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Overview

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§1.1. Foreword

This book covers advanced techniques for the analysis of linear elastic structures by the Finite Element Method (FEM). It has been constructed from Notes prepared for the course **Advanced to Finite Element Methods** or AFEM. This is a doctoral level course that may also be taken as an MS elective. The main prerequisite is an introductory FEM course at the first-year MS level; for example Introduction to Finite Element Methods or IFEM (ASEN 5007); see [247], which is taught every year. AFEM has been taught at the Department of Aerospace Engineering Sciences, University of Colorado at Boulder since 1990. It is offered every 2 or 3 years in the Spring semester.

§1.2. Contents

The AFEM book, as presently configured in its web site, embodies the following parts:

Part 0 **Introduction**. An overview provided in this Chapter.

Part 1 **Variational Methods in Mechanics**. The formulation of problems of engineering and physics in Strong, Weak and Variational Form. The material used here is part of a separate web-posted book in preparation, entitled *Advanced Variational Methods in Mechanics*, with acronym AVMM.

Part 2 **Axisymmetric Solids**. Axisymmetric solids, also called Structures of Revolution, or SOR. This formulation provides a gentle transition between the strict two-dimensional coverage of IFEM and the more realistic 3D world.

Part 3 **General Solids**. Solid elements: bricks, wedges, tetrahedra, pyramids. A first glance at techniques to improve element performance.

Part 4 **Advanced Element Derivation Tools**. The Free Formulation. The Assumed Natural Strain (ANS) formulation and its variants, primarily ANDES. The patch test. Variational crimes. Drilling freedoms.

Part 5 **Thin Plates, Membranes, Templates**. Application of the advanced element derivation techniques of the previous Part to construct plate elements as well as membrane elements with drilling degrees of freedom. Plate bending models: Kirchhoff, and Reissner-Mindlin. Elements based on conventional displacement expansions, as well as the Free Formulation (FF), Assumed Natural Strain (ANS), and Assumed Natural Deviaric Strain (ANDES), culminating with the introduction of finite element templates.

Part 6 **Shell Structures**. Facet and quadrilateral thin shell elements. Treatment of junctures. Transition elements. Thick shell elements.

The course concludes with student presentations on progress and achievements in group projects. Several Chapters posted on the web site are aimed to facilitating these projects, although they are not covered in class.

§1.3. Where the Material Fits

This Section outlines where the book material fits within the vast scope of Mechanics. In the ensuing multilevel classification, topics addressed in some depth in this book are emphasized in **bold** typeface.

§1.3.1. Top Level Classification

Definitions of *Mechanics* in dictionaries usually state two flavors:

- The branch of Physics that studies the effect of forces and energy on physical bodies.[1]
- The practical application of that science to the design, construction or operation of material systems or devices, such as machines, vehicles or structures.

These flavors are science and engineering oriented, respectively. But dictionaries are notoriously archaic. For our objectives it will be convenient to distinguish *four* flavors:

$$\textbf{Mechanics} \begin{cases} \textit{Theoretical} \\ \textit{Applied} \\ \textbf{Computational} \\ \textit{Experimental} \end{cases} \tag{1.1}$$

Theoretical mechanics deals with fundamental laws and principles studied for their intrinsic scientific value. *Applied mechanics* transfers this theoretical knowledge to scientific and engineering applications, especially as regards the construction of mathematical models of physical phenomena. *Computational mechanics* solves specific problems by model-based simulation through numerical methods implemented on digital computers. *Experimental mechanics* subjects the knowledge derived from theory, application and simulation to the ultimate test of observation.

Remark 1.1. Paraphrasing an old joke about mathematicians, one may define a computational mechanician as a person who searches for solutions to given problems, an applied mechanician as a person who searches for problems that fit given solutions, and a theoretical mechanician as a person who can prove the existence of problems and solutions. As regards experimentalists, make up your own joke.

§1.3.2. Computational Mechanics

Computational Mechanics represents the amalgamated integration of several disciplines, as depicted in the "pizza slice" Figure 1.1. It is presently an enormous subject. One useful classification can be done according to the *physical scale* of the focus of attention:

$$\textbf{Computational Mechanics} \begin{cases} \textit{Nanomechanics} \\ \textit{Micromechanics} \\ \textbf{Continuum mechanics} \begin{cases} \textbf{Solids and Structures} \\ \text{Fluids} \\ \text{Multiphysics} \end{cases} \\ \textit{Systems} \end{cases} \tag{1.2}$$

Nanomechanics deals with phenomena at the molecular and atomic levels. As such, it is closely related to particle physics and chemistry. At the atomic scale it transitions to quantum mechanics.

