6

Decomposition of Poisson Problems

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§6.1. Introduction

The Introduction of the Advanced FEM book [231] emphasized that the classical formulation of mathematical models in science and engineering leads to the Strong Form (SF). These *field equations* are (usually) ordinary or partial differential equations in space or space-time expressed in terms of a *primary variable*. They are complemented by *boundary* and/or *initial conditions*. The set of field equations and boundary plus initial conditions are collectively called the *governing equations*. The distinguishing property of the SF is that governing equations and conditions hold *at each point* of the problem domain.

Passing from the Strong Form to Weak and Variational Forms is simplified if the governing equations are presented through a scheme called *Tonti decompositions*.[1] Such schemes introduce two *auxiliary variables* that often have physical significance. One is called the *intermediate variable* and the other the *flux variable*. Examples of such variables are stresses, strains, pressures and heat fluxes. The equations that connect the primary and auxiliary variables in the decomposed SF are called *Strong Links* or *Strong Connectors*.

Tonti decompositions offer two important advantages for further development:

(I) The construction of various types of Weak and Variational Forms can be graphically explained as the result of weakening selected links.

(II) The interpretation of the distinction between *natural* and *essential* boundary conditions is facilitated by visualization.

This Chapter illustrates the construction of the Tonti decomposition for boundary value problems modeled by the scalar Poisson's equation. We start with these problems because the governing equations are considerable simpler than for the elasticity problem of structural and solid mechanics. The simplicity is due to the fact that the primary variable is a *scalar* function whereas the intermediate variables are vectors. On the other hand, in elasticity the primary variable: displacements, is a vector whereas the intermediate variables: strains and stresses, are tensors.

Despite its simplicity, the Poisson equation governs several interesting problems in engineering and physics. The next Chapter illustrates the construction of Weak and Variational Forms for that equation.

§6.2. The Poisson Equation

Many steady-state application problems in solid, fluid and thermo mechanics, as well as electromagnetics, can be modeled by the generalized Poisson's partial differential equation. This includes the famous Laplace equation as a special case.[2]

Suppose that $u = u(x_1, x_2, x_3)$ is a *primary* scalar function that solves a linear, steady-state (time-independent) application problem involving an *isotropic* medium. The problem is posed in a three-dimensional space spanned by the Cartesian coordinates x_1, x_2, x_3. (The physical meaning of u changes with the application; for example in thermal conduction problems it is the temperature).

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[1] A name suggested by a graphical representation introduced by the Italian mathematician and physicist Enzo Tonti.

[2] The Poisson equation is named after the French mathematician, geometer and mechanician Siméon-Denis Poisson. (He is the same person after which Poisson's ratio is named.) The Laplace equation is named after the French mathematician and astronomer Pierre Simon de Laplace. He and Poisson were contemporaneous.

The generalized Poisson's equation for an isotropic media is a PDE:

$$\boxed{\nabla \cdot (k\,\nabla u) = s,} \tag{6.1}$$

in which the first ∇ is the divergence operator, the second ∇ is the gradient operator, s is a given source function, and k is a constitutive coefficient of the medium being modeled by (6.1). This coefficient becomes a second-order tensor k_{ij} in anisotropic media.

Both k and s may depend on the spatial coordinates, that is, $k = k(x_1, x_2, x_3)$ and $s = s(x_1, x_2, x_3)$; if k is not space dependent, the medium (and the equation) are said to be *homogemeous*.Equation (6.1) must be complemented by appropriate boundary conditions. These are examined in further detail in connection with the specific examples iof later sections.

If k is constant in space, (6.1) reduces to the standard Poisson's equation

$$\boxed{k\,\nabla^2 u = s.} \tag{6.2}$$

in which $\nabla^2 = \nabla.\nabla$ is the Laplace operator, often called the *Laplacian*.[3] Furthermore if the source term s vanishes this reduces to the familiar Laplace's equation

$$\boxed{\nabla^2 u = 0.} \tag{6.3}$$

The solutions of (6.3) are called *harmonic functions*. Because of their importance in applications, they have been extensively studied over the past two centuries. The extension of the foregoing equations to an unknown *vector* function $\mathbf{u}$ is straightforward. In such a case the first ∇ in (6.1) is the gradient operator and the second ∇ the divergence operator.

