1
Overview

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§1.1. Book Scope

This is a textbook about *linear structural analysis* using the Finite Element Method (FEM) as a discretization tool. It is intended to support an introductory course at the first-year level of graduate studies in Aerospace, Mechanical, or Civil Engineering.

Basic prerequisites to understanding the material covered here are: (1) a working knowledge of matrix algebra, and (2) an undergraduate structures course at the Materials of Mechanics level. Helpful but not required are previous courses in continuum mechanics and advanced structures.

This Chapter presents an overview of what the book covers, and what finite elements are.

§1.2. Where the Material Fits

This Section outlines where the book material fits within the vast scope of Mechanics. In the ensuing multilevel classification, topics addressed in some depth in this book are emphasized in **bold** typeface.

§1.2.1. Top Level Classification

Definitions of *Mechanics* in dictionaries usually state two flavors:

- The branch of Physics that studies the effect of forces and energy on physical bodies.[1]
- The practical application of that science to the design, construction or operation of material systems or devices, such as machines, vehicles or structures.

These flavors are science and engineering oriented, respectively. But dictionaries are notoriously archaic. For our objectives it will be convenient to distinguish *four* flavors:

$$\textbf{Mechanics} \begin{cases} \textit{Theoretical} \\ \textit{Applied} \\ \textbf{Computational} \\ \textit{Experimental} \end{cases} \tag{1.1}$$

Theoretical mechanics deals with fundamental laws and principles studied for their intrinsic scientific value. *Applied mechanics* transfers this theoretical knowledge to scientific and engineering applications, especially as regards the construction of mathematical models of physical phenomena. *Computational mechanics* solves specific problems by model-based simulation through numerical methods implemented on digital computers. *Experimental mechanics* subjects the knowledge derived from theory, application and simulation to the ultimate test of observation.

Remark 1.1. Paraphrasing an old joke about mathematicians, one may define a computational mechanician as a person who searches for solutions to given problems, an applied mechanician as a person who searches for problems that fit given solutions, and a theoretical mechanician as a person who can prove the existence of problems and solutions. As regards experimentalists, make up your own joke.

[1] Here the term "bodies" includes all forms of matter, whether solid, liquid or gaseous; as well as all physical scales, from subatomic through cosmic.

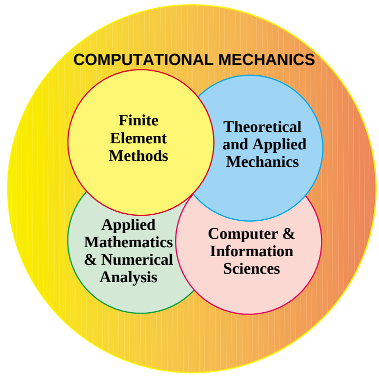

FIGURE 1.1. The "pizza slide:" Computational Mechanics integrates aspects of four disciplines.

§1.2.2. Computational Mechanics

Computational Mechanics represents the integration of several disciplines, as depicted in the "pizza slice" Figure 1.1 Several branches of computational mechanics can be distinguished according to the *physical scale* of the focus of attention:

$$\textbf{Computational Mechanics} \begin{cases} \textit{Nanomechanics} \\ \textit{Micromechanics} \\ \textbf{Continuum mechanics} \begin{cases} \textbf{Solids and Structures} \\ \text{Fluids} \\ \text{Multiphysics} \end{cases} \\ \textit{Systems} \end{cases} \tag{1.2}$$

Nanomechanics deals with phenomena at the molecular and atomic levels. As such, it is closely related to particle physics and chemistry. At the atomic scale it transitions to quantum mechanics.

Micromechanics looks primarily at the crystallographic and granular levels of matter. Its main technological application is the design and fabrication of materials and microdevices.

Continuum mechanics studies bodies at the macroscopic level, using continuum models in which the microstructure is homogenized by phenomenological averaging. The two traditional areas of application are *solid* and *fluid mechanics*. *Structural mechanics* is a conjoint branch of solid mechanics, since structures, for obvious reasons, are fabricated with solids. Computational solid mechanics favors a applied-sciences approach, whereas computational structural mechanics emphasizes technological applications to the analysis and design of structures.

Computational fluid mechanics deals with problems that involve the equilibrium and motion of liquid and gases. Well developed related subareas are hydrodynamics, aerodynamics, atmospheric physics, propulsion, and combustion.

Multiphysics is a more recent newcomer.[2] This area is meant to include mechanical systems that transcend the classical boundaries of solid and fluid mechanics. A key example is interaction

[2] This unifying term is in fact missing from most dictionaries, as it was introduced by computational mechanicians in the 1970s. Several multiphysics problems, however, are older. For example, aircraft aeroelasticity emerged in the 1920s.

between fluids and structures, which has important application subareas such as aeroelasticity and hydroelasticity. Phase change problems such as ice melting and metal solidification fit into this category, as do the interaction of control, mechanical and electromagnetic systems.

Finally, *system* identifies mechanical objects, whether natural or artificial, that perform a distinguishable function. Examples of man-made systems are airplanes, building, bridges, engines, cars, microchips, radio telescopes, robots, roller skates and garden sprinklers. Biological systems, such as a whale, amoeba, virus or pine tree are included if studied from the viewpoint of biomechanics. Ecological, astronomical and cosmological entities also form systems.[3]

In the progression of (1.2), *system* is the most general concept. Systems are studied by *decomposition*: its behavior is that of its components plus the interaction between the components. Components are broken down into subcomponents and so on. As this hierarchical process continues the individual components become simple enough to be treated by individual disciplines, but their interactions may get more complex. Thus there are tradeoff skills in deciding where to stop.[4]

§1.2.3. Statics versus Dynamics

Continuum mechanics problems may be subdivided according to whether inertial effects are taken into account or not:

$$\textbf{Continuum mechanics} \begin{cases} \textbf{Statics} \begin{cases} \textbf{Time Invariant} \\ \textit{Quasi-static} \end{cases} \\ \textit{Dynamics} \end{cases} \tag{1.3}$$

In *statics* inertial forces are ignored or neglected. These problems may be subclassified into *time invariant* and *quasi-static*. For the former time need not be considered explicitly; any time-like response-ordering parameter (should one be needed) will do. In quasi-static problems such as foundation settlements, creep flow, rate-dependent plasticity or fatigue cycling, a more realistic estimation of time is required but inertial forces are ignored as long as motions remain slow.

