11

The HR Variational Principle of Elastostatics

TABLE OF CONTENTS

§11.1. Introduction

In Chapter 7, a *multifield* variational principle was defined as one that has more than one master field. That is, more than one unknown field is subject to independent variations. The present Chapter begins the study of such functionals within the context of elastostatics. Following a classification of the so-called canonical functionals, the Hellinger-Reissner (HR) mixed functional is derived.

The HR principle is applied to the derivation of a couple of 1D elements in the text, and others are provided in the Exercises.

§11.1.1. Mixed Versus Hybrid

The terminology pertaining to multifield functionals is not uniform across applied mechanics and FEM literature. Some authors call all multifield principles mixed; sometimes that term is restricted to specific cases. This book takes a middle ground:

A *Mixed principle* is one where all master fields are internal fields (volume fields in 3D).

A *Hybrid principle* is one where master fields are of different dimensionality. For example one internal volume field and one surface field.

Hybrid principles will be studied in Chapter 12–14. They are intrinsically important for FEM discretizations but have only a limited role outside of FEM.

§11.1.2. The Canonical Functionals

If hybrid functionals are excluded, three unknown internal fields of linear elastostatics are candidates for master fields to be varied: displacements u_i, strains e_{ij}, and stresses σ_{ij}. Seven combinations, listed in Table 11.1, may be chosen as masters. These are called the *canonical* functionals of elasticity.

Table 11.1 The Seven Canonical Functionals of Linear Elastostatics

#	Type	Master fields	Name
(I)	Single-field	Displacements	Total Potential Energy (TPE)
(II)	Single-field	Stresses	Total Complementary Potential Energy (TCPE)
(III)	Single-field	Strains	No name
(IV)	Mixed 2 field	Displacements & stresses	Hellinger-Reissner (HR)
(V)	Mixed 2-field	Displacements & strains	No agreed upon name
(VI)	Mixed 2-field	Strains & stresses	No name
(VII)	Mixed 3-field	Displacements, stresses & strains	Veubeke-Hu-Washizu (VHW)

Four of the canonical functionals: (I), (II), (IV) and (VII), have identifiable names. From the standpoint of finite element development those four, plus (V), are most important although they are not equal in importance. By far (I) and (IV) have been the most seminal, distantly followed by (II), (V) and (VII). Functionals (III) and (VI) are mathematical curiosities.

The construction of mixed functionals involves more expertise than single-field ones. And their FEM implementation requires more care, patience, and luck.[1]

[1] Strang's famous dictum is "mixed elements lead to mixed results." In other words: more master fields are not necessarily better than one. Some general guides as to when mixed functionals pay off will appear as byproduct of examples.

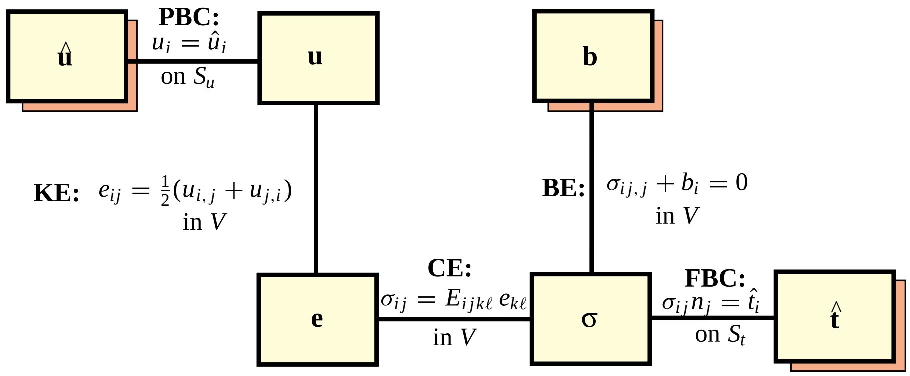

FIGURE 11.1. The Strong Form Tonti diagram for linear elastostatics, reproduced for convenience.

For convenience the Strong Form Tonti diagram of linear elastostatics is shown in Figure 11.1.

§11.2. The Hellinger-Reissner (HR) Principle

§11.2.1. Master Fields, Slave Fields And Weak Links

The Hellinger-Reissner (HR) canonical functional of linear elasticity allows displacements and stresses to be varied separately. This establishes the master fields. Two slave strain fields appear, one coming from displacements and one from stresses:

$$e^u_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \qquad e^\sigma_{ij} = C_{ijk\ell}\, \sigma_{k\ell} \tag{11.1}$$

Here $C_{ijk\ell}$ are the entries of the compliance tensor or strain-stress tensor $\underline{\mathbf{C}}$, which is the inverse of $\underline{\mathbf{E}}$. In matrix form the slave strain fields are given by

$$\mathbf{e}^u = \mathbf{D}\,\mathbf{u}, \qquad \mathbf{e}^\sigma = \mathbf{C}\,\boldsymbol{\sigma}, \tag{11.2}$$

in which $\mathbf{C} = \mathbf{E}^{-1}$ is the 6×6 matrix of elastic compliances.

At the exact solution of the elasticity problem, the two strain fields coalesce point by point. But when these fields are obtained by an approximation procedure such as FEM, strains recovered from displacements and strains computed from stresses will not generally agree.

Three weak links appear: BE and FBC (as in the Total Potential Energy principle derived in the previous Chapter), plus the link between the two slave strain fields, which is identified as EE. Figure 11.2 depicts the resulting Weak Form.