Remark 6.1. In unabridged component notation the Poisson's equation (6.1) in one, two, and three dimensions takes the following forms:

$$\begin{aligned}
\frac{\partial}{\partial x_1}\left(k\,\frac{\partial u}{\partial x_1}\right) &= s,\\
\frac{\partial}{\partial x_1}\left(k\,\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k\,\frac{\partial u}{\partial x_2}\right) &= s,\\
\frac{\partial}{\partial x_1}\left(k\,\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k\,\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(k\,\frac{\partial u}{\partial x_3}\right) &= s.
\end{aligned} \tag{6.4}$$

If k is not space dependent, these reduce to the *homogeneous* forms

$$k\,\frac{\partial^2 u}{\partial x_1^2} = s, \qquad k\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) = s, \qquad k\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right) = s. \tag{6.5}$$

From (6.5) it is plain to see that the Laplacian is $\nabla^2 = \partial^2 u/\partial x_1^2$ in 1D, $\nabla^2 = \partial^2 u/\partial x_1^2 + \partial^2 u/\partial x_2^2$ in 2D, and $\nabla^2 = \partial^2 u/\partial x_1^2 + \partial^2 u/\partial x_2^2 + \partial^2 u/\partial x_3^2$ in 3D. These 3 forms can be contracted to $\nabla^2 = \partial^2 u/(\partial x_i \partial x_i)$ in indicial notation, if Einstein's summation convention is used.

By specializing the primary variable u to various physical quantities, we obtain models for various problems in mechanics, thermomechanics and electromagnetics. Four specific problems: thermal conduction, potential flow, electro and magnetostatics are examined below. Other applications are given as Exercises.

[3] The notation $\Delta = \nabla.\nabla$ for the Laplacian is also commonly used in the literature.

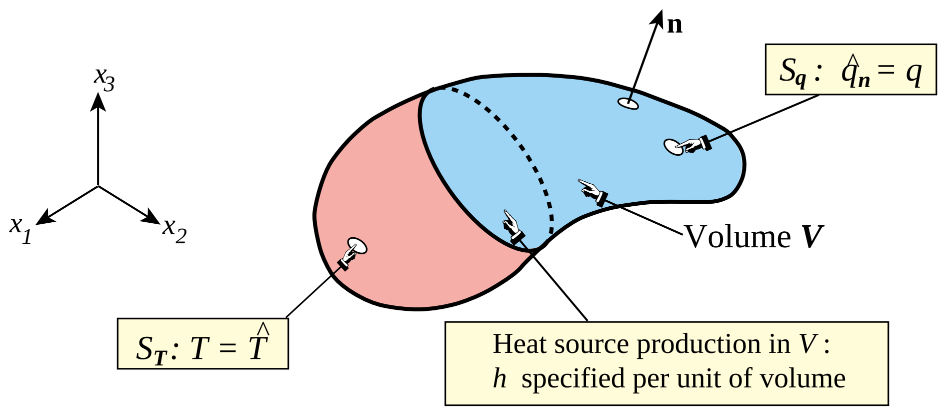

FIGURE 6.1. Tonti diagram for steady-state heat conduction problem, showing only the field equations.

§6.3. Steady Heat Conduction

Consider a thermally conducting isotropic body of volume V that obeys Fourier's (linear) law of heat conduction, as illustrated in Figure 6.1. The body is bounded by a surface S with external unit normal $\mathbf{n}$. The body is in *thermal equilibrium*, meaning that the temperature distribution $T = T(x_1, x_2, x_3)$ is independent of time. The temperature is the *primal* variable of this formulation so we will replace the u of the foregoing section by T. If the body is thermally isotropic, the k of the previous section becomes the thermal conductivity coefficient k, with a $-$ sign to account for the positive flux sense definition. This coefficient may be a function of position.

The *source* field called s in the previous section is the distributed heat production $h = h(x_1, x_2, x_3)$ in V measured per unit of volume. This heat may be generated, for instance, by combustion or by a chemical reaction.[4] A negative h indicates volumetric heat dissipation or "sink."

§6.3.1. Field Equations

The *temperature gradient* vector is called $\mathbf{g} = \nabla T$, which written in full is

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} \partial T/\partial x_1 \\ \partial T/\partial x_2 \\ \partial T/\partial x_3 \end{bmatrix}. \tag{6.6}$$

The *heat flux vector* $\mathbf{q}$ is defined by the constitutive equation $\mathbf{q} = -k\mathbf{g} = -k\nabla T$, which is Fourier's law of heat conduction for a thermally isotropic medium. In full this is

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = -k \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}. \tag{6.7}$$

The heat flux along a direction d defined by the unit vector $\mathbf{d}$ is denoted by $q_d = \mathbf{q} \cdot \mathbf{d} = \mathbf{q}^T \mathbf{d}$. This is a scalar that characterizes the transport of thermal energy along that direction. It is measured in heat units per unit area. As a special case, the boundary-normal heat flux is $q_n = \mathbf{q} \cdot \mathbf{n} = \mathbf{q}^T \mathbf{n}$ evaluated on S.

