# 12

# The Three-Field Mixed Principle of Elastostatics

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## §12.1. Introduction

This Chapter concludes the presentation of canonical variational principles of elastostatics by constructing the Veubeke-Hu-Washizu (VHW) principle. This is used for the development of a special beam element.

## §12.2. The Veubeke-Hu-Washizu Principle

The Veubeke-Hu-Washizu (VHW) principle is the canonical principle of elasticity that allows simultaneous variation of displacements, strains and stresses.[1] The VHW principle is the most general *canonical* principle of elasticity. Contrary to what the literature states, however, this is not the most general variational principle. Within the framework of parametrized variational principles[2] the VHW principle appears as an instance.

### §12.2.1. The Variational Statement

We derive here a slightly generalized version of the VHW principle, in which the displacement boundary condition link (the PBC link) is weakened. This functional will be identified as $\Pi^g_{\rm VHW}$. The Weak Form used as departure point is shown in Figure 12.1.

Because we have picked three masters, in principle we will have three strain fields, one master: $e_{ij}$, and two slaves: $e^u_{ij}$ and $e^\sigma_{ij}$. Similarly there are three stress fields, one master: $\sigma_{ij}$, and two slaves: $\sigma^u_{ij}$ and $\sigma^e_{ij}$. The boxes of $\sigma^u_{ij} = E_{ijk\ell}\, e^u_{k\ell}$ and $e^\sigma_{ij} = C_{ijk\ell}\, \sigma_{k\ell}$ are not shown, however, in Figure 12.1 because those slave fields do not appear in the derivation below. There are five weak connections.[3]

To streamline the derivation we skip the preparatory steps in writing down residuals and Lagrange multiplier fields, and proceed directly to the variational statement in which weak connection residuals are work paired with appropriate variations of the master fields:

$$\begin{aligned}\delta\Pi^g_{\rm VHW} &= \int_V (e^u_{ij} - e_{ij})\,\delta\sigma_{ij}\,dV + \int_V (\sigma^e_{ij} - \sigma_{ij})\,\delta e_{ij}\,dV - \int_V (\sigma_{ij,j} + b_i)\,\delta u_i\,dV \\ &\quad + \int_{S_t} (\sigma_{ij} n_j - \hat t_i)\,\delta u_i\,dS - \int_{S_u} (u_i - \hat u_i)\,n_j\,\delta\sigma_{ij}\,dS.\end{aligned} \tag{12.1}$$

Treat $\sigma_{ij,j}\,\delta u_i$ with the divergence theorem to get rid of stress derivatives:

$$\begin{aligned}-\int_V \sigma_{ij,j}\,\delta u_i &= \int_V \sigma_{ij}\,\delta e^u_{ij} - \int_S \sigma_{ij}\,n_j\,\delta u_i\,dS \\ &= \int_V \sigma_{ij}\,\delta e^u_{ij} - \int_{S_u} \sigma_{ij}\,n_j\,\delta u_i\,dS - \int_{S_t} \sigma_{ij}\,n_j\,\delta u_i\,dS.\end{aligned} \tag{12.2}$$

[1] The VHW functional was published simultaneously in 1955 by H. Hu, “On some variational principles in the theory of elasticity and the theory of plasticity.” *Sci. Sinica* (Peking) **4**, pp. 33–54, 1955, and K. Washizu, “On the variational principles of elasticity and plasticity,” Rept 25-18, Massachusetts Institute of Technology, March 1955. However, four years earlier B. M. Fraeijs de Veubeke had published a version of the principle in 1951 that was overlooked: B. M. Fraeijs de Veubeke, Diffusion des inconnues hyperstatiques dans les voilures à longeron couplés, *Bull. Serv. Technique de L’Aéronautique No. 24*, Imprimerie Marcel Hayez, Bruxelles, 56pp., 1951.

[2] C. A. Felippa, A survey of parametrized variational principles and applications to computational mechanics, *Comp. Meths. Appl. Mech. Engrg.*, **113**, 109–139, 1994.

[3] Other weak connections combinations between strain and stress boxes may be taken, leading to the same result.