In *dynamics* the time dependence is explicitly considered because the calculation of inertial (and/or damping) forces requires derivatives respect to actual time to be taken.

§1.2.4. Linear versus Nonlinear

A classification of static problems that is particularly relevant to this book is

$$\textbf{Statics} \begin{cases} \textbf{Linear} \\ \textit{Nonlinear} \end{cases} \tag{1.4}$$

Linear static analysis deals with static problems in which the *response* is linear in the cause-and-effect sense. For example: if the applied forces are doubled, the displacements and internal stresses also double. Problems outside this domain are classified as *nonlinear*.

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[3] Except that their function may not be clear to us. “What is it that breathes fire into the equations and makes a universe for them to describe? The usual approach of science of constructing a mathematical model cannot answer the questions of why there should be a universe for the model to describe. Why does the universe go to all the bother of existing?” (Stephen Hawking).

[4] Thus in breaking down a car engine, say, the decomposition does not usually proceed beyond the components that may be bought at a automotive shop.

§1.2.5. Discretization Methods

A final classification of computational solid and structural mechanics (CSSM) for static analysis is based on the discretization method by which the continuum mathematical model is *discretized* in space, *i.e.*, converted to a discrete model of finite number of degrees of freedom:

$$
\textbf{CSSM spatial discretization} \begin{cases} \textbf{Finite Element Method (FEM)} \\ \textit{Boundary Element Method (BEM)} \\ \textit{Finite Difference Method (FDM)} \\ \textit{Finite Volume Method (FVM)} \\ \textit{Spectral Method} \\ \textit{Mesh-Free Method} \end{cases} \tag{1.5}
$$

For *linear* problems finite element methods currently dominate the scene, with boundary element methods posting a strong second choice in selected application areas. For *nonlinear* problems the dominance of finite element methods is overwhelming.

Classical *finite difference* methods in solid and structural mechanics have virtually disappeared from practical use. This statement is not true, however, for fluid mechanics, where finite difference discretization methods are still important although their dominance has diminished over time. *Finite-volume methods*, which focus on the direct discretization of conservation laws, are favored in highly nonlinear problems of fluid mechanics. *Spectral methods* are based on global transformations, based on eigendecomposition of the governing equations, that map the physical computational domain to transform spaces where the problem can be efficiently solved.

A recent newcomer to the scene are the *mesh-free methods*. These are finite different methods on arbitrary grids constructed using a subset of finite element techniques

§1.2.6. FEM Formulation Levels

The term *Finite Element Method* actually identifies a broad spectrum of techniques that share common features. Since its emergence in the framework of the Direct Stiffness Method (DSM) over 1956–1964, [765,768] FEM has expanded like a tsunami, surging from its origins in aerospace structures to cover a wide range of nonstructural applications, notably thermomechanics, fluid dynaamics, and electromagnetics. The continuously expanding range makes taxology difficult. Restricting ourselves to applications in computational solid and structural mechanics (CSSM), one classificaation of particular relevance to this book is

$$
\textbf{FEM-CSSM Formulation Level} \begin{cases} \textbf{Mechanics of Materials (MoM) Formulation} \\ \textbf{Conventional Variational Formulation} \\ \textit{Advanced Variational Formulation} \\ \textit{Template Formulation} \end{cases} \tag{1.6}
$$

The MoM formulation is applicable to simple structural elements such as bars and beams, and does not require any knowledge of variational methods. This level is accesible to undergraduate students, as only require some elementary knowledge of linear algebra and makes no use of variational calculus. The second level is characterized by two features: the use of standard work and energy methods (such as the Total Potential Energy principle), and focus on full compliance with the requirements of the classical Ritz-Galerkin direct variational methods (for example, interelement continuity). It is approriate for first year (master level) graduate students with basic exposure to variational methods. The two lower levels were well established by 1970, with no major changes since, and are those used in the present book.

The next two levels are covered in the Advanced Finite Element Methods book [255]. The third one requires a deeper exposure to variational methods in mechanics, notably multifield and hybrid principles. The last level (templates) is the pinnacle "where the rivers of our wisdom flow into one another." Reaching it requires both mastery of advanced variational principles, as well as the confidence and fortitude to discard them along the way to the top.

§1.2.7. FEM Choices

A more down to earth classification considers two key selection attributes: Primary Unknown Variable(s), or PUV, and solution method:[5]

$$\textbf{PUV Choice}\begin{cases}\textbf{Displacement (a.k.a. Primal)}\\ \textit{Force (a.k.a. Dual or Equilibrium)}\\ \textit{Mixed (a.k.a. Primal-Dual)}\\ \textit{Hybrid}\end{cases}\qquad \textbf{Solution Choice}\begin{cases}\textbf{Stiffness}\\ \textit{Flexibility}\\ \textit{Combined}\end{cases}\tag{1.7}$$

Here **PUV Choice** governs the variational framework chosen to develop the discrete equations; if one works at the two middle levels of (1.6). It is possible, however, to develop those completely *outside* a variational framework, as noted there. The solution choice is normally dictated by the PUV, but exceptions are possible.

§1.2.8. Finally: What The Book Is About

Using the classification of (1.1) through (1.5) we can now state the book topic more precisely:

$$\boxed{\begin{array}{l}\text{The model-based simulation of linear static structures discretized by FEM,}\\ \text{formulated at the two lowest levels of (1.6).}\end{array}}\tag{1.8}$$

Of the FEM variants listed in (1.7) emphasis will be placed on the *displacement* PUV choice and *stiffness* solution, just like in [257]. This particular combination is called the *Direct Stiffness Method* or DSM.

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[5] The alternative PUV terms: primal, dual or primal-dual, are those used in FEM non-structural applications, as well as in more general computational methods.