[4] In the human body, the heat source are calories produced from food intake.

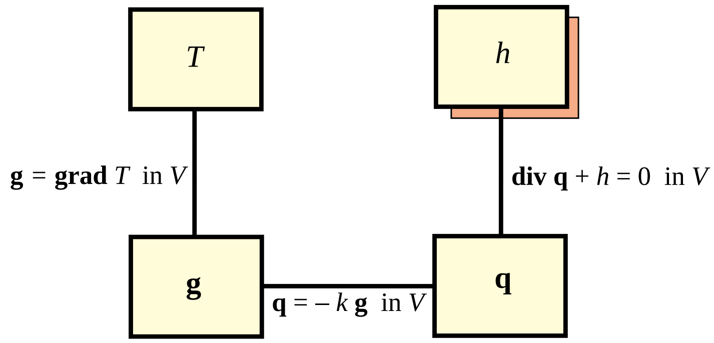

FIGURE 6.2. Tonti diagram for steady-state heat conduction problem, showing only the three field equations.

The *balance equation*, which characterizes steady-state thermal equilibrium, is $\mathbf{div}\,\mathbf{q} + h = 0$. Written in full component notation:

$$\frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} + \frac{\partial q_3}{\partial x_3} + h = 0. \tag{6.8}$$

Equations (6.6), (6.7), and (6.8) complete the *field equations* of the heat conduction problem.

§6.3.2. Boundary Conditions

The classical *boundary conditions* for this problem are of two types:

1. The temperature T is prescribed to be equal to $\hat{T}$ over a portion S_T of the boundary S (S_T is colored red in Figure 6.1.
2. The boundary-normal heat flux $q_n = \mathbf{q}.\mathbf{n} = -k(\partial T/\partial n)$ is prescribed to be equal to $\hat{q}_n$ over the complementary portion S_q of the boundary $S : S_T \cup S_q$ (S_q is colored blue in Figure 6.1).

Other boundary conditions that occur in practice are those associated with *radiation* and *convection*. Those are more complex (in fact, thay are nonlinear) and are not considered here.

§6.3.3. Summary of Governing Equations

The field equations, expressed in direct notation, are now summarized and labeled:

$$\boxed{\begin{array}{lcl} \text{KE:} & \nabla T = \mathbf{g} & \text{in } V, \\ \text{CE:} & -k\mathbf{g} = \mathbf{q} & \text{in } V, \\ \text{BE:} & \nabla\cdot\mathbf{q} + h = 0 & \text{in } V. \end{array}} \tag{6.9}$$

The *kinematic equation* (KE) is simply the definition of the temperature gradient vector $\mathbf{g}$. The constitutive equation (CE) is Fourier's law of thermal conduction. The *balance equation* (BE) is the law of thermal equilibrium: the heat flux gradient must equal to the heat created (or dissipated) per unit volume. These three labels: KE, CE and BE, will be used throughout this course for wide classes of problems governed by differential equations in space variables.

Fields $\mathbf{g}$ and $\mathbf{q}$ are called the *intermediate variable* and the *flux variable*, respectively.

