Let $\underline{w}$ denote the state. A general form of governing equation is assumed to be

$$\underline{r} = \underline{f}(\underline{w}) - \underline{h}(\dot{\underline{w}}) - \underline{g}(\ddot{\underline{w}})$$

The discrete form is assumed to be

$$\underline{r}^n = \underline{f}(\underline{w}^n; \underline{w}^{n-1}) - \underline{h}(\dot{\underline{w}}^n; \dot{\underline{w}}^{n-1}) - \underline{g}(\ddot{\underline{w}}^n; \ddot{\underline{w}}^{n-1})$$

Here $\underline{f}^n, \underline{h}^n, \underline{g}^n$ only depend on $\underline{w}^{n-1}, \dot{\underline{w}}^{n-1}, \ddot{\underline{w}}^{n-1}$ if the time discretization is based on intermediate states $\underline{w}^{n-\alpha}, \dot{\underline{w}}^{n-\beta}, \ddot{\underline{w}}^{n-\gamma}$.

Lagrangian:

$$L = \sum_{K=1}^{N} \left[J^K + \underline{\lambda}^{KT} \underline{r}^K + \underline{\lambda}_1^{KT} \underline{c}_1^K + \underline{\lambda}_2^{KT} \underline{c}_2^K \right]$$ ($\underline{c}_1$ and $\underline{c}_2$ are defined by time discretization)

Adjoint equations at step 'n':

$$\frac{\partial L}{\partial \underline{w}^n} = \frac{\partial J^n}{\partial \underline{w}^n} + \frac{\partial J^{n+1}}{\partial \underline{w}^n} + \underline{\lambda}^{nT} \frac{\partial \underline{r}^n}{\partial \underline{w}^n} + \underline{\lambda}^{n+1\,T} \frac{\partial \underline{r}^{n+1}}{\partial \underline{w}^n} + \underline{\lambda}_1^{nT} \frac{\partial \underline{c}_1^n}{\partial \underline{w}^n} + \underline{\lambda}_1^{n+1\,T} \frac{\partial \underline{c}_1^{n+1}}{\partial \underline{w}^n} + \ldots = 0$$

$$\frac{\partial L}{\partial \dot{\underline{w}}^n} = \frac{\partial J^n}{\partial \dot{\underline{w}}^n} + \frac{\partial J^{n+1}}{\partial \dot{\underline{w}}^n} + \underline{\lambda}^{nT} \frac{\partial \underline{r}^n}{\partial \dot{\underline{w}}^n} + \underline{\lambda}^{n+1\,T} \frac{\partial \underline{r}^{n+1}}{\partial \dot{\underline{w}}^n} + \underline{\lambda}_1^{nT} \frac{\partial \underline{c}_1^n}{\partial \dot{\underline{w}}^n} + \underline{\lambda}_1^{n+1\,T} \frac{\partial \underline{c}_1^{n+1}}{\partial \dot{\underline{w}}^n} + \ldots = 0$$

$$\frac{\partial L}{\partial \ddot{\underline{w}}^n} = \frac{\partial J^n}{\partial \ddot{\underline{w}}^n} + \frac{\partial J^{n+1}}{\partial \ddot{\underline{w}}^n} + \underline{\lambda}^{nT} \frac{\partial \underline{r}^n}{\partial \ddot{\underline{w}}^n} + \underline{\lambda}^{n+1\,T} \frac{\partial \underline{r}^{n+1}}{\partial \ddot{\underline{w}}^n} + \ldots = 0$$

The element needs to provide $\frac{\partial \underline{r}^{(n+i)}}{\partial \underline{w}^n}, \frac{\partial \underline{r}^{(n+i)}}{\partial \dot{\underline{w}}^n}, \frac{\partial \underline{r}^{(n+i)}}{\partial \ddot{\underline{w}}^n}$. The objective function provides the terms containing J and the scheme builds the system at the element level including contributions from c_1 and c_2.

