techniques must be considered. We decide to apply the subgrid scale approach (SGS). In particular, we exploit both the subgrid scale method with an algebraic approximation to the subscales, also known as algebraic subgrid scale (ASGS), and the Orthogoanl Subgrid Scale (OSS) approach. The former approach was firstly introduced in [2], and further discussed in [3, 4]. For the latter we refer to [5]. Both approaches consist of adding a residual-based stabilizing term to the Galerkin formulation.

The subgrid scale approach consists of splitting $\mathcal{Q}$ into the subspaces $\mathcal{Q}_h$ and $\mathcal{Q}_s$, where $\mathcal{Q}_s$ is a space completing $\mathcal{Q}_h$, and represents all components which can not be described by $\mathcal{Q}_h$, due to the mesh size for example. For this reason, it is denoted as the space of subgrid scales. $\mathcal{Q}_h$ is the space of resolvable scales, also known as Finite ELement (FEM) space. The difference between ASGS and OSS is the space on which the subgrid scales are considered. Therefore, we can write $\mathcal{Q}=\mathcal{Q}_h\bigoplus\mathcal{Q}_s$, $\phi=\phi_h+\phi_s$ and $q=q_h+q_s$. As a consequence, equation (17) can be split into two equations

$$\begin{aligned}&\int_\Omega q_h\frac{\partial \phi_h}{\partial t}\,\mathrm{d}\Omega+\int_\Omega q_hv\cdot\nabla\phi_h\,\mathrm{d}\Omega+\int_\Omega q_h\phi_h\nabla\cdot v\,\mathrm{d}\Omega+\int_\Omega k\nabla q_h\cdot\nabla\phi_h\,\mathrm{d}\Omega\\ &\quad+\int_\Omega q_h\frac{\partial \phi_s}{\partial t}\,\mathrm{d}\Omega+\int_\Omega q_hv\cdot\nabla\phi_s\,\mathrm{d}\Omega+\int_\Omega q_h\phi_s\nabla\cdot v\,\mathrm{d}\Omega+\int_\Omega k\nabla q_h\cdot\nabla\phi_s\,\mathrm{d}\Omega\\ &\quad=\int_\Omega q_hf\,\mathrm{d}\Omega\quad\forall q_h\in\mathcal{Q}_h\end{aligned} \tag{19}$$

and

$$\int_\Omega q_s \frac{\partial \phi_h}{\partial t}\,\mathrm{d}\Omega + \int_\Omega q_s v \cdot \nabla \phi_h\,\mathrm{d}\Omega + \int_\Omega q_s \phi_h \nabla \cdot v\,\mathrm{d}\Omega + \int_\Omega k \nabla q_s \cdot \nabla \phi_h\,\mathrm{d}\Omega$$
$$+ \int_\Omega q_s \frac{\partial \phi_s}{\partial t}\,\mathrm{d}\Omega + \int_\Omega q_s v \cdot \nabla \phi_s\,\mathrm{d}\Omega + \int_\Omega q_s \phi_s \nabla \cdot v\,\mathrm{d}\Omega + \int_\Omega k \nabla q_s \cdot \nabla \phi_s\,\mathrm{d}\Omega \quad (20)$$
$$= \int_\Omega q_s f\,\mathrm{d}\Omega \quad \forall q_s \in \mathcal{Q}_s.$$

Let

$$a(\phi, q) := \int_\Omega q \frac{\partial \phi}{\partial t}\,\mathrm{d}\Omega + \int_\Omega q v \cdot \nabla \phi\,\mathrm{d}\Omega + \int_\Omega q \phi \nabla \cdot v\,\mathrm{d}\Omega + \int_\Omega k \nabla q \cdot \nabla \phi\,\mathrm{d}\Omega \quad (21)$$

and

$$l(q) := \int_\Omega q f\,\mathrm{d}\Omega. \quad (22)$$

Then, equations (19) and (20) are equivalent to

$$a(\phi_h, q_h) + a(\phi_s, q_h) = l(q_h) \qquad \forall q_h \in \mathcal{Q}_h \quad (23)$$
$$a(\phi_h, q_s) + a(\phi_s, q_s) = l(q_s) \qquad \forall q_s \in \mathcal{Q}_s. \quad (24)$$

Instead of solving equation (24) for obtaining ϕ_s, we propose to model it. We suppose that its closed-form expression can reproduce its correct behavior, in order to be plugged in equation (23). Different methods exist in literature for modeling the solution on the subgrid scale space and we will consider ASGS and OSS. Another assumption we will make is if considering the solution on the subscales as dynamic or as stationary. Therefore, four different stabilization are considered next:

- quasi-static ASGS,
- quasi-static OSS,
- dynamic ASGS and
- dynamic OSS.

2.1 Quasi-static ASGS

2.1.1 Formulation

As mentioned, we try to give a closed-form expression for ϕ_s which can be plugged-in into equations (23) and (24), in order to obtain ϕ_h on the FEM space.

The modeling assumptions we make are the following. We assume ϕ_s vanishes on the boundaries of each element $\partial\Omega^e$, $e = 1, \ldots, n_{el}$, and we consider the quasi-static hypothesis $\frac{\partial \phi_s}{\partial t} = 0$. As shown in [3, 4], the simplest way to approximate equation (24) is to assume

$$\phi_s \approx \tau \mathcal{P}_s \mathcal{R}_h, \tag{25}$$

where τ is a matrix of stabilization parameters, $\mathcal{P}_s$ the identity matrix and $\mathcal{R}_h$ the finite element residual, defined as

$$\mathcal{R}_h := f - \frac{\partial \phi_h}{\partial t} - v \cdot \nabla \phi_h - \phi_h \nabla \cdot v + \nabla \cdot k \nabla \phi_h. \tag{26}$$

The estimation of τ will be discussed in section 2.6.

Integrating by parts equation (23) within each element domain, neglecting the null boundary terms and applying the quasi-static hypothesis, we get

$$\begin{aligned} a(\phi_h, q_h) + \sum_{e=1}^{n_{el}} \left[- \int_{\Omega^e} \phi_s \nabla \cdot (q_h v) \, \mathrm{d}\Omega + \int_{\Omega^e} q_h \phi_s \nabla \cdot v \, \mathrm{d}\Omega - \int_{\Omega^e} \phi_s \nabla \cdot k \nabla q_h \, \mathrm{d}\Omega \right] \\ = l(q_h) \quad \forall q_h \in \mathcal{Q}_h, \end{aligned} \tag{27}$$

which is equivalent to

$$a(\phi_h, q_h) + \sum_{e=1}^{n_{el}} \int_{\Omega^e} \left(-v \cdot \nabla q_h - \nabla \cdot k \nabla q_h \right) \phi_s \, \mathrm{d}\Omega = l(q_h) \quad \forall q_h \in \mathcal{Q}_h. \tag{28}$$

As known from [3], most stabilization methods for the convection diffusion problem consist in adding a term as

$$r(\phi_h, q_h) := \sum_{e=1}^{n_{el}} \int_{\Omega^e} \mathcal{P}(q_h)^t \tau \mathcal{R}(\phi_h) \tag{29}$$

to the Galerkin weak form, described in equation (18). $\mathcal{P}(q_h)$ is an operator applied to the test functions, and changes according to the stabilization method, τ is a matrix of stabilization parameters, and $\mathcal{R}(\phi_h)$ is the residual of the partial differential equation.

The operators $\mathcal{P}$ and $\mathcal{R}$ we obtained in equation (28) are

$$\mathcal{P}(q_h) := -v \cdot \nabla q_h - \nabla \cdot k \nabla q_h \tag{30}$$

$$\mathcal{R}(\phi_h) := f - \frac{\partial \phi_h}{\partial t} - v \cdot \nabla \phi_h - \phi_h \nabla \cdot v + \nabla \cdot k \nabla \phi_h, \tag{31}$$

which are consistent with literature [3]

$$\mathcal{P}(q_h) := \frac{\partial q_h}{\partial t} + v \cdot \nabla q_h + \nabla \cdot k \nabla q_h \tag{32}$$

$$\mathcal{R}(\phi_h) := \frac{\partial \phi_h}{\partial t} + v \cdot \nabla \phi_h + \phi_h \nabla \cdot v - \nabla \cdot k \nabla \phi_h - f. \tag{33}$$

We give in the next subsection details about how to treat terms of the ASGS stabilized convection diffusion problem, described in equation (28).