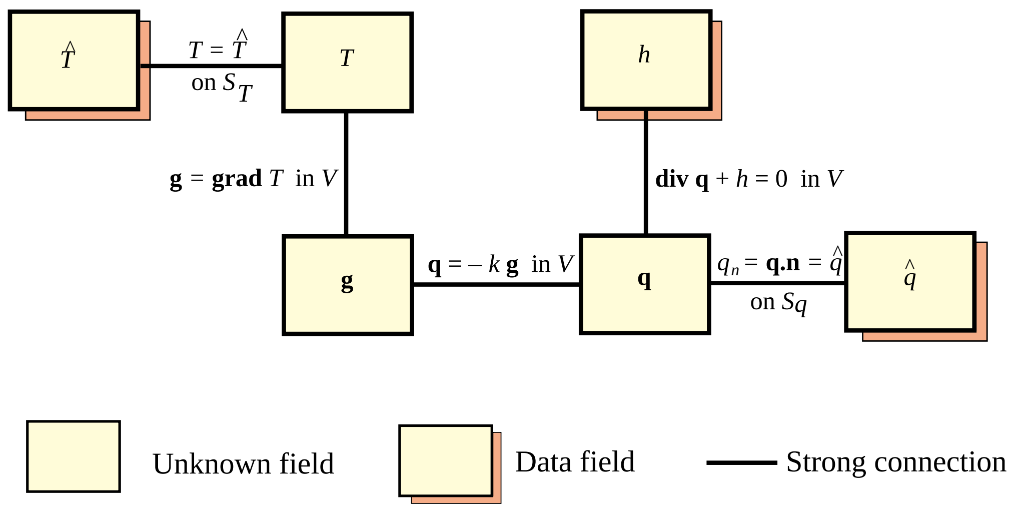

FIGURE 6.3. Tonti diagram for steady-state heat conduction problem, showing both field equations and boundary conditions

Elimination of the intermediate variables **g** and **q** in (6.9) yields

$$\nabla \cdot (k\, \nabla T) = h. \tag{6.10}$$

This shows that steady-state Fourier heat conduction pertains to the "Poisson-problem" class typified by (6.1), in which k remains k, u becomes T and s becomes h.

The two classical boundary conditions are labeled as

$$\boxed{\begin{array}{lcr} \text{PBC:} & T = \hat{T} & \text{on } S_T, \\ \text{FBC:} & \mathbf{q}^T \mathbf{n} = q_n = \hat{q}_n & \text{on } S_q. \end{array}} \tag{6.11}$$

Here labels PBC and FBC denote *primary boundary conditions* and *flux boundary conditions*, respectively. These labels will be used throughout this book.

The set of field equations: KE, CE, BE, and boundary conditions: PBC and FBC, are collectively called the *governing equations*. These equations constitute the statement of the mathematical model for this particular problem. This formulation is collectively called a *boundary value problem*, or BVP, in applied mathematics.

§6.3.4. Tonti Diagrams

A convenient graphical representation of the three field equations is the so-called Tonti-diagram, which is drawn in Figure 6.2. This diagram can be expanded as illustrated in Figure 6.3 to include the boundary conditions. Graphical conventions for this expanded diagram are explained in this figure. The term "strong connection" for a relation means that it applies point by point. A "data field" is one that is given as part of the problem specification.

The expanded Tonti diagram has been found to be more convenient from an instructional standpoint than the reduced diagram, and will be adopted from now on.

Figure 6.4 shows the generic names of the components of the expanded Tonti diagram.

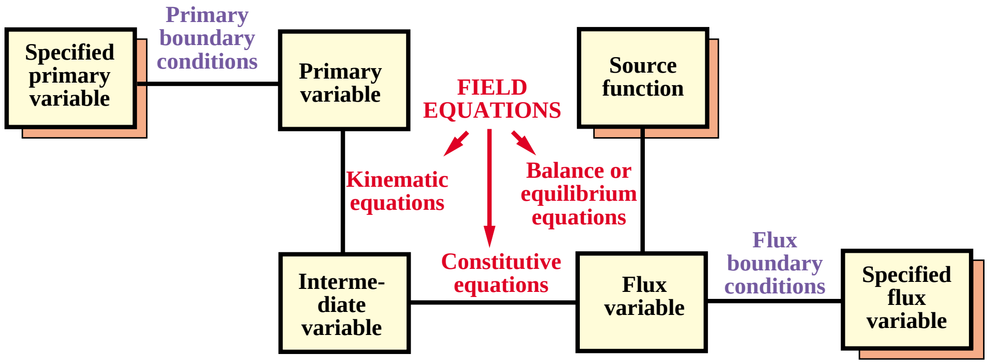

FIGURE 6.4. Generic names for the components (boxes and links) of a Tonti diagram.

§6.3.5. Alternative Notations

To facilitate comparison with the literature, the governing equations are restated below in three alternative forms: in matrix/vector notation (also called 'grad/div' notation), in compact indicial notation and in full (unabridged) component notation. For the second form the summation convention is implied. Indices run from 1 through 3 for the three-dimensional case. The range is reduced to 2 or 1 if the number of space dimensions is reduced to two and one, respectively.