Remark 11.1. The weak connection between $\mathbf{e}^u$ and $\mathbf{e}^\sigma$ could have been replaced by one between $\boldsymbol{\sigma}^u$ and $\boldsymbol{\sigma}$ (the latter being a master field, so no superscript is applied). The results would be the same because the constitutive equation links are strong. The choice of $\mathbf{e}^u$ and $\mathbf{e}^\sigma$ simplifies slightly the derivations below.

§11.2.2. The Weak Equations

We follow the weighting residual technique outlined in the previous Chapter for the TPE derivation. Take the residuals of the three weak connections shown in Figure 11.2, multiply them by weighting functions and integrate over the respective domains:

$$\int_V (e^u_{ij} - e^\sigma_{ij})\, w_{ij}\, dV + \int_V (\sigma_{ij,j} + b_i)\, w^*_i\, dV + \int_S (\sigma_{ij} n_j - \hat{t}_i)\, w^{**}_i\, dS = 0 \tag{11.3}$$

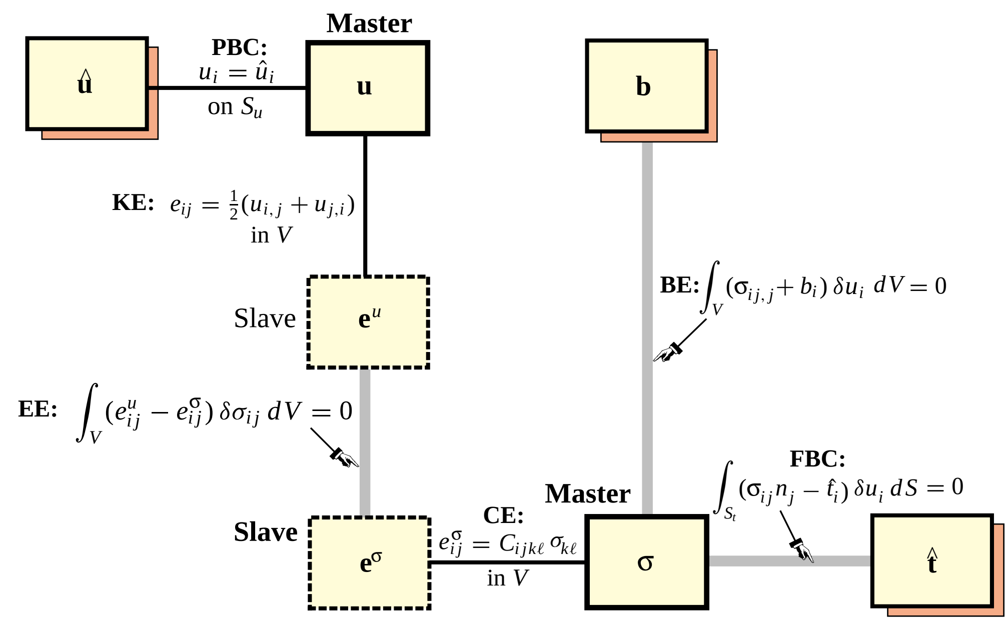

FIGURE 11.2. The starting Weak Form for derivation of the HR principle.

For conformity, w_{ij} must be a second order tensor, whereas w_i^* and w_i^{**} are 3-vectors. These weights must be expressed as variations of either master: either displacements u_i or stresses σ_{ij}, based on work pairing considerations. The residuals of KE are volume forces integrated over V, and those of FBC are surface forces integrated over S. Hence w_i^* and w_i^{**} must be displacement variations to obtain energy density. The residuals of EE are strains integrated over V; consequently w_{ij} must be stress variations. Based on these considerations we set $w_{ij} = \delta\sigma_{ij}$, $w_i^* = -\delta u_i$, $w_i^{**} = \delta u_i$, where the minus sign in the second one is chosen to anticipate eventual cancellation in the surface integrals. Adding the weak link contributions gives

$$\int_V (e^u_{ij} - e^\sigma_{ij})\,\delta\sigma_{ij}\, dV - \int_V (\sigma_{ij,j} + b_i)\,\delta u_i\, dV + \int_S (\sigma_{ij} n_j - \hat{t}_i)\,\delta u_i\, dS = 0. \tag{11.4}$$

Next, integrate the $\sigma_{ij,j}\,\delta u_i$ term by parts to eliminate the stress derivatives, split the surface integral into $S_u \cup S_t$, and enforce the strong link $u_i = \hat{u}_i$ over S_u:

$$\begin{aligned} -\int_V \sigma_{ij,j}\,\delta u_i\, dV &= \int_V \sigma_{ij}\,\delta e^u_{ij}\, dV - \int_S \sigma_{ij} n_j\,\delta u_i\, dS \\ &= \int_V \sigma_{ij}\,\delta e^u_{ij}\, dV - \int_{S_u} \sigma_{ij} n_j\,\cancelto{0}{\delta u_i}\, dS - \int_{S_t} \sigma_{ij} n_j\,\delta u_i\, dS \\ &= \int_V \sigma_{ij}\,\delta e^u_{ij}\, dV - \int_{S_t} \sigma_{ij} n_j\,\delta u_i\, dS. \end{aligned} \tag{11.5}$$

in which δe^u_{ij} means the variation of $\delta\frac{1}{2}(u_{i,j} + u_{j,i}) = \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i})$, as in §10.9.2

Upon simplification of the cancelling terms $\sigma_{ij}\, n_j\, \delta u_i$ on S_t we end up with the following variational statement, written hopefully as the exact variation of a functional $\Pi_{\rm HR}$:

$$\delta\Pi_{\rm HR} = \int_V \big[\, (e^u_{ij} - e^\sigma_{ij})\,\delta\sigma_{ij} + \sigma_{ij}\,\delta e^u_{ij} - b_i\,\delta u_i \,\big]\, dV - \int_{S_t} \hat{t}_i\,\delta u_i\, dS. \tag{11.6}$$