2.1.2 Discretization

Let $\int_{\Omega'} := \sum_{e=1}^{n_{el}} \int_{\Omega^e}$. The Finite Element problem we are interested in is the following. We want to find $\phi_h \in L^2(0,T;\mathcal{Q}_h)$ such that

$$\begin{aligned} &\int_\Omega q_h \frac{\partial \phi_h}{\partial t} \, \mathrm{d}\Omega + \int_\Omega q_h v \cdot \nabla \phi_h \, \mathrm{d}\Omega + \int_\Omega q_h \phi_h \nabla \cdot v \, \mathrm{d}\Omega + \int_\Omega k \nabla q_h \cdot \nabla \phi_h \, \mathrm{d}\Omega \\ &- \int_{\Omega'} \phi_s v \cdot \nabla q_h \, \mathrm{d}\Omega - \int_{\Omega'} \phi_s \nabla \cdot k \nabla q_h \, \mathrm{d}\Omega = \int_\Omega q_h f \, \mathrm{d}\Omega \quad \forall q_h \in \mathcal{Q}_h, \end{aligned} \tag{34}$$

where $\phi_s \approx \tau \mathcal{R}_h$ and q_h are test functions of $\mathcal{Q}_h$. We recall we are exploiting the quasi-static hypothesis $\frac{\partial \phi_s}{\partial t} = 0$, and interelement boundary terms are neglected.

The discrete solution $\phi_h(t,x)$ on the space $\mathcal{Q}_h$ is defined as

$$\phi_h(t,x) = \sum_{i=1}^{n_n} q_{h_i}(x) \phi_i(t), \tag{35}$$

where n_n denotes the number of nodes and ϕ_i the nodal value of the solution on node i. q_{h_i} represents the standard finite element basis function associated to node i.

Each contribution of equation (34) can be discretized by matrices. The global matrices solving the problem can be built assembling the elemental contribution of each operator. In particular, a general matrix $\boldsymbol{A}$ can be constructed from the assembly of its elemental contributions $\boldsymbol{A}^e$. For a finite element with N nodes, $\boldsymbol{A}^e$ is a matrix of size $N \times N$, and A^e_{ij} is the contribution of the pair of nodes (i,j). Using such notation, we can now define the elemental matrices.

Mass term The original dynamic term of the strong equation is

$$\frac{\partial \phi}{\partial t}, \tag{36}$$

which in the weak formulation becomes

$$\int_\Omega q_h \frac{\partial \phi}{\partial t} \, \mathrm{d}\Omega. \tag{37}$$

Then, the discretization on a element e reads

$$M_{ij}^e = \int_\Omega q_{h_i} q_{h_j} \,\mathrm{d}\Omega, \tag{38}$$

which should be multiplied by $\frac{\partial \phi}{\partial t}$ in the global discretized equation. Moreover, since the explicit time integration is applied, the mass matrix multiplying the acceleration is used in the lumped format.

Convection term The convection term of the original problem is

$$v \cdot \nabla\phi + \phi\nabla \cdot v, \tag{39}$$

where the considered framework is Eulerian, and v is the convective velocity. Its weak form reads

$$\int_\Omega q_h v \cdot \nabla\phi_h \,\mathrm{d}\Omega + \int_\Omega q_h \phi_h \nabla \cdot v \,\mathrm{d}\Omega, \tag{40}$$

and the finite element discretization on element e reads

$$C(v)_{ij}^e = \int_{\Omega^e} q_{h_i} v \cdot \nabla q_{h_j} \,\mathrm{d}\Omega + \int_{\Omega^e} q_{h_i} q_{h_j} \nabla \cdot v \,\mathrm{d}\Omega. \tag{41}$$

Diffusion term The diffusive term of the original problem is

$$-\nabla \cdot k\nabla\phi, \tag{42}$$

which leads to a weak formulation of the form

$$\int_\Omega k\nabla q_h \cdot \nabla\phi_h \,\mathrm{d}\Omega, \tag{43}$$

where k is the diffusivity coefficient, and the term has been integrated by parts. The boundary term arising when integrating by parts has been neglected.

The finite element elemental discretization is

$$D(k)_{ij}^e = \int_{\Omega^e} k\nabla q_{h_i} \cdot \nabla q_{h_j} \,\mathrm{d}\Omega. \tag{44}$$

Forcing term The forcing term f of the strong form of the original problem becomes

$$\int_\Omega q_h f \,\mathrm{d}\Omega, \tag{45}$$

in the weak formulation on the FEM subspace $\mathcal{Q}_h$. Its elemental discretization is

$$F_i = \int_{\Omega^e} q_{h_i} f_i \,\mathrm{d}\Omega. \tag{46}$$

ASGS convective term The ASGS convective term of equation (34) is

$$-\int_{\Omega'} \phi_s v \cdot \nabla q_h \,\mathrm{d}\Omega. \tag{47}$$

Applying $\phi_s = \tau \mathcal{R}_h$, we get

$$-\int_{\Omega'} \tau (f - \frac{\partial \phi_h}{\partial t} - v \cdot \nabla \phi_h - \phi_h \nabla \cdot v + \nabla \cdot k \nabla \phi_h) v \cdot \nabla q_h \,\mathrm{d}\Omega. \tag{48}$$

Since we are using linear elements, the term $\nabla \cdot k \nabla \phi_h$ is null, and the ASGS convective term is

$$-\int_{\Omega'} \tau (f - \frac{\partial \phi_h}{\partial t} - v \cdot \nabla \phi_h - \phi_h \nabla \cdot v) v \cdot \nabla q_h \,\mathrm{d}\Omega. \tag{49}$$

The four contributions are

$$-\int_{\Omega'} \tau f (v \cdot \nabla q_h) \,\mathrm{d}\Omega \tag{50}$$

$$\int_{\Omega'} \tau \frac{\partial \phi_h}{\partial t} (v \cdot \nabla q_h) \,\mathrm{d}\Omega \tag{51}$$

$$\int_{\Omega'} \tau (v \cdot \nabla \phi_h)(v \cdot \nabla q_h) \,\mathrm{d}\Omega \tag{52}$$

$$\int_{\Omega'} \tau (\phi_h \nabla \cdot v)(v \cdot \nabla q_h) \,\mathrm{d}\Omega. \tag{53}$$

The elemental discretization is

$$S_{cf}(\tau, v, f)^e_{ij} = -\int_{\Omega^e} \tau f_i (v \cdot \nabla q_{h_j}) \,\mathrm{d}\Omega \tag{54}$$

$$S_{cm}(\tau, v)^e_{ij} = \int_{\Omega^e} \tau q_{h_i} \frac{\partial \phi_{h_i}}{\partial t} (v \cdot \nabla q_{h_j}) \,\mathrm{d}\Omega \tag{55}$$

$$S_{cc}(\tau, v)^e_{ij} = \int_{\Omega^e} \tau (v \cdot \nabla q_{h_i})(v \cdot \nabla q_{h_j}) \,\mathrm{d}\Omega + \int_{\Omega^e} \tau (q_{h_i} \nabla \cdot v)(v \cdot \nabla q_{h_j}) \,\mathrm{d}\Omega. \tag{56}$$

Since the explicit time integration is applied, the stabilization term involving the solution acceleration may be considered in the lumped format, or the old and most recent solution acceleration may be used to approximate it. This last option is followed in this case.

ASGS diffusion term The ASGS diffusion term of equation (34) is

$$-\int_{\Omega'} \phi_s \nabla \cdot k \nabla q_h \,\mathrm{d}\Omega, \tag{57}$$

which we try to integrate by parts, otherwise $\nabla \cdot k\nabla q_h = 0$ due to the linear elements we are using. Then, omitting the boundary term, we obtain

$$\int_{\Omega'} k\nabla\phi_s \cdot \nabla q_h \, \mathrm{d}\Omega, \tag{58}$$

and since we cannot describe $\nabla\phi_h$, we set it equal to zero. Therefore, the ASGS diffusive term is null.

Global system Assembling all contributions, the final discretized system we need to solve is the following:

$$\boldsymbol{M}\frac{\partial \boldsymbol{\phi}}{\partial t} + \left(\boldsymbol{C} + \boldsymbol{D} + \boldsymbol{S_{cc}}\right)\boldsymbol{\phi} = \boldsymbol{F} - \boldsymbol{S_{cf}} - \boldsymbol{S_{cm}}, \tag{59}$$

where $\boldsymbol{\phi}$ denotes the solution vector on the nodes of the discretized domain. We remark all matrices dependencies have been neglected.

2.2 Quasi-static OSS

2.2.1 Formulation

The OSS stabilization models ϕ_s in order to plug-in the solution on the subscales space in equation (23), to obtain ϕ_h on the FEM space. The most important difference with respect to ASGS stabilization is the subscales space considered, which is the space completing the FEM space and orthogonal to the FEM space. In other words, $\mathcal{Q}_s = \mathcal{Q}_h^{\perp}$.

