16

Finite Element Fabrication Overview

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§16.1. Introduction

This Chapter surveys basic and advanced methods for constructing finite elements. A catalog of various formulations for mechanical elements is given, and approaches that may combine several formulations are outlined.

The need for advanced formulations is most acutely fell for plate and shell models, since those fall outside the well tested framework of iso-P elements. Consequently this Chapter may be viewed as a panoramic overview of models appropriate for the next Part, which focuses on plates and shells.

§16.2. Balancing Physics and Mathematics

Chapter 1 of the IFEM Notes [255] discussed two interpretation of the Finite Element method (FEM): physical and mathematical. Both are given equal time in that course.

The physical interpretation emphasizes the breakdown (disassembly, decomposition, separation, tearing) of a complex system into simpler components that eventually reach a primitive level called *elements*. This interpretation is convenient for engineering users since it simplifies modeling. It is also the basis for implementation of element libraries and assemblers in commercial codes.

The mathematical interpretation views FEM as a spatial discretization method for solving partial differential equations. The concept of breakdown and assembly, so integral to the physical interpretation, are no longer necessary.

This interpretation plays a significant role in establishing the mathematical foundations as well as the extension of FEM applications beyond the original focus on structural mechanics.

Chapter 1 of [255] says that the two interpretations synergically complement each other. That sweeping statement is appropriate to an introductory exposition, where to lessen confusion things are painted black and white.

A more realistic perspective is the spectrum pictured in Figure 16.1. Therein "physical modeling dominates" indicates that the fit to a particular application (or even a particular physical system) governs element construction. By constrast, "mathematical virtuosity dominates" conveys the opposite: the use of advanced functional analysis tools takes the upper hand while applications, if any, are downplayed.

Elements that rely heavily on physics and observations[1] tend to be interesting because developers often have to negotiate tradeoffs, and agreement with experience and observation tends to keep them honest.

At the very top of the figure are "experimental elements" defined entirely by test data. For example, a dry-friction contact-impact element with behavior defined entirely by digitized tables, or an actuator element defined by force-extension-rate response functions provided by the manufacturer.[2]

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[1] The disconnection from mathematics should not be taken too literally. A finite element always obeys some rules of mathematics such as matrix algebra and calculus.

[2] In commercial codes these are often implemented as "user-defined elements." The idea of having user entry points to the element library to accomodate unpredictable modeling needs is surprisingly old: it was first discussed by Turner, Martin and Weikel in a landmark 1964 paper [761] that sets out the definitive exposition of the Direct Stiffness Method.

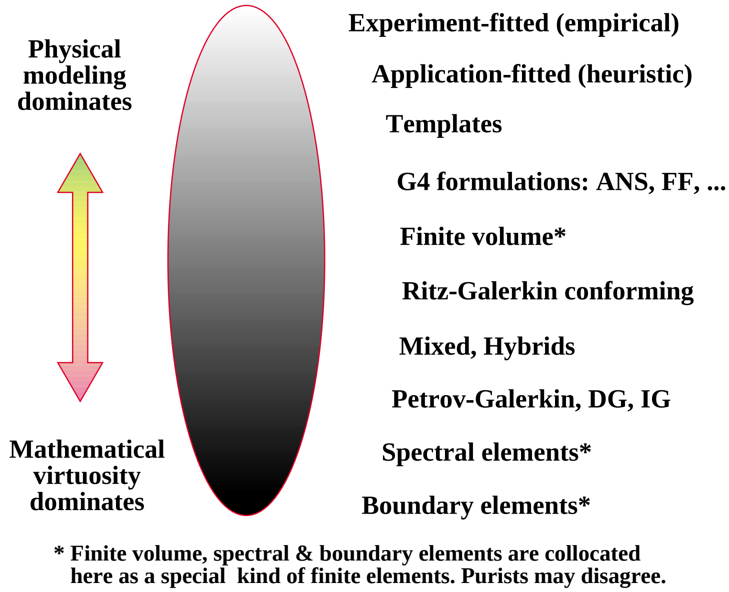

FIGURE 16.1. The spectrum of finite element fabrication methods. Toward the top the physical interpretation, stressing modeling and customized fit-to-application, dominates. Toward the bottom the mathematical interpretation, stressing the use of advanced tools, dominates.