§11.2.3. The Variational Form

And indeed (11.6) is the exact variation of

$$\boxed{\Pi_{\rm HR}[u_i,\sigma_{ij}] = \int_V \left(\sigma_{ij}e^u_{ij} - \tfrac{1}{2}\sigma_{ij}\,C_{ijk\ell}\,\sigma_{k\ell} - b_i u_i\right)\,dV - \int_{S_t} \hat{t}_i\,u_i\,dS.} \qquad (11.7)$$

This is called the *Hellinger-Reissner functional*, abbreviated HR.[2] It is often stated in the literature as

$$\Pi_{\rm HR}[u_i,\sigma_{ij}] = \int_V [-\mathcal{U}^*(\sigma_{ij}) + \sigma_{ij}\tfrac{1}{2}(u_{i,j}+u_{j,i}) - b_i u_i]\,dV - \int_{S_t} \hat{t}_i u_i\,dS, \qquad (11.8)$$

in which

$$\mathcal{U}^*(\sigma_{ij}) = \tfrac{1}{2}\sigma_{ij}C_{ijk\ell}\sigma_{k\ell} = \tfrac{1}{2}\sigma_{ij}e^\sigma_{ij}, \qquad (11.9)$$

is the *complementary energy density* in terms of the master stress field.

In FEM work the functional is usually written in the split form

$$\boxed{\begin{aligned} &\Pi_{\rm HR} = U_{\rm HR} - W_{\rm HR}, \qquad \text{in which}\\ &U_{\rm HR} = \int_V \left(\sigma_{ij}e^u_{ij} - \tfrac{1}{2}\sigma_{ij}C_{ijk\ell}\sigma_{k\ell}\right)\,dV, \quad W_{\rm HR} = \int_V b_i u_i\,dV + \int_{S_t} \hat{t}_i u_i\,dS. \end{aligned}} \qquad (11.10)$$

The HR principle states that stationarity of the total variation

$$\boxed{\delta\Pi_{\rm HR} = 0} \qquad (11.11)$$

provides the BE and EE weak links as Euler-Lagrange equations, whereas the FBC weak link appears as a natural boundary condition.

Remark 11.2. To verify the assertion about (11.6) being the first variation of $\Pi_{\rm HR}$, note that

$$\delta(\sigma_{ij}e^u_{ij}) = e^u_{ij}\,\delta\sigma_{ij} + \sigma_{ij}\,\delta e^u_{ij}, \qquad \delta(\tfrac{1}{2}\sigma_{ij}\,C_{ijk\ell}\,\sigma_{k\ell}) = C_{ijk\ell}\,\sigma_{k\ell}\,\delta\sigma_{ij} = e^\sigma_{ij}\,\delta\sigma_{ij}. \qquad (11.12)$$

§11.2.4. Variational Indices and Continuity Requirements

For a single-field functional, the *variational index* of its primary variable is the highest derivative m of that field that appears in the variational principle. The connection between variational index and required continuity in FEM shape functions was presented (as recipe) in the introductory FEM course (IFEM). That course considered only the single-field TPE functional, in which the primary variable, and only master, is the displacement field. It was stated that displacement shape functions must be C^{m-1} continuous between elements and C^m inside. For the bar and plane stress problem covered in IFEM, $m = 1$, whereas for the Bernoulli-Euler beam $m = 2$.

[2] The basic idea was contained in the work of Hellinger, which appeared in 1914 [**?**]. As a proven theorem for the traction specified problem (no PBC) it was given by Prange in his 1916 thesis [573]. As a complete theorem containing both PBC and FBC it came out much later (1950) in a paper by Reissner [605].

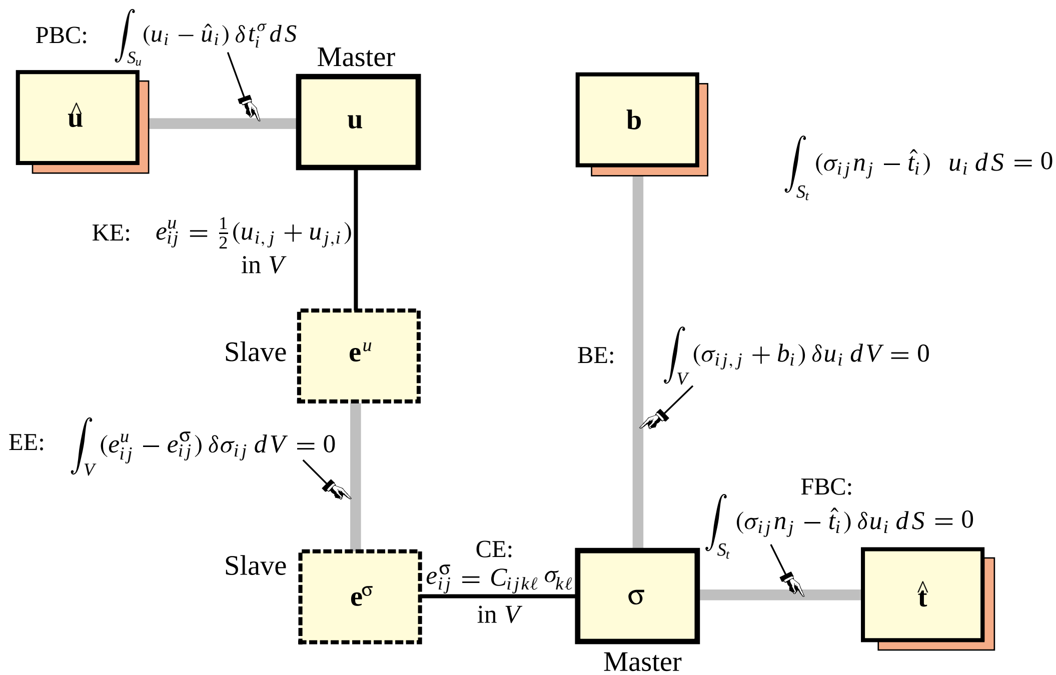

FIGURE 11.3. WF Tonti diagram for displacement-BC-generalized HR, in which the PBC link is weakened.