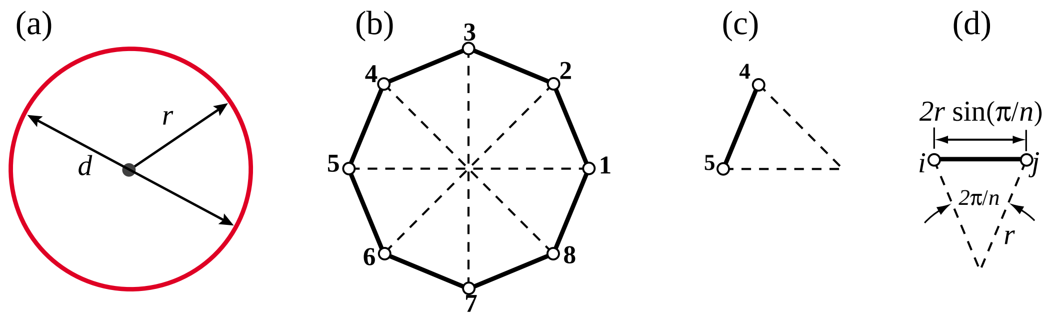

FIGURE 1.2. The “find π” problem treated with FEM concepts: (a) continuum object, (b) a discrete approximation by inscribed regular polygons, (c) disconnected element, (d) generic element.

§1.3. What Does a Finite Element Look Like?

The subject of this book is FEM. But what *is* a finite element? As discussed later, the term admits of two interpretations: physical and mathematical. For now the underlying concept will be partly illustrated through a truly ancient problem: find the perimeter L of a circle of diameter d. Since $L = \pi\, d$, this is equivalent to obtaining a numerical value for π.

Draw a circle of radius r and diameter $d = 2r$ as in Figure 1.2(a). Inscribe a regular polygon of n sides, where $n = 8$ in Figure 1.2(b). Rename polygon sides as *elements* and vertices as *nodes*. Label nodes with integers $1, \ldots 8$. Extract a typical element, say that joining nodes 4–5, as shown in Figure 1.2(c). This is an instance of the *generic element* i–j pictured in Figure 1.2(d). The element length is $L_{ij} = 2r \sin(\pi/n)$. Since all elements have the same length, the polygon perimeter is $L_n = nL_{ij}$, whence the approximation to π is $\pi_n = L_n/d = n \sin(\pi/n)$.

Table 1.1. Rectification of Circle by Inscribed Polygons (“Archimedes FEM”)

n	$\pi_n = n \sin(\pi/n)$	Extrapolated by Wynn-ϵ	Exact π to 16 places
1	0.000000000000000		
2	2.000000000000000		
4	2.828427124746190	3.414213562373096	
8	3.061467458920718		
16	3.121445152258052	3.141418327933211	
32	3.136548490545939		
64	3.140331156954753	3.141592658918053	
128	3.141277250932773		
256	3.141513801144301	3.141592653589786	3.141592653589793

Values of π_n obtained for $n = 1, 2, 4, \ldots 256$ and $r = 1$ are listed in the second column of Table 1.1. As can be seen the convergence to π is fairly slow. However, the sequence can be transformed by Wynn's ϵ algorithm[6] into that shown in the third column. The last value displays 15-place accuracy.

[6] A widely used lozenge extrapolation algorithm that speeds up the convergence of many sequences. See, e.g, [812].

Some key ideas behind the FEM can be identified in this example. The circle, viewed as a *source mathematical object*, is replaced by polygons. These are *discrete approximations* to the circle. The sides, renamed as *elements*, are specified by their end *nodes*. Elements can be separated by disconnecting nodes, a process called *disassembly* in the FEM. Upon disassembly a *generic element* can be defined, *independently of the original circle*, by the segment that connects two nodes i and j. The relevant element property: side length L_{ij}, can be computed in the generic element independently of the others, a property called *local support* in the FEM. The target property: polygon perimeter, is obtained by reconnecting n elements and adding up their length; the corresponding steps in the FEM being *assembly* and *solution*, respectively. There is of course nothing magic about the circle; the same technique can be be used to rectify any smooth plane curve.[7]

This example has been offered in the FEM literature, e.g. in [476], to aduce that finite element ideas can be traced to Egyptian mathematicians from *circa* 1800 B.C., as well as Archimedes' famous studies on circle rectification by 250 B.C. But comparison with the modern FEM, as covered in following Chapters, shows this to be a stretch. The example does not illustrate the concept of degrees of freedom, conjugate quantities and local-global coordinates. It is guilty of circular reasoning: the compact formula $\pi = \lim_{n\to\infty} n\sin(\pi/n)$ uses the unknown π in the right hand side.[8] Reasonable people would argue that a circle is a simpler object than, say, a 128-sided polygon. Despite these flaws the example is useful in one respect: showing a fielder's choice in the replacement of one mathematical object by another. This is at the root of the simulation process described next.

§1.4. The FEM Analysis Process

Processes that use FEM involve carrying out a sequence of steps in some way. Those sequences take two canonical configurations, depending on (i) the environment in which FEM is used and (ii) the main objective: model-based simulation of physical systems, or numerical approximation to mathematical problems. Both are reviewed below to introduce terminology used in the sequel.

§1.4.1. The Physical FEM

A canonical use of FEM is simulation of physical systems. This requires models of such systems. Consequenty the methodology is often called *model-based simulation*.

The process is illustrated in Figure 1.3. The centerpiece is the *physical system* to be modeled. Accordingly, this configuration is called the *Physical FEM*. The processes of idealization and discretization are carried out *concurrently* to produce the discrete model. The solution step is handled by an equation solver often customized to FEM, which delivers a discrete solution (or solutions).

Figure 1.3 also shows an *ideal mathematical model*. This may be presented as a *continuum limit* or "continuification" of the discrete model. For some physical systems, notably those well modeled by continuum fields, this step is useful. For others, such as complex engineering systems (say, a flying aircraft) it makes no sense. Indeed Physical FEM discretizations may be constructed and adjusted *without reference to mathematical models*, simply from experimental measurements.