Next come elements based on rules that have not been fully mathematically blessed, such as the patch test covered on the next Chapter. These includes templates[3] as well as high-performance elements of Generation 4 (1980-date) such as Assumed Natural Strain (ANS) and the Free Formulation (FF). Elements that directly rely on conservation laws, such as finite volume models for Euler and Navier-Stokes fluid flows are placed somewhat closer to the physical side.

In the middle of the spectrum one finds elements based on conventional variational principles (Total Potential Energy, etc). that may be collectively labeled "Ritz-Galerkin conforming." Subclasses of these, such as the isoparametric family, are covered in introductory FEM courses.

Hybrid elements are based on variational principles that are somewhat unconventional because they were designed specifically for FEM. As such, the mathematical tools required are a bit more advanced and have not been studied as extensively as conventional variational formulations.

If no variational principle is available, elements may be still constructed on the basis of the weak form, using the Method of Weighted Residuals (MWR). Although MWR spans a vast class of formulations (including Ritz-Galerkin, finite volumes, least-squares, collocation, subdomain, etc.) a popular choice when a functional is unavailable is Petrov-Galerkin. The mathematical tools needed here are more exotic because lack of self-adjointness removes the "Ritz safety blanket" and physical transparency. Spectral elements are also placed in this region of Figure 16.1 as their formulation is done in transform spaces unrelated to physics.

Boundary elements, if viewed as special case of FEM, are placed at the bottom of Figure 16.1 since they rely on esoteric mathematical tools (for example source distributions) only indirectly verifiable

[3] Templates are parametrized algebraic forms of element operators that include specific elements as instances. Templates for certain applications are discussed later.

by observation. In general an element constructed to numerically solve a PDE completely devoid of physics would be placed at this end.

§16.3. A Catalog of Formulations for Mechanical Elements

Even restricting attention to solid and structural mechanics, the number of finite element formulations has steadily grown over the past five decades. To throw some light into this melange, the following collection outlines several important methodologies for that application, noting advantages and limitations.

§16.3.1. Isoparametric (iso-P) Displacement Formulation

Source — Outgrow of Taig's quadrilateral presented in [719]. Extended by Irons to arbitrary geometries [394,397].

Master field(s) — One internal: displacements. Strains and stresses are slaves.

Var. principle — Total Potential Energy (TPE)

Description — Geometry and displacements interpolated by same shape functions. Numerical integration by Gauss quadrature.

Applicability — Problems governed by functional with variational index one. Only displacements as degrees of freedom.

Popularity — Huge despite limitations noted below.

Strengths — Well established (nearly 46 years old in 2013), widely implemented. Advantages and shortcomings well known by now. Systematic derivation rules, valid for any dimensionality and element complexity. Naturally handles elements with curved sides/faces. Technique easily extended to non-structural elements (thermal, fluids, electromagnetics, etc) as long as variational index in the primary variable stays one. Shape functions useful to express interpolation rules for other applications, such as data fitting.

Limitations — Low order iso-P models may have poor performance in terms of locking and distortion sensitivity. Some fixed-up devices are described in §16.3.2. Simplex elements of this type (linear triangles and tetrahedra) cannot be improved, however, by any device. Not applicable to problems with variational index greater than one in displacements, such as Bernoulli-Euler beams, Kirchhoff plates and thin shells. Cannot handle elements with rotational or derivative-type freedoms without substantial modifications that usually involve restriction to the subP formulation outlined in §16.2.3. "Blows up" for incompressible plane strain, axisymmetric and solid elements.

§16.3.2. Fixed-Up Isoparametric Formulation

Source — Bag of tricks emerging over period 1969–75: [188,724,801,824]. Equivalence to mixed methods: [378]. Comprehensive exposition in Hughes' book [385].

Master field(s) — Same as iso-P.

Var. principle — Total Potential Energy (TPE)

Description This category embraces a number of ad-hoc devices introduced to fix or at least alleviate the poor performance of certain isoparametric models, especially low-order ones, as regards locking (extreme overstiffness) and distortion sensitivity. Most common: (1) reduced integration, (2) selective integration, (3) directional integration, and (4) incompatible displacement modes.

Applicability Same as iso-P formulation.

Popularity High because fixed-up devices are easy to implement. Expected knowledge does not rise to level of variational techniques used in mixed and hybrid formulations.

Strengths Easy to implement. Software reuse (for example, of shape function modules) is facilitated. This is particularly important in legacy and nonlinear codes.

