28

Stress Recovery

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§28.1. Introduction

In this lecture we study the recovery of stress values for two-dimensional plane-stress elements.[1]

This analysis step is sometimes called *postprocessing* because it happens after the main processing step — the calculation of nodal displacements — is completed. Stress calculations are of interest because in structural analysis and design the stresses are often more important to the engineer than displacements.

In the stiffness method of solution discussed in this course, the stresses are obtained from the computed displacements, and are thus *derived quantities*. The accuracy of derived quantities is generally lower than that of primary quantities (the displacements), an informal statement that may be mathematically justified in the theory of finite element methods. For example, if the accuracy level of displacements is 1% that of the stresses may be typically 10% to 20%, and even lower at boundaries.

It is therefore of interest to develop techniques that enhance the accuracy of the computed stresses. The goal is to "squeeze" as much accuracy from the computed displacements while keeping the computational effort reasonable. These procedures receive the generic name *stress recovery techniques* in the finite element literature. In the following sections we cover the simplest stress recovery techniques that have been found most useful in practice.

§28.2. Calculation of Element Strains and Stresses

In elastic materials, stresses $\boldsymbol{\sigma}$ are directly related to strains $\mathbf{e}$ at each point through the elastic constitutive equations $\boldsymbol{\sigma} = \mathbf{E}\mathbf{e}$. It follows that the stress computation procedure begins with strain computations, and that the accuracy of stresses depends on that of strains. Strains, however, are seldom saved or printed.

In the following we focus our attention on two-dimensional isoparametric elements, as the computation of strains, stresses and axial forces in bar elements is strightforward.

Suppose that we have solved the master stiffness equations

$$\mathbf{K}\mathbf{u} = \mathbf{f}, \tag{28.1}$$

for the node displacements $\mathbf{u}$. To calculate strains and stresses we perform a loop over all defined elements. Let e be the element index of a specific two-dimensional isoparametric element encountered during this loop, and $\mathbf{u}^{(e)}$ the vector of computed element node displacements. Recall from §15.3 that the strains at any point in the element may be related to these displacements as

$$\mathbf{e} = \mathbf{B}\mathbf{u}^{(e)}, \tag{28.2}$$

where $\mathbf{B}$ is the strain-displacement matrix (14.18) assembled with the x and y derivatives of the element shape functions evaluated at the point where we are calculating strains. The corresponding stresses are given by

$$\boldsymbol{\sigma} = \mathbf{E}\mathbf{e} = \mathbf{E}\mathbf{B}\mathbf{u} \tag{28.3}$$

[1] This Chapter needs rewriting to show the use of Mathematica for stress computation. To be done in the future.

Table 28.1 Natural Coordinates of Bilinear Quadrilateral Nodes

Corner node	ξ	η	ξ'	η'	Gauss node	ξ	η	ξ'	η'
1	-1	-1	$-\sqrt{3}$	$-\sqrt{3}$	1’	$-1/\sqrt{3}$	$-1/\sqrt{3}$	-1	-1
2	$+1$	-1	$+\sqrt{3}$	$-\sqrt{3}$	2’	$+1/\sqrt{3}$	$-1/\sqrt{3}$	$+1$	-1
3	$+1$	$+1$	$+\sqrt{3}$	$+\sqrt{3}$	3’	$+1/\sqrt{3}$	$+1/\sqrt{3}$	$+1$	$+1$
4	-1	$+1$	$-\sqrt{3}$	$+\sqrt{3}$	4’	$-1/\sqrt{3}$	$+1/\sqrt{3}$	-1	$+1$

Gauss nodes, and coordinates ξ' and η' are defined in §28.4 and Fig. 28.1

In the applications it is of interest to evaluate and report these stresses at the *element nodal points* located on the corners and possibly midpoints of the element. These are called *element nodal point stresses*.

It is important to realize that the stresses computed at the same nodal point from adjacent elements *will not generally be the same*, since stresses are not required to be continuous in displacement-assumed finite elements. This suggests some form of stress averaging can be used to improve the stress accuracy, and indeed this is part of the stress recovery technique further discussed in §28.5. The results from this averaging procedure are called *nodal point stresses*.