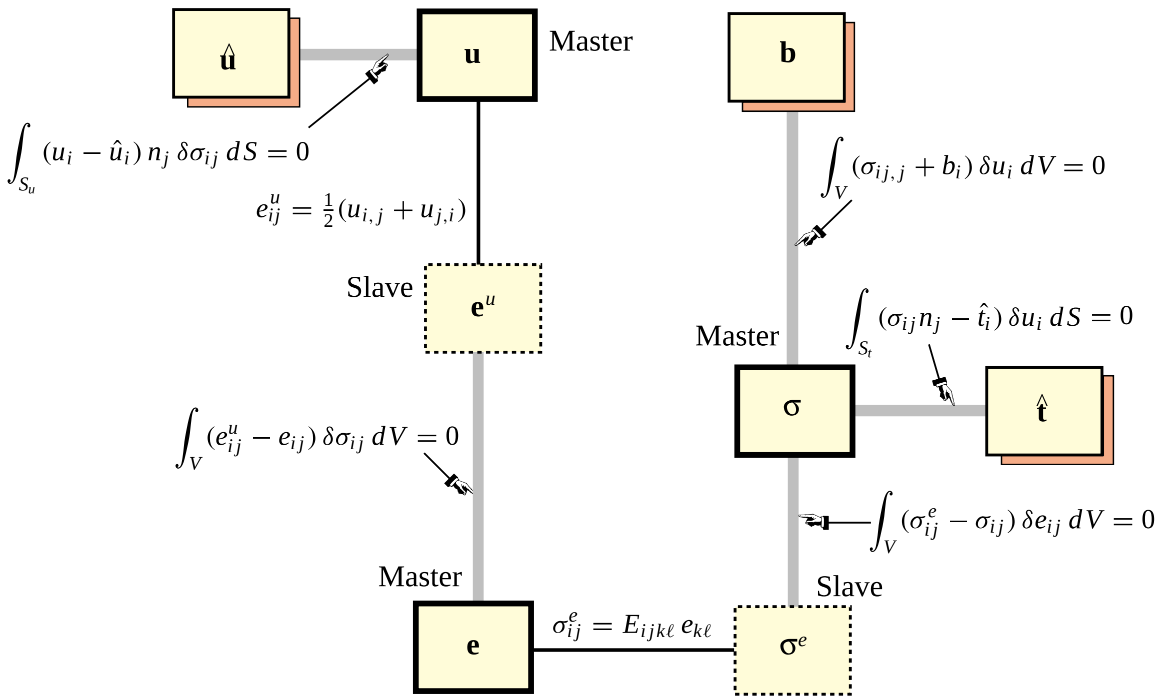


FIGURE 12.1. The Weak Form for derivation of the generalized VHW principle. Preliminary steps are skipped: the diagram shows the appropriate work pairings. The standard form of the principle is obtained if the PBC link is strong.

Substitute (12.2) into (12.1) and collect terms:

$$\begin{aligned}\delta\Pi^g_{\rm VHW} = &\int_V \Big[(e^u_{ij} - e_{ij})\,\delta\sigma_{ij} + (\sigma^e_{ij} - \sigma_{ij})\,\delta e_{ij} + \sigma_{ij}\,\delta e^u_{ij} - b_i\,\delta u_i\Big]\,dV \\ &- \int_{S_t} \hat{t}_i\,\delta u_i\,dS - \int_{S_u} \Big[(u_i - \hat{u}_i)\,n_j\,\delta\sigma_{ij} + \sigma_{ij}\,n_j\,\delta u_i\Big]\,dS.\end{aligned} \tag{12.3}$$

### §12.2.2. The Variational Form

Equation (12.3) can be recognized as the first variation of the functional

$$\boxed{\begin{aligned}\Pi^g_{\rm VHW}[u_i, \sigma_{ij}, e_{ij}] = &\int_V \Big[\sigma_{ij}(e^u_{ij} - e_{ij}) + \mathcal{U}(e_{ij}) - b_i u_i\Big]\,dV - \int_{S_t} \hat{t}_i u_i\,dS \\ &- \int_{S_u} (u_i - \hat{u}_i)\,\sigma_{ij} n_j\,dS.\end{aligned}} \tag{12.4}$$

This is called here "generalized VHW." In this form

$$\mathcal{U}(e_{ij}) = \tfrac{1}{2} e_{ij} E_{ijk\ell} e_{k\ell} = \tfrac{1}{2}\sigma^e_{ij} e_{ij}, \tag{12.5}$$

is the strain energy density in terms of the master (varied) strains. The VHW principle asserts that

$$\boxed{\delta\Pi^g_{\rm VHW} = 0} \tag{12.6}$$

in which the variation is taken simultaneously with respect to displacements, strains and stresses, yields all field equations of elasticity as its Euler-Lagrange equations, and all boundary conditions (displacements and tractions) as its natural boundary conditions.[4]