Micromechanics looks primarily at the crystallographic and granular levels of matter. Its main technological application is the design and fabrication of materials and microdevices.

[1] Here the term "bodies" includes all forms of matter, whether solid, liquid or gaseous; as well as all physical scales, from subatomic through cosmic.

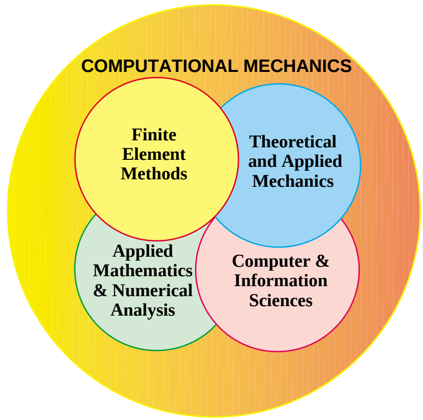

FIGURE 1.1. The "pizza slide:" Computational Mechanics integrates aspects of four disciplines.

Computational continuum mechanics considers bodies at the macroscopic level, using continuum models in which the microstructure is homogenized by phenomenological averaging. The two traditional areas of application are *solid* and *fluid mechanics*. *Structural mechanics* is a conjoint branch of solid mechanics, since structures, for obvious reasons, are fabricated with solids. Computational solid mechanics favors a applied-sciences approach, whereas computational structural mechanics emphasizes technological applications to the analysis and design of structures.

Computational fluid mechanics deals with problems that involve the equilibrium and motion of liquid and gases. Well developed related subareas are hydrodynamics, aerodynamics, atmospheric physics, propulsion, and combustion.

Multiphysics is a more recent newcomer.[2] This area is meant to include mechanical systems that transcend the classical boundaries of solid and fluid mechanics. A key example is interaction between fluids and structures, which has important application subareas such as aeroelasticity and hydroelasticity. Phase change problems such as ice melting and metal solidification fit into this category, as do the interaction of control, mechanical and electromagnetic systems.

Finally, *system* identifies mechanical objects, whether natural or artificial, that perform a distinguishable function. Examples of man-made systems are airplanes, building, bridges, engines, cars, microchips, radio telescopes, robots, roller skates and garden sprinklers. Biological systems, such as a whale, amoeba, virus or pine tree are included if studied from the viewpoint of biomechanics. Ecological, astronomical and cosmological entities also form systems.[3]

In the progression of (1.2), *system* is the most general concept. Systems are studied by *decomposition*: its behavior is that of its components plus the interaction between the components. Components are broken down into subcomponents and so on. As this hierarchical process continues the individual components become simple enough to be treated by individual disciplines, but their

[2] This unifying term is in fact missing from most dictionaries, as it was introduced by computational mechanicians in the 1970s. Several multiphysics problems, however, are older. For example, aircraft aeroelasticity emerged in the 1920s.

[3] Except that their function may not be clear to us. "What is that breathes fire into the equations and makes a universe for them to describe? The usual approach of science of constructing a mathematical model cannot answer the questions of why there should be a universe for the model to describe. Why does the universe go to all the bother of existing?" (Stephen Hawking).

interactions may get more complex. Thus there are tradeoff skills in deciding where to stop.[4]

§1.3.3. Statics versus Dynamics

Continuum mechanics problems may be subdivided according to whether inertial effects are taken into account or not:

$$\textbf{Continuum mechanics} \begin{cases} \textbf{Statics} \begin{cases} \textbf{Time Invariant} \\ \textit{Quasi-static} \end{cases} \\ \textit{Dynamics} \end{cases} \tag{1.3}$$

In *statics* inertial forces are ignored or neglected. These problems may be subclassified into *time invariant* and *quasi-static*. For the former time need not be considered explicitly; any time-like response-ordering parameter (should one be needed) will do. In quasi-static problems such as foundation settlements, creep flow, rate-dependent plasticity or fatigue cycling, a more realistic estimation of time is required but inertial forces are ignored as long as motions remain slow.