Matrix/vector form:

$$\boxed{\begin{array}{lcl} \text{KE:} & \mathbf{grad} T = \mathbf{g} & \text{in } V, \\ \text{CE:} & -k\mathbf{g} = \mathbf{q} & \text{in } V, \\ \text{BE:} & \mathbf{div}\mathbf{q} + h = 0 & \text{in } V, \\ \text{PBC:} & T = \hat{T} & \text{on } S_T, \\ \text{FBC:} & \mathbf{q}.\mathbf{n} = q_n = \hat{q}_n, & \text{on } S_q. \end{array}} \tag{6.12}$$

Indicial form:

$$\boxed{\begin{array}{lcl} \text{KE:} & T_{,i} = g_i & \text{in } V, \\ \text{CE:} & -k g_i = q_i & \text{in } V, \\ \text{BE:} & q_{i,i} + h = 0 & \text{in } V, \\ \text{PBC:} & T = \hat{T} & \text{on } S_T, \\ \text{FBC:} & q_i n_i = q_n = \hat{q}_n, & \text{on } S_q. \end{array}} \tag{6.13}$$

Unabridged:

$$\boxed{\begin{array}{lcl} \text{KE:} & \dfrac{\partial T}{\partial x_1} = g_1, \quad \dfrac{\partial T}{\partial x_2} = g_2, \quad \dfrac{\partial T}{\partial x_3} = g_3, & \text{in } V, \\ \text{CE:} & -k g_1 = q_1, \quad -k g_2 = q_2, \quad -k g_3 = q_3, & \text{in } V, \\ \text{BE:} & \dfrac{\partial q_1}{\partial x_1} + \dfrac{\partial q_2}{\partial x_2} + \dfrac{\partial q_3}{\partial x_3} + h = 0 & \text{in } V, \\ \text{PBC:} & T = \hat{T} & \text{on } S_T, \\ \text{FBC:} & q_1 n_1 + q_2 n_2 + q_3 n_3 = \hat{q}_n & \text{on } S_q. \end{array}} \tag{6.14}$$

§6.4. Steady Potential Flow

As next example consider the potential flow of a fluid of mass density ρ that occupies a volume V.[5] The fluid volume is bounded by a surface S with external unit normal $\mathbf{n}$. The flow is characterized by the *velocity field* 3-vector $\mathbf{v}(x_1, x_2, x_3)$, which is independent of time. For irrotational flow this field can be expressed as the gradient $\mathbf{v} = -\nabla\phi$ of a scalar function $\phi(x_1, x_2, x_3)$ called the *velocity potential*.

This potential is chosen as primal variable. Note the physical contrast with the thermal conduction problem discussed in §6.3. In heat conduction the primal field — the temperature — has immediate physical meaning whereas the temperature gradient $\mathbf{g}$ is a convenient intermediate variable. On the other hand, in potential flow the field of primary significance — fluid velocity — is an intermediate variable whereas the primal field — the velocity potential — has no physical significance. Despite this contrast the two problems share the same mathematical formulation as explained below.

The forcing and boundary conditions are as follows:

1. The source field is σ, the fluid mass production per unit of volume. Such production is rare in applications. Thus for most potential flow problems $\sigma = 0$.
2. The potential ϕ is prescribed to be equal to $\hat{\phi}$ over a portion S_ϕ of the boundary S.
3. The fluid momentum density $m_n = \rho\,\mathbf{v}.\mathbf{n}$ is prescribed to be equal to $\hat{m}_n$ over the complementary portion S_m of the boundary $S : S_\phi) \cap S_m$.

In practice the most common boundary condition is that of prescribed normal velocity $\mathbf{v}.\mathbf{n} = v_n = \hat{v}_n$. This can be easily transformed to the prescribed momentum density B.C. on multiplying by the density ρ. Mathematically the momentum density B.C. is the correct one.

The field equations, expressed in direct notation, are:

$$\boxed{\begin{array}{lcl} \text{KE:} & -\nabla\phi = \mathbf{v} & \text{in } V, \\ \text{CE:} & \rho\,\mathbf{v} = \mathbf{m} & \text{in } V, \\ \text{BE:} & \nabla\cdot\mathbf{m} = \sigma & \text{in } V. \end{array}} \tag{6.15}$$

The *kinematic equation* (KE) is simply the definition of the velocity potential. The constitutive equation (CE) is the definition of momentum density. The *balance equation* (BE) expresses conservation of mass.

Elimination of the intermediate variables $\mathbf{v}$ and $\mathbf{m}$ in (6.15) yields the scalar Poisson's equation

$$\nabla\cdot(\rho\nabla\phi) = \sigma. \tag{6.16}$$

This shows that steady potential flow pertains to the "Poisson-problem" class (6.1), in which $k \to \rho$, $u \to \phi$ and $s \to \sigma$. As previously noted, usually $\sigma = 0$ whereas ρ is constant, whereupon (6.16) reduces to the Laplace's equation $\nabla^2\phi = 0$.

