19

FEM Convergence Requirements

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§19.1. Overview

Chapters 11 through 18 have discussed, in piecemeal fashion, requirements for shape functions of isoparametric elements. These are motivated by *convergence*: as the mesh is refined, the FEM solution should approach the analytical solution of the mathematical model.[1] This attribute is obviously necessary to instill confidence in FEM results from the standpoint of mathematics.

This Chapter provides unified information on convergence requirements. These requirements can be grouped into three:

Completeness. The elements must have enough *approximation power* to capture the analytical solution in the limit of a mesh refinement process. This intuitive statement is rendered more precise below.

Compatibility. The shape functions should provide *displacement continuity* between elements. Physically these insure that no material gaps appear as the elements deform. As the mesh is refined, such gaps would multiply and may absorb or release spurious energy.

Stability. The system of finite element equations must satify certain *well posedness* conditions that preclude nonphysical zero-energy modes in elements, as well as the absence of excessive element distortion.

Completeness and compatibility are two aspects of the so-called **consistency** condition between the discrete and mathematical models. A finite element model that passes both completeness and continuity requirements is called *consistent*. This is the FEM analog of the famous Lax-Wendroff theorem,[2] which says that consistency and stability imply convergence.

Remark 19.1. A deeper mathematical analysis done in more advanced courses shows that completeness is *necessary* for convergence whereas failure of the other requirements does not necessarily precludes it. There are, for example, FEM models in common use that do not satisfy compatibility. Furthermore, numerically unstable models may be used (with caution) in situations where that property is advantageous, as in the modeling of local singularities. Nonetheless, the satisfaction of the three criteria guarantees convergence and may therefore be regarded as a safe choice for the beginner user.

§19.2. The Variational Index

For the mathematical statement of the completeness and continuity conditions, the variational index alluded to in previous sections plays a fundamental role.

The FEM is based on the direct discretization of an energy functional $\Pi[u]$, where u (displacements for the elements considered in this book) is the primary variable, or (equivalently) the function to be varied. Let m be the highest spatial derivative order of u that appears in Π. This m is called the *variational index*.

[1] Of course FEM convergence does not guarantee the correctness of the mathematical model in capturing the physics. As discussed in Chapter 1, *model verification* against experiments is a different and far more difficult problem.

[2] Proven originally for classical finite difference discretizations in fluid mechanics. More precisely, it states that a numerical scheme for the scalar conservation law, $du/dt + df/dx = 0$ converges to a unique (weak) solution, if it is consistent, stable and conservative. There is no equivalent theorem for systems of conservation laws.

Example 19.1. In the bar problem discussed in Chapter 11,

$$\Pi[u] = \int_0^L \left(\tfrac{1}{2}\, u' E A u' - q u\right)\, dx. \tag{19.1}$$

The highest derivative of the displacement $u(x)$ is $u' = du/dx$, which is first order in the space coordinate x. Consequently $m = 1$. This is also the case on the plane stress problem studied in Chapter 14, because the strains are expressed in terms of first order derivatives of the displacements.

Example 19.2. In the plane beam problem discussed in Chapter 12,

$$\Pi[v] = \int_0^L \left(\tfrac{1}{2}\, v'' E I v'' - q v\right)\, dx. \tag{19.2}$$

The highest derivative of the transverse displacement is the curvature $\kappa = v'' = d^2 v/dx^2$, which is of second order in the space coordinate x. Consequently $m = 2$.

§19.3. Consistency Requirements

Using the foregoing definition of variational index, we can proceed to state the two key requirements for finite element shape functions.

§19.3.1. Completeness

> The element shape functions must represent exactly all polynomial terms of order $\leq m$ in the Cartesian coordinates. A set of shape functions that satisfies this condition is called m-complete.

Note that this requirement applies *at the element level* and involves *all* shape functions of the element.

Example 19.3. Suppose a displacement-based element is for a plane stress problem, in which $m = 1$. Then 1-completeness requires that the linear displacement field

$$u_x = \alpha_0 + \alpha_1 x + \alpha_2 y, \qquad u_y = \alpha_0 + \alpha_1 x + \alpha_2 y \tag{19.3}$$

be exactly represented for any value of the α coefficients. This is done by evaluating (19.3) at the nodes to form a displacement vector $\mathbf{u}^e$ and then checking that $\mathbf{u} = \mathbf{N}^e \mathbf{u}^e$ recovers exactly (19.3). Section 16.6 presents the details of this calculation for an arbitrary isoparametric plane stress element. That analysis shows that completeness is satisfied if the *sum of the shape functions is unity* and *the element is compatible*.