For the moment let us see how we can proceed to compute element nodal stresses. Two approaches are followed in practice:

1. Evaluate directly $\boldsymbol{\sigma}$ at the element node locations by substituting the natural coordinates of the nodal points as arguments to the shape function modules. These modules return $\mathbf{q}_x$ and $\mathbf{q}_y$ and direct application of (28.2)-(28.4) yields the strains and stresses at the nodes.

2. Evaluate $\boldsymbol{\sigma}$ at the Gauss integration points used in the element stiffness integration rule and then extrapolate to the element node points.

Empirical evidence indicates that the second approach generally delivers better stress values for *quadrilateral* elements whose geometry departs substantially from the rectangular shape. This is backed up by “superconvergence” results in finite element approximation theory. For rectangular elements there is no difference.

For isoparametric *triangles* both techniques deliver similar results (identical if the elements are straight sided with midside nodes at midpoints) and so the advantages of the second one are marginal. Both approaches are covered in the sequel.

§28.3. Direct Stress Evaluation at Nodes

This approach is straightforward and need not be discussed in detail.

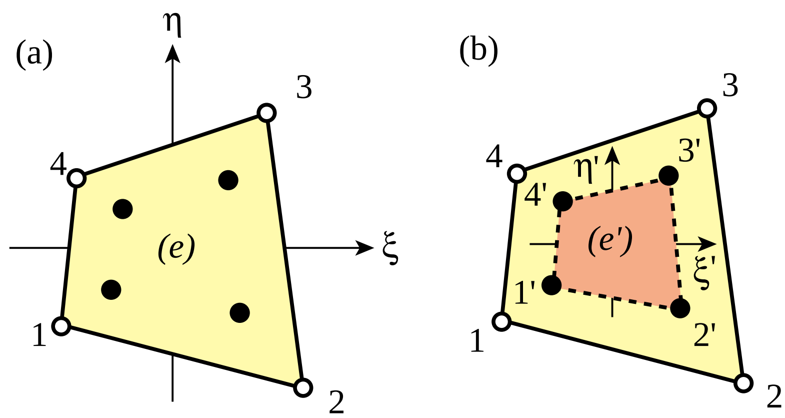

Figure 28.1. Extrapolation from 4-node quad Gauss points: (a) 2×2 rule, (b) Gauss element (e')

§28.4. Extrapolation from Gauss Points

This will again be explained for the four-node bilinear quadrilateral. The normal Gauss integration rule for element stiffness evaluation is 2×2, as illustrated in Figure 28.1.

The stresses are calculated at the Gauss points, which are identified as $1'$, $2'$, $3'$ and $4'$ in Figure 28.1. Point i' is closest to node i so it is seen that Gauss point numbering essentially follows element node numbering in the counterclockwise sense. The natural coordinates of these points are listed in Table 28.1. The stresses are evaluated at these Gauss points by passing these natural coordinates to the shape function subroutine. Then each stress component is "carried" to the corner nodes 1 through 4 through a bilinear extrapolation based on the computed values at $1'$ through $4'$.

To understand the extrapolation procedure more clearly it is convenient to consider the region bounded by the Gauss points as an "internal element" or "Gauss element". This interpretation is depicted in Figure 28.1(b). The Gauss element, denoted by (e'), is also a four-node quadrilateral. Its quadrilateral (natural) coordinates are denoted by ξ' and η'. These are linked to ξ and η by the simple relations

$$\xi = \xi'/\sqrt{3}, \quad \eta = \eta'/\sqrt{3}, \qquad \xi' = \xi\sqrt{3}, \quad \eta' = \eta\sqrt{3}. \tag{28.4}$$