[7] A similar limit process, however, may fail in three dimensions for evaluation of surface areas.

[8] The circularity objection is bypassed if n is advanced as a power of two, as in Table 1.1, by using the half-angle recursion $\sqrt{2}\sin\alpha = \sqrt{1-\sqrt{1-\sin^2 2\alpha}}$, started from $2\alpha = \pi$ for which $\sin\pi = -1$.

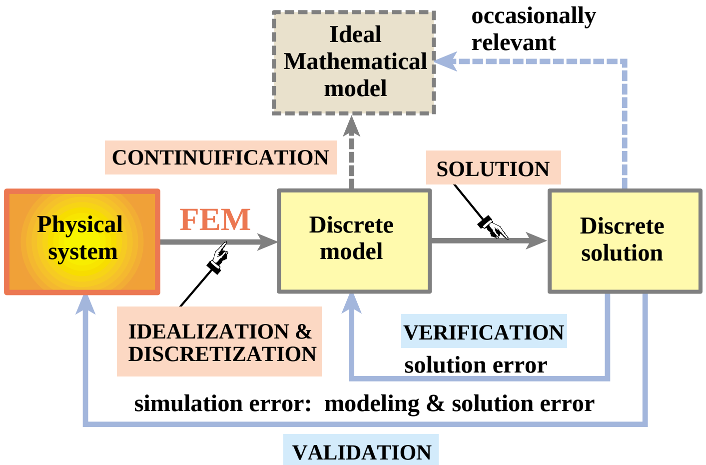

FIGURE 1.3. The Physical FEM. The physical system (left box) is the source of the simulation process. The ideal mathematical model (should one go to the trouble of constructing it) is inessential.

The concept of *error* arises in the Physical FEM in two ways. These are known as *verification* and *validation*, respectively.[9] Verification is done by replacing the discrete solution into the discrete model to get the solution error. This error is not generally important. Substitution in the ideal mathematical model in principle provides the *discretization error*. This step is rarely useful in complex engineering systems, however, because there is no reason to expect that the continuum model exists, and even if it does, that it is more physically relevant than the discrete model.

Validation tries to compare the discrete solution against observation by computing the *simulation error*, which combines modeling and solution errors. As the latter is typically unimportant, the simulation error in practice can be identified with the modeling error. In real-life applications this error overwhelms the others.[10]

One way to adjust the discrete model so that it represents the physics better is called *model updating*. The discrete model is given free parameters. These are determined by comparing the discrete solution against experiments, as illustrated in Figure 1.4. Inasmuch as the minimization conditions are generally nonlinear (even if the model is linear) the updating process is inherently iterative.

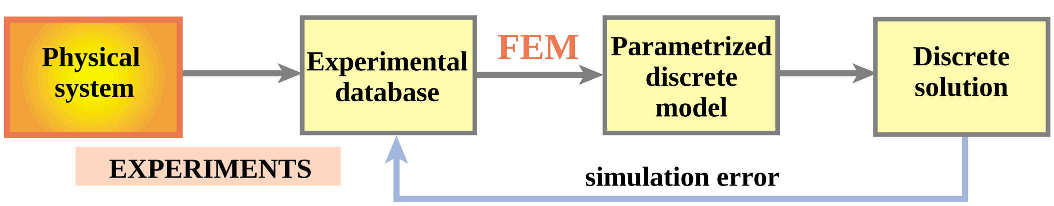

FIGURE 1.4. Model updating process in the Physical FEM.

9 Programming analogs: static and dynamic testing are called *verification* and *validation*, respectively. Static testing is carried at the source level (e.g., code walkthroughs, compilation) whereas dynamic testing is done by running the code.

10 “All models are wrong; some are useful” (George Box)

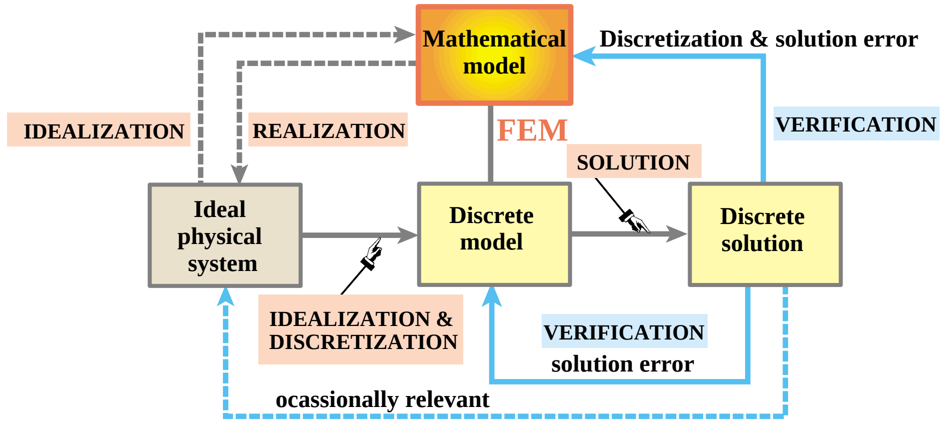

FIGURE 1.5. The Physical FEM. The physical system (left box) is the source of the simulation process. The ideal mathematical model (should one go to the trouble of constructing it) is inessential.

§1.4.2. The Mathematical FEM

The other canonical way of using FEM focuses on the mathematics. The process steps are illustrated in Figure 1.5. The spotlight now falls on the *mathematical model*. This is often an ordinary differential equation (ODE), or a partial differential equation (PDE) in space and time. A discrete finite element model is generated from a variational or weak form of the mathematical model.[11] This is the *discretization* step. The FEM equations are solved as described for the Physical FEM.

On the left, Figure 1.5 shows an *ideal physical system*. This may be presented as a *realization* of the mathematical model. Conversely, the mathematical model is said to be an *idealization* of this system. E.g., if the mathematical model is the Poisson's PDE, realizations may be heat conduction or an electrostatic charge-distribution problem. This step is inessential and may be left out. Indeed Mathematical FEM discretizations *may be constructed without any reference to physics*.