Limitations Errating backstabbing effects. As more tricks are introduced, unpleasant surprises can happen, such as hourglassing and invariance loss. Repair may produce more unexpected side effects: "turn on the light, the shower goes on."[4] Eventually legacy software adorned with these accoutrements may become untouchable for fear of additional side effects.[5]

§16.3.3. Subparametric (subP) Displacement Formulation

Source Assumed-displacement models preceding the iso-P formulation as in [22,484,485].

Master field(s) One internal: displacements. Strains and stresses are slaves.

Var. principle Total Potential Energy (TPE)

Description Element geometry kept simple, typically restricted to the simplest possible shapes. Displacements interpolated by equal or higher order functions than geometry.

Applicability Problems governed by functionals of any variational index. Handles elements with rotational or derivative-type freedoms.

Popularity Medium. Easy to present and teach.

Strengths Applicability wider than that of iso-P elements. Can directly handle C^1 beams, plates and shells.

Limitations Element geometry is restricted, restricting modeling flexibility. No systematic construction rules. As in the iso-P case, performance of low-order elements may be poor. For some configurations full interelement compatibility may be difficult or even impossible to attain.

§16.3.4. Mixed Hellinger-Reissner (HR) Formulation

Source First FEM use proposed by Herrmann [353,354] for incompressible and nearly-incompressible elasticity (underlying motivation was modeling of solid rocket propellants).

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[4] Technically these are called *regression bugs*.

[5] This is no joke. Some commercial FEM codes contain elements empirically constructed and quickly implemented by long departed employees, who were careful not to document them to enahnce job security. Those archeological software pieces are left untouched because of unforeseen effects if removed.

Master field(s)	Two internal: displacements and stresses. Two strain slaves.
Var. principle	Hellinger-Reissner (HR)
Description	Displacement master field may be interpolated with shape functions. Separate master stress interpolation (or pressure) usually done in Cartesian coordinates. Stress DOFs (or stress parameters) may be condensed out at element level if (1) stress (or pressure) may jump between adjacent elements and (2) material is compressible. Else stress (or pressure) DOFs must go into the assembly.
Applicability	Two major uses: (1) treating incompressible or nearly-incompressible materials, and (2) improving the performance of low-order displacement models. For case (1) pressure is assumed in addition to displacements.
Popularity	Low since it requires advanced knowledge. Hampered by the discovery of fixed-up devices for iso-P elements, which under certain restrictions effectively produce the equivalent of mixed elements with simpler tricks.
Strengths	Possible way to improve element performance. Proven useful for the incompressible case. If stress DOFs can be eliminated, condensed model looks like a displacement element to the assembler. Comparing strains from displacements and stresses allows easy construction of an element-level error measure.
Limitations	Requires knowledge of mixed variational principles, not an easy subject. Prone to failure because of rank deficiency if stress (or pressure) and displacement assumptions violated certain stability conditions.[6] Even if element is stable, expected improvements in performance may not necessarily materialize: "mixed elements lead to mixed results" (G. Strang).

§16.3.5. Equilibrium-Stress Hybrid Formulation

Source	Pian [573,574], variational basis by Pian and Tong [575]; linkage to incompatible displacement models established by Pian and Tong in 1986 [577].
Master field(s)	Over element: internal equilibrium stress field, interelement-compatible boundary displacements. Slave Strains. Displacement field inside element known only weakly.
Var. principle	Total Potential Complementary Energy (TCPE) augmented by dislocation potential.
Description	Ingenious motivation: facilitate interelement compatibility by assuming only boundary displacements while mainting internal equilibrium, which should result in better stress recovery.
Applicability	Applicable in principle to any geometrically linear element.
Popularity	Medium despite age.[7] Reason: requires advanced mathematics, way beyond that of most engineering students.

[6] Many papers have been written on this topic, called the Babuska-Brezzi (BB) condition. Of these, 99.9% are forgettable fillers. The physical interpretation of this condition by Fraeijs de Veubeke is called the "limitation principle" [275].

[7] Over 40+ years since original publications. In fact it was published a bit earlier (3 yrs) than the more popular iso-P formulation.

Strengths If workable, a proven way to improve element stress-recovery performance. Especially useful when assumed-displacement interelement compatibility is hard to achieve, as in thin plates and shells. Less prone to stability limitations than HR elements.