As before, the solution on the subscales space $\mathcal{Q}_s$ is approximated as

$$\phi_s \approx \tau \mathcal{P}_s \mathcal{R}_h, \tag{60}$$

where τ is the matrix of stabilization parameters, $\mathcal{P}_s$ a matrix we will discuss next, and $\mathcal{R}_h$ the equation residual on the FEM space. $\mathcal{P}_s$ is a matrix that should ensure ϕ_s belonging to $\mathcal{Q}_s$. As known from [5], the above equation can be reformulated as

$$\phi_s \approx \tau \left(\mathcal{R}_h + p_h\right), \tag{61}$$

where p_h is the component that makes ϕ_s belonging to $\mathcal{Q}_h^{\perp}$, while $p_h \in \mathcal{Q}_h$. The other two modeling assumptions we assume are to consider ϕ_s null on the boundaries of each element, and that the subscales solution is steady-state, namely $\frac{\partial \phi_s}{\partial t} = 0$.

From [5], due to the assumption given by equation (61), the equation for getting p_h is $\int_{\Omega'} q_h \phi_s \, \mathrm{d}\Omega = 0 \; \forall q_h \in \mathcal{Q}_h$. Namely,

$$\int_{\Omega'} \tau q_h \mathcal{R}_h \, \mathrm{d}\Omega + \int_{\Omega'} \tau q_h p_h \, \mathrm{d}\Omega = 0 \quad \forall q_h \in \mathcal{Q}_h, \tag{62}$$

where $\mathcal{R}_h$ is calculated element-wise and τ can be eliminated.

Therefore, before solving equation (23) (with the modeling assumption of equation (61)), we need to estimate the values of p_h. The final formulation for the quasi-static OSS stabilization reads as follows. We want to find $\phi_h \in L^2(0,T;\mathcal{Q}_h)$ such that

$$\begin{aligned}\int_\Omega q_h \frac{\partial \phi_h}{\partial t}\,\mathrm{d}\Omega + \int_\Omega q_h v \cdot \nabla \phi_h \,\mathrm{d}\Omega + \int_\Omega q_h \phi_h \nabla \cdot v \,\mathrm{d}\Omega + \int_\Omega k \nabla q_h \cdot \nabla \phi_h \,\mathrm{d}\Omega \\ - \int_{\Omega'} \phi_s v \cdot \nabla q_h \,\mathrm{d}\Omega - \int_{\Omega'} \phi_s \nabla \cdot k \nabla q_h \,\mathrm{d}\Omega = \int_\Omega q_h f \,\mathrm{d}\Omega \quad \forall q_h \in \mathcal{Q}_h,\end{aligned} \tag{63}$$

where $\phi_s \approx \tau\left(\mathcal{R}_h + p_h\right)$ and q_h are test functions of $\mathcal{Q}_h$.

2.2.2 Discretization

In this part, we analyze the new discretized terms we have with respect to quasi-static ASGS. Since the Galerking formulation remains the same, we refer to section 2.1.2 for details. Then, we focus only on OSS convective discretization term, since OSS diffusive term is null due to linear elements.

OSS convection term The OSS convection term in is equation (63) is

$$-\int_{\Omega'} \phi_s v \cdot \nabla q_h \,\mathrm{d}\Omega. \tag{64}$$

Applying equation (61), we obtain

$$-\int_{\Omega'} \tau(f - \frac{\partial \phi}{\partial t} - v \cdot \nabla \phi_h - \phi_h \nabla \cdot v + \nabla \cdot k \nabla \phi_h + p_h) v \cdot \nabla q_h \,\mathrm{d}\Omega. \tag{65}$$

Since we are using linear elements, $\nabla \cdot k \nabla \phi_h = 0$. Therefore, the five contributes are

$$-\int_{\Omega'} \tau f (v \cdot \nabla q_h) \,\mathrm{d}\Omega \tag{66}$$

$$\int_{\Omega'} \tau \frac{\partial \phi_h}{\partial t} (v \cdot \nabla q_h) \,\mathrm{d}\Omega \tag{67}$$

$$\int_{\Omega'} \tau (v \cdot \nabla \phi_h)(v \cdot \nabla q_h) \,\mathrm{d}\Omega \tag{68}$$

$$\int_{\Omega'} \tau (\phi_h \nabla \cdot v)(v \cdot \nabla q_h) \,\mathrm{d}\Omega \tag{69}$$

$$-\int_{\Omega'} \tau p_h (v \cdot \nabla q_h) \,\mathrm{d}\Omega. \tag{70}$$

The elemental discretization is

$$S_{cf}(\tau, v, f)^e_{ij} = -\int_{\Omega^e} \tau f_i (v \cdot \nabla q_{h_j})\, \mathrm{d}\Omega \tag{71}$$

$$S_{cm}(\tau, v)^e_{ij} = \int_{\Omega^e} \tau q_{h_i} \frac{\partial \phi_{h_i}}{\partial t} (v \cdot \nabla q_{h_j})\, \mathrm{d}\Omega \tag{72}$$

$$S_{cc}(\tau, v)^e_{ij} = \int_{\Omega^e} \tau (v \cdot \nabla q_{h_i})(v \cdot \nabla q_{h_j})\, \mathrm{d}\Omega + \int_{\Omega^e} \tau (q_{h_i} \nabla \cdot v)(v \cdot \nabla q_{h_j})\, \mathrm{d}\Omega \tag{73}$$

$$S_{cp}(\tau, v)^e_{ij} = -\int_{\Omega^e} \tau p_{h_i} (v \cdot \nabla q_{h_j})\, \mathrm{d}\Omega. \tag{74}$$

As discussed before, since the explicit time integration is applied, the stabilization term involving the solution acceleration is approximated considering the solution at previous time step and the solution at current time step.

Global system Assembling all contributions, the final discretized system we need to solve is the following:

$$\boldsymbol{M} \frac{\partial \boldsymbol{\phi}}{\partial t} + (\boldsymbol{C} + \boldsymbol{D} + \boldsymbol{S_{cc}})\, \boldsymbol{\phi} = \boldsymbol{F} - \boldsymbol{S_{cf}} - \boldsymbol{S_{cm}} - \boldsymbol{S_{cp}}, \tag{75}$$

where $\boldsymbol{\phi}$ is the solution vector on the nodes of the discretized domain. We remark all matrices dependencies have been neglected.

2.3 Dynamic ASGS

2.3.1 Formulation

The dynamic ASGS stabilization assumes almost all the modeling hypotheses of the quasi-static ASGS stabilization approach. The only difference is that now the the solution on the subscales space is dynamic, therefore it may happen that $\frac{\partial \phi_s}{\partial t} \neq 0$.

As for the quasi-static ASGS, instead of solving equation (24), we try to model ϕ_s in order to solve equation (23) afterwards. On each element $e \in \Omega$, equation (24) must hold. Therefore

$$\frac{\partial \phi_s}{\partial t} + v \cdot \nabla \phi_s + \phi_s \nabla \cdot v + \nabla \cdot k \nabla \phi_s = \mathcal{R}_h, \tag{76}$$

where $\mathcal{R}_h$ the residual on $\mathcal{Q}_h$ given by equation (26). Then, developing the temporal derivative term, we obtain

$$\frac{\phi_s}{\Delta t} + v \cdot \nabla \phi_s + \phi_s \nabla \cdot v + \nabla \cdot k \nabla \phi_s = \mathcal{R}_h + \frac{\phi_s^n}{\Delta t}, \tag{77}$$

where Δt is the time step length and ϕ_s^n the subscales solution at previous time step. Modeling $\frac{\phi_s}{\Delta t} + v \cdot \nabla \phi_s + \phi_s \nabla \cdot v + \nabla \cdot k \nabla \phi_s$ as $\tau_t^{-1} \phi_s$, the subscales solution on $\mathcal{Q}_s$ is modeled as

$$\phi_s \approx \tau_t \frac{\phi_s^n}{\Delta t} + \tau_t \mathcal{P}_s \mathcal{R}_h, \tag{78}$$

where τ_t is a matrix of stabilization parameters, Δt the time step length, ϕ_s^n the subscales solution at previous time step, $\mathcal{P}_s$ is the identity matrix, and $\mathcal{R}_h$ the residual on $\mathcal{Q}_h$ given by equation (26). We can intuitively see that τ_t differs from τ by a $\frac{1}{\Delta t}$ factor, and further details will be given in section 2.6. Therefore, the modeling assumptions are the model approximating ϕ_s and the null boundary terms.

