21

Kirchhoff Plates: BCs and Variational Forms

TABLE OF CONTENTS

§21.1. Introduction

In this Chapter we continue the discussion of the governing equations of Kirchhoff plates with the consideration of boundary conditions (BCs) and variational forms of the plate equations.

When plates and shells are spatially discretized by FEM the proper modeling of boundary conditions can be a tough subject. Two factors contribute. First, displacement derivatives in the form of rotations are now involved in the kinematic boundary conditions. Second, the correlation between physical support conditions and mathematical B.C.s can be tenuous. Some mathematical BC used in practice are nearly impossible to reproduce in the laboratory, let alone on an actual structure.

§21.2. Boundary Conditions for Kirchhoff Plate

One of the mathematical difficulties associated with the Kirchhoff model is the "Poisson paradox":

- The plate deflection satisfies a fourth order partial differential equation (PDE). For the isotropic homogeneous plate this is the biharmonic equation $\Delta^2\, w = q/D$.
- A fourth order PDE can only have *two* boundary conditions at each boundary point.
- But *three* conjugate quantities: normal moment, twist moment and transverse shear appear naturally at a boundary point.

The reduction from three to two requires variational methods. This was done first by Kirchhoff, thus achieving mathematical closure. But it is not necessary to look at a complete functional such as the TPE. The procedure can be explained directly through virtual work principles.

§21.2.1. Conjugate Quantities

Conside a Kirchhoff plate of general shape as in Figure 21.1(a). Assume that the boundary Γ is smooth, that is, contains no corners. Under those assumptions the exterior normal **n** and tangential direction **s** at each boundary point B are unique, and form a system of local Cartesian axes.

The kinematic quantities referred to these local axes are

$$w, \quad \frac{\partial w}{\partial n} = -\theta_s, \quad \frac{\partial w}{\partial s} = \theta_n, \tag{21.1}$$

where θ_n and θ_s denote the rotations of the midsurface at B about axes n and t, respectively, see Figure 21.1(b). The *work conjugate* static quantities, shown in Figure 21.1(c), are

$$Q_n, \quad M_{nn}, \quad M_{ns}, \tag{21.2}$$

respectively. By 'conjugate' it is meant that the boundary work can be expressed as the line integral

$$W_B = \int_\Gamma \Big(Q_n w + M_{ns}\frac{\partial w}{\partial s} + M_{nn}\frac{\partial w}{\partial n}\Big)\, ds = \int_\Gamma \Big(Q_n w + M_{ns}\theta_n - M_{nn}\theta_s \Big)\, ds \tag{21.3}$$

where $ds \equiv d\Gamma$ denotes the differential boundary arclength. This integral appears naturally in the process of forming the energy functionals of the plate. Given the foregoing configuration of W_B, it appears at first sight as if *three* boundary conditions can be assigned at each boundary point, taken from the conjugate sets (21.1) and (21.2). For example:

$$\begin{array}{llll} \text{Simply supported edge} & w = 0 & \theta_n = 0 & M_{nn} = 0, \\ \text{Free edge} & Q_n = 0 & M_{ns} = 0 & M_{nn} = 0. \end{array} \tag{21.4}$$