Limitations Requires knowledge of hybrid variational principles, a tough subject. Construction of an invariant equilibrated stress field may be difficult or even impossible for arbitrary geometries.[8] Difficult to extend to geometrically nonlinear problems, as long as strong satisfaction of equilibrium equations is enforced.

Remark 16.1. There is a large number of possible hybrid element formulations. They share a common feature: different master fields are assumed over the interior and boundary of the element. The foregoing case is just one of many possibilities. It is listed here on account of historical importance and its implicit presence in the FF, ANDES and template formulations discussed below.

§16.3.6. Assumed Strain Formulations

Source For assumed Cartesian strains: MacNeal [455]. For assumed natural strains (ANS): Bathe-Dvorkin [55] for flat plates, Park-Stanley [555] and Huang-Hinton [375] for shells.

Master field(s) Varies with author. Common feature is that element strain variation is assumed in some form.

Var. principle Varies with author. Most use Total Potential Energy (TPE) with various forms of strain assumptions for certain energy components.

Description The slave connection between displacements and stresses is selectively broken to alleviate locking problems. In the Assumed Natural Strain variant (the most powerful variant) the strain field is expressed in natural coordinate directions, but not necessarily in tensorial form.

Applicability Primarily useful for plates and shells fabricated from degenerated solid elements.

Popularity Medium, since applicability is restricted.

Strengths When done correctly it produces high performance plates and shell elements. Main advantages arise in geometrically nonlinear analysis of shells.

Limitations To make it work correctly requires substantial physical insight, a dab of black magic incantations and plenty of luck.

§16.3.7. Free Formulation (FF)

Source Bergan and coworkers [82,86].

Master field(s) Element response behavior split into basic and higher order. For basic response, same masters as constant-stress equilibrium hybrids. For higher order response,

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8 This difficulty (lack of observer invariance) held up advances in extending Pian's original results for 20 years until the landmark paper by Pian and Sumihara [576].

assumed displacements in generalized coordinates. These are not usually interelement compatible.[9]

Var. principle Not provided in original formulation. Shown to be a mixture of hybrid and TPE functionals in [220].

Description Fundamental idea is splitting of element response. Each component is assigned different but complementary roles. Basic component takes care of convergence and mixability. Higher order component provides stability (rank sufficiency) and accuracy. Orthogonality conditions between those components insure *a priori* satisfaction of the Individual Patch Test (IPT) of [80].

Applicability Any mechanical element.

Popularity Low. Splitting concept is at odds with historical tradition, variational formulation is highly unconventional, and use of generalized coordinates is unfamiliar to many developers.

Strengths Provides high performance elements. Basing the higher order component on assumed displacements taps a well studied subject. Easily extendible to nonlinear statics and structural dynamics.

Limitations Requires substantial physical insight on the part of a developer. Construction of the higher order component not easy, involving many trials to balance orthogonality conditions with rank sufficiency. Mathematical basis incomplete.

§16.3.8. Assumed Natural Deviatoric Strain (ANDES) Formulation

Source Felippa and Militello [225,492].

Master field(s) Element response behavior split into basic and higher order. For basic response, same masters as constant-stress equilibrium hybrid. For higher order response, assumed natural strains.

Var. principle Falls in the context of parametrized variational principles [228].

Description Combines ideas from the Free Formulation (FF) and the Assumed Natural Strain (ANS) formulation. Same response splitting as in FF, and identical basic component. Higher order component built with ANS ideas using assumed deviatoric strains (strains that deviate from a constant state) instead of total strains.

Applicability Any mechanical element.

Popularity Low, as in the case of the FF. Deviates too much from tradition. Paticular case of templates.

Strengths Similar to FF, but construction of the higher order part allows more easy parametrization on the way to templates.

Limitations Requires substantial expertise and a dose of luck since methodology is very new. Trial and error mandatory since mathematical basis is incomplete.

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[9] The qualifier "free" is intended to reflect the fact that these higher order displacement functions need not satisfy interelement compatibility restrictions as in the case of subP and iso-P elements, and thus enjoy greater freedom of choice than those required to do so. It is not used in the English sense of "free of charge."