In *dynamics* the time dependence is explicitly considered because the calculation of inertial (and/or damping) forces requires derivatives respect to actual time to be taken.

§1.3.4. Linear versus Nonlinear

A classification of static problems that is particularly relevant to this book is

$$\textbf{Statics} \begin{cases} \textbf{Linear} \\ \textit{Nonlinear} \end{cases} \tag{1.4}$$

Linear static analysis deals with static problems in which the *response* is linear in the cause-and-effect sense. For example: if the applied forces are doubled, the displacements and internal stresses also double. Problems outside this domain are classified as *nonlinear*.

§1.3.5. Discretization Methods

A final classification of computational solid and structural mechanics (CSSM) for static analysis is based on the discretization method by which the continuum mathematical model is *discretized* in space, *i.e.*, converted to a discrete model of finite number of degrees of freedom:

$$\textbf{CSM spatial discretization} \begin{cases} \textbf{Finite Element Method (FEM)} \\ \textit{Boundary Element Method (BEM)} \\ \textit{Finite Difference Method (FDM)} \\ \textit{Finite Volume Method (FVM)} \\ \textit{Spectral Method} \\ \textit{Mesh-Free Method} \end{cases} \tag{1.5}$$

For *linear* problems finite element methods currently dominate the scene, with boundary element methods posting a strong second choice in selected application areas. For *nonlinear* problems the dominance of finite element methods is overwhelming.

[4] Thus in breaking down a car engine, say, the decomposition does not usually proceed beyond the components that may be bought at a automotive shop.

Classical *finite difference* methods in solid and structural mechanics have virtually disappeared from practical use. This statement is not true, however, for fluid mechanics, where finite difference discretization methods are still important although their dominance has diminished over time. *Finite-volume methods*, which focus on the direct discretization of conservation laws, are favored in highly nonlinear problems of fluid mechanics. *Spectral methods* are based on global transformations, based on eigendecomposition of the governing equations, that map the physical computational domain to transform spaces where the problem can be efficiently solved.

A recent newcomer to the scene are the *mesh-free methods*. These are finite different methods on arbitrary grids constructed using a subset of finite element techniques

§1.3.6. FEM Formulation Levels

The term *Finite Element Method* actually identifies a broad spectrum of techniques that share common features. Since its emergence in the framework of the Direct Stiffness Method (DSM) over 1956–1964, [727,730] FEM has expanded like a tsunami, surging from its origins in aerospace structures to cover a wide range of nonstructural applications, notably thermomechanics, fluid dynamics, and electromagnetics. The continuously expanding range makes taxology difficult. Restricting ourselves to applications in computational solid and structural mechanics (CSSM), one classification of particular relevance to this book is

$$\textbf{FEM-CSSM Formulation Level} \begin{cases} \textit{Mechanics of Materials (MoM) Formulation} \\ \textit{Conventional Variational Formulation} \\ \textbf{Advanced Variational Formulation} \\ \textbf{Template Formulation} \end{cases} \tag{1.6}$$

The MoM formulation is applicable to simple structural elements such as bars and beams, and does not require knowledge of variational methods. This level is accesible to undergraduate students (typically at the junior level) because it only uses basic linear algebra, a subject that is usually learned at the sophomore level.

The second formulation level is characterized by two features:

1. Use of standard work and energy methods, such as the Total Potential Energy principle;

2. Strict compliance with the requirements of the classical Ritz-Galerkin direct variational methods (for example, interelement continuity).

This level is approriate for first-year MS students with basic exposure to variational methods, which actually may be taught as "recipes" if necessary. The two lower levels were well established by 1970, with no major changes since. They are those used in the Introduction to Finite Element Methods (IFEM) book [247].

The next two levels are emphasized in this book. The third one requires deeper exposure to variational methods in mechanics, notably multifield and hybrid principles. The pertinent knowledge is accordingly introduced in Part 1. The template level is the pinnacle "where the rivers of our wisdom flow into one another." Reaching this rarified level requires both mastery of advanced variational principles, as well as the confidence and fortitude to discard them along the way.