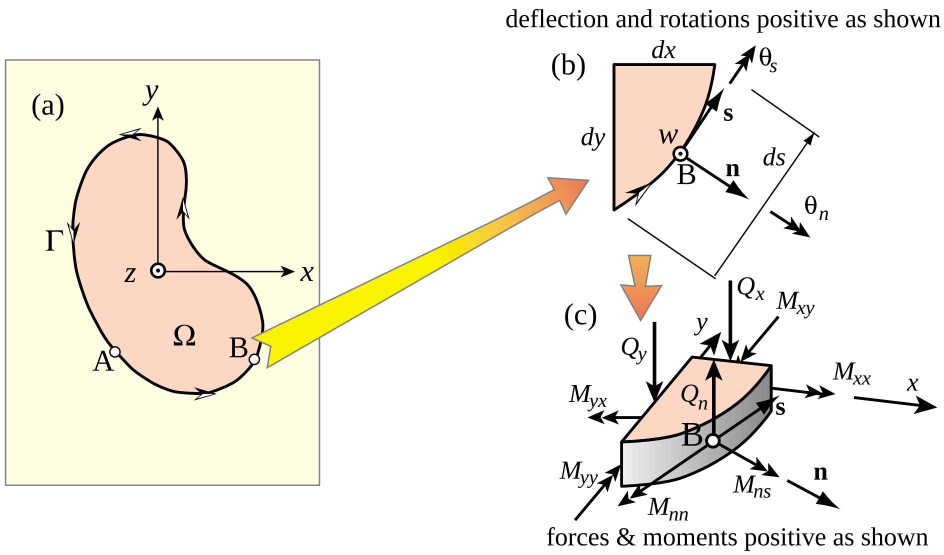

FIGURE 21.1. BCs at a *smooth* boundary point B of a Kirchhoff plate: $\mathbf{n}$ = external normal, $\mathbf{s}$ = tangential direction. (a) boundary traversed in the counterclockwise sense (looking down from $+z$) leaving the plate proper on the left; (b) edge kinematic quantities w, θ_s and θ_n; (c) conjugate force and moment quantities Q_n, M_{nn} and M_{ns} on the boundary face.

The boundary conditions for a free edge were indeed expressed by Poisson in this form.[1] As noted previously, this is inconsistent with the order of the governing PDE. Kirchhoff showed[2] that three conditions are too many and in fact only two are independent.

§21.2.2. The Modified Shear

The reduction to two independent conjugate pairs may be demonstrated through integration of (21.3) by parts with respect to s, along a segment AB of the boundary Γ:

$$W_B|_A^B = \int_A^B \left[\left(Q_n - \frac{\partial M_{ns}}{\partial s} \right) w + M_{nn} \frac{\partial w}{\partial n} \right] dt + M_{ns}\, w|_A^B . \tag{21.5}$$

Introducing the *modified shear*[3]

$$V_n = Q_n - \frac{\partial M_{ns}}{\partial s}, \tag{21.6}$$

we may rewrite (21.5) as

$$W_B|_A^B = \int_A^B \left(V_n w + M_{nn} \frac{\partial w}{\partial n} \right) dt + M_{ns} w|_A^B . \tag{21.7}$$

This transformation reduces the conjugate quantities to two work pairs:

$$V_n,\ w \qquad \text{and} \qquad M_{nn},\ \frac{\partial w}{\partial n} = -\theta_s . \tag{21.8}$$

---

[1] See for example I. Todhunter and K. Pierson, *History of Theory of Elasticity*, Vol. I.

[2] G. Kirchhoff, publications cited in §24.2.

[3] Also called *Kirchhoff equivalent force*, a translation of Kirchhoffische Ersatzkräfte.

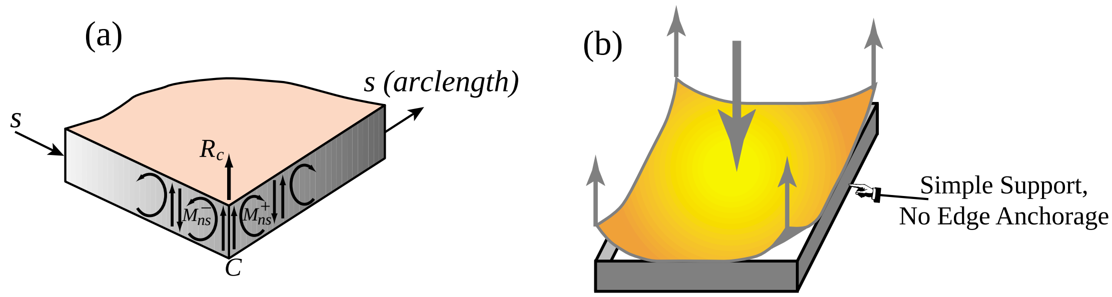

FIGURE 21.2. Effect of modified shear at a plate corner: (a) force-pairs do not cancel, producing a corner force R_c; (b) physical manifestation of modified shear as corner lifting forces.

§21.2.3. Corner Forces

The last term in (21.7) deserves analysis. First consider a plate with *smooth* boundary as in Figure 21.1(a). Assuming that M_{ns} is continuous over Γ, and we go completely "around" the plate so that $A \equiv B$,

$$M_{ns}\, w|_A^B = 0. \tag{21.9}$$

Next consider the case of a plate with a corner C as in Figure 21.2(a). At C the twisting moment jumps from, say, M_{ns}^- to M_{ns}^+. The transverse displacement w must be continuous (if fact, C^1 continuous). Place A and B to each side of C, so that $A \to C$ from the minus side while $C \leftarrow B$ from the plus side. Then

$$M_{ns}|_A^B = R_c w = (M_{ns}^+ - M_{ns}^-)w, \qquad \text{with} \qquad R_c = M_{ns}^+ - M_{ns}^-. \tag{21.10}$$

This jump in the twisting moment is called the *corner force* R_c, as shown in Figure 21.2(a). Note that R_c has the physical dimension of force, because the twisting moment M_{ns} is a moment (force times length) per unit length.