Example 19.4. For the plane beam problem, in which $m = 2$, the quadratic transverse displacement

$$v = \alpha_0 + \alpha_1 x + \alpha_2 x^2 \tag{19.4}$$

must be exactly represented over the element. This is easily verified in for the 2-node beam element developed in Chapter 13, because the assumed transverse displacement is a complete cubic polynomial in x. A complete cubic contains the quadratic (19.4) as special case.

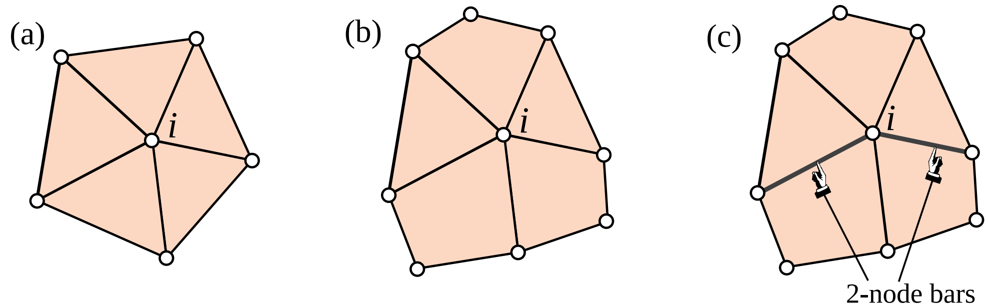

FIGURE 19.1. An element patch is the set of all elements attached to a patch node, labeled i. (a) illustrates a patch of triangles; (b) a mixture of triangles and quadrilaterals; (c) a mixture of triangles, quadrilaterals, and bars.

§19.3.2. Compatibility

To state this requirement succintly, it is convenient to introduce the concept of *element patch*, or simply *patch*. This is the set of all elements attached to a given node, called the *patch node*. The definition is illustrated in Figure 19.1, which shows three different kind of patches attached to patch node i in a plane stress problem. The patch of Figure 19.1(a) contains only one type of element: 3-node linear triangles. The patch of Figure 19.1(b) mixes two plane stress element types: 3-node linear triangles and 4-node bilinear quadrilaterals. The patch of Figure 19.1(c) combines three element types: 3-node linear triangles, 4-node bilinear quadrilaterals, and 2-node bars.

We define a finite element *patch trial function* as the union of shape functions activated by setting a degree of freedom at the patch node to unity, while all other freedoms are zero.

A patch trial function "propagates" only over the patch, and is zero beyond it. This property follows from the local-support requirement stated in §18.1: a shape function for node i should vanish on all sides or faces that do not include i.

With the help of these definitions we can enunciate the compatibility requirement as follows.

> Patch trial functions must be $C^{(m-1)}$ continuous between interconnected elements, and C^m piecewise differentiable inside each element.

If the variational index is $m = 1$, the patch trial functions must be C^0 continuous between elements, and C^1 inside elements.

A set of shape functions that satisfies the first requirement is called *conforming*. A conforming expansion that satisfies the second requirement is said to be of *finite energy*. Note that this condition applies at two levels: individual element, and element patch. An element endowed with conforming shape functions is said to be *conforming*. A conforming element that satisfies the finite energy requirement is said to be *compatible*.[3]

[3] The FEM literature is a bit fuzzy as regards these terms. It seems better to leave the qualifier "conforming" to denote interelement compatibility; informally "an element that gets along with its neighbors." The qualifier "compatible" is used in the stricter sense of conforming while possessing sufficient internal smoothness.