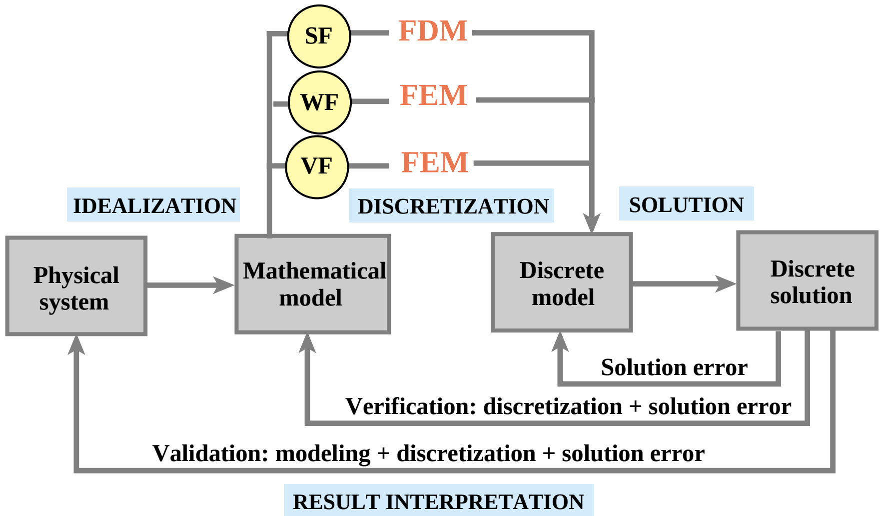

FIGURE 1.2. The main stages of computer-based simulation: idealization, discretization and solution. This is a slightly expanded version of a similar picture shown in Chapter 1 of [247].

§1.3.7. FEM Choices

A more down to earth classification considers two key selection attributes: Primary Unknown Variable(s), or PUV, and solution method:[5]

$$\textbf{PUV Choice} \begin{cases} \textbf{Displacement (a.k.a. Primal)} \\ \textit{Force (a.k.a. Dual or Equilibrium)} \\ \textit{Mixed (a.k.a. Primal-Dual)} \\ \textit{Hybrid} \end{cases} \quad \textbf{Solution Choice} \begin{cases} \textbf{Stiffness} \\ \textit{Flexibility} \\ \textit{Combined} \end{cases} \tag{1.7}$$

The PUV selection governs the variational framework chosen to develop the discrete equations; if one works at the two middle levels of (1.6). It is possible, however, to develop those completely *outside* a variational framework, as noted there. The solution choice is normally dictated by the PUV; exceptions are possible but relatively rare.

§1.3.8. Finally: What The Book Is About

Using the classification of (1.1) through (1.5) we can now state the book subject more precisely:

The model-based simulation of linear static structures discretized by FEM, paying extra attention to formulations at the two highest levels of (1.6).

(1.8)

Of the variants listed in the classification (1.7), emphasis will be placed here on the *displacement* PUV and *stiffness* solution, exactly as in in [247]. (The hybrid choice will be also covered, but limited to cases in which solution unknowns can be reduced to displacements only.) That particular combination is called the *Direct Stiffness Method* or DSM.[6]

[5] Alternative PUV terms shown in (1.7): primal, dual and primal-dual, are those used in FEM non-structural applications, as well as in more general areas of computational mathematics such as optimization.

[6] On first glance this seems just one of 12 possible combinations. However, the DSM combination is that used by the overwhelming majority of public-domain FEM codes, as well as *all* commercial ones. The hybrid PUV will be also covered, but it will be shown that it can be reduced to displacement unknowns.

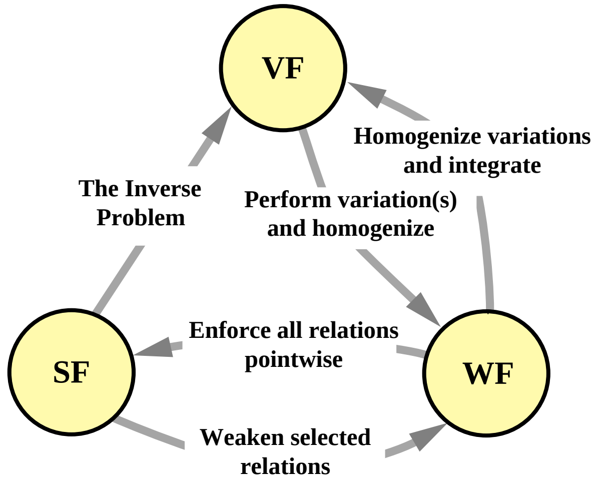

FIGURE 1.3. Diagram sketching Strong, Weak and Variational Forms, and relationships between form pairs. Weak Forms are also called weighted-residual equations, Galerkin equations, variational equations, variational statements, and integral statements in the literature.

