2

Axisymmetric Solids, a.k.a. Structures of Revolution

TABLE OF CONTENTS

§2.1. Introduction

In the Introduction to Finite Element Methods (IFEM) course *two-dimensional* problems were emphasized. The axisymmetric problem considered in this and following two Chapters of this course provides a "bridge" to the treatment of three-dimensional elasticity. Besides its instructional value, the treatment of axisymmetric structures has considerable practical interest in aerospace, civil, mechanical and nuclear engineering.

§2.1.1. The Axisymmetric Problem

The axisymmetric problem deals with the analysis of structures of revolution under axisymmetric loading. A *structure of revolution* or SOR is generated by a *generating cross section* that rotates 360° about an *axis of revolution*, as illustrated in Figure 2.1. Such structures are said to be *rotationally symmetric*.

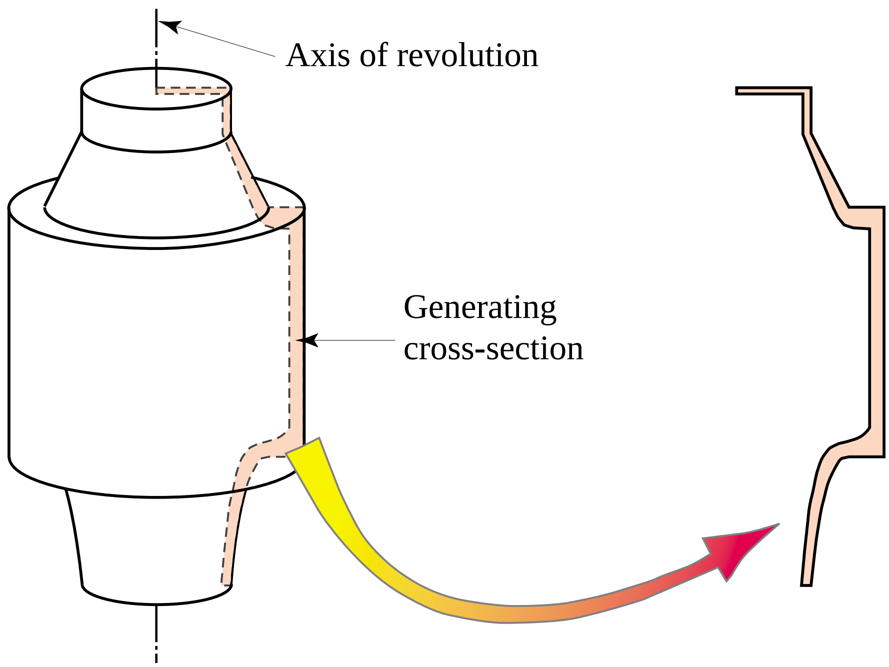


FIGURE 2.1. A structure of revolution is generated by rotating a generating cross section about an axis of revolution.

The technical importance of SOR's is considerable because of the following practical considerations:

1. Fabrication: axisymmetric bodies are usually easier to manufacture than bodies with more complex geometries. Think for example of pipes, piles, axles, wheels, bottles, cans, cups, nails.
2. Strength: axisymmetric configurations are often optimal in terms of strength to weight ratio because of the favorable distribution of the structural material. (Recall that the strongest columns and shafts, if wall buckling is ignored, have annular cross sections.)

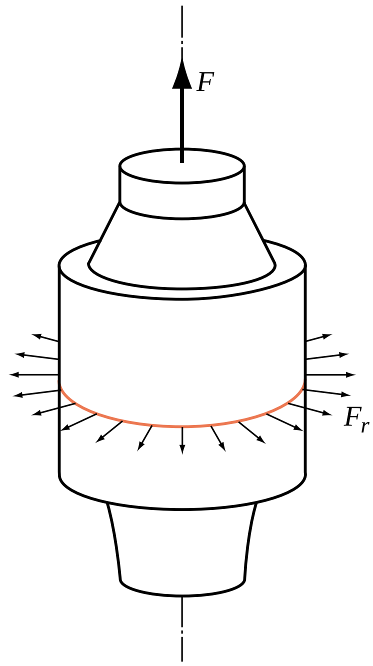


FIGURE 2.2. Axisymmetric loading on a SOR: F = concentrated load, F_r = radial component of “ring” line load.