The concept of *error* arises when the discrete solution is substituted in the "model" boxes. This replacement is generically called *verification*. As in the Physical FEM, the *solution error* is the amount by which the discrete solution fails to satisfy the discrete equations. This error is relatively unimportant when using computers, and in particular direct linear equation solvers, for the solution step. More relevant is the *discretization error*, which is the amount by which the discrete solution fails to satisfy the mathematical model.[12] Replacing into the ideal physical system would in principle quantify modeling errors. In the Mathematical FEM this is largely irrelevant, however, because the ideal physical system is merely that: a figment of the imagination.

§1.4.3. Synergy of Physical and Mathematical FEM

The foregoing canonical sequences are not exclusive but complementary. This synergy[13] is one of the reasons behind the power and acceptance of the method. Historically the Physical FEM was the

[11] The distinction between strong, weak and variational forms is discussed in advanced FEM courses. In the present book such forms will be largely stated (and used) as recipes.

[12] This error can be computed in several ways, the details of which are of no importance here.

[13] Such interplay is not exactly a new idea: "The men of experiment are like the ant, they only collect and use; the reasoners resemble spiders, who make cobwebs out of their own substance. But the bee takes the middle course: it gathers its material from the flowers of the garden and field, but transforms and digests it by a power of its own." (Francis Bacon).

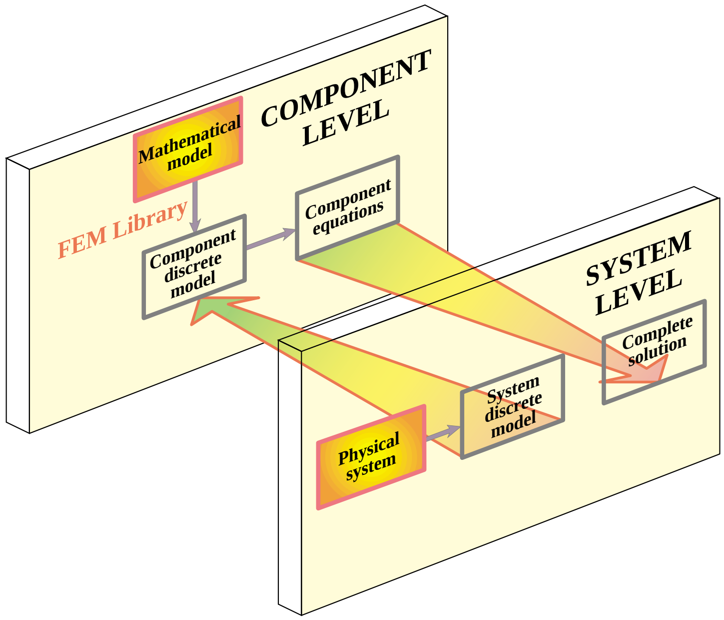

FIGURE 1.6. Combining physical and mathematical modeling through multilevel FEM. Only two levels (system and component) are shown for simplicity.

first one to be developed to model complex physical systems such as aircraft, as narrated in §1.7. The Mathematical FEM came later and, among other things, provided the necessary theoretical underpinnings to extend FEM beyond structural analysis.

A glance at the schematics of a commercial jet aircraft makes obvious the reasons behind the Physical FEM. There is no simple differential equation that captures, at a continuum mechanics level,[14] the structure, avionics, fuel, propulsion, cargo, and passengers eating dinner. There is no reason for despair, however. The time honored *divide and conquer* strategy, coupled with *abstraction*, comes to the rescue.

First, separate the structure out and view the rest as masses and forces. Second, consider the aircraft structure as built up of *substructures* (a part of a structure devoted to a specific function): wings, fuselage, stabilizers, engines, landing gears, and so on.

Take each substructure, and continue to break it down into *components*: rings, ribs, spars, cover plates, actuators, etc. Continue through as many levels as necessary. Eventually those components become sufficiently simple in geometry and connectivity that they can be reasonably well described by the mathematical models provided, for instance, by Mechanics of Materials or the Theory of Elasticity. At that point, *stop*. The component level discrete equations are obtained from a FEM library based on the mathematical model.

The system model is obtained by going through the reverse process: from component equations to substructure equations, and from those to the equations of the complete aircraft. This *system*

[14] Of course at the (sub)atomic level quantum mechanics works for everything, from landing gears to passengers. But it would be slightly impractical to represent the aircraft by, say, 10^{36} interacting particles modeled by the Schrödinger equations. More seriously, Truesdell and Toupin correctly note that "*Newtonian mechanics, while not appropriate to the corpuscles making up a body, agrees with experience when applied to the body as a whole*, except for certain phenomena of astronomical scale" [759, p. 228].

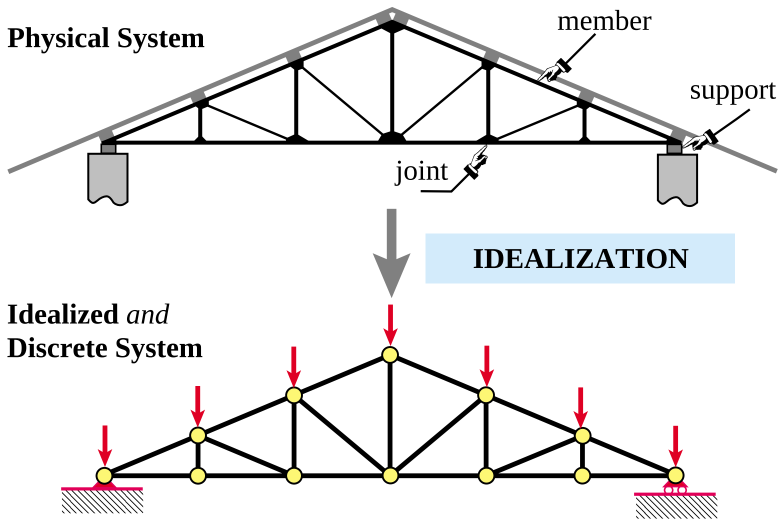

FIGURE 1.7. The idealization process for a simple structure. The physical system — here a conventional roof truss — is directly idealized by the mathematical model: a pin-jointed bar assembly. For this particular structure idealized and discrete models coalesce.

assembly process is governed by the classical principles of Newtonian mechanics, which provide the necessary inter-component "glue." The multilevel decomposition process is diagramed in Figure 1.6, in which intermediate levels are omitted for simplicity

Remark 1.2. More intermediate decomposition levels are used in systems such as offshore and ship structures, which are characterized by a modular fabrication process. In that case multilevel decomposition mimics the way the system is actually fabricated. The general technique, called *superelements*, is discussed in Chapter 10.