In multifield functionals the variational index concept applies to *each varied field*. Thus there are as many variational indices as master fields. In the HR functional (11.10) of 3D elasticity, the variational index m_u of the displacements is 1, because first order derivatives appear in the slave strains e_{ij}^u. The variational index m_σ of the stresses is 0 because no stress derivatives appear. The required continuity of FEM shape functions for displacements and stresses is dictated by these indices. More precisely, if $\Pi_{\rm HR}$ is used as source functional for element derivation:

1. Displacement shape functions must be C^0 (continuous) between elements and C^1 inside (continuous and differentiable).
2. Stress shape functions can be C^{-1} (discontinuous) between elements, and C^0 (continuous) inside.

§11.2.5. Displacement-BC Generalized HR

If the PBC link (displacement BCs) between u_i and $\hat{u}_i$ is weakened as illustrated in Figure 11.3, the functional $\Pi_{\rm HR}$ generalizes to

$$\boxed{\Pi_{\rm HR}^g = \Pi_{\rm HR} - \int_{S_u} \sigma_{ij} n_i (u_i - \hat{u}_i)\, dS = \Pi_{\rm HR} - \int_{S_u} t_i^\sigma (u_i - \hat{u}_i)\, dS.} \tag{11.13}$$

in which $\sigma_{ij} n_j = t_i^\sigma$ is the surface traction associated with the master stress field.

§11.3. Application Example 1: Tapered Bar Element

In this section the use of the HR functional to construct a very simple finite element is illustrated. Consider a tapered bar made up of isotropic elastic material, as depicted in Figure 11.4(a). The

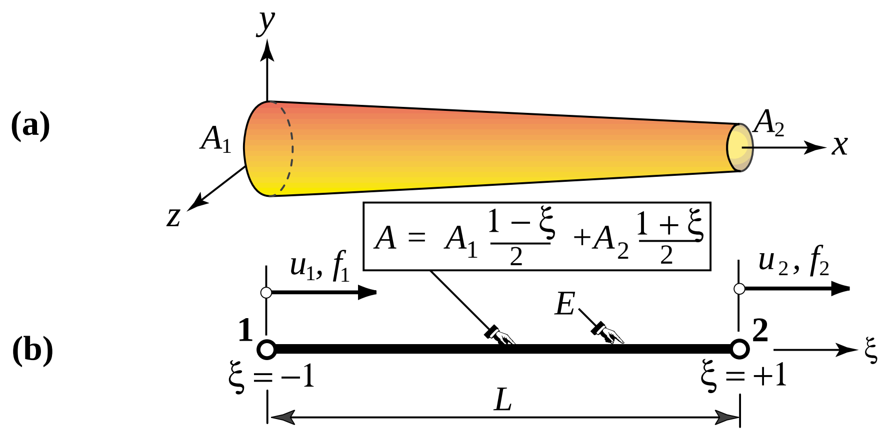

FIGURE 11.4. Two-node tapered bar element by HR: (a) shows the bar as a 3D object and (b) as a FEM model.

$x_1 \equiv x$ axis is placed along the longitudinal direction. The bar cross section area A varies linearly between the end node areas A_1 and A_2. The element has length L and constant elastic modulus E. Body forces are ignored.

The reduction of the HR functional (11.7) to the bar case furnishes an instructive example of the derivation of a *structural* model based on stress resultants and Mechanics of Materials approximations.

In the theory of bars, the only nonzero stress is $\sigma_{11} \equiv \sigma_{xx}$, which will be denoted by σ for simplicity. The only internal force is the bar axial force $N = A\sigma_{xx}$. The only displacement component that participates in the functional is the axial displacement u_x, which is only a function of x and will be simply denoted by $u(x)$. The value of the axial displacement at end sections 1 and 2 is denoted by u_1 and u_2, respectively. The axial strain is $e_{11} \equiv e_{xx}$, which will be denoted by e. The strong links are $e^u = du(x)/dx = u'$, where primes denote derivatives with respect to x, and $e^\sigma = \sigma/E = N/(EA)$. We call $N^u = EA\,e^u = EA\,u'$, etc.

As for as boundary conditions, for a free (unconnected) element S_t embodies the whole surface of the bar. But according to bar theory the lateral surface is traction free and thus drops off from the surface integral. That leaves the two end sections, at which uniform longitudinal surface tractions $\hat{t}_x$ are prescribed whereas the other component vanishes. On assuming a uniform traction distribution over the end cross sections, we find that the node forces are $f_1 = -t_{x1}A_1$ at section 1 and $f_2 = t_{x2}A_2$ at section 2. (The negative sign in the first one arises because at section 1 the external normal points along $-x$.)