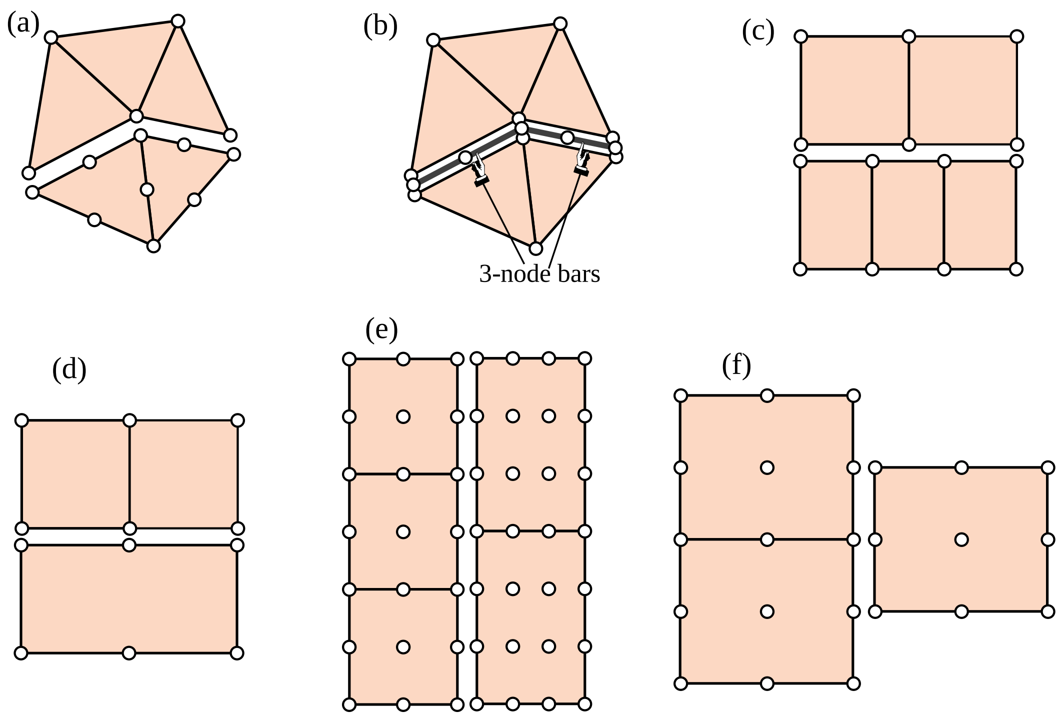

FIGURE 19.2. Examples of 2D non-matching meshes. Interelement boundaries that fail matching conditions are shown offset for visualization convenience. In (a,b,c) some nodes do not match. In (d,e,f) nodes and DOFs match but some sides do not, leading to violations of C^0 continuity.

Figures 19.1(b,c) illustrates the fact that one needs to check the possible connection of *matching elements* of different types and possibly different dimensionality.

§19.3.3. Matching and Non-Matching Meshes

As stated, compatibility refers to the *complete finite element mesh* because mesh trial functions are a combination of patch trial functions, which in turn are the union of element shape functions. This generality poses some logistical difficulties because the condition is necessarily mesh dependent. Compatibility can be checked at the *element level* by restricting attention to *matching meshes*. A matching mesh is one in which adjacent elements share sides, nodes and degrees of freedom, as in the patches shown in Figure 19.1.

For a matching mesh it is sufficient to restrict consideration first to a pair of adjacent elements, and then to the side shared by these elements. Suppose that the variation of a shape function *along that side* is controlled by k nodal values. Then a polynomial variation of order up to $k-1$ in the natural coordinate(s) can be specified uniquely over the side. This is sufficient to verify interelement compatibility for $m=1$, implying C^0 continuity, if the shape functions are polynomials.

This simplified criterion is the one used in previous Chapters. Specific 2D examples were given in Chapters 15 through 18.

Remark 19.2. If the variational index is $m = 2$ and the problem is multidimensional, as in the case of plates and shells, the check is far more involved and trickier because continuity of *normal derivatives along a side* is involved. This practically important scenario is examined in advanced FEM treatments. The case of non-polynomial shape functions is, on the other hand, of little practical interest.

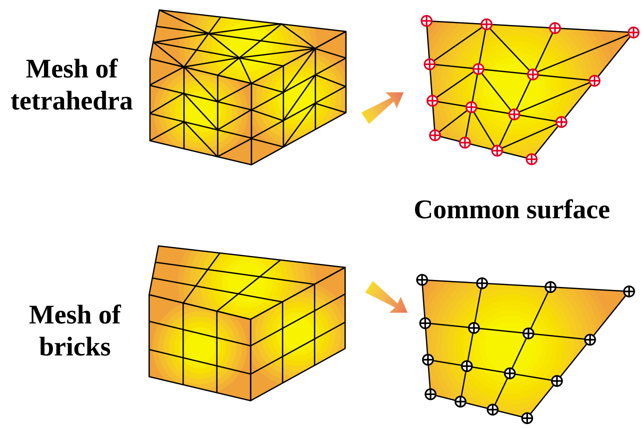

FIGURE 19.3. Example of a 3D non-matching mesh. Top portion discretized with tetrahedra, lower portion with bricks. Nodes and boundary-quad edges and DOFs match, but element types are different, leading to violation of C^0 continuity.