3. Multipurpose: hollow axisymmetric bodies can assume a dual purpose as both *structure* and *shelter*, as in containers, vessels, tanks, rockets, etc.

Perhaps the most important application of SORs is containment and transport of liquid and gasses. Specific examples of such structures are pressure vessels, containment vessels, pipes, cooling towers, and rotating machinery (turbines, generators, shafts, etc.).

But a SOR by itself does not necessarily define an axisymmetric problem. It is also necessary that the loading, as well as the support boundary conditions, be rotationally symmetric. This is illustrated in Figure 2.2 for loads.

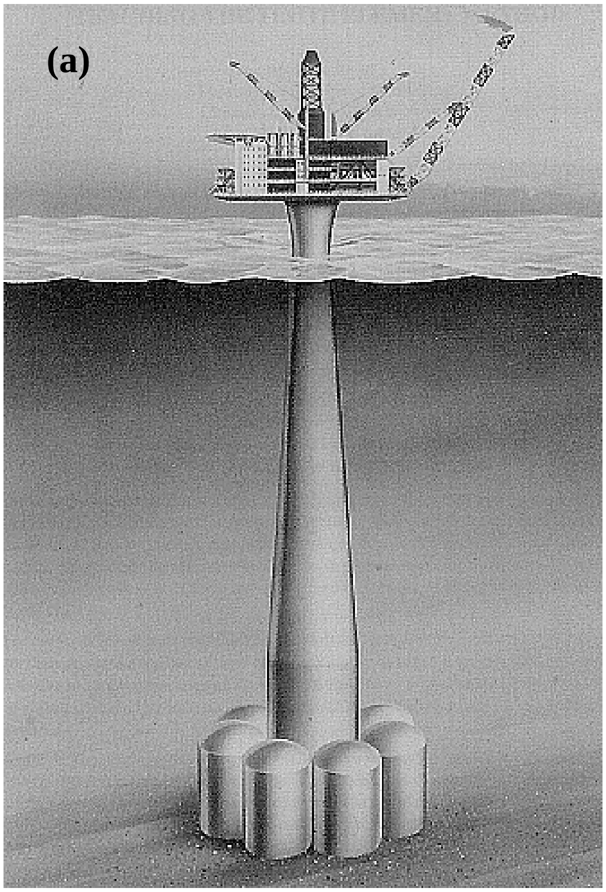
If these two conditions are met:

axisymmetric geometry	and	*axisymmetric loading*

the response of the structure is axisymmetric (also called radially symmetric). By this is meant that all quantities of interest in structural analysis: displacement, strains, and stresses, are *independent of the circumferential coordinate* defined below.

FIGURE 2.3. Axisymmetric FE analysis of a typical rocket nozzle (carried out by E. L. Wilson at Aerojet Corporation, circa 1963). Figure from paper cited in footnote 1.

Remark 2.1. A linear SOR under non-axisymmetric loading can be treated by a Fourier decomposition method. This involves decomposing the load into a Fourier series in the circumferential direction, calculating the response of the structure to each harmonic term retained in the series, and superposing the results. The axisymmetric problem considered here may be viewed as computing the response to the zero-th harmonic. This superposition technique, however, is limited to linear problems.