Remark 21.1. The physical interpretation of modified shears and of corner forces is well covered in Timoshenko and Woinowsky-Krieger.[4] Suffices to say that if a plate corner is constrained not to move laterally, a concentrated force R_c called the corner reaction, appears. If the corner is not held down the reaction cannot be transmitted to the supports and the plate will have a tendency to move away from the support. This is the source of the well known "corner lifting" phenomenon that may be observed on a laterally loaded square plate with simply supported edges that do not prevent lifting. See Figure 21.2(b). Notice that if the boundary is smooth, as in a circular plate, the phenomenon will not be observed.

This effect does not appear if the edges meeting at C are free or clamped, because if so the twist moment M_{ns} on both sides of the corner point are zero.

§21.2.4. Common Boundary Conditions

Below we state *homogeneous* boundary conditions frequently encountered in Kirchhoff plates as selected combinations of the conjugate quantities (21.8). Some BCs are illustrated in the structures depicted in Figure 21.3.

[4] *Theory of Plates and Shells* monograph cited in previous Chapter

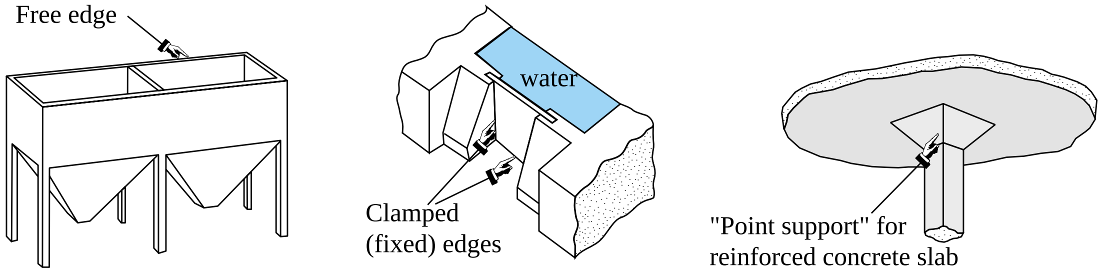

FIGURE 21.3. Boundary condition examples.

Clamped or *Fixed Edge* (with **s** along edge):

$$w = 0, \qquad \frac{\partial w}{\partial n} = -\theta_s = 0. \tag{21.11}$$

Simply Supported Edge (with **s** along edge):

$$w = 0, \qquad M_{nn} = 0. \tag{21.12}$$

Free Edge (with **s** along edge):

$$V_n = 0, \qquad M_{nn} = 0. \tag{21.13}$$

Symmetry Line (with **s** along line):

$$V_n = 0, \qquad \frac{\partial w}{\partial n} = -\theta_s = 0. \tag{21.14}$$

Point Support:

$$w = 0 \tag{21.15}$$

Non-homogeneous boundary conditions of force type involving prescribed normal moment $\hat{M}_{nn}$ or prescribed modified shear $\hat{V}_n$, are also quite common in practice. Non-homogeneous B.C. involving prescribed nonzero transverse displacements or rotations are less common.

§21.2.5. Strong Form Diagram

The Strong Form diagram of the governing equations, including boundary conditions, for the Kirchhoff plate model is shown in Figure 21.4.

We are now ready to present several energy functionals of the Kirchhoff plate that have been used in the construction of finite elements.

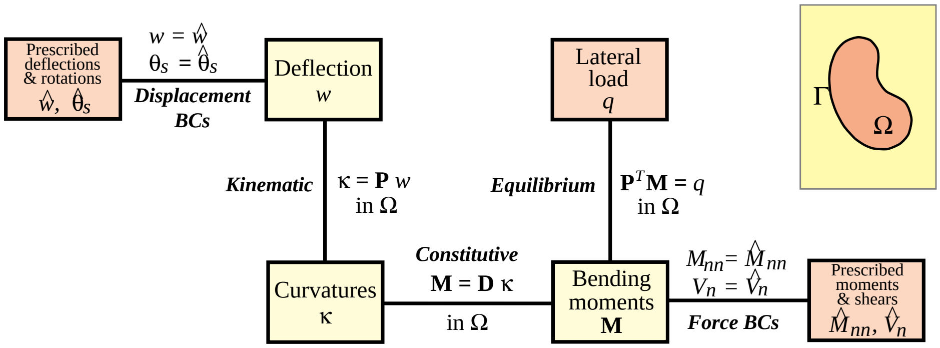

FIGURE 21.4. The Strong Form diagram for the Kirchhoff plate, including boundary conditions.