Integrating by parts equation (23), and applying the null boundary terms hypothesis, we obtain

$$\begin{aligned} a(\phi_h, q_h) &+ \int_{\Omega'} q_h \frac{\partial \phi_s}{\partial t} \, \mathrm{d}\Omega - \int_{\Omega'} \phi_s v \cdot \nabla q_h \, \mathrm{d}\Omega - \int_{\Omega'} \phi_s \nabla \cdot k \nabla q_h \, \mathrm{d}\Omega \\ &= l(q_h) \quad \forall q_h \in \mathcal{Q}_h. \end{aligned} \tag{79}$$

We need now a way to estimate $\frac{\partial \phi_s}{\partial t}$. The temporal derivative is approximated as

$$\frac{\partial \phi_s}{\partial t} \approx \frac{\phi_s - \phi_s^n}{\Delta t}, \tag{80}$$

and applying the modeling assumptions on ϕ_s we obtain

$$\frac{\partial \phi_s}{\partial t} \approx \tau_t \left(\mathcal{R}_h + \frac{\phi_s^n}{\Delta t} \right) - \frac{\phi_s^n}{\Delta t}. \tag{81}$$

Then, adding equation (81) into equation (79), we obtain the final formulation. We want to find $\phi_h \in L^2(0, T; \mathcal{Q}_h)$ such that

$$\begin{aligned} &\int_{\Omega} q_h \frac{\partial \phi_h}{\partial t} \, \mathrm{d}\Omega + \int_{\Omega} q_h v \cdot \nabla \phi_h \, \mathrm{d}\Omega + \int_{\Omega} q_h \phi_h \nabla \cdot v \, \mathrm{d}\Omega + \int_{\Omega} k \nabla q_h \cdot \nabla \phi_h \, \mathrm{d}\Omega \\ &\quad + \int_{\Omega'} q_h \frac{\partial \phi_s}{\partial t} \, \mathrm{d}\Omega - \int_{\Omega'} \phi_s v \cdot \nabla q_h \, \mathrm{d}\Omega - \int_{\Omega'} \phi_s \nabla \cdot k \nabla q_h \, \mathrm{d}\Omega \\ &\quad = \int_{\Omega} q_h f \, \mathrm{d}\Omega \quad \forall q_h \in \mathcal{Q}_h. \end{aligned} \tag{82}$$

We recall $\phi_s \approx \tau_t (\frac{\phi_s^n}{\Delta t} + \mathcal{R}_h)$, $\frac{\partial \phi_s}{\partial t} \approx \tau_t \left(\mathcal{R}_h + \frac{\phi_s^n}{\Delta t} \right) - \frac{\phi_s^n}{\Delta t}$, interelement boundary terms are neglected and $\int_{\Omega'} := \sum_{e=1}^{n_{el}} \int_{\Omega^e}$.

Since ϕ_s^n is needed, we need to keep track of this value. We store the subscales on each integration point used to evaluate integrals. Then, at the end of each time step, we update the subscales as

$$\phi_s^{n+1} = \tau_t \frac{\phi_s^n}{\Delta t} + \tau_t \mathcal{R}_h, \tag{83}$$

where $n+1$ and n are the current and the previous time steps, respectively. As a general rule, the initial value of the subscales on the first time step is zero.

2.3.2 Discretization

In this part, we analyze the new term we introduced with respect to quasi-static ASGS, namely the elemental integrals arising from dynamic subscales assumption. We remark that ASGS diffusive term is null due to linear elements.

Dynamic ASGS term The new terms present in the weak formulation of equation (82) is

$$\int_{\Omega'} q_h \frac{\partial \phi_s}{\partial t} \, \mathrm{d}\Omega. \tag{84}$$

We apply $\frac{\partial \phi_s}{\partial t} \approx \tau_t \left(\mathcal{R}_h + \frac{\phi_s^n}{\Delta t} \right) - \frac{\phi_s^n}{\Delta t}$, and we obtain

$$\begin{aligned} &\int_{\Omega'} \tau_t q_h f \, \mathrm{d}\Omega - \int_{\Omega'} \tau_t q_h \frac{\partial \phi_h}{\partial t} \, \mathrm{d}\Omega - \int_{\Omega'} \tau_t q_h v \cdot \nabla \phi_h \, \mathrm{d}\Omega - \int_{\Omega'} \tau_t q_h \phi_h \nabla \cdot v \, \mathrm{d}\Omega \\ &- \int_{\Omega'} \tau_t k \nabla q_h \cdot \nabla \phi_h \, \mathrm{d}\Omega + \int_{\Omega'} \tau_t q_h \frac{\phi_s^n}{\Delta t} \, \mathrm{d}\Omega - \int_{\Omega'} q_h \frac{\phi_s^n}{\Delta t} \, \mathrm{d}\Omega. \end{aligned} \tag{85}$$

The elemental discretization, on a element e, is

$$S_{df}(\tau_t)_i^e = \int_{\Omega^e} \tau_t q_{h_i} f_i \, \mathrm{d}\Omega \tag{86}$$

$$S_{dm}(\tau_t)_{ij}^e = - \int_{\Omega^e} \tau_t q_{h_i} q_{h_j} \frac{\partial \phi_{h_j}}{\partial t} \, \mathrm{d}\Omega \tag{87}$$

$$S_{dc}(v, \tau_t)_{ij}^e = - \int_{\Omega^e} \tau_t q_{h_i} v \cdot \nabla q_{h_j} \, \mathrm{d}\Omega - \int_{\Omega^e} \tau_t q_{h_i} q_{h_j} \nabla \cdot v \, \mathrm{d}\Omega \tag{88}$$

$$S_{dd}(k, \tau_t)_{ij}^e = - \int_{\Omega^e} \tau_t k \nabla q_{h_i} \cdot \nabla q_{h_j} \, \mathrm{d}\Omega \tag{89}$$

$$S_{ds}(\tau_t)_i^e = \int_{\Omega^e} \tau_t q_{h_i} \frac{\phi_{s_i}^n}{\Delta t} \, \mathrm{d}\Omega - \int_{\Omega^e} q_{h_i} \frac{\phi_{s_i}^n}{\Delta t} \, \mathrm{d}\Omega. \tag{90}$$

The temporal derivative $\frac{\partial \phi_{h_j}}{\partial t}$ is approximated as above, thus computing the acceleration between the solution ϕ_h on the previous time step and on current time step.

ASGS convective term The ASGS convective term of equation (82) is

$$- \int_{\Omega'} \phi_s v \cdot \nabla q_h \, \mathrm{d}\Omega. \tag{91}$$

Applying $\phi_s = \tau_t \left(\frac{\phi_s^n}{\Delta t} + \mathcal{R}_h \right)$, we get

$$-\int_{\Omega'} \tau_t (\frac{\phi_s^n}{\Delta t} + f - \frac{\partial \phi_h}{\partial t} - v \cdot \nabla \phi_h - \phi_h \nabla \cdot v + \nabla \cdot k \nabla \phi_h) v \cdot \nabla q_h \, \mathrm{d}\Omega. \tag{92}$$

Due to linear elements, the term $\nabla \cdot k \nabla \phi_h$ is null, and the ASGS convective term becomes

$$-\int_{\Omega'} \tau_t (\frac{\phi_s^n}{\Delta t} + f - \frac{\partial \phi_h}{\partial t} - v \cdot \nabla \phi_h - \phi_h \nabla \cdot v) v \cdot \nabla q_h \, \mathrm{d}\Omega. \tag{93}$$

The elemental contributions are

$$S_{cd}(\tau_t, v, \phi_s^n)_{ij}^e = -\int_{\Omega^e} \tau_t \frac{\phi_{s_i}^n}{\Delta t} (v \cdot \nabla q_{h_j}) \, \mathrm{d}\Omega \tag{94}$$

$$S_{cf}(\tau_t, v, f)_{ij}^e = -\int_{\Omega^e} \tau_t f_i (v \cdot \nabla q_{h_j}) \, \mathrm{d}\Omega \tag{95}$$

$$S_{cm}(\tau_t, v)_{ij}^e = \int_{\Omega^e} \tau_t q_{h_i} \frac{\partial \phi_{h_i}}{\partial t} (v \cdot \nabla q_{h_j}) \, \mathrm{d}\Omega \tag{96}$$

$$S_{cc}(\tau_t, v)_{ij}^e = \int_{\Omega^e} \tau_t (v \cdot \nabla q_{h_i})(v \cdot \nabla q_{h_j}) \, \mathrm{d}\Omega + \int_{\Omega^e} \tau_t (q_{h_i} \nabla \cdot v)(v \cdot \nabla q_{h_j}) \, \mathrm{d}\Omega. \tag{97}$$

The stabilization term involving the solution acceleration is approximated exploiting the previous time step and the most recent solutions.