§1.4. The Analysis Process

Recall from IFEM that the analysis process by computer methods can be characterized by the stages diagrammed in Figure 1.2. This is an expansion of a similar figure in [247]. The stages are *idealization*, *discretization* and *solution*.

Idealization, also called *mathematical modeling*, leads to a *mathematical model* of the physical system. In Figure 1.3 this model has been subdivided into three broad classes: Strong Form (SF), Weak Form (WF) and Variational Form (VF). These are discussed further in the rest of this Chapter.

§1.5. The Big Picture

Figure 1.3 depicts three alternative forms of a mathematical model. The yellow circles zoom into the three smaller circles of Figure 1.2.

SF **Strong Form**. Presented as a system of *ordinary or partial differential equations* in space and/or time, complemented by appropriate boundary conditions. Ocassionally this form may be presented in integraodifferential form, or reduce to algebraic equations

WF **Weak Form**. Presented as a *weighted integral equation* that "relaxes" the strong form into a domain-averaging statement.

VF **Variational Form**. Presented as a *functional* whose stationary conditions generate the weak and strong forms.

Variational Calculus or VC comprises a set of rules and techniques by which one can pass from one of these forms to another.

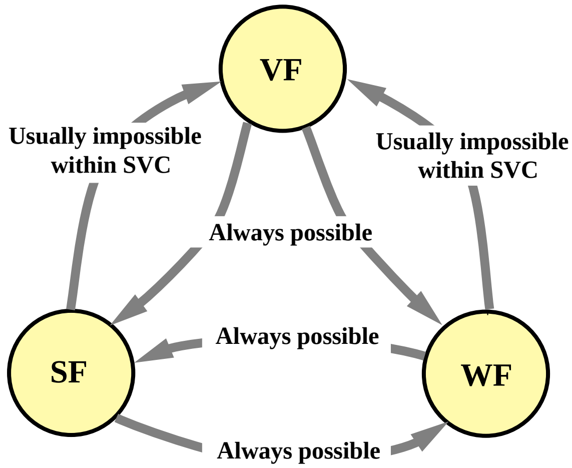

FIGURE 1.4. Feasibility of transformations between SF, WF and VF.

§1.6. Model Form Transformations

Much of variational theory and practice is concerned with the *transformation* of one form into another. As the diagram of Figure 1.4 illustrates, four transformation paths are always possible:

$$\boxed{\text{From SF to WF and vice-versa.}} \qquad \boxed{\text{From VF to WF or from VF to SF.}} \tag{1.9}$$

The last two transformation constitute an important part of standard variational calculus (SVC). The rules to pass from VF to SF essentially represent a generalization of the differentiation rules of ordinary calculus.

The following two transformation paths are generally impossible under the framework of *standard variational calculus* (SVC):

$$\boxed{\text{From SF to VF.}} \qquad \boxed{\text{From WF to VF.}} \tag{1.10}$$

Passing from a given SF to a VF is called the Inverse Problem of Variational Calculus, and may be viewed as a generalization of the problem of integrating arbitrary functions. It is therefore understandable that no general solution to this problem exists. Under *extended variational calculus* (EVC), however, such paths become possible.

§1.7. Why Variational Methods?

The Strong Form (SF) states problems in ordinary or partial differential equation format. This is an old and well studied branch of calculus and mathematical physics. For example the famous Newton's Second Law: $F = ma$, is a Strong Form.

Why then the interest in Weak and Variational Forms? The following reasons may be offered.

1. **Unification**. The functional of the VF embodies *all* properties of the modeled system, including field equations, natural boundary conditions and conservation laws. In the SF and WF versions these may have to be specified as separate pieces.