§21.3. The Total Potential Energy Principle

The only master field is the transverse displacement w. The departure Weak Form is shown in Figure 21.5. The weak links are the internal equilibrium equations and the force boundary conditions.

§21.3.1. The TPE Functional

Starting from the Weak Form flowcharted in Figure 21.5, and proceeding as in previous chapters one finally arrives at the Total Potential Energy (TPE) functional. Processing of the boundary terms is laborious. Timoshenko and Woinowsky-Krieger's book takes 6 pages in getting to the final destination.[5] Here we just state the final result. The TPE functional with the conventional forcing potential is

$$\boxed{\Pi_{\rm TPE}[w] = U_{\rm TPE}[w] - W_{\rm TPE}[w]} \tag{21.16}$$

The internal energy is

$$U_{\rm TPE}[w] = \tfrac{1}{2}\int_\Omega (\mathbf{M}^w)^T \boldsymbol{\kappa}^w \, d\Omega = \tfrac{1}{2}\int_\Omega (\boldsymbol{\kappa}^w)^T \mathbf{D}\,\boldsymbol{\kappa}^w \, d\Omega = \tfrac{1}{2}\int_\Omega (w\mathbf{P}^T)\mathbf{D}\,(\mathbf{P}w)\, d\Omega, \tag{21.17}$$

where $\mathbf{P} = [\,\partial^2/\partial x^2 \;\; \partial^2/\partial y^2 \;\; 2\,\partial^2/\partial x\partial y\,]^T$ is the curvature-displacement operator. The groupings $\mathbf{P}w$ in the last of (21.17) emphasize that $\mathbf{P}$ is to be applied to w to form the slave curvatures $\boldsymbol{\kappa}^w$.

The external work $W_{\rm TPE}[w]$ is more complicated than in plane stress and solids. It is best presented as the sum of three components. These are due to apply lateral loads, applied edge moments and transverse shears, and to corner loads, respectively:

$$W_{\rm TPE}[w] = W_q[w] + W_B[w] + W_C[w], \tag{21.18}$$

The first two terms apply to all plate geometries and are

$$W_q[w] = \int_\Omega q\, w\, d\Omega, \qquad W_B[w] = \int_{\Gamma_{VM}} (\hat{V}_n w - \hat{M}_{nn}\theta_s^w)\, d\Gamma. \tag{21.19}$$

[5] Part of the length is due to use of full notation.

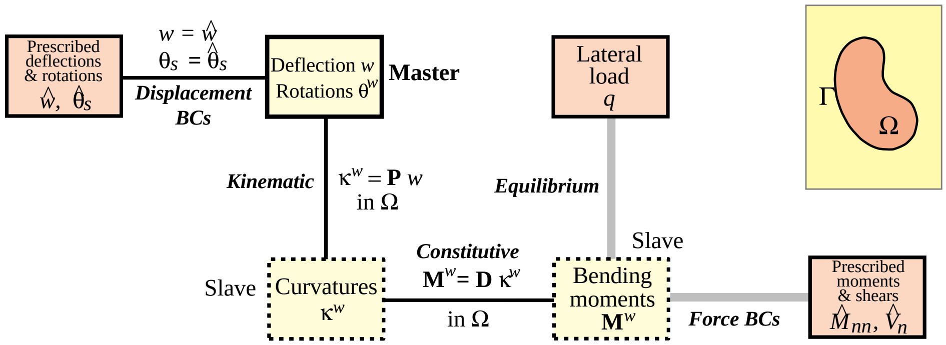

FIGURE 21.5. The Weak Form departure point to derive the TPE variational principle for a Kirchhoff plate.

The last term: W_C arises if the plate has $j = 1, 2, \ldots, n_c$ corners at which the displacement w_j is *not* prescribed. If so,

$$W_C = \sum_{j=1}^{n_c} R_{cj}\, w_j = \sum_{j=1}^{n_c} (\hat{M}^+_{ns} - \hat{M}^-_{ns})\, w. \tag{21.20}$$

If the transverse displacement of a corner is prescribed, the contribution of that corner to the functional vanishes because the displacement variation is zero.