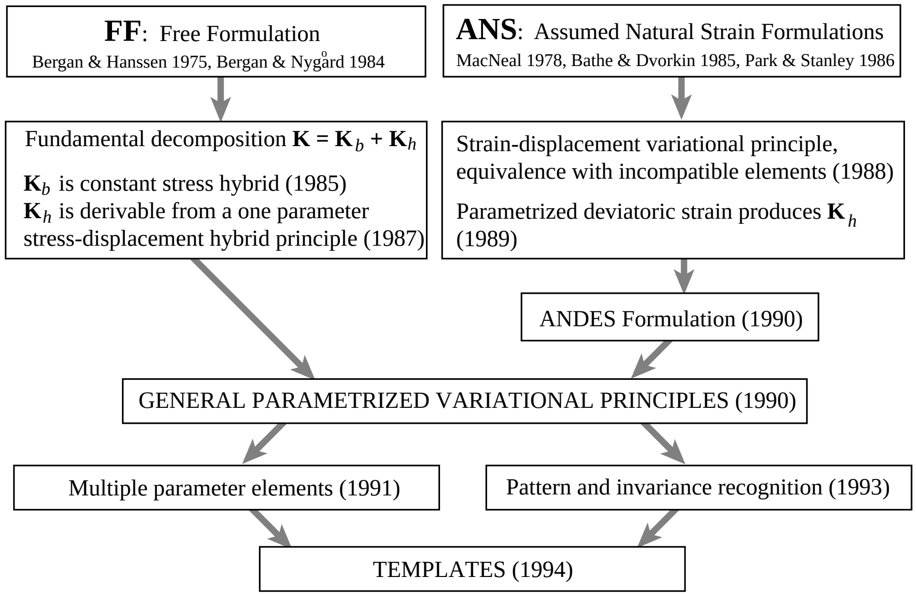

FIGURE 16.2. The road to templates.

§16.3.9. Templates

Source First mentioned in [228]. Developed in [**?**,241]. Recent tutorial: [243]. The roadmap to templates is flowcharted in Figure 16.2.

Master field(s) Element response split as in case of FF and ANDES. Basic part is constant equilibrium-stress hybrid. For higher order component, no predefined masters.

Var. principle For basic component, equilibrium stress hybrid principle. For higher order component, no specific principle; in fact it might not exist.

Description A template is a parametrized algebraic form of element operators (stiffness, mass, etc). This produces an infinity of possible elements that *a priori* satisfy the Individual Element Test (IET). Specific element instances obtained by assigning numerical values to parameters. A *universal template* is one that includes all possible elements that pass the IET.

Applicability In principle any element. But see *Limitations* below.

Popularity None, as it is a recent idea barely off the ground.

Strengths Includes an infinite number of possible elements. Instances can be customized to produce optimal results for envisioned use.

Limitations Development impossible by hand since one must carry along element properties (geometry, constitutive, fabrication, ...) in symbolic form, along with free parameters. Use of a computer algebra system (CAS) mandatory. Limits in current CAS power, however, has restricted the idea to 1D and 2D elements of simple geometry.

§16.4. *Approaches to Element Construction

The term *approach* is taken here to mean a combination of methods and empirical tools to achieve a given objective. In FEM work, isoparametric, stress-assumed-hybrid and ANS (Assumed Natural Strain) formulations are methods and not approaches. An approach may zig-zag through several methods. FEM approaches range from heuristic to highly analytical.

Figure 16.3 makes an implicit assumption: the performance of an element of given geometry, node and freedom configuration can be improved. There are obvious examples where this is not possible. For example, constant-strain elements with translational freedoms only: 2-node bar, 3-node membrane triangle and 4-node elasticity tetrahedron. Those cases are excluded because it makes no sense to talk about high performance or optimality under those conditions.

§16.4.1. *Fixing Up

As noted in §16.2 certain popular element construction methods, such as the iso-P formulation, may produce bad or mediocre low-order elements. If that is the case two questions may be raised:

(i) Can the element be improved?

(ii) Is the improvement worth the trouble?

If the answer to both is yes, the fix-up approach tries to improve the performance by an array of remedies that may be collectively called the FEM pharmacy. Cures range from heuristic tricks such as reduced and selective integration[10] to more scientifically based concoctions.

This approach accounts for most of the current publications in finitelementology. Playing doctor can be fun. But also frustating, as trying to find a black cat in a dark cellar at midnight. Inject these incompatible modes: oops! the patch test is violated. Make the Jacobian constant: oops! it locks in distortion. Reduce the integration order: oops! it lost rank sufficiency. Split the stress-strain equations and integrate selectively: oops! it is not observer invariant. And so on.