[5] In fluid mechanics, *potential flow* is short for steady barotropic irrotational flow of a perfect fluid.

The boundary conditions are:

$$\boxed{\begin{array}{lcl} \text{PBC:} & \phi = \hat{\phi} & \text{on } S_\phi, \\ \text{FBC:} & \mathbf{m}.\mathbf{n} = m_n = \hat{m}_n, & \text{on } S_m. \end{array}} \tag{6.17}$$

It should be obvious now that steady potential flow and steady heat conduction are *mathematically equivalent problems*, despite the great disparity in the physical interpretation of primal and intermediate quantities.

§6.5. Electrostatics

Electrostatics is concerned with the calculation of the steady-state *electrical field* 3-vector $\mathbf{E}(x_1, x_2, x_3)$ in a volume V filled by a dielectric material or medium of permittivity ϵ (this property measures the inductive capacity of the medium; it is also called the dielectric constant).

As in the case of potential flow, $\mathbf{E}$ is not the primal field but is derived from the *electric potential* $\Phi(x_1, x_2, x_3)$ as $\mathbf{E} = -\nabla\Phi$. Thus $\mathbf{E}$ plays the role of intermediate variable. The flux-like variable is the 3-vector $\mathbf{D} = \epsilon\mathbf{E}$, which receives the names of *electric field intensity* or the *electric flux density*.

The forcing and boundary conditions are as follows:

1. The source field is ρ, the electric charge per unit of volume. (This symbol should not be confused with mechanical density, as in the fluid problem of the previous section). For many electrostatic problems all charges migrate to the surface S, thus $\rho = 0$ in the volume.
2. The potential Φ is prescribed to be equal to $\hat{\Phi}$ over a portion S_Φ of the boundary S.
3. The normal electric flux $D_n = \mathbf{D}.\mathbf{n}$ is prescribed to be equal to $\hat{D}_n$ over the complementary portion S_D of the boundary $S : S_\Phi) \cap S_D$.

The electric potential has more physical significance than the (mathematically equivalent) velocity potential in potential flow. In electric circuits this potential can be directly measured as *voltage*. Similarly the flux condition has direct physical interpretation as electric flow, or current. Thus both boundary conditions are physically important.

The field equations, expressed in direct notation, are:

$$\boxed{\begin{array}{lcl} \text{KE:} & -\nabla\Phi = \mathbf{E} & \text{in } V, \\ \text{CE:} & \epsilon\,\mathbf{E} = \mathbf{D} & \text{in } V, \\ \text{BE:} & \nabla\cdot\mathbf{D} = \rho & \text{in } V. \end{array}} \tag{6.18}$$

The *kinematic equation* (KE) is the definition of the electric potential. The constitutive equation (CE) relates electric intensity and flux through the dielectric constant. The *balance equation* (BE) expresses conservation of charge. The last relation is also called Gauss' law and is one of the famous Maxwell equations.

Elimination of the intermediate variables $\mathbf{E}$ and $\mathbf{D}$ in (6.18) yields the scalar Poisson's equation

$$\nabla\cdot(\epsilon\nabla\Phi) = -\rho \tag{6.19}$$

This shows that electrostatics pertains to the "Poisson-problem" class. Often $\rho = 0$ and ϵ is constant, whereupon (6.19) reduces to the Laplace's equation $\nabla^2\Phi = 0$.

The boundary conditions are:

$$\boxed{\begin{array}{lcl} \text{PBC:} & \Phi = \hat{\Phi} & \text{on } S_\Phi, \\ \text{FBC:} & \mathbf{D}.\mathbf{n} = D_n = \hat{D}_n, & \text{on } S_D. \end{array}} \tag{6.20}$$

§6.6. *Magnetostatics

Magnetostatics is concerned with the calculation of the steady-state *magnetic flux density* 3-vector $\mathbf{B}(x_1, x_2, x_3)$ in a volume V filled by a material or medium of permeability μ.