Remark 21.2. If neither the corner displacement nor the twist moments M^-_{ns} and M^+_{ns} at the corner are known, the problem becomes nonlinear because the extent over which the plate lifts from supports is not known *a priori*. This problem becomes one of contact type and will require an iterative method to be solved.

§21.3.2. Finite Element Conditions

The variational index of the TPE functional is $m = 2$ because second derivatives of w (the curvatures) appear in $U_{\rm TPE}$. The classical-Ritz convergence conditions for finite elements derived using this principle are:

Completeness. The assumed w over each element should reproduce exactly all $\{x, y\}$ polynomials of order ≤ 2.

Continuity. The assumed w should be C^2 continuous inside the element and C^1 interelement-continuous.

Stability. Elements should be rank sufficient and the Jacobian determinant everywhere positive.

The C^1 interlement continuity condition is the tough one to crack. It is not easy to satisfy using standard polynomial assumptions. These difficulties have motivated, since the early 1960s, the development of various techniques to alleviate (or totally get rid of) that continuity requirement.

§21.4. The Hellinger-Reissner Principle

The Weak Form useful as departure point for the HR principle is shown in Figure 21.6. Both the transverse displacement w and the bending moment field $\mathbf{M}$ are chosen as master fields. The weak links are the internal equilibrium equations,

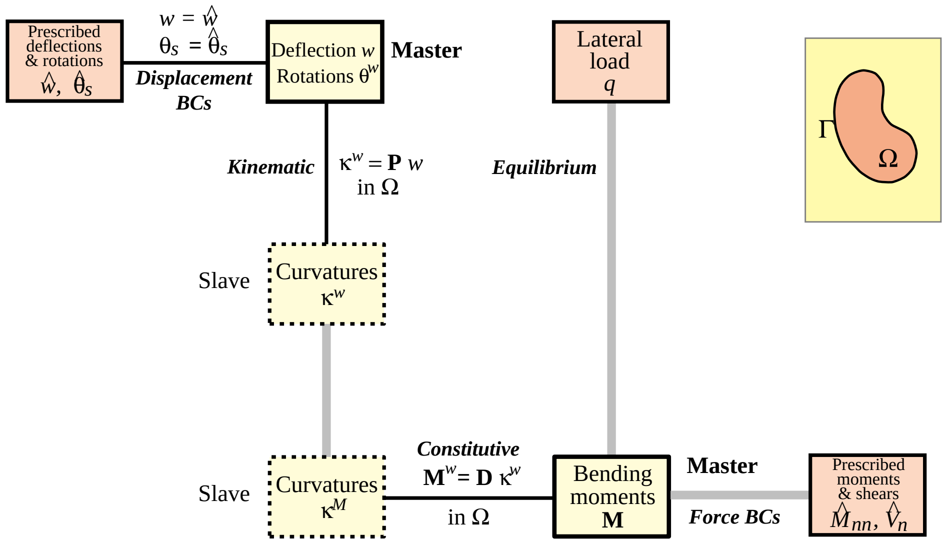

FIGURE 21.6. The Weak Form departure point to derive the Hellinger-Reissner (HR) variational principle for a Kirchhoff plate.

The HR functional with the conventional forcing potential is

$$\boxed{\Pi_{\rm HR}[w, \mathbf{M}] = U_{\rm HR}[w, \mathbf{M}] - W_{\rm HR}[w, \mathbf{M}].} \tag{21.21}$$

The internal energy is

$$U_{\rm HR}[w, \mathbf{M}] = \int_\Omega (\mathbf{M}^T \boldsymbol{\kappa}^w - \tfrac{1}{2}\mathbf{M}^T \mathbf{D}^{-1}\mathbf{M})\, d\Omega = \int_\Omega (\mathbf{M}^T \boldsymbol{\kappa}^w - \mathcal{U}^*)\, d\Omega. \tag{21.22}$$

Here $\mathcal{U}^* = \frac{1}{2}\mathbf{M}^T\mathbf{D}^{-1}\mathbf{M})$ is the complementary energy density (per unit of plate area) written in terms of the bending moments. Integration of this over Ω gives the total complementary energy U^*.

The external work $W_{\rm HR}$ is the same as for the TPE principle treated in the previous section.