Remark 1.3. There is no point in practice in going beyond a certain component level while considering the complete system. The reason is that the level of detail can become overwhelming without adding relevant information. Usually that point is reached when uncertainty impedes further progress. Further refinement of specific components is done by the so-called global-local analysis technique outlined in Chapter 10. This technique is an instance of *multiscale analysis*.

§1.4.4. Streamlined Idealization and Discretization

For sufficiently simple structures, passing to a discrete model is carried out in a single *idealization and discretization* step, as illustrated for the truss roof structure shown in Figure 1.7. Other levels are unnecessary in such cases. Of course the truss may be viewed as a substructure of the roof, and the roof as a a substructure of a building. If so the multilevel process would be more appropriate.

§1.5. Method Interpretations

Just like there are two complementary ways of using the FEM, there are two complementary interpretations for explaining it, a choice that obviously impacts teaching. One interpretation stresses the *physical* significance and is aligned with the Physical FEM. The other focuses on the *mathematical* context, and is aligned with the Mathematical FEM. They are outlined next.

§1.5.1. Physical Interpretation

The physical interpretation focuses on the flowchart of Figure 1.3. This interpretation has been shaped by the discovery and extensive use of the method in the field of structural mechanics. The historical connection is reflected in the use of structural terms such as "stiffness matrix", "force vector" and "degrees of freedom," a terminology that carries over to non-structural applications.

The basic concept in the physical interpretation is the *breakdown* ($\equiv$ disassembly, tearing, partition, separation, decomposition) of a complex mechanical system into simpler, disjoint components called finite elements, or simply *elements*. The mechanical response of an element is characterized in terms of a finite number of degrees of freedom. These degrees of freedoms are represented as the values of the unknown functions as a set of node points. The element response is defined by algebraic equations constructed from mathematical or experimental arguments. The response of the original system is considered to be approximated by that of the *discrete model* constructed by *connecting* or *assembling* the collection of all elements.

The breakdown-assembly concept occurs naturally when an engineer considers many artificial and natural systems. For example, it is easy and natural to visualize an engine, bridge, aircraft or skeleton as being fabricated from simpler parts.

As discussed in §1.4.3, the underlying theme is *divide and conquer*. If the behavior of a system is too complex, the recipe is to divide it into more manageable subsystems. If these subsystems are still too complex the subdivision process is continued until the behavior of each subsystem is simple enough to fit a mathematical model that represents well the knowledge level the analyst is interested in. In the finite element method such "primitive pieces" are called *elements*. The behavior of the total system is that of the individual elements plus their interaction. A key factor in the initial acceptance of the FEM was that the element interaction could be physically interpreted and understood in terms that were eminently familiar to structural engineers.

§1.5.2. Mathematical Interpretation

This interpretation is closely aligned with the flowchart of Figure 1.5. The FEM is viewed as a procedure for obtaining numerical approximations to the solution of boundary value problems (BVPs) posed over a domain Ω. This domain is replaced by the union $\cup$ of disjoint subdomains $\Omega^{(e)}$ called finite elements. In general the geometry of Ω is only approximated by that of $\cup\Omega^{(e)}$.

The unknown function (or functions) is locally approximated over each element by an interpolation formula expressed in terms of values taken by the function(s), and possibly their derivatives, at a set of *node points* generally located on the element boundaries. The states of the assumed unknown function(s) determined by unit node values are called *shape functions*. The union of shape functions "patched" over adjacent elements form a *trial function basis* for which the node values represent the generalized coordinates. The trial function space may be inserted into the governing equations and the unknown node values determined by the Ritz method (if the solution extremizes a variational principle) or by the Galerkin, least-squares or other weighted-residual minimization methods if the problem cannot be expressed in a standard variational form.

Remark 1.4. In the mathematical interpretation the emphasis is on the concept of *local (piecewise) approximation*. The concept of element-by-element breakdown and assembly, while convenient in the computer implementation, is not theoretically necessary. The mathematical interpretation permits a general approach to the questions of convergence, error bounds, trial and shape function requirements, etc., which the physical approach leaves unanswered. It also facilitates the application of FEM to classes of problems that are not so readily amenable to physical visualization as structures; for example electromagnetics and heat conduction.

Remark 1.5. It is interesting to note some similarities in the development of Heaviside's operational methods, Dirac's delta-function calculus, and the FEM. These three methods appeared as ad-hoc computational devices created by engineers and physicists to deal with problems posed by new science and technology (electricity, quantum mechanics, and delta-wing aircraft, respectively) with little help from the mathematical establishment.[15] Only some time after the success of the new techniques became apparent were new branches of mathematics (operational calculus, distribution theory and piecewise-approximation theory, respectively) constructed to justify that success. In the case of the finite element method, the development of a formal mathematical theory started in the late 1960s, and much of it is still in the making.

§1.6. Keeping the Course

The first Part of this book, covered in Chapters 2 through 10, stresses the physical interpretation of FEM within the framework of the Direct Stiffness Method (DSM). This is done on account of its instructional advantages. Furthermore the computer implementation becomes more transparent because the sequence of operations can be placed in close correspondence with the DSM steps.

Chapters 11 through 19 deal specifically with element formulations. Ingredients of the mathematical interpretation are called upon whenever it is felt proper and convenient to do so. Nonetheless excessive entanglement with the mathematical theory is avoided if it may obfuscate the physics.