Global system Assembling all contributions, the final discretized system we need to solve is the following:

$$\begin{aligned} &\boldsymbol{M} \frac{\partial \boldsymbol{\phi}}{\partial t} + (\boldsymbol{C} + \boldsymbol{D} + \boldsymbol{S_{cc}} + \boldsymbol{S_{dc}} + + \boldsymbol{S_{dd}}) \, \boldsymbol{\phi} \\ &\quad = \boldsymbol{F} - \boldsymbol{S_{cd}} - \boldsymbol{S_{cf}} - \boldsymbol{S_{cm}} - \boldsymbol{S_{df}} - \boldsymbol{S_{dm}} - \boldsymbol{S_{ds}}, \end{aligned} \tag{98}$$

where $\boldsymbol{\phi}$ denotes the solution vector on the nodes of the discretized domain.

2.4 Dynamic OSS

2.4.1 Formulation

As for the dynamic ASGS, also for the dynamic OSS we assume all the quasi-static OSS hypotheses, but we remove the quasi-static assumption. Therefore, the only modeling difference with respect quasi-static OSS is that $\frac{\partial \phi_s}{\partial t}$ may be different from 0. As for the dynamic ASGS, ϕ_s is modeled as

$$\phi_s \approx \tau_t \frac{\phi_s^n}{\Delta t} + \tau_t \mathcal{P}_s(\mathcal{R}_h), \tag{99}$$

where τ_t is a matrix of stabilization parameters, Δt the time step length, ϕ_s^n the subscales solution at previous time step, $\mathcal{P}_s$ is the matrix ensuring $\mathcal{Q}_s = \mathcal{Q}_h^\perp$, $\mathcal{R}_h$ the residual on $\mathcal{Q}_h$ given by equation (26). As mentioned above, τ_t differs from τ and it will be defined in section 2.6. As we did above and as shown in [5], the above modeling assumption can be replaced by

$$\phi_s \approx \tau_t \frac{\phi_s^n}{\Delta t} + \tau_t(\mathcal{R}_h + p_h), \tag{100}$$

where p_h is a variable defined on $\mathcal{Q}_h$ that makes $\phi_s \in \mathcal{Q}_h^\perp$. The other modeling assumption is assuming null the integrals on the element boundaries.

The equation we need to solve element-wise to calculate p_h is, as for the quasi-static OSS stabilization, $\int_{\Omega'} q_h \phi_s \, \mathrm{d}\Omega = 0$, which in the case of dynamic subscales hypothesis becomes

$$\int_{\Omega'} q_h \frac{\phi_s^n}{\Delta t} \, \mathrm{d}\Omega + \int_{\Omega'} q_h \mathcal{R}_h \, \mathrm{d}\Omega + \int_{\Omega'} q_h p_h \, \mathrm{d}\Omega = 0. \tag{101}$$

For each time step, once p_h is known we can solve equation (23), exploiting the modeling assumptions.

Moreover, we also need to consider the dynamic subscales. Similarly to what we did for the dynamic ASGS stabilization, we model the dynamic subscales term as

$$\frac{\partial \phi_s}{\partial t} \approx \tau_t \left(\mathcal{R}_h + p_h + \frac{\phi_s^n}{\Delta t} \right) - \frac{\phi_s^n}{\Delta t}. \tag{102}$$

Inserting the modeling assumptions on ϕ_s and $\frac{\partial \phi_s}{\partial t}$ into equation (23), we obtain the following formulation. We want to find $\phi_h \in L^2(0, T; \mathcal{Q}_h)$ such that

$$\begin{aligned} &\int_\Omega q_h \frac{\partial \phi_h}{\partial t} \, \mathrm{d}\Omega + \int_\Omega q_h v \cdot \nabla \phi_h \, \mathrm{d}\Omega + \int_\Omega q_h \phi_h \nabla \cdot v \, \mathrm{d}\Omega + \int_\Omega k \nabla q_h \cdot \nabla \phi_h \, \mathrm{d}\Omega \\ &\quad + \int_{\Omega'} q_h \frac{\partial \phi_s}{\partial t} \, \mathrm{d}\Omega - \int_{\Omega'} \phi_s v \cdot \nabla q_h \, \mathrm{d}\Omega - \int_{\Omega'} \phi_s \nabla \cdot k \nabla q_h \, \mathrm{d}\Omega \\ &\quad = \int_\Omega q_h f \, \mathrm{d}\Omega \quad \forall q_h \in \mathcal{Q}_h. \end{aligned} \tag{103}$$

We recall $\phi_s \approx \tau_t(\frac{\phi_s^n}{\Delta t} + \mathcal{R}_h + p_h)$, $\frac{\partial \phi_s}{\partial t} \approx \tau_t \left(\mathcal{R}_h + p_h + \frac{\phi_s^n}{\Delta t} \right) - \frac{\phi_s^n}{\Delta t}$, interelement boundary terms are neglected and $\int_{\Omega'} := \sum_{e=1}^{n_{el}} \int_{\Omega^e}$.

As before, we need to keep track of ϕ_s^n value. We store the subscales on each integration point used to evaluate integrals. Then, at the end of each time step, we update the subscales as

$$\phi_s^{n+1} = \tau_t \left(\frac{\phi_s^n}{\Delta t} + \mathcal{R}_h + p_h \right), \tag{104}$$

where $n+1$ and n are the current and the previous time steps, respectively. As a general rule, the initial value of the subscales on the first time step is zero.

2.4.2 Discretization

We analyze now the new stabilization elemental integrals of the dynamic OSS stabilization. We recall that OSS diffusive term is null due to linear elements.

Dynamic OSS term The dynamic terms present in the weak formulation of equation (103) is

$$\int_{\Omega'} q_h \frac{\partial \phi_s}{\partial t} \, \mathrm{d}\Omega. \tag{105}$$

Applying the modeling assumptions, we obtain

$$\begin{aligned} &\int_{\Omega'} \tau_t q_h f \, \mathrm{d}\Omega - \int_{\Omega'} \tau_t q_h \frac{\partial \phi_h}{\partial t} \, \mathrm{d}\Omega - \int_{\Omega'} \tau_t q_h v \cdot \nabla \phi_h \, \mathrm{d}\Omega - \int_{\Omega'} q_h \tau_t \phi_h \nabla \cdot v \, \mathrm{d}\Omega \\ &- \int_{\Omega'} \tau_t k \nabla q_h \cdot \nabla \phi_h \, \mathrm{d}\Omega + \int_{\Omega'} \tau_t q_h \frac{\phi_s^n}{\Delta t} \, \mathrm{d}\Omega - \int_{\Omega'} q_h \frac{\phi_s^n}{\Delta t} \, \mathrm{d}\Omega + \int_{\Omega'} \tau_t q_h p_h \, \mathrm{d}\Omega. \end{aligned} \tag{106}$$

The elemental discretization, on a element e, is

$$S_{df}(\tau_t)_i^e = \int_{\Omega^e} \tau_t q_{h_i} f_i \, \mathrm{d}\Omega \tag{107}$$

$$S_{dm}(\tau_t)_{ij}^e = -\int_{\Omega^e} \tau_t q_{h_i} q_{h_j} \frac{\partial \phi_{h_j}}{\partial t} \, \mathrm{d}\Omega \tag{108}$$

$$S_{dc}(v, \tau_t)_{ij}^e = -\int_{\Omega^e} \tau_t q_{h_i} v \cdot \nabla q_{h_j} \, \mathrm{d}\Omega - \int_{\Omega^e} \tau_t q_{h_i} q_{h_j} \nabla \cdot v \, \mathrm{d}\Omega \tag{109}$$

$$S_{dd}(k, \tau_t)_{ij}^e = -\int_{\Omega^e} k \nabla q_{h_i} \cdot \nabla q_{h_j} \, \mathrm{d}\Omega \tag{110}$$

$$S_{ds}(\tau_t)_i^e = \int_{\Omega^e} \tau_t q_{h_i} \frac{\phi_{s_i}^n}{\Delta t} \, \mathrm{d}\Omega - \int_{\Omega^e} q_{h_i} \frac{\phi_{s_i}^n}{\Delta t} \, \mathrm{d}\Omega \tag{111}$$

$$S_{dp}(\tau_t)_i^e = \int_{\Omega^e} \tau_t q_{h_i} p_{h_i} \, \mathrm{d}\Omega. \tag{112}$$

The temporal derivative $\frac{\partial \phi_{h_j}}{\partial t}$ is approximated as above, thus computing the acceleration between the solution ϕ_h on the previous time step and on current time step.