Plugging these relations into the HR functional (11.7) and integrating over the cross section gives

$$\boxed{\Pi_{\mathrm{HR}}[u, N] = \int_L \left(N u' - \frac{N^2}{2EA} \right) dx - f_1 u_1 - f_2 u_2.} \tag{11.14}$$

This is an example of a functional written in term of *stress resultants* rather than actual stresses. The theory of beams, plates and shells leads also to this kind of functionals.

§11.3.1. Formulation of the Tapered Bar Element

We now proceed to construct the two-node bar element (e) depicted in Figure 11.4(b), from the functional (11.14). Define ξ is a natural coordinate that varies from $\xi = -1$ at node 1 to $\xi = 1$ at node 2. Assumptions must be made on the variation of displacements and axial forces. Displacements are taken to vary linearly whereas the axial force will be assumed to be constant over the element:

$$u(x) \approx u_1^{(e)} \frac{1-\xi}{2} + u_2^{(e)} \frac{1+\xi}{2}, \qquad N(x) \approx \bar{N}^{(e)} \tag{11.15}$$

These assumptions comply with the C^0 and C^{-1} continuity requirements for displacements and stresses, respectively, stated in §11.2.4. Inserting (11.14) and (11.15) into the functional (11.13) and carrying out the necessary integral over the element length yields[3]

$$\Pi_{\rm HR}^{(e)} = \tfrac{1}{2} \begin{bmatrix} \bar{N}^{(e)} \\ u_1^{(e)} \\ u_2^{(e)} \end{bmatrix}^T \begin{bmatrix} -\dfrac{\gamma L}{E A_m} & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{N}^{(e)} \\ u_1^{(e)} \\ u_2^{(e)} \end{bmatrix} - \begin{bmatrix} 0 \\ f_1^{(e)} \\ f_2^{(e)} \end{bmatrix}^T \begin{bmatrix} \bar{N}^{(e)} \\ u_1^{(e)} \\ u_2^{(e)} \end{bmatrix} \tag{11.16}$$

in which

$$A_m = \tfrac{1}{2}(A_1 + A_2), \qquad \gamma = \frac{A_m}{A_2 - A_1} \log \frac{A_2}{A_1}. \tag{11.17}$$

Note that if the element is prismatic, $A_1 = A_2 = A_m$, and $\gamma = 1$ (take the limit of the Taylor series for γ).

For this discrete form of $\Pi_{\rm HR}^{(e)}$, the Euler-Lagrange equations are simply the stationarity conditions

$$\frac{\partial \Pi_{\rm HR}^{(e)}}{\partial \bar{N}^{(e)}} = \frac{\partial \Pi_{\rm HR}^{(e)}}{\partial u_1^{(e)}} = \frac{\partial \Pi_{\rm HR}^{(e)}}{\partial u_2^{(e)}} = 0, \tag{11.18}$$

which supply the finite element equations

$$\begin{bmatrix} -\dfrac{\gamma L}{E A_m} & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{N}^{(e)} \\ u_1^{(e)} \\ u_2^{(e)} \end{bmatrix} = \begin{bmatrix} 0 \\ f_1^{(e)} \\ f_2^{(e)} \end{bmatrix} \tag{11.19}$$

This is an example of a *mixed finite element*, where the qualifier “mixed” implies that approximations are made in more than one unknown internal quantity; here axial forces and axial displacements.

Because the axial-force degree of freedom $\bar{N}^{(e)}$ is not continuous across elements (recall that C^{-1} continuity for stress variables is allowed), it may be eliminated or “condensed out” at the element level. The static condensation process studied in IFEM yields

$$\frac{E A_m}{\gamma L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{bmatrix} = \begin{bmatrix} f_1^{(e)} \\ f_2^{(e)} \end{bmatrix}, \tag{11.20}$$

or

$$\boxed{\mathbf{K}^{(e)}\, \mathbf{u}^{(e)} = \mathbf{f}^{(e)}.} \tag{11.21}$$

These are the *element stiffness equations*, obtained here through the HR principle. Had these equations been derived through the TPE principle, one would have obtained a similar expression except that $\gamma = 1$ for any end-area ratio. Thus if the element is prismatic ($A_1 = A_2 = A_m$) the HR and TPE functionals lead to the same element stiffness equations.

[3] Derivation details are worked out in an Exercise.

Table 11.2 Results for one-element analysis of fixed-free tapered bar

Area ratio	u_2 from HR	u_2 from TPE	Exact u_2
$A_1/A_2 = 1$	$PL/(EA_m)$	$PL/(EA_m)$	$PL/(EA_m)$
$A_1/A_2 = 2$	$1.0397PL/(EA_m)$	$PL/(EA_m)$	$1.0397PL/(EA_m)$
$A_1/A_2 = 5$	$1.2071PL/(EA_m)$	$PL/(EA_m)$	$1.2071PL/(EA_m)$

§11.3.2. Numerical Example

To give a simple numerical example, suppose that the bar of Figure 11.4 is fixed at end 1 whereas end 2 is under a given axial force P. Results for sample end area ratios are given in Table 11.2. It can be seen that the HR formulation yields the exact displacement solution for all area ratios.

Also note that the discrepancy of the one-element TPE solution from the exact one grows as the area ratio deviates from one. The TPE elements underestimate the actual deflections, and are therefore on the stiff side. To improve the TPE results we need to divide the bar into more elements.