In Chapters 2 and 3 the time is frozen at about 1965, and the DSM presented as an aerospace engineer of that time would have understood it. This is not done for sentimental reasons, although that happens to be the year in which the writer began thesis work on FEM under Ray Clough. Virtually all FEM commercial codes are now based on the DSM and the computer implementation has not essentially changed since the late 1960s.[16] What has greatly improved since is "marketing sugar": user interaction and visualization.

§1.7. *What is Not Covered

The following topics are not covered in this book:

1. Elements based on equilibrium, mixed and hybrid variational formulations.
2. Flexibility and mixed solution methods.
3. Plate and shell elements.
4. Variational methods in mechanics.
5. General mathematical theory of finite elements.
6. Buckling and stability analysis.
7. General nonlinear response analysis.
8. Structural optimization.

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[15] Oliver Heaviside took heavy criticism from the lotus eaters, which he returned with gusto. His legacy is a living proof that "England is the paradise of individuality, eccentricity, heresy, anomalies, hobbies and humors" (George Santayana). Paul Dirac was luckier: he was shielded as member of the physics establishment and eventually received a Nobel Prize. Gilbert Strang, the first mathematician to dwelve in the real FEM (the one created by engineers) was kind to the founders.

[16] With the gradual disappearance of Fortran as a "live" programming language, noted in §1.7.7, changes at the implementation level have recently accelerated. E.g., C++, Python, Java and Matlab "wrappers" are becoming more common.

9. Error estimates and problem-adaptive discretizations.
10. Non-structural and multiphysics applications of FEM.
11. Designing and building production-level FEM software and use of special hardware (*e.g.* vector and parallel computers)

Topics 1–5 belong to what may be called "Advanced Linear FEM", which is covered in the book [255]. Topics 6–7 pertain to "Nonlinear FEM", which is covered in the book [258]. Topics 8–10 fall into advanced applications, covrede in other books in preparation, whereas 11 is an interdisciplinary topic that interweaves with computer science.

§1.8. The Origins of the Finite Element Method

This section moved to Appendix O to facilitate further expansion.

§1.9. Recommended Books for Linear FEM

The literature is voluminous: over 200 textbooks and monographs have appeared since 1967. Some recommendations for readers interested in further studies within *linear* FEM are offered below.

Basic level (reference): Zienkiewicz and Taylor [837]. This two-volume set is a comprehensive upgrade of the previous edition [835]. Primarily an encyclopœdic reference work that gives a panoramic coverage of FEM applications, as well as a comprehensive list of references. Not a textbook or monograph. Prior editions suffered from loose mathematics, largely fixed in this one. A three-volume fifth edition has appeared recently.

Basic level (textbook): Cook, Malkus and Plesha [149]. The third edition is comprehensive in scope although the coverage is more superficial than Zienkiewicz and Taylor. A fourth edition has appeared recently.

Intermediate level: Hughes [389]. It requires substantial mathematical expertise on the part of the reader. Recently (2000) reprinted as Dover edition.

Mathematically oriented: Strang and Fix [705]. Still the most readable mathematical treatment for engineers, although outdated in several subjects. Out of print.

Best value for the $$$: Przemieniecki's Dover edition [603], list price $15.95 (2003). A reprint of a 1966 McGraw-Hill book. Although woefully outdated in many respects (the word "finite element" does not appear except in post-1960 references), it is a valuable reference for programming simple elements. Contains a fairly detailed coverage of substructuring, a practical topic missing from the other books. Comprehensive bibliography in Matrix Structural Analysis up to 1966.

Most fun (if you appreciate British "humor"): Irons and Ahmad [401]. Out of print.

For buying out-of-print books through web services, check the metasearch engine in `www3.addall.com` (most comprehensive; not a bookseller) as well as that of `www.amazon.com`. A newcomer is `www.campusi.com`

§1.9.1. Hasta la Vista, Fortran

Most FEM books that include programming samples or even complete programs use Fortran. Those face an uncertain future. Since the mid-1990s, Fortran is gradually disappearing as a programming language taught in USA engineering undergraduate programs. (It still survives in some Physics and

Chemistry departments because of large amounts of legacy code.) So one end of the pipeline is drying up. Low-level scientific programming[17] is moving to C and C++, mid-level to Java, Perl and Python, high-level to Matlab, Mathematica and their free-source Linux equivalents. How attractive can a book teaching in a dead language be?

To support this argument with some numbers, here is a September-2003 snapshot of ongoing open source software projects listed in `http://freshmeat.net`. This conveys the relative importance of various languages (a mixed bag of newcomers, going-strongs, have-beens and never-was) in the present environment.

Lang	Projects	Perc	Lang	Projects	Perc	Lang	Projects	Perc
Ada	38	0.20%	APL	3	0.02%	ASP	25	0.13%
Assembly	170	0.89%	Awk	40	0.21%	Basic	15	0.08%
C	5447	28.55%	C#	41	0.21%	C++	2443	12.80%
Cold Fusion	10	0.05%	Common Lisp	27	0.14%	Delphi	49	0.26%
Dylan	2	0.01%	Eiffel	20	0.10%	Emacs-Lisp	33	0.17%
Erlang	11	0.06%	Euler	1	0.01%	Euphoria	2	0.01%
Forth	15	0.08%	Fortran	45	0.24%	Haskell	28	0.15%
Java	2332	12.22%	JavaScript	236	1.24%	Lisp	64	0.34%
Logo	2	0.01%	ML	26	0.14%	Modula	7	0.04%
Object Pascal	9	0.05%	Objective C	131	0.69%	Ocaml	20	0.10%
Other	160	0.84%	Other Scripting Engines	82	0.43%			
Pascal	38	0.20%	Perl	2752	14.42%	PHP	2020	10.59%
Pike	3	0.02%	PL/SQL	58	0.30%	Pliant	1	0.01%
PROGRESS	2	0.01%	Prolog	8	0.04%	Python	1171	6.14%
Rexx	7	0.04%	Ruby	127	0.67%	Scheme	76	0.40%
Simula	1	0.01%	Smalltalk	20	0.10%	SQL	294	1.54%
Tcl	356	1.87%	Unix Shell	550	2.88%	Vis Basic	15	0.08%
Xbasic	1	0.01%	YACC	11	0.06%	Zope	34	0.18%

Total Projects: 19079

Notes and Bibliography

Here is Ray Clough's personal account of how FEM and DSM emerged at Boeing in the early 1950s. (For further historical details, the interested reader may consult Appendices H and O.)