Example:

The VMS element doesn't conform to the above governing equation since stabilization introduces velocity into Mass matrix and $\underline{g}^n = \underline{g}^n(\dot{\underline{w}}^n, \ddot{\underline{w}}^n; \ddot{\underline{w}}^{n-1})$.

$$\underline{r}^n = -\underline{D}(\dot{\underline{w}}^n)\dot{\underline{w}}^{n+\alpha_f-1} - \underline{\underline{M}}(\dot{\underline{w}}^n)\ddot{\underline{w}}^{n+\alpha_m-1} = -\underline{h}(\dot{\underline{w}}^n; \dot{\underline{w}}^{n-1}) - g(\dot{\underline{w}}^n; \ddot{\underline{w}}^n; \ddot{\underline{w}}^{n-1})$$

$$L = \sum_{K=1}^{N} \left[J^K + \underline{\lambda}^{KT}\left(\alpha_f \underline{h}^K + (1-\alpha_f)\underline{h}^{K-1} + \alpha_m \underline{\underline{M}}^K \ddot{\underline{w}}^K + (1-\alpha_m)\underline{\underline{M}}^K \ddot{\underline{w}}^{K-1}\right) + \underline{\lambda}_2^{KT}\left(\ddot{\underline{w}}^{K-1} - \frac{\gamma}{(\gamma-1)}\ddot{\underline{w}}^K + \frac{\dot{\underline{w}}^K - \dot{\underline{w}}^{K-1}}{(\gamma-1)\Delta t}\right)\right]$$

$$\frac{\partial L^T}{\partial \dot{\underline{w}}^n}: \frac{1}{(\gamma-1)\Delta t}(\underline{\lambda}_2^n - \underline{\lambda}_2^{n+1}) + \partial_{\dot{\underline{w}}^n}\underline{h}^{nT}\left[\alpha_f \underline{\lambda}^n + (1-\alpha_f)\underline{\lambda}^{n+1}\right] + \underbrace{\partial_{\dot{\underline{w}}^n}\{\underline{\underline{M}}^n \ddot{\underline{w}}^{n+\alpha_m-1}\}^T}_{\partial_{\dot{\underline{w}}^n} g^{nT}} \underline{\lambda}^n = -\frac{\partial J^n}{\partial \dot{\underline{w}}^n} - \frac{\partial J^{n+1}}{\partial \dot{\underline{w}}^n}$$

$$\frac{\partial L^T}{\partial \ddot{\underline{w}}^n}: -\frac{\gamma}{(\gamma-1)}\underline{\lambda}_2^n + \underline{\lambda}_2^{n+1} + \alpha_m \underbrace{\underline{\underline{M}}^{nT}}_{\partial_{\ddot{\underline{w}}^n} g^n} \underline{\lambda}^n + \underbrace{(1-\alpha_m)\underline{\underline{M}}^{nT}}_{\partial_{\ddot{\underline{w}}^n} g^{n+1}} \underline{\lambda}^{n+1} = -\frac{\partial J^{nT}}{\partial \ddot{\underline{w}}^n} - \frac{\partial J^{n+1\,T}}{\partial \ddot{\underline{w}}^n}$$

If $\alpha_f \neq 1$ we need element functions to get $\partial_{\dot{\underline{w}}^n}\underline{h}^{nT}$ and $\partial_{\dot{\underline{w}}^n}\{\underline{\underline{M}}^n \ddot{\underline{w}}^{n+\alpha_m-1}\}^T$ separately to combine in the scheme (we want it to be possible to keep element and time discretization separate). But, the element function which provides $\partial \underline{r}^{(n+i)}/\partial \dot{\underline{w}}^n = -\partial_{\dot{\underline{w}}^n}\underline{h}^{nT} - \partial_{\dot{\underline{w}}^n}\{\underline{\underline{M}}^n \ddot{\underline{w}}^{n+\alpha_m-1}\}^T$ doesn't do this. This is an exceptional case of the VMS formulation which makes g^n dependent on $\dot{\underline{w}}^n$. If we restrict to Bossak ($\alpha_f = 1$) then it is not a problem. Also the element does not need to know α_m to calculate $\ddot{\underline{w}}^{n+\alpha_m-1}$ since the acceleration is already saved in this format during primal solution.

We can eliminate the 'order' parameter, which is confusing, if we define separate functions:

Element::AdjointMatrix(int step=0) $\longmapsto \frac{\partial \underline{r}^{(n+step)T}}{\partial \underline{w}^n} = \frac{\partial \underline{f}^{(n+step)T}}{\partial \underline{w}^n}$

Element::AdjointDampingMatrix(int step=0) $\longmapsto -\frac{\partial \underline{r}^{(n+step)T}}{\partial \dot{\underline{w}}^n} = \frac{\partial \underline{h}^{(n+step)T}}{\partial \dot{\underline{w}}^n}$

Element::AdjointMassMatrix(int step=0) $\longmapsto -\frac{\partial \underline{r}^{(n+step)T}}{\partial \ddot{\underline{w}}^n} = \frac{\partial \underline{g}^{(n+step)T}}{\partial \ddot{\underline{w}}^n}$