A mesh that does not satisfy the matching criteria stated above is called a *nonmatching mesh*. Several two-dimensional examples are shown in Figure 19.2. As can be seen there is a wide range of possibilities: nonmatching nodes, matching nodes but different element types, etc. Figure 19.3 depicts a three-dimensional example, in which case even more variety can be expected.

Nonmatching meshes are the rule rather than the exception in contact and impact problems (which, being geometrically nonlinear, are outside the scope of this book). See Figure 19.4 illustrates what happens in a problem of slipping contact.

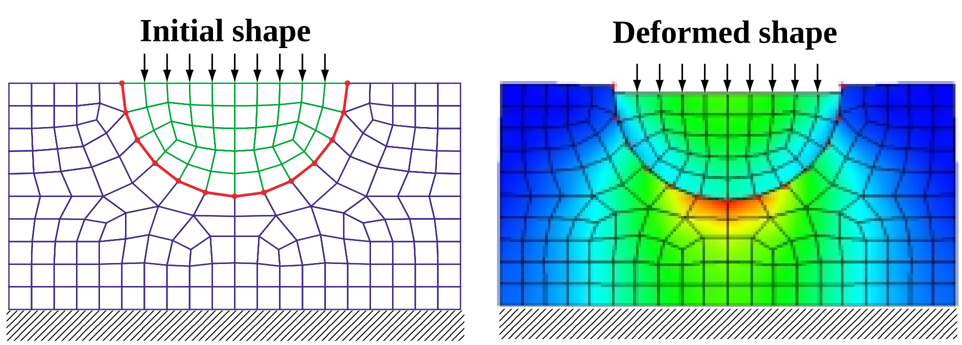

FIGURE 19.4. In contact and impact problems, matching meshes are the exception rather than the rule. Even if the meshes match at initial contact, slipping may produce a nonmatching mesh in the deformed configuration, as illustrated in the figure.

In multiphysics simulations nonmatching meshes are common, since they are often prepared separately for the different physical components, as illustrated in Figure 19.5.

§19.4. Stability

Stability may be informally characterized as ensuring that the finite element model enjoys the same solution uniqueness properties of the analytical solution of the mathematical model. For example, if the only motions that produce zero internal energy in the mathematical model are rigid body motions,

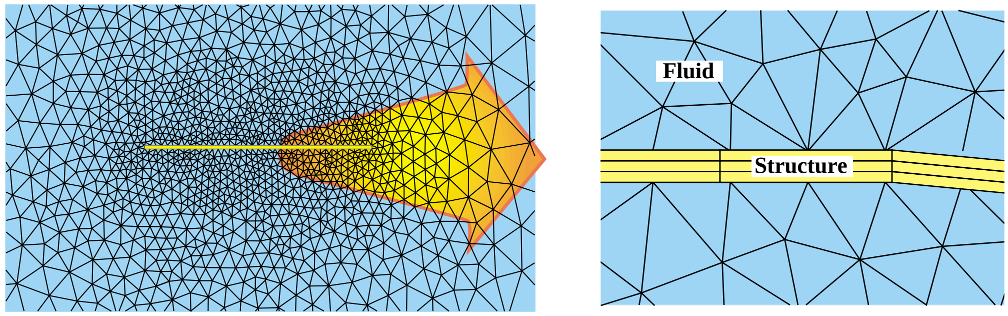

FIGURE 19.5. Nonmatching meshes are common in multiphysics problems, as in this example of fluid-structure interaction (FSI). Two-dimensional model to simulate flow around a thin plate. If the meshes are independently prepared node locations will not generally match.

the finite element model must inherit that property. Since FEM can handle arbitrary assemblies of elements, including individual elements, this property is required to hold at the element level.

In the present outline we are concerned with stability at the element level. Stability is not a property of shape functions *per se* but of the implementation of the element as well as its geometrical definition. It involves two subordinate requirements: rank sufficiency, and Jacobian positiveness. Of these, rank sufficiency is the most important one.

§19.4.1. Rank Sufficiency

The element stiffness matrix must not possess any zero-energy kinematic mode other than rigid body modes.