§16.4.2. *Retrofitting

Retrofitting is a more sedated activity. One begins with a irreproachable parent element, free of obvious defects. Typically this is a higher order iso-P element constructed with a complete or bicomplete polynomial; for example the 6-node quadratic triangle or the 9-node Lagrange quadrilateral. The parent is fine but too complicated to be an HP element. Complexity is reduced by master-slave constraint techniques so as to fit the desired node-freedom configuration pattern.

This approach commonly makes use of node and freedom migration techniques. For example, drilling freedoms may be defined by moving translational midpoint or thirdpoint freedoms to corner rotations by kinematic constraints. Discrete Kirchhoff constraints and degeneration (3D→2D) for plate and shell elements provides another example. Retrofitting has the advantage of being easy to understand and teach. It occasionally produces useful elements but rarely high performance ones.

§16.4.3. *Direct Fabrication

This approach relies on divide and conquer. To give an analogy: upon short training a FEM novice knows that a discrete system is decomposed into elements, which interact only through common freedoms. Going deeper, an element can be constructed as the superposition of components or pieces, with interactions limited through appropriate orthogonality conditions. Components are invisible to the user once the element is implemented.

Fabrication is done in stages. At the start there is nothing: the element is without form, and void. At each stage the developer injects another component (= subspace). Components may be done through different methods.

[10] The FEM equivalent of acupuncture and herbal remedies.

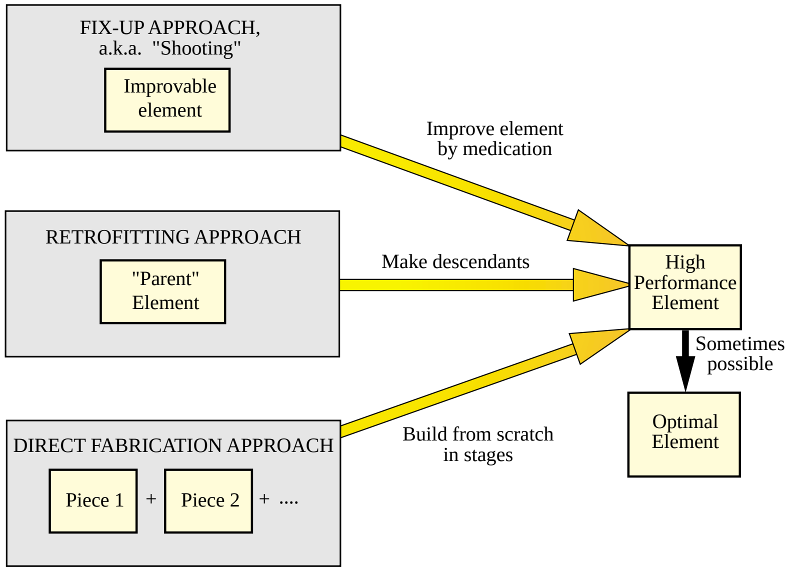

FIGURE 16.3. Three approaches to element construction.

The overarching principle is: correct performance after each stage. If at any stage the element has problems (for example: it locks) no retroactive cure is attempted as in the fix-up approach. Instead the component is trashed and another one picked. One never uses more components than strictly needed: condensation is forbidden. Components may contain free parameters, which may be used to improve performance and eventually to try for optimality. One general scheme for direct fabrication is the template formulation outlined in §16.2.9.

All applications of the direct fabrication method to date have been done in two stages, separating the element response into basic and higher order. This process is further elaborated in further Chapters.

References

Referenced items have been moved to Appendix R.

Homework Exercises for Chapter 16

Finite Element Fabrication Overview

EXERCISE 16.1 [D:5] What are the main restrictions for use of conventional isoparametric elements? In particular, can they be used directly (meaning without modifications) to model Bernoulli-Euler beams and thin plates/shells?

EXERCISE 16.2 [D:10] Thinks about these scenarios:

(1) A “bumper element” for a high speed train is developed from recorded model crash data.

(2) A problem in tank sloshing under gravity (for example, liquid moving inside a partly filled aircraft fuel tank) is modeled using a boundary element method (BEM).

(3) Seismic wave propagation in a stratified soil is simulated using a spectral finite element method.

Where would (1), (2) and (3) fit in the spectrum of Figure 16.1?. That is, which models stress physics against math, and vice-versa?