§11.3.3. The Bar Flexibility

From (11.20) we immediately obtain

$$u_2 - u_1 = \frac{\gamma L}{EA_m}(f_2 - f_1) = F(f_2 - f_1). \tag{11.22}$$

This called a *flexibility equation*. The number $F = \gamma L/(EA_m)$ is the flexibility coefficient or *influence coefficient*. For more complicated elements we would obtain a *flexibility matrix*. Relations such as (11.22) were commonly worked out in older books in matrix structural analysis. The reason is that flexibility equations are closely connected to classical static experiments in which a force is applied, and a displacement or elongation measured.

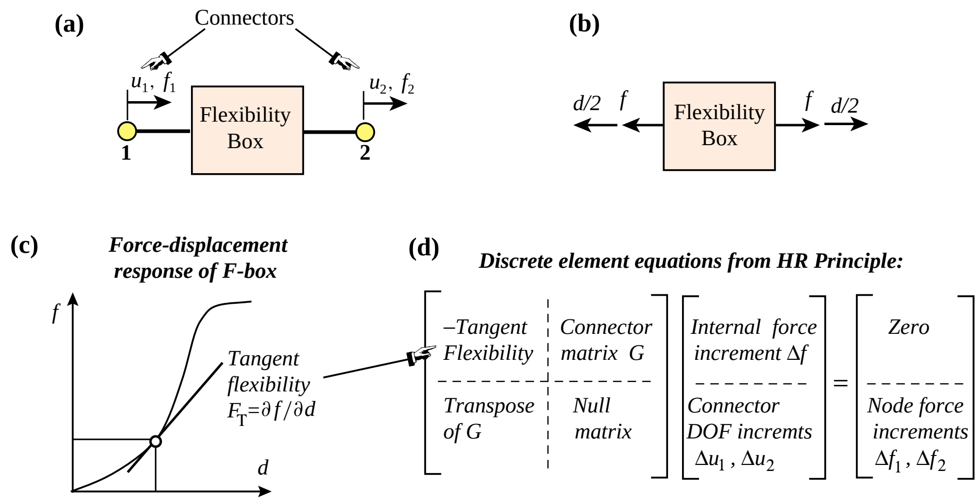

FIGURE 11.5. A connector element (sketch) developed with the help of the HR principle.

§11.4. Application Example 2: Curved Cable Element

§11.4.1. Connector Elements

The HR functional is useful for deriving a class of elements known as *connector elements*.[4] The concept is illustrated in Figure 11.5(a). The *connector nodes* are those through which the element links to other elements through the node displacements. These displacements are the *connector degrees of freedom*, or simple the *connectors*. The box models the intrinsic response of the element; if it is best described in terms of response to forces or stresses, as depicted in Figures 11.5(b,c), it is called a *flexibility box* or F-box.

In many applications the box response is nonlinear. Examples are elements modelling contact, friction and joints. If this is the only place where nonlinear behavior occur, the flexilibity element acts as a device to isolate local nonlinearities. This is an effective way to reuse linear FEM programs.

Consider for simplicity a one-dimensional, 2 node flexibilityelement such as that sketched in Figure AVMM:Ch11:fig:ConnectorElementDevelopedByHR The connector nodes are 1 and 2. The connector DOF are the axial displacements u_1 and u_2. The relative displacement is $\Delta = u_2 - u_1$. The kernel behavior is described by the response to an axial force f, as pictured in Figure 11.5(c):

$$d = F(f). \tag{11.23}$$

The tangent flexibility is

$$F_T = \frac{\partial d}{\partial f} = \frac{\partial F(f)}{\partial f}. \tag{11.24}$$

Application of the HR principle leads to the tangent equation

$$\begin{bmatrix} -F_T & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta f \\ \Delta u_1 \\ \Delta u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix} \tag{11.25}$$

[4] Hybrid elements, covered in Sections 8ff, are also useful in this regard. Often the two approaches lead to identical results.