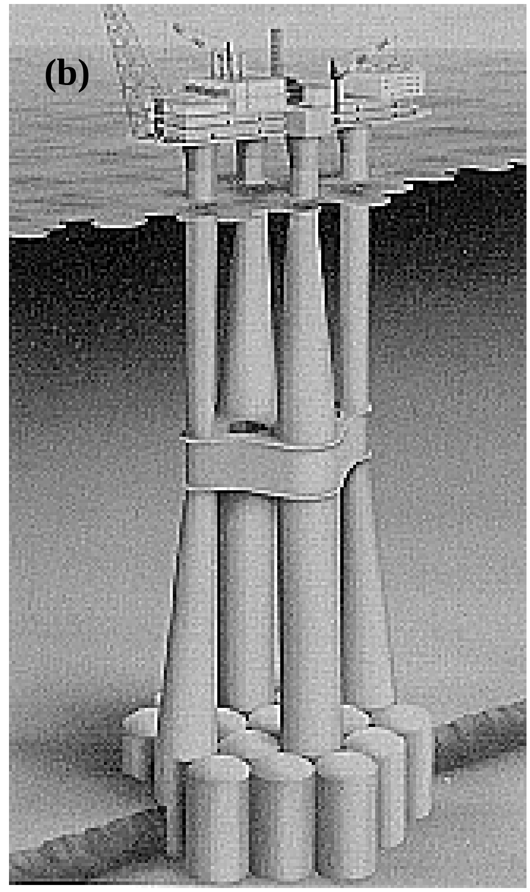


FIGURE 2.4. Two quasi-axisymmetric marine structures. (a) The Draugen oil-drilling platform (artist's sketch). The first monotower concrete platform built by Norwegian Contractors. The concrete structure is 295 m high. First deployed in 1993. The seven cells at the bottom of the sea form a reservoir system that can store up to 1.4 M barrels of oil. (b) The Troll oil-drilling platform (artist sketch). The tallest concrete platform built to date. It is 386 m tall and has 220,000 m^3 of concrete. The foundation consists of 36 m tall concrete skirts that penetrate into the soft seabed.

§2.1.2. Some SOR Examples

Rocket Analysis. The analysis of axisymmetric structures by the Finite Element Method (FEM) has a long history that may be traced back to the early 1960s. Recall that the FEM originated in the aircraft industry in the mid 1950s. Aircraft are not SORs, but several structures of interest in aerospace are, notably rockets. As the FEM began to disseminate throughout the aerospace industry, interest in application to rocket analysis prompted the development of the first *axisymmetric finite elements* during the period 1960-1965. These elements were of shell and solid type. The first archival-journal paper on axisymmetric solid elements, by E. L. Wilson, appeared in 1965.[1] Figure 2.3 shows a realistic application to a rocket nozzle presented in that first paper.

SOR Members as Major Structural Components. Often important structural *components* are have axisymmetric geometry such as pipes, but the entire structure is not SOR. Two examples taken

[1] E. L. Wilson, Structural Analysis of Axisymmetric solids, *AIAA Journal*, Vol. 3, No. 12, 1965.

from the field of petroleum engineering are shown in Figure 2.4.[2] These are two recent designs of oil-drilling platforms intended for water depths of 300 to 400m. As can be observed the main structural members are axisymmetric (reinforced concrete cylindrical shells). This kind of structure is often analyzed by *global-local techniques*. In the global analysis such members are treated with beam or simplified shell models. Forces computed from the global analysis are then applied to individual members for a more detailed 3D analysis that may take advantage of axisymmetry.

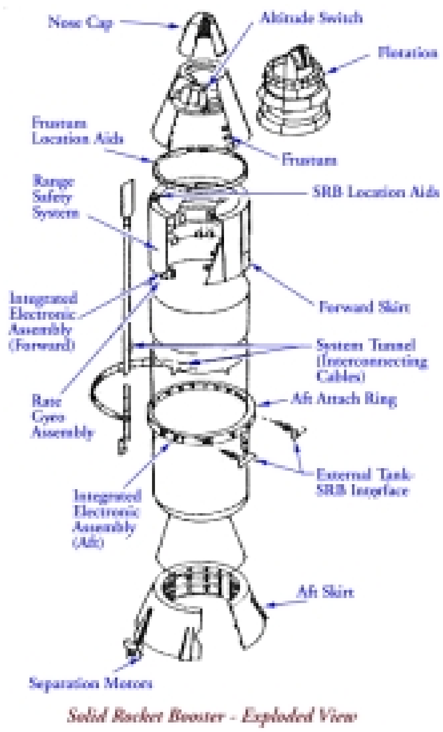


FIGURE 2.5. Solid Rocket Booster (SRB) of Space Shuttle orbiter: a quasi-axisymmetric structure.

Quasi-axisymmetric structures. There is an important class of structures that may be termed “quasi-axisymmetric,” in which the axisymmetric geometry is locally perturbed by non-axisymmetric features such as access openings, foundations and nonstructural attachments. Important examples are cooling towers, container vehicles, jet engines and rockets. See for example the SRB of Figure 2.5. Such structures may benefit from a global-local analysis if the axisymmetric characteristics dominate. In this case the global analysis is axisymmetric but the local analyses are not.

§2.2. The Governing Equations

§2.2.1. Global Coordinate System

[2] From the article by B. Jacobsen, ‘The evolution of the offshore concrete platform,’ in *From Finite Elements to the Troll Platform — a book in honor of Ivar Holand’s 70th Anniversary*, ed. by K. Bell, Dept. of Structural Engineering, The Norwegian Institute of Technology, Torndheim, Norway, 1994.