This can be mathematically expressed as follows. Let n_F be the number of element degrees of freedom, and n_R be the number of independent rigid body modes. Let r denote the rank of $\mathbf{K}^e$. The element is called *rank sufficient* if $r = n_F - n_R$ and *rank deficient* if $r < n_F - n_R$. In the latter case, the *rank deficiency* is defined by

$$d = (n_F - n_R) - r \tag{19.5}$$

If an isoparametric element is numerically integrated, let n_G be the number of Gauss points, while n_E denotes the order of the stress-strain matrix $\mathbf{E}$. Two additional assumptions are made:

(i) The element shape functions satisfy completeness in the sense that the rigid body modes are exactly captured by them.

(ii) Matrix $\mathbf{E}$ is of full rank.

Then each Gauss point adds n_E to the rank of $\mathbf{K}^e$, up to a maximum of $n_F - n_R$. Hence the rank of $\mathbf{K}^e$ will be

$$r = \min(n_F - n_R, n_E\, n_G) \tag{19.6}$$

To attain rank sufficiency, $n_E n_G$ must equal or exceed $n_F - n_R$:

$$\boxed{n_E\, n_G \geq n_F - n_R} \tag{19.7}$$

from which the appropriate Gauss integration rule can be selected.

In the plane stress problem, $n_E = 3$ because $\mathbf{E}$ is a 3×3 matrix of elastic moduli; see equation $(14.5)_2$. Also $n_R = 3$. Consequently $r = \min(n_F - 3, 3n_G)$ and $3n_G \geq n_F - 3$.

Table 19.1 Rank-sufficient Gauss Rules for Some Plane Stress Elements

Element	n	n_F	$n_F - 3$	Min n_G	Recommended rule
3-node triangle	3	6	3	1	centroid*
6-node triangle	6	12	9	3	3-point rules*
10-node triangle	10	20	17	6	6-point rule*
4-node quadrilateral	4	8	5	2	2 x 2
8-node quadrilateral	8	16	13	5	3 x 3
9-node quadrilateral	9	18	15	5	3 x 3
16-node quadrilateral	16	32	29	10	4 x 4

* These triangle integration rules are introduced in §24.2.

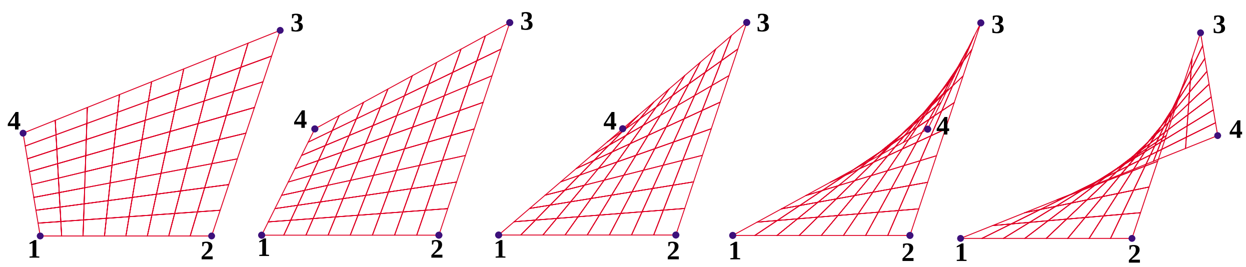

FIGURE 19.6. Effect of displacing node 4 of the four-node bilinear quadrilateral shown on the leftmost picture, to the right.

Remark 19.3. The fact that each Gauss point adds $n_E n_G$ to the rank can be proven considering the following property. Let $\mathbf{B}$ be a $n_E \times n_F$ rectangular real matrix with rank $r_B \leq n_E$, and $\mathbf{E}$ an $n_E \times n_E$ positive-definite (p.d.) symmetric matrix. Then the rank of $\mathbf{B}^T\,\mathbf{E}\,\mathbf{B}$ is r_B. Proof: let $\mathbf{u} \neq \mathbf{0}$ be a non-null n_F-vector. If $\mathbf{B}^T\,\mathbf{E}\,\mathbf{B}\,\mathbf{u} = \mathbf{0}$ then $0 = \mathbf{u}^T\mathbf{B}^T\,\mathbf{E}\,\mathbf{B}\,\mathbf{u} = ||\mathbf{E}^{1/2}\,\mathbf{B}\,\mathbf{u}||$. Therefore $\mathbf{B}\,\mathbf{u} = \mathbf{0}$. Identify now $\mathbf{B}$ and $\mathbf{E}$ with the strain-displacement and stress-strain (constitutive) matrix, respectively. In the plane stress case $n_E = 3$, $n_F = 2n > 3$ is the number of element freedoms. Thus $\mathbf{B}$ has rank 3 and *a fortiori* $\mathbf{B}^T\mathbf{E}\,\mathbf{B}$ must also have rank 3 since $\mathbf{E}$ is p.d. At each Gauss point i a contribution of $w_i\mathbf{B}^T\,\mathbf{E}\,\mathbf{B}$, which has rank 3 if $w_i > 0$, is added to $\mathbf{K}^e$. By a theorem of linear algebra, the rank of $\mathbf{K}^e$ increases by 3 until it reaches $n_F - n_R$.