OSS convective term The OSS convective term of equation (103) is

$$-\int_{\Omega'} \phi_s v \cdot \nabla q_h \, \mathrm{d}\Omega. \tag{113}$$

Applying $\phi_s = \tau_t \left(\frac{\phi_s^n}{\Delta t} + \mathcal{R}_h + p_h \right)$, we get

$$-\int_{\Omega'} \tau_t(\frac{\phi_s^n}{\Delta t} + f - \frac{\partial \phi_h}{\partial t} - v \cdot \nabla \phi_h - \phi_h \nabla \cdot v + \nabla \cdot k \nabla \phi_h + p_h) v \cdot \nabla q_h \, \mathrm{d}\Omega. \quad (114)$$

Due to linear elements, the term $\nabla \cdot k \nabla \phi_h$ is null, and the OSS convective term becomes

$$-\int_{\Omega'} \tau_t(\frac{\phi_s^n}{\Delta t} + f - \frac{\partial \phi_h}{\partial t} - v \cdot \nabla \phi_h - \phi_h \nabla \cdot v + p_h) v \cdot \nabla q_h \, \mathrm{d}\Omega. \quad (115)$$

The elemental contributions are

$$S_{cd}(\tau_t, v, \phi_s^n)_{ij}^e = -\int_{\Omega^e} \tau_t \frac{\phi_{s_i}^n}{\Delta t} (v \cdot \nabla q_{h_j}) \, \mathrm{d}\Omega \quad (116)$$

$$S_{cf}(\tau_t, v, f)_{ij}^e = -\int_{\Omega^e} \tau_t f_i (v \cdot \nabla q_{h_j}) \, \mathrm{d}\Omega \quad (117)$$

$$S_{cm}(\tau_t, v)_{ij}^e = \int_{\Omega^e} \tau_t q_{h_i} \frac{\partial \phi_{h_i}}{\partial t} (v \cdot \nabla q_{h_j}) \, \mathrm{d}\Omega \quad (118)$$

$$S_{cc}(\tau_t, v)_{ij}^e = \int_{\Omega^e} \tau_t (v \cdot \nabla q_{h_i})(v \cdot \nabla q_{h_j}) \, \mathrm{d}\Omega + \int_{\Omega^e} \tau_t (q_{h_i} \nabla \cdot v)(v \cdot \nabla q_{h_j}) \, \mathrm{d}\Omega \quad (119)$$

$$S_{cp}(\tau_t, v, p_h)_{ij}^e = -\int_{\Omega^e} \tau_t p_{h_i} (v \cdot \nabla q_{h_j}) \, \mathrm{d}\Omega. \quad (120)$$

The stabilization term involving the solution acceleration is approximated exploiting the previous time step and the most recent solutions.

Global system Assembling all contributions, the final discretized system we need to solve is the following:

$$\begin{aligned} &\boldsymbol{M} \frac{\partial \boldsymbol{\phi}}{\partial t} + (\boldsymbol{C} + \boldsymbol{D} + \boldsymbol{S_{cc}} + \boldsymbol{S_{dc}} + + \boldsymbol{S_{dd}}) \, \boldsymbol{\phi} \\ &= \boldsymbol{F} - \boldsymbol{S_{cd}} - \boldsymbol{S_{cf}} - \boldsymbol{S_{cm}} - \boldsymbol{S_{cp}} - \boldsymbol{S_{df}} - \boldsymbol{S_{dm}} - \boldsymbol{S_{ds}} - \boldsymbol{S_{dp}}, \end{aligned} \quad (121)$$

where $\boldsymbol{\phi}$ denotes the solution vector on the nodes of the discretized domain.

2.5 Integration with explicit strategy

The chosen explicit strategy is the Runge-Kutta 4 method, which gives an order 4 accuracy in time. The Runge-Kutta strategy is explained above, and here we are interested in showing how we integrate the explicit strategy with the different convection diffusion formulations we explained.

Algorithm 1 reports the integration of the convection diffusion stabilized formulations with the Runge-Kutta 4 explicit strategy. N_g denotes the number of integration points, while k stands for the equation residual, as in equation (7).

Algorithm 1 convection diffusion explicit strategy

$\phi_s(g, t=0) = 0$, $g \in N_g$ **if** D-ASGS or D-OSS
while $t < T$ **do**
  for step $= 0 : 4$ **do**
    $p_h \longleftarrow$ equation (62) or (101) **if** Q-OSS or D-OSS
    $k_{\text{step}} \longleftarrow$ equation (59) or (75) or (98) or (121)
  end for
  $\phi_h \longleftarrow$ equation (3)
  $p_h \longleftarrow$ equation (62) or (101) **if** Q-OSS or D-OSS
  $\phi_s(g, t) \longleftarrow$ equation (83) or (104) , $g \in N_g$ **if** D-ASGS or D-OSS
  $t = t + \Delta t$
end while

2.6 Calibration of stabilization matrix coefficient

2.6.1 Quasi-static stabilization

The matrix of stabilization parameters τ is calibrated element-wise, depending on the convection and diffusion contributes of the element. As shown in [3, 4], a general expression for τ for a steady-state convection diffusion problem is

$$\tau = \left[\frac{c_1 k}{h^2} + \frac{c_2 \|v\|}{h} \right]^{-1}, \tag{122}$$

where for linear elements optimal parameters are $c_1 = 4$ and $c_2 = 2$. Extending to an unsteady system, the general expression we exploit for estimating τ is

$$\tau = \frac{1}{\frac{c_1 k}{h^2} + \frac{c_2 \|v\|}{h} + c_3 \nabla \cdot v + \frac{c_4}{\Delta t}}, \tag{123}$$

where $c_3 = 1$ is the coefficient multiplying the divergence term in the strong equation, and c_4 is a coefficient weighting the dynamic term. Moreover, as general rule, one can set an upper bound τ_{max} for the element-wise stabilization parameter. Therefore

$$\min \left\{ \left(\frac{1}{\frac{c_1 k}{h^2} + \frac{c_2 \|v\|}{h} + c_3 \nabla \cdot v + \frac{c_4}{\Delta t}}, \tau_{max} \right) \right\}. \tag{124}$$

2.6.2 Dynamic stabilization

For the case of dynamic stabilization, we do consider τ_t. From [5], the general expression for τ_t is given by

$$\tau_t = \frac{1}{\frac{c_1 k}{h^2} + \frac{c_2 \|v\|}{h} + \frac{1}{\Delta t}}. \tag{125}$$

Adding the velocity divergence contribution, it becomes

$$\tau_t = \frac{1}{\frac{c_1 k}{h^2} + \frac{c_2 \|v\|}{h} + c_3 \nabla \cdot v + \frac{1}{\Delta t}}, \tag{126}$$

where $c_3 = 1$ since in the strong equation the coefficient multiplying the velocity divergence term is unitary.

Again, we can set an element-wise upper bound for the stabilization parameter

$$\min \left\{ \left(\frac{1}{\frac{c_1 k}{h^2} + \frac{c_2 \|v\|}{h} + c_3 \nabla \cdot v + \frac{1}{\Delta t}}, \tau_{max} \right) \right\}. \tag{127}$$

2.7 Validation

2.7.1 Steady-state benchmark

In order to validate the implementation, a first steady state benchmark with explicit solution has been considered. Let $\Omega = [0,1] \times [0,1]$ and

$$\begin{aligned} &\phi(x,y) = \cos(2\pi x)\sin(2\pi y) \\ &f(x,y) = 2\pi(4\pi k \cos(2\pi x)\sin(2\pi y) - v_x \sin(2\pi x)\sin(2\pi y) + \\ &\quad + v_y \cos(2\pi x)\cos(2\pi y)), \end{aligned} \tag{128}$$

where $k = 2$ $[m^2/s]$ and $v = [v_x, v_y] = [2,3]$ $[m/s]$. The boundary conditions are of Dirichlet type, given by the boundary values of ϕ. The solution is shown in figure 2.1.

It is known that the l^2 norm of the error

$$\varepsilon = \sqrt{\int_\Omega (\phi - \phi_h)^2} \tag{129}$$

decreases as h^2, where h is the mesh size. ϕ and ϕ_h are the analytic and FEM solutions, respectively. Figure 2.2 shows that the error ε converges as expected for both quasi-static ASGS and quasi-static OSS.

2.7.2 Time convergence benchmark

In order to check the explicit time integration order with Runge-Kutta 4 scheme, an unsteady pure convection problem is considered. The benchmark is a two-dimensional bar, with space domain $\Omega = [0.0, 0.05] \times [0.0, 0.35]$, time domain $[0, T = \pi]$, diffusivity $k = 0$, forcing $f = 0$ and convective velocity field $v(t) = \cos(t)$, $t \in [0, T]$. The initial solution field in time $t = 0$ is $\phi_h = x$, where x denotes the horizontal coordinate. We report in figure 2.3 the solution field at $t = \pi$.