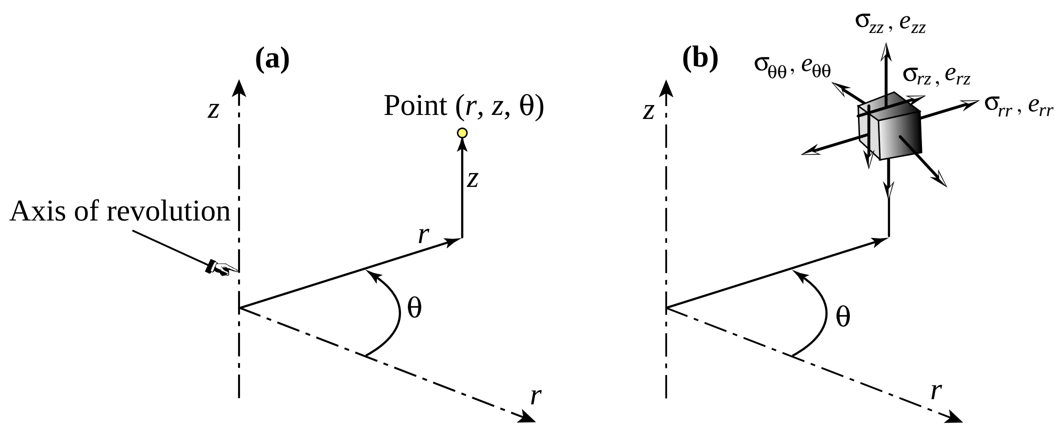


FIGURE 2.6. (a) Global cylindrical coordinate system (r, z, θ) for axisymmetric structural analysis; (b) strains and stresses with respect to cylindrical coordinate system.

To simplify the governing equations of the axisymmetric problem it is natural to use a global *cylindrical* coordinate system (r, z, θ) where

- r the *radial* coordinate: distance from the axis of revolution; always $r \geq 0$.
- z the *axial* coordinate: directed along the axis of revolution.
- θ the *circumferential* coordinate, also called the longitude.

The global coordinate system is sketched in Figure 2.6(a).

Remark 2.2. Note that $\{r, z\}$ form a right-handed Cartesian coordinate system on the $\theta = const$ planes, whereas $\{r, \theta\}$ form a polar coordinate system on the $z = const$ planes.

§2.2.2. Displacement, Strains, Stresses

The displacement field is a function of r and z only, defined by two components:

$$\mathbf{u}(r, z) = \begin{bmatrix} u_r(r, z) \\ u_z(r, z) \end{bmatrix} \tag{2.1}$$

u_r is called the *radial displacement* and u_z is the *axial displacement*. The *circumferential displacement* component, u_θ, is zero on account of rotational symmetry.

The infinitesimal strain tensor in cylindrical coordinates is represented by the symmetric matrix:

$$[\,\mathbf{e}\,] = \begin{bmatrix} e_{rr} & e_{rz} & e_{r\theta} \\ e_{rz} & e_{zz} & e_{z\theta} \\ e_{z\theta} & e_{z\theta} & e_{\theta\theta} \end{bmatrix} \tag{2.2}$$

Because of the assumed axisymmetric state, $e_{r\theta}$ and $e_{z\theta}$ vanish, leaving only four distinct components:

$$[\,\mathbf{e}\,] = \begin{bmatrix} e_{rr} & e_{rz} & 0 \\ e_{rz} & e_{zz} & 0 \\ 0 & 0 & e_{\theta\theta} \end{bmatrix} \tag{2.3}$$

Each of these vanishing components is a function of r and z only. As usual in preparation for finite element work, the nonvanishing components are arranged as a 4×1 strain vector:

$$\mathbf{e} = \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ \gamma_{rz} \end{bmatrix} \tag{2.4}$$

in which $\gamma_{rz} = e_{rz} + e_{zr} = 2e_{rz}$. This differs from the plane stress case considered in the introductory course in the appearance of $e_{\theta\theta}$, the "hoop" or circumferential strain.