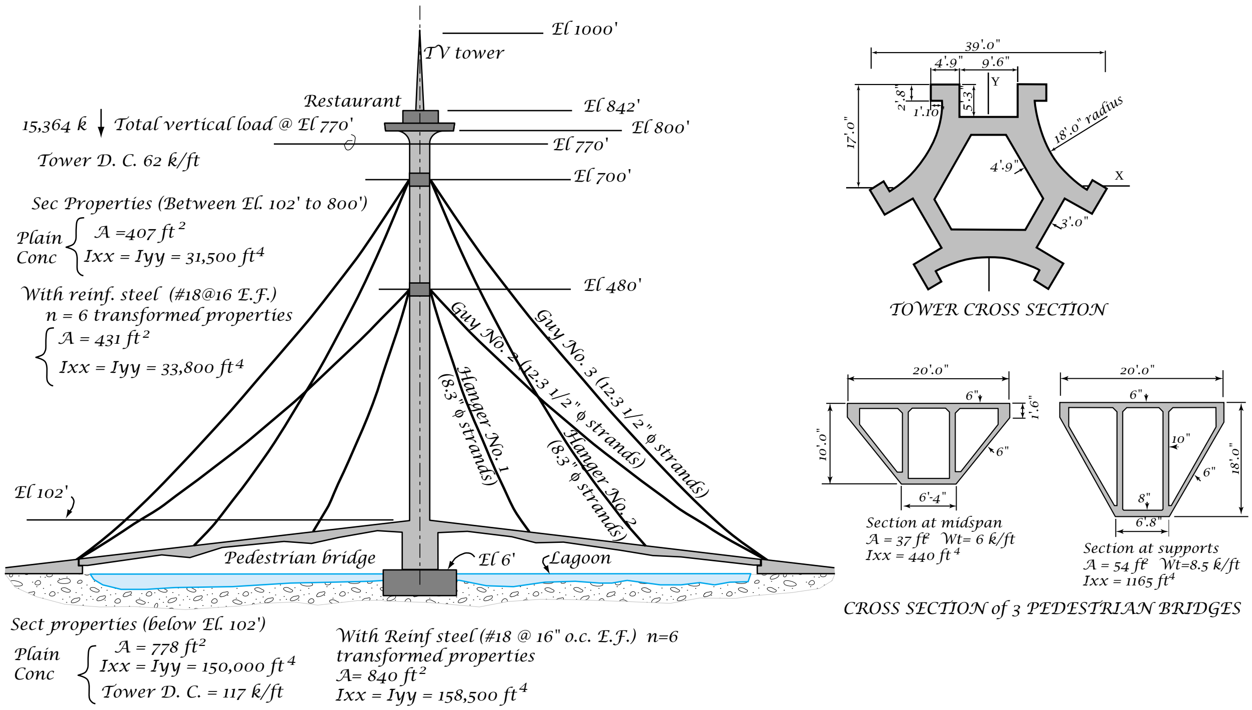

FIGURE 11.6. A connector element (sketch) developed with the help of the HR principle.

where Δ denote increments.[5] Condensation of Δf as internal freedom gives the stiffness matrix

$$\begin{bmatrix} K_T & -K_T \\ -K_T & K_T \end{bmatrix} \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} = \begin{bmatrix} \Delta f_1 \\ \Delta f_2 \end{bmatrix} \tag{11.26}$$

This result could also been obtained directly from physics, or from the displacement formulation. However, the HR approach remains unchanged when passing to 2 and 3 dimensions.

§11.4.2. A Curved Cable Element

As an application consider the development of a curved cable element used to model the guy and hanger members of the tower structure shown in Figure 11.6(a).[6]

Figure 11.7(a) shows a two-dimensional FEM model, with two end nodes and 3 freedoms (all translational displacements) per node.[7] To cut down the number of elements along the cable

[5] The first entry of the right hand side has been set to zero for simplicity. Generally it is not.

[6] A 1000-ft guyed tower proposed for the South Florida coast (near Miami) by a group of rich Cuban expatriates and dubbed the “Tower of Freedom” as it was supposed to serve as a guide beacon for boats escaping Cuba with refugees. The preliminary design shown in Figure 11.6 was made by a well known structural engineering company and dated June 1967. Ray W. Clough and Joseph Penzien were consultants for the verification against hurricane winds. Analyzed using an ad-hoc 2D FEM code by Mike Shears and the writer, who was then a post-doc at UC Berkeley, July–September 1967. Main concern was safety under a hurricane Class 3. The project was canceled as too costly and plans for a 3D cable analysis code shelved.

[7] The structure has 120° circular symmetry. Reduced to one plane of symmetry (plane of the paper) by appropriate projections of the right-side (windward) portion.

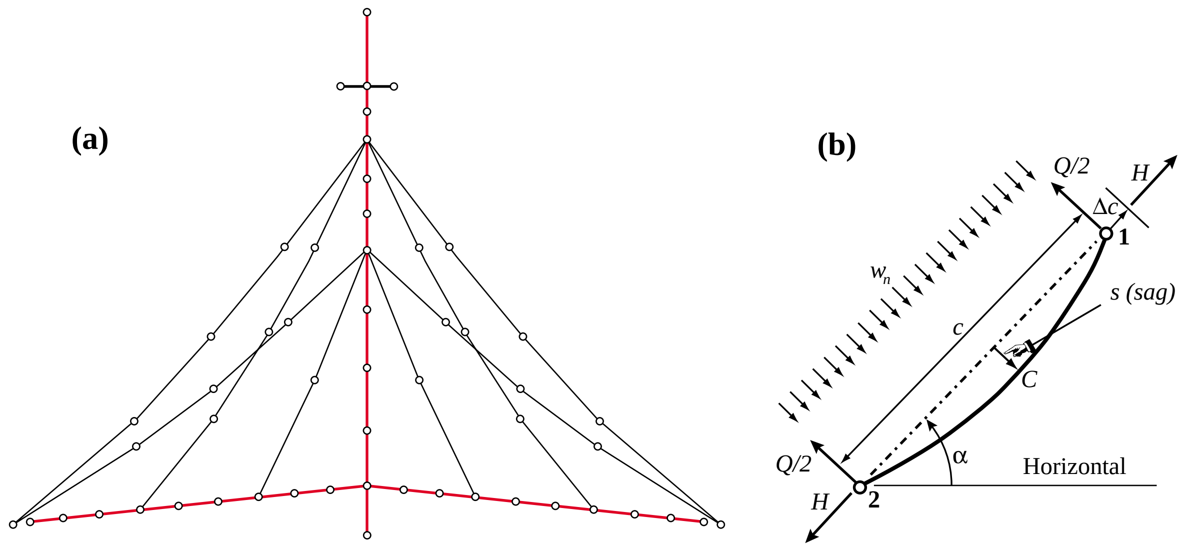

FIGURE 11.7. (a): 2D FEM model of guyed tower of Figure 11.6 for vibration and dynamic analysis under hurricane wind loads (1967); (b): parabolically curved cable element developed to model the guy and hanger cables with few elements along the length.

members, a parabolically curved cable element, pictured in Figure 11.7(b) was developed. The method that follows illustrates the application of the flexibility approach combined with HR to connector elements.