From $\Pi^g_{\rm VHW}$ we may derive other forms that also arise in the applications. For example, if one enforces *a priori* the displacement BCs $u_i = \hat{u}_i$ as a strong link, the integral over $S_u$ drops out and (12.4) reduces to the standard form of the functional:

$$\boxed{\Pi_{\rm VHW}[u_i, \sigma_{ij}, e_{ij}] = \int_V \big[\sigma_{ij}(e^u_{ij} - e_{ij}) + \mathcal{U}(e_{ij}) - b_i u_i\big]\, dV - \int_{S_t} \hat{t}_i u_i \, dS.} \tag{12.7}$$

In FEM work this functional, as was the case with TPE and HR, is often written in the "internal plus external" split form

$$\boxed{\begin{aligned} &\Pi_{\rm VHW} = U_{\rm VHW} - W_{\rm VHW}, \qquad \text{in which} \\ &U_{\rm VHW} = \int_V \big[\sigma_{ij}(e^u_{ij} - e_{ij}) + \mathcal{U}(e_{ij})\big]\, dV, \quad W_{\rm VHW} = \int_V b_i u_i \, dV + \int_{S_t} \hat{t}_i u_i \, dS. \end{aligned}} \tag{12.8}$$

Other reductions are the subject of Exercises.

Additional forms of the functionals (12.5) and (12.7) may be constructed through integration by parts of the $\sigma_{ij} e^u_{ij}\, dV$ term to get rid of displacement derivatives at the cost of introducing stress derivatives. This can be done using

$$\int_V \sigma_{ij} e^u_{ij} \, dV = -\int_V u_i \, \sigma_{ij,j} \, dV + \int_S u_i \sigma_{ij} n_j \, dS. \tag{12.9}$$

This is the subject of an Exercise.

### §12.2.3. Continuity Requirements

Inspection of the functionals (12.5) and (12.7) shows that the variational index of the displacement field is $m_u = 1$ because first order displacement derivatives appear in the slave strain field $e^u_{ij}$. The variational indices $m_\sigma$ and $m_e$ of stresses and strains are zero because no derivatives of these two master fields appear.

From this characterization, it follows that when the VHW principle in the form (12.5) or (12.7) is used to derive finite elements, the assumed displacements should be $C^0$ interelement continuous, whereas assumed stresses and strains can be discontinuous between elements.

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[4] The generalized form (12.5) is actually that derived by Fraeijs de Veubeke in the cited 1951 reference. Hu and Washizu only derived the more restricted form (12.7).

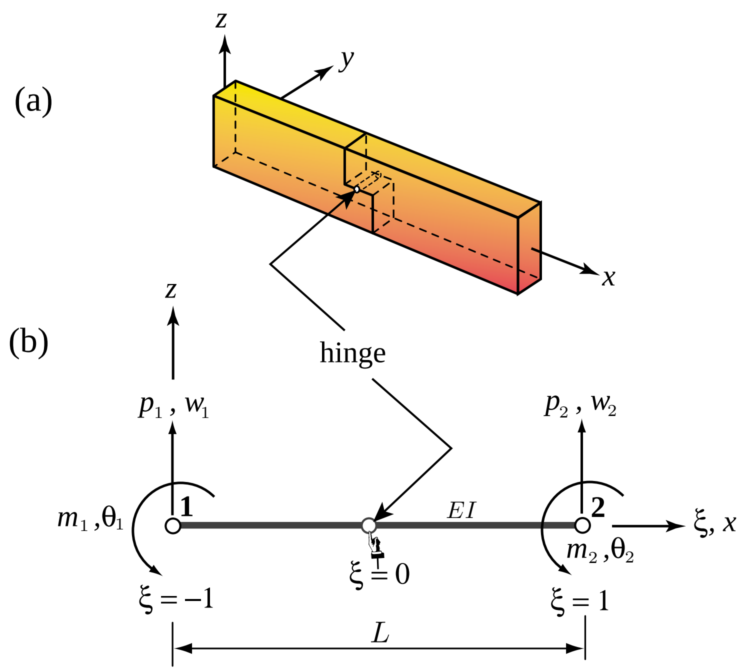


FIGURE 12.2. Hinged plane beam discretization by the VHW principle: (a) hinged beam, (b) two-node finite element model.

## §12.3. VHW Application: A Hinged Plane Beam Element

### §12.3.1. Element Description

The use of the VHW functional to derive a specialized beam element is illustrated next. Consider a two-node prismatic plane beam element of span $L$ with a hinge at its midsection as depicted in Figure 12.2(a). The beam bends in the $xz$ plane. The Euler-Bernoulli (BE) beam model of Chapter 4 is used. The beam is fabricated of isotropic elastic material of elastic modulus $E$. The second moment of inertia with respect to the neutral axis $y$ is $I$. The bending moment at the hinge section is taken to be zero.