The magnetic field $\mathbf{B}$ is a *solenoidal* vector (meaning that its divergence is zero). Thus it can be derived from the 3-vector *magnetic potential* $\mathbf{A}(x_1, x_2, x_3)$ as $\mathbf{B} = \nabla \times \mathbf{A}$. Hence $\mathbf{B}$ plays the role of intermediate variable, but unlike the three previous examples, the primal variable $\mathbf{A}$ is a vector and not a scalar. The flux-like variable is the 3-vector $\mathbf{H} = (1/\mu)\mathbf{B}$, which receives the name of *magnetic field intensity*.

The forcing and boundary conditions are as follows:

1. The source field is $\mathbf{J}$, the electric current density, which is a 3-vector.
2. The quantity $\mathbf{A} \times \mathbf{n} = A_{\times n}$ is prescribed to be equal to $\hat{A}_{\times n}$ over a portion S_A of the boundary S.
3. The quantity $H_{\times n} = \mathbf{H} \times \mathbf{n}$ is prescribed to be equal to $\hat{H}_{\times n}$ over the complementary portion S_H of the boundary $S : S_A \cap S_H$.

The field equations, expressed in direct notation, are:

$$\boxed{\begin{array}{lcl} \text{KE:} & \nabla \times \mathbf{A} = \mathbf{B} & \text{in } V, \\ \text{CE:} & \mu^{-1}\mathbf{B} = \mathbf{H} & \text{in } V, \\ \text{BE:} & \nabla \times \mathbf{H} = \mathbf{J} & \text{in } V. \end{array}} \tag{6.21}$$

The *kinematic equation* (KE) is the definition of the magnetic potential. The constitutive equation (CE) relates magnetic intensity and flux through the permeability constant. The *balance equation* (BE) expresses conservation of current (this last relation is one of the famous Maxwell equations).

Elimination of the intermediate variables $\mathbf{B}$ and $\mathbf{H}$ in (6.21) yields

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{A}) = \mathbf{J}, \tag{6.22}$$

This can be transformed into a vector Poisson's equation given in any book covering field electromagnetics. Finally, the boundary conditions are:

$$\boxed{\begin{array}{lcl} \text{PBC:} & \mathbf{A} \times \mathbf{n} = A_{\times n} = \hat{A}_{\times n} & \text{on } S_A, \\ \text{FBC:} & \mathbf{H} \times \mathbf{n} = H_{\times n} = \hat{H}_{\times n}, & \text{on } S_H. \end{array}} \tag{6.23}$$

Homework Exercises for Chapter 6

Decomposition of Poisson Problems

EXERCISE 6.1 [A:5] Show that **div** is the transpose of **grad** when these operators are treated as vectors.

EXERCISE 6.2 [A:20=15+5] A bar of length L, elastic modulus E and variable cross-sectional area $A(x)$ is aligned along the x axis, extending from $x = 0$ through $x = L$. The bar axial displacement is $u(x)$. It is loaded by a force $q(x)$ along its length. At $x = 0$ the the displacement $u(0)$ is prescribed to be $\hat{u}_0$. At $x = L$ the bar is loaded by axial force $\hat{N}_L$, positive towards $x > 0$. The field equations are $e = du/dx$, $N = EAe$, $dN/dx + q = 0$, and the boundary conditions are $u(0) = \hat{u}_0$ and $N(L) = \hat{N}_L$.

(a) Is this problem governed by the Poisson equation, and if so, what is the correspondence with, say, the first of (6.4).

(b) Draw the expanded Tonti diagram for this problem.

EXERCISE 6.3 [A:25=10+10+5] A steady-state heat conduction problem is posed over the "cylindrical" two-dimensional domain $ABCD$ depicted in Figure E6.1. with dimensions and boundary conditions as shown. (Axis x_3 comes out of the plane of the paper. Domain $ABCD$ extends indefinitely along x_3, and all conditions are independent of that dimension.) The conductivity k is uniform over the domain $ABCD$.

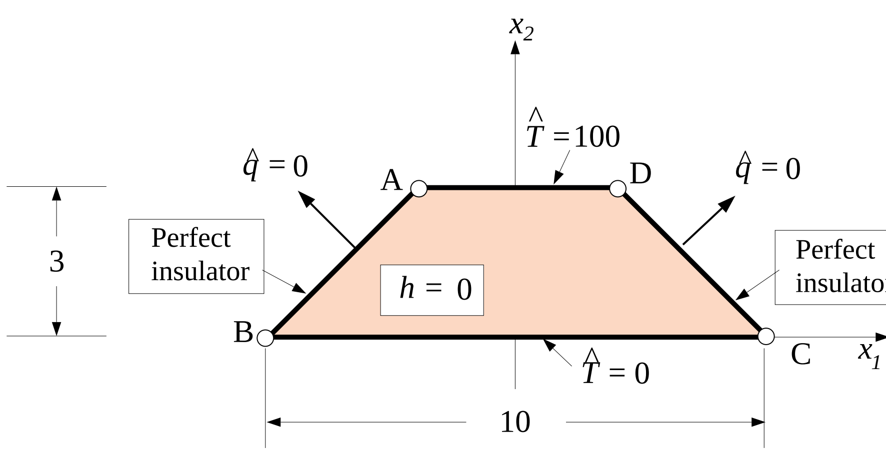

FIGURE E6.1. A steady-state heat conduction problem.