The stress tensor in cylindrical coordinates is represented by the symmetric matrix

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{rr} & \sigma_{rz} & \sigma_{r\theta} \\ \sigma_{rz} & \sigma_{zz} & \sigma_{z\theta} \\ \sigma_{r\theta} & \sigma_{z\theta} & \sigma_{\theta\theta} \end{bmatrix} \tag{2.5}$$

Again because of axisymmetry the components $\sigma_{r\theta}$ and $\sigma_{z\theta}$ vanish, leaving four nontrivial components:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{rr} & \sigma_{rz} & 0 \\ \sigma_{rz} & \sigma_{zz} & 0 \\ 0 & 0 & \sigma_{\theta\theta} \end{bmatrix} \tag{2.6}$$

Each of the nonvanishing components is a function of r and z. Collecting these four components into a stress vector:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{\theta\theta} \\ \sigma_{rz} \end{bmatrix} \tag{2.7}$$

where $\sigma_{rz} \equiv \sigma_{rz}$. The difference with respect to the plane stress problem is again the appearance of the "hoop" or circumferential stress $\sigma_{\theta\theta}$.

The stresses and strains over an infinitesimal volume are depicted in Figure 2.7.

§2.3. Governing Equations

The elasticity equations for the axisymmetric problem are the field equations: strain-displacement, stress-strain, and stress equilibrium equations, complemented by displacement and stress boundary conditions.

§2.3.1. Kinematic Equations

The strain-displacement equations for the axisymmetric problem are:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{zz} = \frac{\partial u_z}{\partial z}, \quad e_{\theta\theta} = \frac{u_r}{r}, \quad \gamma_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} = e_{rz} + e_{zr} = 2e_{rz}. \tag{2.8}$$

In matrix form:

$$\mathbf{e} = \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ \gamma_{rz} \end{bmatrix} = \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ 2e_{rz} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial r} & 0 \\ 0 & \dfrac{\partial}{\partial z} \\ \dfrac{1}{r} & 0 \\ \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} u_r \\ u_z \end{bmatrix} = \mathbf{D}\,\mathbf{u}. \tag{2.9}$$

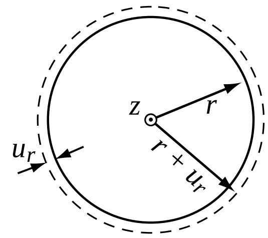


FIGURE 2.7. A uniform radial displacement u_r induces a circumferential strain u_r/r.

where **D** is the 4×2 strain-displacement (symmetric-gradient) operator. A noteworthy difference with respect to the plane stress case is the appearance of the hoop strain $e_{\theta\theta} = u_r/r$. Thus a uniform radial displacement is *no longer a rigid body motion, but produces a circumferential strain*. The physical reason behind this phenomenon is illustrated in Figure 2.7. The length of the original circumference is $2\pi r$, which grows to $2\pi(r + u_r)$, inducing a strain $2\pi u_r/2\pi r = u_r/r$.

§2.3.2. Constitutive Equations

For a linear hyperelastic material, and ignoring thermal and prestress effects, the most general constitutive equation consistent with axisymmetry takes the form:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{\theta\theta} \\ \sigma_{rz} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{12} & E_{22} & E_{23} & E_{24} \\ E_{13} & E_{23} & E_{33} & 0 \\ E_{14} & E_{24} & 0 & E_{44} \end{bmatrix} \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ \gamma_{rz} \end{bmatrix} = \mathbf{E}\,\mathbf{e} \tag{2.10}$$

To retain axisymmetry, the cross-coupling between the shear strain and hoop stress must vanish. Consequently $E_{34} = E_{43} = 0$.

For an isotropic material of elastic modulus E and Poisson's ratio ν,

$$\mathbf{E} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \tag{2.11}$$

Remark 2.3. The coefficients of **E** go to infinity if $\nu \to 1/2$, which characterizes an incompressible material. This behavior is a consequence of the "confinement" effect in solids and appears also in general 3D analysis. On the other hand, the plane stress constitutive matrix remains finite for $\nu = \frac{1}{2}$, a behavior that is characteristic of thin bodies such as plates and shells. Physically, the small transverse dimension of bodies in plane stress (plates) allows the material to freely expand or contract in the z direction.