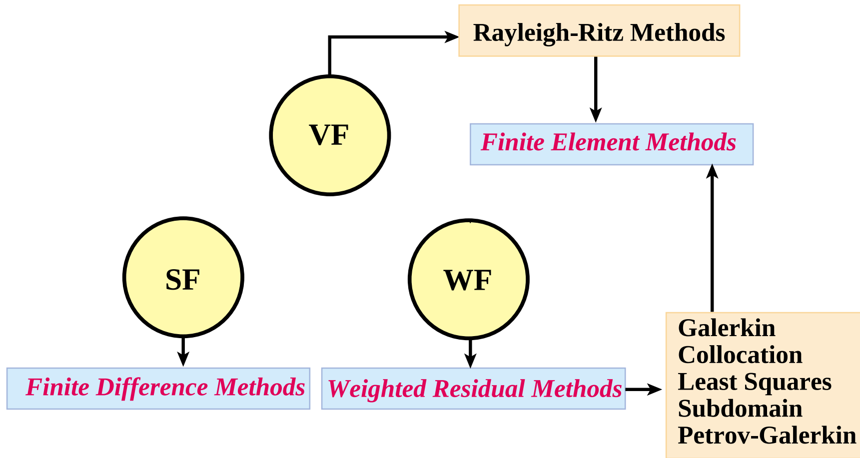

FIGURE 1.5. Strong, Weak and Variational Forms as natural source of numerical approximation methods.

2. **Invariance**. Because functionals are scalars, and scalars are invariant with respect to coordinate transformations, the VF provides automatically for that property between different coordinate frames.[7]

3. **Basis For Approximation**. VFs and WFs provide well-tested platforms for developing computer-based methods of approximation, notably the Finite Element Method (FEM).

4. **Easy Totalization**. VFs, and to less extent WFs, directly characterize "overall" or "agregate" quantities of interest to scientists and engineers. For example: mass, momentum, energy, flux resultants. Mathematically those forms are said to lead naturally into *conservation laws*.

5. **Boundary and Interface Treatment**. VFs clarify and systematize the treatment of boundary and interface conditions, particularly in connection with discretization schemes. [WFs are also useful in handling of BCs, but no so powerful.]

6. **Mathematical Sugar**. VFs permit a deeper and more powerful mathematical treatment of questions of existence, stability, error bounds, convergence of numerical solutions, etc. More importantly, they provide general guidelines on how to achieve desirable behavior of the related discrete schemes. [WFs are better than SFs in this regard, but not as satisfactory as VFs.]

§1.8. Methods of Approximation: Discretization

Transforming a SF to WF or VF does not make a problem easy to solve. Complicated problems still have to be treated by *methods of approximation*. These may be hand-based or (since the advent of the digital computer) computer-based.

The essence of approximation is *discretization*. Continuum mathematical models stated in SF, WF or VF have an infinite number of degrees of freedom. Through a discretization method this is reduced to a finite number, yielding *algebraic* equations than can be solved in a reasonable time.

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[7] In other words, the solution of a variationally formulated problem will not depend on the choice of coordinate system, which may be done simply by convenience considerations. This feature: consistency-across-coordinate-bases is of great importance in applications — can you imagine the reaction of a structural engineer if changing the global coordinate system yields a different solution — and provides a way to check computer implementations by doing exactly that.

Each form: SF, WF and VF has a *natural* class of discretization methods than can be constructed from it. This attribute is illustrated in Figure 1.5, and briefly described below.

§1.8.1. Finite Difference Method

The natural discretization class for SFs is the *finite difference method* (FDM). These are constructed by replacing derivatives by differences. This class is easy to generate and program for regular domains and boundary conditions, but runs into difficulties when geometry or boundary conditions become arbitrary. The other problem with conventional FDM is that the approximate solution is only obtained at the grid points, and extension to other points is not always obvious or even possible. Nevertheless the FDM class is theoretically *general* in that any problem stated in WF or VF can be put into SF.

§1.8.2. Weighted Residual Methods

The natural discretization class for WFs is the *weighted residual method* (WRM). There are well known WRM subclasses: Galerkin, Petrov-Galerkin, collocation, subdomain, finite-volume, least-squares. Sometimes these subclasses, excluding collocation, are collectively called *trial function methods*, an alternative name that accurately reflects the discretization technique. Unlike the FDM, trial-function methods yield approximate solutions defined *everywhere*. Before computers such analytical solutions were obtained by hand, a restriction that limited considerably the scope and accuracy of the approximations. That barrier was overcome with the development of the Finite Element Method (FEM) on high speed computers.