(a) Indicate which portions of the boundary form S_T and S_q. Can something clever be said about the symmetry plane $x_1 = 0$ that may allow the problem to be posed on only half of the domain?

(b) Does the temperature distribution satisfy the Laplace equation $\nabla^2 T = 0$?

(c) Does the "guess solution" $T = 100x_2/3$ satisfy the field equations and boundary conditions?

EXERCISE 6.4 [A:25] The Saint Venant theory of torsion of a cylindrical bar of arbitrary cross section (cf. Figure E6.2) may be posed as follows.[6] The problem domain is the bar cross section A, delimited by boundary B. This domain is assumed simple connected, *i.e.* the section is not hollow. The bar material is isotropic with shear modulus G. The primary variable is the two-dimensional *stress function* $\phi(x_1, x_2)$. This function satisfies the standard Poisson equation

$$\nabla^2\phi = -2G\theta, \tag{E6.1}$$

[6] See, for example, Chapter 11 of Timoshenko and Goodier *Theory of Elasticity*, McGraw-Hill, 1951.

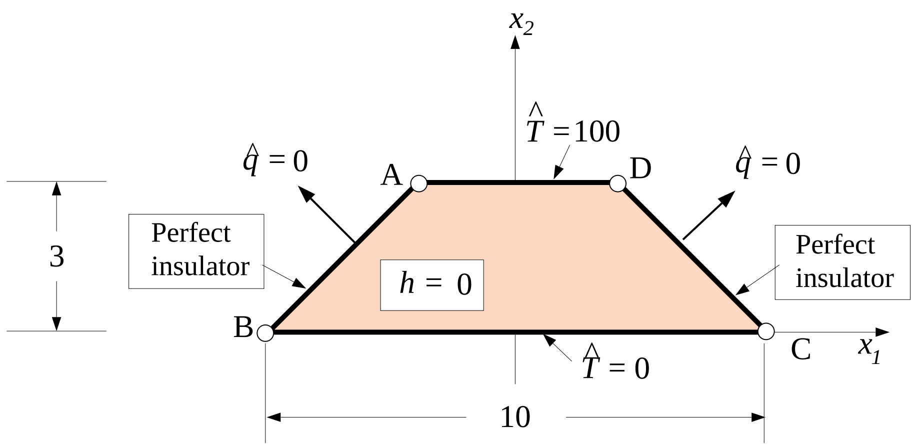

FIGURE E6.2. The Saint-Venant torsion problem for Exercise 6.4.

where θ the torsion angle of rotation (about x_3) per unit length. The function ϕ must be constant over the boundary B: $\phi = C$. In the case of singly-connected cross-section domains (solid bars) this constant can be chosen arbitrarily and for convenience may be taken as zero.

The applied torque is given by $M_t = 2\int_A \phi\, dA = GJ\theta$, where J is the torsional rigidity. The shear stresses are

$$\sigma_{13} = \frac{\partial \phi}{\partial x_2}, \qquad \sigma_{23} = -\frac{\partial \phi}{\partial x_1}. \tag{E6.2}$$

The shear stresses satisfy the equilibrium equations

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} = 0. \tag{E6.3}$$

Draw an expanded Tonti diagram for this problem, in which ϕ is the primary variable, the shear stresses are taken as flux variables, the gradient of ϕ is the intermediate variable, and angle θ is the source. Where would you place the specified-moment condition in the diagram? (Hint: connect it to the source).

EXERCISE 6.5 [A:15] Draw the expanded Tonti diagram for the steady potential fluid flow problem covered in §6.4.

Identify governing equations along the strong links in direct form. Where in the diagram would you specify a prescribed velocity boundary condition?

EXERCISE 6.6 [A:15] Draw the expanded Tonti diagram for the electrostatics problem treated in §6.5. Identify governing equations along the strong links in direct form.