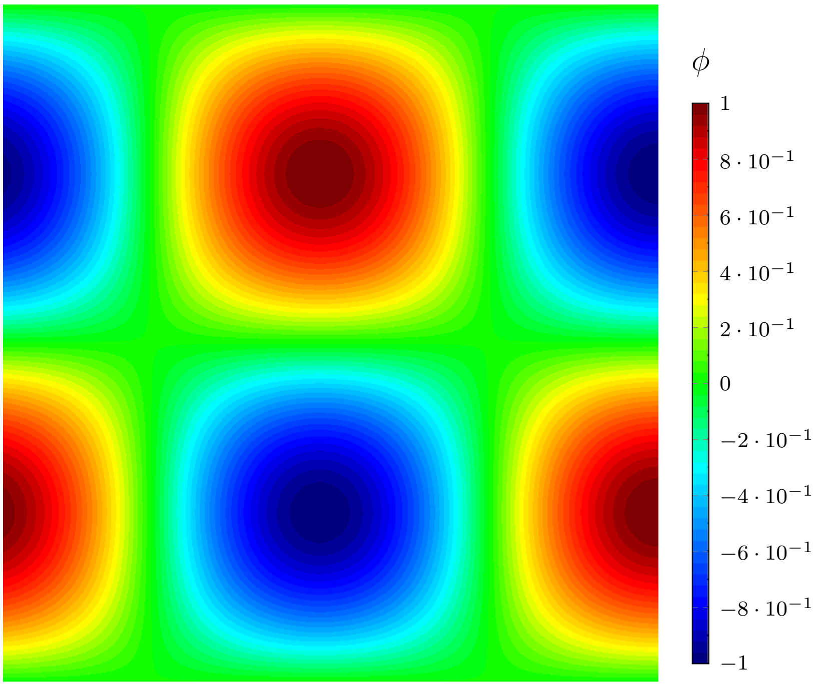


Figure 2.1: Explicit solution ϕ of the steady-state convection diffusion equation.

The problem is solved with standard Galerkin, quasi-static OSS and dynamic OSS formulations. The analytic solution at time t is

$$\phi = x - \sin(t), \tag{130}$$

therefore it is possible to compute the l^2 norm of the error as

$$\varepsilon = \sqrt{\int_\Omega (\phi - \phi_h)^2}, \tag{131}$$

where ϕ_h refers to the Finite Element Method solution. It is expected to obtain an order four accuracy for the Runge-Kutta 4 time integration scheme. We report in figure 2.4 the time accuracy studies for different formulations for the bar problem, which confirm the expected order four.

2.7.3 Pure convection gaussian hill problem

We consider a spatial domain $\Omega = [-1, 1] \times [-1, 1]$ and a temporal domain $[0, T]$, where $T = \frac{\pi}{10}$. We solve a pure convection problem, and we aim to observe that the solution value is transported, but does not spread in the domain. The governing

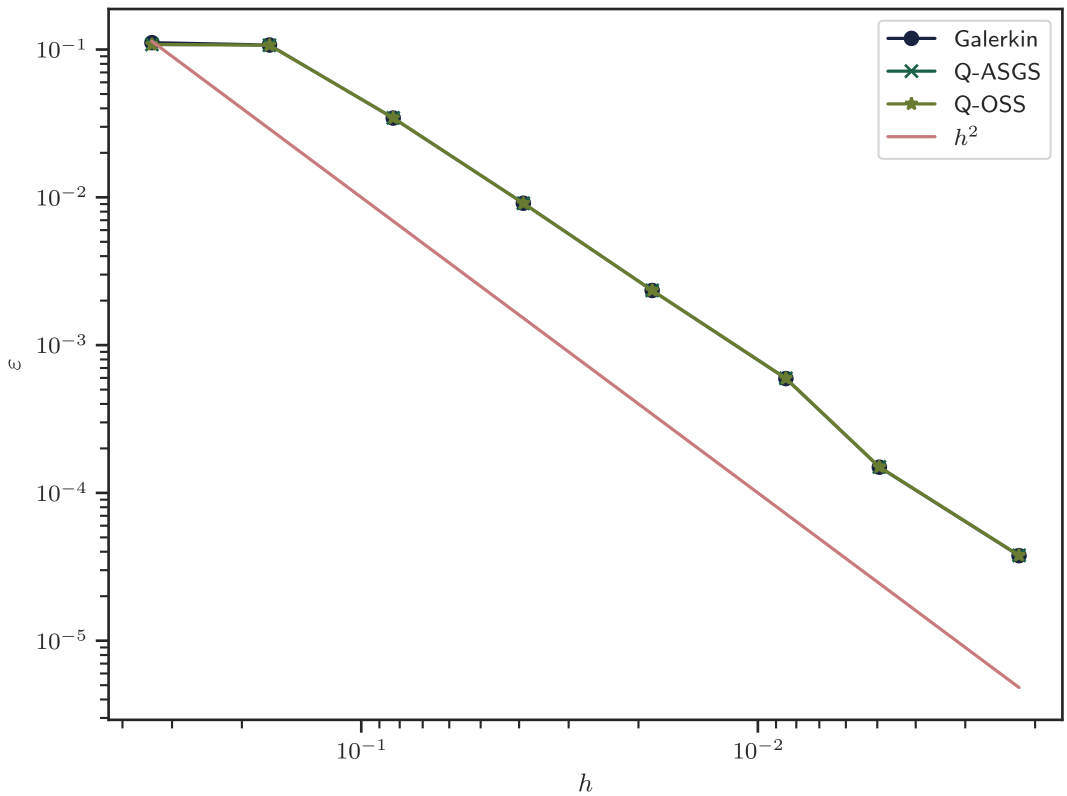


Figure 2.2: l^2 norm of the difference between analytical and numerical solution of the steady-state convection diffusion benchmark. Galerkin refers to the formulation without any stabilization, Q-ASGS to the quasi-static ASGS stabilization formulation, Q-OSS to the quasi-static OSS stabilization formulation.

equation is

$$\frac{\partial \phi}{\partial t} + v \cdot \nabla \phi = 0 \qquad \text{in } \Omega \times [0, T] \tag{132}$$

$$\phi = 0 \qquad \text{in } \partial\Omega \times [0, T] \tag{133}$$

$$\phi = \frac{1}{0.002\pi^2} \exp\left\{ -\left(\frac{r^2}{0.002\pi} \right) \right\} \qquad \text{in } \Omega, t = 0, \tag{134}$$

where, $r^2 = (x + 0.5)^2 + (y)^2$, $v = [-y, x, 0]$ with x and y referring to the horizontal and vertical axes, respectively.

Figure 2.5 shows that the gaussian hill peak is well preserved with all stabilization formulations. The solving mesh has around 277000 nodes and a characteristic length of 0.0023. The initial solution peak in $t = 0$ is 50.638.

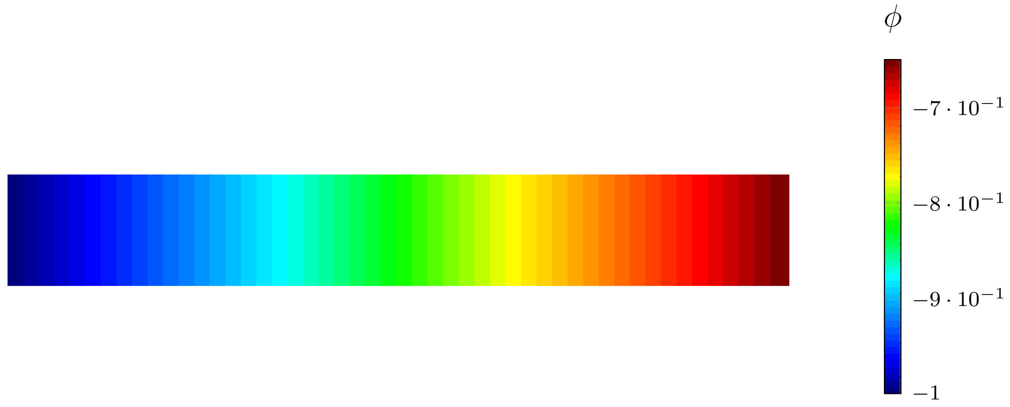


Figure 2.3: Solution field ϕ of the time convergence benchmark at time $t = \pi$.