§2.3.3. Equilibrium Equations

The general (three dimensional) differential equations of equilibrium in cylindrical coordinates are

$$\begin{aligned} &\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{1}{r}\frac{\partial}{\partial\theta}(\sigma_{r\theta}) + \frac{\partial}{\partial z}\sigma_{rz} - \frac{\sigma_{\theta\theta}}{r} + b_r = 0 \\ &\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{zr}) + \frac{1}{r}\sigma_{z\theta} + \frac{\partial}{\partial z}\sigma_{zz} + b_z = 0 \\ &\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\sigma_{\theta r}) + \frac{1}{r}\frac{\partial}{\partial\theta}\sigma_{\theta\theta} + \frac{\partial}{\partial z}\sigma_{\theta z} + b_\theta = 0 \end{aligned} \tag{2.12}$$

where b_r, b_z, b_θ are the components of the body force field in the r, z and θ directions, respectively. For the axisymmetric problem these equations reduce to

$$\begin{aligned} &\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{\partial}{\partial z}\sigma_{rz} - \frac{\sigma_{\theta\theta}}{r} + b_r = 0 \\ &\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{zr}) + \frac{\partial}{\partial z}\sigma_{zz} + b_z = 0 \end{aligned} \tag{2.13}$$

The third equation in (2.12) is identically satified if $b_\theta = 0$, because $\sigma_{\theta r} = \sigma_{\theta z} = 0$ and $\sigma_{\theta\theta}$ is independent of θ. If $b_\theta \neq 0$ the problem cannot be treated as axisymmetric.

§2.3.4. Boundary Conditions

As usual boundary conditions can be of displacement (PBC) or of stress or traction (FBC) type. They are specified on portions S_u and S_t of the boundary, respectively. The reduction of the stress BCs to two dimension is further discussed in §2.4.2 and §2.4.3.

§2.4. Variational Formulation

The variational form of the axisymmetric problem is illustrated with the widely used Total Potential Energy (TPE) form. The delicate part of the formulation is the dimensionality reduction step.

§2.4.1. The TPE Functional

The Total Potential Energy (TPE) functional contains only displacements as master field:

$$\Pi[\mathbf{u}] = U[\mathbf{u}] - W[\mathbf{u}]. \tag{2.14}$$

Here the strain energy functional is

$$U[\mathbf{u}] = \tfrac{1}{2}\int_V \boldsymbol{\sigma}^T\,\mathbf{e}\,dV = \tfrac{1}{2}\int_V \mathbf{e}^T\mathbf{E}\,\mathbf{e}\,dV = \tfrac{1}{2}\int_V \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ 2e_{rz} \end{bmatrix}^T \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{12} & E_{22} & E_{23} & E_{24} \\ E_{13} & E_{23} & E_{33} & 0 \\ E_{14} & E_{24} & 0 & E_{44} \end{bmatrix} \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ 2e_{rz} \end{bmatrix} dV. \tag{2.15}$$

In (2.15) the strains are a slave field are derived from displacements. Superscript u used in Chapter 3–8 to identify the master filed is omitted to reduce clutter.

The external work potential is the sum of contributions due to body force and prescribed surface tractions:

$$\begin{aligned} W[\mathbf{u}] &= W_b[\mathbf{u}] + W_t[\mathbf{u}] \\ W_b[\mathbf{u}] &= \int_V \mathbf{b}^T\mathbf{u}\,dV = \int_V [\,b_r \quad b_z\,] \begin{bmatrix} u_r \\ u_z \end{bmatrix} dV \\ W_t[\mathbf{u}] &= \int_{S_t} \hat{\mathbf{t}}^T\mathbf{u}\,dS = \int_{S_t} [\,\hat{t}_r \quad \hat{t}_z\,] \begin{bmatrix} u_r \\ u_z \end{bmatrix} dS \end{aligned} \tag{2.16}$$

Here $\mathbf{b}$ is the body force vector and $\hat{\mathbf{t}}$ the vector of surface tractions.

§2.4.2. Dimensionality Reduction

The element of volume dV that appears in U and W_b can be expressed as the "ring element"

$$dV = 2\pi r\, dA \tag{2.17}$$

where dA is the element of area in the generating cross section. Insertion in (2.15) and the second of (2.16) reduces U and W_b to area integrals:

$$U = \tfrac{1}{2} 2\pi \int_A r\, \mathbf{e}^T \mathbf{E}\, \mathbf{e}\, dA \tag{2.18}$$

$$W_b = 2\pi \int_A r\, \mathbf{b}^T\, \mathbf{u}\, dA \tag{2.19}$$

Notice the appearance of r in the integrand.