Example 19.5. Consider a plane stress 6-node quadratic triangle. Then $n_F = 2 \times 6 = 12$. To attain the proper rank of $12 - n_R = 12 - 3 = 9$, $n_G \geq 3$. A 3-point Gauss rule, such as the midpoint rule defined in §24.2, makes the element rank sufficient.

Example 19.6. Consider a plane stress 9-node biquadratic quadrilateral. Then $n_F = 2 \times 9 = 18$. To attain the proper rank of $18 - n_R = 18 - 3 = 15$, $n_G \geq 5$. The 2×2 product Gauss rule is insufficient because $n_G = 4$. Hence a 3×3 rule, which yields $n_G = 9$, is required to attain rank sufficiency.

Table 19.1 collects rank-sufficient Gauss integration rules for some widely used plane stress elements with n nodes and $n_F = 2n$ freedoms.

§19.4.2. Jacobian Positiveness

The geometry of the element should be such that the determinant $J = \det \mathbf{J}$ of the Jacobian matrix defined[4] in §17.2, is positive everywhere. As illustrated in Equation (17.20), J characterizes the local metric of the element natural coordinates.

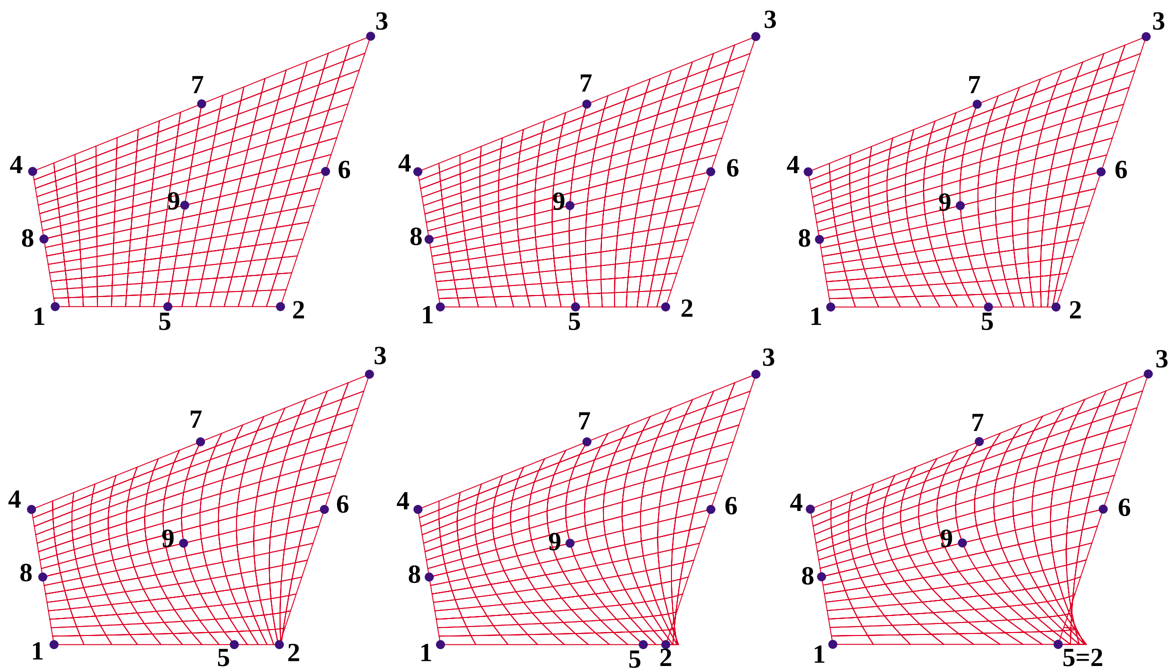

FIGURE 19.7. Effect of moving midpoint 5 of a 9-node biquadratic quadrilateral tangentially toward corner 2.