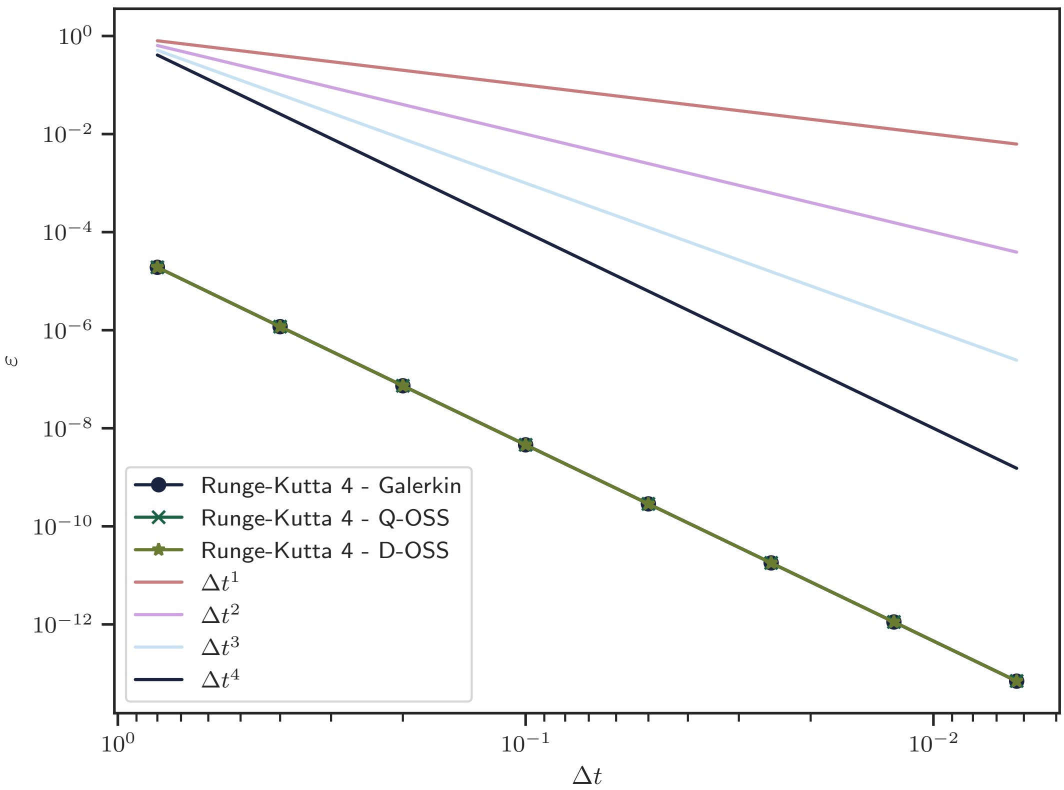
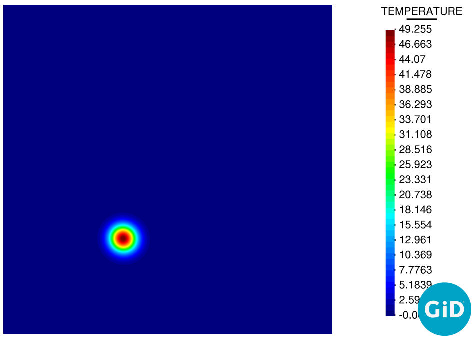
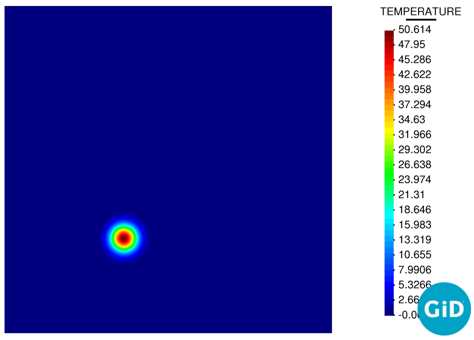


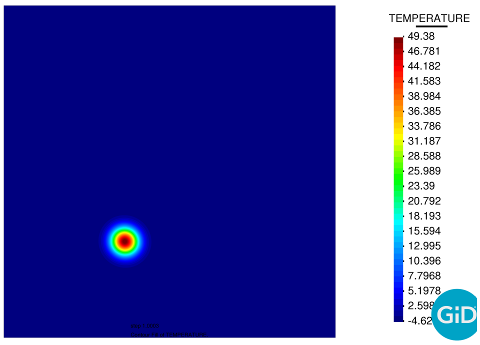
Figure 2.4: l^2 norm of the difference between analytical and numerical solution of the bar benchmark problem, for the same discretization and different time steps Δt. Galerkin refers to the formulation without any stabilization, Q-OSS to the quasi-static OSS stabilization formulation, D-OSS to the dynamic OSS stabilization formulation.



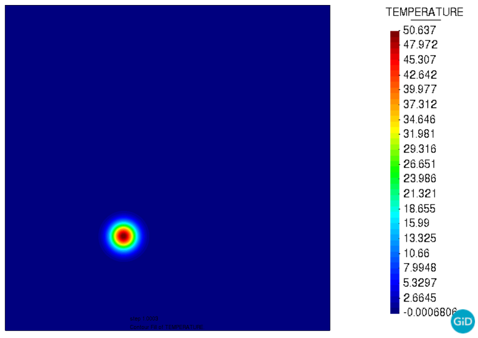
(a) Solution field at $t = \frac{\pi}{10}$ with Q-ASGS. The temperature peak is 49.255.



(b) Solution field at $t = \frac{\pi}{10}$ with Q-OSS. The temperature peak is 50.614.



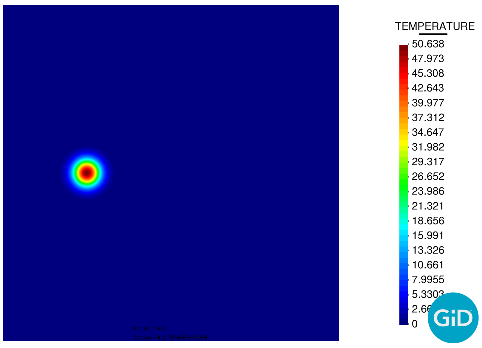
(c) Solution field at $t = 1$ with D-ASGS The temperature peak is 49.38.



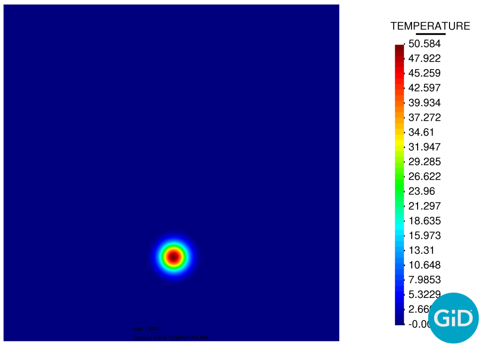
(d) Solution field at $t = 1$ with D-OSS. The temperature peak is 50.637.

Figure 2.5: Pure convection gaussian hill solution field. Comparison with different stabilization formulations. The initial solution peak in $t = 0$ is 50.638.

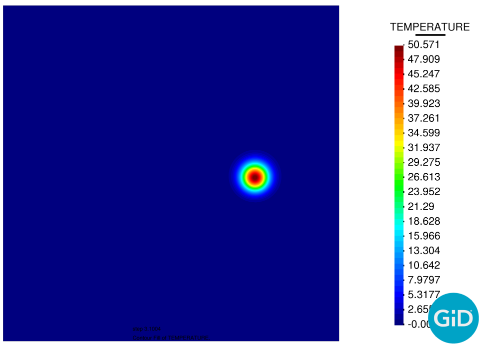
Moreover, one can appreciate in figure 2.6 a complete period of the gaussian hill, with dynamic OSS stabilization. The problem is solved for a uniform mesh with around 277000 nodes and characteristic length 0.0023.



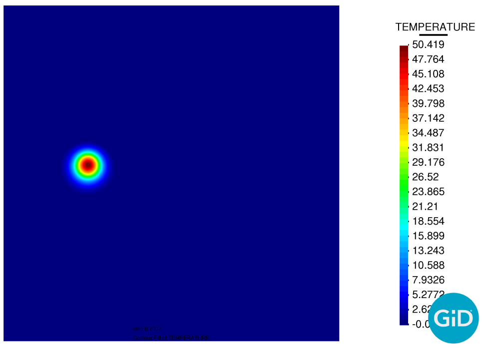
(a) Solution field at $t = 0$ with D-OSS. The temperature peak is 50.638.



(b) Solution field at $t = \frac{\pi}{2}$ with D-OSS. The temperature peak is 50.584.



(c) Solution field at $t = \pi$ with D-OSS. The temperature peak is 50.571.



(d) Solution field at $t = 2\pi$ with D-OSS. The temperature peak is 50.419.

Figure 2.6: Pure convection gaussian hill solution field with dynamic OSS stabilization.

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