> " My involvement with the FEM began when I was employed by the Boeing Airplane Company in Seattle during summer 1952 as a member of their summer faculty program. When I had joined the civil engineering faculty at Berkeley in 1949, I decided to take advantage of my MIT structural dynamics background by taking up the field of Earthquake Engineering. So because the Boeing summer faculty program offered positions with their structural dynamics unit, I seized on that as the best means of advancing my preparation for the earthquake engineering field. I was particularly fortunate in this choice of summer work at Boeing because the head of their structural dynamics unit was Mr. M. J. Turner — a very capable man in dealing with problems of structural vibrations and flutter.
>
> When I arrived for the summer of 1952, Jon Turner asked me to work on the vibration analysis of a delta wing structure. Because of its triangular plan form, this problem could not be solved by procedures based on standard beam theory; so I spent the summer of 1952 trying to formulate a delta wing model built up as an assemblage of one-dimensional beams and struts. However, the results of deflection analyses based on this type of mathematical model were in very poor agreement with data obtained from laboratory tests of a scale model of a delta wing. My final conclusion was that my summer's work was a total failure—however, at least I learned what did not work.

[17] "A programming language is low level when its programs require attention to the irrelevant" (Alan Perlis).

Spurred by this disappointment, I decided to return to Boeing for the summer faculty program in 1953. During the winter, I stayed in touch with Jon Turner so I was able to rejoin the structural dynamics unit in June. The most important development during the winter was that Jon suggested we try to formulate the stiffness property of the wing by assembling plane stress plates of either triangular or rectangular shapes. So I developed stiffness matrices for plates of both shapes, but I decided the triangular form was much more useful because such plates could be assembled to approximate structures of any configuration. Moreover, the stiffness properties of the individual triangular plates could be calculated easily based on assumptions of uniform states of normal stress in the X and the Y directions combined with an uniform state of shear stress. Then the stiffness of the complete structure was obtained by appropriate addition of the contributions from the individual pieces. The Boeing group called this procedure the direct stiffness method.

The remainder of the summer of 1953 was spent in demonstrating that deflections calculated for structures formed as assemblages of triangular elements agreed well with laboratory measurements on the actual physical models. Also, it became apparent that the precision of the calculated results could be improved asymptotically by continued refinement of the finite element mesh. The conclusions drawn from that summer's work were presented in a paper given by Jon Turner at the annual meeting of the Institute of Aeronautical Sciences in January 1954. However, for reasons I never understood Jon did not submit the paper for publication until many months later. So this paper, which often is considered to be the first published description of the FEM, was not published until September 1956 — more than two years after the verbal presentation.

It is important to note that the basic purpose of the work done by Jon Turner's structural dynamics unit was vibration and flutter analysis. They were not concerned with stress analysis because that was the responsibility of the stress analysis unit. However, it was apparent that the model formed by the direct stiffness method could be used for stress analysis as well as for vibration analysis, and I made plans to investigate this stress analysis application as soon as possible. However, because of my other research responsibilities, I was not able to spend any significant time on the stress analysis question until I went on my sabbatical leave to Trondheim, Norway in September 1956. Then, when I arrived in Norway all I could do was to outline the procedures for carrying out the analysis, and to do calculations for very small systems using a desk calculator because the Norwegian Institute of Technology did not yet have an automatic digital computer.

The presentation of the paper to the Institute of Aeronautical Sciences was the first introduction of the principles of the FEM to a technical audience; although some of the basic concepts of the method were stated a short time later in a series of articles published in Aircraft Engineering by Dr. John H. Argyris during October 1954 to May 1955. However, the rectangular element presented in those articles is only a minor part of that contribution. The Argyris work came to my attention during my sabbatical leave in Norway, and I considered it then (as I still do now) to be the most important series of papers ever published in the field of Structural Mechanics. I credit that work for extending the scope of my understanding of structural theory to the level it eventually attained.

From my personal point of view, the next important event in finite element history was the coining of the name FEM. My purpose in choosing that name was to distinguish clearly the relatively large size pieces of the structure that make up a finite element assemblage as contrasted with the infinitesimal contributions that go into evaluation of the displacements of a structure in a typical virtual work analysis. The name first appeared in a publication that was written to demonstrate the finite element procedure for the civil engineering profession. A much more significant application of the method was presented at the Symposium on the use of Computers in Civil Engineering, held in Lisbon, Portugal in 1962, where it was used to evaluate the stress concentrations developed in a gravity dam that had cracked at its mid-section."

References

Referenced items have been moved to Appendix R.

Homework Exercises for Chapter 1
Overview

EXERCISE 1.1 [A:15] Work out Archimedes' problem using a circumscribed regular polygon, with $n = 1, 2, 4, \ldots 256$. Does the sequence converge any faster?

EXERCISE 1.2 [D:20] Select one of the following vehicles: truck, car, motorcycle, or bicycle. Draw a two level decomposition of the structure into substructures, and of selected components of some substructures.

EXERCISE 1.3 [D:30] In one of the earliest articles on the FEM, Clough [139] writes:

"When idealized as an assemblage of appropriately shaped two- and three-dimensional elements in this manner, an elastic continuum can be analyzed by standard methods of structural analysis. It should be noted that the approximation which is employed in this case is of physical nature; a modified structural system is substituted for the actual continuum. There need be no approximation in the mathematical analysis of this structural system. This feature distinguishes the finite element technique from finite difference methods, in which the exact equations of the actual physical system are solved by approximate mathematical procedures."

Discuss critically the contents of this paragraph while placing it in the context of time of writing (early 1960s). Is the last sentence accurate?