For a three-node triangle J is constant and in fact equal to $2A$. The requirement $J > 0$ is equivalent to saying that corner nodes must be positioned and numbered so that a positive area $A > 0$ results. This is called a *convexity condition*. It is easily checked by a finite element program.

But for 2D elements with more than 3 nodes distortions may render *portions* of the element metric negative. This is illustrated in Figure 19.6 for a 4-node quadrilateral in which node 4 is gradually moved to the right. The quadrilateral gradually morphs from a convex figure into a nonconvex one. The center figure is a triangle; note that the metric near node 4 is badly distorted (in fact $J = 0$ there) rendering the element unacceptable. This clearly contradicts the erroneous advice of some FE books, which state that quadrilaterals can be reduced to triangles as special cases, thereby rendering triangular elements unnecessary.

For higher order elements proper location of corner nodes is not enough. The non-corner nodes (midside, interior, etc.) must be placed sufficiently close to their natural locations (midpoints,

[4] This definition applies to quadrilateral elements. The Jacobian determinant of an arbitrary triangular element is defined in §24.2.

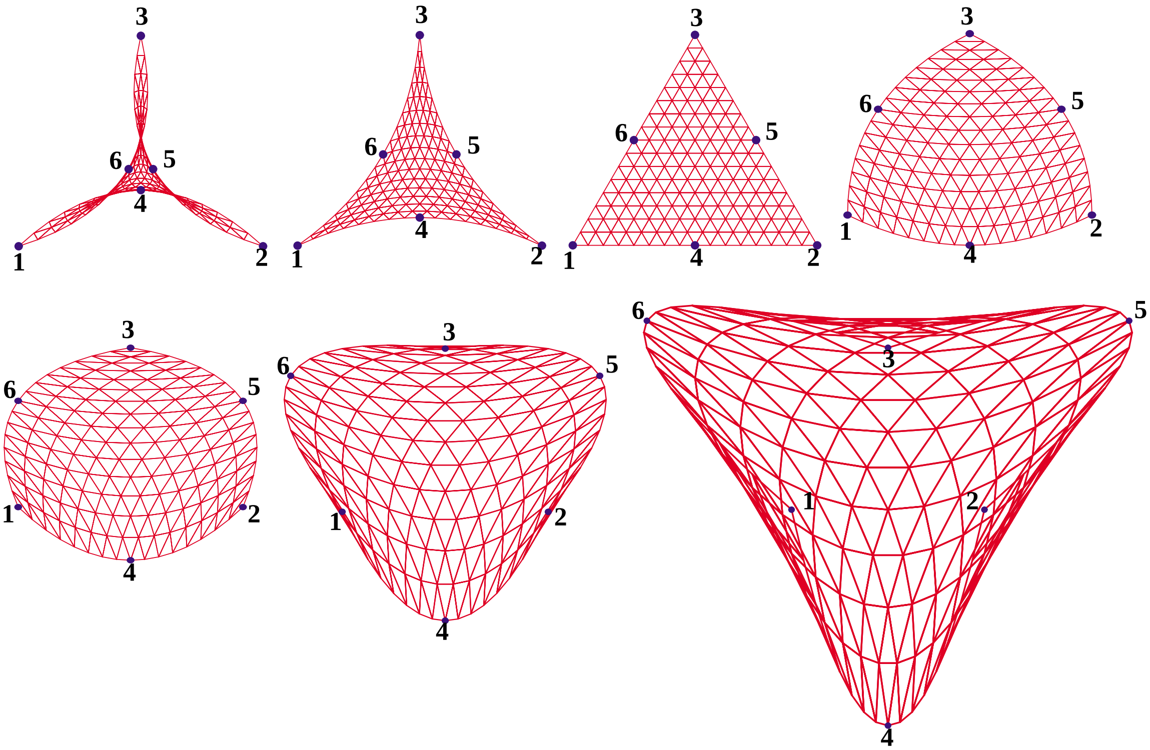

FIGURE 19.8. Effect of displacing midpoints 4, 5 and 6 of an equilateral 6-node triangle along the midpoint normals. Motion is inwards in first two top frames, outwards in the last four. In the lower leftmost picture nodes 1 through 6 lie on a circle.

centroids, etc.) to avoid violent local distortions. The effect of midpoint motions in quadratic elements is illustrated in Figures 19.7 and 19.8.