§21.4.1. Finite Element Conditions

The variational indices of the HR functional are $m_w = 2$ for the transverse deflection and $m_M = 0$ for the bending moments. Consequently the completeness and continuity conditions for w are the same as for the TPE, and nothing is gained by going to the more complicated functional.

It is possible to *balance* the variational indices so that $m_w = m_M = 1$ by integrating the previous form by parts once. The resulting principle was exploited by Herrmann[6] to construct a plate element with linearly varying w, M_{xx}, M_{yy}, M_{xy}. This element, however, was disappointing in accuracy. Furthermore enforcing moment continuity can be physically wrong. Progress in the construction of elements of this type was achieved later using hybrid principles. This advance will not be covered here since the advent of the Post-1980 formulations (FF, ANDES, etc) have taken care of the problem.

[6] L. R. Herrmann, A bending analysis for plates, in *Proceedings 1st Conference on Matrix Methods in Structural Mechanics*, AFFDL-TR-66-80, Air Force Institute of Technology, Dayton, Ohio, pp. 577-604, 1966.

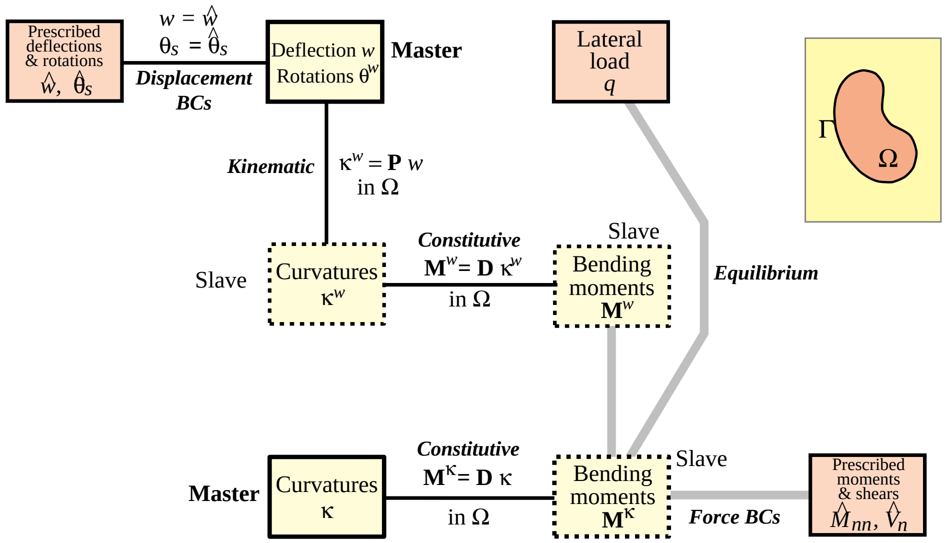

FIGURE 21.7. The Weak Form departure point to derive the Fraeijs de Veubeke curvature-displacement variational principle for a Kirchhoff plate.

§21.5. The Curvature-Displacement Veubeke Principle

This kind of principle (for elastic solids) was introduced by Fraeijs de Veubeke, and functionals will be accordingly identified by a FdV subscript.

§21.5.1. The Freaijs de Veubeke Functional

In this case both the transverse displacement w and the curvatures field $\boldsymbol{\kappa}$ are chosen as master fields. The departure Weak Form is shown in Figure 21.7.

$$\boxed{\Pi_{\rm FdV}[w,\boldsymbol{\kappa}] = U_{\rm FdV}[w,\boldsymbol{\kappa}] - W_{\rm FdV}[w,\boldsymbol{\kappa}].} \tag{21.23}$$

The internal functional is

$$\Pi_{\rm FdV}[w,\boldsymbol{\kappa}] = \int_\Omega (\boldsymbol{\kappa}^T\mathbf{M}^w - \tfrac{1}{2}\boldsymbol{\kappa}^T\mathbf{D}\boldsymbol{\kappa})\,d\Omega. \tag{21.24}$$

The external work is the same as for TPE.

§21.5.2. Finite Element Conditions

The variational indices of the Π_{FdV} functional is $m_w = 2$ for the transverse displacement and $m_\kappa = 0$ for the curvatures. Consequently the completeness and continuity conditions for w are the same as for the TPE, as nothing is gained by going to the more complicated functional. To get a practical scheme that reduces the continuity order one can either integrate by parts the internal energy, or proceed to hybrid principles. Since the latter are implicitly used in the Post-1984 advanced FEM formulations, the procedure will not be covered here.