The element has two nodes, 1 and 2. The distance 1–2 is the chord distance c. The actual length of the strained cable element is L, so $L \geq c$. The force H along the chord is called the *thrust*. H and the chord change Δc play the role of f and d, respectively, in the flexibility response sketched in Figure 11.5. The cable is subjected to a uniform transverse load w_n specified per unit of chord length. (The load is usually a combination of self-weight and wind.) The elastic rigidity of the cable is EA_0, where E is the apparent elastic modulus (which depends on the fabrication of the cable) and A_0 the original structural area.

The following simplifying assumptions are made at the element level:

1. The sag is small compared to chord length: $s < c/10$, which characterizes a taut element.[8]
2. The load w_n is uniform. As a consequence, the transverse reaction loads at nodes are $Q/2$, with $Q = w_n c$. See Figure chapdot7(b).
3. $Q = w_n c$ is fixed even if c changes. This is exact for self weight, and approximately verified for wind loads.
4. The effect of tangential loads (along the chord) on the element deformation is neglected.
5. Hooke's law applies in the form $L - L_0 = H/(EA_0)$, where L_0 is the unstrained length of the element.

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[8] If this property is not realized, the cable member should be divided into more elements. Dividing one element into two cuts c and s approximately by 2 and 4, respectively, so s/c is roughly halved.

Under the foregoing assumptions, the cable deflection profile is parabolic, and we get

$$s = \frac{Qc}{8H}, \qquad L = L_0\left(1 + \frac{H}{EA_0}\right) = c + \frac{8}{3}\frac{s^2}{c^2}, \qquad \frac{c}{L_0} = \frac{1 + \frac{H}{EA_0}}{1 + \frac{Q^2}{24H^2}}. \tag{11.27}$$

The first equation comes from moment equilibrium at the sagged element midpoint C, the second from the shallow parabola-arclength formula, and the third one from eliminating the sag s between the first two. Differentiation gives the tangent flexibility

$$F_T = \frac{\partial c}{\partial H} = \frac{L_0}{EA_0}\frac{1}{1 + \frac{Q^2}{24H^2}} + \frac{\frac{Q^2 L_0}{12H^3}\left(1 + \frac{H}{EA_0}\right)}{\left(1 + \frac{Q^2}{24H^2}\right)^2}. \tag{11.28}$$

For most structural cables, $H << EA_0$ and $(Q/H)^2 << 1$. Accordingly the above formula simplifies to

$$F_T = \frac{L_0}{EA_0} + \frac{Q^2 L_0}{12H^3}, \tag{11.29}$$

which was used in the 1967 dynamic analysis at Berkeley. If $Q \to 0$ or $H \to \infty$, (11.29) reduces to the flexibility $L_0/(EA_0)$ of a linear bar element, as can be expected. Replacing into (11.25) and condensing out ΔH gives the tangent local stiffness matrix of the cable element as (11.26), where u_1 and u_2 are axial displacements at nodes 1 and 2 along the chord, and $K_T = 1/F_T$. This matrix relation can be transformed to the global coordinate system in the usual manner.

Homework Exercises for Chapter 11

The HR Variational Principle of Elastostatics

EXERCISE 11.1 [A:10] Derive the Euler-Lagrange equations and natural BCs of $\Pi_{\rm HR}$ given in (11.-2).

EXERCISE 11.2 [A:20] As shown in Table 11.1, the tapered bar HR model derived in §11.3 gives the exact end displacement with one element. But the assumed linear-displacement variation does not agree with the displacement of the exact solution, which is nonlinear in x if $A_1 \neq A_2$. Explain this contradiction. *Hint*: a variational freak; integrate the Nu' term in (11.13) by parts and use the fact that the exact solution has constant N. (These elements are called *nodally exact* in the FEM literature.)

EXERCISE 11.3 [A:25] Construct the HR functional for the cable element treated in §11.4.2. Hint: construct the complementary energy function of the F-box.

EXERCISE 11.4 [A:15] Show that if the master σ_{ij} is replaced by the slave $\sigma^u_{ij} = E_{ijk\ell}e^u_{ij}$ with $e^u_{ij} = (u_{i,j} + u_{j,i})/2$, the HR functional reduces to the TPE functional.

EXERCISE 11.5 [A:25] Derive the Total Complementary Potential Energy (TCPE) functional of linear elastostatics

$$\Pi_{\rm TCPE}[\sigma_{ij}] = -\tfrac{1}{2}\int_V \sigma_{ij}C_{ijk\ell}\sigma_{k\ell}\,dV + \int_{S_u} \sigma_{ij}n_j\,\hat{u}_i\,dS, \tag{E11.1}$$

by choosing stresses as the only primary field. Choose BE, FBC, and the right-to-left constitutive equations [the strain-stress relations $e^\sigma_{ij} = C_{ijk\ell}\sigma_{k\ell}$] as strong connections whereas KE and PBC are weak connections. Notice that the displacement field u_i does not appear in this functional; only the prescribed displacements $\hat{u}_i$.

EXERCISE 11.6 [A:30] Derive the anonymous functional $\Pi_S[e_{ij}]$ based on strains as only primary field. Take all connections as strong except the constitutive equations. Note: this functional seems to appear in only one book,[9] and therein only as a curiosity. It is weird looking because of its abnormal simplicity:

$$\Pi_{\rm S}[e_{ij}] = \int_V (\sigma_{ij} - \tfrac{1}{2}E_{ijk\ell}e_{k\ell})\,e_{ij}\,dV. \tag{E11.2}$$

That's right, no boundary terms, and σ_{ij} is here a *data* field!

[9] J. T. Oden and J. N. Reddy, *Variational Methods in Theoretical Mechanics*, Springer-Verlag, Berlin (1982).