The beam is referred to a Cartesian coordinate system $(x, y, z)$ with axis $x$ placed along the longitudinal axis of the beam and $z$ along the plane beam transverse direction. Note that for the beam to be plane, the cross section must be symmetric with respect to the $z$ axis, while all applied forces must act on the $xz$ plane.

As discussed in Chapter 4, the field variables that appear in BE plane beam theory are: the internal bending moment $M = M(x)$, the cross-section transverse displacement $w = w(x)$, the cross-section rotation $\theta = \theta(x) = dw/dx = w'$ and the curvature $\kappa = \kappa(x) = d^2w/dx^2 = w''$.

### §12.3.2. Element Formulation

A two-node BE beam element with a midsection hinge is depicted in Figure 12.2(b). The four degrees of freedom are the transverse node displacements $w_1$ and $w_2$, and the about-$y$ end rotations $\theta_1$ and $\theta_2$ (positive counterclockwise when viewed from the $-y$ direction) at the two end nodes. The associated node forces are $f_1$, $f_2$, $m_1$, $m_2$. The natural coordinate $\xi = (2x - 1)/L$ takes the values $-1$, 1 and 0 at the end nodes and at the hinge location, respectively.

Assuming zero body forces, the VHW functional for this beam model reduces to

$$\boxed{\Pi_{\mathrm{VHW}}[w, M, \kappa] = \int_L \left[ M(\kappa^w - \kappa) + \tfrac{1}{2} E I \kappa^2 \right] dx - f_1 w_1 - f_2 w_2 - m_1 \theta_1 - m_2 \theta_2,} \tag{12.10}$$

where $M$ and $\kappa$ are the assumed bending-moment and curvature functions, respectively, $\kappa^w = w''$ is the curvature derived from the assumed transverse displacement, and other quantities are defined in Figure 12.2.

Inspection of (12.10) shows that the variational indices for $M$, $\kappa$ and $w$ are 0, 0, and 2, respectively.[5] It follows that the continuity requirements for these functions are $C^{-1}$, $C^{-1}$ and $C^1$, respectively. Consequently $M$ and $\kappa$ may be discontinuous between elements.

The assumed variation of the bending moment and curvature is linear:

$$M = \bar{M}^{(e)}\,\xi, \qquad \kappa = \bar{\kappa}^{(e)}\,\xi, \tag{12.11}$$

both of which satisfy the hinge condition $M = \kappa = 0$ at $\xi = 0$. As for the transverse displacement we shall take the usual cubic Hermite interpolation for BE beam elements:

$$\begin{aligned} w = {} & \tfrac{1}{4}(1-\xi)^2(2+\xi)\, w_1^{(e)} + \tfrac{1}{8} L (1-\xi)^2 (1+\xi)\, \theta_1^{(e)} + \\ & \tfrac{1}{4}(1+\xi)^2(2-\xi)\, w_2^{(e)} - \tfrac{1}{8} L (1+\xi)^2 (1-\xi)\, \theta_2^{(e)}. \end{aligned} \tag{12.12}$$

The displacement-derived curvature is obtained by differentiating (12.12) twice with respect to $x$:

$$\kappa^w = w'' = \left[ (6\xi/L)\, w_1^{(e)} + (3\xi - 1)\, \theta_1^{(e)} - (6\xi/L)\, w_2^{(e)} + (3\xi + 1)\, \theta_2^{(e)} \right] / L \tag{12.13}$$

Inserting (12.12)-(12.13) into (12.10), integrating over the length to evaluate the internal energy, and equating to zero the partials of the resulting expression of $\Pi_{\mathrm{VHW}}$ with respect to the element degrees of freedom $\bar{M}^{(e)}$, $\bar{\kappa}^{(e)}$, $w_1^{(e)}$, $\theta_1^{(e)}$, $w_2^{(e)}$ and $\theta_2^{(e)}$, we obtain the following finite element equations:

$$\begin{bmatrix} \tfrac{1}{3} E I L & -\tfrac{1}{3} L & 0 & 0 & 0 & 0 \\ -\tfrac{1}{3} L & 0 & \dfrac{2}{L} & 1 & -\dfrac{2}{L} & \tfrac{1}{2} \\ 0 & \dfrac{2}{L} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\dfrac{2}{L} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\kappa}^{(e)} \\ \bar{M}^{(e)} \\ w_1^{(e)} \\ \theta_1^{(e)} \\ w_2^{(e)} \\ \theta_2^{(e)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f_1^{(e)} \\ m_1^{(e)} \\ f_2^{(e)} \\ m_2^{(e)} \end{bmatrix}. \tag{12.14}$$