Similarly, the element of surface dS in W_t can be expressed as

$$dS = 2\pi r\, ds \tag{2.20}$$

where ds is an arclength element. Inserting in the last of (2.16) reduces W_t to a one-dimensional (line) integral

$$W_t = 2\pi \int_{S_t} r\, \mathbf{t}^T \mathbf{u}\, ds \tag{2.21}$$

The common factor 2π in these integrals is (usually) suppressed in the finite element implementation. This should not cause difficulties except for the case of a concentrated load, as discussed in the following subsection.

We summarize the outcome of this dimensionality reduction by saying that the original three-dimensional problem has been reduced to a two-dimensional one.

§2.4.3. Line and Point Forces

Body forces (e.g. gravity or centrifugal forces) and distributed surface forces (e.g. pressure) are handled like in plane elasticity case explained in LFEM, but concentrated loads require more careful treatment. There are two possibilities: a line load and an actual concentrated load.

A line load is actually a "ring" load (see Figure 2.2) acting on a circle described by a point of the generating cross section. If the global components of this load are F_r and F_z, the appropriate energy contribution to the loads potential W is

$$W_F = 2\pi r (F_r u_r + F_z u_z) \tag{2.22}$$

where (u_r, u_z) are the displacements of the "ring" point. Thus the ubiquitous 2π term can be suppressed

A concentrated or point load F, however, can only act along the z direction at points on the axis of revolution as illustrated in Figure 2.2. The corresponding work term is

$$W_F = F u_z \tag{2.23}$$

so the factor 2π is missing. To render this compatible with the other energy terms the load is divided by 2π, so the contribution to the external loads potential is

$$W_F = 2\pi \left(\frac{F}{2\pi} \right) u_z \tag{2.24}$$

This device can be visualized by regarding F as the limit of a z-directed ring load F_z as $r \to 0$.

Remark 2.4. What the last equation means in practice is that if a concentrated force of, say, 1000 lb acts on the z axis, it has to be divided by 2π (that is, $1000/2\pi$) before giving it to a SOR finite element program if the factor of 2π has been suppressed. (It is important to read the users manual to see if that is the case.)

§2.4.4. Other Variational Forms

The Hellinger-Reissner (HR) functional and the equilibrium-stress hybrid functionals are derived in the Exercises.

§2.5. Treating Plane Strain as a Limit Case

The problem of *plane strain* may be viewed as the limit of the axisymmetric case in which the axis of revolution is moved to infinity so that $r \to \infty$, and a "slice" of unit thickness is taken.

Thus a finite element program that handles the axisymmetric problem may be used to solve problems of plane strain with acceptable approximation.

Homework Exercises for Chapter 2

Axisymmetric Solids, a.k.a. Structures of Revolution

EXERCISE 2.1 [A:20] Derive the HR functional for the axisymmetric solid problem. Use compact matrix notation, as done in §2.4.1 for the TPE form, because indicial notation does not fit this particular problem well. In matrix notation, the complementary energy density is $\mathcal{U}^* = \frac{1}{2}\boldsymbol{\sigma}^T \mathbf{C}\,\boldsymbol{\sigma}$, in which $\boldsymbol{\sigma}$ is the stress vector (2.7) and $\mathbf{C} = \mathbf{E}^{-1}$ the 4×4 elastic compliance matrix, with $\mathbf{E}$ given by (2.10).

Is there any difference in the treatment of body forces and surface tractions with respect to the TPE form?

EXERCISE 2.2 [A:20] Derive the equilibrium-stress hybrid functional for the axisymmetric solid problem. Use compact matrix notation, as done in §2.4.1 for the TPE form, because indicial notation does not fit this particular problem well. In matrix notation, the complementary energy density is $\mathcal{U}^* = \frac{1}{2}\boldsymbol{\sigma}^T \mathbf{C}\,\boldsymbol{\sigma}$, in which $\boldsymbol{\sigma}$ is the stress vector (2.7) and $\mathbf{C} = \mathbf{E}^{-1}$ the 4×4 elastic compliance matrix, with $\mathbf{E}$ given by (2.10).

Is there any difference in the treatment of body forces and surface tractions with respect to the TPE form?