Any scalar quantity w whose values w'_i at the Gauss element corners are known can be interpolated through the usual bilinear shape functions now expressed in terms of ξ' and η':

$$w(\xi', \eta') = [\, w'_1 \quad w'_2 \quad w'_3 \quad w'_4 \,] \begin{bmatrix} N_1^{(e')} \\ N_2^{(e')} \\ N_3^{(e')} \\ N_4^{(e')} \end{bmatrix}, \tag{28.5}$$

where (cf. §15.6.2)

$$\begin{aligned} N_1^{(e')} &= \tfrac{1}{4}(1-\xi')(1-\eta'), \\ N_2^{(e')} &= \tfrac{1}{4}(1+\xi')(1-\eta'), \\ N_3^{(e')} &= \tfrac{1}{4}(1+\xi')(1+\eta'), \\ N_4^{(e')} &= \tfrac{1}{4}(1-\xi')(1+\eta'). \end{aligned} \tag{28.6}$$

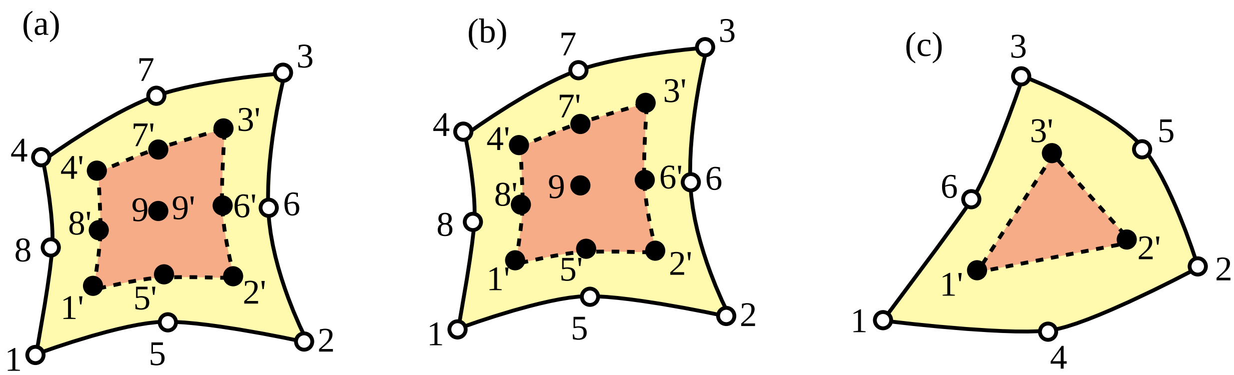

Figure 28.2. Gauss elements for higher order quadrilaterals and triangles: (a) 9-node element with 3 × 3 Gauss rule, (b) 8-node element with 3 × 3 Gauss rule, (c) 6-node element with 3-interior point rule.

To extrapolate w to corner 1, say, we replace its ξ' and η' coordinates, namely $\xi' = \eta' = -\sqrt{3}$, into the above formula. Doing that for the four corners we obtain

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 1+\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 1-\frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ -\frac{1}{2} & 1+\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 1-\frac{1}{2}\sqrt{3} \\ 1-\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 1+\frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ -\frac{1}{2} & 1-\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 1+\frac{1}{2}\sqrt{3} \end{bmatrix} \begin{bmatrix} w'_1 \\ w'_2 \\ w'_3 \\ w'_4 \end{bmatrix} \tag{28.7}$$

Note that the sum of the coefficients in each row is one, as it should be. For stresses we apply this formula taking w to be each of the three stress components, σ_{xx}, σ_{yy} and τ_{xy}, in turn.

Extrapolation in Higher Order Elements

For eight-node and nine-node isoparametric quadrilaterals the usual Gauss integration rule is 3 × 3, and the Gauss elements are nine-noded quadrilaterals that look as in Figure 28.2(a) and (b) above. For six-node triangles the usual quadrature is the 3-point rule with internal sampling points, and the Gauss element is a three-node triangle as shown in Figure 28.2(c).

§28.5. Interelement Averaging

The stresses computed in element-by-element fashion as discussed above, whether by direct evaluation at the nodes or by extrapolation, will generally exhibit jumps between elements. For printing and plotting purposes it is usually convenient to "smooth out" those jumps by computing *averaged nodal stresses*. This averaging may be done in two ways:

(I) Unweighted averaging: assign same weight to all elements that meet at a node;

(II) Weighted averaging: the weight assigned to element contributions depends on the stress component and the element geometry and possibly the element type.

Several weighted average schemes have been proposed in the finite element literature, but they do require additional programming.