One particularly important subclass of WRM is the Finite Volume Method or FVM, which is used extensively in computational gas dynamics.

§1.8.3. Rayleigh-Ritz Methods

The natural discretization class for VFs is the *Rayleigh-Ritz method* (RRM). Although historically this was the first trial-function method, it is in fact a special subclass of the Galerkin weighted-residual method. The Finite Element Method was originally developed (during the 1960s) along these lines, and remains the most powerful computer based RRM.

Note that FEM, like FDM, can be viewed as an *universal* approximation method, because any problem can be placed in WF. This statement is no longer true, however, if one restricts FEM to the subclass of Rayleigh-Ritz method, which relies on the VF.

Remark 1.2. In complex problems treatable within today's computer technology, combinations of these numerical methods, sometimes with a "sprinking" of analytical techniques, are common. Some examples serve to illustrate the richness of possibilities:

1. Fluid-structure interaction: FEM for the structure, FDM or FVM for the fluid.
2. Structural dynamics: FEM in space, FDM in time.
3. Semi-analytical methods: some space directions are treated by FEM (or FDM), while others are treated analytically. The so-called *methods of lines* is a prime example.
4. Finite difference schemes may be constructed from VF and WF in combination with some FEM ideas. The resulting schemes are collectively known as Finite Difference Energy Methods (FDEM). More recently the so-called *mesh free method* has emerged through a blend of FDM and FEM technques.

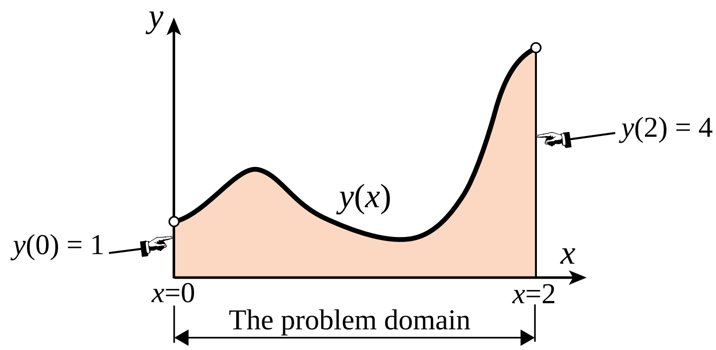

FIGURE 1.6. Function $y(x)$ for the example in §1.10. The drawn function satisfies the boundary conditions $y(0) = 1$ and $y(2) = 4$, which together with the ODE (1.11) defines a *boundary value problem*, or BVP.

§1.9. Boundary Element Methods: Where Are You?

In addition to FEM and FDM, Boundary Element Methods (BEM) represents a third important class of computer-based discretization methods. The BEM is essentially a dimensionality-reducing technique that combines analytical reduction of one space dimension with the FEM discretization of the remaining space dimension(s). It does not have the generality of FEM or FDM, as it is primarily restricted (in its "pure" form) to linear problems with known fundamental solutions.

Originally BEMs were based on a fourth form not shown in Figures 1.3–1.5: the *integro-differential form* or IDF. Over the past three decades substantial attention has been given to "merging" BEMs within the framework of the Finite Element Method. The effort has been motivated by the idea of integrating FEM and BEM in the same programming framework. Thus a subclass of BEM called Variational Boundary Element Methods (VBEM) has emerged. These methods can be constructed from VFs and WFs with nonstandard application of trial functions. As of this writing, the future and importance of such methods is not clear.

§1.10. An Example

The following simple example will serve to illustrate the three forms introduced in §1.5 as well as connecting them with terminology common in applied mathematics.

Consider a function $y = y(x)$, sketched in Figure 1.6 that satisfies the ordinary differential equation

$$\boxed{y'' = y + 2 \quad \text{in} \ \ 0 \le x \le 2.} \tag{1.11}$$

Here primes denote derivative with respect to x. This is a Strong Form because (1.11) is to be satisfied *at each point* of the interval $0 \le x \le 2$. This interval is called the *problem domain* or (after discretization) the *computational domain*. By itself (1.11) is insufficient to determine $y(x)$. It must be complemented with two boundary conditions, two examples of which are

$$y(0) = 1, \qquad y(2) = 4, \tag{1.12}$$

$$y(0) = 1, \qquad y'(0) = 0. \tag{1.13}$$

(The first one is that pictured in Figure 1.6.) Equation (1.11) together with (1.12) defines a *boundary value problem* or BVP. Equation (1.11) together with (1.13) defines an *initial value problem* or IVP. BVPs usually model problems in spatial domains whereas IVPs model problems in the time domain.