[5] The displacement variational index has increased from 1 in (12.28) to 2 because of the introduction of the BE beam theory assumptions.

```
ClearAll[EI,L,n,m]; n=1; m=1;
κw= 6*ξ/L^2*w1+(3*ξ-1)/L*θ1+(-6*ξ/L^2*w2)+(3*ξ+1)/L*θ2;
κ=  κ0*ξ^m;  M=M0*ξ^n;  W=f1*w1+m1*θ1+f2*w2+m2*θ2;
Π =(L/2)*Integrate[M*(κw-κ)+EI*κ^2/2,{ξ,-1,1}]-W;
Π =Simplify[Π]; Print["VHW functional Π=", Π];
r={D[Π,κ0],D[Π,M0],D[Π,w1],D[Π,θ1],D[Π,w2],D[Π,θ2]};
K={D[r,κ0],D[r,M0],D[r,w1],D[r,θ1],D[r,w2],D[r,θ2]};
Print["Full Ke=",K//MatrixForm];
K11=Table[K[[i,j]],{i,1,2},{j,1,2}];
K12=Table[K[[i,j]],{i,1,2},{j,3,6}];
K22=Table[K[[i,j]],{i,3,6},{j,3,6}];
Ke = Simplify[K22-Transpose[K12].Inverse[K11].K12];
Print["Condensed Ke=",Ke//MatrixForm];
Print["Eigenvalues of Cond Ke=",Eigenvalues[Ke]];
```

FIGURE 12.3. *Mathematica* script to derive the stiffness equations of the hinged plane beam element. Exponents $m$ and $n$ in expressions of $\kappa$ and $M$ are for Exercises.

### §12.3.3. The Stiffness Equations

As previously explained both $\bar{\kappa}$ and $\bar{M}$ need not be continuous between elements. Static condensation of these two freedoms yields the following element stiffness equations in terms of the node displacements:

$$\frac{3EI}{L^3}\begin{bmatrix} 4 & 2L & -4 & 2L \\ 2L & L^2 & -2L & L^2 \\ -4 & -2L & 4 & -2L \\ 2L & L^2 & -2L & L^2 \end{bmatrix}\begin{bmatrix} w_1^{(e)} \\ \theta_1^{(e)} \\ w_2^{(e)} \\ \theta_2^{(e)} \end{bmatrix} = \begin{bmatrix} f_1^{(e)} \\ m_1^{(e)} \\ f_2^{(e)} \\ m_2^{(e)} \end{bmatrix}. \tag{12.15}$$

in which the generic element identifier has been inserted; or in compact form

$$\mathbf{K}^{(e)}\mathbf{u}^{(e)} = \mathbf{f}^{(e)}. \tag{12.16}$$

It can be verified that this $4 \times 4$ element stiffness matrix has only rank 1. The element has the two usual rigid-body modes of a plane beam element (rigid translation along $z$ and rigid rotation about $y$), plus one zero-energy mode caused by the presence of the hinge.

The foregoing computations were done with the *Mathematica* script shown in Figure 12.3.

## Homework Exercises for Chapter 12

## The Three-Field Mixed Principle of Elastostatics

**EXERCISE 12.1** [A:20] Explain how to reduce the VWH principle (12.7) to HR.

**EXERCISE 12.2** [A:15] Explain how to reduce the VWH principle (12.7) to TPE. (This is easier than the previous one)

**EXERCISE 12.3** [A:20] Use (12.8) to transform (12.7) keeping PBC strong. Show the form of the variational principle. (This transformation was the subject of a recent Ph. D. thesis in CVEN; 3 years of work for an exercise in variational calculus).

**EXERCISE 12.4** [A/C:15] Can the hinged beam element be derived directly from HR? Does it give the same answers for the stiffness matrix?

**EXERCISE 12.5** [A/C:15] Derive a "shearless" plane beam element by forcing the master moment $M$ to be constant over the element. Compute its $4 \times 4$ condensed stiffness matrix $\mathbf{K}^{(e)}$. *Hint*: if you know how to use *Mathematica* use the script of Figure 12.3, setting `m=n=0`.

**EXERCISE 12.6** [A:15] How would pickmoment and curvature distributions for deriving a *two-hinge* beam element using VHW? (Do not try to derive the element, it will blow up.)

**EXERCISE 12.7** [A:15] The conventional plane beam element of IFEM can be obtained by setting $\kappa = \kappa^w$ in VHW. Why?

**EXERCISE 12.8** [A:25] Transform (12.12) by parts to reduce the variational index of the displacement $w$ to one while raising that of the moment $M$ to one. Which kind of beam element can one derive with this principle? Do you think it is worth a thesis?