Figure 19.7 depicts the effect of moving midside node 5 tangentially in a 9-node quadrilateral element while keeping all other 8 nodes fixed. When the location of 5 reaches the quarter-point of side 1-2, the metric at corner 2 becomes singular in the sense that $J = 0$ there. Although this is disastrous in ordinary FE work, it has applications in the construction of special "crack" elements for linear fracture mechanics.

Displacing midside nodes normally to the sides is comparatively more forgiving, as illustrated in Figure 19.8. This depicts a 6-node equilateral triangle in which midside nodes 4, 5 and 6 are moved inwards and outwards along the normals to the midpoint location. As shown in the lower left picture, the element may be even morphed into a "parabolic circle" (meaning that nodes 1 through 6 lie on a circle) without the metric breaking down.

Notes and Bibliography

The literature on the mathematics of finite element methods has grown exponentially since the monograph of Strang and Fix [705]. This is very readable but out of print. A more up-to-date exposition is the textbook by Szabo and Babuska [721]. The subjects collected in this Chapter tend to be dispersed in recent monographs and obscured by overuse of functional analysis.

References

Referenced items have been moved to Appendix R.

Homework Exercises for Chapter 19
FEM Convergence Requirements

EXERCISE 19.1 [D:20] Explain why the two-dimensional meshes pictured in Figure 19.2(d,e,f) fail interelement compatibility although nodes and DOFs match.

EXERCISE 19.2 [A:20] The isoparametric definition of the straight 3-node bar element in its local system $\bar{x}$ is

$$\begin{bmatrix} 1 \\ \bar{x} \\ \bar{v} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix}. \tag{E19.1}$$

Here ξ is the isoparametric coordinate that takes the values -1, 1 and 0 at nodes 1, 2 and 3, respectively, while N_1^e, N_2^e and N_3^e are the shape functions found in Exercise 16.3 and listed in (E16.2).

For simplicity, take $\bar{x}_1 = 0$, $\bar{x}_2 = L$, $\bar{x}_3 = \frac{1}{2}L + \alpha L$. Here L is the bar length and α a parameter that characterizes how far node 3 is away from the midpoint location $\bar{x} = \frac{1}{2}L$. Show that the minimum α's (minimal in absolute value sense) for which $J = d\bar{x}/d\xi$ vanishes at a point in the element are $\pm 1/4$ (the quarter-points). Interpret this result as a singularity by showing that the axial strain becomes infinite at a an end point. (This result has application in fracture mechanics modeling.)

EXERCISE 19.3 [A:15] Consider one dimensional bar-like elements with n nodes and 1 degree of freedom per node so $n_F = n$. The correct number of rigid body modes is 1. Each Gauss integration point adds 1 to the rank; that is $N_E = 1$. By applying (19.7), find the minimal rank-preserving Gauss integration rules with p points in the longitudinal direction if the number of node points is $n = 2$, 3 or 4.

EXERCISE 19.4 [A:20] Consider three dimensional solid "brick" elements with n nodes and 3 degrees of freedom per node so $n_F = 3n$. The correct number of rigid body modes is 6. Each Gauss integration point adds 6 to the rank; that is, $N_E = 6$. By applying (19.7), find the minimal rank-preserving Gauss integration rules with p points in each direction (that is, $1\times1\times1$, $2\times2\times2$, etc) if the number of node points is $n = 8$, 20, 27, or 64. Partial answer: for $n = 27$ the minimal rank preserving rule is $3 \times 3 \times 3$.

EXERCISE 19.5 [A/C:35] (Requires use of a CAS help to be tractable). Repeat Exercise 19.2 for a 9-node plane stress element. The element is initially a perfect square, nodes 5,6,7,8 are at the midpoint of the sides 1–2, 2–3, 3–4 and 4–1, respectively, and 9 at the center of the square. Displace 5 tangentially towards 2 until the Jacobian determinant at 2 vanishes. This result is important in the construction of "singular elements" for fracture mechanics.

EXERCISE 19.6 [A/C:35] Repeat Exercise 19.5 but moving node 5 along the normal to the side. Discuss the range of motion for which $\det \mathbf{J} > 0$ within the element.

EXERCISE 19.7 [A:20] Discuss whether the deVeubeke triangle presented in Chapter 15 satisfies completeness and interelement-compatbility requirements.