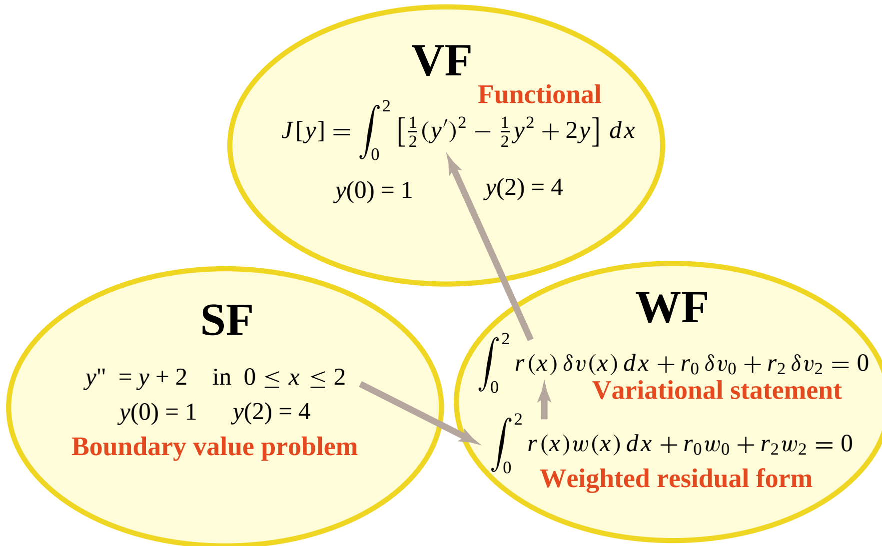

FIGURE 1.7. Diagramatic representation of the SF, WF and VF forms in the example of §1.10.

A *residual function* associated with (1.11) is $r(x) = y'' - y - 2$. The SF (1.11) is equivalent to saying that $r(x) = 0$ at each point of the problem domain $x \in [0, 2]$. The boundary condition residual for (1.12) is the expression pair $r_0 = x(0) - 1$, $r_2 = x(2) - 4$. Multiply the ODE residual $r(x)$ by a *weight function* $w(x)$ and integrate over $[0, 2]$. Multiply r_0 and r_2 by weights w_0 and w_2 and add the three terms to get

$$\boxed{\int_0^2 r(x) w(x)\,dx + r_0 w_0 + r_2 w_2 = 0.} \tag{1.14}$$

This is a *weighted integral form*. It is a Weak Form (WF) statement. Plainly a solution of the BVP (1.11)–(1.12) identically satisfies (1.14). The possibility is open, however, that other functions not satisfying that BVP may verify (1.14). Thus the qualifier "weak."

If w, w_0 and w_2 are formally written as the variations of functions v, v_0 and v_2, respectively, (we have not defined what a variation is, so what follows has to be accepted on faith) then (1.14) becomes

$$\boxed{\int_0^2 r(x)\,\delta v(x)\,dx + r_0\,\delta v_0 + r_2\,\delta v_2 = 0.} \tag{1.15}$$

Here δ denotes the variation symbol. The v's are technically called *test functions*. Equation (1.15) is called a *variational statement*. This equation leads directly to the important Galerkin and Petrov-Galerkin forms.

Finally, for the BVP defined by (1.11) and (1.12) the Inverse Problem of VC has a solution. The functional

$$\boxed{J[y] = \int_0^2 \left[\tfrac{1}{2}(y'')^2 - \tfrac{1}{2}y^2 + 2y\right] dx} \tag{1.16}$$

when restricted to the class of functions satisfying $y(0) = 1$ and $y(2) = 4$ becomes *stationary* in the VC sense when $y(x)$ satisfies (1.11), which is called the *Euler-Lagrange equation* of(1.16). This is an example of a Variational Form. The foregoing forms are diagrammed in Figire 1.7.